



HAL
open science

Théories de cordes : solutions exactes, déformations marginales.

Domenico Orlando

► **To cite this version:**

Domenico Orlando. Théories de cordes : solutions exactes, déformations marginales.. Physique [physics]. Ecole Polytechnique X, 2006. Français. NNT : . pastel-00002178

HAL Id: pastel-00002178

<https://pastel.hal.science/pastel-00002178>

Submitted on 29 Jul 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

String Theory: exact solutions, marginal deformations and hyperbolic spaces

Domenico Orlando

*Thesis presented to obtain the title of
Docteur ès Sciences Physiques de l'école polytechnique*

- to appear in Fortschritte der Physik •

Thesis Committee:

Luis Álvarez-Gaumé (referee)
Édouard Brézin (president)
Costas Kounnas
Marios Petropoulos
Alexander Sevrin (referee)
Philippe Spindel
Arkady Tseytlin

A Lei

Contents

1	Introduction	1
2	Wess-Zumino-Witten Models	5
	<i>Wess-Zumino-Witten models constitute a large class of the exact string theory solutions which we will use as starting points for most of the analysis in the following. In this chapter we see how they can be studied from different perspectives and with different motivations both from a target space and world-sheet point of view.</i>	
2.1	The two-dimensional point of view	5
2.2	The target space point of view	14
3	Deformations	17
	<i>In this rather technical chapter we describe marginal deformations of Wess-Zumino-Witten models. The main purpose for these constructions is to reduce the symmetry of the system while keeping the integrability properties intact, trying to preserve as many nice geometric properties as possible.</i>	
3.1	Deformed WZW models: various perspectives	18
3.2	Background fields for the asymmetric deformation	22
3.3	Geometry of squashed groups	26
3.4	A no-renormalization theorem	28
3.5	Partition functions	30
3.6	The deformation as a gauging	36
4	Applications	43
	<i>In this chapter we present some of the applications for the construction outlined above. After an analysis of the most simple (compact and non-compact) examples, we describe the near-horizon geometry for the Bertotti-Robinson black hole, show some new compactifications and see how Horne and Horowitz's black string can be described in this framework and generalized via the introduction of an electric field.</i>	
4.1	The two-sphere CFT	43
4.2	$SL(2, \mathbb{R})$	46
4.3	Near horizon geometry for the Bertotti-Robinson black hole	60
4.4	The three-dimensional black string revisited	62
4.5	New compactifications	76

5 Squashed groups in type II	85
<i>In this chapter we start deviating from the preceding ones because we will no longer deal with WZW models but with configurations in which the group manifold geometry is sustained by RR fields. In particular, then, we see how the squashed geometries can be obtained in type II theories by T-dualizing black brane configurations.</i>	
5.1 $SL(2, \mathbb{R}) \times SU(2)$ as a D-brane solution	85
5.2 T duality with RR fields	86
5.3 The squashed sphere	88
6 Out of the conformal point: Renormalization Group Flows	91
<i>This chapter is devoted to the study of the relaxation of squashed WZW models further deformed by the insertion of non-marginal operators. The calculation is carried from both the target space and world-sheet points of view, once more highlighting the interplay between the two complementary descriptions. In the last part such techniques are used to outline the connection between the time evolution and the RG-flow which is seen as a large friction limit description; we are hence naturally led to a FRW-type cosmological model.</i>	
6.1 The target space point of view	92
6.2 The CFT approach	102
6.3 RG flow and friction	105
6.4 Cosmological interpretation	110
7 Hyperbolic Spaces	115
<i>In this chapter we investigate type II and M-theory geometries written as direct products of constant-curvature spaces. We find in particular a class of backgrounds with hyperbolic components and we study their stability with respect to small fluctuations.</i>	
7.1 M-theory solutions	115
7.2 Stability	119
7.3 Type IIB backgrounds	128
8 Conclusions and further perspectives	133
A Table of conventions	135
B Explicit parametrizations for some Lie groups	137
B.1 The three-sphere	137
B.2 AdS_3	138
B.3 $SU(3)$	142
B.4 $USp(4)$	144
C Symmetric deformations of $SL(2, \mathbb{R})$	147
D Spectrum of the $SL(2, \mathbb{R})$ super-WZW model	149
Bibliography	153

Acknowledgements

Never had nay man a wit so
presently excellent, as that it
could raise itself; but there must
come both matter, occasion,
commenders, and favourers to it.

Il Volpone
BEN JOHNSON

While writing these acknowledgements I realized how much I'm indebted to so many people. I learned a lot from them all – not just physics – and without them this work simply wouldn't have been possible. I'm pretty sure I'm forgetting someone. I plead guilty. But even if you don't find your name rest assured that you'll always have my personal gratitude.

In any case, I simply can't start without mentioning my parents for – although far – they've always been present in their peculiar way.

I must thank the members of my thesis committee: *Luis Álvarez-Gaumé*, *Edouard Brezin*, *Costas Kounnas*, *Marios Petropoulos*, *Alex Sevrin*, *Philippe Spindel*, *Arkady Tseytlin* for they extremely precious feedback. They turned my defence into a valuable learning experience.

I wish to thank *Patrick Mora* for having welcomed me as a PhD student at the Centre de physique théorique of the École polytechnique. Also I thank *Edouard Brezin* at École normale supérieure, *Dieter Lüst* at Max Planck Institut (Munich) and *Alex Sevrin* at Vrije Universiteit Brussel. During the last year they hosted me for several months at their respective institutions. Each and all of them have made of my stays invaluable experiences from both the professional and human points of view.

Then I thank my advisor *Marios Petropoulos* for all what he taught me and for his continuous support. In these years I learned to think of him as a friend. *Costas Kounnas* for letting me camp in his office at ENS where he would devote me hours and hours of his time, sharing his experience and his amazing physical intuition. *Costas Bachas*, who was among the first professors I had when I arrived in Paris and (unknowingly) pushed me towards string theory. Since then he helped me in many an occasion and I'm almost embarrassed if I think how much I learned from him and in general I owe him.

Doing one's PhD in Paris is a great opportunity for there are not so many places in the world with such a physics community. And it's definitely surpris-

ing to a PhD student, like I was, how everybody is always willing to discuss and makes you feel one of them. I learned a lot from many people and I hope that those I don't remember explicitly won't take it too bad. On the other hand I couldn't not mention *Adel Bilal, Eugène Cremmer, Pierre Fayet, Mariana Graña, Jean Iliopoulos, Bernard Julia, Volodia Kazakov, Ruben Minassian, Hervé Partouche, Michela Petrini, Boris Pioline, Ian Troost, Pierre Vanhove.*

Sometimes discussions would lead to a publication and I'm honored to count among my collaborators *Costas Bachas, Ioannis Bakas, Davide Cassani, Stéphane Detournay, Dan Israël, Marios Petropoulos, Susanne Reffert, Kostas Sfetsos, Philippe Spindel.* I really wish to thank you for what I learned and keep learning from you all.

I must remember my office mates *Luciano Abreu, Stéphane Afchain, Yacine Dolivet, Stéphane Fidanza, Pascal Grange, Claudia Isola, Liguori Jego, Jerome Levie, Xavier Lacroze, Alexey Lokhov, Liuba Mazzanti, Hervé Moutarde, Chloé Papineau, Susanne Reffert, Sylvain Ribault, Maria-Cristina Timirgaziu, Barbara Tumpach* and especially *Claude de Calan.* With all of you I shared more than an office and our never-ending discussions have really been among the most precious moments of my PhD. I hope none of you will get mad at me if in particular I thank *Sylvain* for all the advice he gave me, even before starting my PhD. Not to speak of my brother *Pascal.*

A final thanks (last but not least) goes to the computer people and secretaries at CPHT and ENS, without whom life couldn't be that easy.

Thanks, all of you.

CHAPTER 1

Introduction

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable

The Restaurant at the End of the Universe
DOUGLAS ADAMS

IN THE LAST FIFTY YEARS theoretical physics has been dominated by two apparently incompatible models: the microscopic world being described by quantum field theory and the macroscopic world by general relativity. QFT is by far the most successful theory ever made, allowing to reach an almost incredible level of accuracy in its measurable predictions. But gravity is different from all other interactions; although by far the weakest, it acts on the very structure of the universe at a more fundamental level. Many attempts have been made to obtain a consistent quantum theory of gravity and they all proved unsuccessful so that it has become clear that completely new ideas are needed.

To this day, but this has been true for more than twenty years now, the only promising trail we can follow in the quest for this unification is string theory. Roughly speaking it postulates that the fundamental objects are not point particles as in the standard quantum models but one-dimensional objects – strings. Although their typical size is so small that one might even question the very meaning of distance at this scale, the mere not being pointlike allows to solve an enormous number of theoretical problems and in particular those connected with the severe divergencies that gravity presents due to the local nature of interactions. Field theory particles appear as vibration modes of the fundamental string, spacetime is a semiclassical description for a string condensate, supergravity emerges as a low-energy limit, and the standard model is the result of a compactification in presence of extended objects (D branes).

Of course this is in many ways a wish-list. In its present state string theory is largely incomplete. To begin, only a first-quantization description is known

and as such is intrinsically perturbative, and only the S-matrix elements in a given vacuum are accessible. So, although in principle the very geometry should emerge from string dynamics, in practice we are forced to choose a vacuum, which by itself clips the wings of any hope of attaining a complete quantum gravity theory. Moreover only the perturbative regime is in principle available, even though the existence of a web of dualities can translate strong coupling backgrounds into ones we can deal with.

One thing nevertheless must be kept in mind. One should stay as close as possible to the present knowledge of Nature and try to predict the outcomes of realistic experiments beyond the standard models in particle physics or cosmology by using phenomenological models, but *string theory is not supergravity*. It is reassuring to find it as a low energy limit but most, if not all, of the new physics lies in the regime where the semiclassical approximations break down. From this point of view an important, almost essential, rôle is played by exact models, *i.e.* systems in which the α' corrections can be kept under control and a conformal field theory description is possible. Because of technical issues, not many such backgrounds are known and they are all characterized by a high degree of symmetry. Hence it is not surprising that they in general do not have a direct phenomenological impact. But the reason for their fundamental importance lies elsewhere. They can mostly be used as laboratories to study the extremal conditions – black hole dynamics just to name one – in which general relativity and field theory show their limits. The very reason why string theory was introduced.

Plan of the thesis This thesis is almost entirely devoted to studying string theory backgrounds characterized by simple geometrical and integrability properties. This requires at the same time a good grasp on both the low-energy (supergravity) description in terms of spacetime and on the CFT side controlling all-order-in- α' effects.

The archetype of this type of system is given by Wess-Zumino-Witten models, describing string propagation in a group manifold or, equivalently, a class of conformal field theories with current algebras. Given their prominent rôle we devote the whole Chapter 2 to recall their properties from different points of view, trying to outline some of the most important features.

In Chapter 3 we enter the main subject of these notes, namely we study the moduli space of WZW models by using truly marginal deformations obtained as bilinears in the currents. A vast literature exists on this type of constructions, but we will concentrate on a particular class, which we dub *asymmetric deformations*. They actually present a number of advantages over the more familiar symmetric ones and in particular, although the CFT description is slightly more involved (Sec. 3.5), they enjoy a very nice spacetime interpretation. This can be completely understood in terms of the always-underlying Lie algebra (Sec. 3.3) and can be proven to remain unchanged at all orders in α' (Sec. 3.4).

The following Chapter 4 illustrates some of the obvious applications for our construction. We then start with the simplest $SU(2)$ case, leading to a

CFT on the squashed three-sphere and on the two-sphere (Sec. 4.1). Then we deal with the considerably richer non compact $SL(2, \mathbb{R})$ case (Sec. 4.2). This leads naturally to the description of some black hole geometries such as the near-horizon limit for the Bertotti-Robinson black hole (Sec. 4.3), and the three-dimensional electrically charged black string (Sec. 4.4). Both can be studied in terms of CFT, thus allowing for an evaluation of the spectrum of string primaries. Further applications regard the possibility of introducing new compactification manifolds as part of larger, ten-dimensional backgrounds (Sec. 4.5).

In Chapter 5 we consider an alternative description for the squashed group geometries which are found to be T-duals of the usual type II black brane solutions.

In Chapter 8 we take a slight detour from what we have seen up to this point: instead of exact CFT backgrounds we deal with off-shell systems. Using a renormalization-group approach we describe the relaxation towards the symmetrical equilibrium situation. This same behaviour is studied from different points of view, RG flow in target space (Sec. 6.1), two-dimensional renormalization (Sec. 6.2) and reading the flow as a motion in an extra time direction (Sec. 6.3), thus obtaining Freedman-Robertson-Walker-like metrics that in the most simple case describe an isotropic universe with positive cosmological constant undergoing a big-bang-like expansion (Sec. 6.4).

The final Chapter 7 marks a further deviation from the construction of exact models: we consider in fact backgrounds with Ramond-Ramond fields which still elude a satisfactory CFT interpretation. In particular we analyze direct products of constant-curvature spaces and find solutions with hyperbolic spaces sustained by RR fields.



The themes we treat here have been the subject of the following publications:

- D. Israël, C. Kounnas, D. Orlando and P. M. Petropoulos, *Electric / magnetic deformations of S^3 and $AdS(3)$, and geometric cosets*, Fortsch. Phys. **53**, 73–104 (2005), hep-th/0405213.
- D. Israel, C. Kounnas, D. Orlando and P. M. Petropoulos, *Heterotic strings on homogeneous spaces*, Fortsch. Phys. **53**, 1030–1071 (2005), hep-th/0412220.
- D. Orlando, *$AdS(2) \times S^2$ as an exact heterotic string background*, (2005), hep-th/0502213, Talk given at NATO Advanced Study Institute and EC Summer School on String Theory: From Gauge Interactions to Cosmology, Cargese, France, 7-19 Jun 2004.
- S. Detournay, D. Orlando, P. M. Petropoulos and P. Spindel, *Three-dimensional black holes from deformed anti de Sitter*, JHEP **07**, 072 (2005), hep-th/0504231.
- D. Orlando, *Coset models and D-branes in group manifolds*, (2005), hep-th/0511210. Published in Phys.Lett.B

- D. Orlando, P. M. Petropoulos and K. Sfetsos, *Renormalization-group flows and charge transmutation in string theory*, Fortsch. Phys. **54**, 453–461 (2006), hep-th/0512086.

Some results are on the other hand still unpublished. This concerns in particular Chapter 5 and the second half of Chapter 8 and Chapter 7.

In order to help the reader in quickly browsing through this manuscript some indications are added on the margin:

- the symbol  indicates a technical part which can be skipped but for the final results;
-  means that we report classical results.

CHAPTER 2

Wess-Zumino-Witten Models

Wess-Zumino-Witten models constitute a large class of the exact string theory solutions which we will use as starting points for most of the analysis in the following. In this chapter we see how they can be studied from different perspectives and with different motivations both from a target space and world-sheet point of view.

“To paraphrase Oedipus, Hamlet, Lear, and all those guys,” I said, “I wish I had known this some time ago.”

Amber Chronicles
ROGER ZELAZNY

2.1 The two-dimensional point of view



The classical theory

WESS-ZUMINO-WITTEN models were introduced by Witten in his seminal paper [Wit84] to generalize the usual bosonization of a free fermion to a system of N fermions. It has been known for a long time that the Lagrangian for a free massless Dirac fermion in two dimensions can be mapped to the Lagrangian for a free massless boson as follows:

Free fermion
bosonization

$$L = \psi^* i \not{\partial} \psi \rightarrow \frac{1}{2} \partial_\mu \phi \partial^\mu \phi, \quad (2.1)$$

but the generalization to more complex systems is not straightforward. One of the main motivations for this mapping is given by the fact that bosonic systems admit a semiclassical limit and then allow for simpler intuitive interpretations of their physics.

It is of course possible to rewrite the fields in one description as functions of the fields in the other one but this requires complicated (non-local) expressions. Other quantities remain simple, in particular the currents:

$$J_\mu = \psi^* \gamma_\mu \psi \rightarrow \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \phi \quad (2.2)$$

and as such they are the convenient building blocks for a generalization. A most useful rewriting for their expression is obtained when going to light-cone coordinates

$$J = \frac{i}{2\pi} U^{-1} \partial U = -\frac{1}{\sqrt{\pi}} \partial \phi, \quad \bar{J} = -\frac{i}{2\pi} \bar{\partial} U U^{-1} = \frac{1}{\sqrt{\pi}} \bar{\partial} \phi, \quad (2.3)$$

$U = \exp[i\sqrt{4\pi}\phi]$ being the chiral density $\bar{\psi}(1 + \gamma_5)\psi$. One then finds that the currents are chirally conserved:

$$\partial \bar{J} = \bar{\partial} J = 0, \quad (2.4)$$

which is equivalent to ask for the bosonic field to be harmonic:

$$2\partial\bar{\partial}\phi = 0. \quad (2.5)$$

The generalization of this simple system is given by the theory of $2N$ Dirac fermions.

$$L = \frac{1}{2} \sum_k \bar{\psi}^k i \not{\partial} \psi^k, \quad (2.6)$$

This admits a chiral group $U(N) \times U(N)$ with vector and axial currents written as:

$$V_\mu^a = \bar{\psi} \gamma_\mu T^a \psi, \quad A_\mu^a = \bar{\psi} \gamma_\mu \gamma_5 T^a \psi. \quad (2.7)$$

It is more useful to define the chiral components

$$J^{ij} = -i\psi^i \psi^j, \quad \bar{J}^{ij} = -i\bar{\psi}^i \bar{\psi}^j, \quad (2.8)$$

generating the $U(N) \times U(N)$ symmetry and obeying the same conservation as before:

$$\partial \bar{J} = \bar{\partial} J = 0, \quad (2.9)$$

This is obviously not equivalent to a system of N bosons which would just allow for the diagonal $U(1)^N$ symmetry. What we need is an object $g \in U(N)$ transforming under a couple $(A, B) \in U(N) \times U(N)$ as

$$g \rightarrow A g B^{-1}, \quad A, B \in U(N) \quad (2.10)$$

and express the currents as functions of g just as in the Abelian case we did in terms of the density U :

$$J = \frac{i}{2\pi} g^{-1} \partial g, \quad \bar{J} = -\frac{i}{2\pi} \bar{\partial} g g^{-1}. \quad (2.11)$$

In order for these currents to be conserved as above all we need to find is a Lagrangian admitting the following equations of motion

$$\partial(g^{-1} \bar{\partial} g) = 0. \quad (2.12)$$

The first natural tentative action is given by

$$S_0 = \frac{1}{4\lambda^2} \int_{\Sigma} d^2x \operatorname{tr} \left[\partial^\mu g \partial_\mu g^{-1} \right], \quad (2.13)$$

since this is the only manifestly chirally invariant choice. Unfortunately this can't be the right answer for a number of reasons. In particular it describes an asymptotically free theory with the wrong equations of motion. We are then forced to add another ingredient, the so-called Wess-Zumino term

$$\Gamma = \frac{1}{24\pi} \int_M d^3y \epsilon^{\mu\nu\rho} \operatorname{tr} \left[\tilde{g}^{-1} \partial_\mu \tilde{g} \tilde{g}^{-1} \partial_\nu \tilde{g} \tilde{g}^{-1} \partial_\rho \tilde{g} \right], \quad (2.14)$$

where M is a three-dimensional manifold admitting Σ as border $\partial M = \Sigma$ and \tilde{g} is the extension of the mapping $g : \Sigma \rightarrow G$ to a mapping $\tilde{g} : M \rightarrow G$. Although it might appear a bit surprising at first sight, this is precisely what is needed since the variation of Γ gives only a local term on Σ and the equations of motion for the action $S = S_0 + k\Gamma$ read:

$$\left(\frac{1}{2\lambda^2} + \frac{k}{8\pi} \right) \partial \left(g^{-1} \bar{\partial} g \right) + \left(\frac{1}{2\lambda^2} - \frac{k}{8\pi} \right) \bar{\partial} \left(g^{-1} \partial g \right) = 0, \quad (2.15)$$

which in particular for $\lambda^2 = \frac{4\pi}{k}$ yield precisely the equations we were expecting. It can be shown that this is an infrared fixed point for a renormalization group-flow, and we will expand on this aspect in Ch. 8.

At this point it is on the other hand better to deal more thoughtfully with the interpretation and the consequences of the Wess-Zumino term Γ . First of all it must be remarked that Γ can be put in the form of an ordinary action *ie* an integral over the two-dimensional space-time of a perfectly respectable although non-manifestly chiral-invariant action (which changes by a total derivative under a chiral transformation):

$$\Gamma = \int_{\Sigma} d^2x \epsilon^{\mu\nu} B_{ij}(\phi^k) \partial_\mu \phi^i \partial_\nu \phi^j, \quad (2.16)$$

where B is a (Kalb-Ramond) two-form. Another important aspect is the fact that an ambiguity is present in the definition of Γ for there are infinite topologically inequivalent ways to extend g to \tilde{g} , classified by $\pi_3(G)$. In the case of a compact group $\pi_3(G) = \mathbb{Z}$ and we are led to the same kind of topological argument leading to the quantization of the Dirac monopole. In fact if we consider a different three-manifold M' in the definition of Γ , the string amplitude changes by

$$\exp \left[\iota \int_M H - \iota \int_{M'} H \right], \quad (2.17)$$

where $H = dB$. This implies that the theory is consistent only if

$$\frac{1}{2\pi} \int_{S^3} H \in \mathbb{Z}, \quad (2.18)$$

having remarked that M and M' have the same boundary and then $M - M' \sim S^3$. Using the normalizations above one can show that this is equivalent to asking $k \in \mathbb{Z}$ which can be read as a quantization condition on the radius of the group manifold. In particular, the semiclassical limit is obtained when $k \rightarrow \infty$.

For reasons that will appear evident in the following k is called level of the model and the action is written as:

$$S_k[g] = \frac{k}{16\pi} \int_{\Sigma} d^2z \langle g^{-1} \partial g, g^{-1} \bar{\partial} g \rangle + \frac{k}{24\pi} \int_M \langle \bar{g}^{-1} d\bar{g}, [\bar{g}^{-1} d\bar{g}, \bar{g}^{-1} d\bar{g}] \rangle, \quad (2.19)$$

where $\langle \cdot, \cdot \rangle$ is the Killing form on G .



An exact model

An important feature of WZW models is that they provide exact solutions at all orders in α' or, more precisely, *the effective action is equal to the classical action up to a shift in the overall normalization $k \rightarrow k + g^*$* . The argument goes as follows [KZ84, LS92, Tse93]. Consider the path integral

Quantum
effective action
for WZW models

$$\int \mathcal{D}g e^{-kS[g] + B\bar{J}[g]} = e^{-W[B]}, \quad (2.20)$$

where $S[g]$ is the WZW action at level one and $B\bar{J}[g]$ is the shorthand for

$$B\bar{J}[g] = \frac{k}{\pi} \int dz^2 \langle B \bar{\partial} g g^{-1} \rangle. \quad (2.21)$$

If we write B as $B = u^{-1} \partial u$ we can use the so called Polyakov-Witten identity

$$S[ab] = S[a] + S[b] - \frac{1}{\pi} \int d^2z \langle a^{-1} \partial a \bar{\partial} b b^{-1} \rangle \quad (2.22)$$

and it's then easy to see that W doesn't receive quantum corrections and is simply given by the classical action evaluated on u :

$$W[B] = -kS[u]. \quad (2.23)$$

Indeed, B is an external source coupled with the current \bar{J} so, the effective action for g , $\Gamma[g]$, will be given by the Legendre transformation of $W[B]$, ie by the path integral:

$$\int \mathcal{D}B e^{-W[B] + B\bar{J}[g]} = e^{-\Gamma[g]}. \quad (2.24)$$

This interpretation of effective action for $\Gamma[g]$ is comforted by remarking that combining Eq. (2.20) and Eq. (2.24) one finds that

$$e^{-\Gamma[g]} = \int \mathcal{D}g' e^{-kS[g']} \delta(\bar{J}[g'] - \bar{J}[g]). \quad (2.25)$$

In order to calculate the Legendre transform in Eq. (2.24) one can perform a change of variables from $B = u^{-1} \partial u$ to u : the corresponding Jacobian will give

the announced shift in the effective action. More precisely, as shown in [PW83, Pol88] we have:

$$\frac{\mathcal{D}B}{\mathcal{D}u} = e^{g^*S[u]} \quad (2.26)$$

and putting this back in Eq. (2.24)

$$e^{-\Gamma[g]} = \int \mathcal{D}u e^{(k+g^*)S[u]+B[u]\bar{J}[g]}, \quad (2.27)$$

we can use the same technique as above to derive the celebrated result:

$$\Gamma[g] = (k + g^*) S[g]. \quad (2.28)$$

The CFT approach



CFT with current algebras

Let us forget for a moment the WZW models and consider a more general framework, *ie* two-dimensional conformal field theories with current algebras. Given the stress-energy tensor $T_{\mu\nu}(\zeta)$ in two dimensions one can define

CFT infinitesimal generators

$$T = T_{11} - T_{22} + 2iT_{12} \quad (2.29a)$$

$$\bar{T} = T_{11} - T_{22} - 2iT_{12} \quad (2.29b)$$

so that the conservation $\partial_\mu T^{\mu\nu} = 0$ and the zero trace condition $T^\mu{}_\mu = 0$ translate into analyticity conditions

$$\bar{\partial}T = \partial\bar{T} = 0. \quad (2.30)$$

It is then clear that since the stress-energy tensor is the infinitesimal generator for translations, T and \bar{T} have this role for the conformal transformations.

$$z \rightarrow z + \epsilon(z) \quad (2.31a)$$

$$\bar{z} \rightarrow \bar{z} + \bar{\epsilon}(\bar{z}) \quad (2.31b)$$

which is to say that if we take a local field $A(z, \bar{z})$, this transforms under such variations as

$$\delta_\epsilon A(z, \bar{z}) = \oint_z T(\zeta)\epsilon(\zeta)A(z, \bar{z})d\zeta \quad (2.32)$$

where the contour integral is around z^1 .

This is just the definition of two-dimensional CFT but if the theory is at the same time invariant under a $G(z) \times G(\bar{z})$ action, G being some Lie group, then there are additional generators $J(z)$ and $\bar{J}(\bar{z})$ allowing to express the variation of $A(z, \bar{z})$ as

$$\delta_\omega A(z, \bar{z}) = \oint_z J^a(\zeta)\omega^a(\zeta)A(z, \bar{z})d\zeta \quad (2.33)$$

where $\omega(z)$ is some element in Lie G (the algebra as usual parametrizes the infinitesimal group transformations).

The variations of those generators with respect to ϵ and ω stem from general principles and read

Virasoro \times Kač
Moody

$$\delta_\epsilon T(z) = \epsilon(z)T'(z) + 2\epsilon'(z)T(z) + \frac{1}{12}c\epsilon''(z) \quad (2.34a)$$

$$\delta_\epsilon J(z) = \epsilon(z)J'(z) + \epsilon'(z)J(z) \quad (2.34b)$$

$$\delta_\omega J^a(z) = f^a_{bc}\omega^b(z)J^c(z) + \frac{1}{2}k\omega^{a'}(z) \quad (2.34c)$$

This is just a way of writing the algebra of the generators. Using the definition above in terms of contour integrals it can also be put in terms of operator product expansions

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}T'(w) \quad (2.35a)$$

$$T(z)J(w) = \frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}J'(w) \quad (2.35b)$$

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{f^{ab}_c}{z-w}J^c(w) \quad (2.35c)$$

Any operator in the theory is characterized by a representation for the left and right G groups and its anomalous dimensions Δ and $\bar{\Delta}$, which depend on the behaviour of the operator under dilatation $z \rightarrow \lambda z$. More precisely an operator of weight $(\Delta, \bar{\Delta})$ transforms under $z \rightarrow \lambda z, \bar{z} \rightarrow \bar{\lambda}\bar{z}$ as $A \rightarrow \lambda^\Delta \bar{\lambda}^{\bar{\Delta}} A$, and in particular the primary fields are defined as those who satisfy

$$T(z)\phi(w, \bar{w}) = \frac{\Delta}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\frac{\partial}{\partial w}\phi(w, \bar{w}) \quad (2.36a)$$

$$J^a(z)\phi(w, \bar{w}) = \frac{t^a}{z-w}\phi(w, \bar{w}). \quad (2.36b)$$

Given these relations it is immediate to write the Ward identities satisfied by the correlation functions of primary fields:

$$\langle T(z)\phi_1(z_1, \bar{z}_1) \dots \phi_N(z_N, \bar{z}_N) \rangle = \sum_{j=1}^N \left(\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_j} \frac{\partial}{\partial z_j} \right) \times \langle \phi_1(z_1, \bar{z}_1) \dots \phi_N(z_N, \bar{z}_N) \rangle \quad (2.37a)$$

$$\langle J^a(z)\phi_1(z_1, \bar{z}_1) \dots \phi_N(z_N, \bar{z}_N) \rangle = \sum_{j=1}^N \frac{t^a_j}{z-z_j} \langle \phi_1(z_1, \bar{z}_1) \dots \phi_N(z_N, \bar{z}_N) \rangle \quad (2.37b)$$

¹In the following we will avoid to write the anti-holomorphic counterpart for each relation since it can always be trivially derived.

A further step can be made if one expands the operators T and J in Laurent series obtaining respectively the modes L_n and J_n which by definition act on a local operator to give

$$L_n A(z, \bar{z}) = \oint_z T(\zeta) (\zeta - z)^{n+1} A(z, \bar{z}) d\zeta \quad (2.38a)$$

$$J_n^a A(z, \bar{z}) = \oint_z J^a(\zeta) (\zeta - z)^{n+1} t^a A(z, \bar{z}) d\zeta \quad (2.38b)$$

and in particular the primaries will satisfy

$$L_n \phi = J_n^a \phi = 0 \quad \forall n > 0 \quad (2.39)$$

$$L_0 \phi = \Delta \phi \quad J_0^a \phi = t^a \phi \quad (2.40)$$

The commutation relations among the L_n 's and the J_n^a 's are natural consequences of the OPEs in Eq. (2.35) and read

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{1}{12} c (n^3 - n) \delta_{n+m} \quad (2.41a)$$

$$[L_n, J_m^a] = -m J_{n+m}^a \quad (2.41b)$$

$$[J_n^a, J_m^b] = f_{bc}^a J_{n+m}^c + \frac{1}{2} k n \delta^{ab} \delta_{n+m,0}. \quad (2.41c)$$

This is again a way to write the semi-direct product of the Virasoro (Eq. (2.41a)) and Kač-Moody algebras (Eq. (2.41c)).

The wzw model

As we emphasized above the currents J and \bar{J} are the fundamental building blocks for the construction of WZW models. Their role is even more apparent when we study the symmetries of the theory, which takes us directly to make contact with the conformal field theory description. Hence the importance of these models in giving an explicit realization of the CFT outlined above (among the classical references see *e.g.* [KZ84]).

WZW CFT

The key remark is that the action in Eq. (2.19) is invariant under the transformation

$$g(\zeta) \mapsto \Omega(z) g(\zeta) \bar{\Omega}^{-1}(\bar{z}) \quad (2.42)$$

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are G -valued matrices analytically depending on z and \bar{z} . This gives rise to an infinite set of conserved currents which are precisely those we defined above in Eq. (2.11) and Eq. (2.33). Locally this translates into the fact that for an infinitesimal transformation

$$\Omega(z) = \mathbb{1} + \omega(z) = \mathbb{1} + \omega^a(z) t^a \quad (2.43)$$

the currents change as in Eq. (2.34c)

$$\delta_\omega J = [\omega(z), J(z)] + \frac{1}{2} k \omega'(z) \quad (2.44)$$

which is to say that J and \bar{J} represent an affine Lie algebra with central charge k .

The next step consist in identifying the stress-energy tensor. In the classical theory this is a bilinear of the currents, so it is natural to choose the so-called Sugawara construction [Sug68, Som68, KZ84]

Sugawara
stress-energy
tensor

$$T(z) = \frac{1}{2(k+g^*)} \sum_a J^a(z) J^a(z) \quad (2.45)$$

where the constant factor is fixed by asking a current to be a weight one operator². Note in particular the fact that the level k is shifted to $k+g^*$ which is precisely the same correction we found summing the instanton corrections in the previous section. A simple calculation of the OPE of T with itself gives the value for the central charge of the model

$$c = \frac{k \dim G}{k+g^*} \quad (2.46)$$

Using the definition of primary field (and in particular the fact that they are annihilated by $J_{-1}^a t_1^a - L_{-1}$) one can easily show that each primary is degenerate and has weight

$$\Delta = \frac{c_l}{g^* + k} \quad (2.47)$$

where $c_l = t_1^a t_1^a$ is the quadratic Casimir and in particular coincides with the dual Coxeter number $c_l = g^*$ if the field transforms in the adjoint representation.

Partition function

Affine characters As one would expect, a modular invariant partition function for a WZW group can be build entirely out of group theoretical objects. In particular the building blocks are given by the affine characters, *ie* the generating functions of the weight multiplicities for a given irrep Λ that take into account the conformal dimension of the highest weight of the representation:

$$\chi^\Lambda(\tau, \nu, u) = e^{-2i\pi k u} \text{tr}_{\text{rep}(\Lambda)} \left[q^{L_0 - c/24} e^{2i\pi \kappa(\nu, \mathcal{J})} \right] = e^{2i\pi \tau \frac{\kappa(\Lambda, \Lambda + 2\rho)}{2(k+g^*)} - c/24} ch_\Lambda(\tau, \nu, u) \quad (2.48)$$

where

- $ch_\Lambda(\tau, \nu, u)$ is the usual character for the affine Lie algebra $\hat{\mathfrak{g}}$:

$$ch_\Lambda(\tau, \nu, u) = e^{-2i\pi k u} \sum_{\hat{\lambda} \in \text{Rep}(\Lambda)} \dim V_{\hat{\lambda}} \exp\{2i\pi \tau n + \sum_i \nu_i \kappa(e_i, \hat{\lambda})\}; \quad (2.49)$$

²It follows easily that a Kač-Moody primary is a Virasoro primary but not the other way round. Pictorially Eq. (2.37b) is the “square root” of Eq. (2.41a)

- $V_{\hat{\lambda}}$ is the multiplicity of the weight $\hat{\lambda} = (\lambda, k, n)$;
- $\langle e_i \rangle$ is a basis in the root space;
- $\rho = \sum_{\alpha > 0} \alpha / 2$ is the Weyl vector.

An alternative way of writing the same characters is obtained in terms of theta functions. To each weight one can assign a theta-function defined as:

$$\Theta_{\hat{\lambda}}(\tau, \nu, u) = e^{-2i\pi k u} \sum_{\gamma \in \mathbf{M}_L + \frac{\lambda}{k}} e^{i\pi \tau k \kappa(\gamma, \gamma)} e^{2i\pi k \kappa(\nu, \gamma)} \quad (2.50)$$

where \mathbf{M}_L is the long root lattice. Then, using the Weyl-Kač formula the characters are written as:

$$\chi^{\Lambda}(\tau, \nu, u) = \frac{\sum_{w \in \mathbf{W}} \epsilon(w) \Theta_{w(\hat{\lambda} + \hat{\rho})}(\tau, \nu, u)}{\sum_{w \in \mathbf{W}} \epsilon(w) \Theta_{w(\hat{\rho})}(\tau, \nu, u)}, \quad (2.51)$$

\mathbf{W} being the Weyl group of the algebra and $\epsilon(w)$ the parity of the element w .

Knowing that the affine Lie algebra is the largest chiral symmetry of the theory it is not surprising that the partition function can be written as

$$Z = \sum_{\Lambda, \bar{\Lambda}} M^{\Lambda \bar{\Lambda}} \chi^{\Lambda}(\tau, 0, 0) \bar{\chi}^{\bar{\Lambda}}(\bar{\tau}, 0, 0) \quad (2.52)$$

where the sum runs over left and right representations of \mathfrak{g} with highest weight Λ and $\bar{\Lambda}$ and $M^{\Lambda \bar{\Lambda}}$ is the mass matrix which is chosen so to respect the modular invariance of Z .

A generalization that we will use in the following is obtained for heterotic strings where the $N = (1, 0)$ local supersymmetry requires a super-affine Lie algebra for the left sector. The latter can anyway be decoupled in terms of the bosonic characters above and free fermion characters as to give:

$$Z \begin{bmatrix} a \\ b \end{bmatrix} = \sum_{\Lambda, \bar{\Lambda}} M^{\Lambda \bar{\Lambda}} \chi^{\Lambda}(\tau) \left(\frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau)}{\eta(\tau)} \right)^{\dim(\mathfrak{g})/2} \bar{\chi}^{\bar{\Lambda}} \quad (2.53)$$

where (a, b) are the spin structures of the world-sheet fermions. The characters of the affine algebras can be decomposed according to the generalized parafermionic decomposition, by factorizing the abelian subalgebra of the Cartan torus. For example, we can decompose the left supersymmetric \mathfrak{g}_k characters in terms of characters of the supersymmetric coset, given by the following branching relation (see [KS89b]):

$$\chi^{\Lambda} \left(\frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \right)^{\dim(\mathfrak{j})/2} = \sum_{\lambda \bmod (k+g^*)\mathbf{M}_L} \mathcal{P}_{\lambda}^{\Lambda} \begin{bmatrix} a \\ b \end{bmatrix} \frac{\Theta_{\lambda, k+g^*}}{\eta^{\dim(\mathfrak{k})}} \quad (2.54)$$

in terms of the theta-functions associated to \mathfrak{g}_k .

2.2 The target space point of view

Supergravity appears as a low energy description of string theory, obtained when asking for the Weyl invariance of the σ -model Lagrangian. This amounts, at first order in α' , to the following equations of motion for the metric g , the Kalb-Ramond field B and the dilaton Φ [CMPF85, Tse95]:

$$\begin{cases} \beta_\Phi = -\frac{1}{2}\nabla^\mu\partial_\mu\Phi + \partial_\mu\Phi\partial^\mu\Phi - \frac{1}{24}H_{\mu\rho\sigma}H^{\mu\rho\sigma}, \\ \beta_g = R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + 2\nabla_\mu\nabla_\nu\Phi, \\ \beta_B = \nabla_\mu H^\mu{}_{\nu\rho} + 2\nabla_\mu\Phi H^\mu{}_{\nu\rho}. \end{cases} \quad (2.55)$$

Being a one-loop calculation, the corresponding results should always be checked against higher order corrections in α' . On the other hand, as we have already stressed many times above, WZW models (just like the asymmetric deformations we study in this work) only receive corrections in terms of the level of the algebra (or, in this language, on the overall volume of the manifold). This implies that the target space description at one loop in α' is automatically correct at all orders. From this point of view, Wess–Zumino–Witten models describe the motion of a string on a group manifold geometry. The background fields are completed by a NS-NS three form $H = dB$ (Kalb-Ramond field) and a constant dilaton $\Phi = \Phi_0$.

The target space analysis is greatly simplified by the fact that the geometric quantities are all naturally expressed in terms of group theoretical objects. Let us consider for concreteness the case of a compact group G , whose Lie algebra is generated by $\langle t^\alpha \rangle$ and has structure constants $f^\alpha{}_{\beta\gamma}$. The metric for the group manifold can be chosen as the Killing metric (the choice is unique up to a constant in this case) and it is then natural to use the Maurer–Cartan one-forms as vielbeins. In our conventions, then:

$$g_{\mu\nu} = -\frac{1}{2g^*}f^\alpha{}_{\beta\gamma}f^\gamma{}_{\delta\alpha}J^\beta{}_\mu J^\delta{}_\nu = \delta_{\beta\gamma}J^\beta{}_\mu J^\delta{}_\nu, \quad (2.56)$$

where g^* is the dual Coxeter number and

$$J^\alpha{}_\mu = \langle t^\alpha g^{-1} \partial_\mu g \rangle. \quad (2.57)$$

In this basis the NS-NS 3-form field is written as

$$H_{[3]} = \frac{1}{3!}f_{\alpha\beta\gamma}J^\alpha \wedge J^\beta \wedge J^\gamma. \quad (2.58)$$

Geometry of
group manifolds

The connection one-forms $\omega^\alpha{}_\beta$ can be obtained by asking for the torsion two-form to vanish:

$$dJ^\alpha + \omega^\alpha{}_\beta \wedge J^\beta = T^\alpha = 0, \quad (2.59)$$

and out of them one defines the curvature two form $R^\alpha{}_\beta$ as:

$$R^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta. \quad (2.60)$$

which in turn is given in terms of the Riemann tensor as:

$$R^\alpha{}_\beta = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} J^\gamma \wedge J^\delta. \quad (2.61)$$

In a Lie algebra with structure constants $f^\alpha{}_{\beta\gamma}$ the variation of the currents is given by the Cartan structure equation:

$$dJ^\alpha = -\frac{1}{2} f^\alpha{}_{\beta\gamma} J^\beta \wedge J^\gamma \quad (2.62)$$

whence we can directly read the connection one-forms:

$$\omega^\alpha{}_\beta = -\frac{1}{2} f^\alpha{}_{\beta\gamma} J^\gamma. \quad (2.63)$$

It is then immediate to write:

$$d\omega^\alpha{}_\beta = \frac{1}{4} f^\alpha{}_{\beta\gamma} f^\gamma{}_{\delta\epsilon} J^\delta \wedge J^\epsilon \quad (2.64a)$$

$$\begin{aligned} \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta &= \frac{1}{4} f^\alpha{}_{\gamma\delta} f^\gamma{}_{\beta\epsilon} J^\delta \wedge J^\epsilon = \frac{1}{8} \left(f^\alpha{}_{\gamma\delta} f^\gamma{}_{\beta\epsilon} - f^\alpha{}_{\gamma\epsilon} f^\gamma{}_{\beta\delta} \right) J^\delta \wedge J^\epsilon = \\ &= -\frac{1}{8} f^\alpha{}_{\beta\gamma} f^\gamma{}_{\delta\epsilon} J^\delta \wedge J^\epsilon \end{aligned} \quad (2.64b)$$

where we have antisymmetrized the product of the structure constants and then used a Jacobi identity. The Riemann tensor, the Ricci tensor and the scalar curvature are then given respectively by:

$$R^\alpha{}_{\beta\gamma\delta} = \frac{1}{4} f^\alpha{}_{\beta\kappa} f^\kappa{}_{\gamma\delta}, \quad (2.65a)$$

$$Ric_{\beta\delta} = \frac{1}{4} f^\alpha{}_{\beta\kappa} f^\kappa{}_{\alpha\delta} = \frac{\mathcal{G}^*}{2} g_{\beta\delta}, \quad (2.65b)$$

$$R = \frac{\mathcal{G}^*}{2} \dim G. \quad (2.65c)$$

We are now in a position to show that the metric and H field satisfy the (first order in α') equations of motion in Eq. (2.55). Of course this result is much less powerful than what we obtained in Sec. 2.1 but it is nevertheless an interesting example of how these geometrical calculations are greatly simplified in terms of the underlying algebraic structure. For a system without dilaton the equations reduce to:

$$\beta_{\alpha\beta}^G = R_{\alpha\beta} - \frac{1}{4} H_{\alpha\delta\gamma} H_\beta{}^{\delta\gamma} = 0, \quad (2.66a)$$

$$\beta_{\alpha\beta}^B = (\nabla^\gamma H)_{\gamma\alpha\beta} = 0. \quad (2.66b)$$

The first one is trivially satisfied by using the field in Eq. (2.58); for the second one we just need to remark that in components the Levi-Civita connection is:

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} f^\alpha{}_{\beta\gamma} \quad (2.67)$$

Equations of
motion

and remember that the covariant derivative of a three-form is

$$(\nabla_\alpha H)_{\beta\gamma\delta} = \partial_\alpha H_{\beta\gamma\delta} - \Gamma_{\alpha\beta}^\kappa H_{\kappa\gamma\delta} - \Gamma_{\alpha\gamma}^\kappa H_{\beta\kappa\delta} - \Gamma_{\alpha\delta}^\kappa H_{\beta\gamma\kappa}. \quad (2.68)$$

In Sec. 6.1 we will see from a slightly different perspective how the normalization for the Kalb-Ramond field H can be fixed in terms of renormalization-group flow.

CHAPTER 3

Deformations

In this rather technical chapter we describe marginal deformations of Wess-Zumino-Witten models. The main purpose for these constructions is to reduce the symmetry of the system while keeping the integrability properties intact, trying to preserve as many nice geometric properties as possible.

Mr. Jabez Wilson laughed heavily. "Well, I never!" said he. "I thought at first that you had done something clever, but I see that there was nothing in it, after all." "I begin to think, Watson," said Holmes, "that I make a mistake in explaining. *Omne ignotum pro magnifico*, you know, and my poor little reputation, such as it is, will suffer shipwreck if I am so candid."

The Red Headed League
ARTHUR CONAN DOYLE

THE POWER of WZW models resides in the symmetries of the theory. They impose strong constraints which allow quantum integrability as well as a faithful description in terms of spacetime fields, whose renormalization properties (at every order in α') are easily kept under control, as we have seen in the previous chapter.

It is hence interesting to study their moduli spaces, aiming at finding less symmetric (and richer) structures, that will hopefully enjoy analogous integrability and spacetime properties.

This chapter is devoted to introducing the construction of asymmetric deformations and giving the general results in a formalism adapted to group manifold geometry. For this reason the stress is put on the more mathematical aspects. Physical examples and consequences will be illustrated in greater detail in Ch. 4.

3.1 Deformed WZW models: various perspectives

Truly marginal deformations

Truly marginal deformations In this spirit one can consider marginal deformations of the WZW models obtained in terms of $(1, 1)$ operators built as bilinears in the currents:

$$\mathcal{O}(z, \bar{z}) = \sum_{ij} c_{ij} J^i(z) \bar{J}^j(\bar{z}), \quad (3.1)$$

where $J^i(z)$ and $\bar{J}^j(\bar{z})$ are respectively left- and right-moving currents. It is known [CS89] that this operator represents a *truly marginal deformation*, ie it remains marginal at all orders in the deformation parameter, if the parameter matrix c_{ij} satisfies the following constraints:

$$c_{im} c_{jn} f_k^{ij} = 0, \quad (3.2a)$$

$$c_{mi} c_{nj} \tilde{f}_k^{ij} = 0, \quad (3.2b)$$

where f and \tilde{f} are the structure constants of the algebras generated by J^i and \bar{J}^i . In particular one can remark that if J^i and \bar{J}^i live on a torus then the two equations are automatically satisfied for any value of c_{mn} and hence we get as moduli space, a rank(c)-dimensional hyperplane of exact models¹. The proof of this assertion proceeds as follows: we want to show that \mathcal{O} keeps its conformal dimensions when a term $H\mathcal{O}(z, \bar{z})$ is added to the Lagrangian, H being a coupling constant. The two-point function for $\mathcal{O}(z, \bar{z})$ in the interacting theory with Lagrangian $L + H\mathcal{O}$ can be expanded in powers of H as follows:



$$\begin{aligned} \langle \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) \rangle_H &= \\ &= \frac{\sum_{n=0}^{\infty} (-H)^n / n! \int d^2 z_1 \dots d^2 z_n \langle \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) \mathcal{O}(z_1, \bar{z}_1) \dots \mathcal{O}(z_n, \bar{z}_n) \rangle}{\sum_{n=0}^{\infty} (-H)^n / n! \int d^2 z_1 \dots d^2 z_n \langle \mathcal{O}(z_1, \bar{z}_1) \dots \mathcal{O}(z_n, \bar{z}_n) \rangle}, \end{aligned} \quad (3.3)$$

so, in particular, the H^2 -order term is:

$$\begin{aligned} \langle \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) \rangle_g &= \\ &= \frac{H^2}{2} \int d^2 z_1 d^2 z_2 \langle \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2) \rangle + \\ &\quad - \frac{H^2}{2} \langle \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) \rangle \int d^2 z_1 d^2 z_2 \langle \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2) \rangle. \end{aligned} \quad (3.4)$$

¹Although for special values of the level k the theory contains other operators with the right conformal weights, it is believed that only current-current operators give rise to truly marginal deformations, i.e. operators that remain marginal for finite values of the deformation parameter.

Only the first term can contain logarithmic divergences that can alter the scale dependence of $\mathcal{O}(z, \bar{z})$, so let us study it more closely, by expanding $\mathcal{O}(z, \bar{z})$ in terms of currents:

$$\begin{aligned} & \frac{H^2}{2} \int d^2z_1 d^2z_2 \langle \mathcal{O}(z, \bar{z}) \mathcal{O}(w, \bar{w}) \mathcal{O}(z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2) \rangle = \\ & = \frac{H^2}{2} \int d^2z_1 d^2z_2 \sum_{ghij} \sum_{lmno} c_{gh} c_{hm} c_{in} c_{jo} \langle J_g(z) J_h(w) J_i(z_1) J_j(z_2) \rangle \\ & \qquad \qquad \qquad \langle \bar{J}_l(\bar{z}) \bar{J}_m(\bar{w}) \bar{J}_n(\bar{z}_1) \bar{J}_o(\bar{z}_2) \rangle. \end{aligned} \quad (3.5)$$

Rewriting the four-point functions for the currents in terms of their algebras

$$J_i(z) J_j(w) = \frac{K_{ij}}{(z-w)^2} + \frac{if^k_{ij} J_k(w)}{z-w}, \quad (3.6a)$$

$$\bar{J}_i(\bar{z}) \bar{J}_j(\bar{w}) = \frac{\tilde{K}_{ij}}{(\bar{z}-\bar{w})^2} + \frac{i\tilde{f}^k_{ij} \bar{J}_k(\bar{w})}{\bar{z}-\bar{w}}, \quad (3.6b)$$

one can evaluate the integrals passing to momentum space and introducing some ultraviolet cut-offs $\Lambda_1, \Lambda_2, \Lambda$. In particular, the terms which are interesting from our point of view are those diverging as $|z-w|^{-4}$ and they are:

$$\frac{8\pi^2 H^2 \log \Lambda_1 \log \Lambda_2}{|z-w|^4} \sum_{ghij} \sum_{lmno} \sum_{kp} c_{gl} c_{hm} c_{in} c_{jo} K_{kk} \tilde{K}_{pp} f_{hk}^g f_{jk}^i \tilde{f}_{mp}^l \tilde{f}_{op}^n \quad (3.7a)$$

and

$$\frac{6\pi^2 H^2 \log \Lambda}{|z-w|^4} \sum_{ghij} \sum_{klm} c_{gl} c_{hl} c_{im} c_{jm} K_{kk} \tilde{K}_{ll} \tilde{K}_{mm} f_{ik}^g f_{jk}^h + c_{lg} c_{lh} c_{mi} c_{mj} \tilde{K}_{kk} K_{ll} K_{mm} \tilde{f}_{ik}^g \tilde{f}_{jk}^h. \quad (3.7b)$$

Using the fact that the matrices K_{ij} and \tilde{K}_{ij} are positive-definite it is simple to see that they both vanish if and only if Eq. (3.2) are satisfied (the condition is only sufficient for general semi-simple groups).

Actually there's another piece of information that we learn out of this construction: *the OPE coefficients among the currents used for the deformation do not change with the deformation.* As we will see in the next section, this implies that the effect of the deformation is completely captured by a transformation in the charge lattice of the theory.

Algebraic structure of current-current deformations

The result of the previous section can be recast in more abstract terms: consider a conformal field theory whose holomorphic and anti-holomorphic Kac Moody algebras correspond to Lie algebras \mathfrak{g} and $\bar{\mathfrak{g}}$, which respectively admit the abelian subalgebras \mathfrak{h} and $\bar{\mathfrak{h}}$. Then each pair $\mathfrak{u}(1)^d \subseteq \mathfrak{h}, \mathfrak{u}(1)^{\bar{d}} \subseteq \bar{\mathfrak{h}}$ gives rise to a new family of conformal field theory containing those algebras (the ones

defined in Eqs. (3.6)). It is safe to assume (at least in the compact case) that the CFT remains unitary and that its Hilbert space still decomposes into tensor products of irreducible highest weight representations of $\mathfrak{h} \times \bar{\mathfrak{h}}$ (from now on $\dim \mathfrak{h} = d$ and $\dim \bar{\mathfrak{h}} = \bar{d}$)

$$\mathcal{H} = \sum_{Q, \bar{Q}} \mathcal{H}_{Q, \bar{Q}} \mathcal{V}_Q \otimes \mathcal{V}_{\bar{Q}}, \quad (3.8)$$

Boost on the charge lattice

where we used the fact that those representations are completely characterized by their charges $(Q, \bar{Q}) \in (\mathfrak{h}^*, \bar{\mathfrak{h}}^*)$ and the corresponding conformal weights are given by $h = 1/2\kappa(Q, Q)$ and $\bar{h} = 1/2\bar{\kappa}(\bar{Q}, \bar{Q})$, where κ and $\bar{\kappa}$ are the Killing forms respectively on \mathfrak{g} and $\bar{\mathfrak{g}}$ restricted on \mathfrak{h} and $\bar{\mathfrak{h}}$. This set of charges naturally forms a lattice Λ when equipped with the pairing $\langle \cdot, \cdot \rangle = \kappa - \bar{\kappa}$. Using in example deformation theory as in [FR03] one can see that the effect of the deformation is completely captured by an $O(d, \bar{d})$ pseudo-orthogonal transformation of this charge lattice $\Lambda \subset \mathfrak{h}^* \times \bar{\mathfrak{h}}^*$, ie can be described in terms of the identity component of the group $O(d, \bar{d})$. Moreover, since the charges only characterise the $\mathfrak{h} \times \bar{\mathfrak{h}}$ modules up to automorphisms of the algebras, $O(d) \times O(\bar{d})$ transformations don't change the CFT. Hence the deformation space is given by:

$$D_{\mathfrak{h}, \bar{\mathfrak{h}}} \sim O(d, \bar{d}) / (O(d) \times O(\bar{d})). \quad (3.9)$$

The moduli space is obtained out of $D_{\mathfrak{h}, \bar{\mathfrak{h}}}$ after the identification of the points giving equivalent CFTs².

In the case of WZW models on compact groups, all maximal abelian subgroups are pairwise conjugated by inner automorphisms. This implies that the complete deformation space is $D = O(d, d) / (O(d) \times O(d))$ where d is the rank of the group. The story is different for non-semi-simple algebras, whose moduli space is larger, since we get different $O(d, \bar{d}) / (O(d) \times O(\bar{d}))$ deformation spaces for each (inequivalent) choice of the abelian subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$. We'll see an example of this in the next chapter where deforming a $SL(2, \mathbb{R})$ WZW model (Sec. 4.2 and Sec. 4.4) will give rise to a much richer structure than in the $SU(2)$ case (Sec. 4.1).

Parafermion decomposition

Truly marginal deformations of WZW model single out abelian subalgebras of the model. It is then natural that an important tool in describing these current-current deformations comes from the so-called parafermion decomposition. The highest-weight representation for a $\hat{\mathfrak{g}}_k$ graded algebra can be decomposed into highest-weight modules of a Cartan subalgebra $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}_k$ as follows [GQ87, Gep87]:

$$\mathcal{V}_{\hat{\lambda}} \simeq \bigoplus_{\mu \in \Gamma_k} \mathcal{V}_{\hat{\lambda}, \mu} \otimes \bigoplus_{\delta \in Q_l(\mathfrak{g})} \mathcal{V}_{\mu + k\delta}, \quad (3.10)$$

where $\hat{\lambda}$ is an integrable weight of $\hat{\mathfrak{g}}_k$, $\mathcal{V}_{\hat{\lambda}, \mu}$ is the highest-weight module for the generalized $\hat{\mathfrak{g}}_k / \hat{\mathfrak{h}}$ parafermion, $Q_l(\mathfrak{g})$ is the long-root lattice and $\Gamma_k = P(\mathfrak{g}) / Q_l(\mathfrak{g})$

²Although we will concentrate on WZW models it is worth to emphasize that this construction is more general.

with $P(\mathfrak{g})$ the weight lattice. As a consequence, the WZW model based on $\hat{\mathfrak{g}}_k$ can be represented as an orbifold model:

$$\hat{\mathfrak{g}}_k \simeq (\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}} \otimes t_{\Lambda_k}) / \Gamma_k, \quad (3.11)$$

where t_{Λ_k} is a toroidal CFT with charge lattice, included in the $\hat{\mathfrak{g}}_k$ one, defined as $\Lambda_k = \{(\mu, \bar{\mu}) \in P(\hat{\mathfrak{g}}) \times P(\hat{\mathfrak{g}}) \mid \mu - \bar{\mu} = kQ_l(\hat{\mathfrak{g}})\}$. In our case the advantage given by using this representation relies on the fact that Γ_k acts trivially on the coset and toroidal model algebras; then, if we identify $\hat{\mathfrak{h}}$ and $\bar{\mathfrak{h}}$ with the graded algebras of t_{Λ_k} , the deformation only acts on the toroidal lattice and the deformed model can again be represented as an orbifold:

$$\hat{\mathfrak{g}}_k(\mathcal{O}) \simeq (\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}} \otimes t_{\mathcal{O}\Lambda_k}) / \Gamma_k, \quad (3.12)$$

where \mathcal{O} is an operator in the moduli space. In other words this representation is specially useful because it allows to easily single out the sector of the theory that is affected by the deformation. As we'll see in the next section this simplifies the task of writing the corresponding Lagrangian.

In the following we will separate this kind of deformations into two categories: those who give rise to *symmetric* deformations, *i.e.* the ones where $c_{ij} = \delta_{ij}$ and $J^i(z)$ and $\bar{J}^j(\bar{z})$ represent the same current in the two chiral sectors of the theory and the *asymmetric* ones where the currents are different and in general correspond to different subalgebras. This distinction is somehow arbitrary, since both symmetric and asymmetric deformations act as $O(d, \bar{d})$ rotations on the background fields. It is nonetheless interesting to single out the asymmetric case. In the special situation when one of the two currents belongs to an internal $U(1)$ (coming from the gauge sector in the heterotic or simply from any $U(1)$ subalgebra in the type II), it is in fact particularly simple to study the effect of the deformation, even from the spacetime field point of view; there in fact, the expressions for the background fields are exact (at all order in α' and for every value of the level k) as we will show in Sec. 3.4.

Symmetric and
asymmetric
deformations

Background fields and symmetric deformations

Before moving to the asymmetric deformations we're interested in, let us consider briefly symmetric deformations (also called *gravitational*) which are those that have received by far the most attention in literature [HS93, GK94, F94, FR03, DOPS05]. Specialising Eq. (3.1) to the case of one only current we can write the small deformation Lagrangian as:

$$S = S_{\text{WZW}} + \delta\kappa^2 \int d^2z J(z)\bar{J}(\bar{z}) \quad (3.13)$$

This infinitesimal deformation has to be integrated in order to give a Lagrangian interpretation to the CFT described in the previous section. Different approaches are possible, exploiting the different possible representations described above.

- A possible way consists in implementing an $O(d, d)$ rotation on the background fields [HS93]. More precisely, one has to identify a coordinate system in which the background fields are independent of d space dimensions and metric and B field are written in a block diagonal form. In this way the following matrix is defined:

$$M = \left(\begin{array}{c|c} \hat{g}^{-1} & -\hat{g}^{-1}\hat{B} \\ \hline \hat{B}\hat{g}^{-1} & \hat{g} - \hat{B}\hat{g}^{-1}\hat{B} \end{array} \right), \quad (3.14)$$

where \hat{g} and \hat{B} are the pull-backs of the metric and Kalb–Ramond field on the p selected directions. Then the action of the $O(d, d)$ group on these fields and dilaton is given by:

$$M \rightarrow M' = \Omega M \Omega^t, \quad (3.15a)$$

$$\Phi \rightarrow \Phi' = \Phi + \frac{1}{2} \log \left(\frac{\det \hat{g}}{\det \hat{g}'} \right), \quad (3.15b)$$

where \hat{g}' is the metric after the transformation (3.15a) and $\Omega \in O(d, d)$. It must be emphasized that this transformation rules are valid at the lowest order in α' (but for finite values of the deformation parameters). So, although the model is exact, as we learn from the CFT side, the field expressions that we find only are true at leading order in α' .

- An alternative approach uses the *parafermion representation* Eq. (3.12) (see e.g. [FR03]). In practice this amounts to writing an action as the sum of the G/H parafermion and a deformed H part and finding the appropriate T-duality transformation (realizing the orbifold) such that for zero deformation the WZW on G is recovered, in accordance with Eq. (3.11).
- Finally, another point of view (inspired by the parafermionic representation), consists in *identifying the deformed model with a $(G \times H) / H$ coset model*, in which the embedding of the dividing group has a component in both factors [GK94]. The gauging of the component in G gives the parafermionic sector, the gauging of the component in H gives the deformed toroidal sector and the coupling term (originating from the quadratic structure in the fields introduced for the gauging) corresponds to the orbifold projection³.

3.2 Background fields for the asymmetric deformation

Let us now consider the less-known case of asymmetric deformations, in which the two sets of currents J_i and \bar{J}_j come from distinct sectors of the theory. The

³An instanton-correction-aware technique that should overcome the first order in α' limitation for gauged models has been proposed in [Tse94]. In principle this can be used to get an all-order exact background when we write the deformation as a gauged model. We will not expand further in this direction, that could nevertheless be useful to address issues such as the stability of the black string (see Sec. 4.4).

archetype of such construction is what we get considering an heterotic super-WZW model on a group G at level k and adding an exactly marginal operator built from the total Cartan currents of \mathfrak{g} (so that it preserves the local $N = (1, 0)$ superconformal symmetry of the theory):

$$S = S_{\text{WZW}} + \frac{\sqrt{k k_g}}{2\pi} \int d^2z \sum_a H_a \left(J^a(z) - \frac{i}{k} f_{MN}^a : \psi^M \psi^N : \right) \bar{J}(\bar{z}) \quad (3.16)$$

where the set $\{H_a\}$ are the parameters of the deformation, J^a are currents in the maximal torus $T \subset G$ and $\bar{J}(\bar{z})$ is a right moving current of the Cartan subalgebra of the heterotic gauge group at level k_g . Such a deformation is always truly marginal since the J_a currents commute.

It is not completely trivial to read off the deformed background fields that correspond to the deformed action. A possible way is a method involving a Kaluza–Klein reduction as in [HT95]. For simplicity we will consider the bosonic string with vanishing dilaton and just one operator in the Cartan subalgebra \mathfrak{k} . After bosonization the right-moving gauge current \bar{J} used for the deformation has now a left-moving partner and can hence be written as $\bar{J} = i\bar{\partial}\varphi$, $\varphi(z, \bar{z})$ being interpreted as an internal degree of freedom. The sigma-model action is recast as

$$S = \frac{1}{2\pi} \int d^2z (G_{MN} + B_{MN}) \partial x^M \bar{\partial} x^N, \quad (3.17)$$

where the $x^M, M = 1, \dots, d+1$ embrace the group coordinates $x^\mu, \mu = 1, \dots, d$ and the internal $x^{d+1} \equiv \varphi$:

$$x^M = \left(\frac{x^\mu}{\varphi} \right). \quad (3.18)$$

If we split accordingly the background fields, we obtain the following decomposition:

$$G_{MN} = \left(\begin{array}{c|c} G_{\mu\nu} & G_{\varphi\varphi} A_\mu \\ \hline G_{\varphi\varphi} A_\mu & G_{\varphi\varphi} \end{array} \right), \quad B_{MN} = \left(\begin{array}{c|c} B_{\mu\nu} & B_{\mu\varphi} \\ \hline -B_{\mu\varphi} & 0 \end{array} \right), \quad (3.19)$$

and the action becomes:

$$S = \frac{1}{2\pi} \int d^2z \{ (G_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu + (G_{\varphi\varphi} A_\mu + B_{\mu\varphi}) \partial x^\mu \bar{\partial} \varphi + (G_{\varphi\varphi} A_\mu - B_{\mu\varphi}) \partial \varphi \bar{\partial} x^\mu + G_{\varphi\varphi} \partial \varphi \bar{\partial} \varphi \}. \quad (3.20)$$

We would like to put the previous expression in such a form that spacetime gauge invariance,

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda, \quad (3.21)$$

$$B_{\mu 4} \rightarrow B_{\mu 4} + \partial_\mu \eta, \quad (3.22)$$

is manifest. This is achieved as follows:

$$S = \frac{1}{2\pi} \int d^2z \{ (\hat{G}_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu + B_{\mu\varphi} (\partial x^\mu \bar{\partial} \varphi - \partial \varphi \bar{\partial} x^\mu) + G_{\varphi\varphi} (\partial \varphi + A_\mu \partial x^\mu) (\bar{\partial} \varphi + A_\mu \bar{\partial} x^\mu) \}, \quad (3.23)$$

where $\hat{G}_{\mu\nu}$ is the Kaluza–Klein metric

$$\hat{G}_{\mu\nu} = G_{\mu\nu} - G_{\varphi\varphi} A_\mu A_\nu. \quad (3.24)$$

We can then make the following identifications:

$$\hat{G}_{\mu\nu} = k (\mathcal{J}_\mu \mathcal{J}_\nu - 2H^2 \tilde{\mathcal{J}}_\mu \tilde{\mathcal{J}}_\nu) \quad (3.25a)$$

$$B_{\mu\nu} = k \mathcal{J}_\mu \wedge \mathcal{J}_\nu \quad (3.25b)$$

$$B_{\mu\varphi} = G_{\varphi\varphi} A_\mu = H \sqrt{k k_g} \tilde{\mathcal{J}}_\mu, \quad (3.25c)$$

$$A_\mu = 2H \sqrt{\frac{k}{k_g}} \tilde{\mathcal{J}}_\mu, \quad (3.25d)$$

$$G_{\varphi\varphi} = \frac{k_g}{2}. \quad (3.25e)$$

where $\tilde{\mathcal{J}}$ is the Maurer-Cartan current chosen for the deformation. Let us now consider separately the background fields we obtained so to give a clear geometric interpretation of the deformation, in particular in correspondence of what we will find to be the maximal value for the deformation parameters H_a .

Metric on
squashed group
and decompactifi-
cation
limit



The metric. According to Eq. (3.25a), in terms of the target space metric, the effect of this perturbation amounts to inducing a back-reaction that in the vielbein (current) basis is written as:

$$\langle dg, dg \rangle_H = \sum_M \mathcal{J}_M \otimes \mathcal{J}_M - 2 \sum_a H_a^2 \mathcal{J}_a \otimes \mathcal{J}_a = \sum_\mu \mathcal{J}_\mu \otimes \mathcal{J}_\mu + \sum_a (1 - 2H_a^2) \mathcal{J}_a \otimes \mathcal{J}_a \quad (3.26)$$

where we have explicitly separated the Cartan generators. From this form of the deformed metric we see that there is a “natural” maximal value $H_a = 1/\sqrt{2}$ where the contribution of the $\mathcal{J}_a \otimes \mathcal{J}_a$ term changes its sign and the signature of the metric is thus changed. One could naively think that the maximal value $H_a = 1/\sqrt{2}$ can't be attained since this would correspond to a degenerate manifold of lower dimension; what actually happens is that the deformation selects the maximal torus that decouples in the $H_a = H \rightarrow 1/\sqrt{2}$ limit.

To begin, write the general element $g \in G$ as $g = ht$ where $h \in G/T, t \in T$. Substituting this decomposition in the expression above we find:

$$\begin{aligned} \langle d(ht), d(ht) \rangle_H &= \langle (ht)^{-1} d(ht) (ht)^{-1} d(ht) \rangle - \sum_a 2H_a^2 \langle T_a (ht)^{-1} d(ht)^{-1} \rangle^2 = \\ &= \langle h^{-1} dh h^{-1} dh \rangle + 2 \langle dt t^{-1} h^{-1} dh \rangle + \langle t^{-1} dt t^{-1} dt \rangle + \\ &\quad - \sum_a 2H_a^2 \left(\langle T_a t^{-1} h^{-1} dh \rangle + \langle T_a t^{-1} dt \rangle \right)^2 \end{aligned} \quad (3.27)$$

let us introduce a coordinate system (γ_μ, ψ_a) such as the element in G/T is parametrized as $h = h(\gamma_\mu)$ and t is written explicitly as:

$$t = \exp \left\{ \sum_a \psi_a T_a \right\} = \prod_a e^{\psi_a T_a} \quad (3.28)$$

it is easy to see that since all the T_a commute $t^{-1}dt = dt t^{-1} = \sum_a T_a d\psi_a$. This allows for more simplifications in the above expression that becomes:

$$\begin{aligned} \langle d(ht), d(ht) \rangle_H &= \langle h^{-1}dh h^{-1}dh \rangle + 2 \sum_a \langle T_a h^{-1}dh \rangle d\psi_a + \sum_a d\psi_a d\psi_a + \\ &- \sum_a 2H_a^2 \left(\langle T_a h^{-1}dh \rangle + d\psi_a \right)^2 = \langle h^{-1}dh h^{-1}dh \rangle - \sum_a 2H_a^2 \langle T_a h^{-1}dh \rangle^2 + \\ &+ 2 \sum_a (1 - 2H_a^2) \langle T_a h^{-1}dh \rangle d\psi_a + \sum_a (1 - 2H_a^2) d\psi_a d\psi_a \quad (3.29) \end{aligned}$$

if we reparametrise the ψ_a variables as:

$$\psi_a = \frac{\hat{\psi}_a}{\sqrt{1 - 2H_a}} \quad (3.30)$$

we get a new metric $\langle \cdot, \cdot \rangle'_H$ where we're free to take the $H_a \rightarrow 1/\sqrt{2}$ limit:

$$\begin{aligned} \langle d(ht), d(ht) \rangle'_H &= \langle h^{-1}dh h^{-1}dh \rangle - \sum_a 2H_a^2 \langle T_a h^{-1}dh \rangle^2 + \\ &+ 2 \sum_a \sqrt{1 - 2H_a^2} \langle T_a h^{-1}dh \rangle d\hat{\psi}_a + \sum_a d\hat{\psi}_a d\hat{\psi}_a \quad (3.31) \end{aligned}$$

and get:

$$\langle d(ht), d(ht) \rangle'_{1/\sqrt{2}} = \left[\langle h^{-1}dh h^{-1}dh \rangle - \sum_a \langle T_a h^{-1}dh \rangle^2 \right] + \sum_a d\psi_a d\psi_a \quad (3.32)$$

where we can see the sum of the restriction of the Cartan-Killing metric⁴ on $T_h G/T$ and the metric on $T_t T = T_t U(1)^r$. In other words the coupling terms between the elements $h \in G/T$ and $t \in T$ vanished and the resulting metric $\langle \cdot, \cdot \rangle'_{1/\sqrt{2}}$ describes the tangent space T_{ht} to the manifold $G/T \times T$.

Other Background fields. The asymmetric deformation generates a non-trivial field strength for the gauge field, that from Eq. (3.25d) is found to be:

$$F^a = 2\sqrt{\frac{k}{k_g}} H_a d\mathcal{J}^a = -\sqrt{\frac{k}{k_g}} H_a f^a{}_{\mu\nu} J^\mu \wedge J^\nu \quad (3.33)$$

(no summation implied over a).

On the other hand, the B -field (3.25b) is not changed, but the physical object is now the 3-form $H_{[3]}$:

$$H_{[3]} = dB - G_{\varphi\varphi} A^a \wedge dA^a = \frac{1}{3!} f_{MNP} \mathcal{J}^M \wedge \mathcal{J}^N \wedge \mathcal{J}^P - \sum_a H_a^2 f_{aNP} \mathcal{J}^a \wedge \mathcal{J}^N \wedge \mathcal{J}^P,$$

⁴This always is a left-invariant metric on G/H . A symmetric coset doesn't admit any other metric. For a more complete discussion see Sec. 3.3

$$(3.34)$$

where we have used the Maurer-Cartan structure equations. At the point where the fibration trivializes, $H_a = 1/\sqrt{2}$, we are left with:

$$H_{[3]} = \frac{1}{3!} f_{\mu\nu\rho} \mathcal{J}^\mu \wedge \mathcal{J}^\nu \wedge \mathcal{J}^\rho. \quad (3.35)$$

So only the components of $H_{[3]}$ “living” in the coset G/T survive the deformation. They are not affected of course by the rescaling of the coordinates on T .

A trivial fibration. The whole construction can be reinterpreted in terms of fibration as follows. The maximal torus T is a closed Lie subgroup of the Lie group G , hence we can see G as a principal bundle with fiber space T and base space G/T [Nak]

$$G \xrightarrow{T} G/T \quad (3.36)$$

The effect of the deformation consists then in changing the fiber and the limit value $H_a = 1/\sqrt{2}$ marks the point where the fibration becomes trivial and it is interpreted in terms of a gauge field whose strength is given by the canonical connection on G/T [KN69].

3.3 Geometry of squashed groups



In order to describe the squashed group manifolds that we obtain via asymmetric deformation we need to generalize the discussion on group manifold geometry presented in Sec. 2.2. Let $\{\hat{\theta}^\alpha\}$ be a set of one-forms on a manifold \mathcal{M} satisfying the commutation relations

$$[\hat{\theta}^\beta, \hat{\theta}^\gamma] = f^\alpha_{\beta\gamma} \hat{\theta}^\alpha \quad (3.37)$$

as it is the case when $\hat{\theta}^\alpha$ are the Maurer-Cartan one-forms of Eq. (2.57) and $f^\alpha_{\beta\gamma}$ the structure constants for the algebra. We wish to study the geometry of the Riemann manifold \mathcal{M} endowed with the metric

$$g = g_{\alpha\beta} \hat{\theta}^\alpha \otimes \hat{\theta}^\beta. \quad (3.38)$$

In general such a metric will have a symmetry $G \times G'$ where G is the group corresponding to the structure constants $f^\alpha_{\beta\gamma}$ and $G' \subset G$. The maximally symmetric case, in which $G' = G$ is obtained when g is G -invariant, *i.e.* when it satisfies

$$f^\alpha_{\beta\gamma} g_{\alpha\delta} + f^\alpha_{\delta\gamma} g_{\alpha\beta} = 0. \quad (3.39)$$

for compact groups this condition is fulfilled by the Killing metric in Eq. (2.56).

The connection one-forms ω^α_β are uniquely determined by the compatibility condition and the vanishing of the torsion. Respectively:

$$dg_{\alpha\beta} - \omega^\gamma_\alpha g_{\gamma\beta} - \omega^\gamma_\beta g_{\gamma\alpha} = 0 \quad (3.40)$$

$$d\hat{\theta}^\alpha + \omega^\alpha_\beta \wedge \hat{\theta}^\beta = T^\alpha = 0 \quad (3.41)$$

As it is shown in [MHS88], if $g_{\alpha\beta}$ is constant, the solution to the system can be put in the form

$$\omega^\alpha_\beta = -D^\alpha_{\beta\gamma} \hat{\theta}^\gamma \quad (3.42)$$

where $D^\alpha_{\beta\gamma} = 1/2 f^\alpha_{\beta\gamma} - K^\alpha_{\beta\gamma}$ and $K^\alpha_{\beta\gamma}$ is a tensor (symmetric in the lower indices) given by:

$$K^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\kappa} f^\delta_{\kappa\beta} g_{\gamma\delta} + \frac{1}{2} g^{\alpha\kappa} f^\delta_{\kappa\gamma} g_{\beta\delta}. \quad (3.43)$$

Just as in Sec. 2.2 we define the curvature two-form R^i_j and the Riemann tensor Curvature tensors on squashed groups which now reads:

$$R^\alpha_{\beta\gamma\delta} = D^\alpha_{\beta\kappa} f^\kappa_{\gamma\delta} + D^\alpha_{\kappa\gamma} D^\kappa_{\beta\delta} - D^\alpha_{\kappa\delta} D^\kappa_{\beta\gamma} \quad (3.44)$$

and the corresponding Ricci tensor:

$$Ric_{\beta\delta} = D^\alpha_{\beta\kappa} f^\kappa_{\alpha\delta} - D^\alpha_{\kappa\delta} D^\kappa_{\beta\alpha} \quad (3.45)$$

In particular for $g_{ij} \propto \delta_{ij}$, $K = 0$ so that we recover the usual Maurer–Cartan structure equation Eq. (2.63) and the expressions in Eqs. (2.65).

Let us now specialize these general relations to the case of the conformal model with metric given in Eq. (3.26). The $\hat{\theta}^i$'s are the Maurer–Cartan one-forms for the group G and the metric g_{AB} is

$$g_{AB} = \begin{cases} \delta_{\mu\nu} & \text{if } \mu, \nu \in G/H \\ \left(1 - \frac{H^2}{2}\right) \delta_{ab} & \text{if } a, b \in H \end{cases} \quad (3.46)$$

where H is (a subgroup of) the Cartan torus $H \subset G$. It is quite straightforward to show that the Ricci tensor is given by⁵:

$$Ric_{AB} = \begin{cases} \frac{1}{2} (g^* + H^2) g_{\mu\nu} & \text{if } \mu, \nu \in G/H \\ g^*/2 \left(1 - \frac{H^2}{2}\right) g_{ab} & \text{if } a, b \in H. \end{cases} \quad (3.48)$$

whence we can read the (constant) Ricci scalar

$$R = \frac{g^*}{2} \dim G + \frac{H^2}{2} \left(\dim G - \text{rank } G \left(1 + \frac{g^*}{2}\right) \right) \quad (3.49)$$

Particular attention should be devoted to the limit case $H = \sqrt{2}$ in which

Geometry on G/H cosets

⁵In the $SU(2)$ case this would be

$$g = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 - \frac{H^2}{2} \end{pmatrix} \quad Ric = \begin{pmatrix} 1 + \frac{H^2}{2} & & \\ & 1 + \frac{H^2}{2} & \\ & & \left(1 - \frac{H^2}{2}\right)^2 \end{pmatrix} \quad (3.47)$$

where we chose J_3 as Cartan generator.

the Cartan torus decouples and we are left with the geometry of the G/T coset. In this case it is useful to explicitly write down the commutation relations, separating the generators of T and G/T :

$$[T_m, T_n] = f_{mn}^o T_o \qquad [T_m, T_\nu] = f_{m\nu}^\omega T_\omega \qquad (3.50a)$$

$$[T_\mu, T_\nu] = f_{\mu\nu}^o T_o + f_{\mu\nu}^\omega T_\omega \qquad (3.50b)$$

Of course there are no f_{mn}^ω terms since T is a group. G/T is said to be symmetric if $f_{\mu\nu}^\omega \equiv 0$, *i.e.* if the commutator of any couple of coset elements lives in the dividing subgroup. In this case a classical theorem states that the coset only admits one left-invariant Riemann metric that is obtained as the restriction of the Cartan-Killing metric defined on G (see *eg* [KN69]). This is not the case when T is the maximal torus (except for the most simple case $G = SU(2)$ where maximal torus and maximal subgroup are the trivial $U(1)$) and the coset manifold accepts different structures. From our point of view this means that even when considering deformations and cosets of compact groups where the Cartan subalgebra is unique (up to inner automorphisms), in general we expect different possible outcomes depending on how the gauging is performed (see in particular the $SU(3)$ case studied in detail in Sec. 4.5).

Kähler structure
on G/H

These homogeneous manifolds enjoy many interesting properties. As we pointed out many times already, the best part of them can be interpreted as consequence of the presence of an underlying structure that allows to recast all the geometric problems in Lie algebraic terms. There's however at least one intrinsically geometric property that it is worth to emphasize since it will have many profound implications in the following. All these spaces can be naturally endowed with complex structures by using positive and negative roots as holomorphic and anti-holomorphic generators. This structure doesn't in general correspond to a unique left-invariant Riemann metric. On the other hand there always exists such a metric that is also Kähler. In fact one can easily show that the $(1, 1)$ form defined as:

$$\omega = \frac{i}{2} \sum_{\alpha > 0} c_\alpha \mathcal{J}^\alpha \wedge \mathcal{J}^{\bar{\alpha}} \qquad (3.51)$$

is closed if and only if for each subset of roots $\{\alpha, \beta, \gamma\}$ such as $\alpha = \beta + \gamma$, the corresponding real coefficients c_α satisfy the condition $c_\alpha = c_\beta + c_\gamma$. Of course this is equivalent to say that the tensor

$$g = \sum_{\alpha > 0} c_\alpha \mathcal{J}^\alpha \otimes \mathcal{J}^{\bar{\alpha}} \qquad (3.52)$$

is a Kähler metric on G/T [BH58, Per87].

3.4 A no-renormalization theorem

As we've said many times, WZW models are exact solutions keeping their geometrical description at all orders in α' , the only effect of renormalization being a shift in the level (Sec. 2.1). Here we want to show that this same property is

shared by our asymmetrically deformed models that hence provide a geometric solution at every order in perturbation.

As we emphasized above, in studying symmetrically deformed WZW models, *i.e.* those where the deformation operator is written as the product of two currents belonging to the same sector $\mathcal{O} = \lambda J\bar{J}$, one finds that the Lagrangian formulation only corresponds to a small-deformation approximation. For this reason different techniques have been developed so to read the background fields at every order in λ but, still, the results are in general only valid at first order in α' and have to be modified so to take into account the effect of instanton corrections (. This is not the case for asymmetrically deformed models, for which the background fields in Eqs. (3.25) are exact at all orders in H_a and for which the effect of renormalization only amounts to the usual (for WZW models) shift in the level of the algebra $k \rightarrow k + g^*$.

Consider in example the most simple $SU(2)$ case (which we will review in greater detail in Sec. 4.1). In terms of Euler angles the deformed Lagrangian is written as:

No-
renormalization
for $SU(2)$

$$S = S_{SU(2)}(\alpha, \beta, \gamma) + \delta S = \frac{k}{4\pi} \int d^2z \partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta + \partial\gamma\bar{\partial}\gamma + 2 \cos \beta \partial\alpha\bar{\partial}\gamma + \frac{\sqrt{k k_g} H}{2\pi} \int d^2z (\partial\gamma + \cos \beta \partial\alpha) \bar{I}. \quad (3.53)$$

If we bosonize the right-moving current as $\bar{I} = \bar{\partial}\phi$ and add a standard $U(1)$ term to the action, we get:

$$\begin{aligned} S &= S_{SU(2)}(\alpha, \beta, \gamma) + \delta S(\alpha, \beta, \gamma, \phi) + \frac{k_g}{4\pi} \int d^2z \partial\phi\bar{\partial}\phi = \\ &= S_{SU(2)}\left(\alpha, \beta, \gamma + 2\sqrt{\frac{k_g}{k}} H\phi\right) + \frac{k_g(1 - 2H^2)}{4\pi} \int d^2z \partial\phi\bar{\partial}\phi \end{aligned} \quad (3.54)$$

and in particular at the decoupling limit $H \rightarrow 1/\sqrt{2}$, corresponding to the S^2 geometry, the action is just given by $S = S_{SU(2)}\left(\alpha, \beta, \gamma + 2\sqrt{\frac{k_g}{k}} H\phi\right)$. This implies that our (deformed) model inherits all the integrability and renormalization properties of the standard $SU(2)$ WZW model. In other words the three-dimensional model with metric and Kalb–Ramond field with $SU(2) \times U(1)$ symmetry and a $U(1)$ gauge field is uplifted to an exact model on the $SU(2)$ group manifold (at least locally): the integrability properties are then a consequence of this hidden $SU(2) \times SU(2)$ symmetry that is manifest in higher dimensions.

The generalization of this particular construction to higher groups is easily obtained if one remarks that the Euler parametrization for the $g \in SU(2)$ group representative is written as:

$$g = e^{i\gamma t_3} e^{i\beta t_1} e^{i\alpha t_2}, \quad (3.55)$$

where $t_i = \sigma_i/2$ are the generators of $\mathfrak{su}(2)$ (σ_i being the usual Pauli matrices). As stated above, the limit deformation corresponds to the gauging of the left

action of an abelian subgroup $T \subset SU(2)$. In particular here we chose $T = \{h \mid h = e^{i\phi t_3}\}$, hence it is natural to find (up to the normalization) that:

$$h(\phi) g(\alpha, \beta, \gamma) = g(\alpha, \beta, \gamma + \phi). \quad (3.56)$$

The only thing that one needs to do in order to generalize this result to a general group G consists in finding a parametrization of $g \in G$ such as the chosen abelian subgroup appears as a left factor. In example if in $SU(3)$ we want to gauge the $U(1)^2$ abelian subgroup generated by $\langle \lambda_3, \lambda_8 \rangle$ (Gell-Mann matrices), we can choose the following parametrization for $g \in SU(3)$ [Byr97]:

$$g = e^{i\lambda_8\phi} e^{i\lambda_3c} e^{i\lambda_2b} e^{i\lambda_3a} e^{i\lambda_5\theta} e^{i\lambda_3\gamma} e^{i\lambda_2\beta} e^{i\lambda_3\alpha}. \quad (3.57)$$

The deep reason that lies behind this property (differentiating symmetric and asymmetric deformations) is the fact that not only the currents used for the deformation are preserved (as it happens in both cases), but here their very expression is just modified by a constant factor. In fact, if we write the deformed metric as in Eq. (3.25a) and call \tilde{K}^μ the Killing vector corresponding to the chosen isometry (that doesn't change along the deformation), we see that the corresponding $\tilde{\mathcal{J}}_\mu^{(H)}$ current is given by:

$$\tilde{\mathcal{J}}_\nu^{(H)} = \tilde{K}^\mu g_{\mu\nu}^{(H)} = (1 - 2H^2) \tilde{\mathcal{J}}_\nu^{(0)} \quad (3.58)$$

The most important consequence (from our point of view) of this integrability property is that the SUGRA action is actually *exact* and the only effect of renormalization is the $k \rightarrow k + g^*$ shift.

3.5 Partition functions

Studying the algebraic structure of marginal deformations we have already stressed that they are completely determined by $O(d, \bar{d})$ pseudo-rotations on the charge lattice corresponding to the deforming operator. This means that a modular invariant partition function is simply obtained once we write the initial WZW one, single out those charges and apply the boost. This proves to be a relatively simple exercise for compact groups but presents technical problems even in the most simple non-compact example $SL(2, \mathbb{R})$ which we will study in greater detail in Sec. 4.2.

$SU(2)$

Instead of a general construction, for sake of clearness, we can start with the most simple – but showing some general features – example, taking the $SU(2)$ group (more extensively studied in Sec. 4.1). Our computation will also include the S^2 limiting geometry. To fix the ideas, we will consider the case $k_G = 2$, *i.e.* a $U(1)$ algebra generated by one right-moving complex fermion. As we've seen in Sec. 2.1 the partition function for the supersymmetric $SU(2)$ model can be written as

$$Z \left[\begin{matrix} a; h \\ b; g \end{matrix} \right] = \sum_{j, \bar{j}=0}^{(k-2)/2} M^{j\bar{j}} \chi^j \frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta} \bar{\chi}^{\bar{j}} \frac{\bar{\vartheta} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]}{\bar{\eta}}. \quad (3.59)$$

where the χ^j 's are the characters of bosonic $SU(2)_{k-2}$, (a, b) are the \mathbb{Z}_2 boundary conditions for the left-moving fermions⁶ and (h, g) those of the right-moving – gauge-sector – ones. We can choose any matrix M^{ij} compatible with modular invariance of $SU(2)_{k-2}$. Furthermore, the supersymmetric $SU(2)_k$ characters can be decomposed in terms of those of the $N = 2$ minimal models:

$$\chi^j(\tau) \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau, \nu) = \sum_{m \in \mathbb{Z}_{2k}} C_m^j \begin{bmatrix} a \\ b \end{bmatrix} \Theta_{m,k} \left(\tau, -\frac{2\nu}{k} \right), \quad (3.60)$$

where the $N = 2$ minimal-model characters, determined implicitly by this decomposition, are given in [Kir88, Dob87, Mat87, RY87].

Our aim is to implement the magnetic deformation in this formalism. The deformation acts as a boost on the left-lattice contribution of the Cartan current of the supersymmetric $SU(2)_k$ and on the right current from the gauge sector:

Boost on the charge lattice and partition function

$$\begin{aligned} \Theta_{m,k} \vartheta \begin{bmatrix} h \\ g \end{bmatrix} &= \sum_{n, \bar{n}} e^{-i\pi g(\bar{n} + \frac{h}{2})} q^{\frac{1}{2}(\sqrt{2kn} + \frac{m}{\sqrt{2k}})^2} \bar{q}^{\frac{1}{2}(\bar{n} + \frac{h}{2})^2} \\ &\longrightarrow \sum_{n, \bar{n}} e^{-i\pi g(\bar{n} + \frac{h}{2})} q^{\frac{1}{2}[(\sqrt{2kn} + \frac{m}{\sqrt{2k}}) \cosh x + (\bar{n} + \frac{h}{2}) \sinh x]^2} \\ &\quad \times \bar{q}^{\frac{1}{2}[(\bar{n} + \frac{h}{2}) \cosh x + (\sqrt{2kn} + \frac{m}{\sqrt{2k}}) \sinh x]^2}. \end{aligned} \quad (3.61)$$

The boost parameter x is related to the vacuum expectation value of the gauge field as follows:

$$\cosh x = \frac{1}{1 - 2H^2}. \quad (3.62)$$

We observe that, in the limit $H^2 \rightarrow H_{\max}^2$, the boost parameter diverges ($x \rightarrow \infty$), and the following constraints arise:

$$4(k+2)n + 2m + 2\sqrt{2k}\bar{n} + \sqrt{2k}h = 0. \quad (3.63)$$

Therefore, the limit is well-defined only if the level of the supersymmetric $SU(2)_k$ satisfies a quantization condition:

$$k = 2p^2, \quad p \in \mathbb{Z}. \quad (3.64)$$

This is exactly the charge quantization condition for the flux of the gauge field, Eq. (4.14). Under this condition, the constraints (3.63) lead to

$$m + ph \equiv 0 \pmod{2p} =: 2pN, \quad (3.65a)$$

$$\bar{n} = 2pn + N, \quad N \in \mathbb{Z}_{2p}. \quad (3.65b)$$

As a consequence, the $U(1)$ corresponding to the combination of charges or

Decoupled partition function

⁶We have removed the contribution of the fermion associated to J^3 since it is neutral in the deformation process.

thogonal to (3.63) decouples (its radius vanishes), and can be removed. We end up with the following expression for the S^2 partition function contribution:

$$Z_{S^2} \begin{bmatrix} a; h \\ b; g \end{bmatrix} = \sum_{j, \bar{j}} M^{j\bar{j}} \sum_{N \in \mathbb{Z}_{2p}} e^{i\pi g(N + \frac{h}{2})} \mathcal{C}_{p(2N-h)}^j \begin{bmatrix} a \\ b \end{bmatrix} \bar{\chi}^{\bar{j}}, \quad (3.66)$$

in agreement with the result found in [BJKZ96] by using the coset construction. The remaining charge N labels the magnetic charge of the state under consideration. As a result, the R -charges of the left $N = 2$ superconformal algebra are:

$$\mathcal{Q}_R = n + \frac{a}{2} - \frac{N - h/2}{p} \pmod{2}. \quad (3.67)$$

We now turn to the issue of modular covariance. Under the transformation $\tau \rightarrow -1/\tau$, the minimal-model characters transform as:

$$\mathcal{C}_m^j \begin{bmatrix} a \\ b \end{bmatrix} \left(-\frac{1}{\tau} \right) = e^{i\frac{\pi}{2}ab} \frac{1}{k} \sum_{j'=0}^{(k-2)/2} \sin \left(\frac{\pi(2j+1)(2j'+1)}{k} \right) \sum_{m' \in \mathbb{Z}_{2k}} e^{i\pi \frac{mm'}{k}} \mathcal{C}_{m'}^{j'} \begin{bmatrix} b \\ -a \end{bmatrix} (\tau). \quad (3.68)$$

On the one hand, the part of the modular transformation related to j is precisely compensated by a similar term coming from the transformation of $\bar{\chi}^{\bar{j}}$, in Eq. (3.66). On the other hand, the part of the transformation related to the spin structure (a, b) is compensated by the transformation of the other left-moving fermions in the full heterotic string construction. We can therefore concentrate on the transformation related to the m charge, coming from the transformation of the theta-functions at level k . We have

$$\sum_{N \in \mathbb{Z}_{2p}} e^{-i\pi g(N + \frac{h}{2})} \mathcal{C}_{p(2N-h)}^j \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \frac{1}{\sqrt{2k}} \sum_{m' \in \mathbb{Z}_{4p^2}} \sum_{N \in \mathbb{Z}_{2p}} e^{\frac{i\pi}{2} \left(g - \frac{m'}{p} \right) h} e^{2i\pi \frac{N(m'+pg)}{2p}} \mathcal{C}_{m'}^j \begin{bmatrix} b \\ -a \end{bmatrix}; \quad (3.69)$$

summing over N in \mathbb{Z}_{2p} leads to the constraint:

$$m' + pg \equiv 0 \pmod{2p} := -2pN', \quad N' \in \mathbb{Z}_{2p}. \quad (3.70)$$

So we end up with the sum

$$e^{-\frac{i\pi}{2}hg} \sum_{N' \in \mathbb{Z}_{2p}} e^{-i\pi h(N' + \frac{g}{2})} \mathcal{C}_{p(2N'-g)}^j \begin{bmatrix} b \\ -a \end{bmatrix}. \quad (3.71)$$

combining this expression with the modular transformation of the remaining right-moving fermions of the gauge sector, we obtain a modular invariant result.

In a similar way one can check the invariance of the full heterotic string under $\tau \rightarrow \tau + 1$.

$SU(3)$

As it is often the case, the $SU(2)$ example is illuminating but not exhaustive. In this situation this is due to the fact that $U(1)$ is the Cartan torus and at the same time the maximal subgroup. For this reason we need to work out in detail the next non-trivial example, $SU(3)$. The main difference is that there are two non-equivalent construction leading to the same algebraic structure but to the two possible different metrics on the $SU(3)/U(1)^2$ coset⁷.

The Kazama-Suzuki decomposition of $SU(3)$

We would like to decompose our WZW model in terms of Kazama-Suzuki (KS) cosets, which are conformal theories with extended $N = 2$ superconformal symmetry [KS89b, KS89a].

The simplest of those models are the $N = 2$ minimal models that are given by the quotient: $SU(2)_{k-2} \times SO(2)_1 / U(1)_k$, and their characters come from the branching relation:

$$\chi_{k-2}^j \Xi_2^{s_2} = \sum_{m \in \mathbb{Z}_{2k}} c_m^{j(s_2)} \frac{\Theta_{m,k}}{\eta}. \quad (3.72)$$

For convenience, we write the contribution of the world-sheet fermions in terms of $SO(2n)_1$ characters.

Similarly it is possible to construct an $N = 2$ coset CFT from $SU(3)$ [KS89b, KS89a]:⁸

$$\frac{SU(3)_{k-3} \times SO(4)_1}{SU(2)_{k-2} \times U(1)_{3k}}. \quad (3.73)$$

The characters of this theory are implicitly defined by the branching relation:

$$\chi_{k-3}^\Lambda \Xi_4^{s_4} = \sum_{2j=0}^{k-2} \sum_{n \in \mathbb{Z}_{6k}} c_{jn}^{\Lambda(s_4)} \chi_{k-2}^j \frac{\Theta_{n,3k}}{\eta}. \quad (3.74)$$

Therefore combining the two branching relations, we obtain the decomposition of $SU(3)$ in terms of $N = 2$ KS models:

$$\chi_{k-3}^\Lambda \Xi_4^{s_4} \Xi_2^{s_2} = \sum_{j,m,n} c_{jn}^{\Lambda(s_4)} c_m^{j(s_2)} \frac{\Theta_{m,k}}{\eta} \frac{\Theta_{n,3k}}{\eta} \quad (3.75)$$

This decomposition goes along the following pattern:

$$\begin{aligned} SU(3)_{k-3} \times SO(8)_1 \rightarrow & \frac{SU(3)_{k-3} \times SO(4)_1}{SU(2)_{k-2} \times U(1)_{3k}} \times \frac{SU(2)_{k-2} \times SO(2)_1}{U(1)_k} \times \\ & \times U(1)_{3k} \times U(1)_k \times SO(2)_1 \end{aligned} \quad (3.76)$$

⁷We have already pointed out in Sec. 3.3 that an asymmetric coset in the mathematical sense in general admits more than one left-invariant metric. The two possible choices for $SU(3)/U(1)^2$ will be extensively studied in Sec. 4.5

⁸According to our conventions, the weights of a $U(1)$ at level k are $m^2/4k$, $m \in \mathbb{Z}_{2k}$.

and we shall perform the deformation on the left lattice of $U(1)_{3k} \times U(1)_k$. However the deformation will also act on an appropriate sub-lattice of the right-moving gauge sector. The last $SO(2)_1$ factor corresponds to the fermions which are neutral in the process so they won't be considered afterwards.

The gauge sector To construct the model we assume that the gauge sector of the heterotic strings contain an unbroken $SO(6)_1$, whose contribution to the partition function is, written in terms of $SO(6)_1$ free fermionic characters $\bar{\Xi}_6^{\bar{s}_6}$. Since we decompose the characters of the left-moving sector according to eq. (3.76), a natural choice for the action of the deformation in the right-moving gauge sector is to use a similar Kazama-Suzuki decomposition, but for $k = 3$, in which case the bosonic CFT is trivial:

$$SO(8)_1 \rightarrow \frac{SO(4)_1}{SU(2)_1 \times U(1)_9} \times \frac{SU(2)_1 \times SO(2)_1}{U(1)_3} \times U(1)_3 \times U(1)_1 \times SO(2)_1 \quad (3.77)$$

Since as quoted previously two fermions – the $SO(2)_1$ factor – are neutral it is enough that the gauge sector contains an $SO(6)_1$ subgroup. To achieve this decomposition, first we decompose the $SO(6)_1$ characters in terms of $SO(4)_1 \times SO(2)_1$:

$$\bar{\Xi}_6^{\bar{s}_6} = \sum_{\bar{s}_4, \bar{s}_2 \in \mathbb{Z}_4} C[\bar{s}_6; \bar{s}_4, \bar{s}_2] \bar{\Xi}_4^{\bar{s}_4} \bar{\Xi}_2^{\bar{s}_2} \quad (3.78)$$

where the coefficients of the decomposition $SO(6) \rightarrow SO(4) \times SO(2)$ are either zero or one. And then we perform a coset decomposition for the $SO(4)_1$ characters:

$$\bar{\Xi}_4^{\bar{s}_4} = \sum_{\ell=0,1} \sum_{u \in \mathbb{Z}_{18}} \bar{\omega}_{\ell u}^{\bar{s}_4} \bar{\chi}^\ell \frac{\bar{\Theta}_{u,9}}{\bar{\eta}} \quad (3.79)$$

in terms of $SU(2)_1$ characters $\bar{\chi}^\ell$ and $U(1)$ characters $\bar{\Theta}_{u,9}$. It defines implicitly the coset characters $\bar{\omega}_{\ell u}^{\bar{s}_4}$. Then the $SU(2)_1 \times SO(2)_1$ characters are decomposed as:

$$\bar{\chi}^\ell \bar{\Xi}_2^{\bar{s}_2} = \sum_{v \in \mathbb{Z}_6} \bar{\omega}_v^{\ell, \bar{s}_2} \frac{\bar{\Theta}_{v,3}}{\bar{\eta}}. \quad (3.80)$$

So putting together these branching relations we have the following Kazama-Suzuki decomposition for the free fermions of the gauge sector:

$$\bar{\Xi}_6^{\bar{s}_6} = \sum_{\bar{s}_4, \bar{s}_2 \in \mathbb{Z}_4} \sum_{\ell=0,1} \sum_{u \in \mathbb{Z}_{18}} \sum_{v \in \mathbb{Z}_6} C[\bar{s}_6; \bar{s}_4, \bar{s}_2] \bar{\omega}_{\ell u}^{\bar{s}_4} \bar{\omega}_v^{\ell, \bar{s}_2} \frac{\bar{\Theta}_{u,9}}{\bar{\eta}} \frac{\bar{\Theta}_{v,3}}{\bar{\eta}}. \quad (3.81)$$

The deformation Now we are in position to perform the asymmetric deformation adding a magnetic field to the model. The deformation acts on the following combination of left and right theta functions:

$$\Theta_{n,3k} \bar{\Theta}_{u,9} \times \Theta_{m,k} \bar{\Theta}_{v,3}. \quad (3.82)$$

As for the case of $SU(2)$ [IKOP05a], we have to assume that the level obeys the condition:

$$\sqrt{\frac{k}{3}} = p \in \mathbb{N}, \quad (3.83)$$

to be able to reach the geometric coset point in the moduli space of CFT. Then we have to perform $O(2, 2, \mathbb{R})$ boosts in the lattices of the $U(1)$'s, mixing the left Cartan lattice of the super-WZW model with the right lattice of the gauge sector. These boosts are parametrized in function of the magnetic fields as:

$$\cosh \Omega_a = \frac{1}{1 - 2H_a^2}, \quad a = 1, 2. \quad (3.84)$$

Explicitly we have:

$$\begin{aligned} & \sum_{N_1, N_2 \in \mathbb{Z}} q^{3k(N_1 + \frac{m}{6k})^2} q^{k(N_2 + \frac{n}{2k})^2} \times \sum_{f_1, f_2 \in \mathbb{Z}} \bar{q}^9 (f_1 + \frac{u}{18})^2 \bar{q}^3 (f_2 + \frac{v}{6})^2 \\ & \rightarrow \sum_{N_1, N_2, f_1, f_2 \in \mathbb{Z}} q^9 \left[p \left(N_1 + \frac{m}{18p^2} \right) \cosh \Omega_1 + \left(f_1 + \frac{u}{18} \right) \sinh \Omega_1 \right]^2 q^3 \left[p \left(N_2 + \frac{n}{6p^2} \right) \cosh \Omega_2 + \left(f_2 + \frac{v}{6} \right) \sinh \Omega_2 \right]^2 \\ & \times \bar{q}^9 \left[\left(f_1 + \frac{u}{18} \right) \cosh \Omega_1 + p \left(N_1 + \frac{m}{18p^2} \right) \sinh \Omega_1 \right]^2 \bar{q}^3 \left[\left(f_2 + \frac{v}{6} \right) \cosh \Omega_2 + p \left(N_2 + \frac{n}{6p^2} \right) \sinh \Omega_2 \right]^2. \end{aligned} \quad (3.85)$$

After an infinite deformation, we get the following constraints on the charges:

$$m = p(18\mu - u), \quad \mu \in \mathbb{Z}_p \quad (3.86a)$$

$$n = p(6\nu - v), \quad \nu \in \mathbb{Z}_p \quad (3.86b)$$

and the $U(1)^2$ CFT that has been deformed marginally decouples from the rest and can be safely removed. In conclusion, the infinite deformation gives:

$$\begin{aligned} Z_{F_3}^{(s_4, s_2; \bar{s}_6)}(\tau) &= \sum_{\Lambda} \sum_j \sum_{\mu, \nu \in \mathbb{Z}_p} \sum_{\bar{s}_4, \bar{s}_2 \in \mathbb{Z}_4} C[\bar{s}_6; \bar{s}_4, \bar{s}_2] \\ & \sum_{\ell=0,1} \sum_{u \in \mathbb{Z}_{18}} \sum_{v \in \mathbb{Z}_6} C_{j, p(18\mu-u)}^{\Lambda(s_4)} C_{p(6\nu-v)}^j(s_2) \times \bar{\chi}_{k-3}^{\Lambda} \bar{\omega}_{4; \ell u}^{\bar{s}_4} \bar{\omega}_v^{\ell, \bar{s}_2} \end{aligned} \quad (3.87)$$

where the sum over Λ, j runs over integrable representations. This is the partition function for the $SU(3)/U(1)^2$ coset space. The fermionic charges in the left and right sectors are summed according to the standard rules of Gepner heterotic constructions [Gep88]. The modular properties of this partition function are the same as before the deformation, concerning the \mathbb{Z}_4 indices of the world-sheet fermions.

Alternative approach: direct abelian coset

Here we would like to take a different path, by deforming directly the Cartan lattice of \mathfrak{su}_3 without decomposing the left CFT in terms of KS $N = 2$ theories.

It is possible to perform a generalized (super)parafermionic decomposition of the characters of the $\hat{\mathfrak{su}}_3$ super-algebra at level k (containing a bosonic algebra at level $k - 3$) w.r.t. the Cartan torus:

$$\chi^\Lambda \left(\frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \right)^{\dim(j)/2} = \sum_{\lambda \in \mathbf{M}^* \bmod k\mathbf{M}} \mathcal{P}_\lambda^\Lambda \begin{bmatrix} a \\ b \end{bmatrix} \frac{\Theta_{\lambda,k}}{\eta^{\dim(\mathfrak{k})}} \quad (3.88)$$

where the theta function of the $\hat{\mathfrak{su}}_3$ affine algebra reads, for a generic weight $\lambda = m_i \lambda_f^i$:

$$\Theta_{\lambda,k} = \sum_{\gamma \in \mathbf{M} + \frac{\lambda}{k}} q^{\frac{k}{2} \kappa(\gamma, \gamma)} = \sum_{N^1, N^2 \in \mathbb{Z}} q^{\frac{k}{2} \|N^1 \alpha_1 + N^2 \alpha_2 + \frac{m_1 \lambda_f^1 + m_2 \lambda_f^2}{k}\|^2}. \quad (3.89)$$

To obtain an anomaly-free model (see the discussion at the beginning of Sec. 3.6) it is natural to associate this model with an abelian coset decomposition of an $SU(3)_1$ current algebra made with free fermions of the gauge sector. Thus if the gauge group contains an $SU(3)_1$ unbroken factor their characters can be decomposed as:

$$\tilde{\chi}^{\bar{\Lambda}} = \sum_{\bar{\lambda} = \bar{n}_i \lambda_f^i \in \mathbf{M}^* \bmod \mathbf{M}} \bar{\omega}_{\bar{\lambda}}^{\bar{\Lambda}} \bar{\Theta}_{\bar{\lambda}}. \quad (3.90)$$

Again we will perform the asymmetric deformation as a boost between the Cartan lattices of the left $\hat{\mathfrak{su}}_3$ algebra at level k and the right $\hat{\mathfrak{su}}_3$ lattice algebra at level one coming from the gauge sector. So after the infinite deformation we will get the quantization condition $\sqrt{k} = p$ and the constraint:

$$\lambda + p\bar{\lambda} = 0 \quad \bmod p\mathbf{M} =: p\mu, \quad \mu \in \mathbf{M}. \quad (3.91)$$

So we get a different result compared to the Kazama-Suzuki construction. It is so because the constraints that we get at the critical point force the weight lattice of the $\hat{\mathfrak{su}}_3$ at level k to be projected onto p times the $\hat{\mathfrak{su}}_3$ weight lattice at level one of the fermions. This model does not correspond to a Kählerian manifold and should correspond to the $SU(3)$ -invariant metric on the flag space. Indeed with the KS method we get instead a projection onto p times a lattice of $\hat{\mathfrak{su}}_3$ at level one which is dual to the orthogonal sub-lattice defined by $\alpha_1 \mathbb{Z} + (\alpha_1 + 2\alpha_2) \mathbb{Z}$ — in other words the lattice obtained with the Gell-Mann Cartan generators. In this case it is possible to decompose the model in KS cosets models with $N = 2$ superconformal symmetry.⁹

3.6 The deformation as a gauging

In this section we want to give an alternative construction for our deformed models, this time explicitly based on an asymmetric WZW gauging. The existence of such a construction is not surprising at all since our deformations can

⁹For the symmetrically gauged WZW models, this has been studied in [ESY03].

be seen as a generalization of the ones considered in [GK94]. In these terms, just like $J\bar{J}$ (symmetric) deformations lead to gauged WZW models, our asymmetric construction leads to asymmetrically gauged WZW models, which were studied in [QS03].

A point must be stressed here. The asymmetric deformations admit as limit solutions the usual geometric cosets that one would have expected from field theory, as results of a gauging procedure. So, why do we need to go through this somewhat convoluted procedure? The reason lays in the fact that string theory is not the usual point particle field theory. A left and a right sector are present at the same time and they cannot be considered separately if we don't want to introduce anomalies. Now, gauging the left action of a subgroup, *i.e.* the symmetry $G \sim GH$, which would directly give the geometric coset we are studying, would precisely introduce this kind of problems. Hence we are automatically forced to consider the adjoint action $G \sim H^{-1}GH$ [Wit91]. The key idea then, as it will appear in this section, is that when G is semisimple and written as the product of a group and a copy of its Cartan torus, the left and right action can be chosen such as to act on the two separate sectors and then be equivalent to two left actions.

Instead of a general realization, for sake of clearness, here we will give the explicit construction for the most simple case, the $SU(2)$ model, then introduce a more covariant formalism which will be simpler to generalize to higher groups, in particular for the $SU(3)$ case which we will describe in great detail in the following.

To simplify the formalism we will discuss gauging of bosonic CFTs, and the currents of the gauge sector of the heterotic string are replaced by compact $U(1)$ free bosons. All the results are easily translated into heterotic string constructions.

The $SU(2)/U(1)$ asymmetric gauging

In this section we want to show how the S^2 background described in [IKOP05a] can be directly obtained via an asymmetric gauging of the $SU(2) \times U(1)$ WZW model (a similar construction was first obtained in [Joh95]).

Geometric coset
as an asymmetric
gauging

Consider the WZW model for the group manifold $SU(2)_k \times U(1)_{k'}$. A parametrisation for the general element of this group which is nicely suited for our purposes is obtained as follows:

$$g = \begin{pmatrix} z_1 & z_2 & 0 \\ -\bar{z}_2 & \bar{z}_1 & 0 \\ 0 & 0 & z_3 \end{pmatrix} = \left(\frac{g_2}{0} \middle| \frac{0}{g_1} \right) \in SU(2) \times U(1) \quad (3.92)$$

where g_1 and g_2 correspond to the $SU(2)$ and $U(1)$ parts respectively and (z_1, z_2, z_3) satisfy:

$$SU(2) \times U(1) = \{ (w_1, w_2, w_3) \mid |w_1|^2 + |w_2|^2 = 1, |w_3|^2 = 1 \} \subset \mathbb{C}^3. \quad (3.93)$$

A possible choice of coordinates for the corresponding group manifold is given by the Euler angles:

$$\begin{aligned} & SU(2) \times U(1) \\ &= \left\{ (z_1, z_2, z_3) = \left(\cos \frac{\beta}{2} e^{i(\gamma+\alpha)/2}, \sin \frac{\beta}{2} e^{i(\gamma-\alpha)/2}, e^{i\varphi} \right) \mid 0 \leq \beta \leq \pi, 0 \leq \alpha, \beta, \varphi \leq 2\pi \right\} \end{aligned} \quad (3.94)$$

In order to obtain the coset construction leading to the S^2 background we define two $U(1) \rightarrow SU(2) \times U(1)$ embeddings as follows:

$$\begin{aligned} \epsilon_L : U(1) &\rightarrow SU(2) \times U(1) & \epsilon_R : U(1) &\rightarrow SU(2) \times U(1) \\ e^{i\tau} &\mapsto (e^{i\tau}, 0, 1) & e^{i\tau} &\mapsto (1, 0, e^{i\tau}) \end{aligned} \quad (3.95)$$

so that in terms of the z variables the action of these embeddings boils down to:

$$g \mapsto \epsilon_L(e^{i\tau}) g \epsilon_R(e^{i\tau})^{-1} \quad (3.96)$$

$$(w_1, w_2, w_3) \mapsto (e^{i\tau} w_1, e^{i\tau} w_2, e^{-i\tau} w_3). \quad (3.97)$$

This means that we are free to choose a gauge where w_2 is real or, in Euler coordinates, where $\gamma = \alpha$, the other angular variables just being redefined. To find the background fields corresponding to this gauge choice one should simply write down the Lagrangian where the symmetries corresponding to the two embeddings in (3.95) are promoted to local symmetries, integrate the gauge fields out and then apply a Kaluza-Klein reduction, much in the same spirit as in [IKOP05a].

