Aspects théoriques et algorithmiques de l'optimisation semidéfinie.

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Résumé

Le but de cette thèse est d'étudier des différents sujets de la programmation semidéfinie non linéaire (SDP). Ainsi, dans les deux premiers chapitres nous presentons certains aspects algorithmiques, dans les chapitres 3 et 4 nous travaillons sur des aspects théoriques comme l'analyse de perturbations de ce problème.

Le premier chapitre développe un algorithme global qui étend l'algorithme local *S-SDP*. Cet algorithme est basé sur une fonction de pénalisation de Han et une stratégie de recherche linéaire. Le second chapitre est consacré à l'étude des méthodes de pénalisation ou fonctions barrière pour résoudre des problèmes semidéfinis convexes. Nous démontrons la convergence des suites primale et duale obtenues par cette méthode. De plus, nous étudions l'algorithme à deux paramètres en étendant les résultats connus dans le cadre restreint de la programmation convexe usuelle.

Dans une deuxième partie, constituée des chapitres 3 et 4, nous nous intéressons à la caractérisation de la propriété des solutions fortement régulières en fonction des certaines conditions optimales de deuxième ordre. Ainsi, dans le troisième chapitre nous nous consacrons au problème de second-ordre, lequel est un cas particulier du problème SDP, dont on obtient cette caractérisation. Enfin dans la chapitre 4, nous donnons des conditions nécessaires et suffisantes pour la condition de régularité forte dans le cas SDP, en revanche, sa caractérisation reste un problème ouvert.

Mots Clés. Optimisation Semidéfinie Nonlinéaire, Optimisation de Second-ordre, Régularité Forte, Méthodes de Pénalisation, Reduction des Contraintes, Analyse Convexe.

Abstract

This work deals with different subjects on nonlinear semidefinite programming (SDP). Thus, while in the first two chapters we show some algorithmic aspects, in chapters 3 and 4 we study theoretical aspects as the perturbation analysis of this problem.

In the first chapter we develop a global algorithm that extends the local one *S-SDP*. This algorithm is based on a Han penalty function and a line search strategy. The second chapter is focused on penalty and barrier methods for solving convex semidefinite programming problems. We prove the convergence of primal and dual sequences obtained by this method. Moreover, we study the two parameters algorithm and extend to semidefinite case the results that are known in usual convex programming.

In the second part, that involves chapters 3 and 4, we are interested on the characterization of the strong regularity property in function of second-order optimality conditions. So, in chapter 3, we mainly deal with second-order cone programming problems, whose are a particular instance of semidefinite programming problems. We thus obtain a characterization in this particular case. Finally in chapter 4, we give necessary and sufficient conditions to obtain the strong regularity property in the semidefinite case. However, its characterization is still an open problem.

Keywords. Nonlinear Semidefinite Programming, Second-order Cone Programming, Strong Regularity, Penalty Methods, Reduction Approach, Convex Analysis.

Introduction

Ce travail de thèse est dédié à l'étude de différents sujets en programmation semidéfinie. Ce problème d'optimisation consiste en la minimisation d'une fonction de coût f, dont l'ensemble réalisable sont tous les vecteurs x tel que $G(x)$ soit une matrice semidéfinie négative dans le sens classique de l'analyse matricielle. Ici, l'opérateur G est à valeurs matricielles symétriques.

Concrètement, dans ce travail nous considérons le problème :

$$\min_{x \in I\!R^n} \{f(x)\,;\, G(x) \preceq 0\}, \tag{SDP}$$

où $f : I\!R^n \to I\!R$, $G : I\!R^n \to S^m$, S^m dénote l'espace vectoriel des matrices symétriques de taille m dont on considère le produit scalaire $A \cdot B := \mathrm{trace}(AB) = \sum_{i,j=1}^m A_{ij}B_{ij}$ avec $A = (A_{ij}), B = (B_{ij}) \in S^m$, et $\preceq$ est l'ordre inducé par S^m_- le cône des matrices semidéfinies négatives, c'est à dire, $A \preceq B$ ssi $A - B \in S^m_-$. La régularité des opérateurs f et G sera spécifiée dans chaque chapitre.

L'intérêt apparu par ce type de problème est justifié par ses multiples applications, on peut mentionner par exemple l'optimisation combinatoire, l'optimisation robuste, les applications en statistique, etc. (voir, par exemple, [35, 36]). Pour cela, on a considéré principalement le problème SDP linéaire

$$\min_{x \in I\!R^n} \{f(x) := c^\top x\,;\, G(x) := A_0 + \sum_{i=1}^n x_i A_i \preceq 0\}, \tag{LSDP}$$

où $c \in I\!R^n$ et $A_i \in S^m$ pour tout indice $i = 0, ..., m$. Toutefois, on trouve des applications dont la formulation linéaire (LSDP) n'est plus suffisante pour les modéliser (cf. [15, 19, 24]).

Dans la prochaine section, nous verrons principalement des exemples associés à la formulation linéaire (LSDP). Pour étudier les exemples associés à la formulation nonlinéaire, nous nous adressons au lecteur aux articles mentionnés ci-dessus.

Exemples

Programmation Linéaire

Considérons le problème de programmation linéaire

$$\min_{x \in \mathbb{R}^n} \{c^\top x \,;\, Ax \leq b\}, \tag{LP}$$

où $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ et $A \in \mathbb{R}^{m \times n}$.

En effet, on écrit $A_0 := -\operatorname{diag}(b)$ et $A_i := \operatorname{diag}(a_i)$, $i = 1,...,n$, où les vecteurs a_i sont les colonnes de la matrice A, et pour tout vecteur $a \in \mathbb{R}^m$ on a dénoté par $\operatorname{diag}(a)$ à la matrice diagonal de taille m dont ses composants sont les composants de a. Alors, on retrouve la formulation (LSDP).

Programmation Nonlinéaire Quasi-convexe

Considérons le problème

$$\min_{x \in \mathbb{R}^n} \left\{ \frac{(c^\top x)^2}{d^\top x} \,;\, Ax \leq b \right\}, \tag{QCP}$$

où $c, d \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ et $A \in \mathbb{R}^{m \times n}$, et on a supposé que $d^\top x > 0$ pour tout x point réalisable de (QCP). Ce problème est nonlinéaire, pourtant on verra qu'il existe une modélisation linéaire semidéfinie (cf. (LSDP)).

D'après les *compléments de Schur*, si $d^\top x > 0$ on obtient l'équivalence

$$\eta \geq \frac{(c^\top x)^2}{d^\top x} \qquad \text{ssi} \qquad \begin{pmatrix} \eta & c^\top x \\ c^\top x & d^\top x \end{pmatrix} \succeq 0.$$

Ceci implique que le problème (QCP) peut s'écrire comme le problème semidéfini

$$\min_{x,\eta} \left\{ \eta \,;\, \begin{bmatrix} \operatorname{diag}(Ax - b) & 0 \\ 0 & -\begin{pmatrix} \eta & c^\top x \\ c^\top x & d^\top x \end{pmatrix} \end{bmatrix} \preceq 0 \right\}.$$

Notons que dans ce cas-ci toutes les matrices A_i, dans la formulation (LSDP), sont diagonales par bloc. On a m blocs de taille 1×1 et un seul bloc de taille 2×2.

Programmation de Second Ordre

Nous considérons le problème de programmation sur un cône de second ordre ou cône de Lorentz :

$$\min_{x\in I\!R^n, s^j\in I\!R^{m_j+1}} \{f(x)\,;\, g^j(x) = s^j, (s^j)_0 \geq \|\bar{s}^j\|,\ j = 1,\ldots,J\}, \qquad \text{(SOCP)}$$

où $f : I\!R^n \to I\!R$ et $g^j : I\!R^n \to I\!R^{m_j+1}$, $j = 1,\ldots,J$. Nous utilisons la convention usuelle qui consiste en indexer les composants des vecteurs appartenant à $I\!R^{m_j+1}$ de 0 à m_j, pendant que les vecteurs dans $I\!R^n$ sont indexés de 1 à n. Alors, étant donné $s \in I\!R^{m_j+1}$, on écrit $\bar{s} := (s_1,\ldots,s_{m_j})^\top$. Dénotons le cône du second ordre par

$$Q_{m+1} = \{s \in I\!R^{m+1}\ :\ s_0 \geq \|\bar{s}\|\},$$

et posons $\mathcal{Q} := \Pi_{j=1}^J Q_{m_j+1}$. Le problème (SOCP) peut s'écrire alors comme $\min_{x\in I\!R^n}\{f(x)\,;\, g(x) \in \mathcal{Q}\}$, avec $g(x) := (g^1(x),...,g^J(x))$.

Plusieurs problèmes d'optimisation peuvent être aussi modélisés comme un problème de second ordre (SOCP). Un article récent à ce sujet est Alizadeh et Goldfarb [2].

Le cône de second ordre Q_{m+1} peut être décrit en utilisant une inégalité matricielle grâce à l'équivalence suivante (voir par exemple [2]) :

$$s \in Q_{m+1} \qquad \text{ssi} \qquad \mathrm{Arw}(s) := \begin{pmatrix} s_0 & \bar{s}^\top \\ \bar{s} & s_0 I_m \end{pmatrix} \succeq 0, \qquad (1)$$

où I_m dénote la matrice identité dans $I\!R^{m\times m}$, et $\mathrm{Arw}(s)$ est la *matrice flèche* du vecteur s. Donc, le problème (SOCP) s'écrit trivialement comme le problème semidéfini suivant :

$$\min_{x\in I\!R^n}\{f(x)\,;\, G^j(x) := \mathrm{Arw}(g^j(x)) \succeq 0,\ j = 1,\ldots,J\}.$$

Programmation Linéaire Robuste

L'idée de ce type de problèmes est de traiter des données incertaines. Plus précisément, nous considérons le problème linéaire (LP) dans lequel il y a une certaine incertitude ou variation des paramètres A, b et c. Pour simplifier cette idée, nous supposons que les donnés b et c sont fixes, et que chaque vecteur ligne $a_i^\top$ de A se trouve dans l'ellipsoïde

$$\mathcal{E}_i := \{\bar{a}_i + P_i u\ :\ \|u\| \leq 1\},$$

où $P_i = P_i^\top \succeq 0$. On obtient alors le problème linéaire robuste suivant

$$\min_{x\in I\!R^n}\{c^\top x\,;\, a_i^\top x \leq b_i \quad \forall a_i \in \mathcal{E}_i,\, i = 1,...,m\}. \qquad \text{(RLP)}$$

Nous allons d'abord montrer que le problème (RLP) peut s'écrire comme un problème (SOCP). En effet, la contrainte

$$a_i^\top x \leq b_i \;\text{ pour tout vecteur }\; a_i \in \mathcal{E}_i$$

est équivalente à

$$\max\{a_i^\top x \;:\; a_i \in \mathcal{E}_i\} = \bar{a}_i^\top x + \|P_i x\| \leq b_i,$$

laquelle est une contrainte du type

$$\begin{pmatrix} b_i - \bar{a}_i^\top x \\ P_i x \end{pmatrix} \in Q_{n+1}.$$

Ceci nous montre que le problème (RLP) s'écrit sous la forme (SOCP) dont la fonction f est linéaire et les fonctions g^j, $j = 1,\ldots,J$, sont linéaires affines.

Ensuite, nous réécrivons ce problème-ci comme un problème semidéfini linéaire (LSDP) en utilisant l'équivalence (1).

Optimisation Quadratique Non-convexe

Considérons par exemple le problème quadratique

$$\min_{x \in I\!R^n} \{f_0(x)\,;\; f_i(x) \leq 0,\; i = 1,\ldots,L\}, \tag{QP}$$

où $f_i(x) = x^\top A_i x + 2b_i^\top x + c_i$, $i = 0,\ldots,L$. Ici, les matrices $A_i \in S^m$ peuvent être indéfinies, et donc le problème (QP) est très difficile à résoudre. Par exemple, ce problème-ci inclut tous les problèmes d'optimisation avec une fonction objective et des fonctions contraintes polynomiales (cf. [29, Sect. 6.4.4]).

Dans la pratique, c'est très important d'avoir des bonnes estimations inférieures de la valeur optimale de (QP) qui soient calculables efficacement. Une manière d'obtenir ces estimations est de résoudre le problème semidéfini

$$\min_{t,\, \tau_i \in I\!R} \Big\{ t\,;\; \begin{pmatrix} A_0 & b_0 \\ b_0^\top & c_0 - t \end{pmatrix} + \tau_1 \begin{pmatrix} A_1 & b_1 \\ b_1^\top & c_1 \end{pmatrix} + \ldots + \tau_L \begin{pmatrix} A_L & b_L \\ b_L^\top & c_L \end{pmatrix} \succeq 0, \quad \tau_i \geq 0, \quad i = 1,\ldots,L \Big\}. \tag{$\overline{\text{QP}}$}$$

En effet, supposons que x est réalisable pour le problème (QP), c'est à dire

$$f_i(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^\top \begin{pmatrix} A_i & b_i \\ b_i^\top & c_i \end{pmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0 \quad \forall\, i = 1,\ldots,L$$

Alors, si les variables t et τ_i, $i = 1,\ldots,L$, satisfont les contraintes du problème ($\overline{\text{QP}}$), on déduit que

$$\begin{aligned} 0 &\leq \begin{bmatrix} x \\ 1 \end{bmatrix}^\top \left[\begin{pmatrix} A_0 & b_0 \\ b_0^\top & c_0 - t \end{pmatrix} + \tau_1 \begin{pmatrix} A_1 & b_1 \\ b_1^\top & c_1 \end{pmatrix} + \ldots + \tau_L \begin{pmatrix} A_L & b_L \\ b_L^\top & c_L \end{pmatrix} \right] \begin{bmatrix} x \\ 1 \end{bmatrix} \\ &= f_0(x) - t + \tau_1 f_1(x) + \ldots + \tau_L f_L(x) \\ &\leq f_0(x) - t. \end{aligned}$$

Donc, $t \le f_0(x)$ pour tout point x réalisable pour le problème (QP).

Par ailleurs, le problème *dual* de $(\overline{\mathrm{QP}})$ est donné par [36, Sect. 3] :

$$\min_{x\in \mathbb{R}^n,\, X\in S^n} \Big\{ A_0 \cdot X + 2b_0^\top x + c_0 \,;\; A_i \cdot X + 2b_i^\top x + c_i \le 0, \quad i = 1,\dots,L, \quad \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \succeq 0 \Big\}. \qquad (\overline{\mathrm{DQP}})$$

On note que la contrainte

$$\begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \succeq 0 \qquad (2)$$

équivaut à $X \succeq xx^\top$, qui peut être considérée comme une relaxation de la contrainte $X = xx^\top$. Alors, le problème $(\overline{\mathrm{DQP}})$ peut directement se considérer comme une relaxation du problème

$$\min_{x\in \mathbb{R}^n,\, X\in S^n} \Big\{ A_0 \cdot X + 2b_0^\top x + c_0 \,;\; A_i \cdot X + 2b_i^\top x + c_i \le 0,\; i = 1,\dots,L, \quad X = xx^\top \Big\},$$

qui est une formulation équivalente au problème (QP).

Optimisation Combinatoire

Regardons une application de l'approche décrite dans la section précédente. Considérons le problème quadratique

$$\min_{x\in \mathbb{R}^n} \{ x^\top A x + 2b^\top x \,;\; x_i^2 = 1,\; i = 1,\dots,n \}, \qquad ((\text{-}1,1)\text{–QP})$$

où $A \in S^n$ et $b \in \mathbb{R}^n$. Ce problème *NP-dur*. Pourtant, la contrainte entière $x_i^2 = 1$ (i.e. $x_i = \pm 1$) peut être relaxée comme $x_i^2 \ge 1$, donc, d'après (2), on sait que le problème semidéfini

$$\min_{x\in \mathbb{R}^n,\, X\in S^n} \Big\{ A \cdot X + 2b^\top x \,;\; X_{ii} = 1, \quad i = 1,\dots,n, \; \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \succeq 0 \Big\}. \qquad (\widehat{\mathrm{QP}})$$

nous donne une estimation inférieure de la valeur optimale du problème ((-1,1)–QP).

Goemans et Williamson [21] ont prouvé que pour le *problème de coupe maximum* (MAX-CUT problem), lequel est un cas spécifique du problème ((-1,1)–QP) où $b = 0$ et les composantes de la diagonal de la matrice A sont égaux à 0, l'estimation inférieure donnée par le problème $(\widehat{\mathrm{QP}})$ est au moins 14% sous optimale. Celle-ci est la meilleure estimation inférieure connue jusqu'à présent.

Résolution du Problème (SDP)

Pour le problème (SDP) non-convexe, on peut mentionner entre autres l'algorithme *S-SDP* (Sucessive SemiDefinite Programming) développé par Fares, Noll et Apkarian dans l'article [19], lequel est inspiré de l'algorithme *SQP* (Sequentally Quadratic Programming) pour la programmation mathématique non-linéaire classique (voir par exemple [8]). On prouve que cet algorithme converge quadratiquement si certaines hypothèses sont satisfaites.

Les mêmes auteurs proposent dans [18] un algorithme du type Lagrangien Augmenté pour résoudre un cas particulier du problème (SDP).

D'autre part, on trouve dans l'article de Jarre [24] un algorithme de point intérieur pour résoudre le problème (SDP) non-convexe.

Auslender propose dans l'article [5] une approche unifiée pour résoudre des problèmes convexes en utilisant des différents classes des fonctions barrières ou de penalisation. En particulier, on peut appliquer cette approche au problème (SDP) convexe.

Dans le cas (SDP) convexe, on peut aussi appliquer l'approche de point intérieur de Nesterov et Nemirovskii qui est basée sur des fonctions *auto concordantes* (cf. [11]).

Le problème (LSDP) est convexe, donc il peut être résolu dans un temps polynomial à n'importe quelle précision fixe en utilisant la méthode d'ellipsoïde introduite par Khachiyan [27].

Malheureusement, le temps de fonctionnement de cette méthode-ci est prohibitivement haut dans la pratique.

En revanche, les méthodes de points intérieurs, introduites originalement par Karmarkar [26], se sont avérées être les plus rapides dans la pratique. Dans la prochaine section nous présentons ces méthodes et ses principaux résultats. Pour étudier des autres algorithmes, nous nous adressons au lecteur à l'article de Todd [35], ou le livre "manuel" sur la programmation semidéfinie [37].

Méthode de Points Intérieurs

Les méthodes de point intérieurs pour la programmation semidéfinie linéaire ont été introduits par Nesterov et Nemirovskii (voir [11]). Voir aussi Alizadeh [1].

Réécrivons le problème (LSDP) de la manière suivante :

$$\min_{x\in I\!R^n,\, S\in S^m} \left\{ c^\top x \,;\, G(x) := A_0 + \sum_{i=1}^{n} x_i A_i = -S, \quad S \succeq 0 \right\}. \tag{LSDP}$$

Donc son problème dual est

$$\max_{Y\in S^m} \left\{ A_0 \cdot Y \,;\, A_i \cdot Y + c_i = 0,\ i = 1,\ldots,n, \quad Y \succeq 0 \right\}. \tag{DLSDP}$$

Notons que si on définit la fonction $x \to A(x) := \sum_{i=1}^{n} x_i A_i$ son opérateur adjoint est donné par

$$Z \in S^m \to \mathcal{A}(Z) = (A_1 \cdot Z, \ldots, A_n \cdot Z)^\top.$$

Écrivons alors les conditions d'optimalité associées aux problèmes (LSDP) et (DLSDP) :

$$G(x) + S = A_0 + \sum_{i=1}^{n} x_i A_i + S = 0\,; \quad S \succeq 0, \tag{3.a}$$

$$\mathcal{A}(Y) + c = 0\,; \quad Y \succeq 0, \tag{3.b}$$

$$SY = 0. \tag{3.c}$$

Dans cette section nous supposons que les problèmes (LSDP) et (DLSDP) sont tout les deux *strictement réalisables*, autrement dit, nous supposons qu'il existe (x,S,Y) satisfaisant les conditions (3.a) et (3.b), associées à la réalisabilité de ces problèmes, tel que $S, Y \succ 0$.

Cette hypothèse implique que les valeurs optimales des problèmes (LSDP) et (DLSDP) sont égaux (*dualité forte*), et que les ensembles solutions de ces problèmes sont nonvides et compacts.

Considérons une perturbation de la condition (3.c) de la forme $SY = \mu I_m$ où $\mu > 0$. Si on ignore les contraintes d'inégalité $S, Y \succeq 0$, on obtient le système des equations

$$F_\mu(x,S,Y) := \begin{pmatrix} G(x) + S \\ \mathcal{A}(Y) + c \\ SY - \mu I_m \end{pmatrix} = 0. \tag{4}$$

Sous l'hypothèse de strict réalisabilité, il existe une unique solution (x_μ, S_μ, Y_μ) pour tout $\mu > 0$ (voir par exemple [37, Chapitre 10]). Il est possible aussi de prouver que l'ensemble $\{(x_\mu, S_\mu, Y_\mu) \,:\, \mu > 0\}$ définit une courbe régulière paramétrée par μ, laquelle est usuellement appelée *le chemin central*.

Si nous résolvons (4) par une méthode de Newton, nous obtenons le système des equations linéaires suivant :

$$\sum_{i=1}^{n} \Delta x_i A_i + \Delta S = 0, \tag{5.a}$$

$$\mathcal{A}(\Delta Y) = 0, \tag{5.b}$$

$$\Delta S Y + S \Delta Y = \mu I_m - SY. \tag{5.c}$$

Car la matrice SY n'est pas forcement symétrique, le système (5) est composé par $m(m+1)/2+n+m^2$ équations mais seulement par $m(m+1)+n$ variables. Alors, la solution ΔY risque de ne pas être symétrique et puis $Y+\Delta Y$ ne appartiendra jamais au cône S_+^m des matrices symétriques semidéfinies positives.

Pour surmonter ce problème, Zhang [38] introduit l'opérateur

$$H_P(M) := \frac{1}{2}(PMP^{-1} + (PMP^{-1})^\top), \tag{6}$$

où P est une matrice non-singulière donnée, et il l'utilise pour symétriser l'équation (5.c) en la remplaçant par

$$H_P(\Delta S Y + S \Delta Y + SY) = \mu I_m.$$

Il y a plusieurs possibilités pour choisir la matrice P. Todd étudie des différents variantes dans l'article [34]. Pourtant jusqu'à présent il n'y a pas un clair “vainqueur” dans le sens d'avoir une matrice P qui soit supérieur au niveau théorique et pratique.

En particulier, on peut considérer la direction H.K.M. donnée par $P = S^{\frac{1}{2}}$, laquelle peut être obtenue en remplaçant ΔY par $\frac{1}{2}(\Delta Y + \Delta Y^\top)$ dans (5.c) (voir Helmberg et al. [25]).

Nous allons expliciter l'algorithme de point intérieur en suivant l'interprétation de Helmberg et al. [25] :

Algorithme Primal-Dual de Point Intérieur

Considérons les donnés $A_i, i = 0,...,n$, b et c, et un point initial (x^0, S^0, Y^0) satisfaisant que $F_\mu(x^0, S^0, Y^0) = 0$ et $S^0, Y^0 \succ 0$. Sans perte de généralité nous supposons que $\mu = 1$, c'est à dire que $S^0 Y^0 = I_m$.

Les paramètres initiaux de l'algorithme sont $\mu = 1$, $\tau > 0$ tel que $\delta(x^0, S^0, Y^0) \leq \tau$, et la tolérance $\epsilon > 0$. Ici, on a dénoté par $\delta(x, S, Y)$ une mesure de proximité entre le point (x, S, Y) et le chemin central $\{(x_\mu, S_\mu, Y_\mu) \,:\, \mu > 0\}$.

Pas 1. Réduire le paramètre μ.

Pas 2. Si $\delta(x,S,Y) > \tau$, calculer $(\Delta x, \Delta S, \Delta Y)$ en résolvant (5), et remplacer ΔY par $\frac{1}{2}(\Delta Y + \Delta Y^\top)$.

Pas 3. Trouver $\alpha \in (0{,}1]$ tel que $S + \alpha\Delta S \succ 0$, $Y + \alpha\Delta Y \succ 0$ et la distance $\delta(x,S,Y)$ soit réduite.

Pas 4. Actualiser $(x,S,Y) = (x + \alpha\Delta Y, S + \alpha\Delta S, Y + \alpha\Delta Y)$.

Pas 5. Si $S \cdot Y \leq \epsilon$ alors l'algorithme arrête,
Si non et $\delta(x,S,Y) \leq \tau$ alors on va au pas 1,
Si non, on va au pas 2.

Utilisant cette méthode de point intérieur, on peut résoudre le problème (LSDP) dont les donnés A_i, b et c sont rationnels et la tolérance est égal à ϵ avec $O(\sqrt{m}\log(1/\epsilon))$ itérations réalisables (cf. [34]). Ceci est le même résultat théorique qu'on obtient dans le cas linéaire (LP).

Pour des différents détails sur la complexité associée à chaque itération de l'algorithme de point intérieur, voir l'article Krishnan et Terlaky [28].

Quelques Remarques sur la Complexité du Problème (LSDP)

Nous avons dit que, sous l'hypothèse de strict réalisabilité, les problèmes (LSDP) et son dual (DLSDP) peuvent être résolus à une tolérance fixe dans un temps de fonctionnement polynomial en utilisant la méthode de point intérieur. Pourtant, même si tous les donnés du problème, A_i, b et c, sont rationnels, on ne peut pas établir des bornés polynomiales pour la *longueur de bits* des nombres intermédiaire calculés par l'algorithme de point intérieur. Alors, la méthode de point intérieur pour résoudre (LSDP) n'est polynomial que dans le modèle des nombres réels, car il n'est pas polynomial dans le modèle des nombres de bits utilisés dans ses calculs.

En effet, il existe des problèmes (LSDP) avec des donnés rationnels dont la solution n'est plus rationnel. Par exemple, considérons les contraintes semidéfinies

$$\begin{pmatrix} 1 & x \\ x & 2 \end{pmatrix} \succeq 0 \quad \text{et} \quad \begin{pmatrix} 2x & 2 \\ 2 & x \end{pmatrix} \succeq 0$$

dont l'unique point réalisable est $x = \sqrt{2}$. Clairement, cette solution ne peut pas être décrite en utilisant un nombre polynomial de bits. Cette situation-ci constitue une différence importante entre le problème (LSDP) et le cas linéaire classique (LP).

Une autre situation "pathologique" qui peut arriver dans le cas semidéfini est que tous les points réalisables soient doublement exponentiels. Par exemple, considérons les fonctions $Q_1(x) := (x_1 - 2)$ et

$$Q_i(x) := \begin{pmatrix} 1 & x_{i-1} \\ x_{i-1} & x_i \end{pmatrix} \quad \forall i = 2, \ldots, n.$$

Alors, la contrainte semidéfinie et diagonale par blocs

$$Q(x) := \mathrm{diag}(Q_1(x), Q_2(x), \ldots, Q_n(x)) \succeq 0$$

est satisfaite ssi $Q_i(x) \succeq 0$ pour tout $i = 1, \ldots, n$, ce qui implique que

$$x_i \geq 2^{2^i - 1} \quad \text{pour tout} \;\; i = 1, \ldots, n.$$

Donc, tout point réalisable et rationnel a aussi une longueur exponentielle de nombre de bits.

Plan de la Thèse

Cette thèse est constituée de quatre chapitres :

“A Global Algorithm for Nonlinear Semidefinite Programming”

Dans ce chapitre nous proposons un algorithme global pour la résolution des problèmes semidéfinis nonlinéaires de la forme

$$\min_{x \in \mathbb{R}^n} \{f(x) \, ; \, \mathcal{A}(x) \preceq 0, \, h(x) = 0\}, \tag{P}$$

où $f : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{A} : \mathbb{R}^n \to S^m$ et $h : \mathbb{R}^n \to \mathbb{R}^p$ sont des fonctions suffisamment différentiables. Pour cela, on utilise une stratégie de recherche linéaire, et une fonction de pénalisation nondifférentiable

$$\theta_\sigma(x) = f(x) + \sigma(\lambda_1(\mathcal{A}(x))_+ + \|h(x)\|), \tag{7}$$

où $\sigma > 0$ est le paramètre de pénalisation, $\lambda_1(A) := \max_x x^\top Ax$ dénote la plus grande valeur propre de la matrice A, et $(a)_+ := \max\{0,a\}$ pour tout $a \in \mathbb{R}$.

Récemment, une méthode nommée “Sequentially Semidefinite Programming” (S-SDP) a été introduite dans l’article [19]. Cette méthode résout localement notre problème et est fortement inspirée de la méthode classique “Sequentially Quadratic Programming” (SQP) pour la programmation nonlinéaire. Les résultats de l’article [19] ont été une base pour notre travail, et nous avons suivi certaines de ses idées.

Sur les résultats à remarquer dans ce chapitre, on peut mentionner par exemple que nous prouvons que la fonction Lagrangienne Augmentée $L_\sigma : I\!R^n \times S^m \times I\!R^p \to I\!R$ définie par (voir par exemple la fonction *Lagrangienne Augmentée Proximale* dans [32])

$$\begin{aligned} L_\sigma(x,Z,\lambda) = f(x) + \lambda^T h(x) + \mathrm{Tr}(Z[\mathcal{A}(x) + (-\sigma^{-1}Z - \mathcal{A}(x))_+]) \\ + \frac{\sigma}{2}\Big(\|h(x)\|^2 + \|\mathcal{A}(x) + (-\sigma^{-1}Z - \mathcal{A}(x))_+\|_{Fr}^2\Big), \end{aligned} \tag{8}$$

et la fonction de pénalisation de Han (cf. (7)) sont fonctions de pénalisation exacte pour le problème semidéfini nonlinéaire (P), autrement dit, ces fonctions atteignent un minimum local dans la solution de (P) (voir des différents théorèmes associés à ce sujet dans la Section 3 du Chapitre 1 ou dans [16, Sect. 3]).

Notre résultat principal est la démonstration de la convergence de notre algorithme sous des hypothèses minimales en profitant de la structure matricielle. Ceci est montré dans le Théorème 4.4 du Chapitre 1.

Ces résultats étendent la théorie connue pour la programmation nonlinéaire classique (cf. [10]) et ont inspiré des travaux récents sur le même sujet (voir, par exemple, [20]).

"Penalty and Barrier Methods for Convex Semidefinite Programming "

Dans ce chapitre nous présentons des méthodes de pénalisation et de fonctions barrière pour résoudre des problèmes de programmation semidéfinie convexe. Plus précisément, nous travaillons avec le problème (SDP) dont la fonction de coût f est convexe et l'opérateur G satisfait la propriété de convexité suivante :

$$G(\lambda x + (1-\lambda)y) \preceq \lambda G(x) + (1-\lambda)G(y) \quad \forall x,y \in I\!R^n, \forall \lambda \in [0,1].$$

En effet, nous étudions des méthodes qui consistent en la résolution "approximative" des problèmes de minimisation sans contraintes

$$v_r = \inf\{\phi_r(x) \mid x \in I\!R^n\}, \tag{P$_r$}$$

avec

$$\phi_r(x) := f(x) + \alpha(r)\sum_{i=1}^m \theta\left(\frac{\lambda_i(G(x))}{r}\right),$$

où $\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_m(A)$ sont les valeurs propres de la matrice A, le réel $r > 0$ est un paramètre de pénalisation qui converge vers 0, la fonction $\alpha : I\!R_+ \to I\!R_+$ satisfait les conditions suivantes :

$$\lim_{r\to 0^+} \alpha(r) = 0 \qquad \text{et} \qquad \liminf_{r\to 0^+} \frac{\alpha(r)}{r} > 0,$$

et θ est une fonction de pénalisation qui appartient à la classe suivante [3] :

$$F = \{\theta : I\!R \to I\!R \cup \{+\infty\}, \text{ convexe, sci, propre et non décroissante t.q.} \\ \theta_\infty(1) > 0, \lim_{t\to\eta^-} \theta(t) = +\infty, \text{ et } \operatorname{dom}\theta =]-\infty, \eta[, \text{ où } \eta \in [0, +\infty]\}.$$

Cette classe de méthodes est une extension des méthodes de pénalisation et de fonctions barrière dans le cadre de l'optimisation convexe usuelle (cf. [6]), et dans le cas où l'opérateur convexe G est linéaire (cf. [4]).

Nous donnons un critère d'arrêt implémentable pour obtenir la suite primale $\{x_r\}$ (le point x_r est une solution de (P_r)), et des formules explicites pour la suite duale (ce qui n'avais pas été fait dans le cadre restreint de [4] et [6]). Nous montrons alors la convergence des suites primales et duales obtenues par ces méthodes sous des hypothèses minimales : à savoir, l'ensemble des solutions optimales du problème (SDP) est un compact non vide, et la condition de Slater : il existe x^0 tel que $G(x^0)$ soit une matrice définie négative, est vérifiée. L'analyse de la convergence est ici plus complexe que dans [4] et [6]. En effet, cette dernière repose sur l'Analyse Asymptotique, et ici on doit calculer la fonction asymptote d'une fonction composée où intervient l'opérateur convexe G. Or pour l'instant on n'a aucune notion permettant le calcul d'une fonction asymptote d'un opérateur convexe général.

Finalement, nous étendons aussi l'approche des méthodes de pénalisation à deux paramètres introduite dans [23] à la programmation semidéfinie, lequel consiste en résoudre "approximativement" les problèmes

$$v_r = \inf\{\psi_r(x) \mid x \in I\!R^n\}, \qquad (\hat{\mathrm{P}}_r)$$

avec

$$\psi_r(x) = f(x) + r\beta_r \sum_{i=1}^{m} \theta\left(\frac{\lambda_i(G(x))}{r}\right).$$

Ici, le paramètre $r > 0$ est toujours décroissant pendant que le paramètre $\beta_r > 0$ croît si la solution x_r de $(\hat{\mathrm{P}}_r)$ n'est pas réalisable pour le problème (SDP).

Cette approche est particulièrement intéressante pour des problèmes dans lesquels il est difficile de trouver une solution admissible de départ permettant de calculer la suite primale, et aussi dans lesquels l'admissibilité des solutions approchées est importante. Ces deux difficultés sont ici surmontées et le théorème de convergence de la suite primale montre ici en plus, qu'à partir d'un certain rang la suite primale est admissible. Ceci est démontré sous les conditions minimales usuelles. Les résultats obtenus sont non seulement une extension à la programmation semidéfinie mais améliorent aussi dans le cas usuel les résultats donnés dans [23]. D'abord on donne une règle implémentable de solutions approchées ce qui n'est pas le cas dans [23], ensuite au lieu de supposer que l'ensemble des contraintes est compact, on ne fait cette hypothèse que sur l'ensemble des solutions optimales. Enfin, on associe à la suite primale, une suite duale de multiplicateurs donnés par une formule explicite et l'on démontre que cette suite est bornée et que chacun de ses points limites est une solution optimale du dual. Cet aspect n'apparaît pas dans [23].

"Perturbation Analysis of Second-Order Cone Programming Problems"

Dans ce chapitre nous travaillons avec le problème de programmation sur un cône de second ordre ou cône de Lorentz (SOCP), où les fonctions f et g^j, $j = 1,\dots,J$, sont deux fois continûment différentiables (i.e. C^2).

Dans une première partie nous faisons une comparaison avec sa représentation SDP dont nous montrons que ces problèmes ne sont plus équivalents du point de vue dual. En effet, des propriétés importantes comme l'unicité du multiplicateur dual ne sont plus satisfaites simultanément pour les deux problèmes. Des résultats similaires sont obtenus par Siam et Zhao dans [33] en utilisant une différente approche.

Cette partie de notre travail est développée dans un cadre général et appliquée à nos problèmes. Nous introduisons aussi une notion de partition optimale (cf. Lemme 2.3 dans le Chapitre 3 ou [12, Lemme 3]) laquelle nous permet démontrer certaines propriétés.

Dans une deuxième partie nous donnons le résultat principal de notre article : La caractérisation de la condition de régularité forte [30] pour le problème (SOCP) en fonction des conditions optimales de second ordre. En effet, considérons la fonction lagrangienne associée au problème (SOCP)

$$L(x,y) := f(x) + \sum_{j=1}^{J} g^j(x)^\top y^j \quad \forall x \in I\!R^n, y \in \Pi_{j=1}^{J} I\!R^{m_j+1}, \tag{9}$$

et le *cône de directions critiques* suivant :

$$C(x^*) := \{h \in I\!R^n \,:\, Dg(x^*)h \in T_{\mathcal{Q}}(g(x^*)), \nabla f(x^*)^\top h = 0\}, \tag{10}$$

avec $T_K(A)$ le *cône tangent* à l'ensemble K dans le point $A \in K$. Cette caractérisations vient donnée par le résultat ci-dessous :

Théorème. *Soient x^* une solution local de (SOCP) et y^* son multiplicateur de Lagrange. Alors, (x^*,y^*) est une solution fortement régulière (des conditions de premier ordre associées à (SOCP)) ssi x^* est nondégéneré (définition 4.3 du Chapitre 3), et la condition de second ordre suivante est satisfaite :*

$$h^\top \nabla^2_{xx} L(x^*,y^*)h + h^\top \mathcal{H}(x^*,y^*)h > 0, \quad \forall h \in \mathrm{Sp}(C(x^*)) \setminus \{0\}. \tag{11}$$

où $\mathrm{Sp}(C) := I\!R_+(C - C)$ est l'espace vectoriel généré par l'ensemble C, et la matrice $\mathcal{H}(x^*,y)$ est définie par $\mathcal{H}(x^*,y) = \sum_{j=1}^{J} \mathcal{H}^j(x^*,y^j)$, dont pour $s^j := g^j(x^*)$, $j = 1,...,J$, et I_m la matrice identité dans S^m, on denote

$$\mathcal{H}^j(x^*,y^j) := -\frac{y_0^j}{s_0^j} Dg^j(x^*)^\top \begin{pmatrix} 1 & 0^\top \\ 0 & -I_{m_j} \end{pmatrix} Dg^j(x^*), \tag{12}$$

si $s^j \neq 0$ appartient à la frontière de Q_{m_j+1}, et $\mathcal{H}^j(x^*,y^j) := 0$ si non.

