



# D'Ellsberg à Machina : les modèles de décision dans l'ambiguïté à l'épreuve de l'expérimentation

Laetitia Placido

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**« From Ellsberg to Machina:  
Confronting decision models under ambiguity with experimental evidence »**

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# Introduction

## Choice under uncertainty

Knight (1921) first introduced the distinction between measurable uncertainty, in which probabilities are known; and non measurable uncertainty, in which they are unknown. In both cases, a decision maker has to choose between uncertain alternatives, whose consequences depend on events that can possibly occur. The decision maker endorses a preference relation over all the available actions. The normative principle of decision theory is that the decision maker ought to undertake the best action with respect to his preferences. The model used to predict the behavior will depend on the informational context confronting the decision maker and account for the decision maker's attitude given this context.

The benchmark of decision making under uncertainty is the case of risk. All the events under consideration can be associated with objective probabilities and the decision maker typically applies expected utility theory (von Neumann and Morgenstern, 1944). Allais (1953) shook the predictive power of the theory by pointing out its descriptive limitations in the presence of the certainty effect. Savage (1954) extended (objective) expected utility to informational contexts where the decision maker has no objective probabilistic information at his disposal. Subjective expected utility assumes that the decision maker assigns subjective *additive* probability on events. Then, the decision maker computes the expected utility of each action with respect to his subjective probability and chooses the action associated with the

higher subjective expected utility. If the decision maker more generally behaves in a way that can be analyzed in terms of probabilities without necessarily conforming to the expected utility rule, then he is defined as probabilistically sophisticated (Machina and Schmeidler, 1992). However, an underlying condition of models that are based on subjective probability is that the decision maker effectively treats subjective probabilities as if he is dealing with objective probabilities, independent of the amount of information that leads to the formulation of the personal probability. This condition is far from being respected for some specific choice problems, notably those involving ambiguity.

Ambiguity defines decision contexts where no probabilities are available to (nor can be subjectively revealed by) the decision maker to take one's decision. Ellsberg (1961) proposed thought experiments suggesting that the presence of ambiguity in decision making may affect behavior in a way that deviates from subjective expected utility. He suggested that most people prefer to bet on the color of a ball drawn from a risky urn with known composition (fifty red, fifty black), rather than on the color of a ball drawn from a similar urn containing 100 balls with an unspecified composition. This behavior reveals that the probability of drawing a red (or equivalently black) ball in the unspecified urn is less than one-half. Such a decision maker violates subjective probabilities and (ex ante) probabilistic sophistication which predict that the probabilities of drawing a red or a black ball should sum to one. Ellsberg concluded that decision makers tend to avoid ambiguity, exhibiting what he termed *ambiguity aversion*.

Subsequently, a large experimental literature has empirically confirmed Ellsberg's contradicting examples, and a large theoretical literature has developed alternative models to accommodate non neutral attitudes towards ambiguity. A pioneer approach generalizes subjective expected utility to non additive probability measures (Schmeidler, 1989; Gilboa, 1987; Tversky and Kahneman, 1992). The multiple prior approach (Gilboa and Schmeidler, 1989; Ghirardato, Maccheroni, Marinacci, 2004) modelizes ambiguity through a set of probability

measures. In a final two-stage approach, a decision maker formulates subjective beliefs (first stage) over probability distributions (second stage) but does not necessarily reduce the two stages (Klibanoff, Marinacci and Mukerji, 2005; Seo, 2008; Halevy and Ozdenoren, 2008).

All these models provided successful predictions of the behavior under ambiguity until a recent contribution from Machina (2009). Machina proposed a slight modification of the Allais<sup>1</sup> and Ellsberg examples and predicted that the plausible preferences cannot be explained by the most popular models accounting for these paradoxes, i.e., rank-dependent utility under risk (Quiggin, 1982) and Choquet expected utility for ambiguity (Schmeidler, 1989).

The object of this thesis is to describe and analyze individual decision making in the context of ambiguity. It is based on both Ellsberg and Machina paradoxes and mainly adopts the experimental approach.

### **Contribution and outline of the thesis**

This thesis aims to provide new insights to the understanding of decision behavior under ambiguity. It is composed of three parts.

The first part is a survey of the literature on ambiguity. Chapter 1 presents the modeling of subjective uncertainty and ambiguity. Chapter 2 envisages ambiguity attitudes as a consequence of the joint presence of objective and subjective sources of uncertainty.

The second part explores two specific approaches to the Ellsberg paradox. Different models might be appropriate in describing the individual behavior towards ambiguity, the diversity of approaches reflecting the heterogeneity of human cognitive processes. A common feature of the approaches adopted here is that they allow one to analyze attitudes towards ambiguity as likelihood-dependent. Chapter 3 reconciles ambiguity attitudes with probabilistic sophistication in the Ellsberg two-urn problem. Ambiguity is viewed as a specific source of

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<sup>1</sup>First draft of Machina's paper: July, 2007.

uncertainty (notably as opposed to risk). The method employs Chew and Sagi's (2006, 2008) exchangeability concept to define probabilistic sophistication within a source of uncertainty while not requiring it between sources. It provides a quantitative measurement of behavior through source functions, one for the risky urn and one for the ambiguous urn. The shapes of these functions are found to be different, which empirically confirms the soundness of approaches based on sources of uncertainty (Tversky and Wakker, 1995), and at the same time, reconciles non neutral attitudes towards ambiguity with probabilistic sophistication on condition that exchangeability holds within each source of uncertainty. Chapter 4 adopts a quite different, and perhaps, not contradictory perspective. It studies the idea that decision makers assimilate ambiguity to compound risk. It further investigates a recent empirical finding that establishes equivalence between reduction of compound lotteries and ambiguity neutrality (Halevy, 2007). Our data confirm a link between ambiguity attitudes and compound risk attitudes. However, it puts into perspective Halevy's conclusion, since the equivalence does not match the data: decision makers who fail to reduce compound lotteries are non neutral to ambiguity but those who reduce compound lotteries are not necessarily ambiguity neutral and are even prone to ambiguity aversion. This result does not support recent axiomatizations (Halevy and Ozdenoren, 2008; Seo, 2008) that explicitly relate compound risk and ambiguity.

The third part is entirely based on a recent contribution by Mark Machina (2009). Machina considers a slight modification of the Ellsberg original one-urn problem and convincingly proves that the pattern of preferences that could emerge from his construction is not compatible with Choquet expected utility (Schmeidler, 1989). Chapter 5 provides empirical evidence for the thought experiment proposed by Machina and confirms the possibility of extending the Ellsberg paradox to one of the major models that accounts for ambiguity aversion, i.e, Choquet expected utility. At the same time, Chapter 6 empirically undermines such an extension of the Allais paradox to rank-dependent utility (Quiggin, 1982) in the context of risk. In

Chapter 7, which is theoretical, it is proved that the conclusions of Machina are not restricted to Choquet expected utility but can be extended to four other prominent models of ambiguity as well. Notably, the class of models with sets of priors - including, Gilboa and Schmeidler's maxmin expected utility (1989), its extensions,  $\alpha$ -maxmin expected utility and the variational preferences - and the smooth model of ambiguity aversion (Klibanoff, Marinacci and Mukerji, 2005) are also contradicted by Machina's paradox.

All chapters except those in Part 1 are self-contained in the sense that they are readable independently. Consequently, notations and concepts may appear several times.

The experimental or theoretical results incorporated in this thesis correspond to the following research papers: Chapter 4 refers to a subpart of Abdellaoui, Baillon, Placido and Wakker (2009a), Chapter 5 to Abdellaoui, Klibanoff and Placido (2009b), Chapter 6 and 7 to L'Haridon and Placido (2008, 2009) and Chapter 8 to Baillon, L'Haridon and Placido (2009).



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## Part I

# SUBJECTIVE UNCERTAINTY: MODELS AND PARADOXES

# Chapter 1

## Modeling Subjective Uncertainty

### 1.1 Introduction

Standard economic modeling describes behavior under uncertainty by making the assumption that a decision maker (henceforth, DM) always has either an objective probability at his disposal (Von Neumann and Morgenstern: vNM, 1944) or can formulate subjective additive probability (Savage, 1954) in any decision context. In both cases, the DM is assumed to compute the expected utility of each possible decision with respect to the objective/subjective probability and choose the decision associated with the highest objective/subjective expected utility. However, Ellsberg (1961) remarks that such a rule no longer applies in a specific decision context called ambiguity. This chapter aims at giving an overview of the literature on subjective uncertainty including subjective expected utility and its generalizations to ambiguity.

Section 1.2 presents the modeling of subjective uncertainty introduced by Savage (1954) who first gave all the ingredients for obtaining a purely subjective decision model. Indeed, both tastes - given by the utility function - and beliefs - given by the probability measure - are subjective in the sense that they are simultaneously derived from the preference relation

over decisions. Section 1.3 describes the approach of Anscombe and Aumann (1963), which reintroduces some objective elements to make the axiomatization closer to expected utility for risk. Section 1.4 presents generalizations of subjective uncertainty based on probabilistic sophistication. Section 1.5 focuses on ambiguity attitudes as a contradiction of the classic modeling of subjective uncertainty (Ellsberg, 1961). Eventually, Section 1.6 presents models that have been developed to take into account non neutral ambiguity attitudes.