The starting point is the WZW model, written as:

$$S_{\text{WZW}}(g) = \frac{k}{4\pi} \int dz^2 \langle g_2^{-1} \partial g_2 g_2^{-1} \bar{\partial} g_2 \rangle + \frac{k'}{4\pi} \int dz^2 \langle g_1^{-1} \partial g_1 g_1^{-1} \bar{\partial} g_1 \rangle. \quad (3.98)$$

Its gauge-invariant generalization is given by:

$$\begin{aligned} S &= S_{\text{WZW}} \\ &+ \frac{1}{2\pi} \int d^2z \left[k \bar{A} \langle t_L \partial g g^{-1} \rangle + k' A \langle t_R g^{-1} \bar{\partial} g \rangle + \sqrt{k k'} A \bar{A} \left(-2 + \langle t_L g t_R g^{-1} \rangle \right) \right] \end{aligned} \quad (3.99)$$

where A and \bar{A} are the components of the gauge field, and t_L and t_R are the Lie algebra generators corresponding to the embeddings in (3.95), *i.e.*

$$t_L = \iota \left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & 0 \end{array} \right), \quad t_R = \iota \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & p \end{array} \right), \quad (3.100)$$

σ_3 being the usual Pauli matrix. For such an asymmetric coset to be anomaly free, one has the following constraint on the embeddings:

$$k \langle t_L \rangle^2 = k' \langle t_R \rangle^2 \implies k = k' p^2, \quad \text{with } p \in \mathbb{N}. \quad (3.101)$$

If we pass to Euler coordinates it is simple to give an explicit expression for the action:

$$S(\alpha, \beta, \gamma, \varphi) = \frac{1}{2\pi} \int d^2z \frac{k}{4} (\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta + \partial\gamma\bar{\partial}\gamma + 2\cos\beta\partial\alpha\bar{\partial}\gamma) + \frac{k'}{2} \partial\varphi\bar{\partial}\varphi + ik(\partial\alpha + \cos\beta\partial\gamma)\bar{A} + ik'\sqrt{2}\bar{\partial}\varphi A - 2\sqrt{kk'}A\bar{A}. \quad (3.102)$$

This Lagrangian is quadratic in A, \bar{A} and the quadratic part is constant so we can integrate these gauge fields out and the resulting Lagrangian is:

$$S(\alpha, \beta, \gamma, \varphi) = \frac{1}{2\pi} \int d^2z \frac{k}{4} (\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta + \partial\gamma\bar{\partial}\gamma + 2\cos\beta\partial\alpha\bar{\partial}\gamma) + \frac{k'}{2} \partial\varphi\bar{\partial}\varphi + \frac{\sqrt{2kk'}}{2} (\partial\alpha + \cos\beta\partial\gamma)\bar{\partial}\varphi. \quad (3.103)$$

Now, since we gauged out the symmetry corresponding to the $U(1)$ embeddings, this action is redundant. This can very simply be seen by writing the corresponding metric and remarking that it has vanishing determinant:

$$\det g_{\mu\nu} = \begin{vmatrix} k/4 & & & \\ & k/4 & k/4 \cos\beta & \sqrt{2kk'}/4 \\ & k/4 \cos\beta & k/4 & \sqrt{2kk'}/4 \cos\beta \\ & \sqrt{2kk'}/4 & \sqrt{2kk'}/4 \cos\beta & k'/2 \end{vmatrix} = 0 \quad (3.104)$$

Of course this is equivalent to say that we have a gauge to fix (as we saw above) and this can be chosen by imposing $\gamma = \alpha$, which leads to the following action:

$$S(\alpha, \beta, \varphi) = \frac{1}{2\pi} \int d^2z \frac{k}{4} (2(1 + \cos\beta)\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta) + \frac{k'}{2} \partial\varphi\bar{\partial}\varphi + \frac{\sqrt{2kk'}}{2} (1 + \cos\beta)\partial\alpha\bar{\partial}\varphi \quad (3.105)$$

whence we can read a two dimensional metric by interpreting the $\partial\alpha\bar{\partial}\varphi$ term as a gauge boson and applying the usual Kaluza-Klein reduction. We thus recover the two-sphere as expected:

$$ds^2 = g_{\mu\nu} - G_{\varphi\varphi}A_\mu A_\nu = \frac{k}{4} (d\beta^2 + \sin^2\beta d\alpha^2) \quad (3.106)$$

supported by a (chromo)magnetic field

$$A = \sqrt{\frac{k}{k'}} (1 + \cos\beta) d\alpha \quad (3.107)$$

The current formalism

We now turn to rewrite the above gauging in a more covariant form, simpler to generalize. Since we are interested in the underlying geometry, we will mainly focus on the metric of the spaces we obtain at each step and write these metrics

Asymmetric gauging in the current formalism

in terms of the Maurer-Cartan currents¹⁰. As we have already seen, the metric of the initial group manifold is:

$$ds^2 = \frac{k}{2} \sum \mathcal{J}_i^2 \otimes \mathcal{J}_i^2 + \frac{k'}{2} \mathcal{I} \otimes \mathcal{I} \quad (3.108)$$

where $\{ \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \}$ are the currents of the $SU(2)$ part and \mathcal{I} the $U(1)$ generator. The effect of the asymmetric gauging amounts - at this level - to adding what we can see as an interaction term between the two groups. This changes the metric to:

$$ds^2 = \frac{k}{2} \sum \mathcal{J}_i^2 \otimes \mathcal{J}_i^2 + \frac{k'}{2} \mathcal{I} \otimes \mathcal{I} + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I}. \quad (3.109)$$

Of course if we choose $\langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{I} \rangle$ as a basis we can rewrite the metric in matrix form:

$$g = \frac{1}{2} \begin{pmatrix} k & & & \\ & k & & \\ & & k & \sqrt{kk'} \\ & & \sqrt{kk'} & k' \end{pmatrix} \quad (3.110)$$

where we see that the gauging of the axial symmetry corresponds to the fact that the sub-matrix relative to the $\{ \mathcal{J}_3, \mathcal{I} \}$ generators is singular:

$$\begin{vmatrix} k & \sqrt{kk'} \\ \sqrt{kk'} & k' \end{vmatrix} = 0 \quad (3.111)$$

explicitly this correspond to:

$$k \mathcal{J}_3 \otimes \mathcal{J}_3 + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I} + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I} + k' \mathcal{I} \otimes \mathcal{I} = (k + k') \hat{\mathcal{J}} \otimes \hat{\mathcal{J}} \quad (3.112)$$

where

$$\hat{\mathcal{J}} = \frac{\sqrt{k} \mathcal{J}_3 + \sqrt{k'} \mathcal{I}}{\sqrt{k + k'}} \quad (3.113)$$

is a normalized current. In matrix terms this corresponds to projecting the interaction sub-matrix on its non-vanishing normalized eigenvector:

$$\left(\sqrt{\frac{k}{k+k'}} \quad \sqrt{\frac{k}{k+k'}} \right) \begin{pmatrix} k & \sqrt{kk'} \\ \sqrt{kk'} & k' \end{pmatrix} \begin{pmatrix} \sqrt{\frac{k}{k+k'}} \\ \sqrt{\frac{k}{k+k'}} \end{pmatrix} = k + k' \quad (3.114)$$

and the resulting metric in the $\langle \mathcal{J}_1, \mathcal{J}_2, \hat{\mathcal{J}} \rangle$ basis is:

$$\begin{pmatrix} k & & \\ & k & \\ & & k + k' \end{pmatrix} \quad (3.115)$$

¹⁰One of the advantages of just working on the metrics is given by the fact that in each group one can consistently choose left or right currents as a basis. In the following we will consider the group in the initial WZW model as being generated by the left and the dividing group by the right ones.

This manifold \mathcal{M} (whose metric appears in the action (3.17)) corresponds to an S^1 fibration (the fiber being generated by $\hat{\mathcal{J}}$) over an S^2 base (generated by $\langle \mathcal{J}_1, \mathcal{J}_2 \rangle$).

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & S^2 \end{array} \quad (3.116)$$

It should now appear natural how to generalize this construction so to include all the points in the moduli space joining the unperturbed and gauged model. The decoupling of the $U(1)$ symmetry (that has been “gauged away”) is obtained because the back-reaction of the gauge field (Eq. (3.103)) is such that the interaction sub-matrix is precisely singular. On the other hand we can introduce a parameter that interpolates between the unperturbed and the gauged models so that the interaction matrix now has two non-null eigenvalues, one of which vanishing at the decoupling point.

In practice this is done by adding to the the asymmetrically gauged WZW model an auxiliary $U(1)$ free boson Y at radius $R = (kk')^{1/4}(1/\sqrt{2H-1})^{1/2}$. This $U(1)$ is coupled symmetrically to the gauge fields such that the anomaly cancellation condition is still given by (3.101). In particular if we choose the gauge $Y = 0$, the metric reads:

$$\begin{pmatrix} k & \sqrt{2H}\sqrt{kk'} \\ \sqrt{2H}\sqrt{kk'} & k' \end{pmatrix} \quad (3.117)$$

which is exactly the model studied above. For a generic value of H^2 the two eigenvalues are given by:

$$\lambda_{1,2}(k, k', H) = \frac{k + k' \mp \sqrt{k^2 + k'^2 + 2(4H^2 - 1)kk'}}{2} \quad (3.118)$$

so we can diagonalize the metric in the $\langle \mathcal{J}_1, \mathcal{J}_2, \hat{\mathcal{J}}, \hat{\hat{\mathcal{J}}} \rangle$ basis ($\hat{\mathcal{J}}$ and $\hat{\hat{\mathcal{J}}}$ being the two eigenvectors) and finally obtain:

$$g = \begin{pmatrix} k & & & \\ & k & & \\ & & \lambda_1(k, k', H) & \\ & & & \lambda_2(k, k', H) \end{pmatrix}. \quad (3.119)$$

Of course, in the $H^2 \rightarrow 0$ limit we get the initial WZW model and in the $H^2 \rightarrow 1/2$ limit we recover the asymmetrically gauged model, Eq. (3.115).

It is important to remark that the construction above can be directly generalized to higher groups with non-abelian subgroups, at least for the asymmetric coset part. This is what we will further analyse in the next chapter.

CHAPTER 4

Applications

In this chapter we present some of the applications for the construction outlined above. After an analysis of the most simple (compact and non-compact) examples, we describe the near-horizon geometry for the Bertotti-Robinson black hole, show some new compactifications and see how Horne and Horowitz's black string can be described in this framework and generalized via the introduction of an electric field.

THE TECHNOLOGY we developed in the previous chapter allows for the construction of a large class of exact string theory backgrounds which is one of the main motivations of the present work. This chapter is devoted to the study of some of the most interesting among them. They can be used to provide new CFT models with clear geometric interpretation (Sec. 4.1 and 4.2), to describe near-horizon geometries of four-dimensional black holes (Sec. 4.3), as laboratories for the study of black holes and black strings (Sec. 4.4) or to provide new physically realistic compactification backgrounds (Sec. 4.5).

4.1 The two-sphere CFT

Spacetime fields

The first deformation that we explicitly consider is the marginal deformation of the $SU(2)$ WZW model. This was first obtained in [KK95] that we will closely follow. It is anyway worth to stress that in their analysis the authors didn't study the point of maximal deformation (which was nevertheless identified as a decompactification boundary) that we will here show to correspond to the 2-sphere $S^2 \sim SU(2)/U(1)$. Exact CFT's on this background have already obtained in [BK94] and in [Joh95]. In particular the technique used in the latter, namely the asymmetric gauging of an $SU(2) \times U(1)$ WZW model, bears many resemblances to the one we will describe.

Consider a heterotic string background containing the $SU(2)$ group manifold, times some $(1,0)$ superconformal field theory \mathcal{M} . The sigma model action is:

$$S = kS_{SU(2)}(g) + \frac{1}{2\pi} \int d^2z \left\{ \sum_{a=1}^3 \lambda^a \bar{\partial} \lambda^a + \sum_{n=1}^g \tilde{\chi}^n \partial \tilde{\chi}^n \right\} + S(\mathcal{M}), \quad (4.1)$$

where λ^i are the left-moving free fermions superpartners of the bosonic $SU(2)$ currents, $\tilde{\chi}^n$ are the right-moving fermions of the current algebra and $kS_{SU(2)}(g)$ is the WZW action for the bosonic $SU(2)$ at level k . This theory possesses an explicit $SU(2)_L \times SU(2)_R$ current algebra.

Gauss
decomposition
for $SU(2)$

A parametrization of the $SU(2)$ group that is particularly well suited for our purposes is obtained via the so-called Gauss decomposition that we will later generalize to higher groups (see App. B). A general element $g(z, \psi) \in SU(2)$ where $z \in \mathbb{C}$ and $\psi \in \mathbb{R}$ can be written as:

$$g(z, \psi) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{f} & 0 \\ 0 & \sqrt{f} \end{pmatrix} \begin{pmatrix} 1 & \bar{w} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \quad (4.2)$$

where $w = -z$ and $f = 1 + |z|^2$. In this parametrisation the matrix of invariant one-forms $\Omega = g(z, \psi)^{-1} dg(z, \psi)$ which is projected on the Lie algebra generators to give the expression for the Maurer-Cartan one-forms is:

$$\Omega = \frac{1}{f} \begin{pmatrix} \bar{z}dz - zd\bar{z} & -e^{-i\psi}d\bar{z} \\ e^{i\psi}dz & -\bar{z}dz + zd\bar{z} \end{pmatrix} + i \begin{pmatrix} d\psi & 0 \\ 0 & -d\psi \end{pmatrix} \quad (4.3)$$

(remark that Ω is traceless and anti-Hermitian since it lives in $\mathfrak{su}(2)$). From Ω we can easily derive the Cartan-Killing metric on $T_g SU(2)_k$ as:

$$\begin{aligned} \frac{2}{k} ds^2 = \langle \Omega^\dagger \Omega \rangle &= -\frac{1}{2f^2} \left(\bar{z}^2 dz^2 + z^2 d\bar{z}^2 - 2(2 + |z|^2) dzd\bar{z} \right) + \\ &+ \frac{i}{f} (zd\bar{z} - \bar{z}dz) d\psi + \frac{1}{2} d\psi^2 \quad (4.4) \end{aligned}$$

The left-moving current contains a contribution from the free fermions realizing an $SU(2)_2$ algebra, so that the theory possesses (local) $N = (1, 0)$ superconformal symmetry.

The marginal deformation is obtained by switching on a magnetic field in the $SU(2)$, introducing the following $(1, 0)$ -superconformal-symmetry-compatible marginal operator:

$$\delta S = \frac{\sqrt{k k_g} \mathbb{H}}{2\pi} (J^3 + \lambda^+ \lambda^-) \bar{J} \quad (4.5)$$

where we have picked one particular current \bar{J} from the gauge sector, generating a $U(1)$ at level k_g . For instance, we can choose the level-two current: $\bar{J} = i\tilde{\chi}^1 \tilde{\chi}^2$. As a result the solutions to the deformed σ -model (3.26), (3.33) and

(3.34) read¹:

$$\begin{cases} \frac{1}{k} ds^2 = \frac{dzd\bar{z}}{(1+|z|^2)^2} + (1-2H^2) \left(\frac{izd\bar{z} - i\bar{z}dz}{f} + d\psi \right)^2 \\ dB = \frac{ik}{2} \frac{1}{(1+|z|^2)^2} dz \wedge d\bar{z} \wedge d\psi \\ A = \sqrt{\frac{k}{2k_g}} H \left(-\frac{i}{f} (\bar{z}dz - zd\bar{z}) + d\psi \right). \end{cases} \quad (4.7)$$

It can be useful to write explicitly the volume form on the manifold and the Ricci scalar:

$$\sqrt{\det g} dz \wedge d\bar{z} \wedge d\psi = \frac{k}{2} \frac{\sqrt{k(1-2H^2)}}{(1+|z|^2)^2} dz \wedge d\bar{z} \wedge d\psi \quad (4.8)$$

$$R = \frac{6+4H^2}{k} \quad (4.9)$$

It is quite clear that $H = H_{\max} = 1/\sqrt{2}$ is a special point. In general the three-sphere $SU(2)$ can be seen as a non-trivial fibration of $U(1) \sim S^1$ as fiber and $SU(2)/U(1) \sim S^2$ as base space: the parametrization in (4.7) makes it clear that the effect of the deformation consists in changing the radius of the fiber that naively seems to vanish at H_{\max} . But as we already know the story is a bit different: reparametrizing as in Eq. (3.30):

$$\psi \rightarrow \frac{\hat{\psi}}{\sqrt{1-2H^2}} \quad (4.10)$$

one is free to take the $H \rightarrow 1/\sqrt{2}$ limit where the background fields assume the following expressions:

Kähler structure
on $\mathbb{C}P^1$ model

$$\begin{cases} \frac{1}{k} ds^2 \xrightarrow{H \rightarrow 1/\sqrt{2}} \frac{dzd\bar{z}}{(1+|z|^2)^2} + d\hat{\psi}^2 \\ F \xrightarrow{H \rightarrow 1/\sqrt{2}} \sqrt{\frac{k}{4k_g}} \frac{izd\bar{z} - i\bar{z}dz}{(1+|z|^2)^2} \\ H \xrightarrow{H \rightarrow 1/\sqrt{2}} 0 \end{cases} \quad (4.11)$$

Now we can justify our choice of coordinates: the (z, \bar{z}) part of the metric that decouples from the ψ part is nothing else than the Kähler metric for the manifold $\mathbb{C}P^1$ (which is isomorphic to $SU(2)/U(1)$). In these terms the field strength F is proportional to the Kähler two-form:

$$F = i \sqrt{\frac{k}{k_g}} g_{z\bar{z}} dz \wedge d\bar{z}. \quad (4.12)$$

¹This type of structure is common to $U(1)$ fibrations over Kähler spaces. In example, the line element for S^5 which can be seen as a $U(1)$ fiber over $\mathbb{C}P^2$ is written as

$$ds^2 = ds^2(\mathbb{C}P^2) + (d\psi + A)^2 \quad (4.6)$$

where dA is the Kähler form on $\mathbb{C}P^2$. We will encounter the same structure again in Sec. 4.5 for $SU(3)$ written as the (principal) fibration $U(1)^2 \rightarrow SU(3)/U(1)^2$.

This begs for a remark. It is simple to show that cosets of the form G/H where H is the maximal torus of G can always be endowed with a Kähler structure. The natural hope is then for this structure to pop up out of our deformations, thus automatically assuring the $N = 2$ world-sheet supersymmetry of the model. Actually this is not the case. The Kähler structure is just one of the possible left-invariant metrics that can be defined on a non-symmetric coset (see Sec. 3.3) and the natural generalization of the deformation considered above leads to \mathbb{C} -structures that are not Kähler. From this point of view this first example is an exception because $SU(2)/U(1)$ is a symmetric coset since $U(1)$ is not only the maximal torus in $SU(2)$ but also the maximal subgroup. It is nonetheless possible to define an exact CFT on flag spaces but this requires a slightly different construction, already outlined in Sec. 3.6.

We conclude this section observing that the flux of the gauge field on the two-sphere is given by:

$$\mathcal{Q} = \int_{S^2} F = \sqrt{\frac{k}{k_g}} \int d\Omega_2 = \sqrt{\frac{k}{k_g}} 4\pi \quad (4.13)$$

However one can argue on general grounds that this flux has to be quantized, *e.g.* because the two-sphere appears as a factor of the magnetic monopole solution in string theory [KLL99]. This quantization of the magnetic charge is only compatible with levels of the affine $SU(2)$ algebra satisfying the condition:

$$\frac{k}{k_g} = p^2, \quad p \in \mathbb{Z}. \quad (4.14)$$

4.2 $SL(2, \mathbb{R})$

Anti-de Sitter space in three dimensions is the (universal covering of the) $SL(2, \mathbb{R})$ group manifold. It provides therefore an exact string vacuum with NS background, described in terms of the $SL(2, \mathbb{R})_k$ WZW model, where time is embedded in the non-trivial geometry. We will consider it as part of some heterotic string solution such as $AdS_3 \times S^3 \times T^4$ with NS three-form field in $AdS_3 \times S^3$ (near-horizon NS5/F1 background). The specific choice of a background is however of limited importance for our purpose.

The issue of AdS_3 deformations has been raised in several circumstances. It is richer² than the corresponding S^3 owing to the presence of elliptic, hyperbolic or parabolic elements in $SL(2, \mathbb{R})$. The corresponding generators are time-like, space-like or light-like. Similarly, the residual symmetry of a deformed AdS_3 has $U(1)$ factors, which act in time, space or light direction.

$SL(2, \mathbb{R})$ marginal
symmetric
deformations

Marginal *symmetric* deformations of the $SL(2, \mathbb{R})_k$ WZW are driven by bilinears $J\bar{J}$ where both currents are in $SL(2, \mathbb{R})$ and are of the same kind [F94, IKP03]. These break the $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ affine symmetry to $U(1)_L \times U(1)_R$ and allow to reach, at extreme values of the deformation, gauged $SL(2, \mathbb{R})_k/U(1)$

²As we have already stressed before the Cartan subgroups are not conjugated by inner automorphisms if the group is not simple.

WZW models with an extra free decoupled boson. We can summarize the results as follows:

- (a) $J^3 \bar{J}^3$ These are time-like currents (for conventions see App. B) and the corresponding deformations connect $SL(2, \mathbb{R})_k$ with $U(1) \times SL(2, \mathbb{R})_k / U(1)|_{\text{axial or vector}}$. The $U(1)$ factor stands for a decoupled, non-compact time-like free boson³. The gauged WZW model $SL(2, \mathbb{R})_k / U(1)|_{\text{axial}}$ is the *cigar* (two-dimensional Euclidean black hole) obtained by gauging the $g \rightarrow hgh$ symmetry with the $h = \exp i \frac{\lambda}{2} \sigma^2$ subgroup, whereas $SL(2, \mathbb{R})_k / U(1)|_{\text{vector}}$ corresponds to the $g \rightarrow hgh^{-1}$ gauging. This is the *trumpet* and is T-dual to the *cigar*⁴. The generators of the affine residual symmetry $U(1)_L \times U(1)_R$ are both time-like (the corresponding Killing vectors are not orthogonal though). For extreme deformation, the time coordinate decouples and the antisymmetric tensor is traded for a dilaton. The isometries are time-translation invariance and rotation invariance in the *cigar*/*trumpet*.
- (b) $J^2 \bar{J}^2$ The deformation is now induced by space-like currents. So is the residual affine symmetry $U(1)_L \times U(1)_R$ of the deformed model. Extreme deformation points are T-dual: $U(1) \times SL(2, \mathbb{R})_k / U(1)$ where the $U(1)$ factor is space-like, and the $U(1)$ gauging of $SL(2, \mathbb{R})_k$ corresponds to $g \rightarrow hgh^{(-1)}$ with $h = \exp -\frac{\lambda}{2} \sigma^3$ [DVV92]. The corresponding manifold is (some sector of) the Lorentzian two-dimensional black hole with a non-trivial dilaton.
- (c) $(J^1 + J^3)(\bar{J}^1 + \bar{J}^3)$ This is the last alternative, with both null currents. The deformation connects AdS_3 with $\mathbb{R} \times \mathbb{R}^{1,1}$ plus a dilaton linear in the first factor. The $U(1)_L \times U(1)_R$ left-over current algebra is light-like⁵. Tensorized with an $SU(2)_k$ CFT, this background describes the decoupling limit of the NS5/F1 setup [IKP03], where the fundamental strings regularize the strong coupling regime.

Possible choices for the coordinate systems and the resulting fields are reported in App. C.

Our purpose here is to analyze *asymmetric* deformations of AdS_3 . Following the similar analysis of the previous section for $SU(2)$, we expect those deformations to preserve a $U(1)_L \times SL(2, \mathbb{R})_R$ symmetry appearing as affine algebra from the sigma-model point of view, and as isometry group for the background. The residual $U(1)_L$ factor can be time-like, space-like or null depending on the current that has been used to perturb the WZW model.

It is worth to stress that some deformations of AdS_3 have been studied in the past irrespectively of any conformal sigma-model or string theory anal-

³The extra bosons are always non-compact.

⁴Actually this statement holds only for the vector coset of the *single cover* of $SL(2, \mathbb{R})$. Otherwise, from the n -th cover of the group manifold one obtains the n -th cover of the trumpet [IKP03].

⁵The isometry is actually richer by one (two translations plus a boost), but the extra generator (the boost) is not promoted to an affine symmetry of the sigma-model.

ysis. In particular it was observed in [RS98], following [RT83] that the three-dimensional⁶ Gödel solution of Einstein equations could be obtained as a member of a one-parameter family of AdS₃ deformations that precisely enters the class we discuss here. Gödel space is a constant-curvature Lorentzian manifold. Its isometry group is $U(1) \times SL(2, \mathbb{R})$, and the $U(1)$ factor is generated by a time-like Killing vector. These properties hold for generic values of the deformation parameter. In fact the deformed AdS₃ under consideration can be embedded in a seven-dimensional flat space with appropriate signature, as the intersection of four quadratic surfaces. Closed time-like curves as well as high symmetry are inherited from the multi-time maximally symmetric host space. Another interesting property resulting from this embedding is the possibility for changing the sign of the curvature along the continuous line of deformation, without encountering any singular behaviour (see Eq. (4.16)).

It seems natural to generalize the above results to *new* AdS₃ deformations and promote them to *exact* string backgrounds. Our guideline will be the requirement of a $U(1) \times SL(2, \mathbb{R})$ isometry group, with space-like or light-like $U(1)$'s.

We will first review the time-like (elliptic) deformation of AdS₃ of [RS98] and recently studied from a string perspective in [Isr04]. Hyperbolic (space-like) and parabolic (light-like) deformations will be analyzed in the following. All these deformations are of the type presented in the previous chapter; further generalizations will be obtained in Sec. 4.4. We show in the following how to implement these deformations as exact marginal perturbations in the framework of the $SL(2, \mathbb{R})_k$ WZW model embedded in heterotic string.

Elliptic deformation: magnetic background

Consider AdS₃ in (t, ρ, ϕ) coordinates, with metric given in (B.20). In these coordinates, two manifest Killing vectors are $L_3 \sim \partial_t$ and $R_2 \sim \partial_\phi$, time-like and space-like respectively (see App. B, Tab. B.2).

The deformation studied in [RS98] and quoted as “squashed anti de Sitter” reads, in the above coordinates:

$$ds^2 = \frac{L^2}{4} \left[d\rho^2 + \cosh^2 \rho d\phi^2 - (1 + 2H^2) (dt + \sinh \rho d\phi)^2 \right]. \quad (4.15)$$

It preserves a $U(1) \times SL(2, \mathbb{R})$ isometry group. The $U(1)$ is generated by the *time-like* vector L_3 of one original $SL(2, \mathbb{R})$, while the right-moving $SL(2, \mathbb{R})$ is unbroken (the expressions for the $\{L_3, R_1, R_2, R_3\}$ Killing vectors in Tab. B.2 remain valid at any value of the deformation parameter). The Ricci scalar is constant

$$R = -\frac{2}{L^2} (3 - 2H^2), \quad (4.16)$$

⁶In fact, the original Gödel solution is four-dimensional, but the fourth space dimension is a flat spectator. In the following, we will systematically refer to the three-dimensional non-trivial factor.

while the volume form reads:

$$\omega_{[3]} = \frac{L^3}{8} \sqrt{|1 + 2H^2|} \cosh \rho \, d\rho \wedge d\phi \wedge dt. \quad (4.17)$$

For $H^2 = 1/2$, this deformation coincides with the Gödel metric. It should be stressed, however, that nothing special occurs at this value of the deformation parameter. The properties of Gödel space are generically reproduced at any $H^2 > 0$.

From a physical point of view, as it stands, this solution is pathological because it has topologically trivial closed time-like curves through each point of the manifold, like Gödel space-time which belongs to this family. Its interest mostly relies on the fact that it can be promoted to an exact string solution, with appropriate NS and magnetic backgrounds. The high symmetry of (4.15), is a severe constraint and, as was shown in [Isr04], the geometry at hand does indeed coincide with the unique marginal deformation of the $SL(2, \mathbb{R})_k$ WZW that preserves a $U(1)_L \times SL(2, \mathbb{R})_R$ affine algebra with time-like $U(1)_L$.

It is interesting to observe that, at this stage, the deformation parameter H^2 *needs not be positive*: (4.15) solves the Einstein-Maxwell-scalar equations [RT83] for any H^2 . Furthermore, for $H^2 < 0$, there are no longer closed time-like curves⁷. This statement is based on a simple argument⁸. Consider a time-like curve $x^\mu = x^\mu(\lambda)$. By definition the tangent vector ∂_λ is negative-norm, which, by using Eq. (4.15), translates into

$$\left(\frac{d\rho}{d\lambda}\right)^2 + \cosh^2 \rho \left(\frac{d\phi}{d\lambda}\right)^2 - (1 + 2H^2) \left(\frac{dt}{d\lambda} + \sinh \rho \frac{d\phi}{d\lambda}\right)^2 < 0. \quad (4.18)$$

If the curve is closed, $dt/d\lambda$ must vanish somewhere. At the turning point, the resulting inequality,

$$\left(2H^2 \sinh^2 \rho - 1\right) \left(\frac{d\phi}{d\rho}\right)^2 > 1 \quad (4.19)$$

is never satisfied for $H^2 < 0$, whereas it is for large enough⁹ ρ otherwise.

This apparent *regularization of the causal pathology*, unfortunately breaks down at the string level. In fact, as we will shortly see, in order to be considered as a string solution, the above background requires a (chromo)magnetic field. The latter turns out to be proportional to H , and becomes *imaginary* in the range where the closed time-like curves disappear. Hence, at the string level, unitarity is trade for causality. It seems that no regime exists in the magnetic

Closed time-like curves in the $SL(2, \mathbb{R})$ elliptic deformation

⁷As mentioned previously, the geometry at hand can be embedded in a seven-dimensional flat space, with signature $\epsilon - - - + + +$, $\epsilon = \text{sign}(-H^2)$ [RS98]. This clarifies the origin of the symmetry as well as the presence or absence of closed time-like curves for positive or negative H^2 .

⁸This argument is local and must in fact be completed by global considerations on the manifold (see [RS98]).

⁹This means $\rho > \rho_c$ where ρ_c is the radius where the norm of ∂_ϕ vanishes and switches to negative ($\|\partial_\phi\|^2 = L^2(1 - 2H^2 \sinh^2 \rho)/4$). This never occurs for $H^2 < 0$.

deformation of AdS_3 , where these fundamental requirements are simultaneously fulfilled.

We now turn to the string realization of the above squashed sphere. In the heterotic backgrounds considered here, of the type $\text{AdS}_3 \times S^3 \times T^4$, the two-dimensional $N = (1, 0)$ world-sheet action corresponding to the AdS_3 factor is:

$$S_{SL(2, \mathbb{R})_k} = \frac{1}{2\pi} \int d^2z \left\{ \frac{k}{4} (\partial\rho\bar{\partial}\rho - \partial t\bar{\partial}t + \partial\phi\bar{\partial}\phi - 2\sinh\rho\partial\phi\bar{\partial}t) + \eta_{ab}\psi^a\bar{\partial}\psi^b \right\}, \quad (4.20)$$

where $\eta_{ab} = \text{diag}(+-)$, $a = 1, 2, 3$ and ψ^a are the left-moving superpartners of the $SL(2, \mathbb{R})_k$ currents (see Tab. B.2). The corresponding background fields are the metric (Eq. (B.20)) with radius $L = \sqrt{k}$ and the NS B-field:

$$B = -\frac{k}{4} \sinh\rho d\phi \wedge dt. \quad (4.21)$$

The three-form field strength is $H_{[3]} = dB = -\frac{2}{\sqrt{k}}\omega_{[3]}$ with $\omega_{[3]}$ displayed in Eq. (B.21).

The asymmetric perturbation that preserves a $U(1)_L \times SL(2, \mathbb{R})_R$ affine algebra with time-like $U(1)_L$ is δS given in Eq. (4.5), where J^3 now stands for the left-moving time-like $SL(2, \mathbb{R})_k$ current given in App. B.2, Tab. B.2. This perturbation corresponds to switching on a (chromo)magnetic field, like in the $SU(2)_k$ studied in Sec. 4.1. It is marginal and can be integrated for finite values of H , and is compatible with the $N = (1, 0)$ world-sheet supersymmetry. The resulting background fields, extracted in the usual manner from the deformed action are the metric (4.15) with radius $L = \sqrt{k}$ and the following gauge field:

$$A = H\sqrt{\frac{2k}{k_g}}(dt + \sinh\rho d\phi). \quad (4.22)$$

The NS B-field is not altered by the deformation, (Eq. (4.21)), whereas the three-form field strength depends explicitly on the deformation parameter H , because of the gauge-field contribution:

$$H_{[3]} = dB - \frac{k_G}{4} A \wedge dA = -\frac{k}{4}(1 + 2H^2)\cosh\rho d\rho \wedge d\phi \wedge dt. \quad (4.23)$$

One can easily check that the background fields (4.15), (4.22) and (4.23) solve the lowest-order equations of motion. Of course the solution we have obtained is exact, since it has been obtained as the marginal deformation of an exact conformal sigma-model. The interpretation of the deformed model in terms of background fields $\{G_{ab}, B_{ab}, F_{ab}^G\}$ receives however the usual higher-order correction summarized by the shift $k \rightarrow k + 2$ as we have already explained in Sec. 3.4.

Let us finally mention that it is possible to extract the spectrum and write down the partition function of the above theory [Isr04], since the latter is an

exact deformation of the $SL(2, \mathbb{R})_k$ WZW model. This is achieved by deforming the associated elliptic Cartan subalgebra. The following picture emerges then from the analysis of the spectrum. The short-string spectrum, corresponding to world-sheets trapped in the “center” of the space–time (for some particular choice of coordinates) is well-behaved, because these world-sheets do not feel the closed time-like curves which are “topologically large”. On the contrary, the long strings can wrap the closed time-like curves, and their spectrum contains many tachyons. Hence, the caveats of Gödel space survive the string framework, at any value of $H^2 > 0$. One can circumvent them by slightly deviating from the Gödel line with an extra purely gravitational deformation, driven by $J^3 \bar{J}^3$. This deformation isolates the causally unsafe region, $\rho > \rho_c$ (see [Isr04] for details). It is similar in spirit with the supertube domain-walls of [DFS03] curing the Gödel-like space-times with RR backgrounds.

Hyperbolic deformation: electric background

The background and its CFT realization

We will now focus on a different deformation. We use coordinates (B.22) with metric (B.23), where the manifest Killing vectors are $L_2 \sim \partial_x$ (space-like) and $R_3 \sim \partial_\tau$ (time-like) (see App. B.2, Tab. B.3). This time we perform a deformation that preserves a $U(1) \times SL(2, \mathbb{R})$ isometry. The $U(1)$ corresponds to the space-like Killing vector L_2 , whereas the $SL(2, \mathbb{R})$ is generated by R_1, R_2, R_3 , which are again not altered by the deformation. The resulting metric reads:

$$ds^2 = \frac{L^2}{4} \left[dr^2 - \cosh^2 r d\tau^2 + (1 - 2H^2) (dx + \sinh r d\tau)^2 \right]. \quad (4.24)$$

The scalar curvature of this manifold is constant

$$R = -\frac{2}{L^2} (3 + 2H^2) \quad (4.25)$$

and the volume form

$$\omega_{[3]} = \frac{L^3}{8} \sqrt{|1 - 2H^2|} \cosh^2 r dr \wedge d\tau \wedge dx. \quad (4.26)$$

Following the argument of the previous section, one can check whether closed time-like curves appear. Indeed, assuming their existence, the following inequality must hold at the turning point *i.e.* where $dt/d\lambda$ vanishes (λ being the parameter that describes the curve):

$$(2H^2 - 1) \left(\frac{dx}{dr} \right)^2 > 1. \quad (4.27)$$

The latter cannot be satisfied in the regime $H^2 < 1/2$. Notice that the manifold at hand is well behaved, even for negative H^2 .

Let us now leave aside these questions about the classical geometry, and address the issue of string realization of the above background. As already advertised, this is achieved by considering a world-sheet-supersymmetric marginal

deformation of the $SL(2, \mathbb{R})_k$ WZW model that implements (chromo)electric field. Such a deformation is possible in the heterotic string at hand:

$$\delta S = \frac{\sqrt{k k_G H}}{2\pi} \int d^2 z \left(J^2 + i\psi^1 \psi^3 \right) \bar{J}_G, \quad (4.28)$$

(\bar{J}_G is any Cartan current of the group G and J^2 is given in App. B.2, Tab. B.3), and corresponds, as in previous cases, to an integrable marginal deformation. The deformed conformal sigma-model can be analyzed in terms of background fields. The metric turns out to be (4.24), whereas the gauge field and three-form tensor are

$$A = H \sqrt{\frac{2k}{k_g}} (dx + \sinh r d\tau), \quad (4.29)$$

$$H_{[3]} = \frac{k}{4} (1 - 2H^2) \cosh r dr \wedge d\tau \wedge dx. \quad (4.30)$$

As expected, these fields solve the equations of motion.

The background under consideration is a new string solution generated as a hyperbolic deformation of the $SL(2, \mathbb{R})_k$ WZW model. In contrast to what happens for the elliptic deformation above, the present solution is perfectly sensible, both at the classical and at the string level.

The spectrum of string primaries

String primaries
for $SL(2, \mathbb{R})$
hyperbolic
deformation

The electric deformation of AdS_3 is an exact string background. The corresponding conformal field theory is however more difficult to deal with than the one for the elliptic deformation. In order to write down its partition function, we must decompose the $SL(2, \mathbb{R})_k$ partition function in a hyperbolic basis of characters, where the implementation of the deformation is well-defined and straightforward; this is a notoriously difficult exercise. On the other hand the spectrum of primaries is known¹⁰ from the study of the representations of the Lie algebra in this basis (see *e.g.* [VK91], and [DVV92] for the spectrum of the hyperbolic gauged WZW model, *i.e.* at the extreme value of the deformation parameter). The part of the heterotic spectrum of interest contains the expression for the primaries of $N = (1, 0)$ affine $SL(2, \mathbb{R})$ at purely bosonic level¹¹ $k + 2$, together with some $U(1)$ from the lattice of the heterotic gauge group:

$$L_0 = -\frac{j(j-1)}{k} - \frac{1}{2} \left(n + \frac{a}{2} \right)^2, \quad (4.31)$$

$$\bar{L}_0 = -\frac{j(j-1)}{k} + \frac{1}{2} \left(\bar{n} + \frac{h}{2} \right)^2, \quad (4.32)$$

¹⁰In the following we do not consider the issue of the spectral-flow representations. The spectral-flow symmetry is apparently broken by the deformation considered here.

¹¹More precisely we consider primaries of the purely bosonic affine algebra with an arbitrary state in the fermionic sector.

where the second Casimir of the representation of the $SL(2, \mathbb{R})$ algebra, $-j(j-1)$, explicitly appears. The spectrum contains *continuous representations*, with $j = \frac{1}{2} + is$, $s \in \mathbb{R}_+$. It also contains *discrete representations*, with $j \in \mathbb{R}_+$, lying within the unitarity range $1/2 < j < (k+1)/2$ (see [MO01, Pet90]). In both cases the spectrum of the hyperbolic generator J^2 is $\mu \in \mathbb{R}$. The expression for the left conformal dimensions, Eq. (4.31), also contains the contribution from the world-sheet fermions associated to the $\iota\psi^1\psi^3$ current. The sector (R or NS) is labelled by $a \in \mathbb{Z}_2$. Note that the unusual sign in front of the lattice is the natural one for the fermions of the light-cone directions. In the expression (4.32) we have similarly the contribution of the fermions of the gauge group, where h labels the corresponding sector.

We are now in position to follow the procedure, familiar from the previous examples: we have to (i) isolate from the left spectrum the lattice of the supersymmetric hyperbolic current $J^2 + \iota\psi^1\psi^3$ and (ii) perform a boost between this lattice and the fermionic lattice of the gauge field. We hence obtain the following expressions:

$$L_0 = -\frac{j(j-1)}{k} - \frac{\mu^2}{k+2} - \frac{k+2}{2k} \left(n + \frac{a}{2} + \frac{2\mu}{k+2} \right)^2 + \frac{1}{2} \left[\sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \left(\bar{n} + \frac{h}{2} \right) \sinh x \right]^2, \quad (4.33a)$$

$$\bar{L}_0 = -\frac{j(j-1)}{k} + \frac{1}{2} \left[\left(\bar{n} + \frac{h}{2} \right) \cosh x + \sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x \right]^2. \quad (4.33b)$$

The relation between the boost parameter x and the deformation parameter H^2 is given in Eq. (3.62), as for the case of the $SU(2)_k$ deformation. In particular it is worth to remark that the first three terms of (4.33a) correspond to the left weights of the supersymmetric two-dimensional Lorentzian black hole, *i.e.* the $SL(2, \mathbb{R})/O(1, 1)$ gauged super-WZW model.

This result is less striking that the whole partition function we obtained for the compact $SU(2)$. It is worthwhile to remark that the difference is only due to technical reasons: in principle the very same construction could be applied for the case at hand but it would require the decomposition of the $SL(2, \mathbb{R})$ partition function in terms of hyperbolic characters that at present is not yet known.

Parabolic deformation: the AdS-wave background

In the deformations encountered in the previous sections one $SL(2, \mathbb{R})$ isometry breaks down to a $U(1)$ generated either by a time-like or by a space-like Killing vector. Deformations which preserve a light-like isometry do also exist and are easily implemented in Poincaré coordinates.

We require that the isometry group is $U(1) \times SL(2, \mathbb{R})$ with a null Killing vector for the $U(1)$ factor. Following the by now familiar for the particular

case of light-like residual isometry, we are lead to

$$ds^2 = L^2 \left[\frac{du^2}{u^2} + \frac{dx^+ dx^-}{u^2} - 2H^2 \left(\frac{dx^+}{u^2} \right)^2 \right]. \quad (4.34)$$

The light-like $U(1)$ Killing vector is $L_1 + L_3 \sim \partial_-$ (see App. B.2, Tab. B.4). The remaining $SL(2, \mathbb{R})$ generators are $\{R_1 + R_3, R_1 - R_3, R_2\}$ and remain unaltered after the deformation.

The above deformed anti-de-Sitter geometry looks like a superposition of AdS_3 and of a plane wave (whence the AdS-wave name). As usual, the sign of H^2 is free at this stage, and $H^2 < 0$ are equally good geometries. In the near-horizon region ($|u| \gg |H^2|$) the geometry is not sensitive to the presence of the wave. On the contrary, this plane wave dominates in the opposite limit, near the conformal boundary.

The volume form is not affected by the deformation, and it is still given in (B.27); neither is the Ricci scalar modified:

$$R = -\frac{6}{L^2}. \quad (4.35)$$

Parabolic discrete deformation

Notice also that the actual value of $|H|$ is not of physical significance: it can always be absorbed into a reparametrization $x^+ \rightarrow x^+ / |H|$ and $x^- \rightarrow x^- |H|$. The only relevant values for H^2 can therefore be chosen to be $0, \pm 1$.

We now come to the implementation of the geometry (4.34) in a string background. The only option is to perform an asymmetric exactly marginal deformation of the heterotic $SL(2, \mathbb{R})_k$ WZW model that preserves a $U(1)_L \times SL(2, \mathbb{R})_R$ affine symmetry. This is achieved by introducing

$$\delta S_{\text{electric-magnetic}} = -4\sqrt{kk_G H} \int d^2z \left(J^1 + J^3 + i \left(\psi^1 + \psi^3 \right) \psi^2 \right) \bar{J}_G, \quad (4.36)$$

($J^1 + J^3$ is defined in App. B.2, Tab. B.4). The latter perturbation is integrable and accounts for the creation of a (chromo)electromagnetic field

$$A = 2\sqrt{\frac{2k}{k_G} H} \frac{dx^+}{u^2}. \quad (4.37)$$

It generates precisely the deformation (4.34) and leaves unperturbed the NS field, $H_{[3]} = dB = -\frac{2}{\sqrt{k}} \omega_{[3]}$.

As a conclusion, the AdS_3 plus plane-wave gravitational background is described in terms of an exact conformal sigma model, that carries two extra background fields: an NS three-form and an electromagnetic two-form. Similarly to the symmetric parabolic deformation [IKP03], the present asymmetric one can be used to construct a space-time supersymmetric background. The $SL(2, \mathbb{R})_k$ -CFT treatment of the latter deformation would need the knowledge of the parabolic characters of the affine algebra, not available at present.

Quantum point particles in the AdS-wave background

Further insights of the physics of the AdS-wave background can be gathered if we look at the motion of point particles. Let us start with the sigma model Lagrangian where we keep the H parameter explicitly for sake of consistency:

$$S = \int dz^2 \frac{1}{u^2} \partial u \bar{\partial} u + \frac{1}{u^2} \partial x^+ \bar{\partial} x^- - 2 \frac{H}{u^2} \partial x^- \bar{\partial} \varphi + \frac{1}{2} \partial \varphi \bar{\partial} \varphi, \quad (4.38)$$

where all the fields are function of σ and τ . The point particle limit can be obtained if we let the σ -dependence drop. This leads to:

$$S_{\text{point}} = \frac{1}{2} \int d\tau \left\{ \frac{1}{u^2} \dot{u}^2 + \frac{1}{u^2} \dot{x}^- \dot{x}^+ + \frac{1}{2} \dot{\varphi}^2 - 2 \frac{H}{u^2} \dot{x}^- \dot{\varphi} \right\}, \quad (4.39)$$

where the dot stands for the time derivative. The fourth dimension φ was introduced as a fake direction along which perform a Kaluza–Klein reduction. In this framework the same result is obtained if we consider φ as an auxiliary variable and then substitute its equation of motion:

Kaluza-Klein
reduction as a
partial integration

$$\dot{\varphi} = 2 \frac{H \dot{x}^-}{2u^2}. \quad (4.40)$$

The resulting effective action is then written as:

$$S_{\text{point}} = \frac{1}{2} \int d\tau \left\{ \frac{1}{u^2} \dot{u}^2 + \frac{1}{u^2} \dot{x}^- \dot{x}^+ - \frac{2H^2}{u^4} (x^-)^2 \right\} \quad (4.41)$$

that is exactly the action for a free particle in the 3D AdS-wave metric in Eq. (4.34). Now, out of this we can derive the Hamiltonian:

$$H_{\text{point}} = \frac{1}{16} (c^2 p_+^2 + 2u^2 (p_- p_+ + p_u^2)), \quad (4.42)$$

and with the usual rules of quantization this naturally translates to the Laplacian of the AdS-wave geometry:

$$\Delta = \nabla_\mu \nabla^\mu = 8H^2 \partial_+^2 + 4u^2 \partial_+ \partial_- - u \partial_u + u^2 \partial_u^2. \quad (4.43)$$

A quantum point particle described by the wave function $\Psi(u, x^-, x^+)$ must then obey the Klein-Gordon equation:

$$-\Delta \Psi(u, x^-, x^+) = m^2 \Psi(u, x^-, x^+). \quad (4.44)$$

The fact that only the u variable appears explicitly suggests that we can write the solution as:

$$\Psi(u, x^-, x^+) = \int dp_- dp_+ e^{i(p_- x^- + p_+ x^+)} \tilde{\Psi}(u, p_-, p_+) \quad (4.45)$$

so that the wave equation becomes:

$$u^2 \partial_u^2 \tilde{\Psi}(u, p_-, p_+) - u \partial_u \tilde{\Psi}(u, p_-, p_+) = (-m^2 + 8H^2 p_+^2 + 4u^2 p_- p_+) \tilde{\Psi}(u, p_-, p_+).$$

$$(4.46)$$

This is a modified Bessel equation whose canonical form is:

$$z^2 y''(z) + zy'(z) - (z^2 + \nu^2) y(z) = 0 \quad (4.47)$$

and after some algebra we can write the general solution as:

$$\tilde{\Psi}(u, p_-, p_+) = u I_\nu(2\sqrt{p_- p_+} u) C_1(p_-, p_+) + u K_\nu(2\sqrt{p_- p_+} u) C_2(p_-, p_+) \quad (4.48)$$

where $C_1(p_-, p_+)$ and $C_2(p_-, p_+)$ are arbitrary functions, $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions of the first and second kind and $\nu = \sqrt{1 - m^2 + 8(Hp_+)^2}$.

Limiting geometries: AdS₂ and H₂

We have analyzed in Sec. 4.1 the behaviour of the magnetic deformation of $SU(2)_k$, at some critical (or boundary) value of the modulus H^2 , where the background factorizes as $\mathbb{R} \times S^2$ with vanishing NS three-form and finite magnetic field. We would like to address this question for the asymmetric deformations of the $SL(2, \mathbb{R})_k$ model and show the existence of limiting situations where the geometry indeed factorizes, in agreement with the expectations following the general analysis of Sec. 3.1

What can we expect in the framework of the $SL(2, \mathbb{R})_k$ asymmetric deformations? Any limiting geometry must have the generic $U(1) \times SL(2, \mathbb{R})_k$ isometry that translates the affine symmetry of the conformal model. If a line decouples, it accounts for the $U(1)$, and the remaining two-dimensional surface must be $SL(2, \mathbb{R})$ -invariant. Three different situations may arise: AdS₂, H₂ or dS₂. Anti de Sitter in two dimensions is Lorentzian with negative curvature; the hyperbolic plane H₂ (also called Euclidean anti de Sitter) is Euclidean with negative curvature; de Sitter space is Lorentzian with positive curvature.

Three deformations are available for AdS₃ and these have been analyzed in Sec. 4.2. For unitary string theory, all background fields must be real and consequently $H^2 > 0$ is the only physical regime. In this regime, only the hyperbolic (electric) deformation exhibits a critical behaviour at $H_{\max}^2 = 1/2$. For $H^2 < 1/2$, the deformation at hand is a Lorentzian manifold with no closed time-like curves. When $H^2 > 1/2$, $\det \mathbf{g} > 0$ and *two* time-like directions appear. At $H^2 = H_{\max}^2$, $\det \mathbf{g}$ vanishes, and this is the signature that some direction indeed decompactifies.

Decompactifying
to AdS₂

We proceed therefore as in Sec. 3.1, and define a rescaled coordinate in order to keep the decompactifying direction into the geometry and follow its decoupling:

$$y = \sqrt{\frac{k}{2} \left(\frac{1}{2} - H^2 \right)} x. \quad (4.49)$$

The metric and volume form now read:

$$ds^2 = dy^2 + \frac{k}{4} \left[dr^2 - \left(1 + 2H^2 \sinh^2 r \right) d\tau^2 \right] + \sqrt{k(1 - 2H^2)} \sinh r d\tau dy$$

(4.50)

and

$$\omega_{[3]} = \frac{k}{4} \cosh r \, dr \wedge d\tau \wedge dy. \quad (4.51)$$

For H^2 close to H_{\max}^2 , the y -direction factorizes

$$ds^2 \xrightarrow{H^2 \rightarrow H_{\max}^2} dy^2 + \frac{k}{4} \left[dr^2 - \cosh^2 r \, d\tau^2 \right]. \quad (4.52)$$

The latter expression captures the phenomenon we were expecting:

$$\text{AdS}_3 \xrightarrow{H^2 \rightarrow H_{\max}^2} \mathbb{R} \times \text{AdS}_2. \quad (4.53)$$

It also shows that the two-dimensional anti de Sitter has radius $\sqrt{k/4}$ and supports entirely the curvature of the limiting geometry, $R = -8/k$ (see expression (4.25)).

The above analysis shows that, starting from the $SL(2, \mathbb{R})_k$ WZW model, there is a line of continuous exact deformation (driven by a (chromo)electric field) that leads to a conformal model at the boundary of the modulus H^2 . This model consists of a free non-compact boson times a geometric coset $\text{AdS}_2 \equiv SL(2, \mathbb{R})/U(1)$, with a finite electric field:

$$F = \sqrt{\frac{k}{k_G}} \cosh r \, dr \wedge d\tau \quad (4.54)$$

and vanishing NS three-form background. The underlying geometric structure that makes this phenomenon possible is that AdS_3 can be considered as a non-trivial S^1 fibration over an AdS_2 base. The radius of the fiber couples to the electric field, and vanishes at H_{\max}^2 . The important result is that this enables us to promote the geometric coset AdS_2 to an exact string vacuum.

We would like finally to comment on the fate of dS_2 and H_2 geometries, which are both $SL(2, \mathbb{R})$ -symmetric. De Sitter and hyperbolic geometries are not expected to appear in physical regimes of string theory unless Ramond-Ramond fields are turned on (see Ch. 7). The H_3 sigma-model, for example, is an exact conformal field theory, with imaginary antisymmetric tensor background though [Gaw91, Tes99]. Similarly, imaginary NS background is also required for de Sitter vacua to solve the low-energy equations. It makes sense therefore to investigate regimes with $H^2 < 0$, where the electric or magnetic backgrounds are indeed imaginary.

The elliptic (magnetic) deformation exhibits a critical behaviour in the region of negative H^2 , where the geometry does not contain closed time-like curves. The critical behaviour appears at the minimum value $H_{\min}^2 = -1/2$, below which the metric becomes Euclidean. The vanishing of $\det \mathbf{g}$ at this point of the deformation line, signals the decoupling of the time direction. The remaining geometry is nothing but a two-dimensional hyperbolic plane H_2 . It is Euclidean with negative curvature $R = -8/k$ (see Eq. (4.16) with $L^2 = k$).

Non-unitary H_2
solution

All this can be made more precise by introducing a rescaled time coordinate:

$$T = \sqrt{\frac{k}{2} \left(\frac{1}{2} + H^2 \right)} t. \quad (4.55)$$

The metric and volume form now read:

$$ds^2 = -dT^2 + \frac{k}{4} \left[d\rho^2 + \left(1 - 2H^2 \sinh^2 \rho \right) d\phi^2 \right] - \sqrt{k(1+2H^2)} \sinh \rho d\phi dT \quad (4.56)$$

and

$$\omega_{[3]} = \frac{k}{4} \cosh \rho d\rho \wedge d\phi \wedge dT. \quad (4.57)$$

For H^2 close to H_{\min}^2 , the T -direction factorizes

$$ds^2 \xrightarrow{H^2 \rightarrow H_{\max}^2} -dT^2 + \frac{k}{4} \left[d\rho^2 + \cosh^2 \rho d\phi^2 \right]. \quad (4.58)$$

The latter expression proves the above statement:

$$\text{AdS}_3 \xrightarrow{H^2 \rightarrow H_{\min}^2} \mathbb{R} \times H_2, \quad (4.59)$$

and the two-dimensional hyperbolic plane has radius $\sqrt{k/4}$.

Our analysis finally shows that the continuous line of exactly marginal (chromo)magnetic deformation of the $SL(2, \mathbb{R})$ conformal model has a boundary at $H^2 = -1/2$ where its target space is a free time-like coordinate times a hyperbolic plane. The price to pay for crossing $H^2 = 0$ is an imaginary magnetic field, which at $H^2 = -1/2$ reads:

$$F = \sqrt{-\frac{k}{k_G}} \cosh \rho d\phi \wedge d\rho. \quad (4.60)$$

The NS field strength vanishes at this point, and the geometric origin of the decoupling at hand is again the Hopf fibration of the AdS_3 in terms of an H_2 .

The H_2 spectrum

A $H_2 \times \mathbb{R}_t$ limit geometry can be reached if we allow for negative values of H^2 which in turn imply the presence of an imaginary magnetic field. Although this implies that the corresponding string theory is pathological (in example because of unitarity problems), we obtain a perfectly respectable CFT for which we can write, using the same technique as above, a modular-invariant partition function.

The H_2 CFT Let us start from the deformed partition function (see [Isr04]). The interest-

ing part for us is:

$$\begin{aligned} & \int d^2 t \, Z_{\text{cigar}} \begin{bmatrix} -t_1 \\ -t_2 \end{bmatrix} \sum_{N, W, n, \bar{n} \in \mathbb{Z}} e^{i\pi(2Nt_2 + b(n + \frac{a}{2}) - \delta(\bar{n} + \frac{\gamma}{2}))} \\ & \times q^{-\left[\frac{\cos \zeta}{\sqrt{k+2}} \left(\frac{N}{2} + \frac{k+4}{2}(W+t_1) + n + \frac{a}{2} \right) + \frac{\sin \zeta}{\sqrt{2}} \left(\bar{n} + \frac{\gamma}{2} \right) \right]^2 + \frac{k+4}{2(k+2)} \left(n + \frac{a}{2} + (W+t_1) + \frac{N}{k+4} \right)^2} \\ & \times \bar{q}^{-\frac{1}{k+4} \left(\frac{N}{2} - \frac{k+4}{2}(W+t_1) \right)^2 + \left[\frac{\cos \zeta}{\sqrt{2}} \left(\bar{n} + \frac{\gamma}{2} \right) - \frac{\sin \zeta}{\sqrt{k+2}} \left(N + \frac{k+4}{2}(W+t_1) + n + \frac{a}{2} \right) \right]^2} \end{aligned} \quad (4.61)$$

where

$$\cos \zeta = \frac{1}{1 + 2H^2}. \quad (4.62)$$

If we consider $H^2 < 0$ the trigonometric functions became hyperbolic and there is a critical point $H^2 = -1/2$ where the boost diverges. Consistency then imposes the following constraint on the charges:

$$\frac{1}{\sqrt{k+2}} \left(\frac{N}{2} + \frac{k+4}{2}(W+t_1) + n + \frac{a}{2} \right) + \frac{1}{\sqrt{2}} \left(\bar{n} + \frac{\gamma}{2} \right) = 0 \quad (4.63)$$

introducing, for notation convenience

$$k = 2p^2 - 2 \quad (4.64)$$

where $p \in \mathbb{R}$ (there is no reason for quantization), the constraint can be rewritten as

$$N + 2(p^2 + 1)(W+t_1) + 2n + a + p(\bar{n} + 2\gamma) = 0 \quad (4.65)$$

that is equivalent to asking

$$\begin{cases} N + 2n + a = Q \in \mathbb{Z} \\ 2(p^2 + 1)(W+t_1) + p(\bar{n} + 2\gamma) = -Q \end{cases} \quad (4.66)$$

whence we can rewrite t_1 as

$$t_1 = -\frac{Q + p(2\bar{n} + \gamma)}{2(p^2 + 1)} - W \quad (4.67)$$

and Eq. (4.61) becomes:

$$\begin{aligned} & \int dt_2 \sum_{N, Q, W, \bar{n} \in \mathbb{Z}} Z_{\text{cigar}} \begin{bmatrix} \frac{Q+2p(\bar{n}+\gamma/2)}{2(p^2+1)} + W \\ -t_2 \end{bmatrix} e^{i\pi(2Nt_2 + b\frac{Q-N}{2} - \delta(\bar{n}+\gamma/2))} \times \\ & \times q^{\frac{1}{2(p^2+1)} \left(p\frac{Q-N}{2} - (\bar{n}+\gamma/2) \right)^2} \bar{q}^{-\frac{1}{2(p^2+1)} \left(\frac{Q+N}{2} + p(\bar{n}+\gamma/2) \right)^2} \end{aligned} \quad (4.68)$$

or, introducing the integers A, B as:

$$A = Q + N \quad (4.69)$$

$$B = Q - N \quad (4.70)$$

finally can write the H^2 partition function as follows:

$$\int dt_2 \sum_{A,B,W,\bar{n} \in \mathbb{Z}} Z_{\text{cigar}} \left[\begin{array}{c} \frac{A+B+4p(\bar{n}+\gamma/2)}{4(p^2+1)} + W \\ -t_2 \end{array} \right] e^{i\pi((A-B)t_2 + b\frac{B}{2} - \delta(\bar{n}+\gamma/2))} \times \\ \times q^{\frac{1}{2(p^2+1)}(p\frac{B}{2} - (\bar{n}+\gamma/2))^2} \bar{q}^{-\frac{1}{2(p^2+1)}(\frac{A}{2} + p(\bar{n}+\gamma/2))^2}. \quad (4.71)$$

It is intriguing to find that the partition function for the geometric coset $H_2 = \text{AdS}_3/\mathbb{R}$ is related to the one for the adjoint coset $\text{cigar} = \text{AdS}_3/\mathbb{R}$. One may wonder if this hints at some operation allowing to pass from the former to the latter, but we will not speculate further in this direction.

4.3 Near horizon geometry for the Bertotti-Robinson black hole

The $\text{AdS}_2 \times S^2$ geometry appeared first in the context of Reissner–Nordström black holes. The latter are solutions of Maxwell–Einstein theory in four dimensions, describing charged, spherically symmetric black holes. For a black hole of mass M and charge Q , the solution reads:

$$ds^2 = - \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)} + r^2 d\Omega_2^2, \quad (4.72a)$$

$$F = \frac{Q}{r^2} dt \wedge dr \quad \text{with} \quad r_{\pm} = G_4 \left(M \pm \sqrt{M^2 - Q^2} \right); \quad (4.72b)$$

r_+ and r_- are the outer and inner horizons, and G_4 is Newton's constant in four dimensions.

In the extremal case, $r_+ = r_- = r_0$ ($M^2 = Q^2$), and the metric approaches the $\text{AdS}_2 \times S^2$ geometry in the near-horizon¹² limit $r \rightarrow r_0$. This solution can of course be embedded in various four-dimensional compactifications of string theory, and will be supersymmetric in the extremal case (see e.g. [You99] for a review). In this context we are dealing with some heterotic compactification.

Notice that the $\text{AdS}_2 \times S^2$ geometry also appears in type IIB superstring theory, but with RR backgrounds [FKS95]. The black hole solution is obtained by wrapping D3-branes around 3-cycles of a Calabi–Yau three-fold; in the extremal limit, one obtains the $\text{AdS}_2 \times S^2$ solution, but at the same time the CY moduli freeze to some particular values. A hybrid Green–Schwarz sigma-model action for this model has been presented in [BBH⁺00] (see also [Ver04] for AdS_2). The interest for $\text{AdS}_2 \times S^2$ space–time is motivated by the fact that it provides an interesting simplified laboratory for AdS/CFT correspondence [Mal98]. In the present case the dual theory should correspond to some superconformal quantum mechanics [BPS98, C⁺98, GT99, CCKM01].

¹²With the near-horizon coordinates $U = (1 - r_0/r)^{-1}$ and $T = t/r_0$, the near-horizon geometry is

$$ds^2 = r_0^2 \left(-\frac{dT^2}{U^2} + \frac{dU^2}{U^2} + d\Omega_2^2 \right).$$

Both AdS_2 and S^2 factors have the same radius r_0 .

The spectrum

As a first step in the computation of the $\text{AdS}_2 \times S^2$ string spectrum, we must determine the spectrum of the AdS_2 factor, by using the same limiting procedure as in Sec. 4.1 for the sphere. The spectrum of the electrically deformed AdS_3 , is displayed in Eqs. (4.33a) and (4.33b). The AdS_2 limit is reached for $\cosh x \rightarrow \infty$, which leads to the following constraint on the charges of the primary fields:

Boost on the spectrum of AdS_2 primaries

$$\bar{n} + \frac{h}{2} + \sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) = 0. \quad (4.73)$$

In contrast with the S^2 case, since μ is any real number – irrespectively of the kind of $SL(2, \mathbb{R})$ representation – there is *no extra* quantization condition for the level to make this limit well-defined. In this limit, the extra $U(1)$ decompactifies as usual and can be removed. Plugging the constraint (4.73) in the expressions for the dimensions of the affine primaries, we find

$$L_0 = -\frac{j(j-1)}{k} - \frac{1}{2} \left(\bar{n} + \frac{h}{2} \right)^2 - \frac{1}{2} \left(n + \frac{a}{2} \right)^2, \quad (4.74a)$$

$$\bar{L}_0 = -\frac{j(j-1)}{k}. \quad (4.74b)$$

In addition to the original AdS_3 spectrum, Eqs. (4.31) and (4.32), the right-moving part contain an extra fermionic lattice describing the states charged under the electric field. Despite the absence of $N = 2$ superconformal symmetry due to the Lorentzian signature, the theory has a “fermion-number” left symmetry, corresponding to the current:

$$J = \iota\psi^1\psi^3 + \frac{2}{k} \left(J^2 + \iota\psi^1\psi^3 \right). \quad (4.75)$$

The charges of the primaries (4.74) are

$$\mathcal{Q}_F = n + \frac{a}{2} - \sqrt{\frac{2}{k}} \left(\bar{n} + \frac{h}{2} \right). \quad (4.76)$$

$\text{AdS}_2 \times S^2 \times \mathcal{M}$ and space–time supersymmetry

Let us now consider the complete heterotic string background which consists of the $\text{AdS}_2 \times S^2$ space–time times an $N = 2$ internal conformal field theory \mathcal{M} , that we will assume to be of central charge $\hat{c} = 6$ and with integral R -charges. Examples of thereof are toroidal or flat-space compactifications, as well as Gepner models [Gep88].

The levels k of $SU(2)$ and \hat{k} of $SL(2, \mathbb{R})$ are such that the string background is critical:

$$\hat{c} = \frac{2(k-2)}{k} + \frac{2(\hat{k}+2)}{\hat{k}} = 4 \implies k = \hat{k}. \quad (4.77)$$

Supersymmetry and level quantization for $\text{AdS}_2 \times S^2$

This translates into the equality of the radii of the corresponding S^2 and AdS_2 factors, which is in turn necessary for supersymmetry. Furthermore, the charge quantization condition for the two-sphere (Sec. 4.1) imposes a further restriction on the level to $k = 2p^2$, $p \in \mathbb{N}$.

In this system the total fermionic charge is

$$\mathcal{Q} = n + \frac{a}{2} - \frac{N - h/2}{p} + n' + \frac{a}{2} - \frac{\bar{n}' + h/2}{p} + \mathcal{Q}_{\mathcal{M}}. \quad (4.78)$$

Hence, assuming that the internal $N = 2$ charge $\mathcal{Q}_{\mathcal{M}}$ is integral, further constraints on the electromagnetic charges of the theory are needed in order to achieve space–time supersymmetry. Namely, we must only keep states such that

$$N + \bar{n}' = 0 \pmod{p}. \quad (4.79)$$

This projection is some kind of generalization of Gepner models. Usually, such a projection is supplemented in string theory by new twisted sectors. We then expect that, by adding on top of this projection the usual GSO projection on odd fermion number, one will obtain a space–time supersymmetric background. However, the actual computation would need the knowledge of hyperbolic coset characters of $SL(2, \mathbb{R})$ (i.e. Lorentzian black-hole characters), and of their modular properties. We can already observe that this “Gepner-like” orbifold keeps only states which are “dyonic” with respect to the electromagnetic field background. Notice that, by switching other fluxes in the internal theory \mathcal{M} one can describe more general projections.

4.4 The three-dimensional black string revisited

The AdS_3 moduli space contains black hole geometries. This has been known since the most celebrated of them – the two-dimensional $SL(2, \mathbb{R})/U(1)$ black hole – was found by Witten [Wit91, DVV92]. Generalisations of these constructions to higher dimensions have been considered in [HH92, Ger95, Hor92, KT94]. The three-dimensional black string [HH92, HHS92, HW93] has attracted much attention, for it provides an alternative to the Schwarzschild black hole in three-dimensional asymptotically flat geometries¹³. In this section we want to show how this black string can be interpreted in terms of marginal deformations of $SL(2, \mathbb{R})$, which will enable us to give an expression for its string primary states.

In [HH92] the black string was obtained as an $(SL(2, \mathbb{R}) \times \mathbb{R})/\mathbb{R}$ gauged model. More precisely, expressing $g \in SL(2, \mathbb{R}) \times \mathbb{R}$ as:

$$g = \begin{pmatrix} a & u & 0 \\ -v & b & 0 \\ 0 & 0 & e^x \end{pmatrix}, \quad (4.80)$$

The three-dimensional black string as a current-current deformation

¹³Remember that the *no hair* theorem doesn’t hold in three dimensions [Isr67, Heu98, GIS02].

the left and right embeddings of the \mathbb{R} subgroup are identical and given by:

$$\epsilon_{L/R} : \mathbb{R} \rightarrow SL(2, \mathbb{R}) \times \mathbb{R} \quad (4.81)$$

$$\lambda \mapsto \begin{pmatrix} e^{\frac{1}{\sqrt{\lambda^2+2}}} & 0 & 0 \\ 0 & e^{-\frac{1}{\sqrt{\lambda^2+2}}} & 0 \\ 0 & 0 & e^{\frac{\lambda}{\sqrt{\lambda^2+2}}} \end{pmatrix}. \quad (4.82)$$

From the discussion in Sec. 3.1, we see that performing this gauging is just one of the possible ways to recover the $J^2\bar{J}^2$ symmetrically deformed $SL(2, \mathbb{R})$ geometry. More specifically, since the gauged symmetry is axial ($g \rightarrow hgh$), it corresponds (in our notation) to the $\kappa_2 < 1$ branch of the deformed geometry in Eq. (C.5a)¹⁴. One can find a coordinate transformation allowing to pass from the usual black-string solution

$$\begin{cases} ds^2 = \frac{k}{4} \left[- \left(1 - \frac{1}{r}\right) dt^2 + \left(1 - \frac{\mu^2}{r}\right) dx^2 + \left(1 - \frac{1}{r}\right)^{-1} \left(1 - \frac{\mu^2}{r}\right)^{-1} \frac{dr^2}{r^2} \right], \\ H = \frac{k\mu}{4r} dr \wedge dx \wedge dt, \\ e^{2\Phi} = \frac{\mu}{r} \end{cases} \quad (4.83)$$

to our (local) coordinate system, Eq. (C.5). The attentive reader might now be puzzled by this equivalence between a one-parameter model such as the symmetrically deformed model and a two-parameter one such as the black string in its usual coordinates (in Eqs. (4.83) we redefined the r coordinate as $r \rightarrow r/M$ and then set $\mu = Q/M$ with respect to the conventions in [HH92]). A point that it is interesting to make here is that although, out of physical considerations, the black string is usually described in terms of two parameters (mass and charge), the only physically distinguishable parameter is their ratio $\mu = Q/M$ that coincides with our κ_2 parameter. In the next section we will introduce a different (double) deformation, this time giving rise to a black hole geometry depending on two actual parameters (one of which being related to an additional electric field).

Single physical
parameter for the
black string

As we remarked above, the axial gauging construction only applies for $\mu < 1$, while, in order to obtain the other $\kappa_2 > 1$ branch of the $J^2\bar{J}^2$ deformation, one should perform a vector gauging. On the other hand, this operation, that would be justified by a CFT point of view, is not natural when one takes a more geometrical point of view and writes the black string metric as in Eq. (4.83). In the latter, one can study the signature of the metric as a function of r in the two regions $\mu^2 \gtrless 1$, and find the physically sensible regions (see Tab. 4.1).

Our observations are the following:

- The $\mu^2 < 1$ branch always has the correct $(-, +, +)$ signature for any value of r , with the two special values $r = 1$ and $r = \mu^2$ marking the presence of the horizons that hide the singularity in $r = 0$.

¹⁴The $R \gtrless 1$ convention is not univocal in literature.

μ	name	dt^2	dx^2	dr^2	range	CFT interpretation
$\mu^2 > 1$	(c^+)	-	+	+	$r > \mu^2$	$J^3 \bar{J}^3, \kappa_3 > 1$
	(b^+)	-	-	-	$1 < r < \mu^2$	
	(a^+)	+	-	+	$0 < r < 1$	$J^3 \bar{J}^3, \kappa_3 < 1$
$\mu^2 < 1$	(a^-)	+	-	+	$0 < r < \mu^2$	$J^2 \bar{J}^2, \kappa_2 < 1$
	(b^-)	+	+	-	$\mu^2 < r < 1$	
	(c^-)	-	+	+	$r > 1$	

Table 4.1: Signature for the black-string metric as a function of r , for $\mu^2 \gtrless 1$.

- The $\mu^2 > 1$ branch is different. In particular we see that there are two regions: (a^+) for $0 < r < 1$ and (c^+) for $r > \mu^2$ where the signature is that of a physical space.

A fact deserves to be emphasized here: one should notice that while for $\mu^2 < 1$ we obtain three different regions of the same space, for $\mu^2 > 1$ what we show in Tab. 4.1 really are three different spaces and the proposed ranges for r are just an effect of the chosen parameterization. The $(a^+), \kappa_3 < 1$ and $(c^+), \kappa_3 > 1$ branches are different spaces and not different regions of the same one and one can choose in which one to go when continuing to $\mu > 1$.

But there is more. The $\mu^2 > 1$ region is obtained via an analytic continuation with respect to the other branch, and this analytic continuation is precisely the one that interchanges the roles of the J^2 and the J^3 currents. As a result, we pass from the $J^2 \bar{J}^2$ line to the $J^3 \bar{J}^3$ line. More precisely the (c^+) region describes the “singular” $\kappa_3 > 1$ branch of the $J^3 \bar{J}^3$ deformation (*i.e.* the branch that includes the $r = 0$ singularity) and the (a^+) region describes the regular $\kappa_3 < 1$ branch that has the *cigar* geometry as $\kappa_3 \rightarrow 0$ limit. Also notice that the regions $r < 0$ have to be excluded in order to avoid naked singularities (of the type encountered in the Schwarzschild black hole with negative mass). The black string described in [HH92] covers the regions $(a^-), (b^-), (c^-), (a^+)$.

Our last point concerns the expectation of the genuine AdS_3 geometry as a zero-deformation limit of the black-string metric, since the latter turns out to be a marginal deformation of AdS_3 with parameter μ . The straightforward approach consists in taking the line element in Eq. (4.83) for $\mu = 1$. It is then puzzling that the resulting extremal black-string geometry *is not* AdS_3 . This apparent paradox is solved by carefully looking at the coordinate transformations that relate the black-string coordinates (r, x, t) to either the Euler coordinates (ρ, ϕ, τ) (B.13) for the $J^3 \bar{J}^3$ line, or the hyperbolic coordinates (y, x, t) for the $J^2 \bar{J}^2$ line. These transformations are singular at $\mu = 1$, which therefore corresponds neither to $\kappa_3 = 1$ nor to $\kappa_2 = 1$. Put differently, $\mu = 1$ is not part of a continuous line of deformed models but marks a jump from the $J^2 \bar{J}^2$ to the $J^3 \bar{J}^3$ lines.

The extremal black-string solution is even more peculiar. Comparing Eqs. (4.83) at $\mu = 1$ to Eqs. (C.6), which describe the symmetrically null-deformed $SL(2, \mathbb{R})$, we observe that the two backgrounds at hand are related by a coordinate transformation, provided $\nu = -1$.

The black string background is therefore entirely described in terms of $SL(2, \mathbb{R})$ marginal symmetric deformations, and involves all three of them. The null deformation appears, however, for the extremal black string only and at a negative value of the parameter ν . The latter is the density of fundamental strings, when the deformed AdS_3 is considered within the NS5/F1 system. This might be one more sign pointing towards a Gregory-Laflamme instability in the black string [GL93].

Notice finally that expressions (4.83) receive $1/k$ corrections. Those have been computed in [Sfe93]. Once taken into account, they contribute in making the geometry smoother, as usual in string theory.

An interesting mix

A particular kind of asymmetric deformation is what we will call in the following *double deformation* [KK95, Isr04]. At the Lagrangian level this is obtained by adding the following marginal perturbation to the WZW action:

$$\delta S = \delta \kappa^2 \int d^2z J\bar{J} + H \int d^2z J\bar{I}; \quad (4.84)$$

J is a holomorphic current in the group, \bar{J} the corresponding anti-holomorphic current and \bar{I} an external (to the group) anti-holomorphic current (*i.e.* in the right-moving heterotic sector for example). A possible way to interpret this operator consists in thinking of the double deformation as the superposition of a symmetric – or gravitational – deformation (the first addend) and of an antisymmetric one – the electromagnetic deformation. This mix is consistent because if we perform the κ deformation first, the theory keeps the $U(1) \times U(1)$ symmetry generated by J and \bar{J} that is needed in order to allow for the H deformation. Following this trail, we can read off the background fields corresponding to the double deformation by using at first one of the methods outlined in Sec. 3.1 and then applying the Kaluza-Klein reduction to the resulting background fields.

The final result consists in a metric, a three-form, a dilaton and a gauge field. It is in general valid at any order in the deformation parameters κ and H but only at leading order in α' due to the presence of the symmetric part.

Double deformations of AdS_3 where J is the time-like J^3 operator have been studied in [Isr04]. It was there shown that the extra gravitational deformation allows to get rid of the closed time-like curves, which are otherwise present in the pure J^3 asymmetric deformation (Eq. (4.15)) – the latter includes Gödel space. Here, we will focus instead on the case of double deformation generated by space-like operators, J^2 and \bar{J}^2 .

The hyperbolic double deformation

In order to follow the above prescription for reading the background fields in the double-deformed metric let us start with the fields in Eqs. (C.5). We can introduce those fields in the sigma-model action. Infinitesimal variation

A two-parameter
charged black
string

of the latter with respect to the parameter κ^2 enables us to reach the following expressions for the chiral currents $J_\kappa^2(z)$ and $\bar{J}_\kappa^2(\bar{z})$ at finite values of κ^2 :

$$J_\kappa^2(z) = \frac{1}{\cos^2 t + \kappa^2 \sin^2 t} (\cos^2 t \partial\psi - \sin^2 t \partial\varphi), \quad (4.85)$$

$$\bar{J}_\kappa^2(\bar{z}) = \frac{1}{\cos^2 t + \kappa^2 \sin^2 t} (\cos^2 t \partial\psi + \sin^2 t \partial\varphi). \quad (4.86)$$

Note in particular that the corresponding Killing vectors (that clearly are ∂_φ and ∂_ψ) are to be rescaled as $L_2 = \frac{1}{\kappa^2} \partial_\psi - \partial_\varphi$ and $R_2 = \frac{1}{\kappa^2} \partial_\psi + \partial_\varphi$. Once the currents are known, one has to apply the construction sketched in Sec. 3.2 and write the background fields as follows:

$$\left\{ \begin{aligned} \frac{1}{k} ds^2 &= -dt^2 + \cos^2 t \frac{(\kappa^2 - 2H^2) \cos^2 t + \kappa^4 \sin^2 t}{\Delta_\kappa(t)^2} d\psi^2 - 4H^2 \frac{\cos^2 t \sin^2 t}{\Delta_\kappa(t)^2} d\psi d\varphi + \\ &\quad + \sin^2 t \frac{\cos^2 t + (\kappa^2 - 2H^2) \sin^2 t}{\Delta_\kappa(t)^2} d\varphi^2 \\ \frac{1}{k} B &= \frac{\kappa^2 - 2H^2}{\kappa^2} \frac{\cos^2 t}{\Delta_\kappa(t)} d\varphi \wedge d\psi \\ F &= 2H \sqrt{\frac{2k}{k_g}} \frac{\sin(2t)}{\Delta_\kappa(t)^2} (\kappa^2 d\psi \wedge dt + dt \wedge d\varphi) \\ e^{-\Phi} &= \frac{\sqrt{\kappa^2 - 2H^2}}{\Delta_\kappa(t)} \end{aligned} \right. \quad (4.87)$$

where $\Delta_\kappa(t) = \cos^2 t + \kappa^2 \sin^2 t$ as in App. C. In particular the dilaton, that can be obtained by imposing the one-loop beta equation is proportional to the ratio of the double deformed volume form and the AdS₃ one.

A first observation about the above background is in order here. The electric field is bounded from above since $H^2 \leq \frac{\kappa^2}{2}$. As usual in string theory, tachyonic instabilities occur at large values of electric or magnetic fields, which is just a way of interpreting the decompactification boundary value for the deformation parameter. At the critical value of H , one dimension degenerates and the B -field vanishes. We are left with a two-dimensional space (with non-constant curvature) plus electric field.

The expression (4.87) here above of the metric provides only a local description of the space-time geometry. To discuss the global structure of the whole space it is useful to perform several coordinate transformations. Firstly let us parametrize by $\kappa^2 = \lambda / (1 + \lambda)$ the deformation parameter (with $\kappa < 1$ for $\lambda > 0$ and $\kappa > 1$ for $\lambda < -1$) and introduce a radial coordinate *à la* Horne and Horowitz:

$$r = \lambda + \cos^2 t, \quad (4.88)$$

which obviously varies between λ and $\lambda + 1$. The expression of the metric

(4.87) becomes in terms of this new coordinate:

$$\begin{aligned}
 ds^2 = & - \left[\left(2H^2 (1 + \lambda)^2 - \lambda \right) + \frac{\lambda \left(\lambda - 4H^2 (1 + \lambda)^2 \right)}{r} + \frac{2\lambda^2 H^2 (1 + \lambda)^2}{r^2} \right] d\psi^2 + \\
 & - (1 + \lambda) \left[2H^2 (1 + \lambda) + 1 - \frac{(1 + \lambda) \left(1 + 4H^2 (1 + \lambda)^2 \right)}{r} + \frac{2(1 + \lambda)^3 H^2}{r^2} \right] d\varphi^2 + \\
 & + 4H^2 (1 + \lambda)^2 \left[1 - \frac{1 + 2\lambda}{r} + \frac{\lambda(1 + \lambda)}{r^2} \right] d\psi d\varphi + \frac{1}{4(r - \lambda)(r - \lambda - 1)} dr^2. \quad (4.89)
 \end{aligned}$$

This expression looks close to the one discussed by Horne and Horowitz. It also represents a black string. However, it depends on more physical parameters as the expression of the scalar curvature shows:

$$\mathcal{R} = 2 \frac{2r(1 + 2\lambda) - 7\lambda(1 + \lambda) - 2H^2(1 + \lambda)^2}{r^2}. \quad (4.90)$$

Obviously this metric can be extended behind the initial domain of definition of the r variable. But before to discuss it, it is interesting to note that the Killing vector $\mathbf{k} = (1 + \lambda) \partial_\psi + \lambda \partial_\varphi \propto R_2$ is of constant square length

$$\mathbf{k} \cdot \mathbf{k} = \lambda(1 + \lambda) - 2H^2(1 + \lambda)^2 := \omega. \quad (4.91)$$

Note that as H^2 is positive, we have the inequality $\omega < \lambda(1 + \lambda)$. Moreover, in order to have a Lorentzian signature we must impose $\omega > 0$. The fact that the Killing vector \mathbf{k} is space-like and of constant length makes it a candidate to perform identifications. We shall discuss this point at the end of this section.

The constancy of the length of the Killing vector \mathbf{k} suggests to make a new coordinate transformation (such that $\mathbf{k} = \partial_x$):

$$\psi = (1 + \lambda)x + t, \quad (4.92a)$$

$$\varphi = t + \lambda x, \quad (4.92b)$$

which leads to the much simpler expression of the line element:

$$ds^2 = - \frac{(r - \lambda)(r - \lambda - 1)}{r^2} dt^2 + \omega \left(dx + \frac{1}{r} dt \right)^2 + \frac{1}{4(r - \lambda)(r - \lambda - 1)} dr^2. \quad (4.93)$$

This metric is singular at $r = 0, \lambda, \lambda + 1$; $r = 0$ being a curvature singularity. On the other hand, the volume form is $\sqrt{\omega}/(2r) dt \wedge dx \wedge dr$, which indicates that the singularities at $r = \lambda$ and $r = \lambda + 1$ may be merely coordinate singularities, corresponding to horizons. Indeed, it is the case. If we expand the metric, around $r = \lambda + 1$, for instance, at first order (*i.e.* for $r = \lambda + 1 + \epsilon$) we obtain:

$$\begin{aligned}
 ds^2 = & \frac{\omega}{(1 + \lambda)^2} (dt + (1 + \lambda) dx)^2 - \frac{\epsilon}{(1 + \lambda)^2} dt \left[dt + 2 \frac{\omega}{1 + \lambda} (dt + (1 + \lambda) dx) \right] + \\
 & + \frac{1}{4\epsilon} dr^2 \quad (4.94)
 \end{aligned}$$

indicating the presence of a horizon. To eliminate the singularity in the metric, we may introduce Eddington–Finkelstein like coordinates:

$$t = (1 + \lambda) \left(u \pm \frac{1}{2} \ln \epsilon \right) - \omega \xi, \quad (4.95a)$$

$$x = \left(1 + \frac{\omega}{1 + \lambda} \right) \xi - \left(u \pm \frac{1}{2} \ln \epsilon \right). \quad (4.95b)$$

The same analysis can also be done near the horizon located at $r = \lambda$. Writing $r = \lambda + \epsilon$, the corresponding regulating coordinate transformation to use is given by:

$$t = \lambda \left(u \pm \frac{1}{2} \ln \epsilon \right) + \omega \xi, \quad (4.96a)$$

$$x = \left(1 - \frac{\omega}{\lambda} \right) \xi - \left(u \pm \frac{1}{2} \ln \epsilon \right). \quad (4.96b)$$

In order to reach the null Eddington–Finkelstein coordinates, we must use null rays. The geodesic equations read, in terms of a function $\Sigma^2[E, P, \epsilon; r] = (Er - P)^2 - (P^2/\omega) - \epsilon(r - \lambda)(r - \lambda - 1)$:

$$\sigma = \int \frac{1}{4\Sigma[E, P, \epsilon; r]} dr \quad (4.97a)$$

$$t = \int \frac{(Er - P)r}{2(r - \lambda)(r - \lambda - 1)\Sigma[E, P, \epsilon; r]} dr \quad (4.97b)$$

$$x = - \int \frac{(Er - P) + P/\omega}{2(r - \lambda)(r - \lambda - 1)\Sigma[E, P, \epsilon; r]} dr \quad (4.97c)$$

where E and P are the constant of motion associated to ∂_t and ∂_x , σ is an affine parameter and ϵ , equal to $1, 0, -1$, characterizes the time-like, null or space-like nature of the geodesic. Comparing these equations (with $\epsilon = 0$ and $P = 0$) with the coordinates introduced near the horizons, we see that regular coordinates in their neighbourhoods are given by

$$t = T \pm \frac{1}{2} \left((1 + \lambda) \ln |r - \lambda - 1| - \lambda \ln |r - \lambda| \right), \quad (4.98a)$$

$$x = X \mp \frac{1}{2} \left(\ln |r - \lambda - 1| - \ln |r - \lambda| \right), \quad (4.98b)$$

which leads to the metric

$$ds^2 = \left(-1 + \frac{1 + 2\lambda}{r} - \frac{\lambda(1 + \lambda) - \omega}{r^2} \right) dT^2 + 2\frac{\omega}{r} dXdT + \omega dX^2 \mp \frac{1}{r} dTdr \quad (4.99)$$

According to the sign, we obtain incoming or outgoing null coordinates; to build a Kruskal coordinate system we have still to exponentiate them.

Obviously, we may choose the X coordinate in the metric (4.99) to be periodic without introducing closed causal curves. The question of performing more general identifications in these spaces will be addressed below.

We end this section by computing the conserved charges associated to the asymptotic symmetries of our field configurations. As is well known, their expressions provide solutions of the equations of motion derived from the low energy effective action

$$S = \int d^d x \sqrt{-g} e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 - \frac{k_g}{8}F^2 + \frac{\delta c}{3} \right] , \quad (4.100)$$

in which we have chosen the units such that $\delta c = 12$.

The expression (4.93) for the metric is particularly appropriate to describe the asymptotic properties of the solution. In these coordinates, the various, non gravitational, fields read as

$$F = \pm \frac{\sqrt{2}H(1+\lambda)}{r^2\sqrt{k_g}} dt \wedge dr, \quad (4.101a)$$

$$H = \mp \frac{\omega}{r^2} dt \wedge dx \wedge dr, \quad (4.101b)$$

$$\Phi = \Phi_* - \frac{1}{2} \ln r, \quad (4.101c)$$

By setting $\sqrt{\omega}x = \bar{x}$ and $r = e^{2\bar{\rho}}$, near infinity ($\bar{\rho} \rightarrow \infty$), the metric asymptotes the standard flat metric: $ds^2 = -dt^2 + d\bar{x}^2 + d\bar{\rho}^2$, while the fields F and H vanish and the dilaton reads $\Phi = \Phi_* - \bar{\rho}$. This allows to interpret the asymptotic behavior of our solution (4.87) as a perturbation around the solution given by $F = 0$, $H = 0$, the flat metric and a linear dilaton: $\Phi = \Phi_* + f_\alpha X^\alpha$, (with here $f_\alpha = (0, 0, -1)$). Accordingly, we may define asymptotic charges associated to each asymptotic reductibility parameter (see [BB02]).

For the gauge symmetries we obtain as charges, associated to the H field

$$Q_H = \pm 2e^{-2\Phi_*} \sqrt{\omega} \quad (4.102)$$

and to the F field

$$Q_F = \pm \frac{2\sqrt{2}e^{-2\Phi_*}H(1+\lambda)}{\sqrt{k_g}}. \quad (4.103)$$

The first one reduces (up to normalization) for $H = 0$ to the result given in [HH92], while the second one provides an interpretation of the deformation parameter H .

Moreover, all the Killing vectors of the flat metric defining isometries preserving the dilaton field allow to define asymptotic charges. These charges are obtained by integrating on the surface at infinity the antisymmetric tensor:

$$k_\xi^{[\mu\nu]} = e^{-2\Phi} \left(\xi_\sigma \partial_\lambda \mathcal{H}^{\sigma\lambda\mu\nu} + \frac{1}{2} \partial_\lambda \xi_\sigma \mathcal{H}^{\sigma\lambda\mu\nu} + 2(\xi^\mu h_\lambda^\nu f^\lambda - \xi^\nu h_\lambda^\mu f^\lambda) \right) \quad (4.104)$$

where

$$\mathcal{H}^{\sigma\lambda\mu\nu} = \bar{h}^{\sigma\nu} \eta^{\lambda\mu} + \bar{h}^{\lambda\mu} \eta^{\sigma\nu} - \bar{h}^{\sigma\mu} \eta^{\lambda\nu} - \bar{h}^{\lambda\nu} \eta^{\sigma\mu} \quad (4.105)$$

Asymptotic
charges for the
charged black
string

is the well known tensor sharing the symmetries of the Riemann tensor and $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}$, while the Killing vector ζ has to verify the invariance condition $\zeta_\alpha f^\alpha = 0$. The expression of the tensor $k_\zeta^{[\mu\nu]}$ depends only on the perturbation $h_{\mu\nu}$ of the metric tensor because, on the one hand, the F and H fields appear quadratically in the lagrangian, and their background values are zero, while, on the other hand, the perturbation field for the dilaton vanishes: $\Phi = \bar{\Phi}$.

Restricting ourselves to constant Killing vectors, we obtain the momenta (defined for the indice $\sigma = t$ and \bar{x})

$$P^\sigma = \int d\bar{x} e^{-2\Phi} \left(\partial_\lambda \mathcal{H}^{\sigma\lambda\bar{t}\bar{p}} - 2\eta^{\sigma t} h_{\bar{p}}^v \right) \quad (4.106)$$

i.e. the density of mass (μ) and momentum (ω) per unit length:

$$\mu = 2e^{-2\Phi_*}(1 + 2\lambda) \text{ and} \quad \omega = -2e^{-2\Phi_*}\sqrt{\omega}. \quad (4.107)$$

Of course, if we perform identifications such that the string acquires a finite length, the momenta (4.106) become also finite.

To make an end let us notice that the expressions of μ and ω that we obtain differ from those given in [HH92] by a normalization factor but also in their dependance with respect to λ , even in the limit $H = 0$; indeed, the asymptotic Minkowskian frames used differ from each other by a boost.

Discrete identifications

In the same spirit as the original BTZ construction reminded in the previous section, we would like to investigate to what extent discrete identifications could be performed in the deformed background. Necessary conditions for a solution (4.99) to remain “viable” black hole can be stated as follows:

- the identifications are to be performed along the orbits of some Killing vector ζ of the deformed metric
- there must be causally safe asymptotic regions (at spatial infinity)
- the norm of ζ has to be positive in some region of space-time, and chronological pathologies have to be hidden with respect to an asymptotic safe region by a horizon.

The resulting quotient space will exhibit a black hole structure if, once the regions where $\|\zeta\| < 0$ have been removed, we are left with an almost geodesically complete space, the only incomplete geodesics being those ending on the locus $\|\zeta\| = 0$. It is nevertheless worth emphasizing an important difference with the BTZ construction. In our situation, unlike the undeformed AdS_3 space, the initial space-time where we are to perform identifications do exhibit curvature singularities.

The BTZ black hole

In the presence of isometries, discrete identifications provide alternatives for creating new backgrounds. Those have the *same* local geometry, but differ with respect to their global properties. Whether these identifications can be implemented as orbifolds at the level of the underlying two-dimensional string model is very much dependent on each specific case.

For AdS₃, the most celebrated geometry obtained by discrete identification is certainly the BTZ black hole [BTZ92]. The discrete identifications are made along the integral lines of the following Killing vectors (see Eqs. (B.15)):

$$\text{non-extremal case : } \xi = (r_+ + r_-) R_2 - (r_+ - r_-) L_2, \quad (4.108a)$$

$$\text{extremal case : } \xi = 2r_+ R_2 - (R_1 - R_3) - (L_1 + L_3). \quad (4.108b)$$

In the original BTZ coordinates, the metric reads:

$$ds^2 = L^2 \left[-f^2(r) dt^2 + f^{-2}(r) dr^2 + r^2 \left(d\varphi - \frac{r_+ r_-}{r^2} dt \right)^2 \right], \quad (4.109)$$

with

$$f(r) = \frac{1}{r} \sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}. \quad (4.110)$$

In this coordinate system,

$$\partial_\varphi \equiv \xi, \quad \partial_t \equiv -(r_+ + r_-) R_2 - (r_+ - r_-) L_2 \quad \text{and} \quad r^2 \equiv \|\xi\|. \quad (4.111)$$

In AdS₃ φ is not a compact coordinate. The discrete identification makes φ an angular variable, $\varphi \cong \varphi + 2\pi$, which imposes to remove the region with $r^2 < 0$. The BTZ geometry describes a three-dimensional black hole, with mass M and angular momentum J , in a space-time that is locally (and asymptotically) anti-de Sitter. The chronological singularity at $r = 0$ is hidden behind an inner horizon at $r = r_-$, and an outer horizon at $r = r_+$. Between these two horizons, r is time-like. The coordinate t becomes space-like inside the ergosphere, when $r^2 < r_{\text{erg}}^2 \equiv r_+^2 + r_-^2$. The relation between M, J and r_\pm is as follows:

$$r_\pm^2 = \frac{ML}{2} \left[1 \pm \sqrt{1 - \left(\frac{J}{ML} \right)^2} \right]. \quad (4.112)$$

Extremal black holes have $|J| = ML$ ($r_+ = r_-$). In the special case $J = ML = 0$ one finds the near-horizon geometry of the five-dimensional NS5/F1 stringy black hole in its ground state. Global AdS₃ is obtained for $J = 0$ and $ML = -1$.

Many subtleties arise, which concern *e.g.* the appearance of closed time-like curves in the excised region of negative r^2 (where ∂_φ would have been time-like) or the geodesic completion of the manifold; a comprehensive analysis of these issues can be found in [BHTZ93]. At the string-theory level, the BTZ identification is realized as an *orbifold* projection, which amounts to keeping invariant states and adding twisted sectors [NS98, HKV02].

Besides the BTZ solution, other locally AdS_3 geometries are obtained, by imposing identification under purely left (or right) isometries, referred to as self-dual (or anti-self-dual) metrics. These were studied in [CH94]. Their classification and isometries are exactly those of the asymmetric deformations studied in the present chapter. The Killing vector used for the identification is (A) time-like (elliptic), (B) space-like (hyperbolic) or (C) null (parabolic), and the isometry group is $U(1) \times SL(2, \mathbb{R})$. It was pointed out in [CH94] that the resulting geometry was free of closed time-like curves only in the case (B).

Discrete identifications in asymmetric deformations

Our analysis of the residual isometries in purely asymmetric deformations (Sec. 3.1) shows that the vector ζ (Eq. (4.108a)) survives only in the hyperbolic deformation, whereas ζ in Eq. (4.108b) is present in the parabolic one. Put differently, non-extremal BTZ black holes allow for electric deformation, while in the extremal ones, the deformation can only be induced by an electro-magnetic wave. Elliptic deformation is not compatible with BTZ identifications.

The question that we would like to address is the following: how much of the original black hole structure survives the deformation? The answer is simple: a new chronological singularity appears in the asymptotic region of the black hole. Evaluating the norm of the Killing vector shows that a naked singularity appears. Thus the deformed black hole is no longer a viable gravitational background. Actually, whatever the Killing vector we consider to perform the identifications, we are always confronted to such pathologies.

The fate of the *asymmetric parabolic* deformation of AdS_3 is similar: there is no region at infinity free of closed time-like curves after performing the identifications.

Discrete identifications in symmetric deformations

Let us consider the *symmetric hyperbolic* deformation, whose metric is given by (4.93) with $\mathbb{H} = 0$, i.e. $\omega = \lambda(1 + \lambda)$. This metric has two residual Killing vectors, manifestly given by ∂_t and ∂_x . We may thus, in general, consider identifications along integral lines of

$$\zeta = a \partial_t + \partial_x. \quad (4.113)$$

This vector has squared norm:

$$\|\zeta\|^2 = (\lambda(1 + \lambda) - a^2) + \frac{a\lambda(1 + \lambda) + a^2(1 + 2\lambda)}{r}. \quad (4.114)$$

To be space-like at infinity the vector ζ must verify the inequality $\lambda(1 + \lambda) > a^2$. If $a > 0$, or $-\sqrt{\lambda(1 + \lambda)} < a < -2\lambda(1 + \lambda)/(1 + 2\lambda)$, ζ is everywhere space-like. Otherwise, it becomes time-like behind the inner horizon ($r = \lambda$), or on this horizon if $a = -\lambda$. In this last situation, the quotient space will exhibit a structure similar to that of the black string, with a time-like singularity (becoming light-like for $a = -\lambda$) and two horizons.

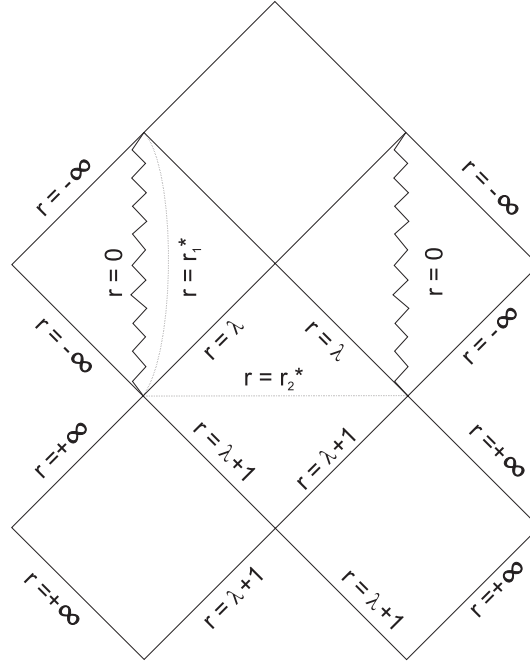


Figure 4.1: Penrose diagram exhibiting the global structure of the double hyperbolic deformation. The time-like curvature singularities $r = 0$ are represented, as well as the horizons, located at $r = \lambda$ and $r = \lambda + 1$. When performing identifications along orbits of a Killing vector allowing for a causally safe region at infinity, there appears chronological singularities, which can be time-like and hidden behind an outer and an inner horizon ($r = r_1^*$), or space-like and hidden behind a single horizon ($r = r_2^*$), while the regions where $r < r^*$ have to be removed.

Discrete identifications in double deformations

The norm squared of the identification vector (4.113) in the metric (4.93) is

$$\|\xi\|^2 = (\omega - a^2) + 2 \frac{a\omega + a^2(1 + 2\lambda)}{r} - \frac{a^2(\lambda(1 + \lambda) - \omega)}{r^2}. \quad (4.115)$$

Between $r = 0$ and $r = \infty$, this scalar product vanishes once and only once (if $a \neq 0$). To be space-like at infinity we have to restrict the time component of ξ to $|a| < \omega$. Near $r = 0$ it is negative, at the outer horizon ($r = \lambda + 1$) it takes the positive value $\omega(1 + \lambda + a)^2 / (1 + \lambda)^2$ and near the inner horizon ($r = \lambda$) the non-negative value $\omega(\lambda + a)^2 / \lambda^2$. Accordingly, by performing identifications using this Killing vector, we will encounter a chronological singularity, located at $r = r^*$, with $0 < r^* < \lambda + 1$. When $r^* < \lambda$, the singularity will be of the same type as the one in the symmetric case. But when $\lambda < r^* < \lambda + 1$, the chronological singularity will be space-like, and the causal structure we get is much like that of the Schwarzschild black hole, as is shown in Fig. 4.1.

Towards the exact spectra

Let us consider the algebraic point of view. Again as in the electric deformation of $SL(2, \mathbb{R})$ we can't write the partition function but we must content ourselves with the spectrum which will generalize what we found in Eqs. (4.33a) and (4.33b).

Deformed Spectrum

Consider the double deformation described above for a $SL(2, \mathbb{R})_k$ super-WZW model where J is the hyperbolic (space-like) J_2 current.

The evaluation of the spectrum for our deformed model is pretty straightforward once one realizes that the deformations act as $O(2, 2)$ pseudo-orthogonal transformations on the charge lattice corresponding to the abelian subgroup of the $\mathfrak{sl}(2, \mathbb{R})$ heterotic model (as described in Sec. 3.1). Left and right weights for the relevant lattices are (see Eqs. (D.20) and (D.21)):

$$L_0 = \frac{1}{k} \left(\mu + n + \frac{a}{2} \right)^2, \quad (4.116a)$$

$$\bar{L}_0 = \frac{\bar{\mu}^2}{k+2} + \frac{1}{k_g} \left(\bar{n} + \frac{\bar{a}}{2} \right)^2, \quad (4.116b)$$

where the anti-holomorphic part contains the contribution coming from a $u(1)$ subgroup of the heterotic gauge group.

At the Lagrangian level, the infinitesimal deformation we want to describe is given by the following marginal operator:

$$\mathcal{O} = \kappa^2 \frac{(J^2 + i\psi_1\psi_3)}{\sqrt{k}} \frac{\bar{J}^2}{\sqrt{k+2}} + \text{H} \frac{(J^2 + i\psi_1\psi_3)}{\sqrt{k}} \frac{\bar{I}}{\sqrt{k_g}}. \quad (4.117)$$

This suggests that the actual $O(2, 2)$ transformation should be obtained as a boost between the holomorphic part and the result of a rotation between the two anti-holomorphic components. The deformed lattices then read:

$$L_0^{\text{dd}} = \left\{ \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \left(\frac{\bar{\mu}}{\sqrt{k+2}} \cos \alpha + \frac{1}{\sqrt{k_g}} \left(\bar{n} + \frac{\bar{a}}{2} \right) \sin \alpha \right) \sinh x \right\}^2, \quad (4.118a)$$

$$\bar{L}_0^{\text{dd}} = \left\{ \left(\frac{\bar{\mu}}{\sqrt{k+2}} \cos \alpha + \frac{1}{\sqrt{k_g}} \left(\bar{n} + \frac{\bar{a}}{2} \right) \sin \alpha \right) \cosh x + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x \right\}^2, \quad (4.118b)$$

where the parameters x and α can be expressed as functions of κ and H as follows:

$$\begin{cases} \kappa^2 = \sinh(2x) \cos \alpha, \\ \text{H} = \sinh(2x) \sin \alpha. \end{cases} \quad (4.119)$$

Of course this is a generalization of the expressions in Eq. (4.33).

Twisting

The identification operation we performed in the symmetrically and double-deformed metric (as in Sec. 4.4) is implemented in the string theory framework by the orbifold construction. This was already obtained in [NS98, HKV02] for the “standard” BTZ black hole that was described as a $SL(2, \mathbb{R})/\mathbb{Z}$ orbifold.

In order to write the spectrum that will contain the twisted sectors, the first step consists in writing explicitly the primary fields in our theory, distinguishing between the holomorphic and anti-holomorphic parts (as it is natural to do since the construction is intrinsically heterotic).

- The holomorphic part is written by introducing the charge boost of Eq. (4.118a) in Eq. (D.16):

$$\Phi_{j\bar{\mu}\bar{\nu}}^{\text{dd}}(z) = U_{j\mu}(z) \exp \left[i \left(\sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \sqrt{2} \bar{Q}_\alpha \sinh x \right) \vartheta_2 \right], \quad (4.120)$$

where $Q_\alpha = \bar{\mu} \sqrt{\frac{2}{k+2}} \cos \alpha + \bar{\nu} \sqrt{\frac{2}{k_g}} \sin \alpha$ and the dd superscript stands for double deformed

- To write the anti-holomorphic part we need at first to implement the rotation between the \bar{J}^3 and gauge current components:

$$\begin{aligned} \bar{\Phi}_{j\bar{\mu}\bar{\nu}}(\bar{z}) &= V_{j\mu}(\bar{z}) e^{i\bar{\mu}\sqrt{2/k+2}\bar{\vartheta}_2} e^{i\bar{\nu}\sqrt{2/k_g}\bar{X}} = \\ &= V_{j\mu}(\bar{z}) e^{i\sqrt{2}\bar{Q}_\alpha(\bar{\vartheta}_2 \cos \alpha + \bar{X} \sin \alpha)} e^{i\sqrt{2}\bar{Q}_\alpha - \pi/2(-\bar{\vartheta}_2 \sin \alpha + \bar{X} \cos \alpha)}, \end{aligned} \quad (4.121)$$

and then realize the boost in Eq. (4.118b) on the involved part:

$$\begin{aligned} \bar{\Phi}_{j\bar{\mu}\bar{\nu}}^{\text{dd}}(\bar{z}) &= V_{j\mu} e^{i\sqrt{2}\bar{Q}_\alpha - \pi/2(-\bar{\vartheta}_2 \sin \alpha + \bar{X} \cos \alpha)} \times \\ &\times \exp \left[i \left(\sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x + \sqrt{2} \bar{Q}_\alpha \cosh x \right) (\bar{\vartheta}_2 \cos \alpha + \bar{X} \sin \alpha) \right]. \end{aligned} \quad (4.122)$$

Now that we have the primaries, consider the operator $W_w(z, \bar{z})$ defined as follows:

$$W_w(z, \bar{z}) = e^{-i\frac{k}{2}w\Delta_- \vartheta_2 + i\frac{k+2}{2}w\Delta_+ \bar{\vartheta}_2}, \quad (4.123)$$

where $w \in \mathbb{Z}$ and $\bar{\vartheta}_2$ the boson corresponding to the \bar{J}_2 current. It is easy to show that the following OPE's hold:

$$\vartheta_2(z) W_n(0, \bar{z}) \sim -i w \Delta_- \log z W_w(0, \bar{z}), \quad (4.124)$$

$$\bar{\vartheta}_2(\bar{z}) W_n(z, 0) \sim i w \Delta_+ \log \bar{z} W_w(z, 0), \quad (4.125)$$

showing that $W_w(z, \bar{z})$ acts as twisting operator with winding number w (θ_2 and θ_1 shift by $2\pi\Delta_-w$ and $2\pi\Delta_+w$ under $z \rightarrow e^{2\pi i}z$). This means that the general primary field in the $SL(2, \mathbb{R})_k / \mathbb{Z}$ theory can be written as:

$$\Phi_{j\bar{\mu}\bar{\nu}v\bar{w}}^{\text{tw}}(z, \bar{z}) = \Phi_{j\bar{\mu}\bar{\nu}v\bar{w}}^{\text{dd}}(z, \bar{z}) W_w(z, \bar{z}). \quad (4.126)$$

where the tw superscript stands for twisted.

Having the explicit expression for the primary field, it is simple to derive the scaling dimensions which are obtained, as before, via the GKO decomposition of the Virasoro algebra $T[\mathfrak{sl}(2, \mathbb{R})] = T[\mathfrak{sl}(2, \mathbb{R}) / \mathfrak{o}(1, 1)] + T[\mathfrak{o}(1, 1)]$. Given that the $T[\mathfrak{sl}(2, \mathbb{R}) / \mathfrak{o}(1, 1)]$ part remains invariant (and equal to $L_0 = -j(j+1)/k - \mu^2/(k+2)$ as in Eq. (D.18)), the deformed weights read:

$$L_0^{\text{tw}} = \left\{ \frac{k}{2\sqrt{2}} w \Delta_- + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \bar{Q}_\alpha \sinh x \right\}^2, \quad (4.127a)$$

$$\begin{aligned} \bar{L}_0^{\text{tw}} = & \left\{ -\frac{k+2}{2\sqrt{2}} w \Delta_+ \cos \alpha + \bar{Q}_\alpha \cosh x + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x \right\}^2 + \\ & + \left\{ \frac{k+2}{2\sqrt{2}} w \Delta_+ \sin \alpha + \bar{Q}_{\alpha-\pi/2} \right\}^2. \end{aligned} \quad (4.127b)$$

4.5 New compactifications

Up to this point we have focused on the squashed and coset models under the underlying hypothesis that they act as parts of larger ten-dimensional backgrounds. In this section we will study other examples which are likely to be part of physically sound models. In particular we will closely study the $SU(3)/U(1)^2$ coset that can be used as the six-dimensional compact counterpart of an AdS_4 background.

The $SU(3)/U(1)^2$ flag space

Let us now consider the next example in terms of coset dimensions, $SU(3)/U(1)^2$. As a possible application for this construction we may think to associate this manifold to a four-dimensional $(1, 0)$ superconformal field theory \mathcal{M} so to compactify a critical string theory since $\dim[SU(3)/U(1)^2] = 8 - 2 = 6$. Our construction gives rise to a whole family of CFT's depending on two parameters (since $\text{rank}[SU(3)] = 2$) but in this case we are mainly interested to the point of maximal deformation, where the $U(1)^2$ torus decouples and we obtain an exact theory on the $SU(3)/U(1)^2$ coset. Before giving the explicit expressions for the objects in our construction it is hence useful to recall some properties of this manifold. The first consideration to be made is the fact that $SU(3)/U(1)^2$ is an asymmetric coset in the mathematical sense defined in Sec. 3.3 (as we show below). This allows for the existence of more than one left-invariant Riemann metric. In particular, in this case, if we just consider

structures with constant Ricci scalar, we find, together with the restriction of the Cartan-Killing metric on $SU(3)$, the Kähler metric of the flag space F^3 . The construction we present in this section will lead to the first one of these two metrics. This is known to admit a nearly-Kähler structure and has already appeared in the superstring literature as a basis for a cone of G_2 holonomy [AW03].