Ce sujet est bien développé dans la programmation mathématique nonlinéaire classique. On peut citer par exemple deux différents approches : [14] et [17], lesquelles montrent la caractérisation de la propriété de régularité forte en fonction de conditions optimales de second ordre, mais ce travail-ci est le premier où on donne une caractérisation précise pour un problème d'optimisation sur un cône différent d'un cône polyédral.

"A note on Strong Regularity for Semidefinite Programming"

Nous considérons un problème de programmation semidéfinie nonlinéaire (cf. Problème (SDP)) et analysons le comportement des solutions de ce problème quand une petite perturbation est appliquée. En particulier, nous étudions la propriété des "solutions fortement régulières" (dans le sens de Robinson [30]) et sa relation avec des conditions optimales de second ordre. Comme on a déjà mentionné, ce genre de résultats est bien connu pour la programmation nonlinéaire classique.

Dans cet article nous donnons des conditions nécessaires et suffisantes pour le problème (SDP), en revanche, sa caractérisation est encore un problème ouvert. Pour cela, on a utilisé des résultats connus dans le contexte d'optimisation sur un cône convexe et fermé quelconque, ainsi que des techniques matricielles bien précises.

En effet, il est bien connu que la régularité forte est satisfaite ssi la condition de croissance quadratique uniforme est satisfaite et la solution primal $\bar{x}$ est nondégéneré (cf. [13]). Cette dernière condition veut dire que la fonction linéaire $\psi_{\bar{x}} : I\!R^n \to S^{m-r}$ définie par

$$\psi_{\bar{x}}(h) := E^\top DG(\bar{x})hE$$

est surjective. Ici, le réel r est le rang de $G(\bar{x})$, et on denote par $E \in I\!R^{m \times m-r}$ une matrice dont ses colonnes sont une base orthonormale de $\operatorname{Ker} G(\bar{x})$.

Il suffit donc de caractériser la condition de croissance quadratique uniforme en fonction des conditions optimales de second ordre sous l'hypothèse de nondégénérescence.

Dans ce contexte, notre condition optimale nécessaire améliore les conditions nécessaires connues sur deux points : elle considère un terme quadratique additionnel associé à la géométrie du cône des matrices semidéfinies négatives, et le cône des directions critiques dont la condition est satisfaite est plus grand que ceux considérés antérieurement dans la littérature. En effet, considérons la fonction lagrangienne associée au problème (SDP)

$$L(x,Y) := f(x) + G(x) \cdot Y \quad \forall x \in I\!R^n, Y \in S^m, \tag{13}$$

et le *cône de directions critiques* suivant :

$$C(\bar{x}) := \{h \in I\!R^n \ : \ DG(\bar{x})h \in T_{S^m_-}(G(\bar{x})), \nabla f(\bar{x})^\top h = 0\}, \tag{14}$$

avec $T_K(A)$ le *cône tangent* à l'ensemble K dans le point $A \in K$.

Alors, la condition nécessaire est donnée par le théorème ci-dessous :

Théorème. *Soient $\bar{x}$ une solution local de (SDP) et $\bar{Y}$ son multiplicateur de Lagrange. Si $(\bar{x},\bar{Y})$ est une solution fortement régulière (des conditions de premier ordre associées à (SDP)), alors $\bar{x}$ est nondégeneré et la condition du second ordre suivante est satisfaite :*

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h + h^\top \mathcal{H}(\bar{x},\bar{Y})h > 0, \quad \forall h \in \operatorname{Sp}(C(\bar{x})) \setminus \{0\}. \tag{15}$$

où $\operatorname{Sp}(C) := I\!R_+(C - C)$ est l'espace vectoriel généré par l'ensemble C, et les composants de la matrice $\mathcal{H}(\bar{x},\bar{Y})$ sont

$$\mathcal{H}(\bar{x},\bar{Y})_{ij} := -2\bar{Y} \cdot ([D_{x_i} G(\bar{x})] G(\bar{x})^\dagger [D_{x_j} G(\bar{x})]). \tag{16}$$

avec $A^\dagger := \sum_{i;\lambda_i \neq 0} \lambda_i^{-1} q_i q_i^\top$ la matrice pseudo-inverse *de $A = \sum_i \lambda_i q_i q_i^\top$ (sa décomposition spectrale).*

Nous pensons que la condition (15) est aussi suffisante, pourtant, la condition optimale suffisante qu'on a montré ne considère pas le terme quadratique décrit par (16), et en plus, le cône des directions critiques de cette condition suffisante est plus petit que celui de la condition nécessaire. Voyons cette condition suffisante.

Théorème. *Soient $\bar{x}$ une solution local de (SDP) et $\bar{Y}$ son multiplicateur de Lagrange. Si $\bar{x}$ est nondégeneré et si la condition de second ordre suivante est satisfaite :*

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h > 0, \quad \forall h \neq 0; \, DG(\bar{x})h \cdot \bar{Y} = 0, \tag{17}$$

alors $(\bar{x},\bar{Y})$ est une solution fortement régulière.

On a donc un écart entre les conditions optimales nécessaires et suffisantes.

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CHAPITRE I

A Global Algorithm for Nonlinear Semidefinite Programming

A Global Algorithm for Nonlinear Semidefinite Programming [1]

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Abstract. In this paper we propose a global algorithm for solving nonlinear semidefinite programming problems. This algorithm, inspired in the classic SQP (sequentially quadratic programming) method, modifies the S-SDP (sequentially semidefinite programming) local method by using a nondifferentiable merit function combined with a line search strategy.

1 Introduction

We consider the nonlinear programming problem

$$(\mathrm{P}) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \mathcal{A}(x) \preceq 0, \\ & h(x) = 0, \end{array}$$

where $x \in I\!R^n, \mathcal{A}$ is a smooth function whose values are symmetric matrices, $\preceq$ denotes the negative semidefinite order (that is, $A \preceq B$ if and only if $A - B$ is a negative semidefinite matrix), h is a smooth vector function with values in $I\!R^p$, and f is the smooth objective function. The smoothness of all these functions is specified at each statement.

This problem becomes interesting when the linear matrix formulation [23]

$$(\mathrm{LMI}) \qquad \begin{array}{ll} \text{minimize} & f(x) = c^T x \\ \text{subject to} & \mathcal{A}(x) = A_0 + \sum_{i=1}^m x_i A_i \preceq 0 \end{array}$$

does not give a satisfactory model for certain problems, particularly those from control theory [1, 5, 8, 9].

This paper is organized as follows. In section 2 the optimality and constraint qualification conditions for problem (P) are presented. The results contained in this section are adaptations of known results (see [17, 21]). Here only the optimality conditions that are useful in our context are discussed. Other conditions

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can be found in [2, 6]. In Section 3 we demonstrate some exactness results associated with the Lagrangian, the augmented Lagrangian and the Han penalty function. The first one is well known and we review it to make our exposition self-contained. The exactness of the augmented Lagrangian and the Han penalty function are extensions of the corresponding classical mathematical programming results [3, 15]. In Section 4 we propose a global S-SDP (sequentially semidefinite programming) algorithm and prove its convergence. The convergence of a local S-SDP algorithm has been proved by Fares, Noll, and Apkarian [9]. Other works concerning the global convergence of methods for solving optimization programs with nonlinear matrix inequalities constraints are [8, 16].

1.1 Notations

Throughout we denote by S^m the set of all symmetric matrices of dimension m, by S^m_+ the set of all symmetric positive semidefinite matrices, and by S^m_{++} the set of all symmetric positive definite matrices. The sets S^m_- and S^m_{--} are defined similarly. For all these sets of matrices we use the trace product $\langle A,B\rangle = \mathrm{Tr}(AB)$, and the Frobenius norm $\|A\|_{Fr} = \sqrt{\mathrm{Tr}(A^2)}$. For a given matrix A, $\lambda_j(A)$ denotes its jth eigenvalue in nonincreasing order and A_+ denotes the matrix defined by

$$A_+ := P\,\mathrm{diag}((\lambda_1)_+,\dots,(\lambda_m)_+)P^T, \tag{1.1}$$

where $(\lambda)_+ = \max\{0,\lambda\}$ and P is the matrix in the spectral decomposition $A = P\,\mathrm{diag}(\lambda_1,\dots,\lambda_m)P^T$. It is easy to see that A_+ is the orthogonal projection of A on S^m_+.

Given a matrix-valued function $\mathcal{A}(\cdot)$ we will use the notation

$$D\mathcal{A}(x_*) = \left(\frac{\partial\mathcal{A}(x_*)}{\partial x_i}\right)_{i=1}^n = \left(\frac{\partial\mathcal{A}(x_*)}{\partial x_1},\dots,\frac{\partial\mathcal{A}(x_*)}{\partial x_n}\right)^T$$

for its differential operator evaluated at x_*. This notation comes from the fact that

$$D\mathcal{A}(x_*)y = \sum_{i=1}^n y_i\frac{\partial\mathcal{A}(x_*)}{\partial x_i} \quad \forall y \in I\!R^n. \tag{1.2}$$

Finally, if we define the linear operator V from $I\!R^n$ to S^m by $Vy = \sum_{i=1}^n y_iV_i$, where $V_i \in S^m$ for all $i \in \{1,\dots,n\}$, we have for the adjoint operator V^* the formula

$$V^*Z = (\mathrm{Tr}(V_1Z),\dots,\mathrm{Tr}(V_nZ))^T \quad \forall Z \in S^m. \tag{1.3}$$

2 Optimality Conditions Review

In this section we state the first- and second-order optimality conditions for (P) and discuss their implications. To this end, we consider the Lagrangian

$L : \mathbb{R}^n \times S^m \times \mathbb{R}^p \to \mathbb{R}$ of problem (P) defined by

$$L(x,Z,\lambda) = f(x) + \mathrm{Tr}(Z\mathcal{A}(x)) + \lambda^T h(x). \tag{2.1}$$

2.1 First order optimality conditions

The Karush–Kuhn–Tucker necessary optimality conditions for a feasible point x_* of (P) are given by the existence of $Z_* \in S^m$ and $\lambda_* = (\lambda_{*1},...,\lambda_{*p})^T \in \mathbb{R}^p$ such that

$$\text{(KKT)} \qquad \begin{array}{rcl} \nabla f(x_*) + D\mathcal{A}(x_*)^* Z_* + \displaystyle\sum_{j=1}^{p} \lambda_{*j} \nabla h_j(x_*) & = & 0 \\ \mathrm{Tr}(Z_*\mathcal{A}(x_*)) & = & 0 \\ Z_* & \succeq & 0. \end{array}$$

The pair (Z_*,λ_*) is called the (KKT)-multiplier associated with x_*. The complementarity condition $\mathrm{Tr}(Z_*\mathcal{A}(x_*)) = 0$ has the following two useful equivalent forms:

$$\lambda_j(Z_*)\lambda_j(\mathcal{A}(x_*)) = 0 \quad \forall\ j \in \{1,...,m\} \tag{2.2}$$

and

$$Z_*\mathcal{A}(x_*) = 0. \tag{2.3}$$

Both forms are easily obtained from the *Von Neumann–Theobald inequality*:

$$\mathrm{Tr}(AB) \le \sum_{j=1}^{m} \lambda_j(A)\lambda_j(B), \tag{2.4}$$

where the equality holds if and only if there is a matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal [22, 24].

Condition (2.2) allows us to define the *strict complementarity condition* in (KKT) as follows

$$\lambda_j(Z_*) = 0 \quad \text{if and only if} \quad \lambda_j(\mathcal{A}(x_*)) < 0 \quad \forall\ j \in \{1,...,m\}. \tag{2.5}$$

As is well known, the (KKT) conditions are not a consequence of the optimality of x_*, and to ensure this consequence, we must assume an extra condition. In this paper, we will use *Robinson's constraint qualification condition* [18]

$$0 \in \mathrm{int} \left\{ \begin{pmatrix} \mathcal{A}(x_*) \\ h(x_*) \end{pmatrix} + \begin{pmatrix} D\mathcal{A}(x_*) \\ \nabla h(x_*) \end{pmatrix} \mathbb{R}^n - \begin{pmatrix} S^m_- \\ \{0\} \end{pmatrix} \right\}, \tag{2.6}$$

where $\mathrm{int}\, C$ denotes the topological interior of the set C. A direct consequence of [12, Chapter 3, Prop. 2.1.12] is the equivalence between condition (2.6) and the *Mangasarian–Fromovitz constraint qualification condition*

$$\{\nabla h_j(x_*)\} \qquad \text{is linearly independent, and} \tag{2.7a}$$

$$\exists \bar{d} \in I\!R^n \text{ s. t.} \qquad \begin{cases} \nabla h(x_*)\bar{d} = 0 \\ \text{and } \mathcal{A}_*(\bar{d}) \prec 0, \end{cases} \tag{2.7b}$$

where $\mathcal{A}_* : I\!R^n \to S^m$ is the linear affine function defined by $\mathcal{A}_*(y) := \mathcal{A}(x_*) + \sum_{i=1}^n y_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i}$. It can be shown that under (2.6) the set of (KKT) Lagrange multipliers is nonempty and also bounded [14].

We will also consider the *transversality condition* which asks that the function $\psi : I\!R^n \to I\!R^p \times S_r$ defined by

$$\psi(d) := \left((\nabla h(x_*)d)^T, N^T \sum_{i=1}^n d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N \right)^T \tag{2.8}$$

be surjective, where

$$N = [v_1 ... v_r] \tag{2.9}$$

is the matrix whose columns v_i are an orthonormal basis of $Ker\ \mathcal{A}(x_*)$. We set $N = 0$ if $Ker\ \mathcal{A}(x_*) = \{0\}$. This condition has originally been defined in the context of smooth manifolds [10] and implies the Robinson's constraint qualification condition (2.7); moreover, (2.8) guarantees the uniqueness of the (KKT)-multiplier. Unfortunately, this condition can be very strong, because it forces $n \geq p + r(r+1)/2$, where $r = \dim[Ker\, \mathcal{A}(x_*)]$.

It is clear that the transversality condition (2.8) cannot hold when the matrix $\mathcal{A}(x_*)$ has a diagonal block structure. Indeed, in this case the multiplier Z_* is not unique, and therefore the transversality condition does not hold. This difficulty can be easily avoided if we assume the transversality condition for each block of $\mathcal{A}(x_*)$. For example, if $\mathcal{A}(x_*)$ has two diagonal block structure with sizes m_1 and m_2, then the mapping ψ should be considered into the cross-product space $S^{m_1} \times S^{m_2}$ rather then the larger space $S^{m_1+m_2}$. For simplicity of notation we only consider the case where $\mathcal{A}(x_*)$ is a one-block matrix. More details about the transversality condition in the semidefinite programming context can be seen in [21] and the references within.

2.2 Second-Order sufficient conditions

In this section we introduce only the second-order sufficient conditions that will be used in this paper as well as results that involve transversality condition (2.8). We assume that f, h, and $\mathcal{A}$ are twice differentiable at x_*.

Given a set $B \subseteq I\!R^m$ we define

$$S^m_-(B) := \{M \in S^m : w^T M w \leq 0 \quad \forall w \in B\}. \tag{2.10}$$

Proposition 2.1. *A sufficient condition to obtain the isolated optimality of x_* for problem* (P)*, is the existence of $(Z_*,\lambda_*) \in S^m \times I\!R^p$ such that (x_*,Z_*,λ_*) satisfies* (KKT) *and*

$$d^T \nabla^2_{xx} L(x_*,Z_*,\lambda_*) d > 0 \tag{2.11}$$

for all nonzero vectors $d \in C(x_*)$, where

$$\begin{aligned} C(x_*) = \Big\{ d \in I\!R^n : \sum_{i=1}^n d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} \in S^m_-(\operatorname{Ker} \mathcal{A}(x_*)),& \\ \nabla h(x_*) d = 0 \text{ and } \nabla f(x_*)^T d = 0 \Big\}& \end{aligned} \tag{2.12}$$

is a cone of critical directions for problem (P) *at the point x_*.*

Proof. See, for example, [19, Thm. 2.2] and note that

$$T_{S^m_-}(\mathcal{A}(x_*)) = S^m_-(\operatorname{Ker} \mathcal{A}(x_*)).$$

■

Remark 2.2. *Condition* (2.11) *can be far from necessary. For instance, in the problem* (LMI)*, mentioned in the introduction, we always have $\nabla^2_{xx} L = 0$; thus, if $C(x_*) \neq \{0\}$, condition* (2.11) *never holds. This is because condition* (2.11) *does not consider the geometry of S^m_-. This kind of problem was the motivation for works such as [2, 6, 21] in the 1990s. We will just consider the nonlinear problem* (P)*, where the algorithm S-SDP makes sense.*

Let us define now a larger cone of critical directions $C'(x_*,Z_*)$, which considers the (KKT)-multiplier Z_* associated with the matrix inequality $\mathcal{A}(x) \preceq 0$, as follows:

$$C'(x_*,Z_*) := \left\{ d \in I\!R^n : \operatorname{Im} Z_* \subseteq \operatorname{Ker} \operatorname{Pr} \sum_{i=1}^n d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} \text{ and } \nabla h(x_*) d = 0 \right\}, \tag{2.13}$$

where Pr is the orthogonal projection in $I\!R^m$ over Ker $\mathcal{A}(x_*)$. Note that $\operatorname{Pr} = NN^T$ with N defined in (2.9).

The next proposition relates both cones of critical directions and the function ψ used in the tranversality condition.

Proposition 2.3. *Let x_* be a solution of* (P) *and (Z_*,λ_*) be a* (KKT)*-multiplier. Let us also consider the function ψ, defined in* (2.8)*, and the cones of critical directions defined above. Then*

$$\operatorname{Ker} \psi \subseteq C(x_*) \subseteq C'(x_*,Z_*), \tag{2.14}$$

with equality when the strict complementarity condition (2.5) *holds.*

Proof. First, note that we can write (2.3) in the equivalent form

$$Z_* = N \phi_* N^T, \tag{2.15}$$

with $\phi_* \in S_r^+$ and $r = \dim \text{Ker } \mathcal{A}(x_*)$. Then, to prove the first inclusion in (2.14), it is sufficient to show that $\nabla f(x_*)^T d = 0$. This comes from the first equation in (KKT) and the equality

$$\text{Tr}\left(Z_* \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i}\right) = \text{Tr}\left(\phi_* N^T \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N\right). \tag{2.16}$$

For the second inclusion, if $d \in C(x_*)$ then $\nabla f(x_*)^T d = 0$ and $\nabla h(x_*) d = 0$, and we obtain from (2.16) and the first equation in (KKT) that

$$\text{Tr}(\phi_* N^T \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N) = 0.$$

Since $\sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} \in S_-^m(\text{Ker } \mathcal{A}(x_*))$, we see that $N^T \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N \in S_r^-$, and using (2.4), we deduce from the last equality that

$$N^T \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N \phi_* = 0, \tag{2.17}$$

which is equivalent to

$$\text{Pr} \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} Z_* = N N^T \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N \phi_* N^T = 0, \tag{2.18}$$

and we conclude that $d \in C'(x_*, Z_*)$.

If, in addition, we assume the strict complementarity condition (2.5), we have that ϕ_* is nonsingular, and from the equivalence between (2.18) and (2.17) we deduce the converse inclusion $C'(x_*, Z_*) \subseteq \text{Ker } \psi$. ■

A direct consequence of propositions 2.1 and 2.3 is the following stronger second-order sufficient condition for optimality.

Proposition 2.4. *Under the hypotheses of Proposition* 2.1*, where the critical cone $C(x_*)$ is replaced by $C'(x_*, Z_*)$, the point x_* is an isolated local minimum of* (P).

3 Exact Penalty Functions

A pair (x_*, y_*) in the product set $X \times Y$ is said to be a *saddle-point* of the function $\varphi : X \times Y \to I\!R$ on $X \times Y$ if

$$\varphi(x_*, y) \leq \varphi(x_*, y_*) \leq \varphi(x, y_*), \quad \forall x \in X, \forall y \in Y.$$

We say that a function $\Phi : I\!R^n \to I\!R$ is an *exact penalty function* for a local minimum x_* of (P) if x_* is a local minimum of Φ too.

In this section we study different penalty functions associated with problem (P) and state necessary and sufficient conditions for exactness. A general approach for the study of exact penalty functions can be found in [4, Sect. 3.4.2].

3.1 The Lagrangian Function in the Convex Case

Let us consider the particular case of (P) when $h : I\!R^n \to I\!R^p$ is an affine function $h(x) = h_0 + Hx$, with $H \in I\!R^{p\times n}$ and $h_0 \in I\!R^p$, $f : I\!R^n \to I\!R$ is convex, and $\mathcal{A}(\cdot)$ is convex in the sense of the semidefinite order, that is,

$$\mathcal{A}(tx + (1-t)y) \preceq t\mathcal{A}(x) + (1-t)\mathcal{A}(y) \quad \forall t \in [0,1], \forall x,y \in I\!R^n.$$

With these assumptions, (P) will be denoted by ($\mathrm{P_C}$) (*convex problem*). For (x_*,Z_*,λ_*) satisfying (KKT), it can be shown that the function $L(\cdot,Z_*,\lambda_*)$, defined in (2.1), is an exact penalty function for ($\mathrm{P_C}$). This is an immediate consequence of the fact that (x_*,Z_*,λ_*) is a saddle-point of the Lagrangian function on $I\!R^n \times S^m_+ \times I\!R^p$ (see [25, Thm. 4.1.3]). However, it is known that the Lagrangian is no longer an exact penalty function in the nonconvex case, which is the reason other penalty functions are introduced to obtain exactness results for our general problem (P).

3.2 The Augmented Lagrangian

We define the augmented Lagrangian function L_σ associated with problem (P) as

$$\begin{aligned} L_\sigma(x,Z,\lambda) = f(x) + \lambda^T h(x) + \mathrm{Tr}(Z[\mathcal{A}(x) + (-\sigma^{-1}Z - \mathcal{A}(x))_+]) \\ + \frac{\sigma}{2}\Big(\|h(x)\|^2 + \|\mathcal{A}(x) + (-\sigma^{-1}Z - \mathcal{A}(x))_+\|^2_{Fr}\Big), \end{aligned} \tag{3.1}$$

where $\sigma > 0$ is the penalty parameter. In [20], L_σ is called the *proximal augmented Lagrangian.*

If (x_*,Z_*,λ_*) is any point satisfying (KKT), from (2.2) it can be shown that

$$L_\sigma(x_*,Z_*,\lambda_*) = f(x_*). \tag{3.2}$$

In the next theorem we prove that $L_\sigma(\cdot,Z_*,\lambda_*)$ is an exact penalty function when σ is sufficiently large.

Theorem 3.1. *Let us assume that f, h, and $\mathcal{A}$ are twice differentiable at x_* and that (x_*,Z_*,λ_*) satisfies* (KKT) *conditions and the second-order sufficient condition* (2.11)*. Then, there is a neighborhood V of x_* and a real $\bar\sigma > 0$ such that for all $\sigma \geq \bar\sigma$, (x_*,Z_*,λ_*) is a saddle-point of L_σ on $V \times (S^m \times I\!R^p)$. Moreover,*

$$L_\sigma(x,Z_*,\lambda_*) > L_\sigma(x_*,Z_*,\lambda_*) \geq L_\sigma(x_*,Z,\lambda)$$

for all $(x,Z,\lambda) \in V \times S^m \times I\!R^p$ *with* $x \neq x_*$.

Proof. Since the operator $(\cdot)_+$, defined in (1.1), is the projection on S^m_+, we have that

$$\|(-\sigma^{-1}Z - \mathcal{A}(x))_+ - (-\sigma^{-1}Z - \mathcal{A}(x))\|^2_{Fr} \leq \|W - (-\sigma^{-1}Z - \mathcal{A}(x))\|^2_{Fr} \quad (3.3)$$

for all $W \in S^m_+$, and then

$$\begin{aligned}\frac{\sigma}{2}\|(-\sigma^{-1}Z - \mathcal{A}(x))_+ + \mathcal{A}(x)\|^2_{Fr} + \mathrm{Tr}(Z[(-\sigma^{-1}Z - \mathcal{A}(x))_+ + \mathcal{A}(x)])\\ \leq \mathrm{Tr}(Z[W + \mathcal{A}(x)]) + \frac{\sigma}{2}\|W + \mathcal{A}(x)\|^2_{Fr},\end{aligned} \quad (3.4)$$

for all $W \in S^m_+$. Taking $x = x_*$ and $W = -\mathcal{A}(x_*)$ (which belongs to S^m_+), we get

$$\frac{\sigma}{2}\|(-\sigma^{-1}Z - \mathcal{A}(x_*))_+ + \mathcal{A}(x_*)\|^2_{Fr} + \mathrm{Tr}(Z[(-\sigma^{-1}Z - \mathcal{A}(x_*))_+ + \mathcal{A}(x_*)]) \leq 0;$$

hence

$$L_\sigma(x_*,Z,\lambda) \leq f(x_*) = L_\sigma(x_*,Z_*,\lambda_*) \quad \forall Z \in S^m, \forall\lambda \in I\!R^p, \forall\sigma > 0.$$

Let us now prove the second inequality. Let $\bar{B}_\varepsilon(x_*)$ be a closed ball with center x_* and radius ε such that $f(x) > f(x_*)$ for all feasible points $x \in \bar{B}_\varepsilon(x_*)$, $x \neq x_*$. We prove that for all $\sigma > 0$ sufficiently large, x_* is the unique point satisfying $\inf_{x\in\bar{B}_\varepsilon(x_*)} L_\sigma(x,Z_*,\lambda_*) = f(x_*)$. For this purpose, we define the problem:

$$\psi_\sigma := \inf_{\substack{(x,W) \in \bar{B}_\varepsilon(x_*) \times S^m \\ \mathcal{A}(x) \preceq W}} \left\{ f(x) + \mathrm{Tr}(Z_*W) + \lambda_*^T h(x) + \frac{\sigma}{2}(\|h(x)\|^2 + \|W\|^2_{Fr}) \right\}, \quad (3.5)$$

and from inequality (3.4) we can deduce that

$$\psi_\sigma = \inf_{x\in\bar{B}_\varepsilon(x_*)} L_\sigma(x,Z_*,\lambda_*). \quad (3.6)$$

To conclude, we show that $(x_*,0,Z_*,0)$ is a point that satisfies the Karush–Kuhn–Tucker and the second-order sufficient conditions for the optimization problem (3.5). The Lagrangian associated with minimization problem (3.5) is

$$\begin{aligned}\tilde{L}(x,W,\Omega,\alpha) := f(x) + \mathrm{Tr}(Z_*W) + \lambda_*^T h(x) + \frac{\sigma}{2}\|h(x)\|^2\\ + \frac{\sigma}{2}\|W\|^2_{Fr} + \frac{\alpha}{2}(\|x - x_*\|^2 - \varepsilon^2) + \mathrm{Tr}(\Omega(\mathcal{A}(x) - W)),\end{aligned} \quad (3.7)$$

and the (KKT) conditions are

$$\begin{aligned}\nabla f(x) + \nabla h(x)^T \lambda_* + \sigma \nabla h(x)^T h(x) + \alpha(x - x_*) + D\mathcal{A}(x)^* \Omega = 0\\ Z_* + \sigma W = \Omega\\ \tfrac{\alpha}{2}(\|x - x_*\|^2 - \varepsilon^2) = 0\\ \mathrm{Tr}(\Omega(\mathcal{A}(x) - W)) = 0\\ \alpha \geq 0, \quad \Omega \succeq 0\\ \|x - x_*\| - \varepsilon \leq 0\\ \mathcal{A}(x) - W \preceq 0.\end{aligned}$$

It can be easily seen that $(x,W,\Omega,\alpha) = (x_*,0,Z_*,0)$ satisfies all these conditions.

In what follows we will state the second-order sufficient condition

$$(d^T,U)\nabla^2 \tilde{L}(x_*,0,Z_*,0) \begin{pmatrix} d \\ U \end{pmatrix} > 0, \tag{3.8}$$

for any nonzero vector $(d,U) \in \tilde{C}(x_*,0)$.

The Hessian of $\tilde{L}$ with respect to the variables (x,W) at $(x_*,0,Z_*,0)$ is given by

$$\nabla^2_{(x,W)} \tilde{L}(x_*,0,Z_*,0) = \left(\begin{array}{c|c} \tilde{\mathcal{H}} & 0 \\ \hline 0 & \sigma I_m \end{array} \right), \tag{3.9}$$

with

$$\begin{aligned}\tilde{\mathcal{H}} := \nabla^2 f(x_*) + \sum_{j=1}^{p} \lambda_{*j} \nabla^2 h_j(x_*) + \left[\mathrm{Tr}(Z_* \frac{\partial^2 \mathcal{A}(x_*)}{\partial x_i \partial x_j}) \right]_{i,j}\\ + \sigma \sum_{j=1}^{p} h_j(x_*) \nabla^2 h_j(x_*) + \sigma \nabla h(x_*)^T \nabla h(x_*),\end{aligned} \tag{3.10}$$

and the cone of critical directions for problem (3.5) is

$$\begin{aligned}\tilde{C}(x_*,0) = \Big\{ (d,U) \in I\!R^n \times S^m :\ \nabla f(x_*)^T d + \mathrm{Tr}(Z_* U) + \lambda_*^T \nabla h(x_*) d = 0,\\ \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} - U \in S^m_-(\mathrm{Ker}\ \mathcal{A}(x_*)) \Big\}.\end{aligned}$$

Thus, condition (3.8) can be written as

$$d^T \nabla^2 f(x_*) d + \sum_{j=1}^{p} \lambda_{*j} d^T \nabla^2 h_j(x_*) d + d^T \mathcal{H} d + \sigma \|\nabla h(x_*) d\|^2 + \sigma \|U\|^2_{Fr} > 0 \tag{3.11}$$

for any nonzero vector $(d,U) \in \tilde{C}(x_*,0)$, where $\mathcal{H} := \left[\mathrm{Tr}(Z_* \frac{\partial^2 \mathcal{A}(x_*)}{\partial x_i \partial x_j}) \right]_{i,j}$.

The case when $d = 0$ and $U \neq 0$ is trivial. Another easy case is when we take $(d,U) \in \tilde{C}(x_*,0)$ satisfying either $\|U\|_{Fr} > \delta\|d\|$ or $\|\nabla h(x_*)d\| > \delta\|d\|$ for some fixed $\delta > 0$, indeed

$$\begin{aligned} d^T\nabla^2 f(x_*)d + &\sum_{j=1}^{p} \lambda_{*j} d^T \nabla^2 h_j(x_*)d + d^T\mathcal{H}d + \sigma(\|\nabla h(x_*)d\|^2 + \|U\|_{Fr}^2) \\ &> d^T\nabla^2 f(x_*)d + \sum_{j=1}^{p} \lambda_{*j} d^T \nabla^2 h_j(x_*)d + d^T\mathcal{H}d + \sigma\delta^2\|d\|^2 \\ &\geq -\|\nabla^2 f(x_*) + \sum_{j=1}^{p} \lambda_{*j}\nabla^2 h_j(x_*) + \mathcal{H}\|\|d\|^2 + \sigma\delta^2\|d\|^2, \end{aligned}$$

and (3.11) is verified by taking $\sigma \geq \sigma_\delta := \frac{1}{\delta^2}\|\nabla^2 f(x_*) + \sum_{j=1}^{p}\lambda_{*j}\nabla^2 h_j(x_*) + \mathcal{H}\|$.

Finally, we show that such a $\delta > 0$ always exists. We proceed by contradiction. Let us suppose that there is a sequence $\{(d_k,U_k)\}$ in $\tilde{C}(x_*,0)$ such that

$$\|U_k\|_{Fr} \leq \frac{1}{k}\|d_k\|, \tag{3.12}$$

$$\|\nabla h(x_*)d_k\| \leq \frac{1}{k}\|d_k\| \tag{3.13}$$

and

$$d_k^T\nabla^2 f(x_*)d_k + \sum_{j=1}^{p} \lambda_{*j} d_k^T \nabla^2 h_j(x_*)d_k + d_k^T\mathcal{H}d_k \leq 0 \quad \forall k. \tag{3.14}$$

If we divide (3.14) by $\|d_k\|^2$ and suppose that $\frac{d_k}{\|d_k\|} \to \hat{d}$, by taking the limit in this inequality we get

$$\hat{d}^T\nabla^2_{xx}L(x_*,Z_*,\lambda_*)\hat{d} \leq 0, \tag{3.15}$$

which means, by Proposition 2.1, that $\hat{d} \notin C(x_*)$.

On the other hand, since $(d_k,U_k) \in \tilde{C}(x_*,0)$, we have that

$$v^T \sum_{i=1}^{n} d_{ki}\frac{\partial \mathcal{A}(x_*)}{\partial x_i} v \leq v^T U_k v \quad \forall v \in \text{Ker } \mathcal{A}(x_*), \forall k,$$

and using the fact that $\|U_k\|_{Fr} \geq \frac{v^T U_k v}{\|v\|^2}$, for all $v \neq 0$, together with (3.12) we obtain

$$v^T \sum_{i=1}^{n} d_{ki}\frac{\partial \mathcal{A}(x_*)}{\partial x_i} v \leq \frac{1}{k}\|d_k\|\|v\|^2 \quad \forall v \in \text{Ker } \mathcal{A}(x_*) \forall k,$$

which implies that

$$\sum_{i=1}^{n} \hat{d}_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} \in S^m_-(\text{Ker } \mathcal{A}(x_*)).$$

The equality $\nabla h(x_*)\hat{d} = 0$ follows directly from (3.13) and since $\nabla f(x_*)^T d_k = -\operatorname{Tr}(Z_* U_k) - \lambda_*^T \nabla h(x_*) d_k$, from (3.12) and (3.13) we obtain that $\nabla f(x_*)^T \hat{d} = 0$. In this way we conclude that $\hat{d} \in C(x_*)$, which contradicts (3.15). ■

3.3 The Han penalty function

We now define another penalty function associated with problem (P), which will be a key issue in the global algorithm that we will describe in section 4. For $\sigma > 0$ we define

$$\theta_\sigma(x) = f(x) + \sigma(\lambda_1(\mathcal{A}(x))_+ + \|h(x)\|). \tag{3.16}$$

This function comes from the Han penalty function in mathematical programming [3, 11]. In the rest of this section we will prove some properties of θ_σ and its exactness.

In order to compute the directional derivative $\theta'_\sigma(x; d)$, we start by recalling a particular chain rule.

Lemma 3.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a function with directional derivative $\varphi'(x; d) = \lim_{t\to 0^+} t^{-1}(\varphi(x + td) - \varphi(x))$, and let $\phi : \mathbb{R}^m \to \mathbb{R}^p$ be a Lipschitz function in a neighborhood of $\varphi(x)$ with directional derivative $\phi'(\varphi(x); \varphi'(x; d))$. Then, the function $(\phi \circ \varphi)$ has a directional derivative at x in the direction d given by*

$$(\phi \circ \varphi)'(x; d) = \phi'(\varphi(x); \varphi'(x; d)). \tag{3.17}$$

Proof. By using the usual notation $o(t)$ for a function verifying $\lim_{t\to 0} t^{-1} o(t) = 0$, we can write for $t > 0$

$$\begin{aligned} t^{-1}[(\phi \circ \varphi)(x + td) - (\phi \circ \varphi)(x)] &= t^{-1}[\phi(\varphi(x) + t\varphi'(x; d) + o(t)) - (\phi \circ \varphi)(x)] \\ &= t^{-1}[\phi(\varphi(x) + t\varphi'(x; d)) - \phi(\varphi(x))] + t^{-1}o(t), \end{aligned}$$

and we can conclude by taking the limit when $t \to 0^+$. ■

As a consequence of this result we give in the next lemma the directional derivative of the penalty function θ_σ.

Lemma 3.3. *If f, h, and $\mathcal{A}$ in* (3.16) *have directional derivatives at x in the direction d, where x is a feasible point for* (P)*, then θ_σ also has a directional derivative that can be characterized by*

$$\theta'_\sigma(x; d) = f'(x; d) + \sigma(\lambda_1(N^T \mathcal{A}'(x; d) N)_+ + \|h'(x; d)\|),$$

where N is the matrix defined in (2.9).

Proof. Let x be a feasible point. From Lemma 3.2, we have that

$$\begin{aligned} \theta'_\sigma(x; d) &= f'(x; d) + \sigma([\lambda_1(\mathcal{A}(\cdot))_+]'(x; d) + \|h'(x; d)\|) \\ &= f'(x; d) + \sigma([\lambda_1(\cdot)_+]'(\mathcal{A}(x); \mathcal{A}'(x; d)) + \|h'(x; d)\|). \end{aligned}$$

For $A \in S_-^m$ and $B \in S^m$, we easily check that

$$[\lambda_1(\cdot)_+]'(A;B) = \begin{cases} 0 & \text{if } \lambda_1(A) < 0 \\ \lambda_1'(A;B)_+ & \text{if } \lambda_1(A) = 0, \end{cases}$$

and using formula (2.8) in [7] for the calculus of the directional derivative of $\lambda_1(A) = \max\{x^T A x : \|x\| = 1\}$, we obtain that $\lambda_1'(A;B) = \max\{x^T B x : \|x\| = 1 \text{ and } x^T A x = \lambda_1(A)\}$. Then, if $\lambda_1(A) = 0$, we can write $\lambda_1'(A;B) = \lambda_1(N^T B N)$ where N is a matrix whose columns are an orthonormal base of Ker A.

We conclude replacing A by $\mathcal{A}(x)$, B by $\mathcal{A}'(x;d)$ and recalling that N is the matrix 0 when $\lambda_1(A) < 0$. ■

Remark 3.4. *If f, h, and $\mathcal{A}$ are differentiable at x, then*

$$\theta_\sigma'(x;d) = \nabla f(x) d + \sigma \left(\lambda_1 \left(N^T \sum_{i=1}^n d_i \frac{\partial \mathcal{A}(x)}{\partial x_i} N \right)_+ + \|\nabla h(x) d\| \right).$$

In the following proposition we give a lower bound for the parameter σ in order to obtain the exactness of θ_σ.