### 1.2 Savage's approach

Savage's (1954) subjective expected utility (SEU) consists of an extension of vNM expected utility to decision context where objective probabilities are not available. It provides a set of technical and behavioral axioms that are sufficient to characterize both a utility function and a probability measure. Because both are derived from conditions on preferences, the theory is said to be fully subjective.

#### 1.2.1 Framework and notations

Savage's formulation is based on three elements: the state space that modelizes uncertainty, the set of outcomes that describes the possible consequences a DM can undergo, and the set of acts that relates the two, and on which the DM has a preference relation.

##### The state space

The state space (or the *world*)  $S$  is "the object about which the person is concerned" and contains the states of the world. We assume  $S$  finite. A *state of the world*  $s$  is an element of  $S$  and is "a description of the world, leaving no relevant aspect undescribed". The resolution of uncertainty relies on the properties of the states of the world: (i) *exhaustive*: a DM is able to completely describe the world in which he lives (he can list all the states), (ii) *mutually*

*exclusive*: two distinct states cannot simultaneously occur, (iii) *only one state is true*: "the state that does in fact obtain" (Savage, p. 9).

Events  $E_i$  are subsets of  $S$ , and  $2^S$  is the set of all the subsets of  $S$ . The universal event  $S$  is the event having every state of the world as element. The vacuous event  $\emptyset$  has no state as element. A collection of events  $\{E_1, \dots, E_n\}$  with  $n$  belonging to the natural number, forms a partition of the state space.

***Conceptual and descriptive limitations.*** The main limitations of Savage's state space are due to the assumption of exogeneity that bears on it.

First, each state represents nature's exogenous uncertainty. This implies that it should be possible to reconstruct on the basis of a DM's observed choices, the unique state space underlying his decisions. However, as argued by Machina (2003), there is no guarantee that the state space thus obtained corresponds to the state space that preexists and is observed, or to an endogenous construction of the DM.

Second, the exhaustivity requirement imposes that the DM have a complete representation of the world. As shown by Newcomb's paradox (see Gilboa, 2003), evidence can lead the DM to conclude that his initial image of the world is incomplete. More concretely, the exhaustivity of the state space seems impossible to guarantee in practice due to objective complexity and human cognitive limitation. As pointed out by Karni (2006): "the depiction of the relevant state space is often unintuitive and too complex to be compatible with DMs' perception of choice problems".

Third, exogeneity implicitly supposes that the DM is aware of all the states that can occur. Consequently, the state space leaves no room for unforeseen contingencies, and at best "when the DM has reason to 'expect the unexpected' (...) the best one can do is specify a final, catch-all state, with a label like 'none of the above', and a very ill-defined consequence" (Machina, 2003).

Finally, exogeneity can be conceptually called into question for a theory that claims to be subjectivist. Epstein and Zhang (2001) argue that a fully subjective theory should derive both the domain and the subjective probability from preferences. They tend to remedy this conceptual limitation by proposing an axiomatization that endogenously defines a domain on which a DM has subjective probabilities from the rest of the world on which the DM ought not to have such probabilities.

The two following remarks express more descriptive concerns since they point out behavioral assumptions contained in Savage's axioms. The realization of the state is independent of the action undertaken by the DM. As argued by Karni (2006), this assumption seems unrealistic and implies unconceivable fatalism. Eventually, the evaluation of a consequence is independent on the state in which it is received (the utility is state independent).

These remarks reveal Savage's construction to be conceptually limited and descriptively inadequate for the representation of many decision problems. The literature provides some extensions that aim to remedy to such limitations.

*Extensions.* Extensions have been provided that allow for a more flexible and realistic description of the world. Indeed, a DM facing a decision problem is generally not provided with the background structure.

Ghirardato (2001) envisages that events which are relevant for the result of the DM's choices may have been left out of the description of the state space and refers to 'unforeseen contingencies'. The DM is aware of his ignorance and perceives the state space as under-specified; this is formally represented by correspondences: each state  $s$  under consideration is a collection of possible states, differing in aspects which have not been included in the description of  $s$ .

In Dekel, Lipman and Rustichini (2001), the state space reflects the DM's subjective understanding of the world. The DM's uncertainty about the future can be measured by the size of



his endogenously defined subjective state space.

Karni (2006) proposes a theory that dispenses with the state space and accommodates both the presence of moral hazard considerations as well as the possibility that the evaluation of the consequences of decisions are effect-dependent. He obtains subjective expected utility with unique and action-dependent subjective probability.

Chew and Sagi (2008) points out that, although Savage envisaged a big world as defined above, he, at the same time observed that decisions are generally made in smaller worlds, which contain events summarizing the relevant aspects of the contingencies pertaining to specific decision situations. In Savage's formulation, events in any small world are comparable to events in any other small world and they all remain similar to the big world. Hence, Savage implicitly assumes that the DM's attitude is independent of the small world in which the decision is taken. Chew and Sagi refine Savage's small world approach, basing it on the intuition of similarity among events. They introduce the concept of *small world events domain* defined as a collection of comparable events; they provide consistency conditions that explain the presence of distinct attitudes between small worlds.

### **The outcome space**

To each event that occurs there follows a *consequence*, i.e., "anything that may happen to the person". The set of consequences  $X$  is finite and will refer to monetary outcomes, although Savage defines more generally the "states of the person".

### **The choice space**

Objects of choice are called *acts*. The set of acts is  $\mathcal{A} = \{f : S \rightarrow X\}$ , i.e., maps from the state space to the outcome space.  $f(s)$  represents the consequence of choosing  $f$  if the state of the world is  $s$ . An act  $h$  noted  $[f \text{ on } E; g \text{ on non } E]$  means that  $h(s) = f$  for all  $s \in E$  and  $h(s) = g$

otherwise. A *constant act* gives the same outcome over all the states the world ( $f(s) = x$  for all  $s \in S$ ) and is shortly designated by the unique outcome  $x$  it is associated with. We assume that acts have only finite consequences. Hence, an act will be alternatively written  $[x_1 \text{ on } E_1; \dots; x_n \text{ on } E_n]$  with the understanding that  $x_i$  is obtained if  $E_i$  is true. The DM has a preference relation  $\succsim$  over  $\mathcal{A}$ .  $\succsim$  denotes weak preference with  $\succ$  and  $\sim$  the strict preference and indifference, respectively ( $\precsim$  denote the reverse preference).

$E$  is a *null event* if indifference holds between all pairs of acts that only differ on  $E$ :

**Definition 1** (Null event).  $E \subseteq S$  is a null event if for all  $f, g, h \in \mathcal{A}$ ,

$$[f \text{ on } E; h \text{ on non } E] \sim [g \text{ on } E; h \text{ on non } E].$$

Null events will turn to be those with zero probability.

### 1.2.2 Savage's axiomatization

Savage provides a set of postulates for preference among acts. According to Savage, a rational DM ought to satisfy these postulates and he shows that conforming to these postulates is equivalent to agreeing with a ranking of acts in term of subjective expected utility.

**P 1 (Ordering)** The preference relation  $\succsim$  is a weak order (complete and transitive)

P1 says that preferences should be transitive: if a DM prefers act  $f$  to act  $g$ , and act  $g$  to act  $h$ , then he should prefer act  $f$  to act  $h$ . The transitivity condition is itself normatively desirable and not controversial as a rationality requirement. However, as argued by Shafer (1986), Savage goes one step further by making the assumption of completeness, that imposes a DM should always have well-defined preferences between acts  $f$ ,  $g$  and  $h$ . Imagine a DM who actually does not have preferences between these particular three acts, in words, he is indecisive; it comes from Savage's P1 that the DM is obliged to construct these preferences. Moreover, even if they exist, preferences are often unstable and non robust to procedural invariance as emphasized by Tversky (1972):

"When faced with a choice among several alternatives, people often experience uncertainty and exhibit inconsistency. That is, people are often not sure which alternative they should select, nor do they always make the same choice under seemingly identical conditions."

The three following axioms provide together the disentanglement between subjective probabilities and the subjective values of consequences (utilities).

**P 2 (Sure-thing principle)** for all events  $E$  and for all acts  $f, g, h, h^* \in \mathcal{A}$ ,

$$[f \text{ on } E; h \text{ on } \text{non } E] \succsim [g \text{ on } E; h \text{ on } \text{non } E] \Rightarrow [f \text{ on } E; h^* \text{ on } \text{non } E] \succsim [g \text{ on } E; h^* \text{ on } \text{non } E]$$

P2 says that if two acts are equal on a given event  $\text{non } E$ , then the preference ranking over these acts should not depend on what they are equal to on  $\text{non } E$ . In words, the DM does not care about what is sure,  $h$  or  $h^*$ , when choosing between  $f$  and  $g$ . Intuitively, this condition implies that preferences are separable on mutually exclusive events. The sure-thing principle constitutes the weak point of Savage's theory and is violated as soon as DMs exhibit non neutral ambiguity attitudes (see Chapter 2).