A suitable parametrisation for the $SU(3)$ group is obtained via the Gauss decomposition described in App. B.3. In these terms the general group element is written as:

Gauss
decomposition
for $SU(3)$

$$g(z_1, z_2, z_3, \psi_1, \psi_2) = \begin{pmatrix} \frac{e^{i\psi_1/2}}{\sqrt{f_1}} & -\frac{\bar{z}_1 + z_2 \bar{z}_3}{\sqrt{f_1 f_2}} e^{i(\psi_1 - \psi_2)/2} & -\frac{\bar{z}_3 - \bar{z}_1 \bar{z}_2}{\sqrt{f_2}} e^{-i\psi_2/2} \\ \frac{z_1 e^{i\psi_1/2}}{\sqrt{f_1}} & -\frac{1 + |z_3|^2 - z_1 z_2 \bar{z}_3}{\sqrt{f_1 f_2}} e^{i(\psi_1 - \psi_2)/2} & -\frac{\bar{z}_2}{\sqrt{f_2}} e^{-i\psi_2/2} \\ \frac{z_3 e^{i\psi_1/2}}{\sqrt{f_1}} & -\frac{z_2 - \bar{z}_1 z_3 + z_2 |z_1|^2}{\sqrt{f_1 f_2}} e^{i(\psi_1 - \psi_2)/2} & \frac{1}{\sqrt{f_2}} e^{-i\psi_2/2} \end{pmatrix} \quad (4.128)$$

where z_i are three complex parameters, ψ_j are two real parameters and $f_1 = 1 + |z_1|^2 + |z_3|^2$, $f_2 = 1 + |z_2|^2 + |z_3 - z_1 z_2|^2$. As for the group, we need also an explicit parametrisation for the $\mathfrak{su}(3)$ algebra, such as the one provided by the Gell-Mann matrices in Eq. (B.40). It is a well known result that if a Lie algebra is semi-simple (or, equivalently, if its Killing form is negative-definite) then all Cartan subalgebras are conjugated by some inner automorphism¹⁵. This leaves us the possibility of choosing any couple of commuting generators, knowing that the final result won't be influenced by such a choice. In particular, then, we can pick the subalgebra generated by $\mathfrak{k} = \langle \lambda_3, \lambda_8 \rangle$.¹⁶

The holomorphic currents of the bosonic $SU(3)_k$ corresponding to the two operators in the Cartan are:

$$\mathcal{J}^3 = -\langle \lambda_3 g(z_\mu, \psi_a)^{-1} dg(z_\mu, \psi_a) \rangle \quad \mathcal{J}^8 = -\langle \lambda_8 g(z_\mu, \psi_a)^{-1} dg(z_\mu, \psi_a) \rangle \quad (4.129)$$

and in these coordinates they read:

$$\mathcal{J}^3 = -\frac{i}{\sqrt{2}} \left\{ \left(\frac{\bar{z}_1}{f_1} + \frac{z_2(-\bar{z}_1 \bar{z}_2 + \bar{z}_3)}{2f_2} \right) dz_1 - \frac{\bar{z}_2(1 + |z_1|^2) - z_1 \bar{z}_3}{2f_2} dz_2 + \left(\frac{\bar{z}_3}{f_1} + \frac{\bar{z}_1 \bar{z}_2 - \bar{z}_3}{2f_2} \right) dz_3 \right\} \\ + \text{c.c.} + \frac{d\psi_1}{\sqrt{2}} - \frac{d\psi_2}{2\sqrt{2}} \quad (4.130)$$

$$\mathcal{J}^8 = -i\sqrt{\frac{3}{2}} \left\{ \frac{\bar{z}_1 \bar{z}_2 - \bar{z}_3}{2f_2} z_2 dz_1 + \frac{\bar{z}_2 + |z_1|^2 \bar{z}_2 - z_1 \bar{z}_3}{2f_2} dz_2 + \frac{-\bar{z}_1 \bar{z}_2 + \bar{z}_3}{2f_2} dz_3 \right\} + \text{c.c.} + \frac{1}{2} \sqrt{\frac{3}{2}} d\psi_2.$$

¹⁵This is the reason why the study of non-semi-simple Lie algebra deformation constitutes a richer subject. In example the $SL(2, \mathbb{R})$ group admits for 3 different deformations, leading to 3 different families of exact CFT's with different physics properties. On the other hand the 3 possible deformations in $SU(3)$ are equivalent.

¹⁶In this explicit parametrisation it is straightforward to show that the coset we're considering is not symmetric. It suffices to pick two generators, say λ_2 and λ_4 , and remark that their commutator $[\lambda_2, \lambda_4] = -1/\sqrt{2} \lambda_6$ doesn't live in the Cartan subalgebra.

(4.131)

Those currents appear in the expression of the exactly marginal operator that we can add to the $SU(3)$ WZW model action:

$$V = \frac{\sqrt{k\bar{k}_g}}{2\pi} \int dz^2 H_3 \left(J^3 - \frac{1}{\sqrt{2}k} (2 : \psi_2 \psi_1 : + : \psi_5 \psi_4 : + : \psi_7 \psi_6 :) \right) \bar{J}^3 + H_8 \left(J^8 - \frac{1}{k} \sqrt{\frac{3}{2}} (: \psi_5 \psi_4 : + : \psi_7 \psi_6 :) \right) \bar{J}^8 \quad (4.132)$$

where ψ^i are the bosonic current superpartners and \bar{J}^3, \bar{J}^8 are two currents from the gauge sector both generating a $U(1)_{k_g}$.

Since $\text{rank}[SU(3)] = 2$ we have a bidimensional family of deformations parametrised by the two moduli H_3 and H_8 . The back-reaction on the metric is given by:

$$ds^2 = g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + (1 - 2H_3^2) \mathcal{J}^3 \otimes \mathcal{J}^3 + (1 - 2H_8^2) \mathcal{J}^8 \otimes \mathcal{J}^8 \quad (4.133)$$

where $g_{\alpha\bar{\beta}}$ is the restriction of the $SU(3)$ metric on $SU(3)/U(1)^2$. It is worth to remark that for any value of the deformation parameters H_3 and H_8 the deformed metric is Einstein with constant Ricci scalar.

With a procedure that has by now become familiar we introduce the following reparametrization:

$$\psi_1 = \frac{\hat{\psi}_1}{\sqrt{1 - 2H^2}} \quad \psi_2 = \frac{\hat{\psi}_2}{\sqrt{1 - 2H^2}} \quad (4.134)$$

and take the $H_3 \rightarrow 1/\sqrt{2}, H_8 \rightarrow 1/\sqrt{2}$ limit. The resulting metric is:

$$ds^2 = g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + \frac{d\hat{\psi}_1 \otimes d\hat{\psi}_1 - d\hat{\psi}_1 \otimes d\hat{\psi}_2 + d\hat{\psi}_2 \otimes d\hat{\psi}_2}{2} \quad (4.135)$$

that is the metric of the tangent space to the manifold $SU(3)/U(1)^2 \times U(1) \times U(1)$. The coset metric hence obtained has a \mathbb{C} -structure, is Einstein and has constant Ricci scalar $R = 15/k$. The other background fields at the boundary of the moduli space read:

$$F = d\mathcal{J}^3 + d\mathcal{J}^8 \quad (4.136)$$

$$H_{[3]} = -3\sqrt{2} \left\{ \mathcal{J}^1 \wedge (\mathcal{J}^4 \wedge \mathcal{J}^5 - \mathcal{J}^6 \wedge \mathcal{J}^7) + \sqrt{3} \mathcal{J}^2 \wedge (\mathcal{J}^4 \wedge \mathcal{J}^5 + \mathcal{J}^6 \wedge \mathcal{J}^7) \right\} \quad (4.137)$$

Supersymmetry
properties of
 $SU(3)/U(1)^2$

If we consider the supersymmetry properties along the deformation line we can remark the presence of an interesting phenomenon. The initial $SU(3)$ model has $N = 2$ but this symmetry is naively broken to $N = 1$ by the deformation. This is true for any value of the deformation parameter but for the boundary point $H_3^2 = H_8^2 = 1/2$ where the $N = 2$ supersymmetry is restored. Following [GHR84, KS89b, KS89a] one can see that a G/T coset admits $N = 2$ supersymmetry if it possesses a complex structure and the corresponding algebra can be decomposed as $\mathfrak{j} = \mathfrak{j}_+ \oplus \mathfrak{j}_-$ such as $[j_+, j_+] = j_+$ and

$[j_-, j_-] = j_-$. Explicitly, this latter condition is equivalent (in complex notation) to $f_{ijk} = f_{i\bar{j}\bar{k}} = f_{aij} = f_{a\bar{i}\bar{j}} = 0$. These are easily satisfied by the $SU(3)/U(1)^2$ coset (and actually by any G/T coset) since the commutator of two positive (negative) roots can only be proportional to the positive (negative) root obtained as the sum of the two or vanish. Having $N = 2$ supersymmetry is equivalent to asking for the presence of two complex structures. The first one is trivially given by considering positive roots as holomorphic and negative roots as anti-holomorphic, the other one by interchanging the role in one out of the three positive/negative couples (the same flip on two couples would give again the same structure and on all the three just takes back to the first structure). The metric is Hermitian with respect to both structures since it is $SU(3)$ invariant. It is worth to remark that such background is different from the ones described in [KS89a] because it is not Kähler and can't be decomposed in terms of Hermitian symmetric spaces.

Different constructions on $SU(3)/U(1)^2$

To study the $SU(3)$ case we will use the "current" approach of Sec. 3.6, since a direct computation in coordinates would be impractical. As one could expect, the study of $SU(3)$ deformation is quite richer because of the presence of an embedded $SU(2)$ group that can be gauged. Basically this means that we can choose two different deformation patterns that will lead to the two possible Einstein structures that can be defined on the $SU(3)/U(1)^2$ manifold.

Direct gauging.

The first possible choice leads to the same model as before by simply gauging the $U(1)^2$ Cartan torus. Consider the initial $SU(3)_k \times U(1)_{k'} \times U(1)_{k''}$ model. In the $\langle \mathcal{J}_1, \dots, \mathcal{J}_8, \mathcal{I}_1, \mathcal{I}_2 \rangle$ base ($\{\mathcal{J}_i\}$ being the $SU(3)$ generators and $\{\mathcal{I}_k\}$ the 2 $U(1)$'s), the initial metric is written as:

$$g = \left(\begin{array}{c|c} k\mathbb{I}8 & 0 \\ \hline 0 & \begin{array}{c} k' \\ k'' \end{array} \end{array} \right) \quad (4.138)$$

the natural choice for the Cartan torus is given by the usual $\langle \mathcal{J}_3, \mathcal{J}_8 \rangle$ generators, so we can proceed as before and write the deformed metric as:

$$g = \left(\begin{array}{cccc} k\mathbb{I}2 & & & \\ & \lambda_1(k, k', H_3) & & \\ & & k\mathbb{I}4 & \\ & & & \lambda_1(k, k'', H_8) \\ & & & & \lambda_2(k, k', H_3) \\ & & & & & \lambda_2(k, k'', H_8) \end{array} \right) \quad (4.139)$$

where H_3 and H_8 are the deformation parameters and λ_1 and λ_2 are the eigenvalues for the interaction matrices, given in Eq. (3.118). In particular, then, in

the $H_3^2 \rightarrow 1/2$, $H_8^2 \rightarrow 1/2$ limit two eigenvalues vanish, the corresponding directions decouple and we are left with the following (asymmetrically gauged) model:

$$g = \left(\begin{array}{c|c} k\mathbb{I}6 & \\ \hline & k+k' \\ & k+k'' \end{array} \right) \quad (4.140)$$

in the $\langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \sqrt{k'}\mathcal{I}_1 + \sqrt{k}\mathcal{J}_3, \sqrt{k''}\mathcal{I}_2 + \sqrt{k}\mathcal{J}_8 \rangle$ basis that can be seen as a $U(1)^2$ fibration over an $SU(3)/U(1)^2$ base with metric $\text{diag}(1, 1, 1, 1, 1, 1)$ (in the current basis). This is precisely the same result we obtained in the previous section when we read the fibration as a gauge field living on the base.

$$\begin{array}{ccc} U(1)^2 & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & SU(3)/U(1)^2 \end{array} \quad (4.141)$$

As in the previous example all this construction is valid only if the asymmetrically gauged WZW model is anomaly-free.

The F_3 flag space

Let us now turn to the other possible choice for the $SU(3)$ gauging, namely the one where we take advantage of the $SU(2)$ embedding. Let us then consider the $SU(3)_{k_3} \times SU(2)_{k_2} \times U(1)_{k'} \times U(1)_{k''}$ WZW model whose metric is

$$g = \left(\begin{array}{c|c|c} k_3\mathbb{I}8 & & \\ \hline & k_2\mathbb{I}3 & \\ \hline & & k' \\ & & k'' \end{array} \right) \quad (4.142)$$

in the $\langle \mathcal{J}_1, \dots, \mathcal{J}_8, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{K}_1, \mathcal{K}_2 \rangle$ basis, where $\langle \mathcal{J}_i \rangle$ generate the $SU(3)$, $\langle \mathcal{I}_i \rangle$ generate the $SU(2)$ and $\langle \mathcal{K}_i \rangle$ generate the $U(1)^2$.

The first step in this case consists in an asymmetric gauging mixing the $\{ \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \}$ and $\{ \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \}$ currents respectively. At the gauging point, a whole 3-sphere decouples and we obtain the following metric:

$$g = \left(\begin{array}{c|c|c} k_3\mathbb{I}5 & & \\ \hline & (k_2 + k_3)\mathbb{I}3 & \\ \hline & & k' \\ & & k'' \end{array} \right) \quad (4.143)$$

where we have to remember that in order to have an admissible embedding $k_2 = k_3 = k$. Our result is again – not surprisingly – a $SU(2)$ fibration over a $SU(3)/SU(2)$ base (times the two $U(1)$'s).

$$\begin{array}{ccc} SU(2) & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & SU(3)/SU(2) \end{array} \quad (4.144)$$

Of course one could be tempted to give \mathcal{M} the same interpretation as before, namely an $SU(3)/SU(2)$ space supported by a chromo-magnetic $SU(2)$ field (or, even better, gauging an additional $U(1)$, of a $\mathbb{C}\mathbb{P}^2$ background with an $SU(2) \times U(1)$ chromo-magnetic field). Actually this is not the case. The main point is the fact that this $SU(3) \times SU(2)$ model is essentially different from the previous ones because the $U(1)$ factors were the result of the bosonization of the right-moving gauge current which in this way received a (fake) left-moving partner as in Sec. 3.1. This is not possible in the non-abelian case since one can't obtain an $SU(2)$ at arbitrary level k out of the fermions of the theory¹⁷. In other words, the $SU(2)$ factor is in this case truly a constituent of the theory and there is no reason why it should be decoupled or be given a different interpretation from the $SU(3)$ part. This is why the structure obtained by the $SU(2)$ asymmetric gauging is to be considered an eight-dimensional space admitting an $SU(2) \rightarrow SU(3)/SU(2)$ fibration structure, or, equivalently, a deformed $SU(3)$ where an embedded $SU(2)$ is at a level double with respect to the other generators.

On the other hand we are still free to gauge away the two $U(1)$ factors just as before. This time we can choose to couple K_1 with the \mathcal{J}_8 factor that was left untouched in the initial $SU(3)$ and \mathcal{K}_2 with the $\mathcal{J}_3 + \mathcal{I}_3$ generator. Again we find a two-parameter family of deformations whose metric can be written as:

$$g = \left(\begin{array}{c|c|c|c} k\mathbb{I}4 & & & \\ \hline & \mu_1 & & \\ \hline & & 2k\mathbb{I}2 & \\ \hline & & & \nu_1 \\ & & & \mu_2 \\ & & & \nu_2 \end{array} \right) \quad (4.145)$$

where:

$$\mu = \lambda(k, k', H') \quad (4.146)$$

$$\nu = \lambda(2k, k'', H''). \quad (4.147)$$

In particular now we can take the decoupling $H' = H'' \rightarrow 1/2$ limit where we obtain:

$$g = \left(\begin{array}{c|c|c} k\mathbb{I}4 & & \\ \hline & 2k\mathbb{I}2 & \\ \hline & & k + k' \\ & & 2k + k'' \end{array} \right) \quad (4.148)$$

this structure is once more a $U(1)^2 \rightarrow SU(3)/U(1)^2$ fibration but in this case it is perfectly fine to separate the space components from the gauge field ones. So we can read out our final background fields as the Kähler metric on F_3 supported by a $U(1)^2$ (chromo)magnetic field.

¹⁷This would be of course be possible if we limited ourselves to small values of k , but in this case the whole geometric interpretation of the background would be questionable. However for Gepner-like string compactifications this class of models is relevant.

To summarize our results we can say that the two Einstein structures that one can define on $SU(3)/U(1)^2$ are both exact string theory backgrounds:

- The first one, obtained as the asymmetric coset $\frac{SU(3) \times U(1)^2}{U(1)^2}$ is supported by an NS-NS field strength and a magnetic field;
- The second, corresponding to the $\frac{SU(3) \times SU(2) \times U(1)^2}{SU(2) \times U(1)^2}$ asymmetric coset is Kähler and hence supported by the (chromo-)magnetic field alone.

Kähler form for $SU(3)/U(1)^2$

This Kähler structure has been deeply studied both from the mathematical and physical points of view. In particular the Kähler form can be written as in App. B.3:

$$K(\gamma_\mu, \bar{\gamma}_\mu) = \log \left[1 + |\gamma_1|^2 + |\gamma_3|^2 \right] + \log \left[1 + |\gamma_2|^2 + |\gamma_3 - \gamma_1 \gamma_2|^2 \right]. \quad (4.149)$$

It is immediate to show that this manifold is Einstein and in particular its Ricci scalar is $R = 12$. Being Kähler, F_3 is torsionless, that means in turn that there is no NS-NS form¹⁸. Moreover there is no dilaton by construction¹⁹. The only other field that supports the background comes from the $U(1)^2$ fibration. Since the manifold is Kähler it is useful to take advantage of the complex structure and write our background fields in complex formalism. In these terms the metric is written as:

$$g = \frac{k}{2} \left(\mathcal{J}^1 \otimes \mathcal{J}^{\bar{1}} + \mathcal{J}^2 \otimes \mathcal{J}^{\bar{2}} + 2\mathcal{J}^3 \otimes \mathcal{J}^{\bar{3}} \right) \quad (4.150)$$

where \mathcal{J}^i and $\mathcal{J}^{\bar{i}}$ are the Maurer-Cartan corresponding to positive and negative roots respectively and the field strength is given by:

$$F^a = \sqrt{\frac{k}{2k_g}} f^a{}_{\mu\bar{\nu}} C^{\bar{\rho}\sigma} R_{\sigma\bar{\nu}} \mathcal{J}^\mu \wedge \mathcal{J}^{\bar{\nu}} \quad (4.151)$$

where C is the following tensor

$$C = \sum_{\alpha} \mathcal{J}^\alpha \otimes \mathcal{J}^{\bar{\alpha}} \quad (4.152)$$

New linear dilaton backgrounds of Heterotic strings

These left-coset superconformal field theories can be used to construct various supersymmetric exact string backgrounds. The first class are generalizations of Gepner models [Gep88] and Kazama-Suzuki constructions [KS89b] using the left cosets as building blocks for the internal SCFT. This has already been

¹⁸To be precise one could define a B field but this would have to be closed

¹⁹The dilaton would basically measure the difference between the asymmetric coset volume form and the homogeneous space one as it is shown in [Tse94]

considered in [BJKZ96] for the S^2 coset but can be extended using the new theories constructed above. In this case there is no geometric interpretation from the sigma model point of view since these theories have no semi-classical limit. Indeed the levels of the cosets are frozen because their central charge must add up to $c = 9$ (in the case of four-dimensional compactification). However we expect that they correspond to special points in the moduli spaces of supersymmetric compactifications, generalizing the Gepner points of the CY manifolds.

Another type of models are the left cosets analogues of the NS5-branes solutions [CHS91, KPR91] and of their extensions to more generic supersymmetric vacua with a dilaton background. It was shown in [GKP99] that a large class of these linear dilaton theories are dual to singular CY manifolds in the decoupling limit. An extensive review of the different possibilities in various dimensions has been given in [ESY03] with all the possible G/H cosets. The left cosets that we constructed allows to extend all these solutions to heterotic strings, with a different geometrical interpretation since our cosets differ from ordinary gauged WZW model. However the superconformal structure of the left sector of our models is exactly the same as for the corresponding gauged WZW – except that the values of the $N = 2$ R-charges that appear in the spectrum are constrained – so we can carry over all the known constructions to the case of the geometric cosets.

In the generic case these constructions involve non-abelian cosets, and as we showed the asymmetric deformations and gaugings apply only to the abelian components. Thus in general we will get mixed models which are gauged WZW models w.r.t. the non-abelian part of H and geometric cosets w.r.t. the abelian components of H . Below we will focus on purely abelian examples, i.e. corresponding to geometric cosets. The dual interpretation of these models, in terms of the decoupling limit of some singular compactification manifolds, is not known. Note however that by construction there are about \sqrt{k} times less massless states in our models than in the standard left-right symmetric solutions. Therefore they may correspond to some compactifications with fluxes, for which the number of moduli is reduced. It would be very interesting to investigate this issue further.

Six-dimensional model. Let us take as a first example the critical superstring background:

$$\mathbb{R}^{5,1} \times \frac{SL(2, \mathbb{R})_{k+2} \times SO(2)_1}{U(1)_k} \times \left[U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right] \quad (4.153)$$

the first factor is an ordinary gauged model while the second one is a left coset CFT as discussed in this paper. This is the direct analogue of the five-brane solution, or more precisely of the double scaling limit of NS5-branes on a circle [GK99, IKPT04], in the present case with magnetic flux. This theory has $N = 2$ charges but, in order to achieve spacetime supersymmetry one must project onto odd-integral $N = 2$ charges on the left-moving side, as in the type II construction [IKPT04]. This can be done in the standard way by orbifoldizing the left $N = 2$ charges of the two cosets.

Four-dimensional model. A simple variation of the six-dimensional theory is provided by

$$\mathbb{R}^{3,1} \times \frac{SL(2, \mathbb{R})_{k/2+2} \times SO(2)_1}{U(1)_{2k}} \times \left[U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right] \\ \times \left[U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right] \quad (4.154)$$

which is the magnetic analogue of the (double scaling limit of) intersecting five-branes solution. Also here an orbifoldization of the left $N = 2$ charges is needed to achieve space-time supersymmetry.

Three-dimensional models: the flagbrane[Ⓞ]. We can construct the following background of the G_2 holonomy type, as in the case of symmetric coset [ES01]:

$$\mathbb{R}^{2,1} \times \mathbb{R}_Q \times \left[U(1)_k \times U(1)_{3k} \backslash SU(3)_{k-3} \times SO(6)_1 \right] \quad (4.155)$$

and the non-trivial part of the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + \frac{k}{4r^2} [dr^2 + 4r^2 ds^2(SU(3)/U(1)^2)]. \quad (4.156)$$

Without the factor of four it would be a direct analogue of the NS5-brane, being conformal to a cone over the flag space.

Another possibility in three dimensions is to lift the $SL(2, \mathbb{R})/U(1)$ coset to the group manifold $SL(2, \mathbb{R})$. In this case, as for the standard gauged WZW construction [AGS00] we will get the following anti-de Sitter background:

$$SL(2, \mathbb{R})_{k/4+2} \times \left[U(1)_{3k} \backslash SU(3)_{k-3} \times SO(6)_1 \right] \quad (4.157)$$

and the left moving sector of this worldsheet CFT defines an $N = 3$ superconformal algebra in spacetime.

Two-dimensional model. In this case we can construct the background:

$$\mathbb{R}^{1,1} \times \frac{SL(2, \mathbb{R})_{k/4+2} \times SO(2)_1}{U(1)_{4k}} \times \frac{U(1)_{3k} \backslash SU(3)_{k-3} \times SO(6)_1}{U(1)_k} \quad (4.158)$$

which corresponds in the classification of [ESY03] to a non-compact manifold of $SU(4)$ holonomy once the proper projection is done on the left $N = 2$ charges. This solution can be also be thought as conformal to a cone over the Einstein space $SU(3)/U(1)$. Using the same methods as for the NS5-branes in [IKPT04], we can show that the full solution corresponding to the model (4.158) can be obtained directly as the null super-coset:

$$\frac{SL(2, \mathbb{R})_{k/4} \times U(1) \backslash SU(3)_k}{U(1)_L \times U(1)_R} \quad (4.159)$$

where the action is along the elliptic generator in the $SL(2, \mathbb{R})$, with a normalization $\langle t^3 \rangle^2 = -4$, and along the direction $\alpha_1 + 2\alpha_2$ in the coset space $U(1) \backslash SU(3)$, with a canonical normalization. For $r \rightarrow \infty$ the solution asymptotes the cone but when $r \rightarrow 0$ the strong coupling region is smoothly capped by the cigar.

CHAPTER 5

Squashed groups in type II

In this chapter we start deviating from the preceding ones because we will no longer deal with WZW models but with configurations in which the group manifold geometry is sustained by RR fields. In particular, then, we see how the squashed geometries can be obtained in type II theories by T-dualizing black brane configurations.

THE MODELS that we have studied so far are intrinsically heterotic. In fact it is the very presence of a heterotic electromagnetic field that allows for the solution of the equations of motion. On the other hand, and this is especially true for purely asymmetric deformations that only have a constant dilaton, we can expect them to be mapped via S-duality to type II solutions. In this chapter we will build such solutions but using a slightly different path: in particular we will see how using a T-duality it is possible to modify a fibration geometry in the same way as we did before by adding a marginal operator, thus recovering the same geometries as above, but this time in presence of Ramond-Ramond fields.

5.1 $SL(2, \mathbb{R}) \times SU(2)$ as a D-brane solution



Up to this point we have considered WZW models for the sake of their self-consistency. In other words we have used group manifolds as part of critical string backgrounds on the ground of the existence of an underlying CFT. On the other hand, at low energies we should obtain a SUGRA description, so it is plausible that a description for the same geometry is available in terms of diverse ten-dimensional sources.

The starting point is a higher-dimensional generalization of the usual four-dimensional charged black hole. The natural, most symmetric, ansatz for the geometry in presence of a p -dimensional extended object (a black D_p -brane) consists in keeping the Lorentz symmetry in $(p + 1)$ dimensions and a spherical symmetry in $(9 - p)$. In other words breaking the $SO(1, 9)$ symmetry to $SO(1, p) \times SO(9 - p)$. Moreover we expect a $C_{[(p+1)]}$ form naturally coupled to the p -brane. It is possible to show [Ste98] that the solution also contains a dilaton $\Phi(r)$ and is completely determined in terms of a harmonic function

Black brane
ansatz

by

$$\begin{aligned}
e^{-1}L_{IIA} = & R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\mathcal{F}_{[1]}^{(12)})^2 e^{\frac{3}{2}\phi + \frac{\sqrt{7}}{2}\varphi} + \\
& - \frac{1}{48}(F_{[4]})^2 e^{\frac{1}{2}\phi + \frac{3}{2\sqrt{7}}\varphi} - \frac{1}{12}(F_{[3]}^{(1)})^2 e^{-\phi + \frac{1}{\sqrt{7}}\varphi} - \frac{1}{12}(F_{[3]}^{(2)})^2 e^{\frac{1}{2}\phi - \frac{5}{2\sqrt{7}}\varphi} + \\
& - \frac{1}{4}(F_{[2]}^{(12)})^2 e^{-\phi - \frac{3}{\sqrt{7}}\varphi} - \frac{1}{4}(\mathcal{F}_{[2]}^{(1)})^2 e^{\frac{3}{2}\phi + \frac{1}{2\sqrt{7}}\varphi} - \frac{1}{4}(\mathcal{F}_{[2]}^{(2)})^2 e^{\frac{4}{\sqrt{7}}\varphi} + \\
& - \frac{1}{2e}\tilde{F}_{[4]} \wedge \tilde{F}_{[4]} \wedge A_{[1]}^{(12)} - \frac{1}{e}\tilde{F}_{[3]}^{(1)} \wedge \tilde{F}_{[3]}^{(2)} \wedge A_{[3]}, \quad (5.4)
\end{aligned}$$

where ϕ is the original dilaton, φ is a scalar measuring the compact circle, defined by the reduction (in string frame)

$$ds^2 = e^{\phi/2} ds_{10}^2 = e^{\phi/2} \left(e^{-\varphi/(2\sqrt{7})} ds_9^2 + e^{\sqrt{7}\varphi/2} (dz + \mathcal{A}_{[1]})^2 \right) \quad (5.5)$$

and $F_{[n]}$ are n -form field strengths defined as

$$F_{[4]} = \tilde{F}_{[4]} - \tilde{F}_{[3]}^{(1)} \wedge \mathcal{A}_{[1]}^{(1)} - \tilde{F}_{[3]}^{(2)} \wedge \mathcal{A}_{[1]}^{(2)} - \frac{1}{2}\tilde{F}_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(1)} \wedge \mathcal{A}_{[1]}^{(2)} \quad (5.6a)$$

$$F_{[3]}^{(1)} = \tilde{F}_{[3]}^{(1)} - \tilde{F}_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(2)} \quad (5.6b)$$

$$F_{[3]}^{(2)} = \tilde{F}_{[3]}^{(2)} + F_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(1)} - \mathcal{A}_{[0]}^{(12)} \left(\tilde{F}_{[3]}^{(1)} - F_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(2)} \right) \quad (5.6c)$$

$$F_{[2]}^{(12)} = \tilde{F}_{[2]}^{(12)} \quad (5.6d)$$

$$\mathcal{F}_{[2]}^{(1)} = \mathcal{F}_{[2]}^{(1)} + \mathcal{A}_{[0]}^{(12)} \mathcal{F}_{[1]}^{(2)} \quad (5.6e)$$

$$\mathcal{F}_{[2]}^{(2)} = \tilde{\mathcal{F}}_{[2]}^{(2)} \quad (5.6f)$$

$$\mathcal{F}_{[1]}^{(12)} = \tilde{\mathcal{F}}_{[1]}^{(12)}. \quad (5.6g)$$

In the same way, starting from the IIB action one obtains the following nine-dimensional IIB Lagrangian:

$$\begin{aligned}
e^{-1}L_{IIB} = & R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 + \\
& - \frac{1}{48}e^{-\frac{2}{\sqrt{7}}\varphi} F_{[4]}^2 - \frac{1}{12}e^{-\phi + \frac{1}{\sqrt{7}}\varphi} (F_{[3]}^{(NS)})^2 - \frac{1}{2}e^{\phi + \frac{1}{\sqrt{7}}\varphi} (F_{[3]}^{(R)})^2 + \\
& - \frac{1}{4}e^{\frac{4}{\sqrt{7}}\varphi} (\mathcal{F}_{[2]})^2 - \frac{1}{4}e^{\phi - \frac{3}{\sqrt{7}}\varphi} (F_{[2]}^{(R)})^2 - \frac{1}{4}e^{-\phi - \frac{3}{\sqrt{7}}\varphi} (F_{[2]}^{(NS)})^2 + \\
& - \frac{1}{2e}\tilde{F}_{[4]} \wedge \tilde{F}_{[4]} \wedge \mathcal{A}_{[1]} - \frac{1}{e}\tilde{F}_3^{(NS)} \wedge \tilde{F}_{[3]}^{(R)} \wedge A_{[3]}. \quad (5.7)
\end{aligned}$$

Knowing that both describe the same theory we easily obtain the conversion table in Tab. 5.1 which acts as a dictionary between IIA and IIB in ten dimensions plus the following relation between the scalar fields

$$\begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{IIA} = \begin{pmatrix} 3/4 & -\sqrt{7}/4 \\ -\sqrt{7}/4 & -3/4 \end{pmatrix} \begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{IIB} \quad (5.8)$$

This completes the T-duality relations generalizing the usual ones [Bus87] valid in the NS-NS sector.

	IIA		T-duality	IIB	
	$D = 10$	$D = 9$		$D = 9$	$D = 10$
R-R fields	A_3	A_3	\longleftrightarrow	A_3	B_4
		$A_2^{(2)}$	\longleftrightarrow	A_2^R	A_2^R
	$\mathcal{A}_1^{(1)}$	$\mathcal{A}_1^{(1)}$	\longleftrightarrow	A_1^R	
		$\mathcal{A}_0^{(12)}$	\longleftrightarrow	χ	χ
NS-NS fields	$G_{\mu\nu}$	$\mathcal{A}_1^{(2)}$	\longleftrightarrow	A_1^{NS}	A_2^{NS}
	$A_2^{(1)}$	$A_2^{(1)}$	\longleftrightarrow	A_2^{NS}	
		$A_1^{(12)}$	\longleftrightarrow	\mathcal{A}_1	$G_{\mu\nu}$

Table 5.1: T-duality dictionary with RR fields

5.3 The squashed sphere

Start with a $D_1 - D_5$ system in type IIB described in Sec. 5.1. The near-horizon geometry is

$$ds_{10}^2 = \text{AdS}_3 \times S^3 \times \mathbb{R}^4, \quad (5.9)$$

with a three-form flux

$$F_3 = \sqrt{2}m (\omega_{\text{AdS}} + \omega_S), \quad (5.10)$$

where ω is the volume form and the constant m is fixed by demanding:

$$\text{Ric}|_{\text{AdS}} = -m^2 g|_{\text{AdS}} \quad (5.11)$$

$$\text{Ric}|_S = m^2 g|_S. \quad (5.12)$$

Now, introduce the coordinates $(\vartheta, \varphi, \psi, x)$ on $S^3 \times S^1$ and write explicitly:

$$ds_{10}^2 = \text{AdS}_3 \times \mathbb{R}^3 + \frac{1}{2m^2} [d\vartheta^2 + d\varphi^2 + d\psi^2 + 2 \cos \vartheta d\varphi d\psi] + dx^2 \quad (5.13)$$

$$F_3 = m\sqrt{2}\omega_{\text{AdS}} + \frac{\sin \vartheta}{2m^2} d\vartheta \wedge d\varphi \wedge d\psi, \quad (5.14)$$

where x is a periodic variable with period

$$x \sim x + \frac{4\pi}{\lambda}n. \quad (5.15)$$

If we change the variable ψ to $\psi = \alpha + \lambda x$ we still have a 4π -periodic direction α and can rewrite the metric as:

$$ds_{10}^2 = \text{AdS}_3 \times \mathbb{R}^3 + \frac{1}{2m^2} [d\vartheta^2 + d\varphi^2 + d\alpha^2 + 2 \cos \vartheta d\varphi d\alpha] + \left(1 + \frac{\lambda^2}{2m^2}\right) dx^2 + \frac{\lambda}{m^2} (d\alpha + \cos \vartheta d\varphi) dx \quad (5.16)$$

$$F_3 = m\sqrt{2}\omega_{\text{AdS}} + \frac{\sin \vartheta}{2m^2} d\vartheta \wedge d\varphi \wedge d\alpha + \lambda \frac{\sin \vartheta}{2m^2} d\vartheta \wedge d\varphi \wedge dx. \quad (5.17)$$

Redefining

$$z = \sqrt{1 + \frac{\lambda^2}{2m^2}x} \quad h = \frac{\lambda}{2m} \frac{1}{\sqrt{1 + \lambda^2/2m^2}}, \quad (5.18)$$

the fields read

$$\begin{aligned} ds_{10}^2 = & \text{AdS}_3 \times \mathbb{R}^3 + \frac{1}{2m^2} \left[d\vartheta^2 + \sin^2 \vartheta d\varphi^2 + (1 - 2h^2) (d\alpha + \cos \vartheta d\varphi)^2 \right] + \\ & + \left[dz + \frac{h}{m} (d\alpha + \cos \vartheta d\varphi) \right]^2 \end{aligned} \quad (5.19)$$

$$F_3 = m\sqrt{2}\omega_{\text{AdS}} + \frac{\sin \vartheta}{2m^2} d\vartheta \wedge d\varphi \wedge d\alpha + \frac{h}{m} \sin \vartheta d\vartheta \wedge d\varphi \wedge dz, \quad (5.20)$$

and we can perform a Kaluza-Klein reduction on z and go to nine dimensions. The metric reads:

$$ds_9^2 = \text{AdS}_3 \times \mathbb{R}^3 + \frac{1}{2m^2} \left[d\vartheta^2 + \sin^2 \vartheta d\varphi^2 + (1 - 2h^2) (d\alpha + \cos \vartheta d\varphi)^2 \right], \quad (5.21)$$

and the gauge fields are obtained from:

$$F_3 = F_3^{(3)} + F_2^{(3)} \wedge (dz + A), \quad (5.22)$$

where $F_m^{(n)}$ is the m -form obtained from the reduction of a n -form and A is the one-form

$$A = \frac{h}{m} (d\alpha + \cos \vartheta d\varphi). \quad (5.23)$$

Explicitly, adding the extra Kaluza-Klein two-form:

$$F_3^{(3)} = m\sqrt{2}\omega_{\text{AdS}} + (1 - 2h^2) \frac{\sin \vartheta}{2m^2} d\vartheta \wedge d\varphi \wedge d\alpha \quad (5.24)$$

$$F_2^{(3)} = \frac{h}{m} \sin \vartheta d\vartheta \wedge d\varphi \quad (5.25)$$

$$F_2^{(g)} = dA = \frac{h}{m} \sin \vartheta d\vartheta \wedge d\varphi. \quad (5.26)$$

For the moment this is just a rewriting. Now, let us perform a T-duality to go to type IIA. The fields keep their expressions but the interpretation changes according to Tab. 5.1: $F_3^{(3)}$ now comes from the reduction of a four-form in ten dimensions, $F_2^{(2)}$ from a two-form and $F_2^{(g)}$ is now obtained as the result of the reduction of the Kalb-Ramond field:

$$F_3^{(4)} = F_3^{(3)} \quad F_2^{(2)} = F_2^{(3)} \quad F_2^{(B)} = F_2^{(g)}. \quad (5.27)$$

We can oxidise back to ten dimensions and get a IIA background:

$$ds_{10}^2 = \text{AdS}_3 \times \mathbb{R}^3 + \frac{1}{2m^2} \left[d\vartheta^2 + \sin^2 \vartheta d\varphi^2 + (1 - 2h^2) (d\alpha + \cos \vartheta d\varphi)^2 \right] + d\zeta^2 \quad (5.28)$$

$$F_4 = F_3^{(4)} \wedge d\zeta = \left[m\sqrt{2}\omega_{\text{AdS}} + (1 - 2h^2) \frac{\sin \vartheta}{2m^2} d\vartheta \wedge d\varphi \wedge d\alpha \right] \wedge d\zeta \quad (5.29)$$

$$F_2 = F_2^{(2)} = \frac{h}{m} \sin \vartheta d\vartheta \wedge d\varphi \quad (5.30)$$

$$H_3 = F_2^B \wedge d\zeta = \frac{h}{m} \sin \vartheta d\vartheta \wedge d\varphi \wedge d\zeta. \quad (5.31)$$

It is worthwhile to emphasize that by construction α is 4π -periodic and then the geometry is the one of a respectable squashed three-sphere. A very similar construction was considered in [DLP99]. In that case, though, the authors start with the same $\text{AdS}_3 \times S^3$ geometry with both RR and NS-NS fields and then by reducing on one of the sphere isometries, find the Lens space S^3/\mathbb{Z}_p or a squashed version, where p and the squashing depend on the values of the charges. This is clearly an orbifold of the solutions above.

In principle these constructions can be extended to other group manifold geometries (*e.g.* the obvious choice leading to a squashed AdS_3) but in any case one should start from a configuration with RR fields (typically S-dual to the WZW models we described in great detail previously), since the absence of NS-NS antisymmetric fields is the key ingredient for the trivialization of the fiber bundle. More general geometries can be obtained by starting with a mixed RR-NS-NS configuration.

CHAPTER 6

Out of the conformal point: Renormalization Group Flows

This chapter is devoted to the study of the relaxation of squashed WZW models further deformed by the insertion of non-marginal operators. The calculation is carried from both the target space and world-sheet points of view, once more highlighting the interplay between the two complementary descriptions. In the last part such techniques are used to outline the connection between the time evolution and the RG-flow which is seen as a large friction limit description; we are hence naturally led to a FRW-type cosmological model.

STRING THEORY, at least in its world-sheet formulation, is most easily studied on-shell. Thanks to the power of conformal field theory, this permits a profound analysis of specific backgrounds. At the same time, though, it makes it difficult to describe more general effects that require a less local knowledge of the theory and its moduli space. In particular it is not obvious how to describe transitions between two different solutions or even the relaxation of an unstable background towards an on-shell solution.

In this chapter we deviate from conformality by adding non-marginal deformations on the top of exact solutions, such as WZW-models or squashed-group models. The resulting RG-flow then drives those systems back to or away from the conformal point, depending on the character of the deformation. As it is usually the case, these calculations can be faced from two complementary points of view: either in terms of the target space description or in terms of world-sheet two-dimensional theory. We will consider both approaches and show how they do really complete each other, in the sense that they can be considered as two different series expansions of the same quantity. As such, each side contains more information than the other at any given order in perturbation. This will allow us in particular to make a prediction on the outcome of a technically involved one-loop calculation in the WZW and squashed group CFTs on the basis of a two-loop result on target space renormalization.

In the last part of this chapter we use the technology developed so far to show how an RG-flow analysis can allow for further insights on the issue of time-dependent solutions. More precisely we will see how for a given class of

systems whose geometry is the warped product of a constant curvature space and a time direction the RG-flow equations are a sort of large-friction approximation with the central charge deficit playing the rôle of an effective friction coefficient.

6.1 The target space point of view

Renormalization in a dimensional regularization scheme

Consider the σ -model with Lagrangian density

$$\mathcal{L} = \frac{1}{2\lambda} (g_{ij} + B_{ij}) \Xi^{ij}, \quad (6.1)$$

where g_{ij} is a metric, B_{ij} a two-form and $\Xi^{ij} = \partial_\mu X^i \partial^\mu X^j + \epsilon_{\mu\nu} \partial^\mu X^i \partial^\nu X^j$. We will say that the model is renormalizable if the corresponding counterterms at any given order in the loop expansion can be reabsorbed into a renormalization of the coupling constant and other parameters that appear in the expressions for g_{ij} and B_{ij} .¹

The standard technique for dealing with this kind of Lagrangian consists in incorporating the Kalb–Ramond field (or, equivalently, the WZ term) into the geometry by reading it as a torsion. This means that instead of the usual Levi-Civita connection one uses the connection Γ^- defined as

$$\Gamma^{-i}{}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - \frac{1}{2} H^i{}_{jk}. \quad (6.2)$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ is the Christoffel symbol and with respect to this connection one defines the Riemann and Ricci tensors R^- and Ric^- .

Two-loop target
space RG-flow

Now, using the background field method in a dimensional regularization scheme (see [Osb90, AGFM81, Fri85, Fri80, BCZ85, HT87] and for various applications [BFHP96, BFM⁺98, Sfe98, Sfe99]) we can calculate the one- and two-loop counterterms that turn out to be:

$$\mu^\varepsilon \mathcal{L}^{(1)} = \frac{1}{\pi\varepsilon} T^{(1)} = \frac{\alpha'}{2\varepsilon\lambda} Ric^-{}_{ij} \Xi^{ij}, \quad (6.3)$$

$$\mu^\varepsilon \mathcal{L}^{(2)} = \frac{\lambda}{8\pi^2\varepsilon} T^{(2)} = \frac{\alpha'^2}{16\varepsilon\lambda} Y^{lmk}{}_j R^-{}_{iklm} \Xi^{ij}, \quad (6.4)$$

where Y is given by

$$Y_{ijkl} = -2R^-{}_{ijkl} + 3R^-{}_{[kij]l} + \frac{1}{2} (H^2{}_{ki}g_{jl} - H^2{}_{kj}g_{il}), \quad (6.5)$$

and

$$H^2{}_{ij} = H_{[3]ilm} H_{[3]j}{}^{lm}. \quad (6.6)$$

¹When this is not the case the model might nevertheless be renormalizable in a more general sense, in the infinite-dimensional space of metrics and torsions

In general the metric and the Kalb–Ramond field depend on a set of bare parameters $a_k^{(0)}$. In this case we can convert the counterterms given above into coupling and parameter renormalizations if we write perturbatively the bare quantities as:

$$\begin{cases} \lambda^{(0)} = \mu^\varepsilon \lambda \left(1 + \frac{J_1(a)}{\pi\varepsilon} \lambda + \dots \right) = \mu^\varepsilon \lambda \left(1 + \frac{y_\lambda}{\varepsilon} + \dots \right), \\ a_k^{(0)} = a_k + \frac{a_k^{(1)}(a)}{\pi\varepsilon} + \dots = a_k \left(1 + \frac{y_{a_k}}{\varepsilon} + \dots \right), \\ X^{(0)\mu} = X^\mu + \frac{X^{(1)\mu}(X,a)}{\pi\varepsilon} + \dots, \end{cases} \quad (6.7)$$

where we allowed for a slight generalization with respect to the definition of renormalization given above in terms of a coordinate reparametrization of the target space². Then the one- and two-loop β -functions are given by:

$$\begin{cases} \beta_\lambda = \frac{d\lambda}{dt} = \lambda^2 \frac{\partial y_\lambda}{\partial \lambda} = \frac{\lambda^2}{\pi} \left(J^{(1)}(a) + \frac{\lambda}{4\pi} J^{(2)}(a) \right), \\ \beta_{a_k} = \frac{da_k}{dt} = \lambda a_k \frac{\partial y_{a_k}}{\partial \lambda} = \frac{\lambda}{\pi} \left(a_k^{(1)}(a) + \frac{\lambda}{4\pi} a_k^{(2)}(a) \right). \end{cases} \quad (6.8)$$

The unknown functions $J^{(i)}, a^{(i)}, X^{(i)\mu}$ are determined by the equation

$$T^{(i)} = -J^{(i)} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial a_k} a_k^{(i)} + \frac{\partial \mathcal{L}}{\partial X^\mu} X^{(i)\mu}. \quad (6.9)$$

This corresponds to demanding the generalized quantum effective action $\Gamma[X]$ to be finite order by order.

Two-loop β equations for a WZW model.

As we have already announced in Ch. 2, the normalization for the WZ term in a WZW action can be fixed by an RG-flow calculation. This is precisely what we will do in the following at two-loop order using the dimensional regularization scheme outlined above. In this way we will find a new apparent non-trivial solution that doesn't show up at first order (and which will prove to be an artifact as we'll see in the following, by using a CFT description in Sec. 6.2). Moreover we will see how the roles of the IR and UV limits are interchanged between the compact and the non-compact case, *ie* how the same kind of deformation is relevant or irrelevant depending on the compactness of the starting model.

Consider the following action

$$S_{\lambda,H} = \frac{1}{2\lambda} \int_{\Sigma} d^2z (g_{ab} + \text{HB}_{ab}) J_\mu^a J_\nu^b \partial X^\mu \bar{\partial} X^\nu \quad (6.10)$$

where J^a are the Maurer–Cartan one-forms for a group G whose algebra has structure constants f_{abc} , $g_{ab} = -1/(2g^*) f_{as}^t f_{bt}^s$ (g^* is the dual Coxeter number)

²This redefinition is in general non-linear; in the special case when the $X^{(i)}$'s depend linearly on X the last equation of the system 6.7 reduces to a multiplicative wavefunction renormalization. The only condition is that $X^{(i)}$ shouldn't depend on the derivatives of X . In more geometric terms we are just using the diffeomorphism invariance of the renormalized theory.

and B_{ab} is an antisymmetric matrix satisfying $d(B_{ab}J^a \wedge J^b) = 1/3! f_{abc}J^a \wedge J^b \wedge J^c$ as in Sec. 2.2. Since the deformation (parameterized by having $H \neq 1$) doesn't affect the geometric part (but for the overall normalization) we can still express the curvature in terms of the Lie algebra structure constants. In particular it is easy to recover that the Riemann tensor is written as:

$$R^a{}_{bcd} = \frac{1}{4} f^a{}_{be} f^e{}_{cd} \quad (6.11)$$

and the Ricci tensor is obtained by contracting:

$$Ric_{ab} = \frac{g^*}{2} g_{ab}. \quad (6.12)$$

as in Eqs. (2.65).

In order to write the beta equations as described above we need to incorporate the WZ term (or, more precisely, the Kalb–Ramond field) into the geometry. The most natural approach is to consider $H^a{}_{bc}$ as a torsion and include it in the connection [BCZ85]. We hence define:

$$\Gamma^{-a}{}_{bc} = \left\{ \begin{matrix} a \\ bc \end{matrix} \right\} - \frac{1}{2} H^a{}_{bc}. \quad (6.13)$$

The covariant derivative of a one-form is then defined as:

$$\nabla^-{}_a V_b = \partial_a V_b - \Gamma^{-c}{}_{ab} V_c = \nabla_a V_b + \frac{1}{2} H^c{}_{ab} V_c \quad (6.14)$$

where ∇_a is the covariant derivative with respect to the Levi–Civita connection. Similarly we define the curvature:

$$[\nabla^-{}_a, \nabla^-{}_b] V_c = R^{-d}{}_{cab} V_d + H^d{}_{ab} \nabla^-{}_d V_c \quad (6.15)$$

and it is straightforward to show that:

$$R^-{}_{abcd} = R_{abcd} + \frac{1}{2} \nabla_c H_{abd} - \frac{1}{2} \nabla_d H_{abc} + \frac{1}{4} H_{fac} H^f{}_{db} - \frac{1}{4} H_{fad} H^f{}_{cb}. \quad (6.16)$$

Let us now specialize this general relation to our particular deformation. Since $H_{abc} = H f_{abc}$ it is immediate that $\nabla_a H_{bcd} = 0$ and that the Jacobi identity holds. We then derive:

$$R^-{}_{abcd} = (1 - H^2) R_{abcd}, \quad (6.17)$$

whence in particular:

$$Ric^-{}_{ab} = (1 - H^2) Ric_{ab} = (1 - H^2) \frac{g^*}{2} g_{ab}. \quad (6.18)$$

The one-loop counterterm becomes:

$$T_{ab}^{(1)} = \frac{g^*}{4} (1 - H^2) g_{ab}. \quad (6.19)$$

The evaluation of the two-loop counterterm is lengthy but straightforward once R_{abcd} is written in terms of the structure constants. The result is:

$$T_{ab}^{(2)} = \frac{g^{*2}}{8} (1 - H^2) (1 - 3H^2) g_{ab}. \quad (6.20)$$

Substituting these expressions in Eq. (6.9) (and using the fact that g_{ab} and ϵ_{ab} are orthogonal) one sees that they become identities for the following choice of parameters:

$$\begin{cases} J^{(1)} = -\frac{g^*}{4} (1 - H^2) \\ a_H^{(1)} = -\frac{g^*}{4} H (1 - H^2) \end{cases} \quad \begin{cases} J^{(2)} = -\frac{g^{*2}}{8} (1 - H^2) (1 - 3H^2) \\ a_H^{(2)} = -\frac{g^{*2}}{8} H (1 - H^2) (1 - 3H^2) \end{cases} \quad (6.21)$$

corresponding to the following beta equations:

$$\beta_H = -\frac{1}{4\pi} \lambda^* H (1 - H^2) \left(1 + \frac{1}{8\pi} \lambda^* (1 - 3H^2) \right) \quad (6.22)$$

$$\beta_{\lambda^*} = -\frac{1}{4\pi} \lambda^{*2} (1 - H^2) \left(1 + \frac{1}{8\pi} \lambda^* (1 - 3H^2) \right) \quad (6.23)$$

where $\lambda^* = g^* \lambda$ is the effective coupling constant (this is precisely the fixed parameter in a 't Hooft limit since for a $SU(N)$ group $g^* = N$). The difference between a compact and a non-compact group lies in the sign of the dual Coxeter number that is respectively positive/negative. In both cases we remark that H/λ^* remains constant, which is a nice check of our construction, since in the notation of Ch. 2 this is just the level of the model that, in the compact case, is quantized and hence is not expected to receive any perturbative correction. On the other hand non perturbative effects do eventually lead to the $k \rightarrow k + g^*$ shift which is the reason for the two-loop behaviour of the flow.

Let us analyze the flow in detail:

- The flow diagram for the compact case is drawn in Fig.6.1 where we see the presence of three phases:
 - region ① is the basin of attraction for the WZW model ($z = 1$);
 - the points in region ② describe systems that flow towards asymptotic freedom;
 - region ③ seems to be the basin of attraction for a different theory, always with a group manifold geometry but with a differently-normalized WZ term.

only a discrete set of trajectories is allowed and, in particular, region ③ – separated from region ① by the line $\lambda^* = 4\pi H$ – is only accessible for levels $k < g^*/2$.

- The flow diagram for the non-compact case is drawn in Fig...; again we see three different phases:
 - region ① describe theories flowing to asymptotic freedom;

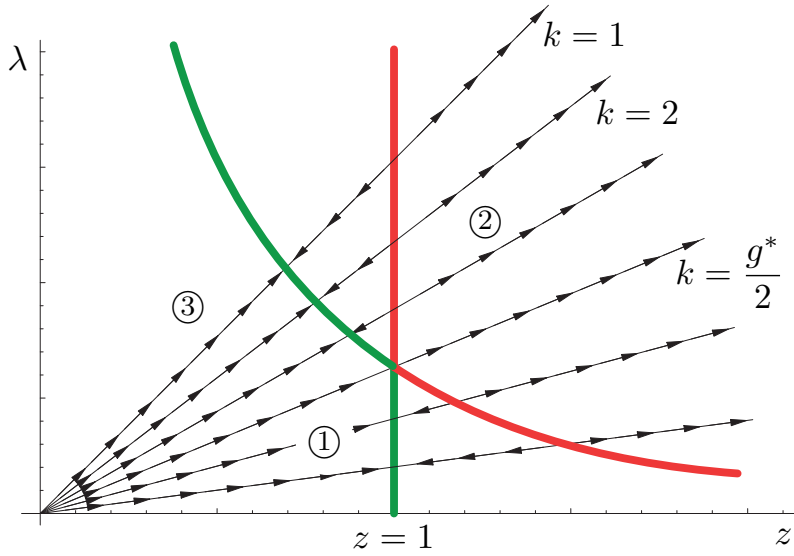


Figure 6.1: Two-loop RG-flow diagram for compact groups.

- region ② looks like the basin of attraction for the non-trivial solution with the group manifold metric and a new normalization topological term;
- region ③ describe theories flowing to a strong coupling regime.

In particular it is interesting to remark that the roles of the UV and IR are somehow inverted. The WZW model appears as a UV fixed point and thus an unstable solution from the point of view of dynamical systems.

Renormalization group-flow in squashed compact groups

The models that we have presented in Ch. 3 are conformal; for this reason we expect to find them as fixed points in an RG flow. To verify this claim let us introduce a two-parameter family of σ models generalizing the exact backgrounds; a possible choice consists in adding an extra magnetic field on the top of the one responsible for the squashing, but now coming from a higher-dimensional right sector. Explicitly

$$\left\{ \begin{array}{l} ds^2 = \sum_{\mu \in G/T} J^\mu J^\mu + (1 - H^2) \sum_{a \in T} J^a J^a, \\ H_{[3]} = \frac{\tilde{H}}{2\tilde{H}} f_{\mu\nu\rho} J^\mu \wedge J^\nu \wedge J^\rho \\ F^a = \frac{H+\tilde{H}}{2} \sqrt{\frac{k}{k_g}} f^a_{\mu\nu} J^\mu \wedge J^\nu \\ \bar{F}^a = \frac{H-\tilde{H}}{2} \sqrt{\frac{k}{k_g}} f^a_{\mu\nu} J^\mu \wedge J^\nu \end{array} \right. \quad \begin{array}{l} \mu \in G/T, \\ \mu \in G/T, a \in T, \\ \mu \in G/T, a \in T \end{array} \quad (6.24)$$

and in particular for $SU(2)$:

$$\begin{cases} ds^2 = d\theta^2 + d\psi^2 + d\phi^2 + \cos\theta d\psi d\phi - H^2 (d\psi + \cos\theta d\phi)^2, \\ B = \frac{\bar{H}}{H} \cos\theta d\psi \wedge d\phi, \\ A = (H + \bar{H}) (d\psi + \cos\theta d\phi), \\ \bar{A} = (H - \bar{H}) (d\psi + \cos\theta d\phi), \end{cases} \quad (6.25)$$

where \bar{H} is a new parameter, describing the deviation from the conformal point. It is clear that the above background reduces to the one we are used to in the $\bar{H} \rightarrow H$ limit. In particular we see that the metric is unchanged, the Kalb–Ramond field has a different normalization and a new field \bar{A} appears. This configuration can be described in a different way: the geometry of a squashed sphere supports two covariantly constant magnetic fields with charge $Q = H + \bar{H}$ and $\bar{Q} = H - \bar{H}$; the RG flow will describe the evolution of these two charges from a generic (Q, \bar{Q}) to $(2H, 0)$, while the sum $Q + \bar{Q} = 2H$ remains constant. In this sense the phenomenon can be interpreted as a charge transmutation of \bar{Q} into Q . The conservation of the total charge is in fact a consequence of having chosen a perturbation that keeps the metric and only changes the antisymmetric part of the background.

We can also see the background in Eq.(6.24) from a higher dimensional perspective where only the metric and the Kalb–Ramond field are switched on. Pictorially:

$$g = \left(\begin{array}{c|c} g_{WZW} & HJ_a \\ \hline HJ_a & 1 \end{array} \right) \quad B = \left(\begin{array}{c|c} \frac{\bar{H}}{H} B_{WZW} & \bar{H}J_a \\ \hline -\bar{H}J_a & 0 \end{array} \right) \quad (6.26)$$

where g_{WZW} and B_{WZW} are the usual metric and Kalb–Ramond fields for the WZW model on the group G . More explicitly in the $SU(2)$ case:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \cos\theta & H \\ 0 & \cos\theta & 1 & H \cos\theta \\ 0 & H & H \cos\theta & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\bar{H}}{H} \cos\theta & \bar{H} \\ 0 & -\frac{\bar{H}}{H} \cos\theta & 0 & \bar{H} \cos\theta \\ 0 & -\bar{H} & -\bar{H} \cos\theta & 0 \end{pmatrix} \quad (6.27)$$

where the fourth entry represents the bosonized internal current. In particular this clarifies the stated right-sector origin for the new gauge field \bar{A} . This higher dimensional formalism is the one we will use in the following RG analysis.

The beta-equations at two-loop order in the expansion in powers of the overall coupling constant λ and the field redefinitions for the internal coordi-

nates X^i turn out to be:

$$\begin{cases} \beta_{\lambda^*} = \frac{d\lambda^*}{dt} = -\frac{\lambda^{*2}}{4\pi} \left(1 - \frac{\bar{H}^2}{H^2}\right) \left(1 + \frac{\lambda^*}{8\pi} \left(1 - 3\frac{\bar{H}^2}{H^2}\right)\right), \\ \beta_H = \frac{dH}{dt} = \frac{\lambda^* H}{8\pi} \left(1 - H^2\right) \left(1 - \frac{\bar{H}^2}{H^2}\right) \left(1 + \frac{\lambda^*}{8\pi} \left(1 - 3\frac{\bar{H}^2}{H^2}\right)\right), \\ \beta_{\bar{H}} = \frac{d\bar{H}}{dt} = -\frac{\lambda^* \bar{H}}{8\pi} \left(1 + H^2\right) \left(1 - \frac{\bar{H}^2}{H^2}\right) \left(1 + \frac{\lambda^*}{8\pi} \left(1 - 3\frac{\bar{H}^2}{H^2}\right)\right), \\ X^i = X^i - \frac{\lambda^*}{16} \left(1 - H^2\right) \left(1 - 4\frac{\bar{H}^2}{H^2} + 3\frac{\bar{H}^4}{H^4}\right), \end{cases} \quad (6.28)$$

where $\lambda^* = \lambda g^*$, g^* being the dual Coxeter number, is the effective coupling constant ($\lambda^* = N\lambda$ for $G = SU(N)$). The contributions at one- and two-loop order are clearly separated. In the following we will concentrate on the one-loop part and we will comment on the two-loop result later. Let us then consider the system:

$$\begin{cases} \beta_{\lambda^*} = \frac{d\lambda^*}{dt} = -\frac{\lambda^{*2}}{4\pi} \left(1 - \frac{\bar{H}^2}{H^2}\right), \\ \beta_H = \frac{dH}{dt} = \frac{\lambda^* H}{8\pi} \left(1 - H^2\right) \left(1 - \frac{\bar{H}^2}{H^2}\right), \\ \beta_{\bar{H}} = \frac{d\bar{H}}{dt} = -\frac{\lambda^* \bar{H}}{8\pi} \left(1 + H^2\right) \left(1 - \frac{\bar{H}^2}{H^2}\right). \end{cases} \quad (6.29)$$

This can be integrated by introducing the parameter $z = \bar{H}/H$ which makes one of the equations redundant. The other two become:

$$\begin{cases} \dot{\lambda}^* = -\frac{\lambda^{*2}}{4\pi} (1 - z^2), \\ \dot{z} = -\frac{\lambda^* z}{4\pi} (1 - z^2). \end{cases} \quad (6.30)$$

By inspection one easily sees that $\dot{\lambda}^*/\lambda^* = \dot{z}/z$, implying $\lambda^*(t) = Cz(t)$, where C is a constant. This was to be expected since C is proportional to the normalization of the topological WZ term. Since we are dealing with a compact group it turns out that C is, as in [Wit84], quantized with:

$$C_k = \frac{2\pi}{k}, \quad k \in \mathbb{N}. \quad (6.31)$$

Now it's immediate to separate the system and find that $z(t)$ is defined as the solution to the implicit equation:

$$-\frac{t}{2k} = \frac{1}{z_0} - \frac{1}{z(t)} + \log \left[\frac{(z(t) + 1)(z_0 - 1)}{(z(t) - 1)(z_0 + 1)} \right] \quad (6.32)$$

with the initial condition $z(0) = z_0$. A similar expression was found in [BCZ85, Wit84]. The reason for this is, as pointed out previously [KK95], that the conformal model ($\bar{H} = H$) in its higher-dimensional representation (the one in Eq. (6.26)) coincides with a $G \times H$ WZW model after a suitable local field redefinition.

As it is usually the case in the study of non-linear dynamics, a better understanding of the solution is obtained by drawing the RG flow. In a (z, λ^*) plane, the trajectories are straight lines through the origin and only a discrete set of them are allowed. Moreover the line $z = 1$ is an IR fixed-point locus. This

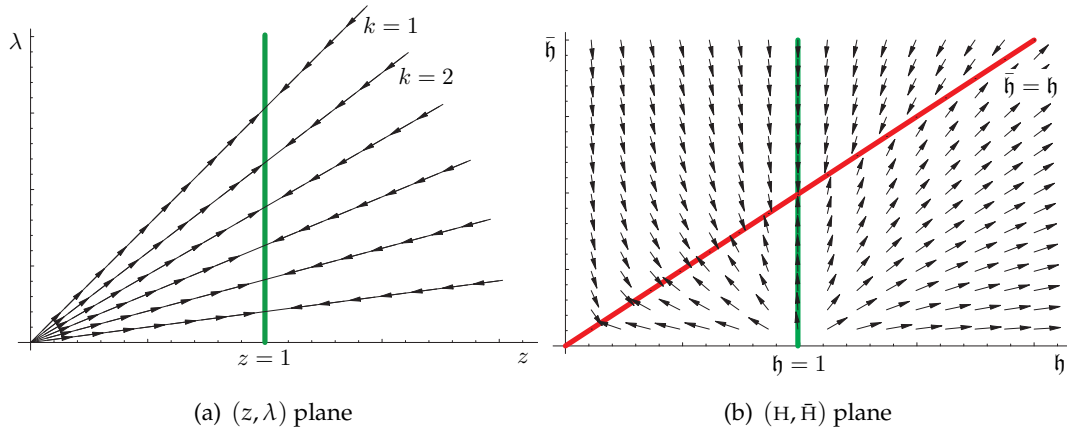


Figure 6.2: Flow lines for the deformed (non-conformal) squashed WZW model in (a) the (z, λ) and (b) the (H, \bar{H}) planes. The arrows point in the negative t direction, *i.e.* towards the infrared; in (a) we see how the squashed WZW model $z = 1$ appears as an IR fixed point, in (b) how perturbing the conformal $\bar{H} = H$ model by increasing \bar{H} leads to a new fixed point corresponding to a value of H closer to 1.

situation is sketched in Fig. 6.2(a). Just as expected the $z = \bar{H}/H = 1$ point, corresponding to the initial exact model described in Ch. 3, is an IR fixed point for the RG flow.

Further insights can be gained if we substitute the condition $\lambda^* = C_k \bar{H}/H$ into the system (6.29) thus getting:

$$\begin{cases} \frac{dH}{dt} = \frac{\bar{H}}{4k} (1 - H^2) \left(1 - \frac{\bar{H}^2}{H^2}\right), \\ \frac{d\bar{H}}{dt} = -\frac{\bar{H}^2}{4kH} (1 + H^2) \left(1 - \frac{\bar{H}^2}{H^2}\right). \end{cases} \quad (6.33)$$

The flow diagram for this system in the (H, \bar{H}) plane, Fig. 6.2(b), shows how the system relaxes to equilibrium after a perturbation. In particular we can see how increasing \bar{H} leads to a new fixed point corresponding to a value of H closer to 1.

We would like to pause for a moment and put the above results in perspective. Consider for simplicity the $SU(2)$ case: the target-space of the sigma-model under consideration is a squashed three-sphere with two different magnetic fields. Along the flow, a transmutation of the two magnetic charges occurs: the system is driven to a point where one of the magnetic charges vanishes. This fixed point is an ordinary squashed-WZW (of the type studied in Ch. 3), that supports a single magnetic charge.

As we pointed out in Ch. 3, in the squashed-WZW, the magnetic field is bounded by a critical value, $H = 1$. As long as $H \leq 1$, the geometry is a genuine squashed three-sphere. For $H > 1$, the signature becomes Lorentzian and the geometry exhibits closed time-like curves. Although of limited physical interest, such a background can be used as a laboratory for investigating the fate of chronological pathologies along the lines described above. In the case

under consideration and under the perturbation we are considering the model shows a symmetry between the $H > 1$ and $H < 1$ regions. The presence of closed time-like curves doesn't seem to effect the stability (note that regions with different signatures are disconnected, *i.e.* the signature of the metric is preserved under the RG flow). It is clear however that these results are preliminary. To get a more reliable picture for the fate of closed time-like curves, one should repeat the above analysis in a wider parameter space, where other RG motions might appear and deliver a more refined stability landscape.

A final remark concerns the fact that we find the same RG flow behaviour as for a compact (non-squashed) group. We have already made extensive use of the fact that formally the squashed $SU(2)$ behaves like a $SU(2) \times U(1)$ WZW model, in particular in Sec. 3.4 where this was at the root of the non-renormalization theorem. In some sense, then, the present calculation is just a perturbative confirmation of that statement.

Renormalization group-flow in squashed anti de Sitter

As we've already discussed in Sec. 4.2, sigma models based on non-compact group offer richer (*i.e.* more complex) phase diagrams than the compact ones. In our particular models this is because the possible choices for a Cartan torus are not pairwise conjugated by inner automorphisms and this is why different choices correspond to inequivalent backgrounds, exhibiting different physical properties. If we concentrate our attention on the $SL(2, \mathbb{R})$ WZW model (that is the only non-compact case with just one time direction), we see that the three possible choices for the Cartan generator (elliptic, parabolic, hyperbolic) respectively lead to the exact backgrounds we introduced in Sec. 4.2 and we report here for convenience:

$$\begin{cases} ds^2 = d\rho^2 - dt^2 + d\phi^2 - 2 \sinh \rho dt d\phi - H^2 (dt + \sinh \rho d\phi)^2, \\ B = \sinh \rho dt \wedge d\phi, \\ A = 2H (dt + \sinh \rho d\phi). \end{cases} \quad (6.34)$$

$$\begin{cases} ds^2 = \frac{du^2}{u^2} + \frac{dx^+ dx^-}{u^2} - H^2 \frac{dx^+ dx^-}{u^4}, \\ B = \frac{dx^+ \wedge dx^-}{u^2}, \\ A = 2H \frac{dx^+}{u^2}. \end{cases} \quad (6.35)$$

$$\begin{cases} ds^2 = dr^2 + dx^2 - d\tau^2 + 2 \sinh r dx d\tau - H^2 (dx + \sinh r d\tau)^2 \\ B = \sinh r dx \wedge d\tau, \\ A = 2H (dx + \sinh r d\tau). \end{cases} \quad (6.36)$$

Since these solutions are exact CFT backgrounds, we expect them to appear as fixed points for an RG flow, like the compact configuration described in the previous section. As we will see in the following this is actually the case, but with a difference regarding the role of the UV and IR which is proper to non-compact groups (as explained in Sec. 6.1). Using the same technique as

above, the first step consists in generalizing the three backgrounds by introducing the following three families of low energy configurations:

$$\begin{cases} ds^2 = d\rho^2 - dt^2 + d\phi^2 - 2 \sinh \rho dt d\phi - H^2 (dt + \sinh \rho d\phi)^2 \\ B = \frac{\bar{H}}{H} \sinh \rho dt \wedge d\phi \\ A = (H + \bar{H}) (dt + \sinh \rho d\phi) \\ \bar{A} = (H - \bar{H}) (dt + \sinh \rho d\phi) \end{cases} \quad (6.37)$$

$$\begin{cases} ds^2 = \frac{du^2}{u^2} + \frac{dx^+ dx^-}{u^2} - H^2 \frac{dx^+ dx^-}{u^4} \\ B = \frac{\bar{H}}{H} \frac{dx^+ \wedge dx^-}{u^2} \\ A = (H + \bar{H}) \frac{dx^+}{u^2} \\ \bar{A} = (H - \bar{H}) \frac{dx^+}{u^2} \end{cases} \quad (6.38)$$

$$\begin{cases} ds^2 = dr^2 + dx^2 - d\tau^2 + 2 \sinh r dx d\tau - H^2 (dx + \sinh r d\tau)^2 \\ B = \frac{\bar{H}}{H} \sinh r dx \wedge d\tau \\ A = (H + \bar{H}) (dx + \sinh r d\tau) \\ \bar{A} = (H - \bar{H}) (dx + \sinh r d\tau) \end{cases} \quad (6.39)$$

The guiding principle remains the same, *i.e.* keep the same geometry, rescale the KR field and introduce a new electromagnetic field, coming (in a four-dimensional perspective) from the right-moving sector. Again we will observe the same charge-transmutation effect as before, this time in terms of charge density (or charge at infinity).

The backgrounds above can be equivalently described in four dimensions by a metric and a KR field as follows:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -\sinh \rho & H \\ 0 & -\sinh \rho & 1 & H \sinh \rho \\ 0 & H & H \sinh \rho & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\bar{H}}{H} \sinh \rho & \bar{H} \\ 0 & -\frac{\bar{H}}{H} \sinh \rho & 0 & \bar{H} \sinh \rho \\ 0 & -\bar{H} & -\bar{H} \sinh \rho & 0 \end{pmatrix} \quad (6.40)$$

$$g = \begin{pmatrix} \frac{1}{u^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2u^2} & \frac{H}{u^2} \\ 0 & -\frac{1}{2u^2} & 0 & 0 \\ 0 & \frac{H}{u^2} & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\bar{H}}{H} \frac{1}{2u^2} & \frac{\bar{H}}{u^2} \\ 0 & \frac{\bar{H}}{H} \frac{1}{2u^2} & 0 & 0 \\ 0 & -\frac{\bar{H}}{u^2} & 0 & 0 \end{pmatrix} \quad (6.41)$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \sinh r & H \\ 0 & \sinh r & -1 & H \sinh r \\ 0 & H & H \sinh r & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\bar{H}}{H} \sinh r & \bar{H} \\ 0 & -\frac{\bar{H}}{H} \sinh r & 0 & \bar{H} \sinh r \\ 0 & -\bar{H} & -\bar{H} \sinh r & 0 \end{pmatrix} \quad (6.42)$$

We must now evaluate the R^- tensor (*i.e.* the Ricci tensor with respect to the connection $\Gamma^- = \Gamma + 1/2H$) and read the counterterms in a dimensional

regularization scheme as described in Eq. (6.7):

$$\left\{ \begin{array}{l} J^{(1)} = \frac{1}{4} \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ a_H^{(1)} = -\frac{1+H^2}{8} H \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ a_{\bar{H}}^{(1)} = \frac{1-H^2}{8} \bar{H} \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ X_X^{(1)} = \frac{1+H^2}{8} \left(1 - \frac{\bar{H}^2}{H^2} \right) X. \end{array} \right. \quad \left\{ \begin{array}{l} J^{(1)} = \frac{1}{4} \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ a_H^{(1)} = -\frac{1}{8} H \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ a_{\bar{H}}^{(1)} = \frac{1}{8} \bar{H} \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ X_X^{(1)} = \frac{1}{8} \left(1 - \frac{\bar{H}^2}{H^2} \right) X. \end{array} \right. \quad \left\{ \begin{array}{l} J^{(1)} = \frac{1}{4} \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ a_H^{(1)} = -\frac{1-H^2}{8} H \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ a_{\bar{H}}^{(1)} = \frac{1+H^2}{8} \bar{H} \left(1 - \frac{\bar{H}^2}{H^2} \right), \\ X_X^{(1)} = \frac{1-H^2}{8} \left(1 - \frac{\bar{H}^2}{H^2} \right) X. \end{array} \right. \quad (6.43)$$

The analogies among the three configurations are clear, but become striking when we introduce the parameter $z = \bar{H}/H$ and all three β -functions systems all reduce to the following:

$$\left\{ \begin{array}{l} \dot{\lambda} = \frac{\lambda^2}{4\pi} (1 - z^2), \\ \dot{z} = \frac{\lambda z}{4\pi} (1 - z^2). \end{array} \right. \quad (6.44)$$

This is (up to a sign) the same system we found in the compact case and it is hence immediate to write the solution

$$\lambda(t) = Cz(t) \quad (6.45)$$

$$\frac{Ct}{4\pi} = \frac{1}{z_0} - \frac{1}{z(t)} + \log \left[\frac{(z(t) + 1)(z_0 - 1)}{(z(t) - 1)(z_0 + 1)} \right]. \quad (6.46)$$

Although, as expected, $z = 1$ is a fixed point (corresponding to the conformal points) some differences are important. First of all the background is non-compact, so C is not quantized and, although the flow trajectories are still straight lines through the origin, the angular parameter is now arbitrary. The other difference is that $z = 1$ is a fixed point, but it doesn't correspond to a IR stable configuration but to a UV stable one. This is precisely the same behaviour that one encounters for non-compact WZW models when varying the normalization of the WZ term (as in Sec. 6.1). Again the flow diagram is the same as for the original $SL(2, \mathbb{R})$ group and is summarized in Fig. 6.3.

6.2 The CFT approach

In order to make contact with genuine CFT techniques, we must identify the relevant operators which are responsible for the (H, \bar{H}) deformation of the $G \times H$ original WZW model ($H = U(1)^{\text{rank } G}$). At lowest approximation, all we need is their conformal dimensions in the unperturbed theory.

Following [Zam86], let S_0 be the unperturbed (conformal) action and \mathcal{O}_i the relevant operators of conformal dimension $\Delta_i = 1 - \epsilon_i$. Consider the perturbed model,

$$S = S_0 + g^i \mathcal{O}_i. \quad (6.47)$$

The tree-level beta-functions read:

$$\beta^i(g) = -\epsilon_i g^i, \quad (6.48)$$

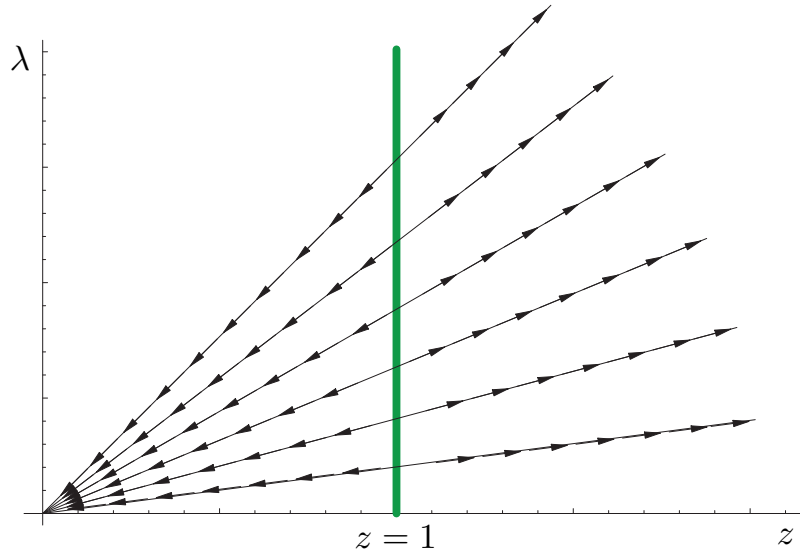


Figure 6.3: Flow diagram for the system in Eq. (6.44). The arrows point from the UV to the IR and $z = 1$ appears as a UV stable solution locus.

where g^i is supposed to be small, for the perturbative expansion of β^i to hold³.

The $G \times H$ primary operator we need can be written as follows:

$$\mathcal{O} = \sum_{A,B} \langle t^A g t^B g^{-1} \rangle \langle t^A \partial g g^{-1} \rangle \langle t^B g^{-1} \bar{\partial} g \rangle = \sum_{A,B} \Phi^{AB} J^A \bar{J}^B, \quad (6.49)$$

where Φ^{AB} is a primary field transforming in the adjoint representation of the left and right groups G . As such, the total conformal dimensions (as we've seen in Sec. 2.1) are

$$\Delta = \bar{\Delta} = 1 + \frac{g^*}{g^* + k'}, \quad (6.50)$$

where g^* is the dual Coxeter number and as such the operator is irrelevant (in the infrared).

Specializing this general construction to our case we find that the action for the fields in Eq. (3.17) is:

$$S = \frac{k}{4\pi} \left\{ S_0 + \left(\frac{H}{\bar{H}} - 1 \right) \sum_{A,B} \Phi^{AB} J^A \bar{J}^B + \frac{H}{\bar{H}} (H + \bar{H}) \sum_i J^{a_i} \bar{J}^i + \frac{H}{\bar{H}} (H - \bar{H}) \sum_{i,A} J^i \Phi^{a_i A} \bar{J}^A \right\}. \quad (6.51)$$

where A runs over all currents, i over the internal currents (in H) and J^{a_i} is the WZW current of the Cartan subalgebra of G coupled to the internal \bar{J}^i . The extra

³One should be very careful in the choice of signs in these formulae. In [Zam86] the time variable, in fact, describes the evolution of the system towards the infrared and as such it is opposite with respect to the $t = \log \mu$ convention that we used in the previous section (as in [Wit84]).

terms can be interpreted as relevant combinations of operators in the $G \times H$ model. The beta-functions are thus computed following Eq. (6.48):

$$\begin{cases} \left. \frac{d}{dt} \log \left[\left(\frac{H}{\bar{H}} - 1 \right) \right] \right|_{\bar{H}=H} = \frac{g^*}{g^*+k} = \frac{g^*}{k} - \frac{g^{*2}}{k^2} + \mathcal{O}\left(\frac{1}{k^3}\right), \\ \left. \frac{d}{dt} \log \left[\frac{H}{\bar{H}} (H + \bar{H}) \right] \right|_{\bar{H}=H} = \mathcal{O}\left(\frac{1}{k^3}\right), \\ \left. \frac{d}{dt} \log \left[\frac{H}{\bar{H}} (H - \bar{H}) \right] \right|_{\bar{H}=H} = \frac{g^*}{g^*+k} = \frac{g^*}{k} - \frac{g^{*2}}{k^2} + \mathcal{O}\left(\frac{1}{k^3}\right). \end{cases} \quad (6.52)$$

Equations (6.52) agree with the results of the field-theoretical approach (up to the overall normalization), at least in the regime where (6.52) are valid, namely for small H and \bar{H} perturbations. But there's more: as pointed out before the conformal model ($\bar{H} = H$) is exact because it coincides with a $G \times H$ WZW model after a suitable field redefinition for any value of H . As a consequence the equations remain valid for any finite H . This is reassuring both for the validity of the geometrical approach⁴ and for the conclusions on the stability picture of the models under consideration.

The extra information that we obtain from this calculation is about the interpretation for the two-loop β -function we described in the previous section. In fact it is now clear that with the target-space approach we just describe the Taylor expansion of the tree-level CFT result:

$$\frac{g^*}{g^*+k} = \frac{g^*}{k} - \frac{g^{*2}}{k^2} + \mathcal{O}\left(\frac{1}{k^3}\right). \quad (6.53)$$

This is not surprising since the would-be non-trivial fixed point of the two-loop expansion lay out of the validity range for our approximation. If we really want to go beyond the large k limit, we need to push the analysis from this, CFT, side.

Target space vs
CFT
renormalization

From the target space view point, the renormalization approach remains valid in the large k limit for any value of H/\bar{H} . This enables us to use Eq. (6.28) and push (for $k \rightarrow \infty$) Eq. (6.52) at least to the next leading order in $(1 - H/\bar{H})$ so to get

$$\left. \frac{d}{dt} \left(\frac{H}{\bar{H}} - 1 \right) \right|_{\bar{H}=H} = \left(\frac{g^*}{k} - \frac{g^{*2}}{k^2} \right) \left(\frac{H}{\bar{H}} - 1 \right) + \frac{1}{2} \left(-\frac{g^*}{k} + 7\frac{g^{*2}}{k^2} \right) \left(\frac{H}{\bar{H}} - 1 \right)^2 + \dots \quad (6.54)$$

that obviously agrees to first order in the coupling $(H/\bar{H} - 1)$ with the expression above.

The extra information that we obtain from this calculation is about the interpretation for the two-loop beta-function we described in the previous section. The one-loop corrections to (6.48) are of the form $C_{ijk} g^i g^j$, where C_{ijk} are related to the three-point function of the unperturbed theory [Zam86]. This coefficient is a measure of the dimension of the operator \mathcal{O}_i in the theory perturbed by the set of all operators. Eq.(6.54), based on the target-space

⁴There is no doubt on the method itself. It could simply fail to describe the desired phenomenon due to an inappropriate ansatz for the off-criticality excursion in parameter space.

approach, precisely predicts the coefficient of the term $(H/\bar{H} - 1)^2$ to second order in the $1/k$ -expansion. It seems that such a computation is feasible from the CFT viewpoint at least as a series expansion for large k . This would allow for a genuine two-loop comparison of the two methods, and is left for future investigation.

6.3 RG flow and friction

It has already been noted in literature [GHMS03] that a deep link exists between the equations of motion and the RG-flow. In an oversimplified toy model one can consider the equations of motion for a system with friction:

$$\frac{d^2 r}{dt^2} = -V'(r) - k \frac{dr}{dt}. \quad (6.55)$$

Large friction corresponds to the $k \rightarrow \infty$ limit where the dynamics described by this second order equation is well approximated by a first order one:

$$\frac{dr}{dt'} = -V'(r), \quad t' = \frac{t}{k}. \quad (6.56)$$

At least in some cases the same link exists between the second order equations of motion and the first order RG flow equation: the latter provide a good approximation for the dynamics of the system in some region of the moduli space. In this section we will provide a class of systems (with constant-curvature metrics and no dilaton) where this can be explicitly verified and the “friction” identified with the expectation value for the dilaton which appears out of equilibrium. More precisely we will consider the RG flow for the coupling constant of the metric with respect to the Kalb-Ramond field and then show that the equations that one obtains in this way are an approximation of those for a system in which the constant is a field depending on an extra time direction.

The RG-flow approach

As announced above we would like to study the RG-flow for the coupling of the metric in a system without dilaton, that is for the sigma model

$$S = \frac{1}{2\lambda} \int d^2 z (c g_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu \quad (6.57)$$

knowing that for $c = 1$ the model is conformal. Using the geometric RG-flow approach developed in Sec. 6.1 we find that Riemann tensor with respect to the connection of Eq. (6.13):

$$R^{-\mu}{}_{\nu\rho\sigma} = \left(1 - \frac{1}{c^2}\right) R^\mu{}_{\nu\rho\sigma}. \quad (6.58)$$

It follows that the one-loop counterterm is given by

$$T_{\mu\nu} = \frac{R}{d} \left(1 - \frac{1}{c^2}\right) g_{\mu\nu} \quad (6.59)$$

where for simplicity we supposed the manifold to be Einstein, which is consistent with the fact that the conformal model with fields g and B doesn't include a dilaton. Hence we immediately find the parameters

$$J(c) = 0 \quad a(c) = \frac{R}{d} \left(1 - \frac{1}{c^2} \right) \quad (6.60)$$

and the corresponding beta equations

$$\begin{cases} \beta_\lambda = 0, \\ \beta_c = \frac{\lambda}{\pi} a_c = \frac{\lambda R}{d\pi} \left(1 - \frac{1}{c^2} \right). \end{cases} \quad (6.61)$$

In order to compare this result with what we will find in the following we can write

$$c(\mu) = e^{2\sigma(\mu)} \quad (6.62)$$

where μ is the energy scale. Then the energy evolution of $\sigma(\mu)$ (going towards the infrared) gives:

$$\frac{d\sigma}{d\mu} = -\frac{\lambda R}{2d\pi} e^{-2\sigma(\mu)} \left(1 - e^{-4\sigma(\mu)} \right) = -V'(\sigma(\mu)) \quad (6.63)$$

which admits the implicit solution

$$\log \mu = -\frac{1}{4} \left(2e^{2\sigma(\mu)} + \log(\tanh \sigma(\mu)) \right). \quad (6.64)$$

This is for us the equivalent of Eq. (6.56). Now we move to the $(d+1)$ -dimensional spacetime to find the corresponding Eq. (6.55).

Spacetime interpretation

Equations of motion. As we said above we want to describe the same system by introducing an extra time dimension and reading the coupling as a time-dependent field. In other words we would like to write the equations of motion for the following sigma model:

$$S = \int d^2z \left[-\partial t \bar{\partial} t + (c(t)g_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu \right] \quad (6.65)$$

where, g and B are background fields solving the low-energy string equations of motion. In order to write the equations of motion let us rewrite the $d+1$ dimensional metric in terms of a Weyl rescaling as:

$$\bar{g}_{MN} = e^{2\sigma(t)} \begin{pmatrix} -e^{-2\sigma(t)} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix} = e^{2\sigma(t)} g_{MN} \quad (6.66)$$

where $c(t) = e^{2\sigma(t)}$. This means in particular that the Ricci tensor (this time with respect to the standard Levi-Civita connection) can be written as

$$\bar{Ric}_{MN} = Ric_{MN} - g_{MN} K_L^L - (d-1) K_{MN} \quad (6.67)$$

where K_{MN} is defined as

$$K_M^N = -\partial_M \sigma g^{NL} \partial_L \sigma + g^{NL} (\partial_M \partial_L \sigma - \Gamma_{ML}^P \partial_P \sigma) + \frac{1}{2} g^{LP} \partial_L \sigma \partial_P \sigma \delta_M^N \quad (6.68)$$

$$K_{MN} = g_{NL} B_M^L \quad (6.69)$$

After some algebra one finds that

$$\Gamma_{tt}^t = -\dot{\sigma}(t) \quad \Gamma_{t\mu}^t = 0 \quad \Gamma_{\mu\nu}^t = 0 \quad (6.70)$$

$$K_t^t = -e^{2\sigma(t)} \left(\ddot{\sigma}(t) + \frac{\dot{\sigma}^2(t)}{2} \right) \quad K_\mu^{\nu} = -\frac{1}{2} e^{2\sigma(t)} \dot{\sigma}^2(t) \delta_\mu^{\nu} \quad (6.71a)$$

$$K_{tt} = \left(\ddot{\sigma}(t) + \frac{\dot{\sigma}^2(t)}{2} \right) \quad K_{\mu\nu} = -\frac{e^{2\sigma(t)}}{2} \dot{\sigma}^2(t) g_{\mu\nu} \quad (6.71b)$$

where $\dot{\sigma}(t)$ is the notation for

$$\dot{\sigma}(t) = \frac{d\sigma(t)}{dt} \quad (6.72)$$

In particular this implies that

$$K_L^L = -e^{2\sigma(t)} \left(\frac{d+1}{2} \dot{\sigma}^2(t) + \ddot{\sigma}(t) \right). \quad (6.73)$$

It then easily follows that

$$\overline{Ric}_{tt} = -d (\ddot{\sigma}(t) + \dot{\sigma}^2(t)) \quad (6.74a)$$

$$\overline{Ric}_{t\mu} = 0 \quad (6.74b)$$

$$\overline{Ric}_{\mu\nu} = Ric_{\mu\nu} + \bar{g}_{\mu\nu} (d\dot{\sigma}^2(t) + \ddot{\sigma}(t)). \quad (6.74c)$$

The other terms in the equations of motion read

$$\bar{H}_{\mu\nu}^2 = H_{\mu\alpha\beta} H_{\nu\gamma\delta} \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} = e^{-4\sigma(t)} H_{\mu\nu}^2 \quad (6.75)$$

$$\bar{\nabla}_M \bar{\nabla}_N \Phi = \partial_M \partial_N \Phi - \bar{\Gamma}_{MN}^\lambda \partial_\lambda \Phi \quad (6.76)$$

now, $\bar{\Gamma}_{\mu\nu}^t = -\dot{\sigma}(t) \bar{g}_{\mu\nu}$ so

$$\bar{\nabla}_t \bar{\nabla}_t \Phi = \ddot{\Phi}(t) \quad (6.77a)$$

$$\bar{\nabla}_\mu \bar{\nabla}_t \Phi = 0 \quad (6.77b)$$

$$\bar{\nabla}_\mu \bar{\nabla}_\nu \Phi = \dot{\sigma}(t) \dot{\Phi}(t) \bar{g}_{\mu\nu}. \quad (6.77c)$$

These are all the ingredients we need to write the equations of motion:

$$\overline{Ric}_{MN} - \frac{1}{4} \bar{H}_{MN}^2 + 2 \bar{\nabla}_M \bar{\nabla}_N \Phi = 0. \quad (6.78)$$

Splitting the time component we obtain

$$Ric_{tt} + 2 \partial_t \partial_t \Phi(t) = -d (\ddot{\sigma}(t) + \dot{\sigma}^2(t)) + 2 \ddot{\Phi}(t) = 0 \quad (6.79a)$$

$$Ric_{\mu\nu} (1 - e^{-4\sigma(t)}) + \bar{g}_{\mu\nu} (d\dot{\sigma}^2(t) + \ddot{\sigma}(t) - 2\dot{\sigma}(t) \dot{\Phi}(t)) = 0 \quad (6.79b)$$

where we have used the equations of motion for the system in $\sigma = 0$:

$$Ric_{\mu\nu} = \frac{1}{4}H_{\mu\nu}^2 \quad (6.80)$$

The system admits a solution if and only if $g_{\mu\nu}$ is Einstein (since the original system didn't have any dilaton). Taking the trace with \bar{g}^{MN} we obtain the system:

$$\begin{cases} d(\ddot{\sigma}(t) + \dot{\sigma}^2(t)) - 2\ddot{\Phi}(t) = 0 \\ Re^{-2\sigma(t)}(1 - e^{-4\sigma(t)}) + d(d\dot{\sigma}^2(t) + \ddot{\sigma}(t) - 2\dot{\sigma}(t)\dot{\Phi}(t)) = 0 \end{cases} \quad (6.81)$$

Introducing

$$Q(t) = -\dot{\Phi}(t) + \frac{d}{2}\dot{\sigma}(t) \quad (6.82)$$

the equations become:

$$\begin{cases} \dot{Q}(t) = -d\dot{\sigma}^2(t) \\ \ddot{\sigma}(t) = -\frac{R}{d}e^{-2\sigma(t)}(1 - e^{-4\sigma(t)}) - 2\dot{\sigma}(t)Q(t) \end{cases} \quad (6.83)$$

This second equation has precisely the structure of the motion in a potential

$$V(\sigma) - V(0) = \frac{R}{6d}e^{-6\sigma}(1 - 3e^{4\sigma}) \sim -\frac{R}{3d} + \frac{2R}{d}\sigma^2. \quad (6.84)$$

and with a time-dependent friction coefficient $Q(t)$. In the limit of $Q \rightarrow \infty$ we clearly recover Eq. (6.63) with the same potential $V(\sigma)$ when identifying the *energy scale* μ for the off-shell system with the *time direction* here following

$$\log \mu = \frac{\pi\bar{Q}}{\lambda}t. \quad (6.85)$$

Linearization. The system (6.83) can be solved numerically and typical results for large $Q(0)$ and small $Q(0)$ are presented in Fig. 6.4.

A further step can be made by linearization. Introduce

$$\Sigma(t) = \dot{\sigma}(t) \quad (6.86)$$

the system becomes a first order one:

$$\begin{cases} \dot{Q}(t) = -d\Sigma^2(t) \\ \dot{\sigma}(t) = \Sigma(t) \\ \dot{\Sigma}(t) = -V'(\sigma(t)) - 2\Sigma(t)Q(t) \end{cases} \quad (6.87)$$

which has a fixed point for $(Q, \sigma, \Sigma) = (\bar{Q}, 0, 0)$ where \bar{Q} is a constant. Around this point the equations read:

$$\begin{cases} \dot{Q}(t) = 0 \\ \dot{\sigma}(t) = \Sigma(t) \\ \dot{\Sigma}(t) = -V''(0)\sigma(t) - 2\bar{Q}\Sigma(t) = -\frac{4R}{d}\sigma(t) - 2\bar{Q}\Sigma(t) \end{cases} \quad (6.88)$$

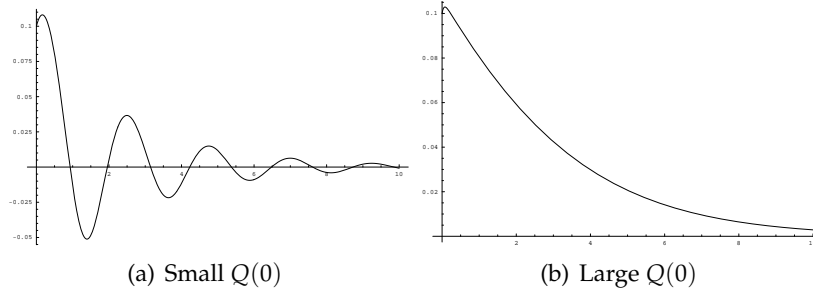


Figure 6.4: Typical behaviour for $\sigma(t)$ in the system (6.83) for (a) small and (b) large (positive) initial values of $Q(t)$.

so, Q decouples (and remains constant) and the equation of motion around the fixed point is

$$\frac{d^2\sigma(t)}{dt^2} = -\frac{4R}{d}\sigma(t) - 2\bar{Q}\frac{d\sigma(t)}{dt}, \quad (6.89)$$

which can be integrated giving

$$\sigma(t) = C_1 \exp \left[- \left(\bar{Q} + \sqrt{\bar{Q}^2 - \frac{4R}{d}} \right) t \right] + C_2 \exp \left[- \left(\bar{Q} - \sqrt{\bar{Q}^2 - \frac{4R}{d}} \right) t \right] \quad (6.90)$$

with C_1 and C_2 integration constants.

For positive \bar{Q} the solution converges to $\sigma = 0$ with or without oscillations if $\bar{Q}^2 \gtrless 4R/d$. In terms of $\sigma(t)$ and $\Phi(t)$ this limit solution is

$$\sigma(t) \xrightarrow[t \rightarrow \infty]{} 0 \quad \Phi(t) \sim -\bar{Q}t, \quad (6.91)$$

which is not surprisingly the initial conformal model in Eq. (6.57) plus a linear dilaton.

The meaning of \bar{Q} . \bar{Q} is linked to the dilaton: larger values correspond to negative and larger absolute values for Φ , *i.e.* moving further inside the perturbative regime. On the other hand, negative values of \bar{Q} give diverging solutions, but in this case the dilaton grows (see Eq. 6.82) and the very underlying perturbative approach collapses. It is worth to remark that if we make an hypothesis of uniqueness for the system (6.83), Q can't change sign because $Q(t) = 0, \sigma(t) = 1$ is a solution (the starting conformal model with constant dilaton).

A better understanding of the actual meaning of this parameter can be obtained if we consider the limiting situation of linear dilaton. In this case, in fact, it is immediate to derive the central-charge of the overall system:

$$c = (d + 1) - 3\bar{Q}^2 - c_d + c_I = 0, \quad (6.92)$$

where c_d is the central-charge of the conformal system in Eq. (6.57) (e.g. $6/(k+2)$ for the $SU(2)$ WZW model) and c_I is the internal central-charge. It follows that for a critical model

$$\bar{Q}^2 = \frac{1}{3}(d+1 - c_d + c_I) \quad (6.93)$$

and \bar{Q} is essentially a measure of the deficit.

A final remark regards the consistency of the approximation for the dynamics one obtains from the RG-flow equation (6.63), corresponding to a $Q \rightarrow \infty$ limit. The linearized system (6.88) provides a justification for such limit: in fact the time scale for $Q(t)$ is comparably larger than $\sigma(t)$'s – to the point that the former decouples around the fixed point. For this reason it can be taken as a constant (fixed by the initial conditions) if we just concentrate on the evolution of the warping factor $\sigma(t)$.

6.4 Cosmological interpretation

The type of backgrounds we are studying are time-dependent and as such can have a cosmological interest. For this reason, since there is a non-trivial dilaton, one should better move to the Einstein frame (as opposed to the string frame we've been using thus far). This means that the metric is written as:

$$\tilde{g}_{MN} = e^{-\Phi(t)/2} \bar{g}_{MN} \quad (6.94)$$

and after a coordinate change

$$\tau(t) = \int e^{-\Phi(t)/4} dt \quad (6.95)$$

can be put back to the same warped product form as in Eq. (6.66):

$$\begin{aligned} \widetilde{ds}^2 &= \tilde{g}_{MN} dx^M dx^N = -d\tau^2 + e^{2\sigma(t) - \Phi(t)/2} \Big|_{t=\tau} (g_{\mu\nu} dx^\mu dx^\nu) = \\ &= -d\tau^2 + w(\tau) (g_{\mu\nu} dx^\mu dx^\nu). \end{aligned} \quad (6.96)$$

Cosmologically interesting solutions are obtained when $d = 3$. In this case the H field is proportional to the volume form on g . This implies that $H_{\mu\nu}^2 \propto g_{\mu\nu}$ and then the equations reduce to

$$Ric_{\mu\nu} = \Lambda^2 g_{\mu\nu} \quad (6.97)$$

ie $g_{\mu\nu}$ is to be the metric of an Einstein three-manifold (the most simple case being a three-sphere). What we get then is a typical example of Friedmann-Robertson-Walker (FRW) spacetime such as those already studied in [Tse92b, Tse92a, GP94, CLW94]. As such it describes the time evolution of an isotropic spacetime (or more in general of a spacetime with the symmetries of the conformal theory in Eq. (6.57)). Some intuition about the time evolution can be developed if we take the linearized system in Eq. (6.88) and consider the large t limit. In fact, as remarked above the solution asymptotically approaches a

linear dilaton background (which was already studied from this point of view in [ABEN89]):

$$\sigma(t) \xrightarrow[t \rightarrow \infty]{} 0 \quad Q(t) = \bar{Q} \quad \Phi(t) \sim -\bar{Q}t \quad (6.98)$$

hence one verifies that the metric in the Einstein frame is asymptotically

$$\widetilde{ds}^2 \sim -d\tau^2 + \bar{Q}^2 \tau^2 (g_{\mu\nu} dx^\mu dx^\nu) \quad (6.99)$$

which corresponds to an expanding universe with curvature

$$\bar{R} \sim \frac{R + \bar{Q}^2 d(d-1)}{\bar{Q}^2 \tau^2}. \quad (6.100)$$

A similar result, with a polynomial expansion is found if we consider an exponential decrease for $\sigma(t)$, or better for $c(t)$ (in the linear limit $c(t) - 1$ obeys the same equations as $\sigma(t)$). After a redefinition of the variables we can let

$$c(t) = e^{-t} + 1. \quad (6.101)$$

It is easy to check that in general⁵

$$\tau(t) = \int c(t)^{-d/16} e^{1/4 \int Q(t') dt'} dt \quad (6.102)$$

and in this linearized approximation the latter becomes

$$\tau(t) = \int (e^{-t} + 1)^{-d/16} e^{\bar{Q}t/4} dt. \quad (6.103)$$

This integral can be solved analytically:

$$\tau(u) = \frac{16}{d+4\bar{Q}} \left(1 + \frac{1}{u}\right)^{-d/16} u^{\bar{Q}/4} (1+u)^{d/16} {}_2F_1\left(\frac{d}{16}, \frac{d+4\bar{Q}}{16}; \frac{d+4\bar{Q}}{16} + 1, -u\right), \quad (6.104)$$

where $u = e^t$ and ${}_2F_1$ is an hypergeometric function⁶. It is better however to consider the asymptotic behaviours. For $u \rightarrow \infty$ one finds that $\tau(u)$ and the warping factor $w(u)$ go like:

$$\tau(u) \sim \frac{4}{\bar{Q}} u^{\bar{Q}/4}, \quad w(u) \sim u^{\bar{Q}/2}, \quad (6.107)$$

⁵On a side note, since $c(t) > 0$ by construction the relation $\tau = \tau(t)$ is always invertible.

⁶The hypergeometric function ${}_2F_1$ is defined as follows:

$${}_2F_1(a, b; c, u) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \quad (6.105)$$

where $(a)_k$ is the Pochhammer symbol

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (6.106)$$

and consistently with the results above for the linear dilaton case (which is precisely the large- u limit):

$$w(\tau) \sim \tau^2; \quad (6.108)$$

similarly for small u :

$$\tau(u) \sim \frac{16}{d+4\bar{Q}} u^{(d+4\bar{Q})/16}, \quad w(u) \sim u^{d/4+2+\bar{Q}/2} \quad (6.109)$$

and then

$$w(\tau) \sim \tau^{4+8(4-\bar{Q})/(d+4\bar{Q})}. \quad (6.110)$$

Note that this behaviour precisely measures the effect of a finite value for \bar{Q} and in fact for $\bar{Q} \rightarrow \infty$ we recover again $w(\tau) \sim \tau^2$. Summarizing, just as advertised, we get again a polynomially expanding universe (a so-called big-bang solution).

The analysis for the small- \bar{Q} regime is clearly more difficult to be carried out analytically. Apart from numerical solutions (see Fig. 6.5), in general we can study $w(\tau)$ as a parametric curve in the (w, τ) plane defined by $(w(t), \tau(t))$. Then $\tau(t)$ appears to be a monotonically increasing function since $c(t) > 0$ which implies that $w(\tau)$ has an extremum for each extremum in $w(t)$. This means that we expect the superposition of a polynomial expansion and a damped oscillation. The limiting situation is obtained when \bar{Q} is small (but not vanishing), and for large t , $\tau(t) \sim t$ so that $w(\tau)$ slowly converges, oscillating, to a constant value.

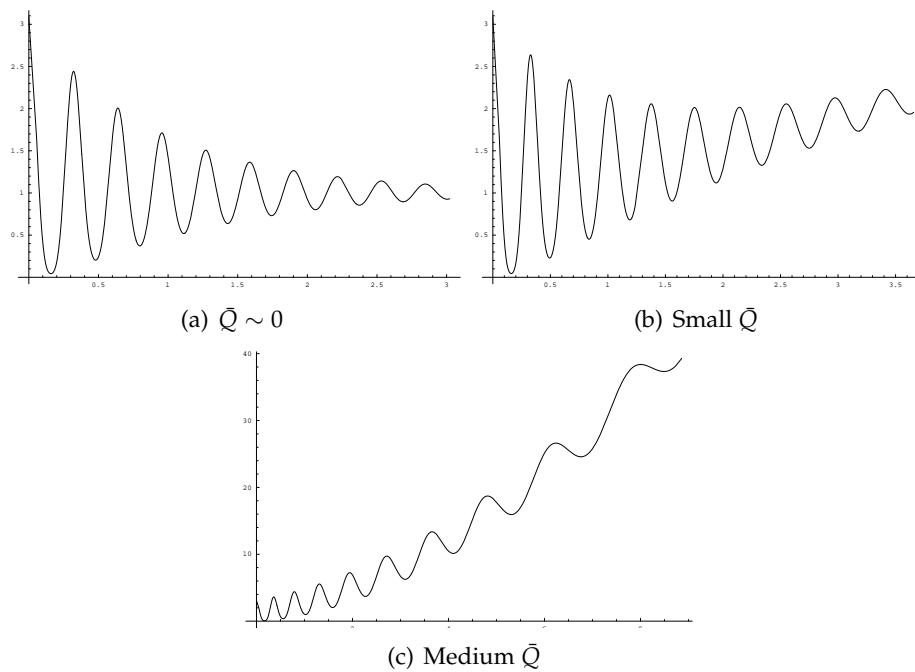


Figure 6.5: Typical behaviour for the warping factor in the small- \bar{Q} regime. We consider (a) \bar{Q} very small but not vanishing, (b) small \bar{Q} and (c) larger \bar{Q} (but still compatible with oscillations).

CHAPTER 7

Hyperbolic Spaces

In this chapter we investigate type II and M-theory geometries written as direct products of constant-curvature spaces. We find in particular a class of backgrounds with hyperbolic components and we study their stability with respect to small fluctuations.

THIS CHAPTER does in some sense deviate from the main theme we developed in this thesis. In fact we will deal with type II theories in presence of Ramond-Ramond fields, that – to this moment – still elude a precise CFT treatment. For this reason our analysis will be mostly confined to supergravity considerations. On the other hand we still continue to follow one of the main guiding threads, *i.e.* look for backgrounds with simple geometric interpretation, which in this case means (maximally) symmetric spaces, with special emphasis on hyperbolic, negative curvature, Poincaré spaces. We will show in fact that these spaces can be used as building blocks for M-theory and type II backgrounds, in genuinely perturbative configurations or in presence of D branes, both in the non-compact and in the compact part after discrete identifications. In particular we find a series of M-theory solutions that can be obtained starting with the usual $\text{AdS}_7 \times S^4$ by splitting the anti-de Sitter in a $\text{AdS} \times H$ product and verify their stability with respect to small fluctuations. Finally we also show how obtaining negative-curvature Euclidean signature spaces is not in general an easy task and in particular show that the presence of orientifold planes, giving a negative contribution to the stress-energy tensor is not enough to allow for H_3 spaces in type IIB theories.

7.1 M-theory solutions

Let us start with the action of the bosonic sector of eleven-dimensional supergravity:

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{2} |F_{[4]}|^2 \right) + \frac{1}{12\kappa_{11}^2} \int A_{[3]} \wedge F_{[4]} \wedge F_{[4]}, \quad (7.1)$$

and the corresponding equations of motion

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} |F_{[4]}^2|_{\mu\nu} + \frac{1}{4} |F_{[4]}^2| g_{\mu\nu} = 0, \quad (7.2)$$

where with the notation $|F_{[n]}|^2$ we mean

$$|F_{[p]}|^2 = \frac{1}{p!} F_{\mu_1 \mu_2 \dots \mu_p} F^{\mu_1 \mu_2 \dots \mu_p}. \quad (7.3)$$

The ansatz we want to study is the following. We consider direct products of symmetric spaces of the form $M_{11} = M_0 \times M_1 \times M_2 \times \dots$ where M_0 has signature $-, +, \dots, +$ and all the other spaces are Riemann. Since we assume that they are all symmetric spaces, we can split the Ricci tensor in blocks and each block will be proportional to the metric of the corresponding submanifold. To fix the notation we can introduce the parameters Λ_i as

$$R_{\mu\nu}|_i = \Lambda_i g_{\mu\nu}|_i, \quad (7.4)$$

so that the Ricci scalars are given by

$$R_i = \Lambda_i \dim M_i = \Lambda_i d_i. \quad (7.5)$$

In particular we can raise an index and rewrite

$$R^\mu{}_\nu|_i = \Lambda_i \delta^\mu{}_\nu|_i. \quad (7.6)$$

The Poincaré invariance constraint fixes the allowed gauge fields to be proportional to the volume form of each submanifold. It is always possible to perform an electric/magnetic duality so that there's no field on the Minkowskian submanifold. This means that we can consider gauge fields having the form:

$$F_{[d_i]} = F_i = Q_i \tilde{\omega}_i, \quad (7.7)$$

where F_i is a d_i -form, $Q_i \in \mathbb{N}$ and $\tilde{\omega}_i = \widetilde{\omega}_{M_i}$ is the volume form on M_i , normalized to one. It is useful to rewrite the expression above as

$$F_i = \sqrt{2} k_i Q_i |\Lambda_i|^{d_i/2} \omega_i \quad (7.8)$$

where ω_i is the volume form on M_i and k_i is a constant whose value is

- $k_i = \Gamma(i/2) / (2\sqrt{2}\pi^{i/2})$ for a sphere S^i
- $k_i = 1 / (4\pi(g-1))$ for a genus g Riemann surface H^2/Γ
- some value that completely identifies the lattice in a H^3/Γ compactification (rigidity theorem for three-manifolds [Thurston]).

In coordinates

$$F_i|_{\mu_1 \dots \mu_{d_i}} = k_i Q_i \Lambda_i^{d_i/2} \sqrt{2 \det g_i} \epsilon_{\mu_1 \dots \mu_{d_i}} \quad (7.9)$$

which implies (as one can verify in a non-coordinate basis):

$$F_i|_{\mu_1 \dots \mu_{d_i}} F_i|_{\nu_1 \dots \nu_{d_i}} = 2(n-1)! k_i^2 Q_i^2 \Lambda_i^{d_i} \delta^\mu{}_\nu \quad (7.10)$$

Furthermore

$$|F_i^2|^\mu{}_\nu = 2k_i^2 Q_i^2 |\Lambda_i|^{d_i} \delta^\mu{}_\nu \quad (7.11)$$

$$|F_i^2| = 2k_i^2 Q_i^2 |\Lambda_i|^{d_i}. \quad (7.12)$$

We are now in position to write the equations of motion that will simply translate into an algebraic system for the (Λ_0, Λ_i) :

$$\begin{cases} \Lambda_0 - \frac{1}{2}R = -\frac{1}{2} \sum_j k_j^2 Q_j^2 |\Lambda_j|^{d_j}, \\ \Lambda_i - \frac{1}{2}R = -\frac{1}{2} \sum_j (-)^{\delta_{ij}} k_j^2 Q_j^2 |\Lambda_j|^{d_j}, \end{cases} \quad (7.13)$$

where

$$R = d_0 \Lambda_0 + \sum_i d_i \Lambda_i. \quad (7.14)$$

Let us now turn to study some particular examples.

$$M^{1,6-d} \times M^d \times M^4$$

Let us consider as an example the case $M_{11} = M_0 \times M_1 \times M_2$ where $(d_0, d_1, d_2) = (7-d, d, 4)$. If we turn F_2 in the equations of motion read:

$$\begin{cases} \Lambda_0 - \frac{1}{2}R = -\frac{1}{2}k_2^2 Q_2^2 \Lambda_2^4, \\ \Lambda_1 - \frac{1}{2}R = -\frac{1}{2}k_2^2 Q_2^2 \Lambda_2^4, \\ \Lambda_2 - \frac{1}{2}R = \frac{1}{2}k_2^2 Q_2^2 \Lambda_2^4, \end{cases} \quad (7.15)$$

with $R = (7-d)\Lambda_0 + d\Lambda_1 + 4\Lambda_2$. The solution is:

$$\Lambda_0 = \Lambda_1 = -\frac{1}{2} \left(\frac{3}{2}\right)^{1/3} \frac{1}{(k_2 Q_2)^{2/3}}, \quad \Lambda_2 = \left(\frac{3}{2}\right)^{1/3} \frac{1}{(k_2 Q_2)^{2/3}}, \quad (7.16)$$

and given the curvatures, this describes an $\text{AdS}_{7-d} \times H_d \times S^4$ space.