Proposition 3.5. *If x_* is a feasible point of* (P) *and θ_σ has a (strict) local minimum at x_*, then x_* is a (strict) local minimum of* (P). *Furthermore, if f, h and $\mathcal{A}$ are differentiable at x_* and if the transversality condition* (2.8) *is verified, then $\sigma \geq \max\{\mathrm{Tr}(Z_*), \|\lambda_*\|\}$.*

Proof. If x_* is a local minimum of θ_σ, there is a neighborhood V of x_* such that for all $x \in V$ we have that $\theta_\sigma(x_*) \leq \theta_\sigma(x)$, and since x_* is feasible we obtain

$$\begin{aligned} f(x_*) = \theta_\sigma(x_*) &\leq \theta_\sigma(x) \quad \forall x \in V \\ &= f(x) \quad \forall x \in V,\ x \text{ feasible}, \end{aligned}$$

which means that x_* is a local minimum of (P). When the minimum x_* is strict, the proof is identical.

Now, let us assume that f, h and $\mathcal{A}$ are differentiable at x_* and that the transversality condition holds. Since x_* is a local minimum of θ_σ, we have that $\theta_\sigma'(x_*;d) \geq 0$ for all directions d, and using Lemma 3.3 we can write

$$0 \leq \nabla f(x_*)^T d + \sigma \left(\lambda_1 \left(N^T \sum_{i=1}^n d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N \right)_+ + \|\nabla h(x_*) d\| \right) \quad \forall d \in I\!R^n. \tag{3.18}$$

Let us first consider the case when Ker $\mathcal{A}(x_*) = \{0\}$. This implies $Z_* = 0$ and $N = 0$; hence from inequality (3.18) and the first equation in (KKT), we see that $\sigma \geq \frac{\lambda_*^T \nabla h(x_*) d}{\|\nabla h(x_*) d\|}$ for all nonzero $d \in I\!R^n$. The surjectivity of $\nabla h(x_*)$ shows that $\sigma \geq \|\lambda_*\|$.

Let us suppose now that Ker $\mathcal{A}(x_*) \neq \{0\}$. From the first equation in (KKT), inequality (3.18) and equality (2.15), we can write for all $d \in I\!R^n$

$$\begin{aligned}\sigma \left(\|\nabla h(x_*)d\| + \lambda_1 \left(N^T \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N \right)_+ \right) \\ \geq \lambda_*^T \nabla h(x_*)d + \operatorname{Tr}\left(Z_* \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} \right) \\ = \lambda_*^T \nabla h(x_*)d + \operatorname{Tr}\left(\phi_* N^T \sum_{i=1}^{n} d_i \frac{\partial \mathcal{A}(x_*)}{\partial x_i} N \right).\end{aligned}$$

Then the transversality condition (surjectivity of ψ) allows us to say that

$$\sigma(\|v\| + \lambda_1(W)_+) \geq \lambda_*^T v + \operatorname{Tr}(\phi_* W) \tag{3.19}$$

for all $(v,W) \in I\!R^p \times S_r$, and we can conclude from the inequality

$$\|W\|_2 := \sqrt{\max\{\lambda^2 : \lambda \text{ is an eigenvalue of } W\}} \geq \lambda_1(W)_+$$

and the equality

$$\|(\lambda_*,\phi_*)\|_D := \sup\{|\lambda_*^T v + \operatorname{Tr}(\phi_* W)| : \|v\| + \|W\|_2 = 1\} = \max\{\|\lambda_*\|, \operatorname{Tr}(Z_*)\}.$$

■

We conclude this section establishing sufficient conditions for exactness of the Han penalty function θ_σ. In Proposition 3.7 we consider the convex case and in theorem 3.8 the general one.

The following useful lemma is a direct consequence of inequality (2.4).

Lemma 3.6. *If $Z \succeq 0$ and $\sigma \geq \max\{\operatorname{Tr}(Z),\|\lambda\|\}$, then $L(\cdot,Z,\lambda) \leq \theta_\sigma(\cdot)$.*

Proposition 3.7. *Let us consider the convex problem* ($\mathrm{P_C}$)*, defined in section* 3.1*, and let us suppose that f, h and $\mathcal{A}$ are differentiable at a solution x_* of* ($\mathrm{P_C}$)*. Then, if (Z_*,λ_*) are* (KKT)*-multipliers associated with x_* and $\sigma \geq \max\{\operatorname{Tr}(Z_*),\|\lambda_*\|\}$, we have that θ_σ has a global minimum in x_*.*

Proof. Let us suppose that (x_*,Z_*,λ_*) satisfies the (KKT) conditions. For the convex problem (P_C), it can be easily seen that $L(\cdot,Z_*,\lambda_*)$ has a global minimum at x_*, that is, $\theta_\sigma(x_*) = f(x_*) = L(x_*,Z_*,\lambda_*) \leq L(x,Z_*,\lambda_*)$ for all x. If $\sigma \geq \max\{\operatorname{Tr}(Z_*),\|\lambda_*\|\}$, from lemma 3.6 we have $L(\cdot,Z_*,\lambda_*) \leq \theta_\sigma(\cdot)$, which leads to the desired result. ■

Theorem 3.8. *Let us suppose that f, h, and $\mathcal{A}$ are differentiable at x_*. Let (x_*,Z_*,λ_*) be a point that satisfies the* (KKT) *conditions and the second-order sufficient condition* (2.11)*. If $\sigma > \max\{\operatorname{Tr}(Z_*),\|\lambda_*\|\}$, then θ_σ has a strict local minimum in x_*.*

Proof. Taking $Z = Z_*$ and $W = (-\mathcal{A}(x))_+ = -(\mathcal{A}(x))_-$ in (3.4) we have that

$$\frac{r}{2}\left\|\left(-\frac{Z_*}{r} - \mathcal{A}(x)\right)_+ + \mathcal{A}(x)\right\|_{Fr}^2 + \operatorname{Tr}\left(Z_*\left[\left(-\frac{Z_*}{r} - \mathcal{A}(x)\right)_+ + \mathcal{A}(x)\right]\right) \leq \operatorname{Tr}(Z_*\mathcal{A}(x)_+) + \frac{r}{2}\|\mathcal{A}(x)_+\|_{Fr}^2. \tag{3.20}$$

Hence, using the Cauchy–Schwarz inequality and the Von Neumann–Theobald inequality (2.4), we obtain

$$\begin{aligned} L_r(x,Z_*,\lambda_*) &\leq f(x) + \lambda_*^T h(x) + \frac{r}{2}\|h(x)\|^2 + \operatorname{Tr}(Z_*\mathcal{A}(x)_+) + \frac{r}{2}\|\mathcal{A}(x)_+\|_{Fr}^2 \\ &\leq f(x) + \|h(x)\|\left(\|\lambda_*\| + \frac{r}{2}\|h(x)\|\right) \\ &\qquad + \lambda_1(\mathcal{A}(x))_+\left(\operatorname{Tr}(Z_*) + \frac{r}{2}\sum_{j=1}^{m}\lambda_j(\mathcal{A}(x)_+)\right). \end{aligned}$$

The last inequality follows from $\lambda_1(\mathcal{A}(x)_+) = \lambda_1(\mathcal{A}(x))_+$. Since $\sigma > \max\{\operatorname{Tr}(Z_*), \|\lambda_*\|\}$, for any fixed $r > 0$ there is a neighborhood V_r of x_* such that

$$L_r(x,Z_*,\lambda_*) \leq f(x) + \sigma(\|h(x)\| + \lambda_1(\mathcal{A}(x))_+) = \theta_\sigma(x) \quad \forall x \in V_r.$$

From Theorem 3.1, we know that there is an $\bar{r} > 0$ and a neighborhood $\bar{V}$ of x_* where x_* is a strict minimum of $L_{\bar{r}}(\cdot,Z_*,\lambda_*)$. This implies that x_* is a strict minimum of θ_σ on $\bar{V}\bigcap V_{\bar{r}}$. ■

4 Sequentially Semidefinite Programming

In this section we propose a global S-SDP algorithm for solving problem (P). This algorithm is inspired by the classical sequentially quadratic programming (SQP). We begin by recalling the local S-SDP algorithm proposed in [9] and its convergence theorem.

Given an initial point (x_0,Z_0,λ_0) close to a point (x_*,Z_*,λ_*) that satisfies the (KKT) conditions, we generate a sequence (x_k,Z_k,λ_k) by solving the linearized problem:

$$(\mathrm{T_k}) \quad \begin{array}{ll} \text{minimize}_{d\in \mathbb{R}^n} & \nabla f(x_k)^T d + \frac{1}{2} d^T M_k d \\ \text{subject to} & \mathcal{A}_k(d) \preceq 0, \\ & h(x_k) + \nabla h(x_k) d = 0, \end{array}$$

where $\mathcal{A}_k(d) := \mathcal{A}(x_k) + \sum_{i=1}^n d_i \frac{\partial \mathcal{A}(x_k)}{\partial x_i}$ and the matrix M_k replaces the Hessian $\nabla^2_{xx} L(x_k,Z_k,\lambda_k)$, emulating the so-called quasi-Newton methods.

If d_k is the solution of problem (T_k), we define $x_{k+1} = x_k + d_k$. The point $(d_k,Z_{k+1},\lambda_{k+1})$ is obtained from the (KKT) conditions for the minimization problem (T_k), that is

$$\nabla f(x_k) + D\mathcal{A}(x_k)^* Z_{k+1} + \nabla h(x_k)^T \lambda_{k+1} + M_k d_k = 0 \tag{4.1a}$$
$$\mathcal{A}_k(d_k) \preceq 0 \tag{4.1b}$$
$$h(x_k) + \nabla h(x_k) d_k = 0 \tag{4.1c}$$
$$Z_{k+1} \succeq 0 \tag{4.1d}$$
$$\mathrm{Tr}(Z_{k+1}\mathcal{A}_k(d_k)) = 0 \tag{4.1e}$$

These equations will be called (KKT_k) in the sequel.

Theorem 4.1. *Let (x_*,Z_*,λ_*) be a point satisfying the (KKT) conditions and the second-order sufficient condition (2.11). Suppose that $(D\mathcal{A}(x_*),\nabla h(x_*)^T)^T$ has full rank and that $M_k \to \nabla^2_{xx} L(x_*,Z_*,\lambda_*)$. Then there is $\delta > 0$ such that if $\|x_0 - x_*\| < \delta$, $\|(Z_0,\lambda_0) - (Z_*,\lambda_*)\| < \delta$ and $\|M_k - \nabla^2_{xx} L(x_*,Z_*,\lambda_*)\| < \delta$ for all k, the sequence (x_k,Z_k,λ_k) generated by the algorithm S-SDP is well defined and converges superlinearly to (x_*,Z_*,λ_*). The convergence is even quadratic if $M_k - \nabla^2_{xx} L(x_*,Z_*,\lambda_*) = O(\|x_k - x_*\| + \|(Z_k,\lambda_k) - (Z_*,\lambda_*)\|$.*

Our purpose here is to extend the S-SDP algorithm to obtain global convergence. For this, we consider the Han penalty function, defined in (3.16), and an *Armijo line search.*

In the following proposition we prove that the solution d_k of (T_k) is a descent direction for θ_σ at the point x_k when M_k is positive definite and σ is sufficiently large.

Proposition 4.2. *Suppose that f,h, and $\mathcal{A}$ are C^1 functions and that their derivatives are locally Lipschitz at x_k. Using the penalty function θ_σ, defined in (3.16), if the point $(d_k,Z_{k+1},\lambda_{k+1})$ verifies the (KKT_k) conditions, written in (4.1), then*

$$\begin{aligned}\theta'_\sigma(x_k;d_k) \leq & \nabla f(x_k)^T d_k - \sigma(\lambda_1(\mathcal{A}(x_k))_+ + \|h(x_k)\|) \\ = & -d_k^T M_k d_k + \mathrm{Tr}(Z_{k+1}\mathcal{A}(x_k)) \\ & + \lambda_{k+1}^T h(x_k) - \sigma(\lambda_1(\mathcal{A}(x_k))_+ + \|h(x_k)\|).\end{aligned} \tag{4.2}$$

Furthermore, if $\sigma \geq \max\{\mathrm{Tr}(Z_{k+1}),\|\lambda_{k+1}\|\}$ we obtain

$$\theta'_\sigma(x_k;d_k) \leq -d_k^T M_k d_k. \tag{4.3}$$

Proof. Let us fix $t \in [0,1]$. By (4.1c) we have that

$$\|h(x_k) + t\nabla h(x_k) d_k\| = (1-t)\|h(x_k)\|,$$

and by the convexity of $\lambda_1(\cdot)_+$ and (4.1b) we obtain

$$\lambda_1(\mathcal{A}_k(td_k))_+ = \lambda_1\left(\mathcal{A}(x_k) + t\sum_{i=1}^{n} d_{ki}\frac{\partial \mathcal{A}(x_k)}{\partial x_i}\right)_+$$
$$\leq (1-t)\lambda_1(\mathcal{A}(x_k))_+ + t\lambda_1(\mathcal{A}_k(d_k))_+ = (1-t)\lambda_1(\mathcal{A}(x_k))_+.$$

From these relations we have that

$$\|\cdot\|'(h(x_k);\nabla h(x_k)d_k) = -\|h(x_k)\|, \tag{4.4a}$$

$$(\lambda_1(\cdot)_+)'\left(\mathcal{A}(x_k);\sum_{i=1}^{n} d_{ki}\frac{\partial \mathcal{A}(x_k)}{\partial x_i}\right) \leq -\lambda_1(\mathcal{A}(x_k))_+. \tag{4.4b}$$

and applying Lemma 3.2 we get

$$\theta'_\sigma(x_k;d_k) \leq \nabla f(x_k)^T d_k - \sigma(\lambda_1(\mathcal{A}(x_k))_+ + \|h(x_k)\|),$$

and from (4.1a), (4.1c) and (4.1e) we obtain

$$\theta'_\sigma(x_k;d_k) \leq \mathrm{Tr}(Z_{k+1}\mathcal{A}(x_k)) + h(x_k)^T\lambda_{k+1} - \sigma(\|h(x_k)\| + \lambda_1(\mathcal{A}(x_k))_+) - d_k^T M_k d_k.$$

Finally, if $\sigma \geq \max\{\mathrm{Tr}(Z_{k+1}),\|\lambda_{k+1}\|\}$, the Cauchy–Schwarz inequality and the Von Neumann–Theobald inequality (2.4) lead to the result. ■

We are now ready to describe the iteration k of the global algorithm for solving problem (P). We suppose that x_k is known and that M_k is positive definite.

Step 1. Compute a point $(d_k,Z_{k+1},\lambda_{k+1})$ satisfying (KKT_k) in (4.1).

Step 2. Compute σ_k satisfying $\sigma_k \geq \max\{\mathrm{Tr}(Z_{k+1}),\|\lambda_{k+1}\|\}$ in such a way that the sequence $\{\sigma_k\}$ satisfies the following properties:

$$\begin{array}{ll} (a) & \sigma_k \geq \max\{\mathrm{Tr}(Z_{k+1}),\|\lambda_{k+1}\|\} + \bar{\sigma}. \\ (b) & \text{For all } k \geq k_1, \\ & \text{if } \sigma_{k-1} \geq \max\{\mathrm{Tr}(Z_{k+1}),\|\lambda_{k+1}\|\} + \bar{\sigma}, \text{ then } \sigma_k = \sigma_{k-1}. \\ (c) & \text{If } \{\sigma_k\} \text{ is bounded, then } \sigma_k \text{ is modified just finitely many times,} \end{array} \tag{4.5}$$

where $k_1 \in \mathbb{N}$ and $\bar{\sigma} > 0$ are fixed parameters. A simple way to update σ_k verifying (4.5) is defining $\sigma_k = \max\{1.5\sigma_{k-1},\ \max\{\mathrm{Tr}(Z_{k+1}),\|\lambda_{k+1}\|\}+\bar{\sigma}\}$ when (b) fails.

Step 3. The step length α_k is computed by using an *Armijo search rule*, that is, α_k is an approximation of the maximum $\alpha \in (0,1]$ which verifies

$$\theta_{\sigma_k}(x_k + \alpha d_k) \leq \theta_{\sigma_k}(x_k) + w\alpha\Delta_k, \tag{4.6}$$

where $0 < w < 1$ and Δ_k is the upper bound of $\theta'_{\sigma_k}(x_k;d_k)$ given in (4.2). More precisely, α_k can be computed as follows:

Step 0. $j := 0, \quad r_j := 1.$

Step 1. If (4.6) is satisfied with $\alpha = r_j$ then $\alpha_k = r_j$ and stop the line search.

Step 2. If not, take $r_{j+1} = \beta r_j$, increase j by one and go to Step 1 with $\beta \in (0,1)$ a fixed constant.

Step 4. Define $x_{k+1} = x_k + \alpha_k d_k$.

Remark 4.3. *The existence of the Armijo step α satisfying* (4.6) *is a consequence of inequality* (4.3) *and the fact that M_k is chosen positive definite.*

Theorem 4.4. *Let us suppose that $f, h,$ and $\mathcal{A}$ are C^1 functions and that their derivatives are Lipschitz. If we consider the global algorithm described in the steps* 1 *to* 4 *and suppose that the matrices M_k are chosen positive definite such that the sequence $\{M_k\}$ is bounded together with the sequence $\{M_k^{-1}\}$. Then one of the following situations occurs for the sequence $\{(x_k,Z_{k+1},\lambda_{k+1})\}$:*

1. *The sequences $\{\sigma_k\}$ and $\{(Z_{k+1},\lambda_{k+1})\}$ are unbounded.*
2. *There is an index k_2 such that σ_k is constant for all $k \geq k_2$. In this case one of the following situations occurs:*
   (a) $\theta_{\sigma_k}(x_k) \to -\infty$, *or*
   (b) $\nabla_x L(x_k,Z_{k+1},\lambda_{k+1}) \to 0, h(x_k) \to 0, \lambda_1(\mathcal{A}(x_k))_+ \to 0,$ *and* $\mathrm{Tr}(Z_{k+1}\mathcal{A}(x_k)) \to 0$.

Proof. 1. The equivalence between the unboundedness of $\{\sigma_k\}$ and $\{(Z_{k+1},\lambda_{k+1})\}$ is direct from (4.5) Parts (a) and (b).

2. Let us suppose that $\{\sigma_k\}$ is bounded. By (4.5)(c) we know that there is an index k_2 such that $\sigma_k = \sigma := \sigma_{k_2}$ for all $k \geq k_2$.

To conclude we prove that if (a) is not true then (b) holds. From (4.6), with $\alpha = \alpha_k$, we know that the sequence $\{\theta_{\sigma_k}(x_k)\}$ is decreasing for all $k \geq k_2$, and then $\theta_{\sigma_k}(x_k) \geq C$ for some constant C, obtaining again from (4.6) the limit $\alpha_k \Delta_k \to 0$.

All limits in (b) are consequences of the existence of $\bar{\alpha} > 0$ such that $\alpha_k \geq \bar{\alpha}$ for all $k \geq k_2$, which implies from the limit above that $\Delta_k \to 0$. Indeed, inequalities

$$\begin{aligned}\Delta_k \leq & - d_k^T M_k d_k + \lambda_1(\mathcal{A}(x_k))_+ \,\mathrm{Tr}(Z_{k+1}) \\ & \quad + \|\lambda_{k+1}\|\|h(x_k)\| - \sigma_k(\lambda_1(\mathcal{A}(x_k))_+ + \|h(x_k)\|) \\ \leq & - d_k^T M_k d_k + (\sigma_k - \bar{\sigma})(\lambda_1(\mathcal{A}(x_k))_+ + \|h(x_k)\|) \\ & \quad - \sigma_k(\lambda_1(\mathcal{A}(x_k))_+ + \|h(x_k)\|)\end{aligned}$$

prove that

$$\Delta_k \leq -d_k^T M_k d_k - \bar{\sigma}(\lambda_1(\mathcal{A}(x_k))_+ + \|h(x_k)\|) \leq 0, \tag{4.7}$$

which implies that if $\Delta_k \to 0$, then

$$\lambda_1(\mathcal{A}(x_k))_+ \to 0 \quad \text{and} \quad h(x_k) \to 0.$$

Inequality (4.7) also shows that $-d_k^T M_k d_k \to 0$, and together with the fact that $\{M_k^{-1}\}$ is bounded, it is easy to conclude that $d_k \to 0$. This, together with (4.1a) and the boundedness of $\{M_k\}$, implies that

$$\nabla_x L(x_k, Z_{k+1}, \lambda_{k+1}) \to 0.$$

Finally, by definition of Δ_k we know that

$$\nabla f(x_k)^T d_k = \Delta_k + \sigma_k(\lambda_1(\mathcal{A}(x_k))_+ + \|h(x_k)\|) \to 0,$$

and from (4.1a) we see that

$$\operatorname{Tr}\left(Z_{k+1}\sum_{i=1}^n d_{ki}\frac{\partial \mathcal{A}(x_k)}{\partial x_i}\right) + \lambda_{k+1}^T \nabla h(x_k) d_k = -d_k^T M_k d_k - \nabla f(x_k)^T d_k \to 0.$$

This limit and the equalities (4.1e) and (4.1c), together with the boundedness of $\{\lambda_{k+1}\}$, allow us to write

$$\begin{aligned}
\lim_{k\to+\infty} \operatorname{Tr}(Z_{k+1}\mathcal{A}(x_k)) &= \lim_{k\to+\infty} -\operatorname{Tr}\left(Z_{k+1}\sum_{i=1}^n d_{ki}\frac{\partial \mathcal{A}(x_k)}{\partial x_i}\right)\\
&= \lim_{k\to+\infty} \lambda_{k+1}^T \nabla h(x_k) d_k = \lim_{k\to+\infty} -\lambda_{k+1}^T h(x_k) = 0.
\end{aligned}$$

Let us prove now that $\alpha_k \geq \bar{\alpha} > 0$. If $\alpha_k < 1$, by the Armijo search rule, there is an $r_j \in (0,1]$ such that $\alpha_k = \beta r_j$ and

$$\theta_{\sigma_k}(x_k + r_j d_k) > \theta_{\sigma_k}(x_k) + w r_j \Delta_k. \tag{4.8}$$

Let us consider the first-order Taylor expansion

$$\begin{aligned}
f(x_k + r_j d_k) &= f(x_k) + r_j \nabla f(x_k)^T d_k + O(r_j^2\|d_k\|^2),\\
h(x_k + r_j d_k) &= h(x_k) + r_j \nabla h(x_k) d_k + O(r_j^2\|d_k\|^2),\\
&= (1 - r_j)h(x_k) + r_j(h(x_k) + \nabla h(x_k) d_k) + O(r_j^2\|d_k\|^2),\\
\mathcal{A}(x_k + r_j d_k) &= \mathcal{A}(x_k) + r_j \sum_{i=1}^n d_{ki}\frac{\partial \mathcal{A}(x_k)}{\partial x_i} + O(r_j^2\|d_k\|^2)\\
&= (1 - r_j)\mathcal{A}(x_k) + r_j\left(\mathcal{A}(x_k) + \sum_{i=1}^n d_{ki}\frac{\partial \mathcal{A}(x_k)}{\partial x_i}\right) + O(r_j^2\|d_k\|^2).
\end{aligned}$$

Since $r_j \leq 1$, from the convexity of $\lambda_1(\cdot)_+$ and the relations (4.1b) and (4.1c) we obtain

$$\|h(x_k + r_j d_k)\| = (1 - r_j)\|h(x_k)\| + O(r_j^2\|d_k\|^2),$$
$$\lambda_1(\mathcal{A}(x_k + r_j d_k))_+ \leq (1 - r_j)\lambda_1(\mathcal{A}(x_k))_+ + O(r_j^2\|d_k\|^2),$$

which imply from (4.8) the inequality

$$\theta_{\sigma_k}(x_k) + r_j\Delta_k + C_1 r_j^2\|d_k\|^2 > \theta_{\sigma_k}(x_k) + w r_j \Delta_k,$$

that is, $-(1-w)r_j\Delta_k < C_1 r_j^2\|d_k\|^2$ for some constant $C_1 > 0$. Due to inequality (4.7), the boundedness of $\{M_k^{-1}\}$ and the fact that M_k is positive definite, we see that $\Delta_k \leq -C_2\|d_k\|^2$ for some constant $C_2 > 0$. The last two inequalities show that

$$r_j \geq \frac{C_2}{C_1}(1 - w) > 0,$$

and the proof is complete. ■

Remark 4.5. *The notion of "global convergence" characterized by the situations* 1 *and* 2 *in Theorem 4.4 is fairly standard. However, it should remark that the "pathological" situations* 1 *and* 2(a) *can happen.*

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CHAPITRE II

Penalty and Barrier Methods for Convex Semidefinite Programming

Penalty and Barrier Methods for Convex Semidefinite Programming[1]

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Abstract. In this paper we present penalty and barrier methods for solving general convex semidefinite programming problems. More precisely, the constraint set is described by a convex operator that takes its values in the cone of negative semidefinite symmetric matrices. This class of methods is an extension of penalty and barrier methods for convex optimization to this setting. We provide implementable stopping rules and prove the convergence of the primal and dual paths obtained by these methods under minimal assumptions. The two parameters approach for penalty methods is also extended. As for usual convex programming, we prove that after a finite number of steps all iterates will be feasible.

1 Introduction.

Let S^m be the space of symmetric real $m \times m$ matrices endowed with the inner product $A \cdot B = \text{trace}(AB)$ denoting the trace of the matrix product AB, and let S^m_+ be the cone of positive semidefinite symmetric matrices. Related to S^m_+ we define the partial ordering $\succeq$ via

$$A \succeq B \quad \Leftrightarrow \quad B \preceq A \quad \Leftrightarrow \quad A - B \in S^m_+ \quad \forall A,B \in S^m.$$

We denote $A \succ 0$ or $0 \prec A$ if $A \in S^m_{++}$, the cone of positive definite symmetric $m \times m$ matrices. Similar relations can be established for S^m_- and S^m_{--}, the cones of negative semidefinite and definite symmetric $m \times m$ matrices, respectively.

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Throughout the general development, we denote by $I\!R^n$ an arbitrary finite real dimensional space, and by $\langle \cdot,\cdot \rangle$ an arbitrary inner product on $I\!R^n$.

This paper is focused on convex optimization with constraint sets described mainly by $\succeq$ convex maps, which are defined as follows: let X be a convex set in $I\!R^n$, a map $G : X \to S^m$ is said to be $\succeq$ convex if

$$G(\lambda x + (1-\lambda)y) \preceq \lambda G(x) + (1-\lambda)G(y) \quad \forall x,y \in X, \forall \lambda \in [0,1].$$

Simple examples of $\succeq$ convex maps that show the interest of this notion are affine maps as $G(x) = B + \sum_{j=1}^{n} x_j A_j$ with B, $A_j \in S^m$, or functions of the form $G(x) = B + \sum_{j=1}^{p} g_j(x) A_j$ where the $g_j(\cdot)$'s are convex functions while the A_j's are positive semidefinite matrices. Similarly, matrix convex functions, for instance $x^2 : S^m \to S^m$ and $-\log x : S^m_{++} \to S^m_{++}$, are $\succeq$ convex maps defined on a matrix space. Other examples, properties and applications of such maps can be found in the books of Bathia [5, Chapter 5], Bonnans and Shapiro [6, Chapter 5], and Ben-Tal and Nemirowski [7, Chapter 4].

Throughout this paper, we suppose that G is a $\succeq$ convex map, continuously differentiable (C^1) on $I\!R^n$, and $f_i : I\!R^n \to I\!R \cup \{+\infty\}$, $i = 0,1,\ldots,p$, are convex, lower semicontinuous(lsc) functions. Thus, we define

$$D = \{x \in I\!R^n : \ f_i(x) \leq 0, \forall i = 1,\ldots,p\}, E = \{x \in I\!R^n : \ G(x) \preceq 0\}, C = D \cap E,$$

and consider the optimization problem

$$(\mathrm{P}) \qquad v = \inf\{f_0(x) \mid x \in C\}.$$

The aim of this paper is to propose penalty and barrier methods for solving (P). Methods of this kind has been widely developed in nonlinear optimization (i.e. $C = D$). In this context, Auslender, Cominetti and Haddou [3] have proposed a unified framework containing most of the methods given in the literature. The article [3] also provides a systematic way to generate penalty and barrier methods.

In the case when $C = D \cap E$ and G is an affine map into S^m, Auslender [1] proposed a general framework for solving (P). Roughly speaking, a systematic way for building penalty and barrier functions ϕ_r with parameter $r > 0$ going ultimately to 0 was presented. These functions are defined in order to solve a family of unconstrained minimization problems of the form

$$(\mathrm{P}_r) \qquad v_r = \inf\{f_0(x) + \phi_r(x) \mid x \in I\!R^n\}.$$

In [1], the existence of optimal solutions x_r of (P_r) is guaranteed by supposing Slater's condition and the usual hypothesis that the optimal set S of (P) is nonempty and compact. Then, it was proven that the generalized sequence $\{x_r\}_{r>0}$ is bounded with all its limit points in S.

In the first part of this paper our objective is to improve the results established in [1] in three directions. Firstly, we give an implementable stopping rule that ensures the obtainment of x_r in a finite numbers of steps by any usual unconstrained descent method. This avoids the exact minimization used in [1] to obtain x_r.

Secondly, here G is no longer affine but $\succeq$ convex. Hence, the convergence analysis is now much more complicated than in the affine case. Indeed, the computation of the recession function of ϕ_r by a useful formula is actually no longer available when G is $\succeq$ convex, contrary to the case when G is affine. Unfortunately, the recession functional analysis is a key element in our approach. The only known result when G is $\succeq$ convex appears in Graña-Drummond and Peterzil [16], where they use the classical log-barrier function in semidefinite programming (SDP) composed with $G(x)$ instead of a more general penalty or barrier function $\phi_r(x)$. In [16], convergence properties are obtained under a very restrictive assumption (cf. [16, Assumption 2]). Here, the convergence is proved for general penalty and barrier functions assuming the two usual hypotheses in constrained convex programming, that is, the optimal set is nonempty and compact, and Slater's condition holds.

A third direction is the improvement of the duality results given in [1] and [3], where the exact solution of the Fenchel dual problem of (P_{r_k}) is supposed to be computed ($\{r_k\}$ is a sequence of positive real numbers going to 0). Obviously this is a theoretical result. Here we associate with x_{r_k} a multiplier Y_k given by an explicit formula. Then we prove that the sequence $\{Y_k\}$ is bounded and that each limit point of this sequence is an optimal solution of the usual Lagrangian dual of (P).

Penalty and barrier methods introduced in Section 3 are based on a smooth procedure and depend on a single parameter. This smooth procedure involves two possible classes of penalty functions. The first class deals with the indicator function of $\mathbb{R}^p_- \times S^m_-$, while the second class concerns an exact penalty function. However, when $C = D$, i.e. when we only consider the classical convex constrained programming problem, a second approach can be used. This approach is only applied to functions of the second class mentioned above and its basic idea consists of distinguishing two parameters: the "smoothness parameter" r and the penalty weight β. This two-parameter approach has been firstly developed by Xavier [23] for a specific hyperbolic function and has been also the base of a recent work of C. Gonzaga and R. A. Castillo [15]. Indeed, C. Gonzaga and R. A. Castillo introduce a method that uses a smooth approximation $\theta(\cdot)$ of the exact penalty function $t \to \max\{0,t\}$ and two parameters, r and β, so that the penalized function $\psi_{r,\beta}(x) := f_0(x) + \beta r \sum_{i=1}^m \theta\left(f_i(x)/r\right)$ is minimized at each iteration. The parameters play different roles: r always decreases in order to improve the precision of the approximation, and β increases to penalize an infeasible iteration. Thus, the aim of the second part of this article is to extend this approach to more general feasible sets C. Particularly, we consider

$C = D \cap E$ instead of $C = D$, that is, a feasible set that involves semidefinite constraints. Nevertheless, our results are an improvement of those in [15] even in the nonlinear programming case where $C = D$. Indeed firstly, we only work in the convex case which allows us to give an implementable stopping rule (this is not the case in [15]). Secondly, we do not suppose neither the assumption named "Hypothesis" in [15] nor the compactness of the feasible set. Finally, we associate with the primal sequence a dual sequence of multipliers given by an explicit formula. Hence we prove that this dual sequence is bounded with each limit point being an optimal solution of the usual Lagrangian dual of (P). Such a result is not given in [15].

The outline of this paper is as follows. In the next section we recall material concerning recession functions, convex analysis in SDP and matrix properties which will be needed in the sequel. In Section 3 we present the penalty and barrier methods, including the convergence analysis concerning the primal path. Section 4 deals primarily with the dual path. Finally in Section 5 we consider the penalty approach with two parameters.

2 Preliminaries

2.1 Asymptotic cones and functions.

We recall some basic notions about asymptotic cones and functions (see for more details the books of Auslender and Teboulle [4] and of Rockafellar [20]).

The asymptotic cone of a set $Q \subseteq I\!R^n$ is defined to be

$$Q_\infty = \{y \ : \exists t_k \to +\infty, x_k \in Q \quad \text{with } y = \lim_{k\to\infty} \frac{x_k}{t_k}\}. \tag{2.1}$$

When Q is convex and closed, it coincides with its recession cone

$$0^+(Q) := \{y \ : x + \lambda y \in Q \ \ \forall \lambda > 0, \forall x \in Q\}. \tag{2.2}$$

Let $f : I\!R^n \to I\!R \cup \{+\infty\}$ be lower-semicontinuous (lsc) and proper (i.e., $\exists x \in \operatorname{dom} f := \{x : f(x) < +\infty\}$). We recall that the asymptotic function f_∞ of f is defined by the relation

$$\text{epi} f_\infty = (\text{epi} f)_\infty,$$

where $\text{epi} f := \{(x,r) : \ f(x) \le r\}$. As a straightforward consequence, we get (cf. [4, Theorem 2.5.1])

$$f_\infty(y) = \inf \left\{ \liminf_{k\to+\infty} \frac{f(x_k t_k)}{t_k} \ : \ t_k \to +\infty, x_k \to y \right\} \tag{2.3}$$

where the sequences $\{t_k\}$ and $\{x_k\}$ belong to $I\!R$ and $I\!R^n$, respectively.

Remark 2.1. *This formula is fundamental in the convergence analysis of unbounded sequences and is often used in the following way: let $\{x_k\}$ be an unbounded sequence satisfying*

$$\lim_{k\to\infty}\|x_k\| = +\infty, \quad \lim_{k\to\infty}\frac{x_k}{\|x_k\|} = d \neq 0.$$

Suppose that $f_\infty(d) > -\infty$, and let $\alpha \in I\!R$ so that $f_\infty(d) > \alpha$. Then it follows from (2.3) that for all k sufficiently large we have

$$f(x_k) = f\left(\frac{x_k}{\|x_k\|}\|x_k\|\right) \geq \alpha\|x_k\|.$$

Note also that f_∞ is positively homogeneous, that is

$$f_\infty(\lambda d) = \lambda f_\infty(d) \quad \forall d, \forall\lambda > 0. \tag{2.4}$$

When f is a convex, lsc, proper function its asymptotic function coincides with its recession function

$$0^+ f(y) = \lim_{\lambda\to+\infty}\frac{f(x+\lambda y) - f(x)}{\lambda} \quad \forall x \in \operatorname{dom} f, \tag{2.5}$$

deducing immediately that

$$f_\infty(y) = \lim_{t\to+\infty}\frac{f(ty)}{t} \quad \forall y \in \operatorname{dom} f. \tag{2.6}$$

Furthermore, if $\partial f(x)$ denotes the (convex) subdifferential of f at x, we also have

$$f_\infty(y) = \sup\{\langle c,y\rangle \mid c \in \partial f(x),\, x \in dom\ \partial f\}. \tag{2.7}$$

Now consider the lsc functions f, $g : I\!R^n \to I\!R \cup \{+\infty\}$ satisfying $f_\infty(d) > -\infty$ and $g_\infty(d) > -\infty$. Then

$$(f+g)_\infty(d) \geq f_\infty(d) + g_\infty(d), \tag{2.8}$$

with equality in the convex case. Recall that $f_\infty(d) > -\infty$ always holds when f is convex, lsc and proper.

When f is convex, a useful consequence of (2.2) and (2.5) is the following

$$\{x \ : \ f(x) \leq \lambda\}_\infty = \{d \ : \ f_\infty(d) \leq 0\}, \tag{2.9}$$

for any λ such that $\{x \ : \ f(x) \leq \lambda\}$ is nonempty.

The following proposition is crucial in the convergence analysis. The reader can see a proof in [4, Chapter 3].

Proposition 2.2. *Let C be a closed convex set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex, lsc, proper function such that* $\operatorname{dom} f \cap C$ *is nonempty. Consider the optimization problem*

$$(\mathrm{P}) \qquad \alpha = \inf\{f(x) \mid x \in C\},$$

and let S be the optimal set of (P)*. Then a necessary and sufficient condition for S to be nonempty and compact is given by*

$$f_\infty(d) > 0 \quad \forall d \in C_\infty,\, d \neq 0,$$

or equivalently

$$\lim_{||x|| \to \infty,\, x \in C} f(x) = +\infty.$$

In this case (P) *is said to be coercive.*

In our analysis, the asymptotic function of a composite function is of a particular interest. More precisely, we will consider the composition between a penalty or barrier function θ and the $\succeq$ convex function $G(\cdot)$.