**P 3 (Eventwise monotonicity)** for all act  $f \in \mathcal{A}$ , for all outcomes  $x, y \in X$ , for all non null event  $E$ ,  $[x \text{ on } E; f \text{ on } \text{non } E] \succsim [y \text{ on } E; f \text{ on } \text{non } E] \Leftrightarrow x \succsim y$

The DM who weakly prefers the sure consequence  $x$  to the sure consequence  $y$  will choose the right act because the alternative yields less on the event on which the acts differ. This axiom implies that the tastes concerning outcomes do not depend on the events under which they are received. Hence, utility is not state dependent.

**P 4 (Weak comparative probability)** for all events  $E, F$  and outcomes  $x \succ y$  and  $x' \succ y'$ ,

$$[x \text{ on } E; y \text{ on } \text{non } E] \succsim [x \text{ on } F; y \text{ on } \text{non } F] \Rightarrow [x' \text{ on } E; y' \text{ on } \text{non } E] \succsim [x' \text{ on } F; y' \text{ on } \text{non } F]$$

P4 says that, since  $x$  is more desirable than  $y$ , the first act is a win if  $E$  occurs and the second is a win if  $F$  occurs. The first is weakly preferred to the second if event  $E$  is judged at least as likely as event  $F$ . Then it should also be the case for the two last acts since  $x'$  is more

desirable than  $y'$ . In brief, beliefs on events do not depend on the consequences. P3 adds that this ranking in term of likelihood does not depend on the consequences used. Thus, P4 allows to infer an ordering of events in terms of likelihood and is then crucial for the existence of subjective probabilities.

**P 5 (Non degeneracy)**  $\exists$  two outcomes  $x, y \in X$  such that  $x \succ y$

P5 guarantees the existence of the probability measure. If this axiom is not true ( $\forall x, y \in X, x \sim y$ ), the DM is indifferent to all consequences so there is no longer a decision problem. Note that it also means that  $S$  is not a null event.

**P 6 (Small event continuity)** For any acts  $f \succ g$  and outcome  $x \in X$  there exists a finite set of events  $\{E_1, \dots, E_n\}$  partitioning  $S$  such that  $\forall i \in \{1, \dots, n\}$ :

$$f \succ [x \text{ on } E_i; g \text{ on non } E_i] \text{ and } [x \text{ on } E_i; f \text{ on non } E_i] \succ g$$

P6 implies that  $S$  can be partitioned in sufficiently small events so that a modification of each act (by putting the best or the worst outcome on one of these small events) is not sufficient for reversing the original preference order.

**Theorem 1** (Savage, 1954). *Under P1-P6 there exists a unique finitely additive and non atomic probability measure  $P(\cdot)$  on  $2^S$  and a state-independent utility function  $u(\cdot)$  on the set of outcomes  $X$  such that the subjective expected value of act  $f$  is:*

$$SEU(f) = \sum_{s \in S} P(s)u(f(s)) \quad (1.1)$$

Moreover,  $u(\cdot)$  is unique up to a positive linear transformation.

Act  $f$  is preferred to another act  $g$  if the expected utility calculated with respect to the subjective probability measure  $P$  is higher for  $f$  than for  $g$ . The subjective measure  $P$  is derived from the preference of the DM and is thus personal. Consequently, two DMs might not to reveal the same subjective probability on the state space.

### 1.3 Anscombe-Aumann's approach

Anscombe and Aumann (1963) propose an alternative derivation of SEU. An act  $f$  is a map from the state space to the set of lotteries over consequences (and no longer to the set of consequences as in Savage).

#### 1.3.1 Framework and notations

The preference relation  $\succsim$  is defined on the set of acts  $\mathcal{F} = \{f : S \rightarrow \mathcal{L}\}$ , where  $\mathcal{L}$  is the set of simple lotteries (with finite support) on  $X$ . An element of  $\mathcal{L}$  is a lottery  $l = (x_1, p_1; \dots; x_n, p_n)$  which gives the consequences  $x_i$  with probability  $p_i$ . Hence, each act  $f$  in  $\mathcal{F}$  combines "horse race lotteries" (i.e., Savagean acts) and "roulette lotteries" (i.e., objective lotteries) and can be written in the following way  $[\dots; (\dots; x_i, p_i; \dots) \text{ on } E_j; \dots]$ .  $f$  can be interpreted as a bet on a horse race, but instead of receiving the winnings of the bet directly in money, the DM is actually given a ticket for a lottery with objective probabilities.

The  $\lambda$ -mixture of acts  $f = (l_1, \dots, l_n)$  and  $g = (l'_1, \dots, l'_n)$  with  $\lambda \in [0, 1]$  noted  $\lambda f + (1 - \lambda)g$  yields  $\lambda f(i) + (1 - \lambda)g(i) = \lambda l_i + (1 - \lambda)l'_i$  on state  $i$ . Thanks to mixture, Anscombe-Aumann's axiomatization will be very similar to vNM's.

#### 1.3.2 Anscombe-Aumann's axiomatization

**A 1 (Weak order)** The preference relation  $\succsim$  is a weak order (transitive and complete)

Bewley (2002) proposes a theory of choice under subjective uncertainty that removes the completeness axiom from the Anscombe-Aumann setting but he needs to introduce an inertia assumption to deliver a representation theorem.

**A 2 (Independence)** for all  $f, g, h \in \mathcal{F}$ , and for all  $\lambda \in [0, 1]$ ,

$$f \succsim g \Rightarrow \lambda f + (1 - \lambda)h \succsim \lambda g + (1 - \lambda)h$$

**A 3 (Jensen continuity)** for all  $f, g, h \in \mathcal{F}$ , if  $f \succ g \succ h$ , then  $\exists \lambda, \mu \in ]0, 1[$  such that,  
 $\lambda f + (1 - \lambda)h \succ g \succ \mu f + (1 - \mu)h$

The lottery is only played after a particular state  $s \in S$  occurs; hence, axioms A1-A3 only deliver a state-dependent expected utility function  $U_s : \mathcal{L} \rightarrow X$  and the preferences are represented by:

$$A(f) = \sum_{s \in S} U_s(l_s)$$

Intuitively, for obtaining state independent expected utility we need something that says that a preference between two lotteries  $l$  and  $l'$  has to be preserved whatever the state is. First we need to introduce one definition:

**Definition 2** (Null state).  $s \in S$  is a null state if for all  $q, q' \in \mathcal{L}$

$$(l_1, \dots, l_{s-1}, q, l_{s+1}, \dots, l_n) \sim (l'_1, \dots, l'_{s-1}, q', l'_{s+1}, \dots, l'_n).$$

If a DM is indifferent between these two acts, then effectively state  $s$  does not matter, i.e. it is equivalent to stating that the DM believes  $s$  will never happen. It will be assumed that there are at least some states that are non-null states. To establish this, the following axiom is needed:

**A 4 (Non degeneracy)**  $\exists f, g \in \mathcal{F}$  such that  $f \succ g$

A4 guarantees the existence of non-null states.

**A 5 (State independence)**  $s \in S$  is a non-null state and  $q, q' \in \mathcal{L}$ . If

$$(l_1, \dots, l_{s-1}, q, l_{s+1}, \dots, l_n) \succ (l'_1, \dots, l'_{s-1}, q', l'_{s+1}, \dots, l'_n) \text{ then, for every non-null state } t \in S,$$

$$(l_1, \dots, l_{t-1}, q, l_{t+1}, \dots, l_n) \succ (l'_1, \dots, l'_{t-1}, q', l'_{t+1}, \dots, l'_n)$$

**Theorem 2** (Anscombe-Aumann, 1963). Under A1 - A5 there exists of a unique probability measure  $P(\cdot)$  on  $S$  and a state-independent expected utility function  $U(\cdot)$  on  $\mathcal{L}$  such that act  $f$  is evaluated through:

$$AA(f) = \sum_{s \in S} P(s)U(l_s) \tag{1.2}$$

Moreover,  $U(\cdot)$  is unique up to a positive linear transformation.

As in Savage, the subjective probabilities  $P(s)$  are derived from preferences over actions and not imposed externally.

Savage and Anscombe-Aumann axiomatizations both result in expected utility functional forms. In the following section, we present two derivations of subjective probabilities that do not constrain to an expected utility form.

### 1.4 Probabilistic sophistication

Probabilistic sophistication (PS) is weaker than SEU since it assumes that the DMs formulate probabilistic beliefs over events without requiring the expected utility form.

#### 1.4.1 Machina and Schmeidler's PS

Machina and Schmeidler (1992) abandon the expected utility form of Savage's theory but keep the idea that DMs should have subjective additive probabilistic beliefs. They establish the conditions to obtain a probabilistically sophisticated non (necessarily) expected utility maximizer. They remove P2 from the Savage settings and strengthen P4 in the following way:

**P4\* (Strong comparative probability).** for all (disjoint) events  $E, F$ , for all  $f, g \in \mathcal{A}$  and for all consequences  $x \succ y$  and  $x' \succ y'$ ,

$$\begin{aligned} & [x \text{ on } E; x' \text{ on } F; f \text{ on non } (E \cup F)] \succsim [x' \text{ on } E; x \text{ on } F; f \text{ on non } (E \cup F)] \\ \Rightarrow & [y \text{ on } E; y' \text{ on } F; g \text{ on non } (E \cup F)] \succsim [y' \text{ on } E; y \text{ on } F; g \text{ on non } (E \cup F)] \end{aligned}$$

They find that P1, P3, P4\*, P5 and P6 are equivalent to the existence of the preference functional  $W_{PS}(\cdot)$  over acts which takes the form of a composition of a preference function  $V(\cdot)$  over lotteries and a subjective probability measure  $\mu(\cdot)$  over events as follows:

$$W_{PS}(f(\cdot)) = W_{PS}([x_1 \text{ on } E_1; \dots; x_n \text{ on } E_n]) = V(x_1, \mu(E_1); \dots; x_n, \mu(E_n))$$

where  $V$  is a (non necessarily expected utility) preference function  $V(P) = V(x_1, p_1; \dots; x_n, p_n)$  over lotteries. Hence, a DM is said to be probabilistically sophisticated if her beliefs can be completely summarized by a subjective probability  $\mu(\cdot)$  and she evaluates an act on the sole basis of the implied probability distribution  $(x_1, \mu(E_1); \dots; x_n, \mu(E_n))$  over the consequences.