A few remarks are in order. First of all, the result doesn't depend on d and in particular it would be the same for $d = 0$ (which is some sort of limit case). In other words, at the level of the equations of motion, we can't distinguish between an AdS_7 space and any product of the form $\text{AdS}_{7-d} \times H_d$ once the respective curvatures are such as

$$\frac{R_{\text{AdS}}}{7-d} = \frac{R_H}{d}. \quad (7.17)$$

This calculation is generalizable to any product of the form $M_0 \times M_1 \times \dots \times M_n$ with dimensions $(d_0, d_1, \dots, d_{n-1}, 4)$. In fact the equations of motion read

$$\begin{cases} \Lambda_0 - \frac{1}{2}R = -\frac{1}{2}k_n^2 Q_n^2 \Lambda_n^4, \\ \Lambda_1 - \frac{1}{2}R = -\frac{1}{2}k_n^2 Q_n^2 \Lambda_n^4, \\ \dots \\ \Lambda_{n-1} - \frac{1}{2}R = -\frac{1}{2}k_n^2 Q_n^2 \Lambda_n^4, \\ \Lambda_n - \frac{1}{2}R = \frac{1}{2}k_n^2 Q_n^2 \Lambda_n^4, \end{cases} \quad (7.18)$$

with $R = d_0\Lambda_0 + d_1\Lambda_1 + \dots + d_n\Lambda_n$. From the system above we conclude that $\Lambda_0 = \Lambda_1 = \dots = \Lambda_{n-1}$ and then we are back to the situation above:

$$\Lambda_0 = \Lambda_1 = \dots = \Lambda_{n-1} = -\left(\frac{3}{2}\right)^{1/3} \frac{1}{2(k_n Q_n)^{2/3}}, \quad \Lambda_n = \left(\frac{3}{2}\right)^{1/3} \frac{1}{(k_n Q_n)^{2/3}}. \quad (7.19)$$

We have thus found a series of possible M-theory backgrounds where the anti-de Sitter component is split into two or more subspaces of the form

$$\text{AdS}_n \rightarrow \text{AdS}_{n-p} \times H_p, \quad (7.20)$$

with Ricci scalars obeying

$$\frac{R^{(n)}}{n} = \frac{R^{(n-p)}}{n-p} = \frac{R^{(p)}}{p}. \quad (7.21)$$

In particular we get the direct products $\text{AdS}_2 \times H_2 \times H_3 \times S^4$, $\text{AdS}_2 \times H_5 \times S^4$, $\text{AdS}_3 \times H_2 \times H_2 \times S^4$, $\text{AdS}_3 \times H_4 \times S^4$, $\text{AdS}_4 \times H_3 \times S^4$, $\text{AdS}_5 \times H_2 \times S^4$.

$$M^{1,3} \times M^d \times M^{7-d}$$

The dual situation is obtained for $(d_0, d_1, d_2) = (4, d, 7-d)$. In this case we can turn on the 7-form field

$$F_{[7]} = \sqrt{2}k_0 Q_0 \Lambda_1^{d/2} \Lambda_2^{(7-d)/2} \quad (7.22)$$

and the equations of motion read:

$$\begin{cases} \Lambda_0 - \frac{1}{2}R = -\frac{1}{2}k_0^2 Q_0^2 \left| \Lambda_1^d \Lambda_2^{7-d} \right|, \\ \Lambda_1 - \frac{1}{2}R = \frac{1}{2}k_0^2 Q_0^2 \left| \Lambda_1^d \Lambda_2^{7-d} \right|, \\ \Lambda_2 - \frac{1}{2}R = \frac{1}{2}k_0^2 Q_0^2 \left| \Lambda_1^d \Lambda_2^{7-d} \right|, \end{cases} \quad (7.23)$$

with $R = 4\Lambda_0 + d\Lambda_1 + (7-d)\Lambda_2$. Again the solution is easily found

$$\Lambda_0 = -2 \frac{3^{1/6}}{(k_0 Q_0)^{2/3}}, \quad \Lambda_1 = \Lambda_2 = \frac{3^{1/6}}{(k_0 Q_0)^{2/3}}. \quad (7.24)$$

Just as before this does not depend on d and shows that at this level an $\text{AdS}_4 \times S^d \times S^{7-d}$ space is not distinguishable from a $\text{AdS}_4 \times S^7$ one. Again we can consider more general configurations with dimensions $(4, d_1, \dots, d_n)$ to find that the solution remains the same

$$\Lambda_0 = -2 \frac{3^{1/6}}{(k_0 Q_0)^{2/3}}, \quad \Lambda_1 = \Lambda_2 = \dots = \Lambda_n = \frac{3^{1/6}}{(k_0 Q_0)^{2/3}}. \quad (7.25)$$

This describes a second series of solutions in which a sphere is split into the product of two smaller spheres according to

$$S^n \rightarrow S^{n-p} \times S^p \quad (7.26)$$

with Ricci scalars obeying as above

$$\frac{R^{(n)}}{n} = \frac{R^{(n-p)}}{n-p} = \frac{R^{(p)}}{p}, \quad (7.27)$$

and then we recover $\text{AdS}_4 \times S^2 \times S^2 \times S^3$, $\text{AdS}_4 \times S^2 \times S^5$, $\text{AdS}_4 \times S^3 \times S^2 \times S^2$, $\text{AdS}_4 \times S^3 \times S^4$, $\text{AdS}_4 \times S^5 \times S^2$.

Finally we can combine the two types of splitting and put $\text{AdS}_2 \times H^2$ instead of AdS_4 and $S^2 \times S^2$ instead of S^4 in the former series.

The key ingredient to these constructions is the fact that with a careful choice of radii the product spaces still remain Einstein and this is all one needs to satisfy the equations of motion. This means of course that in general one can use any Einstein manifold with the proper curvature. In particular, then, instead of the five-sphere S^5 one can put a generalization of the $S^3 \times S^2$ product, such as any of the representatives of the two-parameter class of spaces $T^{p,q}$ obtained as S^1 fibrations over $S^2 \times S^2$ or equivalently as the coset $(SU(2) \times SU(2)) / U(1)$, the parameter being the embedding indexes of $U(1)$ in the $SU(2)$ 's¹. Similarly, instead of AdS^5 one can put a space written as a time-fibration over $H_2 \times H_2$ or as the coset $(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})) / \mathbb{R}$. The metric of such L_{Q_1, Q_2} space can be written as

$$ds^2 = Q_1 \left(\frac{dx^2}{1+x^2} + (1+x^2) du^2 \right) + Q_2 \left(\frac{dy^2}{1+y^2} + (1+y^2) dv^2 \right) + \frac{2Q_1(Q_1 - 2Q_2)}{Q_2} \left(dt + xdu + \frac{Q_2}{Q_1} \sqrt{\frac{Q_2 - 2Q_1}{Q_1 - 2Q_2}} ydv \right)^2. \quad (7.28)$$

Such cosets were studied in [PZT00] where they were found to be exact string backgrounds by using a construction very close to the asymmetric cosets of Ch. 3.

Of course since in general these geometries don't preserve any supersymmetry we should address the problem of their stability, which we do in the following section.

7.2 Stability

The set of solutions we found above are not in general protected by supersymmetry. This implies in particular that we should care about their stability. In our analysis we will deal with the breathing modes of the compact H_n and S^n internal manifolds which, in an effective action, are to be described by scalar fields. The stability (with respect to small fluctuations) will then translate into the positivity of the squared mass for such fields, condition that can be relaxed into the respect of a Breitenlohner-Freedman bound when the spacetime is of the anti-de Sitter type.

¹This clearly is the basis of the conifold.

First of all then we give an explicit derivation for the bound, so to completely fix the notation and then construct the general expression for the space-time effective action for the breathing modes – hence finding again the same solutions as above as stationary points for a potential whose Hessian matrix encodes the stability for the background.

Breitenlohner-Freedman bound

Anti-de Sitter Consider an action of the kind

$$S = \int d^d x \sqrt{-g^d} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) \quad (7.29)$$

the equation of motion for ϕ reads

$$-\nabla_\mu \partial^\mu \phi = -\frac{\partial V}{\partial \phi} \quad (7.30)$$

The relations that we obtain are all tensorial so we can just choose a suitable coordinate system, knowing that the result will remain invariant. In AdS_d a good choice would be

$$ds^2 = dr^2 + e^{2Hr} (-dt^2 + dx_1^2 + \dots + dx_{d-2}^2) \quad (7.31)$$

and we can consider a potential V with a minimum in $\phi = 0$:

$$V(\phi) = V_0 + \frac{m^2}{2} \phi^2. \quad (7.32)$$

The equation of motion for an r -dependent field ϕ reads

$$\phi'' + (d-1)H\phi' - m^2\phi = 0. \quad (7.33)$$

Solving it one can see that the presence of the friction term effectively changes the mass to

$$M^2 = \left(\frac{d-1}{2} H \right)^2 + m^2 \geq 0 \quad (7.34)$$

or, given that $R = -d(d-1)H^2$

$$M^2 = -\frac{d-1}{4d} R + m^2 \geq 0 \quad (7.35)$$

or, again, in terms of minimum of the potential:

$$M^2 = -\frac{d-1}{4(d-2)} \langle V \rangle + m^2 \geq 0. \quad (7.36)$$

Positivity of the effective mass squared, and thus stability, translate therefore into a less stringent constraint for m^2 . This is the BF bound.

L_{Q_1, Q_2} spaces. The presence of the bound is due to the curvature of the manifold. It can be restated by saying that in an appropriate coordinate system the Klein–Gordon equation can be put in the form

$$-\square\phi + m^2\phi = (-\tilde{\square} + \Delta)\phi + m^2\phi = -\tilde{\square}\phi + M^2\phi = 0, \quad (7.37)$$

where \square is the d'Alembertian for the curved space, $\tilde{\square}$ is the d'Alembertian for flat Minkowski space and Δ is some constant depending on the curvature and other details of the geometry ($\Delta = -(d-1)/(4d)R$ in the case of AdS_d). It is natural to expect a similar behaviour for other negative-curvature spaces, but the precise value of Δ will depend on the details. In particular it is interesting to consider the L_{Q_1, Q_2} spaces introduced above. Again, as before, we can choose a coordinate system and then use the fact that the equations we get are tensor relations and hence invariant. Take into example the following metric:

$$ds^2 = Q_1 \left(\frac{dx^2}{1+x^2} + (1+x^2) du^2 \right) + Q_2 \left(\frac{dy^2}{1+y^2} + (1+y^2) dv^2 \right) + \frac{2Q_1(Q_1 - 2Q_2)}{Q_2} \left(dt + xdu + \frac{Q_2}{Q_1} \sqrt{\frac{Q_2 - 2Q_1}{Q_1 - 2Q_2}} ydv \right)^2, \quad (7.38)$$

which describes an L_{Q_1, Q_2} space with Ricci scalar

$$R = -\frac{5}{3} \frac{Q_1 + Q_2}{Q_1 Q_2}. \quad (7.39)$$

The d'Alembertian on $\phi(x, y)$ gives:

$$\square\phi(x, y) = \frac{1}{Q_1} [(1+x^2)\phi_{xx} + 2x\phi_x] + \frac{1}{Q_2} [(1+y^2)\phi_{yy} + 2y\phi_y]. \quad (7.40)$$

This is the same expression we would have got by considering the d'Alembertian in an $\text{AdS}_3 \times \text{AdS}_3$ space with coordinates

$$ds^2 = \frac{1}{Q_1} \left[\frac{dx^2}{1+x^2} - dt^2 - 2xdtdu + du^2 \right] + \frac{1}{Q_2} \left[\frac{dy^2}{1+y^2} - d\tau^2 - 2yd\tau dv + dv^2 \right]. \quad (7.41)$$

The two subspaces have curvature

$$R_i = -\frac{3}{2Q_i} \quad (7.42)$$

and since the shifts compose linearly this gives an overall shift

$$M^2 = m^2 + \frac{1}{4Q_1} + \frac{1}{4Q_2} = m^2 + \frac{1}{4} \frac{Q_1 + Q_2}{Q_1 Q_2}, \quad (7.43)$$

which is therefore the BF bound for L_{Q_1, Q_2} space. This can be compared with an AdS_5 space with the same curvature as in Eq. (7.39), where the shift would be given by

$$M^2 = m^2 + \frac{1}{3} \frac{Q_1 + Q_2}{Q_1 Q_2}. \quad (7.44)$$

In this case the shift is larger: in some sense then, as one might have expected, an anti de-Sitter space is more stable with respect to small fluctuations than a L_{Q_1, Q_2} space with the same scalar curvature.

Effective low-dimensional description

We want to write the d -dimensional effective action for a $\text{AdS}_d \times M_1 \times M_2$ background, where M_i are constant curvature spaces, starting from the following sector of the M-theory action

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g^{(11)}} (R^{11} - V) \quad (7.45)$$

where V takes into account the presence of the fluxes. In order to study the stability of the product background at hand let us consider the following ansatz:

$$ds_{11}^2 = ds^2(M^{(d)}) + e^{2\varphi_1(x)} ds^2(M^{(1)}) + e^{2\varphi_2(x)} ds^2(M^{(2)}) \quad (7.46)$$

where the fields φ_i depend only on the coordinates of $M^{(d)}$. The strategy is the following:

- (a) write the curvature $R^{(11)}$ in terms of the curvature for $M^{(d)}$
- (b) separate the determinant $g^{(11)}$ in its compact and non-compact part
- (c) Weyl-rescale the four-dimensional metric to get a canonical Einstein-Hilbert action
- (d) rescale the scalar fields φ_i to get a canonical kinetic term
- (e) study the potential and verify the stability taking into account the BF bound.

Step a For a warped product

$$ds^2 = ds^2(M^{(d)}(x)) + e^{2\varphi(x)} ds^2(M^{(1)}) \quad (7.47)$$

where $M^{(1)}$ has dimension d_1 , the Ricci scalar is

$$R = R^{(1)} + e^{-2\varphi(x)} R^{(2)} - 2d_1 \nabla_\mu \partial^\mu \varphi(x) + d_1 (d_1 - 1) \partial_\mu \varphi(x) \partial^\mu \varphi(x) \quad (7.48)$$

where the covariant derivative and the raising is done with the warped product metric. Note that the result only depends on d_1 and not on d . This easily generalizes to a warped product with three factors like the ones we study

$$ds^2 = ds^2(M^{(d)}(x)) + e^{2\varphi_1(x)} ds^2(M^{(1)}) + e^{2\varphi_2(x)} ds^2(M^{(2)}) \quad (7.49)$$

and one obtains

$$\begin{aligned} R^{(11)} = & R^{(d)} + e^{-2\varphi_1(x)} R^{(1)} + e^{-2\varphi_2(x)} R^{(2)} - 2d_1 \nabla_\mu \partial^\mu \varphi_1(x) + \\ & + d_1 (d_1 - 1) \partial_\mu \varphi_1(x) \partial^\mu \varphi_1(x) + -2d_2 \nabla_\mu \partial^\mu \varphi_2(x) + \\ & + d_2 (d_2 - 1) \partial_\mu \varphi_2(x) \partial^\mu \varphi_2(x) + 2d_1 d_2 \partial_\mu \varphi_1(x) \partial^\mu \varphi_2(x) \end{aligned} \quad (7.50)$$

where $d_i = \dim M^{(i)}$.

Since the covariant derivative is with respect to the whole $g^{(11)}$ metric, two of the terms above are total derivatives and the action is then equivalent to:

$$\begin{aligned} S &= \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g^{(11)}} \left(R^{(11)} - V \right) \sim \\ &\sim \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g^{(11)}} \left(R^{(d)} + e^{-2\varphi_1(x)} R^{(1)} + e^{-2\varphi_2(x)} R^{(2)} + \right. \\ &+ d_1 (d_1 - 1) \partial_\mu \varphi_1(x) \partial^\mu \varphi_1(x) + d_2 (d_2 - 1) \partial_\mu \varphi_2(x) \partial^\mu \varphi_2(x) + \\ &\left. 2d_1 d_2 \partial_\mu \varphi_1(x) \partial^\mu \varphi_2(x) - V \right) \quad (7.51) \end{aligned}$$

Step b The eleven-dimensional determinant can be written as

$$\det g^{(11)} = \det g^{(1)} \det g^{(2)} \det g^{(d)} e^{d_1 \varphi_1 + d_2 \varphi_2} \quad (7.52)$$

in particular we can integrate over the internal coordinates and get:

$$\begin{aligned} S &= \frac{V_1 V_2}{2\kappa_{11}^2} \int d^d x \sqrt{-g^{(d)}} e^{d_1 \varphi_1 + d_2 \varphi_2} \left(R^{(d)} + e^{-2\varphi_1(x)} R^{(1)} + e^{-2\varphi_2(x)} R^{(2)} + \right. \\ &+ d_1 (d_1 - 1) \partial_\mu \varphi_1(x) \partial^\mu \varphi_1(x) + d_2 (d_2 - 1) \partial_\mu \varphi_2(x) \partial^\mu \varphi_2(x) + \\ &\left. + 2d_1 d_2 \partial_\mu \varphi_1(x) \partial^\mu \varphi_2(x) - V \right) \quad (7.53) \end{aligned}$$

We will introduce

$$\Psi = d_1 \varphi_1 + d_2 \varphi_2 \quad (7.54)$$

for future purposes.

Step c The action above is not in the usual Hilbert-Einstein form because of the $e^{d_1 \varphi_1 + d_2 \varphi_2}$ factor. For this reason we perform a Weyl rescaling

$$\bar{g}_{\mu\nu} = e^{2\sigma(x)} g_{\mu\nu} \quad (7.55)$$

under which the curvature becomes:

$$\bar{R} = e^{-2\sigma(x)} R - 2(d-1) \bar{\nabla}_\mu \partial^\mu \sigma(x) + (d-1)(d-2) \partial_\mu \sigma(x) \partial^\mu \sigma(x) \quad (7.56)$$

where $\bar{\nabla}$ is the covariant derivative with respect to \bar{g} and μ is raised with \bar{g} .

In d dimensions a term $\sqrt{\det \bar{g}} e^\Psi R$ is brought to the standard form by the Weyl rescaling

$$\bar{g}_{\mu\nu} = \exp \left[\frac{2}{d-2} \Psi \right] g_{\mu\nu} \quad (7.57)$$

and

$$\sqrt{\bar{g}} e^\Psi R = \sqrt{\bar{g}} \left(\bar{R} + 2 \frac{d-1}{d-2} \bar{\nabla}_\mu \partial^\mu \Psi - \frac{d-1}{d-2} \partial_\mu \Psi \partial^\mu \Psi \right). \quad (7.58)$$

Discarding the total derivatives, the action now reads

$$\begin{aligned}
S = & \frac{V_1 V_2}{2\kappa_{11}^2} \int d^d x \sqrt{-\bar{g}^{(d)}} \left[\bar{R}^{(d)} - \frac{d-1}{d-2} \partial_\mu \Psi \partial^\mu \Psi + \right. \\
& + d_1 (d_1 - 1) \partial_\mu \varphi_1(x) \partial^\mu \varphi_1(x) + d_2 (d_2 - 1) \partial_\mu \varphi_2(x) \partial^\mu \varphi_2(x) + \\
& \left. + 2d_1 d_2 \partial_\mu \varphi_1(x) \partial^\mu \varphi_2(x) + e^{-2\Psi_1/(d-2)} \left(e^{-2\varphi_1(x)} R^{(1)} + e^{-2\varphi_2(x)} R^{(2)} - V(\varphi_1, \varphi_2) \right) \right] \quad (7.59)
\end{aligned}$$

where indices are raised with $\bar{g}^{(d)}$.

Step d Now let us collect all the φ terms in the action:

$$\begin{aligned}
S = & \frac{V_1 V_2}{2\kappa_{11}^2} \int d^d x \sqrt{-\bar{g}^{(d)}} \left[\bar{R}^{(d)} - d_1 \left(\frac{d_1}{d-2} + 1 \right) \partial_\mu \varphi_1 \partial^\mu \varphi_1 + \right. \\
& \left. - d_2 \left(\frac{d_2}{d-2} + 1 \right) \partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{2d_1 d_2}{d-2} \partial_\mu \varphi_1 \partial^\mu \varphi_2 - \bar{V}(\varphi_1, \varphi_2) \right] \quad (7.60)
\end{aligned}$$

where

$$\bar{V}(\varphi_1, \varphi_2) = e^{-2(d_1 \varphi_1 + d_2 \varphi_2)/(d-2)} \left(-e^{-2\varphi_1(x)} R^{(1)} - e^{-2\varphi_2(x)} R^{(2)} + V(\varphi_1, \varphi_2) \right). \quad (7.61)$$

To bring the kinetic terms to the standard form we introduce:

$$\begin{cases} \Phi_1 = \sqrt{\frac{2(D-2)}{(d_1+d_2)(d-2)}} (d_1 \varphi_1 + d_2 \varphi_2) \\ \Phi_2 = \sqrt{\frac{2d_1 d_2}{d_1+d_2}} (\varphi_1 - \varphi_2) \end{cases} \quad (7.62)$$

or, the other way round:

$$\begin{cases} \varphi_1 = \frac{1}{\sqrt{2(d_1+d_2)}} \left(\sqrt{\frac{d-2}{D-2}} \Phi_1 + \sqrt{\frac{d_2}{d_1}} \Phi_2 \right) \\ \varphi_2 = \frac{1}{\sqrt{2(d_1+d_2)}} \left(\sqrt{\frac{d-2}{D-2}} \Phi_1 - \sqrt{\frac{d_1}{d_2}} \Phi_2 \right) \end{cases} \quad (7.63)$$

where $D = d + d_1 + d_2 = 11$, and we finally obtain

$$S = \frac{V_1 V_2}{2\kappa_{11}^2} \int d^d x \sqrt{-g^{(d)}} \left[R^{(d)} - \frac{1}{2} \partial_\mu \Phi_1 \partial^\mu \Phi_1 - \frac{1}{2} \partial_\mu \Phi_2 \partial^\mu \Phi_2 - \bar{V}(\Phi_1, \Phi_2) \right] \quad (7.64)$$

Step e The type of backgrounds we obtain after compactification are AdS. Therefore, negative- m^2 modes are not tachyonic if they don't cross the Breitenlohner-Freedman bound.

Using the results of the previous section this means:

$$M^2 = -\frac{d-1}{4(d-2)} \langle V \rangle + m^2 \geq 0 \quad (7.65)$$

that must be respected by each eigenvalue m^2 of the Hessian matrix

$$m_{ij}^2 = \left. \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} \right|_{V=V_0} \quad (7.66)$$

These general considerations apply to any choice of background gauge fields. Then one should write the potential for each case at hand in terms of Φ_1 and Φ_2 (and possibly the dilaton in type II) and check the mass matrix against the BF bound.

$\text{AdS}_4 \times H^3 \times S^4$

As a first example let us consider $\text{AdS}_4 \times H^3 \times S^4$. The potential V in Eq. (7.45) is due to the presence of a four-form field on the S^4 part.

$$V(\varphi_1, \varphi_2) = \frac{Q^2}{2} e^{-8\varphi_2} \quad (7.67)$$

the dimensions are $d = 4$, $d_1 = 3$ and $d_2 = 4$ and the curvatures read $R^{(1)} = -3/2$ and $R^{(2)} = 2$, so the expression in Eq. (7.61) becomes:

$$\bar{V}(\varphi_1, \varphi_2) = e^{-3\varphi_1 - 4\varphi_2} \left(\frac{3}{2} e^{-2\varphi_1} - 2e^{\varphi_2} + \frac{Q^2}{2} e^{-8\varphi_2} \right). \quad (7.68)$$

To get the canonical scalar fields Φ_i and Φ_2 we can use Eq. (7.63):

$$\begin{cases} \varphi_1 = \frac{1}{3\sqrt{7}} (\Phi_1 + \sqrt{6}\Phi_2) \\ \varphi_2 = \frac{1}{12\sqrt{7}} (4\Phi_1 - 3\sqrt{6}\Phi_2) \end{cases} \quad (7.69)$$

and this leads to the effective potential

$$V(\Phi_1, \Phi_2) = \frac{1}{2} e^{-5\Phi_1/\sqrt{7} - 2\Phi_2\sqrt{2/21}} \left(3e^{2\Phi_1/\sqrt{7}} - 4e^{2\Phi_1/\sqrt{7} + \Phi_2\sqrt{7/6}} + Q^2 e^{8\Phi_2\sqrt{2/21}} \right) \quad (7.70)$$

which has a minimum for

$$\Phi_1 = \sqrt{7} \log \frac{2^{8/7} Q}{\sqrt{3}} \quad \Phi_2 = \sqrt{\frac{6}{7}} \log 2 \quad (7.71)$$

and in correspondence of this point

$$\langle V \rangle = -\frac{3\sqrt{3}}{16Q^3}. \quad (7.72)$$

The Hessian matrix on the minimum is:

$$\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} = \frac{3}{28Q^3} \begin{pmatrix} \frac{15\sqrt{3}}{4} & -\frac{9}{\sqrt{2}} \\ -\frac{9}{\sqrt{2}} & \frac{17\sqrt{3}}{2} \end{pmatrix}. \quad (7.73)$$

Both eigenvalues are positive

$$m^2|_1 = \frac{3\sqrt{3}}{16Q^3} \quad m^2|_2 = \frac{9\sqrt{3}}{8Q^3} \quad (7.74)$$

and the solution is stable.

d_1	$\langle \Phi_1 \rangle$	$\langle \Phi_2 \rangle$	$\langle V \rangle$	$\frac{m^2 _1}{ \langle V \rangle }$	$\frac{m^2 _2}{ \langle V \rangle }$
2	$\log \frac{4Q^2}{3}$	$\sqrt{\frac{2}{3}} \log 2$	$-\frac{9}{8 \times 2^{2/3} Q^2}$	2/3	4
3	$\frac{\sqrt{7}}{2} \log \frac{4^{8/7} Q^2}{3}$	$\sqrt{\frac{6}{7}} \log 2$	$-\frac{3\sqrt{3}}{16Q^3}$	1	6
4	$2 \log \frac{4\sqrt{2} Q^2}{3}$	$\log 2$	$-\frac{27}{512Q^6}$	2	12

Table 7.1: Minima and masses for AdS \times H backgrounds

AdS $_{7-d_1} \times H^{d_1} \times S^4$

All the backgrounds of the form AdS $_{7-d_1} \times H^{d_1} \times S^4$ can be treated similarly. The potential in Eq. (7.61) becomes:

$$\bar{V}(\varphi_1, \varphi_2) = e^{-2(d_1\varphi_1 - 4\varphi_2)/(5-d_1)} \left(-\frac{d_1}{2} e^{-2\varphi_1} - 2e^{\varphi_2} + \frac{Q^2}{2} e^{-8\varphi_2} \right) \quad (7.75)$$

and one finds that in each case there is a minimum (stable) solution. The actual values are in Tab. 7.1. In any case, not surprisingly one can see that for any d_1 the warping factors φ_1 and φ_2 always take the values

$$\varphi_1 = \frac{1}{6} \left(\log \frac{16Q^2}{3} \right) \quad \varphi_2 = \frac{1}{6} \left(\log \frac{3Q^2}{2} \right) \quad (7.76)$$

which agrees with Λ_2 and Λ_3 being

$$\Lambda_2 = - \left(\frac{3}{2} \right)^{1/3} \frac{1}{2Q^{2/3}} \quad \Lambda_3 = \left(\frac{3}{2} \right)^{1/3} \frac{1}{Q^{2/3}} \quad (7.77)$$

as we have already seen by directly solving the equations of motion in eleven dimensions.

AdS $_4 \times S^3 \times S^4$

Consider now AdS $_4 \times S^3 \times S^4$. The potential V in Eq. (7.45) is due to the presence of a seven-form field on the $S^3 \times S^4$ factor.

$$V(\varphi_1, \varphi_2) = \frac{Q^2}{2} e^{-6\varphi_1 - 8\varphi_2}. \quad (7.78)$$

The dimensions are $d = 4$, $d_1 = 3$ and $d_2 = 4$ and the curvatures read $R^{(1)} = 3/2$ and $R^{(2)} = 2$, so the expression in Eq. (7.61) becomes:

$$\bar{V}(\varphi_1, \varphi_2) = e^{-3\varphi_1 - 4\varphi_2} \left(-\frac{3}{2} e^{-2\varphi_1} - 2e^{\varphi_2} + \frac{Q^2}{2} e^{-6\varphi_1 - 8\varphi_2} \right). \quad (7.79)$$

To get the canonical scalar fields Φ_i and Φ_2 we can use Eq. (7.63):

$$\begin{cases} \varphi_1 = \frac{1}{3\sqrt{7}} \left(\Phi_1 + \sqrt{6}\Phi_2 \right) \\ \varphi_2 = \frac{1}{12\sqrt{7}} \left(4\Phi_1 - 3\sqrt{6}\Phi_2 \right) \end{cases} \quad (7.80)$$

and this leads to the effective potential

$$V(\Phi_1, \Phi_2) = \frac{1}{2} e^{-\sqrt{7}\Phi_1} \left(-3e^{4\Phi_1/\sqrt{7}-2\Phi_2\sqrt{2/21}} - 4e^{4\Phi_1/\sqrt{7}+\Phi_2\sqrt{3/14}} + Q^2 \right). \quad (7.81)$$

In this case there is an extremum for

$$\Phi_1 = \frac{\sqrt{7}}{4} \log \frac{Q^2}{3} \quad \Phi_2 = 0 \quad (7.82)$$

and in correspondence of this point:

$$\langle V \rangle = -2 \frac{3^{3/4}}{Q^{3/2}}. \quad (7.83)$$

To verify the stability we write the Hessian matrix:

$$\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} = 2 \frac{3^{3/4}}{Q^{3/2}} \begin{pmatrix} 3 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad (7.84)$$

which possesses a negative eigenvalue

$$m^2|_1 = 3 |\langle V \rangle| \quad m^2|_2 = -\frac{1}{2} |\langle V \rangle|. \quad (7.85)$$

This must be confronted with the BF bound we found in Eq. (7.65):

$$M^2 = \frac{3}{8} |\langle V \rangle| + m^2 = \left(\frac{3}{8} - \frac{1}{2} \right) |\langle V \rangle| = -\frac{1}{8} |\langle V \rangle| < 0. \quad (7.86)$$

The mode, corresponding to the ratio of the two radii Φ_2 , is actually unstable. This is not so surprising since in this case the flux is proportional to the total volume of the compact directions and the system is intrinsically unstable with respect to a perturbation that would keep this volume constant while changing the ratio between the two radii.

$\text{AdS}_4 \times S^{d_1} \times S^{7-d_1}$

The field choice we made above proves to be very useful when dealing with general $\text{AdS}_4 \times S^{d_1} \times S^{7-d_1}$ backgrounds. In fact the potential reads:

$$V(\Phi_1, \Phi_2) = \frac{1}{2} e^{-\sqrt{7}\Phi_1} \left(-d_1 e^{4\Phi_1/\sqrt{7}-\Phi_2\sqrt{(-2d_1+14)/(7d_1)}} + \right. \\ \left. - (7-d_1) e^{4\Phi_1/\sqrt{7}+\Phi_2\sqrt{2d_1/(7(d_1-7))}} + Q^2 \right) \quad (7.87)$$

where d_1 appears only as a coefficient for Φ_2 . Therefore the solution is independent of the dimension d_1 and the stationary point is again

$$\Phi_1 = \frac{\sqrt{7}}{4} \log \frac{Q^2}{3} \quad \Phi_2 = 0 \quad (7.88)$$

with

$$\langle V \rangle = -2 \frac{3^{3/4}}{Q^{3/2}}. \quad (7.89)$$

This solution is again unstable, since the square masses are again

$$m^2|_1 = 3 |\langle V \rangle| \quad m^2|_2 = -\frac{1}{2} |\langle V \rangle| \quad (7.90)$$

and the BF bound is always for $-3/8 |\langle V \rangle|$ since it only depends on the non-compact dimensions that are $d = 4$.

7.3 Type IIB backgrounds

Some of the solutions we found thus far can be naturally reduced to type II. This is the case when they contain factors of AdS_3 or odd-dimensional spheres for they can be respectively written as space-like fibrations over AdS_2 or complex projective planes and as such, when compactified on the fiber don't give rise to dilaton fields.

On the other hand one might also directly look for type II solutions with the same type of geometry factorized in constant curvature spaces. In this section we will in particular concentrate on type IIB solutions with structure $V_4 \times M_3 \times \tilde{M}_3$. Before starting one can try to make some educated guesses about the expected kind of solutions. It would then appear rather natural to expect perturbative $\text{AdS}_4 \times S^3 \times S^3$ solutions that might prove to be unstable (in the same spirit as in [DFG⁺02]) and one might further imagine that adding non-perturbative objects with negative tension – orientifold planes – the no-go theorem for de Sitter [MN01] can be contoured, thus allowing for internal hyperbolic manifolds and for de Sitter solutions (as it was suggested in [SS06]). In fact we will prove that both guesses are ultimately wrong by carefully studying the effective potential in four dimensions obtained by taking into account dilaton, RR zero-form and the two breathing modes for the internal manifolds. More precisely we will show that:

- no truly perturbative solution exists, *i.e.* the presence of D-branes is necessary;
- in presence of orientifolds the only possible compactification happens on a T^6 ;
- no solution is allowed with hyperbolic or de Sitter components;
- the only allowed solution is $\text{AdS}_4 \times S^3 \times S^3$ and this solution is perturbatively stable, thanks to the BF bound, although it can't be completely trusted since it belongs to an intermediate-coupling regime.

Let us consider a type IIB background of the form $V_4 \times M_3 \times \tilde{M}_3$. The action in Einstein frame reads:

$$S = S_0 + S_{\text{loc}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g^{(10)}} \left[R^{(10)} - \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{2(\text{Im } \tau)^2} - \frac{1}{2} M_{ij} F_3^i F_3^j + \right. \\ \left. - \frac{1}{2} C_4 \wedge F_3^1 \wedge F_3^2 \right] + S_{\text{loc}} \quad (7.91)$$

where we didn't include the 1-form and the 5-form which are not compatible with the symmetries of the metric ansatz. As usual $\tau = C_0 + i e^{-\Phi}$ is the dilaton-axion field, M_{ij} is

$$M_{ij} = \frac{1}{\text{Im } \tau} \begin{pmatrix} |\tau|^2 & -\text{Re } \tau \\ -\text{Re } \tau & 1 \end{pmatrix} \quad (7.92)$$

and S_{loc} is the contribution due to D_3 -branes and O_3 -planes:

$$S_{\text{loc}} = NT_3 \int d^4x \sqrt{-g^{(4)}}. \quad (7.93)$$

As before we look for a solution of the kind

$$ds_{(10)}^2 = ds_{(4)}^2 + e^{2\varphi(x)} ds_3^2 + e^{2\tilde{\varphi}(x)} ds_{(3)}^2 \quad (7.94)$$

where for the moment being the two internal manifolds can have positive or negative curvature. With respect to the M-theory situation, in this case we will have to pay attention to the extra τ complex scalar field and to the extra-term in the action S_{loc} .

Let's start with the latter. This already has the form of a four-dimensional integral, so to evaluate the corresponding contribution to the potential we only need to take into account the Weyl rescaling in Eq. (7.57) and the presence of the internal volumes and Newton constant:

$$S_{\text{loc}} = NT_3 \int d^4x \sqrt{-g^{(4)}} = \frac{V_3 \tilde{V}_3}{2\kappa_{10}^2} \frac{2NT_3 \kappa_{10}^2}{V_3 \tilde{V}_3} \int d^4x e^{-2\Psi_1} \sqrt{\tilde{g}^{(4)}} \quad (7.95)$$

and then

$$V_{\text{loc}} = \frac{2NT_3 \kappa_{10}^2}{V_3 \tilde{V}_3} e^{-6\varphi - 6\tilde{\varphi}}. \quad (7.96)$$

For reasons that will appear more clear in the following let us rescale the τ field as:

$$z = \frac{\tau}{k} \quad (7.97)$$

k being a real constant. This does not affect the kinetic term:

$$\frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{2(\text{Im } \tau)^2} = \frac{\partial_\mu z \partial^\mu \bar{z}}{2(\text{Im } z)^2} \quad (7.98)$$

but makes it canonical when studying the small perturbations around the equilibrium solutions.

The most general three-field configuration compatible with the symmetries is

$$F_3^i = 4\pi^2 \alpha' (n_i \omega_3 + \tilde{n}_i \tilde{\omega}_3) \quad (7.99)$$

where n_i and \tilde{n}_i are integers and ω_3 and $\tilde{\omega}_3$ are the normalized volume forms. The subscript 1 stands for RR field, 2 for NS. The corresponding potential reads:

$$V_F = \frac{(4\pi^2)^2 \alpha'^2}{2k \operatorname{Im} z V_3^2} |n_1 k z - n_2|^2 e^{-6\varphi} + \frac{(4\pi^2)^2 \alpha'^2}{2k \operatorname{Im} z \tilde{V}_3^2} |\tilde{n}_1 k z - \tilde{n}_2|^2 e^{-6\tilde{\varphi}} \quad (7.100)$$

in order to clean the notation let us write the volumes as

$$V_3 = 2x\pi^2 \quad \tilde{V}_3 = 2\tilde{x}\pi^2. \quad (7.101)$$

Adding the contribution from the curvatures of the internal manifolds:

$$R = \frac{3}{2}\epsilon \quad \tilde{R} = \frac{3}{2}\tilde{\epsilon} \quad (7.102)$$

where $\epsilon = 1$ for spheres and $\epsilon = -1$ for hyperbolic spaces, we can write the potential as

$$V(\varphi, \tilde{\varphi}, z) = e^{-3\varphi-3\tilde{\varphi}} \left(-\frac{3\epsilon}{2} e^{-2\varphi} - \frac{3\tilde{\epsilon}}{2} e^{-2\tilde{\varphi}} + \frac{2\alpha'^2}{x^2 k \operatorname{Im} z} |n_1 k z - n_2|^2 e^{-6\varphi} + \frac{2\alpha'^2}{x^2 k \operatorname{Im} z} |\tilde{n}_1 k z - \tilde{n}_2|^2 e^{-6\tilde{\varphi}} \right) + \frac{2NT_3 \kappa_{10}^2}{x\tilde{x} (2\pi^2)^2} e^{-6\varphi-6\tilde{\varphi}}. \quad (7.103)$$

As a first step we normalize the scalars φ and $\tilde{\varphi}$:

$$\varphi = \frac{\Phi_1 + 2\Phi_2}{4\sqrt{3}} \quad \tilde{\varphi} = \frac{\Phi_1 - 2\Phi_2}{4\sqrt{3}} \quad (7.104)$$

so to rewrite the potential as

$$V(\Phi_1, \Phi_2, z) = e^{-\Phi_1 \sqrt{3}/2} \left(-\frac{3}{2} \epsilon e^{-(\Phi_1+2\Phi_2)/(2\sqrt{3})} - \frac{3}{2} \tilde{\epsilon} e^{-(\Phi_1-2\Phi_2)/(2\sqrt{3})} + \frac{2\alpha'^2}{x^2 k \operatorname{Im} z} |n_1 k z - n_2|^2 e^{-(\Phi_1+2\Phi_2)\sqrt{3}/2} + \frac{2\alpha'^2}{x^2 k \operatorname{Im} z} |\tilde{n}_1 k z - \tilde{n}_2|^2 e^{-(\Phi_1-2\Phi_2)\sqrt{3}/2} \right) + \frac{2NT_3 \kappa_{10}^2}{x\tilde{x} (2\pi^2)^2} e^{-\Phi_1 \sqrt{3}}. \quad (7.105)$$

We have to take into account the anomaly cancellation condition

$$N = n_1 \tilde{n}_2 - n_2 \tilde{n}_1 \quad (7.106)$$

and express all the constants in terms of α' :

$$T_3 = |\mu_3| = \frac{1}{(2\pi)^2 \alpha'^2} \quad 2\kappa_{10}^2 = (2\pi)^7 \alpha'^4 \quad (7.107)$$

so that $2\kappa_{10}^2 T_3 = (2\pi)^4 \alpha'^2$.

Up to this moment the sign of N is arbitrary, so one might think that different kinds of solution are possible. Actually this is not the case. If we differentiate with respect to $\text{Im } z$ and $\text{Re } z$ we obtain the values corresponding to the stationary point

$$\text{Im } z = \frac{e^{\Phi_2 \sqrt{3}} |N| x \tilde{x}}{k \left(e^{2\Phi_2 \sqrt{3}} \tilde{n}_1^2 x^2 + n_1^2 \tilde{x}^2 \right)} \quad \text{Re } z = \frac{e^{2\Phi_2 \sqrt{3}} \tilde{n}_1 \tilde{n}_2 x^2 + n_1 n_2 \tilde{x}^2}{k \left(e^{2\Phi_2 \sqrt{3}} \tilde{n}_1^2 x^2 + n_1^2 \tilde{x}^2 \right)}. \quad (7.108)$$

Putting these back into the potential one gets:

$$V(\Phi_1, \Phi_2) = \frac{e^{(3\Phi_1 + \Phi_2)/\sqrt{3}}}{2x\tilde{x}} \left(8e^{\Phi_2/\sqrt{3}} (N + |N|) - 3x\tilde{x}e^{\Phi_1/\sqrt{3}} \left(\epsilon + \tilde{\epsilon}e^{(\Phi_1 + 2\Phi_2)/\sqrt{3}} \right) \right) \quad (7.109)$$

and for $N \geq 0$ this gives respectively:

$$V^+(\Phi_1, \Phi_2) = \frac{e^{(3\Phi_1 + \Phi_2)/\sqrt{3}}}{2x\tilde{x}} \left(16e^{\Phi_2/\sqrt{3}} N - 3x\tilde{x}e^{\Phi_1/\sqrt{3}} \left(\epsilon + \tilde{\epsilon}e^{(\Phi_1 + 2\Phi_2)/\sqrt{3}} \right) \right) \quad (7.110)$$

$$V^-(\Phi_1, \Phi_2) = -\frac{3}{2}e^{-(2\Phi_1 + \Phi_2)/\sqrt{3}} \left(\epsilon + \tilde{\epsilon}e^{2\Phi_2/\sqrt{3}} \right). \quad (7.111)$$

The second potential only admits a solution if $\epsilon = \tilde{\epsilon} = 0$, but this corresponds to a flat internal space (Polchinski).

Choosing the first $N > 0$ case one finds a stationary point for

$$\langle \Phi_1 \rangle = \sqrt{3} \log \frac{4N\alpha'^2}{x\tilde{x}\tilde{\epsilon}} \sqrt{\frac{\tilde{\epsilon}}{\epsilon}} \quad \langle \Phi_2 \rangle = \frac{\sqrt{3}}{2} \log \frac{\epsilon}{\tilde{\epsilon}} \quad (7.112)$$

which imposes $\epsilon = \tilde{\epsilon} = 1$, i.e. the only solution has an internal $S^3 \times S^3$ space, as advertized above.

To summarize, the stationary point corresponds to:

$$\langle \text{Im } z \rangle = \frac{Nx\tilde{x}}{k \left(\tilde{n}_1^2 x^2 + n_1^2 \tilde{x}^2 \right)} \quad \langle \text{Re } z \rangle = \frac{\tilde{n}_1 \tilde{n}_2 x^2 + n_1 n_2 \tilde{x}^2}{k \left(\tilde{n}_1^2 x^2 + n_1^2 \tilde{x}^2 \right)} \quad (7.113)$$

$$\langle \Phi_1 \rangle = \sqrt{3} \log \frac{4N\alpha'^2}{x\tilde{x}} \quad \langle \Phi_2 \rangle = 0 \quad (7.114)$$

now we can choose

$$k = \frac{Nx\tilde{x}}{\tilde{n}_1^2 x^2 + n_1^2 \tilde{x}^2} \quad (7.115)$$

and putting $x = \tilde{x} = 1$ (as we must in the case of spheres) we have

$$\langle \text{Im } z \rangle = 1 \quad \langle \text{Re } z \rangle = \frac{\tilde{n}_1 \tilde{n}_2 + n_1 n_2}{n_1 \tilde{n}_2 - n_2 \tilde{n}_1} \quad \langle \Phi_1 \rangle = \sqrt{3} \log (4N\alpha'^2) \quad \langle \Phi_2 \rangle = 0 \quad (7.116)$$

and the corresponding potential is

$$\langle V \rangle = -\frac{1}{16N^2\alpha^4}. \quad (7.117)$$

To check the stability we write the Hessian matrix and compute its eigenvalues. They turn out to be

$$m^2|_1 = |\langle V \rangle| \quad m^2|_2 = 2|\langle V \rangle| \quad (7.118)$$

$$m^2|_3 = \frac{\sqrt{13}+3}{2} |\langle V \rangle| \quad m^2|_4 = -\frac{\sqrt{13}-3}{2} |\langle V \rangle| \quad (7.119)$$

and the only negative one $m^2|_4$ doesn't cause any instability since it doesn't cross the BF bound:

$$m^2|_4 + \frac{3}{8} |\langle V \rangle| = \frac{15-4\sqrt{13}}{8} |\langle V \rangle| > 0 \quad (7.120)$$

This result about the stability of the product of two three-spheres can at first sight be puzzling since, after analyzing the results of Sec. 7.2, we've grown to expect such configurations to be unstable under the mode in which one of the two spheres shrinks and the other grows, keeping constant the overall volume. Here it is not the case and this can be easily understood in terms of gauge fields. In the M-theory configurations, in fact, only one field was turned on and it spanned over the whole internal manifold, so that it effected only the total volume. Here, on the other hand, RR and NS-NS fields are turned on separately on the two submanifolds, thus contributing to the stabilization of each of the radii.

CHAPTER 8

Conclusions and further perspectives

Beauty is truth, truth beauty, – that is all
Ye know on earth, and all ye need to know.

Ode on a Grecian Urn
JOHN KEATS

The search for exact string solutions is a fascinating subject by itself. It is based, as most of the wonders of string theory, on the interplay between the two-dimensional conformal field theory description on the world-sheet and the ten-dimensional low-energy interpretation in terms of spacetime fields.

In this thesis we have dealt with a new class of string backgrounds living in the moduli space of WZW models. They enjoy at the same time nice supergravity properties, all geometrical quantities being naturally expressed in terms of algebraic invariants, and a clear CFT characterization, inherited from the beautiful theory of group manifolds.

Apart from their intrinsic elegance, those new backgrounds also find interesting physical applications as compactification manifolds, laboratories for the analysis of string propagation in classically pathological backgrounds with closed time-like curves, in black hole configurations with non-trivial topology. Laboratories in which we can keep higher order effects under control and write down a modular-invariant partition function, or at least the spectra of primary operators.

Starting from this solid CFT ground we are then allowed to peep into the off-shell physics using RG techniques, both from a two-dimensional and field-theoretical perspective. We can thus observe the relaxation of out-of-equilibrium vacua described by charge transmutation, *i.e.* two gauge fields eventually collapsing into a single one, while the total charge is conserved. A new change of viewpoint then allows us to recast the problem in terms of a cosmological time-dependent solution. The RG dynamics becomes an approximate description – valid in a certain region of the moduli space depending on the central charge deficit – for the behaviour of a Big-bang-like isotropic FRW universe.

We have emphasized many times the importance of having a CFT description and exact models, but this is not possible in general. For example we

completely lack such a kind of interpretation for type II string or M-theory, but – and maybe for this very reason – it is important to look for new insights in these frameworks. In this spirit we have studied compactifications involving maximally symmetric spaces which in general do not preserve supersymmetry, looking in particular for hyperbolic solutions, in the not-too-concealed hope of reaching de Sitter-like spacetimes.

So, what remains to do? By its very nature this is a work in progress. The path to further developments is full of technical and conceptual obstacles but one can easily name some of the natural possible directions. First of all it would be interesting to have a non-Abelian counterpart for our asymmetric gauging. This would be allowed by the heterotic string framework and it does indeed work at the supergravity level; on the other hand it is not clear how it could be implemented in a CFT framework – the evidence at hand pointing towards a discrete structure for the deformations. Then, one would like to reach a better understanding of non-rational CFTs and, in particular, of the $SL(2, \mathbb{R})$ WZW model; this would allow us to write the partition function for AdS_2 spacetime, Bertotti-Robinson black hole and (charged) black string. This, in turn, might prove useful for obtaining a non-trivial microscopic description for the thermodynamics of such singular objects. Another more phenomenological direction would be to study the low energy field theory consequence of a compactification on geometric cosets, again using the complete knowledge which we have of the spectrum and partition function in this case. As soon as we move away from the familiar CFT framework things become more difficult and much more interesting. Even if a theory for the two-dimensional RG flow is established, there are very few cases in which one can really work out non-trivial examples such as the flow one expects to link WZW models at different levels. The spacetime description might then prove useful for getting new hints at the world-sheet physics.

APPENDIX A

Table of conventions

Christoffel symbol	$\{\begin{smallmatrix} \kappa \\ \mu\nu \end{smallmatrix}\}$	$\frac{1}{2}g^{\kappa\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$
covariant derivative	∇	$\nabla_\nu t_{\mu_1 \dots \mu_p}^{\lambda_1 \dots \lambda_p} = \partial_\nu t_{\mu_1 \dots \mu_p}^{\lambda_1 \dots \lambda_p} + \Gamma^{\lambda_1}_{\nu\kappa} t_{\mu_1 \dots \mu_p}^{\kappa\lambda_2 \dots \lambda_p} + \dots - \Gamma^\kappa_{\nu\mu_1} t_{\kappa\mu_2 \dots \mu_p}^{\lambda_1 \dots \lambda_p} - \dots$
dual Coxeter number	g^*	$g^* = \frac{1}{2 \dim G} f_{\alpha\beta\gamma} f^{\alpha\beta\gamma}$
exterior derivative	d	$d\omega = \frac{1}{n!} \frac{\partial \omega_{\mu_1 \mu_2 \dots \mu_n}}{\partial x^\lambda} dx^\lambda \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}$
Maurer-Cartan one-form	\mathcal{J}^a	$\mathcal{J}^a = \langle t^a g^{-1} dg \rangle$
n -form	ω	$\omega = \frac{1}{n!} \omega_{\mu_1 \mu_2 \dots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}$
structure constants	$f_{\beta\gamma}^\alpha$	$[t^\alpha, t^\beta] = f_{\beta\gamma}^\alpha t^\gamma$
trace	$\langle \rangle$	$\langle M_{ij} \rangle = \sum_i M_{ii}$

APPENDIX B

Explicit parametrizations for some Lie groups

In this appendix we collect the explicit parametrizations used for the $SU(2)$, $SL(2, \mathbb{R})$, $SU(3)$ and $USp(4)$ groups.

B.1 The three-sphere

The commutation relations for the generators of $SU(2)$ are

$$[J^1, J^2] = iJ^3 \quad [J^2, J^3] = iJ^1 \quad [J^3, J^1] = iJ^2. \quad (\text{B.1})$$

A two-dimensional realization is obtained by using the standard Pauli matrices¹ σ^a : $J^a = \sigma^a/2$.

The Euler-angle parameterization for $SU(2)$ is defined as:

$$g = e^{i\frac{\gamma}{2}\sigma^3} e^{i\frac{\beta}{2}\sigma^1} e^{i\frac{\alpha}{2}\sigma^3}. \quad (\text{B.2})$$

The $SU(2)$ group manifold is a unit-radius three-sphere. A three-sphere can be embedded in flat Euclidean four-dimensional space with coordinates (x^1, x^2, x^3, x^4) , as $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = L^2$. The corresponding $SU(2)$ element g is the following:

$$g = L^{-1} \begin{pmatrix} x^4 + ix^2 & x^3 + ix^1 \\ -x^3 + ix^1 & x^4 - ix^2 \end{pmatrix}. \quad (\text{B.3})$$

In general, the invariant metric of a group manifold can be expressed in terms of the left-invariant Cartan–Maurer one-forms. In the $SU(2)$ case under consideration (unit-radius S^3),

$$\mathcal{J}^1 = \frac{1}{2} \text{tr} \left(\sigma^1 g^{-1} dg \right), \quad \mathcal{J}^2 = \frac{1}{2} \text{tr} \left(\sigma^2 g^{-1} dg \right), \quad \mathcal{J}^3 = \frac{1}{2} \text{tr} \left(\sigma^3 g^{-1} dg \right) \quad (\text{B.4})$$

and

$$ds^2 = \sum_{i=1}^3 \mathcal{J}^i \otimes \mathcal{J}^i \quad (\text{B.5})$$

¹The normalization of the generators with respect to the Killing product in $\mathfrak{su}(2)$: $\kappa(X, Y) = \text{tr}(XY)$ is such that $\kappa(J^a, J^b) = 1/2$ and correspondingly the root has length squared $\psi = 2$.

The volume form reads:

$$\omega_{[3]} = \mathcal{J}^1 \wedge \mathcal{J}^2 \wedge \mathcal{J}^3. \quad (\text{B.6})$$

In the Euler-angle parameterization, Eq. (B.5) reads (for a radius- L three-sphere):

$$ds^2 = \frac{L^2}{4} \left(d\alpha^2 + d\gamma^2 + 2 \cos \beta d\alpha d\gamma + d\beta^2 \right), \quad (\text{B.7})$$

whereas (B.6) leads to

$$\omega_{[3]} = \frac{L^3}{8} \sin \beta d\alpha \wedge d\beta \wedge d\gamma. \quad (\text{B.8})$$

The Levi-Civita connection has scalar curvature $R = 6/L^2$.

The isometry group of the $SU(2)$ group manifold is generated by left or right actions on g : $g \rightarrow hg$ or $g \rightarrow gh \forall h \in SU(2)$. From the four-dimensional point of view, it is generated by the rotations $\zeta_{ab} = i(x_a \partial_b - x_b \partial_a)$ with $x_a = \delta_{ab} x^b$. We list here explicitly the six generators, as well as the group action they correspond to:

$$L_1 = \frac{1}{2} (-\zeta_{32} + \zeta_{41}), \quad g \rightarrow e^{-i\frac{\lambda}{2}\sigma^1} g, \quad (\text{B.9a})$$

$$L_2 = \frac{1}{2} (-\zeta_{43} - \zeta_{12}), \quad g \rightarrow e^{i\frac{\lambda}{2}\sigma^2} g, \quad (\text{B.9b})$$

$$L_3 = \frac{1}{2} (-\zeta_{31} - \zeta_{42}), \quad g \rightarrow e^{i\frac{\lambda}{2}\sigma^3} g, \quad (\text{B.9c})$$

$$R_1 = \frac{1}{2} (\zeta_{41} + \zeta_{32}), \quad g \rightarrow g e^{i\frac{\lambda}{2}\sigma^1}, \quad (\text{B.9d})$$

$$R_2 = \frac{1}{2} (-\zeta_{43} + \zeta_{12}), \quad g \rightarrow g e^{i\frac{\lambda}{2}\sigma^2}, \quad (\text{B.9e})$$

$$R_3 = \frac{1}{2} (\zeta_{31} - \zeta_{42}), \quad g \rightarrow g e^{i\frac{\lambda}{2}\sigma^3}. \quad (\text{B.9f})$$

Both sets satisfy the algebra (B.1). The norms squared of the Killing vectors are all equal to $L^2/4$.

The currents of the $SU(2)_k$ WZW model are easily obtained as:

$$J^i = -k \operatorname{tr} \left(i\sigma^i \partial g g^{-1} \right) \quad \bar{J}^i = -k \operatorname{tr} \left(i\sigma^i g^{-1} \bar{\partial} g \right), \quad (\text{B.10})$$

where $L = \sqrt{k}$, at the classical level. Explicit expressions are given in Tab. B.1.

B.2 AdS₃

The commutation relations for the generators of the $SL(2, \mathbb{R})$ algebra are

$$[J^1, J^2] = -iJ^3 \quad [J^2, J^3] = iJ^1 \quad [J^3, J^1] = iJ^2. \quad (\text{B.11})$$

The sign in the first relation is the only difference with respect to the $SU(2)$ in Eq. (B.1).

The three-dimensional anti-de-Sitter space is the universal covering of the $SL(2, \mathbb{R})$ group manifold. The latter can be embedded in a Lorentzian flat space with signature $(-, +, +, -)$ and coordinates (x^0, x^1, x^2, x^3) :

$$g = L^{-1} \begin{pmatrix} x^0 + x^2 & x^1 + x^3 \\ x^1 - x^3 & x^0 - x^2 \end{pmatrix}, \quad (\text{B.12})$$

sector	Killing vector	Current
left moving	$\frac{\sin \gamma}{\sin \beta} \partial_\alpha + \cos \gamma \bar{\partial}_\beta - \frac{\sin \gamma}{\tan \beta} \partial_\gamma$	$k (\sin \beta \sin \gamma \partial_\alpha + \cos \gamma \partial b)$
	$\frac{\cos \gamma}{\sin \beta} \partial_\alpha - \sin \gamma \bar{\partial}_\beta - \frac{\cos \gamma}{\tan \beta} \partial_\gamma$	$k (\cos \gamma \sin \beta \partial_\alpha - \sin \gamma \partial \beta)$
	∂_γ	$k (\partial \gamma + \cos \beta \partial \alpha)$
right moving	$-\frac{\sin \alpha}{\tan \beta} \partial_\alpha + \cos \alpha \bar{\partial}_\beta + \frac{\sin \alpha}{\sin \beta} \partial_\gamma$	$k (\cos \alpha \bar{\partial} \beta + \sin \alpha \sin \beta \bar{\partial} \gamma)$
	$\frac{\cos \alpha}{\tan \beta} \partial_\alpha + \sin \alpha \bar{\partial}_\beta - \frac{\cos \alpha}{\sin \beta} \partial_\gamma$	$k (\sin \alpha \bar{\partial} \beta - \cos \alpha \sin \beta \bar{\partial} \gamma)$
	∂_α	$k (\bar{\partial} \alpha + \cos \beta \bar{\partial} \gamma)$

Table B.1: Killing vectors $\{\iota L_1, \iota L_2, \iota L_3\}$ and $\{\iota R_1, \iota R_2, \iota R_3\}$, and holomorphic and anti-holomorphic currents for $SU(2)$ in Euler angles.

where L is the radius of AdS₃. One can again introduce Euler-like angles

$$g = e^{i(\tau+\phi)\sigma_2/2} e^{\rho\sigma_1} e^{i(\tau-\phi)\sigma_2/2}, \quad (\text{B.13})$$

which provide good global coordinates for AdS₃ when $\tau \in (-\infty, +\infty)$, $\rho \in [0, \infty)$, and $\phi \in [0, 2\pi)$.

An invariant metric (see Eq. (B.5)) can be introduced on AdS₃. In Euler angles, the latter reads:

$$ds^2 = L^2 \left[-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2 \right]. \quad (\text{B.14})$$

The Ricci scalar of the corresponding Levi-Civita connection is $R = -6/L^2$.

The isometry group of the $SL(2, \mathbb{R})$ group manifold is generated by left or right actions on g : $g \rightarrow hg$ or $g \rightarrow gh \forall h \in SL(2, \mathbb{R})$. From the four-dimensional point of view, it is generated by the Lorentz boosts or rotations $\zeta_{ab} = i(x_a \partial_b - x_b \partial_a)$ with $x_a = \eta_{ab} x^b$. We list here explicitly the six generators, as well as the group action they correspond to:

$$L_1 = \frac{1}{2} (\zeta_{32} - \zeta_{01}), \quad g \rightarrow e^{-\frac{\lambda}{2}\sigma^1} g, \quad (\text{B.15a})$$

$$L_2 = \frac{1}{2} (-\zeta_{31} - \zeta_{02}), \quad g \rightarrow e^{-\frac{\lambda}{2}\sigma^3} g, \quad (\text{B.15b})$$

$$L_3 = \frac{1}{2} (\zeta_{03} - \zeta_{12}), \quad g \rightarrow e^{i\frac{\lambda}{2}\sigma^2} g, \quad (\text{B.15c})$$

$$R_1 = \frac{1}{2} (\zeta_{01} + \zeta_{32}), \quad g \rightarrow g e^{\frac{\lambda}{2}\sigma^1}, \quad (\text{B.15d})$$

$$R_2 = \frac{1}{2} (\zeta_{31} - \zeta_{02}), \quad g \rightarrow g e^{-\frac{\lambda}{2}\sigma^3}, \quad (\text{B.15e})$$

$$R_3 = \frac{1}{2} (\zeta_{03} + \zeta_{12}), \quad g \rightarrow g e^{i\frac{\lambda}{2}\sigma^2}. \quad (\text{B.15f})$$

Both sets satisfy the algebra (B.11). The norms of the Killing vectors are the following:

$$\|\iota L_1\|^2 = \|\iota R_1\|^2 = \|\iota L_2\|^2 = \|\iota R_2\|^2 = -\|\iota L_3\|^2 = -\|\iota R_3\|^2 = \frac{L^2}{4}. \quad (\text{B.16})$$

Moreover $L_i \cdot L_j = 0$ for $i \neq j$ and similarly for the right set. Left vectors are not orthogonal to right ones.

The isometries of the $SL(2, \mathbb{R})$ group manifold turn into symmetries of the $SL(2, \mathbb{R})_k$ WZW model, where they are realized in terms of conserved currents²:

$$J^1(z) \pm J^3(z) = -k \operatorname{tr} \left(\left(\sigma^1 \mp i\sigma^2 \right) \partial g g^{-1} \right), \quad J^2(z) = -k \operatorname{tr} \left(\sigma^3 \partial g g^{-1} \right), \quad (\text{B.17a})$$

$$\bar{J}^1(\bar{z}) \pm \bar{J}^3(\bar{z}) = k \operatorname{tr} \left(\left(\sigma^1 \pm i\sigma^2 \right) g^{-1} \bar{\partial} g \right), \quad \bar{J}^2(\bar{z}) = -k \operatorname{tr} \left(\sigma^3 g^{-1} \bar{\partial} g \right). \quad (\text{B.17b})$$

At the quantum level, these currents, when properly normalized, satisfy the following affine $SL(2, \mathbb{R})_k$ OPA³:

$$J^3(z)J^3(0) \sim -\frac{k}{2z^2}, \quad (\text{B.18a})$$

$$J^3(z)J^\pm(0) \sim \pm \frac{J^\pm}{z}, \quad (\text{B.18b})$$

$$J^+(z)J^-(0) \sim \frac{2J^3}{z} - \frac{k}{z^2}, \quad (\text{B.18c})$$

and similarly for the right movers. The central charge of the enveloping Virasoro algebra is $c = 3 + 6/(k-2)$.

We will introduce three different coordinate systems where the structure of AdS_3 as a Hopf fibration is more transparent. They are explicitly described in the following.

- The (ρ, t, ϕ) coordinate system used to describe the magnetic deformation is defined as follows:

$$\begin{cases} \frac{x_0}{L} &= \cosh \frac{\rho}{2} \cosh \frac{\phi}{2} \cos \frac{t}{2} - \sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \sin \frac{t}{2} \\ \frac{x_1}{L} &= -\sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} - \cosh \frac{\rho}{2} \sinh \frac{\phi}{2} \sin \frac{t}{2} \\ \frac{x_2}{L} &= -\cosh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} + \sinh \frac{\rho}{2} \cosh \frac{\phi}{2} \sin \frac{t}{2} \\ \frac{x_3}{L} &= -\sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} - \cosh \frac{\rho}{2} \cosh \frac{\phi}{2} \sin \frac{t}{2}. \end{cases} \quad (\text{B.19})$$

The metric (B.5) reads:

$$ds^2 = \frac{L^2}{4} \left(d\rho^2 + d\phi^2 - dt^2 - 2 \sinh \rho dt d\phi \right) \quad (\text{B.20})$$

and the corresponding volume form is:

$$\omega_{[3]} = \frac{L^3}{8} \cosh \rho \cdot d\rho \wedge d\phi \wedge dt \quad (\text{B.21})$$

Killing vectors and currents are given in Tab. B.2. It is worth to remark that this coordinate system is such that the t -coordinate lines coincide with the integral curves of the Killing vector tL_3 , whereas the ϕ -lines are the curves of tR_2 .

²When writing actions a choice of gauge for the NS potential is implicitly made, which breaks part of the symmetry: boundary terms appear in the transformations. These must be properly taken into account in order to reach the conserved currents. Although the expressions for the latter are not unique, they can be put in an improved-Noether form, in which they have only holomorphic (for L_i 's) or anti-holomorphic (for R_j 's) components.

³In some conventions the level is $x = -k$. This allows to unify commutation relations for the affine $SL(2, \mathbb{R})_x$ and $SU(2)_x$ algebras. Unitarity demands $x < -2$ for the former and $0 < x$ with integer x for the latter.

- The (r, x, τ) coordinate system used to describe the electric deformation is defined as follows:

$$\begin{cases} \frac{x_0}{L} = \cosh \frac{r}{2} \cosh \frac{x}{2} \cos \frac{\tau}{2} + \sinh \frac{r}{2} \sinh \frac{x}{2} \sin \frac{\tau}{2} \\ \frac{x_1}{L} = -\sinh \frac{r}{2} \cosh \frac{x}{2} \cos \frac{\tau}{2} + \cosh \frac{r}{2} \sinh \frac{x}{2} \sin \frac{\tau}{2} \\ \frac{x_2}{L} = -\cosh \frac{r}{2} \sinh \frac{x}{2} \cos \frac{\tau}{2} - \sinh \frac{r}{2} \cosh \frac{x}{2} \sin \frac{\tau}{2} \\ \frac{x_3}{L} = \sinh \frac{r}{2} \sinh \frac{x}{2} \cos \frac{\tau}{2} - \cosh \frac{r}{2} \cosh \frac{x}{2} \sin \frac{\tau}{2}. \end{cases} \quad (\text{B.22})$$

For $\{r, x, \tau\} \in \mathbb{R}^3$, this patch covers exactly once the whole AdS₃, and is regular everywhere [CH94]. The metric is then given by

$$ds^2 = \frac{L^2}{4} (dr^2 + dx^2 - d\tau^2 + 2 \sinh r dx d\tau) \quad (\text{B.23})$$

and correspondingly the volume form is

$$\omega_{[3]} = \frac{L^3}{8} \cosh r dr \wedge dx \wedge d\tau. \quad (\text{B.24})$$

Killing vectors and currents are given in Tab. B.3. In this case the x -coordinate lines coincide with the integral curves of the Killing vector ιR_2 , whereas the τ -lines are the curves of ιR_3 .

- The Poincaré coordinate system used to obtain the electromagnetic-wave background is defined by

$$\begin{cases} x^0 + x^2 = \frac{L}{u} \\ x^0 - x^2 = Lu + \frac{Lx^+x^-}{u} \\ x^1 \pm x^3 = \frac{Lx^\pm}{u}. \end{cases} \quad (\text{B.25})$$

For $\{u, x^+, x^-\} \in \mathbb{R}^3$, the Poincaré coordinates cover once the $SL(2\mathbb{R})$ group manifold. Its universal covering, AdS₃, requires an infinite number of such patches. Moreover, these coordinates exhibit a Rindler horizon at $|u| \rightarrow \infty$; the conformal boundary is at $|u| \rightarrow 0$. Now the metric reads:

$$ds^2 = \frac{L^2}{u^2} (du^2 + dx^+ dx^-) \quad (\text{B.26})$$

and the volume form:

$$\omega_{[3]} = \frac{L^3}{2u^3} du \wedge dx^- \wedge dx^+. \quad (\text{B.27})$$

In these coordinates it is simple to write certain a linear combination of the Killing vector so to obtain explicitly a light-like isometry generator. For this reason in Tab. B.4 we report the $\{L_1 + L_3, L_1 - L_3, L_2, R_1 + R_3, R_1 - R_3, R_2\}$ isometry generators and the corresponding $\{J_1 + J_3, J_1 - J_3, J_2, \bar{J}_1 + \bar{J}_3, \bar{J}_1 - \bar{J}_3, \bar{J}_2\}$ currents.

Finally, another useful although not global, set of coordinates is defined by

$$g = e^{\frac{\psi-\varphi}{2}\sigma^3} e^{t\sigma^1} e^{\frac{\psi+\varphi}{2}\sigma^3}, \quad (\text{B.28})$$

(ψ and φ are not compact coordinates). The metric reads:

$$ds^2 = L^2 [\cos^2 t d\psi^2 - dt^2 + \sin^2 t d\varphi^2], \quad (\text{B.29})$$

with volume form

$$\omega_{[3]} = \frac{L^3}{2} \sin 2t dt \wedge d\psi \wedge d\varphi. \quad (\text{B.30})$$

Now $L_2 = \frac{1}{2} (\partial_\psi - \partial_\varphi)$ and $R_2 = \frac{1}{2} (\partial_\psi + \partial_\varphi)$.

sector	Killing vector	Current
left moving	$\cos t \partial_\rho + \frac{\sin t}{\cosh \rho} \partial_\phi - \sin t \tanh \rho \partial_t$ $-\sin t \partial_\rho + \frac{\cos t}{\cosh \rho} \partial_\phi - \cos t \tanh \rho \partial_t$ $-\partial_t$	$k (\cos t \partial_\rho + \cosh \rho \sin t \partial_\phi)$ $k (\cos t \cosh \rho \partial_\phi - \sin t \partial_\rho)$ $k (\partial_t + \sinh \rho \partial_\phi)$
right moving	$\cosh \phi \partial_\rho - \sinh \phi \tanh \rho \partial_\phi - \frac{\sinh \phi}{\cosh \rho} \partial_t$ ∂_ϕ $\sinh \phi \partial_\rho - \cosh \phi \tanh \rho \partial_\phi - \frac{\cosh \phi}{\cosh \rho} \partial_t$	$-k (\cosh \phi \partial_\rho + \cosh \rho \sinh \phi \partial_t)$ $k (\partial_\phi - \sinh \rho \partial_t)$ $k (\cosh \rho \cosh \phi \partial_t + \sinh \phi \partial_\rho)$

Table B.2: Killing vectors $\{lL_1, lL_2, lL_3\}$ and $\{lR_1, lR_2, lR_3\}$, and holomorphic and anti-holomorphic currents for the (ρ, t, ϕ) coordinate system (elliptic base).

sector	Killing vector	Current
left moving	$\cosh x \partial_r - \sinh x \tanh r \partial_x + \frac{\sinh x}{\cosh r} \partial_\tau$ ∂_x $-\sinh x \partial_r + \cosh x \tanh r \partial_x - \frac{\cosh x}{\cosh r} \partial_\tau$	$k (\cosh x \partial_r - \cosh r \sinh x \partial_\tau)$ $k (\partial_x + \sinh r \partial_\tau)$ $k (\cosh r \cosh x \partial_\tau - \sinh x \partial_r)$
right moving	$-\cos \tau \partial_r + \frac{\sin \tau}{\cosh \tau} \partial_x - \sin \tau \tanh \tau \partial_\tau$ $\frac{(\cos \tau + \sin \tau \tanh r) \partial_x + (\cos \tau \sinh r - \frac{\sin \tau}{\cosh r}) \partial_\tau}{\cosh r}$ $-\partial_\tau$	$k (-\cos \tau \partial_r + \cosh r \sin \tau \partial_\tau)$ $k (\cos \tau \cosh r \partial_x + \sin \tau \partial_r)$ $k (\partial_\tau - \sinh r \partial_x)$

Table B.3: Killing vectors $\{lL_1, lL_2, lL_3\}$ and $\{lR_1, lR_2, lR_3\}$, and holomorphic and anti-holomorphic currents for the (r, x, τ) coordinate system (hyperbolic base).

B.3 $SU(3)$

To obtain the the Cartan-Weyl basis $\{H_a, E^{\alpha_j}\}$ for the $\mathfrak{su}(3)$ algebra we need to choose the positive roots as follows:

$$\alpha_1 = [\sqrt{2}, 0] \quad \alpha_2 = [-1/\sqrt{2}, \sqrt{3}/2] \quad \alpha_3 = [1/\sqrt{2}, \sqrt{3}/2] \quad (\text{B.31})$$

The usual choice for the defining representation is:

$$\begin{aligned}
 H_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & H_2 &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} & E_1^+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 E_2^+ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & E_3^+ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
 \end{aligned} \quad (\text{B.32})$$

and $E_j^- = (E_j^+)^t$.

sector	Killing vector	Current
left moving	$-\partial_-$	$-2k \frac{\partial x^+}{u^2}$
	$ux^- \partial_u - u^2 \partial_+ + (x^-)^2 \partial_-$	$2k \left(2x^- \frac{\partial u}{u} - \partial x^- + (x^-)^2 \frac{\partial x^+}{u^2} \right)$
	$\frac{u}{2} \partial_u + x^- \partial_-$	$2k \left(\frac{\partial u}{u} + x^- \frac{\partial x^+}{u^2} \right)$
right moving	∂_+	$2k \frac{\bar{\partial} x^-}{u^2}$
	$-ux^+ \partial_u - (x^+)^2 \partial_+ + u^2 \partial_-$	$2k \left(-2x^+ \frac{\bar{\partial} u}{u} + \bar{\partial} x^+ - (x^+)^2 \frac{\bar{\partial} x^-}{u^2} \right)$
	$\frac{u}{2} \partial_u + x^+ \partial_+$	$2k \left(\frac{\bar{\partial} u}{u} + x^+ \frac{\bar{\partial} x^-}{u^2} \right)$

Table B.4: Killing vectors, and holomorphic and anti-holomorphic currents in Poincaré coordinates (parabolic base). The $\{iL_1 + iL_3, iL_1 - iL_3, iL_2, iR_1 + iR_3, iR_1 - iR_3, iR_2\}$ isometry generators and the corresponding $\{J_1 + J_3, J_1 - J_3, J_2, \bar{J}_1 + \bar{J}_3, \bar{J}_1 - \bar{J}_3, \bar{J}_2\}$ currents are represented so to explicitly obtain light-like isometry generators.

A good parametrisation for the $SU(3)$ group can be obtained via the Gauss decomposition: every matrix $g \in SU(3)$ is written as the product:

$$g = b_- d b_+ \quad (\text{B.33})$$

where b_- is a lower triangular matrix with unit diagonal elements, b_+ is a upper triangular matrix with unit diagonal elements and d is a diagonal matrix with unit determinant. The element g is written as:

$$g(z_1, z_2, z_3, \psi_1, \psi_2) = \exp \left[z_1 E_1^- + z_2 E_3^- + \left(z_3 - \frac{z_1 z_2}{2} \right) E_2^- \right] \times \\ \times e^{-F_1 H_1 - F_2 H_2} \exp \left[\bar{w}_1 E_1^+ + \bar{w}_2 E_3^+ + \left(\bar{w}_3 - \frac{\bar{w}_1 \bar{w}_2}{2} \right) E_2^+ \right] e^{i\psi_1 H_1 + i\psi_2 H_2} \quad (\text{B.34})$$

where z_μ are 3 complex parameters, ψ_i are two real and F_1 and F_2 are positive real functions of the z_μ 's:

$$\begin{cases} F_1 = \log f_1 = \log \left(1 + |z_1|^2 + |z_3|^2 \right) \\ F_2 = \log f_2 = \log \left(1 + |z_2|^2 + |z_3 - z_1 z_2|^2 \right) \end{cases} \quad (\text{B.35})$$

By imposing $g(z_\mu, \psi_a)$ to be unitary we find that the w_μ 's are complex functions of the z_μ 's:

$$\begin{cases} w_1 = -\frac{z_1 + \bar{z}_2 z_3}{\sqrt{f_2}} \\ w_2 = \frac{\bar{z}_1 z_3 - z_2 (1 + |z_1|^2)}{\sqrt{f_1}} \\ w_3 = -(z_3 - z_1 z_2) \sqrt{\frac{f_1}{f_2}} \end{cases} \quad (\text{B.36})$$

and the defining element $g(z_\mu, \psi_a)$ can then be written explicitly as:

$$g(z_1, z_2, z_3, \psi_1, \psi_2) = \begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_3 & z_2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{f_1}} & 0 & 0 \\ 0 & \sqrt{f_1/f_2} & 0 \\ 0 & 0 & \sqrt{f_2} \end{pmatrix} \times \\ \times \begin{pmatrix} 1 & \bar{w}_1 & \bar{w}_3 \\ 0 & 1 & \bar{w}_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\psi_1/2} & 0 & 0 \\ 0 & e^{-i(\psi_1-\psi_2)/2} & 0 \\ 0 & 0 & e^{i\psi_2/2} \end{pmatrix} \quad (\text{B.37})$$

Now, to build a metric for the tangent space to $SU(3)$ we can define the 1-form $\Omega(\mathbf{z}, -) = g^{-1}(\mathbf{z}, -) dg(\mathbf{z}, -)$ and write the Killing-Cartan metric tensor as $g_{KC} = \text{tr}(\Omega^\dagger \Omega) = -\text{tr}(\Omega \Omega)$ where we have used explicitly the property of anti-Hermiticity of Ω (that lives in the $\mathfrak{su}(3)$ algebra). The explicit calculation is lengthy but straightforward. The main advantage of this parametrization from our point of view is that it allows for a “natural” embedding of the $SU(3)/U(1)^2$ coset (see e.g. [GK98] or [KT00]): in fact in these coordinates the Kähler potential is

$$K(z_\mu, \bar{z}_\mu) = \log(f_1(z_\mu) f_2(z_\mu)) = \\ = \log \left[\left(1 + |z_1|^2 + |z_3|^2\right) \left(1 + |z_2|^2 + |z_3 - z_1 z_2|^2\right) \right] \quad (\text{B.38})$$

and the coset Kähler metric is hence simply obtained as:

$$g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} K(z_\mu, \bar{z}_\mu) dz^\alpha \otimes d\bar{z}^\beta \quad (\text{B.39})$$

Another commonly used $\mathfrak{su}(3)$ basis is given by the Gell-Mann matrices:

$$\begin{aligned} \gamma_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \gamma_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \gamma_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \gamma_4 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \gamma_5 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} & \gamma_6 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \\ \gamma_7 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & \gamma_8 &= \frac{1}{\sqrt{6}} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix} \end{aligned} \quad (\text{B.40})$$

which presents the advantage of being orthonormal $\kappa(\lambda_i, \lambda_j) = \delta_{ij}$. In this case the Cartan subalgebra is generated by $\mathfrak{k} = \langle \lambda_3, \lambda_8 \rangle$.

B.4 $USp(4)$

The symplectic group $Sp(4, \mathbb{C})$ is the set of 4×4 complex matrices that preserve the symplectic form J :

$$J = \left(\begin{array}{c|c} 0 & \mathbb{I}_{2 \times 2} \\ \hline -\mathbb{I}_{2 \times 2} & 0 \end{array} \right) \quad (\text{B.41})$$

that is

$$Sp(4, \mathbb{C}) = \{ g \mid g^\dagger J g = J \} \quad (\text{B.42})$$

The unitary symplectic group $USp(4, \mathbb{C})$ is the compact group obtained as the intersection of $Sp(4, \mathbb{C})$ with $U(4)$:

$$USp(4, \mathbb{C}) = Sp(4, \mathbb{C}) \cap U(4) \quad (\text{B.43})$$

It follows easily that the Lie algebra $\mathfrak{usp}(4)$ is the set of complex matrices X such that:

$$\mathfrak{usp}(4) = \{ X \mid X^t J + J X^t = 0 \} \quad (\text{B.44})$$

To obtain the the Cartan-Weyl basis $\{ H_a, E^{\alpha_j} \}$ we need to choose the positive roots

$$\alpha_1 = [\sqrt{2}/2, -\sqrt{2}/2] \quad \alpha_2 = [0, \sqrt{2}] \quad \alpha_3 = [\sqrt{2}/2, \sqrt{2}/2] \quad \alpha_4 = [\sqrt{2}, 0] \quad (\text{B.45})$$

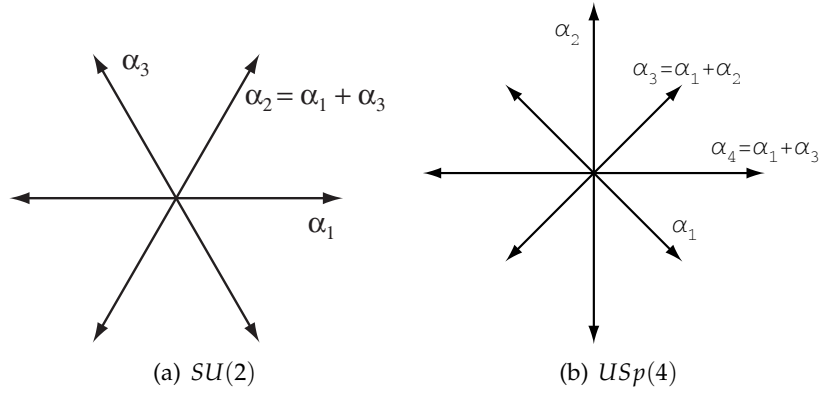


Figure B.1: Root system for $\mathfrak{su}(3)$ and $\mathfrak{sp}(4)$.

and the $N_{\mu, \nu}$ coefficients:

$$N_{1,2} = 1 \quad N_{1,3} = 2 \quad (\text{B.46})$$

The defining realization is given by the following choice:

$$\begin{aligned} H_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & H_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & E_1^+ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ E_2^+ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & E_3^+ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & E_4^+ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (\text{B.47})$$

and $E^{-\alpha_\mu} = (E^{\alpha_\mu})^t$.

Just like in the case of $SU(3)$, the general element in $USp(4)$ is written as:

$$\begin{aligned}
g(\gamma_\mu, \psi_a) &= \exp \left[\gamma_1 E_1^- + \frac{\gamma_2}{\sqrt{2}} E_2^- + \frac{2\gamma_3 - \gamma_1\gamma_2}{2} E_3^- + \frac{\gamma_1^2\gamma_2 - \gamma_1\gamma_3 + \gamma_4}{\sqrt{2}} E_4^- \right] e^{-F_1 H_1 - F_2 H_2} \\
&\exp \left[\bar{\beta}_1 E_1^+ + \frac{\bar{\beta}_2}{\sqrt{2}} E_2^+ + \frac{2\bar{\beta}_3 - \bar{\beta}_1\bar{\beta}_2}{2} E_3^+ + \frac{\bar{\beta}_1^2\bar{\beta}_2 - \bar{\beta}_1\bar{\beta}_3 + \bar{\beta}_4}{\sqrt{2}} E_4^+ \right] e^{i\psi_1 H_1 + i\psi_2 H_2} = \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \gamma_1 & 1 & 0 & 0 \\ \gamma_4 & -\gamma_1\gamma_2 + \gamma_3 & 1 & -\gamma_1 \\ \gamma_3 & \gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & 1/f_1 & 0 \\ 0 & 0 & 0 & 1/f_2 \end{pmatrix} \begin{pmatrix} 1 & \bar{\beta}_1 & \bar{\beta}_4 & \bar{\beta}_3 \\ 0 & 1 & -\bar{\beta}_1\bar{\beta}_2 + \bar{\beta}_3 & \bar{\beta}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\bar{\beta}_1 & 1 \end{pmatrix} \times \\
&\quad \times \begin{pmatrix} e^{i\psi_1} & 0 & 0 & 0 \\ 0 & e^{i\psi_2} & 0 & 0 \\ 0 & 0 & e^{-i\psi_1} & 0 \\ 0 & 0 & 0 & e^{-i\psi_2} \end{pmatrix} \quad (\text{B.48})
\end{aligned}$$

A orthonormal basis for the $\mathfrak{usp}(4)$, similar to the Gell-Mann matrices system is given by the following set of matrices:

$$\begin{aligned}
T_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & T_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} & T_3 &= \frac{1}{2} \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \\
T_4 &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} & T_5 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} & T_6 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
T_7 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} & T_8 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} & T_9 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
T_{10} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

(B.49)

APPENDIX C

Symmetric deformations of $SL(2, \mathbb{R})$

The group manifold of $SL(2, \mathbb{R})$ is anti de Sitter in three dimensions. Metric and anti-symmetric tensor read (in Euler coordinates, see App. B):

$$ds^2 = L^2 \left[d\rho^2 + \sinh^2 \rho d\phi^2 - \cosh^2 \rho d\tau^2 \right], \quad (\text{C.1a})$$

$$H_{[3]} = L^2 \sinh 2\rho d\rho \wedge d\phi \wedge d\tau, \quad (\text{C.1b})$$

with L related to the level of $SL(2, \mathbb{R})_k$ as usual: $L = \sqrt{k+2}$. In the case at hand, three different lines of symmetric deformations arise due to the presence of time-like (J^3, \bar{J}^3), space-like ($J^1, \bar{J}^1, J^2, \bar{J}^2$), or null generators [FR03, F94, IKP03]. The residual isometry is $U(1) \times U(1)$ that can be time-like (L_3, R_3), space-like (L_2, R_2) or null ($L_1 + L_3, R_1 + R_3$) depending on the deformation under consideration.

The *elliptic deformation* is driven by the $J^3 \bar{J}^3$ bilinear. At first order in α' the background fields are given by¹:

$$ds^2 = k \left[d\rho^2 + \frac{\sinh^2 \rho d\phi^2 - \kappa_3^2 \cosh^2 \rho d\tau^2}{\Theta_{\kappa_3}(\rho)} \right], \quad (\text{C.2a})$$

$$H_{[3]} = k \frac{\kappa_3^2 \sinh 2\rho}{\Theta_{\kappa_3}(\rho)^2} d\rho \wedge d\phi \wedge d\tau, \quad (\text{C.2b})$$

$$e^\Phi = \frac{\Theta_{\kappa_3}(\rho)}{\kappa_3}. \quad (\text{C.2c})$$

where $\Theta_{\kappa_3}(\rho) = \cosh^2 \rho - \kappa_3 \sinh^2 \rho$. At extreme deformation ($\kappa_3^2 \rightarrow 0$), a time-like direction decouples and we are left with the axial² $SL(2, \mathbb{R})_k/U(1)_{\text{time}}$. The target space of the latter is the *cigar* geometry (also called Euclidean two-dimensional black hole):

$$e^\Phi \sim \cosh^2 \rho, \quad (\text{C.3})$$

$$ds^2 = k \left[d\rho^2 + \tanh^2 \rho d\phi^2 \right], \quad (\text{C.4})$$

($0 \leq \rho < \infty$ and $0 \leq \phi \leq 2\pi$).

¹The extra index "3" in the deformation parameter κ reminds that the deformation refers here to $J^3 \bar{J}^3$.

²The deformation parameter has two T-dual branches. The extreme values of deformation correspond to the axial or vector gaugings. The vector gauging leads to the *trumpet*. For the $SU(2)_k/U(1)$, both gaugings correspond to the *bell*.

Similarly, with $J^2 \bar{J}^2$ one generates the *hyperbolic deformation*. This allows to reach the Lorentzian two-dimensional black hole times a free space-like line. Using the coordinates defined in Eq. (B.28), we find:

$$ds^2 = k \left[-dt^2 + \frac{\sin^2 t d\varphi^2 + \kappa_2^2 \cos^2 t d\psi^2}{\Delta_{\kappa_2}(t)} \right], \quad (\text{C.5a})$$

$$H_{[3]} = k \frac{\kappa_2^2 \sin 2t}{\Delta_{\kappa_2}(t)^2} dt \wedge d\psi \wedge d\phi, \quad (\text{C.5b})$$

$$e^\Phi = \frac{\Delta_{\kappa_2}(t)}{\kappa_2}, \quad (\text{C.5c})$$

where $\Delta_{\kappa_2}(t) = \cos^2 t + \kappa_2^2 \sin^2 t$. This coordinate patch does not cover the full AdS_3 . We will expand on this line in Sec. 4.4.

Finally, the bilinear $(J^1 + J^3)(\bar{J}^1 + \bar{J}^3)$ generates the *parabolic deformation*. Using Poincaré coordinates in Eq. (B.26)³ we obtain:

$$ds^2 = k \left[\frac{du^2}{u^2} + \frac{dX^2 - dT^2}{u^2 + 1/\nu} \right], \quad (\text{C.6a})$$

$$H_{[3]} = k \frac{2u}{(u^2 + 1/\nu)^2} du \wedge dT \wedge dX, \quad (\text{C.6b})$$

$$e^\Phi = \frac{u^2 + 1/\nu}{u^2}. \quad (\text{C.6c})$$

The deformation parameter is $1/\nu$. At infinite value of the parameter ν , we recover pure AdS_3 ; for $\nu \rightarrow 0$, a whole light-cone decouples and we are left with a single direction and a dilaton field, linear in this direction.

The physical interpretation of the parabolic deformation is far reaching, when AdS_3 is considered in the framework of the NS5/F1 near-horizon background, $\text{AdS}_3 \times S^3 \times T^4$. In this physical set-up, the parameter ν is the density of F1's (number of fundamental strings over the volume of the four-torus T^4) [IKP03, KKPR03]⁴. At infinite density, the background is indeed $\text{AdS}_3 \times S^3 \times T^4$. At null density, the geometry becomes $\mathbb{R}^{1,2} \times S^3 \times T^4$ plus a linear dilaton and a three-form on the S^3 .

³Note that $x^\pm = X \pm T$.

⁴Our present convention for the normalization of the dilaton results from Eq. (3.15b). It differs by a factor -2 with respect to the one used in those papers.

APPENDIX D

Spectrum of the $SL(2, \mathbb{R})$ super-WZW model

In this appendix we give a reminder of the superconformal WZW model on $SL(2, \mathbb{R})_k$ (for a recent discussion see [GKPS03]). The affine extension of the $\mathfrak{sl}(2, \mathbb{R})$ algebra at level k is obtained by considering two sets of holomorphic and anti-holomorphic currents of dimension one, defined as

$$J^M(z) = k \langle T^M, \text{Ad}_g g^{-1} \partial g \rangle, \quad \bar{J}^M(\bar{z}) = k \langle T^M, g^{-1} \bar{\partial} g \rangle, \quad (\text{D.1})$$

where $\langle \cdot, \cdot \rangle$ is the scalar product (Killing form) in $\mathfrak{sl}(2, \mathbb{R})$, $\{T^M\}$ is a set of generators of the algebra that for concreteness we can choose as follows:

$$T^1 = \sigma^1, \quad T^2 = \sigma^3, \quad T^3 = \sigma^2. \quad (\text{D.2})$$

Each set satisfies the OPE

$$J^M(z) J^N(w) \sim \frac{k \delta^{MN}}{2(z-w)^2} + \frac{f^{MN}_P J^P(w)}{z-w}, \quad (\text{D.3})$$

where f^{MN}_P are the structure constants of the $\mathfrak{sl}(2, \mathbb{R})$ algebra. The chiral algebra contains the Virasoro operator (stress tensor) obtained by the usual Sugawara construction:

$$T(z) = \sum_M \frac{:J^M J^M:}{k-2}. \quad (\text{D.4})$$

A heterotic model is built if we consider a left-moving $\mathcal{N} = 1$ extension, obtained by adding 3 free fermions which transform in the adjoint representation. More explicitly:

$$T(z) = \sum_M \frac{:J^M J^M:}{k-2} + : \psi_M \partial \psi_M :, \quad (\text{D.5})$$

$$G(z) = \frac{2}{k} \left(\sum_M J^M \psi_M - \frac{1}{3k} \sum_{MNP} f^{MNP} : \psi_M \psi_N \psi_P : \right). \quad (\text{D.6})$$

On the right side, instead of superpartners, we add a right-moving current with total central charge $c = 16$.

Let us focus on the left-moving part. The supercurrents are given by $\psi_M + \theta\sqrt{2/k}\mathcal{J}_M$ where:

$$\mathcal{J}_M = J^M - \frac{i}{2} \sum_{NP} \epsilon^{MNP} \psi_N \psi_P; \quad (\text{D.7})$$

it should be noted that the bosonic J^M currents generate an affine $\mathfrak{sl}(2, \mathbb{R})$ algebra at level $k+2$, while the level for the total \mathcal{J}_M currents is k .

Let us now single out the operator that we used for both the deformation (Eqs. (4.87)) and the identifications (Sec. 4.4):

$$\mathcal{J}_2 = J^2 + i\psi_1\psi_3. \quad (\text{D.8})$$

Let us now bosonize these currents as follows:

$$\mathcal{J}_2 = -\sqrt{\frac{k}{2}} \partial\theta_2, \quad (\text{D.9})$$

$$J^2 = -\sqrt{\frac{k+2}{2}} \partial\theta_2, \quad (\text{D.10})$$

$$\psi_1\psi_3 = \partial H, \quad (\text{D.11})$$

and introduce a fourth free boson X so to separate the θ_2 components both in θ_2 and H :

$$iH = \sqrt{\frac{2}{k}} \theta_2 + i\sqrt{\frac{k+2}{k}} X, \quad (\text{D.12})$$

$$\theta_2 = \sqrt{\frac{2}{k}} \left(\sqrt{\frac{k+2}{2}} \theta_2 + iX \right). \quad (\text{D.13})$$

A primary field $\Phi_{j\mu\bar{\mu}}$ of the bosonic $SL(2, \mathbb{R})_{k+2}$ with eigenvalue μ with respect to J^2 and $\bar{\mu}$ with respect to \bar{J}^2 obeys by definition

$$J^2(z) \Phi_{j\mu\bar{\mu}}(w, \bar{w}) \sim \frac{\mu \Phi_{j\mu\bar{\mu}}(w, \bar{w})}{z-w}, \quad (\text{D.14a})$$

$$\bar{J}^2(\bar{z}) \Phi_{j\mu\bar{\mu}}(w, \bar{w}) \sim \frac{\bar{\mu} \Phi_{j\mu\bar{\mu}}(w, \bar{w})}{\bar{z}-\bar{w}}. \quad (\text{D.14b})$$

Since $\Phi_{j\mu\bar{\mu}}$ is purely bosonic, the same relation holds for the supercurrent:

$$\mathcal{J}_2(z) \Phi_{j\mu\bar{\mu}}(w, \bar{w}) \sim \frac{\mu \Phi_{j\mu\bar{\mu}}(w, \bar{w})}{z-w}. \quad (\text{D.15})$$

Consider now the holomorphic part of $\Phi_{j\mu\bar{\mu}}(z, \bar{z})$. If $\Phi_{j\mu}$ is viewed as a primary in the SWZW model, we can use the parafermion decomposition as follows:

$$\Phi_{j\mu}(z) = U_{j\mu}(z) e^{i\mu\sqrt{2/k}\theta_2}, \quad (\text{D.16})$$

where $U_{j\mu}(z)$ is a primary of the superconformal $SL(2, \mathbb{R})_k/U(1)$. On the other hand, we can just consider the bosonic WZW and write:

$$\Phi_{j\mu}(z) = V_{j\mu}(z) e^{i\mu\sqrt{2/(k+2)}\theta_2} = V_{j\mu}(z) e^{i\frac{2m}{k+2}\sqrt{\frac{k+2}{k}}X + i\mu\sqrt{2/k}\theta_2}, \quad (\text{D.17})$$

where now $V_{j\mu}(z)$ is a primary of the bosonic $SL(2, \mathbb{R})_{k+2}/U(1)$. The scaling dimension for this latter operator (*i.e.* its eigenvalue with respect to L_0) is then given by:

$$\Delta(V_{j\mu}) = -\frac{j(j+1)}{k} - \frac{\mu^2}{k+2}. \quad (\text{D.18})$$

An operator in the full supersymmetric $SL(2, \mathbb{R})_k$ theory is then obtained by adding the $\psi^1\psi^3$ fermionic superpartner contribution:

$$\Phi_{j\mu\nu}(z) = \Phi_{j\mu}(z) e^{i\nu H} = V_{j\mu}(z) e^{i\left(\frac{2\mu}{k+2} + \nu\right) \sqrt{\frac{k+2}{k}} X} e^{i\sqrt{2/k}(\mu+\nu)\vartheta_2} \quad (\text{D.19})$$

that is an eigenvector of \mathcal{J}_2 with eigenvalue $\mu + \nu$ where $\mu \in \mathbb{R}$ and ν can be decomposed as $\nu = n + a/2$ with $n \in \mathbb{N}$ and $a \in \mathbb{Z}_2$ depending on whether we consider the NS or R sector. The resulting spectrum can be read directly as:

$$\begin{aligned} \Delta(\Phi_{j\mu\nu}(z)) &= -\frac{j(j+1)}{k} - \frac{\mu^2}{k+2} - \frac{k+2}{2k} \left(\frac{2\mu}{k+2} + n + \frac{a}{2} \right)^2 + \frac{1}{k} \left(\mu + n + \frac{a}{2} \right)^2 = \\ &= -\frac{j(j+1)}{k} - \frac{1}{2} \left(n + \frac{a}{2} \right)^2. \end{aligned} \quad (\text{D.20})$$

Of course the last expression was to be expected since it is the sum of the $\mathfrak{sl}(2, \mathbb{R})_{k+2}$ Casimir and the contribution of a light-cone fermion. Nevertheless the preceding construction is useful since it allowed us to isolate the \mathcal{J}_2 contribution to the spectrum $(\mu + \nu)^2/k$.

The right-moving part of the spectrum is somewhat simpler since there are no superpartners. This means that we can repeat our construction above and the eigenvalue of the \bar{L}_0 operator is simply obtained by adding to the dimension in Eq. (D.18) the contribution of the \bar{J}^2 operator and of some $U(1)$ coming from the gauge sector:

$$\bar{\Delta}(\bar{\Phi}_{j\bar{\mu}\bar{n}}(\bar{z})) = -\frac{j(j+1)}{k} - \frac{\bar{\mu}^2}{k+2} + \left\{ \frac{\bar{\mu}^2}{k+2} + \frac{1}{k_g} \left(\bar{n} + \frac{\bar{a}}{2} \right)^2 \right\}, \quad (\text{D.21})$$

where again $\bar{n} \in \mathbb{N}$ and $\bar{a} \in \mathbb{Z}_2$ depending on the sector.

Bibliography

- [ABEN89] I. Antoniadis, C. Bachas, J. R. Ellis and D. V. Nanopoulos, *an expanding universe in string theory*, Nucl. Phys. **B328**, 117–139 (1989).
- [AGFM81] L. Alvarez-Gaume, D. Z. Freedman and S. Mukhi, *the background field method and the ultraviolet structure of the supersymmetric nonlinear sigma model*, Ann. Phys. **134**, 85 (1981).
- [AGS00] R. Argurio, A. Giveon and A. Shomer, *Superstring theory on $AdS(3) \times G/H$ and boundary $N = 3$ superconformal symmetry*, JHEP **04**, 010 (2000), hep-th/0002104.
- [AW03] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of $G(2)$ holonomy*, Adv. Theor. Math. Phys. **6**, 1–106 (2003), hep-th/0107177.
- [BB02] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, Nucl. Phys. **B633**, 3–82 (2002), hep-th/0111246.
- [BBH⁺00] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, *Superstring theory on $AdS(2) \times S(2)$ as a coset supermanifold*, Nucl. Phys. **B567**, 61–86 (2000), hep-th/9907200.
- [BCZ85] E. Braaten, T. L. Curtright and C. K. Zachos, *TORSION AND GEOMETROSTASIS IN NONLINEAR SIGMA MODELS*, Nucl. Phys. **B260**, 630 (1985).
- [BFHP96] J. Balog, P. Forgacs, Z. Horvath and L. Palla, *Perturbative quantum (in)equivalence of dual sigma models in two-dimensions*, Nucl. Phys. Proc. Suppl. **49**, 16–26 (1996), hep-th/9601091.
- [BFM⁺98] J. Balog, P. Forgacs, N. Mohammadi, L. Palla and J. Schnittger, *On quantum T-duality in sigma models*, Nucl. Phys. **B535**, 461–482 (1998), hep-th/9806068.
- [BH58] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces. I*, Amer. J. Math. **80**, 458 (1958).
- [BHTZ93] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, Phys. Rev. **D48**, 1506–1525 (1993), gr-qc/9302012.
- [BJKZ96] P. Berglund, C. V. Johnson, S. Kachru and P. Zaugg, *Heterotic Coset Models and (0,2) String Vacua*, Nucl. Phys. **B460**, 252–298 (1996), hep-th/9509170.
- [BK94] C. Bachas and E. Kiritsis, *Exact string theory instantons by dimensional reduction*, Phys. Lett. **B325**, 103–114 (1994), hep-th/9311185.

- [BPS98] H. J. Boonstra, B. Peeters and K. Skenderis, *Brane intersections, anti-de Sitter spacetimes and dual superconformal theories*, Nucl. Phys. **B533**, 127–162 (1998), hep-th/9803231.
- [BTZ92] M. Banados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, Phys. Rev. Lett. **69**, 1849–1851 (1992), hep-th/9204099.
- [Bus87] T. H. Buscher, *a symmetry of the string background field equations*, Phys. Lett. **B194**, 59 (1987).
- [Byr97] M. Byrd, *The Geometry of SU(3)*, (1997), physics/9708015.
- [C⁺98] P. Claus et al., *Black holes and superconformal mechanics*, Phys. Rev. Lett. **81**, 4553–4556 (1998), hep-th/9804177.
- [CCKM01] M. Cadoni, P. Carta, D. Klemm and S. Mignemi, *AdS(2) gravity as conformally invariant mechanical system*, Phys. Rev. **D63**, 125021 (2001), hep-th/0009185.
- [CH94] O. Coussaert and M. Henneaux, *Self-dual solutions of 2+1 Einstein gravity with a negative cosmological constant*, (1994), hep-th/9407181.
- [CHS91] J. Callan, Curtis G., J. A. Harvey and A. Strominger, *World sheet approach to heterotic instantons and solitons*, Nucl. Phys. **B359**, 611–634 (1991).
- [CLW94] E. J. Copeland, A. Lahiri and D. Wands, *Low-energy effective string cosmology*, Phys. Rev. **D50**, 4868–4880 (1994), hep-th/9406216.
- [CMPF85] J. Callan, Curtis G., E. J. Martinec, M. J. Perry and D. Friedan, *strings in background fields*, Nucl. Phys. **B262**, 593 (1985).
- [CS89] S. Chaudhuri and J. A. Schwartz, *a criterion for integrably marginal operators*, Phys. Lett. **B219**, 291 (1989).
- [DFG⁺02] O. DeWolfe, D. Z. Freedman, S. S. Gubser, G. T. Horowitz and I. Mitra, *Stability of AdS(p) x M(q) compactifications without supersymmetry*, Phys. Rev. **D65**, 064033 (2002), hep-th/0105047.
- [DFS03] N. Drukker, B. Fiol and J. Simon, *Goedel's universe in a supertube shroud*, Phys. Rev. Lett. **91**, 231601 (2003), hep-th/0306057.
- [DLP98] M. J. Duff, H. Lu and C. N. Pope, *AdS(5) x S(5) untwisted*, Nucl. Phys. **B532**, 181–209 (1998), hep-th/9803061.
- [DLP99] M. J. Duff, H. Lu and C. N. Pope, *AdS(3) x S**3 (un)twisted and squashed, and an O(2,2,Z) multiplet of dyonic strings*, Nucl. Phys. **B544**, 145–180 (1999), hep-th/9807173.
- [Dob87] V. K. Dobrev, *characters of the unitarizable highest weight modules over the n=2 superconformal algebras*, Phys. Lett. **B186**, 43 (1987).
- [DOPS05] S. Detournay, D. Orlando, P. M. Petropoulos and P. Spindel, *Three-dimensional black holes from deformed anti de Sitter*, JHEP **07**, 072 (2005), hep-th/0504231.
- [DVV92] R. Dijkgraaf, H. Verlinde and E. Verlinde, *String propagation in a black hole geometry*, Nucl. Phys. **B371**, 269–314 (1992).

- [ES01] T. Eguchi and Y. Sugawara, *CFT description of string theory compactified on non-compact manifolds with $G(2)$ holonomy*, Phys. Lett. **B519**, 149–158 (2001), hep-th/0108091.
- [ESY03] T. Eguchi, Y. Sugawara and S. Yamaguchi, *Supercoset CFT's for string theories on non-compact special holonomy manifolds*, Nucl. Phys. **B657**, 3–52 (2003), hep-th/0301164.
- [F94] S. Förste, *A Truly marginal deformation of $SL(2, R)$ in a null direction*, Phys. Lett. **B338**, 36–39 (1994), hep-th/9407198.
- [FKS95] S. Ferrara, R. Kallosh and A. Strominger, *$N=2$ extremal black holes*, Phys. Rev. **D52**, 5412–5416 (1995), hep-th/9508072.
- [FR03] S. Förste and D. Roggenkamp, *Current current deformations of conformal field theories, and WZW models*, JHEP **05**, 071 (2003), hep-th/0304234.
- [Fri80] D. Friedan, *nonlinear models in two epsilon dimensions*, Phys. Rev. Lett. **45**, 1057 (1980).
- [Fri85] D. H. Friedan, *NONLINEAR MODELS IN TWO + epsilon DIMENSIONS*, Ann. Phys. **163**, 318 (1985).
- [Gaw91] K. Gawedzki, *Noncompact WZW conformal field theories*, (1991), hep-th/9110076.
- [Gep87] D. Gepner, *NEW CONFORMAL FIELD THEORIES ASSOCIATED WITH LIE ALGEBRAS AND THEIR PARTITION FUNCTIONS*, Nucl. Phys. **B290**, 10 (1987).
- [Gep88] D. Gepner, *space-time supersymmetry in compactified string theory and superconformal models*, Nucl. Phys. **B296**, 757 (1988).
- [Ger95] D. Gershon, *Exact solutions of four-dimensional black holes in string theory*, Phys. Rev. **D51**, 4387–4393 (1995), hep-th/9202005.
- [GHMS03] M. Gutperle, M. Headrick, S. Minwalla and V. Schomerus, *Space-time energy decreases under world-sheet RG flow*, JHEP **01**, 073 (2003), hep-th/0211063.
- [GHR84] J. Gates, S. J., C. M. Hull and M. Rocek, *twisted multiplets and new supersymmetric nonlinear sigma models*, Nucl. Phys. **B248**, 157 (1984).
- [GIS02] G. W. Gibbons, D. Ida and T. Shiromizu, *Uniqueness and non-uniqueness of static black holes in higher dimensions*, Phys. Rev. Lett. **89**, 041101 (2002), hep-th/0206049.
- [GK94] A. Giveon and E. Kiritsis, *Axial vector duality as a gauge symmetry and topology change in string theory*, Nucl. Phys. **B411**, 487–508 (1994), hep-th/9303016.
- [GK98] S. Gnutzmann and M. Kuś, *Coherent states and the classical limit on irreducible $SU(3)$ representations*, J.Phys. A **31**, 9871 (1998).
- [GK99] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, JHEP **10**, 034 (1999), hep-th/9909110.
- [GKP99] A. Giveon, D. Kutasov and O. Pelc, *Holography for non-critical superstrings*, JHEP **10**, 035 (1999), hep-th/9907178.

- [GKPS03] A. Giveon, A. Konechny, A. Pakman and A. Sever, *Type 0 strings in a 2-d black hole*, JHEP **10**, 025 (2003), hep-th/0309056.
- [GL93] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, Phys. Rev. Lett. **70**, 2837–2840 (1993), hep-th/9301052.
- [GP94] D. S. Goldwirth and M. J. Perry, *String dominated cosmology*, Phys. Rev. **D49**, 5019–5025 (1994), hep-th/9308023.
- [GQ87] D. Gepner and Z.-a. Qiu, *modular invariant partition functions for parafermionic field theories*, Nucl. Phys. **B285**, 423 (1987).
- [GT99] G. W. Gibbons and P. K. Townsend, *Black holes and Calogero models*, Phys. Lett. **B454**, 187–192 (1999), hep-th/9812034.
- [Heu98] M. Heusler, *Stationary black holes: Uniqueness and beyond*, Living Rev. Rel. **1**, 6 (1998).
- [HH92] J. H. Horne and G. T. Horowitz, *Exact black string solutions in three-dimensions*, Nucl. Phys. **B368**, 444–462 (1992), hep-th/9108001.
- [HHS92] J. H. Horne, G. T. Horowitz and A. R. Steif, *An Equivalence between momentum and charge in string theory*, Phys. Rev. Lett. **68**, 568–571 (1992), hep-th/9110065.
- [HKV02] S. Hemming and E. Keski-Vakkuri, *The spectrum of strings on BTZ black holes and spectral flow in the $SL(2,R)$ WZW model*, Nucl. Phys. **B626**, 363–376 (2002), hep-th/0110252.
- [Hor92] P. Horava, *Some exact solutions of string theory in four-dimensions and five-dimensions*, Phys. Lett. **B278**, 101–110 (1992), hep-th/9110067.
- [HS93] S. F. Hassan and A. Sen, *Marginal deformations of WZNW and coset models from $O(d,d)$ transformation*, Nucl. Phys. **B405**, 143–165 (1993), hep-th/9210121.
- [HT87] C. M. Hull and P. K. Townsend, *the two loop beta function for sigma models with torsion*, Phys. Lett. **B191**, 115 (1987).
- [HT95] G. T. Horowitz and A. A. Tseytlin, *A New class of exact solutions in string theory*, Phys. Rev. **D51**, 2896–2917 (1995), hep-th/9409021.
- [HW93] G. T. Horowitz and D. L. Welch, *Exact three-dimensional black holes in string theory*, Phys. Rev. Lett. **71**, 328–331 (1993), hep-th/9302126.
- [IKOP05a] D. Israël, C. Kounnas, D. Orlando and P. M. Petropoulos, *Electric / magnetic deformations of S^{*3} and $AdS(3)$, and geometric cosets*, Fortsch. Phys. **53**, 73–104 (2005), hep-th/0405213.
- [IKOP05b] D. Israel, C. Kounnas, D. Orlando and P. M. Petropoulos, *Heterotic strings on homogeneous spaces*, Fortsch. Phys. **53**, 1030–1071 (2005), hep-th/0412220.
- [IKP03] D. Israël, C. Kounnas and M. P. Petropoulos, *Superstrings on NS5 backgrounds, deformed $AdS(3)$ and holography*, JHEP **10**, 028 (2003), hep-th/0306053.
- [IKPT04] D. Israël, C. Kounnas, A. Pakman and J. Troost, *The partition function of the supersymmetric two-dimensional black hole and little string theory*, JHEP **06**, 033 (2004), hep-th/0403237.

- [Isr67] W. Israel, *Event horizons in static vacuum space-times*, Phys. Rev. **164**, 1776–1779 (1967).
- [Isr04] D. Israël, *Quantization of heterotic strings in a Gödel/anti de Sitter space-time and chronology protection*, JHEP **01**, 042 (2004), hep-th/0310158.
- [Joh95] C. V. Johnson, *Heterotic Coset Models*, Mod. Phys. Lett. **A10**, 549–560 (1995), hep-th/9409062.
- [Kir88] E. Kiritsis, *CHARACTER FORMULAE AND THE STRUCTURE OF THE REPRESENTATIONS OF THE $N=1$, $N=2$ SUPERCONFORMAL ALGEBRAS*, Int. J. Mod. Phys. **A3**, 1871 (1988).
- [KK95] E. Kiritsis and C. Kounnas, *Infrared behavior of closed superstrings in strong magnetic and gravitational fields*, Nucl. Phys. **B456**, 699–731 (1995), hep-th/9508078.
- [KKPR03] E. Kiritsis, C. Kounnas, P. M. Petropoulos and J. Rizos, *Five-brane configurations, conformal field theories and the strong-coupling problem*, (2003), hep-th/0312300.
- [KLL99] D. Kutasov, F. Larsen and R. G. Leigh, *String theory in magnetic monopole backgrounds*, Nucl. Phys. **B550**, 183–213 (1999), hep-th/9812027.
- [KN69] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Wiley, 1969.
- [KPR91] C. Kounnas, M. Porrati and B. Rostand, *ON $N=4$ extended superLiouville theory*, Phys. Lett. **B258**, 61–69 (1991).
- [KS89a] Y. Kazama and H. Suzuki, *characterization of $N=2$ superconformal models generated by coset space method*, Phys. Lett. **B216**, 112 (1989).
- [KS89b] Y. Kazama and H. Suzuki, *new $N=2$ superconformal field theories and superstring compactification*, Nucl. Phys. **B321**, 232 (1989).
- [KT94] C. Klimcik and A. A. Tseytlin, *Exact four-dimensional string solutions and Toda like sigma models from 'null gauged' WZNW theories*, Nucl. Phys. **B424**, 71–96 (1994), hep-th/9402120.
- [KT00] K.-I. Kondo and Y. Taira, *Non-Abelian Stokes theorem and quark confinement in $SU(N)$ Yang-Mills gauge theory*, Prog. Theor. Phys. **104**, 1189–1265 (2000), hep-th/9911242.
- [KZ84] V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl. Phys. **B247**, 83–103 (1984).
- [LP96] H. Lu and C. N. Pope, *p -brane Solitons in Maximal Supergravities*, Nucl. Phys. **B465**, 127–156 (1996), hep-th/9512012.
- [LPS96] H. Lu, C. N. Pope and K. S. Stelle, *Weyl Group Invariance and p -brane Multiplets*, Nucl. Phys. **B476**, 89–117 (1996), hep-th/9602140.
- [LS92] H. Leutwyler and M. A. Shifman, *Perturbation theory in the Wess-Zumino-Novikov-Witten model*, Int. J. Mod. Phys. **A7**, 795–842 (1992).
- [Mal98] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2**, 231–252 (1998), hep-th/9711200.