Let us consider the following class of functions F introduced in [3] by Auslender, Cominetti, and Haddou

$$F = \{\theta : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}, \text{ lsc, convex, proper and non-decreasing with}$$
$$\theta_\infty(1) > 0,\ \lim_{t \to \eta^-} \theta(t) = +\infty, \text{ and } \operatorname{dom}\theta =]-\infty, \eta[\text{ where } \eta \in [0, +\infty]\}.$$

We divide F into two subclasses F_1 and F_2 (cf. [3] and [9], respectively) defined by

$$F_1 = \{\theta \in F :\ \theta \text{ is } C^1 \text{ on } \operatorname{dom}\theta,\, \theta_\infty(1) = +\infty,\, \theta_\infty(-1) = 0\},$$
$$F_2 = \{\theta \in F :\ \operatorname{dom}\theta = \mathbb{R},\, \theta \text{ is } C^1,\, \theta_\infty(1) = 1,\ \lim_{t \to -\infty} \theta(t) = 0\}.$$

For example, the functions

$\theta_1(u) = \exp(u)$,	$\operatorname{dom}\theta = \mathbb{R}$	$\to$	exponential penalty [10],
$\theta_2(u) = -\log(1-u)$,	$\operatorname{dom}\theta =]-\infty, 1[$,	$\to$	modified log barrier [19],
$\theta_3(u) = \frac{u}{1-u}$,	$\operatorname{dom}\theta =]-\infty, 1[$,	$\to$	hyperbolic modified barrier [8],
$\theta_4(u) = -\log(-u)$,	$\operatorname{dom}\theta =]-\infty, 0[$,	$\to$	log barrier [12],
$\theta_5(u) = -u^{-1}$,	$\operatorname{dom}\theta =]-\infty, 0[$	$\to$	inverse barrier method [11],

belong to the class F_1, while the functions

$$\theta_6(u) = \log(1 + \exp(u)), \quad \theta_7(u) = 2^{-1}(u + \sqrt{u^2 + 4})$$

belong to F_2. Furthermore, systematic ways to generate classes of functions θ belonging either to F_1 or to F_2 are described in [3] and [9].

The following result was proved in [3].

Proposition 2.3. *Let $\theta \in F$, f be a convex, lsc, proper function with* $\operatorname{dom}\theta \cap f(I\!R^n) \neq \emptyset$ *and consider the composite function*

$$g(x) = \theta(f(x)) \quad \textit{if } x \in \operatorname{dom} f, \quad +\infty \quad \textit{otherwise.}$$

Then the function g is a convex, lsc, proper function and we have

$$g_\infty(d) = \theta_\infty(f_\infty(d)) \quad \textit{if } d \in \operatorname{dom} f_\infty, \quad +\infty \quad \textit{otherwise.}$$

2.2 Convex Analysis over the cone of symmetric semidefinite positive matrices

Let S^m be equipped with the inner product $A \cdot B := \operatorname{trace}(AB)$ where $\operatorname{trace}(A)$ denotes the trace of the matrix A. Let $A \in S^m$ with the eigenvalue decomposition $A = Q\Lambda Q^t$. Thus Q is an orthogonal matrix whose columns q_i, $i = 1, \ldots, m$, are the orthonormalized eigenvectors of A, and Λ is a diagonal matrix whose entries $\lambda_i(A)$, $i = 1, \ldots, m$, are the eigenvalues of A in nonincreasing order.

Let $c_i(A) := q_i q_i^t$. The spectral decomposition of A can be written as

$$A = \sum_{i=1}^{m} \lambda_i(A) c_i(A).$$

Now let $g : I\!R \to I\!R \cup \{+\infty\}$. For any $A \in S^m$ such that $\lambda_i(A) \in \operatorname{dom} g$ for each i, we set

$$g^\circ(A) := \sum_{i=1}^{m} g(\lambda_i(A)) c_i(A), \tag{2.10}$$

the usual matrix function associated with g. We are particularly interested here in the function $\Psi_g : S^m \to I\!R \cup \{+\infty\}$ defined by

$$\Psi_g(A) = \sum_{i=1}^{m} g(\lambda_i(A)) \ \text{ if } \lambda_i(A) \in \operatorname{dom} g \text{ for each } i, \quad +\infty \ \text{ otherwise,} \tag{2.11}$$

or equivalently

$$\Psi_g(A) = \operatorname{trace}(g^\circ(A)) \ \text{ if } \lambda_i(A) \in \operatorname{dom} g \text{ for each } i, \quad +\infty \ \text{ otherwise.}$$

The function Ψ_g is a spectrally defined function and the following properties hold (see e.g. [2, Proposition 2.2])

Proposition 2.4. *Suppose that $g \in F$. Then*

(i) Ψ_g *is a proper, lsc, convex function.*

(ii) $\operatorname{dom}\Psi_g$ *is open.*

(iii) *If g is C^1 on* $\operatorname{dom} g$*, then Ψ_g is C^1 on $dom\Psi_g$ with $\nabla\Psi_g(A) = (g')^\circ(A)$, for all $A \in \operatorname{dom}\Psi_g$.*

(iv) $(\Psi_g)_\infty(D) = \Psi_{g_\infty}(D)$*, for all D.*

(v) *For $g \in F$, Ψ_g is isotone, i.e., $A \succeq B \Rightarrow \Psi_g(A) \geq \Psi_g(B)$.*

(vi) *For all $D \in S^m$ it holds that*

$$(\Psi_\theta)_\infty(D) = \delta_{S^m_-}(D), \quad \text{if} \quad \theta \in F_1, \tag{2.12}$$

$$= \Psi_{a^+}(D), \quad \text{if} \quad \theta \in F_2, \tag{2.13}$$

where $\delta_{S^m_-}$ is the indicator function of $S^m_- = -S^m_+$ and where $a^+ =$ $\max(0,a)$ *with $a \in I\!R$.*

Consider the functions $\theta \in F$ given in Section 2.1 and set $L := \Psi_\theta$. For $\theta \in F_1$, we have the following examples from [1]:

$L_1(D) = \operatorname{trace}(\exp D)$,
$L_2(D) = -\log(\det(I - D))$ if $D \prec I$, $+\infty$ otherwise,
$L_3(D) = \operatorname{trace}((I - D)^{-1}D)$ if $D \prec I$, $+\infty$ otherwise,
$L_4(D) = -\log(\det(-D))$ if $D \prec 0$, $+\infty$ otherwise,
$L_5(D) = \operatorname{trace}(-D^{-1})$ if $D \prec 0$, $+\infty$ otherwise.

And for $\theta \in F_2$ we get

$$L_6(D) = \log(\det(I + \exp D)), \quad L_7(D) = \operatorname{trace}\left(\frac{D + \sqrt{D^2 + 4I}}{2}\right).$$

It is worthwhile to note that L_4 is the classical log-barrier function used in semidefinite programming (see, for example, [13]).

To end this subsection we recall two characterizations of $\succeq$ convexity. First, it is easy to show that $G : I\!R^n \to S^m$ is $\succeq$ convex iff for each $u \in I\!R^m$ the map $x \to u^t G(x) u$ is convex. Then, if in addition G is continuously differentiable (C^1), these last assertions are also equivalent to

$$u^t G(y) u \geq u^t G(x) u + u^t DG(x)(y - x) u \quad \forall x,y \in I\!R^n, \forall u \in I\!R^m. \tag{2.14}$$

2.3 Matrix Properties Review

We start this section recalling the well-known Debreu's lemma.

Lemma 2.5. (Debreu's lemma) *Let $A \preceq 0$, we have that $v^t B v < 0$, for all $v \in \operatorname{Ker} A \setminus \{0\}$ if and only if there exists $r > 0$ such that $B + rA \prec 0$.*

Consider a symmetric matrix $A \in S^m$. Let $l_0(A)$ and $l_+(A)$ be the number of their null and nonnegative eigenvalues, respectively, and let $E(A) \in I\!R^{m \times l_0(A)}$ and $E^+(A) \in I\!R^{m \times l_+(A)}$ be matrices whose columns are orthonormalized eigenvectors of A associated with their null and nonnegative eigenvalues, respectively. The following relations are directly established

$$\operatorname{Im} E(A) = \operatorname{Ker} A \subseteq \operatorname{Im} E^+(A) = \operatorname{Im} A^+ = \operatorname{Ker} A^-,$$

and hence

$$l_0(A) = \dim(\operatorname{Ker} A) \leq l_+(A) = \dim(\operatorname{Im} A^+) = \dim(\operatorname{Ker} A^-),$$

where A^+ (A^-) denotes the orthogonal projection of $A \in S^m$ onto the cone S^m_+ (S^m_-) of $m \times m$ positive (negative) semidefinite symmetric matrices. This is given by

$$A^+ := Q \ \mathrm{diag}(\lambda_1(A)^+,...,\lambda_m(A)^+)Q^t,$$

where Q is an orthogonal matrix such that its i-th column is an eigenvector of A associated with $\lambda_i(A)$. Matrix A^- is similarly stated.

So, if $A \preceq 0$, then $A = A^-$ obtaining that $\operatorname{Im} E(A) = \operatorname{Im} E^+(A)$ and $l_0(A) = l_+(A) = \dim(\operatorname{Ker} A)$.

When $x \in I\!R^n$, similar relations hold for $E(G(x))$ and $E^+(G(x))$.

Lemma 2.6. *Consider a matrix $\tilde{A} \preceq 0$. If $A_k \to \tilde{A}$, then for all k sufficiently large, we have that $l_+(A_k) \leq l_0(\tilde{A})$.*

Proof. It is a direct consequence of the continuity of the eigenvalue function $\lambda_i(\cdot)$. Indeed, by definition we have $\lambda_i(\tilde{A}) < 0$ for all $i = l_0(\tilde{A}) + 1,...,m$. Since $A_k \to \tilde{A}$, it follows that $\lambda_i(A_k) < 0$ for all $i = l_0(\tilde{A}) + 1,...,m$ and for all k sufficiently large, i.e., the matrix A_k has at least $m - l_0(\tilde{A})$ negatives eigenvalues, that is, $l_+(A_k) \leq l_0(\tilde{A})$. ■

The next lemma will be very useful in the rest of this article. Its proof appears in [6, Ex. 3.140] and is included here in order to make this work as selfcontained as possible.

Lemma 2.7. *Consider a matrix $\tilde{A} \preceq 0$. If $A_k \to \tilde{A}$, then we can construct a matrix $E_k \in I\!R^{m \times l_0(\tilde{A})}$ whose columns are an orthonormal basis of the space spanned by the eigenvectors associated with the $l_0(\tilde{A})$ biggest eigenvalues of A_k, such that $E_k \to E(\tilde{A})$.*

Proof. Consider $\tilde{E} := E(\tilde{A})$ and $\tilde{l} := l_0(\tilde{A}) = l_+(\tilde{A})$ (because $\tilde{A} \preceq 0$). For a given A, let $e_1(A),...,e_{\tilde{l}}(A)$ be a set of orthonormal eigenvectors of A associated with their $\tilde{l}$ biggest eigenvalues $\lambda_1(A) \geq ... \geq \lambda_{\tilde{l}}(A)$. Denote by $L(A)$ the space spanned by the eigenvectors $e_1(A),...,e_{\tilde{l}}(A)$ and let $P(A)$ be the orthogonal projection matrix onto $L(A)$. Note that $L(\tilde{A}) = \operatorname{Im} \tilde{E} = \operatorname{Ker} \tilde{A}$.

It is known that the projection matrix $P(A)$ is a continuous (and even analytic) function of A in a sufficiently small neighborhood of $\tilde{A}$ (see, for example, [17, Theorem 1.8] and [14, Corollary 8.1.11]). Consequently the function $F(A) := P(A)\tilde{E}$ is also a continuous function of A in a neighborhood of $\tilde{A}$, and moreover $F(\tilde{A}) = \tilde{E}$. It follows that for all A sufficiently close to $\tilde{A}$, the rank of $F(A)$ is equal to the rank of $F(\tilde{A}) = \tilde{E}$, i.e., $\operatorname{rank} F(A) = \tilde{l}$. It means that the $\tilde{l}$ columns of $F(A)$ are linearly independent when A is sufficiently close to $\tilde{A}$. Now let $U(A)$ be a matrix whose columns are obtained by applying the Gram-Schmidt orthonormalization process to the columns of $F(A)$. The matrix

$U(A)$ is well defined and continuous in a neighborhood of $\tilde{A}$. Even more, the matrices $U(A)$ satisfy that their columns are orthonormalized, i.e. $U(A)^t U(A) = I_{\tilde{l}}$, and $\operatorname{Im} U(A) = L(A)$, for all A sufficiently close to $\tilde{A}$. We also have that $U(\tilde{A}) = F(\tilde{A}) = \tilde{E}$. Hence the theorem follows by setting $E_k := U(A_k)$. ■

From Lemmas 2.2 and 2.3 we get directly the following corollary concerning a feasible set $C = \{x : \; G(x) \preceq 0\}$ where $G : \mathbb{R}^n \to S^m$ is $\succeq$ convex and continuous.

Corollary 2.8. *Consider a point $\bar{x}$ such that $G(\bar{x}) \preceq 0$. If $x_k \to \bar{x}$, then for all k sufficiently large, we have that $l_+(G(x_k)) \leq l_0(G(\bar{x}))$. Furthermore, we can construct a matrix $E_k \in \mathbb{R}^{m \times l_0(G(\bar{x}))}$ whose columns are an orthonormal basis of the space spanned by the eigenvectors associated with the $l_0(G(\bar{x}))$ biggest eigenvalues of $G(x_k)$, such that $E_k \to E(G(\bar{x}))$.*

The notions introduced in this subsection allow us to characterize Slater's condition: there exists x^0 such that $G(x^0) \prec 0$, as follows.

Proposition 2.9. *Suppose that G is a $\succeq$ convex map C^1 on $\mathbb{R}^n$. Then Slater's condition is equivalent to* Robinson's constraint qualification *condition*

$$\text{for all } \bar{x} \text{ such that } G(\bar{x}) \preceq 0 \text{ there exists } \bar{h} \in \mathbb{R}^n \text{ such that } G(\bar{x}) + DG(\bar{x})\bar{h} \prec 0. \tag{2.15}$$

Moreover, Robinson's condition (2.15) *is always equivalent to*

$$\begin{aligned} &\text{for all } \bar{x} \text{ such that } G(\bar{x}) \preceq 0 \text{ there exists } \bar{h} \in \mathbb{R}^n \text{ such that} \\ &\qquad\qquad E(G(\bar{x}))^t DG(\bar{x})\bar{h} E(G(\bar{x})) \prec 0. \end{aligned} \tag{2.16}$$

Proof. That Robinson's condition (2.15) implies Slater's condition is well-known and follows directly from the differentiability of G and the convexity of the set S^m_-. This is true even when G is not $\succeq$ convex. Conversely, Slater's condition and inequality (2.14) implies in a straightforward way condition (2.15). Finally, the equivalence between conditions (2.15) and (2.16) is due to Debreu's lemma 2.5. ■

3 Penalty and barrier methods: description and convergence analysis

For the sake of simplicity we consider here the optimization problem (P) described in the introduction when $C = E$, i.e., problem (P) only contains semidefinite constraints. Then throughout this paper $G : \mathbb{R}^n \to S^m$ is a $\succeq$ convex map C^1 on $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 convex function, and we consider the optimization problem

$$(\text{P}) \qquad v = \inf\{f(x) \mid x \in C\},$$

where $C = \{x \in \mathbb{R}^n : \; G(x) \preceq 0\}$.

Indeed, if we define $D = \{x \in I\!R^n \ : \ F(x) \preceq 0\}$ when $F(x)$ is the diagonal matrix whose entries are given by the functions f_i's (obviously $F(\cdot)$ is a $\preceq$ convex map), then the constraint set $C = D \cap E$ is given by a convex operator that takes its values in S^m_-.

From now on we assume

(A_1) The optimal set of (P), denoted by S, is nonempty and compact,

(A_2) Slater's condition holds, i.e. there exists x^0 such that $G(x^0) \prec 0$.

Let $r > 0$ be a penalty parameter which will ultimately go to 0 and $\alpha : I\!R_+ \to I\!R_+$ such that

$$\lim_{r\to 0^+} \alpha(r) = 0 \quad \text{and} \quad \liminf_{r\to 0^+} \frac{\alpha(r)}{r} > 0. \tag{3.1}$$

We associate with each $\theta \in F$ the function $\Psi_\theta : S^m \to I\!R \cup \{+\infty\}$ given by formula (2.11), and define the function $H^r : I\!R^n \to I\!R \cup \{+\infty\}$ by

$$H^r(x) = \Psi_\theta\left(\frac{G(x)}{r}\right) = \sum_{i=1}^{m} \theta\left(\frac{\lambda_i(G(x))}{r}\right), \tag{3.2}$$

where $\lambda_i(A)$ denotes the i-th eigenvalue in nonincreasing order of A ($\lambda_1(A)$ is the largest eigenvalue of A).

In this section, we study methods that consist of solving "approximatively" the unconstrained minimization problems

$$(\mathrm{P}_r) \qquad v_r = \inf\{\phi_r(x) \mid x \in I\!R^n\}, \quad \text{where } \phi_r(x) = f(x) + \alpha(r) H^r(x). \tag{3.3}$$

It is worthwhile to note that when $C = D$ we recover the methods introduced in [3].

As in [1, 3], we consider two classes of methods; $\theta \in F_1$ and $\theta \in F_2$.

Throughout we denote by S_r the optimal set of (P_r) and assume that

$$\alpha(r) = r, \quad \text{if } \theta \in F_1 \quad \text{and} \quad \lim_{r\to 0^+} \frac{\alpha(r)}{r} = +\infty, \quad \text{if } \theta \in F_2. \tag{3.4}$$

More precisely, we set

$$r_k > 0, \epsilon_k \geq 0, \gamma_k > 0 \ \text{ with } \lim_{k\to\infty} \epsilon_k = \lim_{k\to\infty} \gamma_k = \lim_{k\to\infty} r_k = 0. \tag{3.5}$$

Solving approximatively (P_{r_k}) means to compute $x_k \in I\!R^n$ such that if we set $\eta_k := \nabla \phi_{r_k}(x_k) = \nabla f(x_k) + \alpha(r_k) \nabla H^{r_k}(x_k)$ then

$$||\eta_k|| \leq \epsilon_k, \quad ||\eta_k|| \cdot ||x_k|| \leq \gamma_k. \tag{3.6}$$

Note that if the optimal set S_k of (P_{r_k}) is nonempty and compact, then any usual descent method (gradient type, Newton or quasi-Newton type method) provides in a finite number of steps a point x_k satisfying the stopping rule (3.6). Consequently, we will prove first that S_k is nonempty and compact. Indeed, it is true for all k when $\theta \in F_1$, and for k sufficiently large when $\theta \in F_2$. The next proposition will be a key result on this subject, and also for other purposes.

Proposition 3.1. *For $i = 1, \ldots, m$ and $r > 0$, let $\tilde{\lambda}_i(x) = \lambda_i(G(x))$, $h_i^r(x) = \theta\left(\frac{\tilde{\lambda}_i(x)}{r}\right)$. Then*

(i) $\tilde{\lambda}_i(\cdot)$ and $h_i^r(\cdot)$ are continuous functions on $\mathbb{R}^n$.
(ii) $(\tilde{\lambda}_i)_\infty(d) > -\infty$, for all d.
(iii) $(h_i^r)_\infty(d) \geq 0$, for all d.
(iv) $\tilde{\lambda}_1$ is a convex continuous function on $\mathbb{R}^n$ and

$$(\tilde{\lambda}_1)_\infty(d) \leq 0 \quad \textit{iff} \quad d \in C_\infty. \tag{3.7}$$

Furthermore, h_1^r is an lsc proper convex function, and for each $d \in \mathbb{R}^n$ we have

$$(h_1^r)_\infty(d) = \frac{(h_1^1)_\infty(d)}{r} = \begin{cases} \delta_{\mathbb{R}_-}((\tilde{\lambda}_1)_\infty(d)) & \text{if } \theta \in F_1, \\ \frac{((\tilde{\lambda}_1)_\infty(d))^+}{r} & \text{if } \theta \in F_2. \end{cases} \tag{3.8}$$

Proof. (i) Since $\lambda_i(\cdot)$ and $G(\cdot)$ are continuous, their composition $\tilde{\lambda}_i(\cdot)$ is also continuous. In order to prove that $h_1^r(\cdot)$ is continuous, let $y = \lim_{k\to\infty} y_k$, then since $\tilde{\lambda}_i(\cdot)$ is continuous we have $\frac{\tilde{\lambda}_i(y_k)}{r} \to \frac{\tilde{\lambda}_i(y)}{r}$. If $\frac{(\tilde{\lambda}_i)(y)}{r} \notin \partial \operatorname{dom}\theta$ then, by continuity of θ on $\operatorname{int}\operatorname{dom}\theta$, we have $h_i^r(y_k) \to h_i^r(y)$. If $\frac{\tilde{\lambda}_i(y)}{r} \in \partial \operatorname{dom}\theta$, that is, $\frac{\tilde{\lambda}_i(y)}{r} = \eta$, the same limit holds thanks to the property $\lim_{u\to\eta^-} \theta(u) = +\infty$.

(ii) Let $d' \to d, t \to +\infty$, and let x^0 satisfy Slater's condition (A_2). Since G is $\succeq$ convex, for each $u \in \mathbb{R}^n$ we get (cf. (2.14))

$$u^t G(td')u \geq u^t G(x^0)u + u^t DG(x^0)(td' - x^0)u.$$

Taking $u = u_i$ such that $||u_i|| = 1$ and $G(td')u_i = \lambda_i(G(td'))u_i$, this last inequality yields

$$\frac{\lambda_i(G(td'))}{t} \geq -\frac{||G(x^0)||}{t} - ||DG(x^0)|| \cdot \left\| d' - \frac{x^0}{t} \right\|. \tag{3.9}$$

Passing to the liminf in (3.9) we obtain

$$(\tilde{\lambda}_i)_\infty(d) = \liminf_{t\to\infty,\, d'\to d} \frac{\lambda_i(G(td'))}{t} \geq -||DG(x^0)|| \cdot ||d||.$$

(iii) Since θ is nondecreasing we have from (3.9) with $G(\cdot)/r$ instead of $G(\cdot)$ that

$$\frac{1}{t} h_i^r(td') \geq \frac{1}{t}\theta\left(\frac{t}{r}\left[-\frac{||G(x^0)||}{t} - ||DG(x^0)|| \cdot \left\| d' - \frac{x^0}{t} \right\|\right]\right).$$

Passing to the liminf in this last inequality and using formula (2.3) we get

$$(h_i^r)_\infty(d) = \liminf_{t\to\infty,\ d'\to d} \frac{h_i^r(td')}{t} \geq$$
$$\liminf_{t\to\infty,\ u\to -\frac{1}{r}||DG(x^0)||\cdot||d||} \frac{\theta(tu)}{t} = \theta_\infty\left(-\frac{1}{r}||DG(x^0)||\cdot||d||\right),$$

and, by virtue of the inequality $\theta_\infty \geq 0$, it follows that $(h_i^r)_\infty(d) \geq 0$.

(iv) Since $\tilde{\lambda}_1(x) = \max\{u^t G(x) u\ ;\ \|u\| = 1,\ u \in I\!R^m\}$ and since G is $\succeq$ convex, we have that $\tilde{\lambda}_1(\cdot)$ is convex as a supremum of convex functions. Furthermore since $C = \{x :\ \tilde{\lambda}_1(x) \leq 0\}$, it follows from (2.9) that $C_\infty = \{d : (\tilde{\lambda}_1)_\infty(d) \leq 0\}$ and then equivalence (3.7) holds.

So, by Proposition 2.3 we get that $h_i^r(\cdot)$ is lsc, convex and proper. Moreover, since θ_∞ is positively homogeneous, and $\operatorname{dom}\theta_\infty$ is either equal to $I\!R_-$ or $I\!R$, using again Proposition 2.3 we obtain

$$(h_1^r)_\infty(d) = \frac{1}{r}\theta_\infty((\tilde{\lambda}_1)_\infty(d)) \ \text{ if } (\tilde{\lambda}_1)_\infty(d) \in \operatorname{dom}\theta_\infty, \quad +\infty \ \text{ otherwise},$$

so that

$$(h_1^r)_\infty(d) = \frac{(h_1^1)_\infty(d)}{r}.$$

Finally, equality (3.8) is a immediately consequence of these formulas and the definition of θ_∞. ■

Now we proceed to prove that the optimal set S_r is nonempty and compact. As we mentioned before, this condition is enough to show that the rule defining the point x_k is implementable.

Theorem 3.1. (i) *Suppose that either $\theta \in F_1$, or $\theta \in F_2$ and $f_\infty(d) \geq 0$ for all d. Then S_r is nonempty and compact for all $r > 0$.*
(ii) *If $\theta \in F_2$ then S_r is nonempty and compact for all $r > 0$ sufficiently small.*

Proof. (i) By Proposition 3.1 we have $(h_i^r)_\infty(d) \geq 0$, for all d, $i = 1,\ldots,m$ and $r > 0$, and since $\phi_r(x) = f(x) + \alpha(r)\sum_{i=1}^m h_i^r(x)$ we have from inequality (2.8) and formula (3.8) that

$$(\phi_r)_\infty(d) \geq f_\infty(d) + \frac{\alpha(r)}{r}(h_1^1)_\infty(d) \quad \forall d. \tag{3.10}$$

Suppose that $\theta \in F_1$. We get from (3.10) and Proposition 3.1 Part (iv) that

$$(\phi_r)_\infty(d) \geq f_\infty(d) \ \text{ if } \ d \in C_\infty, \quad (\phi_r)_\infty(d) = +\infty \ \text{ otherwise.}$$

Hence, since S is nonempty and compact it follows from Proposition 2.1 that $(\phi_r)_\infty(d) > 0$, for all $d \neq 0$, which is equivalent to saying that S_r is nonempty and compact.

Now suppose that $\theta \in F_2$ and $f_\infty(d) \geq 0$, for all d. Inequality (3.10) and Proposition 3.1 Part (iv) imply again that $(\phi_r)_\infty(d) > 0$, for all $d \neq 0$, and the same conclusion holds.

(ii) Assume that $\theta \in F_2$. We shall prove that S_r is nonempty and compact for $r > 0$ sufficiently small. By contradiction, suppose the existence of sequences $r_k \to 0^+$, $d_k \to d \neq 0$ such that

$$f_\infty(d_k) + \frac{\alpha(r_k)}{r_k}(h_1^1)_\infty(d_k) \leq 0.$$

Due to the lower semicontinuity of f_∞ and $(h_1^1)_\infty$, and the fact that $\liminf_{k\to\infty} \frac{\alpha(r_k)}{r_k} = +\infty$, we apply liminf to the last inequality to obtain $(h_1^1)_\infty(d) = 0$ and $f_\infty(d) \leq 0$. However, Proposition 3.1 tells us that $(h_1^1)_\infty(d) = 0$ is equivalent to $d \in C_\infty$ implying that $f_\infty(d) \leq 0$ for some $d \in C_\infty$, $d \neq 0$, which is impossible because S is nonempty and compact. ■

Remark 3.2. (i) *Note that if f is an extended lsc function satisfying that $\inf\{f(x) \mid x \in \mathbb{R}^n\} > -\infty$, then condition $f_\infty(d) \geq 0$, for all d, always holds.*

(ii) *When $\theta \in F_2$ and is strictly increasing (which is the case of all the current examples), we can suppose, without loss of generality, that $f_\infty(d) \geq 0$ for all d. Indeed, if we set $g(x) := \theta(f(x))$, then problem* (P) *is equivalent to convex problem*

$$(\mathrm{P}_s) \qquad \alpha = \inf\{g(x) \mid x \in C\}$$

in the sense that problems (P) *and* (P_s) *share the same optimal set. This is due to the strict monotonicity of function θ. Hence condition $g_\infty(d) \geq 0$ for all d, follows from the fact that θ is nonnegative.*

Theorem 3.2. *Let $\{x_k\}$ be a sequence satisfying relations (3.6). Then, this sequence is bounded and each limit point of this sequence is an optimal solution of* (P).

Proof. Let x^0 be an arbitrary interior point of C (i.e. x^0 satisfies Slater's condition (A_2)). Since function $x \to \phi_r(x) = f(x) + \alpha(r)H^r(x)$ is convex, it follows from the definition of x_k and $\eta_k = \nabla\phi_{r_k}(x_k)$ (cf. (3.6)) that

$$f(x_k) + \alpha(r_k)H^{r_k}(x_k) \leq f(x^0) + \alpha(r_k)H^{r_k}(x^0) + \langle \eta_k, x_k - x^0\rangle,$$

Hence, as a consequence of the monotonicity of θ we get for k sufficiently large

$$f(x_k) + \frac{\alpha(r_k)}{r_k}\sum_{i=1}^{m} r_k\theta\left(\frac{\lambda_i(G(x_k))}{r_k}\right) \leq f(x^0) + m\alpha(r_k)\theta(\lambda_1(G(x^0))) + \langle \eta_k, x_k - x^0\rangle. \tag{3.11}$$

First, we proceed to prove that the sequence $\{x_k\}$ is bounded. We argue by contradiction. Without loss of generality we can assume that

$$||x_k|| \to +\infty, \quad \lim_{k\to\infty}\frac{x_k}{||x_k||} = d \neq 0,$$

Proposition 3.1 Part (ii) says that $(\tilde{\lambda}_i)_\infty(d) > -\infty$. So, we define $\epsilon_i < (\tilde{\lambda}_i)_\infty(d)$. By formula (2.3) (see Remark 2.1) we have for all k sufficiently large

$$\tilde{\lambda}_i(x_k) = \tilde{\lambda}_i\left(\frac{x_k}{||x_k||}||x_k||\right) \geq \epsilon_i ||x_k||.$$

By dividing (3.11) by $||x_k||$ we obtain from the last inequality

$$\frac{1}{||x_k||} f\left(\frac{x_k}{||x_k||}||x_k||\right) + \frac{\alpha(r_k)}{r_k}\sum_{i=1}^m \frac{r_k}{||x_k||}\,\theta\left(\frac{\epsilon_i ||x_k||}{r_k}\right) \leq$$
$$\frac{f(x^0)}{||x_k||} + m\frac{\alpha(r_k)}{||x_k||}\theta(\lambda_1(G(x^0)) + \frac{\langle \eta_k, x_k - x^0\rangle}{||x_k||}.$$

Taking the limit when $k \to +\infty$ and using relations (3.5)-(3.6) and formula (2.3) we get

$$f_\infty(d) + \lim_{k\to\infty}\frac{\alpha(r_k)}{r_k}\sum_{i=1}^m \theta_\infty(\epsilon_i) \leq 0. \tag{3.12}$$

Now recall that if $\theta \in F_1$, then $\alpha(r) = r$, $\theta_\infty(-1) = \theta_\infty(0) = 0$ and $\theta_\infty(1) = +\infty$. Then we obtain from (3.12) that

$$\theta_\infty(\epsilon_i) = 0. \tag{3.13}$$

In the case when $\theta \in F_2$, we have $\lim_{k\to\infty}\frac{\alpha(r_k)}{r_k} = +\infty$, $\theta_\infty(-1) = \theta_\infty(0) = 0$ and $\theta_\infty(1) = 1$, and therefore (3.13) also holds. Thus, inequality (3.12) implies that $f_\infty(d) \leq 0$. Furthermore, since θ_∞ is positively homogeneous it follows from (3.13) that $\epsilon_i \leq 0$. Hence, letting $\epsilon_1 \to (\tilde{\lambda}_1)_\infty(d)$ we get that $(\tilde{\lambda}_1)_\infty(d) \leq 0$, which is equivalent to $d \in C_\infty$ (cf. Proposition 3.1). This together with $f_\infty(d) \leq 0$ and $d \neq 0$ implies a contradiction with the fact that the optimal solution set S is nonempty and compact.

We have proved that the sequence $\{x_k\}$ is bounded. Now let x be an limit point of the sequence $\{x_k\}$. For simplicity of notation, we suppose that $x = \lim_{k\to\infty} x_k$. We shall show that x is an optimal solution of (P).

Let $\delta < f(x)$, $\delta_i < \lambda_i(G(x))$ for all $i = 1, \ldots, m$. By continuity of functions f and $\lambda_i(G(\cdot))$, we have for all k sufficiently large that

$$\delta < f(x_k),\, \delta_i < \lambda_i(G(x_k)) \quad \forall i = 1, \ldots, m.$$

Then, from inequalities (3.6) and (3.11), and the monotonicity of θ it follows

$$\delta + \frac{\alpha(r_k)}{r_k}\sum_{i=1}^m r_k\theta\left(\frac{\delta_i}{r_k}\right) \leq f(x^0) + m\alpha(r_k)\theta(\lambda_1(G(x^0))) + (\epsilon_k||x^0|| + \gamma_k). \tag{3.14}$$

On the other hand, the following relations are satisfied (cf. (3.1) and (3.5))

$$\lim_{k\to\infty}\epsilon_k = \lim_{k\to\infty}\gamma_k = \lim_{k\to\infty}\alpha(r_k) = \lim_{k\to\infty} r_k = 0.$$

So, passing to the liminf in (3.14) we get

$$\delta + \lim_{k\to\infty} \frac{\alpha(r_k)}{r_k} \sum_{i=1}^{m} \theta_\infty(\delta_i) \le f(x^0),$$

which implies that $\theta_\infty(\delta_i) = 0$, for all i, and also $\delta \le f(x^0)$. In particular, $\theta_\infty(\delta_1) = 0$ which means that $\delta_1 \le 0$. Hence, by letting $\delta \to f(x)$ and $\delta_1 \to \lambda_1(G(x))$ we deduce that

$$x \in C \quad \text{and} \quad f(x) \le f(x^0) \quad \forall x^0 \in \operatorname{int} C.$$

Finally, continuity of f implies that $f(x) \le f(u)$ for all $u \in C$, that is, x is an optimal solution of (P). We thus obtain the desired result. ■

4 Duality Results

We associate with problem (P), defined in Section 3, the following Lagrangian functional

$$L(x,Y) = f(x) + Y \cdot G(x) \quad \forall x \in I\!R^n, \forall Y \in S^m,$$

as well as the following dual functional

$$p(Y) = -\inf\{L(x,Y) \mid x \in I\!R^n\} \;\text{ if } Y \succeq 0, \quad +\infty \;\text{ otherwise.}$$

Thus the (Lagrangian) dual problem of (P) is given by

$$\text{(D)} \qquad \gamma = \inf\{p(Y) \mid Y \in S^m\}.$$

As in Section 3, we suppose that f is a C^1 convex function, G is $\succeq$ convex and that Assumptions $(A_1) - (A_2)$ and (3.4) still hold. Thus, if the primal path $\{x_k\}$ satisfies the stopping rule (3.6), the convergence Theorem 3.2 still tells us that the sequence $\{x_k\}$ is bounded and that each of its limit points is an optimal solution of (P). It is also well known that there is no duality gap between (P) and (D), and that the set T of optimal solutions of (D) is nonempty and compact under these assumptions (see e.g. [6, Theorem 5.81]). Furthermore, the matrix $\bar{Y} \succeq 0$ will be an optimal solution of (D) iff there exists $\bar{x} \in C$ such that

$$\nabla_x L(\bar{x},\bar{Y}) = \nabla f(\bar{x}) + DG(\bar{x})^t \bar{Y} = 0 \quad \text{and} \quad G(\bar{x}) \cdot \bar{Y} = 0. \tag{4.1}$$

Note that, for a linear operator $Ay := \sum_{i=1}^{n} y_i A_i$ with $A_i \in S^m$, as $DG(x)$, we have for its adjoint operator A^t the formula:

$$A^t Z = (A_1 \cdot Z, \ldots, A_n \cdot Z)^t \quad \forall Z \in S^m. \tag{4.2}$$

Let

$$Y_k = \frac{\alpha(r_k)}{r_k}(\theta')^0\left(\frac{G(x_k)}{r_k}\right) = \frac{\alpha(r_k)}{r_k}\sum_{i=1}^{m}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)e_i^k(e_i^k)^t, \tag{4.3}$$

where $(\theta')^0$ is the matrix function associated with θ', defined in (2.10), and e_i^k's are orthonormal eigenvectors of $G(x_k)$ associated with the eigenvalues $\lambda_i(G(x_k))$.

Using the derivation rule given in Proposition 2.4 Part (iii), we get

$$\eta_k = \nabla f(x_k) + DG(x_k)^t Y_k. \tag{4.4}$$

The aim of this section is to prove that the sequence $\{Y_k\}$ is bounded and that each limit point of this sequence is an optimal solution of the dual problem (D).

Theorem 4.1. *Consider a sequence $\{x_k\}$ satisfying relations (3.6), and let $\{Y_k\}$ be the sequence defined by formula* (4.3)*. Then, $\{Y_k\}$ is bounded and each of its limit points is an optimal solution of* (D).

Proof. It was proven in theorem 3.2 that the sequence $\{x_k\}$ is bounded and that each of its limit points is an optimal solution of (P). Let $\bar{x}$ be a limit point of $\{x_k\}$ and $\bar{l} := l_0(\bar{x})$ be the number of the null eigenvalues of $G(\bar{x})$. For simplicity we suppose without lost of generality that $\lim_{k\to+\infty} x_k = \bar{x}$.

Now by Lemma 2.7 there exist sequences of orthonormal vectors $\{e_i^k\}$, $i = 1,...,m$, which are eigenvectors of $G(x_k)$ associated with $\lambda_i(G(x_k))$, converging toward $\bar{e}_i$ such that the set $\{\bar{e}_i : i = 1,...,m\}$ is an orthonormal eigenbasis of the matrix $G(\bar{x})$.

In order to prove that the sequence $\{Y_k\}$ is bounded, we will show that each sequence $\left\{\frac{\alpha(r_k)}{r_k}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)\right\}$ is bounded. Particularly, we will show that, for all $i = \bar{l}+1,...,m$, these sequences converge to 0. This will be very useful to conclude that any limit point of $\{Y_k\}$ is a solution of (D).

First let us prove that

$$\lim_{t\to-\infty}\theta'(t) = 0. \tag{4.5}$$

Indeed, since θ' is nonnegative and nondecreasing it follows that $\lim_{t\to-\infty}\theta'(t) = \epsilon \geq 0$ and $\theta'(u) \geq 0$, for all $u \in \operatorname{dom}\theta$. Now formula (2.7) implies

$$\theta_\infty(-1) = \sup\{\langle -1,\theta'(t)\rangle :\ t \in \operatorname{dom}\theta\} = -\epsilon,$$

which together with the equality $\theta_\infty(-1) = 0$ allows us to conclude (4.5).