### 1.4.2 Chew and Sagi's PS

Chew and Sagi (2006) provide a derivation of probabilistic sophistication from event exchangeability. Let  $\Sigma$  be an algebra of events over  $S$ .

**Definition 3** (Event Exchangeability). *Two events  $E$  and  $F$  (disjoint)  $\in \Sigma$  are exchangeable ( $E \approx F$ ) if for all  $x, y \in X$  and act  $f$ ,  $[x \text{ on } E; y \text{ on } F; f \text{ on } S-(E \cup F)] \sim [y \text{ on } E; x \text{ on } F; f \text{ on } S-(E \cup F)]$ .*

A DM is always indifferent in permutting the payoffs between exchangeable events. Exchangeability results in equal likelihood: two exchangeable events are revealed equally likely by the DM.

**Definition 4** (Exchangeability-Based Comparative Likelihood). *For any events  $E$  and  $F \in \Sigma$ ,  $E$  is 'at least as likely as'  $F$  whenever  $E-F$  contains a subevent  $e \approx (F-E)$ .*

The three following axioms imply that the state space can be partitionned into equally likely events, in such a way that the DM is indifferent in betting on an event of two different partitions with the constraint that these partitions contain the same number of elements.

**EA (Event Archimedean Property).** Any sequence of pairwise disjoint and non null events  $\{e_i\}_{i=0}^{\infty} \subseteq \Sigma$  such that  $e_i \approx e_{i+1}$  for every  $i = 0, \dots$  is necessarily finite.

**C (Completeness).** Given any disjoint pair of events, one of the two must contain a subevent that is exchangeable with the other.

**N (Event Nonsatiation).** For any pairwise disjoint events  $E, F, A \in \Sigma$  if  $E \approx F$  and  $A$  non null, no subevent of  $F$  is exchangeable with  $E \cup A$ .



EA, C and N are equivalent to the existence of a unique and finitely additive probability measure on  $\Sigma$  that represents the 'at least as likely' relation.

## 1.5 Ellsberg paradox

Ellsberg (1961) proposes two main thought experiments that disturbed the usual way of modeling behavior under uncertainty. Notably, the following examples violate SEU and Machina and Schmeidler's PS.

### 1.5.1 Two-urn experiment

Imagine two urns; the known urn contains fifty red balls and fifty black balls; the unknown urn contains 100 balls that can be red or black but the proportion is not specified. The state space is  $S = \{0, 1, \dots, 100\}$  where  $s$  corresponds to the state in which exactly  $s$  balls are red. The payments of the DM are described by the set of possible consequences  $X = \{0, x\}$ , with  $x > 0$ .

Let us formalize Ellsberg paradox in the Anscombe-Aumann framework. The bet on red (black) in the unknown urn is modeled through the horse race lottery  $f (f') : S \rightarrow \mathcal{L}$  as follows:  $f(s) = \frac{s}{100}\delta_x + \frac{100-s}{100}\delta_0$  and  $f'(s) = \frac{100-s}{100}\delta_x + \frac{s}{100}\delta_0$  with  $\delta_x$  the degenerate lottery that gives payoff  $x$  with certainty. The bet on red (black) in the known urn is represented through  $g (g')$ :  $g(s) = \frac{1}{2}\delta_x + \frac{1}{2}\delta_0$  and  $g'(s) = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_x$ . Ellsberg shows that, while being indifferent between betting on red or black within each urn ( $f \sim f'$  and  $g \sim g'$ ), DMs may reasonably prefer a bet on red (resp. black) in the known urn rather than a bet on red (resp. black) in the unknown one ( $g \succ f$  and  $g' \succ f'$ ). This preference for known (risky) over unknown (ambiguous) bets is referred to by Ellsberg as ambiguity aversion.

Suppose now that preferences are represented by a subjective probability  $P$  on  $S$  and a utility function  $u$  and assume  $u(0) = 0$  and  $u(x) = 1$ . Thus,  $f \sim f'$  means  $\sum_{s \in S} P(s) \frac{s}{100} =$

$\sum_{s \in S} P(s) \frac{100-s}{100}$  which is equivalent to  $\sum_{s \in S} P(s) \frac{s}{100} = \frac{1}{2}$  meaning  $f \sim g$ . This contradicts  $g \succ f$ .

Ellsberg preference cannot be represented with subjective additive probability. Thus, PS and SEU are violated.

### 1.5.2 One-urn experiment

One urn contains 90 balls; 30 are known to be red and the remaining 60 are known to be black or yellow but the information about the precise proportion is missing. Ellsberg shows that a DM who prefers a bet that gives a positive amount of money on {red} to an identical bet on {black} (i.e., who is ambiguity averse in the same way as in the two-urn experiment), will however prefer a bet that wins on {black and yellow} to a bet that wins on {red and yellow}. Indeed, adding a winning event yellow to the initial bets should not affect the ranking between them (by P2). Choices are again driven by ambiguity avoidance and violate SEU and PS.

Ellsberg's examples show that SEU and Machina and Schmeidler's PS are violated as soon as DMs exhibit ambiguity attitudes. Ambiguity attitudes contradict the Bayesian approach that makes no difference between probabilities that are based on objective information and probabilities that are built on a default in information. As argued by Schmeidler (1989) the main limitation of SEU is that "The probability attached to an uncertain event does not reflect the heuristic amount of information that led to the assignement of that probability." The models presented in the next section provide extension of SEU in order to account for ambiguity attitudes.

## 1.6 The modeling of ambiguity

Ellsberg experiments (1961) rule out the possibility of modeling behavior under uncertainty through consistent probability measures. The aim of this section is to present the models that have been constructed in response to the paradox raised by Ellsberg. Each category of models proposes a different way to solve the Ellsberg paradox. We distinguish four approaches in the modeling of ambiguity.

The rank dependent approach, including Schmeidler (1989) and cumulative prospect theory (Tversky and Kahneman, 1992), abandons subjective probabilities, allowing the probabilities of events to be non-necessarily additive. The multiple prior approach preserves the expected utility form but describes ambiguity through by means of a set of priors. The multi-stage approach represents ambiguity as compound risk where the different stages of the compound lottery are not necessarily reduced. The source of uncertainty approach envisages that each uncertainty subjectively covers different features and consequently, that all uncertainties cannot be treated similarly in the decision process.

### 1.6.1 Choquet expected utility

The point of departure of the modeling of non additive probability is illustrated by the following example provided by Schmeidler (1989). Imagine a DM can bet on two coins. The first coin has been extensively tested and was found to be fair. No information exists regarding the second coin. While the first coin carries enough evidence to be assigned with a fifty-fifty distribution, the second will be naturally assigned the same distribution but by invoking some other rule, typically Laplace's principle of insufficient reason. The two distributions, while being the same, feel different. Actually, the DM tends to consider that the two bets are not equivalent, and he would be willing to bet less on the second coin (Gilboa et al., 2008). Savage's approach fails to distinguish between probabilities based on information and

probabilities that result from the absence of information.

Schmeidler (1989) finds the conditions to extend SEU to non additive probabilities. He keeps Anscombe-Aumann's A1, A3 and A4 and introduces the comonotonic independence axiom and a monotonicity axiom.

**Definition 5** (Comonotonicity). *Two acts  $f$  and  $g \in \mathcal{F}$  are comonotonic if there are no states  $s$  and  $s'$  such that  $f(s) \succ f(s')$  and  $g(s') \succ g(s)$ .*

The following comonotonic independence condition replaces Anscombe-Aumann's A2.

**A'2 (Comonotonic Independence)** For all  $f, g, h \in \mathcal{F}$ , if  $f \succ g$ ,  $f$  and  $g$  are comonotonic with  $h$ , then  $\lambda f + (1 - \lambda)h \succ \lambda g + (1 - \lambda)h$

**A'5 (Monotonicity)** for all  $f, g \in \mathcal{F}$ , if  $f(s) \succeq g(s)$  for all  $s \in S$ , then  $f \succeq g$

We need to introduce the definition of a non additive probability measure (also called capacity):

**Definition 6** (Capacity). *The function  $\nu : S \rightarrow [0, 1]$  is a capacity if*

(i)  $\nu(\emptyset) = 0$ ,  $\nu(S) = 1$  and (ii)  $E \subseteq F \Rightarrow \nu(E) \leq \nu(F)$ .