- [Mat87] Y. Matsuo, *character formula of $c < 1$ unitary representation of $n=2$ superconformal algebra*, Prog. Theor. Phys. **77**, 793 (1987).
- [MHS88] F. Mueller-Hoissen and R. Stuckl, *coset spaces and ten-dimensional unified theories*, Class. Quant. Grav. **5**, 27 (1988).
- [MN01] J. M. Maldacena and C. Nunez, *Supergravity description of field theories on curved manifolds and a no go theorem*, Int. J. Mod. Phys. **A16**, 822–855 (2001), hep-th/0007018.
- [MO01] J. M. Maldacena and H. Ooguri, *Strings in $AdS(3)$ and $SL(2,R)$ WZW model. I*, J. Math. Phys. **42**, 2929–2960 (2001), hep-th/0001053.
- [Nak] M. Nakahara, *Geometry, topology and physics*, Bristol, UK: Hilger (1990) 505 p. (Graduate student series in physics).
- [NS98] M. Natsuume and Y. Satoh, *String theory on three dimensional black holes*, Int. J. Mod. Phys. **A13**, 1229–1262 (1998), hep-th/9611041.
- [OPS06] D. Orlando, P. M. Petropoulos and K. Sfetsos, *Renormalization-group flows and charge transmutation in string theory*, Fortsch. Phys. **54**, 453–461 (2006), hep-th/0512086.
- [Orl05a] D. Orlando, *$AdS(2) \times S^{*2}$ as an exact heterotic string background*, (2005), hep-th/0502213, Talk given at NATO Advanced Study Institute and EC Summer School on String Theory: From Gauge Interactions to Cosmology, Cargese, France, 7-19 Jun 2004.
- [Orl05b] D. Orlando, *Coset models and D-branes in group manifolds*, (2005), hep-th/0511210.
- [Os90] H. Osborn, *general bosonic sigma models and string effective actions*, Ann. Phys. **200**, 1 (1990).
- [Per87] A. M. Perelomov, *CHIRAL MODELS: GEOMETRICAL ASPECTS*, Phys. Rept. **146**, 135–213 (1987).
- [Pet90] P. M. S. Petropoulos, *comments on $su(1,1)$ string theory*, Phys. Lett. **B236**, 151 (1990).
- [Pol88] A. M. Polyakov, *Two-dimensional quantum gravity: Superconductivity at high T/c* , (1988), In *Les Houches 1988, Proceedings, Fields, strings and critical phenomena* 305-368.
- [PW83] A. M. Polyakov and P. B. Wiegmann, *Theory of nonabelian Goldstone bosons in two dimensions*, Phys. Lett. **B131**, 121–126 (1983).
- [PZT00] L. A. Pando Zayas and A. A. Tseytlin, *Conformal sigma models for a class of $T(p,q)$ spaces*, Class. Quant. Grav. **17**, 5125–5131 (2000), hep-th/0007086.
- [QS03] T. Quella and V. Schomerus, *Asymmetric cosets*, JHEP **02**, 030 (2003), hep-th/0212119.
- [RS98] M. Rooman and P. Spindel, *Goedel metric as a squashed anti-de Sitter geometry*, Class. Quant. Grav. **15**, 3241–3249 (1998), gr-qc/9804027.
- [RT83] M. J. Reboucas and J. Tiomno, *on the homogeneity of riemannian space-times of godel type*, Phys. Rev. **D28**, 1251–1264 (1983).

- [RY87] F. Ravanini and S.-K. Yang, *modular invariance in $n=2$ superconformal field theories*, Phys. Lett. **B195**, 202 (1987).
- [Sfe93] K. Sfetsos, *Conformally exact results for $SL(2,R) \times SO(1,1)(d-2) / SO(1,1)$ coset models*, Nucl. Phys. **B389**, 424–444 (1993), hep-th/9206048.
- [Sfe98] K. Sfetsos, *Poisson-Lie T-duality beyond the classical level and the renormalization group*, Phys. Lett. **B432**, 365–375 (1998), hep-th/9803019.
- [Sfe99] K. Sfetsos, *Duality-invariant class of two-dimensional field theories*, Nucl. Phys. **B561**, 316–340 (1999), hep-th/9904188.
- [Som68] C. M. Sommerfield, *Currents as dynamical variables*, Phys. Rev. **176**, 2019–2025 (1968).
- [SS06] A. Saltman and E. Silverstein, *A new handle on de Sitter compactifications*, JHEP **01**, 139 (2006), hep-th/0411271.
- [Ste98] K. S. Stelle, *BPS branes in supergravity*, (1998), hep-th/9803116.
- [Sug68] H. Sugawara, *A Field theory of currents*, Phys. Rev. **170**, 1659–1662 (1968).
- [Tes99] J. Teschner, *On structure constants and fusion rules in the $SL(2,C)/SU(2)$ WZNW model*, Nucl. Phys. **B546**, 390–422 (1999), hep-th/9712256.
- [Tse92a] A. A. Tseytlin, *Cosmological solutions with dilaton and maximally symmetric space in string theory*, Int. J. Mod. Phys. **D1**, 223–245 (1992), hep-th/9203033.
- [Tse92b] A. A. Tseytlin, *Dilaton, winding modes and cosmological solutions*, Class. Quant. Grav. **9**, 979–1000 (1992), hep-th/9112004.
- [Tse93] A. A. Tseytlin, *Effective action of gauged WZW model and exact string solutions*, Nucl. Phys. **B399**, 601–622 (1993), hep-th/9301015.
- [Tse94] A. A. Tseytlin, *Conformal sigma models corresponding to gauged Wess-Zumino-Witten theories*, Nucl. Phys. **B411**, 509–558 (1994), hep-th/9302083.
- [Tse95] A. A. Tseytlin, *Exact solutions of closed string theory*, Class. Quant. Grav. **12**, 2365–2410 (1995), hep-th/9505052.
- [Ver04] H. Verlinde, *Superstrings on $AdS(2)$ and superconformal matrix quantum mechanics*, (2004), hep-th/0403024.
- [VK91] N. J. Vilenkin and A. Klimyk, *Representation of Lie groups and special functions*, Kluwer Academic Publishers, Dordrecht, 1991.
- [Wit84] E. Witten, *Nonabelian bosonization in two dimensions*, Commun. Math. Phys. **92**, 455–472 (1984).
- [Wit91] E. Witten, *On string theory and black holes*, Phys. Rev. **D44**, 314–324 (1991).
- [You99] D. Youm, *Black holes and solitons in string theory*, Phys. Rept. **316**, 1–232 (1999), hep-th/9710046.
- [Zam86] A. B. Zamolodchikov, *'irreversibility' of the flux of the renormalization group in a 2-d field theory*, JETP Lett. **43**, 730–732 (1986).

Electric/magnetic deformations of S^3 and AdS_3 , and geometric cosets

Published in Fortsch. Phys. 53 (2005) 73-104 [hep-th/0405213].

Preprint typeset in JHEP style - HYPER VERSION

LPTENS-04/26
CPTH-RR014.0404
hep-th/yymmnnn

Electric/magnetic deformations of S^3 and AdS_3 , and geometric cosets*

Dan Israël[◇], Costas Kounnas[◇], Domenico Orlando[♣], P. Marios Petropoulos[♣]

[◇] *Laboratoire de Physique Théorique de l'Ecole Normale Supérieure[†]*

24 rue Lhomond, 75231 Paris Cedex 05, France

[♣] *Centre de Physique Théorique, Ecole Polytechnique[‡]*

91128 Palaiseau, France

*E-mail: israel@lpt.ens.fr, kounnas@lpt.ens.fr,
orlando@cpht.polytechnique.fr, marios@cpht.polytechnique.fr*

ABSTRACT: We analyze asymmetric marginal deformations of $SU(2)_k$ and $SL(2, \mathbb{R})_k$ wzw models. These appear in heterotic string backgrounds with non-vanishing Neveu–Schwarz three-forms plus electric or magnetic fields, depending on whether the deformation is elliptic, hyperbolic or parabolic. Asymmetric deformations create new families of exact string vacua. The geometries which are generated in this way, deformed S^3 or AdS_3 , include in particular *geometric cosets* such as S^2 , AdS_2 or H_2 . Hence, the latter are consistent, exact conformal sigma models, with electric or magnetic backgrounds. We discuss various geometric and symmetry properties of the deformations at hand as well as their spectra and partition functions, with special attention to the supersymmetric $\text{AdS}_2 \times S^2$ background. We also comment on potential holographic applications.

*Research partially supported by the EEC under the contracts HPRN-CT-2000-00122, HPRN-CT-2000-00131, HPRN-CT-2000-00148.

[†]Unité mixte du CNRS et de l'Ecole Normale Supérieure, UMR 8549.

[‡]Unité mixte du CNRS et de l'Ecole Polytechnique, UMR 7644.

Contents

1. Introduction	1
2. Magnetic deformation of S^3	3
2.1 Squashing the three-sphere	4
2.2 Critical magnetic field and the geometric coset	6
2.3 Character formulas and modular invariance	8
3. Electric/magnetic deformations of AdS_3	10
3.1 Elliptic deformation: magnetic background	12
3.2 Hyperbolic deformation: electric background	14
3.2.1 The background and its CFT realization	14
3.2.2 The spectrum of string primaries	15
3.3 Parabolic deformation: electromagnetic-wave background	16
3.4 A remark on discrete identifications	18
4. Limiting geometries: AdS_2 and H_2	19
5. $AdS_2 \times S^2$	22
5.1 The spectrum	22
5.2 $AdS_2 \times S^2 \times \mathcal{M}$ and space-time supersymmetry	23
6. Outlook	24
A. Geometric deformations	27
B. The three-sphere	28
C. AdS_3	30
D. Equations of motion	33

1. Introduction

Near-horizon geometries of NS5-branes [CHS91], NS5/F1 or S-dual versions of those [ABS90, BPS98] have been thoroughly analyzed over the past years. These involve AdS_3 or S^3 spaces and turn out to be exact string backgrounds, tractable beyond the supergravity approximation. They offer a unique setting in which to analyze AdS/CFT correspondence, black-hole physics, little-string theory, . . .

An important, and not yet unravelled aspect of such configurations is the investigation of their moduli space. String propagation in the above backgrounds is described in terms of some exact two-dimensional conformal field theory. Hence, marginal deformations of the latter provide the appropriate tool for exploring the moduli of the corresponding string vacua.

A well-known class of marginal deformations for WZW models are those driven by left-right current bilinears [CS89, FR03]: $\int d^2z J\bar{J}$. These can be “symmetric” in the sense that both J and \bar{J} are generators of the affine algebra of the model. However, asymmetric deformations can also be considered, where either J or \bar{J} correspond to some other $U(1)$, living outside of the chiral algebra of the model. These deformations describe the response of the system to a finite (chromo) electric or magnetic background field and since these deformations are exactly marginal, the gravitational back-reaction is properly taken into account at any magnitude of the external field [KK95].

The purpose of this note is to report on the asymmetric deformations of the $SL(2, \mathbb{R})_k$ heterotic string background. Since $SL(2, \mathbb{R})$ has time-like, null and space-like generators, three distinct asymmetric deformations are possible, corresponding respectively to magnetic, electromagnetic and electric field backgrounds. The former case has been recently analyzed from this perspective [Isr04] and the corresponding deformation was shown to include Gödel space-time, which is therefore promoted to an exact string background (despite the caveats of closed time-like curves). The latter case, on the other hand, corresponds to a new deformation, which connects AdS_3 with $\mathbb{R} \times AdS_2$.

This observation is far reaching: while symmetric deformations usually connect WZW models to some $U(1)$ -gauged version of them [GK94], asymmetric deformations turn out to connect the original theory to some *geometric coset*, with *electric* or *magnetic* background fields. This holds for the $SU(2)_k$, where the limiting magnetic deformation has $\mathbb{R} \times S^2$ geometry, and can be generalized to any WZW model: geometric cosets with electric or magnetic background fields provide thus *exact* string vacua. Here we focus on the S^2 and AdS_2 examples, previously discussed as heterotic coset constructions in [Joh95] (see also [LS94]). Moreover, they both enter in the near-horizon geometry of the four-dimensional Reissner–Nordström extremal black hole, $AdS_2 \times S^2$, which is here shown to be an exact string vacuum. We also show how H_2 appears as an exact CFT, although this background is of limited interest for string theory because of lack of unitarity.

The paper is organized as follows. First we review the magnetic deformation of S^3 , appearing in the framework of the $SU(2)_k$ WZW model. The appearance of the two-sphere plus magnetic field as exact string background is described in Sec. 2, where we also determine the corresponding partition function. The AdS_3 case is analyzed in Sec. 3, where its asymmetric deformations are described in detail from geometrical and two-dimensional-CFT points of view. We also investigate their spectra. Limiting deformations are discussed in Sec. 4. There, we show how to reach the $AdS_2 = SL(2, \mathbb{R})/U(1)$ geometric coset with electric field. These backgrounds are consistent and exact string vacua.

The H_2 geometric coset of AdS_3 is also shown to appear on the line of magnetic deformation, with imaginary magnetic field though. The near-horizon geometry of the four-dimensional Reissner–Nordström extremal black hole is further discussed in Sec. 5. Section

6 contains a collection of final comments, where we sort various geometries that should be investigated in order to get a comprehensive picture of the general AdS₃ landscape, and its connection to other three-dimensional geometries. Four appendices provide some complementary/technical support. Appendix A sets the general framework for geometric deformations of a metric, designed to keep part of its original isometry. Appendices B and C contain material about $SU(2)$ and $SL(2, \mathbb{R})$ groups. A reminder of low-energy field equations for the bosonic degrees of freedom of heterotic string is given in App. D.

2. Magnetic deformation of S^3

$SU(2)_k$ WZW model magnetic deformations were analyzed in [KK95], for both type II and heterotic string backgrounds. For concreteness, we will concentrate here on the latter case. In contrast to what happens in flat space-time, these deformations are truly marginal in the background of a three-sphere plus NS flux, and preserve $N = (1, 0)$ world-sheet supersymmetry.

Consider heterotic string on $\mathbb{R}^{1,3} \times S^3 \times T^4$. The theory is critical provided we have a linear dilaton living on the $\mathbb{R}^{1,3}$, with background charge $Q = 1/\sqrt{k+2}$, where k is the level of the $SU(2)_L \times SU(2)_R$ affine algebra. The target-space geometry is the near-horizon limit of the solitonic NS5-brane [CHS91].

The two-dimensional $N = (1, 0)$ world-sheet action corresponding to the S^3 factor is

$$S_{SU(2)_k} = \frac{1}{2\pi} \int d^2z \left\{ \frac{k}{4} (\partial\alpha \bar{\partial}\alpha + \partial\beta \bar{\partial}\beta + \partial\gamma \bar{\partial}\gamma + 2 \cos\beta \partial\alpha \bar{\partial}\gamma) + \sum_{a=1}^3 \psi^a \bar{\partial}\psi^a \right\}, \quad (2.1)$$

where ψ^a are the left-moving free fermions, superpartners of the bosonic $SU(2)_k$ currents, and (α, β, γ) are the usual Euler angles parameterizing the $SU(2)$ group manifold (see App. B for a reminder). In this parameterization, the chiral currents of the Cartan subalgebra read:

$$J^3 = k (\partial\gamma + \cos\beta \partial\alpha), \quad \bar{J}^3 = k (\bar{\partial}\alpha + \cos\beta \bar{\partial}\gamma) \quad (2.2)$$

(Tab. 1) with the following short-distance expansion:

$$J^3(z)J^3(0) = \frac{k}{2z^2} + \text{reg.} \quad (2.3)$$

and similarly for the right-moving one. The left-moving fermions transform in the adjoint of $SU(2)$. There are no right-moving superpartners but a right-moving current algebra with total central charge $c = 16$ (realized *e.g.* in terms of right-moving free fermions). The currents of the latter are normalized so that the Cartan generators \bar{J}_G^i of the group factor G satisfy the following short-distance expansion:

$$\bar{J}_G^i(z)\bar{J}_G^j(0) = \frac{k_G h^{ij}}{2z^2} + \text{reg.}, \quad i, j = 1, \dots, \text{rank}(G) \quad (2.4)$$

with $h^{ij} = f_\ell^{ik} f_k^{\ell j} / g^*$, f_k^{ij} and g^* being the structure constants and dual Coxeter number of the group G .

The background metric and NS two-form are read off directly from (2.1):

$$ds^2 = \frac{k}{4} \left[d\beta^2 + \sin^2 \beta d\alpha^2 + (d\gamma + \cos \beta d\alpha)^2 \right], \quad (2.5)$$

$$B = \frac{k}{4} \cos \beta d\alpha \wedge d\gamma. \quad (2.6)$$

They describe a three-sphere of radius $L = \sqrt{k}$ and a NS three-form whose field strength is $dB = \frac{2}{\sqrt{k}} \omega_{[3]}$ ($\omega_{[3]}$ stands for the volume form given in Eq. (B.8)).

A comment is in order here. In general, the background fields G_{ab} , B_{ab} , ... receive quantum corrections due to two-dimensional renormalization effects controlled by α' (here set equal to one). This holds even when the world-sheet theory is exact. Conformal invariance requires indeed the background fields to solve Eqs. (D.11) that receive higher-order α' corrections. The case of WZW models is peculiar in the sense that the underlying symmetry protects G_{ab} and B_{ab} from most corrections; these eventually boil down to the substitution $k \rightarrow k + 2$ in Eqs. (2.5) and (2.6), see *e.g.* [Tse94].

Notice finally that the $SU(2)_k$ plus linear dilaton background introduces a mass gap with respect to flat space: $\mu^2 = 1/(k + 2)$. This plays the role of infra-red regulator, consistent with all string requirements including supersymmetry.

2.1 Squashing the three-sphere

We now turn to the issue of conformal deformations. As already advertised, we will not consider left-right symmetric ones, which are purely gravitational. Instead, we will switch the following $N = (1, 0)$ world-sheet supersymmetry compatible perturbation on

$$\delta S_{\text{magnetic}} = \frac{\sqrt{k k_G} H}{2\pi} \int d^2z (J^3 + \psi^1 \psi^2) \bar{J}_G; \quad (2.7)$$

\bar{J}_G being any Cartan current of the group factor G .

Although one may easily show the *integrability* of this marginal perturbation out of general arguments (see *e.g.* [CS89]) it is instructive to pause and write an explicit proof. If we limit ourselves to the bosonic sector, we can bosonize the \bar{J}_G current as $\bar{J}_G = i\bar{\partial}\varphi$ and interpret $\varphi(z, \bar{z})$ as an internal degree of freedom (see App. D for a more precise discussion). Incorporating the kinetic term for the φ field, the deformed action reads:

$$S = S_{SU(2)_k}(\alpha, \beta, \gamma) + \delta S_{\text{magnetic}} + \frac{k_G}{4\pi} \int d^2z \partial\varphi \bar{\partial}\varphi, \quad (2.8)$$

where $S_{SU(2)_k}$ (Eq. (2.1)) now contains the bosonic degrees of freedom only. The terms in previous expression can be recollected so to give:

$$S = S_{SU(2)_k} \left(\alpha, \beta, \gamma + \sqrt{\frac{k_G}{k}} H\varphi \right) + \frac{k_G(1 - 2H^2)}{4\pi} \int d^2z \partial\varphi \bar{\partial}\varphi, \quad (2.9)$$

which is manifestly exact. As a corollary, we observe that in the present setting, $\mathcal{O}(\alpha')$ solutions of Eqs. (D.11) are automatically promoted to all-order *exact* solutions by simply shifting $k \rightarrow k + 2$, just like for an “ordinary” WZW model.

The effect of the deformation at hand is to turn a (chromo)magnetic field on along some Cartan direction inside G , which in turn induces a gravitational back-reaction on the metric and the three-form antisymmetric tensor. Following the previous discussion and App. D (*i.e.* by using Kaluza–Klein reduction), it is straightforward to read off the space–time backgrounds from (2.1) and (2.7). We obtain:

$$ds^2 = \frac{k}{4} \left[d\beta^2 + \sin^2 \beta d\alpha^2 + (1 - 2H^2) (d\gamma + \cos \beta d\alpha)^2 \right] \quad (2.10)$$

and

$$A = \sqrt{\frac{2k}{k_G}} H (d\gamma + \cos \beta d\alpha) \quad (2.11)$$

for the metric and gauge field, whereas neither the B -field nor the dilaton are altered. The three-form field strength is however modified owing to the presence of the gauge field (see Eqs. (D.11)):

$$H_{[3]} = dB - \frac{k_G}{4} A \wedge dA = \frac{k}{4} (1 - 2H^2) \sin \beta d\alpha \wedge d\beta \wedge d\gamma \quad (2.12)$$

(the non-abelian structure of the gauge field plays no role since the non-vanishing components are in the Cartan subalgebra¹).

The deformed geometry (2.10) is a *squashed* three-sphere: its volume decreases with respect to the original S^3 while its curvature increases. These properties are captured in the expressions of the volume form and Ricci scalar:

$$\omega_{[3]} = \left(\frac{k}{4}\right)^{3/2} \sqrt{|1 - 2H^2|} \sin \beta d\alpha \wedge d\beta \wedge d\gamma, \quad (2.13)$$

$$R = \frac{2}{k} (3 + 2H^2). \quad (2.14)$$

The latter is constant and the background under consideration has $U(1) \times SU(2)$ isometry generated by the Killing vectors $\{L_3, R_1, R_2, R_3\}$ whose explicit expression is reported in App. B, Tab. 1.

This situation should be compared to the symmetric deformation generated by the marginal operator $(J^3 + i\psi^1\psi^2) \bar{J}^3$. This is purely gravitational and alters the metric, the B -field and the dilaton [GK94]. The isometry is in that case broken to $U(1) \times U(1)$ and the curvature is not constant.

At this point one might wonder to what extent the constant curvature of the asymmetric deformation is due to the large (almost maximal) residual isometry $U(1) \times SU(2)$. This question is answered in App. A, where it is shown, in a general framework, that the isometry requirement is not stringent enough to substantially reduce the moduli space of deformations. In particular, the resulting curvature is in general not constant.

In the case under consideration, however, the geometric deformation is driven by an integrable marginal perturbation of the sigma-model. Combined with the left-over isometry, this requirement leads to the above geometry with constant curvature, Eq. (2.14). From

¹Similarly, the (chromo)magnetic field strength is given by $F = dA$.

a purely geometrical point of view (i.e. ignoring the CFT origin), such a deformation joins the subclass of one-parameter families described in App. A, obtained by demanding stability *i.e.* integrability on top of the symmetry.

Notice finally that the $U(1) \times SU(2)$ isometry originates from an affine symmetry at the level of the sigma-model. The asymmetric marginal deformation under consideration breaks the original affine $SU(2)_L$ down to $U(1)_L$, while it keeps the affine $SU(2)_R$ unbroken.

It is worthwhile stressing that this asymmetric and the previously quoted symmetric deformations of the three-sphere background are mutually compatible. They can be performed simultaneously, although in that case the magnetic field can lead to (tachyonic) instabilities before reaching its maximal value [KK95].

All the above discussion about integrability, geometry and isometries of the $SU(2)_k$ magnetic perturbation is valid for the various asymmetric deformations of $SL(2, \mathbb{R})$ that will be analysed in Sec. 3.

2.2 Critical magnetic field and the geometric coset

It was made clear in [GK94] that the symmetric deformation of $SU(2)_k$ WZW is a well-defined theory for any value of the deformation parameter. For an infinite deformation, the sigma-model becomes a gauged WZW model $SU(2)_k/U(1)$ (*bell* geometry) times a decoupled boson. In some sense, the two $U(1)$ isometries present on the deformation line act, for extreme deformation, on two disconnected spaces: the bell and the real line.

As already stressed, the magnetic deformation of $SU(2)_k$ preserves a larger symmetry, namely a $U(1) \times SU(2)$, and has constant curvature. We are in the framework discussed in App. A, Eqs. (A.1) and (A.3) with $h = 2H^2$. This deformation has an end-point where the space is expected to factorize into a line with $U(1)$ isometry and a two-dimensional constant-curvature space with $SU(2)$ isometry, which can only be a two-sphere.

These statements can be made more precise by considering the background (2.10). The deformation parameter H^2 is clearly bounded: $H^2 \leq 1/2$ (the boundary $H^2_{\max} = 1/2$ of the moduli space is reminiscent of the $\text{Im}(U) \rightarrow \infty$ limit in a two-dimensional toroidal compactification). In general, a three sphere can be seen as a S^1 Hopf fibration over a base S^2 . It is clear from expressions (2.10) and (2.13) that the effect of the magnetic field consists in changing the radius of the fiber. At $H^2 = H^2_{\max}$, this radius vanishes and the states coupling to the magnetic field become infinitely massive and decouple. The corresponding dimension decompactifies, and factorizes from the three-dimensional geometry:

$$S^3 \xrightarrow{H^2 \rightarrow H^2_{\max}} \mathbb{R} \times S^2, \quad (2.15)$$

where S^2 is the *geometric coset* $SU(2)/U(1)$. The $M_H \rightarrow SU(2)/U(1)$ fibration trivializes in this limit. This can be made more transparent by introducing a new coordinate:

$$y = \sqrt{\frac{k}{2} \left(\frac{1}{2} - H^2 \right)} \gamma. \quad (2.16)$$

The metric and volume form now read:

$$ds^2 = \frac{k}{4} [d\beta^2 + \sin^2 \beta d\alpha^2] + dy^2 + 2\sqrt{\frac{k}{2} \left(\frac{1}{2} - H^2\right)} \cos \beta d\alpha dy + \frac{k}{2} \left(\frac{1}{2} - H^2\right) \cos^2 \beta d\alpha^2 \quad (2.17)$$

and

$$\omega_{[3]} = \frac{k}{4} \sin \beta d\alpha \wedge d\beta \wedge dy. \quad (2.18)$$

For H^2 close to H_{\max}^2 , the y -direction factorizes

$$ds^2 \xrightarrow{H^2 \rightarrow H_{\max}^2} dy^2 + \frac{k}{4} [d\beta^2 + \sin^2 \beta d\alpha^2], \quad (2.19)$$

while the curvature ($R \rightarrow 8/k$) is entirely supported by the remaining two-sphere of radius $\sqrt{k}/4$. The other background fields read:

$$F = \sqrt{\frac{2k}{k_G}} H \sin \beta d\alpha \wedge d\beta \xrightarrow{H^2 \rightarrow H_{\max}^2} \sqrt{\frac{k}{k_G}} \sin \beta d\alpha \wedge d\beta, \quad (2.20)$$

$$H_{[3]} = \sqrt{\frac{k}{2} \left(\frac{1}{2} - H^2\right)} \sin \beta d\alpha \wedge d\beta \wedge dy \xrightarrow{H^2 \rightarrow H_{\max}^2} 0. \quad (2.21)$$

The above analysis deserves several comments. Our starting point was a marginal deformation of the $SU(2)_k$ WZW model embedded in heterotic strings and induced by a space-time (chromo)magnetic field. Our observation is here that the corresponding moduli space has a boundary, where the background is $\mathbb{R} \times S^2$, with finite magnetic field and no three-form NS background. Being a marginal deformation, this background is *exact*, showing thereby that the geometric coset is as good as a gauged WZW model background. The latter appears similarly as the end-point of a purely gravitational deformation; it carries neither magnetic field nor $H_{[3]}$, but has a non-trivial dilaton.

Notice also that it was observed in the past that S^2 could provide part of a string vacuum in the presence of RR fluxes [FKS95], but as usual when dealing with RR fluxes, no exact conformal field theory description is available.

The procedure we have developed so far for obtaining the two-sphere as an exact background in the presence of a magnetic field is easily generalizable to other geometric cosets of compact or non-compact groups. We will focus on the latter case in Sec. 3, and analyze the electric/magnetic deformations of AdS_3 .

Our last comment concerns the quantization of the magnetic flux. At the limiting value of the deformation, the flux of the gauge field through the two-sphere is given by:

$$\mathcal{Q} = \int_{S^2} F = \sqrt{\frac{k}{k_G}} \int_{S^2} \hat{\omega}_2 = \sqrt{\frac{k}{k_G}} 4\pi, \quad (2.22)$$

where $\hat{\omega}_2$ stands for the volume form of a unit-radius two-sphere. Therefore, a quantization of the magnetic charge is only compatible with levels of the affine algebras such that:

$$\frac{k}{k_G} = p^2, \quad p \in \mathbb{Z}. \quad (2.23)$$

Actually, it was shown in [Joh95] that the model corresponding to the critical magnetic field can be obtained directly with the following asymmetric gauged WZW model:

$$\frac{SU(2)_k \times U(1)_{k_G}}{U(1)}, \quad (2.24)$$

where the left gauging lies in the $SU(2)_k$ WZW model and the right gauging in the $U(1)_{k_G}$ of the gauge sector. The cancellation of the anomalies of the coset dictates a condition on the level of $SU(2)_k$ similar to (2.23).

2.3 Character formulas and modular invariance

We will here construct the contribution of the squashed three-sphere to the partition function. This contribution is modular-covariant, and combines with the remaining degrees of freedom into a modular-invariant result. Our computation will also include the S^2 limiting geometry. We will consider the case $k_G = 2$, *i.e.* a $U(1)$ algebra generated by one right-moving complex fermion. We begin with the following combination of $SU(2)_{k-2}$ supersymmetric characters and fermions from the gauge sector:

$$Z \begin{bmatrix} a; h \\ b; g \end{bmatrix} = \sum_{j, \bar{j}=0}^{(k-2)/2} M^{j\bar{j}} \chi^j \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \bar{\chi}^{\bar{j}} \frac{\bar{\vartheta} \begin{bmatrix} h \\ g \end{bmatrix}}{\bar{\eta}}. \quad (2.25)$$

where the χ^j 's are the characters of bosonic $SU(2)_{k-2}$, (a, b) are the \mathbb{Z}_2 boundary conditions for the left-moving fermions² and (h, g) those of the right-moving – gauge-sector – ones. We can choose any matrix $M^{j\bar{j}}$ compatible with modular invariance of $SU(2)_{k-2}$. Furthermore, the supersymmetric $SU(2)_k$ characters can be decomposed in terms of those of the $N = 2$ minimal models:

$$\chi^j(\tau) \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau, \nu) = \sum_{m \in \mathbb{Z}_{2k}} \mathcal{C}_m^j \begin{bmatrix} a \\ b \end{bmatrix} \Theta_{m,k} \left(\tau, -\frac{2\nu}{k} \right), \quad (2.26)$$

where the $N = 2$ minimal-model characters, determined implicitly by this decomposition, are given in [Kir88, Dob87, Mat87, RY87].

Our aim is to implement the magnetic deformation in this formalism. The deformation acts as a boost on the left-lattice contribution of the Cartan current of the supersymmetric $SU(2)_k$ and on the right current from the gauge sector:

$$\begin{aligned} \Theta_{m,k} \bar{\vartheta} \begin{bmatrix} h \\ g \end{bmatrix} &= \sum_{n, \bar{n}} e^{-i\pi g(\bar{n} + \frac{h}{2})} q^{\frac{1}{2}(\sqrt{2kn} + \frac{m}{\sqrt{2k}})^2} \bar{q}^{\frac{1}{2}(\bar{n} + \frac{h}{2})^2} \\ &\longrightarrow \sum_{n, \bar{n}} e^{-i\pi g(\bar{n} + \frac{h}{2})} q^{\frac{1}{2}[(\sqrt{2kn} + \frac{m}{\sqrt{2k}}) \cosh x + (\bar{n} + \frac{h}{2}) \sinh x]^2} \\ &\quad \times \bar{q}^{\frac{1}{2}[(\bar{n} + \frac{h}{2}) \cosh x + (\sqrt{2kn} + \frac{m}{\sqrt{2k}}) \sinh x]^2}. \end{aligned} \quad (2.27)$$

²We have removed the contribution of the fermion associated to J^3 since it is neutral in the deformation process.

The boost parameter x is related to the vacuum expectation value of the gauge field as follows:

$$\cosh x = \frac{1}{1 - 2H^2}. \quad (2.28)$$

We observe that, in the limit $H^2 \rightarrow H_{\max}^2$, the boost parameter diverges ($x \rightarrow \infty$), and the following constraints arise:

$$4(k+2)n + 2m + 2\sqrt{2k\bar{n}} + \sqrt{2kh} = 0. \quad (2.29)$$

Therefore, the limit is well-defined only if the level of the supersymmetric $SU(2)_k$ satisfies a quantization condition:

$$k = 2p^2, \quad p \in \mathbb{Z}. \quad (2.30)$$

This is exactly the charge quantization condition for the flux of the gauge field, Eq. (2.23). Under this condition, the constraints (2.29) lead to

$$m + ph \equiv 0 \pmod{2p} =: 2pN, \quad (2.31a)$$

$$\bar{n} = 2pn + N, \quad N \in \mathbb{Z}_{2p}. \quad (2.31b)$$

As a consequence, the $U(1)$ corresponding to the combination of charges orthogonal to (2.29) decouples (its radius vanishes), and can be removed. We end up with the following expression for the S^2 partition function contribution:

$$Z_{S^2} \left[\begin{matrix} a; h \\ b; g \end{matrix} \right] = \sum_{j, \bar{j}} M^{j\bar{j}} \sum_{N \in \mathbb{Z}_{2p}} e^{i\pi g(N + \frac{h}{2})} \mathcal{C}_{p(2N-h)}^j \left[\begin{matrix} a \\ b \end{matrix} \right] \bar{\chi}^{\bar{j}}, \quad (2.32)$$

in agreement with the result found in [BJKZ96] by using the coset construction. The remaining charge N labels the magnetic charge of the state under consideration. As a result, the R -charges of the left $N = 2$ superconformal algebra are:

$$\mathcal{Q}_R = n + \frac{a}{2} - \frac{N - h/2}{p} \pmod{2}. \quad (2.33)$$

We now turn to the issue of modular covariance. Under the transformation $\tau \rightarrow -1/\tau$, the minimal-model characters transform as:

$$\mathcal{C}_m^j \left[\begin{matrix} a \\ b \end{matrix} \right] \left(-\frac{1}{\tau} \right) = e^{i\frac{\pi}{2}ab} \frac{1}{k} \sum_{j'=0}^{(k-2)/2} \sin \left(\frac{\pi(2j+1)(2j'+1)}{k} \right) \sum_{m' \in \mathbb{Z}_{2k}} e^{i\pi \frac{mm'}{k}} \mathcal{C}_{m'}^{j'} \left[\begin{matrix} b \\ -a \end{matrix} \right] (\tau). \quad (2.34)$$

On the one hand, the part of the modular transformation related to j is precisely compensated by a similar term coming from the transformation of $\bar{\chi}^{\bar{j}}$, in Eq. (2.32). On the other hand, the part of the transformation related to the spin structure (a, b) is compensated by the transformation of the other left-moving fermions in the full heterotic string construction. We can therefore concentrate on the transformation related to the m charge, coming from the transformation of the theta-functions at level k . We have

$$\sum_{N \in \mathbb{Z}_{2p}} e^{-i\pi g(N + \frac{h}{2})} \mathcal{C}_{p(2N-h)}^j \left[\begin{matrix} a \\ b \end{matrix} \right] \rightarrow \frac{1}{\sqrt{2k}} \sum_{m' \in \mathbb{Z}_{4p^2}} \sum_{N \in \mathbb{Z}_{2p}} e^{i\frac{\pi}{2} \left(g - \frac{m'}{p} \right) h} e^{2i\pi \frac{N(m'+pg)}{2p}} \mathcal{C}_{m'}^j \left[\begin{matrix} b \\ -a \end{matrix} \right]; \quad (2.35)$$

summing over N in \mathbb{Z}_{2p} leads to the constraint:

$$m' + pg \equiv 0 \pmod{2p} := -2pN', \quad N' \in \mathbb{Z}_{2p}. \quad (2.36)$$

So we end up with the sum

$$e^{-\frac{i\pi}{2}hg} \sum_{N' \in \mathbb{Z}_{2p}} e^{-i\pi h(N' + \frac{g}{2})} \mathcal{C}_{p(2N'-g)}^j \begin{bmatrix} b \\ -a \end{bmatrix}. \quad (2.37)$$

combining this expression with the modular transformation of the remaining right-moving fermions of the gauge sector, we obtain a modular invariant result.

In a similar way one can check the invariance of the full heterotic string under $\tau \rightarrow \tau+1$.

3. Electric/magnetic deformations of AdS_3

Anti-de-Sitter space in three dimensions is the (universal covering of the) $SL(2, \mathbb{R})$ group manifold. It provides therefore an exact string vacuum with NS background, described in terms of the $SL(2, \mathbb{R})_k$ WZW model, where time is embedded in the non-trivial geometry. We will consider it as part of some heterotic string solution such as $\text{AdS}_3 \times S^3 \times T^4$ with NS three-form field in $\text{AdS}_3 \times S^3$ (near-horizon NS5/F1 background). The specific choice of a background is however of limited importance for our purpose.

The issue of AdS_3 deformations has been raised in several circumstances. It is richer than the corresponding S^3 owing to the presence of elliptic, hyperbolic or parabolic elements in $SL(2, \mathbb{R})$. The corresponding generators are time-like, space-like or light-like. Similarly, the residual symmetry of a deformed AdS_3 has $U(1)$ factors, which act in time, space or light direction.

Marginal *symmetric* deformations of the $SL(2, \mathbb{R})_k$ WZW are driven by bilinears $J\bar{J}$ where both currents are in $SL(2, \mathbb{R})$ and are of the same kind [For94, IKP03]. These break the $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ affine symmetry to $U(1)_L \times U(1)_R$ and allow to reach, at extreme values of the deformation, gauged $SL(2, \mathbb{R})_k/U(1)$ WZW models with an extra free decoupled boson. We can summarize the results as follows:

- (a) $J^3\bar{J}^3$ These are time-like currents (for conventions see App. C) and the corresponding deformations connect $SL(2, \mathbb{R})_k$ with $U(1) \times SL(2, \mathbb{R})_k/U(1)|_{\text{axial or vector}}$. The $U(1)$ factor stands for a decoupled, non-compact time-like free boson³. The gauged WZW model $SL(2, \mathbb{R})_k/U(1)|_{\text{axial}}$ is the *cigar* (two-dimensional Euclidean black hole) obtained by gauging the $g \rightarrow hgh$ symmetry with the $h = \exp i\frac{\lambda}{2}\sigma^2$ subgroup, whereas $SL(2, \mathbb{R})_k/U(1)|_{\text{vector}}$ corresponds to the $g \rightarrow hgh^{-1}$ gauging. This is the *trumpet* and is T-dual to the *cigar*⁴. The generators of the affine residual symmetry $U(1)_L \times U(1)_R$ are both time-like (the corresponding Killing vectors are not orthogonal though). For extreme deformation, the time coordinate decouples and the antisymmetric tensor is trade for a dilaton. The isometries are time-translation invariance and rotation invariance in the *cigar/trumpet*.

³The extra bosons are always non-compact.

⁴Actually this statement holds only for the vector coset of the *single cover* of $SL(2, \mathbb{R})$. Otherwise, from the n -th cover of the group manifold one obtains the n -th cover of the *trumpet* [IKP03].

- (b) $J^2\bar{J}^2$ The deformation is now induced by space-like currents. So is the residual affine symmetry $U(1)_L \times U(1)_R$ of the deformed model. Extreme deformation points are T-dual: $U(1) \times SL(2, \mathbb{R})_k / U(1)$ where the $U(1)$ factor is space-like, and the $U(1)$ gauging of $SL(2, \mathbb{R})_k$ corresponds to $g \rightarrow hgh^{(-1)}$ with $h = \exp -\frac{\lambda}{2}\sigma^3$ [DVV92]. The corresponding manifold is (some sector of) the Lorentzian two-dimensional black hole with a non-trivial dilaton.
- (c) $(J^1 + J^3)(\bar{J}^1 + \bar{J}^3)$ This is the last alternative, with both null currents. The deformation connects AdS_3 with $\mathbb{R} \times \mathbb{R}^{1,1}$ plus a dilaton linear in the first factor. The $U(1)_L \times U(1)_R$ left-over current algebra is light-like⁵. Tensorized with an $SU(2)_k$ CFT, this background describes the decoupling limit of the NS5/F1 setup [IKP03], where the fundamental strings regularize the strong coupling regime.

Our purpose here is to analyze *asymmetric* deformations of AdS_3 . Following App. A and the similar analysis of Sec. 2.1 for S^3 , we expect those deformations to preserve a $U(1)_L \times SL(2, \mathbb{R})_R$ symmetry appearing as affine algebra from the sigma-model point of view, and as isometry group for the background. The residual $U(1)_L$ factor can be time-like, space-like or null depending on the current that has been used to perturb the WZW model.

It is worth to stress that some deformations of AdS_3 have been studied in the past irrespectively of any conformal sigma-model or string theory analysis. In particular it was observed in [RS98], following [RT83] that the three-dimensional⁶ Gödel solution of Einstein equations could be obtained as a member of a one-parameter family of AdS_3 deformations that precisely enters the class we discuss in App. A. Gödel space is a constant-curvature Lorentzian manifold. Its isometry group is $U(1) \times SL(2, \mathbb{R})$, and the $U(1)$ factor is generated by a time-like Killing vector. These properties hold for generic values of the deformation parameter. In fact the deformed AdS_3 under consideration can be embedded in a seven-dimensional flat space with appropriate signature, as the intersection of four quadratic surfaces. Closed time-like curves as well as high symmetry are inherited from the multi-time maximally symmetric host space. Another interesting property resulting from this embedding is the possibility for changing the sign of the curvature along the continuous line of deformation, without encountering any singular behaviour (see Eq. (3.2)).

It seems natural to generalize the above results to *new* AdS_3 deformations and promote them to *exact* string backgrounds. Our guideline will be the requirement of a $U(1) \times SL(2, \mathbb{R})$ isometry group, with space-like or light-like $U(1)$'s, following the procedure developed in App. A.

We will first review the time-like (elliptic) deformation of AdS_3 of [RS98] and recently studied from a string perspective in [Isr04]. Hyperbolic (space-like) and parabolic (light-like) deformations will be analyzed in Secs. 3.2 and 3.3. All these deformations are of the type (A.1) and (A.3) or (A.4). We show in the following how to implement these

⁵The isometry is actually richer by one (two translations plus a boost), but the extra generator (the boost) is not promoted to an affine symmetry of the sigma-model.

⁶In fact, the original Gödel solution is four-dimensional, but the fourth space dimension is a flat spectator. In the following, we will systematically refer to the three-dimensional non-trivial factor.

deformations as exact marginal perturbations in the framework of the $SL(2, \mathbb{R})_k$ WZW model embedded in heterotic string.

3.1 Elliptic deformation: magnetic background

Consider AdS_3 in (ρ, t, ϕ) coordinates, with metric given in (C.10). In these coordinates, two manifest Killing vectors are $L_3 \sim \partial_t$ and $R_2 \sim \partial_\phi$, time-like and space-like respectively (see App. C, Tab. 2).

The deformation studied in [RS98] and quoted as “squashed anti de Sitter” reads, in the above coordinates:

$$ds^2 = \frac{L^2}{4} \left[d\rho^2 + \cosh^2 \rho d\phi^2 - (1 + 2H^2) (dt + \sinh \rho d\phi)^2 \right]. \quad (3.1)$$

It preserves a $U(1) \times SL(2, \mathbb{R})$ isometry group. The $U(1)$ is generated by the *time-like* vector L_3 of one original $SL(2, \mathbb{R})$, while the right-moving $SL(2, \mathbb{R})$ is unbroken (the expressions for the $\{L_3, R_1, R_2, R_3\}$ Killing vectors in Tab. 2 remain valid at any value of the deformation parameter). The Ricci scalar is constant

$$R = -\frac{2}{L^2} (3 - 2H^2), \quad (3.2)$$

while the volume form reads:

$$\omega_{[3]} = \frac{L^3}{8} \sqrt{|1 + 2H^2|} \cosh \rho d\rho \wedge d\phi \wedge dt. \quad (3.3)$$

For $H^2 = 1/2$, this deformation coincides with the Gödel metric. It should be stressed, however, that nothing special occurs at this value of the deformation parameter. The properties of Gödel space are generically reproduced at any $H^2 > 0$.

From a physical point of view, as it stands, this solution is pathological because it has topologically trivial closed time-like curves through each point of the manifold, like Gödel space-time which belongs to this family. Its interest mostly relies on the fact that it can be promoted to an exact string solution, with appropriate NS and magnetic backgrounds. The high symmetry of (3.1), is a severe constraint and, as was shown in [Isr04], the geometry at hand does indeed coincide with the unique marginal deformation of the $SL(2, \mathbb{R})_k$ WZW that preserves a $U(1)_L \times SL(2, \mathbb{R})_R$ affine algebra with time-like $U(1)_L$.

It is interesting to observe that, at this stage, the deformation parameter H^2 *needs not be positive*: (3.1) solves the Einstein-Maxwell-scalar equations [RT83] for any H^2 . Furthermore, for $H^2 < 0$, there are no longer closed time-like curves⁷. This statement is based on a simple argument⁸. Consider a time-like curve $x^\mu = x^\mu(\lambda)$. By definition the tangent vector ∂_λ is negative-norm, which, by using Eq. (3.1), translates into

$$\left(\frac{d\rho}{d\lambda} \right)^2 + \cosh^2 \rho \left(\frac{d\phi}{d\lambda} \right)^2 - (1 + 2H^2) \left(\frac{dt}{d\lambda} + \sinh \rho \frac{d\phi}{d\lambda} \right)^2 < 0. \quad (3.4)$$

⁷As mentioned previously, the geometry at hand can be embedded in a seven-dimensional flat space, with signature $\varepsilon - - - + +$, $\varepsilon = \text{sign}(-H^2)$ [RS98]. This clarifies the origin of the symmetry as well as the presence or absence of closed time-like curves for positive or negative H^2 .

⁸This argument is local and must in fact be completed by global considerations on the manifold (see [RS98]).

If the curve is closed, $dt/d\lambda$ must vanish somewhere. At the turning point, the resulting inequality,

$$(2H^2 \sinh^2 \rho - 1) \left(\frac{d\phi}{d\rho} \right)^2 > 1 \quad (3.5)$$

is never satisfied for $H^2 < 0$, whereas it is for large enough⁹ ρ otherwise.

This apparent *regularization of the causal pathology*, unfortunately breaks down at the string level. In fact, as we will shortly see, in order to be considered as a string solution, the above background requires a (chromo)magnetic field. The latter turns out to be proportional to H , and becomes *imaginary* in the range where the closed time-like curves disappear. Hence, at the string level, unitarity is traded for causality. It seems that no regime exists in the magnetic deformation of AdS_3 , where these fundamental requirements are simultaneously fulfilled.

In the heterotic backgrounds considered here, of the type $\text{AdS}_3 \times S^3 \times T^4$, the two-dimensional $N = (1, 0)$ world-sheet action corresponding to the AdS_3 factor is

$$S_{SL(2, \mathbb{R})_k} = \frac{1}{2\pi} \int d^2z \left\{ \frac{k}{4} (\partial\rho \bar{\partial}\rho - \partial t \bar{\partial}t + \partial\phi \bar{\partial}\phi - 2 \sinh \rho \partial\phi \bar{\partial}t) + \eta_{ab} \psi^a \bar{\partial}\psi^b \right\}, \quad (3.6)$$

where $\eta_{ab} = \text{diag}(+ + -)$, $a = 1, 2, 3$ and ψ^a are the left-moving superpartners of the $SL(2, \mathbb{R})_k$ currents (see Tab. 2). The corresponding background fields are the metric (Eq. (C.10)) with radius $L = \sqrt{k}$ and the NS B-field:

$$B = -\frac{k}{4} \sinh \rho d\phi \wedge dt. \quad (3.7)$$

The three-form field strength is $H_{[3]} = dB = -\frac{2}{\sqrt{k}} \omega_{[3]}$ with $\omega_{[3]}$ displayed in Eq. (C.11).

The asymmetric perturbation that preserves a $U(1)_L \times SL(2, \mathbb{R})_R$ affine algebra with time-like $U(1)_L$ is $\delta S_{\text{magnetic}}$ given in Eq. (2.7), where J^3 now stands for the left-moving time-like $SL(2, \mathbb{R})_k$ current given in App. C, Tab. 2. This perturbation corresponds to switching on a (chromo)magnetic field, like in the $SU(2)_k$ studied in Sec. 2. It is marginal and can be integrated for finite values of H , and is compatible with the $N = (1, 0)$ world-sheet supersymmetry. The resulting background fields, extracted in the usual manner from the deformed action are the metric (3.1) with radius $L = \sqrt{k}$ and the following gauge field:

$$A = H \sqrt{\frac{2k}{k_g}} (dt + \sinh \rho d\phi). \quad (3.8)$$

The NS B -field is not altered by the deformation, (Eq. (3.7)), whereas the three-form field strength depends explicitly on the deformation parameter H , because of the gauge-field contribution:

$$H_{[3]} = dB - \frac{k_G}{4} A \wedge dA = -\frac{k}{4} (1 + 2H^2) \cosh \rho d\rho \wedge d\phi \wedge dt. \quad (3.9)$$

⁹This means $\rho > \rho_c$ where ρ_c is the radius where the norm of ∂_ϕ vanishes and switches to negative ($\|\partial_\phi\|^2 = L^2 (1 - 2H^2 \sinh^2 \rho) / 4$). This never occurs for $H^2 < 0$.

One can easily check that the background fields (3.1), (3.8) and (3.9) solve the lowest-order equations of motion (D.11). Of course the solution we have obtained is exact, since it has been obtained as the marginal deformation of an exact conformal sigma-model. The interpretation of the deformed model in terms of background fields $\{G_{ab}, B_{ab}, F_{ab}^G\}$ receives however the usual higher-order correction summarized by the shift $k \rightarrow k + 2$ as we have already seen for the sphere in Sec. 2.1.

Let us finally mention that it is possible to extract the spectrum and write down the partition function of the above theory [Isr04], since the latter is an exact deformation of the $SL(2, \mathbb{R})_k$ WZW model. This is achieved by deforming the associated elliptic Cartan subalgebra. The following picture emerges then from the analysis of the spectrum. The short-string spectrum, corresponding to world-sheets trapped in the center of the space-time (for some particular choice of coordinates) is well-behaved, because these world-sheets do not feel the closed time-like curves which are “topologically large”. On the contrary, the long strings can wrap the closed time-like curves, and their spectrum contains many tachyons. Hence, the caveats of Gödel space survive the string framework, at any value of $H^2 > 0$. One can circumvent them by slightly deviating from the Gödel line with an extra purely gravitational deformation, driven by $J^3 \bar{J}^3$. This deformation isolates the causally unsafe region, $\rho > \rho_c$ (see [Isr04] for details). It is similar in spirit with the supertubes domain-walls of [DFS03b] curing the Gödel-like space-times with RR backgrounds.

As already stressed, one could alternatively switch to negative H^2 . Both metric and antisymmetric tensor are well-defined and don't suffer of causality problems. The string picture however breaks down because the magnetic field (Eq. (3.8)) becomes imaginary.

3.2 Hyperbolic deformation: electric background

3.2.1 The background and its CFT realization

We will now focus on a different deformation. We use coordinates (C.12) with metric (C.13), where the manifest Killing vectors are $L_2 \sim \partial_x$ (space-like) and $R_3 \sim \partial_\tau$ (time-like) (see App. C, Tab. 3). This time we perform a deformation that preserves a $U(1) \times SL(2, \mathbb{R})$ isometry. The $U(1)$ corresponds to the space-like Killing vector L_2 , whereas the $SL(2, \mathbb{R})$ is generated by R_1, R_2, R_3 , which are again not altered by the deformation. This is achieved by implementing Eqs. (A.1) and (A.3) in the present set up, with $\xi = \partial_x$ and $h = 2H^2$. The resulting metric reads:

$$ds^2 = \frac{L^2}{4} \left[dr^2 - \cosh^2 r d\tau^2 + (1 - 2H^2) (dx + \sinh r d\tau)^2 \right]. \quad (3.10)$$

The scalar curvature of this manifold is constant

$$R = -\frac{2}{L^2} (3 + 2H^2) \quad (3.11)$$

and the volume form

$$\omega_{[3]} = \frac{L^3}{8} \sqrt{|1 - 2H^2|} \cosh^2 r dr \wedge d\tau \wedge dx. \quad (3.12)$$

Following the argument of Sec. 3.1, one can check whether closed time-like curves appear. Indeed, assuming their existence, the following inequality must hold at the turning point *i.e.* where $dt/d\lambda$ vanishes (λ being the parameter that describes the curve):

$$(2H^2 - 1) \left(\frac{dx}{dr} \right)^2 > 1. \quad (3.13)$$

The latter cannot be satisfied in the regime $H^2 < 1/2$. Notice that the manifold at hand is well behaved, even for negative H^2 .

Let us now leave aside these questions about the classical geometry, and address the issue of string realization of the above background. As already advertised, this is achieved by considering a world-sheet-supersymmetric marginal deformation of the $SL(2, \mathbb{R})_k$ WZW model that implements (chromo)electric field. Such a deformation is possible in the heterotic string at hand:

$$\delta S_{\text{electric}} = \frac{\sqrt{k k_G} H}{2\pi} \int d^2z (J^2 + i\psi^1\psi^3) \bar{J}_G, \quad (3.14)$$

(\bar{J}_G is any Cartan current of the group G and J^2 is given in App. C, Tab. 3), and corresponds, as in previous cases, to an integrable marginal deformation. The deformed conformal sigma-model can be analyzed in terms of background fields. The metric turns out to be (3.10), whereas the gauge field and three-form tensor are

$$A = H \sqrt{\frac{2k}{k_g}} (dx + \sinh r d\tau), \quad (3.15)$$

$$H_{[3]} = \frac{k}{4} (1 - 2H^2) \cosh r dr \wedge d\tau \wedge dx. \quad (3.16)$$

As expected, these fields solve Eqs. (D.11).

The background under consideration is a new string solution generated as a hyperbolic deformation of the $SL(2, \mathbb{R})_k$ WZW model. In contrast to what happens for the elliptic deformation (magnetic background analyzed in Sec. 3.1), the present solution is perfectly sensible, both at the classical and at the string level.

3.2.2 The spectrum of string primaries

The electric deformation of AdS_3 is an exact string background. The corresponding conformal field theory is however more difficult to deal with than the one for the elliptic deformation. In order to write down its partition function, we must decompose the $SL(2, \mathbb{R})_k$ partition function in a hyperbolic basis of characters, where the implementation of the deformation is well-defined and straightforward; this is a notoriously difficult exercise. On the other hand the spectrum of primaries is known¹⁰ from the study of the representations of the Lie algebra in this basis (see *e.g.* [VK91], and [DVV92] for the spectrum of the hyperbolic gauged WZW model). The part of the heterotic spectrum of interest contains the

¹⁰In the following we do not consider the issue of the spectral-flow representations. The spectral-flow symmetry is apparently broken by the deformation considered here.

expression for the primaries of $N = (1, 0)$ affine $SL(2, \mathbb{R})$ at purely bosonic level¹¹ $k + 2$, together with some $U(1)$ from the lattice of the heterotic gauge group:

$$L_0 = -\frac{j(j-1)}{k} - \frac{1}{2} \left(n + \frac{a}{2} \right)^2, \quad (3.17)$$

$$\bar{L}_0 = -\frac{j(j-1)}{k} + \frac{1}{2} \left(\bar{n} + \frac{h}{2} \right)^2, \quad (3.18)$$

where the second Casimir of the representation of the $SL(2, \mathbb{R})$ algebra, $-j(j-1)$, explicitly appears. The spectrum contains *continuous representations*, with $j = \frac{1}{2} + is$, $s \in \mathbb{R}_+$. It also contains *discrete representations*, with $j \in \mathbb{R}_+$, lying within the unitarity range $1/2 < j < (k+1)/2$ (see [MO01, Pet90]). In both cases the spectrum of the hyperbolic generator J^2 is $\mu \in \mathbb{R}$. The expression for the left conformal dimensions, Eq. (3.17), also contains the contribution from the world-sheet fermions associated to the $\psi^1\psi^3$ current. The sector (R or NS) is labelled by $a \in \mathbb{Z}_2$. Note that the unusual sign in front of the lattice is the natural one for the fermions of the light-cone directions. In the expression (3.18) we have similarly the contribution of the fermions of the gauge group, where h labels the corresponding sector.

We are now in position to follow the procedure, familiar from the previous examples: we have to (i) isolate from the left spectrum the lattice of the supersymmetric hyperbolic current $J^2 + \psi^1\psi^3$ and (ii) perform a boost between this lattice and the fermionic lattice of the gauge field. We hence obtain the following expressions:

$$L_0 = -\frac{j(j-1)}{k} - \frac{\mu^2}{k+2} - \frac{k+2}{2k} \left(n + \frac{a}{2} + \frac{2\mu}{k+2} \right)^2 + \frac{1}{2} \left[\sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \left(\bar{n} + \frac{h}{2} \right) \sinh x \right]^2, \quad (3.19)$$

$$\bar{L}_0 = -\frac{j(j-1)}{k} + \frac{1}{2} \left[\left(\bar{n} + \frac{h}{2} \right) \cosh x + \sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x \right]^2. \quad (3.20)$$

The relation between the boost parameter x and the deformation parameter H^2 is given in Eq. (2.28), as for the case of the $SU(2)_k$ deformation. In particular it is worth to remark that the first three terms of (3.19) correspond to the left weights of the supersymmetric two-dimensional Lorentzian black hole, *i.e.* the $SL(2, \mathbb{R})/O(1, 1)$ gauged super-WZW model.

3.3 Parabolic deformation: electromagnetic-wave background

In the deformations of Secs. 3.1 and 3.2, one $SL(2, \mathbb{R})$ isometry breaks down to a $U(1)$ generated either by a time-like or by a space-like Killing vector. Deformations which preserve a light-like isometry do also exist and are easily implemented in Poincaré coordinates.

We require that the isometry group is $U(1) \times SL(2, \mathbb{R})$ with a null Killing vector for the $U(1)$ factor. Following the deformation procedure described in App. A for the particular

¹¹More precisely we consider primaries of the purely bosonic affine algebra with an arbitrary state in the fermionic sector.

case of light-like residual isometry, Eq. (A.4) with $h = 2H^2$, we are lead to

$$ds^2 = L^2 \left[\frac{du^2}{u^2} + \frac{dx^+ dx^-}{u^2} - 2H^2 \left(\frac{dx^+}{u^2} \right)^2 \right]. \quad (3.21)$$

The light-like $U(1)$ Killing vector is $L_1 + L_3 \sim \partial_-$ (see App. C, Tab. 4). The remaining $SL(2, \mathbb{R})$ generators are $\{R_1 + R_3, R_1 - R_3, R_2\}$ and remain unaltered after the deformation.

The above deformed anti-de-Sitter geometry looks like a superposition of AdS_3 and of a plane wave. As usual, the sign of H^2 is free at this stage, and $H^2 < 0$ are equally good geometries. In the near-horizon region ($|u| \gg |H^2|$) the geometry is not sensitive to the presence of the wave. On the contrary, this plane wave dominates in the opposite limit, near the conformal boundary.

The volume form is not affected by the deformation, and it is still given in (C.17); neither is the Ricci scalar modified:

$$R = -\frac{6}{L^2}. \quad (3.22)$$

Notice also that the actual value of $|H|$ is not of physical significance: it can always be absorbed into a reparameterization $x^+ \rightarrow x^+ / |H|$ and $x^- \rightarrow x^- |H|$. The only relevant values for H^2 can therefore be chosen to be $0, \pm 1$.

We now come to the implementation of the geometry (3.21) in a string background. The only option is to perform an asymmetric exactly marginal deformation of the heterotic $SL(2, \mathbb{R})_k$ WZW model that preserves a $U(1)_L \times SL(2, \mathbb{R})_R$ affine symmetry. This is achieved by introducing

$$\delta S_{\text{electric-magnetic}} = -4\sqrt{k k_G} H \int d^2z (J^1 + J^3 + i(\psi^1 + \psi^3)\psi^2) \bar{J}_G, \quad (3.23)$$

($J^1 + J^3$ is defined in App. C, Tab. 4). The latter perturbation is integrable and accounts for the creation of an (chromo)electromagnetic field

$$A = 2\sqrt{\frac{2k}{k_G}} H \frac{dx^+}{u^2}. \quad (3.24)$$

It generates precisely the deformation (3.21) and leaves unperturbed the NS field, $H_{[3]} = dB = -\frac{2}{\sqrt{k}} \omega_{[3]}$.

As a conclusion, the AdS_3 plus plane-wave gravitational background is described in terms of an exact conformal sigma model, that carries two extra background fields: a NS three-form and an electromagnetic two-form. Similarly to the symmetric parabolic deformation [IKP03], the present asymmetric one can be used to construct a space-time supersymmetric background. The $SL(2, \mathbb{R})_k$ -CFT treatment of the latter deformation would need the knowledge of the parabolic characters of the affine algebra, not available at present.

As already stressed for the elliptic deformation (end of Sec. 3.1), the residual affine symmetry leaves the possibility for an extra, purely gravitational, symmetric marginal deformation. Although the systematic analysis of the full AdS_3 landscape is beyond the

present scope, we would like to quote the effect of such a deformation on the parabolic line. The perturbation which is turned on is $\sim \int d^2z J^+ \bar{J}^+$ (the currents are given in Eqs. (C.7) and Tab. 4), with parameter $1/M^2$. In the absence of electromagnetic background [IKP03], this deformation connects the NS5/F1 background to the pure NS5 dilatonic solution. Here it is performed on top of the asymmetric one, which introduces an electromagnetic wave, and we find:

$$ds^2 = k \left[\frac{du^2}{u^2} + \frac{dx^+ dx^-}{u^2 + 1/M^2} - 2H^2 \left(\frac{dx^+}{u^2 + 1/M^2} \right)^2 \right], \quad (3.25a)$$

$$B = \frac{k}{2} \frac{1}{u^2 + 1/M^2} dx^+ \wedge dx^-, \quad (3.25b)$$

$$e^{2\Phi} = k g_s^2 \frac{u}{u^2 + 1/M^2}, \quad (3.25c)$$

plus an electromagnetic field:

$$A = \sqrt{\frac{2k}{k_G}} H \frac{dx^+}{u^2 + 1/M^2}. \quad (3.25d)$$

At $M^2 \rightarrow \infty$, we recover the solution (3.21) and (3.24), whereas at $M^2 \rightarrow 0$ the present solution asymptotes the linear dilaton background. Therefore, in an NS5/F1 setup, the deformation at hand may be relevant for investigating the holography of little string theories [ABKS98].

3.4 A remark on discrete identifications

Before closing the chapter on AdS₃, we would like to discuss briefly the issue of discrete identifications. So far we have focused on continuous deformations as a procedure for generating new backgrounds. It appeared that under specific symmetry and integrability requirements, the moduli of such deformations are unique, and the corresponding backgrounds are described in terms of exact two-dimensional conformal models.

In the presence of isometries, discrete identifications provide alternatives for creating new backgrounds. Those have the *same* local geometry, but differ with respect to their global properties. Whether these identifications can be implemented as orbifolds, at the level of the underlying two-dimensional model is very much dependent on each specific case.

For AdS₃, the most celebrated geometry obtained by discrete identification is certainly the BTZ black hole [BTZ92]. The discrete identifications are made along the integral lines of the following Killing vectors (defined in Eqs. (C.5)):

$$\text{extremal case : } \xi = 2vr_+ R_2 - \iota(R_1 - R_3) - \iota(L_1 + L_3), \quad (3.26a)$$

$$\text{non-extremal case : } \xi' = \iota(r_+ + r_-) R_2 - \iota(r_+ - r_-) L_2, \quad (3.26b)$$

where r_+ and r_- are the outer and inner horizons, coinciding for the extremal black hole. Many subtleties arise, which concern *e.g.* the appearance of closed time-like curves; a comprehensive analysis of these issues can be found in [BHTZ93]. At the string theory

level, this projection is realized as an *orbifold*, which amounts to realize the projection of the string spectrum onto invariant states and to add twisted sectors [NS98, HKV02].

Besides the BTZ solution, other locally AdS_3 geometries are obtained, by imposing identification under purely left (or right) isometries, refereed to as self-dual (or anti-self-dual) metrics. These were studied in [CH94]. Their classification and isometries are exactly those of the asymmetric deformations studied in the present chapter. The Killing vector used for the identification is (A) time-like (elliptic), (B) space-like (hyperbolic) or (C) null (parabolic), and the isometry group is $U(1) \times SL(2, \mathbb{R})$. It was pointed out in [CH94] that the resulting geometry was free of closed time-like curves only in the case (B).

We could clearly combine the continuous deformations with the discrete identifications – whenever these are compatible – and generate thereby new backgrounds. This offers a large variety of possibilities that deserve further investigation (issue of horizons, closed time-like curves . . .). One can *e.g.* implement the non-extremal BTZ identifications (3.26b) on the hyperbolic continuous deformation (3.10) since the isometry group of the latter contains the vectors of the former.

Furthermore, it can be used to generate new interesting solutions of Einstein equations by performing discrete identifications in the spirit of [CH94]. In the latter, the residual isometry group was precisely the one under consideration here, so that our deformation is compatible with their discrete identification.

Similarly, the extremal BTZ identifications (3.26a) are compatible with the isometries of the parabolic deformation (3.21). One could thus create extremal black holes out of this AdS/plane-wave solution.

4. Limiting geometries: AdS_2 and H_2

We have analyzed in Sec. 2.2 the behaviour of the magnetic deformation of $SU(2)_k$, at some critical (or boundary) value of the modulus H^2 , where the background factorizes as $\mathbb{R} \times S^2$ with vanishing NS three-form and finite magnetic field. We would like to address this question for the asymmetric deformations of the $SL(2, \mathbb{R})_k$ model and show the existence of limiting situations where the geometry indeed factorizes, in agreement with the expectations following the general analysis of App. A.

In general, exact deformations of string backgrounds as those we are considering here, are carried by a modulus that controls the string spectrum. The modulus might exhibit critical or boundary values, where a whole sector of states becomes massless or infinitely massive, and decouples. Such a phenomenon corresponds to the decompactification of some compact coordinate, which decouples from the remaining geometry. This is exactly what happens for the magnetic deformation of $SU(2)_k$ WZW where the S^3 is more and more squashed, and eventually shrinks to a $\mathbb{R} \times S^2$. Not only is the geometry affected, but the antisymmetric tensor disappears in this process, and the S^2 is left with a finite magnetic field that ensures the consistency of the string theory.

What can we expect in the framework of the $SL(2, \mathbb{R})_k$ asymmetric deformations? Any limiting geometry must have the generic $U(1) \times SL(2, \mathbb{R})_k$ isometry that translates the affine symmetry of the conformal model. If a line decouples, it accounts for the $U(1)$,

and the remaining two-dimensional surface must be $SL(2, \mathbb{R})$ -invariant. Three different situations may arise: AdS_2 , H_2 or dS_2 . Anti de Sitter in two dimensions is Lorentzian with negative curvature; the hyperbolic plane H_2 (also called Euclidean anti de Sitter) is Euclidean with negative curvature; de Sitter space is Lorentzian with positive curvature.

Three deformations are available for AdS_3 and these have been analyzed in Sec. 3. For unitary string theory, all background fields must be real and consequently $H^2 > 0$ is the only physical regime. In this regime, only the hyperbolic (electric) deformation exhibits a critical behaviour at $H_{\max}^2 = 1/2$. For $H^2 < 1/2$, the deformation at hand is a Lorentzian manifold with no closed time-like curves (see Sec. 3.2). When $H^2 > 1/2$, $\det \mathbf{g} > 0$ and *two* time-like directions appear. At $H^2 = H_{\max}^2$, $\det \mathbf{g}$ vanishes, and this is the signature that some direction indeed decompactifies.

We proceed therefore as in Sec. 2.2, and define a rescaled coordinate in order to keep the decompactifying direction into the geometry and follow its decoupling:

$$y = \sqrt{\frac{k}{2} \left(\frac{1}{2} - H^2 \right)} x . \quad (4.1)$$

The metric and volume form now read:

$$ds^2 = dy^2 + \frac{k}{4} [dr^2 - (1 + 2H^2 \sinh^2 r) d\tau^2] + \sqrt{k(1 - 2H^2)} \sinh r d\tau dy \quad (4.2)$$

and

$$\omega_{[3]} = \frac{k}{4} \cosh r dr \wedge d\tau \wedge dy. \quad (4.3)$$

For H^2 close to H_{\max}^2 , the y -direction factorizes

$$ds^2 \xrightarrow{H^2 \rightarrow H_{\max}^2} dy^2 + \frac{k}{4} [dr^2 - \cosh^2 r d\tau^2]. \quad (4.4)$$

The latter expression captures the phenomenon we were expecting:

$$AdS_3 \xrightarrow{H^2 \rightarrow H_{\max}^2} \mathbb{R} \times AdS_2. \quad (4.5)$$

It also shows that the two-dimensional anti de Sitter has radius $\sqrt{k/4}$ and supports entirely the curvature of the limiting geometry, $R = -8/k$ (see expression (3.11)).

The above analysis shows that, starting from the $SL(2, \mathbb{R})_k$ WZW model, there is a line of continuous exact deformation (driven by a (chromo)electric field) that leads to a conformal model at the boundary of the modulus H^2 . This model consists of a free non-compact boson times a geometric coset $AdS_2 \equiv SL(2, \mathbb{R})/U(1)$, with a finite electric field:

$$F = \sqrt{\frac{k}{k_G}} \cosh r dr \wedge d\tau \quad (4.6)$$

and vanishing NS three-form background. The underlying geometric structure that makes this phenomenon possible is that AdS_3 can be considered as a non-trivial S^1 fibration over an AdS_2 base. The radius of the fiber couples to the electric field, and vanishes at H_{\max}^2 .

The important result is that this enables us to promote the geometric coset AdS_2 to an exact string vacuum.

We would like finally to comment on the fate of dS_2 and H_2 geometries, which are both $SL(2, \mathbb{R})$ -symmetric. De Sitter and hyperbolic geometries are not expected to appear in physical regimes of string theory. The H_3 sigma-model, for example, is an exact conformal field theory, with imaginary antisymmetric tensor background though [Gaw91, Tes99]. Similarly, imaginary NS background is also required for de Sitter vacua to solve the low-energy equations (D.11). It makes sense therefore to investigate regimes with $H^2 < 0$, where the electric or magnetic backgrounds are indeed imaginary.

The elliptic (magnetic) deformation studied in Sec. 3.1 exhibits a critical behaviour in the region of negative H^2 , where the geometry does not contain closed time-like curves. The critical behaviour appears at the minimum value $H_{\min}^2 = -1/2$, below which the metric becomes Euclidean. The vanishing of $\det \mathbf{g}$ at this point of the deformation line, signals the decoupling of the time direction. The remaining geometry is nothing but a two-dimensional hyperbolic plane H_2 . It is Euclidean with negative curvature $R = -8/k$ (see Eq. (3.2) with $L^2 = k$).

All this can be made more precise by introducing a rescaled time coordinate:

$$T = \sqrt{\frac{k}{2} \left(\frac{1}{2} + H^2 \right)} t. \quad (4.7)$$

The metric and volume form now read:

$$ds^2 = -dT^2 + \frac{k}{4} [d\rho^2 + (1 - 2H^2 \sinh^2 \rho) d\phi^2] - \sqrt{k(1 + 2H^2)} \sinh \rho d\phi dT \quad (4.8)$$

and

$$\omega_{[3]} = \frac{k}{4} \cosh \rho d\rho \wedge d\phi \wedge dT. \quad (4.9)$$

For H^2 close to H_{\min}^2 , the T -direction factorizes

$$ds^2 \xrightarrow{H^2 \rightarrow H_{\min}^2} -dT^2 + \frac{k}{4} [d\rho^2 + \cosh^2 \rho d\phi^2]. \quad (4.10)$$

The latter expression proves the above statement:

$$\text{AdS}_3 \xrightarrow{H^2 \rightarrow H_{\min}^2} \mathbb{R} \times H_2, \quad (4.11)$$

and the two-dimensional hyperbolic plane has radius $\sqrt{k/4}$.

Our analysis finally shows that the continuous line of exactly marginal (chromo)magnetic deformation of the $SL(2, \mathbb{R})$ conformal model, studied in Sec. 3.1, has a boundary at $H^2 = -1/2$ where its target space is a free time-like coordinate times a hyperbolic plane. The price to pay for crossing $H^2 = 0$ is an imaginary magnetic field, which at $H^2 = -1/2$ reads:

$$F = \sqrt{-\frac{k}{k_G}} \cosh \rho d\phi \wedge d\rho. \quad (4.12)$$

The NS field strength vanishes at this point, and the geometric origin of the decoupling at hand is again the Hopf fibration of the AdS_3 in terms of an H_2 .

5. $\text{AdS}_2 \times S^2$

The $\text{AdS}_2 \times S^2$ geometry appeared first in the context of Reissner–Nordström black holes. The latter are solutions of Maxwell–Einstein theory in four dimensions, describing charged, spherically symmetric black holes. For a black hole of mass M and charge Q , the solution reads:

$$ds^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)} + r^2 d\Omega_2^2, \quad (5.1a)$$

$$F = \frac{Q}{r^2} dt \wedge dr \quad \text{with} \quad r_{\pm} = G_4 \left(M \pm \sqrt{M^2 - Q^2} \right); \quad (5.1b)$$

r_+ and r_- are the outer and inner horizons, and G_4 is Newton’s constant in four dimensions.

In the extremal case, $r_+ = r_- = r_0$ ($M^2 = Q^2$), and the metric approaches the $\text{AdS}_2 \times S^2$ geometry in the near-horizon¹² limit $r \rightarrow r_0$. This solution can of course be embedded in various four-dimensional compactifications of string theory, and will be supersymmetric in the extremal case (see e.g. [You99] for a review). In this paper we are dealing with some heterotic compactification.

Notice that the $\text{AdS}_2 \times S^2$ geometry also appears in type IIB superstring theory, but with RR backgrounds [FKS95]. The black hole solution is obtained by wrapping D3-branes around 3-cycles of a Calabi–Yau three-fold; in the extremal limit, one obtains the $\text{AdS}_2 \times S^2$ solution, but at the same time the CY moduli freeze to some particular values. A hybrid Green–Schwarz sigma-model action for this model has been presented in [BBH⁺00] (see also [Ver04] for AdS_2). The interest for $\text{AdS}_2 \times S^2$ space–time is motivated by the fact that it provides an interesting candidate for AdS/CFT correspondence [Mal98]. In the present case the dual theory should correspond to some superconformal quantum mechanics [BPS98, C⁺98, GT99, CCKM01].

5.1 The spectrum

As a first step in the computation of the $\text{AdS}_2 \times S^2$ string spectrum, we must determine the spectrum of the AdS_2 factor, by using the same limiting procedure as in Sec. 2.3 for the sphere. The spectrum of the electrically deformed AdS_3 , is displayed in Eqs. (3.19) and (3.20). The AdS_2 limit is reached for $\cosh x \rightarrow \infty$, which leads to the following constraint on the charges of the primary fields:

$$\bar{n} + \frac{h}{2} + \sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) = 0. \quad (5.2)$$

In contrast with the S^2 case, since μ is any real number – irrespectively of the kind of $SL(2, \mathbb{R})$ representation – there is *no extra* quantization condition for the level to make

¹²With the near-horizon coordinates $U = (1 - r_0/r)^{-1}$ and $T = t/r_0$, the near-horizon geometry is

$$ds^2 = r_0^2 \left(-\frac{dT^2}{U^2} + \frac{dU^2}{U^2} + d\Omega_2^2 \right).$$

Both AdS_2 and S^2 factors have the same radius r_0 .

this limit well-defined. In this limit, the extra $U(1)$ decompactifies as usual and can be removed. Plugging the constraint (5.2) in the expressions for the dimensions of the affine primaries, we find

$$L_0 = -\frac{j(j-1)}{k} - \frac{1}{2} \left(\bar{n} + \frac{h}{2} \right)^2 - \frac{1}{2} \left(n + \frac{a}{2} \right)^2, \quad (5.3a)$$

$$\bar{L}_0 = -\frac{j(j-1)}{k}. \quad (5.3b)$$

In addition to the original AdS_3 spectrum, Eqs. (3.17) and (3.18), the right-moving part contain an extra fermionic lattice corresponding to the states charged under the electric field. Despite the absence of $N = 2$ superconformal symmetry due to the Lorentzian signature, the theory has a “fermion-number” left symmetry, corresponding to the current:

$$J = \psi^1 \psi^3 + \frac{2}{k} (J^2 + \psi^1 \psi^3). \quad (5.4)$$

The charges of the primaries (5.3) are

$$\mathcal{Q}_F = n + \frac{a}{2} - \sqrt{\frac{2}{k}} \left(\bar{n} + \frac{h}{2} \right). \quad (5.5)$$

5.2 $\text{AdS}_2 \times S^2 \times \mathcal{M}$ and space–time supersymmetry

Let us now consider the complete heterotic string background which consists of the $\text{AdS}_2 \times S^2$ space–time times an $N = 2$ internal conformal field theory \mathcal{M} , that we will assume to be of central charge $\hat{c} = 6$ and with integral R -charges. Examples of thereof are toroidal or flat-space compactifications, as well as Gepner models [Gep88].

The levels k of $SU(2)$ and \hat{k} of $SL(2, \mathbb{R})$ are such that the string background is critical:

$$\hat{c} = \frac{2(k-2)}{k} + \frac{2(\hat{k}+2)}{\hat{k}} = 4 \implies k = \hat{k}. \quad (5.6)$$

This translates into the equality of the radii of the corresponding S^2 and AdS_2 factors, which is in turn necessary for supersymmetry. Furthermore, the charge quantization condition for the two-sphere (Sec. 2.2) restricts further the level to $k = 2p^2$, $p \in \mathbb{N}$.

In this system the total fermionic charge is

$$\mathcal{Q} = n + \frac{a}{2} - \frac{N - h/2}{p} + n' + \frac{a}{2} - \frac{\bar{n}' + h/2}{p} + \mathcal{Q}_{\mathcal{M}}. \quad (5.7)$$

Hence, assuming that the internal $N = 2$ charge $\mathcal{Q}_{\mathcal{M}}$ is integral, further constraints on the electromagnetic charges of the theory are needed in order to achieve space–time supersymmetry. Namely, we must only keep states such that

$$N + \bar{n}' = 0 \pmod{p}. \quad (5.8)$$

This projection is some kind of generalization of Gepner models. Usually, such a projection is supplemented in string theory by new twisted sectors. We then expect that, by adding on

top of this projection the usual GSO projection on odd fermion number, one will obtain a space–time supersymmetric background. However, the actual computation would need the knowledge of hyperbolic coset characters of $SL(2, \mathbb{R})$ (i.e. Lorentzian black-hole characters), and of their modular properties. We can already observe that this “Gepner-like” orbifold keeps only states which are “dyonic” with respect to the electromagnetic field background. Notice that, by switching other fluxes in the internal theory \mathcal{M} one can describe more general projections.

6. Outlook

The main motivation of this work was to analyze the landscape of the S^3 and AdS_3 deformation. This analysis is performed from the geometrical viewpoint with the symmetry as a guideline. The deformations obtained in that way are then shown to be target spaces of exact marginal perturbations of the $SU(2)_k$ and $SL(2, \mathbb{R})_k$ WZW models.

An important corollary of our analysis is that geometric cosets like $S^2 = SU(2)/U(1)$ or $\text{AdS}_2 = SL(2, \mathbb{R})/U(1)_{\text{space-like}}$ can be realized, with appropriate (chromo) magnetic or electric fields, as exact conformal models, which hence provide new string backgrounds. They appear as limiting asymmetric marginal deformations of WZW models, and are therefore tractable conformal field theories leading to unitary strings.

The two-dimensional hyperbolic space $H_2 = SL(2, \mathbb{R})/U(1)_{\text{time-like}}$ does also appear in the same manner, although the accompanying magnetic field is imaginary. We display in Fig. 1 the summary of the various situations analyzed here.

Figure 1: We have summarized the various regimes appearing in magnetic/electric deformations of the $SU(2)_k$ and $SL(2, \mathbb{R})_k$ WZW models. Lighter bars indicate the signature, darker ones the sign of the curvature. Regions with $H^2 < 0$ have well-defined geometries with imaginary gauge field. Their string interpretation is therefore questionable.

We have presented the spectrum and the partition function of the $SU(2)_k$ magnetic deformation with $S^2 \times \mathbb{R}$ as extreme target space reached at $H_{\text{max}}^2 = 1/2$. At this value the fiber S^1 decompactifies, as we have seen in Sec. 2.2. We can go across and explore the deformation for $H^2 > 1/2$. The geometry (metric given in Eq. (2.10)), is now Lorentzian and all other fields are real. This exact solution is still a marginal deformation of S^3 , of a peculiar type though: it appears after a signature flip, has positive constant curvature (see Fig. 1) and $U(1) \times SU(2)$ isometry, where some of the $SU(2)$ Killing vectors are time-like¹³. In order to avoid trivial closed time-like curves, we must promote the angle γ to $\gamma \in \mathbb{R}$. Unfortunately this is not enough to ensure consistency, and closed time-like curves *à la* Gödel appear whenever $\cos^2 \beta > 1/2H^2$, as one can check by using standard arguments (Sec. 3).

The space–time under consideration is a *compact Gödel universe* (“compact” refers to the angle β), already discussed in the context of general relativity [DFS03a]. Our approach

¹³In the regime $H^2 > 1/2$, ∂_γ is time-like since $\|\partial_\gamma\|^2 = -k(2H^2 - 1)/4$, whereas ∂_α becomes time-like for $\cos^2 \beta > 1/2H^2$.

promotes it to the level of exact string background, and we have in principle the tools to investigate its spectrum. The latter may be obtained from the partition function (2.27) with the replacement:

$$\cosh x = \frac{\iota}{2H^2 - 1}, \quad \sinh x = \iota \sqrt{\frac{2H^2}{2H^2 - 1}}. \quad (6.1)$$

Such a straightforward approach is however questionable since it involves a sort of analytic continuation. We'll not expand further on this issue.

The hyperbolic deformation of $SL(2, \mathbb{R})_k$, leading precisely to AdS_2 , is an important achievement of this work. It is however technically involved: no description is presently available for quantities like the partition function, and our handling over the spectrum remains incomplete¹⁴. For the elliptic deformation, the partition function is available in the ‘‘Gödel regime’’ ($H^2 > 0$) where the geometry is spoiled by closed time-like curves. The continuation to the negative- H^2 region, free of closed time-like curves, is not straightforward since it requires an analytic continuation to imaginary magnetic field. The string theory is no longer unitary, but it might still be useful to investigate this regime for pure CFT purposes.

Several other non-unitary regimes appear in the deformed geometries at hand, either with Euclidean or with Lorentzian signature, and positive or negative curvature (see Fig. 1). Whether these could be connected to the H_3 or dS_3 spaces is an open question. Three-dimensional hyperbolic plane, H_3 , and de Sitter, dS_3 , are respectively negative and positive constant-curvature symmetric spaces, with Euclidean and Lorentzian signatures. The differences with respect to the S^3 or AdS_3 are that they have the opposite signature/curvature combination, and are not group manifolds: H_3 and dS_3 are $SL(2, \mathbb{C})$ cosets. A non-unitary conformal sigma model can be defined on H_3 , and considerable progress has been made to understand its structure. In the case of dS_3 , nothing similar is available, despite the growing interest for cosmological applications. Any hint in that direction could be important.

Connections of our families of spaces to H_3 or dS_3 could arise by introducing extra deformations. Since the residual affine symmetry of the asymmetric marginal deformations is $U(1)_L \times SL(2, \mathbb{R})_R$, there is room, at any point, for a extra modulus generated by a $U(1)_L \times U(1)_R \subset U(1)_L \times SL(2, \mathbb{R})_R$ bilinear. This freedom has been used *e.g* in [Isr04] in order to cure the causal problem of the magnetic deformation, by slightly deviating from the pure magnetic line. One can readily repeat this analysis in the new regimes we have presented here, as already sketched for the parabolic deformation – end of Sec. 3.3 – and further investigate the AdS_3 landscape and its connections to other three-dimensional constant-curvature spaces of physical interest.

Our analysis deserves some extra comments. We are in position now to describe $\text{AdS}_2 \times S^2$ as the target space of a two-dimensional conformal model. This supersymmetric setting is the near-horizon geometry of the extremal Reissner–Nordström four-dimensional

¹⁴Much like in the compact Gödel universe quoted previously, one may try to give sense to the electric deformation for $H^2 > 1/2$. There, we have two times (see Fig. 1). We can however perform an overall flip to reverse the signature $\{- - +\} \rightarrow \{+ + -\}$ and obtain a well-defined geometry. As usual, the important issue is to implement all the sign flips and analytic continuations at the level of the CFT.

black hole. In our approach, this geometry arises as a marginal deformation of the near-horizon geometry of the NS5/F1 set up ($\text{AdS}_3 \times S^3$). The crucial question that remains to be answered is how to deviate from this supersymmetric geometry towards a non-extremal configuration of the charged four-dimensional black hole.

We should also stress that the new string backgrounds we have constructed may have interesting applications for holography. For example, the AdS/plane-wave background (3.21) is a superposition of two solutions with known holographic interpretation. The line of geometries relating AdS_3 and AdS_2 (hyperbolic deformation), which are both candidates for AdS/CFT, can possibly have some holographic dual for all values of the modulus H^2 . It is important to remark that all the families of conformal field theories considered in this paper preserve one chiral $SL(2, \mathbb{R})$ affine algebra, hence allow to construct one space-time Virasoro algebra by using the method given in [GKS98].

Finally, concerning the general technique that we have used, it clearly opens up new possibilities for compactifications. Asymmetric deformations can be performed on any group- G manifold. The high residual symmetry $G \times U(1)^{\text{rank } G}$ forces the geometry along the line as well as at the boundary of the modulus. There, we are naturally led to the geometric coset $G/U(1)^{\text{rank } G}$ times rank G decoupled, non compact, free bosons. A magnetic field drives this asymmetric deformation in the case of compact groups, whereas in non-compact cases a variety of situations can arise as we showed for the $SL(2, \mathbb{R})$.

Acknowledgments

We have enjoyed very useful discussions with C. Bachas, I. Bakas, E. Cremmer, B. Julia, E. Kiritsis and J. Troost.

A. Geometric deformations

In this appendix, we would like to present some selected results about geometric deformations. Since most of the material discussed in this paper deals with deformations of backgrounds that preserve part of the original isometries, it is interesting to understand how such deformations can be implemented in a controllable way. Whether these are the most general, or can be promoted to exact marginal lines in the underlying two-dimensional sigma model, are more subtle issues that we analyze in the main part of the paper for S^3 and AdS_3 .

Consider a manifold with metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. We assume the existence of an isometry generated by a Killing vector ξ . The components of its dual form are $\varpi_\alpha = g_{\alpha\beta} \xi^\beta$. We will consider the following family of geometries:

$$g_{\alpha\beta}^{\text{def}} = g_{\alpha\beta} + f(\|\xi\|^2) \varpi_\alpha \varpi_\beta \quad (\text{A.1})$$

with f an arbitrary function of the norm of the Killing vector. Each member of the family corresponds to a specific choice of f .

Deformation (A.1) provides a simple and straightforward recipe for perturbing a background in a way that keeps under control its isometries. Indeed, the vector ξ is still an

isometry generator ($\mathcal{L}_\xi \|\xi\|^2 = 0$; $\mathcal{L}_\xi \varpi = 0$). So are all Killing vectors which commute with ξ , whereas for a general $f(\|\xi\|^2)$ those of the original isometry group which do not commute with ξ are no longer Killing vectors. Therefore, the “deformed” isometry group contains only a subgroup of the simple component to which ξ belongs, times the other simple components, if any. These symmetry properties hold for generic functions $f(\|\xi\|^2)$. Accidental enhancements can occur though, where a larger subgroup of the original group is restored (at least locally).

The deformed dual form of ξ is computed by using (A.1):

$$\varpi_\alpha^{\text{def}} = \varpi_\alpha (1 + \|\xi\|^2 f(\|\xi\|^2)). \quad (\text{A.2})$$

Clearly, the symmetry requirement is not strong enough to reduce substantially the freedom of the deformation. We would like to focus on a specific subset of deformations for which

$$h = -\|\xi\|^2 f(\|\xi\|^2) \quad (\text{A.3})$$

is a real constant. This family is interesting for several reasons. It is stable under repetition of the deformation: by repeating this deformation we stay in the same class, but reach a different point of it. This is a sort of integrability property that leads to a *one-parameter family* of continuous deformations. The value $h = 1$ is a *critical* value of the deformation parameter. At this value the metric degenerates. This is a sign for the decoupling of one dimension. For $h > 1$, the signature flips and the decoupled dimension reenters the geometry, with reversed signature though.

In the presence of Lorentzian geometries, light-like Killing vectors can also be used. For those, Eq. (A.1) is *necessarily* of the form

$$g_{\alpha\beta}^{\text{def}} = g_{\alpha\beta} - h\varpi_\alpha\varpi_\beta \text{ if } \|\xi\|^2 = 0. \quad (\text{A.4})$$

The symmetry constraint is here very powerful. Furthermore, since now $\varpi_\alpha^{\text{def}} = \varpi_\alpha$, no critical phenomenon occurs in this case, at least for *finite* values of h .

Deformations of the kind (A.1) with (A.3) arise naturally as background geometries of integrable marginal deformations of WZW models. The integrability requirement for the CFT deformation selects the above one-parameter family of geometries. In turn, these families exhibit limiting spaces.

It is useful to notice that deformations (A.1) with (A.3) become trivial if ξ is of the form $\xi = \partial_1$ with $g_{\alpha\beta}$ block diagonal: $g_{\alpha 1} = g_{11} \delta_{\alpha 1}$. In this case, $\varpi = g_{11} dx^1$, and $g_{\alpha\beta}$ is unaffected for $(\alpha, \beta) \neq (1, 1)$, whereas $g_{11}^{\text{def}} = g_{11} (1 - h)$. The net effect of the deformation is a rescaling of the coordinate x^1 . The isometry group remains unaltered, at least locally. In fact, if x^1 is an angle (ξ is a compact generator), the deformation amounts to introducing an angle deficit, which brakes globally the original isometry group to the subgroup described previously and introduces a conical singularity.

We can illustrate the latter “degenerate” situation with a simple example: the case of the two-dimensional Euclidean plane. The isometry group has three generators: one rotation and two translations. Consider the rotation and perform the deformation as in

(A.1), (A.3). The resulting manifold is a cone. Although translation generators do not commute with the rotation generator, translation invariance is still a symmetry, locally – since the only effect of the rotation is to rescale the polar angle. Translation is however broken globally. The critical value of the deformation ($h = 1$) corresponds to the maximal angle deficit, where the cone degenerates into a half-line.

Many deformations driven by $\xi_{(1)}, \xi_{(2)}, \dots$ can be performed simultaneously along the previous lines of thought. In order to keep $\xi_{(1)}, \xi_{(2)}, \dots$ in the isometry group of the deformed metric, these must commute, hence must belong to the Cartan subgroup of the original isometry group. The full deformed group also contains those among the original Killing vectors, which commute with the set $\{\xi_{(1)}, \xi_{(2)}, \dots\}$.

Curvature tensors (Riemann, Ricci, Gauss) can be computed for the deformed geometries under consideration (Eq. (A.1)), in terms of the original ones. How much they are altered depends on the left-over symmetry and on the function f . We will not expand further in this direction.

B. The three-sphere

The commutation relations for the generators of $SU(2)$ are

$$[J^1, J^2] = \imath J^3 \quad [J^2, J^3] = \imath J^1 \quad [J^3, J^1] = \imath J^2. \quad (\text{B.1})$$

A two-dimensional realization is obtained by using the standard Pauli matrices¹⁵ $J^a = \sigma^a/2$.

The Euler-angle parameterization for $SU(2)$ is defined as:

$$g = e^{\imath \frac{\gamma}{2} \sigma^3} e^{\imath \frac{\beta}{2} \sigma^1} e^{\imath \frac{\alpha}{2} \sigma^3}. \quad (\text{B.2})$$

The $SU(2)$ group manifold is a unit-radius three-sphere. A three-sphere can be embedded in flat Euclidean four-dimensional space with coordinates (x^1, x^2, x^3, x^4) , as $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = L^2$. The corresponding $SU(2)$ element g is the following:

$$g = L^{-1} \begin{pmatrix} x^4 + \imath x^2 & x^3 + \imath x^1 \\ -x^3 + \imath x^1 & x^4 - \imath x^2 \end{pmatrix}. \quad (\text{B.3})$$

In general, the invariant metric of a group manifold can be expressed in terms of the left-invariant Cartan–Maurer one-forms. In the $SU(2)$ case under consideration (unit-radius S^3),

$$\mathcal{J}^1 = \frac{1}{2} \text{tr} (\sigma^1 g^{-1} dg), \quad \mathcal{J}^2 = \frac{1}{2} \text{tr} (\sigma^2 g^{-1} dg), \quad \mathcal{J}^3 = \frac{1}{2} \text{tr} (\sigma^3 g^{-1} dg) \quad (\text{B.4})$$

and

$$ds^2 = \sum_{i=1}^3 \mathcal{J}^i \otimes \mathcal{J}^i \quad (\text{B.5})$$

¹⁵The normalization of the generators with respect to the Killing product in $\mathfrak{su}(2)$: $\kappa(X, Y) = \text{tr}(XY)$ is such that $\kappa(J^a, J^b) = 1/2$ and correspondingly the root has length squared $\psi = 2$.

The volume form reads:

$$\omega_{[3]} = \mathcal{J}^1 \wedge \mathcal{J}^2 \wedge \mathcal{J}^3. \quad (\text{B.6})$$

In the Euler-angle parameterization, Eq. (B.5) reads (for a radius- L three-sphere):

$$ds^2 = \frac{L^2}{4} (d\alpha^2 + d\gamma^2 + 2 \cos \beta d\alpha d\gamma + d\beta^2), \quad (\text{B.7})$$

whereas (B.6) leads to

$$\omega_{[3]} = \frac{L^3}{8} \sin \beta d\alpha \wedge d\beta \wedge d\gamma. \quad (\text{B.8})$$

The Levi-Civita connection has scalar curvature $R = 6/L^2$.

The isometry group of the $SU(2)$ group manifold is generated by left or right actions on g : $g \rightarrow hg$ or $g \rightarrow gh \forall h \in SU(2)$. From the four-dimensional point of view, it is generated by the rotations $\zeta_{ab} = i(x_a \partial_b - x_b \partial_a)$ with $x_a = \delta_{ab} x^b$. We list here explicitly the six generators, as well as the group action they correspond to:

$$L_1 = \frac{1}{2} (-\zeta_{32} + \zeta_{41}), \quad g \rightarrow e^{-i\frac{\lambda}{2}\sigma^1} g, \quad (\text{B.9a})$$

$$L_2 = \frac{1}{2} (-\zeta_{43} - \zeta_{12}), \quad g \rightarrow e^{i\frac{\lambda}{2}\sigma^2} g, \quad (\text{B.9b})$$

$$L_3 = \frac{1}{2} (-\zeta_{31} - \zeta_{42}), \quad g \rightarrow e^{i\frac{\lambda}{2}\sigma^3} g, \quad (\text{B.9c})$$

$$R_1 = \frac{1}{2} (\zeta_{41} + \zeta_{32}), \quad g \rightarrow g e^{i\frac{\lambda}{2}\sigma^1}, \quad (\text{B.9d})$$

$$R_2 = \frac{1}{2} (-\zeta_{43} + \zeta_{12}), \quad g \rightarrow g e^{i\frac{\lambda}{2}\sigma^2}, \quad (\text{B.9e})$$

$$R_3 = \frac{1}{2} (\zeta_{31} - \zeta_{42}), \quad g \rightarrow g e^{i\frac{\lambda}{2}\sigma^3}. \quad (\text{B.9f})$$

Both sets satisfy the algebra (B.1). The norms squared of the Killing vectors are all equal to $L^2/4$.

The currents of the $SU(2)_k$ WZW model are easily obtained as:

$$J^i = -k \operatorname{tr} (\iota \sigma^i \partial g g^{-1}) \quad \bar{J}^i = -k \operatorname{tr} (\iota \sigma^i g^{-1} \bar{\partial} g), \quad (\text{B.10})$$

where $L = \sqrt{k}$, at the classical level. Explicit expressions are given in Tab. 1.

C. AdS₃

The commutation relations for the generators of the $SL(2, \mathbb{R})$ algebra are

$$[J^1, J^2] = -\iota J^3 \quad [J^2, J^3] = \iota J^1 \quad [J^3, J^1] = \iota J^2. \quad (\text{C.1})$$

The sign in the first relation is the only difference with respect to the $SU(2)$ in Eq. (B.1).

The three-dimensional anti-de-Sitter space is the universal covering of the $SL(2, \mathbb{R})$ group manifold. The latter can be embedded in a Lorentzian flat space with signature $(-, +, +, -)$ and coordinates (x^0, x^1, x^2, x^3) :

$$g = L^{-1} \begin{pmatrix} x^0 + x^2 & x^1 + x^3 \\ x^1 - x^3 & x^0 - x^2 \end{pmatrix}, \quad (\text{C.2})$$

sector	Killing vector	Current
left moving	$\frac{\sin \gamma}{\sin \beta} \partial_\alpha + \cos \gamma \partial_\beta - \frac{\sin \gamma}{\tan \beta} \partial_\gamma$	$k (\sin \beta \sin \gamma \partial_\alpha + \cos \gamma \partial_\beta)$
	$\frac{\cos \gamma}{\sin \beta} \partial_\alpha - \sin \gamma \partial_\beta - \frac{\cos \gamma}{\tan \beta} \partial_\gamma$	$k (\cos \gamma \sin \beta \partial_\alpha - \sin \gamma \partial_\beta)$
	∂_γ	$k (\partial_\gamma + \cos \beta \partial_\alpha)$
right moving	$-\frac{\sin \alpha}{\tan \beta} \partial_\alpha + \cos \alpha \partial_\beta + \frac{\sin \alpha}{\sin \beta} \partial_\gamma$	$k (\cos \alpha \bar{\partial}_\beta + \sin \alpha \sin \beta \bar{\partial}_\gamma)$
	$\frac{\cos \alpha}{\tan \beta} \partial_\alpha + \sin \alpha \partial_\beta - \frac{\cos \alpha}{\sin \beta} \partial_\gamma$	$k (\sin \alpha \bar{\partial}_\beta - \cos \alpha \sin \beta \bar{\partial}_\gamma)$
	∂_α	$k (\bar{\partial}_\alpha + \cos \beta \bar{\partial}_\gamma)$

Table 1: Killing vectors $\{\iota L_1, \iota L_2, \iota L_3\}$ and $\{\iota R_1, \iota R_2, \iota R_3\}$, and holomorphic and anti-holomorphic currents (as defined in Eqs. (B.9) and (B.10)) in Euler angles.

where L is the radius of AdS_3 . One can again introduce Euler-like angles

$$g = e^{i(\tau+\phi)\sigma_2/2} e^{\rho\sigma_1} e^{i(\tau-\phi)\sigma_2/2}, \quad (\text{C.3})$$

which provide good global coordinates for AdS_3 when $\tau \in]-\infty, +\infty[$, $\rho \in [0, \infty[$, and $\phi \in [0, 2\pi[$.

An invariant metric (see Eq. (B.5)) can be introduced on AdS_3 . In Euler angles, the latter reads:

$$ds^2 = L^2 [-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2]. \quad (\text{C.4})$$

The Ricci scalar of the corresponding Levi-Civita connection is $R = -6/L^2$.

The isometry group of the $SL(2, \mathbb{R})$ group manifold is generated by left or right actions on g : $g \rightarrow hg$ or $g \rightarrow gh \forall h \in SL(2, \mathbb{R})$. From the four-dimensional point of view, it is generated by the Lorentz boosts or rotations $\zeta_{ab} = i(x_a \partial_b - x_b \partial_a)$ with $x_a = \eta_{ab} x^b$. We list here explicitly the six generators, as well as the group action they correspond to:

$$L_1 = \frac{1}{2} (\zeta_{32} - \zeta_{01}), \quad g \rightarrow e^{-\frac{\lambda}{2} \sigma^1} g, \quad (\text{C.5a})$$

$$L_2 = \frac{1}{2} (-\zeta_{31} - \zeta_{02}), \quad g \rightarrow e^{-\frac{\lambda}{2} \sigma^3} g, \quad (\text{C.5b})$$

$$L_3 = \frac{1}{2} (\zeta_{03} - \zeta_{12}), \quad g \rightarrow e^{i\frac{\lambda}{2} \sigma^2} g, \quad (\text{C.5c})$$

$$R_1 = \frac{1}{2} (\zeta_{01} + \zeta_{32}), \quad g \rightarrow g e^{\frac{\lambda}{2} \sigma^1}, \quad (\text{C.5d})$$

$$R_2 = \frac{1}{2} (\zeta_{31} - \zeta_{02}), \quad g \rightarrow g e^{-\frac{\lambda}{2} \sigma^3}, \quad (\text{C.5e})$$

$$R_3 = \frac{1}{2} (\zeta_{03} + \zeta_{12}), \quad g \rightarrow g e^{i\frac{\lambda}{2} \sigma^2}. \quad (\text{C.5f})$$

Both sets satisfy the algebra (C.1). The norms of the Killing vectors are the following:

$$\|\iota L_1\|^2 = \|\iota R_1\|^2 = \|\iota L_2\|^2 = \|\iota R_2\|^2 = -\|\iota L_3\|^2 = -\|\iota R_3\|^2 = \frac{L^2}{4}. \quad (\text{C.6})$$

Moreover $L_i \cdot L_j = 0$ for $i \neq j$ and similarly for the right set. Left vectors are not orthogonal to right ones.

The isometries of the $SL(2, \mathbb{R})$ group manifold turn into symmetries of the $SL(2, \mathbb{R})_k$ WZW model, where they are realized in terms of conserved currents¹⁶:

$$J^1(z) \pm J^3(z) = -k \operatorname{tr} \left((\sigma^1 \mp i\sigma^2) \partial g g^{-1} \right), \quad J^2(z) = -k \operatorname{tr} \left(\sigma^3 \partial g g^{-1} \right), \quad (\text{C.7a})$$

$$\bar{J}^1(\bar{z}) \pm \bar{J}^3(\bar{z}) = k \operatorname{tr} \left((\sigma^1 \pm i\sigma^2) g^{-1} \bar{\partial} g \right), \quad \bar{J}^2(\bar{z}) = -k \operatorname{tr} \left(\sigma^3 g^{-1} \bar{\partial} g \right). \quad (\text{C.7b})$$

At the quantum level, these currents, when properly normalized, satisfy the following affine $SL(2, \mathbb{R})_k$ OPA¹⁷:

$$J^3(z)J^3(0) \sim -\frac{k}{2z^2}, \quad (\text{C.8a})$$

$$J^3(z)J^\pm(0) \sim \pm \frac{J^\pm}{z}, \quad (\text{C.8b})$$

$$J^+(z)J^-(0) \sim \frac{2J^3}{z} - \frac{k}{z^2}, \quad (\text{C.8c})$$

and similarly for the right movers. The central charge of the enveloping Virasoro algebra is $c = 3 + 6/(k - 2)$.

We will introduce three different coordinate systems where the structure of AdS_3 as a Hopf fibration is more transparent. They are explicitly described in the following.

- The (ρ, t, ϕ) coordinate system used to describe the magnetic deformation in Sec. 3.1 is defined as follows:

$$\begin{cases} \frac{x_0}{L} = \cosh \frac{\rho}{2} \cosh \frac{\phi}{2} \cos \frac{t}{2} - \sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \sin \frac{t}{2} \\ \frac{x_1}{L} = -\sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} - \cosh \frac{\rho}{2} \sinh \frac{\phi}{2} \sin \frac{t}{2} \\ \frac{x_2}{L} = -\cosh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} + \sinh \frac{\rho}{2} \cosh \frac{\phi}{2} \sin \frac{t}{2} \\ \frac{x_3}{L} = -\sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} - \cosh \frac{\rho}{2} \cosh \frac{\phi}{2} \sin \frac{t}{2}. \end{cases} \quad (\text{C.9})$$

The metric (B.5) reads:

$$ds^2 = \frac{L^2}{4} (d\rho^2 + d\phi^2 - dt^2 - 2 \sinh \rho dt d\phi) \quad (\text{C.10})$$

and the corresponding volume form is:

$$\omega_{[3]} = \frac{L^3}{8} \cosh \rho \cdot d\rho \wedge d\phi \wedge dt \quad (\text{C.11})$$

Killing vectors and currents are given in Tab. 2. It is worth to remark that this coordinate system is such that the t -coordinate lines coincide with the integral curves of the Killing vector ιL_3 , whereas the ϕ -lines are the curves of ιR_2 .

¹⁶When writing actions a choice of gauge for the NS potential is implicitly made, which breaks part of the symmetry: boundary terms appear in the transformations. These must be properly taken into account in order to reach the conserved currents. Although the expressions for the latter are not unique, they can be put in an improved-Noether form, in which they have only holomorphic (for L_i 's) or anti-holomorphic (for R_j 's) components.

¹⁷In some conventions the level is $x = -k$. This allows to unify commutation relations for the affine $SL(2, \mathbb{R})_x$ and $SU(2)_x$ algebras. Unitarity demands $x < -2$ for the former and $0 < x$ with integer x for the latter.

- The (r, x, τ) coordinate system used to describe the electric deformation in Sec. 3.2 is defined as follows:

$$\begin{cases} \frac{x_0}{L} &= \cosh \frac{r}{2} \cosh \frac{x}{2} \cos \frac{\tau}{2} + \sinh \frac{r}{2} \sinh \frac{x}{2} \sin \frac{\tau}{2} \\ \frac{x_1}{L} &= -\sinh \frac{r}{2} \cosh \frac{x}{2} \cos \frac{\tau}{2} + \cosh \frac{r}{2} \sinh \frac{x}{2} \sin \frac{\tau}{2} \\ \frac{x_2}{L} &= -\cosh \frac{r}{2} \sinh \frac{x}{2} \cos \frac{\tau}{2} - \sinh \frac{r}{2} \cosh \frac{x}{2} \sin \frac{\tau}{2} \\ \frac{x_3}{L} &= \sinh \frac{r}{2} \sinh \frac{x}{2} \cos \frac{\tau}{2} - \cosh \frac{r}{2} \cosh \frac{x}{2} \sin \frac{\tau}{2}. \end{cases} \quad (\text{C.12})$$

For $\{r, x, \tau\} \in \mathbb{R}^3$, this patch covers exactly once the whole AdS_3 , and is regular everywhere [CH94]. The metric is then given by

$$ds^2 = \frac{L^2}{4} (dr^2 + dx^2 - d\tau^2 + 2 \sinh r dx d\tau) \quad (\text{C.13})$$

and correspondingly the volume form is

$$\omega_{[3]} = \frac{L^3}{8} \cosh r dr \wedge dx \wedge d\tau. \quad (\text{C.14})$$

Killing vectors and currents are given in Tab. 3. In this case the x -coordinate lines coincide with the integral curves of the Killing vector ιR_2 , whereas the τ -lines are the curves of ιR_3 .

- The Poincaré coordinate system used in Sec. 3.3 to obtain the electromagnetic-wave background is defined by

$$\begin{cases} x^0 + x^2 &= \frac{L}{u} \\ x^0 - x^2 &= Lu + \frac{Lx^+x^-}{u} \\ x^1 \pm x^3 &= \frac{Lx^\pm}{u}. \end{cases} \quad (\text{C.15})$$

For $\{u, x^+, x^-\} \in \mathbb{R}^3$, the Poincaré coordinates cover once the $SL(2\mathbb{R})$ group manifold. Its universal covering, AdS_3 , requires an infinite number of such patches. Moreover, these coordinates exhibit a Rindler horizon at $|u| \rightarrow \infty$; the conformal boundary is at $|u| \rightarrow 0$. Now the metric reads:

$$ds^2 = \frac{L^2}{u^2} (du^2 + dx^+ dx^-) \quad (\text{C.16})$$

and the volume form:

$$\omega_{[3]} = \frac{L^3}{2u^3} du \wedge dx^- \wedge dx^+. \quad (\text{C.17})$$

In these coordinates it is simple to write certain a linear combination of the Killing vector so to obtain explicitly a light-like isometry generator. For this reason in Tab. 4 we report the $\{L_1 + L_3, L_1 - L_3, L_2, R_1 + R_3, R_1 - R_3, R_2\}$ isometry generators and the corresponding $\{J_1 + J_3, J_1 - J_3, J_2, \bar{J}_1 + \bar{J}_3, \bar{J}_1 - \bar{J}_3, \bar{J}_2\}$ currents.

sector	Killing vector	Current
left moving	$\cos t \partial_\rho + \frac{\sin t}{\cosh \rho} \partial_\phi - \sin t \tanh \rho \partial_t$ $- \sin t \partial_\rho + \frac{\cos t}{\cosh \rho} \partial_\phi - \cos t \tanh \rho \partial_t$ $- \partial_t$	$k (\cos t \partial_\rho + \cosh \rho \sin t \partial_\phi)$ $k (\cos t \cosh \rho \partial_\phi - \sin t \partial_\rho)$ $k (\partial t + \sinh \rho \partial \phi)$
right moving	$\cosh \phi \partial_\rho - \sinh \phi \tanh \rho \partial_\phi - \frac{\sinh \phi}{\cosh \rho} \partial_t$ ∂_ϕ $\sinh \phi \partial_\rho - \cosh \phi \tanh \rho \partial_\phi - \frac{\cosh \phi}{\cosh \rho} \partial_t$	$-k (\cosh \phi \bar{\partial} \rho + \cosh \rho \sinh \phi \bar{\partial} t)$ $k (\bar{\partial} \phi - \sinh \rho \bar{\partial} t)$ $k (\cosh \rho \cosh \phi \bar{\partial} t + \sinh \phi \bar{\partial} \rho)$

Table 2: Killing vectors $\{\iota L_1, \iota L_2, \iota L_3\}$ and $\{\iota R_1, \iota R_2, \iota R_3\}$, and holomorphic and anti-holomorphic currents (as defined in Eqs. (C.5) and (C.7)) for the (ρ, t, ϕ) coordinate system (elliptic base).

sector	Killing vector	Current
left moving	$\cosh x \partial_r - \sinh x \tanh r \partial_x + \frac{\sinh x}{\cosh r} \partial_\tau$ ∂_x $- \sinh x \partial_r + \cosh x \tanh r \partial_x - \frac{\cosh x}{\cosh r} \partial_\tau$	$k (\cosh x \partial r - \cosh r \sinh x \partial \tau)$ $k (\partial x + \sinh r \partial \tau)$ $k (\cosh r \cosh x \partial \tau - \sinh x \partial r)$
right moving	$- \cos \tau \partial_r + \frac{\sin \tau}{\cosh \tau} \partial_x - \sin \tau \tanh \tau \partial_\tau$ $\frac{(\cos \tau + \sin \tau \tanh r) \partial_x + (\cos \tau \sinh r - \frac{\sin \tau}{\cosh r}) \partial_\tau}{\cosh r}$ $- \partial_\tau$	$k (-\cos \tau \bar{\partial} r + \cosh r \sin \tau \bar{\partial} r)$ $k (\cos \tau \cosh r \bar{\partial} x + \sin \tau \bar{\partial} r)$ $k (\bar{\partial} \tau - \sinh r \bar{\partial} x)$

Table 3: Killing vectors $\{\iota L_1, \iota L_2, \iota L_3\}$ and $\{\iota R_1, \iota R_2, \iota R_3\}$, and holomorphic and anti-holomorphic currents (as defined in Eqs. (C.5) and (C.7)) for the (r, x, τ) coordinate system (hyperbolic base).