Now we proceed to show that

$$\frac{\alpha(r_k)}{r_k}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right) \to 0 \quad \forall i = \bar{l}+1,...,m. \tag{4.6}$$

Lemma 2.6 tells us that $l_+(G(x_k)) \leq l_0(G(\bar{x})) =: \bar{l}$. This implies that for k sufficiently large we have

$$\lambda_i(G(x_k)) \leq \frac{\lambda_i(G(\bar{x}))}{2} < 0 \quad \forall i = \bar{l}+1,\dots,m.$$

In the case when $\theta \in F_1$, we know that $\alpha(r) = r$ and limit (4.6) follows directly from (4.5). Suppose then that $\theta \in F_2$. Since θ' is nonnegative and nondecreasing the last inequality yields to

$$0 \leq \frac{\alpha(r_k)}{r_k}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right) \leq \frac{\alpha(r_k)}{r_k}\theta'\left(\frac{\lambda_i(G(\bar{x}))}{2r_k}\right). \tag{4.7}$$

Also from the fact that θ is nonnegative and convex we get

$$0 \leq \frac{\alpha(r_k)}{2r_k}\theta'\left(\frac{\lambda_i(G(\bar{x}))}{2r_k}\right)(-\lambda_i(G(\bar{x}))) \leq \alpha(r_k)\left[\theta(0) - \theta\left(\frac{\lambda_i(G(\bar{x}))}{2r_k}\right)\right] \leq \alpha(r_k)\theta(0),$$

which together with $\lim_{k\to\infty}\alpha(r_k) = 0$ and inequality (4.7) implies condition (4.6).

Now let us prove that, for all $i = 1,\dots,\bar{l}$, the sequences $\left\{\frac{\alpha(r_k)}{r_k}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)\right\}$ are bounded. We argue by contradiction. Since $\theta'(\cdot) \geq 0$ we can suppose without lost of generality that

$$\lim_{k\to\infty}\mu_k = +\infty \;\text{ with }\; \mu_k := \sum_{i=1}^{\bar{l}}\frac{\alpha(r_k)}{r_k}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right).$$

Then set

$$\begin{aligned}\hat{\eta}_k = \frac{1}{\mu_k}\nabla f(x_k) + DG(x_k)^t\left(\sum_{i=1}^{\bar{l}}\xi_k^i e_i^k (e_i^k)^t\right),\\ \text{with }\; \xi_k^i := \frac{\alpha(r_k)}{\mu_k r_k}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right) \in [0,1].\end{aligned} \tag{4.8}$$

Dividing (4.4) by μ_k and using (4.6) we get

$$\lim_{k\to\infty}\hat{\eta}_k = 0. \tag{4.9}$$

We can consider, passing to a subsequence if necessary, that each sequence $\{\xi_k^i\}$ converges to some $\bar{\xi}^i \in [0,1]$. Moreover, since $\sum_{i=1}^{\bar{l}}\xi_k^i = 1$ for all k, it follows that $\sum_{i=1}^{\bar{l}}\bar{\xi}^i = 1$.

Letting $k \to +\infty$ in (4.8) and using that $e_i^k \to \bar{e}_i$, condition (4.9) implies that

$$DG(\bar{x})^t\left(\sum_{i=1}^{\bar{l}}\bar{\xi}^i\bar{e}_i(\bar{e}_i)^t\right) = 0, \tag{4.10}$$

with $\bar{\xi}^i \geq 0$ satisfying that $\sum_{i=1}^{\bar{l}} \bar{\xi}^i = 1$. We will verify that (4.10) contradicts Robinson's condition (2.16) (which is equivalent to Slater's condition). Indeed, by definition of the adjoint operator, condition (4.10) can be written as

$$\sum_{i=1}^{\bar{l}} \bar{\xi}^i \left(\bar{e}_i(\bar{e}_i)^t\right) \cdot DG(\bar{x})h = \sum_{i=1}^{\bar{l}} \bar{\xi}^i (\bar{e}_i)^t \left[DG(\bar{x})h\right] \bar{e}_i = 0 \quad \forall h \in I\!R^n. \tag{4.11}$$

Let $\bar{h}$ be the direction appearing in Robinson's condition (2.16). Since $\bar{\xi}^i \geq 0$ and $(\bar{e}_i)^t \left[DG(\bar{x})\bar{h}\right] \bar{e}_i < 0$ for all $i = 1,...,\bar{l}$, we immediately get that every term of the sum in (4.11) is equal to 0, and consequently $\bar{\xi}^i = 0$ for all $i = 1,...,\bar{l}$. This contradicts the equality $\sum_{i=1}^{\bar{l}} \bar{\xi}^i = 1$. Hence, we have proved that the sequences $\left\{\frac{\alpha(r_k)}{r_k}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)\right\}$ are bounded for all $i = 1, \ldots, \bar{l}$. This together with (4.6) implies the boundedness of $\{Y_k\}$.

Finally, let $\bar{Y}$ be a limit point of $\{Y_k\}$. Since $Y_k \succeq 0$ (because θ is nondecreasing), it directly follows that $\bar{Y} \succeq 0$. On the other hand, condition (4.4) implies that $\nabla_x L(\bar{x},\bar{Y}) = \nabla f(\bar{x}) + DG(\bar{x})^t \bar{Y} = 0$. Furthermore, from (4.6) and since the sequences $\left\{\frac{\alpha(r_k)}{r_k}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)\right\}$ are bounded for all $i = 1, \ldots, \bar{l}$, it follows that $\bar{Y} = \sum_{i=1}^{\bar{l}} \bar{\delta}_i \bar{e}_i \bar{e}_i^t$ with $\bar{\delta}_i \geq 0$, which implies

$$G(\bar{x}) \cdot \bar{Y} = 0.$$

Hence $\bar{Y}$ satisfies optimality conditions (4.1). We thus conclude that $\bar{Y}$ is an optimal solution of (D). ■

5 Penalty methods with two parameters

We consider again in this section the convex optimization problem (P) defined in Section 3 and suppose assumptions (A_1) and (A_2). Additionally, we will also suppose

$$(A_3) \qquad f_\infty(d) \geq 0 \quad \forall d.$$

It was noted in Remark 3.1 that there is no lost of generality to do such an assumption.

In this section we will only work with penalty functions θ that belong to F_2. In this way, for any real r_k, $\beta_k > 0$ we consider

$$p^{r_k}(x) = r_k H^{r_k}(x) = r_k \sum_{i=1}^{m} \theta\left(\frac{\lambda_i(G(x))}{r_k}\right),$$

and we define

$$\psi_k(x) = f(x) + \beta_k p^{r_k}(x).$$

The main computation of the forthcoming algorithm will be to solve approximatively, at each iteration k, the unconstrained optimization problem

$$(\mathrm{P}_k) \qquad v_k = \inf\{\psi_k(x) \mid x \in I\!R^n\}.$$

Let S_k be the optimal set of (P_k), and let $\{\epsilon_k\}$ and $\{\gamma_k\}$ be sequences such that

$$\forall k: \quad \epsilon_k > 0, \gamma_k > 0, \quad \lim_{k\to\infty} \epsilon_k = \lim_{k\to\infty} \gamma_k = 0. \tag{5.1}$$

As in Theorem 3.1 we can show that S_k is nonempty and compact for each k. Hence, following the discussion of Section 3, we can compute for each k a point x_k satisfying

$$||\eta_k|| \leq \epsilon_k, \quad ||\eta_k||||x_k|| \leq \gamma_k, \quad \text{where } \eta_k = \nabla\psi_k(x_k). \tag{5.2}$$

As we seen before this can be done in a finite number of steps with any usual descent method.

Now we proceed similarly to [15]. The parameters r_k and β_k play two different roles: r_k always decreases in order to improve the approximation of the function $t \to t^+$ by the mapping $t \to r_k\theta(t/r_k)$, while β_k is a penalty weight that increases only at an infeasible iteration point x_k.

The algorithm proposed in this article is the following

1. Let $\beta_0 = r_0 = 1$ and $k = 0$.
2. Compute x_k satisfying (5.2).
3. Update $r_{k+1} = \frac{r_k}{2}$, and if x_k is feasible then set $\beta_{k+1} = \beta_k$, otherwise set $\beta_{k+1} = 2\beta_k$. Finally set $k = k + 1$.

When $C = D$ and $\epsilon_k = 0$ (that is, x_k is an exact minimizer of (P_k)), our algorithm coincides with the proposed one by C. Gonzaga and R. A. Castillo in [15]. We refer the reader to this article for a detailed discussion of this scheme.

In addition to the hypothesis made in this section we denote by $\{x_k\}, \{r_k\}$ and $\{\beta_k\}$ the sequences generated by our algorithm. In this context, the following convergence result holds.

Theorem 5.1. *The sequence $\{x_k\}$ is bounded and all its limit points are optimal solutions of* (P).

Proof. We start this proof establishing five conditions that will be important in the sequel. First, by construction of the algorithm we have

$$1 \leq \beta_k, \quad \beta_k r_k \leq 1 \quad \forall k. \tag{5.3}$$

Second, since $\lim_{t\to-\infty} \theta(t) = 0$ we obtain

$$\lim_{k\to\infty} \theta\left(\frac{\lambda_i(G(x^0))}{r_k}\right) = 0 \quad \forall i = 1,2,\ldots,m, \forall x^0 \in \operatorname{int} C.$$

Consequently

$$\lim_{k\to\infty} \beta_k p^{r_k}(x^0) = \lim_{k\to\infty} \beta_k r_k \sum_{i=1}^{m} \theta\left(\frac{\lambda_i(G(x^0))}{r_k}\right) = 0 \quad \forall x^0 \in \operatorname{int} C. \tag{5.4}$$

Third, since for all $t > 0$ the function $r \to r(\theta(t/r) - \theta(0))$ is nondecreasing on $I\!R_{++}$, and since $\theta(0) \geq 0$, we deduce that

$$r\theta(t/r) \geq \theta(t) - \theta(0) \quad \forall t \in I\!R, \forall r \in (0,1]. \tag{5.5}$$

Fourth, convexity of the function ψ_k and the definition of $\eta_k := \nabla\psi_k(x_k)$ imply that

$$f(x_k) + \beta_k p^{r_k}(x_k) \leq f(x^0) + \beta_k p^{r_k}(x^0) + \langle \eta_k, x_k - x^0\rangle \quad \forall x^0 \in \operatorname{int} C. \tag{5.6}$$

Finally, since θ is nonnegative we get from (5.2)–(5.4) and (5.6)

$$\begin{gathered} f(x_k) + r_k\theta\left(\frac{\lambda_1(G(x_k))}{r_k}\right) \leq f(x^0) + \mu_k(x^0), \\ \text{with } \lim_{k\to\infty} \mu_k(x^0) = 0 \quad \forall x^0 \in \operatorname{int} C. \end{gathered} \tag{5.7}$$

Now let us show that the sequence $\{x_k\}$ is bounded. By contradiction, we can suppose, passing to a subsequence if necessary, that

$$||x_k|| \to +\infty, \quad \lim_{k\to\infty} \frac{x_k}{||x_k||} = d \neq 0.$$

By Proposition 3.1 Part (ii) it follows that $(\tilde{\lambda}_1)_\infty(d) > -\infty$. Set $\alpha_1 < (\tilde{\lambda}_1)_\infty(d)$. From formula (2.3) (see Remark 2.1) we have for all k sufficiently large

$$\tilde{\lambda}_1(x_k) = \tilde{\lambda}_1\left(\frac{x_k}{||x_k||}||x_k||\right) \geq \alpha_1 ||x_k||.$$

This together with the monotonicity of θ and inequality (5.7) yields to

$$\frac{f(x_k)}{||x_k||} + \frac{r_k}{||x_k||}\theta\left(\frac{\alpha_1||x_k||}{r_k}\right) \leq \frac{f(x^0)}{||x_k||} + \frac{\mu_k(x^0)}{||x_k||}.$$

By passing to the liminf in this last inequality we get

$$f_\infty(d) + \theta_\infty(\alpha_1) \leq 0. \tag{5.8}$$

Since f_∞ and θ_∞ are nonnegative we obtain that $\theta_\infty(\alpha_1) = 0$, and consequently $f_\infty(d) = 0$. Furthermore, due to relations $\theta_\infty(-1) = 0$ and $\theta_\infty(1) = 1$ it follows that $\alpha_1 \leq 0$. Then letting $\alpha_1 \uparrow (\tilde{\lambda}_1)_\infty(d)$ it follows that $(\tilde{\lambda}_1)_\infty(d) \leq 0$, or equivalently, $d \in C_\infty$. This together with $f_\infty(d) \leq 0$, $d \neq 0$ contradicts the fact that S is nonempty and compact.

Let $\bar{x}$ be a limit point of the sequence $\{x_k\}$. For the sake of simplicity, we can suppose that $\bar{x} = \lim_{k\to\infty} x_k$.

Firstly, we proceed to prove that $\bar{x}$ is feasible. This is obviously true if for all k sufficiently large the iteration point x_k is feasible for problem (P), i.e. $x_k \in C$. If this is not the case, we have from the construction of the algorithm

$$\lim_{k\to\infty} \beta_k = +\infty. \tag{5.9}$$

At the first iteration, the convexity of function ψ_0 implies

$$f(x_0) + p^1(x_0) \leq f(x_k) + p^1(x_k) + \langle \eta_0, x_0 - x_k \rangle. \tag{5.10}$$

Using inequality (5.5) we get

$$r_k \theta(\lambda_i(G(x_k))/r_k) \geq \theta(\lambda_i(G(x_k))) - \theta(0) \quad \forall i = 1, \ldots, m,$$

which yields to

$$p^{r_k}(x_k) \geq p^1(x_k) - m\theta(0).$$

Adding this last inequality to (5.10) we obtain

$$f(x_0) + p^1(x_0) - m\theta(0) \leq f(x_k) + p^{r_k}(x_k) + \langle \eta_0, x_0 - x_k \rangle,$$

deducing from relation (5.6) that

$$(\beta_k - 1)p^{r_k}(x_k) \leq \beta_k r_k \sum_{i=1}^{m} \theta\left(\frac{\lambda_i(G(x^0))}{r_k}\right) + \|\eta_0\|\|x_k\| + \|\eta_k\|\|x_k - x^0\| + K,$$

where K is a constant. Hence, from the boundedness of $\{x_k\}$ and relations (5.1), (5.2) and (5.4) we can give an upper bound $\hat{K}$ for the right hand side of the last inequality. Thus, from the fact that θ is nonnegative it follows for all k sufficiently large that

$$r_k \theta\left(\frac{\lambda_1(G(x_k))}{r_k}\right) \leq \frac{\hat{K}}{(\beta_k - 1)},$$

Passing to the liminf and using formula (2.3) and (5.9) we get $\theta_\infty(\lambda_1(G(\bar{x}))) \leq 0$. As a consequence we conclude that $\lambda_1(G(\bar{x})) \leq 0$, that is, $\bar{x}$ is feasible for problem (P).

Secondly, we shall prove that $\bar{x}$ is an optimal solution of (P). Since $\theta(\cdot) \geq 0$ and inequality (5.7) we have

$$f(x_k) \leq f(x^0) + \mu_k(x^0) \quad \forall x^0 \in \operatorname{int} C.$$

We thus obtain at the limit that $f(\bar{x}) \leq f(x^0)$ for all $x^0 \in \operatorname{int} C$. Hence, continuity of function f implies that $\bar{x}$ is an optimal solution of (P). ■

In the next theorem we extend to our semidefinite framework the main result of the article [15] (cf. Theorem 1). For this purpose, we denote by F_2^* the subset of functions $\theta \in F_2$ satisfying the inequality $\theta'(0) > 0$. We remark that θ_6 and θ_7 belong to F_2^*.

The following theorem says that for $\theta \in F_2^*$ and k sufficiently large, the point x_k will be feasible. This result is important for optimization problems where feasibility is a key issue. Of course, there are some examples of $\theta \in F_1$ ($-\log(x)$, $1/x$,...) for which x_k is strictly feasible, but in these cases the starting point of the numerical methods used to obtain x_k must also be strictly feasible, which can be a difficult task for some problems. Thanks to the next theorem this difficulty is avoided when $\theta \in F_2^*$.

Theorem 5.2. *Suppose in addition to hypothesis of Theorem 5.1 that $\theta \in F_2^*$. Then, there exists k_0 such that for all $k \geq k_0$, x_k is feasible.*

Proof. We argue by contradiction. So, since $\{x_k\}$ is bounded, we can assume the existence of a convergent but infeasible subsequence of $\{x_k\}$ (which for simplicity will be also called $\{x_k\}$). Hence, by construction of our algorithm, $\beta_k \to +\infty$. Let $\bar{x} := \lim_{k\to\infty} x_k$. It follows from Theorem 5.1 that $\bar{x}$ is an optimal solution of (P).

In the rest of this proof we consider that k is large enough. If $G(\bar{x}) \prec 0$ then by smoothness of the function G we get $G(x_k) \prec 0$, obtaining directly a contradiction. We then suppose $\operatorname{Im} E(G(\bar{x})) = \operatorname{Ker} G(\bar{x}) \neq \{0\}$, that is, $G(\bar{x})$ is singular.

By Proposition 2.9, Slater's condition (A_2) is equivalent to Robinson's condition (2.16), which can be written at $\bar{x}$ as follows

There exists $\bar{h} \in I\!R^n$ and $\rho > 0$ such that $E(G(\bar{x}))^t DG(\bar{x})\bar{h}E(G(\bar{x})) \prec -\rho I_m$,

where I_m is the identity matrix in S^m. Hence, continuity of $DG(\cdot)$ implies that

$$E_k^t DG(x_k)\bar{h}E_k \prec -\frac{1}{2}\rho I_m, \tag{5.11}$$

where $E_k \in I\!R^{m\times l_0(G(\bar{x}))}$ are the matrices given by Corollary 2.8, that is, the columns of matrices E_k are the orthonormalized eigenvectors of $G(x_k)$ associated with their $l_0(G(\bar{x}))$ largest eigenvalues, and $E_k \to E(G(\bar{x}))$. Corollary 2.8 also tells us that $l_+(G(x_k)) \leq l_0(G(\bar{x}))$. Actually we have

$$\lambda_i(G(x_k)) \leq \mu < 0 \quad \forall\, i = l_0(G(\bar{x})) + 1, \ldots, m, \tag{5.12}$$

where $\mu > \bar{\mu} := \max\{\lambda_i\,;\, \lambda_i = \lambda_i(G(\bar{x})) < 0\}$.

We proceed to compute the inner product $\langle \eta_k, \bar{h}\rangle = \eta_k^t \bar{h}$, where $\eta_k = \nabla\psi_k(x_k)$ and $\bar{h}$ is the vector appearing in (5.11).

From the derivation rule given in Proposition 2.4 Part (iii) we get

$$\begin{aligned}\nabla p^{r_k}(x_k)^t\bar{h} &= \sum_{i=1}^{m}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)c_i(x_k)\cdot DG(x_k)\bar{h}\\ &= \sum_{i=1}^{m}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)(e_i^k)^t(DG(x_k)\bar{h})e_i^k,\end{aligned} \tag{5.13}$$

where $c_i(x_k) := e_i^k(e_i^k)^t$ and vectors e_i^k's are the columns of E_k such that each e_i^k corresponds to the eigenvector of $G(x_k)$ associated with $\lambda_i(G(x_k))$.

Condition (5.13) implies that

$$\langle\eta_k,\bar{h}\rangle = \nabla f(x_k)^t\bar{h} + \beta_k\sum_{i=1}^{m}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)(e_i^k)^t(DG(x_k)\bar{h})e_i^k,$$

which can be rewritten as

$$\begin{aligned}-\frac{\langle\eta_k,\bar{h}\rangle}{\beta_k}+\frac{\nabla f(x_k)^t\bar{h}}{\beta_k}+\sum_{i=l_0(G(\bar{x}))+1}^{m}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)(e_i^k)^T(DG(x_k)\bar{h})e_i^k =\\ -\sum_{i=1}^{l_0(G(\bar{x}))}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)(e_i^k)^T(DG(x_k)\bar{h})e_i^k.\end{aligned} \tag{5.14}$$

Taking the limit when $k\to+\infty$ we have that the terms $-\frac{\langle\eta_k,\bar{h}\rangle}{\beta_k}$ and $\frac{\nabla f(x_k)^t\bar{h}}{\beta_k}$ converge toward 0 due to relations (5.1) and (5.2), and $\beta_k\to+\infty$. By (5.12), we obtain $\lambda_i(G(x_k)/r_k)\to-\infty$ for all $i=l_0(G(\bar{x}))+1,\ldots,m$. This together with the limit $\lim_{t\to-\infty}\theta'(t)=0$ implies that $\theta'(\lambda_i(G(x_k)/r_k))\to 0$ for all $i=l_0(G(\bar{x}))+1,\ldots,m$. Then we deduce that the entire left hand side of (5.14) converges toward 0.

We will obtain a contradiction by showing that the right hand side of (5.14) is strictly positive. Indeed, condition (5.11) implies that $(e_i^k)^t(DG(x_k)\bar{h})e_i^k < -\rho/2$ for $i=1,\ldots,l_0(G(\bar{x}))$, and since θ is nondecreasing, $\theta'(\cdot)\geq 0$ and $l_+(G(x_k))\leq l_0(G(\bar{x}))$ it follows that

$$\begin{aligned}-\sum_{i=1}^{l_0(G(\bar{x}))}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)(e_i^k)^t(DG(x_k)\bar{h})e_i^k &\geq \frac{\rho}{2}\sum_{i=1}^{l_0(G(\bar{x}))}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)\\ &\geq \frac{\rho}{2}\sum_{i=1}^{l_+(G(x_k))}\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)\\ &\geq \frac{\rho}{2}\theta'(0)l_+(x_k).\end{aligned}$$

But $\theta'(0)>0$ (because $\theta\in F_2^*$) and x_k is infeasible, i.e. $l_+(x_k)\geq 1$. Hence the right hand side of (5.14) has a strictly positive lower bound. The theorem follows. ■

As for penalty and barrier methods with one parameter we can associate with the sequence $\{x_k\}$ a sequence $\{Y_k\}$ of dual multipliers defined by

$$Y_k = \beta_k(\theta')^0\left(\frac{G(x_k)}{r_k}\right) = \beta_k \sum_{i=1}^{m} \theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right) e_i^k (e_i^k)^t, \tag{5.15}$$

where $(\theta')^0$ is the matrix function associated with θ', defined in (2.10), and e_i^k's are orthonormal eigenvectors of $G(x_k)$ associated with the eigenvalues $\lambda_i(G(x_k))$. Then we have

$$\eta_k = \nabla\psi_k(x_k) = \nabla f(x_k) + DG(x_k)^t Y_k. \tag{5.16}$$

As in Section 4, we prove in the next thorem that the sequence $\{Y_k\}$ is bounded with each of its limit points being an optimal solution of (D).

Theorem 5.3. *Suppose that the assumptions of Theorem 5.2 are satisfied. Consider a sequence $\{x_k\}$ satisfying relations* (5.2)*, and let $\{Y_k\}$ be the sequence defined by formula* (5.15)*. Then, $\{Y_k\}$ is bounded and each of its limit points is an optimal solution of* (D)*.*

Proof. By Theorems 5.1 and 5.2 we can assume, without lost of generality, that the sequence $\{x_k\}$ converges to an optimal solution $\bar{x}$ of (P) and for k sufficiently large x_k is feasible and $\beta_k = \beta \geq 1$. Since x_k is feasible and by the monotonicity of $\theta'(\cdot)$, we have that $\theta'(\lambda_i(G(x_k))/r_k) \leq \theta'(0)$ for all i, which proves that the sequence $\{Y_k\}$ is bounded.

Let $\bar{Y}$ be a limit point of $\{Y_k\}$. The proof is now similar to the one given in Theorem 4.1. Since $Y_k \succeq 0$, it directly follows that $\bar{Y} \succeq 0$. On the other hand, condition (5.16) implies that $\nabla_x L(\bar{x},\bar{Y}) = \nabla f(\bar{x}) + DG(\bar{x})^t\bar{Y} = 0$.

Let $\bar{l} := l_0(\bar{x})$ be the number of null eigenvalues of $G(\bar{x})$. Since $\lim_{t\to-\infty}\theta'(t) = 0$ (cf. (4.5)), we get

$$\beta_k\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right) \to 0 \quad \forall i = \bar{l}+1,...,m, \tag{5.17}$$

and since the sequences $\left\{\beta_k\theta'\left(\frac{\lambda_i(G(x_k))}{r_k}\right)\right\}$ are bounded for all $i = 1,\ldots,\bar{l}$, it follows that $\bar{Y} = \sum_{i=1}^{\bar{l}} \bar{\delta}_i\bar{e}_i\bar{e}_i^t$ with $\bar{\delta}_i \geq 0$, which implies that $G(\bar{x})\cdot\bar{Y} = 0$. Hence $\bar{Y}$ satisfies optimality conditions (4.1). We thus conclude that $\bar{Y}$ is an optimal solution of (D). ■

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CHAPITRE III

Perturbation Analysis of Second-Order Cone Programming Problems

Perturbation Analysis of Second-Order Cone Programming Problems [1]

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Abstract. We discuss first and second order optimality conditions for nonlinear second-order cone programming problems, and their relation with semidefinite programming problems. For doing this we extend in an abstract setting the notion of optimal partition. Then we state a characterization of strong regularity in terms of second order optimality conditions.

1 Introduction

Consider the nonlinear *second-order cone programming problem*

$$\operatorname*{Min}_{x\in\mathbb{R}^n, s^j\in\mathbb{R}^{m_j+1}} f(x)\,;\; g^j(x) = s^j, (s^j)_0 \geq \|\bar{s}^j\|,\; j = 1,\dots,J, \tag{SOCP}$$

where f and g^j, $j = 1,\dots,J$ are C^1 mappings from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^{m_j+1}$, respectively. We use the standard convention of indexing components of vectors of $\mathbb{R}^{m_j+1}$ from 0 to m_j, while vectors in $\mathbb{R}^n$ are indexed from 1 to n. Given $s \in \mathbb{R}^{m_j+1}$, we also denote $\bar{s} := (s_1,\dots,s_{m_j})^\top$.

The second-order cone (or ice-cream cone, or Lorentz cone) of dimension $m+1$ is defined as

$$Q_{m+1} := \{s \in \mathbb{R}^{m+1}\,;\, s_0 \geq \|\bar{s}\|\},$$

and the order relation $\succeq_{Q_{m+1}}$ induced by Q_{m+1} is given by

$$s \succeq_{Q_{m+1}} 0 \qquad \text{iff} \qquad s \in \mathbb{R}^{m+1}, s_0 \geq \|\bar{s}\|.$$

The interior of this cone is the set of $s \in \mathbb{R}^{m+1}$ such that $s_0 > \|\bar{s}\|$. In that case we say that $s \succ_{Q_{m+1}} 0$. We also denote $\mathcal{Q} := \Pi_{j=1}^J Q_{m_j+1}$. A second-order cone $Q = Q_{m+1}$ can be described as a linear matrix inequality by using the known equivalence (e.g. [1])

$$s \succeq_Q 0 \qquad \text{iff} \qquad \operatorname{Arw}(s) := \begin{pmatrix} s_0 & \bar{s}^\top \\ \bar{s} & s_0 I_m \end{pmatrix} \succeq 0, \tag{1.1}$$

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where I_m denotes the identity matrix in $I\!R^{m\times m}$, $\mathrm{Arw}(s)$ is the *arrow matrix* of the vector s, and $\succeq$ denotes the positive semidefinite order, that is, $A \succeq B$ iff A, B are symmetric matrices and $A - B$ is a positive semidefinite matrix. We also denote the set of $m+1$ by $m+1$ symmetric matrices by $\mathcal{S}^{m+1}$, indexed from 0 to m, equipped with the inner product $A \cdot B := \mathrm{Tr}(AB) = \sum_{i,j=0}^{m} A_{ij}B_{ij}$; the subset of symmetric positive semidefinite matrices is denoted $\mathcal{S}_+^{m+1}$. Finally, for two arbitrary vectors x and z of any dimension we set $x \cdot z := x^\top z = \sum_i x_i z_i$ the corresponding Euclidian inner product, and for an arbitrary optimization problem (P) we denote by $S(P)$, $F(P)$ and $\mathrm{val}(P)$ its solution set, feasible set and optimal value, respectively. The equivalence (1.1) implies that (SOCP) is *SDP-representable*, i.e., can be written as the nonlinear semidefinite problem

$$\min_{x\in I\!R^n} f(x)\,;\; G^j(x) := \mathrm{Arw}(g^j(x)) \succeq 0,\;\; j = 1,\ldots,J. \qquad \text{(SDP)}$$

For a general view of semidefinite programming problems, see [18, 19]. A first objective in this paper is to compare the linear second-order programming problem (see (LSOCP) below) and its linear SDP-representation (see (LSDP) below) in terms of duality results. We show that their dual problems are no longer equivalent, and some important notions as the uniqueness of Lagrange multipliers (or equivalently, dual problems solutions) do not simultaneously hold for both problems (LSOCP) and (LSDP). We perform this analysis in an abstract framework. When specialized to second order cone problems, we recover some of the results of Sim and Zhao [17]. Still our main result is the characterization of the *strong regularity property* for SOCP problems in terms of second-order optimality conditions. This is a well studied subject in nonlinear programming and the reader can see two different approaches in the articles of Bonnans and Sulem [7], and Dontchev and Rockafellar [9]. Nevertheless, it is still an open problem in a general conic optimization framework, even in particular instances as semidefinite programming. Necessary and sufficient second-order conditions to obtain the strong regularity property in SDP are studied by the authors in [4].

The paper is organized as follows. Section 2 breaks into three subsections. In the first one, we review the main duality results concerning the linear second-order programming problem (LSOCP) and their comparison to linear SDP problems. Section 2.2 deals with an abstract framework involving two equivalent linear conic optimization problems with constraints in product form, that are related by a linear mapping (as in relation (1.1)). It introduces a notion of optimal partition of active constraints. It allows us to deduce several duality statements and related properties. Subsection 2.3 applies this abstract framework to linear problems (LSOCP) and (LSDP). In Section 3 we discuss briefly the duality theory for nonlinear SOCP problems. Section 4 recalls some key notions as the *nondegeneracy condition* and the *reduction approach*, mainly for their use in Section 5 where is stated our main result: the characterization of the strong regularity property for SOCP problems in terms of second-order optimality conditions. For this, we use the concepts given in Section 4 as well some suitable known theorems and SOCP techniques.

2 Duality theory for linear SOCP problems

2.1 Dual linear SOCP problems

We assume in this section that $f(x) = c{\cdot}x$ and $g^j(x) = A^j x - b^j$, $j = 1,\ldots,J$, where $c \in I\!R^n$ and A^j are $(m_j+1)\times n$ matrices. In that case we speak of a linear SOCP problem:

$$\operatorname*{Min}_{x\in I\!R^n, s^j\in I\!R^{m_j+1}} c\cdot x\,;\; A^j x - b^j = s^j, (s^j)_0 \geq \|\bar{s}^j\|,\;\; j = 1,\ldots,J. \qquad \text{(LSOCP)}$$

The dual problem of (LSOCP) is given by

$$\operatorname*{Max}_{y^j\in I\!R^{m_j+1}} \sum_{j=1}^{J} b^j\cdot y^j;\;\; \sum_{j=1}^{J}(A^j)^\top y^j = c,\;\; (y^j)_0 \geq \|\bar{y}^j\|,\;\; j = 1,\ldots,J. \quad \text{(LSOCP}^*\text{)}$$

Since both the primal and dual problems are convex, we have the following results of convex analysis (cf. Rockafellar [14]). The *weak duality* inequality val (LSOCP) $\geq$ val (LSOCP*) holds, with the convention that the optimal value (val) of problem (LSOCP) (resp. (LSOCP*)) is equal to $+\infty$ (resp. $-\infty$) if this problem is infeasible. If the value of (LSOCP) is finite, it is known that (LSOCP) is *strictly feasible*, i.e., there exists a point $\hat{x}$ such that $A^j\hat{x} - b^j \in \operatorname{int} Q_{m_j+1}$ for all $j = 1,\ldots,J$, iff the set of solutions of the dual problem is nonempty and bounded. In that case we have the *strong duality* property, i.e., val (LSOCP) = val (LSOCP*). A symmetric statement holds by permuting the words "primal" and "dual" (we will see in lemma 2.2 a refinement of this statement). If the strong duality property holds, then a pair of primal-dual solution $(x^*,y^*) \in I\!R^n \times \Pi_{j=1}^J I\!R^{m_j+1}$ is characterized by the following optimality system

$$A^\top y^* = c, \quad Ax^* - b \in \mathcal{Q}, \quad y^* \in \mathcal{Q}, \quad (Ax^* - b)\circ y^* = 0, \qquad (2.1)$$

where we have defined $A := (A^1;\cdots;A^J)$ as the matrix whose rows are those of A^1 to A^J and whose columns a_i are equal to $\operatorname{vec}(a_i^1,\ldots,a_i^J)$, with a_i^j the i-th column of A^j, $b := \operatorname{vec}(b^1,\ldots,b^J)$ and the operation $\circ$ (e.g. [1]) is given by

$$x\circ s := \operatorname{Arw}(x)s = \begin{pmatrix} x^\top s \\ x_0\bar{s} + s_0\bar{x}\end{pmatrix}, \text{ for all } x,s \in I\!R^{m+1},$$

and for x, s in $\Pi_{j=1}^J I\!R^{m_j+1}$ we set

$$x\circ s := \operatorname{vec}(x^1\circ s^1,\ldots,x^J\circ s^J).$$

We may write $(Ax^* - b)\cdot y^* = 0$ instead of the last relation in (2.1), in view of the well known property (e.g. [1, Lemma 15])

$$\text{For all } x,s \in Q_{m+1}, \quad x\circ s = 0 \text{ iff } x\cdot s = 0. \qquad (2.2)$$

In fact it is easily checked that relations in (2.2) are satisfied iff x and s belong to Q_{m+1} and

$$\text{Either } x = 0 \text{ or } s = 0, \text{ or there exists } \alpha > 0 \text{ s.t. } s_0 = \alpha x_0 \text{ and } \bar{s} = -\alpha \bar{x}. \tag{2.3}$$

Similar duality results hold for the linear semidefinite problem, which can be written as

$$\underset{x \in \mathbb{R}^n}{\text{Min}}\ c \cdot x\,;\ \sum_{i=1}^{n} x_i G_i^j \succeq G_0^j,\ j = 1, \ldots, J, \tag{LSDP}$$

where we have set

$$G_0^j := \text{Arw}(b^j) \qquad \text{and} \qquad G_i^j := \text{Arw}(a_i^j), \quad i = 1, \ldots, n. \tag{2.4}$$

In this case, the dual problem of (LSDP) is

$$\underset{Y^j \in \mathcal{S}^{m_j+1}}{\text{Max}} \left\{ \sum_{j=1}^{J} G_0^j \cdot Y^j\,;\ \sum_{j=1}^{J} \mathcal{G}^j(Y^j) + c = 0,\ Y^j \succeq 0,\ j = 1, \ldots, J, \right\}, \tag{LSDP*}$$

where the mappings $Y \in \mathcal{S}^{m_j+1} \to \mathcal{G}^j(Y) := (G_1^j \cdot Y, \ldots, G_n^j \cdot Y)^\top$ are the adjoint operators of G^j, and a primal-dual solution $(x, Y) \in \mathbb{R}^n \times \Pi_{j=1}^J \mathcal{S}^{m_j+1}$ is characterized by

$$\sum_{j=1}^{J} \mathcal{G}^j(Y^*)^j + c = 0, \quad G^j(x) \succeq 0, Y^j \succeq 0, \quad G^j(x) Y^j = 0,\ j = 1, \ldots, J. \tag{2.5}$$

In the sequel we denote $\mathcal{G}(Y) := \sum_{j=1}^{J} \mathcal{G}^j(Y^*)^j$.

Note that a linear second-order cone programming problem as (LSOCP) satisfies the strong duality property if both problems (LSOCP) and its dual (LSOCP*) are feasible, see Shapiro and Nemirovski [16], whereas this is no longer true for a linear semidefinite programming problem, see [18, page 65].

2.2 An abstract framework

The aim of this section is to clarify some properties of optimization problems with constraints in product form, as well as relations between the dual solutions of (LSOCP) and (LSDP). For this, we consider a general linear conic optimization problem with constraints in product form, i.e.,

$$\underset{x \in \mathbb{R}^n}{\text{Min}}\ c \cdot x\,;\ A^j x - b^j \in K_j,\ j = 1, \ldots, J, \tag{COP}$$

where K_j are closed convex cones in $\mathbb{R}^{q_j}$. We set $K := K_1 \times \cdots \times K_J$, and define $A = (A^1; \cdots; A^J)$ as the matrix whose rows are those of A^1 to A^J, and

$b := \mathrm{vec}(b^1, \dots, b^J)$ so that (COP) is equivalent to $\mathrm{Min}_{x \in I\!R^n} \{c \cdot x;\ Ax - b \in K\}$. The dual problem is

$$\operatorname*{Max}_{y^1,\dots,y^J} \sum_{j=1}^{J} b^j \cdot y^j; \quad \sum_{j=1}^{J} (A^j)^\top y^j = c,\ y^j \in K_j^+,\ j = 1, \dots, J, \qquad \text{(COP}^*\text{)}$$

where the (positive) polar of a set $C \subset I\!R^m$ is defined as $C^+ := \{y \in I\!R^m;\ y \cdot z \geq 0,\ \text{ for all } z \in C\}$. If the primal and dual values are equal, a pair (x,y) of the primal and dual problems is characterized by the optimality system

$$A^j x - b^j \in K_j, \quad y^j \in K_j^+, \quad y^j \cdot (A^j x - b^j) = 0, j = 1, \dots, J; \quad A^\top y = c. \qquad \text{(COPOS)}$$

We denote by S(COPOS) the set of solutions of relations (COPOS). In the sequel we introduce notions of componentwise strict feasibility and strict complementarity.