In the following representation, outcomes of act  $f = [x_1 \text{ on } E_1; \dots; x_n \text{ on } E_n]$  expressed in monetary amounts are rank-ordered from worst to best  $x_1 \leq \dots \leq x_n$ . Note that the ranking of outcomes imply that the partition  $\{E_1; \dots; E_n\}$  of  $S$  is also rank-ordered.

**Theorem 3** (Schmeidler, 1989). *Preferences satisfying axioms A1, A'2, A3, A4, A'5 have the following Choquet Expected Utility (CEU) representation:*

$$CEU(f) = \sum_{j=1}^n \pi(j) u(x_j) \quad (1.3)$$

where the decision weight  $\pi(j)$  is equal to  $\nu(\cup_{i=j}^n E_i) - \nu(\cup_{i=j+1}^n E_i)$ . Moreover, the capacity  $\nu(\cdot)$  on  $S$  is unique and the utility  $u(\cdot)$  is defined up to a positive linear transformation.

The difference between CEU and SEU consists in the weights that precede utility when evaluating an act. The weights  $P(E_j)$  under SEU are generated by an additive probability measure on  $S$  while the weights  $\pi(j)$  under CEU are generated by a possible nonadditive measure. The decision weight  $\pi(j)$  of event  $E_j$  depends on the event  $E_j$  and its ranking position. Note that when  $\nu(\cdot)$  is additive,  $\nu(\cup_{i=j}^n E_j) - \nu(\cup_{i=j+1}^n E_j)$  reduces to  $\nu(E_j)$  and SEU is obtained.

### 1.6.2 Multiple Prior models

The multiple prior approach assumes that the DM may not hold a unique belief on the states of the world. Consequently, ambiguity is reflected by the multiplicity of priors. The extensions of the original Maximin Expected utility (Gilboa and Schmeidler, 1989) consist in:

1. enlarging the attitudes towards ambiguity by allowing a continuum from extreme pessimism to extreme optimism in (i) combining the two extreme behaviors (min and max) or (ii) adding an extraneous ambiguity index.
2. taking into account the prior information available to the DM.

A similar but not equivalent approach consists in considering probability intervals for events. Eventually, multiple priors can be related to possible incomplete preferences.

#### Maxmin expected utility

Gilboa & Schmeidler (1989) establish the axiomatization of Wald's (1950) idea of maxmin expected utility (MEU). The idea is as follows: when no information is available, it may be too difficult for the DM to formulate a unique prior. In a way, it is less demanding to allow the DM to consider a set of priors. This set is subjectively defined. For instance, an extreme case would be to envisage all the possible probability distributions: in Ellsberg's two-urn experiment it would correspond to all the possible compositions of the ambiguous urn.

The axiomatic keeps Anscombe-Aumann's A1, A3 and A4, adds A'5, replaces the independence axiom (A2) by the certainty independence axiom (A"2) and adds an uncertainty aversion axiom (A6).

$\mathcal{F}_C$  is the set of constant acts, i.e, acts that give the same lottery whatever the state is; elements of  $\mathcal{F}_C$  are indexed by  $c$ . A"2 is weaker than A2 since it requires that independence holds whenever acts are mixed with a constant act  $h_c$ .

**A" 2 (Certainty-independence)** for all  $f, g \in \mathcal{F}$  and  $h_c \in \mathcal{F}_C$ , and for all  $\lambda \in ]0, 1]$ ,  
 $f \succ g \Leftrightarrow \lambda f + (1 - \lambda)h_c \succ \lambda g + (1 - \lambda)h_c$

This following axiom captures the hedging phenomena. The DM should prefer a mixture of two indifferent acts to each of these two acts.

**A 6 (Uncertainty Aversion)** for all  $f, g \in \mathcal{F}$  and  $\lambda \in ]0, 1[$ ,  $f \sim g \Rightarrow \lambda f + (1 - \lambda)g \succeq f$

The mixture operation reduces the uncertainty separately born by each act. We observe that this axiom ex ante imposes a constraint on the DM's reaction to ambiguity and thus MEU will be able to describe Ellsberg type behavior.

It is worth remarking that adding this axiom to the CEU theorem results in a nonadditive probability  $\nu$  that satisfies convexity, i.e.,  $\nu(E) + \nu(F) \leq \nu(E \cup F) + \nu(E \cap F)$ . Conversely, if  $\nu$  is convex, then the CEU preference relation satisfies A6.

**Theorem 4** (Gilboa and Schmeidler, 1989). *Preferences satisfying axioms A1, A"2, A3, A4, A'5 and A6 have a the following MEU representation:*

$$MEU(f) = \min_{P \in \mathcal{C}} \sum_{s \in S} P(s)u(f(s)) \quad (1.4)$$

with  $u$  the utility function which is unique up to a positive linear transformation, and  $\mathcal{C}$  the unique (closed and convex) set of priors  $P$ . Uniqueness of  $\mathcal{C}$  is given by A4.

A min expected utility maximizer ranking two acts computes the expected utility of each

act for each prior probability distribution and then chooses the act that yields the highest evaluation with respect to the worst prior distribution. We have seen that *one* prior distribution (the singleton set  $\mathcal{C} = \{P\}$ , which corresponds to SEU) is not able to explain Ellsberg preferences. However, we can see that a *set* of priors explains Ellsberg preferences. Take the case where the set of priors  $\mathcal{C}$  corresponds to  $\mathcal{L}$ . With the usual normalization conditions for utility ( $u(0) = 0$  and  $u(x) = 1$ ), Ellsberg preferences  $f \sim f'$  and  $g \sim g'$  imply  $MEU(f) = MEU(f') = 0$  and  $MEU(g) = MEU(g') = \frac{1}{2}$  and these equalities are completely consistent with  $g \succ f$  and  $g' \succ f'$ . It is worth noticing that the same result would be obtained with a smaller set of priors (for instance, if  $\mathcal{C} = \{\text{priors such that } s \in 48, 49, 50\}$ ).

Maxmin is often viewed as associating the modeling of ambiguity to pessimism because of the presence of the axiom of uncertainty aversion. However, nothing is said about the nature of the set of beliefs that are revealed by the representation theorem. For instance, this set may be assumed to comprise only optimistic probability distributions so that a DM who acts pessimistically relative to his optimistic beliefs may finally behave in a less pessimistic way than a true pessimist would have behaved. In short, behavioral traits that are not necessarily due to ambiguity can be contained in the set  $\mathcal{C}$ .

Moreover, it seems natural to interpret the size of  $\mathcal{C}$  as a representation of the ambiguity that the DM may perceive in the decision problem, but one problem with such interpretation is the fact that the set  $\mathcal{C}$  appears in Gilboa and Schmeidler's analysis only as a result of the assumption of ambiguity hedging. It therefore seems that the DM's revealed ambiguity cannot be disentangled from his behavioral response to such ambiguity. That is precisely what  $\alpha$ -MEU aims to achieve.

### $\alpha$ -MEU

Ghirardato, Maccheroni and Marinacci (2004) provide the axiomatization of the Hurwitz criterion and extend MEU to allow for a more varied descriptions of ambiguity attitudes.  $\alpha$ -MEU combines both the maxmin and its extreme opposite maxmax (where the best probability is considered) approaches. This combination permits us to account for all ambiguity attitudes between maxmin and maxmax.

They distinguish between (i) the ambiguity perceived by the DM, which is given by a set of probabilities  $\mathcal{C}$  and (ii) the reaction of the DM to it, his ambiguity attitude, which is captured by a unique coefficient  $\alpha$ .

They first derive an "unambiguous preference" relation denoted by  $\succsim^*$  from the preferences of the DM that is built on a unanimity criterion: a DM unambiguously prefers ( $\succsim^*$ ) an act  $f$  to an act  $g$  if the expected utility of act  $f$  is higher than the expected utility of  $g$  with respect to *every* probability measure in the set  $\mathcal{C}$ . The set  $\mathcal{C}$  describes the DM's *revealed perception of ambiguity*<sup>1</sup>. A DM 1 perceives more ambiguity than a DM 2 if for all  $f, g$   $f \succsim_1^* g \Rightarrow f \succsim_2^* g$ . When  $\mathcal{C} = \{P\}$  then  $\succsim^*$  is complete and corresponds to  $\succsim$ . Hence, the DM 2 has a richer unambiguous preference because she behaves *as if* she is better informed. Moreover, the size of the set of priors gives information on the ambiguity attitude of a DM. Thus, in the previous case, the DM 1 is more ambiguity averse than the second because she considers that more probability distribution can occur ( $\mathcal{C}_1$  is larger than  $\mathcal{C}_2$ ).

In a second step, ambiguity attitudes are represented through a parameter that captures ambiguity attitudes. The following axiom guarantees that the certainty equivalent of  $f$  with respect to  $\succsim^*$  (noted  $E^*(f)$ ) contains all the information the DM uses in evaluating  $f$ .

**A 7** For every  $f, g \in L$ ,  $E^*(f) = E^*(g) \Rightarrow f \sim g$

**Theorem 5** (Ghirardato, Maccheroni, Marinacci, 2004). *Preferences satisfying axioms A1, A"2,*

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<sup>1</sup>As in MEU, no objective meaning is attached to  $\mathcal{C}$ .