D. Equations of motion

The general form for the marginal deformations we have studied is given by:

$$S = \frac{1}{2\pi} \int d^2 z (G_{\mu\nu}^{\text{WZW}} + B_{\mu\nu}^{\text{WZW}}) \partial x^\mu \bar{\partial} x^\nu + \frac{\sqrt{k k_G H}}{2\pi} \int d^2 z J \bar{J}_G. \quad (\text{D.1})$$

It is not completely trivial to read off the deformed background fields that correspond to this action. In this appendix we will present a method involving a Kaluza–Klein reduction, following [HT95]. For simplicity we will consider the bosonic string with vanishing dilaton. The right-moving gauge current \bar{J}_G used for the deformation has now a left-moving partner and can hence be bosonized as $\bar{J}_G = \iota \bar{\partial} \varphi$, $\varphi(z, \bar{z})$ being interpreted as an internal degree of freedom. The sigma-model action is recast as

$$S = \frac{1}{2\pi} \int d^2 z (G_{MN} + B_{MN}) \partial x^M \bar{\partial} x^N, \quad (\text{D.2})$$

sector	Killing vector	Current
left moving	$-\partial_-$	$-2k \frac{\partial x^+}{u^2}$
	$ux^- \partial_u - u^2 \partial_+ + (x^-)^2 \partial_-$	$2k \left(2x^- \frac{\partial u}{u} - \partial x^- + (x^-)^2 \frac{\partial x^+}{u^2} \right)$
	$\frac{u}{2} \partial_u + x^- \partial_-$	$2k \left(\frac{\partial u}{u} + x^- \frac{\partial x^+}{u^2} \right)$
right moving	∂_+	$2k \frac{\bar{\partial} x^-}{u^2}$
	$-ux^+ \partial_u - (x^+)^2 \partial_+ + u^2 \partial_-$	$2k \left(-2x^+ \frac{\bar{\partial} u}{u} + \bar{\partial} x^+ - (x^+)^2 \frac{\bar{\partial} x^-}{u^2} \right)$
	$\frac{u}{2} \partial_u + x^+ \partial_+$	$2k \left(\frac{\bar{\partial} u}{u} + x^+ \frac{\bar{\partial} x^-}{u^2} \right)$

Table 4: Killing vectors, and holomorphic and anti-holomorphic currents (as defined in Eqs. (C.5) and (C.7)) in Poincaré coordinates (parabolic base). The $\{iL_1 + iL_3, iL_1 - iL_3, iL_2, iR_1 + iR_3, iR_1 - iR_3, iR_2\}$ isometry generators and the corresponding $\{J_1 + J_3, J_1 - J_3, J_2, \bar{J}_1 + \bar{J}_3, \bar{J}_1 - \bar{J}_3, \bar{J}_2\}$ currents are represented so to explicitly obtain light-like isometry generators.

where the $x^M, M = 1, \dots, 4$ embrace the group coordinates $x^\mu, \mu = 1, 2, 3$ and the internal $x^4 \equiv \varphi$:

$$x^M = \begin{pmatrix} x^\mu \\ x^4 \end{pmatrix}. \quad (\text{D.3})$$

If we split accordingly the background fields, we obtain the following decomposition:

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & A_\mu \\ A_\mu & G_{\varphi\varphi} \end{pmatrix}, \quad B_{MN} = \begin{pmatrix} B_{\mu\nu} & B_{\mu 4} \\ -B_{\mu 4} & 0 \end{pmatrix}, \quad (\text{D.4})$$

and the action becomes:

$$S = \frac{1}{2\pi} \int dz^2 \left\{ (G_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu + (A_\mu + B_{\mu 4}) \partial x^\mu \bar{\partial} \varphi + (A_\mu - B_{\mu 4}) \partial \varphi \bar{\partial} x^\mu + G_{\varphi\varphi} \partial \varphi \bar{\partial} \varphi \right\}. \quad (\text{D.5})$$

We would like to put the previous expression in such a form that space-time gauge invariance,

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda, \quad (\text{D.6})$$

$$B_{\mu 4} \rightarrow B_{\mu 4} + \partial_\mu \eta, \quad (\text{D.7})$$

is manifest. This is achieved as follows:

$$S = \frac{1}{2\pi} \int d^2 z \left\{ (\hat{G}_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu + B_{\mu 4} (\partial x^\mu \bar{\partial} \varphi - \partial \varphi \bar{\partial} x^\mu) + G_{\varphi\varphi} (\partial \varphi + A_\mu \partial x^\mu) (\bar{\partial} \varphi + A_\mu \bar{\partial} x^\mu) \right\}, \quad (\text{D.8})$$

where $\hat{G}_{\mu\nu}$ is the Kaluza–Klein metric

$$\hat{G}_{\mu\nu} = G_{\mu\nu} - G_{\varphi\varphi} A_\mu A_\nu. \quad (\text{D.9})$$

Expression (D.2) coincides with (D.1) after the following identifications:

$$\hat{G}_{\mu\nu} = G_{\mu\nu}^{\text{wzw}} - \frac{k_G}{2} A_\mu A_\nu, \quad (\text{D.10a})$$

$$B_{\mu\nu} = B_{\mu\nu}^{\text{wzw}}, \quad (\text{D.10b})$$

$$A_\mu = B_{\mu A} = H \sqrt{\frac{k}{2k_G}} J_\mu, \quad (\text{D.10c})$$

$$G_{\varphi\varphi} = \frac{k_G}{2}. \quad (\text{D.10d})$$

The various backgrounds we have found throughout this paper correspond to truly marginal deformations of wzw models. Thus, they are target spaces of exact CFT's. They solve, however, the lowest-order (in α') equations since all higher-order effects turn out to be captured in the shift $k \rightarrow k + 2$. These $\mathcal{O}(\alpha')$ equations are obtained by using the bosonic action (D.2), or equivalently by writing the heterotic string equations of motion for the fields in (D.10). They read [FT85]:

$$\delta\mathcal{C} = -R + \frac{k_G}{16} F_{\mu\nu} F^{\mu\nu}, \quad (\text{D.11a})$$

$$\beta_{\mu\nu}^G = R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} - \frac{k_G}{4} F_{\mu\rho} F_{\nu}{}^{\rho} = 0, \quad (\text{D.11b})$$

$$\beta_{\nu\rho}^B = \nabla^\mu H_{\mu\nu\rho} = 0, \quad (\text{D.11c})$$

$$\beta_{\mu}^A = \nabla^\nu F_{\mu\nu} - \frac{1}{2} F^{\nu\rho} H_{\mu\nu\rho} = 0, \quad (\text{D.11d})$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (\text{D.12})$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{k_G}{4} A_\mu F_{\nu\rho} + \text{cyclic}. \quad (\text{D.13})$$

References

- [ABKS98] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, JHEP **10**, 004 (1998), hep-th/9808149.
- [ABS90] I. Antoniadis, C. Bachas and A. Sagnotti, *GAUGED SUPERGRAVITY VACUA IN STRING THEORY*, Phys. Lett. **B235**, 255 (1990).
- [BBH⁺00] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, *Superstring theory on AdS(2) x S(2) as a coset supermanifold*, Nucl. Phys. **B567**, 61–86 (2000), hep-th/9907200.
- [BHTZ93] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, Phys. Rev. **D48**, 1506–1525 (1993), gr-qc/9302012.

- [BJKZ96] P. Berglund, C. V. Johnson, S. Kachru and P. Zaugg, *Heterotic Coset Models and (0,2) String Vacua*, Nucl. Phys. **B460**, 252–298 (1996), hep-th/9509170.
- [BPS98] H. J. Boonstra, B. Peeters and K. Skenderis, *Brane intersections, anti-de Sitter spacetimes and dual superconformal theories*, Nucl. Phys. **B533**, 127–162 (1998), hep-th/9803231.
- [BTZ92] M. Banados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, Phys. Rev. Lett. **69**, 1849–1851 (1992), hep-th/9204099.
- [C⁺98] P. Claus et al., *Black holes and superconformal mechanics*, Phys. Rev. Lett. **81**, 4553–4556 (1998), hep-th/9804177.
- [CCKM01] M. Cadoni, P. Carta, D. Klemm and S. Mignemi, *AdS(2) gravity as conformally invariant mechanical system*, Phys. Rev. **D63**, 125021 (2001), hep-th/0009185.
- [CH94] O. Coussaert and M. Henneaux, *Self-dual solutions of 2+1 Einstein gravity with a negative cosmological constant*, (1994), hep-th/9407181.
- [CHS91] J. Callan, Curtis G., J. A. Harvey and A. Strominger, *World sheet approach to heterotic instantons and solitons*, Nucl. Phys. **B359**, 611–634 (1991).
- [CS89] S. Chaudhuri and J. A. Schwartz, *A CRITERION FOR INTEGRABLY MARGINAL OPERATORS*, Phys. Lett. **B219**, 291 (1989).
- [DFS03a] N. Drukker, B. Fiol and J. Simon, *Goedel-type universes and the Landau problem*, (2003), hep-th/0309199.
- [DFS03b] N. Drukker, B. Fiol and J. Simon, *Goedel’s universe in a supertube shroud*, Phys. Rev. Lett. **91**, 231601 (2003), hep-th/0306057.
- [Dob87] V. K. Dobrev, *CHARACTERS OF THE UNITARIZABLE HIGHEST WEIGHT MODULES OVER THE N=2 SUPERCONFORMAL ALGEBRAS*, Phys. Lett. **B186**, 43 (1987).
- [DVV92] R. Dijkgraaf, H. Verlinde and E. Verlinde, *String propagation in a black hole geometry*, Nucl. Phys. **B371**, 269–314 (1992).
- [FKS95] S. Ferrara, R. Kallosh and A. Strominger, *N=2 extremal black holes*, Phys. Rev. **D52**, 5412–5416 (1995), hep-th/9508072.
- [For94] S. Forste, *A Truly marginal deformation of SL(2, R) in a null direction*, Phys. Lett. **B338**, 36–39 (1994), hep-th/9407198.
- [FR03] S. Forste and D. Roggenkamp, *Current current deformations of conformal field theories, and WZW models*, JHEP **05**, 071 (2003), hep-th/0304234.
- [FT85] E. S. Fradkin and A. A. Tseytlin, *QUANTUM STRING THEORY EFFECTIVE ACTION*, Nucl. Phys. **B261**, 1–27 (1985).
- [Gaw91] K. Gawedzki, *Noncompact WZW conformal field theories*, (1991), hep-th/9110076.
- [Gep88] D. Gepner, *SPACE-TIME SUPERSYMMETRY IN COMPACTIFIED STRING THEORY AND SUPERCONFORMAL MODELS*, Nucl. Phys. **B296**, 757 (1988).
- [GK94] A. Giveon and E. Kiritsis, *Axial vector duality as a gauge symmetry and topology change in string theory*, Nucl. Phys. **B411**, 487–508 (1994), hep-th/9303016.

- [GKS98] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS(3)*, Adv. Theor. Math. Phys. **2**, 733–780 (1998), hep-th/9806194.
- [GT99] G. W. Gibbons and P. K. Townsend, *Black holes and Calogero models*, Phys. Lett. **B454**, 187–192 (1999), hep-th/9812034.
- [HKV02] S. Hemming and E. Keski-Vakkuri, *The spectrum of strings on BTZ black holes and spectral flow in the SL(2,R) WZW model*, Nucl. Phys. **B626**, 363–376 (2002), hep-th/0110252.
- [HT95] G. T. Horowitz and A. A. Tseytlin, *A New class of exact solutions in string theory*, Phys. Rev. **D51**, 2896–2917 (1995), hep-th/9409021.
- [IKP03] D. Israël, C. Kounnas and M. P. Petropoulos, *Superstrings on NS5 backgrounds, deformed AdS(3) and holography*, JHEP **10**, 028 (2003), hep-th/0306053.
- [Isr04] D. Israël, *Quantization of heterotic strings in a Gödel/anti de Sitter spacetime and chronology protection*, JHEP **01**, 042 (2004), hep-th/0310158.
- [Joh95] C. V. Johnson, *Heterotic Coset Models*, Mod. Phys. Lett. **A10**, 549–560 (1995), hep-th/9409062.
- [Kir88] E. Kiritsis, *CHARACTER FORMULAE AND THE STRUCTURE OF THE REPRESENTATIONS OF THE N=1, N=2 SUPERCONFORMAL ALGEBRAS*, Int. J. Mod. Phys. **A3**, 1871 (1988).
- [KK95] E. Kiritsis and C. Kounnas, *Infrared behavior of closed superstrings in strong magnetic and gravitational fields*, Nucl. Phys. **B456**, 699–731 (1995), hep-th/9508078.
- [LS94] D. A. Lowe and A. Strominger, *Exact four-dimensional dyonic black holes and Bertotti-Robinson space-times in string theory*, Phys. Rev. Lett. **73**, 1468–1471 (1994), hep-th/9403186.
- [Mal98] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2**, 231–252 (1998), hep-th/9711200.
- [Mat87] Y. Matsuo, *CHARACTER FORMULA OF C < 1 UNITARY REPRESENTATION OF N=2 SUPERCONFORMAL ALGEBRA*, Prog. Theor. Phys. **77**, 793 (1987).
- [MO01] J. M. Maldacena and H. Ooguri, *Strings in AdS(3) and SL(2,R) WZW model. I*, J. Math. Phys. **42**, 2929–2960 (2001), hep-th/0001053.
- [NS98] M. Natsuume and Y. Satoh, *String theory on three dimensional black holes*, Int. J. Mod. Phys. **A13**, 1229–1262 (1998), hep-th/9611041.
- [Pet90] P. M. S. Petropoulos, *COMMENTS ON SU(1,1) STRING THEORY*, Phys. Lett. **B236**, 151 (1990).
- [RS98] M. Rooman and P. Spindel, *Goedel metric as a squashed anti-de Sitter geometry*, Class. Quant. Grav. **15**, 3241–3249 (1998), gr-qc/9804027.
- [RT83] M. J. Reboucas and J. Tiomno, *ON THE HOMOGENEITY OF RIEMANNIAN SPACE-TIMES OF GODEL TYPE*, Phys. Rev. **D28**, 1251–1264 (1983).
- [RY87] F. Ravanini and S.-K. Yang, *MODULAR INVARIANCE IN N=2 SUPERCONFORMAL FIELD THEORIES*, Phys. Lett. **B195**, 202 (1987).

- [Tes99] J. Teschner, *On structure constants and fusion rules in the $SL(2,C)/SU(2)$ WZNW model*, Nucl. Phys. **B546**, 390–422 (1999), hep-th/9712256.
- [Tse94] A. A. Tseytlin, *Conformal sigma models corresponding to gauged Wess-Zumino-Witten theories*, Nucl. Phys. **B411**, 509–558 (1994), hep-th/9302083.
- [Ver04] H. Verlinde, *Superstrings on $AdS(2)$ and superconformal matrix quantum mechanics*, (2004), hep-th/0403024.
- [VK91] N. J. Vilenkin and A. Klimyk, *Representation of Lie groups and special functions*, Kluwer Academic Publishers, Dordrecht, 1991.
- [You99] D. Youm, *Black holes and solitons in string theory*, Phys. Rept. **316**, 1–232 (1999), hep-th/9710046.

Heterotic strings on homogeneous spaces

Published in Fortsch. Phys. 53 (2005) 1030-1071, [hep-th/0412220].

Fortschr. Phys. **53**, No. 9, 1030–1071 (2005) / DOI 10.1002/prop.200510250

Heterotic strings on homogeneous spaces*

Dan Israël^{1,3}, Costas Kounnas², Domenico Orlando^{3,**}, and P. Marios Petropoulos^{3,***}

¹ Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

² Laboratoire de Physique Théorique de l'École Normale Supérieure, Unité mixte du CNRS et de l'École Normale Supérieure, UMR 8549, 24 rue Lhomond, 75231 Paris Cedex 05, France

³ Centre de Physique Théorique, École Polytechnique, Unité mixte du CNRS et de l'École Polytechnique, UMR 7644, 91128 Palaiseau, France

Received 30 May 2005, accepted 3 June 2005

Published online 16 August 2005

Key words Superstring vacua, conformal field models in string theory, p-branes.

PACS 11.25.-w, 11.25.Hf, 11.25.Mj, 11.25.Sq, 11.40.qn

We construct heterotic string backgrounds corresponding to families of homogeneous spaces as exact conformal field theories. They contain left cosets of compact groups by their maximal tori supported by NS-NS 2-forms and gauge field fluxes. We give the general formalism and modular-invariant partition functions, then we consider some examples such as $SU(2)/U(1) \sim S^2$ (already described in a previous paper) and the $SU(3)/U(1)^2$ flag space. As an application we construct new supersymmetric string vacua with magnetic fluxes and a linear dilaton.

© 2005 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

Contents

1 Motivations and summary	1031
2 Compact coset spaces: general formalism	1032
2.1 String theory on group manifolds: a reminder	1033
2.2 Asymmetric deformations	1034
2.3 Equations of motion	1037
3 Some examples	1039
3.1 The two-sphere CFT	1039
3.2 The $SU(3)/U(1)$ flag space	1041
4 Gauging	1043
4.1 The $SU(2)/U(1)$ asymmetric gauging	1044
4.2 $SU(3)/U(1)^2$	1047
5 Exact construction: partition functions	1051
5.1 The $SU(3)/U(1)^2$ flag space CFT	1051
5.2 Generalization	1054
5.3 Kazama-Suzuki decomposition vs. abelian quotient	1058
6 New linear dilaton backgrounds of heterotic strings	1059

* Research partially supported by the EEC under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104, MRTN-CT-2004-503369, and HPRN-CT-2000-00122.

** Corresponding author E-mail: orlando@cpht.polytechnique.fr

*** E-mail: marios@cpht.polytechnique.fr

© 2005 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

Fortschr. Phys. 53, No. 9 (2005) / www.fp-journal.org	1031
A Coset space geometry	1061
B Equations of motion	1062
B.1 Explicit derivation of some terms	1062
B.2 Equations of motion for the F_3 flag space	1064
C The $SU(3)$ group: an explicit parametrization	1066
D Characters of affine Lie algebras	1067
D.1 The example of $SU(3)$	1069
D.2 Modular transformations	1069
D.3 Fermionic characters	1070
References	1070

1 Motivations and summary

The search for exact string backgrounds has been a major motivation in the field for many years. Gravitational backgrounds with a clear geometric interpretation are even more important since they may provide a handle on quantum gravitational phenomena, black holes, and ultimately cosmology – for those which are time-dependent. Wess–Zumino–Witten models provide such a class of solutions, with remarkable properties. The target space is in that case a group manifold and, together with the metric, the Neveu–Schwarz antisymmetric tensor is the only background field. Both of these fields are exactly known to all orders in α' . So are the spectrum, partition function, two- and three-point functions, . . .

Wess–Zumino–Witten models appear in many physical set-ups, as near-horizon geometries of specific brane configurations. The three-sphere is part of the near-horizon geometry of N_5 NS5-branes. This is the target space of an $SU(2)_k$ super-WZW model at bosonic level $k = N_5 - 2$. Another celebrated example is that of AdS_3 . The latter appears in the NS5-brane/fundamental-string background, together with S^3 , at equal radius $L = \sqrt{\alpha' N_5}$; it is realized in terms of the $SL(2, \mathbb{R})_{\tilde{k}}$ WZW at level $\tilde{k} = k + 4$. These are important examples because of their role in the study of decoupling limits, little-string theory, holographic dualities etc. The knowledge of exact spectra, amplitudes, . . . is crucial for better understanding of these issues.

Despite the many assets of WZW models, the major limitation comes from the dimension and signature of their target spaces. When dealing with compact groups, the dimension often exceeds six (e.g. $SU(3)$ is eight-dimensional), while for non-compact groups, $SL(2, \mathbb{R})_{\tilde{k}}$ is the only example with a single time direction.

In order to reduce the dimensionality and the number of symmetries of the target space, while keeping two-dimensional conformal invariance and tractability, the usual procedure is the gauging. Gauged WZW models are realized algebraically, at the level of the chiral currents and energy–momentum tensor, by following the GKO construction [1]. Alternatively, one can work directly on the action and gauge symmetrically a subgroup $H \subset G$. For $H = U(1)$, the gauged model can even be obtained as an extreme marginal deformation of the original model, driven by a $\int d^2z J\bar{J}$ perturbation, where J and \bar{J} are the currents associated with the $U(1) \subset G$.

Target spaces of gauged WZW models are *not* usual geometric cosets G/H . Firstly, the background fields of gauged WZW receive non-trivial α' corrections¹, while geometric cosets can be assigned a well-defined metric. Secondly, the isometry groups are different. For geometric cosets, the isometry group is G , while it is H for the target space of the gauged WZW.

¹ The higher-order α' corrections are trivial for WZW models: they boil down to shifting $k \rightarrow k + g^*$ in the classical backgrounds (g^* is the dual Coxeter number of the group G).

Geometric cosets could provide alternative backgrounds, with different properties and new possibilities for accommodating six or less space dimensions, or a single time direction (in the non-compact case). Unfortunately, they have not been systematically analyzed, and were even thought to be, at most, leading-order solutions to the string equations. Although some exact solutions were identified in the past [2–4], no generic pattern for generalization was known.

The issue of geometric cosets as exact backgrounds has been recently revisited in [5]. There, it was shown that $S^2 \equiv SU(2)/U(1)$ and $\text{AdS}_2 \equiv SL(2, \mathbb{R})/U(1)_{\text{space}}$, with magnetic and electric fluxes and no dilaton, can be obtained as extreme marginal deformations of the $SU(2)_k$ and $SL(2, \mathbb{R})_{\tilde{k}}$ WZW models. In this case the background fields are exact up to the usual finite renormalization of the radius ($k \rightarrow k+2$ and $\tilde{k} \rightarrow \tilde{k}-2$) and spectra, partition functions, . . . are within reach. The marginal deformations are *asymmetric* because the right current that appears in the bilinear does not belong to the right-moving affine algebra of the group at hand.

Asymmetric marginal deformations apply to any group. The aim of the present paper is to investigate on several interesting generalizations of this method, in the case of compact groups, and make contact with *asymmetrically gauged* WZW models. We will focus in particular on the $SU(3)$ group. In this case the asymmetric marginal deformation leads to the $SU(3)/U(1)$ geometric coset, with magnetic fluxes and no dilaton. In turn, this coset is identified with the asymmetric gauging of a $U(1)^2$ in the original WZW models.

In the cases under consideration, however, more possibilities exist, which we further exploit. We examine the asymmetric gauging of the full Cartan torus $U(1)^2$. The geometric cosets obtained in this way, can be assigned two different metrics depending on the precise manner the gauging is performed, in combination with the extreme asymmetric marginal deformation. One is Kählerian and consequently no NS form survives: we obtain the flag space $F_3 = SU(3)/U(1)^2$, recognized many years ago [6] to be a leading-order solution, thanks to its Kählerian structure. The other metric is not Kählerian, and the background has both magnetic and NS fluxes. It enters into the construction of non-compact manifolds of G_2 holonomy [7].

All our solutions are exact string backgrounds with no dilaton – contrary to the usual symmetrically gauged WZW models. We can determine their spectra as well as their full partition functions.

The paper is organized as follows: first we fix the notation by reviewing some known facts about WZW models and then show how to read the background fields corresponding to an asymmetric marginal deformation of such models. We emphasize in particular the decompactification of the Cartan torus that takes place at the extremal points in moduli space (Sect. refsec:geom-constr). This formalism is then used to study the deformation of the $SU(2)$ and $SU(3)$ models (Sect. 3). In the following we introduce a different construction in which the limit deformations are identified to asymmetrically gauged WZW models [8] and the deformation is generalized so to reach the different constant-curvature structures admitted by an asymmetric G/T coset, with particular emphasis on the $SU(3)/U(1)^2$ case (Sect. 4). The next section (Sect. 5) deals with the computation of the one-loop partition functions for the asymmetric deformations leading to geometric cosets. Two different methods are proposed, one using the Kazama–Suzuki decomposition in terms of Hermitian symmetric spaces, the other via the direct deformation of the Cartan lattice of the Lie algebra corresponding to the group. In the final section (Sect. 6) we give an example of application by using these SCFT's to construct other supersymmetric exact string backgrounds such as the left-coset analogues of the NS5-branes solutions [9, 10]. They provide new holographic backgrounds of the Little String Theory type [11–13], and may be dual to non-trivial supersymmetric compactifications on manifolds with singularities. The concluding appendices contain some facts about the geometry of coset spaces, partition functions, and characters of affine Lie algebras.

2 Compact coset spaces: general formalism

In this section we will fix the notation by reviewing some well known facts about conformal field theories on group manifolds (WZW models) and give the general formalism for the truly marginal deformations leading to exact CFT's on left coset spaces.

2.1 String theory on group manifolds: a reminder

Let \mathfrak{g} be the (semi-simple) Lie algebra of the (compact) group \mathcal{G} and $\{T_M\}$ a set generators that satisfy the usual commutation relations $[T_M, T_N] = \sum_P f_{MN}^P T_P$ and are normalized with respect to the Killing product $\kappa(T_M, T_N) = -\text{tr}(T_M T_N) = \delta_{MN}$. We can always write \mathfrak{g} as the direct sum $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{k}$ where \mathfrak{k} is the Cartan subalgebra and correspondingly distinguish between the Cartan generators $\{T_a\}$ and the generators of \mathfrak{j} , $\{T_\mu\}$.

The generators are in one-to-one correspondence with the Maurer–Cartan left-invariant one-forms defined by

$$\mathcal{J}_M = \kappa(T_M, g^{-1} dg) = -\text{tr}(T_M g^{-1} dg) \quad (2.1)$$

where g is the general element of the group \mathcal{G} . It is a well known fact that the scalar product on \mathfrak{g} naturally induces a scalar product $\langle \cdot, \cdot \rangle$ on the tangent space T_g to \mathcal{G} that can be written by decomposing the induced metric (the so-called Cartan–Killing metric) in terms of the currents as follows:

$$\langle dg, dg \rangle = \kappa(g^{-1} dg, g^{-1} dg) = \sum_{MN} \delta_{MN} \kappa(T_M, g^{-1} dg) \kappa(T_N, g^{-1} dg) = \sum_{MN} \delta_{mn} \mathcal{J}^M \otimes \mathcal{J}^N. \quad (2.2)$$

Now let us consider the affine extension of the Lie algebra $\hat{\mathfrak{g}}_k$, at level k . We have two sets of holomorphic and anti-holomorphic currents of dimension one, naturally related to the Maurer–Cartan right- and left-invariant one-forms

$$J_M(z) = -k\kappa(T_M, \partial g g^{-1}), \quad \bar{J}_M(\bar{z}) = k\kappa(T_M, g^{-1} \bar{\partial} g). \quad (2.3)$$

Each set satisfies the following operator product expansion

$$J_M(z) J_N(w) = \frac{k\delta_{MN}}{2(z-w)^2} + \frac{f_{MN}^P J_P(w)}{z-w} + \mathcal{O}((z-w)^0). \quad (2.4)$$

This chiral algebra contains the Virasoro operator, given by the Sugawara construction

$$T(z) = \sum_M \frac{:J_M J_M:}{k + g^*} \quad (2.5)$$

where g^* is the dual Coxeter number and the corresponding central charge is given by

$$c = \frac{k \dim(\mathfrak{g})}{k + g^*}. \quad (2.6)$$

An $N = 1$ superconformal extension is obtained by adding $(\dim \mathfrak{g})$ free fermions transforming in the adjoint representation

$$T(z) = \sum_M \frac{:J_M J_M:}{k + g^*} + : \psi_M \partial \psi_M :, \quad (2.7)$$

$$G(z) = \frac{2}{k} \left(\sum_M J_M \psi_M - \frac{i}{3k} \sum_{MNP} f_{MNP} : \psi_M \psi_N \psi_P : \right). \quad (2.8)$$

An heterotic model is provided by considering a left-moving $N = 1$ current algebra and a right-moving $N = 0$ one. The Lagrangian (σ -model) description of this model is given by the linear combination of the following WZW-model and the action for free fermions transforming in the adjoint representation

$$S = \frac{k}{4\pi} \int_{\partial B} \text{Tr} (g^{-1} dg \wedge *g^{-1} dg) + \frac{k}{12\pi} \int_B \text{Tr} (g^{-1} dg)^3 + \frac{1}{2\pi} \int d^2z \psi_M \bar{\partial} \psi_M \quad (2.9)$$

(the exterior derivative is here understood as acting on the worldsheet coordinates). The background fields corresponding to this action are the Cartan-Killing metric eq.(2.2) and the NS-NS two-form field, coming from the WZ term

$$H = dB = \text{Tr} (g^{-1} dg)^3 = \frac{1}{2} f_{MNP} \mathcal{J}^M \wedge \mathcal{J}^N \wedge \mathcal{J}^P. \quad (2.10)$$

2.2 Asymmetric deformations

Truly marginal deformations of WZW models were already studied in [14, 15]. In particular in heterotic strings we can consider a deformation obtained with the following exactly marginal operator V built from the total Cartan currents of \mathfrak{g} (so that it preserves the local $N = (1, 0)$ superconformal symmetry of the theory)

$$V = \int d^2z \sum_a \frac{\sqrt{k k_g^a}}{2\pi} H_a \left(J^a(z) - \frac{i}{k} f_{MN}^a : \psi^M \psi^N : \right) \bar{I}^a(\bar{z}) \quad (2.11)$$

(where the set $\{H_a\}$ are the parameters of the deformation and $\bar{I}^a(\bar{z})$ are right moving currents of the Cartan subalgebra of the heterotic gauge group at level k_g^a). Such a deformation is always truly marginal since the J_a currents commute.

It is not completely trivial to read off the deformed background fields that correspond to the $S + V$ deformed action. A possible way is a method involving a Kaluza–Klein reduction as in [16]. For simplicity we will consider the bosonic string with vanishing dilaton and just one operator in the Cartan subalgebra \mathfrak{k} . The right-moving gauge current J used for the deformation has now a left-moving partner and can hence be bosonized as $\bar{J} = i\bar{\partial}\varphi$, $\varphi(z, \bar{z})$ being interpreted as an internal degree of freedom. The sigma-model action is recast as

$$S = \frac{1}{2\pi} \int d^2z (G_{MN} + B_{MN}) \partial x^M \bar{\partial} x^N, \quad (2.12)$$

where the $x^M, M = 1, \dots, 4$ embrace the group coordinates $x^\mu, \mu = 1, 2, 3$ and the internal $x^4 \equiv \varphi$:

$$x^M = \begin{pmatrix} x^\mu \\ x^4 \end{pmatrix}. \quad (2.13)$$

If we split accordingly the background fields, we obtain the following decomposition

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\varphi\varphi} A_\mu \\ G_{\varphi\varphi} A_\mu & G_{\varphi\varphi} \end{pmatrix}, \quad B_{MN} = \begin{pmatrix} B_{\mu\nu} & B_{\mu 4} \\ -B_{\mu 4} & 0 \end{pmatrix}, \quad (2.14)$$

and the action becomes

$$S = \frac{1}{2\pi} \int dz^2 \{ (G_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu + (G_{\varphi\varphi} A_\mu + B_{\mu 4}) \partial x^\mu \bar{\partial} \varphi + (G_{\varphi\varphi} A_\mu - B_{\mu 4}) \partial \varphi \bar{\partial} x^\mu + G_{\varphi\varphi} \partial \varphi \bar{\partial} \varphi \}. \quad (2.15)$$

We would like to put the previous expression in such a form that space–time gauge invariance,

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda, \quad (2.16)$$

$$B_{\mu 4} \rightarrow B_{\mu 4} + \partial_\mu \eta, \quad (2.17)$$

is manifest. This is achieved as follows:

$$S = \frac{1}{2\pi} \int d^2z \left\{ (\hat{G}_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu + B_{\mu 4} (\partial x^\mu \bar{\partial} \varphi - \partial \varphi \bar{\partial} x^\mu) + G_{\varphi\varphi} (\partial \varphi + A_\mu \partial x^\mu) (\bar{\partial} \varphi + A_\mu \bar{\partial} x^\mu) \right\}, \quad (2.18)$$

where $\hat{G}_{\mu\nu}$ is the Kaluza–Klein metric

$$\hat{G}_{\mu\nu} = G_{\mu\nu} - G_{\varphi\varphi} A_\mu A_\nu. \quad (2.19)$$

We can then make the following identifications

$$\hat{G}_{\mu\nu} = \frac{k}{2} (\mathcal{J}_\mu \mathcal{J}_\nu - 2\mathbb{H}^2 \tilde{\mathcal{J}}_\mu \tilde{\mathcal{J}}_\nu), \quad (2.20a)$$

$$B_{\mu\nu} = \frac{k}{2} J_\mu \wedge J_\nu, \quad (2.20b)$$

$$B_{\mu 4} = G_{\varphi\varphi} A_\mu = \mathbb{H} \sqrt{\frac{k k_g}{2}} \tilde{\mathcal{J}}_\mu, \quad (2.20c)$$

$$A_\mu = \mathbb{H} \sqrt{\frac{2k}{k_g}} \tilde{\mathcal{J}}_\mu, \quad (2.20d)$$

$$G_{\varphi\varphi} = \frac{k_g}{2}. \quad (2.20e)$$

Let us now consider separately the background fields we obtained so to give a clear geometric interpretation of the deformation, in particular in correspondence of what we will find to be the maximal value for the deformation parameters \mathbb{H}_a .

The metric. According to eq. (2.20a), in terms of the target space metric, the effect of this perturbation amounts to inducing a back-reaction that in the basis of eq. (2.2) is written as

$$\langle dg, dg \rangle_{\mathbb{H}} = \sum_M \mathcal{J}_M \otimes \mathcal{J}_M - 2 \sum_a \mathbb{H}_a^2 \mathcal{J}_a \otimes \mathcal{J}_a = \sum_\mu \mathcal{J}_\mu \otimes \mathcal{J}_\mu + \sum_a (1 - 2\mathbb{H}_a^2) \mathcal{J}_a \otimes \mathcal{J}_a, \quad (2.21)$$

where we have explicitly separated the Cartan generators. From this form of the deformed metric we see that there is a “natural” maximal value $\mathbb{H}_a = 1/\sqrt{2}$ where the contribution of the $\mathcal{J}_a \otimes \mathcal{J}_a$ term changes its sign and the signature of the metric is thus changed. One could naively think that the maximal value $\mathbb{H}_a = 1/\sqrt{2}$ can’t be attained since this would correspond to a degenerate manifold of lower dimension; what actually happens is that the deformation selects the maximal torus that decouples in the $\mathbb{H}_a = \mathbb{H} \rightarrow 1/\sqrt{2}$ limit as it was shown in [5] for the $SU(2)$ and $SL(2, \mathbb{R})$ algebras.

To begin, write the general element $g \in \mathcal{G}$ as $g = ht$ where $h \in \mathcal{G}/\mathcal{T}$, $t \in \mathcal{T}$. Substituting this decomposition in the expression above we find

1036

D. Israël et al.: Heterotic strings on homogeneous spaces

$$\begin{aligned}
\langle d(ht), d(ht) \rangle_H &= \text{tr} \left((ht)^{-1} d(ht) (ht)^{-1} d(ht) \right) - \sum_a 2H_a^2 \text{tr} \left(T_a (ht)^{-1} d(ht)^{-1} \right)^2 \\
&= \text{tr} \left(h^{-1} dh h^{-1} dh \right) + 2 \text{tr} \left(dt t^{-1} h^{-1} dh \right) + \text{tr} \left(t^{-1} dt t^{-1} dt \right) \\
&\quad - \sum_a 2H_a^2 \left(\text{tr} \left(T_a t^{-1} h^{-1} dh t \right) + \text{tr} \left(T_a t^{-1} dt \right) \right)^2. \tag{2.22}
\end{aligned}$$

Let us introduce a coordinate system (γ_μ, ψ_a) such as the element in \mathcal{G}/\mathcal{T} is parametrized as $h = h(\gamma_\mu)$ and t is written explicitly as

$$t = \exp \left\{ \sum_a \psi_a T_a \right\} = \prod_a e^{\psi_a T_a}. \tag{2.23}$$

It is easy to see that since all the T_a commute $t^{-1} dt = dt t^{-1} = \sum_a T_a d\psi_a$. This allows for more simplifications in the above expression that becomes

$$\begin{aligned}
\langle d(ht), d(ht) \rangle_H &= \text{tr} \left(h^{-1} dh h^{-1} dh \right) + 2 \sum_a \text{tr} \left(T_a h^{-1} dh \right) d\psi_a + \sum_a d\psi_a d\psi_a \\
&\quad - \sum_a 2H_a^2 \left(\text{tr} \left(T_a h^{-1} dh \right) + d\psi_a \right)^2 \\
&= \text{tr} \left(h^{-1} dh h^{-1} dh \right) - \sum_a 2H_a^2 \left(\text{tr} \left(T_a h^{-1} dh \right) \right)^2 \\
&\quad + 2 \sum_a \left(1 - 2H_a^2 \right) \text{tr} \left(T_a h^{-1} dh \right) d\psi_a + \sum_a \left(1 - 2H_a^2 \right) d\psi_a d\psi_a. \tag{2.24}
\end{aligned}$$

If we reparametrise the ψ_a variables as

$$\psi_a = \frac{\hat{\psi}_a}{\sqrt{1 - 2H_a^2}} \tag{2.25}$$

we get a new metric $\langle \cdot, \cdot \rangle'_H$ where we're free to take the $H_a \rightarrow 1/\sqrt{2}$ limit

$$\begin{aligned}
\langle d(ht), d(ht) \rangle'_H &= \text{tr} \left(h^{-1} dh h^{-1} dh \right) - \sum_a 2H_a^2 \left(\text{tr} \left(T_a h^{-1} dh \right) \right)^2 \\
&\quad + 2 \sum_a \sqrt{1 - 2H_a^2} \text{tr} \left(T_a h^{-1} dh \right) d\hat{\psi}_a + \sum_a d\hat{\psi}_a d\hat{\psi}_a \tag{2.26}
\end{aligned}$$

and get

$$\langle d(ht), d(ht) \rangle'_{1/\sqrt{2}} = \left[\text{tr} \left(h^{-1} dh h^{-1} dh \right) - \sum_a \left(\text{tr} \left(T_a h^{-1} dh \right) \right)^2 \right] + \sum_a d\psi_a d\psi_a \tag{2.27}$$

where we can see the sum of the restriction of the Cartan-Killing metric² on $T_h\mathcal{G}/\mathcal{T}$ and the metric on $T_t\mathcal{T} = T_tU(1)^r$. In other words the coupling terms between the elements $h \in \mathcal{G}/\mathcal{T}$ and $t \in \mathcal{T}$ vanished and the resulting metric $\langle \cdot, \cdot \rangle'_{1/\sqrt{2}}$ describes the tangent space T_{ht} to the manifold $\mathcal{G}/\mathcal{T} \times \mathcal{T}$.

² This always is a left-invariant metric on G/H . A symmetric coset doesn't admit any other metric. For a more complete discussion see App.A.

These homogeneous manifolds enjoy many interesting properties. The best part of them can be interpreted as consequence of the presence of an underlying structure that allows to recast all the geometric problems in Lie algebraic terms (see App. A for some constructions). There's however at least one intrinsically geometric property that it is worth to emphasize since it will have many profound implications in the following. All these spaces can be naturally endowed with complex structures by using positive and negative roots as holomorphic and anti-holomorphic generators. Moreover for each space there is not in general only one of these structures (but for the lowest dimensional $SU(2)$ case) and there always exists one of them which is Kähler [17].

Other background fields. The asymmetric deformation of eq. (2.11) generates a non-trivial field strength for the gauge field, that from eq. (2.20d) is found to be

$$F^a = \sum_a \sqrt{\frac{2k}{k_g}} H_a d\mathcal{J}^a = - \sum_a \sqrt{\frac{k}{2k_g}} H_a f^a_{\mu\nu} J^\mu \wedge J^\nu \quad (2.28)$$

(no summation implied over a).

On the other hand, the B -field (2.20b) is not changed, but the physical object is the modified field strength

$$H_{[3]} = dB - \frac{1}{k_g} A^a \wedge dA^a = \frac{1}{2} f_{MNP} \mathcal{J}^M \wedge \mathcal{J}^N \wedge \mathcal{J}^P - 2 \sum_a H_a^2 f_{aNP} \mathcal{J}^a \wedge \mathcal{J}^N \wedge \mathcal{J}^P, \quad (2.29)$$

where we have used the Maurer-Cartan structure equations. At the point where the fibration trivializes, $H_a = 1/\sqrt{2}$, we are left with

$$H_{[3]} = \frac{1}{2} f_{\mu\nu\rho} \mathcal{J}^\mu \wedge \mathcal{J}^\nu \wedge \mathcal{J}^\rho. \quad (2.30)$$

So only the components of $H_{[3]}$ "living" in the coset \mathcal{G}/\mathcal{T} survive the deformation. They are not affected of course by the rescaling of the coordinates on \mathcal{T} .

A trivial fibration. The whole construction can be reinterpreted in terms of fibration as follows. The maximal torus \mathcal{T} is a closed Lie subgroup of the Lie group \mathcal{G} , hence we can see \mathcal{G} as a principal bundle with fiber space \mathcal{T} and base space \mathcal{G}/\mathcal{T} [18]

$$\mathcal{G} \xrightarrow{\mathcal{T}} \mathcal{G}/\mathcal{T}. \quad (2.31)$$

The effect of the deformation consists then in changing the fiber and the limit value $H_a = 1/\sqrt{2}$ marks the point where the fibration becomes trivial and it is interpreted in terms of a gauge field whose strength is given by the canonical connection on \mathcal{G}/\mathcal{T} [19].

2.3 Equations of motion

In this section we want to explicitly show that the background fields we found on the left coset space are solution to the first order (in α') equations of motion [20].

For a vanishing dilaton they read

$$\delta c = -R + \frac{k_g}{16} F^a_{\mu\nu} F^{a\mu\nu}, \quad (2.32a)$$

$$\beta^{(G)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} - \frac{k_g}{4} F^a_{\mu\rho} F^a_{\nu}{}^{\rho} = 0, \quad (2.32b)$$

$$\beta^{(B)}_{\nu\rho} = \nabla^\mu H_{\mu\nu\rho} = 0, \quad (2.32c)$$

$$\beta^{(A)}_{\mu} = \nabla^{\nu} F^a_{\mu\nu} - \frac{1}{2} F^{a\nu\rho} H_{\mu\nu\rho} = 0, \tag{2.32d}$$

after applying the proper normalizations³ our fields are given by

$$g_{\mu\nu} = \frac{k}{2} \delta_{\mu\nu}, \tag{2.33a}$$

$$F^a_{\mu\nu} = -\sqrt{\frac{2k}{k_g}} f^a_{\mu\nu}, \tag{2.33b}$$

$$H_{\mu\nu\rho} = -\frac{k}{2} f_{\mu\nu\rho}. \tag{2.33c}$$

- The $\beta^{(B)} = 0$ equation (2.32c) is just the restriction of the same equation for the initial WZW model,
- the two terms in the $\beta^{(A)} = 0$ equation (2.32d) vanish separately: the first one because F is closed (or, equivalently because $f^a_{\mu\nu}$ seen as a two form in G/T satisfies the condition stated below eq. (A.13)); the second because it is proportional to

$$\sum_{\nu, \rho \in \mathfrak{g}/\mathfrak{h}} f_{a\nu\rho} f_{\mu\nu\rho} = \sum_{M, R \in \mathfrak{g}} f_{aMR} f_{\mu MR} = 2g^* \delta_{a\mu} = 0, \tag{2.34}$$

- to solve the $\beta^{(G)} = 0$ equation (2.32b) we need some more work. Using the results in App. B.1, for a general algebra, we obtain

$$R_{\mu\nu} = \frac{1}{4} \sum_{\rho, \sigma} f_{\mu\rho\sigma} f_{\nu\rho\sigma} + \sum_{a, \rho} f_{a\mu\rho} f_{a\nu\rho} \tag{2.35}$$

that is consistent with the result in eq. (A.12).

If we introduce the orthonormal basis described in (B.1) the Ricci tensor can be explicitly written as

$$R_{\mu\nu} = \frac{1}{4} \sum_{\rho, \sigma} f_{\mu\rho\sigma} f_{\nu\rho\sigma} + \sum_{a, \rho} f_{a\mu\rho} f_{a\nu\rho} = \frac{1}{2} g^* \delta_{\mu\nu} + \frac{1}{2} \delta_{\mu\nu} \begin{cases} |\alpha_{(\mu+1)/2}|^2 & \text{if } \mu \text{ is odd,} \\ |\alpha_{\mu/2}| & \text{if } \mu \text{ is even.} \end{cases} \tag{2.36}$$

In particular for a simply laced algebra reduces to

$$R_{\mu\nu} = \frac{g^* + 2}{2} \delta_{\mu\nu} = \frac{g^* + 2}{k} g_{\mu\nu}. \tag{2.37}$$

This result can be read by saying that the metric we obtain on a simply laced algebra is Einstein with the following Ricci scalar

$$R = \frac{g^* + 2}{k} (\dim \mathfrak{g} - \dim \mathfrak{k}). \tag{2.38}$$

For example in the case of $G = SU(N)$, $g^* = N$, $(\dim \mathfrak{g} - \dim \mathfrak{k}) = N(N - 1)$ and then

$$R = \frac{(N + 2) N (N - 1)}{k}. \tag{2.39}$$

³ Unless explicitly stated we consider $\alpha' = 1$ and the highest root $\psi = 2$.

3 Some examples

In this section we will give some explicit examples of our construction. In particular we will consider the deformation leading from the $SU(2)$ background to the $SU(2)/U(1) \sim S^2$ coset (which already appeared in [5] as part of the $AdS_2 \times S^2$ background) and the superconformal field theory on $SU(3)/U(1)^2$. Although our construction is quite general and can in principle be applied to any group there is a limited number of examples giving critical heterotic string theory backgrounds with a clear geometrical meaning. This is just because of dimensional reasons: $SU(2)/U(1)$ is two-dimensional, $SU(3)/U(1)^2$ is 6-dimensional and $USp(4)/U(1)^2$ is 8-dimensional; higher groups on the other hand would lead to cosets of dimension greater than 10 (in example $SU(4)/U(1)^3$ has dimension $15 - 3 = 12$). On the other hand these higher-dimensional cosets can be used e.g. to obtain non-trivial compactifications generalizing the constructions of [21, 22], if the level of these CFTs are kept small.

3.1 The two-sphere CFT

The first deformation that we explicitly consider is the marginal deformation of the $SU(2)$ WZW model. This was first obtained in [23] that we will closely follow. It is anyway worth to stress that in their analysis the authors didn't study the point of maximal deformation (which was nevertheless identified as a decompactification boundary) that we will here show to correspond to the 2-sphere $S^2 \sim SU(2)/U(1)$. Exact CFT's on this background have already obtained in [24] and in [2]. In particular the technique used in the latter, namely the asymmetric gauging of a $SU(2) \times U(1)$ WZW model, bears many resemblances to our own.

Consider an heterotic string background containing the $SU(2)$ group manifold, times some $(1, 0)$ superconformal field theory \mathcal{M} . The sigma model action is

$$S = kS_{SU(2)}(g) + \frac{1}{2\pi} \int d^2z \left\{ \sum_{a=1}^3 \lambda^a \bar{\partial} \lambda^a + \sum_{n=1}^g \tilde{\chi}^n \partial \tilde{\chi}^n \right\} + S(\mathcal{M}), \quad (3.1)$$

where λ^i are the left-moving free fermions superpartners of the bosonic $SU(2)$ currents, $\tilde{\chi}^n$ are the right-moving fermions realizing the current algebra in the gauge sector and $kS_{SU(2)}(g)$ is the WZW action for the bosonic $SU(2)$ at level k . This theory possesses an explicit $SU(2)_L \times SU(2)_R$ current algebra.

A parametrization of the $SU(2)$ group that is particularly well suited for our purposes is obtained via the so-called Gauss decomposition that we will later generalize to higher groups (see App C). A general element $g(z, \psi) \in SU(2)$ where $z \in \mathbb{C}$ and $\psi \in \mathbb{R}$ can be written as

$$g(z, \psi) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{f} & 0 \\ 0 & \sqrt{f} \end{pmatrix} \begin{pmatrix} 1 & \bar{w} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \quad (3.2)$$

where $w = -z$ and $f = 1 + |z|^2$. In this parametrisation the matrix of invariant one-forms $\Omega = g(z, \psi)^{-1} dg(z, \psi)$ appearing in the expression for the Maurer-Cartan one-forms (2.1) is

$$\Omega_{11} = \frac{\bar{z} dz - z d\bar{z} + i f d\psi}{2f}, \quad \Omega_{12} = -\frac{e^{-i\psi}}{f} d\bar{z}, \quad (3.3)$$

$$\Omega_{21} = -\bar{\Omega}_{12}, \quad \Omega_{22} = -\Omega_{11} \quad (3.4)$$

(remark that Ω is traceless and anti-Hermitian since it lives in $\mathfrak{su}(2)$). From Ω we can easily derive the Cartan-Killing metric on $T_g SU(2)_k$ as

$$\begin{aligned} \frac{2}{k} ds^2 &= \text{tr} \left(\Omega^\dagger \Omega \right) \\ &= -\frac{1}{2f^2} \left(\bar{z}^2 dz \otimes dz + z^2 d\bar{z} \otimes d\bar{z} - 2 \left(2 + |z|^2 \right) dz \otimes d\bar{z} \right) \\ &\quad + \frac{\imath}{f} \left(z d\bar{z} - \bar{z} dz \right) \otimes d\psi + \frac{1}{2} d\psi \otimes d\psi. \end{aligned} \tag{3.5}$$

The left-moving current contains a contribution from the free fermions realizing an $SU(2)_2$ algebra, so that the theory possesses (local) $N = (1, 0)$ superconformal symmetry.

The marginal deformation is obtained by switching on a magnetic field in the $SU(2)$, introducing the following $(1, 0)$ -superconformal-symmetry-compatible marginal operator

$$\delta S = \frac{\sqrt{k k_g} H}{2\pi} \left(J^3 + \lambda^+ \lambda^- \right) \bar{J} \tag{3.6}$$

where we have picked one particular current \bar{J} from the gauge sector, generating a $U(1)$ at level k_g . For instance, we can choose the level two current: $\bar{J} = i\tilde{\chi}^1 \tilde{\chi}^2$. As a result the solutions to the deformed σ -model (2.21), (2.28) and (2.29) read

$$\frac{1}{k} ds^2 = \frac{dz \otimes d\bar{z}}{\left(1 + |z|^2 \right)^2} + \left(1 - 2H^2 \right) \frac{1}{f^2} \left(\imath z d\bar{z} - \imath \bar{z} dz + f d\psi \right) \otimes \left(\imath z d\bar{z} - \imath \bar{z} dz + f d\psi \right), \tag{3.7}$$

$$dB = \frac{\imath k}{2} \frac{1}{\left(1 + |z|^2 \right)^2} dz \wedge d\bar{z} \wedge d\psi, \tag{3.8}$$

$$A = \sqrt{\frac{k}{2k_g}} H \left(-\frac{\imath}{f} \left(\bar{z} dz - z d\bar{z} \right) + d\psi \right). \tag{3.9}$$

It can be useful to write explicitly the volume form on the manifold and the Ricci scalar

$$\sqrt{\det g} dz \wedge d\bar{z} \wedge d\psi = \frac{k}{2} \frac{\sqrt{k(1-2H^2)}}{\left(1 + |z|^2 \right)^2} dz \wedge d\bar{z} \wedge d\psi, \tag{3.10}$$

$$R = \frac{6 + 4H^2}{k}. \tag{3.11}$$

It is quite clear that at $H = H_{\max} = 1/\sqrt{2}$ something happens as it was already remarked in [23]. In general the three-sphere $SU(2)$ can be seen a non-trivial fibration of $U(1) \sim S^1$ as fiber and $SU(2)/U(1) \sim S^2$ as base space: the parameterization in (3.7) makes it clear that the effect of the deformation consists in changing the radius of the fiber that naively seems to vanish at H_{\max} . But as we already know the story is a bit different: reparameterising as in eq. (2.25)

$$\psi \rightarrow \frac{\hat{\psi}}{\sqrt{1-2H^2}} \tag{3.12}$$

one is free to take the $H \rightarrow 1/\sqrt{2}$ limit where the background fields assume the following expressions

$$\frac{1}{k} ds^2 \xrightarrow{H \rightarrow 1/\sqrt{2}} \frac{dz \otimes d\bar{z}}{\left(1 + |z|^2 \right)^2} + d\hat{\psi} \otimes d\hat{\psi}, \tag{3.13}$$

$$F \xrightarrow{H \rightarrow 1/\sqrt{2}} \sqrt{\frac{k}{4k_g}} \iota dz \wedge d\bar{z}, \quad (3.14)$$

$$H \xrightarrow{H \rightarrow 1/\sqrt{2}} 0. \quad (3.15)$$

Now we can justify our choice of coordinates: the (z, \bar{z}) part of the metric that decouples from the ψ part is nothing else than the Kähler metric for the manifold $\mathbb{C}\mathbb{P}^1$ (which is isomorphic to $SU(2)/U(1)$). In this terms the field strength F is proportional to the Kähler two-form

$$F = \iota \sqrt{\frac{k}{k_g}} g_{z\bar{z}} dz \wedge d\bar{z}. \quad (3.16)$$

This begs for a remark. It is simple to show that cosets of the form G/H where H is the maximal torus of G can always be endowed with a Kähler structure. The natural hope is then for this structure to pop up out of our deformations, thus automatically assuring the $N = 2$ world-sheet supersymmetry of the model. Actually this is not the case. The Kähler structure is just one of the possible left-invariant metrics that can be defined on a non-symmetric coset (see App. A) and the obvious generalization of the deformation considered above leads to \mathbb{C} -structures that are not Kähler. From this point of view this first example is an exception because $SU(2)/U(1)$ is a symmetric coset since $U(1)$ is not only the maximal torus in $SU(2)$ but also the maximal subgroup. It is nonetheless possible to define exact an CFT on flag spaces but this will require a slightly different construction, that we will introduce in Sect. 4.

We conclude this section observing that the flux of the gauge field on the two-sphere is given by

$$Q = \int_{S^2} F = \sqrt{\frac{k}{k_g}} \int d\Omega_2 = \sqrt{\frac{k}{k_g}} 4\pi. \quad (3.17)$$

However one can argue on general grounds that this flux has to be quantized, e.g. because the two-sphere appears as a factor of the magnetic monopole solution in string theory [25]. This quantization of the magnetic charge is only compatible with levels of the affine $SU(2)$ algebra satisfying the condition

$$\frac{k}{k_g} = p^2, \quad p \in \mathbb{Z}. \quad (3.18)$$

3.2 The $SU(3)/U(1)$ flag space

Let us now consider the next example in terms of coset dimensions, $SU(3)/U(1)^2$. As a possible application for this construction we may think to associate this manifold to a $(1, 0)$ superconformal field theory \mathcal{M} with 3+1 dimensional target space so to compactify a critical string theory since $\dim[SU(3)/U(1)^2] = 8 - 2 = 6$. Our construction gives rise to a whole family of CFT 's depending on two parameters (since $\text{rank}[SU(3)] = 2$) but as before we are mainly interested to the point of maximal deformation, where the $U(1)^2$ torus decouples and we obtain an exact theory on the $SU(3)/U(1)^2$ coset. Before giving the explicit expressions for the objects in our construction it is hence useful to remember some properties of this manifold. The first consideration to be made is the fact that $SU(3)/U(1)^2$ is an asymmetric coset in the mathematical sense defined in App. A (as we show below). This allows for the existence of more than one left-invariant Riemann metric. In particular, in this case, if we just consider structures with constant Ricci scalar, we find, together with the restriction of the Cartan-Killing metric on $SU(3)$, the Kähler metric of the flag space F^3 . The construction we present in this section will lead to the first one of these two metrics. This is known to admit a nearly-Kähler structure and has already appeared in the superstring literature as a basis for a cone of G_2 holonomy [7].

A suitable parametrisation for the $SU(3)$ group is obtained via the Gauss decomposition described in App. C. In these terms the general group element is written as

$$g(z_1, z_2, z_3, \psi_1, \psi_2) = \begin{pmatrix} \frac{e^{i\psi_1/2}}{\sqrt{f_1}} & -\frac{\bar{z}_1+z_2\bar{z}_3}{\sqrt{f_1f_2}} e^{i(\psi_1-\psi_2)/2} & -\frac{\bar{z}_3-\bar{z}_1\bar{z}_2}{\sqrt{f_2}} e^{-i\psi_2/2} \\ \frac{z_1 e^{i\psi_1/2}}{\sqrt{f_1}} & -\frac{1+|z_3|^2-z_1z_2\bar{z}_3}{\sqrt{f_1f_2}} e^{i(\psi_1-\psi_2)/2} & -\frac{\bar{z}_2}{\sqrt{f_2}} e^{-i\psi_2/2} \\ \frac{z_3 e^{i\psi_1/2}}{\sqrt{f_1}} & -\frac{z_2-\bar{z}_1z_3+z_2|z_1|^2}{\sqrt{f_1f_2}} e^{i(\psi_1-\psi_2)/2} & \frac{1}{\sqrt{f_2}} e^{-i\psi_2/2} \end{pmatrix} \quad (3.19)$$

where z_i are three complex parameters, ψ_j are two real parameters, and $f_1 = 1 + |z_1|^2 + |z_3|^2$, $f_2 = 1 + |z_2|^2 + |z_3 - z_1z_2|^2$. As for the group, we need also an explicit parameterisation for the $\mathfrak{su}(3)$ algebra, such as the one provided by the Gell-Mann matrices in eq. (C.10). It is a well known result that if a Lie algebra is semi-simple (or, equivalently, if its Killing form is negative-definite) then all Cartan subalgebras are conjugated by some inner automorphism⁴. This leaves us the possibility of choosing any couple of commuting generators, knowing that the final result won't be influenced by such a choice. In particular, then, we can pick the subalgebra generated by $\mathfrak{k} = \langle \lambda_3, \lambda_8 \rangle$.⁵

We can now specialize the general expressions given in Sect. 2. The holomorphic currents (2.3) of the bosonic $SU(3)_k$ corresponding to the two operators in the Cartan are

$$\mathcal{J}^3 = -\text{tr} \left(\lambda_3 g(z_\mu, \psi_a)^{-1} dg(z_\mu, \psi_a) \right), \quad \mathcal{J}^8 = -\text{tr} \left(\lambda_8 g(z_\mu, \psi_a)^{-1} dg(z_\mu, \psi_a) \right), \quad (3.20)$$

that in these coordinates read

$$\mathcal{J}^3 = -\frac{i}{\sqrt{2}} \left\{ \left(\frac{\bar{z}_1}{f_1} + \frac{z_2(-\bar{z}_1\bar{z}_2 + \bar{z}_3)}{2f_2} \right) dz_1 - \frac{\bar{z}_2(1 + |z_1|^2) - z_1\bar{z}_3}{2f_2} dz_2 + \left(\frac{\bar{z}_3}{f_1} + \frac{\bar{z}_1\bar{z}_2 - \bar{z}_3}{2f_2} \right) dz_3 \right\} + \text{c.c.} + \frac{d\psi_1}{\sqrt{2}} - \frac{d\psi_2}{2\sqrt{2}}, \quad (3.21)$$

$$\mathcal{J}^8 = -i\sqrt{\frac{3}{2}} \left\{ \frac{\bar{z}_1\bar{z}_2 - \bar{z}_3}{2f_2} z_2 dz_1 + \frac{\bar{z}_2 + |z_1|^2\bar{z}_2 - z_1\bar{z}_3}{2f_2} dz_2 + \frac{-\bar{z}_1\bar{z}_2 + \bar{z}_3}{2f_2} dz_3 + \text{c.c.} + \frac{1}{2}\sqrt{\frac{3}{2}} d\psi_2 \right\}, \quad (3.22)$$

they appear in the expression of the exactly marginal operator (2.11) that we can add to the $SU(3)$ wzw model action is

$$V = \frac{\sqrt{k\bar{k}g}}{2\pi} \int dz^2 \text{H}_3 \left(\mathcal{J}^3 - \frac{i}{\sqrt{2}k} (2 : \psi_2\psi_1 : + : \psi_5\psi_4 : + : \psi_7\psi_6 :) \right) \bar{\mathcal{J}}^3 + \text{H}_8 \left(\mathcal{J}^8 - \frac{i}{k} \sqrt{\frac{3}{2}} (: \psi_5\psi_4 : + : \psi_7\psi_6 :) \right) \bar{\mathcal{J}}^8 \quad (3.23)$$

where ψ^i are the bosonic current superpartners and $\bar{\mathcal{J}}$ are two currents from the gauge sector both generating a $U(1)_{k_g}$.

⁴ This is the reason why the study of non-semi-simple Lie algebra deformation constitutes a richer subject. In example the $SL(2, \mathbb{R})$ group admits for 3 different deformations, leading to 3 different families of exact CFT's with different physics properties. On the other hand the 3 possible deformations in $SU(3)$ are equivalent.

⁵ In this explicit parameterisation it is straightforward to show that the coset we're considering is not symmetric. It suffices to pick two generators, say λ_2 and λ_4 , and remark that their commutator $[\lambda_2, \lambda_4] = -1/\sqrt{2}\lambda_6$ doesn't live in the Cartan subalgebra.

Since $\text{rank}[SU(3)] = 2$ we have a bidimensional family of deformations parameterised by the two moduli H_3 and H_8 . The back-reaction on the metric is given by

$$ds^2 = g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + (1 - 2H_3^2) \mathcal{J}^3 \otimes \mathcal{J}^3 + (1 - 2H_8^2) \mathcal{J}^8 \otimes \mathcal{J}^8 \quad (3.24)$$

where $g_{\alpha\bar{\beta}}$ is the restriction of the $SU(3)$ metric on $SU(3)/U(1)^2$. It is worth to remark that for any value of the deformation parameters H_3 and H_8 the deformed metric is Einstein with constant Ricci scalar.

With a procedure that has by now become familiar we introduce the following reparametrisation

$$\psi_1 = \frac{\hat{\psi}_1}{\sqrt{1 - 2H_3^2}}, \quad \psi_2 = \frac{\hat{\psi}_2}{\sqrt{1 - 2H_8^2}} \quad (3.25)$$

and take the $H_3 \rightarrow 1/\sqrt{2}$, $H_8 \rightarrow 1/\sqrt{2}$ limit. The resulting metric is

$$ds^2 = g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + \frac{d\hat{\psi}_1 \otimes d\hat{\psi}_1 - d\hat{\psi}_1 \otimes d\hat{\psi}_2 + d\hat{\psi}_2 \otimes d\hat{\psi}_2}{2}, \quad (3.26)$$

that is the metric of the tangent space to the manifold $SU(3)/U(1)^2 \times U(1) \times U(1)$. As shown in App.A the coset metric hence obtained has a \mathbb{C} -structure, is Einstein, and has constant Ricci scalar $R = 15/k$. The other background fields at the boundary of the moduli space read

$$F = d\mathcal{J}^3 + d\mathcal{J}^8, \quad (3.27)$$

$$H_{[3]} = -3\sqrt{2} \left\{ \mathcal{J}^1 \wedge (\mathcal{J}^4 \wedge \mathcal{J}^5 - \mathcal{J}^6 \wedge \mathcal{J}^7) + \sqrt{3} \mathcal{J}^2 \wedge (\mathcal{J}^4 \wedge \mathcal{J}^5 + \mathcal{J}^6 \wedge \mathcal{J}^7) \right\}. \quad (3.28)$$

If we consider the worldsheet supersymmetry properties along the deformation line we can remark the presence of an interesting phenomenon. The initial $SU(3)$ model has $N = 2$ superconformal symmetry but this symmetry is naively broken to $N = 1$ by the deformation. This is true for any value of the deformation parameter but for the boundary point $H_3^2 = H_8^2 = 1/2$ where the $N = 2$ supersymmetry is restored. Following [22, 26, 27] one can see that a G/T coset admits $N = 2$ supersymmetry if it possesses a complex structure and the corresponding algebra can be decomposed as $\mathfrak{j} = \mathfrak{j}_+ \oplus \mathfrak{j}_-$ such as $[\mathfrak{j}_+, \mathfrak{j}_+] = \mathfrak{j}_+$ and $[\mathfrak{j}_-, \mathfrak{j}_-] = \mathfrak{j}_-$. Explicitly, this latter condition is equivalent (in complex notation) to $f_{ijk} = f_{\bar{i}\bar{j}\bar{k}} = f_{aij} = f_{\bar{a}\bar{i}\bar{j}} = 0$. These are easily satisfied by the $SU(3)/U(1)^2$ coset (and actually by any G/T coset) since the commutator of two positive (negative) roots can only be proportional to the positive (negative) root obtained as the sum of the two or vanish, as shown in eq. (B.3). Having $N = 2$ supersymmetry is equivalent to asking for the presence of two complex structures. The first one is trivially given by considering positive roots as holomorphic and negative roots as anti-holomorphic, the other one by interchanging the role in one out of the three positive/negative couples (the same flip on two couples would give again the same structure and on all the three just takes back to the first structure). The metric is hermitian with respect to both structures since it is $SU(3)$ invariant. It is worth to remark that such background is different from the ones described in [27] because it is not Kähler and can't be decomposed in terms of Hermitian symmetric spaces.

4 Gauging

In this section we want to give an alternative construction for our deformed models, this time explicitly based on an asymmetric WZW gauging. The existence of such a construction is not surprising at all since our deformations can be seen as a generalization of the ones considered in [28]. In these terms, just like $J\bar{J}$ (symmetric) deformations lead to gauged WZW models, our asymmetric construction leads to asymmetrically gauged WZW models, which were studied in [8].

First of all we will give the explicit construction for the most simple case, the $SU(2)$ model, then introduce a more covariant formalism which will be simpler to generalize to higher groups, in particular

for the $SU(3)$ case, whose gauging will lead, this time, to two different exact models corresponding to the two possible Einstein complex structures admitted by the $SU(3)/U(1)^2$ manifold.

To simplify the formalism we will discuss gauging of bosonic CFTs, and the currents of the gauge sector of the heterotic string are replaced by compact $U(1)$ free bosons. It is obvious that all the results are easily translated into heterotic string constructions.

4.1 The $SU(2)/U(1)$ asymmetric gauging

In this section we want to show how the S^2 background described in [5] can be directly obtained via an asymmetric gauging of the $SU(2) \times U(1)$ WZW model (a similar construction was first obtained in [2]).

Consider the WZW model for the group manifold $SU(2)_k \times U(1)_{k'}$. A parametrisation for the general element of this group which is nicely suited for our purposes is obtained as follows

$$g = \begin{pmatrix} z_1 & z_2 & 0 \\ -\bar{z}_2 & \bar{z}_1 & 0 \\ 0 & 0 & z_3 \end{pmatrix} = \left(\begin{array}{c|c} g_2 & 0 \\ \hline 0 & g_1 \end{array} \right) \in SU(2) \times U(1) \quad (4.1)$$

where g_1 and g_2 correspond to the $SU(2)$ and $U(1)$ parts respectively and (z_1, z_2, z_3) satisfy

$$SU(2) \times U(1) = \{(w_1, w_2, w_3) \mid |w_1|^2 + |w_2|^2 = 1, |w_3|^2 = 1\} \subset \mathbb{C}^3. \quad (4.2)$$

A possible choice of coordinates for the corresponding group manifold is given by the Euler angles:

$$SU(2) \times U(1) = \left\{ (z_1, z_2, z_3) = \left(\cos \frac{\beta}{2} e^{i(\gamma+\alpha)/2}, \sin \frac{\beta}{2} e^{i(\gamma-\alpha)/2}, e^{i\varphi} \right) \mid 0 \leq \beta \leq \pi, 0 \leq \alpha, \beta, \varphi < 2\pi \right\}. \quad (4.3)$$

In order to obtain the coset construction leading to the S^2 background we define two $U(1) \rightarrow SU(2) \times U(1)$ embeddings as follows

$$\begin{aligned} \epsilon_L : U(1) &\rightarrow SU(2) \times U(1), & \epsilon_R : U(1) &\rightarrow SU(2) \times U(1), \\ e^{i\tau} &\mapsto (e^{i\tau}, 0, 1), & e^{i\tau} &\mapsto (1, 0, e^{i\tau}), \end{aligned} \quad (4.4)$$

so that in terms of the z variables the action of these embeddings boils down to

$$\begin{aligned} g &\mapsto \epsilon_L(e^{i\tau}) g \epsilon_R(e^{i\tau})^{-1}, \\ (w_1, w_2, w_3) &\mapsto (e^{i\tau} w_1, e^{i\tau} w_2, e^{-i\tau} w_3). \end{aligned} \quad (4.5)$$

This means that we are free to choose a gauge where w_2 is real or, in Euler coordinates, where $\gamma = \alpha$, the other angular variables just being redefined. To find the background fields corresponding to this gauge choice one should simply write down the Lagrangian where the symmetries corresponding to the two embeddings in (4.4) are promoted to local symmetries, integrate the gauge fields out, and then apply a Kaluza-Klein reduction, much in the same spirit as in [5].

The starting point is the WZW model, written as

$$S_{\text{WZW}}(g) = \frac{k}{4\pi} \int dz^2 \text{Tr} (g_2^{-1} \partial g_2 g_2^{-1} \bar{\partial} g_2) + \frac{k'}{4\pi} \int dz^2 \text{Tr} (g_1^{-1} \partial g_1 g_1^{-1} \bar{\partial} g_1) + \text{WZ term}. \quad (4.6)$$

Its gauge-invariant generalization is given by

$$S = S_{\text{WZW}} + \frac{1}{4\pi} \int d^2z [2k\bar{A} \text{Tr} (t_L \partial g g^{-1}) + 2k'A \text{Tr} (t_R g^{-1} \bar{\partial} g) - 2kA\bar{A} \text{Tr} (t_L g_1^{-1} t_R g_1)]$$

$$-2k' A \bar{A} \text{Tr}(t_L g_2^{-1} t_R g_2) + k A \bar{A} \text{Tr}_1(t_L)^2 + k' A \bar{A} \text{Tr}_2(t_R)^2] \quad (4.7)$$

where A and \bar{A} are the components of the gauge field, and t_L and t_R are the Lie algebra generators corresponding to the embeddings in (4.4), i.e.

$$t_L = i \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad t_R = i\sqrt{2} \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \quad (4.8)$$

σ_3 being the usual Pauli matrix. For such an asymmetric coset to be anomaly free, one has the following constraint on the embeddings

$$k \text{Tr}(t_L)^2 = k' \text{Tr}(t_R)^2 \implies k = k' p^2, \text{ with } p \in \mathbb{N}. \quad (4.9)$$

If we pass to Euler coordinates it is simple to give an explicit expression for the action

$$S(\alpha, \beta, \gamma, \varphi) = \frac{1}{2\pi} \int d^2 z \frac{k}{4} (\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta + \partial\gamma\bar{\partial}\gamma + 2\cos\beta\partial\alpha\bar{\partial}\gamma) + \frac{k'}{2} \partial\varphi\bar{\partial}\varphi \\ + k(\partial\alpha + \cos\beta\partial\gamma)\bar{A} + k'\sqrt{2}p\bar{\partial}\varphi A - 2kA\bar{A}. \quad (4.10)$$

This Lagrangian is quadratic in A, \bar{A} and the quadratic part is constant so we can integrate these gauge fields out and the resulting Lagrangian is

$$S(\alpha, \beta, \gamma, \varphi) = \frac{1}{2\pi} \int d^2 z \frac{k}{4} (\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta + \partial\gamma\bar{\partial}\gamma + 2\cos\beta\partial\alpha\bar{\partial}\gamma) + \frac{k'}{2} \partial\varphi\bar{\partial}\varphi \\ + \frac{\sqrt{2kk'}}{2} (\partial\alpha + \cos\beta\partial\gamma)\bar{\partial}\varphi. \quad (4.11)$$

Now, since we gauged out the symmetry corresponding to the $U(1)$ embeddings, this action is redundant. This can very simply be seen by writing the corresponding metric and remarking that it has vanishing determinant

$$\det g_{\mu\nu} = \begin{vmatrix} k/4 & & & \\ & k/4 & k/4 \cos\beta & \sqrt{2kk'}/4 \\ & k/4 \cos\beta & k/4 & \sqrt{2kk'}/4 \cos\beta \\ & \sqrt{2kk'}/4 & \sqrt{2kk'}/4 \cos\beta & k'/2 \end{vmatrix} = 0. \quad (4.12)$$

Of course this is equivalent to say that we have a gauge to fix (as we saw above) and this can be chosen by imposing $\gamma = \alpha$, which leads to the following action

$$S(\alpha, \beta, \varphi) = \frac{1}{2\pi} \int d^2 z \frac{k}{4} (2(1 + \cos\beta)\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta) + \frac{k'}{2} \partial\varphi\bar{\partial}\varphi + \frac{\sqrt{2kk'}}{2} (1 + \cos\beta)\partial\alpha\bar{\partial}\varphi \quad (4.13)$$

whence we can read a two dimensional metric by interpreting the $\partial\alpha\bar{\partial}\varphi$ term as a gauge boson and applying the usual Kaluza-Klein reduction. We thus recover the two-sphere we expect.

$$ds^2 = g_{\mu\nu} - G_{\varphi\varphi} A_\mu A_\nu = \frac{k}{4} (d\beta^2 + \sin^2\beta d\alpha^2) \quad (4.14)$$

supported by a (chromo)magnetic field

$$A = \sqrt{\frac{k}{k'}} (1 + \cos\beta) d\alpha. \quad (4.15)$$

As advertised above we now turn to rewrite the above gauging in a more covariant form, simpler to generalize. Since we're interested in the underlying geometry, we'll mainly focus on the metric of the spaces we obtain at each step and write these metrics in terms of the Maurer-Cartan currents⁶. As we've already seen in eq.(2.2), the metric of the initial group manifold is

$$ds^2 = \frac{k}{2} \sum \mathcal{J}_i^2 \otimes \mathcal{J}_i^2 + \frac{k'}{2} \mathcal{I} \otimes \mathcal{I} \tag{4.16}$$

where $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ are the currents of the $SU(2)$ part and \mathcal{I} the $U(1)$ generator. The effect of the asymmetric gauging amounts - at this level - to adding what we can see as an interaction term between the two groups. This changes the metric to

$$ds^2 = \frac{k}{2} \sum \mathcal{J}_i^2 \otimes \mathcal{J}_i^2 + \frac{k'}{2} \mathcal{I} \otimes \mathcal{I} + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I}. \tag{4.17}$$

Of course if we choose $\langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{I} \rangle$ as a basis we can rewrite the metric in matrix form

$$g = \frac{1}{2} \begin{pmatrix} k & & & \\ & k & & \\ & & k & \sqrt{kk'} \\ & & \sqrt{kk'} & k' \end{pmatrix} \tag{4.18}$$

where we can see that the gauging of the axial symmetry corresponds to the fact that the sub-matrix relative to the $\{\mathcal{J}_3, \mathcal{I}\}$ generators is singular

$$\begin{vmatrix} k & \sqrt{kk'} \\ \sqrt{kk'} & k' \end{vmatrix} = 0 \tag{4.19}$$

explicitly this corresponds to

$$k\mathcal{J}_3 \otimes \mathcal{J}_3 + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I} + \sqrt{kk'} \mathcal{J}_3 \otimes \mathcal{I} + k'\mathcal{I} \otimes \mathcal{I} = (k + k') \hat{\mathcal{J}} \otimes \hat{\mathcal{J}} \tag{4.20}$$

where

$$\hat{\mathcal{J}} = \frac{\sqrt{k}\mathcal{J}_3 + \sqrt{k'}\mathcal{I}}{\sqrt{k + k'}} \tag{4.21}$$

is a normalized current. So we project the interaction sub-matrix on its non-vanishing normalized eigenvector

$$\left(\sqrt{\frac{k}{k+k'}} \quad \sqrt{\frac{k}{k+k'}} \right) \begin{pmatrix} k & \sqrt{kk'} \\ \sqrt{kk'} & k' \end{pmatrix} \begin{pmatrix} \sqrt{\frac{k}{k+k'}} \\ \sqrt{\frac{k}{k+k'}} \end{pmatrix} = k + k' \tag{4.22}$$

and the resulting metric in the $\langle \mathcal{J}_1, \mathcal{J}_2, \hat{\mathcal{J}} \rangle$ basis is

$$\begin{pmatrix} k & & \\ & k & \\ & & k + k' \end{pmatrix}. \tag{4.23}$$

⁶ One of the advantages of just working on the metrics is given by the fact that in each group one can consistently choose holomorphic or anti-holomorphic currents as a basis. In the following we will consider the group in the initial WZW model as being generated by the holomorphic and the dividing group by the anti-holomorphic ones.

This manifold \mathcal{M} (whose metric appears in the action (2.12)) corresponds to a S^1 fibration (the fiber being generated by $\hat{\mathcal{J}}$) over a S^2 base (generated by $\langle \mathcal{J}_1, \mathcal{J}_2 \rangle$).

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & S^2 \end{array} \quad (4.24)$$

It should now appear obvious how to generalize this construction so to include all the points in the moduli space joining the unperturbed and gauged model. The decoupling of the $U(1)$ symmetry (that has been “gauged away”) is obtained because the back-reaction of the gauge field eq.(4.11) is such that the interaction sub-matrix is precisely singular. On the other hand we can introduce a parameter that interpolates between the unperturbed and the gauged models so that the interaction matrix now has two non-null eigenvalues, one of which will vanish at the decoupling point.

In practice this is done by adding to the the asymmetrically gauged WZW model an auxiliary $U(1)$ free boson Y at radius $R = (kk')^{1/4}(1/\sqrt{2H} - 1)^{1/2}$. This $U(1)$ is coupled symmetrically to the gauge fields such that the anomaly cancellation condition is still given by (4.9). In particular if we choose the gauge $Y = 0$, the metric reads

$$\begin{pmatrix} k & \sqrt{2H}\sqrt{kk'} \\ \sqrt{2H}\sqrt{kk'} & k' \end{pmatrix} \quad (4.25)$$

which is exactly the model studied above. For a generic value of H^2 the two eigenvalues are given by

$$\lambda_{\pm}(k, k', H) = \frac{k + k' \mp \sqrt{k^2 + k'^2 + 2(4H^2 - 1)kk'}}{2} \quad (4.26)$$

so we can diagonalize the metric in the $\langle \mathcal{J}_1, \mathcal{J}_2, \hat{\mathcal{J}}, \hat{\mathcal{J}} \rangle$ basis ($\hat{\mathcal{J}}$ and $\hat{\mathcal{J}}$ being the two eigenvectors) and finally obtain

$$g = \begin{pmatrix} k & & & \\ & k & & \\ & & \lambda_1(k, k', H) & \\ & & & \lambda_2(k, k', H) \end{pmatrix}. \quad (4.27)$$

Of course, in the $H^2 \rightarrow 0$ limit we get the initial WZW model and in the $H^2 \rightarrow 1/2$ limit we recover the asymmetrically gauged model eq.(4.23).

It is important to remark that the construction above can be directly generalized to higher groups with non-abelian subgroups, at least for the asymmetric coset part. This is what we will do in the next section.

4.2 $SU(3)/U(1)^2$

To study the $SU(3)$ case we will use the “current” approach, since a direct computation in coordinates would be impractical. As one could expect, the study of $SU(3)$ deformation is quite richer because of the presence of an embedded $SU(2)$ group that can be gauged. Basically this means that we can choose two different deformation patterns that will lead to the two possible Einstein structures that can be defined on the $SU(3)/U(1)^2$ manifold (see App.A).

4.2.1 Direct gauging

The first possible choice consists in the obvious generalization of the $SU(2)/U(1)$ construction above, i.e. simply gauging the $U(1)^2$ Cartan torus. Consider the initial $SU(3)_k \times U(1)_{k'} \times U(1)_{k''}$ model. In the $\langle \mathcal{J}_1, \dots, \mathcal{J}_8, \mathcal{I}_1, \mathcal{I}_2 \rangle$ base ($\{\mathcal{J}_i\}$ being the $SU(3)$ generators and $\{\mathcal{I}_k\}$ the 2 $U(1)$'s), the initial metric is written as

$$g = \left(\begin{array}{c|cc} k \mathbb{1}_{8 \times 8} & & 0 \\ \hline & k' & \\ \hline 0 & & k'' \end{array} \right), \tag{4.28}$$

the natural choice for the Cartan torus is given by the usual $\langle \mathcal{J}_3, \mathcal{J}_8 \rangle$ generators, so we can proceed as before and write the deformed metric as

$$g = \left(\begin{array}{ccc|cc} k \mathbb{1}_{2 \times 2} & & & & \\ & \lambda_1(k, k', H_3) & & & \\ & & k \mathbb{1}_{4 \times 4} & & \\ & & & \lambda_1(k, k'', H_8) & \\ & & & & \lambda_2(k, k', H_3) \\ & & & & & \lambda_2(k, k'', H_8) \end{array} \right) \tag{4.29}$$

where H_3 and H_8 are the deformation parameters and λ_1 and λ_2 are the eigenvalues for the interaction matrices, given in eq. 4.26. In particular, then, in the $H_3^2 \rightarrow 1/2, H_8^2 \rightarrow 1/2$ limit two eigenvalues vanish, the corresponding directions decouple and we're left with the following (asymmetrically gauged) model

$$g = \left(\begin{array}{c|cc} k \mathbb{1}_{6 \times 6} & & \\ \hline & k + k' & \\ \hline & & k + k'' \end{array} \right) \tag{4.30}$$

in the $\langle \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_6, \mathcal{J}_7, \sqrt{k'}\mathcal{I}_1 + \sqrt{k}\mathcal{J}_3, \sqrt{k''}\mathcal{I}_2 + \sqrt{k}\mathcal{J}_8 \rangle$ basis that can be seen as a $U(1)^2$ fibration over a $SU(3)/U(1)^2$ base with metric $\text{diag}(1, 1, 1, 1, 1, 1)$ (in the notation of App A). This is precisely the same result we obtained in Sect. 3.2 when we read the fibration as a gauge field living on the base.

$$\begin{array}{ccc} U(1)^2 & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & SU(3)/U(1)^2 \end{array} \tag{4.31}$$

As in the previous example all this construction is valid only if the asymmetrically gauged WZW model is anomaly-free. This will be explained in detail in Sect. 5.

4.2.2 The F_3 flag space

Let us now turn to the other possible choice for the $SU(3)$ gauging, namely the one where we take advantage of the $SU(2)$ embedding. Let us then consider the $SU(3)_{k_3} \times SU(2)_{k_2} \times U(1)_{k'} \times U(1)_{k''}$ WZW model whose metric is

$$g = \left(\begin{array}{c|cc|cc} k_3 \mathbb{1}_{8 \times 8} & & & & & \\ \hline & & & & & \\ \hline & & k_2 \mathbb{1}_{3 \times 3} & & & \\ \hline & & & k' & & \\ \hline & & & & & k'' \end{array} \right) \tag{4.32}$$

in the $\langle \mathcal{J}_1, \dots, \mathcal{J}_8, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{K}_1, \mathcal{K}_2 \rangle$ basis, where $\langle \mathcal{J}_i \rangle$ generate the $SU(3)$, $\langle \mathcal{I}_i \rangle$ generate the $SU(2)$ and $\langle \mathcal{K}_i \rangle$ generate the $U(1)^2$.

The first step in this case consists in an asymmetric gauging mixing the $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ and $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ currents respectively. At the gauging point, a whole 3-sphere decouples and we obtain the following metric

$$g = \begin{pmatrix} k_3 \mathbb{1}_{5 \times 5} & & & \\ & (k_2 + k_3) \mathbb{1}_{3 \times 3} & & \\ & & k' & \\ & & & k'' \end{pmatrix} \tag{4.33}$$

where we have to remember that in order to have an admissible embedding $k_2 = k_3 = k$. Our result is again – not surprisingly – a $SU(2)$ fibration over a $SU(3)/SU(2)$ base (times the two $U(1)$'s).

$$\begin{array}{ccc} SU(2) & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & SU(3)/SU(2) \end{array} \tag{4.34}$$

Of course one could be tempted to give \mathcal{M} the same interpretation as before, namely a $SU(3)/SU(2)$ space supported by a chromo-magnetic $SU(2)$ field (or, even better, gauging an additional $U(1)$, of a $\mathbb{C}\mathbb{P}^2$ background with a $SU(2) \times U(1)$ chromo-magnetic field). Actually this is not the case. The main point is the fact that this $SU(3) \times SU(2)$ model is essentially different from the previous ones because the $U(1)$ factors were the result of the bosonisation of the right-moving gauge current which in this way received a (fake) left-moving partner as in Sect. 2.2. This is not possible in the non-abelian case since one can't obtain a $SU(2)$ at arbitrary level k out of the fermions of the theory⁷. In other words, the $SU(2)$ factor is in this case truly a constituent of the theory and there is no reason why it should be decoupled or be given a different interpretation from the $SU(3)$ part. This is why the structure obtained by the $SU(2)$ asymmetric gauging is to be considered a 8-dimensional space admitting a $SU(2) \rightarrow SU(3)/SU(2)$ fibration structure, or, equivalently, a deformed $SU(3)$ where an embedded $SU(2)$ is at a level double with respect to the other generators.

On the other hand we are still free to gauge away the two $U(1)$ factors just as before. This time we can choose to couple \mathcal{K}_1 with the \mathcal{J}_8 factor that was left untouched in the initial $SU(3)$ and \mathcal{K}_2 with the $\mathcal{J}_3 + \mathcal{I}_3$ generator. Again we find a two-parameter family of deformations whose metric can be written as

$$g = \begin{pmatrix} k \mathbb{1}_{4 \times 4} & & & & \\ & \mu_1 & & & \\ & & 2k \mathbb{1}_{2 \times 2} & & \\ & & & \nu_1 & \\ & & & & \mu_2 \\ & & & & & \nu_2 \end{pmatrix} \tag{4.35}$$

where

$$\mu = \lambda(k, k', \mathbb{H}') \tag{4.36}$$

$$\nu = \lambda(2k, k'', \mathbb{H}'') \tag{4.37}$$

⁷ This would be of course be possible if we limited ourselves to small values of k , but in this case the whole geometric interpretation of the background would be questionable. However for Gepner-like string compactifications this class of models is relevant.

In particular now we can take the decoupling $\mathfrak{h}' = \mathfrak{h}'' \rightarrow 1/2$ limit where we obtain

$$g = \left(\begin{array}{c|c|c} k \mathbb{1}_{4 \times 4} & & \\ \hline & 2k \mathbb{1}_{2 \times 2} & \\ \hline & & k + k' \\ & & \hline & & 2k + k'' \end{array} \right), \tag{4.38}$$

this structure is once more a $U(1)^2 \rightarrow SU(3)/U(1)^2$ fibration but in this case it is perfectly fine to separate the space components from the gauge field ones. So we can read out our final background fields as the Kähler metric on F_3 (see App.A) supported by a $U(1)^2$ (chromo)magnetic field.

To summarize our results we can say that the two Einstein structures that one can define on $SU(3)/U(1)^2$ are both exact string theory backgrounds

- The first one, obtained as the asymmetric coset $\frac{SU(3) \times U(1)^2}{U(1)^2}$ is supported by an NS-NS field strength and a magnetic field;
- The second, corresponding to the $\frac{SU(3) \times SU(2) \times U(1)^2}{SU(2) \times U(1)^2}$ asymmetric coset is Kähler and hence supported by the (chromo-)magnetic field alone.

These two backgrounds are obtained from two different marginal deformations of $SU(3)$.

This Kähler structure has been deeply studied both from the mathematical and physical points of view. In particular the Kähler form on F_3 can be written as in App. C:

$$K(\gamma_\mu, \bar{\gamma}_\mu) = \log [1 + |\gamma_1|^2 + |\gamma_3|^2] + \log [1 + |\gamma_2|^2 + |\gamma_3 - \gamma_1 \gamma_2|^2]. \tag{4.39}$$

It is immediate to show that this manifold is Einstein and in particular its Ricci scalar is $R = 12$. Being Kähler, F_3 is torsionless, that means in turn that there is no NS-NS flux⁸. Moreover there is no dilaton by construction⁹. The only other field that supports the background comes from the $U(1)^2$ fibration. Since the manifold is Kähler it is useful to take advantage of the complex structure and write our background fields in complex formalism. In these terms the metric is written as

$$g = \frac{k}{2} (\mathcal{J}^1 \otimes \mathcal{J}^{\bar{1}} + \mathcal{J}^2 \otimes \mathcal{J}^{\bar{2}} + 2\mathcal{J}^3 \otimes \mathcal{J}^{\bar{3}}) \tag{4.40}$$

where \mathcal{J}^i and $\mathcal{J}^{\bar{i}}$ are the Maurer-Cartan corresponding to positive and negative roots respectively and the field strength is given by

$$F^\alpha = \sqrt{\frac{k}{2k_g}} f_{\mu\bar{\nu}}^\alpha C^{\bar{\rho}\sigma} R_{\sigma\bar{\nu}} \mathcal{J}^\mu \wedge \mathcal{J}^{\bar{\nu}} \tag{4.41}$$

where C is the following tensor

$$C = \sum_\alpha \mathcal{J}^\alpha \otimes \mathcal{J}^{\bar{\alpha}}. \tag{4.42}$$

In Sect. B.2 we show that the metric and (chromo)magnetic field solve the first order in α' equations of motion.

⁸ To be precise one could define a B field but this would have to be closed.

⁹ The dilaton would basically measure the difference between the asymmetric coset volume form and the homogeneous space one as it is shown in [29].

5 Exact construction: partition functions

In this section we will compute the one-loop partition functions for the various asymmetric deformations leading to geometric cosets. We consider the part of the partition function of the CFT affected by the deformation. We have holomorphic supersymmetric characters and anti-holomorphic bosonic characters of the affine Lie algebra $\hat{\mathfrak{g}}_k$, times some anti-holomorphic fermionic characters from the gauge sector

$$Z \left[\begin{matrix} a; \{h\} \\ b; \{g\} \end{matrix} \right] = \sum_{\Lambda, \bar{\Lambda}} M^{\Lambda \bar{\Lambda}} \chi^{\Lambda}(\tau) \left(\frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (\tau)}{\eta(\tau)} \right)^{\dim(\mathfrak{g})/2} \bar{\chi}^{\bar{\Lambda}} \prod_{\ell} \bar{\vartheta} \left[\begin{matrix} h_{\ell} \\ g_{\ell} \end{matrix} \right] \quad (5.1)$$

where (a, b) and (h_{ℓ}, g_{ℓ}) are the spin structures of the (left and right) worldsheet fermions. Useful formulas about characters are provided in appendix D. Starting from the CFTs defined by these partition functions we will perform the asymmetric deformation that has been discussed in the previous sections from the geometrical point of view.

5.1 The $SU(3)/U(1)^2$ flag space CFT

The partition function for the asymmetric deformation of $SU(2)$ has already been given in [5]. We can hence begin with the next non-trivial example of $SU(3)$. In this case we will compare explicitly two possible constructions, the Kazama-Suzuki method and the direct deformation along the Cartan torus to show that they give the two inequivalent metrics on the geometric coset.

5.1.1 The Kazama-Suzuki decomposition of $SU(3)$

We would like to decompose our WZW model in terms of Kazama-Suzuki (KS) cosets, which are conformal theories with extended $N = 2$ superconformal symmetry [22, 27].

The simplest of those models are the $N = 2$ minimal models that are given by the quotient: $SU(2)_{k-2} \times SO(2)_1/U(1)_k$, and their characters come from the branching relation

$$\chi_{k-2}^j \Xi_2^{s_2} = \sum_{m \in \mathbb{Z}_{2k}} C_m^{j(s_2)} \frac{\Theta_{m,k}}{\eta} \quad (5.2)$$

For convenience, we write the contribution of the worldsheet fermions in terms of $SO(2n)_1$ characters, see appendix D.

Similarly it is possible to construct an $N = 2$ coset CFT from $SU(3)$ [22, 27]:¹⁰

$$\frac{SU(3)_{k-3} \times SO(4)_1}{SU(2)_{k-2} \times U(1)_{3k}} \quad (5.3)$$

The characters of this theory are implicitly defined by the branching relation

$$\chi_{k-3}^{\Lambda} \Xi_4^{s_4} = \sum_{2j=0}^{k-2} \sum_{n \in \mathbb{Z}_{6k}} C_{jn}^{\Lambda(s_4)} \chi_{k-2}^j \frac{\Theta_{n,3k}}{\eta} \quad (5.4)$$

Therefore combining the two branching relations, we obtain the decomposition of the $SU(3)$ supersymmetric WZW model in terms of $N = 2$ KS models

$$\chi_{k-3}^{\Lambda} \Xi_4^{s_4} \Xi_2^{s_2} = \sum_{j,m,n} C_{jn}^{\Lambda(s_4)} C_m^{j(s_2)} \frac{\Theta_{m,k}}{\eta} \frac{\Theta_{n,3k}}{\eta} \quad (5.5)$$

¹⁰ According to our conventions, the weights of a $U(1)$ at level k are $m^2/4k$, $m \in \mathbb{Z}_{2k}$.

This decomposition follows the following pattern

$$SU(3)_{k-3} \times SO(8)_1 \rightarrow \frac{SU(3)_{k-3} \times SO(4)_1}{SU(2)_{k-2} \times U(1)_{3k}} \times \frac{SU(2)_{k-2} \times SO(2)_1}{U(1)_k} \times U(1)_{3k} \times U(1)_k \times SO(2)_1 \quad (5.6)$$

and we shall perform the deformation on the left lattice of $U(1)_{3k} \times U(1)_k$. However the deformation will also act on an appropriate sub-lattice of the right-moving gauge sector. The last $SO(2)_1$ factor corresponds to the fermions which are neutral in the process so they won't be considered afterwards.

5.1.2 The gauge sector

To construct the model we assume that the gauge sector of the heterotic strings contain an unbroken $SO(6)_1$, whose contribution to the partition function is, written in terms of $SO(6)_1$ free fermionic characters $\bar{\Xi}_6^{\bar{s}_6}$, see App.D. Since we decompose the characters of the left-moving sector according to eq.(5.6), a natural choice for the action of the deformation in the right-moving gauge sector is to use a similar Kazama-Suzuki decomposition, but for $k = 3$, in which case the bosonic CFT is trivial

$$SO(8)_1 \rightarrow \frac{SO(4)_1}{SU(2)_1 \times U(1)_9} \times \frac{SU(2)_1 \times SO(2)_1}{U(1)_3} \times U(1)_3 \times U(1)_1 \times SO(2)_1. \quad (5.7)$$

Since as quoted previously two fermions – the $SO(2)_1$ factor – are neutral it is enough that the gauge sector contains an $SO(6)_1$ subgroup. To achieve this decomposition, first we decompose the $SO(6)_1$ characters in terms of $SO(4)_1 \times SO(2)_1$

$$\bar{\Xi}_6^{\bar{s}_6} = \sum_{\bar{s}_4, \bar{s}_2 \in \mathbb{Z}_4} C[\bar{s}_6; \bar{s}_4, \bar{s}_2] \bar{\Xi}_4^{\bar{s}_4} \bar{\Xi}_2^{\bar{s}_2} \quad (5.8)$$

where the coefficients of the decomposition $SO(6) \rightarrow SO(4) \times SO(2)$ are either zero or one. And then we perform a coset decomposition for the $SO(4)_1$ characters

$$\bar{\Xi}_4^{\bar{s}_4} = \sum_{\ell=0,1} \sum_{u \in \mathbb{Z}_{18}} \bar{\omega}_{\ell u}^{\bar{s}_4} \bar{\chi}^\ell \frac{\bar{\Theta}_{u,9}}{\bar{\eta}} \quad (5.9)$$

in terms of $SU(2)_1$ characters $\bar{\chi}^\ell$ and $U(1)$ characters $\bar{\Theta}_{u,9}$. It defines implicitly the coset characters $\bar{\omega}_{\ell u}^{\bar{s}_4}$ of $SO(4)_1/SU(2)_1 \times U(1)_9$. Then the $SU(2)_1 \times SO(2)_1$ characters are decomposed as

$$\bar{\chi}^\ell \bar{\Xi}_2^{\bar{s}_2} = \sum_{v \in \mathbb{Z}_6} \bar{\omega}_v^{\ell, \bar{s}_2} \frac{\bar{\Theta}_{v,3}}{\bar{\eta}}, \quad (5.10)$$

in terms of coset characters of $SU(2)_1 \times SO(2)_1/U(1)_3$. So putting together these branching relations we have the following Kazama-Suzuki decomposition for the free fermions of the gauge sector

$$\bar{\Xi}_6^{\bar{s}_6} = \sum_{\bar{s}_4, \bar{s}_2 \in \mathbb{Z}_4} \sum_{\ell=0,1} \sum_{u \in \mathbb{Z}_{18}} \sum_{v \in \mathbb{Z}_6} C[\bar{s}_6; \bar{s}_4, \bar{s}_2] \bar{\omega}_{\ell u}^{\bar{s}_4} \bar{\omega}_v^{\ell, \bar{s}_2} \frac{\bar{\Theta}_{u,9}}{\bar{\eta}} \frac{\bar{\Theta}_{v,3}}{\bar{\eta}}. \quad (5.11)$$

5.1.3 The deformation

Now we are in position to perform the asymmetric deformation adding a magnetic field to the model. The deformation acts on the following combination of left and right theta functions

$$\Theta_{n,3k} \bar{\Theta}_{u,9} \times \Theta_{m,k} \bar{\Theta}_{v,3}. \quad (5.12)$$

As for the case of $SU(2)$ [5], we have to assume that the level obeys the condition

$$\sqrt{\frac{k}{3}} = p \in \mathbb{N}, \tag{5.13}$$

to be able to reach the geometric coset point in the moduli space of CFT. Then we have to perform $O(2, 2, \mathbb{R})$ boosts in the lattices of the $U(1)$'s, mixing the left Cartan lattice of the super-WZW model with the right lattice of the gauge sector. These boosts are parameterized in function of the magnetic fields as

$$\cosh \Omega_a = \frac{1}{1 - 2H_a^2}, \quad a = 1, 2. \tag{5.14}$$

Explicitly we have

$$\begin{aligned} & \sum_{N_1, N_2 \in \mathbb{Z}} q^{3k(N_1 + \frac{m}{6k})^2} q^{k(N_2 + \frac{n}{2k})^2} \times \sum_{f_1, f_2 \in \mathbb{Z}} \bar{q}^{9(f_1 + \frac{u}{18})^2} \bar{q}^{3(f_2 + \frac{v}{6})^2} \\ \rightarrow & \sum_{N_1, N_2, f_1, f_2 \in \mathbb{Z}} q^9 \left[p \left(N_1 + \frac{m}{18p^2} \right) \cosh \Omega_1 + \left(f_1 + \frac{u}{18} \right) \sinh \Omega_1 \right]^2 q^3 \left[p \left(N_2 + \frac{n}{6p^2} \right) \cosh \Omega_2 + \left(f_2 + \frac{v}{6} \right) \sinh \Omega_2 \right]^2 \\ & \times \bar{q}^9 \left[\left(f_1 + \frac{u}{18} \right) \cosh \Omega_1 + p \left(N_1 + \frac{m}{18p^2} \right) \sinh \Omega_1 \right]^2 \bar{q}^3 \left[\left(f_2 + \frac{v}{6} \right) \cosh \Omega_2 + p \left(N_2 + \frac{n}{6p^2} \right) \sinh \Omega_2 \right]^2. \end{aligned} \tag{5.15}$$

After an infinite deformation, we get the following constraints on the charges

$$m = p(18\mu - u), \quad \mu \in \mathbb{Z}_p, \tag{5.16a}$$

$$n = p(6\nu - v), \quad \nu \in \mathbb{Z}_p, \tag{5.16b}$$

and the $U(1)^2$ CFT that has been deformed marginally decouples from the rest and can be safely removed. In conclusion, the infinite deformation gives

$$\begin{aligned} Z_{F_3}^{(s_4, s_2; \bar{s}_6)}(\tau) = & \sum_{\Lambda} \sum_j \sum_{\mu, \nu \in \mathbb{Z}_p} \sum_{\bar{s}_4, \bar{s}_2 \in \mathbb{Z}_4} C[\bar{s}_6; \bar{s}_4, \bar{s}_2] \\ & \sum_{\ell=0,1} \sum_{u \in \mathbb{Z}_{18}} \sum_{v \in \mathbb{Z}_6} C_{j, p(18\mu-u)}^{\Lambda(s_4)} C_{p(6\nu-v)}^{j(s_2)} \times \chi_{k-3}^{\Lambda} \bar{\varpi}_{4; \ell u}^{\bar{s}_4} \bar{\varpi}_v^{\ell, \bar{s}_2} \end{aligned} \tag{5.17}$$

where the sum over Λ, j runs over integrable representations, see appendix D. This is the partition function for the $SU(3)/U(1)^2$ coset space. The fermionic charges in the left and right sectors are summed according to the standard rules of Gepner heterotic constructions [21]. The modular properties of this partition function are the same as before the deformation, concerning the \mathbb{Z}_4 indices of the worldsheet fermions.

5.1.4 Alternative approach: direct abelian coset

Here we would like to take a different approach, by deforming directly the Cartan lattice of $\widehat{\mathfrak{su}}_3$ without decomposing the left CFT in terms of $KS\ N = 2$ theories. As explained in the App.D, it is possible to perform a generalized (super)parafermionic decomposition of the characters of the $\widehat{\mathfrak{su}}_3$ super-algebra at level k (containing a bosonic algebra at level $k - 3$) w.r.t. the Cartan torus

$$\chi^{\Lambda} \left(\frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \right)^{\dim(i)/2} = \sum_{\lambda \in \mathbf{M}^* \bmod k\mathbf{M}} \mathcal{P}_{\lambda}^{\Lambda} \begin{bmatrix} a \\ b \end{bmatrix} \frac{\Theta_{\lambda, k}}{\eta^{\dim(\mathfrak{t})}} \tag{5.18}$$

where the theta function of the $\widehat{\mathfrak{su}}_3$ affine algebra reads, for a generic weight $\lambda = m_i \lambda_f^i$ (see App. D):

$$\Theta_{\lambda,k} = \sum_{\gamma \in \mathbf{M} + \frac{\lambda}{k}} q^{\frac{k}{2} \kappa(\gamma, \gamma)} = \sum_{N^1, N^2 \in \mathbb{Z}} q^{\frac{k}{2} \left\| N^1 \alpha_1 + N^2 \alpha_2 + \frac{m_1 \lambda_f^1 + m_2 \lambda_f^2}{k} \right\|^2}. \tag{5.19}$$

To obtain an anomaly-free model it is natural to associate this model with an abelian coset decomposition of an $SU(3)_k$ current algebra made with free fermions of the gauge sector. Thus if the gauge group contains e.g. an $SU(3)_1$ unbroken factor their characters can be decomposed as

$$\bar{\chi}^{\bar{\lambda}} = \sum_{\bar{\lambda} = \bar{n}_i \lambda_f^i \in \mathbf{M}^* \bmod \mathbf{M}} \bar{\omega}_{\bar{\lambda}}^{\bar{\lambda}} \bar{\Theta}_{\bar{\lambda}}. \tag{5.20}$$

Again we will perform the asymmetric deformation as a boost between the Cartan lattices of the left $\widehat{\mathfrak{su}}_3$ algebra at level k and the right $\widehat{\mathfrak{su}}_3$ lattice algebra at level one coming from the gauge sector. So after the infinite deformation we will get the quantization condition $\sqrt{k} = p$ and the constraint

$$\lambda + p \bar{\lambda} \equiv 0 \bmod p \mathbf{M} =: p \mu, \quad \mu \in \mathbf{M}. \tag{5.21}$$

So we get a different result compared to the Kazama-Suzuki construction. It is so because the constraints that we get at the critical point force the weight lattice of the $\widehat{\mathfrak{su}}_3$ at level k to be projected onto p times the $\widehat{\mathfrak{su}}_3$ weight lattice at level one of the fermions. This model does not correspond to a Kählerian manifold and should correspond to the $SU(3)$ -invariant metric on the flag space. With the KS method we get instead a projection onto p times a lattice of $\widehat{\mathfrak{su}}_3$ at level one which is dual to the orthogonal sublattice defined by $\alpha_1 \mathbb{Z} + (\alpha_1 + 2\alpha_2) \mathbb{Z}$ – in other words the lattice obtained with the Gell-Mann Cartan generators. In this case it is possible to decompose the model in KS cosets models corresponding to Hermitean symmetric spaces.¹¹

We have seen in Sect. 4 that, in the gauging approach, ones obtains the Kähler metric automatically when one starts from the $SU(2)$ fibration over $SU(3)/SU(2)$ rather than from the WZW model $SU(3)$. It is now very easy to understand why it is the case. Indeed once the $SU(2)$ has been taken out of the $SU(3)$, the only $U(1)$ that can be gauged (or deformed) is the $U(1)$ orthogonal to the root α_1 of the $SU(2)$ subalgebra, thus must be along $\alpha_1 + 2\alpha_2$. This allows to decompose the abelian coset into KS models for Hermitean symmetric spaces, and the model corresponds to the Kählerian metric on the flag space.

5.2 Generalization

The previous construction can be easily generalized to any affine Lie algebra, but the formalism gets a little bit bulky. We will consider separately all the families of simple Lie algebras, since the construction differ significantly. We will mainly focus below on the KS decomposition method.

5.2.1 A_n algebras

For an $SU(n+1)$ WZW model we use the following decomposition in terms of $N = 2$ Kazama-Suzuki models:

$$\begin{aligned} & SU(n+1)_{k-n-1} \times SO(n^2 + 2n)_1 \\ & \rightarrow \frac{SU(n+1)_{k-n-1} \times SO(2n)_1}{SU(n)_{k-n} \times U(1)_{n(n+1)k/2}} \times \frac{SU(n)_{k-n} \times SO(2(n-1))_1}{SU(n-1)_{k-n+1} \times U(1)_{(n-1)nk/2}} \times \dots \times \frac{SU(2)_{k-2} \times SO(2)_1}{U(1)_k} \\ & \times SO(n)_1 \times U(1)_{n(n+1)k/2} \times U(1)_{(n-1)nk/2} \times \dots \times U(1)_k. \end{aligned} \tag{5.22}$$

¹¹ For the symmetrically gauged WZW models, this has been studied in [30].

So the left worldsheet fermions of $SO(n^2 + 2n)_1$ are decomposed into

$$SO(n^2 + 2n) \rightarrow SO(2n)_1 \times SO(2(n-1))_1 \times \dots \times SO(2)_1 \times SO(n)_1 \tag{5.23}$$

where n fermions, corresponding to the $SO(n)_1$ factor, are neutral. The Kazama-Suzuki decomposition of the characters reads

$$\chi^{\Lambda} \Xi_{2n}^{s_{2n}} \Xi_{2(n-1)}^{s_{2(n-1)}} \dots \Xi_2^{s_2} \Xi_n^{s_1} = \sum_{\Lambda^1, \Lambda^2, \dots, j} \sum_{m_1 \in \mathbb{Z}_{n(n+1)k}} \sum_{m_2 \in \mathbb{Z}_{(n-1)nk}} \dots \sum_{m_n \in \mathbb{Z}_{2k}} \tag{5.24}$$

$$C_{\Lambda^1, m_1}^{\Lambda, (s_{2n})} C_{\Lambda^2, m_2}^{\Lambda^1, (s_{2(n-1)})} \dots C_{m_n}^{j, (s_2)} \Xi_n^{s_1} \times \frac{\Theta_{m_1, \frac{n(n+1)k}{2}}}{\eta} \frac{\Theta_{m_2, \frac{(n-1)nk}{2}}}{\eta} \dots \frac{\Theta_{m_n, k}}{\eta}$$

where the sum on $\Lambda^1, \Lambda^2, \dots, j$ is taken over integrable representations (see App. D) of $SU(n), SU(n-1), \dots, SU(2)$. For the right fermions of the gauge sector the story is the same as for the $SU(3)$ example. We will need $n(n+1)$ free fermions realizing an $SO(n^2 + n)_1$ algebra, in order to use the Kazama-Suzuki decomposition for the A_n model at level $k = n + 1$, such that the bosonic part trivializes

$$SO(n^2 + n)_1 \rightarrow \frac{SO(2n)_1}{SU(n)_1 \times U(1)_{\frac{n(n+1)^2}{2}}} \times \frac{SU(n)_1 \times SO(2(n-1))_1}{SU(n-1)_2 \times U(1)_{\frac{(n-1)n(n+1)}{2}}} \times \dots \times \frac{SU(2)_{n-1} \times SO(2)_1}{U(1)_{n+1}} \times U(1)_{\frac{n(n+1)^2}{2}} \times U(1)_{\frac{(n-1)n(n+1)}{2}} \times \dots \times U(1)_{n+1}. \tag{5.25}$$

So we can write the decomposition in terms of coset characters as

$$\Xi_{n(n+1)}^{\bar{s}_{n(n+1)}} = \sum_{\bar{s}_{2n}, \bar{s}_{2(n-1)} \dots \bar{s}_2 \in \mathbb{Z}_4} C[\bar{s}_{n(n+1)}; \bar{s}_{2n}, \bar{s}_{2(n-1)}, \dots, \bar{s}_2] \sum_{u_1 \in \mathbb{Z}_{n(n+1)^2}} \sum_{u_2 \in \mathbb{Z}_{(n-1)n(n+1)}} \dots \sum_{u_n \in \mathbb{Z}_{2(n+1)}} \tag{5.26}$$

$$\bar{\omega}_{\Lambda^1, u_1}^{\bar{s}_{n(n+1)}} \bar{\omega}_{\Lambda^2, u_2}^{\bar{s}_{2n}} \dots \bar{\omega}_{u_n}^{j, \bar{s}_2} \times \frac{\Theta_{u_1, \frac{n(n+1)^2}{2}}}{\bar{\eta}} \frac{\Theta_{u_2, \frac{(n-1)n(n+1)}{2}}}{\bar{\eta}} \dots \frac{\Theta_{u_n, n+1}}{\bar{\eta}}.$$

For the left coset to exist one has to assume the following constraint on the level of the A_n affine algebra

$$\sqrt{\frac{k}{n+1}} = p \in \mathbb{Z}. \tag{5.27}$$

Then the decomposition can be carried out straightforwardly, by mixing the lattices of the holomorphic theta function for the decomposition (5.24) and the decomposition (5.25). We get the following constraints

$$\begin{cases} m_1 = p [n(n+1)^2 \mu_1 - u_1], & \mu_1 \in \mathbb{Z}_p, \\ m_2 = p [(n-1)n(n+1) \mu_2 - u_2], & \mu_2 \in \mathbb{Z}_p, \\ \dots \\ m_n = p [2(n+1) \mu_n - u_n], & \mu_n \in \mathbb{Z}_p. \end{cases} \tag{5.28}$$

So at the end we can remove the $U(1)^n$ free CFT contribution and we get the following ‘‘partition function’’ for the $SU(n+1)/U(1)^n$ left coset, with $N = 2$ worldsheet superconformal symmetry

$$Z_{F_{n+1}}^{(s_{2n}, \dots, s_2; \bar{s}_{n(n+1)})}(\tau) = \sum_{\Lambda} \sum_{\Lambda^1, \Lambda^2, \dots, j} \sum_{m_1 \in \mathbb{Z}_{n(n+1)k}} \sum_{m_2 \in \mathbb{Z}_{(n-1)nk}} \dots \sum_{m_n \in \mathbb{Z}_k}$$

$$\sum_{\bar{s}_{2n}, \bar{s}_{2(n-1)} \dots \bar{s}_2 \in \mathbb{Z}_4} C[\bar{s}_{n(n+1)}; \bar{s}_{2n}, \bar{s}_{2(n-1)}, \dots, \bar{s}_2] \sum_{u_1 \in \mathbb{Z}_{n(n+1)^2}} \sum_{u_2 \in \mathbb{Z}_{(n-1)(n+1)}} \dots \sum_{u_n \in \mathbb{Z}_{2(n+1)}} \\ \sum_{\mu_1, \dots, \mu_n \in \mathbb{Z}_p} C_{\Lambda^1, p[n(n+1)^2 \mu_1 - u_1]}^{\Lambda, (s_{2n})} C_{\Lambda^2, p[(n-1)n(n+1)\mu_2 - u_2]}^{\Lambda^1, (s_{2(n-1)})} \dots C_{p[2(n+1)\mu_n - u_n]}^{j, (s_2)} \\ \times \bar{\chi}^{\Lambda} \bar{\omega}_{\Lambda_1, u_1}^{\bar{s}_{n(n+1)}} \bar{\omega}_{\Lambda_2, u_2}^{\bar{s}_{2n}} \dots \bar{\omega}_{u_n}^{j, \bar{s}_2}. \tag{5.29}$$

As in the previous example this characters combination behaves covariantly under modular transformation, i.e. is modular invariant up to the transformation of the fermionic indices $\{s_i\}$ and $\bar{s}_{n(n+1)}$. The modular invariance of the complete heterotic string background will be ensured by an appropriate Gepner construction.