Definition 2.1. *We say that strict primal (resp. dual) feasibility holds for* $j \in \{1, \dots, J\}$ *if there exists* $x \in F(COP)$ *such that* $A^j x - b^j \in \operatorname{int} K_j$ *(resp.* $y \in F(COP^*)$ *such that* $y^j \in \operatorname{int} K_j^+$*).*

Lemma 2.2. *Let j be strictly primal (resp. dual) feasible. Then the set* $\{y^j; y \in S(COP^*)\}$ *(resp.* $\{A^j x - b^j; x \in S(COP)\}$*) is bounded.*

Proof. If j is strictly primal feasible, there exists $\varepsilon > 0$ such that $s = Ax - b$ satisfies $s^j + \varepsilon B \subset K_j$, or equivalently $\varepsilon B \subset s^j - K_j$. Let $y \in S(COP^*)$. Since $y^j \in K_j^+$, it follows that $\varepsilon \|y^j\| \leq y^j \cdot s^j$. Using also $y^{j'} \cdot s^{j'} \geq 0$, for all j', we get

$$0 = x \cdot (c - A^\top y) = c \cdot x - y \cdot Ax = c \cdot x - b \cdot y - y \cdot s \leq c \cdot x - b \cdot y - \varepsilon \|y^j\|.$$

In other words, $\varepsilon \|y^j\| \leq c \cdot x - b \cdot y = c \cdot x - \mathrm{val}(COP^*)$, which gives the desired estimate. The proof for the dual statement is similar. ■

One says (e.g., [6, Def. 4.74]) that the *strict complementarity hypothesis* holds for problem (COP) if there exists a pair (x,y) solution of the optimality system, such that $-y \in \operatorname{ri} N_K(Ax - b)$, where N_K is the normal cone of convex analysis. Since K is a closed convex cone, we have for $s \in K$ that

$$N_K(s) = (-K^+) \cap s^\perp, \qquad (2.6)$$

(where $s^\perp$ denotes the set of all orthogonal vectors to s) and $N_K(s) = \emptyset$ if $s \notin K$.

For problems with constraints in product form, it is worthwhile to introduce the concept of *componentwise strict complementarity hypothesis*, which for each component j means that there exists a pair $(x,y) \in S$(COPOS), such that $-y^j \in \operatorname{ri} N_{K_j}(A^j x - b^j)$.

We can extend and refine for this framework the notion of optimal partition, well known for linear programming and monotone linear complementarity problems, see e.g. [3, Section 18.2.4].

Lemma 2.3. *If $S(\text{COPOS})$ is not empty, there exists a partition (B,N,R,T) of $\{1,\dots,J\}$ such that,* (i) *The set B is the union of j such that there exists $(x(j),y(j)) \in S(\text{COPOS})$ satisfying $A^j x(j) - b^j \in \operatorname{int} K_j$,* (ii) *The set N is the union of j such that there exists $(x(j),y(j)) \in S(\text{COPOS})$ satisfying $y^j(j) \in \operatorname{int} K_j^+$,* (iii) *The set R is the union of j, not belonging to B or N, such that there exists $(x(j),y(j)) \in S(\text{COPOS})$ with $-y^j(j) \in \operatorname{ri} N_{K_j}(A^j x(j) - b^j)$, and* (iv) *for all $j \in T$, every $(x,y) \in S(\text{COPOS})$ does not satisfy strict complementarity for component j.*

Proof. Let (B,N,R,T) be defined as in the lemma; we have to check that this is a partition. The definition of T implies that their union equals $\{1,\dots,J\}$, and by definition of R and T, we have that $(B \cup N,R,T)$ is a partition of $\{1,\dots,J\}$. It remains to prove that $B \cap N = \emptyset$. Since $S(\text{COPOS})$ is not empty, we know that $S(\text{COPOS}) = S(COP) \times S(COP^*)$. Therefore $\hat{x} := |B|^{-1} \sum_{j \in B} x(j)$ satisfy $\hat{x} \in S(COP)$. We see that $A^j \hat{x} - b^j \in \operatorname{int} K_j$, for all $j \in B$. Therefore any $y \in S(COP^*)$ is such that $y^j = 0$, for all $j \in B$. This proves that $B \cap N = \emptyset$. ■

Remark 2.4. *Note that, for monotone linear complementarity problems the optimal partition is of the form (B,N,T), since in that case a strictly complementary component belongs either to B or N. Therefore the main novelty consists in introducing the set R.*

Definition 2.5. *Any pair $(x,y) \in S(\text{COPOS})$ satisfying the relations below is said to be of maximal complementarity:*

$$\begin{cases} \text{(i) } A^i x - b^i \in \operatorname{int} K_i, \ \forall i \in B, \quad \text{(ii) } y^i \in \operatorname{int} K_i^+, \ \forall i \in N, \\ \text{(iii) } -y^i \in \operatorname{ri} N_{K_i}(A^i x - b^i), \ \forall i \in R. \end{cases} \tag{2.7}$$

Let $x(j)$ and $y(j)$ be as in lemma 2.3. We define

$$\hat{x} := (|B| + |R|)^{-1} \sum_{j \in B \cup R} x(j); \quad \hat{y} := (|N| + |R|))^{-1} \sum_{j \in N \cup R} y(j).$$

Let us state some properties of the set of maximal complementarity solutions. We need a preliminary lemma.

Lemma 2.6. *Let K be a closed convex cone. Let $s^i \in K$, for $i = 1,2$, $-y^1 \in N_K(s^1)$, and $-y^2 \in \operatorname{ri} N_K(s^2)$. Given $\alpha \in]0,1[$, set $(s,y) := \alpha(s^1,y^1) + (1-\alpha)(s^2,y^2)$. If $-y \in N_K(s)$, then $-y \in \operatorname{ri} N_K(s)$.*

Proof. Since $-N_K(s) = K^+ \cap s^\perp$, we have that $-y \in \operatorname{ri} N_K(s)$ iff, for all $z \in N_K(s)$, $y \pm \varepsilon z \in K^+$ for small enough $\varepsilon > 0$. As K^+ is a cone, $y + \varepsilon z \in K^+$ always holds. Therefore we have to prove that for $z \in N_K(s)$, $y - \varepsilon z \in K^+$ for small enough $\varepsilon > 0$. Using $N_K(s) = N_K(s^1) \cap N_K(s^2)$, obtain $z \in N_K(s^2)$, and hence, $y^2 - \varepsilon' z \in K^+$ for some $\varepsilon' > 0$. Let $\varepsilon := (1-\alpha)\varepsilon'$. Then $y - \varepsilon z = \alpha y^1 + (1-\alpha)(y^2 - \varepsilon' z)$ belongs to K^+. The conclusion follows. ■

Lemma 2.7. (i) *The pair $(\hat{x},\hat{y})$ is of maximal complementarity.* (ii) *Any pair $(\hat{x},\hat{y}) \in \operatorname{ri} S(\text{COPOS})$ (set equal to $\operatorname{ri} S(COP) \times \operatorname{ri} S(COP^*)$) is of maximal complementarity.*

Proof. (i) That $A^j\hat{x}-b^j \in \operatorname{int} K_j$, for all $j \in B$, is a classical property. Similarly, $\hat{y}^j \in \operatorname{int} K_j^+$, for all $j \in N$. Finally, that $-\hat{y}^j \in \operatorname{ri} N_{K_j}(A^j\hat{x} - b^j)$, for all $j \in R$, is consequence of lemma 2.6.
(ii) Let $(\hat{x},\hat{y}) \in \operatorname{ri} S(\text{COPOS})$, and $(\tilde{x},\tilde{y}) \in S(\text{COPOS})$ be of maximal complementarity. Then there exists $\varepsilon > 0$ such that $(\hat{x},\hat{y}) - \varepsilon(\tilde{x},\tilde{y}) \in S(\text{COPOS})$. Set $\alpha = 1/(1+\varepsilon) \in (0,1)$. We may write

$$\alpha(\hat{x},\hat{y}) = \alpha[(\hat{x},\hat{y}) - \varepsilon(\tilde{x},\tilde{y})] + (1-\alpha)(\tilde{x},\tilde{y}).$$

Similarly, setting $\hat{s} := A\hat{x} - b$ and $\tilde{s} := A\tilde{x} - b$, we have that

$$\alpha(\hat{s}^j,\hat{y}^j) = \alpha[(\hat{s}^j,\hat{y}^j) - \varepsilon(\tilde{s}^j,\tilde{y}^j)] + (1-\alpha)(\tilde{s}^j,\tilde{y}^j).$$

We conclude by applying lemma 2.6 to the above relation. ■

We now introduce another problem related to (COP), having in mind the relations between SOCP and SDP problems. Let $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_J$ be another finite family of closed convex cones in $I\!R^{r_j}$, $j = 1$ to J, and M^j be $r_j \times q_j$ matrices such that

$$s^j \in K_j \quad \text{iff} \quad M^j s^j \in \mathcal{K}_j,\, j = 1,\ldots,J. \tag{2.8}$$

Let $M = (M^1;\cdots;M^J)$ be the matrix whose rows are those of M^j. Then (COP) is equivalent to the linear conic problem

$$\operatorname*{Min}_{x\in I\!R^n} c\cdot x\,;\; M^j(A^jx - b^j) \in \mathcal{K}_j,\, j = 1,\ldots,J, \tag{MCOP}$$

whose dual is

$$\operatorname*{Max}_{z\in\mathcal{K}^+} \sum_{j=1}^{J} b^j \cdot M^\top z^j;\;\; \sum_{j=1}^{J}(A^j)^\top(M^j)^\top z = c;\;\; z^j \in \mathcal{K}_j^+,\, j = 1,\ldots,J. \tag{MCOP*}$$

If the primal and dual values are equal, a pair (x,y) of the primal and dual problems is characterized by the optimality system

$$\begin{cases} M^j(A^jx - b^j) \in \mathcal{K}_j,\; z^j \in \mathcal{K}_j^+,\; z^j\cdot M^j(A^jx - b^j) = 0,\, j = 1,\ldots,J; \\ \sum_{j=1}^J (A^j)^\top(M^j)^\top z^j = c. \end{cases} \tag{2.9}$$

We first state two lemmas that deal with properties that do not involve explicitly the product form.

Lemma 2.8. *The following relations hold:* (i) $MS(COP) = S(MCOP)$, $M^\top\mathcal{K}^+ \subset K^+$*, and* $M^\top S(MCOP^*) \subset S(COP^*)$. (ii) *If* $M^\top\mathcal{K}^+$ *is closed, then* $M^\top\mathcal{K}^+ = K^+$ *and* $M^\top S(MCOP^*) = S(COP^*)$. (iii) *Closeness of* $M^\top\mathcal{K}^+$ *holds if* $M^\top$ *is coercive on* $\mathcal{K}^+$*, i.e., if* $\|M^\top z\| \geq c\|z\|$ *for all* $z \in \mathcal{K}^+$*. In that case,* $S(MCOP^*)$ *is bounded iff* $S(COP^*)$ *is bounded.*

Proof. (i) That $MS(COP) = S(MCOP)$ is a consequence of (2.8). Since $MK \subset \mathcal{K}$, any $z \in \mathcal{K}^+$ is such that $M^\top z \in K^+$. It follows from the expression

of dual problems that $M^\top S(MCOP^*) \subset S(COP^*)$.
(ii) Assume now that $\hat{K} := M^\top \mathcal{K}^+$ is closed. Since we know that $M^\top \mathcal{K}^+ \subset K^+$, we have to prove the converse inclusion. If this is not true, then there exists $y \in K^+$, $y \notin \hat{K}$. By the separation theorem there exists $h \in \hat{K}^+$ such that $h^\top y < 0$. That $h \in \hat{K}^+$ is equivalent to $Mh \in \mathcal{K}$, hence to $h \in K$; but since $y \in K^+$, this contradicts $h^\top y < 0$. This proves $M^\top \mathcal{K}^+ \subset K^+$, from which $M^\top S(MCOP^*) = S(COP^*)$ follows easily.
(iii) Finally, that the closeness of $M^\top \mathcal{K}^+$ is a consequence of coercivity of $M^\top$ is easy and left to the reader, as well as the equivalence of boundedness of $S(MCOP^*)$ and $S(COP^*)$. ■

Lemma 2.9. *Assume that M is one to one. Then the following holds.* (i) *The mapping $M^\top$ is onto, and $M^\top \operatorname{int} \mathcal{K}^+ \subset \operatorname{int} K^+$.* (ii) *If in addition $M^\top \mathcal{K}^+$ is closed, then $M^\top \operatorname{int} \mathcal{K}^+ = \operatorname{int} K^+$ and $M^\top \operatorname{int} S(MCOP^*) = \operatorname{int} S(COP^*)$.* (iii) *Under the same assumptions as in* (ii) *we also have that, for all $s \in K$, $M^\top \operatorname{ri}(\mathcal{K}^+ \cap (Ms)^\perp) \subset \operatorname{ri}(K^+ \cap s^\perp)$.*

Proof. (i) That the transposition of an injective mapping is surjective is well-known. If $z \in \operatorname{int} \mathcal{K}^+$, then there exists $\varepsilon > 0$ such that $z + \varepsilon B \subset \mathcal{K}^+$ (where B denotes the Euclidean ball). Since $M^\top$ is onto, $M^\top B \supset \alpha B$ for some $\alpha > 0$, and hence, $K^+ \supset M^\top(z + \varepsilon B) \supset M^\top z + \varepsilon\alpha B$, which proves that $M^\top z \in \operatorname{int} K^+$.
(ii) Since $M^\top$ is onto, $M^\top \operatorname{int} \mathcal{K}^+$ is an open set. As $M^\top \mathcal{K}^+$ is closed, the closure of $M^\top \operatorname{int} \mathcal{K}^+$ is $M^\top \mathcal{K}^+$, and the latter is equal to K^+ by lemma 2.8. This means that $M^\top \operatorname{int} \mathcal{K}^+ = \operatorname{int} K^+$. The equality between $M^\top \operatorname{int} S(MCOP^*)$ and $\operatorname{int} S(COP^*)$ is proved in a similar manner.
(iii) We know that $M^\top \mathcal{K}^+ = K^+$, and that for all $x \in K^+$, $z \cdot Mx = 0$ iff $(M^\top z) \cdot x = 0$. It follows that $M^\top(\mathcal{K}^+ \cap (Ms)^\perp) = (K^+ \cap s^\perp)$.

Let $z \in \operatorname{ri}(\mathcal{K}^+ \cap (Ms)^\perp)$, and set $y = M^\top z$. Let $y' \in K^+ \cap s^\perp$. We know that there exists $z' \in \mathcal{K}^+ \cap (Ms)^\perp$ such that $y' = M^\top z'$. Since $z \in \operatorname{ri}(\mathcal{K}^+ \cap (Ms)^\perp)$, there exists $\varepsilon' > 0$ such that $z \pm \varepsilon z' \in \mathcal{K}^+ \cap (Ms)^\perp$. It follows that $y \pm \varepsilon y' \in K^+ \cap s^\perp$. The conclusion follows. ■

We denote by $(B_{COP}, N_{COP}, R_{COP}, T_{COP})$ and $(B_{MCOP}, N_{MCOP}, R_{MCOP}, T_{MCOP})$ the optimal partitions of (COP) and $(MCOP)$, respectively.

Lemma 2.10. *Assume that $M^\top \mathcal{K}^+$ is closed, that M is one to one, and that*

$$\text{For all } s^j \in K_j, M^j s^j \in \partial\mathcal{K}_j \text{ iff } s^j \in \partial K_j. \tag{2.10}$$

Then the following relations hold between the optimal partitions of problems (COP) and $(MCOP)$:

$$B_{COP} = B_{MCOP}, \quad N_{COP} = N_{MCOP}, \quad R_{COP} \supset R_{MCOP}, \quad T_{COP} \subset T_{MCOP}. \tag{2.11}$$

In particular, the strict complementarity hypothesis holds for (COP) if it holds for $(MCOP)$.

Proof. That $B_{COP} = B_{MCOP}$ is an immediate consequence of (2.8) and (2.10). Applying the first part of lemma 2.9(ii) to $(K_i, \mathcal{K}_i, M^i)$ we deduce that $N_{COP} =$

N_{MCOP}. Finally that $R_{COP} \supset R_{MCOP}$ follows from lemma 2.9(iii) applied to $(K_i,\mathcal{K}_i,M^i)$. The relation $T_{COP} \subset T_{MCOP}$ follows from the three others.

As a consequence, if T_{MCOP} is empty then T_{COP} is also empty, which means that the strict complementarity hypothesis holds for (COP) if it holds for $(MCOP)$. ■

2.3 Application of the abstract framework

We apply the results of the above section. Here $K_j = Q_{m_j+1}$, $\mathcal{K}_j := \mathcal{S}_+^{m_j+1}$, and $M^j s^j = \operatorname{Arw} s^j$. Note that we can write

$$\operatorname{Arw}(s) = (s_0 - \|\bar{s}\|)I_{m+1} + \begin{pmatrix} \|\bar{s}\| & \bar{s}^\top \\ \bar{s} & \|\bar{s}\| I_m \end{pmatrix}. \tag{2.12}$$

This shows that for $s \in Q_{m+1} \setminus \{0\}$, $\operatorname{Arw}(s)$ is of rank m iff $s \in \partial Q_{m+1}$, and of rank $m+1$ otherwise. In particular, $\operatorname{Arw} \partial Q_{m+1} \subset \partial \mathcal{S}_+^{m+1}$, and $\operatorname{Arw} \operatorname{int} Q_{m+1} \subset \operatorname{int} \mathcal{S}_+^{m+1}$. Therefore (2.10) holds. Let us decompose any matrix $Y \in \mathcal{S}^{m+1}$ as follows

$$Y = \begin{pmatrix} Y_{00} & \bar{Y}_0^\top \\ \bar{Y}_0 & \bar{Y} \end{pmatrix}, \tag{2.13}$$

where $Y_{00} \in I\!R$, $\bar{Y}_0 \in I\!R^m$ and $\bar{Y} \in \mathcal{S}^m$. We note that for any $s \in I\!R^{m+1}$ we get

$$\operatorname{Arw}(s) \cdot Y = s_0 \operatorname{Tr}(Y) + 2\bar{s} \cdot \bar{Y}_0. \tag{2.14}$$

It follows that $\operatorname{Arw}^\top : \mathcal{S}^{m+1} \to I\!R^{m+1}$ is nothing but

$$\operatorname{Arw}^\top Y := \begin{pmatrix} \operatorname{Tr}(Y) \\ 2\bar{Y}_0 \end{pmatrix}. \tag{2.15}$$

Consequently

$$M^\top(Y^1,\ldots,Y^J) = \operatorname{vec}\left(\begin{pmatrix} \operatorname{Tr}(Y^1) \\ 2\bar{Y}_0^1 \end{pmatrix}, \ldots, \begin{pmatrix} \operatorname{Tr}(Y^J) \\ 2\bar{Y}_0^J \end{pmatrix} \right). \tag{2.16}$$

Proposition 2.11. (i) *We have that y is solution of $(LSOCP^*)$ iff there exists z solution of $(LSDP^*)$ such that $y = M^\top z$.* (ii) *One of these dual problems has a bounded set of solutions iff the other one has the same property.* (iii) *One of these dual problems has an interior feasible point iff the other one has the same property.* (iv) *Problems $(LSOCP)$ and $(LSDP)$ have the same optimal partition.*

Proof. Since $\operatorname{Arw}^\top$ is coercive on $\mathcal{S}_+^{m+1}$, $M^\top$ is also coercive. By lemma 2.8, we have that $S(LSOCP^*) = M^\top S(LSDP^*)$ and $S(LSDP^*)$ is bounded iff $S(LSOCP^*)$ is bounded. This proves points (i) and (ii). Point (iii) is consequence of lemma 2.9(ii). We now prove (iv). By lemma 2.10, $B_{LSOCP} = B_{LSDP}$,

$N_{LSOCP} = N_{LSDP}$, and $R_{LSOCP} \supset R_{LSDP}$; it remains to prove that $R_{LSOCP} \subset R_{LSDP}$ since (B,N,R,T) is a partition. Let $j \in R_{LSOCP}$. Then there is a pair (x,y) solution of (2.1) such that $s^j \neq 0 \neq y^j$, and both s^j and y^j belong to the boundary of Q_{m^j+1}. As observed after (2.12), this implies that $\operatorname{Arw} s^j$ is of rank m^j, and hence, the corresponding set of normals is a half line (of rank one semidefinite positive matrices, orthogonal to $\operatorname{Arw} s^j$). Since the corresponding multiplier Y for problem $(LSDP)$ is such that $0 \neq y^j = \operatorname{Arw}^\top Y^j$, we have that $Y^j \neq 0$, proving that $-Y^j$ belongs to the relative interior of the normal cone (to the set of semidefinite positive matrices) at $\operatorname{Arw} s^j$. ■

The above analysis shows that strong duality holds for problem (LSOCP) iff it holds for problem (LSDP). The next proposition states an interesting relation between the solutions of (LSOCP*) and (LSDP*).

Proposition 2.12. *Let the strong duality property hold for problem* (LSOCP)*. Let I be the set of indexes in $1,\ldots,J$ such that there exists $x^* \in S$*(LSOCP) *satisfying $A^j x^* \neq b^j$. Then every $Y \in S$*(LSDP*) *is such that, for some $y \in S$*(LSOCP*)*, the following relation holds:*

$$Y^j = 0, \text{ if } y^j = 0; \qquad Y^j = \frac{1}{2}\begin{pmatrix} \|\bar{y}^j\| & (\bar{y}^j)^\top \\ \bar{y}^j & \bar{y}^j(\bar{y}^j)^\top/\|\bar{y}^j\| \end{pmatrix}, \text{ otherwise.} \tag{2.17}$$

Proof. Let $j \in I$, x^* be the associated solution of (LSOCP), and let $Y \in S$(LSDP*). We claim that

$$Y_{00}^j \bar{Y}^j - (\bar{Y}_0^j)(\bar{Y}_0^j)^\top = 0, \tag{2.18}$$

where Y_{00}^j, $\bar{Y}^j$ and $\bar{Y}_0^j$ are given by (2.13). Since $Y^j \in \mathcal{S}_+^{m_j+1}$, by Schur complement the matrix $Y_{00}^j \bar{Y}^j - (\bar{Y}_0^j)(\bar{Y}_0^j)^\top$ is positive semidefinite, and hence, it is enough to show that

$$\operatorname{Tr}\left(Y_{00}^j \bar{Y}^j - (\bar{Y}_0^j)(\bar{Y}_0^j)^\top\right) \leq 0. \tag{2.19}$$

By strong duality, any primal-dual solution (x^*,y^*) of (LSOCP) is solution of (2.1). Since $A^j x^* \neq b^j$, the complementarity condition implies that any $y \in S$(LSOCP*) satisfies $y_0^j = \|\bar{y}^j\|$. Taking $y^j = \operatorname{Arw}^\top Y^j$, we deduce $\operatorname{Tr}(Y^j) = y_0^j = \|\bar{y}^j\| = 2\|\bar{Y}_0^j\|$, which implies

$$\begin{aligned}\operatorname{Tr}\left(Y_{00}^j \bar{Y}^j - (\bar{Y}_0^j)(\bar{Y}_0^j)^\top\right) &= Y_{00}^j \operatorname{Tr}\left(Y^j\right) - (Y_{00}^j)^2 - \|\bar{Y}_0^j\|^2 \\ &= -\left(Y_{00}^j - \|\bar{Y}_0^j\|\right)^2 \leq 0,\end{aligned} \tag{2.20}$$

proving (2.19) and therefore also (2.18). Combining (2.15) and (2.20), obtain

$$Y_{00}^j = \|\bar{Y}_0^j\| = \frac{1}{2}\|\bar{y}^j\|. \tag{2.21}$$

Now, we distinguish two cases: a) If $Y^j_{00} = 0$, we obtain from (2.21) that $\bar{Y}^j_0 = \bar{y}^j = 0$ and then $\text{Tr}(Y^j) = y^j_0 = 0$. Hence, since Y^j is positive semidefinite this implies $Y^j = 0$. b) Else if $Y^j_{00} \neq 0$, we get directly from (2.18) and (2.21) that

$$\bar{Y}^j = (Y^j_{00})^{-1}(\bar{Y}^j_0)(\bar{Y}^j_0)^\top = \frac{2}{\|\bar{y}^j\|}(\bar{y}^j/2)(\bar{y}^j/2)^\top = \frac{1}{2}(\bar{y}^j)(\bar{y}^j)^\top/\|\bar{y}^j\|,$$

which, combined with (2.21), allows to conclude the proof. ■

3 Duality theory for nonlinear SOCP problems

The Lagrangian function associated with problem (SOCP) (stated in the introduction) is $L(x,y) := f(x) - \sum_{j=1}^m y^j \cdot g^j(x)$, and the dual problem is

$$\operatorname*{Max}_{y\in\mathcal{Q}} \inf_x L(x,y), \tag{DSOCP}$$

where we have set $\mathcal{Q} := \Pi_j Q_{m_j+1}$. If problems (SOCP) and (DSOCP) have the same finite value, then a pair (x,y) of primal and dual solution is characterized by the optimality system

$$\begin{aligned} & L(x,y) = \min_{x'} L(x',y); \\ & g^j(x) \in Q_{m_j+1}; \quad y^j \in Q_{m_j+1}; \quad y^j \circ g^j(x) = 0,\ j = 1,\dots,J. \end{aligned} \tag{3.1}$$

The above statement is of special interest when problem (SOCP) is convex, i.e. (see e.g. [6, Def 2.163]) if $f(x)$ is convex, and the mapping $g(x)$ is convex with respect to the set $\mathcal{Q}' := -\mathcal{Q}$. The latter means [6, Section 2.3.5] that

$$g(tx + (1-t)x') \succeq_{\mathcal{Q}} tg(x) + (1-t)g(x'), \quad \forall\, x,x' \in I\!R^n \text{ and } t \in [0,1]. \tag{3.2}$$

Since $\mathcal{Q}$ is in product form, this is equivalent to say that $g^j(x)$ is convex w.r.t. Q_{m_j+1} for all j, that is, $x \to \|\bar{g}^j(x)\| - g^j_0(x)$ is convex for all j. This holds, for instance, if $\bar{g}^j(x)$ is affine and $g^j_0(x)$ is concave for all j.

The results of the previous sections have a natural extension to nonlinear second order cone problems. Since, for smooth problems, Lagrange multipliers are solutions of the dualization of the linearized problems we have that, for a nonconvex problem, there is a natural notion of optimal partition of constraints (B,N,R,T). For convex nonlinear second order cone problems, we can in the same way define the optimal partition of constraints (B,N,R,T), defined as follows. The set B is the union of j such that there exists $x \in S(SOCP)$ satisfying $g^j_0(x) > \bar{g}^j(x)$, the set N is the union of j such that there exists $y \in S(DSOCP)$ satisfying $y^j_0 > \|\bar{y}^j\|$, the set R is the union of j, such that there exists $x \in S(SOCP)$ and $y \in S(DSOCP)$ satisfying $g^j(x) \neq 0 \neq y^j$, and for all $j \in T$, $x \in S(SOCP)$ and $y \in S(DSOCP)$, either $g^j(x)$ or y^j is equal to 0, or both are zero, and neither $g^j(x)$ or y^j belong to the interior of Q_{m_j+1}.

Remark 3.1. *For second order cone problems we can even partition T as T_0, T_P and T_D, with T_0 the set of j for which, if $x \in S(SOCP)$ and $y \in S(DSOCP)$,* *then* $g^j(x) = 0 = y^j$, T_P *is the set of* $j \in T$ *such that there exists* $x \in S(SOCP)$ *with* $\|g^j(x)\| > 0$, *and* T_D *is the set of* $j \in T$ *such that there exists* $y \in S(DSOCP)$ *with* $\|y^j\| > 0$. *It is easy to see that such a refined partition is invariant under the reformulation as a semidefinite programming problem.*

4 Nondegeneracy Condition and Reduction Approach

We recall the basic concepts of the *reduction approach*, see [6, Sec. 3.4.4].

Definition 4.1. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be two finite dimensional spaces. Let* $K \subseteq \mathbb{X}$ *and* $\hat{K} \subseteq \mathbb{Y}$ *be closed, convex sets. We say that the set* K *is* reducible *to* $\hat{K}$ *at* $s^* \in K$ *if there exist a neighborhood* V *of* s^* *and a smooth mapping* $\phi : V \to \mathbb{Y}$ *such that: i) for all* $s \in V$, $s \in K$ *iff* $\phi(s) \in \hat{K}$, *and ii)* $D\phi(s^*) : \mathbb{X} \to \mathbb{Y}$ *is onto. If the set* K *is* reducible *to* $\hat{K}$ *at all* $s^* \in K$, *we just say that the set* K *is reducible to* $\hat{K}$. *If in addition* $\phi(s^*) = 0$, *and* $\hat{K}$ *is a pointed cone, we say that* K *is cone reducible.*

For our purposes, a smooth mapping will be a twice continuously differentiable (C^2) mapping. For problems with constraints in product form, i.e. $K = K_1 \times \cdots \times K_J$, the reduction approach has the following obvious decomposition property: cone reducibility holds whenever it holds for each set K_j, $j = 1$ to J.

Lemma 4.2. *The second-order cone* Q_{m+1} *is cone reducible at every point* $\hat{s} \in Q_{m+1}$, *in the following way:* (i) *If* $\hat{s} = 0$, *take* $\hat{K} = Q_{m+1}$ *and* $\phi(s) = s$, (ii) *If* $\hat{s}_0 > \|\bar{\hat{s}}\|$, *take* $\hat{K} = \{0\}$ *and* $\phi(s) = 0$, (iii) *If* $0 \neq \bar{s}_0 = \|\bar{s}\|$, *take* $\hat{K} = \mathbb{R}_-$ *and* $\phi(s) = \|\bar{s}\| - s_0$.

Definition 4.3. *Consider an arbitrary problem* (P) $\mathrm{Min}_{x\in\mathbb{X}}\{f(x)\,;\, g(x) \in K\}$, *where* f, g *are smooth functions,* $\mathbb{X}$, $\mathbb{Y}$ *and* $\mathbb{Z}$ *are finite dimensional spaces and* $K \subseteq \mathbb{Y}$ *is a closed convex cone, reducible to a closed convex cone* $\hat{K} \subseteq \mathbb{Z}$ *at* $g(x^*) \in K$ *by a mapping* ϕ. *We say that* x^* *is* nondegenerate *(with respect to the reduction given by* ϕ*) if the derivative* $D\mathcal{A}(x^*)$ *of the function* $\mathcal{A} := \phi \circ g$ *is onto.*

This notion, introduced in [5], generalizes to problems with general constraints the corresponding concept used in linear or nonlinear programming. Note that there are other definitions of nondegeneracy, e.g. [1, Def. 18] and references therein. In the case of second order cones all these definitions are essentially equivalent.

One of the main implication of nondegeneracy is stated in the next proposition, proved in [6, Prop. 4.75].

Proposition 4.4. *Consider the problem* (P) *given in definition 4.3. Let* x^* *be a solution of* (P) *and suppose that the set* K *is reducible to a pointed closed convex cone* $\hat{K}$ *at the point* $g(x^*)$. *If* x^* *is nondegenerate then there exists a unique Lagrange multiplier* y^* *associated. Conversely, if the pair* (x^*, y^*) *is* *strictly complementarity, and y^* is the unique Lagrange multiplier associated with x^*, then x^* is nondegenerate.*

Proposition 4.5. *Let x^* be a solution of the second-order problem (LSOCP) with $J = 1$. Set $s^* = Ax^* - b$ and $m = m_1$. Then, x^* is* nondegenerate *if and only if one of the following conditions holds: a) $s^* \in \operatorname{int} Q_{m+1}$, b) $s^* = 0$ and the matrix A is onto, c) $A^\top R_m(Ax^* - b) \neq 0$, where $R_m := \begin{pmatrix} 1 & 0^\top \\ 0 & -I_m \end{pmatrix}$.*

Proof. The result is a direct consequence of lemma 4.2. ■

We extend the above result to the case $J > 1$.

Proposition 4.6. *Let x^* be a solution of the second-order problem (LSOCP), and set $s^j = A^j x^* - b^j$. Set $I^* = \{1 \leq j \leq J; s^j \in \operatorname{int} Q_{m_j+1}\}$, $Z^* = \{1 \leq j \leq J; s^j = 0\}$, and $B^* = \{1 \leq j \leq J; s^j \in \partial Q_{m_j+1} \setminus \{0\}\}$, where ∂Q_{m_j+1} is the boundary of Q_{m_j+1}. Then, x^* is* nondegenerate *if and only if the following conditions holds: The matrix $\mathbb{A}$ whose rows are the union of those of A^j, for $j \in Z^*$, and the vectors rows $(A^j x^* - b^j)^\top R_{m_j} A^j$, for $j \in B^*$, is onto.*

Proof. This is once again a consequence of lemma 4.2. Indeed, for $\mathcal{A}^j(x) := \phi(g^j(x)) = \phi(A^j x - b^j)$, where ϕ is the reduction map of lemma 4.2, its derivative at x^* is given by

$$D\mathcal{A}^j(x^*) = \Big\{0, \text{ if } j \in I^*;\ A^j, \text{ if } j \in Z^*; \\ -(s_0^j)^{-1}(A^j x^* - b^j)^\top R_{m_j} A^j, \text{ if } j \in B^*\Big\}.$$

So, the derivative $D\mathcal{A}(x^*)$ of function $\mathcal{A} := (\mathcal{A}^1; \ldots; \mathcal{A}^J)$ is onto iff the matrix $\mathbb{A}$ is onto. ■

Remark 4.7. *We recover the result of [1, Thm 20]. Obviously, if $(A^1; \cdots; A^J)$ is onto, then any feasible point is nondegenerate.*

For problem (LSOCP), the Lagrange multipliers y^* are the solutions of (LSOCP*), so, if x^* is nondegenerate then proposition 4.4 tells us that the dual problem (LSOCP*) has a unique solution y^*.On the other hand, we know from proposition 2.11 that any $Y^* \in S(\text{LSDP}^*)$ is such that $y^* = Arw^\top Y^* \in S(\text{LSOCP}^*)$. By proposition 2.12, if $A^j x^* \neq b^j$ for all j, uniqueness of solution of (LSOCP*) implies uniqueness of the solution of (LSDP*). Yet it may happen that $S(\text{LSDP}^*)$ is not a singleton, even when x^* is nondegenerate for problem (LSOCP), as the next example shows.

Example 4.8. *Consider just one block $J = 1$. Let $A = I_3 \in \mathbb{R}^{3\times 3}$ the identity matrix, $m = 2$, $b = 0$ and $c = (1,0,0)^\top$. It follows that $x^* = 0$ is the unique solution of* (LSOCP) *(and then of* (LSDP)*), which is actually nondegenerate, and $y^* = (1,0,0)^\top$ is the unique solution of* (LSOCP*). *Using proposition 2.11(i) and* (2.15), *and since $Ax^* - b = 0$, we see that $Y \in S(\text{LSDP}^*)$ iff $Y \succeq 0$, $\operatorname{Tr}(Y) = 1$ and $\bar{Y}_0 = 0$. For instance, $y^*(y^*)^\top$ and $Y^* = \frac{1}{3} I_3$ belong to $S(\text{LSDP}^*)$.*

5 Strongly Regular Solutions of SOCP

In this section we consider the problem (SOCP) defined in the introduction as follows:

$$\operatorname*{Min}_{x\in I\!R^n, s^j\in I\!R^{m_j+1}} f(x);\;\; g^j(x) = s^j \succeq_{Q_{m_j+1}} 0,\, j = 1,\ldots,J, \tag{SOCP}$$

where $f : I\!R^n \to I\!R$ and $g^j : I\!R^n \to I\!R^{m_j+1}$ are smooth functions (at least C^2). The first-order optimality system is

$$D_x L(x^*,y) = Df(x^*) - \sum_{j=1}^{J} Dg^j(x^*)^\top y^j = 0, \tag{5.1a}$$

$$g^j(x) = s^j \succeq_{Q_{m_j+1}} 0,\;\; y^j \succeq_{Q_{m_j+1}} 0,\;\; s^j \circ y^j = 0,\;\; j = 1,\ldots,J, \tag{5.1b}$$

where $L : I\!R^n \times I\!R^{m+1} \to I\!R$ is the Lagrangian function of problem (SOCP)

$$L(x,y) := f(x) - \sum_{j=1}^{J} y^j \cdot g^j(x). \tag{5.2}$$

If (x^*,y^*) satisfies (5.1), then x^* will be called a critical or stationary point of (SOCP). Let us recall the definition of *strongly regular* solutions [13]:

Definition 5.1. *We say that* (x^*,y^*) *is a* strongly regular *solution of KKT-conditions* (5.1) *if there exists a neighborhood* V *of* (x^*,y^*) *such that for every* $\delta := (\delta_1,\delta_2) \in I\!R^n \times \Pi_{j=1}^{J} I\!R^{m_j+1}$ *close enough to 0, the "linearized" system:*

$$D^2_{xx}L(x^*,y^*)(x - x^*) - Dg(x^*)^\top(y - y^*) = \delta_1, \tag{5.3a}$$

$$g(x^*) \circ y + Dg(x^*)(x - x^*) \circ y = \delta_2 \circ y, \tag{5.3b}$$

$$g(x^*) + Dg(x^*)(x - x^*) - \delta_2 \succeq_{\mathcal{Q}} 0, y \succeq_{\mathcal{Q}} 0, \tag{5.3c}$$

has a unique solution $(x,y) = (x^*(\delta),y^*(\delta))$ *in* V*, which is a Lipschitz continuous map of* δ*.*

It can be shown that the strong regularity condition implies *Robinson's constraint qualification* condition:

$$\text{There exists } h^* \in I\!R^n \text{ such that } g(x^*) + Dg(x^*)h^* \in \operatorname{int} \mathcal{Q}, \tag{5.4}$$

which coincides with the *Slater* (or *primal strict feasibility*) condition for linear problem (LSOCP). This condition is discussed in [6, Section 2.3.4].