$A3, A4, A'5$  and  $A7$  have the following  $\alpha$ -MEU representation:

$$\alpha - MEU(f) = \alpha \min_{P \in \mathcal{C}} \sum_S u(f(s)) dP(s) + (1 - \alpha) \max_{P \in \mathcal{C}} \sum_S u(f(s)) dP(s) \quad (1.5)$$

Moreover,  $\mathcal{C}$  is unique,  $u(\cdot)$  is unique up to an affine transformation and  $\alpha \in [0, 1]$  is unique if  $\mathcal{C}$  is not a singleton.

The DM's reaction to ambiguity is captured by the ambiguity aversion coefficient  $\alpha$ . The set  $\mathcal{C}$  is shown to be equal to the set of priors that Gilboa and Schmeidler derived in their representation for  $\alpha$  equal to 1. If  $\alpha$  is equal to 0, maxmax is obtained. The set  $\mathcal{C}$  yields the smallest set of possible probability distributions that can be obtained, i.e., the closest approximation of SEU.

Let us show that an ambiguity averse DM in the Ellsberg 90-ball urn experiment has an  $\alpha$  superior to  $1/2$ . Let  $i$  defines the number of yellow balls and  $60 - i$  the number of black balls in the urn.  $i_{min}$  (resp.  $i_{max}$ ) is the minimum (resp. maximum) of  $i$  with respect to the set of beliefs. A DM who prefers a bet on red ( $R$ ) to a bet on yellow ( $Y$ ) or similary black ( $B$ ) reveals (with the normalization conditions  $u(x) = 1$  and  $u(0) = 0$ ):

$$R \succ Y \Rightarrow \frac{30}{90} > \frac{\alpha i_{min}}{90} + \frac{(1 - \alpha) i_{max}}{90}$$

since the worth distribution is  $i_{min}$  and

$$R \succ B \Rightarrow \frac{30}{90} > \frac{\alpha(60 - i_{max})}{90} + \frac{(1 - \alpha)(60 - i_{min})}{90}$$

since the worth distribution is  $i_{max}$ .

By definition  $i_{max} \geq i_{min}$ . Ellsberg type preferences  $R \succ Y$  and  $R \succ B$  imply that  $i_{min} = i_{max}$  is not possible. It follows that  $i_{min} < i_{max}$  then  $\alpha > 1/2$ .

**Variational preferences**

The variational preferences model (Maccheroni, Marinacci and Rustichini, 2006) is an alternative generalization of the multiple prior model.

The axiomatization conserves all MEU's axioms except that the certainty independence (A''2) is replaced by the weak certainty independence axiom (A2''').

**A 2''' (Weak Certainty independence)** if  $f, g \in \mathcal{F}$  and  $h_c, h'_c \in \mathcal{F}_c$ , and  $\lambda \in ]0, 1[$  then,  
 $\lambda f + (1 - \lambda)h_c \succsim \lambda g + (1 - \lambda)h_c \Rightarrow \lambda f + (1 - \lambda)h'_c \succsim \lambda g + (1 - \lambda)h'_c$

The certainty independence axiom actually involves two types of independence: independence relative to mixing with constant acts and independence relative to the weights used in such mixing. A''2 retains the first form of independence, but not the second one. They allow for preference reversals in mixing with constants unless the weights themselves are kept constant.

**Theorem 6** (Maccheroni, Marinacci and Rustichini, 2006). *Preferences satisfying axioms A1, A''2, A3, A4, A'5, A6 are variational and have the following representation:*

$$VP(f) = \min_{P \in D} \int_S u(f(s)) dP(s) + b(P) \quad (1.6)$$

where  $b$  is an ambiguity aversion index from  $D(\Sigma)$  to  $(0, +\infty)$  where  $D(\Sigma)$  is the set of all probability distributions on  $\Sigma$ , an algebra over  $S$ .

Variational preferences are ambiguity averse due to A6. The lower is  $b$ , the higher is the ambiguity aversion exhibited by the DM.  $b$  associates a weight to each probability distribution  $P$ . A relation  $\succsim_1$  more ambiguity averse than  $\succsim_2$  is equivalent to  $(u_1 = u_2 \text{ and } b_1 \leq b_2)$ . MEU with the set of priors  $\mathcal{C}$  is a special case of the variational preferences where  $b(P) = 0$  if  $P \in \mathcal{C}$  and  $b(P) = \infty$  otherwise.

### Ambiguity as imprecise information

In the previous models, nothing is said about the informational structure of the decision problem and the models themselves do not envisage the case where the DM possesses objective information in the sense where it does not explicitly appear in their construction. Informational aspects, if they exist, appear in the revealed set of beliefs as an output of the decision process. However, in most decision situations data is often available to the DM even vague or imprecise.

For instance, in Ellsberg's three-color problem, the prior information is the set of probability distributions that admit the probability  $1/3$  on the event 'drawing a red ball'. Thus, the DM has information since the probability interval for drawing a black or yellow ball is smaller than the unit interval that would correspond to full ambiguity. Gajdos, Hayashi, Tallon and Vergnaud (2008) specifically aim at modeling information as a part of ambiguity.

They describe the informational structure of a decision context by  $P$  that represents the objective a priori information and  $r$  a reference prior also called anchor ( $r$  belongs to the convex hull of  $P$ ). For instance, in the three-color problem, the set of priors appropriate to model available information is the set of all probability distributions that place  $1/3$  on red. The reference prior of this set is the natural distribution  $(1/3, 1/3, 1/3)$  (by symmetry).

The concept of reference prior is central in this model since aversion towards imprecision is built around it. Indeed, one situation is considered more imprecise than another if the set of probability distributions considered possible in the second situation is included in the set of the first. In Ellsberg's two-color example, having one red and ninety-nine black is more precise than no information about the proportion of the two colors, but it seems reasonable to assume that a DM would prefer to bet on red in the ambiguous urn than in the more precise known urn. Thus, a proper description of information requires a condition on anchor.

They define  $[P_1, r_1]$  as a weak center preserving increase in imprecision of  $[P_2, r_2]$  if (i)  $P_1$

is more imprecise than  $P_2$  and (ii)  $r_1 = r_2$  (they have the same center). For instance, in the two-color problem, if there are two balls in the ambiguous urn, the information is described by  $P_1 = \{(1,0), (0,1)\}$  and the center is  $(1/2, 1/2)$ ; if there are three balls in the ambiguous urn, the information  $P_2 = \{(1,0), (2/3, 1/3), (1/3, 2/3), (0,1)\}$  has the same center  $(1/2, 1/2)$ . Hence, the imprecision of the two situations is the same (the number of balls considered as immaterial).

In this model, the DM's preferences are defined on both act and information. The representation theorem allows two acts to be compared in two different informational situations.

$$\begin{aligned}
 (f, [P_1, r_1]) \succeq (g, [P_2, r_2]) &\Leftrightarrow \\
 \alpha \min_{p \in co(P_1)} \int_S u(f(s)) dp_1(s) + (1 - \alpha) \int_S u(f(s)) dr_1 &\geq \\
 \alpha \min_{p \in co(P_2)} \int_S u(g(s)) dp_2(s) + (1 - \alpha) \int_S u(g(s)) dr_2 &
 \end{aligned} \tag{1.7}$$

The functional form is a convex combination of the minimum expected utility with respect to probability distributions in the set of objectively admissible probability distribution and of the expected utility with respect to the anchor. The revealed set is a subset of the set of admissible probability distributions. Hence, an extremely pessimistic DM will keep the entire initial set of admissible priors. Conversely, a DM not affected by imprecision reduces any prior set of probability distribution to the anchor distribution.  $\alpha$  measures the degree of pessimism. If  $\alpha$  is equal to zero, the DM is EU with respect to the anchor, and if  $\alpha$  is equal to one, he is MEU with respect to all distributions compatible with information.

### **Ambiguity and indecision**

For many authors, (Mandler 2005, among others), the completeness assumption - which requires that any two acts are comparable (that is preferred to the other or equivalent in the

preference ordering) - does not appear to be a condition of rationality. In certain cases, it is even rational not to decide as argued by Gilboa, Maccheroni, Marinacci and Schmeidler (2008):

"In the absence of information, it appears more rational to be silent than to pretend to have knowledge that one does not have".

Hence, a DM may stay indecisive when ambiguity is unbearable. While their paper deals with the incompleteness of beliefs defined as the incompleteness of preferences that is due to the absence of information (the DM does not know what the probabilities of various states of the world are), they do not especially focus on ambiguity. However, a previous work of Bewley could shed some light on the possible link between indecision and ambiguity.

Bewley (2002) removes the completeness assumption from the Anscombe and Aumann formulation of Savage's theory and introduces an extraneous inertia assumption. The inertia hypothesis states that a person never accepts an uncertain alternative unless he prefers acceptance to rejection. Inertia implies the determination of a status quo. The status quo is the position that serves as comparison with another alternative. The inertia assumption is necessary since it avoids the arbitrary selection between incomparable alternative, and in this sense, contains a form of aversion to uncertainty. Bewley obtains a set of subjective probability distributions  $\mathcal{C}$  and his decision rule requires the unanimity of priors as shown by the following equation.

$$f \succ g \Leftrightarrow \sum_{s \in S} p(s)u(f(s)) > \sum_{s \in S} p(s)u(g(s)) \quad \forall p \in \mathcal{C} \quad (1.8)$$

$f$  is more desirable than  $g$  if and only if the expected utility is higher for  $f$  than for  $g$  when considering each and every prior in the set  $\mathcal{C}$ . Bewley's model is not considered as a response to Ellsberg paradox since it cannot per se predict Ellsberg preferences, but it is still compatible

with Ellsberg type behavior.