Now let us consider the other simple Lie algebras. For sake of brevity we will only sketch the method, which is quite parallel to the present case.

5.2.2 B_n algebras

In this case, the relevant Kazama-Suzuki $N = 2$ coset model is

$$\frac{SO(2n+1)_{k-2n+1} \times SO(4n-2)_1}{SO(2n-1)_{k-2n+3} \times U(1)_{2k}} \tag{5.30}$$

therefore the decomposition in $N = 2$ models of the group manifold is

$$SO(2n+1)_{k-2n+1} \times SO(n(2n+1))_1 \\ \rightarrow \frac{SO(2n+1)_{k-2n+1} \times SO(4n-2)_1}{SO(2n-1)_{k-2n+3} \times U(1)_{2k}} \times \frac{SO(2n-1)_{k-2n+3} \times SO(4n-6)_1}{SO(2n-3)_{k-2n+5} \times U(1)_{2k}} \\ \times \dots \times \frac{SO(3)_{k-1} \times SO(2)_1}{U(1)_{2k}} \times SO(n)_1 \times (U(1)_{2k})^n. \tag{5.31}$$

So there are no specific constraints on the right fermions of the gauge sector. We only need to pick up n complex fermions with arbitrary boundary conditions, realizing an $[SO(2)_1]^n$ algebra¹². The level of the $SO(2n+1)$ has to be quantized as $\sqrt{k} \in \mathbb{N}$. Under this condition the deformation can be carried out straightforwardly.

5.2.3 C_n algebras

We consider here the KS cosets

$$\frac{Sp(2n)_{k-n-1} \times SO(n(n+1))_1}{SU(n)_{2k-n} \times U(1)_{nk}}. \tag{5.32}$$

So apart from the first step the decomposition follows the pattern for A_n algebras

$$Sp(2n)_{k-n-1} \times SO(n(2n+1))_1 \\ \rightarrow \frac{Sp(2n)_{k-n-1} \times SO(n(n+1))_1}{SU(n)_{2k-n} \times U(1)_{nk}} \times \frac{SU(n)_{2k-n} \times SO(2(n-1))_1}{SU(n-1)_{2k-n+1} \times U(1)_{(n-1)nk}}$$

¹² Of course this algebra may be enhanced in the specific model at hand but this is not necessary. Note also that there is another construction when ones starts with and $SO(2n^2)_1$ algebra in the gauge sector and decompose it in terms of the B_n Kazama-Suzuki model at level $2n - 1$.

$$\begin{aligned} & \times \dots \times \frac{SU(2)_{2k-2} \times SO(2)_1}{U(1)_{2k}} \\ & \times SO(n)_1 \times U(1)_{nk} \times U(1)_{(n-1)nk} \times U(1)_{(n-2)(n-1)k} \times \dots \times U(1)_{2k}. \end{aligned} \quad (5.33)$$

Then one need in the gauge sector an $SO(2n^2)_1$ algebra that will be split according to the purely fermionic Kazama-Suzuki decomposition for C_n , together with the quantization condition

$$\sqrt{\frac{k}{n+1}} \in \mathbb{N}. \quad (5.34)$$

Then the deformation will lead to the partition function for the abelian coset space.

5.2.4 D_n algebras

We consider here the KS cosets

$$\frac{SO(2n)_{k-2n+2} \times SO(n(n-1))_1}{SU(n)_{k-n} \times U(1)_{2nk}}. \quad (5.35)$$

This case is very close to the last one. We have the decomposition

$$\begin{aligned} & SO(2n)_{k-2n+2} \times SO(n(2n-1))_1 \\ & \rightarrow \frac{SO(2n)_{k-2n+2} \times SO(n(n-1))_1}{SU(n)_{k-n} \times U(1)_{2nk}} \times \frac{SU(n)_{k-n} \times SO(2(n-1))_1}{SU(n-1)_{k-n+1} \times U(1)_{(n-1)nk/2}} \\ & \times \dots \times \frac{SU(2)_{k-2} \times SO(2)_1}{U(1)_k} \\ & \times SO(n)_1 \times U(1)_{2nk} \times U(1)_{(n-1)nk/2} \times U(1)_{(n-2)(n-1)k/2} \times \dots \times U(1)_k. \end{aligned} \quad (5.36)$$

So the fermions of the gauge sector have to realize an $SO[2n(n-1)]_1$ algebra, together with the quantization condition

$$\sqrt{\frac{k}{2n-2}} \in \mathbb{N}. \quad (5.37)$$

5.2.5 Exceptional algebras

The two exceptional algebras corresponding to WZW models that can be decomposed into KS models are E_6 and E_7 . In the first case, we have the decomposition

$$\begin{aligned} & (E_6)_{k-12} \times SO(78)_1 \\ & \rightarrow \frac{(E_6)_{k-12} \times SO(32)_1}{SO(10)_k \times U(1)_{6k}} \times \frac{SO(10)_k \times SO(20)_1}{SU(5)_{k+3} \times U(1)_{10(k+8)}} \times \frac{SU(5)_{k+3} \times SO(8)_1}{SU(4)_{k+4} \times U(1)_{10(k+8)}} \\ & \times \dots \times \frac{SU(2)_{k+6} \times SO(2)_1}{U(1)_{k+8}} \times SO(6)_1 \times U(1)_{6k} \times U(1)_{10(k+8)} \\ & \times U(1)_{10(k+8)} \times U(1)_{6(k+8)} \times U(1)_{3(k+8)} \times U(1)_{k+8}. \end{aligned} \quad (5.38)$$

However we would need an $SO(72)_1$ algebra from the gauge sector, which doesn't "fit" into the $SO(32)$ or $E_8 \times E_8$ algebra of the heterotic strings. The same holds for E_7 ; in this case we have the decomposition

$$(E_7)_{k-18} \times SO(133)_1 \rightarrow \frac{(E_7)_{k-18} \times SO(54)_1}{(E_6)_k \times U(1)_{3k}} \times \frac{(E_6)_k \times SO(32)_1}{SO(10)_{k+12} \times U(1)_{6(k+12)}} \times \dots \quad (5.39)$$

5.3 Kazama-Suzuki decomposition vs. abelian quotient

In this section we would like to stress the ambiguity in defining an abelian coset of WZW models. We will consider the A_n case in the discussion, although it's pretty much the same for the other classical Lie algebras.

An abelian super-coset $G \times SO(\# \mathfrak{g} - d)/U(1)^d$, (with $\hat{\mathfrak{g}}$ at level $k - g^*$) must be supplemented with the definition of the action of the abelian subgroup in g , corresponding to a choice of a particular sub-lattice of $\Gamma \in \sqrt{k}M$ (these issues have been discussed in [30] for symmetric supercosets of type II superstrings). In our construction, the left-coset structure will require that, in order to achieve modular invariance, the lattice behaves covariantly as some combination of right-moving fermions of the gauge sector of the heterotic string. It will be possible only if the level of the $\hat{\mathfrak{g}}$ affine algebra obeys a special quantization condition. In the KS construction we define with these right-moving fermions an orthogonal lattice; therefore we have also to choose an orthogonal sub-lattice of the root lattice for the WZW model in order to make this construction possible.

For the A_n algebra, the relevant orthogonal basis is written as follows¹³:

$$\begin{cases} \nu_1 = \sqrt{k}\alpha_1 & (\nu_1, \nu_1) = 2k, \\ \nu_2 = \sqrt{k}(\alpha_1 + 2\alpha_2) & (\nu_2, \nu_2) = 6k, \\ \nu_3 = \sqrt{k}(\alpha_1 + 2\alpha_2 + 3\alpha_3) & (\nu_3, \nu_3) = 12k, \\ \dots \\ \nu_n = \sqrt{k}(\alpha_1 + 2\alpha_2 + \dots + n\alpha_n) & (\nu_n, \nu_n) = n(n+1)k, \end{cases} \tag{5.40}$$

and is of course a sub-lattice of the complete root lattice. More precisely it corresponds to

$$\sqrt{k}\Gamma = \sqrt{k} \bigoplus_{a=1}^n a\mathbb{Z}\alpha_a \subset \sqrt{k} \bigoplus_{a=1}^n \mathbb{Z}\alpha_a. \tag{5.41}$$

Then the associated theta-functions of $\hat{\mathfrak{su}}_{n+1}$ can be written as a product of usual $\hat{\mathfrak{su}}_2$ theta functions

$$\Theta_{\lambda, k}^{(\Gamma)} = \prod_{a=1}^n \Theta_{m_a, a(a+1)k/2} \quad \text{with} \quad \lambda = \sum m_a \nu_a^*. \tag{5.42}$$

This choice of orthogonal basis allows actually to decompose the abelian coset into a chain of $N = 2$ Kazama-Suzuki models. Indeed we have to choose the lattice of the $U(1)$ in $SU(n+1)/SU(n) \times U(1)$ to be $\mathbb{Z}\nu_n$, such that it will be orthogonal to the root lattice of \mathfrak{su}_{n-1} given by $\sum_{a=1}^{n-1} \mathbb{Z}\alpha_a$, thus allowing to gauge it.

The left coset corresponding to this choice of abelian subgroup is obtained by a marginal deformation with the operator $\bigoplus_a (\nu_a, \mathbb{H})$. Its partition function is composed of the coset characters obtained through the branching relation

$$\chi^\Lambda \prod_{r=1}^{n^2+n} \Theta_{s_r, 2} = \sum_{\lambda = \lambda_a \nu_a^* \in \Gamma^* \text{ mod } k\Gamma} C_\lambda^\Lambda(s_1, \dots, s_{n^2+n}) \prod_{a=1}^n \Theta_{\lambda_a, a(a+1)k/2}. \tag{5.43}$$

On the other hand, in the standard $N = 1$ abelian coset construction the characters of the affine Lie algebra are decomposed according to the full $\sqrt{k}M$ lattice. The left coset is obtained by a marginal deformation with the operator $\bigoplus_a (\alpha_a, \mathbb{H})$. The relevant coset characters are given by

$$\chi^\Lambda \prod_{r=1}^{n^2+n} \Theta_{s_r, 2} = \sum_{\lambda \in M^* \text{ mod } kM} \tilde{C}_\lambda^\Lambda(s_1, \dots, s_{n^2+n}) \Theta_{\lambda, k}. \tag{5.44}$$

¹³ In the case of A_2 , we find the Gell-Mann matrices of $SU(3)$ (C.10).

As in the A_3 case, we can show that the *left cosets* corresponding to these two classes of models are different. They are in correspondence with the different possible metrics (Kählerian and non-Kählerian) on asymmetric cosets spaces discussed in appendix A.

6 New linear dilaton backgrounds of heterotic strings

These left-coset superconformal field theories can be used to construct various supersymmetric exact string backgrounds. The first class are generalizations of Gepner models [21] and Kazama-Suzuki constructions [22] using the left cosets as building blocks for the internal SCFT. For instance to construct a four-dimensional string theory one takes a product of left cosets such that the central charges add up to $c = 9$. This has already been considered in [3] for the S^2 coset¹⁴ but can be extended using the new theories constructed above. In this case there is no geometric interpretation from the sigma model point of view since these theories have no semi-classical limit. Indeed the levels of the cosets are frozen because their central charge must add up to $c = 9$ (in the case of four-dimensional compactification). However we expect that they correspond to special points in the moduli spaces of supersymmetric compactifications, generalizing the Gepner points of the CY manifolds.

Another type of models are the left cosets analogues of the NS5-branes solutions [9, 10] and of their extensions to more generic supersymmetric vacua with a dilaton background. It was shown in [13] that a large class of these linear dilaton theories are dual to singular CY manifolds in the decoupling limit. An extensive review of the different possibilities in various dimensions has been given in [30] with all the possible G/H cosets. The left cosets that we constructed allow to find new solutions of this type in heterotic strings, with a different geometrical interpretation since our cosets differ from ordinary gauged WZW model. However the superconformal structure of the left sector of our models is exactly the same as for the corresponding gauged WZW – except that the values of the $N=2$ R-charges that appear in the spectrum are constrained – so we can carry over all the known constructions to the case of the geometric cosets.

In the generic case these constructions involve non-abelian cosets, and as we showed the asymmetric deformations and gaugings apply only to the abelian components. Thus in general we will get mixed models which are gauged WZW models w.r.t. the non-abelian part of H and geometric cosets w.r.t. the abelian components of H . Below we will focus on purely abelian examples, i.e. corresponding to geometric cosets. The dual interpretation of these models, in terms of the decoupling limit of some singular compactification manifolds, is not known. Note however that by construction there are about \sqrt{k} times less massless states in our models than in the standard left-right symmetric solutions. Therefore they may correspond to some compactifications with (NS-NS and magnetic) fluxes, for which the number of moduli is reduced. It would be very interesting to investigate this issue further.

Six-dimensional model. We consider here the critical superstring background

$$\mathbb{R}^{5,1} \times \frac{SL(2, \mathbb{R})_{k+2} \times SO(2)_1}{U(1)_k} \times \left[U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right] \quad (6.1)$$

the second factor being a left coset CFT as discussed in this paper. This is the direct analogue of the five-brane solution, or more precisely of the double scaling limit of NS5-branes on a circle [12, 31], in the present case with magnetic flux. This theory has $N = 2$ charges but, in order to achieve spacetime supersymmetry one must project onto odd-integral $N = 2$ charges on the left-moving side, as in the type II construction [31]. This can be done in the standard way by orbifoldizing the left $N = 2$ charges of the two cosets.

¹⁴ Only in the simplest case $k_g = 2$.

Four-dimensional model. A simple variation of the six-dimensional theory is given by

$$\mathbb{R}^{3,1} \times \frac{SL(2, \mathbb{R})_{k/2+2} \times SO(2)_1}{U(1)_{2k}} \times \left[U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right] \times \left[U(1)_k \backslash SU(2)_{k-2} \times SO(2)_1 \right] \quad (6.2)$$

which is the magnetic analogue of the (deformation of) intersecting five-branes solution. Also here an orbifoldization of the left $N = 2$ charges is needed to achieve space-time supersymmetry.

Three-dimensional models: the flagbrane. We can construct the following background of the G_2 holonomy type, as in the case of symmetric coset [32]

$$\mathbb{R}^{2,1} \times \mathbb{R}_Q \times \left[U(1)_k \times U(1)_{3k} \backslash SU(3)_{k-3} \times SO(6)_1 \right] \quad (6.3)$$

and the non-trivial part of the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + \frac{k}{4r^2} [dr^2 + 4r^2 ds^2(SU(3)/U(1)^2)] . \quad (6.4)$$

Without the factor of four it would be a direct analogue of the NS5-brane, being conformal to a cone over the flag space.

Another possibility in three dimensions is to lift the $SL(2, \mathbb{R})/U(1)$ coset to the group manifold $SL(2, \mathbb{R})$. In this case, as for the standard gauged WZW construction [33] we will get the following anti-de Sitter background

$$SL(2, \mathbb{R})_{k/4+2} \times \left[U(1)_{3k} \backslash SU(3)_{k-3} \times SO(6)_1 \right] \quad (6.5)$$

and the left moving sector of this worldsheet CFT allows to construct an $N = 3$ superconformal algebra in spacetime.

Two-dimensional model. In this case we can construct the background

$$\mathbb{R}^{1,1} \times \frac{SL(2, \mathbb{R})_{\frac{k+2}{4}} \times SO(2)_1}{U(1)_{4k}} \times \frac{U(1)_{3k} \backslash SU(3)_{k-3} \times SO(6)_1}{U(1)_k} \quad (6.6)$$

which corresponds in the classification of [30] to a non-compact manifold of $SU(4)$ holonomy once the proper projection is done on the left $N = 2$ charges. This solution is asymptotically conformal to a cone over the Einstein space $SU(3)/U(1)$. Using the same methods as for the NS5-branes in [31], we can show that the full solution corresponding to the model (6.6) can be obtained directly as the null super-coset

$$\frac{SL(2, \mathbb{R})_{k/4} \times U(1) \backslash SU(3)_k}{U(1)_L \times U(1)_R} \quad (6.7)$$

where the action is along the elliptic generator in the $SL(2, \mathbb{R})$, with a normalization $\text{Tr}[(t^3)^2] = -4$, and along the direction $\alpha_1 + 2\alpha_2$ in the coset space $U(1) \backslash SU(3)$, with a canonical normalization. For $r \rightarrow \infty$ the solution asymptotes the cone but when $r \rightarrow 0$ the strong coupling region is smoothly capped by the cigar.

Acknowledgements We would like to thank E. Cremmer, B. Julia, and J. Troost for interesting discussions. The work of D. I. has been supported by a Golda Meir Fellowship, the ISF Israel Academy of Sciences and Humanities and the GIF German-Israel Binational Science Found.

A Coset space geometry

Coset spaces have been extensively studied in the mathematical literature of the last fifty years. In this appendix we limit ourselves to collect some classical results mainly dealing with the geometric interpretation. In particular we will follow the notations of [34].

Let G be a semisimple Lie group and $H \in G$ a subgroup. As in the rest of the paper, upper-case indices $\{M, N, O\}$ refer to the whole group (algebra) G , lower-case indices $\{m, n, o\}$ to the subgroup (subalgebra) and Greek indices $\{\mu, \nu, \omega\}$ to the coset.

It is useful to explicitly write down the commutation relations, separating the generators of H and G/H

$$[T_m, T_n] = f_{mn}^o T_o, \quad [T_m, T_\nu] = f_{m\nu}^\omega T_\omega, \quad (\text{A.1a})$$

$$[T_\mu, T_\nu] = f_{\mu\nu}^\rho T_\rho + f_{\mu\nu}^\omega T_\omega. \quad (\text{A.1b})$$

Of course there are no f_{mn}^ω terms since H is a group. G/H is said to be symmetric if $f_{\mu\nu}^\omega \equiv 0$, i.e. if the commutator of any couple of coset elements lives in the dividing subgroup. In this case a classical theorem states that the coset only admits one left-invariant Riemann metric that is obtained as the restriction of the Cartan-Killing metric defined on G (see e.g. [19]). This is not the case when H is the maximal torus (except for the most simple case $G = SU(2)$) and the coset manifold accepts different structures.

Any metric (or, more generally, any degree-2 covariant tensor) on G/H can be put in the form

$$g = g_{\mu\nu}(x) J^\mu \otimes J^\nu. \quad (\text{A.2})$$

One can show that the G invariance of g is equivalent to

$$f_{a\mu}^\kappa g_{\kappa\nu}(x) + f_{a\nu}^\kappa g_{\kappa\mu}(x) = 0, \quad (\text{A.3})$$

and the homogeneity imposes

$$g_{ij} = \text{constant}. \quad (\text{A.4})$$

Both conditions are easily satisfied by $g_{\mu\nu} \propto \delta_{\mu\nu}$ (this is the metric on G/H that we obtained in eq. (2.27)). The Levi-Civita connection 1-forms ω_{ν}^{μ} of g are determined by

$$d g_{\mu\nu} - \omega_{\mu}^{\kappa} g_{\kappa\nu} - \omega_{\nu}^{\kappa} g_{\mu\kappa} = 0, \quad (\text{A.5a})$$

$$d J^\mu + \omega_{\nu}^{\mu} \wedge J^\nu = 0, \quad (\text{A.5b})$$

and are explicitly written in terms of the structure constants as

$$\omega_{\nu}^{\mu} = f_{a\nu}^{\mu} J^a + D_{\rho\nu}^{\mu} J^\rho \quad (\text{A.6})$$

where $D_{\rho\nu}^{\mu}$ can be separated into its symmetric and antisymmetric parts as follows

$$D_{\rho\nu}^{\mu} = \frac{1}{2} f_{\rho\nu}^{\mu} + K_{\rho\nu}^{\mu}, \quad (\text{A.7a})$$

$$K_{\rho\nu}^{\mu} = \frac{1}{2} \left(g^{\mu\sigma} f_{\sigma\rho}^{\omega} g_{\omega\nu} + g^{\mu\sigma} f_{\sigma\nu}^{\omega} g_{\omega\rho} \right). \quad (\text{A.7b})$$

We can then derive the curvature 2-form $\Omega = d\omega + \omega \wedge \omega$

$$\Omega_{\nu}^{\mu} = \left(D_{\sigma\nu}^{\rho} D_{\kappa\rho}^{\mu} - D_{\kappa\nu}^{\rho} D_{\sigma\rho}^{\mu} - f_{\kappa\sigma}^a f_{a\nu}^{\nu} - f_{\kappa\sigma}^{\rho} D_{\rho\nu}^{\mu} \right) \frac{J^\kappa \wedge J^\sigma}{2}. \quad (\text{A.8})$$

the Riemann tensor

$$R_{\nu\kappa\sigma}^{\mu} = -f_{\kappa\sigma}^a f_{a\nu}^{\mu} - \frac{1}{2} f_{\kappa\sigma}^{\rho} f_{\rho\nu}^{\mu} + \frac{1}{4} f_{\nu\kappa}^{\rho} f_{\sigma\rho}^{\mu} + \frac{1}{4} f_{\nu\sigma}^{\rho} f_{\rho\kappa}^{\mu} + \frac{1}{2} f_{\nu\kappa}^{\rho} K_{\sigma\rho}^{\mu} + \frac{1}{2} f_{\nu\sigma}^{\rho} K_{\rho\kappa}^{\mu}$$

$$-\frac{1}{2}f^\mu_{\rho\kappa}K^\rho_{\sigma\nu} - \frac{1}{2}f^\mu_{\rho\sigma}K^\rho_{\kappa\nu} - f^\rho_{\kappa\sigma}K^\mu_{\rho\nu} + K^\rho_{\sigma\nu}K^\mu_{\kappa\rho} - K^\rho_{\kappa\nu}K^\mu_{\sigma\rho} \tag{A.9}$$

and the Ricci tensor

$$\begin{aligned} Ric_{\nu\sigma} = R^\mu_{\nu\mu\sigma} = & -f^\alpha_{\mu\nu}f^\mu_{\alpha\nu} - \frac{1}{2}f^\rho_{\mu\sigma}f^\mu_{\rho\nu} + \frac{1}{4}f^\rho_{\nu\mu}f^\mu_{\sigma\rho} + \frac{1}{2}f^\rho_{\nu\mu}K^\mu_{\sigma\rho} + \frac{1}{2}f^\mu_{\rho\sigma}K^\rho_{\mu\nu} \\ & - \frac{1}{2}f^\rho_{\mu\sigma}K^\mu_{\rho\nu} - \frac{1}{2}K^\rho_{\mu\nu}K^\mu_{\sigma\rho}. \end{aligned} \tag{A.10}$$

In particular, in the case of $g_{\mu\nu} = \delta_{\mu\nu}$ the expressions are greatly simplified because the antisymmetric part $K^\mu_{\nu\rho}$ vanishes and then the Riemann and Ricci tensors are respectively given by

$$R^\mu_{\nu\kappa\sigma} = -f^\alpha_{\kappa\sigma}f^\mu_{\alpha\nu} - \frac{1}{2}f^\rho_{\kappa\sigma}f^\mu_{\rho\nu} + \frac{1}{4}f^\rho_{\nu\kappa}f^\mu_{\sigma\rho} + \frac{1}{4}f^\rho_{\nu\sigma}f^\mu_{\kappa\rho}, \tag{A.11}$$

$$Ric_{\nu\sigma} = -f^\alpha_{\mu\sigma}f^\mu_{\alpha\nu} - \frac{1}{4}f^\rho_{\mu\sigma}f^\mu_{\rho\nu}. \tag{A.12}$$

Another fact that we used in the paper about G/H cosets is a construction due to Borel [17, 35] of a Kähler structure over G/T where T is the maximal torus. First of all we remark that such a coset can be given a \mathbb{C} structure when associating holomorphic and anti-holomorphic sectors to positive and negative roots respectively. One can then show that the $(1, 1)$ form defined as

$$\omega = \frac{i}{2} \sum_{\alpha>0} c_\alpha \mathcal{J}^\alpha \wedge \mathcal{J}^{\bar{\alpha}} \tag{A.13}$$

is closed if and only if for each subset of roots $\{\alpha, \beta, \gamma\}$ such as $\alpha = \beta + \gamma$, the corresponding real coefficients c_α satisfy the condition $c_\alpha = c_\beta + c_\gamma$. Of course this is equivalent to say that the tensor

$$g = \sum_{\alpha>0} c_\alpha \mathcal{J}^\alpha \otimes \mathcal{J}^{\bar{\alpha}} \tag{A.14}$$

is a Kähler metric on G/T .

In particular, if we consider the $SU(3)$ group, for the $\mathfrak{su}(3)$ algebra we can choose the Gell-Mann λ matrices (C.10) as a basis. In this case if we divide by the $U(1) \times U(1)$ subgroup generated by $\langle \lambda_3, \lambda_8 \rangle$, the most general metric satisfying (A.3) has the form $g = \text{diag}\{a, a, b, b, c, c\}$ i.e. $SU(3)/U(1) \times U(1)$ admits a three parameter family of metrics. Among them, the moduli space lines $a = b = c$ (the metric obtained in Sect. 3.2) and $a = b = c/2$ (the metric in Sect. 4.2) represent Einstein structures (with Ricci scalar $15/a$ and $12/a$ respectively). In both cases the manifold can be endowed with complex structures (positive and negative roots respectively generating the holomorphic and anti-holomorphic sectors) but only the latter admits a Kähler structure (in this way we obtain the so-called flag space F_3).

B Equations of motion

B.1 Explicit derivation of some terms

In this appendix we explicitly derive the expressions for the $F^a_{\mu\rho}F^a_{\nu}{}^{\rho}$ and $H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma}$ terms appearing in the equations of motion (2.32b).

Gauge field strength. Consider the term coming from the gauge field strength. First of all we can build an orthonormal basis out of the Weyl-Cartan basis by complexifying the Cartan generators and combining opposite ladder operators as follows

$$\begin{cases} T^a = iH^a, \\ T^{2\mu-1} = i \frac{|\alpha_\mu|}{2} (E^{\alpha_\mu} - E^{-\alpha_\mu}), \\ T^{2\mu} = \frac{|\alpha_\mu|}{2} (E^{\alpha_\mu} + E^{-\alpha_\mu}), \end{cases} \tag{B.1}$$

if we write explicitly the $(F^2)_{\mu\nu}$ term as follows

$$(F^2)_{\mu\nu} \propto \sum_{m,\omega} f_{\nu\omega}^m f_{\pi\omega}^m = \sum_{m,\omega} \kappa(T^m, [T^\nu, T^\omega]) \kappa(T^m, [T^\pi, T^\omega]). \quad (\text{B.2})$$

We can see why rewriting everything this choice of basis simplifies the calculation: the only commutators that will give a non-vanishing result when projected on the Cartan generators are the ones involving opposite ladder operators¹⁵, that is $[T^{2\mu-1}, T^{2\mu}]$ which are explicitly given by

$$[T^{2\mu-1}, T^{2\mu}] = i \frac{|\alpha_\mu|^2}{4} 2 [E^{\alpha_\mu}, E^{-\alpha_\mu}] = \alpha_\mu \cdot (iH), \quad (\text{B.4})$$

this means that

$$\begin{cases} \kappa(T^m, [T^\nu, T^\omega]) = |\alpha_\mu|^m \delta_{\nu+1,\omega} & \text{if } \nu = 2\mu - 1, \\ \kappa(T^m, [T^\nu, T^\omega]) = -|\alpha_\mu|^m \delta_{\nu-1,\omega} & \text{if } \nu = 2\mu, \end{cases} \quad (\text{B.5})$$

putting this back in eq.(B.2) we find

$$\sum_{m,\omega} f_{\nu\omega}^m f_{\pi\omega}^m = \delta_{\nu\pi} \begin{cases} |\alpha_{(\nu+1)/2}|^2 & \text{if } \nu \text{ is odd,} \\ |\alpha_{\nu/2}|^2 & \text{if } \nu \text{ is even,} \end{cases} \quad (\text{B.6})$$

if \mathfrak{g} is simply laced then we can fix the normalizations to $|\alpha_\mu|^2 = \psi^2 \equiv 2$ and the above expression is greatly simplified

$$\sum_{m,\omega} f_{\nu\omega}^m f_{\pi\omega}^m = 2\delta_{\nu\pi} \quad (\text{B.7})$$

and by applying the right normalizations (see eq.(2.33)) we find that for a general algebra

$$F_{\nu\omega}^m g^{\omega\varpi} F_{\pi\varpi}^m = \frac{4}{k_g} \delta_{\nu\pi} \begin{cases} |\alpha_{(\nu+1)/2}|^2 & \text{if } \nu \text{ is odd,} \\ |\alpha_{\nu/2}|^2 & \text{if } \nu \text{ is even,} \end{cases} \quad (\text{B.8})$$

and for a simply laced one

$$F_{\nu\omega}^m g^{\omega\varpi} F_{\pi\varpi}^m = \frac{8}{k_g} \delta_{\nu\pi}. \quad (\text{B.9})$$

¹⁵ We remember that the Cartan-Weyl basis is defined by

$$[H^m, H^n] = 0, \quad (\text{B.3a})$$

$$[H^a, E^{\alpha_\mu}] = \alpha_\mu^a E^{\alpha_\mu}, \quad (\text{B.3b})$$

$$[E^{\alpha_\mu}, E^{\alpha_\nu}] = \begin{cases} N_{\mu,\nu} E^{\alpha_\mu + \alpha_\nu} & \text{if } \alpha_\mu + \alpha_\nu \in \Delta, \\ \frac{2}{|\alpha_\mu|^2} \alpha_\mu \cdot H & \text{if } \alpha_\mu = -\alpha_\nu, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B.3c})$$

NS-NS flux. From the definition of Casimir of the algebra we easily derive that

$$Q = - \sum_M \sum_O f_{NO}^M f_{OP}^M = 2g^* \delta_{NP} \tag{B.10}$$

where g^* is the dual Coxeter number. Limit N and P to j (and call them ν and π) and separate the two sums (that span over the entire algebra) into the components over j and \mathfrak{k}

$$\sum_{m \in \mathfrak{k}} \left(\sum_{o \in \mathfrak{k}} f_{\nu o}^m f_{o\pi}^m + \sum_{\omega \in j} f_{\nu\omega}^m f_{\omega\pi}^m \right) + \sum_{\mu \in j} \left(\sum_{o \in \mathfrak{k}} f_{\nu o}^\mu f_{o\pi}^\mu + \sum_{\omega \in j} f_{\nu\omega}^\mu f_{\omega\pi}^\mu \right) = -2g^* \delta_{\nu\pi} \tag{B.11}$$

now,

- the term with two elements in the Cartan is identically vanishing $f_{\nu o}^m \equiv 0$ (for two generators in \mathfrak{k} always commute)
- the terms with one component in \mathfrak{k} can be collected and interpreted as field strengths

$$\sum_{m,\omega} f_{\nu\omega}^m f_{\omega\pi}^m + \sum_{o,\mu} f_{\nu o}^\mu f_{o\pi}^\mu \tag{B.12}$$

and at the end of the day

$$\sum_{\mu,\omega} f_{\nu\mu\omega} f_{\pi\mu\omega} = 2g^* \delta_{\nu\pi} - 2 \sum_{m,\omega} f_{\nu\omega}^m f_{\pi\omega}^m \tag{B.13}$$

so that for a general algebra, using (B.6)

$$\sum_{\mu,\omega} f_{\nu\mu\omega} f_{\pi\mu\omega} = 2g^* \delta_{\nu\pi} - 2\delta_{\nu\pi} \begin{cases} |\alpha_{(\nu+1)/2}|^2 & \text{if } \nu \text{ is odd,} \\ |\alpha_{\nu/2}| & \text{if } \nu \text{ is even,} \end{cases} \tag{B.14}$$

that reduces in the simply laced case to

$$\sum_{\mu,\omega} f_{\nu\mu\omega} f_{\pi\mu\omega} = 2(g^* - 2) \delta_{\nu\pi} \tag{B.15}$$

and with the proper normalizations

$$H_{\nu\mu\omega} g^{\mu\nu} g^{\omega\varpi} H_{\pi\nu\varpi} = 2g^* \delta_{\nu\pi} - 2\delta_{\nu\pi} \begin{cases} |\alpha_{(\nu+1)/2}|^2 & \text{if } \nu \text{ is odd,} \\ |\alpha_{\nu/2}|^2 & \text{if } \nu \text{ is even,} \end{cases} \tag{B.16}$$

which reads in the simply laced case

$$H_{\nu\mu\omega} g^{\mu\nu} g^{\omega\varpi} H_{\pi\nu\varpi} = 2(g^* - 2) \delta_{\nu\pi} . \tag{B.17}$$

B.2 Equations of motion for the F_3 flag space

To verify that the background fields that we obtained in Sect. 4.2 solve the equations of motion at first order in α' it is convenient to consider the complex structure defined on the $SU(3)/U(1)^2$ coset by considering positive and negative roots as holomorphic and anti-holomorphic generators respectively.

To fix the notation let the two simple roots be

$$\alpha_1 = [\sqrt{2}, 0], \quad \alpha_2 = [-1/\sqrt{2}, \sqrt{3}/2], \tag{B.18}$$

and the third positive root $\alpha_3 = \alpha_1 + \alpha_2 = [1/\sqrt{2}, \sqrt{3}/2]$. We already know that in the complex formalism the metric is diagonal and the coefficient relative to the non-simple root is given by the sum of the two others as in eq. (A.14). With the right normalization we have the following metric and Ricci tensor

$$g_{\mu\bar{\nu}} = \frac{k}{2} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}, \quad R_{\mu\bar{\nu}} = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 4 \end{pmatrix}. \quad (\text{B.19})$$

To write the structure constants we just have to remember the defining relations for the Cartan–Weyl basis eq. (B.3): it is immediate to see that $f^1_{\mu\nu}$ and $f^2_{\mu\nu}$ are non-vanishing only if α_μ and α_ν are opposite roots (which means in turn that in our complex formalism they are represented by diagonal matrices) and, given the above choice of roots, we have

$$f^1_{\mu\bar{\nu}} = \begin{pmatrix} \sqrt{2} & & \\ & -1/\sqrt{2} & \\ & & 1/\sqrt{2} \end{pmatrix}, \quad f^2_{\mu\bar{\nu}} = \begin{pmatrix} 0 & & \\ & \sqrt{3}/2 & \\ & & \sqrt{3}/2 \end{pmatrix}. \quad (\text{B.20})$$

Let us now introduce a new tensor C that in this basis assumes the form of the unit matrix (this is indeed shown to be a tensor in App. A)

$$C_{\mu\bar{\nu}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \quad (\text{B.21})$$

We can use this tensor to define the $U(1)^2$ gauge field that supports the F_3 background as¹⁶

$$F^a_{\mu\bar{\nu}} = \sqrt{\frac{k}{2k_g}} f^a_{\mu\bar{\rho}} C^{\bar{\rho}\sigma} R_{\sigma\bar{\nu}}. \quad (\text{B.22})$$

The only non-trivial equation of motion is $\beta^G = 0$ (2.32b)

$$\beta^G = R_{\mu\bar{\nu}} - \frac{k_g}{4} F^a_{\mu\bar{\sigma}} g^{\bar{\sigma}\rho} F^a_{\rho\bar{\nu}} \quad (\text{B.23})$$

in our basis all the tensors are diagonal matrices. For this reason it is useful to pass to matrix notation. Let

$$\mathbf{G} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} \quad (\text{B.24})$$

so that the metric and the Ricci tensor are given by $\mathbf{g} = \frac{k}{2} \mathbf{G}$ and $\mathbf{R} = 2\mathbf{G}$. In this notation the above equation reads

$$\begin{aligned} \beta^G &= \mathbf{R} - \frac{k_g}{4} \sum_{a=1}^2 \sqrt{\frac{k}{2k_g}} \mathbf{f}^a \mathbf{R} \mathbf{g}^{-1} \sqrt{\frac{k}{2k_g}} \mathbf{f}^a \mathbf{R} = \mathbf{R} - \frac{k}{8} \sum_{a=1}^2 \mathbf{f}^a (2\mathbf{G}) \left(\frac{2}{k} \mathbf{G}^{-1} \right) \mathbf{f}^a \mathbf{R} \\ &= \mathbf{R} - \frac{1}{2} \sum_{a=1}^2 \mathbf{f}^a \mathbf{f}^a \mathbf{R} = 0, \end{aligned} \quad (\text{B.25})$$

since $\sum_{a=1}^2 \mathbf{f}^a \mathbf{f}^a = 2 \mathbb{1}_{3 \times 3}$ as one can see by direct inspection.

¹⁶ One can read this additional term with respect to the expression in eq. (2.28) as a way to keep track of the fact that the embedded $SU(2)$ subalgebra is at a different level with respect to the remaining currents. Actually this expression can be seen just as a generalisation of the initial one where we were restricting to cosets in which the currents played the rôle of vielbeins, i.e. in this formalism the metric was proportional to the unit matrix.

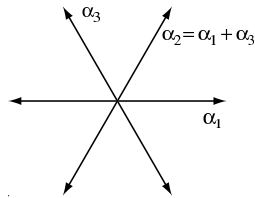


Fig. 1 Root system for $\mathfrak{su}(3)$.

C The $SU(3)$ group: an explicit parametrization

In this section we summarize some known facts about the representation of the $SU(3)$ group so to get a consistent set of conventions.

To obtain the the Cartan-Weyl basis $\{H_a, E^{\alpha_j}\}$ (defined in eq. (B.3)) for the $\mathfrak{su}(3)$ algebra we need to choose the positive roots as follows:

$$\alpha_1 = [\sqrt{2}, 0], \quad \alpha_2 = [-1/\sqrt{2}, \sqrt{3}/2], \quad \alpha_3 = [1/\sqrt{2}, \sqrt{3}/2]. \tag{C.1}$$

The usual choice for the defining representation is

$$H_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{C.2}$$

$$E_1^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $E_j^- = (E_j^+)^t$.

A good parametrisation for the $SU(3)$ group can be obtained via the Gauss decomposition: every matrix $g \in SU(3)$ is written as the product

$$g = b_- db_+ \tag{C.3}$$

where b_- is a lower triangular matrix with unit diagonal elements, b_+ is a upper triangular matrix with unit diagonal elements, and d is a diagonal matrix with unit determinant. The element g is written as

$$g(z_1, z_2, z_3, \psi_1, \psi_2) = \exp\left(z_1 E_1^- + z_2 E_3^- + \left(z_3 - \frac{z_1 z_2}{2}\right) E_2^-\right) \exp(-F_1 H_1 - F_2 H_2) \\ \times \exp\left(\bar{w}_1 E_1^+ + \bar{w}_2 E_3^+ + \left(\bar{w}_3 - \frac{\bar{w}_1 \bar{w}_2}{2}\right) E_2^+\right) \exp(i\psi_1 H_1 + i\psi_2 H_2) \tag{C.4}$$

where z_μ are 3 complex parameters, ψ_i are two real, and F_1 and F_2 are positive real functions of the z_μ 's:

$$\begin{cases} F_1 = \log f_1 = \log\left(1 + |z_1|^2 + |z_3|^2\right), \\ F_2 = \log f_2 = \log\left(1 + |z_2|^2 + |z_3 - z_1 z_2|^2\right). \end{cases} \tag{C.5}$$

By imposing $g(z_\mu, \psi_a)$ to be unitary we find that the w_μ 's are complex functions of the z_μ 's

$$\begin{cases} w_1 = -\frac{z_1 + \bar{z}_2 z_3}{\sqrt{f_2}}, \\ w_2 = \frac{\bar{z}_1 z_3 - z_2(1 + |z_1|^2)}{\sqrt{f_1}}, \\ w_3 = -(z_3 - z_1 z_2) \sqrt{\frac{f_1}{f_2}}, \end{cases} \quad (\text{C.6})$$

and the defining element $g(z_\mu, \psi_a)$ can then be written explicitly as

$$g(z_1, z_2, z_3, \psi_1, \psi_2) = \begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_3 & z_2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{f_1}} & 0 & 0 \\ 0 & \sqrt{f_1/f_2} & 0 \\ 0 & 0 & \sqrt{f_2} \end{pmatrix} \begin{pmatrix} 1 & \bar{w}_1 & \bar{w}_3 \\ 0 & 1 & \bar{w}_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\psi_1/2} & 0 & 0 \\ 0 & e^{-i(\psi_1 - \psi_2)/2} & 0 \\ 0 & 0 & e^{i\psi_2/2} \end{pmatrix}. \quad (\text{C.7})$$

Now, to build a metric for the tangent space to $SU(3)$ we can define the 1-form $\Omega(\mathbf{z}, \psi) = g^{-1}(\mathbf{z}, \psi) dg(\mathbf{z}, \psi)$ and write the Killing-Cartan metric tensor as $g_{\text{KC}} = \text{tr}(\Omega^\dagger \Omega) = -\text{tr}(\Omega \Omega)$ where we have used explicitly the property of anti-Hermiticity of Ω (that lives in the $\mathfrak{su}(3)$ algebra). The explicit calculation is lengthy but straightforward. The main advantage of this parametrization from our point of view is that it allows for a "natural" embedding of the $SU(3)/U(1)^2$ coset (see e.g. [36] or [37]): in fact in these coordinates the Kähler potential is

$$K(z_\mu, \bar{z}_\mu) = \log(f_1(z_\mu) f_2(z_\mu)) = \log\left[\left(1 + |z_1|^2 + |z_3|^2\right)\left(1 + |z_2|^2 + |z_3 - z_1 z_2|^2\right)\right] \quad (\text{C.8})$$

and the coset Kähler metric is hence simply obtained as

$$g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta = \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} K(z_\mu, \bar{z}_\mu) dz^\alpha \otimes d\bar{z}^\beta. \quad (\text{C.9})$$

Another commonly used $\mathfrak{su}(3)$ basis is given by the Gell-Mann matrices

$$\begin{aligned} \gamma_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \imath & 0 \\ \imath & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \gamma_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \gamma_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} \imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \gamma_4 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \imath \\ 0 & 0 & 0 \\ \imath & 0 & 0 \end{pmatrix}, & \gamma_5 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, & \gamma_6 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \imath \\ 0 & \imath & 0 \end{pmatrix}, \\ \gamma_7 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, & \gamma_8 &= \frac{1}{\sqrt{6}} \begin{pmatrix} \imath & 0 & 0 \\ 0 & \imath & 0 \\ 0 & 0 & -2\imath \end{pmatrix} \end{aligned} \quad (\text{C.10})$$

which presents the advantage of being orthonormal $\kappa(\lambda_i, \lambda_j) = \delta_{ij}$. In this case the Cartan subalgebra is generated by $\mathfrak{k} = \langle \lambda_3, \lambda_8 \rangle$.

D Characters of affine Lie algebras

In this section we will recall some facts about the partition functions and characters of affine Lie algebras. The characters of an affine Lie algebra $\hat{\mathfrak{g}}$ are the generating functions of the weights multiplicities in a given

irreducible representation of highest weight Λ

$$ch_{\Lambda}(\tau, \nu, u) = e^{-2i\pi k u} \sum_{\hat{\lambda} \in \text{Rep}(\Lambda)} \dim V_{\hat{\lambda}} \exp \left\{ 2i\pi \tau n + \sum_i \nu_i \kappa(e_i, \hat{\lambda}) \right\}, \quad (D.1)$$

where $\dim V_{\hat{\lambda}}$ is the multiplicity of the affine weight $\hat{\lambda} = (\lambda, k, n)$ and $\{e_i\}$ an orthonormal basis of the root space. In the framework CFT we define slightly different characters, weighted by the conformal dimension of the highest weight of the representation

$$\chi^{\Lambda}(\tau, \nu, u) = e^{-2i\pi k u} \text{Tr}_{\text{rep}(\Lambda)} \left[q^{L_0 - c/24} e^{2i\pi \kappa(\nu, \mathcal{J})} \right] = e^{2i\pi \tau \frac{\kappa(\Lambda, \Lambda + 2\rho)}{2(k + \mathfrak{g}^*)} - c/24} ch_{\Lambda}(\tau, \nu, u) \quad (D.2)$$

where $\rho = \sum_{\alpha > 0} \alpha/2$ and \mathfrak{g}^* the dual Coxeter number. To each affine weight $\hat{\lambda}$ we shall assign a theta-function as follows

$$\Theta_{\hat{\lambda}}(\tau, \nu, u) = e^{-2i\pi k u} \sum_{\gamma \in M_L + \frac{\hat{\lambda}}{k}} e^{i\pi \tau k \kappa(\gamma, \gamma)} e^{2i\pi k \kappa(\nu, \gamma)} \quad (D.3)$$

with M_L the the long roots lattice. We can write the affine characters in terms of the theta-function with the Weyl-Kač formula

$$\chi^{\Lambda}(\tau, \nu, u) = \frac{\sum_{w \in \mathbf{W}} \epsilon(w) \Theta_{w(\hat{\lambda} + \rho)}(\tau, \nu, u)}{\sum_{w \in \mathbf{W}} \epsilon(w) \Theta_{w(\rho)}(\tau, \nu, u)}, \quad (D.4)$$

\mathbf{W} being the Weyl group of the algebra and $\epsilon(w)$ the parity of the element w .

These affine characters are the building blocks of the modular invariant partition function for the WZW model, since the affine Lie algebra is the largest chiral symmetry of the theory

$$Z = \sum_{\Lambda, \bar{\Lambda}} M^{\Lambda \bar{\Lambda}} \chi^{\Lambda}(\tau, 0, 0) \bar{\chi}^{\bar{\Lambda}}(\bar{\tau}, 0, 0) \quad (D.5)$$

where the sum runs over left and right representations of \mathfrak{g} with highest weight Λ and $\bar{\Lambda}$. The representations appearing in this partition function are the *integrable* ones, which are such that

$$\text{Rep}(\Lambda) \text{ integrable} \iff \frac{2}{\kappa(\theta, \theta)} [k - \kappa(\Lambda, \theta)] \in \mathbb{N}, \quad (D.6)$$

where θ is the highest root. The matrix $M^{\Lambda \bar{\Lambda}}$ is such that the partition function of \mathfrak{g}_k is modular invariant; at least, the diagonal $\delta_{\Lambda, \bar{\Lambda}}$ exists since the characters form an unitary representation of the modular group.

In the heterotic strings, the worldsheet has a local $N = (1, 0)$ local supersymmetry so the left algebra is lifted to a super-affine Lie algebra. However the characters can be decoupled as characters of the bosonic algebra times characters of free fermions

$$Z \begin{bmatrix} a \\ b \end{bmatrix} = \sum_{\Lambda, \bar{\Lambda}} M^{\Lambda \bar{\Lambda}} \chi^{\Lambda}(\tau) \left(\frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau)}{\eta(\tau)} \right)^{\dim(\mathfrak{g})/2} \bar{\chi}^{\bar{\Lambda}} \quad (D.7)$$

where (a, b) are the spin structures of the worldsheet fermions.

The characters of the affine algebras can be decomposed according to the generalized parafermionic decomposition, by factorizing the abelian subalgebra of the Cartan torus. For example, we can decompose

the left supersymmetric \mathfrak{g}_k characters in terms of characters of the supersymmetric coset, given by the following branching relation (see [22])

$$\chi^\Lambda \left(\frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta} \right)^{\dim(\mathfrak{g})/2} = \sum_{\lambda \bmod (k+g^*)\mathbf{M}_L} \mathcal{P}_\lambda^\Lambda \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] \frac{\Theta_{\lambda, k+g^*}}{\eta^{\dim(\mathfrak{t})}} \tag{D.8}$$

in terms of the theta-functions associated to \mathfrak{g}_k .

D.1 The example of $SU(3)$

In an orthonormal basis, the simple roots of $SU(3)$ are

$$\alpha_1 = (\sqrt{2}, 0), \quad \alpha_2 = (-1/\sqrt{2}, \sqrt{3}/2). \tag{D.9}$$

The dual basis of the fundamental weights, defined by $(\lambda_f^i, \alpha_j) = \delta_j^i$ is given by

$$\lambda_f^1 = (1/\sqrt{2}, 1/\sqrt{6}), \quad \lambda_f^2 = (0, \sqrt{2}/3). \tag{D.10}$$

As they should the simple roots belong to the weight lattice

$$\alpha_1 = 2\lambda_f^1 - \lambda_f^2, \quad \alpha_2 = 2\lambda_f^2 - \lambda_f^1. \tag{D.11}$$

The theta function of the $\widehat{\mathfrak{su}}_3$ affine algebra reads, for a generic weight $\lambda = m_i \lambda_f^i$:

$$\Theta_{\lambda, k} = \sum_{\gamma \in M} q^{\frac{k}{2} \|\gamma + \frac{\lambda}{k}\|^2} = \sum_{n^1, n^2} q^{\frac{k}{2} \|n^1 \alpha_1 + n^2 \alpha_2 + \frac{m_1 \lambda_f^1 + m_2 \lambda_f^2}{k}\|^2}. \tag{D.12}$$

So the vector appearing in the theta function is

$$\left\{ \sqrt{k} (2n^1 - n^2) + \frac{m_1}{\sqrt{k}} \right\} \frac{e_1}{2} + \left\{ \sqrt{k} n^2 + \frac{m_1 + 2m_2}{3\sqrt{k}} \right\} \frac{\sqrt{3}}{2} e_2. \tag{D.13}$$

D.2 Modular transformations

We have the following modular transformations for the theta-functions

$$\Theta_{\lambda, k}(-1/\tau) = (-i\tau)^{\dim(\mathfrak{t})/2} \left| \frac{\mathbf{M}^*}{k\mathbf{M}_L} \right|^{-1/2} \sum_{\mu \in \mathbf{M}^* \bmod k\mathbf{M}_L} e^{2i\pi(\lambda, \mu)/k} \Theta_{\mu, k}(\tau), \tag{D.14}$$

where \mathbf{M}^* is the lattice dual to \mathbf{M}_L , $|\mathbf{M}_L|$ is the size of the basic cell of \mathbf{M}_L and for the affine characters

$$\chi^\Lambda(-1/\tau) = \left| \frac{\mathbf{M}^*}{(k+g^*)\mathbf{M}_L} \right|^{-1/2} i^{|\Delta_+|} \sum_{\Lambda'} \sum_{w \in W} \epsilon(w) e^{\frac{2i\pi}{k+g^*}(\Lambda+\rho)w(\Lambda'+\rho)} \chi^{\Lambda'}(\tau). \tag{D.15}$$

In this formula, $|\Delta_+|$ is the number of positive roots. From these two formulas we deduce the modular transformation of the characters of the super-coset under $\tau \rightarrow -1/\tau$:

$$\begin{aligned} & \mathcal{C}_\lambda^\Lambda \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right](-1/\tau) \\ &= e^{\frac{i\pi}{4} ab \dim(\mathfrak{g})} i^{|\Delta_+|} \sum_{\mu \in \mathbf{M}^* \bmod k\mathbf{M}_L} e^{2i\pi \frac{(\lambda, \mu)}{k+g^*}} \sum_{\Lambda'} \sum_{w \in W} \epsilon(w) e^{\frac{2i\pi}{k+g^*}(\Lambda+\rho)w(\Lambda'+\rho)} \mathcal{C}_\mu^{\Lambda'} \left[\begin{smallmatrix} b \\ -a \end{smallmatrix} \right](\tau). \end{aligned} \tag{D.16}$$

D.3 Fermionic characters

For an even number of fermions it is possible to express the characters in terms of representations of the $SO(2n)_1$ affine algebra. The characters are labelled by $s = (0, 1, 2, 3)$ for the trivial, spinor, vector, and conjugate spinor representations

$$\begin{aligned}\Xi_{2n}^0 &= \frac{1}{2\eta^n} \left[\theta_{[0]}^{[0]n} + \theta_{[1]}^{[0]n} \right] && \text{trivial,} \\ \Xi_{2n}^2 &= \frac{1}{2\eta^n} \left[\theta_{[0]}^{[0]n} - \theta_{[1]}^{[0]n} \right] && \text{vector,} \\ \Xi_{2n}^1 &= \frac{1}{2\eta^n} \left[\theta_{[0]}^{[1]n} + i^{-n} \theta_{[1]}^{[1]n} \right] && \text{spinor,} \\ \Xi_{2n}^3 &= \frac{1}{2\eta^n} \left[\theta_{[0]}^{[1]n} - i^{-n} \theta_{[1]}^{[1]n} \right] && \text{conjugate spinor.}\end{aligned}\tag{D.17}$$

Their modular matrices are

$$T = e^{-i\pi/12} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{pmatrix}\tag{D.18}$$

and

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-n} & -i^{-n} \\ 1 & -1 & -i^{-n} & i^{-n} \end{pmatrix}.\tag{D.19}$$

References

- [1] F. Gliozzi, J. Scherk, and D. I. Olive, Nucl. Phys. B **122**, 253–290 (1977).
- [2] C. V. Johnson, Mod. Phys. Lett. A **10**, 549–560 (1995) [hep-th/9409062].
- [3] P. Berglund, C. V. Johnson, S. Kachru, and P. Zaug, Nucl. Phys. B **460**, 252–298 (1996) [hep-th/9509170].
- [4] S. Ferrara, R. Kallosh, and A. Strominger, Phys. Rev. D **52**, 5412–5416 (1995), [hep-th/9508072].
- [5] D. Israël, C. Kounnas, D. Orlando, and P.M. Petropoulos, Fortschr. Phys. **53**, 73–104 (2005) [hep-th/0405213].
- [6] M. C. Prati and A. M. Perelomov, Nucl. Phys. B **258**, 647–660 (1985).
- [7] M. Atiyah and E. Witten, Adv. Theor. Math. Phys. **6**, 1–106 (2003), [hep-th/0107177].
- [8] T. Quella and V. Schomerus, J. High Energy Phys. **02**, 030 (2003) [hep-th/0212119].
- [9] J. Callan, G. Curtis, J.A. Harvey, and A. Strominger, Nucl. Phys. B **359**, 611–634 (1991).
- [10] C. Kounnas, M. Porrati, and B. Rostand, Phys. Lett. B **258**, 61–69 (1991).
- [11] O. Aharony, M. Berkooz, D. Kutasov, and N. Seiberg, J. High Energy Phys. **10**, 004 (1998), [hep-th/9808149].
- [12] A. Giveon and D. Kutasov, J. High Energy Phys. **10**, 034 (1999), [hep-th/9909110].
- [13] A. Giveon, D. Kutasov, and O. Pelc, J. High Energy Phys. **10**, 035 (1999), [hep-th/9907178].
- [14] S. Chaudhuri and J.A. Schwartz, Phys. Lett. B **219**, 291 (1989).
- [15] S. Forste and D. Roggenkamp, J. High Energy Phys. **05**, 071 (2003), [hep-th/0304234].
- [16] G. T. Horowitz and A. A. Tseytlin, Phys. Rev. D **51**, 2896–2917 (1995), [hep-th/9409021].
- [17] A. Borel and F. Hirzebruch, Am. J. Math. **80**, 458 (1958). ISSN 0002-9327.
- [18] M. Nakahara, Geometry, Topology, and Physics, Graduate Student Series in Physics (Hilger, Bristol, 1990).

- [19] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, (Wiley, New York, Chichester, 1969).
- [20] J. Callan, G. Curtis, E. J. Martinec, M. J. Perry, and D. Friedan, *Nucl. Phys. B* **262**, 593 (1985).
- [21] D. Gepner, *Nucl. Phys. B* **296**, 757 (1988).
- [22] Y. Kazama and H. Suzuki, *Nucl. Phys. B* **321**, 232 (1989).
- [23] E. Kiritsis and C. Kounnas, *Nucl. Phys. B* **456**, 699–731 (1995) [hep-th/9508078].
- [24] C. Bachas and E. Kiritsis, *Phys. Lett. B* **325**, 103–114 (1994), [hep-th/9311185].
- [25] D. Kutasov, F. Larsen, and R. G. Leigh, *Nucl. Phys. B* **550**, 183–213 (1999), [hep-th/9812027].
- [26] J. Gates, S. J., C. M. Hull, and M. Rocek, *Nucl. Phys. B* **248**, 157 (1984).
- [27] Y. Kazama and H. Suzuki, *Phys. Lett. B* **216**, 112 (1989).
- [28] A. Giveon and E. Kiritsis, *Nucl. Phys. B* **411**, 487–508 (1994) [hep-th/9303016].
- [29] A. A. Tseytlin, *Nucl. Phys. B* **411**, 509–558 (1994) [hep-th/9302083].
- [30] T. Eguchi, Y. Sugawara, and S. Yamaguchi, *Nucl. Phys. B* **657**, 3–52 (2003) [hep-th/0301164].
- [31] D. Israel, C. Kounnas, A. Pakman, and J. Troost, *J. High Energy Phys.* **06**, 033 (2004), [hep-th/0403237].
- [32] T. Eguchi and Y. Sugawara, *Phys. Lett. B* **519**, 149–158 (2001) [hep-th/0108091].
- [33] R. Argurio, A. Giveon, and A. Shomer, *J. High Energy Phys.* **04**, 010 (2000), [hep-th/0002104].
- [34] F. Mueller-Hoissen and R. Stuckl, *Class. Quantum Gravity* **5**, 27 (1988).
- [35] A. M. Perelomov, *Phys. Rep.* **146**, 135–213 (1987).
- [36] S. Gnutzmann and M. Kuś, *J. Phys. A* **31**, 9871 (1998).
- [37] K.-I. Kondo and Y. Taira, *Prog. Theor. Phys.* **104**, 1189–1265 (2000) [hep-th/9911242].

$\text{AdS}_2 \times S^2$ as an exact heterotic string background

Published in Cargese 2004, String theory: From gauge interactions to cosmology 345-349

$AdS_2 \times S^2$ AS AN EXACT HETEROTIC STRING BACKGROUND

Domenico Orlando
CPHT - École Polytechnique
 91128 Palaiseau, France
 domenico.orlando@cphpt.polytechnique.fr

Abstract An exact heterotic string theory on an $AdS_2 \times S^2$ background is found as deformation of an $SL(2, \mathbb{R}) \times SU(2)$ wzw model. Based on ?.

1. INTRO

Anti de Sitter in three dimensions and S^3 are among the most simple and yet interesting string backgrounds. They are exact solutions to the string equations beyond the supergravity approximation and, at the same time, are simple to deal with although non-trivial thanks to the presence of non-vanishing curvatures. For this reason they constitute a unique setting in which to analyze AdS/CFT correspondence, black-hole physics, little-string theory.

String propagation in these backgrounds is described in terms of wzw models for the $SL(2, \mathbb{R})$ and $SU(2)$ groups, hence marginal deformations of such models allow to study moduli space of the string vacua. In particular well-known class of marginal deformations for wzw models are those driven by left-right current bilinears ?; ?. On the other hand S^3 and AdS_3 are embedded in larger structures so one can consider marginal deformations where just one of the currents belongs to the $SU(2)$ or AdS_3 algebra, the other belonging to some other $U(1)$ corresponding to an internal magnetic or electric field.

This kind of deformation generates a continuous line of exact CFT's. In this note we will show how with an appropriate choice for the deforming current we obtain a boundary in moduli space and that this boundary can be given a simple geometric interpretation in terms of the $AdS_2 \times S^2$ near-horizon geometry of the Bertotti-Robinson black hole ?; ?.

2

2. $SU(2)$ ASYMMETRIC DEFORMATION

In the $SU(2)$ case, there exists just one possible choice for the deforming current the two other being related by inner automorphisms, since the group has rank one, is compact and its Lie algebra simple. Take the WZW model for $SU(2)$:

$$S_{SU(2)_k} = \frac{1}{2\pi} \int d^2z \left\{ \frac{k}{4} (\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta + \partial\gamma\bar{\partial}\gamma + 2\cos\beta\partial\alpha\bar{\partial}\gamma) + \sum_{a=1}^3 \psi^a \bar{\partial}\psi^a \right\} \quad (1.1)$$

where ψ^a are the left-moving free fermions, superpartners of the bosonic $SU(2)_k$ currents, and (α, β, γ) are the usual Euler angles parameterizing the $SU(2)$ group manifold. The left-moving fermions transform in the adjoint of $SU(2)$; there are no right-moving superpartners but a right-moving current algebra of total charge $c = 16$ can be realized in terms of right-moving free fermions. This means that we can build a $\mathcal{N} = (1, 0)$ world-sheet supersymmetry-compatible deformation given by:

$$\delta S_{\text{magnetic}} = \frac{\sqrt{k k_G H}}{2\pi} \int d^2z (J^3 + \imath\psi^1\psi^2) \bar{J}_G; \quad (1.2)$$

where J^3 belongs to the $SU(2)$ algebra and \bar{J}_G is the current of the algebra at level k_g realized by the right-moving free fermions. An exact CFT is obtained for any value of the deformation parameter H .

2.1 GEOMETRY

These new backgrounds all present a constant dilaton, a magnetic field, a NS-NS field proportional to the unperturbed one and a metric retaining a residual $SU(2) \times U(1)$ isometry ?. The most remarkable property is that the deformation line in moduli space has a boundary corresponding to a critical value of the deformation parameter $H^2 = 1/2$. At this point the $U(1)$ subgroup decompactifies and the resulting geometry is the left coset $SU(2)/U(1) \sim S^2$ which is thus found to be an exact CFT background only supported by a magnetic field (the dilaton remains constant and NS field vanishes). A geometrical interpretation for this process can be given as follows: the initial S^3 sphere is a Hopf fibration of an S^1 fiber generated by the J^3 current over an S^2 base; the deformation only acts on the fiber, changing its radius up to the point where this seems to vanish, actually marking the trivialization of the fibration:

$$S^3 \xrightarrow{H^2 \rightarrow H_{\text{max}}^2} \mathbb{R} \times S^2, \quad (1.3)$$

If we turn our attention to the gauge field one can show that a quantization of the magnetic charge is only compatible with levels of the affine

AdS₂ × S² as an exact heterotic string background 3

algebras such that $\frac{k}{k_G} = p^2$, $p \in \mathbb{Z}$. We will find the same condition in terms of the partition function for the boundary deformation.

Although this construction has been implicitly carried on for first order in α' background fields, it is important to stress that the resulting metric is nevertheless exact at all orders since the renormalization boils down to the redefinition of the level k that is simply shifted by the dual Coxeter number (just as in the WZW case).

2.2 PARTITION FUNCTION

Consider the case of $k_g = 2$ (one right-moving \mathbb{C} fermion). The relevant components of the initial partition function are given by a $SU(2)_{k-2}$ -modular-invariance-compatible combination of $SU(2)_{k-2}$ supersymmetric characters and fermions from the gauge sector. For our purposes it is useful to further decompose the supersymmetric $SU(2)_k$ characters in terms of those of the $\mathcal{N} = 2$ minimal models:

$$\chi^j(\tau) \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau, \nu) = \sum_{m \in \mathbb{Z}_{2k}} c_m^j \begin{bmatrix} a \\ b \end{bmatrix} \Theta_{m,k} \left(\tau, -\frac{2\nu}{k} \right). \quad (1.4)$$

The deformation acts as a boost on the left-lattice contribution of the Cartan current of the supersymmetric $SU(2)_k$ and on the right current from the gauge sector:

$$\begin{aligned} \Theta_{m,k} \bar{\vartheta} \begin{bmatrix} h \\ g \end{bmatrix} &= \sum_{n, \bar{n}} e^{-i\pi g(\bar{n} + \frac{h}{2})} q^{\frac{1}{2}(\sqrt{2kn} + \frac{m}{\sqrt{2k}})^2} \bar{q}^{\frac{1}{2}(\bar{n} + \frac{h}{2})^2} \\ &\longrightarrow \sum_{n, \bar{n}} e^{-i\pi g(\bar{n} + \frac{h}{2})} q^{\frac{1}{2}[(\sqrt{2kn} + \frac{m}{\sqrt{2k}}) \cosh x + (\bar{n} + \frac{h}{2}) \sinh x]^2} \\ &\quad \times \bar{q}^{\frac{1}{2}[(\bar{n} + \frac{h}{2}) \cosh x + (\sqrt{2kn} + \frac{m}{\sqrt{2k}}) \sinh x]^2}, \quad (1.5) \end{aligned}$$

where the boost parameter x is given by $\cosh x = \frac{1}{1-2H^2}$.

Although an exact CFT is obtained for any value of the deformation parameter H we will concentrate, as before, on the boundary value $H^2 = 1/2$. In this case the boost parameter diverges thus giving the following constraints: $4(k+2)n + 2m + 2\sqrt{2k}\bar{n} + \sqrt{2k}h = 0$. Therefore, the limit is well-defined only if the level of the supersymmetric $SU(2)_k$ satisfies the quantization condition $k = 2p^2$, $p \in \mathbb{Z}$ *i.e.* the charge quantization for the flux of the gauge field. Under these constraints the $U(1)$ corresponding to the combination of charges orthogonal our condition decouples and can be removed. In this way we end up with the

4

expression for the S^2 partition function:

$$Z_{S^2} \begin{bmatrix} a; h \\ b; g \end{bmatrix} = \sum_{j, \bar{j}} M^{j\bar{j}} \sum_{N \in \mathbb{Z}_{2p}} e^{i\pi g(N + \frac{h}{2})} C_{p(2N-h)}^j \begin{bmatrix} a \\ b \end{bmatrix} \bar{\chi}^{\bar{j}} \quad (1.6)$$

in agreement with the result found in [?] by using the coset construction. The remaining charge N labels the magnetic charge of the state under consideration.

3. $SL(2, \mathbb{R})$ DEFORMATION

The same construction as above can be repeated for the $SL(2, \mathbb{R})$ WZW model. In this case the moduli space is somewhat richer for it is possible to realize three different asymmetric deformations using the three generators of the group. These are not equivalent ($SL(2, \mathbb{R})$ is not compact) and in fact they lead to three physically different backgrounds. The elliptic deformation line, in example, contains the Gödel universe [?], the parabolic deformation gives the superposition of AdS_3 and a gravitational plane wave. Two of these deformation lines present the same boundary effect as the $SU(2)$ deformation. In particular the elliptic deformation leads to the hyperbolic space $H_2 = SL(2, \mathbb{R})/U(1)$ supported by an imaginary magnetic field, *ie* an exact but non-unitary CFT. The hyperbolic deformation, on the other hand, leads to $AdS_2 = SL(2, \mathbb{R})/U(1)$ supported by an electric field. No charge quantization is present in this case, because of the non-compact nature of the background.

In this latter case it is not yet possible to give the same construction for the partition function as for the $SU(2)$ case since this would require the decomposition of the initial partition function in a basis of hyperbolic characters which is not a simple exercise. Nevertheless by following the same procedure as before it is possible to evaluate the effect of the deformation on the spectrum of primaries and hence give the resulting AdS_2 background spectrum.

4. $AdS_2 \times S^2$

The S^2 and AdS_2 backgrounds can be combined so to give an exact CFT corresponding to the $AdS_2 \times S^2$ near-horizon geometry of the BR black-hole.

Let us now consider the complete heterotic string background which consists of the $AdS_2 \times S^2$ space-time times an $\mathcal{N} = 2$ internal conformal field theory \mathcal{M} , that we will assume to be of central charge $\hat{c} = 6$ and with integral R -charges. The levels k of $SU(2)$ and \hat{k} of $SL(2, \mathbb{R})$ are

AdS₂ × S² as an exact heterotic string background 5

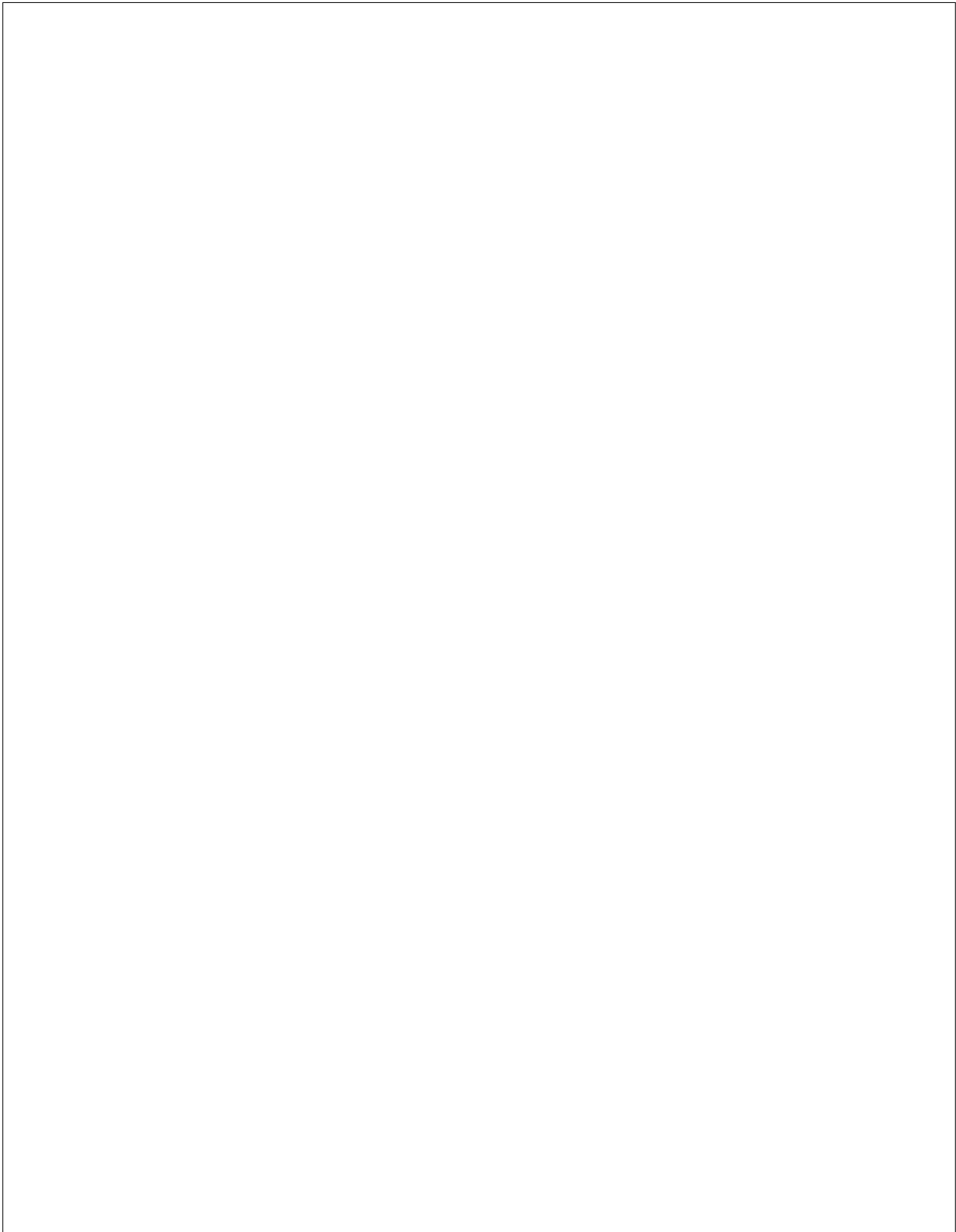
such that the string background is critical:

$$\hat{c} = \frac{2(k-2)}{k} + \frac{2(\hat{k}+2)}{\hat{k}} = 4 \implies k = \hat{k}. \quad (1.7)$$

This translates into the equality of the radii of the corresponding S^2 and AdS_2 factors, which is in turn necessary for supersymmetry. Furthermore, the charge quantization condition for the two-sphere restricts further the level to $k = 2p^2$, $p \in \mathbb{N}$.

The combined $\text{AdS}_2 \times S^2$ background can give new insights about the physics of the BR black hole in particular by analyzing the Schwinger-pair production in such background, or the study of the stability and propagation of D-branes.

Research partially supported by the EEC under the contracts HPRN-CT-2000-00131, HPRN-CT-2000-00148, MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369.



References

- D. Israel, C. Kounnas, D. Orlando, and P. M. Petropoulos, *Electric / magnetic deformations of s^{**3} and $ads(3)$, and geometric cosets*, [[hep-th/0405213](#)].
- S. Chaudhuri and J. A. Schwartz, *A criterion for integrably marginal operators*, *Phys. Lett.* **B219** (1989) 291.
- S. Forste and D. Roggenkamp, *Current current deformations of conformal field theories, and wzw models*, *JHEP* **05** (2003) 071, [[hep-th/0304234](#)].
- B. Bertotti, *Uniform Electromagnetic Field In The Theory Of General Relativity*, *Phys. Rev.* **116** (1959) 1331.
- I. Robinson, *Bull. Acad. Polon. Sci.* **7** (1959) 351.
- E. Kiritsis and C. Kounnas, *Infrared behavior of closed superstrings in strong magnetic and gravitational fields*, *Nucl. Phys.* **B456** (1995) 699–731, [[hep-th/9508078](#)].
- D. Israël, *Quantization of heterotic strings in a Gödel/anti de Sitter spacetime and chronology protection*, *JHEP* **01** (2004) 042, [[hep-th/0310158](#)].
- P. Berglund, C. V. Johnson, S. Kachru and P. Zaugg, *Heterotic Coset Models and (0,2) String Vacua*, *Nucl. Phys. B* **460** (1996) 252, [[hep-th/9509170](#)].

Three-dimensional black holes from deformed anti de Sitter

Published in JHEP 07 (2005) 072, [hep-th/0504231]



PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA

RECEIVED: June 6, 2005

ACCEPTED: July 9, 2005

PUBLISHED: July 28, 2005

Three-dimensional black holes from deformed anti de Sitter*

Stéphane Detournay,^{a†} Domenico Orlando,^b P. Marios Petropoulos^b and Philippe Spindel^a

^a*Mécanique et Gravitation, Université de Mons-Hainaut
20 Place du Parc, 7000 Mons, Belgique*

^a*Centre de Physique Théorique, École Polytechnique[‡]
91128 Palaiseau, France*

*E-mail: Stephane.Detournay@umh.ac.be, orlando@cpht.polytechnique.fr,
marios@cpht.polytechnique.fr, spindel@umh.ac.be*

ABSTRACT: We present new exact three-dimensional black-string backgrounds, which contain both NS-NS and electromagnetic fields, and generalize the BTZ black holes and the black string studied by Horne and Horowitz. They are obtained as deformations of the $SL(2, \mathbb{R})$ WZW model. Black holes resulting from purely continuous deformations possess true curvature singularities. When discrete identifications are introduced, extra chronological singularities appear, which under certain circumstances turn out to be naked. The backgrounds at hand appear in the moduli space of the $SL(2, \mathbb{R})$ WZW model. Hence, they provide exact string backgrounds and allow for a more algebraical (CFT) description. This makes possible the determination of the spectrum of primaries.

KEYWORDS: Conformal Field Models in String Theory Black Holes in String Theory.

*Research partially supported by the EEC under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369.

[†]“Chercheur FRIA”, Belgium.

[‡]Unité mixte du CNRS et de l'École Polytechnique, UMR 7644.

JHEP07(2005)072

Contents

1. Introduction	1
2. Deformed wzw models: various perspectives	3
2.1 Algebraic structure of current-current deformations	3
2.2 Background fields and symmetric deformations	5
2.3 Background fields and asymmetric deformations	7
3. The three-dimensional black string revisited	10
4. The two-parameter deformations	12
4.1 An interesting mix	12
4.2 The hyperbolic double deformation	13
5. Discrete identifications	17
5.1 Discrete identifications in asymmetric deformations	18
5.2 Discrete identifications in symmetric deformations	18
5.3 Discrete identifications in double deformations	19
6. Towards the exact spectra	20
6.1 Deformed spectrum	20
6.2 Twisting	21
7. Summary	22
A. AdS₃ coordinate patches	23
A.1 AdS ₃ SL(2, ℝ)	23
A.2 “Symmetric” coordinates	24
A.3 “Asymmetric” coordinates	25
B. The BTZ black hole	26
C. Spectrum of the SL(2, ℝ) super-wzw model	27

1. Introduction

The search for exact string backgrounds has been pursued over the past years from various perspectives. Those investigations are motivated by phenomenology, background-geometry analysis or, more recently, for understanding holography beyond the usual supergravity approximation.

Anti-de Sitter backgrounds have played an important role in many respects. Together with the spheres, they are the only maximally symmetric spaces appearing naturally in string theory. They arise as near-horizon geometries of distinguished brane configurations and offer the appropriate set up for studying little-string theory, black-hole physics, . . .

The realization of anti-de Sitter spaces or spheres as string backgrounds requires non-vanishing fluxes, which account for the cosmological constant term in the low-energy equations of motion. In general, those fluxes are of the Ramond-Ramond type, hence no two-dimensional sigma-model is available. This happens indeed for $\text{AdS}_5 \times S^5$ in type IIB or $\text{AdS}_4 \times S^7$ in M-theory. For $\text{AdS}_3 \times S^3 \times T^4$ (type IIA, B or heterotic), however, we have the option to switch on a Neveu-Schwarz antisymmetric tensor only. In this framework, the $\text{AdS}_3 \times S^3$ is the target space of the $\text{SL}(2, \mathbb{R}) \times \text{SU}(2)$ Wess-Zumino-Witten model. The latter has been studied extensively [1–6].

Three-dimensional anti-de Sitter space provides a good laboratory for studying many aspects of gravity and strings, including black-hole physics. Locally anti-de Sitter three-dimensional black holes are obtained by performing identifications in the original AdS_3 under discrete isometry subgroups [7–10]. Those black holes (BTZ) have mass and angular momentum. Generically, two horizons (inner and outer) mask the singularity, which turns out to be a chronological singularity rather than a genuine curvature singularity.

The two-dimensional sigma-model description of the AdS_3 plus Kalb-Ramond field background allows for exact conformal deformations, driven by integrable marginal operators [11–14, 6, 15–17]. In general, a subgroup of the original isometry group survives along those lines. Identification under discrete isometries is thus legitimate and provides a tool for investigating new and potentially interesting “deformed BTZ” geometries. The latter may or may not be viable black holes, whereas black holes may also appear by just deforming AdS_3 without further surgery [18].

The aim of the present work is to clarify those issues, and reach a global point of view on the geometries that emerge from the $\text{SL}(2, \mathbb{R})$ WZW model, by using the above techniques. This will allow us to introduce new three-dimensional black hole backgrounds that in general involve the presence of an electric field. For these theories we give a complete CFT description, including an explicit expression for the spectrum of string primaries. In particular, the usual black string background [18] will appear in this terms as a special vanishing-field limit. Carrying on identifications *à la* BTZ on these geometries will let us obtain more black string and/or black hole backgrounds, generalizing the one in [7] and in [18], for which we again provide a CFT description. Not all the backgrounds could be adapted to support the discrete identifications. This will be stated in terms of a consistency condition that has to be satisfied in order to avoid the presence of naked (causal) singularities.

We will start with a quick overview of various distinct methods based on Wess-Zumino-Witten models and aiming at generating new exact CFTs, that turn out to be equivalent to each other. We will in particular exhibit their effect on the $\text{SL}(2, \mathbb{R})$ WZW model. These results enable us to recast in section 3 the three-dimensional black-string solution of [18], as a patchwork of marginal deformations of the $\text{SL}(2, \mathbb{R})$ WZW model. We clarify in this way the role of the mass and charge parameters of the black string.

Section 4 is devoted to a two-parameter deformation of $SL(2, \mathbb{R})$. This leads to a new family of black strings, with NS-NS and electric field. We study the causal structure of these black holes as well as their various charges. They exhibit genuine curvature singularity hidden behind horizons. In section 5 we proceed with discrete identifications as a solution-generating procedure applied to the deformed AdS_3 — wherever it is allowed by residual symmetries.

After having stated the consistency conditions to be fulfilled in order to avoid naked singularities, we find that time-like chronological singularities protected by two horizons are possible, while light-like singularities with only one horizon appear as a limiting case. Finally, in section 6 we determine the spectrum of primaries, using standard CFT techniques.

2. Deformed wzw models: various perspectives

The power of WZW models resides in the symmetries of the theory. Those impose strong constraints which allow quantum integrability as well as a faithful description in terms of space-time fields whose renormalization properties (at every order in α') are easily kept under control [19–21].

It is hence interesting to study the moduli space for these models, aiming at finding less symmetric (and more interesting) structures, that will hopefully enjoy analogous integrability and space-time properties.

2.1 Algebraic structure of current-current deformations

In this spirit one can consider marginal deformations of the WZW models obtained in terms of $(1, 1)$ operators built as bilinears in the currents:

$$\mathcal{O}(z, \bar{z}) = \sum_{ij} c_{ij} J^i(z) \tilde{J}^j(\bar{z}), \quad (2.1)$$

where $J^i(z)$ and $\tilde{J}^j(\bar{z})$ are respectively left- and right-moving currents. It is known that this operator represents a truly marginal deformation if the parameter matrix c_{ij} satisfies appropriate constraints [11], which are automatically satisfied for any value of c_{ij} , whatever the algebra, if J^i and \tilde{J}^j live on a torus. Hence, we get as moduli space continuous surfaces of exact models.¹

From the CFT point of view, it is known [14] that the effect of the deformation is completely captured by an $O(d, \bar{d})$ pseudo-orthogonal transformation of the charge lattice $\Lambda \subset \mathfrak{h}^* \times \bar{\mathfrak{h}}^*$ of the abelian sector of the theory ($\mathfrak{h} \subset \mathfrak{g}$ and $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$ being abelian subalgebras of the undeformed WZW model $\mathfrak{g} \times \bar{\mathfrak{g}}$ algebra). Moreover, since the charges only characterize the $\mathfrak{h} \times \bar{\mathfrak{h}}$ modules up to automorphisms of the algebras, $O(d) \times O(\bar{d})$ transformations don't

¹Although for special values of the level k the theory contains other operators with the right conformal weights, it is believed that only current-current operators give rise to truly marginal deformations, i.e. operators that remain marginal for finite values of the deformation parameter.

change the CFT. Hence the deformation space is given by:

$$D_{\mathfrak{h}, \bar{\mathfrak{h}}} \sim O(d, \bar{d}) / (O(d) \times O(\bar{d})). \quad (2.2)$$

The moduli space is obtained out of $D_{\mathfrak{h}, \bar{\mathfrak{h}}}$ after the identification of the points giving equivalent CFTs.²

In the case of WZW models on compact groups, all maximal abelian subgroups are pairwise conjugated by inner automorphisms. This implies that the complete deformation space is $D = O(d, d) / (O(d) \times O(d))$ where d is the rank of the group. The story is different for non-semisimple algebras, whose moduli space is larger, since we get different $O(d, \bar{d}) / (O(d) \times O(\bar{d}))$ deformation spaces for each (inequivalent) choice of the abelian subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$.

An alternative way of describing current-current deformations comes from the so-called parafermion decomposition. The highest-weight representation for a $\hat{\mathfrak{g}}_k$ graded algebra can be decomposed into highest-weight modules of a Cartan subalgebra $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}_k$ as follows [22, 23]:

$$\mathcal{V}_{\hat{\lambda}} \simeq \bigoplus_{\mu \in \Gamma_k} \mathcal{V}_{\hat{\lambda}, \mu} \otimes \bigoplus_{\delta \in Q_l(\mathfrak{g})} \mathcal{V}_{\mu + k\delta}, \quad (2.3)$$

where $\hat{\lambda}$ is an integrable weight of $\hat{\mathfrak{g}}_k$, $\mathcal{V}_{\hat{\lambda}, \mu}$ is the highest-weight module for the generalized $\hat{\mathfrak{g}}_k / \hat{\mathfrak{h}}$ parafermion, $Q_l(\mathfrak{g})$ is the long-root lattice and $\Gamma_k = P(\mathfrak{g}) / Q_l(\mathfrak{g})$ with $P(\mathfrak{g})$ the weight lattice. As a consequence, the WZW model based on $\hat{\mathfrak{g}}_k$ can be represented as an orbifold model:

$$\hat{\mathfrak{g}}_k \simeq \left(\hat{\mathfrak{g}}_k / \hat{\mathfrak{h}} \otimes t_{\Lambda_k} \right) / \Gamma_k, \quad (2.4)$$

where t_{Λ_k} is a toroidal CFT with charge lattice, included in the $\hat{\mathfrak{g}}_k$ one, defined as $\Lambda_k = \{(\mu, \bar{\mu}) \in P(\hat{\mathfrak{g}}) \times P(\hat{\mathfrak{g}}) | \mu - \bar{\mu} = kQ_l(\hat{\mathfrak{g}})\}$. The advantage given by using this representation relies on the fact that Γ_k acts trivially on the coset and toroidal model algebras; then, if we identify $\hat{\mathfrak{h}}$ and $\bar{\hat{\mathfrak{h}}}$ with the graded algebras of t_{Λ_k} , the deformation only acts on the toroidal lattice and the deformed model can again be represented as an orbifold:

$$\hat{\mathfrak{g}}_k(\mathcal{O}) \simeq \left(\hat{\mathfrak{g}}_k / \hat{\mathfrak{h}} \otimes t_{\mathcal{O}\Lambda_k} \right) / \Gamma_k, \quad (2.5)$$

where \mathcal{O} is an operator in the moduli space. This representation is specially useful because it allows to easily single out the sector of the theory that is affected by the deformation. As we'll see in the next section this simplifies the task of writing the corresponding lagrangian.

In the following we will separate (somehow arbitrarily) this kind of deformations into two categories: those who give rise to *symmetric* deformations, i.e. the ones where $c_{ij} = \delta_{ij}$ and $J^i(z)$ and $\bar{J}^j(\bar{z})$ represent the same current in the two chiral sectors of the theory and the *asymmetric* ones where the currents are different and in general correspond to different subalgebras. In some ways this distinction is arbitrary, since both symmetric and asymmetric deformations act as $O(d, \bar{d})$ rotations on the background fields. It is nonetheless interesting to single out the asymmetric case. In the particular situation,

²Although we will concentrate on WZW models it is worth to emphasize that this construction is more general.

when one of the two currents belongs to an internal $U(1)$ (coming from the gauge sector in the heterotic or simply from any $U(1)$ subalgebra in the type II), it is particularly simple to study the effect of the deformation, even from the space-time field point of view; there, the expressions for the background fields are exact (at all order in α' and for every value of the level k) [16].

2.2 Background fields and symmetric deformations

General construction. Symmetric deformations (also called *gravitational*) are those that have received by far the most attention in literature. Specializing eq. (2.1) to the case of one only current we can write the small deformation lagrangian as:

$$S = S_{\text{WZW}} + \delta\kappa^2 \int d^2z J(z)\bar{J}(\bar{z}). \quad (2.6)$$

This infinitesimal deformation has to be integrated in order to give a lagrangian interpretation to the CFT described in the previous section. Different approaches are possible, exploiting the different possible representations described above.

- A possible way consists in implementing an $O(d, d)$ rotation on the background fields [12]. More precisely, one has to identify a coordinate system in which the background fields are independent of d space dimensions and metric and B field are written in a block diagonal form. In this way the following matrix is defined:

$$M = \left(\begin{array}{c|c} \hat{g}^{-1} & -\hat{g}^{-1}\hat{B} \\ \hline \hat{B}\hat{g}^{-1} & \hat{g} - \hat{B}\hat{g}^{-1}\hat{B} \end{array} \right), \quad (2.7)$$

where \hat{g} and \hat{B} are the pull-backs of the metric and Kalb-Ramond field on the p selected directions. Then the action of the $O(d, d)$ group on these fields and dilaton is given by:

$$M \rightarrow M' = \Omega M \Omega^t, \quad (2.8)$$

$$\Phi \rightarrow \Phi' = \Phi - \frac{1}{4} \log \left(\frac{\det \hat{g}}{\det \hat{g}'} \right), \quad (2.9)$$

where \hat{g}' is the metric after the transformation (2.8) and $\Omega \in O(d, d)$. It must be emphasized that this transformation rules are valid at the lowest order in α' (but at all orders in the deformation parameters). So, although the model is exact, as we learn from the CFT side, the field expressions that we find only are true at leading order in α' .

- An alternative approach uses the parafermion representation eq. (2.5) (see e.g. [14]). In practice this amounts to writing an action as the sum of the G/H parafermion and a deformed H part and finding the appropriate T-duality transformation (realizing the orbifold) such that for zero deformation the WZW on G is recovered, in accordance with eq. (2.4).

- Finally, another point of view (inspired by the parafermionic representation), consists in identifying the deformed model with a $(G \times H)/H$ coset model, in which the embedding of the dividing group has a component in both factors [13]. The gauging of the component in G gives the parafermionic sector, the gauging of the component in H gives the deformed toroidal sector and the coupling term (originating from the quadratic structure in the fields introduced for the gauging) corresponds to the orbifold projection.³

The $SL(2, \mathbb{R})$ case. In the present work, we want to concentrate on the deformations of $SL(2, \mathbb{R})$. Symmetric deformations of this WZW model are known in the literature. The group manifold of $SL(2, \mathbb{R})$ is anti de Sitter in three dimensions. Metric and antisymmetric tensor read (in Euler coordinates, see appendix A.2):

$$ds^2 = L^2 [d\rho^2 + \sinh^2 \rho d\phi^2 - \cosh^2 \rho d\tau^2], \quad (2.10a)$$

$$H_{[3]} = L^2 \sinh 2\rho d\rho \wedge d\phi \wedge d\tau, \quad (2.10b)$$

with L related to the level of $SL(2, \mathbb{R})_k$ as usual: $L = \sqrt{k+2}$. In the case at hand, three different lines of symmetric deformations arise due to the presence of time-like (J^3, \bar{J}^3), space-like ($J^1, \bar{J}^1, J^2, \bar{J}^2$), or null generators [14, 25, 6]. The residual isometry is $U(1) \times U(1)$ that can be time-like (L_3, R_3), space-like (L_2, R_2) or null ($L_1 + L_3, R_1 + R_3$) depending on the deformation under consideration.

The *elliptic deformation* is driven by the $J^3 \bar{J}^3$ bilinear. At first order in α' the background fields are given by:⁴

$$ds^2 = k \left[d\rho^2 + \frac{\sinh^2 \rho d\phi^2 - \kappa_3^2 \cosh^2 \rho d\tau^2}{\Theta_{\kappa_3}(\rho)} \right], \quad (2.11a)$$

$$H_{[3]} = k \frac{\kappa_3^2 \sinh 2\rho}{\Theta_{\kappa_3}(\rho)^2} d\rho \wedge d\phi \wedge d\tau, \quad (2.11b)$$

$$e^{-2\Phi} = \frac{\Theta_{\kappa_3}(\rho)}{\kappa_3}. \quad (2.11c)$$

where $\Theta_{\kappa_3}(\rho) = \cosh^2 \rho - \kappa_3^2 \sinh^2 \rho$ and, of course, Φ is defined up to an additive constant. At extreme deformation ($\kappa_3^2 \rightarrow 0$), a time-like direction decouples and we are left with the axial⁵ $SL(2, \mathbb{R})_k/U(1)_{\text{time}}$. The target space of the latter is the *cigar* geometry (also called euclidean two-dimensional black hole):

$$e^{-2\Phi} \sim \cosh^2 \rho, \quad (2.12)$$

$$ds^2 = k [d\rho^2 + \tanh^2 \rho d\phi^2], \quad (2.13)$$

($0 \leq \rho < \infty$ and $0 \leq \phi \leq 2\pi$).

³An instanton-correction-aware technique that should overcome the first order in α' limitation for gauged models has been proposed in [24]. In principle this can be used to get an all-order exact background when we write the deformation as a gauged model. We will not expand further in this direction, that could nevertheless be useful to address issues such as the stability of the black string (see section 3).

⁴The extra index “3” in the deformation parameter κ reminds that the deformation refers here to $J^3 \bar{J}^3$.

⁵The deformation parameter has two T-dual branches. The extreme values of deformation correspond to the axial or vector gaugings. The vector gauging leads to the *trumpet*. For the $SU(2)_k/U(1)$, both gaugings correspond to the *bell*.

Similarly, with $J^2 \bar{J}^2$ one generates the *hyperbolic deformation*. This allows to reach the lorentzian two-dimensional black hole times a free space-like line. Using the coordinates defined in eq. (A.8), we find:

$$ds^2 = k \left[-d\beta^2 + \frac{\sin^2 \beta d\varphi^2 + \kappa_2^2 \cos^2 \beta d\psi^2}{\Delta_{\kappa_2}(\beta)} \right], \quad (2.14a)$$

$$H_{[3]} = k \frac{\kappa_2^2 \sin 2\beta}{\Delta_{\kappa_2}(\beta)^2} d\beta \wedge d\psi \wedge d\phi, \quad (2.14b)$$

$$e^{-2\Phi} = \frac{\Delta_{\kappa_2}(\beta)}{\kappa_2}, \quad (2.14c)$$

where $\Delta_{\kappa_2}(\beta) = \cos^2 \beta + \kappa_2^2 \sin^2 \beta$. This coordinate patch does not cover the full AdS₃. We will expand on this line in section 3.

Finally, the bilinear $(J^1 + J^3)(\bar{J}^1 + \bar{J}^3)$ generates the *parabolic deformation*. Using Poincaré coordinates (eqs. (A.11)–(A.13))⁶ we obtain:

$$ds^2 = k \left[\frac{du^2}{u^2} + \frac{dX^2 - dT^2}{u^2 + 1/\nu} \right], \quad (2.15a)$$

$$H_{[3]} = k \frac{2u}{(u^2 + 1/\nu)^2} du \wedge dT \wedge dX, \quad (2.15b)$$

$$e^{-2\Phi} = \frac{u^2 + 1/\nu}{u^2}. \quad (2.15c)$$

The deformation parameter is $1/\nu$. At infinite value of the parameter ν , we recover pure AdS₃; for $\nu \rightarrow 0$, a whole light-cone decouples and we are left with a single direction and a dilaton field, linear in this direction.

The physical interpretation of the parabolic deformation is far reaching, when AdS₃ is considered in the framework of the NS5/F1 near-horizon background, AdS₃ \times S³ \times T⁴. In this physical set-up, the parameter ν is the density of F1's (number of fundamental strings over the volume of the four-torus T⁴) [6, 26]. At infinite density, the background is indeed AdS₃ \times S³ \times T⁴. At null density, the geometry becomes $\mathbb{R}^{1,2} \times S^3 \times T^4$ plus a linear dilaton and a three-form on the S³.

2.3 Background fields and asymmetric deformations

General construction. Consider the case of $G = G' \times U(1)^r$, $H = U(1)^r$ where $r = \text{rank}(G)$ embedded such as $\epsilon_L(H) \subset G'$ and $\epsilon_R(H) = G'' = U(1)^r$. To clarify the notation we can write the deformation operator as:

$$\mathcal{O}(z, \bar{z}) = \sum_{a=1}^r H_a J^a(z) \bar{\partial} X^a \quad (2.16)$$

where $X^a(z, \bar{z})$ results from the bosonisation of the right current. Using e.g. Kaluza-Klein reduction [27, 28, 17], one shows that the effect of the deformation on the background fields

⁶Note that $x^\pm = X \pm T$.

(identified as those living in the G' sector) is the following:

$$G_{\mu\nu} = \mathring{G}_{\mu\nu} - 2 \sum_{i=a}^r \mathbb{H}_a^2 J_\mu^i J_\nu^i, \quad (2.17a)$$

$$B_{\mu\nu} = \mathring{B}_{\mu\nu}, \quad (2.17b)$$

$$A_\mu^a = \mathbb{H}_a \sqrt{\frac{2k}{k_g}} J_\mu^a, \quad (2.17c)$$

where $\mathring{G}_{\mu\nu}$ and $\mathring{B}_{\mu\nu}$ are the initial, unperturbed background fields that are expressed in terms of the $g \in G'$ group element as follows:

$$\mathring{G}_{\mu\nu} dx^\mu dx^\nu = \langle g^{-1} dg, g^{-1} dg \rangle, \quad (2.18a)$$

$$\mathring{B}_{\mu\nu} dx^\mu \wedge x^\nu = g^{-1} dg \wedge g^{-1} dg. \quad (2.18b)$$

No dilaton is present (as a consequence of the fact that the Ricci scalar for these deformed systems remains constant) and these semiclassical solutions can be promoted to exact ones just by remarking that the effect of the renormalisation simply boils down to the shift $k \rightarrow k + c_{G'}$ where $c_{G'}$ is the dual Coxeter number, just as in the case of the unperturbed WZW model.

The $SL(2, \mathbb{R})$ case. We now apply the above to the $SL(2, \mathbb{R})$ case. As previously, three asymmetric deformations are available: the elliptic, the hyperbolic and the parabolic.

The *elliptic deformation* is generated by a bilinear where the left current is an $SL(2, \mathbb{R})_k$ time-like current. The background field is magnetic and the residual symmetry is $U(1)_{\text{time}} \times SL(2, \mathbb{R})$ generated by $\{L_3, R_1, R_2, R_3\}$ (see appendix A.2). The metric reads (in elliptic coordinates):

$$ds^2 = \frac{k}{4} \left[d\rho^2 + \cosh^2 \rho d\phi^2 - (1 + 2H^2) (dt + \sinh \rho d\phi)^2 \right], \quad (2.19)$$

where ∂_t is the Killing vector associated with the $U(1)_{\text{time}}$. This AdS_3 deformation was studied in [29] as a *squashed anti de Sitter* and in [15, 16] from the string theory point of view. It has curvature

$$\mathcal{R} = -\frac{2}{k}(3 - 2H^2). \quad (2.20)$$

Here, it comes as an *exact string solution* (provided $k \rightarrow k + 2$) together with an NS three-form and a magnetic field:

$$H_{[3]} = dB - \frac{k_g}{4} A \wedge dA = -\frac{k}{4} (1 + 2H^2) \cosh \rho d\rho \wedge d\phi \wedge dt, \quad (2.21a)$$

$$A = \mathbb{H} \sqrt{\frac{2k}{k_g}} (dt + \sinh \rho d\phi). \quad (2.21b)$$

For $H^2 > 0$ (unitary region), the above metric is pathological because it has topologically trivial closed time-like curves passing through any point of the manifold. Actually, for $H^2 = 1/2$ we recover exactly the Gödel space, which is a well-known example of pathological solution of Einstein-Maxwell equations.

The *hyperbolic deformation* can be studied in a similar fashion, where the left current in the bilinear is an $\mathrm{SL}(2, \mathbb{R})_k$ space-like current. In hyperbolic coordinates:

$$ds^2 = \frac{k}{4} \left[dr^2 - \cosh^2 r d\tau^2 + (1 - 2H^2) (dx + \sinh r d\tau)^2 \right], \quad (2.22)$$

where ∂_x generates a $\mathrm{U}(1)_{\text{space}}$. The total residual symmetry is $\mathrm{U}(1)_{\text{space}} \times \mathrm{SL}(2, \mathbb{R})$, generated by $\{L_2, R_1, R_2, R_3\}$, and

$$\mathcal{R} = -\frac{2}{k} (3 + 2H^2). \quad (2.23)$$

The complete string background now has an NS three-form and an electric field:

$$H_{[3]} = \frac{k}{4} (1 - 2H^2) \cosh r dr \wedge d\tau \wedge dx, \quad (2.24a)$$

$$A = H \sqrt{\frac{2k}{k_g}} (dx + \sinh r d\tau). \quad (2.24b)$$

The background at hand is free of closed time-like curves. The squashed AdS_3 is now obtained by going to the AdS_3 picture as an S^1 fibration over an AdS_2 base, and modifying the S^1 fiber. The magnitude of the electric field is limited at $H_{\text{max}}^2 = 1/2$, where it causes the degeneration of the fiber, and we are left with an AdS_2 background with an electric monopole; in other words, a geometric coset $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)_{\text{space}}$.

The string spectrum of the above deformation is accessible by conformal-field-theory methods. It is free of tachyons and a whole tower of states decouples at the critical values of the electric fields. Details are available in [16].

Finally, the *parabolic deformation* is generated by a null $\mathrm{SL}(2, \mathbb{R})_k$ current times some internal right-moving current. The deformed metric reads, in Poincaré coordinates:

$$ds^2 = k \left[\frac{du^2}{u^2} + \frac{dx^+ dx^-}{u^2} - 2H^2 \left(\frac{dx^+}{u^2} \right)^2 \right], \quad (2.25)$$

and the curvature remains unaltered $\mathcal{R} = -6/k$. This is not surprising since the resulting geometry is a plane-wave like deformation of AdS_3 . The residual symmetry is $\mathrm{U}(1)_{\text{null}} \times \mathrm{SL}(2, \mathbb{R})$, where the $\mathrm{U}(1)_{\text{null}}$ is generated by $\partial_- = -L_1 - L_3$.

The parabolic deformation is somehow peculiar. Although it is continuous, the deformation parameter can always be re-absorbed by a redefinition of the coordinates:⁷ $x^+ \rightarrow x^+ / |H|$ and $x^- \rightarrow x^- |H|$. Put differently, there are only three truly different options: $H^2 = 0, 1$. No limiting geometry emerges in the case at hand.

As expected, the gravitational background is accompanied by an NS three-form (unaltered) and an electromagnetic wave:

$$A = 2 \sqrt{\frac{2k}{k_g}} H \frac{dx^+}{u^2}. \quad (2.26)$$

⁷This statement holds as long as these coordinates are not compact. After discrete identifications have been imposed (see section 5.1), H becomes a genuine continuous parameter.

A final remark is in order here, which holds for all three asymmetric deformations of $\mathrm{SL}(2, \mathbb{R})$. The background electric or magnetic fields that appear in these solutions (eqs. (2.21b), (2.24b) and (2.26)) diverge at the boundary of the corresponding spaces. Hence, these fields cannot be considered as originating from localized charges.

3. The three-dimensional black string revisited

The AdS_3 moduli space contains black hole geometries. This has been known since the most celebrated of them — the two-dimensional $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$ black hole — was found by Witten [30, 31]. Generalisations of these constructions to higher dimensions have been considered in [18, 32–34]. The three-dimensional black string [18, 35, 36] has attracted much attention, for it provides an alternative to the Schwarzschild black hole in three-dimensional asymptotically flat geometries.⁸ In this section we want to show how this black string can be interpreted in terms of marginal deformations of $\mathrm{SL}(2, \mathbb{R})$, which will enable us to give an expression for its string primary states (section 6).

In [18] the black string was obtained as an $(\mathrm{SL}(2, \mathbb{R}) \times \mathbb{R})/\mathbb{R}$ gauged model. More precisely, expressing $g \in \mathrm{SL}(2, \mathbb{R}) \times \mathbb{R}$ as:

$$g = \begin{pmatrix} a & u & 0 \\ -v & b & 0 \\ 0 & 0 & e^x \end{pmatrix}, \quad (3.1)$$

the left and right embeddings of the \mathbb{R} subgroup are identical and given by:

$$\epsilon_{L/R} : \mathbb{R} \rightarrow \mathrm{SL}(2, \mathbb{R}) \times \mathbb{R} \quad (3.2)$$

$$\lambda \mapsto \begin{pmatrix} e^{\frac{1}{\sqrt{\lambda^2+2}}} & 0 & 0 \\ 0 & e^{-\frac{1}{\sqrt{\lambda^2+2}}} & 0 \\ 0 & 0 & e^{\frac{\lambda}{\sqrt{\lambda^2+2}}} \end{pmatrix}. \quad (3.3)$$

From the discussion in section 2.2, we see that performing this gauging is just one of the possible ways to recover the $J^2 \bar{J}^2$ symmetrically deformed $\mathrm{SL}(2, \mathbb{R})$ geometry. More specifically, since the gauged symmetry is axial ($g \rightarrow hgh$), it corresponds (in our notation) to the $\kappa_2 < 1$ branch of the deformed geometry⁹ in eq. (2.14a). One can find a coordinate transformation allowing to pass from the usual black-string solution

$$ds^2 = \frac{k}{4} \left[- \left(1 - \frac{1}{r}\right) dt^2 + \left(1 - \frac{\mu^2}{r}\right) dx^2 + \left(1 - \frac{1}{r}\right)^{-1} \left(1 - \frac{\mu^2}{r}\right)^{-1} \frac{dr^2}{r^2} \right], \quad (3.4a)$$

$$H = \frac{k\mu}{4r} dr \wedge dx \wedge dt, \quad (3.4b)$$

$$e^{2\Phi} = \frac{\mu}{r} \quad (3.4c)$$

⁸Remember that the *no hair* theorem doesn't hold in three dimensions [37–39].

⁹The $R \gtrsim 1$ convention is not univocal in literature.

μ	name	dt^2	dx^2	dr^2	range	CFT interpretation
		$-(1 - \frac{1}{r})$	$1 - \frac{\mu^2}{r}$	$(1 - \frac{1}{r})^{-1} (1 - \frac{\mu^2}{r})^{-1}$		
$\mu^2 > 1$	(c^+)	---	+	+	$r > \mu^2$	$J^3 \bar{J}^3, \kappa_3 > 1$
	(b^+)	---	---	---	$1 < r < \mu^2$	
	(a^+)	+	---	+	$0 < r < 1$	$J^3 \bar{J}^3, \kappa_3 < 1$
$\mu^2 < 1$	(a^-)	---	-	+	$0 < r < \mu^2$	$J^2 \bar{J}^2, \kappa_2 < 1$
	(b^-)	---	+	-	$\mu^2 < r < 1$	
	(c^-)	-	+	+	$r > 1$	

Table 1: Signature for the black-string metric as a function of r , for $\mu^2 \gtrless 1$.

to our (local) coordinate system, eq. (2.14). The attentive reader might now be puzzled by this equivalence between a one-parameter model such as the symmetrically deformed model and a two-parameter one such as the black string in its usual coordinates (in eqs. (3.4) we redefined the r coordinate as $r \rightarrow r/M$ and then set $\mu = Q/M$ with respect to the conventions in [18]). A point that it is interesting to make here is that although, out of physical considerations, the black string is usually described in terms of two parameters (mass and charge), the only physically distinguishable parameter is their ratio $\mu = Q/M$ that coincides with our κ_2 parameter. In section 4 we will introduce a different (double) deformation, this time giving rise to a black hole geometry depending on two actual parameters (one of which being related to an additional electric field).

As we remarked above, the axial gauging construction only applies for $\mu < 1$, while, in order to obtain the other $\kappa_2 > 1$ branch of the $J^2 \bar{J}^2$ deformation, one should perform a vector gauging. On the other hand, this operation, that would be justified by a CFT point of view, is not natural when one takes a more geometrical point of view and writes the black string metric as in eq. (3.4a). In the latter, one can study the signature of the metric as a function of r in the two regions $\mu^2 \gtrless 1$, and find the physically sensible regions (see table 1).

Our observations are the following:

- The $\mu^2 < 1$ branch always has the correct $(-, +, +)$ signature for any value of r , with the two special values $r = 1$ and $r = \mu^2$ marking the presence of the horizons that hide the singularity in $r = 0$.
- The $\mu^2 > 1$ branch is different. In particular we see that there are two regions: (a^+) for $0 < r < 1$ and (c^+) for $r > \mu^2$ where the signature is that of a physical space.

A fact deserves to be emphasized here: one should notice that while for $\mu^2 < 1$ we obtain three different regions of the same space, for $\mu^2 > 1$ what we show in table 1 really are three different spaces and the proposed ranges for r are just an effect of the chosen parameterization. The $(a^+), \kappa_3 < 1$ and $(c^+), \kappa_3 > 1$ branches are different spaces and not different regions of the same one and one can choose in which one to go when continuing to $\mu > 1$.

But there is more. The $\mu^2 > 1$ region is obtained via an analytic continuation with respect to the other branch, and this analytic continuation is precisely the one that interchanges the roles of the J^2 and the J^3 currents. As a result, we pass from the $J^2\bar{J}^2$ line to the $J^3\bar{J}^3$ line. More precisely the (c^+) region describes the “singular” $\kappa_3 > 1$ branch of the $J^3\bar{J}^3$ deformation (i.e. the branch that includes the $r = 0$ singularity) and the (a^+) region describes the regular $\kappa_3 < 1$ branch that has the *cigar* geometry as $\kappa_3 \rightarrow 0$ limit. Also notice that the regions $r < 0$ have to be excluded in order to avoid naked singularities (of the type encountered in the Schwarzschild black hole with negative mass). The black string described in [18] covers the regions $(a^-), (b^-), (c^-), (a^+)$.

Our last point concerns the expectation of the genuine AdS₃ geometry as a zero-deformation limit of the black-string metric, since the latter turns out to be a marginal deformation of AdS₃ with parameter μ . The straightforward approach consists in taking the line element in eq. (3.4a) for $\mu = 1$. It is then puzzling that the resulting extremal black-string geometry *is not* AdS₃. This apparent paradox is solved by carefully looking at the coordinate transformations that relate the black-string coordinates (r, x, t) to either the Euler coordinates (ρ, ϕ, τ) (A.5) for the $J^3\bar{J}^3$ line, or the hyperbolic coordinates (y, x, t) (A.8) for the $J^2\bar{J}^2$ line. These transformations are singular at $\mu = 1$, which therefore corresponds neither to $\kappa_3 = 1$ nor to $\kappa_2 = 1$. Put differently, $\mu = 1$ is not part of a continuous line of deformed models but marks a jump from the $J^2\bar{J}^2$ to the $J^3\bar{J}^3$ lines.

The extremal black-string solution is even more peculiar. Comparing eqs. (3.4) at $\mu = 1$ to eqs. (2.15), which describe the symmetrically null-deformed SL(2, \mathbb{R}), we observe that the two backgrounds at hand are related by a coordinate transformation, provided $\nu = -1$.