In this section we characterize the strong regularity in the context of problem (SOCP) by using second order optimality conditions. This characterization is a consequence of a well developed theory in a general conic optimization framework given by problem (P) stated in definition 4.3. Note that the strong regularity condition (definition 5.1) can be written in this general framework as

$$D^2_{xx}L(x^*,y^*)(x-x^*) - Dg(x^*)^\top(y-y^*) = \delta_1, \tag{5.5a}$$
$$(g(x^*) + Dg(x^*)(x-x^*) - \delta_2)\cdot y = 0, \tag{5.5b}$$
$$g(x^*) + Dg(x^*)(x-x^*) - \delta_2 \in K, y \in K^-. \tag{5.5c}$$

In order to establish our main result we will recall some key notions and theorems. For instance, a useful definition involved in this section is the following *uniform second order growth* condition [12]. For this, we define a family of perturbation of (P), denoted (P_u), as follows

$$\operatorname*{Min}_{x\in\mathbb{X}}\{f(x,u)\,;\, g(x,u)\in K\}, \tag{5.6}$$

where $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{U}$ are finite dimensional spaces, $u \in \mathbb{U}$ (perturbation space) is the perturbation parameter and the functions $f(x,u) : \mathbb{X}\times\mathbb{U} \to I\!R$ and $g(x,u) : \mathbb{X}\times\mathbb{U} \to \mathbb{Y}$ are at least twice continuously differentiable and satisfy $f(\cdot,0) := f(\cdot)$ as well as $g(\cdot,0) := g(\cdot)$.

Definition 5.2. *Let x^* be a stationary (or critical) point of problem* (P). *It is said that the* uniform second order growth *condition holds at x^* if there exist $\alpha > 0$ and a neighborhood $\mathcal{N}$ of x^* such that for any $u \in \mathbb{U}$ (perturbation space) close enough to 0 and any stationary (or critical) point $x^*(u) \in \mathcal{N}$ of the perturbed problem* (P_u), *we have that*

$$f(x,u) \geq f(x^*(u),u) + \alpha\|x - x^*(u)\|^2, \quad \forall x \in \mathcal{N}, g(x,u) \in K. \tag{5.7}$$

We say that the second order growth *condition holds at x^* if* (5.7) *holds for problem* (P)*, that is, there exist $\alpha > 0$ and a neighborhood $\mathcal{N}$ of x^* such that condition* (5.7) *is satisfied at $u = 0$ and $x^*(0) = x^*$.*

We need the next result, obtained in [6, Th. 5.24], that states a first characterization which is valid in a general context.

Theorem 5.3. *Let x^* be a local solution of problem (P) and y^* its corresponding Lagrange multiplier. Suppose that K is reducible to a pointed closed convex cone $\hat{K} \subseteq \mathbb{Z}$ at the point $g(x^*)$. Then (x^*,y^*) is a strongly regular solution of the Karush-Kuhn-Tucker conditions if and only if x^* is nondegenerate (definition 4.3) and the uniform second order growth condition holds at x^*.*

Theorem 5.3 means that we can completely characterize the strong regularity condition by giving sufficient and necessary conditions to obtain the uniform second order growth condition, under a nondegeneracy hypothesis. Unfortunately, such a characterization (in terms of derivatives of data at the nominal point) is known only in very specific examples as for nonlinear programming problems with C^2 data, see e.g. Bonnans and Sulem [7] and Dontchev and Rockafellar [9] and their references. For conic optimization problems, such a characterization is not known. In fact the (non uniform) second order growth condition itself can be characterized essentially in two situations. The first is when the cone is second order regular, see Bonnans, Cominetti and Shapiro [2], and the second is when reduction to a pointed cone holds. We will apply this second approach later in this section. Let us denote by $\Lambda(x^*)$ the set of Lagrange multiplier associated with x^* for problem (P), i.e., $y^* \in \Lambda(x^*)$ iff $D_x L(x^*,y^*) = 0$ and $-y^* \in N_K(g(x^*))$ (the *normal cone* to K at $g(x^*)$), where $L(x,y) := f(x) - y \cdot g(x)$ is the Lagrangian function of problem (P). We define the *tangent cone* to the set $K \subseteq \mathbb{Y}$ at the point $y \in K$ as

$$T_K(s) := \{d \in \mathbb{Y} \,:\, s + td + o(t) \in K, \forall t > 0\}, \tag{5.8}$$

and the *critical directions cone* at x^* for problem (P) as follows

$$C(x^*) := Df(x^*)^{\perp} \cap Dg(x^*)^{-1} T_K(g(x^*)) \tag{5.9}$$

or equivalently, if $\Lambda(x^*)$ is not empty, say contains some y^*:

$$C(x^*) := \{h \in \mathbb{X} \,:\, Dg(x^*)h \in T_K(g(x^*)) \cap (y^*)^{\perp}\}.$$

Lemma 5.4. *Consider the second order cone $Q := Q_{m+1}$ and let $s \in Q$. Then,*

$$T_Q(s) = \left\{ \begin{array}{ll} I\!R^{m+1}, & s \in \operatorname{int} Q, \\ Q, & s = 0, \\ d \in I\!R^{m+1} \,:\, \bar{d} \cdot \bar{s} - s_0 d_0 \leq 0, & s \in \partial Q \setminus \{0\}. \end{array} \right\} \tag{5.10}$$

Proof. The cases when $s \in \operatorname{int} Q$ and $s = 0$ follow directly from the definition of $T_Q(s)$ and the fact that Q is a cone. Suppose then that $s \in \partial Q \setminus \{0\}$, that is, $s_0 = \|\bar{s}\| \neq 0$.

Since the set Q can be written in the form $Q = \{s \in I\!R^{m+1} \,:\, \phi(s) \leq 0\}$, where $\phi(s) := \|\bar{s}\| - s_0$ is a convex differentiable function at all s such that $\bar{s} \neq 0$, by [6, Prop. 2.61] the tangent cone $T_Q(s)$ is given by

$$T_Q(s) = \{d \in I\!R^{m+1} \,:\, \phi'(s;d) \leq 0\}.$$

Therefore, we conclude by noting that the directional derivative $\phi'(s;d)$ when $\bar{s} \neq 0$ is equal to $\phi'(s;d) = D\phi(s) \cdot d = \bar{s} \cdot \bar{d}/\|\bar{s}\| - d_0$, and using $0 \neq s_0 = \|\bar{s}\|$. ■

Corollary 5.5. *Let x^* be a stationary (or critical) point of problem (SOCP) and $y \in \Lambda(x^*)$. Given $h \in I\!R^n$, denote $d^j(h) := Dg^j(x^*)h$, as well as $s^j = g^j(x^*)$. Then, the critical directions cone $C(x^*)$ is given by*

$$C(x^*) = \left\{ \begin{array}{ll} h \in I\!R^n \,:\, \text{ for all } j = 1, \ldots, J, & \\ d^j(h) \in T_{Q_{m_j+1}}(s^j), & y^j = 0, \\ d^j(h) = 0, & y^j \in \operatorname{int} Q_{m_j+1}, \\ d^j(h) \in I\!R_+(y_0^j, -\bar{y}^j), & y^j \in \partial Q_{m_j+1} \setminus \{0\}, s^j = 0, \\ d^j(h) \cdot y^j = 0, & y^j, s^j \in \partial Q_{m_j+1} \setminus \{0\}. \end{array} \right. \tag{5.11}$$

Proof. Since the constraints are in product form, the critical cone has the following decomposition property:

$$C(x^*) = \Big\{h \in I\!R^n;\ d^j(h) \in T_{Q_{m_j+1}}(s^j),\, d^j(h)\cdot y^j = 0,\, j = 1,\dots,J\Big\}. \quad (5.12)$$

It suffices to establish the equivalence between the relations in (5.11) and (5.12) concerning a given j. The case when $y^j = 0$ is obvious. If $y^j \in \operatorname{int} Q_{m_j+1}$, then $s^j = 0$ (by (5.1b)), and hence, $T_{Q_{m_j+1}}(s^j) = Q_{m_j+1}$, concluding that $T_{Q_{m_j+1}}(s^j) \cap (y^j)^\perp = Q_{m_j+1} \cap (y^j)^\perp = \{0\}$ and the result follows.

Suppose now that $y^j \in \partial Q_{m_j+1} \setminus \{0\}$. If $s^j = 0$ then, $T_{Q_{m_j+1}}(s^j) = Q_{m_j+1}$ again. Using (2.2), we obtain after elementary computations that $Q_{m_j+1} \cap (y^j)^\perp$ is the set of d^j satisfying $d_0^j(h) = \|\bar{d}^j(h)\|$ as well as $\bar{d}^j(h) \in I\!R_-\bar{y}^j$. If $s^j \neq 0$, we obtain by similar computations that $T_{Q_{m_j+1}}(s^j) \cap (y^j)^\perp$ is the set of d^j satisfying $\bar{d}^j(h)\cdot \bar{s}^j - s_0^j d_0^j(h) = 0$. The conclusion follows. ■

For the second-order analysis we need the notion of *(outer) second order tangent set* at $s \in K$ in the direction $d \in T_K(s)$, defined as follows

$$T_K^2(s,d) := \{w \in \mathbb{Y}\,;\ \exists t_n \downarrow 0 \text{ s.t. } s + t_n d + \frac{1}{2}t_n^2 w + o(t_n^2) \in K\}. \quad (5.13)$$

Let us characterize this set when $K = Q$.

Lemma 5.6. *Let $s \in Q = Q_{m+1}$, and $d \in T_Q(s)$. Then,*

$$T_Q^2(s,d) = \begin{cases} I\!R^{m+1}, & d \in \operatorname{int} T_Q(s),\\ T_Q(d), & s = 0,\\ \{w \in I\!R^{m+1} : \bar{w}\cdot\bar{s} - w_0 s_0 \le d_0^2 - \|\bar{d}\|^2\}, & \textit{otherwise.} \end{cases} \quad (5.14)$$

Note that the last case in (5.14) is when $s \in \partial Q \setminus \{0\}$ and $d \in \partial T_Q(s)$, the latter being, by lemma 5.4, equivalent to $\bar{d}\cdot\bar{s} - s_0 d_0 = 0$.

Proof. The first two cases follow directly from the definitions of second order tangent set, and the fact that Q is a cone. Suppose now that $s \in \partial Q \setminus \{0\}$ and $d \in \partial T_Q(s)$. As in lemma 5.4, since Q has the form $Q = \{s \in I\!R^{m+1} : \phi(s) \le 0\}$, where $\phi(s) := \|\bar{s}\| - s_0$, by [6, Prop. 3.30], the set $T_Q^2(s,d)$ is given by

$$T_Q^2(s,d) = \{d \in I\!R^{m+1} : \phi''(s;d,w) \le 0\},$$

where

$$\phi''(s;d,w) := \lim_{t\downarrow 0} \frac{\phi(s + td + \frac{1}{2}t^2 w) - \phi(s) - t\phi'(s;d)}{\frac{1}{2}t^2}$$

is the *(parabolic) second order directional derivative* of ϕ. But ϕ is twice differentiable at all s such that $\bar{s} \neq 0$ which implies that (e.g. [6, Eq. 2.81])

$$\phi''(s;d,w) = D\phi(s)w + D^2\phi(s)(d,d) = \frac{\bar{s}\cdot\bar{w}}{\|\bar{s}\|} - w_0 + \frac{\|\bar{d}\|^2}{\|\bar{s}\|} - \frac{(\bar{d}\cdot\bar{s})^2}{\|\bar{s}\|^3}, \quad (5.15)$$

and the desired result follows using $s_0 = \|\bar{s}\|$ and $d_0 s_0 = \bar{d} \cdot \bar{s}$ (the latter being consequence of lemma 5.4 and the fact that $d \in \partial T_Q(s)$). ■

Roughly speaking, the characterization of the second order growth condition (definition 5.2), established in [2, Th. 3.2], assumes a notion of set regularity on K, called *second order regularity*, that holds under the hypothesis that the set K is reducible to a cone $\hat{K}$ (e.g. [6, Prop.3.136]). The result presented below is a simplified version of this characterization. (cf. [6, Th. 3.137].)

Theorem 5.7. *Let x^* be a feasible point of problem (P) satisfying Robinson's constraint qualification condition*

$$0 \in \operatorname{int}\{\, g(x^*) + Dg(x^*)\mathbb{X} - K\} \tag{5.16}$$

Suppose that the set K is reducible to a closed convex cone $\hat{K}$ at the point $g(x^*)$. Then, the second order growth condition holds at x^* iff the next second order condition holds:

$$\sup_{y^* \in \Lambda(x^*)} D^2_{xx} L(x^*,y^*)(h,h) - \sigma(-y^*; \mathcal{T}^2) > 0, \quad \forall h \in C(x^*) \setminus \{0\}, \tag{5.17}$$

where $\sigma(\cdot\,; \mathcal{T}^2)$ denotes the support function of the set $\mathcal{T}^2 := T^2_K(g(x^*),Dg(x^*)h)$.

In the case of problem (SOCP) (i.e., $K = \mathcal{Q}$), the set

$$\mathcal{T}^2 := T^2_{\mathcal{Q}}(g(x^*),Dg(x^*)h)$$

can be written in the product form $\mathcal{T}^2 = \mathcal{T}^2_1 \times \ldots \times \mathcal{T}^2_J$ such that each $\mathcal{T}^2_j$ is given by formula (5.14) where $Q = Q_{m_j+1}$, $s = s^{*j}$ and $d = d^j(h)$. We have set $s^* := g(x^*)$ and $d(h) := Dg(x^*)h$. Since $-y^* \in N_{\mathcal{Q}}(s^*) \cap d^{\perp}$, we always have that $y^* \cdot w \geq 0$, for all $w \in \mathcal{T}^2$. So, formula (5.14) implies that $0 \in \mathcal{T}^2$ and hence $\sigma(-y^*; \mathcal{T}^2) = 0$, except in the case when $s^{*j} \in \partial Q_{m_j+1} \setminus \{0\}$ and $d^j(h) \in \partial T_{Q_{m_j+1}}(s^{*j}) \setminus \{0\}$, for some index $j \in \{1, \ldots, J\}$. Dealing with the latter case means, thanks to (5.14), to maximize $-(y_0 w_0 + \bar{y} \cdot \bar{w})$ over the set of w satisfying $\bar{w} \cdot \bar{s} - w_0 s_0 \leq d_0^2 - \|\bar{d}\|^2$, where we have considered the notation $y = y^{*j}$, and s and d given above, with j given by the case. Since $\bar{y} = -(y_0/s_0)\bar{s}$, we have that $-(y_0 w_0 + \bar{y} \cdot \bar{w}) = (y_0/s_0)(\bar{w} \cdot \bar{s} - w_0 s_0)$. It follows that

$$\sigma(-y^*; \mathcal{T}^2) = \sum_{j \in \mathcal{J}} (y_0^{*j}/s_0^{*j})(d^j(h)_0^2 - \|\bar{d}^j(h)\|^2), \tag{5.18}$$

where $\mathcal{J}$ is the set of index j s.t. $s^{*j} \in \partial Q_{m_j+1} \setminus \{0\}$ and $d^j(h) \in \partial T_{Q_{m_j+1}}(s^{*j}) \setminus \{0\}$. On the other hand, we know that $\mathcal{Q}$ is reducible, (cf. lemma 4.2), so we can apply theorem 5.7 to problem (SOCP) and state the following theorem.

Theorem 5.8. *Let x^* be a feasible point of the problem (SOCP) satisfying Robinson's constraint qualification condition* (5.4). *Then, the second order growth condition holds at x^* iff the following second order condition holds:*

$$\sup_{y \in \Lambda(x^*)} D^2_{xx} L(x^*,y)(h,h) + h^{\top} \mathcal{H}(x^*,y) h > 0, \quad \forall h \in C(x^*) \setminus \{0\}, \tag{5.19}$$

where the critical directions cone $C(x^*)$ *is established in* (5.11)*, and the* $n \times n$ *matrix* $\mathcal{H}(x^*,y)$ *is defined by* $\mathcal{H}(x^*,y) = \sum_{j=1}^{J} \mathcal{H}^j(x^*,y^j)$*, where for* $s^j = g^j(x^*)$*,* $j = 1$ *to* J*,*

$$\mathcal{H}^j(x^*,y^j) := -\frac{y_0^j}{s_0^j} Dg^j(x^*)^\top R_{m_j} Dg^j(x^*) = -\frac{y_0^j}{s_0^j} Dg^j(x^*)^\top \begin{pmatrix} 1 & 0^\top \\ 0 & -I_{m_j} \end{pmatrix} Dg^j(x^*), \tag{5.20}$$

if $s^j \in \partial Q_{m_j+1} \setminus \{0\}$*, and* $\mathcal{H}^j(x^*,y^j) := 0$ *otherwise.*

In the next theorem we give a characterization of the strong regularity condition.

Theorem 5.9. *Let* x^* *be a local solution of problem (SOCP) and* y^* *its corresponding Lagrange multiplier. Then,* (x^*,y^*) *is a strongly regular solution of optimality conditions* (5.1) *iff* x^* *is nondegenerate (definition 4.3) and the next second order condition holds at* x^**:*

$$Q_0(h) := D^2_{xx}L(x^*,y^*)(h,h) + h^\top \mathcal{H}(x^*,y^*)h > 0, \quad \forall h \in \operatorname{Sp}(C(x^*))\setminus\{0\}. \tag{5.21}$$

Proof. a) We establish some preliminary results. By theorem 5.3 we know that (x^*,y^*) is a strongly regular solution of (5.1) iff x^* is nondegenerate and the uniform growth condition holds at x^* for problem (SOCP). So, under the nondegeneracy hypothesis, we just need to prove that second order condition (5.21) is equivalent to the uniform growth condition. It is not difficult to check that, under this hypothesis, the linear space generated by the critical cone has the following expression:

$$\operatorname{Sp}(C(x^*)) = \left\{ \begin{array}{ll} h \in I\!R^n : & \text{for all } j = 1,\ldots,J, \\ d^j(h) = 0, & y^j \in \operatorname{int} Q_{m_j+1}, \\ d^j(h) \in I\!R(y_0^j, -\bar{y}^j), & y^j \in \partial Q_{m_j+1} \setminus \{0\}, s^j = 0, \\ d^j(h) \cdot y^j = 0, & y^j,\, s^j \in \partial Q_{m_j+1} \setminus \{0\}, \end{array} \right. \tag{5.22}$$

where throughout this proof we will denote by y^j the j-th vector block of y^*. (In particular, there is no condition on $d^j(h)$ if $y^j = 0$.)

b) Let us prove that the uniform growth condition implies (5.21). Consider the vector space E defined by

$$E := \left\{ \begin{array}{ll} h \in I\!R^n : & \text{for all } j = 1,\ldots,J, \\ d^j(h) = 0, & y^j \in \operatorname{int} Q_{m_j+1}, \\ d^j(h) \cdot y^j = 0, & y^j \in \partial Q_{m_j+1} \setminus \{0\}. \end{array} \right. \tag{5.23}$$

(Again, there is no restriction of $d^j(h)$ if $y^j = 0$.) We have that $\operatorname{Sp}(C(x^*)) \subset E$. The key idea is to consider a perturbed version of problem (SOCP) in such a way that x^* is still a local solution with the same Lagrange multiplier y^*, but with a bigger critical cone, equal to E. This perturbed problem is of the form

$$\operatorname*{Min}_{x \in I\!R^n} f(x)\,;\; g^j(x,u) := g^j(x) + u\delta^j \succeq_{Q_{m_j+1}} 0, \; j = 1,\ldots,J, \tag{SOCP$_u$}$$

where for all j, e_1^j denotes the first element of the natural basis of $\mathbb{R}^{m_j+1}$, $u > 0$ is the perturbation parameter, and

$$\delta^j = \begin{cases} e_1^j & \text{if } y^j = 0, \\ (y_0^j, -\bar{y}^j) & \text{if } \ s^j := g^j(x^*) = 0, y^j \in \partial Q_{m_j+1} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.24}$$

This means that, if $y^j = 0$, the constraint $g^j(x) \succeq_{Q_{m_j+1}} 0$ is made inactive (in a neighborhood of x^*), and if $s^j = 0$ and $y^j \in \partial Q_{m_j+1} \setminus \{0\}$, then the constraint $g^j(x) \succeq_{Q_{m_j+1}} 0$ is still active, but at a point different from 0 where the set of tangent directions to Q_{m_j+1} is a half space. The point (x^*,y^*) is still solution of the optimality system of (SOCP$_u$). It is easily seen that the expression of the critical cone for problem (SOCP$_u$) at the point (x^*,y^*) is given by (5.23).

Define

$$I := \{1 \le j \le J;\ g^j(x^*) = 0,\ y^j \in \partial Q_{m_j+1} \setminus \{0\}\}.$$

Let $\mathcal{H}(x^*,y^j,u)$ denote the matrices in the expression of second order conditions, for the perturbed problem. We have that $\mathcal{H}(x^*,y^j,u) = \mathcal{H}(x^*,y^j)$ for all $j \notin I$, whereas for $j \in I$ we obtain

$$\begin{aligned} \mathcal{H}^j(x^*,y^j,u) &= \frac{1}{u}\hat{\mathcal{H}}^j(x^*,y^j), \\ \text{where } \ \hat{\mathcal{H}}^j(x^*,y^j) &:= -Dg^j(x^*)^\top \begin{pmatrix} 1 & 0^\top \\ 0 & -I_{m_j} \end{pmatrix} Dg^j(x^*). \end{aligned} \tag{5.25}$$

Set

$$Q_1(h) := \sum_{j\in I} h^\top \hat{\mathcal{H}}^j(x^*,y^j)h = \sum_{j\in I} \left(\|\bar{d}^j(h)\|^2 - (d^j(h)_0)^2 \right). \tag{5.26}$$

Note that, if $h \in E$, then since $d^j(h) \cdot y^j = 0$ and $y_0^j = \|\bar{y}^j\|$:

$$|d^j(h)_0| = |\bar{d}^j(h) \cdot \bar{y}^j| / y_0^j \le \|\bar{d}^j(h)\|, \tag{5.27}$$

with equality iff $d^j(h) \in \mathbb{R}(y_0^j, -\bar{y}^j)$. Combining with (5.26), we obtain that, for all $h \in E$, $Q_1(h) \ge 0$, and that $Q_1(h) = 0$ iff $h \in \mathrm{Sp}(C(x^*))$.

We see that the uniform second-order growth for the perturbed problem implies

$$Q_0(h) + \frac{1}{u} Q_1(h) > 0 \quad \text{ for all } \ h \in E \setminus \{0\}, \tag{5.28}$$

for u small enough. This implies that $Q_0(h) > 0$, for all $h \in E$ such that $Q_1(h) = 0$. Therefore, the uniform second-order growth condition implies (5.21).
c) Conversely, assume that the second order condition (5.21) holds. If the uniform second order growth condition at x^* is not satisfied, then there exists a family of perturbed functions $f(x,u)$ and $g(x,u)$ such that, for some sequences

$u_n \to 0$, there exist (x_n,y_n) solution of the optimality system (5.1) of the perturbed problem satisfying $x_n \to x^*$, $h_n \to 0$ in $I\!R^n$, with $h_n \neq 0$, such that $x_n + h_n$ is a feasible point of (P_{u_n}) (cf. (5.6)) (that is, $g(x_n + h_n,u_n) \in \mathcal{Q}$) and they also satisfy that

$$f(x_n + h_n,u_n) \leq f(x_n,u_n) + o(\|h_n\|^2). \tag{5.29}$$

The nondegeneracy condition being stable under small perturbations, for large enough n, there exists a unique Lagrange multiplier y_n associated with each stationary (or critical primal) point x_n of (P_{u_n}), and since $x_n \to x^*$, we have that $y_n \to y^*$.

Extracting if necessary a subsequence, we may assume that $h_n/\|h_n\|$ converges to some $h^* \neq 0$. Let us check that $h^* \in \mathrm{Sp}(C(\bar{x}))$. Since $g^j(x_n+h_n,u_n) \in Q_{m_j+1}$ we have that

$$g^j(x_n + h_n,u_n) = g^j(x_n,u_n) + D_x g^j(x_n,u_n)h_n + o(\|h_n\|) \succeq_{Q_{m_j+1}} 0. \tag{5.30}$$

Since $g^j(x_n,u_n)$ and $(y_n)^j$ are orthogonal this implies

$$(y_n)^j \cdot D_x g^j(x_n,u_n)h_n + o(\|h_n\|) \geq 0. \tag{5.31}$$

Dividing by $\|h_n\|$, setting $d^j(h^*) := Dg^j(x^*)h^*$, and passing to the limit, obtain $y^j \cdot Dg^j(x^*)h^* \geq 0$ for all j. Passing to the limit in (5.29) and combining with (5.1a), we obtain $0 \geq \nabla f(x^*) \cdot h^* = y \cdot Dg(x^*)h^* = \sum_{j=1}^J y^j \cdot Dg^j(x^*)h^*$. We have proved that

$$d^j(h^*) \cdot y^j = 0, \quad j = 1,\ldots,J. \tag{5.32}$$

Consider the case when $y^j \in \operatorname{int} Q_{m_j+1}$. Since $y_n^j \to y^j$, we have that $g^j(x_n,u_n) = 0$ for large enough n. Let $\varepsilon > 0$ be such that $y^j + 2\varepsilon B \subset Q_{m_j+1}$. Then for all unit vector z, $y_n^j + \varepsilon z \in Q_{m_j+1}$ for large enough n. Computing the scalar product of (5.30) by $y_n^j + \varepsilon z$, and passing to the limit as was done before, obtain $(y^j + \varepsilon z) \cdot Dg^j(x^*)h^* \geq 0$. Using (5.32), since this is true for any unit norm z, we get

$$d^j(h^*) = 0, \quad \text{for all } j;\ y^j \in \operatorname{int} Q_{m_j+1}. \tag{5.33}$$

Now in the case when $y^j \in \partial Q_{m_j+1} \setminus \{0\}$ and $g^j(x^*) = 0$, we have that $g^j(x_n,u_n) \in \partial Q_{m_j+1}$ for all n large enough (otherwise we obtain from complementarity condition that $y_n^j = 0$ for some sequence $y_n^j \to y^j \neq 0$). Let us set $g_n^j := g^j(x_n,u_n)$ and $d_n^j := D_x g^j(x_n,u_n)h_n$. Of course $d_n^j/\|h_n\| \to d^j(h^*) := Dg^j(x^*)h^*$. By the very definition of Q_{m_j+1}, condition (5.30) can be equivalently written as follows

$$(g_n^j)_0 + (d_n^j)_0 \geq \|\bar{g}_n^j + \bar{d}_n^j\| + o(\|h_n\|).$$

Since $g_n \in \partial Q_{m_j+1}$, that is $(g_n^j)_0 = \|\bar{g}_n^j\|$, we obtain that

$$(d_n^j)_0 \geq \|\bar{g}_n^j + \bar{d}_n^j\| - \|\bar{g}_n^j\| + o(\|h_n\|) \geq \|\bar{d}_n^j\| + o(\|h_n\|).$$

Hence, by dividing by $\|h_n\|$ and tending $n \to +\infty$, we deduce that $d^j(h^*) \in Q_{m_j+1} = T_{Q_{m_j+1}}(g^j(x^*))$. This together with relations (5.32)-(5.33) proves that $h^* \in \mathrm{Sp}(C(x^*))$.

We now use the same reduction argument as in lemma 4.2. It suffices for indexes in

$$I := \{1 \le j \le J : \ \ g^j(x^*) \ne 0 \ne y^j\}. \tag{5.34}$$

to change the formulation of corresponding constraint of the perturbed problem, that is, $g^j(x,u) \succeq_{Q_{m_j+1}} 0$, into $\phi(g^j(x,u)) \le 0$, where $\phi(s) := \|\bar{s}\| - s_0$. The corresponding component of Lagrange multiplier is y_0^j (see the discussion of relation between Lagrange multipliers before and after reduction in [6, Section 3.4.4], especially equation (3.267)). We have that, for each feasible point of the perturbed problem (P_{u_n}), and denoting by y_n the Lagrange multiplier associated with x_n,

$$\sum_{j \notin I} (y_n)^j \cdot g^j(x,u) + \sum_{j \in I} (y_n)_0^j \phi(g^j(x,u)) \ge 0. \tag{5.35}$$

Writing this inequality at point $(x_n + h_n, u_n)$ and noticing that equality holds at (x_n, u_n) in view of the complementarity conditions, obtain

$$\begin{aligned} &\sum_{j \notin I} (y_n)^j \cdot (g^j(x_n + h_n, u_n) - g^j(x_n, u_n)) \\ &\qquad + \sum_{j \in I} (y_n)_0^j (\phi(g^j(x_n + h_n, u_n) - \phi(g^j(x_n, u_n)) \ge 0. \end{aligned} \tag{5.36}$$

Adding it to (5.29), in order to get a difference of Lagrangian functions, and after a second-order expansion (using the fact that the derivative of Lagrangian function w.r.t. x, at (x_n, u_n), is zero), it follows that

$$\begin{aligned} &D^2_{xx} f(x_n, u_n)(h_n, h_n) - \sum_{j \notin I} (y_n)^j \cdot D^2_{xx} g^j(x_n, u_n)(h_n, h_n) \\ &\qquad - \sum_{j \in I} (y_n)_0^j D^2_{xx} \phi(g^j(x_n, u_n))(d_n^j(h_n), d_n^j(h_n)) \le o(\|h_n\|), \end{aligned} \tag{5.37}$$

where $d_n^j(h_n) := D_x g^j(x_n, u_n) h_n$. Using the expression of the expansion of ϕ, computed in (5.15), and passing to the limit in n, obtain $Q_0(h^*) \le 0$. Since $h^* \in \mathrm{Sp}(C(x^*)) \setminus \{0\}$, this contradicts (5.21). The conclusion follows. ■

Remark 5.10. *A related result is [6, Thm 5.25], where it is proved that a necessary condition for uniform quadratic growth, assuming uniqueness of the Lagrange multiplier, is that the Hessian of Lagrangian function is positive definite over the space spanned by radial critical directions. By contrast, our result is a characterization involving additional terms in the quadratic form, and space spanned by all critical directions. There is also a second part in [6, Thm 5.25] that involves the space spanned by all critical directions, but under a certain "strong extended polyhedricity condition" that is not satisfied here.*

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CHAPITRE IV

A note on Strong Regularity for Semidefinite Programming

A note on Strong Regularity for Semidefinite Programming

J. Frédéric Bonnans [1] & Héctor Ramírez C. [2]

Abstract. In this work, we consider a semidefinite programming problem that minimizes a nonlinear objective function subject to a nonlinear matrix inequality constraint. We assume that these functions are at least twice continuously differentiable.

We state some necessary or sufficient conditions for strong regularity, in the sense of Robinson, in terms of nonnegativity or positivity of some quadratic forms on some subspaces. Although this improves the known results, there is still a gap between the necessary and sufficient conditions.

1 Introduction

We consider the following problem

$$\min_{x \in \mathbb{R}^n} \{f(x)\,;\, G(x) \preceq 0\}, \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to S^m$ are C^2-functions, S^m denotes the linear space of $m \times m$ symmetric matrices equipped with the inner product $A \cdot B := \operatorname{trace}(AB) = \sum_{i,j=1}^m A_{ij}B_{ij}$ for all matrices $A = (A_{ij}), B = (B_{ij}) \in S^m$, and $\preceq$ denotes the negative semidefinite order, that is, $A \preceq B$ iff $A - B$ is a negative semidefinite matrix. The order relations $\prec$, $\succeq$ and $\succ$ are defined similarly.

Throughout this article we denote by $(\bar{x},\bar{Y}) \in \mathbb{R}^n \times S^m$ a solution of the following Karush-Kuhn-Tucker (KKT) conditions:

$$\nabla_x L(\bar{x},\bar{Y}) = \nabla f(\bar{x}) + DG(\bar{x})^* \bar{Y} = 0, \tag{1.1a}$$

$$G(\bar{x})\bar{Y} = 0, \tag{1.1b}$$

$$G(\bar{x}) \preceq 0, \bar{Y} \succeq 0, \tag{1.1c}$$

where $L : \mathbb{R}^n \times S^m \to \mathbb{R}$ is the Lagrangian function of problem (P)

$$L(x,Y) := f(x) + Y \cdot G(x). \tag{1.2}$$

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We say that $\bar{Y}$ is a Lagrange multiplier associated with $\bar{x}$. Note that, for a linear operator $Ay := \sum_{i=1}^{m} y_i A_i$ with $A_i \in S^m$, as $DG(x)$, we have for its adjoint operator A^* the formula:

$$A^* Z = (A_1 \cdot Z, \ldots, A_n \cdot Z)^\top, \quad \forall Z \in S^m. \tag{1.3}$$

A pair $(\bar{x},\bar{Y})$ satisfying (1.1) will be also called critical pair or KKT-point of problem (P), and the set of Lagrange multipliers associated with $\bar{x}$ will be denoted by $\Lambda(\bar{x})$. Finally, $\bar{x}$ is called a critical point or critical solution of (P) if $\Lambda(\bar{x}) \neq \emptyset$.

In this work, we investigate the behavior of the pair $(\bar{x},\bar{Y})$ when a perturbation $u \in I\!R^k$ is applied to problem (P), obtaining then the perturbed problem

$$\min_{x \in I\!R^n} \{f(x,u)\,;\, G(x,u) \preceq 0\}, \tag{P$_u$}$$

where $f : I\!R^n \times I\!R^k \to I\!R$ and $G : I\!R^n \times I\!R^k \to S^m$ are C^2-functions satisfying that $f(\cdot,0) := f(\cdot)$ and $G(\cdot,0) := G(\cdot)$.

We recall that *Robinson's constraint qualification* condition holds at a feasible point $\bar{x}$ of (P) if

$$\text{There exists } \bar{h} \in I\!R^n \text{ such that } G(\bar{x}) + DG(\bar{x})\bar{h} \prec 0. \tag{1.4}$$

Since $\bar{x}$ is assumed to be a local solution of (P), condition (1.4) is equivalent to say that the set of Lagrange multipliers $\Lambda(\bar{x})$ is nonempty and compact. Obviously, condition (1.4) is stable under small perturbations of problem (P), and hence, (1.4) also implies the existence of a Lagrange multiplier (and such multipliers are uniformly bounded) associated with a solution $\bar{x}(u)$ of (P$_u$), close enough to $\bar{x}$, i.e., for all u close enough to 0, there exists a pair $(\bar{x}(u),\bar{Y}(u))$ satisfying the KKT-conditions of problem (P$_u$):

$$\nabla_x L(\bar{x}(u),\bar{Y}(u),u) = \nabla_x f(\bar{x}(u),u) + D_x G(\bar{x}(u),u)^* \bar{Y}(u) = 0, \tag{1.5a}$$

$$G(\bar{x}(u),u)\bar{Y}(u) = 0, \tag{1.5b}$$

$$G(\bar{x}(u),u) \preceq 0, \bar{Y}(u) \succeq 0, \tag{1.5c}$$

where the Lagrangian function $L : I\!R^n \times S^m \times I\!R^k \to I\!R$ of the perturbed problem (P$_u$) is defined as $L(x,Y,u) := f(x,u) + Y \cdot G(x,u)$.

The outline of this paper is as follows. In Section 2 we present a review of optimality conditions for our semidefinite problem and introduce the hypotheses that will be used in the sequel. Section 3 deals with the main subject of this paper: the strong regularity condition (in the sense of Robinson). Here we recall some known results and establish a necessary condition to obtain the strong regularity condition in the SDP case. Finally in Section 4, a sufficient condition is shown.

2 Review of optimality conditions

2.1 General properties of critical points

In this subsection we state some properties that are valid without any hypothesis on a solution $(\bar{x},\bar{Y})$ of the KKT system.

With $G(\bar{x})$ and a Lagrange multiplier $\bar{Y}$ is associated a certain decomposition of $\mathbb{R}^n$ that we present now. Let r be the rank of $G(\bar{x})$, and denote by $E \in \mathbb{R}^{m\times m-r}$ a matrix whose columns are an orthonormal basis of $\operatorname{Ker} G(\bar{x})$, i.e., $E^\top E = I_{m-r}$ and $G(\bar{x})E = 0$. Define also $\hat{E} \in \mathbb{R}^{m\times r}$ as the matrix whose columns are an orthonormal basis of $\operatorname{Im} G(\bar{x})$. Such a matrix is characterized by the relations $\hat{E}^\top \hat{E} = I_r$ and $\hat{E}^\top E = 0$.

Since $\bar{Y}$ and $-G(\bar{x})$ are positive semidefinite, the complementarity relation $\bar{Y} \cdot G(\bar{x}) = 0$ is equivalent to $\operatorname{Im} \bar{Y} \subset (\operatorname{Im} G(\bar{x}))^\perp = \operatorname{Ker} G(\bar{x})$. Therefore, we can write

$$\bar{Y} = E\bar{Y}_{11}E^\top = EW\bar{\phi}W^\top E^\top, \tag{2.1}$$

where $\bar{Y}_{11} := E^\top \bar{Y} E$ is an $(m-r)\times(m-r)$ matrix, $\bar{r} := dim(\operatorname{Ker} \bar{Y})$, and the matrix $W \in \mathbb{R}^{(m-r)\times(m-\bar{r})}$ satisfies that $W^\top W = I_{m-\bar{r}}$ and that $\bar{\phi} := W^\top \bar{Y}_{11} W$ is positive definite.

We also define $\hat{W} \in \mathbb{R}^{(m-r)\times(\bar{r}-r)}$ by the relations $\hat{W}^\top \hat{W} = I_{\bar{r}-r}$ and $\hat{W}^\top W = 0$.

We next need some basic concepts of convex analysis. The *inner tangent cone*, to the set $K \subseteq \mathbb{Y}$ at the point $A \in K$, is defined as

$$T_K(A) := \{M \in \mathbb{Y} : \operatorname{dist}(A + tM, K) = o(t), \forall t > 0\}. \tag{2.2}$$

When $K = S^m_-$ we have the following characterization

$$T_{S^m_-}(A) = \{M \in S^m : v^\top M v \leq 0, \forall v \in \operatorname{Ker} A\}.$$

Therefore, $T_{S^m_-}(G(\bar{x})) = \{M \in S^m : E^\top M E \preceq 0\}$.