### **Others approaches**

In parallel research, ambiguity is assimilated to probability intervals. See for instance Budescu and Wallsten (1987) for a review of this literature. Because it is essentially psychological, no behavioral foundation underlies this approach. In the statistics field, uncertainty is also modeled through imprecise probability (see Walley, 1991). Note that approaches based on probability intervals are mathematically different from the multiple prior ones (Wakker , 2008).

### **Limitations**

Multiple prior models present certain limitations especially related to the problematic observability of the set of priors (Wakker, 2008). First, specifying the set of probability distributions is more complex than specifying only one distribution. Until now, no assessment of the set of priors exists in empirical literature which tends to confirm the robustness of this limitation. An underlying difficulty related to the previous point is the above mentionned distinction between probability measures that are contained in the set (subjectively considered as possible) and those which are not (subjectively considered as impossible). Because the underlying decision process is selective, the sole observation of choice behavior is not sufficient to infer the hidden set of priors the DM had in mind.

Finally, by construction, multiple priors models do not allow one to consider the plausible cognitive state in which different probabilities are assigned to the different priors of the set. An approach that envisages such a mental procedure is not far from the two-stage models we present below.

### 1.6.3 Multiple Stage models

The point of departure of multi-stage models is that DMs have a representation of uncertainty in terms of compound risk and ambiguity attitudes are modeled through the non reduction of multi-stage lotteries. This line of research has been initiated by Segal (1987) who assumes that a DM formulates subjective probability over the possible probability distributions in the second stage but does not impose the reduction of compound lotteries. A DM satisfies the reduction of compound lotteries (ROCL) if her preferences depend only on the probability of the final outcomes in a multi-stage lottery. DM who satisfies ROCL multiplies the probabilities of the consequences between the different stages according to classic probability laws.

In the Ellsberg two-urn experiment, having a multi-stage representation of uncertainty corresponds to having a belief over of the set of all the possible compositions of the ambiguous urn.

#### Klibanoff, Marinacci and Mukerji (2005)

Preferences characterized by Klibanoff, Marinacci and Mukerji (2005) are represented by a functional of double expectational form where the DM has a SEU preference on the set of objective probability distributions.

$$KMM(f) = \sum_D \Phi \left( \sum_S u(f(s)) dP(s) \right) d\mu \quad (1.9)$$

The functional form allows distinguishing between ambiguity, that refers to the properties of the DM's subjective beliefs  $\mu$  and ambiguity attitude contained in  $\Phi$ .  $\mu$  is the subjective prior over  $D$ , the set of probability measures over  $S$  considered by the DM.  $\mu$  can be thought as a measure of the subjective relevance of a particular  $P(\in D)$  to be the "right" probability. ROCL corresponds to  $\Phi$  linear. Concavity of  $\Phi$  corresponds to ambiguity aversion: each expected

utility with respect to a given probability measure  $P$  is reduced. Ambiguity aversion is thus equivalent to aversion to mean preserving spread.

An advantage of KMM is that it allows for smooth rather than kinked preferences; thus comparative statics should be used to study the effects of a change in ambiguity or ambiguity attitude *ceteris paribus*.

### **Seo (2008)**

Seo (2008) provides a characterization of second-order subjective expected utility (SOSEU) in an Anscombe-Aumann framework. He shows that the Anscombe-Aumann approach implicitly assumes a reversal of order property according to which a DM is indifferent between mixtures of acts before and after a realization of a state. He points out that mixing, before or after the state is realized, results alternatively in ambiguous or non ambiguous prospects. Relaxing the reversal of order axiom permits us to accommodate Ellsberg type behavior.

### **Halevy & Ozdenoren (2008)**

Halevy and Ozdenoren (2008) define preferences of DMs on a domain that includes both Savagean acts and compound lotteries. They define second-order probabilistic sophistication (SOPS) that generalizes PS to two-stage lotteries, and allows for behaviors that are non-neutral with respect to ambiguity. DM's preferences satisfy SOPS if there exists a set of probability measures over the state space, and a single probability measure over the measures in this set (representing the DMs belief over the measures), such that the DM is indifferent to acts that induce the same compound lottery. Hence, DMs who satisfy SOPS reduce all uncertainty to compound risk. Moreover, DMs who reduce all uncertainty to simple risk are PS.



### **Ergin & Gul (2009)**

Ergin & Gul (2009) construct a theory in which a DM identifies an uncertain prospect with subjective compound lotteries. They introduce the issue preference assumption according to which a DM may not be indifferent among gambles that yield the same probability distribution if they depend on different issues. They assume two issues in an Ellsbergian framework: the first issue concerns the uncertainty regarding the identity of the ball drawn; the second issue concerns the uncertainty about the composition of the urn. They assume the second issue is resolved first. A DM who is indifferent between acts that imply one or both issues treats the uncertainty as risk and is second order probabilistically sophisticated. Hence, Ellsberg paradox is a consequence of a greater aversion to the risk associated with the composition of the urn, also called second order risk (which resolves the first stage of a compound lottery).

### **1.6.4 Sources of uncertainty**

A series of contributions admit there can be source preference (Tversky and Wakker, 1995; Tversky and Fox, 1995). Source preference argues that behavior cannot be exclusively explained on the basis of likelihood and payoff considerations. The events that bear the uncertainty are themselves a source of preference. Hence, distinguishing between sources of uncertainty allows us to envisage DMs' attitudes as source-dependent.

The two following approaches aim to endogenously determine domains of events (sources) that are consistent with probabilistic sophistication.

### **Epstein and Zhang (2001)**

Epstein and Zhang's (2001) approach consists in identifying events on which the DM is probabilistically sophisticated (called unambiguous events) from those on which he is not (called ambiguous). They propose the following preference-based definition of an unambiguous

event:

**Definition 7** (Unambiguous event). *An event  $T$  is unambiguous if for all  $E$  and  $F$  in  $\text{non } T$ , acts  $f$ , and outcomes  $x, x^*, y, y' \in X$ ,*

(i)  $[x^* \text{ on } E; x \text{ on } F; f(s) \text{ on } \text{non } T - (E \cup F), y \text{ on } T] \succsim [x \text{ on } E; x^* \text{ on } F; f(s) \text{ on } \text{non } T - (E \cup F), y \text{ on } T] \Rightarrow [x^* \text{ on } E; x \text{ on } F; f(s) \text{ on } \text{non } T - (E \cup F), y' \text{ on } T] \succsim [x \text{ on } E; x^* \text{ on } F; f(s) \text{ on } \text{non } T - (E \cup F), y' \text{ on } T],$

(ii) *The condition (i) is also true when replacing  $T$  by  $\text{non } T$ .*

They end up with a unique source of "subjectively unambiguous events" consistent with probabilistic sophistication.

### Source dependence and small worlds

This approach goes further than Epstein and Zhang's mere distinction between un/ambiguous events and, contrary to the latter, allows to accomodate Ellsberg two-urn paradox.

Chew and Sagi (2008) use PS based on exchangeability of events (Chew and Sagi, 2006) to define self-contained groups of events. So-called *small world events domains* each modelizes a single and subjectively determined source of uncertainty. Each small world events domain corresponds to a subjectively distinct source of uncertainty. Hence, their approach allows them to model behavior as source dependent. Hence, Ellsberg behavior is interpreted as a difference of attitudes between two small worlds (the risky and the ambiguous urn) on the condition that PS holds within each source. However, PS is not necessarily required between small worlds.

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## Chapter 2

# Sources of Uncertainty and Ambiguity

## Attitudes

### 2.1 Introduction

A *source of uncertainty* refers to a group of events that are generated by a common mechanism of uncertainty. Classic modeling of subjective uncertainty makes the implicit assumption of *source neutrality*, i.e., a decision maker (DM) is indifferent to the mechanism that generates the uncertainty. Subjective expected utility (SEU: Savage 1954) reduces all uncertainties to risk by supposing each DM can formulate subjective probabilities in any decision context. Probabilistic sophistication (PS: Machina and Schmeidler, 1992) imposes that a DM ought to be indifferent between decisions that imply the same subjective probability distribution on consequences, independently of the mechanisms that govern the resolution of the uncertainty. In these models, subjective probabilities share the same strong features: first, they have to be additive; second, they are not corrected by the (possible absence of) objective information on which they are based, neither by the subjective degree of confidence they are associated with.

However, literature provides substantial evidence in favour of *source preference*. A DM

exhibits a preference for source  $S_a$  over source  $S_b$  when she is indifferent between a gamble that gives a positive payoff on an event  $a$  belonging to  $S_a$  and a gamble giving it on an event  $b$  belonging  $S_b$ , and has an higher evaluation of the gamble that gives the same payoff on  $S - a$  than the one giving it on  $S - b$  (Tversky and Fox, 1995; Tversky and Wakker, 1995). For instance, DMs may prefer bets on events generated by a chance mechanism over events associated with no information, revealing ambiguity aversion (Ellsberg, 1961); conversely, DMs may prefer bets on events for which they feel more competent, to bets involving pure chance, exhibiting ambiguity loving (Heath and Tversky, 1991). Hence, ambiguity attitudes can be envisaged as a result of non neutrality towards sources of uncertainty.