The black string background is therefore entirely described in terms of SL(2, \mathbb{R}) marginal symmetric deformations, and involves all three of them. The null deformation appears, however, for the extremal black string only and at a negative value of the parameter ν . The latter is the density of fundamental strings, when the deformed AdS₃ is considered within the NS5/F1 system. This might be one more sign pointing towards a possible instability in the black string [40].

Notice finally that expressions (3.4) receive $1/k$ corrections. Those have been computed in [41]. Once taken into account, they contribute in making the geometry smoother, as usual in string theory.

4. The two-parameter deformations

4.1 An interesting mix

A particular kind of asymmetric deformation is what we will call in the following *double deformation* [15, 42]. At the lagrangian level this is obtained by adding the following marginal perturbation to the WZW action:

$$\delta S = \delta\kappa^2 \int d^2z J\bar{J} + \text{H} \int d^2z J\bar{I}; \quad (4.1)$$

J is a holomorphic current in the group, \bar{J} is the corresponding anti-holomorphic current and \bar{I} an external (to the group) anti-holomorphic current (i.e. in the right-moving heterotic

sector for example). A possible way to interpret this operator consists in thinking of the double deformation as the superposition of a symmetric — or gravitational — deformation (the first addend) and of an antisymmetric one — the electromagnetic deformation. This mix is consistent because if we perform the κ deformation first, the theory keeps the $U(1) \times U(1)$ symmetry generated by J and \bar{J} that is needed in order to allow for the \mathbb{H} deformation. Following this trail, we can read off the background fields corresponding to the double deformation by using at first one of the methods outlined in section 2.2 and then applying the reduction in eq. (2.17) to the resulting background fields.

The final result consists in a metric, a three-form, a dilaton and a gauge field. It is in general valid at any order in the deformation parameters κ and \mathbb{H} but only at leading order in α' due to the presence of the symmetric part.

Double deformations of AdS_3 where J is the time-like J^3 operator have been studied in [15]. It was there shown that the extra gravitational deformation allows to get rid of the closed time-like curves, which are otherwise present in the pure J^3 asymmetric deformation (eq. (2.19)) — the latter includes Gödel space. Here, we will focus instead on the case of double deformation generated by space-like operators, J^2 and \bar{J}^2 .

4.2 The hyperbolic double deformation

In order to follow the above prescription for reading the background fields in the double-deformed metric let us start with the fields in eqs. (2.14). We can introduce those fields in the sigma-model action. Infinitesimal variation of the latter with respect to the parameter κ^2 enables us to reach the following expressions for the chiral currents $J_\kappa^2(z)$ and $\bar{J}_\kappa^2(\bar{z})$ at finite values of κ^2 :

$$J_\kappa^2(z) = \frac{1}{\cos^2 \beta + \kappa^2 \sin^2 \beta} (\cos^2 \beta \partial \psi - \sin^2 \beta \partial \varphi), \quad (4.2)$$

$$\bar{J}_\kappa^2(\bar{z}) = \frac{1}{\cos^2 \beta + \kappa^2 \sin^2 \beta} (\cos^2 \beta \partial \psi + \sin^2 \beta \partial \varphi). \quad (4.3)$$

Note in particular that the corresponding Killing vectors (that clearly are ∂_φ and ∂_ψ) are to be rescaled as $L_2 = \frac{1}{\kappa^2} \partial_\psi - \partial_\varphi$ and $R_2 = \frac{1}{\kappa^2} \partial_\psi + \partial_\varphi$. Once the currents are known, one just has to apply the construction sketched in section 2.3 and write the background fields as follows:

$$\begin{aligned} \frac{1}{k} ds^2 = & -d\beta^2 + \cos^2 \beta \frac{(\kappa^2 - 2\mathbb{H}^2) \cos^2 \beta + \kappa^4 \sin^2 \beta}{\Delta_\kappa(\beta)^2} d\psi^2 - 4\mathbb{H}^2 \frac{\cos^2 \beta \sin^2 \beta}{\Delta_\kappa(\beta)^2} d\psi d\varphi + \\ & + \sin^2 \beta \frac{\cos^2 \beta + (\kappa^2 - 2\mathbb{H}^2) \sin^2 \beta}{\Delta_\kappa(\beta)^2} d\varphi^2, \end{aligned} \quad (4.4a)$$

$$\frac{1}{k} B = \frac{\kappa^2 - 2\mathbb{H}^2}{\kappa^2} \frac{\cos^2 \beta}{\Delta_\kappa(\beta)} d\varphi \wedge d\psi, \quad (4.4b)$$

$$F = 2\mathbb{H} \sqrt{\frac{2k}{k_g}} \frac{\sin(2\beta)}{\Delta_\kappa(\beta)^2} (\kappa^2 d\psi \wedge d\beta + d\beta \wedge d\varphi), \quad (4.4c)$$

$$e^{2\Phi} = \frac{\sqrt{\kappa^2 - 2\mathbb{H}^2}}{\Delta_\kappa(\beta)}, \quad (4.4d)$$

where $\Delta_\kappa(\beta) = \cos^2 \beta + \kappa^2 \sin^2 \beta$ as in section 2.2. In particular the dilaton, that can be obtained by imposing the one-loop beta equation is proportional to the ratio of the double deformed volume form and the AdS_3 one.

A first observation about the above background is in order here. The electric field is bounded from above since $H^2 \leq \frac{\kappa^2}{2}$. As usual in string theory, tachyonic instabilities occur at large values of electric or magnetic fields, and we already observed that phenomenon in section 2.3, for purely asymmetric ($\kappa^2 = 1$) deformations. At the critical value of the parameter H , one dimension degenerates and the B -field vanishes. We are left with a two-dimensional space (with non-constant curvature) plus electric field.

The expression (4.4a) here above of the metric provides only a local description of the space-time geometry. To discuss the global structure of the whole space it is useful to perform several coordinate transformations. Firstly let us parametrize by $\kappa^2 = \lambda/(1 + \lambda)$ the deformation parameter (with $\kappa < 1$ for $\lambda > 0$ and $\kappa > 1$ for $\lambda < -1$) and introduce a radial coordinate *à la* Horne and Horowitz:

$$r = \lambda + \cos^2 \beta, \quad (4.5)$$

which obviously varies between λ and $\lambda + 1$. The expression of the metric (4.4a) becomes in terms of this new coordinate:

$$\begin{aligned} ds^2 = & - \left[\left(2H^2 (1 + \lambda)^2 - \lambda \right) + \frac{\lambda \left(\lambda - 4H^2 (1 + \lambda)^2 \right)}{r} + \frac{2\lambda^2 H^2 (1 + \lambda)^2}{r^2} \right] d\psi^2 - \\ & - (1 + \lambda) \left[2H^2 (1 + \lambda) + 1 - \frac{(1 + \lambda) \left(1 + 4H^2 (1 + \lambda)^2 \right)}{r} + \frac{2(1 + \lambda)^3 H^2}{r^2} \right] d\varphi^2 + \\ & + 4H^2 (1 + \lambda)^2 \left[1 - \frac{1 + 2\lambda}{r} + \frac{\lambda(1 + \lambda)}{r^2} \right] d\psi d\varphi + \frac{1}{4(r - \lambda)(r - \lambda - 1)} dr^2. \quad (4.6) \end{aligned}$$

This expression looks close to the one discussed by Horne and Horowitz. It also represents a black string. However, it depends on more physical parameters as the expression of the scalar curvature shows:

$$\mathcal{R} = 2 \frac{2r(1 + 2\lambda) - 7\lambda(1 + \lambda) - 2H^2(1 + \lambda)^2}{r^2}. \quad (4.7)$$

This result may seem strange at first sight since, for $\kappa = 1$ and $H = 0$, the metric (4.4a) is of constant Ricci (and thus scalar) curvature, corresponding to a local patch of AdS_3 while here, in the same limit, the curvature vanishes for large r . The absence of contradiction follows from the definition of the r -coordinate, becoming ill-defined for $\kappa = 1$, as it corresponds to $\lambda = \infty$.

Obviously this metric can be extended behind the initial domain of definition of the r variable. But before to discuss it, it is interesting to note that the Killing vector $\mathbf{k} = (1 + \lambda) \partial_\psi + \lambda \partial_\varphi \propto R_2$ is of constant square length

$$\mathbf{k} \cdot \mathbf{k} = \lambda(1 + \lambda) - 2H^2(1 + \lambda)^2 := \omega. \quad (4.8)$$

Note that as H^2 is positive, we have the inequality $\omega < \lambda(1 + \lambda)$. Moreover, in order to have a lorentzian signature we must impose $\omega > 0$. The fact that the Killing vector \mathbf{k} is space-like and of constant length makes it a candidate to perform identifications. We shall discuss this point at the end of this section.

The constancy of the length of the Killing vector \mathbf{k} suggests to make a new coordinate transformation (such that $\mathbf{k} = \partial_x$):

$$\psi = (1 + \lambda)x + t, \quad (4.9a)$$

$$\varphi = t + \lambda x, \quad (4.9b)$$

which leads to the much simpler expression of the line element:

$$ds^2 = -\frac{(r - \lambda)(r - \lambda - 1)}{r^2} dt^2 + \omega \left(dx + \frac{1}{r} dt \right)^2 + \frac{1}{4(r - \lambda)(r - \lambda - 1)} dr^2. \quad (4.10)$$

This metric is singular at $r = 0, \lambda, \lambda + 1$; $r = 0$ being a curvature singularity. On the other hand, the volume form is $\sqrt{\omega}/(2r) dt \wedge dx \wedge dr$, which indicates that the singularities at $r = \lambda$ and $r = \lambda + 1$ may be merely coordinate singularities, corresponding to horizons. Indeed, it is the case. If we expand the metric, around $r = \lambda + 1$, for instance, at first order (i.e. for $r = \lambda + 1 + \epsilon$) we obtain:

$$ds^2 = \frac{\omega}{(1 + \lambda)^2} (dt + (1 + \lambda) dx)^2 - \frac{\epsilon}{(1 + \lambda)^2} dt \left[dt + 2\frac{\omega}{1 + \lambda} (dt + (1 + \lambda) dx) \right] + \frac{1}{4\epsilon} dr^2 \quad (4.11)$$

indicating the presence of an horizon. To eliminate the singularity in the metric, we may introduce Eddington-Finkelstein like coordinates:

$$t = (1 + \lambda) \left(u \pm \frac{1}{2} \ln \epsilon \right) - \omega \xi, \quad (4.12a)$$

$$x = \left(1 + \frac{\omega}{1 + \lambda} \right) \xi - \left(u \pm \frac{1}{2} \ln \epsilon \right). \quad (4.12b)$$

The same analysis can also be done near the horizon located at $r = \lambda$. Writing $r = \lambda + \epsilon$, the corresponding regulating coordinate transformation to use is given by:

$$t = \lambda \left(u \pm \frac{1}{2} \ln \epsilon \right) + \omega \xi, \quad (4.13a)$$

$$x = \left(1 - \frac{\omega}{\lambda} \right) \xi - \left(u \pm \frac{1}{2} \ln \epsilon \right). \quad (4.13b)$$

In order to reach the null Eddington-Finkelstein coordinates, we must use null rays. The geodesic equations read, in terms of a function $\Sigma^2[E, P, \epsilon; r] = (Er - P)^2 - (P^2/\omega) - \epsilon(r - \lambda)(r - \lambda - 1)$:

$$\sigma = \int \frac{1}{4\Sigma[E, P, \epsilon; r]} dr, \quad (4.14a)$$

$$t = \int \frac{(Er - P)r}{2(r - \lambda)(r - \lambda - 1)\Sigma[E, P, \epsilon; r]} dr, \quad (4.14b)$$

$$x = - \int \frac{(Er - P) + P/\omega}{2(r - \lambda)(r - \lambda - 1)\Sigma[E, P, \epsilon; r]} dr, \quad (4.14c)$$

where E and P are the constant of motion associated to ∂_t and ∂_x , σ is an affine parameter and ε , equal to 1, 0, -1 , characterizes the time-like, null or space-like nature of the geodesic. Comparing these equations (with $\varepsilon = 0$ and $P = 0$) with the coordinates introduced near the horizons, we see that regular coordinates in their neighbourhoods are given by

$$t = T \pm \frac{1}{2} ((1 + \lambda) \ln |r - \lambda - 1| - \lambda \ln |r - \lambda|), \quad (4.15a)$$

$$x = X \mp \frac{1}{2} (\ln |r - \lambda - 1| - \ln |r - \lambda|), \quad (4.15b)$$

which leads to the metric

$$ds^2 = \left(-1 + \frac{1 + 2\lambda}{r} - \frac{\lambda(1 + \lambda) - \omega}{r^2} \right) dT^2 + 2\frac{\omega}{r} dX dT + \omega dX^2 \mp \frac{1}{r} dT dr. \quad (4.16)$$

According to the sign, we obtain incoming or outgoing null coordinates; to build a Kruskal coordinate system we have still to exponentiate them.

Obviously, we may choose the X coordinate in the metric (4.16) to be periodic without introducing closed causal curves. The question of performing more general identifications in these spaces will be discussed addressed now.

We end this section by computing the conserved charges associated to the asymptotic symmetries of our field configurations (4.4). As is well known, their expressions provide solutions of the equations of motion derived from the low-energy effective action

$$S = \int d^d x \sqrt{-g} e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 - \frac{k_g}{8} F^2 + \frac{\delta c}{3} \right], \quad (4.17)$$

in which we have chosen the units such that $\delta c = 12$.

Expression (4.10) for the metric is particularly appropriate to describe the asymptotic properties of the solution. In these coordinates, the various non-gravitational fields read as

$$F = \pm \frac{\sqrt{2}H(1 + \lambda)}{r^2 \sqrt{k_g}} dt \wedge dr, \quad (4.18)$$

$$H = \mp \frac{\omega}{r^2} dt \wedge dx \wedge dr, \quad (4.19)$$

$$\Phi = \Phi_* - \frac{1}{2} \ln r. \quad (4.20)$$

By setting $\sqrt{\omega}x = \bar{x}$ and $r = e^{2\bar{\rho}}$, near infinity ($\bar{\rho} \rightarrow \infty$), the metric asymptotes the standard flat metric: $ds^2 = -dt^2 + d\bar{x}^2 + d\bar{\rho}^2$, while the fields F and H vanish and the dilaton reads $\Phi = \Phi_* - \bar{\rho}$. This allows to interpret the asymptotic behavior of our solution (4.4) as a perturbation around the solution given by $F = 0$, $H = 0$, the flat metric and a linear dilaton: $\bar{\Phi} = \Phi_* + f_\alpha X^\alpha$ (here $f_\alpha = (0, 0, -1)$). Accordingly, we may define asymptotic charges associated to each asymptotic reductibility parameter (see [43]).

For the gauge symmetries we obtain as charges, associated to the H field

$$Q_H = \pm 2e^{-2\Phi_*} \sqrt{\omega} \quad (4.21)$$

and to the F field

$$Q_F = \pm \frac{2\sqrt{2}e^{-2\Phi_*}H(1+\lambda)}{\sqrt{k_g}}. \quad (4.22)$$

The first one reduces (up to normalization) for $H = 0$ to the result given in [18], while the second one provides an interpretation of the deformation parameter H .

Moreover, all the Killing vectors of the flat metric defining isometries that preserve the dilaton field allow to define asymptotic charges. These charges are obtained by integrating the antisymmetric tensor on the surface at infinity:

$$k_\xi^{[\mu\nu]} = e^{-2\Phi} \left(\xi_\sigma \partial_\lambda \mathcal{H}^{\sigma\lambda\mu\nu} + \frac{1}{2} \partial_\lambda \xi_\sigma \mathcal{H}^{\sigma\lambda\mu\nu} + 2 \left(\xi^\mu h_\lambda^\nu f^\lambda - \xi^\nu h_\lambda^\mu f^\lambda \right) \right), \quad (4.23)$$

where

$$\mathcal{H}^{\sigma\lambda\mu\nu} = \bar{h}^{\sigma\nu} \eta^{\lambda\mu} + \bar{h}^{\lambda\mu} \eta^{\sigma\nu} - \bar{h}^{\sigma\mu} \eta^{\lambda\nu} - \bar{h}^{\lambda\nu} \eta^{\sigma\mu} \quad (4.24)$$

is the well known tensor sharing the symmetries of the Riemann tensor and $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta} h_{\alpha\beta}$, while the Killing vector ξ has to fulfill the invariance condition $\xi_\alpha f^\alpha = 0$. The expression of the tensor $k_\xi^{[\mu\nu]}$ depends only on the perturbation $h_{\mu\nu}$ of the metric tensor because, on the one hand, the F and H fields appear quadratically in the lagrangian, and their background values are zero, while, on the other hand, the perturbation field for the dilaton vanishes: $\Phi = \bar{\Phi}$.

Restricting ourselves to constant Killing vectors, we obtain the momenta (defined for the indice $\sigma = t$ and \bar{x})

$$P^\sigma = \int d\bar{x} e^{-2\bar{\Phi}} \left(\partial_\lambda \mathcal{H}^{\sigma\lambda\bar{\rho}} - 2\eta^{\sigma t} h_{\bar{\rho}}^\nu \right) \quad (4.25)$$

i.e. the density of mass (μ) and momentum (ϖ) per unit length:

$$\mu = 2e^{-2\Phi_*}(1+2\lambda) \quad \text{and} \quad \varpi = -2e^{-2\Phi_*}\sqrt{\omega}. \quad (4.26)$$

Of course, if we perform identifications such that the string acquires a finite length, the momenta (4.25) become also finite.

To make an end let us notice that the expressions of μ and ϖ that we obtain differ from those given in [18] by a normalization factor but also in their dependance with respect to λ , even in the limit $H = 0$; indeed, the asymptotic minkowskian frames used differ from each other by a boost.

5. Discrete identifications

In the same spirit as the original BTZ construction reminded in appendix B, we would like to investigate to what extent discrete identifications could be performed in the deformed background. Necessary conditions for a solution (4.16) to remain “viable” black hole can be stated as follows:

- the identifications are to be performed along ‘the orbits of some Killing vector ξ of the deformed metric
- there must be causally safe asymptotic regions (at spatial infinity)

- the norm of ξ has to be positive in some region of space-time, and chronological pathologies have to be hidden with respect to an asymptotic safe region by a horizon.

The resulting quotient space will exhibit a black hole structure if, once the regions where $\|\xi\| < 0$ have been removed, we are left with an almost geodesically complete space, the only incomplete geodesics being those ending on the locus $\|\xi\| = 0$. It is nevertheless worth emphasizing an important difference with the BTZ construction. In our situation, unlike the undeformed AdS₃ space, the initial space-time where we are to perform identifications do exhibit curvature singularities.

5.1 Discrete identifications in asymmetric deformations

Our analysis of the residual isometries in purely asymmetric deformations (section 2.3) shows that the vector ξ (eq. (B.1a)) survives only in the hyperbolic deformation, whereas ξ in eq. (B.1b) is present in the parabolic one. Put differently, non-extremal BTZ black holes allow for electric deformation, while in the extremal ones, the deformation can only be induced by an electro-magnetic wave. Elliptic deformation is not compatible with BTZ identifications.

The question that we would like to address is the following: how much of the original black hole structure survives the deformation? The answer is simple: a new chronological singularity appears in the asymptotic region of the black hole. Evaluating the norm of the Killing vector shows that a naked singularity appears. Thus the deformed black hole is no longer a viable gravitational background. Actually, whatever the Killing vector we consider to perform the identifications, we are always confronted to such pathologies.

The fate of the *asymmetric parabolic* deformation of AdS₃ is similar: there is no region at infinity free of closed time-like curves after performing the identifications.

5.2 Discrete identifications in symmetric deformations

Let us consider the *symmetric hyperbolic* deformation, whose metric is given by (4.10) with $H = 0$, i.e. $\omega = \lambda(1 + \lambda)$. This metric has two residual Killing vectors, manifestly given by ∂_t and ∂_x . We may thus, in general, consider identifications along integral lines of

$$\xi = a \partial_t + \partial_x. \quad (5.1)$$

This vector has squared norm:

$$\|\xi\|^2 = (\lambda(1 + \lambda) - a^2) + \frac{2a\lambda(1 + \lambda) + a^2(1 + 2\lambda)}{r}. \quad (5.2)$$

To be space-like at infinity the vector ξ must verify the inequality $a^2 < \lambda(1 + \lambda)$. For definiteness, we will hereafter consider $\lambda > 0$ and $r > 0$ (the case $\lambda < -1$, $r < 0$ leads to similar conclusions, while the two other situations have to be excluded in order to avoid naked singularities, see eq. (4.10)). If $a > 0$, or $-\sqrt{\lambda(1 + \lambda)} < a < -2\lambda(1 + \lambda)/(1 + 2\lambda)$, ξ is everywhere space-like. Otherwise, it becomes time-like behind the inner horizon ($r = \lambda$), or on this horizon if $a = -\lambda$. In this situation, the quotient space will exhibit a structure similar to that of the black string, with a time-like chronological singularity (becoming light-like for $a = -\lambda$) hidden behind two horizons (or a single one for $a = -\lambda$).

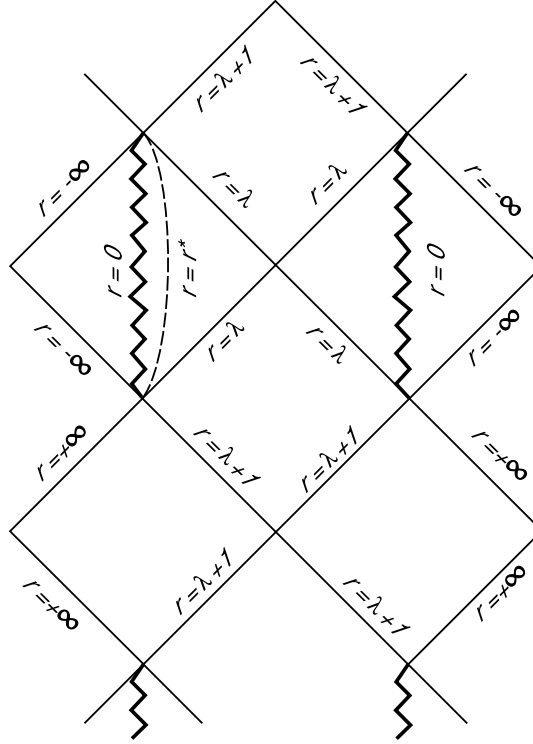


Figure 1: Penrose diagram exhibiting the global structure of the double hyperbolic deformation. The time-like curvature singularities $r = 0$ are represented, as well as the horizons, located at $r = \lambda$ and $r = \lambda + 1$. When performing identifications along orbits of Killing vectors that allow for a causally safe region at infinity, a time-like chronological singularity may appear at $r = r^*$, with $0 < r^* \leq \lambda$.

5.3 Discrete identifications in double deformations

The norm squared of the identification vector (5.1) in the metric (4.10) is

$$\|\xi\|^2 = (\omega - a^2) + 2\frac{a\omega + a^2(1 + 2\lambda)}{r} - \frac{a^2(\lambda(1 + \lambda) - \omega)}{r^2}. \quad (5.3)$$

Between $r = 0$ and $r = \infty$, this scalar product vanishes once and only once (if $a \neq 0$). To be space-like at infinity we have to restrict the time component of ξ to $|a| < \omega$. Near $r = 0$ it is negative, while near the inner horizon ($r = \lambda$) it takes the non-negative value $\omega(\lambda + a)^2/\lambda^2$. Accordingly, by performing identifications using this Killing vector, we will encounter a chronological singularity, located at $r = r^*$, with $0 < r^* \leq \lambda$, the singularity being of the same type as the one in the symmetric case (see figure 1).

6. Towards the exact spectra

The main guideline for exploring the black hole geometries that we have so far considered has been the presence of an underlying CFT description. This allows us to identify the background fields as the lagrangian counterparts of exact conformal field theories. In this section we will give a look to the other — algebraic — aspect of these models, showing how it is possible to write an explicit expression for the spectrum of primary operators.

Since this kind of construction has already been carried on in [15] for the J_3 double deformation of $SL(2, \mathbb{R})$, here we will focus of the J_2 deformations, giving the spectrum for the deformed theory (section (6.1)) and for a deformed theory with discrete identifications (section (6.2)). We will limit ourselves to giving the spectrum for the theory: the evaluation of the partition function, although straightforward in principle, would require the decomposition of the $SL(2, \mathbb{R})$ partition function in a hyperbolic basis of characters, a still unresolved problem.

6.1 Deformed spectrum

Consider the double deformation described in section 4 for a $SL(2, \mathbb{R})_k$ super-WZW model where J is the hyperbolic (space-like) J_2 current.

The evaluation of the spectrum for our deformed model is pretty straightforward once one realizes that the deformations act as $O(2, 2)$ pseudo-orthogonal transformations on the charge lattice corresponding to the abelian subgroup of the $\mathfrak{sl}(2, \mathbb{R})$ heterotic model (as described in section 2). Left and right weights for the relevant lattices are (see eqs. (C.20) and (C.21)):

$$L_0 = \frac{1}{k} \left(\mu + n + \frac{a}{2} \right)^2, \quad (6.1a)$$

$$\bar{L}_0 = \frac{\bar{\mu}^2}{k+2} + \frac{1}{k_g} \left(\bar{n} + \frac{\bar{a}}{2} \right)^2, \quad (6.1b)$$

where the anti-holomorphic part contains the contribution coming from a $\mathfrak{u}(1)$ subgroup of the heterotic gauge group.

At the lagrangian level, the infinitesimal deformation we want to describe is given by the following marginal operator:

$$\mathcal{O} = \kappa^2 \frac{(J^2 + \psi_1 \psi_3)}{\sqrt{k}} \frac{\bar{J}^2}{\sqrt{k+2}} + \text{H} \frac{(J^2 + \psi_1 \psi_3)}{\sqrt{k}} \frac{\bar{I}}{\sqrt{k_g}}. \quad (6.2)$$

This suggests that the actual $O(2, 2)$ transformation should be obtained as a boost between the holomorphic part and the result of a rotation between the two anti-holomorphic components. The deformed lattices then read:

$$L_0^{\text{dd}} = \left\{ \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \left(\frac{\bar{\mu}}{\sqrt{k+2}} \cos \alpha + \frac{1}{\sqrt{k_g}} \left(\bar{n} + \frac{\bar{a}}{2} \right) \sin \alpha \right) \sinh x \right\}^2, \quad (6.3a)$$

$$\bar{L}_0^{\text{dd}} = \left\{ \left(\frac{\bar{\mu}}{\sqrt{k+2}} \cos \alpha + \frac{1}{\sqrt{k_g}} \left(\bar{n} + \frac{\bar{a}}{2} \right) \sin \alpha \right) \cosh x + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x \right\}^2, \quad (6.3b)$$

where the parameters x and α can be expressed as functions of ζ and ξ as follows:

$$\begin{cases} \kappa^2 = \sinh(2x) \cos \alpha, \\ \text{H} = \sinh(2x) \sin \alpha. \end{cases} \quad (6.4)$$

6.2 Twisting

The identification operation we performed in the symmetrically and double-deformed metric (as in section 5) is implemented in the string theory framework by the orbifold construction. This was already obtained in [44, 45] for the “standard” BTZ black hole that was described as a $\text{SL}(2, \mathbb{R})/\mathbb{Z}$ orbifold.

In order to write the spectrum that will contain the twisted sectors, the first step consists in writing explicitly the primary fields in our theory, distinguishing between the holomorphic and anti-holomorphic parts (as it is natural to do since the construction is intrinsically heterotic).

- The holomorphic part is simply written by introducing the charge boost of eq. (6.3a) in eq. (C.16):

$$\Phi_{j\bar{\mu}\bar{\nu}}^{\text{dd}}(z) = U_{j\mu}(z) \exp \left[i \left(\sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \sqrt{2} \bar{Q}_\alpha \sinh x \right) \vartheta_2 \right], \quad (6.5)$$

where $Q_\alpha = \bar{\mu} \sqrt{\frac{2}{k+2}} \cos \alpha + \bar{\nu} \sqrt{\frac{2}{k_g}} \sin \alpha$ and the dd superscript stands for double deformed

- To write the anti-holomorphic part we need at first to implement the rotation between the \bar{J}^3 and gauge current components:

$$\begin{aligned} \bar{\Phi}_{j\bar{\mu}\bar{\nu}}(\bar{z}) &= V_{j\mu}(\bar{z}) e^{i\bar{\mu}\sqrt{2/k+2}\bar{\theta}_2} e^{i\bar{\nu}\sqrt{2/k_g}\bar{X}} = \\ &= V_{j\mu}(\bar{z}) e^{i\sqrt{2}\bar{Q}_\alpha(\bar{\theta}_2 \cos \alpha + \bar{X} \sin \alpha)} e^{i\sqrt{2}\bar{Q}_{\alpha-\pi/2}(-\bar{\theta}_2 \sin \alpha + \bar{X} \cos \alpha)}, \end{aligned} \quad (6.6)$$

and then realize the boost in eq. (6.3b) on the involved part:

$$\begin{aligned} \bar{\Phi}_{j\bar{\mu}\bar{\nu}}^{\text{dd}}(\bar{z}) &= V_{j\mu} e^{i\sqrt{2}\bar{Q}_{\alpha-\pi/2}(-\bar{\theta}_2 \sin \alpha + \bar{X} \cos \alpha)} \times \\ &\times \exp \left[i \left(\sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x + \sqrt{2} \bar{Q}_\alpha \cosh x \right) (\bar{\theta}_2 \cos \alpha + \bar{X} \sin \alpha) \right]. \end{aligned} \quad (6.7)$$

Now that we have the primaries, consider the operator $W_w(z, \bar{z})$ defined as follows:

$$W_w(z, \bar{z}) = e^{-i\frac{k}{2}w\Delta_- \vartheta_2 + i\frac{k+2}{2}w\Delta_+ \bar{\theta}_2}, \quad (6.8)$$

where $w \in \mathbb{Z}$ and $\bar{\theta}_2$ the boson corresponding to the \bar{J}_2 current. It is easy to show that the following OPE's hold:

$$\vartheta_2(z) W_n(0, \bar{z}) \sim -w \Delta_- \log z W_w(0, \bar{z}), \quad (6.9)$$

$$\bar{\theta}_2(\bar{z}) W_n(z, 0) \sim w \Delta_+ \log \bar{z} W_w(z, 0), \quad (6.10)$$

showing that $W_w(z, \bar{z})$ acts as twisting operator with winding number w (ϑ_2 and $\bar{\theta}_2$ shift by $2\pi\Delta_-w$ and $2\pi\Delta_+w$ under $z \rightarrow e^{2\pi i}z$). This means that the general primary field in the $\text{SL}(2, \mathbb{R})_k/\mathbb{Z}$ theory can be written as:

$$\Phi_{j\mu\bar{\mu}\nu w}^{\text{tw}}(z, \bar{z}) = \Phi_{j\mu\bar{\mu}\nu}^{\text{dd}}(z, \bar{z}) W_w(z, \bar{z}). \quad (6.11)$$

where the tw superscript stands for twisted.

Having the explicit expression for the primary field, it is simple to derive the scaling dimensions which are obtained, as before, via the GKO decomposition of the Virasoro algebra $T[\mathfrak{sl}(2, \mathbb{R})] = T[\mathfrak{sl}(2, \mathbb{R})/\mathfrak{o}(1, 1)] + T[\mathfrak{o}(1, 1)]$. Given that the $T[\mathfrak{sl}(2, \mathbb{R})/\mathfrak{o}(1, 1)]$ part remains invariant (and equal to $L_0 = -j(j+1)/k - \mu^2/(k+2)$ as in eq. (C.18)), the deformed weights read:

$$L_0^{\text{tw}} = \left\{ \frac{k}{2\sqrt{2}} w \Delta_- + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \bar{Q}_\alpha \sinh x \right\}^2, \quad (6.12a)$$

$$\begin{aligned} \bar{L}_0^{\text{tw}} = & \left\{ -\frac{k+2}{2\sqrt{2}} w \Delta_+ \cos \alpha + \bar{Q}_\alpha \cosh x + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x \right\}^2 + \\ & + \left\{ \frac{k+2}{2\sqrt{2}} w \Delta_+ \sin \alpha + \bar{Q}_{\alpha-\pi/2} \right\}^2. \end{aligned} \quad (6.12b)$$

7. Summary

The main motivation for this work has been a systematic search of black-hole structures in the moduli space of AdS_3 , via marginal deformations of the $\text{SL}(2, \mathbb{R})$ WZW model and discrete identifications. This allows to reach three-dimensional geometries with black-hole structure that generalize backgrounds such as the BTZ black hole [7] or the three-dimensional black string [18].

The backgrounds under consideration include a (singular) metric, a Kalb-Ramond field, a dilaton and an electric field. The latter is always bounded from above, as usual in string theory, where tachyonic instabilities are expected for large electric or magnetic fields.

We have computed parameters such as mass or charge. For backgrounds obtained by performing marginal deformations, those parameters are related to the deformation parameters. Singularities are true curvature singularities, hidden behind horizons. This is to be opposed to the BTZ black-holes, where masses and momenta are introduced by the Killing vector of the discrete identification, and where the singularity is a chronological singularity.

Discrete identifications *à la* BTZ can be superimposed to the black holes obtained by continuous deformations of AdS_3 . Extra chronological singularities appear in that case,

which force us to excise some part of the original space. This part turns out to contain the locus of the curvature singularity. It is worth stressing that for certain range of the deformation parameters, *naked* singularities appear.

Although the geometrical view point has been predominating, the guideline for our study comes from the underlying CFT structure. This has enabled us to provide both a geometrical and an algebraical description in terms of the spectrum of the string primaries.

Since we are dealing with the extension of AdS_3 one may wonder about a possible holographic interpretation for the exact string backgrounds at hand, aiming at generalizing the usual AdS/CFT correspondence. A major obstruction to this is due to the asymptotic flatness of the geometries. Hence, it is not clear how to find a suitable boundary map.

Interesting questions that we did not address, which are in principle within reach, are those dealing with the thermodynamical properties of the above black holes, for which a microscopic interpretation in terms of string states should be tractable.

Acknowledgments

D.O. and M.P. thank D. Israël, E. Kiritsis, C. Kounnas and K. Sfetsos for useful discussions. S.D. and Ph.S. are grateful to G. Compère for its patient explanation of the general method for computing charges; S.D. thanks P. Aliani and D. Haumont for their advice. S.D. and Ph.S. acknowledge support from the Fonds National de la Recherche scientifique through an F.R.F.C. grant. D.O. and M.P. thank U.M.H. and U.L.B. for kind hospitality at various stages of the project, and acknowledge support from the E.U. under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369.

A. AdS_3 coordinate patches

A.1 AdS_3 $\text{SL}(2, \mathbb{R})$

The commutation relations for the generators of the $\text{SL}(2, \mathbb{R})$ algebra are

$$[J^1, J^2] = -iJ^3 \quad [J^2, J^3] = iJ^1 \quad [J^3, J^1] = iJ^2. \quad (\text{A.1})$$

The three-dimensional anti-de Sitter space is the universal covering of the $\text{SL}(2, \mathbb{R})$ group manifold. The latter can be embedded in a lorentzian flat space with signature $(-, +, +, -)$ and coordinates (x^0, x^1, x^2, x^3) :

$$g = L^{-1} \begin{pmatrix} x^0 + x^2 & x^1 + x^3 \\ x^1 - x^3 & x^0 - x^2 \end{pmatrix}, \quad (\text{A.2})$$

where L is the radius of AdS_3 .

The isometry group of the $\text{SL}(2, \mathbb{R})$ group manifold is generated by left or right actions on g : $g \rightarrow hg$ or $g \rightarrow gh \forall h \in \text{SL}(2, \mathbb{R})$. From the four-dimensional point of view, it is generated by the Lorentz boosts or rotations $\zeta_{ab} = i(x_a \partial_b - x_b \partial_a)$ with $x_a = \eta_{ab} x^b$. We

list here explicitly the six Killing vectors, as well as the group action they correspond to:

$$L_1 = \frac{\imath}{2} (\zeta_{32} - \zeta_{01}), \quad g \rightarrow e^{-\frac{\lambda}{2}\sigma^1} g, \quad (\text{A.3a})$$

$$L_2 = \frac{\imath}{2} (-\zeta_{31} - \zeta_{02}), \quad g \rightarrow e^{-\frac{\lambda}{2}\sigma^3} g, \quad (\text{A.3b})$$

$$L_3 = \frac{\imath}{2} (\zeta_{03} - \zeta_{12}), \quad g \rightarrow e^{\imath\frac{\lambda}{2}\sigma^2} g, \quad (\text{A.3c})$$

$$R_1 = \frac{\imath}{2} (\zeta_{01} + \zeta_{32}), \quad g \rightarrow g e^{\frac{\lambda}{2}\sigma^1}, \quad (\text{A.3d})$$

$$R_2 = \frac{\imath}{2} (\zeta_{31} - \zeta_{02}), \quad g \rightarrow g e^{-\frac{\lambda}{2}\sigma^3}, \quad (\text{A.3e})$$

$$R_3 = \frac{\imath}{2} (\zeta_{03} + \zeta_{12}), \quad g \rightarrow g e^{\imath\frac{\lambda}{2}\sigma^2}. \quad (\text{A.3f})$$

Both sets satisfy the algebra (A.1) (once multiplied by $-\imath$). The norms of the Killing vectors are the following:

$$\|L_1\|^2 = \|R_1\|^2 = \|L_2\|^2 = \|R_2\|^2 = -\|L_3\|^2 = -\|R_3\|^2 = \frac{L^2}{4}. \quad (\text{A.4})$$

Moreover $L_i \cdot L_j = 0$ for $i \neq j$ and similarly for the right set. Left vectors are not orthogonal to right ones.

The isometries of the $\text{SL}(2, \mathbb{R})$ group manifold turn into symmetries of the $\text{SL}(2, \mathbb{R})_k$ WZW model, where they are realized in terms of conserved currents.¹⁰ The reader will find details on those issues in the appendices of [16].

A.2 “Symmetric” coordinates

One introduces Euler-like angles by

$$g = e^{\imath\frac{\tau+\phi}{2}\sigma^2} e^{\rho\sigma^1} e^{\imath\frac{\tau-\phi}{2}\sigma^2}, \quad (\text{A.5})$$

which provide good global coordinates for AdS_3 when $\tau \in]-\infty, +\infty[$, $\rho \in [0, \infty[$, and $\phi \in [0, 2\pi]$. In Euler angles, the invariant metric reads:

$$ds^2 = L^2 [-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2]. \quad (\text{A.6})$$

The Ricci scalar of the corresponding Levi-Civita connection is $\mathcal{R} = -6/L^2$. The volume form reads:

$$\omega_{[3]} = \frac{L^3}{2} \sinh 2\rho d\rho \wedge d\phi \wedge d\tau, \quad (\text{A.7})$$

whereas $L_3 = \frac{1}{2} (\partial_\tau + \partial_\phi)$ and $R_3 = \frac{1}{2} (\partial_\tau - \partial_\phi)$.

Another useful, although not global, set of coordinates is defined by

$$g = e^{\frac{\psi-\varphi}{2}\sigma^3} e^{\imath\beta\sigma^1} e^{\frac{\psi+\varphi}{2}\sigma^3}, \quad (\text{A.8})$$

¹⁰When writing actions a choice of gauge for the NS potential is implicitly made, which breaks part of the symmetry: boundary terms appear in the transformations. These must be properly taken into account in order to reach the conserved currents. Although the expressions for the latter are not unique, they can be put in an improved-Noether form, in which they have only holomorphic (for L_i 's) or anti-holomorphic (for R_j 's) components.

(ψ and φ are not compact coordinates). The metric reads:

$$ds^2 = L^2 [\cos^2 \beta d\psi^2 - d\beta^2 + \sin^2 \beta d\varphi^2], \quad (\text{A.9})$$

with volume form

$$\omega_{[3]} = \frac{L^3}{2} \sin 2\beta d\beta \wedge d\psi \wedge d\varphi. \quad (\text{A.10})$$

Now $L_2 = \frac{1}{2}(\partial_\psi - \partial_\varphi)$ and $R_2 = \frac{1}{2}(\partial_\psi + \partial_\varphi)$.

Finally, the Poincaré coordinate system is defined by

$$\begin{cases} x^0 + x^2 = \frac{L}{u}, \\ x^0 - x^2 = Lu + \frac{Lx^+x^-}{u}, \\ x^1 \pm x^3 = \frac{Lx^\pm}{u}. \end{cases} \quad (\text{A.11})$$

For $\{u, x^+, x^-\} \in \mathbb{R}^3$, the Poincaré coordinates cover once the $SL(2\mathbb{R})$ group manifold. Its universal covering, AdS_3 , requires an infinite number of such patches. Moreover, these coordinates exhibit a Rindler horizon at $|u| \rightarrow \infty$; the conformal boundary is at $|u| \rightarrow 0$. Now the metric reads:

$$ds^2 = \frac{L^2}{u^2} (du^2 + dx^+ dx^-), \quad (\text{A.12})$$

and the volume form:

$$\omega_{[3]} = \frac{L^3}{2u^3} du \wedge dx^+ \wedge dx^-. \quad (\text{A.13})$$

We also have $L_1 + L_3 = -\partial_-$ and $R_1 + R_3 = \partial_+$.

A.3 “Asymmetric” coordinates

The above three sets of AdS_3 coordinates are suitable for implementing symmetric parabolic, elliptic or hyperbolic deformations, respectively driven by $(J^1 + J^3)$ ($\bar{J}^1 + \bar{J}^3$), $J^3 \bar{J}^3$ or $J^2 \bar{J}^2$. For asymmetric elliptic or hyperbolic deformations, we must use different coordinate systems, where the structure of AdS_3 as a Hopf fibration is more transparent. They are explicitly described in the following.

- The coordinate system used to describe the elliptic asymmetric deformation is defined as follows:

$$\begin{cases} \frac{x_0}{L} = \cosh \frac{\rho}{2} \cosh \frac{\phi}{2} \cos \frac{t}{2} - \sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \sin \frac{t}{2}, \\ \frac{x_1}{L} = -\sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} - \cosh \frac{\rho}{2} \sinh \frac{\phi}{2} \sin \frac{t}{2}, \\ \frac{x_2}{L} = -\cosh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} + \sinh \frac{\rho}{2} \cosh \frac{\phi}{2} \sin \frac{t}{2}, \\ \frac{x_3}{L} = -\sinh \frac{\rho}{2} \sinh \frac{\phi}{2} \cos \frac{t}{2} - \cosh \frac{\rho}{2} \cosh \frac{\phi}{2} \sin \frac{t}{2}. \end{cases} \quad (\text{A.14})$$

The metric now reads:

$$ds^2 = \frac{L^2}{4} (d\rho^2 + d\phi^2 - dt^2 - 2 \sinh \rho dt d\phi), \quad (\text{A.15})$$

and the corresponding volume form is

$$\omega_{[3]} = \frac{L^3}{8} \cosh \rho \, d\rho \wedge d\phi \wedge dt. \quad (\text{A.16})$$

This coordinate system is such that the t -coordinate lines coincide with the integral curves of the Killing vector $L_3 = -\partial_t$, whereas the ϕ -lines are the curves of $R_2 = \partial_\phi$.

- The coordinate system used to describe the asymmetric hyperbolic deformation is defined as follows:

$$\begin{cases} \frac{x_0}{L} = \cosh \frac{r}{2} \cosh \frac{x}{2} \cos \frac{\tau}{2} + \sinh \frac{r}{2} \sinh \frac{x}{2} \sin \frac{\tau}{2}, \\ \frac{x_1}{L} = -\sinh \frac{r}{2} \cosh \frac{x}{2} \cos \frac{\tau}{2} + \cosh \frac{r}{2} \sinh \frac{x}{2} \sin \frac{\tau}{2}, \\ \frac{x_2}{L} = -\cosh \frac{r}{2} \sinh \frac{x}{2} \cos \frac{\tau}{2} - \sinh \frac{r}{2} \cosh \frac{x}{2} \sin \frac{\tau}{2}, \\ \frac{x_3}{L} = \sinh \frac{r}{2} \sinh \frac{x}{2} \cos \frac{\tau}{2} - \cosh \frac{r}{2} \cosh \frac{x}{2} \sin \frac{\tau}{2}. \end{cases} \quad (\text{A.17})$$

For $\{r, x, \tau\} \in \mathbb{R}^3$, this patch covers exactly once the whole AdS_3 , and is regular everywhere [46]. The metric is then given by

$$ds^2 = \frac{L^2}{4} (dr^2 + dx^2 - d\tau^2 + 2 \sinh r \, dx \, d\tau), \quad (\text{A.18})$$

and correspondingly the volume form is

$$\omega_{[3]} = \frac{L^3}{8} \cosh r \, dr \wedge dx \wedge d\tau. \quad (\text{A.19})$$

In this case the x -coordinate lines coincide with the integral curves of the Killing vector $L_2 = \partial_x$, whereas the τ -lines are the curves of $R_3 = -\partial_\tau$.

B. The BTZ black hole

In the presence of isometries, discrete identifications provide alternatives for creating new backgrounds. Those have the *same* local geometry, but differ with respect to their global properties. Whether these identifications can be implemented as orbifolds at the level of the underlying two-dimensional string model is very much dependent on each specific case.

For AdS_3 , the most celebrated geometry obtained by discrete identification is certainly the BTZ black hole [7]. The discrete identifications are made along the integral lines of the following Killing vectors (see eqs. (A.3)):

$$\text{non-extremal case : } \quad \xi = (r_+ + r_-) R_2 - (r_+ - r_-) L_2, \quad (\text{B.1a})$$

$$\text{extremal case : } \quad \xi = 2r_+ R_2 - (R_1 - R_3) - (L_1 + L_3). \quad (\text{B.1b})$$

In the original BTZ coordinates, the metric reads:

$$ds^2 = L^2 \left[-f^2(r) \, dt^2 + f^{-2}(r) \, dr^2 + r^2 \left(d\varphi - \frac{r_+ r_-}{r^2} \, dt \right)^2 \right], \quad (\text{B.2})$$

with

$$f(r) = \frac{1}{r} \sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}. \quad (\text{B.3})$$

In this coordinate system,

$$\partial_\varphi \equiv \xi, \quad \partial_t \equiv -(r_+ + r_-) R_2 - (r_+ - r_-) L_2 \quad \text{and} \quad r^2 \equiv \|\xi\|. \quad (\text{B.4})$$

In AdS₃ φ is not a compact coordinate. The discrete identification makes φ an angular variable, $\varphi \cong \varphi + 2\pi$, which imposes to remove the region with $r^2 < 0$. The BTZ geometry describes a three-dimensional black hole, with mass M and angular momentum J , in a space-time that is locally (and asymptotically) anti-de Sitter. The chronological singularity at $r = 0$ is hidden behind an inner horizon at $r = r_-$, and an outer horizon at $r = r_+$. Between these two horizons, r is time-like. The coordinate t becomes space-like inside the ergosphere, when $r^2 < r_{\text{erg}}^2 \equiv r_+^2 + r_-^2$. The relation between M, J and r_\pm is as follows:

$$r_\pm^2 = \frac{ML}{2} \left[1 \pm \sqrt{1 - \left(\frac{J}{ML} \right)^2} \right]. \quad (\text{B.5})$$

Extremal black holes have $|J| = ML$ ($r_+ = r_-$). In the special case $J = ML = 0$ one finds the near-horizon geometry of the five-dimensional NS5/F1 stringy black hole in its ground state. Global AdS₃ is obtained for $J = 0$ and $ML = -1$.

Many subtleties arise, which concern e.g. the appearance of closed time-like curves in the excised region of negative r^2 (where ∂_φ would have been time-like) or the geodesic completion of the manifold; a comprehensive analysis of these issues can be found in [8]. At the string-theory level, the BTZ identification is realized as an *orbifold* projection, which amounts to keeping invariant states and adding twisted sectors [44, 45].

Besides the BTZ solution, other locally AdS₃ geometries are obtained, by imposing identification under purely left (or right) isometries, referred to as self-dual (or anti-self-dual) metrics. These were studied in [46]. Their classification and isometries are exactly those of the asymmetric deformations studied in the present chapter. The Killing vector used for the identification is (A) time-like (elliptic), (B) space-like (hyperbolic) or (C) null (parabolic), and the isometry group is $U(1) \times \text{SL}(2, \mathbb{R})$. It was pointed out in [46] that the resulting geometry was free of closed time-like curves only in the case (B).

C. Spectrum of the $\text{SL}(2, \mathbb{R})$ super-wzw model

In this appendix we give a reminder of the superconformal wzw model on $\text{SL}(2, \mathbb{R})_k$ (for a recent discussion see [47]). The affine extension of the $\mathfrak{sl}(2, \mathbb{R})$ algebra at level k is obtained by considering two sets of holomorphic and anti-holomorphic currents of dimension one, defined as

$$J^M(z) = k \langle T^M, \text{Ad}_g g^{-1} \partial g \rangle, \quad \bar{J}^M(\bar{z}) = k \langle T^M, g^{-1} \bar{\partial} g \rangle, \quad (\text{C.1})$$

where $\langle \cdot, \cdot \rangle$ is the scalar product (Killing form) in $\mathfrak{sl}(2, \mathbb{R})$, $\{T^M\}$ is a set of generators of the algebra that for concreteness we can choose as follows:

$$T^1 = \sigma^1, \quad T^2 = \sigma^3, \quad T^3 = \sigma^2. \quad (\text{C.2})$$

Each set satisfies the OPE

$$J^M(z)J^N(w) \sim \frac{k\delta^{MN}}{2(z-w)^2} + \frac{f_{\text{P}}^{MN}J^{\text{P}}(w)}{z-w}, \quad (\text{C.3})$$

where f_{P}^{MN} are the structure constants of the $\mathfrak{sl}(2, \mathbb{R})$ algebra. The chiral algebra contains the Virasoro operator (stress tensor) obtained by the usual Sugawara construction:

$$T(z) = \sum_{\text{M}} \frac{:J^{\text{M}}J^{\text{M}}:}{k-2}. \quad (\text{C.4})$$

A heterotic model is built if we consider a left-moving $\mathcal{N} = 1$ extension, obtained by adding 3 free fermions which transform in the adjoint representation. More explicitly:

$$T(z) = \sum_{\text{M}} \frac{:J^{\text{M}}J^{\text{M}}:}{k-2} + :\psi_{\text{M}}\partial\psi_{\text{M}}:, \quad (\text{C.5})$$

$$G(z) = \frac{2}{k} \left(\sum_{\text{M}} J^{\text{M}}\psi_{\text{M}} - \frac{i}{3k} \sum_{\text{MNP}} f^{\text{MNP}} : \psi_{\text{M}}\psi_{\text{N}}\psi_{\text{P}} : \right). \quad (\text{C.6})$$

On the right side, instead of superpartners, we add a right-moving current with total central charge $c = 16$.

Let us focus on the left-moving part. The supercurrents are given by $\psi_{\text{M}} + \theta\sqrt{2/k}\mathcal{J}_{\text{M}}$ where:

$$\mathcal{J}_{\text{M}} = J^{\text{M}} - \frac{i}{2} \sum_{\text{NP}} \epsilon^{\text{MNP}} \psi_{\text{N}}\psi_{\text{P}}; \quad (\text{C.7})$$

it should be noted that the bosonic J^{M} currents generate an affine $\mathfrak{sl}(2, \mathbb{R})$ algebra at level $k+2$, while the level for the total \mathcal{J}_{M} currents is k .

Let us now single out the operator that we used for both the deformation (eqs. (4.4a)) and the identifications (section 5.3):

$$\mathcal{J}_2 = J^2 + i\psi_1\psi_3. \quad (\text{C.8})$$

Let us now bosonize these currents as follows:

$$\mathcal{J}_2 = -\sqrt{\frac{k}{2}}\partial\vartheta_2, \quad (\text{C.9})$$

$$J^2 = -\sqrt{\frac{k+2}{2}}\partial\theta_2, \quad (\text{C.10})$$

$$\psi_1\psi_3 = \partial H, \quad (\text{C.11})$$

and introduce a fourth free boson X so to separate the ϑ_2 components both in θ_2 and H :

$$iH = \sqrt{\frac{2}{k}}\vartheta_2 + i\sqrt{\frac{k+2}{k}}X, \quad (\text{C.12})$$

$$\theta_2 = \sqrt{\frac{2}{k}} \left(\sqrt{\frac{k+2}{2}}\vartheta_2 + iX \right). \quad (\text{C.13})$$

A primary field $\Phi_{j\mu\bar{\mu}}$ of the bosonic $\text{SL}(2, \mathbb{R})_{k+2}$ with eigenvalue μ with respect to J^2 and $\bar{\mu}$ with respect to \bar{J}^2 obeys by definition

$$\mathcal{J}^2(z) \Phi_{j\mu\bar{\mu}}(w, \bar{w}) \sim \frac{\mu \Phi_{j\mu\bar{\mu}}(w, \bar{w})}{z - w}, \quad (\text{C.14a})$$

$$\bar{\mathcal{J}}^2(\bar{z}) \Phi_{j\mu\bar{\mu}}(w, \bar{w}) \sim \frac{\bar{\mu} \Phi_{j\mu\bar{\mu}}(w, \bar{w})}{\bar{z} - \bar{w}}. \quad (\text{C.14b})$$

Since $\Phi_{j\mu\bar{\mu}}$ is purely bosonic, the same relation holds for the supercurrent:

$$\mathcal{J}_2(z) \Phi_{j\mu\bar{\mu}}(w, \bar{w}) \sim \frac{\mu \Phi_{j\mu\bar{\mu}}(w, \bar{w})}{z - w}. \quad (\text{C.15})$$

Consider now the holomorphic part of $\Phi_{j\mu\bar{\mu}}(z, \bar{z})$. If $\Phi_{j\mu}$ is viewed as a primary in the SWZW model, we can use the parafermion decomposition as follows:

$$\Phi_{j\mu}(z) = U_{j\mu}(z) e^{i\mu\sqrt{2/k}\vartheta_2}, \quad (\text{C.16})$$

where $U_{j\mu}(z)$ is a primary of the superconformal $\text{SL}(2, \mathbb{R})_k / \text{U}(1)$. On the other hand, we can just consider the bosonic WZW and write:

$$\Phi_{j\mu}(z) = V_{j\mu}(z) e^{i\mu\sqrt{2/(k+2)}\vartheta_2} = V_{j\mu}(z) e^{i\frac{2m}{k+2}\sqrt{\frac{k+2}{k}}X + i\mu\sqrt{2/k}\vartheta_2}, \quad (\text{C.17})$$

where now $V_{j\mu}(z)$ is a primary of the bosonic $\text{SL}(2, \mathbb{R})_{k+2} / \text{U}(1)$. The scaling dimension for this latter operator (i.e. its eigenvalue with respect to L_0) is then given by:

$$\Delta(V_{j\mu}) = -\frac{j(j+1)}{k} - \frac{\mu^2}{k+2}. \quad (\text{C.18})$$

An operator in the full supersymmetric $\text{SL}(2, \mathbb{R})_k$ theory is then obtained by adding the $\psi^1\psi^3$ fermionic superpartner contribution:

$$\Phi_{j\mu\nu}(z) = \Phi_{j\mu}(z) e^{\nu H} = V_{j\mu}(z) e^{i\left(\frac{2\mu}{k+2} + \nu\right)\sqrt{\frac{k+2}{k}}X} e^{i\sqrt{2/k}(\mu+\nu)\vartheta_2} \quad (\text{C.19})$$

that is an eigenvector of \mathcal{J}_2 with eigenvalue $\mu + \nu$ where $\mu \in \mathbb{R}$ and ν can be decomposed as $\nu = n + a/2$ with $n \in \mathbb{N}$ and $a \in \mathbb{Z}_2$ depending on whether we consider the NS or R sector. The resulting spectrum can be read directly as:

$$\begin{aligned} \Delta(\Phi_{j\mu\nu}(z)) &= -\frac{j(j+1)}{k} - \frac{\mu^2}{k+2} - \frac{k+2}{2k} \left(\frac{2\mu}{k+2} + n + \frac{a}{2} \right)^2 + \frac{1}{k} \left(\mu + n + \frac{a}{2} \right)^2 \\ &= -\frac{j(j+1)}{k} - \frac{1}{2} \left(n + \frac{a}{2} \right)^2. \end{aligned} \quad (\text{C.20})$$

Of course the last expression was to be expected since it is the sum of the $\mathfrak{sl}(2, \mathbb{R})_{k+2}$ Casimir and the contribution of a light-cone fermion. Nevertheless the preceding construction is useful since it allowed us to isolate the \mathcal{J}_2 contribution to the spectrum $(\mu + \nu)^2/k$.

The right-moving part of the spectrum is somewhat simpler since there are no superpartners. This means that we can repeat our construction above and the eigenvalue of the \bar{L}_0 operator is simply obtained by adding to the dimension in eq. (C.18) the contribution of the \bar{J}^2 operator and of some $\text{U}(1)$ coming from the gauge sector:

$$\bar{\Delta}(\bar{\Phi}_{j\mu\bar{n}}(\bar{z})) = -\frac{j(j+1)}{k} - \frac{\bar{\mu}^2}{k+2} + \left\{ \frac{\bar{\mu}^2}{k+2} + \frac{1}{k_g} \left(\bar{n} + \frac{\bar{a}}{2} \right)^2 \right\}, \quad (\text{C.21})$$

where again $\bar{n} \in \mathbb{N}$ and $\bar{a} \in \mathbb{Z}_2$ depending on the sector.

References

- [1] I. Antoniadis, C. Bachas and A. Sagnotti, *Gauged supergravity vacua in string theory*, *Phys. Lett.* **B 235** (1990) 255.
- [2] P.M.S. Petropoulos, *Comments on SU(1,1) string theory*, *Phys. Lett.* **B 236** (1990) 151.
- [3] H.J. Boonstra, B. Peeters and K. Skenderis, *Brane intersections, anti-de Sitter spacetimes and dual superconformal theories*, *Nucl. Phys.* **B 533** (1998) 127 [[hep-th/9803231](#)].
- [4] J.M. Maldacena and A. Strominger, *AdS₃ black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [[hep-th/9804085](#)].
- [5] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS₃*, *Adv. Theor. Math. Phys.* **2** (1998) 733 [[hep-th/9806194](#)].
- [6] D. Israël, C. Kounnas and M.P. Petropoulos, *Superstrings on NS5 backgrounds, deformed AdS₃ and holography*, *JHEP* **10** (2003) 028 [[hep-th/0306053](#)].
- [7] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)].
- [8] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys. Rev.* **D 48** (1993) 1506 [[gr-qc/9302012](#)].
- [9] P. Bieliavsky, M. Rooman and P. Spindel, *Regular poisson structures on massive non-rotating BTZ black holes*, *Nucl. Phys.* **B 645** (2002) 349 [[hep-th/0206189](#)].
- [10] P. Bieliavsky, S. Detournay, M. Herquet, M. Rooman and P. Spindel, *Global geometry of the 2+1 rotating black hole*, *Phys. Lett.* **B 570** (2003) 231 [[hep-th/0306293](#)].
- [11] S. Chaudhuri and J.A. Schwartz, *A criterion for integrably marginal operators*, *Phys. Lett.* **B 219** (1989) 291.
- [12] S.F. Hassan and A. Sen, *Marginal deformations of WZNW and coset models from O(d,d) transformation*, *Nucl. Phys.* **B 405** (1993) 143 [[hep-th/9210121](#)].
- [13] A. Giveon and E. Kiritsis, *Axial vector duality as a gauge symmetry and topology change in string theory*, *Nucl. Phys.* **B 411** (1994) 487 [[hep-th/9303016](#)].
- [14] S. Förste and D. Roggenkamp, *Current current deformations of conformal field theories and WZW models*, *JHEP* **05** (2003) 071 [[hep-th/0304234](#)].
- [15] D. Israël, *Quantization of heterotic strings in a Gödel/anti de Sitter spacetime and chronology protection*, *JHEP* **01** (2004) 042 [[hep-th/0310158](#)].
- [16] D. Israël, C. Kounnas, D. Orlando and P.M. Petropoulos, *Electric/magnetic deformations of S³ and AdS₃ and geometric cosets*, *Fortschr. Phys.* **53** (2005) 73 [[hep-th/0405213](#)].
- [17] D. Israël, C. Kounnas, D. Orlando and P.M. Petropoulos, *Heterotic strings on homogeneous spaces*, [hep-th/0412220](#).
- [18] J.H. Horne and G.T. Horowitz, *Exact black string solutions in three-dimensions*, *Nucl. Phys.* **B 368** (1992) 444 [[hep-th/9108001](#)].
- [19] V.G. Knizhnik and A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, *Nucl. Phys.* **B 247** (1984) 83.
- [20] H. Leutwyler and M.A. Shifman, *Perturbation theory in the Wess-Zumino-Novikov-Witten model*, *Int. J. Mod. Phys.* **A 7** (1992) 795.

JHEP07(2005)072

- [21] A.A. Tseytlin, *Effective action of gauged WZW model and exact string solutions*, *Nucl. Phys.* **B 399** (1993) 601 [[hep-th/9301015](#)].
- [22] D. Gepner and Z.-A. Qiu, *Modular invariant partition functions for parafermionic field theories*, *Nucl. Phys.* **B 285** (1987) 423.
- [23] D. Gepner, *New conformal field theories associated with Lie algebras and their partition functions*, *Nucl. Phys.* **B 290** (1987) 10.
- [24] A.A. Tseytlin, *Conformal sigma models corresponding to gauged Wess-Zumino-Witten theories*, *Nucl. Phys.* **B 411** (1994) 509 [[hep-th/9302083](#)].
- [25] S. Förste, *A truly marginal deformation of $SL(2, \mathbb{R})$ in a null direction*, *Phys. Lett.* **B 338** (1994) 36 [[hep-th/9407198](#)].
- [26] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, *Five-brane configurations, conformal field theories and the strong-coupling problem*, [hep-th/0312300](#).
- [27] G.T. Horowitz and A.A. Tseytlin, *A new class of exact solutions in string theory*, *Phys. Rev.* **D 51** (1995) 2896 [[hep-th/9409021](#)].
- [28] E. Kiritsis and C. Kounnas, *Infrared regularization of superstring theory and the one loop calculation of coupling constants*, *Nucl. Phys.* **B 442** (1995) 472 [[hep-th/9501020](#)].
- [29] M. Rooman and P. Spindel, *Goedel metric as a squashed anti-de Sitter geometry*, *Class. and Quant. Grav.* **15** (1998) 3241 [[gr-qc/9804027](#)].
- [30] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D 44** (1991) 314.
- [31] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *String propagation in a black hole geometry*, *Nucl. Phys.* **B 371** (1992) 269.
- [32] D. Gershon, *Exact solutions of four-dimensional black holes in string theory*, *Phys. Rev.* **D 51** (1995) 4387 [[hep-th/9202005](#)].
- [33] P. Hořava, *Some exact solutions of string theory in four-dimensions and five-dimensions*, *Phys. Lett.* **B 278** (1992) 101 [[hep-th/9110067](#)].
- [34] C. Klimčík and A.A. Tseytlin, *Exact four-dimensional string solutions and Toda like sigma models from ‘null gauged’ WZNW theories*, *Nucl. Phys.* **B 424** (1994) 71 [[hep-th/9402120](#)].
- [35] J.H. Horne, G.T. Horowitz and A.R. Steif, *An equivalence between momentum and charge in string theory*, *Phys. Rev. Lett.* **68** (1992) 568 [[hep-th/9110065](#)].
- [36] G.T. Horowitz and D.L. Welch, *Exact three-dimensional black holes in string theory*, *Phys. Rev. Lett.* **71** (1993) 328 [[hep-th/9302126](#)].
- [37] W. Israel, *Event horizons in static vacuum space-times*, *Phys. Rev.* **164** (1967) 1776.
- [38] M. Heusler, *Stationary black holes: uniqueness and beyond*, *Living Rev. Rel.* **1** (1998) 6.
- [39] G.W. Gibbons, D. Ida and T. Shiromizu, *Uniqueness and non-uniqueness of static black holes in higher dimensions*, *Phys. Rev. Lett.* **89** (2002) 041101 [[hep-th/0206049](#)].
- [40] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837 [[hep-th/9301052](#)].
- [41] K. Sfetsos, *Conformally exact results for $SL(2, \mathbb{R}) \otimes SO(1, 1)^{d-2}/SO(1, 1)$ coset models*, *Nucl. Phys.* **B 389** (1993) 424 [[hep-th/9206048](#)].

- [42] E. Kiritsis and C. Kounnas, *Infrared behavior of closed superstrings in strong magnetic and gravitational fields*, *Nucl. Phys.* **B 456** (1995) 699 [[hep-th/9508078](#)].
- [43] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, *Nucl. Phys.* **B 633** (2002) 3 [[hep-th/0111246](#)].
- [44] M. Natsuume and Y. Satoh, *String theory on three dimensional black holes*, *Int. J. Mod. Phys.* **A 13** (1998) 1229 [[hep-th/9611041](#)].
- [45] S. Hemming and E. Keski-Vakkuri, *The spectrum of strings on BTZ black holes and spectral flow in the $SL(2, \mathbb{R})$ WZW model*, *Nucl. Phys.* **B 626** (2002) 363 [[hep-th/0110252](#)].
- [46] O. Coussaert and M. Henneaux, *Self-dual solutions of 2+1 Einstein gravity with a negative cosmological constant*, [hep-th/9407181](#).
- [47] A. Giveon, A. Konechny, A. Pakman and A. Sever, *Type-0 strings in a 2-d black hole*, *JHEP* **10** (2003) 025 [[hep-th/0309056](#)].

JHEP07(2005)072

Coset models and D-branes in group manifolds

Published in Phys.Lett.B [hep-th/0511210]

*15

Coset models and D-branes in group manifolds

Domenico Orlando*

Centre de Physique Théorique de l'École Polytechnique[†]
91128 Palaiseau, France andLaboratoire de Physique Théorique de l'École Normale Supérieure[‡]
24 rue Lhomond, 75231 Paris Cedex 05, France

(Dated: July 16, 2006)

We conjecture the existence of a duality between heterotic closed strings on homogeneous spaces and symmetry-preserving D -branes on group manifolds, based on the observation about the coincidence of the low-energy field description for the two theories. For the closed string side we also give an explicit proof of a no-renormalization theorem as a consequence of a hidden symmetry and infer that the same property should hold true for the higher order terms of the DBI action.

One of the main technical advantages provided by the study of models on group manifolds is that the geometrical analysis can be recast in Lie algebraic terms. At the same time the underlying conformal symmetry makes it possible to explicitly study the integrability properties that, in general, allow for extremely nice behaviours under renormalization. Wess-Zumino-Witten models can be used as starting points for many interesting models: the main challenge in this case consists in partially removing the symmetry while retaining as many algebraic and integrability properties as possible.

In this note we aim at pointing out an analogy (or, as we will say, a duality) between two – in principle disconnected – constructions based on WZW models: closed string (heterotic) backgrounds obtained via asymmetric deformations and symmetry-preserving D -branes on group manifolds. As we will show, in fact, the low-energy field contents for both theories are the same, although they minimize different effective actions (SUGRA for the former and DBI for the latter). For one of the sides of the duality (the closed string one) we will also show a no-renormalization theorem stating that the effect of higher-order terms can be resummed to a shift in the radii of the manifold. A similar behaviour can also be conjectured from the D -brane side, and this would be consistent with some remark in literature about the coincidence between the DBI and CFT results concerning mass spectra, ... up to the said shift [1, 2].

Let us start with the open-string side of this duality, by reminding some known facts about the geometric description of D -branes in WZW models on compact groups, pointing out in particular the low-energy field configuration. Natural boundary conditions on WZW models are those in which the gluing between left- and right-moving

currents can be expressed in terms of automorphisms ω of the current algebra. The corresponding world-volumes are then given by (twisted) conjugacy classes on the group [3]:

$$\mathcal{C}^\omega(g) = \{ hg\omega(h^{-1}) \mid h \in G \}. \quad (1)$$

As it was pointed out in [1], one can use Weyl's theory of conjugacy classes so to give a geometric description of $\mathcal{C}^\omega(g)$. For a given automorphism ω we can always find an ω -invariant maximal torus $T \subset G$ (such as $\omega(T) = T$). Let $T^\omega \subset T$ be the set of elements $t \in T$ invariant under ω ($T^\omega = \{ t \in T \mid \omega(t) = t \}$) and $T_0^\omega \subset T^\omega$ the connected component to the unity. When ω is inner $T = T^\omega = T_0^\omega$ while in general (*i.e.* if we allow ω to be outer) $\dim(T_0^\omega) \leq \text{rank } G$.

Let ω be inner. Define a map:

$$q : G/T \times T \rightarrow G \\ ([g], t) \mapsto q([g], t) = gtg^{-1}. \quad (2)$$

One can show that this map is surjective, so that each element in G is conjugated to some element in T . This implies in particular that the conjugacy classes are characterized by elements in T , or, in other words, fixing $t \in T$ (so to take care of the action of the Weyl group), we find that the (regular) conjugacy classes $\mathcal{C}^\omega(g)$ are isomorphic to the homogeneous space G/T . A similar result holds for twisted classes, but in this case

$$\mathcal{C}^\omega(g) \simeq G/T_0^\omega. \quad (3)$$

The description of the D -brane is completed by the $U(1)$ gauge field that lives on it. The possible $U(1)$ fluxes are elements in $H^2(G/T_0^\omega, \mathbb{R})$ and one can show that

$$H^2(G/T_0^\omega) \simeq \mathbb{Z}^{\dim T_0^\omega}. \quad (4)$$

Summarizing we find that the gauge content of the low energy theory is given by:

[†]Unité mixte du CNRS et de l'École Polytechnique, UMR 7644.

[‡]Unité mixte du CNRS et de l'École Normale Supérieure, UMR 8549.

- the metric on G/T_0^ω (in particular G/T for un-twisted branes),
- the pull-back of the Kalb–Ramond field on G/T_0^ω ,
- $\dim T_0^\omega$ independent $U(1)$ fluxes (rank G for un-twisted branes).

These fields extremize the DBI action

$$S = \int dx \sqrt{\det(\mathbf{g} + \mathbf{B} + 2\pi\mathbf{F})}. \quad (5)$$

and according to some coincidence with known exact CFT results there are reasons to believe that the fields only receive a normalization shift when computed at all loops.

Let us now move to the other – closed string – side of the advertised duality. A good candidate for a deformation of a WZW model that reduces the symmetry, at the same time preserving the integrability and renormalization properties, is obtained via the introduction of a truly marginal operator written as the product of a holomorphic and an antiholomorphic current

$$\mathcal{O} = \sum_{ij} c_{ij} J^i \bar{J}^j. \quad (6)$$

As it was shown in [4], a necessary and sufficient condition for this marginal operator to be integrable is that the left and right currents both belong to abelian groups. If we consider the heterotic super-WZW model, a possible choice consists in taking the left currents in the Cartan torus and the right currents from the heterotic gauge sector [5, 6]:

$$\mathcal{O} = \sum_{a=1}^N H_a J^a \bar{I}^a. \quad (7)$$

where $J^a \in H \subset T$, T being the maximal torus in G .

Using a construction bearing many resemblances to a Kaluza–Klein reduction it is straightforward to show that the background fields corresponding to this kind of deformation consist in a metric, a Kalb–Ramond field and a $U(1)^N$ gauge field. Their explicit expressions are simply given in terms of Maurer–Cartan one-forms on G as follows:

$$g = \frac{k}{2} \delta_{MN} \mathcal{J}^M \otimes \mathcal{J}^N - k \delta_{ab} H_a^2 \tilde{\mathcal{J}}^a \otimes \tilde{\mathcal{J}}^b, \quad (8a)$$

$$H_{[3]} = dB - \frac{1}{k_g} A^a \wedge dA^a = \frac{k}{2} f_{MNP} \mathcal{J}^M \wedge \mathcal{J}^N \wedge \mathcal{J}^P - k H_a^2 f_{aMN} \tilde{\mathcal{J}}^a \wedge \mathcal{J}^M \wedge \mathcal{J}^N, \quad (8b)$$

$$A^a = H_a \sqrt{\frac{2k}{k_g}} \tilde{\mathcal{J}}^a \text{ (no summation over } a \text{ implied)}, \quad (8c)$$

where $\tilde{\mathcal{J}}_\mu^a$ are the currents that have been selected for the deformation operator. In this way we get an N -dimensional space of exact models. Here we will concentrate on a special point in this space, namely the one that corresponds to $\{H_a = 1/\sqrt{2}, \forall a = 1, 2, \dots, N\}$. This point is remarkable for it corresponds to a decompactification limit where N dimensions decouple and we're left with the homogeneous G/H space times N non-compact dimensions[. More precisely, when H coincides with the maximal torus T , the background fields read:

$$G = \frac{k}{2} \sum_{\mu} \mathcal{J}^\mu \otimes \mathcal{J}^\mu, \quad (9a)$$

$$H_{[3]} = dB = \frac{1}{2} f_{\mu\nu\rho} \mathcal{J}^\mu \wedge \mathcal{J}^\nu \wedge \mathcal{J}^\rho, \quad (9b)$$

$$F^a = -\sqrt{\frac{k}{2k_g}} H_a f_{\mu\nu}^a \mathcal{J}^\mu \wedge \mathcal{J}^\nu \quad (9c)$$

(no summation over a). Geometrically:

- g is the metric on G/T obtained as the restriction of the Cartan–Killing metric on G
- $H_{[3]}$ is the pullback of the usual Kalb–Ramond field present in the WZW model on the group G
- F^a are rank(G) independent $U(1)$ gauge fluxes that satisfy some quantization conditions and hence naturally live in $H^2(G/T, \mathbb{Z})$

Having chosen a truly marginal operator for the deformation we know that this model is conformal. This implies in particular that the background fields solve the usual β equations that stem from the variation of the effective SUGRA action:

$$S = \int dx \sqrt{g} \left(R - \frac{1}{12} H_{\mu\rho\sigma} H^{\mu\rho\sigma} - \frac{k_g}{8} F_{\mu\nu}^a F^{\alpha\mu\nu} + \frac{\delta c}{3} \right) \quad (10)$$

In example if we consider $G = SU(2)$, then $T = U(1)$ and the decompactification limit $H \rightarrow 1/\sqrt{2}$ we get the exact $S^2 = SU(2)/U(1)$ background supported by a $U(1)$ magnetic monopole field (see *e.g.* [7, 8]).

Our conjecture stems precisely from this: the gauge field above exactly match the ones we found before for symmetry-preserving D-branes. Moreover both sides of the duality are derived from WZW models that enjoy a no-renormalization property which would make this correspondence true at all orders. In this spirit we now pass to prove that a similar theorem holds for closed heterotic strings on coset models inferring that the duality, when proven, would give a direct way to deduce the same feature for the D-brane action.

In studying symmetrically deformed WZW models, *i.e.* those where the deformation operator is written as the

product of two currents belonging to the same sector $\mathcal{O} = \lambda J \bar{J}$, one finds that the Lagrangian formulation only corresponds to a small-deformation approximation. For this reason different techniques have been developed so to read the background fields at every order in λ [9–13] but, still, the results are in general only valid at first order in α' and have to be modified so to take into account the effect of instanton corrections. In this section we want to show that this is not the case for asymmetrically deformed models, for which the background fields in Eqs. (8) are exact at all orders in H_a and for which the effect of renormalization only amounts to the usual (for WZW models) shift in the level of the algebra $k \rightarrow k + c_G$ where c_G is the dual Coxeter number.

Consider in example the most simple $SU(2)$ case. In terms of Euler angles the deformed Lagrangian is written as:

$$S = S_{SU(2)}(\alpha, \beta, \gamma) + \delta S = \frac{k}{4\pi} \int d^2z \partial\alpha \bar{\partial}\alpha + \partial\beta \bar{\partial}\beta + \partial\gamma \bar{\partial}\gamma + 2 \cos\beta \partial\alpha \bar{\partial}\gamma + \frac{\sqrt{k k_g H}}{2\pi} \int d^2z (\partial\gamma + \cos\beta \partial\alpha) \bar{I}. \quad (11)$$

If we bosonize the right-moving current as $\bar{I} = \bar{\partial}\phi$ and add a standard $U(1)$ term to the action, we get:

$$S = S_{SU(2)}(\alpha, \beta, \gamma) + \delta S(\alpha, \beta, \gamma, \phi) + \frac{k_g}{4\pi} \int d^2z \partial\phi \bar{\partial}\phi = S_{SU(2)}\left(\alpha, \beta, \gamma + 2\sqrt{\frac{k_g}{k}} H\phi\right) + \frac{k_g(1 - 2H^2)}{4\pi} \int d^2z \partial\phi \bar{\partial}\phi \quad (12)$$

and in particular at the decoupling limit $H \rightarrow 1/\sqrt{2}$, corresponding to the S^2 geometry, the action is just given by $S = S_{SU(2)}\left(\alpha, \beta, \gamma + 2\sqrt{\frac{k_g}{k}} H\phi\right)$. This implies that our (deformed) model inherits all the integrability and renormalization properties of the standard $SU(2)$ WZW model. In other words the three-dimensional model with metric and Kalb–Ramond field with $SU(2) \times U(1)$ symmetry and a $U(1)$ gauge field is uplifted to an exact model on the $SU(2)$ group manifold (at least locally): the integrability properties are then a consequence of this hidden $SU(2) \times SU(2)$ symmetry that is manifest in higher dimensions.

The generalization of this particular construction to higher groups is easily obtained if one remarks that the Euler parametrization for the $g \in SU(2)$ group representative is written as:

$$g = e^{i\gamma t_3} e^{i\beta t_1} e^{i\alpha t_2}, \quad (13)$$

where $t_i = \sigma_i/2$ are the generators of $\mathfrak{su}(2)$ (σ_i being

the usual Pauli matrices). As stated above, the limit deformation corresponds to the gauging of the left action of an abelian subgroup $T \subset SU(2)$. In particular here we chose $T = \{h \mid h = e^{i\phi t_3}\}$, hence it is natural to find (up to the normalization) that:

$$h(\phi) g(\alpha, \beta, \gamma) = g(\alpha, \beta, \gamma + \phi). \quad (14)$$

The only thing that one needs to do in order to generalize this result to a general group G consists in finding a parametrization of $g \in G$ such as the chosen abelian subgroup appears as a left factor. In example if in $SU(3)$ we want to gauge the $U(1)^2$ abelian subgroup generated by $\langle \lambda_3, \lambda_8 \rangle$ (Gell-Mann matrices), we can choose the following parametrization for $g \in SU(3)$ [14]:

$$g = e^{i\lambda_8\phi} e^{i\lambda_3c} e^{i\lambda_2b} e^{i\lambda_3a} e^{i\lambda_5\theta} e^{i\lambda_3\gamma} e^{i\lambda_2\beta} e^{i\lambda_3\alpha}. \quad (15)$$

The deep reason that lies behind this property (differentiating symmetric and asymmetric deformations) is the fact that not only the currents used for the deformation

are preserved (as it happens in both cases), but here their very expression is just modified by a constant factor. In fact, if we write the deformed metric as in Eq. (8a) and call \tilde{K}^μ the Killing vector corresponding to the chosen isometry (that doesn't change along the deformation), we see that the corresponding $\tilde{\mathcal{J}}_\mu^{(H)}$ current is given by:

$$\tilde{\mathcal{J}}_\nu^{(H)} = \tilde{K}^\mu g_{\mu\nu}^{(H)} = (1 - 2H^2) \tilde{\mathcal{J}}_\nu^{(0)} \quad (16)$$

The most important consequence (from our point of view) of this integrability property is that the SUGRA action in Eq. (10) is *exact* and the only effect of renormalization is the $k \rightarrow k + c_G$ shift.

It is very tempting to extend this no-renormalization theorem to the D -brane side. Of course this would require an actual proof of the duality we conjecture. Nevertheless we think that this kind of approach might prove (at least for these highly symmetric systems) more fruitful than adding higher loop corrections to the DBI action, which on the other hand remains an interesting directions of study by itself.

Is this duality just a coincidence, due to the underlying Lie algebraic structures that both sides share, or is it a sign of the presence of some deeper connection? Different aspects of the profound meaning of the DBI effective action are still poorly understood and it is possible that this approach – pointing to one more link to conformal field theory – might help shedding some new light.

I would like to thank C. Bachas, C. Kounnas and S. Ribault for illuminating discussions and especially thank M. Petropoulos for encouragement during different stages of this work and for the careful reading of the manuscript.

* domenico.orlando@cpht.polytechnique.fr; Research partially supported by the EEC under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369.

- [1] C. Bachas, M. R. Douglas, and C. Schweigert, JHEP **05**, 048 (2000), hep-th/0003037.
 - [2] P. Bordalo, S. Ribault, and C. Schweigert, JHEP **10**, 036 (2001), hep-th/0108201.
 - [3] A. Y. Alekseev and V. Schomerus, Phys. Rev. **D60**, 061901 (1999), hep-th/9812193.
 - [4] S. Chaudhuri and J. A. Schwartz, Phys. Lett. **B219**, 291 (1989).
 - [5] E. Kiritsis and C. Kounnas, Nucl. Phys. **B456**, 699 (1995), hep-th/9508078.
 - [6] D. Israel, C. Kounnas, D. Orlando, and P. M. Petropoulos (2004), hep-th/0412220.
 - [7] C. V. Johnson, Mod. Phys. Lett. **A10**, 549 (1995), hep-th/9409062.
 - [8] P. Berglund, C. V. Johnson, S. Kachru, and P. Zaugg, Nucl. Phys. **B460**, 252 (1996), hep-th/9509170.
 - [9] S. F. Hassan and A. Sen, Nucl. Phys. **B405**, 143 (1993), hep-th/9210121.
 - [10] A. Giveon and E. Kiritsis, Nucl. Phys. **B411**, 487 (1994), hep-th/9303016.
 - [11] S. Förste, Phys. Lett. **B338**, 36 (1994), hep-th/9407198.
 - [12] S. Förste and D. Roggenkamp, JHEP **05**, 071 (2003), hep-th/0304234.
 - [13] S. Detournay, D. Orlando, P. M. Petropoulos, and P. Spindel (2005), hep-th/0504231.
 - [14] M. Byrd (1997), physics/9708015.
- One can see the initial group manifold G as a principal fibration of H over a G/H basis: the deformation changes the radii of the fiber and eventually trivializes in correspondence of this special point.

Renormalization group flows and charge transmutation in string theory

Published in Fortsch. Phys. 54 (2006) 453-461 [*hep-th/0512086*].

Fortschr. Phys. 54, No. 5–6, 453–461 (2006) / DOI 10.1002/prop.200510290

Renormalization-group flows and charge transmutation in string theory*

Domenico Orlando^{1,2,**}, P. Marios Petropoulos¹, and Konstadinos Sfetsos^{3,1}

¹ Centre de Physique Théorique, École Polytechnique, Unité mixte du CNRS et de l'École Polytechnique, UMR 7644, 91128 Palaiseau, France

² Laboratoire de Physique Théorique de l'École Normale Supérieure, Unité mixte du CNRS et de l'École Normale Supérieure, UMR 8549, 24 rue Lhomond, 75231 Paris Cedex 05, France

³ Department of Engineering Sciences, University of Patras, 26110 Patras, Greece

Received 15 December 2005, accepted 15 December 2005

Published online 18 April 2006

Key words Superstring vacua, conformal field models, string theory

PACS 11.25.Hf, 11.40.-q, 11.15.Kc

We analyze the behaviour of heterotic squashed-Wess–Zumino–Witten backgrounds under renormalization-group flow. The flows we consider are driven by perturbation creating extra gauge fluxes. We show how the conformal point acts as an attractor from both the target-space and world-sheet points of view. We also address the question of instabilities created by the presence of closed time-like curves in string backgrounds.

© 2006 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

1 Introduction

The purpose of this note is to analyze the behavior of certain string backgrounds under world-sheet renormalization-group flows. In our set-up, these flows are driven by world-sheet operators, which create, in the target space–time, extra $U(1)$, electric or magnetic, gauge fields and push the string off criticality.

World-sheet renormalization group has been investigated both from the general conformal field theory (CFT) and from the geometrical, target space viewpoints [1,2]. The motivations are diverse: study the stability of the background against off-critical excursions, search for new critical string backgrounds, eventually explore string theory off-shell, etc.

In the presence of “impurities” such as branes or orbifold fixed points in non-compact target spaces, or when background electric or magnetic fields are switched on, tachyons may in general appear [3,4]. World-sheet renormalization-group techniques are then useful for investigating the relaxation process of the original unstable vacuum, towards a new, stable infrared fixed point. Such relaxation is usually a tachyon condensation [5–7], that can be accompanied by emission of particles (in the form *e.g.* of charged pairs) [8,9].

The presence of closed time-like curves can also trigger decays. It was argued years ago [10] that gravitational solutions with such chronological pathologies might naturally evolve towards chronologically safe backgrounds. This has been recast more recently in the framework of string vacua [11,12] with some preliminary results. It is clear that one would gain insight by studying renormalization-group flows in an appropriately chosen parameter space for families of string backgrounds.

World-sheet renormalization group can be studied directly at the level of the two-dimensional CFT. Any relevant operator can be used to leave the conformal point, and the necessary tools are in principle

* Partially supported by the EU under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369

** Corresponding author E-mail: domenico.orlando@cpht.polytechnique.fr, Phone: +00 331 69 33 47 21

available for computing the beta-functions and determining the flows. This procedure is usually perturbative. It turns out that it is trustful [1] in determining the new conformal point in the IR only when the operators responsible for the flow are marginally relevant (conformal dimensions $\Delta = \bar{\Delta} = 1$, but only at first order in the deformation parameter), or almost relevant (conformal dimensions $\Delta = \bar{\Delta} = 1 - \varepsilon$). In the case of deformations with irrelevant operators we return back to the conformal point towards the IR. In the framework of Wess–Zumino–Witten (WZW) models [13] (which capture *e.g.* the S^3 and AdS_3 spaces with NS background fluxes), such operators exist only at large level k . Hence, their operator product expansions involve a plethora of fields, and the actual computation of their beta function is very intricate. In order to overcome this difficulty, we will here use an alternative method, more geometric and based on target-space techniques.

This note is organized as follows. In Sect. 2 we present a quick review of the heterotic squashed WZW models [14, 15]. Then in Sect. 3 we introduce a perturbation and study the system as the RG flow, in the corresponding two-dimensional σ -model, takes it back to the conformal point. In Sect. 4 we show that this is consistent with the CFT results.

2 Squashed wzw models

One of the most appealing properties of WZW models is that they allow for both an exact CFT bidimensional description and a simple spacetime interpretation in terms of group manifolds. Current-current deformations allow to explore their moduli space, leading in general to models that keep the integrability properties but may lack a nice spacetime description. Special attention is deserved by the asymmetric deformations in which the two currents come from different sectors of the theory; in this case, in fact, together with the nice CFT properties, the spacetime geometry remains simple to describe in terms of squashed groups.

To be more concrete consider a heterotic SWZW model on a group G of dimension d and rank r . The asymmetric current-current deformation is realized by adding the operator

$$\mathcal{O} = \frac{\sqrt{k k_g}}{2\pi} \int d^2z \sum_{a,b=1}^r c_{ab} \left(J^a(z) - \frac{i}{k} f_{MN}^a \psi^M \psi^N \right) \bar{J}^b(\bar{z}), \quad (1)$$

where J^a are currents in the Cartan torus $T \subset G$, ψ^M are the fermionic superpartners and \bar{J}^a are anti-holomorphic currents belonging to the gauge sector. The engineering dimension of the operator is obviously $(1, 1)$ and, as it has been shown in [16], \mathcal{O} is truly marginal (*i.e.* at every order in deformation) for any value of the parameter matrix c_{ab} , since the currents commute. In other words, with the aid of \mathcal{O} , we reach an r -dimensional space of exact CFT's.

As described in [15], the background fields corresponding to the new sigma-model can be read using a technique bearing many resemblances to a Kaluza–Klein reduction¹ and consist in a metric, a Kalb–Ramond field and a $U(1)^r$ (chromo-)magnetic field. As announced above the description remains simple and the all-order exact expression can be given in terms of Maurer–Cartan currents J^M on G as follows:

$$\begin{cases} ds^2 = \sum_{\mu \in G/T} J^\mu J^\mu + (1 - h^2) \sum_{a \in T} J^a J^a, \\ H_{[3]} = \frac{1}{2} f_{\mu\nu\rho} J^\mu \wedge J^\nu \wedge J^\rho, & \mu \in G/T, \\ F^a = h \sqrt{\frac{k}{k_g}} f_{\mu\nu}^a J^\mu \wedge J^\nu, & \mu \in G/T, a \in T, \end{cases} \quad (2)$$

where we chose $c_{ab} = h \delta_{ab}$. In particular we see that the metric is the one of a squashed group *i.e.* we still have the structure of a T fibration over G/T but the radius of the fiber changes with h . A special value

¹ It would be a genuine reduction if we had done the construction in type II or in a bosonic theory. In this case the current \bar{J}^a would just be the anti-holomorphic derivative of an internal coordinate X^a .

of the deformation parameter is singled out: for $h < 1$ the metric is positive definite, while for $h > 1$ the signature changes. The apparently singular $h = 1$ value can nevertheless be reached by a limiting procedure whose geometrical interpretation is the trivialization of the fiber. We end up with an exact CFT on a G/T background sustained by a (chromo-)magnetic field.

The simplest example is given by $G = SU(2)$ where we have (in Euler coordinates) the following background fields:

$$\begin{cases} ds^2 = d\theta^2 + d\psi^2 + d\phi^2 + \cos\theta d\psi d\phi - h^2 (d\psi + \cos\theta d\phi)^2, \\ B = \cos\theta d\psi \wedge d\phi, \\ A = 2h (d\psi + \cos\theta d\phi), \end{cases} \quad (3)$$

corresponding, in the $h \rightarrow 1$ limit, to a S^2 geometry.

3 RG-flows for compact groups: geometric approach

We present here the geometric, target-space techniques for analyzing RG flows in two-dimensional theories. These techniques apply to any compact group. We will however expand on the case of $SU(2)$ since it captures all the relevant features.

3.1 The parameter space

The model that we have presented in the previous section is conformal; for this reason we expect to find it as a fixed point in an RG flow. To verify this claim let us introduce a two-parameter family of σ models generalizing the exact backgrounds of eq. (2); a possible choice consists in adding a new magnetic field, this time coming from a higher dimensional right sector. Explicitly

$$\begin{cases} ds^2 = \sum_{\mu \in G/T} J^\mu J^\mu + (1 - h^2) \sum_{a \in T} J^a J^a, \\ H_{[3]} = \frac{\bar{h}}{2h} f_{\mu\nu\rho} J^\mu \wedge J^\nu \wedge J^\rho, & \mu \in G/T, \\ F^a = \frac{h + \bar{h}}{2} \sqrt{\frac{k}{k_g}} f_{\mu\nu}^a J^\mu \wedge J^\nu, & \mu \in G/T, \quad a \in T, \\ \bar{F}^a = \frac{h - \bar{h}}{2} \sqrt{\frac{k}{k_g}} f_{\mu\nu}^a J^\mu \wedge J^\nu, & \mu \in G/T, \quad a \in T \end{cases} \quad (4)$$

and in particular for $SU(2)$:

$$\begin{cases} ds^2 = d\theta^2 + d\psi^2 + d\phi^2 + \cos\theta d\psi d\phi - h^2 (d\psi + \cos\theta d\phi)^2, \\ B = \frac{\bar{h}}{h} \cos\theta d\psi \wedge d\phi, \\ A = (h + \bar{h}) (d\psi + \cos\theta d\phi), \\ \bar{A} = (h - \bar{h}) (d\psi + \cos\theta d\phi), \end{cases} \quad (5)$$

where \bar{h} is a new parameter, describing the deviation from the conformal point. It is clear that the above background reduces to the one in eq. (3) in the $\bar{h} \rightarrow h$ limit. In particular we see that the metric is unchanged,

the Kalb–Ramond field has a different normalization and a new field \bar{A} appears. This configuration can be described in a different way: the geometry of a squashed sphere supports two covariantly constant magnetic fields with charge $Q = h + \bar{h}$ and $\bar{Q} = h - \bar{h}$; the RG flow will describe the evolution of these two charges from a generic (Q, \bar{Q}) to $(2h, 0)$, while the sum $Q + \bar{Q} = 2h$ remains constant. In this sense the phenomenon can be interpreted as a charge transmutation of \bar{Q} into Q . The conservation of the total charge is in fact a consequence of having chosen a perturbation that keeps the metric and only changes the antisymmetric part of the background.

We can also see the background in eq. (4) from a higher dimensional perspective where only the metric and the Kalb–Ramond field are switched on. Pictorially:

$$g = \left(\begin{array}{c|c} g_{WZW} & hJ_a \\ \hline hJ_a & 1 \end{array} \right), \quad B = \left(\begin{array}{c|c} \frac{\bar{h}}{h}B_{WZW} & \bar{h}J_a \\ \hline -\bar{h}J_a & 0 \end{array} \right) \quad (6)$$

where g_{WZW} and B_{WZW} are the usual metric and Kalb–Ramond fields for the WZW model on the group G . More explicitly in the $SU(2)$ case:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \cos\theta & h \\ 0 & \cos\theta & 1 & h \cos\theta \\ 0 & h & h \cos\theta & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\bar{h}}{h} \cos\theta & \bar{h} \\ 0 & -\frac{\bar{h}}{h} \cos\theta & 0 & \bar{h} \cos\theta \\ 0 & -\bar{h} & -\bar{h} \cos\theta & 0 \end{pmatrix} \quad (7)$$

where the fourth entry represents the bosonized internal current. In particular this clarifies the stated right-sector origin for the new gauge field \bar{A} . This higher dimensional formalism is the one we will use in the following RG analysis.

3.2 The renormalization group flow

The σ -model in eq. (6) is not conformal for generic values of the parameters h and \bar{h} ; this is why it makes sense to study its behaviour under the RG flow. Following a dimensional-regularization scheme (see [2, 17, 18] and for various applications [19–22]) we consider the action

$$S = \frac{1}{2\lambda} \int d^2z (g_{MN} + B_{MN}) \partial X^M \bar{\partial} X^N, \quad (8)$$

where g and B are the fields in eq. (6). The beta-equations at two-loop order in the expansion in powers of the overall coupling constant l and the field redefinitions for the internal coordinates X^i turn out to be:

$$\left\{ \begin{array}{l} \beta_{\lambda^*} = \frac{d\lambda^*}{dt} = -\frac{\lambda^{*2}}{4\pi} \left(1 - \frac{\bar{h}^2}{h^2}\right) \left(1 + \frac{\lambda^*}{8\pi} \left(1 - 3\frac{\bar{h}^2}{h^2}\right)\right), \\ \beta_h = \frac{dh}{dt} = \frac{\lambda^* h}{8\pi} (1 - h^2) \left(1 - \frac{\bar{h}^2}{h^2}\right) \left(1 + \frac{\lambda^*}{8\pi} \left(1 - 3\frac{\bar{h}^2}{h^2}\right)\right), \\ \beta_{\bar{h}} = \frac{d\bar{h}}{dt} = -\frac{\lambda^* \bar{h}}{8\pi} (1 + h^2) \left(1 - \frac{\bar{h}^2}{h^2}\right) \left(1 + \frac{\lambda^*}{8\pi} \left(1 - 3\frac{\bar{h}^2}{h^2}\right)\right), \\ X^i = X^i - \frac{\lambda^*}{16} (1 - h^2) \left(1 - 4\frac{\bar{h}^2}{h^2} + 3\frac{\bar{h}^4}{h^4}\right), \end{array} \right. \quad (9)$$

where $\lambda^* = \lambda g^*$, g^* being the dual Coxeter number, is the effective coupling constant ($\lambda^* = N\lambda$ for $G = SU(N)$). The contributions at one- and two-loop order are clearly separated. In the following we will

concentrate on the one-loop part and we will comment on the two-loop result later. Let us then consider the system:

$$\begin{cases} \beta_{\lambda^*} = \frac{d\lambda^*}{dt} = -\frac{\lambda^{*2}}{4\pi} \left(1 - \frac{\bar{h}^2}{h^2}\right), \\ \beta_h = \frac{dh}{dt} = \frac{\lambda^* h}{8\pi} (1 - h^2) \left(1 - \frac{\bar{h}^2}{h^2}\right), \\ \beta_{\bar{h}} = \frac{d\bar{h}}{dt} = -\frac{\lambda^* \bar{h}}{8\pi} (1 + h^2) \left(1 - \frac{\bar{h}^2}{h^2}\right). \end{cases} \quad (10)$$

This can be integrated by introducing the parameter $z = \bar{h}/h$ which makes one of the equations redundant. The other two become:

$$\begin{cases} \dot{\lambda}^* = -\frac{\lambda^{*2}}{4\pi} (1 - z^2), \\ \dot{z} = -\frac{\lambda^* z}{4\pi} (1 - z^2). \end{cases} \quad (11)$$

By inspection one easily sees that $\dot{\lambda}/\lambda = \dot{z}/z$, implying $\lambda(t) = Cz(t)$, where C is a constant. This was to be expected since C is proportional to the normalization of the topological WZ term. Since we are dealing with a compact group it turns out that C is, as in [13], quantized with:

$$C_k = \frac{2\pi}{k}, \quad k \in \mathbb{N}. \quad (12)$$

Now it's immediate to separate the system and find that $z(t)$ is defined as the solution to the implicit equation:

$$-\frac{t}{2k} = \frac{1}{z_0} - \frac{1}{z(t)} + \log \left[\frac{(z(t)+1)(z_0-1)}{(z(t)-1)(z_0+1)} \right] \quad (13)$$

with the initial condition $z(0) = z_0$. A similar expression was found in [13, 23]. The reason for this is, as pointed out previously [24], that the conformal model ($\bar{h} = h$) in its higher-dimensional representation (the one in eq.(6)) coincides with a $G \times H$ WZW model after a suitable local field redefinition.

As it is usually the case in the study of non-linear dynamics, a better understanding of the solution is obtained by drawing the RG flow. In a (z, λ^*) plane, the trajectories are straight lines through the origin and only a discrete set of them are allowed. Moreover the line $z = 1$ is an IR fixed-point locus. This situation is sketched in Fig. 1a. Just as expected the $z = \bar{h}/h = 1$ point, corresponding to the initial exact model described in Sect. 2, is an IR fixed point for the RG flow.

Further insights can be gained if we substitute the condition $\lambda^* = C_k \bar{h}/h$ into the system (10) thus getting:

$$\begin{cases} \frac{dh}{dt} = \frac{\bar{h}}{4k} (1 - h^2) \left(1 - \frac{\bar{h}^2}{h^2}\right), \\ \frac{d\bar{h}}{dt} = -\frac{\bar{h}^2}{4k\bar{h}} (1 + h^2) \left(1 - \frac{\bar{h}^2}{h^2}\right). \end{cases} \quad (14)$$

The flow diagram for this system in the (h, \bar{h}) plane, Fig. 1b, shows how the system relaxes to equilibrium after a perturbation. In particular we can see how increasing \bar{h} leads to a new fixed point corresponding to a value of h closer to 1.

We would like to pause for a moment and put the above results in perspective. The target-space of the sigma-model under consideration is a squashed three-sphere with two different magnetic fields. Along the

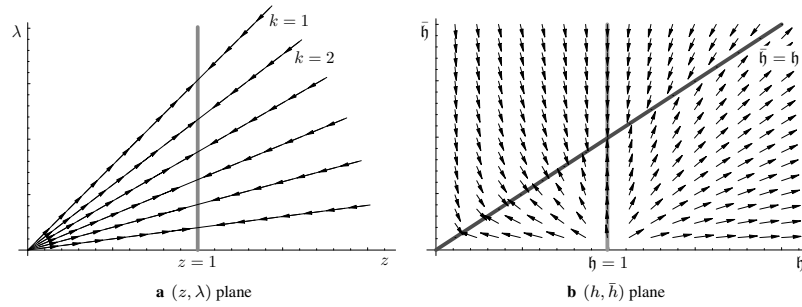


Fig. 1 (online colour at: www.fp-journal.org) Flow lines for the deformed (non-conformal) squashed WZW model in **a** the (z, λ) and **b** the (h, \bar{h}) planes. The arrows point in the negative t direction, *i.e.* towards the infrared; in **a** we see how the squashed WZW model $z = 1$ appears as an IR fixed point, in **b** how perturbing the conformal $\bar{h} = h$ model by increasing \bar{h} leads to a new fixed point corresponding to a value of h closer to 1.

flow, a transmutation of the two magnetic charges occurs: the system is driven to a point where one of the magnetic charges vanishes. This fixed point is an ordinary squashed-WZW (of the type studied in Sect. 2), that supports a single magnetic charge.

As we pointed out in Sect. 2, in the squashed-WZW, the magnetic field is bounded by a critical value, $h = 1$. As long as $h \leq 1$, the geometry is a genuine squashed three-sphere. For $h > 1$, the signature becomes Lorentzian and the geometry exhibits closed time-like curves. Although of limited physical interest, such a background can be used as a laboratory for investigating the fate of chronological pathologies along the lines described above. In particular we see that under the perturbation we are considering the model shows a symmetry between the $h > 1$ and $h < 1$ regions. In fact the presence of closed time-like curves doesn't seem to make any difference, but for the fact that regions with different signatures are disconnected, *i.e.* the signature of the metric is preserved under the RG flow. It is clear that these results are preliminary. To get a more reliable picture for closed time-like curves, one should repeat the above analysis in a wider parameter space, where other RG motions might appear and deliver a more refined stability landscape.

4 RG-flows for compact groups: CFT approach

In order to make contact with genuine CFT techniques, we must identify the relevant operators which are responsible for the (h, \bar{h}) deformation of the $G \times H$ original WZW model ($H = U(1)^{\text{rank } G}$). At lowest approximation, all we need is their conformal dimensions in the unperturbed theory.

Following [1], let \mathcal{L}_0 be the unperturbed (conformal) action and \mathcal{O}_i the operators of conformal dimension Δ_i . Consider the perturbed model, with Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + g^i \mathcal{O}_i. \quad (15)$$

The tree-level beta-functions read:

$$\beta^i(g) = (\Delta_i - 1)g^i, \quad (16)$$

where g^i is supposed to be small, for the perturbative expansion of β^i to hold².

² One should be very careful in the choice of signs in these formulae. In [1] the time variable, in fact, describes the evolution of the system towards the infrared and as such it is opposite with respect to the $t = \log \mu$ convention that we used in the previous section (as in [13]).

The $G \times H$ primary operator we need can be written as follows:

$$\mathcal{O} = \sum_{A,B} \langle t^A g t^B g^{-1} \rangle \langle t^A \partial g g^{-1} \rangle \langle t^B g^{-1} \bar{\partial} g \rangle = \sum_{A,B} \Phi^{AB} J^A \bar{J}^B, \quad (17)$$

where Φ^{AB} is a primary field transforming in the adjoint representation of the left and right groups G . As such, the total conformal dimensions are [25]

$$\Delta = \bar{\Delta} = 1 + \frac{g^*}{g^* + k}, \quad (18)$$

where g^* is the dual Coxeter number and as such the operator is irrelevant (in the infrared).

Specializing this general construction to our case we find that the action for the fields in eq. (6) is:

$$\mathcal{L} = \frac{k}{4\pi} \left\{ \mathcal{L}_0 + \left(\frac{h}{\bar{h}} - 1 \right) \sum_{A,B} \Phi^{AB} J^A \bar{J}^B + \frac{h}{\bar{h}} (h + \bar{h}) \sum_i J^{a_i} \bar{J}^i + \frac{h}{\bar{h}} (h - \bar{h}) \sum_{i,A} J^i \Phi^{a_i A} \bar{J}^A \right\}, \quad (19)$$

where A runs over all currents, i over the internal currents in H and J^{a_i} is the WZW current of the Cartan subalgebra of G coupled to the internal \bar{J}^i . The extra terms can be interpreted as combinations of operators in the $G \times H$ model. The beta-functions are thus computed following eq. (16) with the coupling $g = h/\bar{h} - 1$. We obtain

$$\left. \frac{d}{dt} \left(\frac{h}{\bar{h}} - 1 \right) \right|_{\bar{h}=h} = \frac{g^*}{g^* + k} \left(\frac{h}{\bar{h}} - 1 \right) + \dots = \left(\frac{g^*}{k} - \frac{g^{*2}}{k^2} \right) \left(\frac{h}{\bar{h}} - 1 \right) + \dots \quad (20)$$

where the dots after the first equality denote higher order terms in the $(h/\bar{h} - 1)$ -expansion and after the second equality, in addition to that, higher order terms in the $1/k$ -expansion. This result is the same as the one in [25] since, as we have mentioned, there is the local field redefinition that maps this model at the conformal point to the $G \times H$ WZW model. The above result is to be compared with the results following from eq. (20) when they are expanded around $h = \bar{h}$. We obtain:

$$\left. \frac{d}{dt} \left(\frac{h}{\bar{h}} - 1 \right) \right|_{\bar{h}=h} = \left(\frac{g^*}{k} - \frac{g^{*2}}{k^2} \right) \left(\frac{h}{\bar{h}} - 1 \right) + \frac{1}{2} \left(-\frac{g^*}{k} + 7\frac{g^{*2}}{k^2} \right) \left(\frac{h}{\bar{h}} - 1 \right)^2 + \dots \quad (21)$$

We see that these results agree to first order in the coupling $h/\bar{h} - 1$.

The extra information that we obtain from this calculation is about the interpretation for the two-loop beta-function we described in the previous section. The one-loop corrections to (16) are of the form $C_{ijk} g^i g^j$, where C_{ijk} are related to the three-point function of the unperturbed theory [1]. This coefficient is a measure of the dimension of the operator \mathcal{O}_i in the theory perturbed by the set of all operators. Such a computation goes beyond the scope of the present note. Nevertheless, (21) predicts the coefficient of the term $(h/\bar{h} - 1)^2$ to second order in the $1/k$ -expansion and it seems that such a computation is feasible from the CFT viewpoint at least as a series expansion in $1/k$.

5 Conclusions

In this work, we have analyzed the phase space of squashed WZW models, away from the original conformal point. Our analysis is given in detail for the compact group $SU(2)$ and can be generalized to any compact group. We have restricted ourselves to deviations from the conformal point, generated by switching on simultaneously two distinct magnetic (or electric) fields. The corresponding backgrounds may have interesting interpretation in terms of NS5-branes. We have investigated the phase diagram using geometrical,

target-space techniques, as well as standard CFT renormalization-group methods. Our results can be summarized as follows: the squashed-WZW models are found, as expected, as IR fixed points in the RG flow, and this result is confirmed from both a target space and world-sheet point of view. The field theory interpretation of this flow consists in what we have called charge transmutation. One $U(1)$ charge transforms into another $U(1)$ while the total charge is conserved. For large values of the parameter h the backgrounds under consideration contain closed time-like curves. These do not seem to change the behaviour of the flow and the model remains stable, at least under the deformation we consider.

This charge transmutation enters the class of phenomena that are expected to take place when a metastable string background jumps to a stable one through a non-critical path. These include tachyon condensation, particle production and other interesting physical phenomena: the RG flow around the conformal point is a tool to get information on the dynamics of the relaxation. Our geometrical tools are well-fitted to describe the latter provided we allow for more parameters in the phase space. A generalization of our approach may also allow to address more thoroughly the issue of instabilities triggered by the presence of closed time-like curves. Of course this is all very preliminary and in particular much still remains to be done in clarifying the link between energy minimization and time evolution in non-compact and time-dependent backgrounds. A first step in this direction consists in investigating non-compact groups, like $SL(2, \mathbb{R})$, for which some aspects (e.g. related to Zamolodchikov's C-theorem) of the underlying theory remain obscure.

Acknowledgements We would like to thank C. de Calan, E. Kiritsis, C. Kounnas, N. Toumbas, J. Troost and especially V. Dotsenko for illuminating discussions.

P.M.P. and D.O. acknowledge Patras University for kind hospitality as well partial financial support by the INTAS contract 03-51-6346 *Strings, branes and higher-spin gauge fields* and the EU under the contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369.

K.S. acknowledges partial financial support by the CNRS, the EU under the contract MRTN-CT-2004-005104, the INTAS contract 03-51-6346 *Strings, branes and higher-spin gauge fields*, as well as the Greek Ministry of Education programs ΠΥΘΑΓΟΡΑΣ with contract 89194 and ΕΠΙΛΑΝ with code-number B.545.

References

- [1] A. B. Zamolodchikov, JETP Lett. (USA) **43**, 730–732 (1986).
- [2] D. H. Friedan, Ann. Phys. (USA) **163**, 318 (1985).
- [3] J. G. Russo and A. A. Tseytlin, Nucl. Phys. B **611**, 93–124 (2001), [hep-th/0104238].
- [4] E. Kiritsis and C. Kounnas, Nucl. Phys. B **442**, 472–493 (1995), [hep-th/9501020].
- [5] J. R. David, M. Gutperle, M. Headrick, and S. Minwalla, J. High Energy Phys. **02**, 041 (2002), [hep-th/0111212].
- [6] J. A. Harvey, D. Kutasov, E. J. Martinec, and G. W. Moore, Localized Tachyons and rg Flows, hep-th/0111154.
- [7] D. Z. Freedman, M. Headrick, and A. Lawrence, On Closed String Tachyon Dynamics, hep-th/0510126.
- [8] C. Bachas and M. Porrati, Phys. Lett. B **296**, 77–84 (1992), [hep-th/9209032].
- [9] B. Pioline and J. Troost, J. High Energy Phys. **03**, 043 (2005), [hep-th/0501169].
- [10] S. W. Hawking, Phys. Rev. D **46**, 603–611 (1992).
- [11] D. Israël, J. High Energy Phys. **01**, 042 (2004), [hep-th/0310158].
- [12] M. S. Costa, C. A. R. Herdeiro, J. Penedones, and N. Sousa, Nucl. Phys. B **728**, 148–178 (2005), [hep-th/0504102].
- [13] E. Witten, Commun. Math. Phys. **92**, 455–472 (1984).
- [14] D. Israël, C. Kounnas, D. Orlando, and P. M. Petropoulos, Fortschr. Phys. **53**, 73–104 (2005), [hep-th/0405213].
- [15] D. Israel, C. Kounnas, D. Orlando, and P. M. Petropoulos, Fortschr. Phys. **53**, 1030–1071 (2005), [hep-th/0412220].
- [16] S. Chaudhuri and J. A. Schwartz, Phys. Lett. B **219**, 291 (1989).
- [17] H. Osborn, Ann. Phys. (USA) **200**, 1 (1990).

- [18] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, *Ann. Phys. (USA)* **134**, 85 (1981).
- [19] J. Balog, P. Forgacs, Z. Horvath, and L. Palla, *Nucl. Phys. Proc. Suppl.* **49**, 16–26 (1996), [hep-th/9601091].
- [20] J. Balog, P. Forgacs, N. Mohammadi, L. Palla, and J. Schnittger, *Nucl. Phys. B* **535**, 461–482 (1998), [hep-th/9806068].
- [21] K. Sfetsos, *Phys. Lett. B* **432**, 365–375 (1998), [hep-th/9803019].
- [22] K. Sfetsos, *Nucl. Phys. B* **561**, 316–340 (1999), [hep-th/9904188].
- [23] E. Braaten, T. L. Curtright, and C. K. Zachos, *Nucl. Phys. B* **260**, 630 (1985).
- [24] E. Kiritsis and C. Kounnas, *Nucl. Phys. B* **456**, 699–731 (1995), [hep-th/9508078].
- [25] V. G. Knizhnik and A. B. Zamolodchikov, *Nucl. Phys. B* **247**, 83–103 (1984).