The *critical cone* is defined to be

$$C(\bar{x}) := \{h \in \mathbb{R}^n : DG(\bar{x})h \in T_{S^m_-}(G(\bar{x})), \nabla f(\bar{x})^\top h = 0\}. \tag{2.3}$$

Using the KKT conditions and (2.1), we have that, if h is critical,

$$0 = -\nabla f(\bar{x})^\top h = DG(\bar{x})h \cdot \bar{Y} = \bar{\phi} \cdot W^\top E^\top DG(\bar{x})hEW. \tag{2.4}$$

Since $\bar{\phi} \succ 0$ and $W^\top E^\top DG(\bar{x})hEW \preceq 0$, (2.4) is equivalent to

$$W^\top E^\top DG(\bar{x})hEW = 0. \tag{2.5}$$

It follows that

$$C(\bar{x}) = \{h \in I\!R^n \,:\, E^\top DG(\bar{x})hE \preceq 0;\ W^\top E^\top DG(\bar{x})hEW = 0\}. \tag{2.6}$$

A negative semidefinite matrix with a null diagonal block is characterized by null corresponding non diagonal blocks and the other diagonal block negative semidefinite. Therefore, using the columns of W and $\hat{W}$ as a base, we obtain an equivalent expression of the critical cone:

$$C(\bar{x}) = \{h \in I\!R^n \,:\, E^\top DG(\bar{x})hEW = 0;\ \hat{W}^\top E^\top DG(\bar{x})hE\hat{W} \preceq 0\}. \tag{2.7}$$

Given two points A, B of a vector space $\mathbb{Y}$, let $[A,B] = \{\alpha A + (1-\alpha)B; \alpha \in [0,1]\}$ denote the segment from A to B. The *radial cone* to a set $K \subseteq \mathbb{Y}$ at the point $A \in K$ is defined by

$$\mathcal{R}_K(A) := \{M \in \mathbb{Y} \,:\, \exists\, t > 0, [A, A + tM] \in K\}. \tag{2.8}$$

It is known [4, Prop. 5.68] that $H \in T_{S^m_-}(G(\bar{x}))$ belongs to $\mathcal{R}_{S^m_-}(G(\bar{x}))$ iff

$$F^\top E^\top H G^\dagger H E F = 0, \tag{2.9}$$

where $F \in I\!R^{(m-r)\times k}$ is a matrix whose columns are an orthonormal basis of Ker $E^\top HE$, and $M^\dagger$ denotes the Moore-Penrose inverse of the matrix M, defined by $M^\dagger := \sum_{j=1}^r \lambda_j^{-1} m_j m_j^\top$, where λ_j are the nonzero eigenvalues of M and m_j the associated orthonormal eigenvectors.

Define the *radial critical cone* [4, Def. 3.52] as

$$C_R(\bar{x}) := \{h \in I\!R^n \,:\, DG(\bar{x})h \in \mathcal{R}_{S^m_-}(G(\bar{x}));\ \nabla f(\bar{x})^\top h = 0\}. \tag{2.10}$$

Radial critical directions satisfy (2.7) as well as $F_h^\top E^\top H_h G^\dagger H_h E F_h = 0$, where now $H_h = DG(\bar{x})h$ and F_h is a matrix whose columns are an orthonormal basis of $\operatorname{Ker} E^\top DG(\bar{x})hE$. Since $E^\top$ is a basis of the range space of $G(\bar{x})$, (2.9) is equivalent to $\hat{E}^\top H_h E F_h = 0$. Since $\operatorname{Im} F_h \supset \operatorname{Im} W$ by complementarity, $C_R(\bar{x})$ is the set of directions $h \in I\!R^n$ satisfying

$$H_h E W = 0;\ \hat{E}^\top H_h E F_h = 0;\ \hat{W}^\top E^\top H_h E \hat{W} \preceq 0. \tag{2.11}$$

Using $E^\top H_h E F_h = 0$, we obtain finally

$$C_R(\bar{x}) = \{h \in I\!R^n \,:\, H_h E F_h = 0;\ \hat{W}^\top E^\top H_h E \hat{W} \preceq 0\}. \tag{2.12}$$

Remark 2.1. *That $C_R(\bar{x})$ is convex is not immediate from expression* (2.12). *However, we have that, if $h_i \in C_R(\bar{x})$ for $i = 1,2$, and $\alpha \in (0,1)$, then $h := \alpha h_1 + (1-\alpha)h_2$ is such that* $\operatorname{Im}(F_h) = \operatorname{Im}(F_{h_1}) \cap \operatorname{Im}(F_{h_2})$ *(this is in fact true for all critical directions defined above and follows from the first relation in* (2.6)*). Therefore $H_h E F_h = 0$, whereas the second relation in the r.h.s. of* (2.12) *is clearly satisfied.*

We will need in the sequel expressions of the spaces spanned by the critical cone and the radial critical cone. Therefore we prove the following general lemma.

Lemma 2.2. *Let $\mathcal{C}$ be the cone defined by*

$$\mathcal{C} := \{h \in I\!R^n \,:\, \mathcal{A}(h) = 0;\ \mathcal{B}(h) \preceq 0\}, \tag{2.13}$$

where $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional linear mappings, with $\mathcal{B}$ having its range in a space of symmetric matrices. Let $\bar{h} \in \mathrm{ri}(\mathcal{C})$ (relative interior of $\mathcal{C}$), and D matrix whose columns are an orthonormal basis of the kernel of $\mathcal{B}(\bar{h})$. Then

$$\mathrm{Sp}(\mathcal{C}) = \{h \in I\!R^n \,:\, \mathcal{A}(h) = 0;\ D^\top \mathcal{B}(h) = 0\}, \tag{2.14}$$

where $\mathrm{Sp}(\mathcal{C}) := I\!R_+(\mathcal{C} - \mathcal{C})$ is the linear space generated by $\mathcal{C}$.

Proof. Let $h \in \mathcal{C}$. Since $\bar{h} \in \mathrm{ri}(\mathcal{C})$, there exists $\varepsilon > 0$ such that $\bar{h} \pm \varepsilon h \in \mathcal{C}$. In particular, $\mathcal{B}(\bar{h}) \preceq \varepsilon \mathcal{B}(h) \preceq 0$. This implies $\mathrm{Ker}\, \mathcal{B}(\bar{h}) \subset \mathrm{Ker}\, \mathcal{B}(h)$, and hence, $D^\top \mathcal{B}(h) = 0$ for all $h \in \mathcal{C}$, implying that $\mathrm{Sp}(\mathcal{C})$ is included in the r.h.s. of (2.14).

Conversely, let h belong to the r.h.s. of (2.14). We want to show that $h \in \mathrm{Sp}(\mathcal{C})$. Since $h = \varepsilon^{-1}(\bar{h} - (\bar{h} - \varepsilon h))$, it is sufficient to check that $\bar{h} - \varepsilon h \in \mathcal{C}$ for small enough ε. Let $\hat{D}$ be a matrix whose columns are an orthonormal basis of the range of $\mathcal{B}(\bar{h})$. Then $\hat{D}^\top \mathcal{B}(\bar{h}) \hat{D}$ is negative definite, and hence, $\hat{D}^\top \mathcal{B}(\bar{h}) \hat{D} \preceq \varepsilon \hat{D}^\top \mathcal{B}(h) \hat{D} \preceq 0$ for small enough ε. Since $D^\top \mathcal{B}(h) = 0$, this implies $\mathcal{B}(\bar{h}) \preceq \varepsilon \mathcal{B}(h)$, hence $\bar{h} - \varepsilon h \in \mathcal{C}$. The conclusion follows. ■

This lemma allows to compute the space spanned by the critical cone.

Corollary 2.3. *The existence of a critical direction $\bar{h} \in C(\bar{x})$ such that $\hat{W}^\top E^\top DG(\bar{x}) \bar{h} E \hat{W} \prec 0$ characterizes the equality*

$$\mathrm{Sp}(C(\bar{x})) = \{h \in I\!R^n \,:\, E^\top DG(\bar{x}) h E W = 0\}. \tag{2.15}$$

We cannot apply lemma 2.2 for the computation of the space spanned by the radial critical cone, since (2.12) is not of the form (2.13). However, we can get the following result.

Lemma 2.4. (i)*Let $\bar{h} \in \mathrm{ri}(C_R(\bar{x}))$, $\bar{H} := DG(\bar{x})\bar{h}$, and $\bar{F}$ be the matrix whose columns are an orthonormal basis of* $\mathrm{Ker}\, E^\top \bar{H} E$. *Then*

$$\mathrm{Sp}(C_R(\bar{x})) \subset \{h \in I\!R^n \,:\, H_h E \bar{F} = 0\}. \tag{2.16}$$

(ii) *Conversely, if in addition $\hat{W}^\top E^\top \bar{H} E \hat{W} \prec 0$, (i.e., if $\bar{F} = W$), then*

$$\mathrm{Sp}(C_R(\bar{x})) = \{h \in I\!R^n \,:\, H_h E W = 0\}. \tag{2.17}$$

Proof. Let $h \in C_R(\bar{x})$. Since $\bar{h} - \varepsilon h \in C_R(\bar{x})$ for some $\varepsilon > 0$, we have that $\hat{W}^\top E^\top \bar{H} E \hat{W} \preceq \varepsilon \hat{W}^\top E^\top H E \hat{W} \preceq 0$; since $\bar{H} E W = \varepsilon H E W = 0$, we have in fact $E^\top \bar{H} E \preceq \varepsilon E^\top H E \preceq 0$. This implies $\operatorname{Im} \bar{F} \subset \operatorname{Im} F$. Using (2.12), we deduce that $\operatorname{Sp}(C_R(\bar{x}))$ belongs to the r.h.s. of (2.16).

Conversely, let h belong to the r.h.s. of (2.16). Since $h = \varepsilon^{-1}((\bar{h} + \varepsilon h) - \bar{h})$, it suffices to prove that $h_\varepsilon := \bar{h} + \varepsilon h \in C_R(\bar{x})$ for small enough ε. Set $H_\varepsilon := DG(\bar{x}) h_\varepsilon$; the associated null space $\operatorname{Im} F_\varepsilon$ is equal to $\operatorname{Im} \bar{F} = \operatorname{Im} W$. Obviously $H_\varepsilon E W = 0$. Since $E^\top \bar{H} E$ is negative definite on the orthogonal of $\operatorname{Im} \bar{F}$, which is the space spanned by the columns of $\hat{W}$, we have that $\hat{W}^\top E^\top H_\varepsilon E \hat{W} \prec 0$ for small enough ε. The conclusion follows. ■

Let us define now another set of directions related to a particular Lagrange multiplier $\bar{Y} \in \Lambda(\bar{x})$, defined as follows

$$\hat{C}(\bar{x}, \bar{Y}) := \{h \in I\!R^n \;:\; [\operatorname{Pr} DG(\bar{x})h]\bar{Y} = 0\}, \tag{2.18}$$

where Pr is the matrix representation of the orthogonal projection over $\operatorname{Im} \bar{Y} \subseteq I\!R^m$. The cone $\hat{C}(\bar{x}, \bar{Y})$ defined in (2.18) can be equivalently written as

$$\hat{C}(\bar{x}, \bar{Y}) := \{h \in I\!R^n : \operatorname{Im} \bar{Y} \subseteq \operatorname{Ker}[\operatorname{Pr} DG(\bar{x})h]\}.$$

We set $\hat{C}(\bar{x}) := \hat{C}(\bar{x}, \bar{Y})$ if $\bar{Y}$ is unique.

We note that $\operatorname{Pr} = EWW^\top E^\top$, so, if $h \in \hat{C}(\bar{x})$, then

$$\begin{aligned} [\operatorname{Pr} DG(\bar{x})h]\bar{Y} &= [EWW^\top E^\top DG(\bar{x})h]\bar{Y} \\ &= [EWW^\top E^\top DG(\bar{x})h] EW \bar{\phi} W^\top E^\top = 0. \end{aligned}$$

Multiplying the last equality at the left side by $W^\top E^\top$ and at the right side by $EW\bar{\phi}^{-1}$, we obtain the characterization

$$\hat{C}(\bar{x}) = \{h \in I\!R^n : W^\top E^\top DG(\bar{x}) h E W = 0\}. \tag{2.19}$$

Comparing with (2.7) we see that

$$C(\bar{x}) \subset \hat{C}(\bar{x}). \tag{2.20}$$

Note that we also have

$$C(\bar{x}) \subset \bar{C}(\bar{x}) := \{h \in I\!R^n : E^\top DG(\bar{x}) h E W = 0\}. \tag{2.21}$$

2.2 Specific hypotheses

In this article we will sometimes use the following assumptions:

A1 Strict Complementarity Condition We say that the *strict complementarity* condition holds at $\bar{x}$ if there exists a Lagrange multiplier $\bar{Y}$ associated with $\bar{x}$ satisfying that

$$G(\bar{x}) - \bar{Y} \prec 0, \tag{2.22}$$

or equivalently (when $\operatorname{Ker} G(\bar{x}) \neq \{0\}$)

$$\text{The matrix } E^\top \bar{Y} E \text{ is nonsingular.} \tag{2.23}$$

There are some other equivalent ways to define the strict complementarity condition in our context but conditions (2.22) and (2.23) are the most useful.

In that case we can take $W = I_{m-r}$ and $\hat{W} = 0$. Denote the linear space generated by the set S by $\operatorname{Sp}(S) := I\!R_+(S - S)$. It follows from (2.7) and (2.11) that, under the strict complementarity hypothesis,

$$\operatorname{Sp}(C(\bar{x})) = C(\bar{x}) \;=\; \{h \in I\!R^n \;:\; E^\top DG(\bar{x})hE = 0\}, \tag{2.24}$$
$$\operatorname{Sp}(C_R(\bar{x})) = C_R(\bar{x}) \;=\; \{h \in I\!R^n \;:\; DG(\bar{x})hE = 0\}. \tag{2.25}$$

A2 Nondegeneracy (or Transversality) Condition We say that the point $\bar{x}$, feasible for problem (P) of section 1, is *nondegenerate* if the mapping $\psi_{\bar{x}} : I\!R^n \to S^{m-r}$ defined by

$$\psi_{\bar{x}}(h) := E^\top DG(\bar{x})hE \tag{2.26}$$

is onto. This notion was introduced by Shapiro and Fan in [13, Sec. 2].

The KKT system implies

$$-\nabla f(\bar{x})^\top h = \bar{Y} \cdot DG(\bar{x})h = \bar{Y}_{11} \cdot E^\top DG(\bar{x})hE, \text{ for all } h \in I\!R^n.$$

The tranversality condition implies uniqueness of the solution $\bar{Y}_{11}$ of this infinite system of equations. Therefore, the tranversality condition implies uniqueness of the Lagrange multiplier.

Obviously

$$\operatorname{Ker} \psi_{\bar{x}} \subseteq C(\bar{x}) \subseteq \hat{C}(\bar{x},\bar{Y}), \tag{2.27}$$

with equality when the strict complementarity condition (2.22) holds (see e.g. [8, Prop. 2]).

Note that Robinson's constraint qualification condition (1.4) can be written as

$$\text{There exists } \bar{h} \in I\!R^n \text{ such that } \quad \psi_{\bar{x}}(\bar{h}) \prec 0.$$

Thus, nondegeneracy (or transversality) condition is stronger than Robinson's constraint qualification condition. Hence, if $\bar{x}$ is nondegenerate then there exists a (necessarily unique) Lagrange multiplier Moreover, under the strict complementarity condition (2.22), the existence and uniqueness of the Lagrange multiplier $\bar{Y}$ is equivalent to the nondegeneracy condition [4, Prop. 4.75]. For more details about the nondegeneracy condition in the semidefinite programming context see e.g. [4, 21], and in a general cone optimization framework see [2].

It is known that assumptions A1 and A2 are stable under small perturbations of problem (P), as shown by the lemma below:

Lemma 2.5. *Let $(\bar{x},\bar{Y})$ be a critical or KKT-point of problem (P), and $(\bar{x}(u),\bar{Y}(u))$, solution of (1.5), converge to $(\bar{x},\bar{Y})$ as $u \downarrow 0$.*

i) *We can construct a matrix $E_u \in I\!R^{m\times m-r}$ whose columns are an orthonormal basis of the space spanned by the eigenvectors associated with the $m-r$ biggest eigenvalues of $G(\bar{x}(u))$, such that $E_u \to E$.*

ii) *If A1 holds, then $(\bar{x}(u),\bar{Y}(u))$ satisfies the* strict complementarity condition *for problem (P_u), when u is close to 0, i.e.,*

$$\text{The matrix } E_u^\top \bar{Y}(u) E_u \text{ is nonsingular,} \tag{2.28}$$

iii) *If A2 holds, then any ((P_u)-feasible) point $\bar{x}(u)$, which converges to $\bar{x}$ as $u \downarrow 0$, is* nondegenerate *for problem (P_u) when the perturbation u is small enough, i.e., the following mapping is onto:*

$$\psi_u(h) := E_u^\top D_x G(\bar{x}(u),u) h E_u \tag{2.29}$$

Proof. Part i) was shown in [4, Ex. 3.140]. Parts ii) and iii) are a direct consequence of the continuity of E_u and $\bar{Y}(u)$ as functions of u, and the smoothness of G. ■

A3 Second Order Optimality Conditions

Let us state for future reference various conditions for necessary or sufficient optimality, involving second order derivatives of data. We will relate them later to local optimality.

Let $(\bar{x},\bar{Y})$ be a solution of the KKT system. These second order conditions involve the following matrix, introduced in [11]:

$$\mathcal{H}(\bar{x},\bar{Y})_{ij} := -2\bar{Y} \cdot ([D_{x_i}G(\bar{x})]G(\bar{x})^\dagger[D_{x_j}G(\bar{x})]). \tag{2.30}$$

We call *standard second order sufficient* condition the following [2, Th. 3.2]:

$$\sup_{\bar{Y}\in\Lambda(\bar{x})} h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h + h^\top \mathcal{H}(\bar{x},\bar{Y})h > 0, \quad \forall h \in C(\bar{x}) \setminus \{0\}, \tag{2.31}$$

where the cone of critical directions $C(\bar{x})$ was defined in (2.3).

The term $h^\top \mathcal{H}(\bar{x},\bar{Y})h$, related to the geometry (or curvature) of the cone S^m_- of $m \times m$ symmetric negative semidefinite matrices, being nonnegative, implies that (2.31) is weaker than the classical sufficient second order condition (e.g. [19, Th. 2.2]): there exists $\bar{Y} \in \Lambda(\bar{x})$ such that

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h > 0, \quad \forall h \in C(\bar{x}) \setminus \{0\}. \tag{2.32}$$

See also [4, §5.3] and [11]. Condition (2.31) is a sufficient condition for optimality of $\bar{x}$ when Robinson's constraint qualification condition (1.4) holds (See theorem 3.2 below). On the other hand, a *second order necessary* condition to be $\bar{x}$ optimal (when (1.4) holds) is the following [21, Th. 8]

$$\sup_{\bar{Y}\in\Lambda(\bar{x})} h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h + h^\top \mathcal{H}(\bar{x},\bar{Y})h \geq 0, \quad \forall h \in C(\bar{x}). \tag{2.33}$$

There is a *no-gap* relation between the necessary condition (2.33) and the sufficient one (2.31), in the sense that an inequality is changed into a strict equality.

Condition (2.31) and the second inclusion in (2.27) allows us to state a *stronger second order sufficient condition* given by the existence of $\bar{Y} \in \Lambda(\bar{x})$ such that

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h + h^\top \mathcal{H}(\bar{x},\bar{Y})h > 0, \quad \forall h \in \hat{C}(\bar{x},\bar{Y}) \setminus \{0\}. \tag{2.34}$$

This condition will be particularly useful when the multiplier $\bar{Y}$ is unique (for example, if the nondegeneracy assumption A2 is satisfied). A direct consequence of (2.27) is the following lemma (see [4, Ex. 3.140]).

Lemma 2.6. *Assume that the Lagrange multiplier $\bar{Y} \in \Lambda(\bar{x})$ is unique (this is the case when A2 holds) and the strict complementarity condition* (2.22) *(i.e. A1). Then we have that second order sufficient conditions* (2.31) *and* (2.34) *are equivalent to*

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h + h^\top \mathcal{H}(\bar{x},\bar{Y})h > 0, \quad \forall h \in \operatorname{Ker} \psi_{\bar{x}} \setminus \{0\}. \tag{2.35}$$

Sufficient condition (2.35) was established by Shapiro in [21, Th. 9]. Finally, under A1 and A2 condition (2.35) will be called *assumption* **A3**.

On the other hand, the second order sufficient condition (2.35) at the critical point $(\bar{x}(u),\bar{Y}(u))$ for problem (P_u) is written as follows

$$h^\top \nabla^2_{xx} L(\bar{x}(u),\bar{Y}(u),u)h + h^\top \mathcal{H}(\bar{x}(u),\bar{Y}(u),u)h > 0, \forall h \in \operatorname{Ker} \psi_u \setminus \{0\}, \tag{2.36}$$

where $\mathcal{H}(\bar{x}(u),\bar{Y}(u),u) \in S^m$ is the matrix whose components are

$$\mathcal{H}(\bar{x}(u),\bar{Y}(u),u)_{ij} := \\ -2\bar{Y}(u) \cdot ([D_{x_i} G(\bar{x}(u),u)] G(\bar{x}(u),u)^\dagger [D_{x_j} G(\bar{x}(u),u)]).$$

3 Strong Regularity Condition

Let us recall the definition of *strong regular* solution [10].

Definition 3.1. *We say that* $(\bar{x},\bar{Y})$ *is a* strong regular *solution of KKT-conditions* (1.1) *if there exists a neighborhood* $\mathcal{V}$ *of* $(\bar{x},\bar{Y})$ *such that for every* $\delta := (\delta_1,\delta_2) \in I\!R^n \times S^m$ *close to 0, the "linearized" system:*

$$\nabla^2_{xx}L(\bar{x},\bar{Y})(x-\bar{x}) + DG(\bar{x})^*(Y-\bar{Y}) = \delta_1, \tag{3.1a}$$

$$G(\bar{x})Y + DG(\bar{x})(x-\bar{x})Y = \delta_2 Y, \tag{3.1b}$$

$$G(\bar{x}) + DG(\bar{x})(x-\bar{x}) - \delta_2 \preceq 0, Y \succeq 0, \tag{3.1c}$$

has a unique solution $(x,Y) = (\bar{x}(\delta),\bar{Y}(\delta))$ *in* $\mathcal{V}$*, which is a Lipschitz continuous function of* δ*.*

Remark 3.2. *It is known that the strong regularity condition implies Robinson's constraint qualification condition* (1.4)*. See for example [4, pp. 416] or [3].*

In this section we present the results concerning the strong regularity in the semidefinite context of problem (P). Some of these results are also true in a general cone optimization framework

$$\min_{x\in\mathbb{X}}\{f(x)\,;\,G(x)\in K\}, \tag{3.2}$$

where $K \subseteq \mathbb{Y}$ is a convex cone and $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. We can refer the reader to [2, 3] in order to find out the different results available for problem (3.2).

A useful definition involved in this work is the *(uniform) second order growth quadratic* condition [10] stated below.

Definition 3.3. *Let* $\bar{x}$ *be a critical point of problem (P). It is said that the* uniform second order growth *condition holds at* $\bar{x}$ *if, for any smooth perturbation of the form* (P_u)*, there exist* $\alpha > 0$ *and a neighborhood* $\mathcal{N}$ *of* $\bar{x}$ *such that for any* $u \in I\!R^k$ *close enough to 0 and any critical point* $\bar{x}(u) \in \mathcal{N}$ *of the perturbed problem* (P_u)*, we have that*

$$f(x,u) \geq f(\bar{x}(u),u) + \alpha\|x - \bar{x}(u)\|^2, \quad \forall x \in \mathcal{N}, G(x,u) \preceq 0. \tag{3.3}$$

We say that the second order growth *condition holds at* $\bar{x}$ *if condition* (3.3) *just holds for problem (P), it means, there exist* $\alpha > 0$ *and a neighborhood* $\mathcal{N}$ *of* $\bar{x}$ *such that condition* (3.3) *is satisfied at* $u = 0$ *and* $\bar{x}(0) = \bar{x}$*.*

The following characterization (cf. [4, Th. 5.24]) plays an important role in the rest of this section. It is also valid for the general optimization problem (3.2) when the convex cone K is C^2*-reducible* to a pointed cone (cf. [4, Sec. 3.4.4]).

Theorem 3.1. *Let* $\bar{x}$ *be a local solution of problem (P) and* $\bar{Y}$ *its corresponding Lagrange multiplier. We have that* $(\bar{x},\bar{Y})$ *is a strongly regular solution of KKT system* (1.1) *if and only if* $\bar{x}$ *is nondegenerate (assumption A2) and the uniform second order growth quadratic condition holds at* $\bar{x}$*.*

The next result was introduced by Bonnans, Cominetti and Shapiro in [1] using *Fritz John* conditions instead of KKT-conditions (1.1) (i.e. without assuming Robinson's constraint qualification condition (1.4)) in a more general context (see problem (3.2)). The reader can see also [4, Ch. 3] for more details.

Theorem 3.2. *Let $\bar{x}$ be a critical point of problem (P) satisfying Robinson's constraint qualification condition* (1.4)*. Then, the second order growth condition (definition 3.3) holds at $\bar{x}$ iff the second order sufficient condition* (2.31) *is satisfied.*

Theorem 3.1 shows us that we can completely characterize the strong regularity condition by giving sufficient and necessary conditions to obtain the uniform second order growth condition, as it was done for the second order growth condition in theorem 3.2.

We start by recalling a necessary condition for uniform second order growth condition, valid in a more general context. Remind that we have stated in lemma 2.4 some relations satisfied by $\mathrm{Sp}(C_R(\bar{x}))$.

Theorem 3.3. *Let $\bar{x}$ be a local solution of problem (P), and $\bar{Y}$ the unique Lagrange multiplier associated with $\bar{x}$. If the uniform second order growth condition holds at $\bar{x}$, then*

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y}) h > 0, \forall h \in \mathrm{Sp}(C_R(\bar{x})) \setminus \{0\}. \tag{3.4}$$

Proof. This is an application to SDP problems of [4, Th. 5.25]. ■

Let us now state a stronger necessary condition for uniform second order growth. We recall an easy extension of lemmas on pair of quadratic forms, see Hestenes [9].

Lemma 3.4. *Let $C \subseteq I\!R^m$ be a closed, nonempty and convex cone, and P and Q two quadratics forms satisfying that $Q(x) \geq 0$ for all $x \in C$. Then the next two conditions are equivalent:*

$$P(x) + rQ(x) > 0, \forall x \in C \setminus \{0\}, \textit{ for all large enough } r, \tag{3.5}$$

$$P(x) > 0, \quad \textit{for all nonzero } x \textit{ in } C \cap Q^{-1}(0). \tag{3.6}$$

By lemma 2.2, the nondegeneracy assumption A2 implies

$$\mathrm{Sp}(C(\bar{x})) = \{h \in I\!R^n \; : \; E^\top DG(\bar{x})hEW = 0\}. \tag{3.7}$$

Theorem 3.4. *Let $\bar{x}$ be a local solution of problem (P) and $\bar{Y}$ its corresponding Lagrange multiplier. If $(\bar{x},\bar{Y})$ is a strongly regular solution of KKT system* (1.1)*, then $\bar{x}$ is nondegenerate (assumption A2) and the following second order condition holds at $\bar{x}$:*

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y}) h + h^\top \mathcal{H}(\bar{x},\bar{Y}) h > 0, \quad \forall h \in \mathrm{Sp}(C(\bar{x})) \setminus \{0\}. \tag{3.8}$$

Proof. Consider the perturbed problem (P_u) where

$$f(x,u) := f(x) \text{ and } G(x,u) := G(x) - uE\hat{W}\hat{W}^\top E^\top, \tag{3.9}$$

for $u > 0$ small enough. Then $(\bar{x},\bar{Y})$ is a solution of the KKT system for (P_u). Recall that the columns of $\hat{W}$ is an orthonormal basis of the orthogonal of $\operatorname{Im}\bar{Y}$ in $\operatorname{Ker} G(\bar{x})$, and hence, the columns of EW are a basis of $\operatorname{Ker} G(\bar{x},u)$, and the strict complementarity hypothesis is satisfied. Therefore the critical cone of the perturbed problem is the set $\hat{C}(\bar{x})$ defined in (2.19).

By theorem 3.1, $\bar{x}$ is nondegenerate and the uniform second order growth condition holds at $(\bar{x},\bar{Y})$.

Thus, the uniform second order growth condition implies that condition (3.3) holds at $x(u) = \bar{x}$ when u is small enough. By lemma 2.5.ii), the perturbed nondegeneracy condition (2.29) holds. Using theorem 3.2, we see that condition (3.3) is equivalent to the second order sufficient condition (2.31) for (P_u). Since this pertubed problem satisfies the strict complementarity condition (2.28), the quadratic growth is equivalent to (2.36). Due to the special structure of (3.9)), this may be written as

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h + h^\top \mathcal{H}(u)h > 0, \quad \forall h \in \hat{C}(\bar{x}) \setminus \{0\}, \tag{3.10}$$

where $\mathcal{H}(u) \in S^n$ is defined by its components as

$$\mathcal{H}(u)_{ij} := -2\bar{Y} \cdot ([D_{x_i} G(\bar{x})](G(\bar{x}) - uE\hat{W}\hat{W}^\top E^\top)^\dagger [D_{x_j} G(\bar{x})]).$$

We claim that condition (3.10) is equivalent to (3.8). Indeed, since $(G(\bar{x}) - uE\hat{W}\hat{W}^\top E^\top)^\dagger = G(\bar{x})^\dagger - u^{-1}E\hat{W}\hat{W}^\top E^\top$, we obtain that

$$h^\top \mathcal{H}(u)h = h^\top \mathcal{H}(\bar{x},\bar{Y})h + \frac{2}{u}\bar{Y} \cdot (DG(\bar{x})hE\hat{W}\hat{W}^\top E^\top DG(\bar{x})h). \tag{3.11}$$

On the other hand, since inequality (3.10) holds for the small values of $u > 0$ and the second term in (3.11)

$$\begin{aligned}&\tfrac{2}{u}\bar{Y} \cdot (DG(\bar{x})hE\hat{W}\hat{W}^\top E^\top DG(\bar{x})h) = \\ &\qquad \tfrac{2}{u}\bar{\phi} \cdot ([W^\top E^\top DG(\bar{x})hE\hat{W}][\hat{W}^\top E^\top DG(\bar{x})hEW])\end{aligned}$$

is a nonnegative quadratic form on h, by using lemma 3.4, we obtain that the second order necessary condition (3.10) holds if and only if

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h + h^\top \mathcal{H}(\bar{x},\bar{Y})h > 0,$$

for all nonzero direction $h \in \hat{C}(\bar{x})$ (i.e., $W^\top E^\top DG(\bar{x})hEW = 0$) satisfying that

$$\bar{\phi} \cdot ([W^\top E^\top DG(\bar{x})hE\hat{W}][\hat{W}^\top E^\top DG(\bar{x})hEW]) = 0. \tag{3.12}$$

Since $\bar{\phi}$ is positive definite, this is equivalent to $\hat{W}^\top E^\top DG(\bar{x})hEW = 0$. In view of the expression of the critical cone for the perturbed problem, we see that a necessary condition for uniform quadratic growth is

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h + h^\top \mathcal{H}(\bar{x},\bar{Y})h > 0, \quad \forall h \neq 0, E^\top DG(\bar{x})hEW = 0,$$

from which the conclusion follows. ■

Remark 3.5. *Theorem 3.4 implies theorem 3.3, since the set of directions is larger, and the additional term of the quadratic form (which is nonnegative) is zero on* $\mathrm{Sp}(C_R(\bar{x}))$ *as we show now. Indeed, let* $h \in \mathrm{Sp}(C_R(\bar{x}))$. *Relation* (2.30) *implies*

$$h^\top \mathcal{H}(\bar{x},\bar{Y})h = -2H_h^\top \bar{Y} H_h \cdot G(\bar{x})^\dagger. \tag{3.13}$$

From the expression (2.12) *of* $C_R(\bar{x})$*, and the fact already noticed that* $\mathrm{Im}\, F_h \supset \mathrm{Im}\, W$ *by complementarity if* $h \in C_R(\bar{x})$*, it follows that* $H_h E W = 0$ *for each* $h \in C_R(\bar{x})$*, and hence, for each* $h \in \mathrm{Sp}(C_R(\bar{x}))$*. Replacing* $\bar{Y}$ *in* (3.13) *by its expression given in* (2.1)*, we obtain that* $h^\top \mathcal{H}(\bar{x},\bar{Y})h = 0$*, as was to be shown.*

4 Sufficient condition for strong regularity

Here is the main result of this section.

Theorem 4.1. *Let* $\bar{x}$ *be a local solution of problem (P) and* $\bar{Y}$ *its corresponding Lagrange multiplier. If* $\bar{x}$ *is nondegenerate (assumption A2) and the next second order sufficient condition holds*

$$h^\top \nabla^2_{xx} L(\bar{x},\bar{Y})h > 0, \quad \forall h \in \hat{C}(\bar{x}) \setminus \{0\}, \tag{4.1}$$

then $(\bar{x},\bar{Y})$ *is a strong regular solution of KKT-system* (1.1)*, where* $\hat{C}(\bar{x})$ *was defined in* (2.18) *and characterized in* (2.19).

Proof. We argue by contradiction. By Theorem 3.1 we know that the nondegeneracy condition always holds. If the uniform second order growth condition does not hold at $\bar{x}$, then there exist sequences $u_n \to 0$, $x_n \to \bar{x}$, $h_n \to 0$, with $h_n \neq 0$, such that x_n and $x_n + h_n$ are feasible points of the perturbed problem (P_{u_n}), and

$$f(x_n + h_n,u_n) \leq f(x_n,u_n) + o(\|h_n\|^2). \tag{4.2}$$

We can suppose (passing to a subsequence if necessary) that $h_n/\|h_n\| \to \bar{h}$. Feasibility of $x_n + h_n$ implies

$$G(x_n + h_n,u_n) = G(x_n,u_n) + D_x G(x_n,u_n)h_n + o(\|h_n\|) \preceq 0. \tag{4.3}$$

Let the columns of E_n be an orthonormal basis of kernel of $G(x_n,u_n)$, then $E_n^\top D_x G(x_n,u_n)h_n E_n \preceq o(\|h_n\|)$.

Since Robinson's condition (1.4) is stable under small perturbations, we know that there exists a Lagrange multiplier Y_n associated with x_n for problem (P_{u_n}). Even more, since $x_n \to \bar{x}$ it follows that $Y_n \to \bar{Y}$.

Note that $\mathrm{Im}\, \bar{Y} = \mathrm{Im}(EW)$ and $\mathrm{Im}\, Y_n \subset \mathrm{Im}\, E_n$. Let $y \in \mathrm{Im}\, \bar{Y}$, i.e., $y = \bar{Y}z$ for a certain z. Set $y_n := Y_n z$; since $y_n \in \mathrm{Im}\, E_n$, we have that $y_n^\top D_x G(x_n,u_n)h_n y_n$

$\leq o(\|h_n\|)$. Dividing by $\|h_n\|$ and passing to the limit, obtain $y^\top DG(\bar{x})\bar{h}y \leq 0$. Since $\operatorname{Im}\bar{Y} = \operatorname{Im}(EW)$, it follows that

$$W^\top E^\top DG(\bar{x})hEW \preceq 0. \tag{4.4}$$

On the other hand, passing to the limit in (4.2) we get that $Df(\bar{x})\bar{h} \leq 0$. This together with the first KKT-condition (1.1a) implies that $\bar{Y} \cdot DG(\bar{x})\bar{h} \geq 0$, obtaining the equality

$$\bar{Y} \cdot DG(\bar{x})\bar{h} = 0,$$

which can be written as

$$\bar{\phi} \cdot W^\top E^\top DG(\bar{x})\bar{h}EW = 0. \tag{4.5}$$

Since $\bar{\phi}$ is positive semidefinite and (4.4) holds, this implies

$$W^\top E^\top DG(\bar{x})\bar{h}EW = 0. \tag{4.6}$$

We now prove that $\bar{h}^\top \nabla^2_{xx} L(\bar{x},\bar{Y})\bar{h}$ is nonpositive. By using the first (perturbed) KKT-condition (1.5a) and a Taylor's expansion we have

$$L(x_n + h_h, Y_n, u_n) - L(x_n, Y_n, u_n) = h_n^\top D^2_{xx} L(\bar{x},\bar{Y})h_n + o(\|h_n\|^2). \tag{4.7}$$

Since $L(x_n,Y_n,u_n) = f(x_n,u_n)$ and $L(x_n + h_h, Y_n, u_n) \leq f(x_n + h_n, u_n)$, relations (4.2) and (4.7) yield to

$$h_n^\top D^2_{xx} L(\bar{x},\bar{Y})h_n \leq o(\|h_n\|^2).$$

Dividing by $\|h_n\|^2$ and passing to the limit $n \to +\infty$, we conclude that $\bar{h}^\top \nabla_{xx} L(\bar{x},\bar{Y})\bar{h} \leq 0$. Since (4.6) holds, this contradicts (4.1). ■

Remark 4.1. *We could have taken for y_n an arbitrary element of $\operatorname{Im} E_n$. Then* (4.4) *improves to*

$$\tilde{W}^\top E^\top DG(\bar{x})hE\tilde{W} \preceq 0, \tag{4.8}$$

where the columns of $\tilde{W}$ span the vector space of limit points of such y_n. Obviously $\operatorname{Im}(\tilde{W}) \supset \operatorname{Im}(W)$*. Since* (4.6) *holds, the additional information is equivalent to*

$$\tilde{\tilde{W}}^\top E^\top DG(\bar{x})hE\tilde{\tilde{W}} \preceq 0, \tag{4.9}$$

where the columns of $\tilde{\tilde{W}}$ span the subspace of $\operatorname{Im}(\tilde{W})$ *orthogonal to* $\operatorname{Im}(W)$*. This is not, unfortunately, so useful since we cannot say much on $\tilde{W}$.*

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