More specifically, this chapter examines ambiguity attitudes as a consequence of the presence of two basic sources of uncertainty: (i) *source of objective uncertainty*, which refers to the part of the decision problem that is formulated in terms of exogenous probabilities; the mechanism generating the uncertainty is objective because it depends on an extraneous chance device (like casino games) and (ii) *source of subjective uncertainty*, which refers to the part of the decision problem which deals with events, i.e., eventualities that do not contain enough information or give rise to enough confidence to be definitely formulated in terms of probabilities.

A DM who respects source neutrality is indifferent between the two above sources and is said to be ambiguity neutral<sup>1</sup>. While it is admitted that DMs are generally neutral to different sources of objective uncertainty (the throw of a coin or the draw of a ball in a fifty-fifty two-color urn), DMs need not to be neutral between different sources of subjective uncertainty. It is worth noticing that while sources of subjective uncertainty can, most of the time, be clearly identified by an outside observer (for instance, "the temperature tomorrow" versus

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<sup>1</sup>Literature proposes two alternative definitions of an ambiguity neutrality. Being ambiguity neutral is equivalent to SEU according to Ghirardato and Marinacci (2002); being ambiguity neutral is equivalent to PS according to Epstein and Zang (2001).



"the evolution of the Nikkei"), identifying more subtle sources is ultimately a subjective affair: an american expert of the stock market will consider "the evolution of the Dow Jones" and "the evolution of the Nikkei" as two distinct sources of uncertainty while an european expert will probably not operate such a fine distinction and will indistinctly consider "the evolution of the foreign stock market" in general.

In this chapter, we show how paradoxes of behavior under uncertainty are due to non neutrality towards sources of uncertainty. The examples presented in this chapter show that the joint presence (juxtaposition or combination) of both objective and subjective sources of uncertainty in decision contexts generates (non neutral) ambiguity attitudes, attitudes that are precisely at the incipience of the paradoxes<sup>2</sup>. We distinguish between two majors contributions that are particularly relevant for the analysis of ambiguity attitudes, namely the Ellsberg (1961) and Machina (2009b) paradoxes.

### **Bringing ambiguity attitudes to light through separate sources of uncertainty ...**

In his two-urn example, Ellsberg (1961) *juxtaposes* a source of objective uncertainty (represented by an 100-ball urn with a fifty-fifty proportion of black and red balls) and a source of subjective uncertainty (represented by a similar urn with no proportion prespecified) and simply highlights the existence of ambiguity attitudes, i.e., a preference for the former over the latter. This example is particularly convenient since it enables us to define ambiguity attitudes in a clear-cut and comparative manner: two *separate* sources of uncertainty - objective and subjective - are sufficient for the existence of ambiguity attitudes. With regards to the modeling of uncertainty, the two-urn paradox provides the first falsification<sup>3</sup> of SEU and

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<sup>2</sup>The Allais paradox (1953), which exclusively deals with objective uncertainty, is consequently excluded from this chapter.

<sup>3</sup>The term 'falsification' has to be understood in its Popperian meaning: a theory can be falsified if at least one example can be found that cannot be explained by the theory. Hence, 'falsification' does not endorse a pejorative meaning; on the contrary, the (experimental) possibility of falsifying theories is a guarantee of their scientism. Conversely, validating theory would be neither constructive nor scientific from a methodological perspective.

the ex ante falsification of Machina and Schmeidler's PS. The discovery of the existence of ambiguity attitudes has initiated the development of a new range of models sufficiently sophisticated to account for it, such as for instance Choquet Expected Utility (CEU: Schmeidler, 1989).

Section 2.2 focuses on the existence of ambiguity attitudes as resulting from the presence of separate sources of uncertainty; it is shown that non neutral ambiguity attitudes contradict classic models of subjective uncertainty; different interpretations of the existence of ambiguity attitudes are reviewed.

### **...and playing with them via mixed sources of uncertainty**

Ellsberg (1961) and Machina (2009b) one-urn examples are more complex because they entail a salient informational structure: they *combine* objective and subjective uncertainties. The resulting *mixed source* is a good candidate for ambiguity attitudes since the nature of certain events or group of events can be modified (from objective to subjective and vice versa) through simple manipulations of consequences. The key point is that such manipulations are permitted by the models that these examples end up by contradicting. Ellsberg's one-urn paradox provides the simplest framework to explain outcome-manipulation effects, however it does not reveal more than the already known violations of SEU and PS.

A contrario, Machina's one-urn paradox provides a falsification of CEU<sup>4</sup>. Unexpectedly, as soon as more complex mixtures between sources are envisaged, ambiguity attitudes fail to be correctly predicted by ambiguity models.

Section 2.3 focuses on ambiguity attitudes in mixed sources problems and shows how paradoxes become ad hoc when mixed sources are properly separated.

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<sup>4</sup>It has been proved that maxmin expected utility, alpha-maxmin expected utility, the variational preference and the smooth model of ambiguity are also falsified by Machina's example (Baillon et al., 2009).

### Notations

Preferences are described as usual with  $\succsim$  designating the weak preference;  $\succ$  is the strict preference and  $\sim$  the indifference ( $\precsim$  denotes the reverse preference); preferences are defined on bets. For decisions implying a source of objective uncertainty, bets are entirely identified by the probability distribution  $P = (\dots, p_i, \dots)$  on payoffs  $(\dots, x_i, \dots)$  with  $x_i \geq 0$ ; the corresponding lotteries are designated by brackets  $(\dots; x_i, p_i; \dots)$ . For decisions implying a source of subjective uncertainty, bets are identified by acts that associate payoffs  $(\dots, x_i, \dots)$  to a partition of events  $\{\dots, E_i, \dots\}$ ; such subjective acts are designated by brackets  $[\dots; x_i \text{ on } E_i; \dots]$ . For decisions implying a mixed uncertainty,  $[\dots; (\dots; x_i, p_i; \dots) \text{ on } E_j; \dots]$  denotes a subjective act with lotteries as consequences (an Anscombe-Aumann act) while  $(\dots; [\dots; x_i \text{ on } E_i; \dots], p_j; \dots)$  denotes a lottery with subjective acts as consequences.

## 2.2 Separate sources of uncertainty

There has been much debate about the nature of probability (Hacking, 1975), especially as regards the distinction between objective probabilities which are based on empirical frequency or mathematical calculus, and subjective probabilities which are based on another, less reliable kind of evidence. Ellsberg (1961) provides a design that juxtaposes two sources of uncertainty - objective and subjective - and shows a preference for objective probabilities that result in DMs exhibiting ambiguity attitudes. Throughout this section, interpretations of ambiguity attitudes and implications for the modeling of behavior are discussed.

### 2.2.1 Ellsberg two-urn paradox

The existence of ambiguity attitudes is simply highlighted in Ellsberg's two-color paradox. A DM faces the two urns represented in Figure 2.1. The known urn (K) has a known composition

## 2.2. SEPARATE SOURCES OF UNCERTAINTY

of 50 red (R) and 50 black (B) balls. The unknown<sup>5</sup> urn (U) contains 100 red and black balls in an unknown proportion. One ball is randomly drawn from each urn. We designate by  $C_S$  the event "the ball drawn from urn  $S$  is of color  $C$ ",  $S \in \{K, U\}$ .

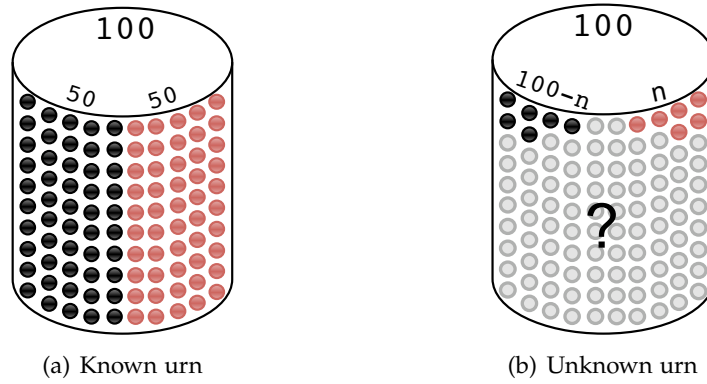


Figure 2.1: Separate sources of uncertainty

The DM chooses between the four bets of Table 2.1. The bet  $e_1$  ( $e_2$ ) gives the payoff 100 on event  $R_K$  ( $B_K$ ) and nothing otherwise. The bet  $e_3$  ( $e_4$ ) gives the payoff 100 on event  $R_U$  ( $B_U$ ) and nothing otherwise.

	50 balls			100 balls	
	red	black		red	black
$e_1$	100	0	$e_3$	100	0
$e_2$	0	100	$e_4$	0	100

Table 2.1: Separate objective and subjective bets

Ellsberg suggests that the DM may reasonably be indifferent between a bet on red and a bet on black when considering each urn separately ( $e_1 \sim e_2$  and  $e_3 \sim e_4$ ). However, he may also reasonably express a strict preference for a bet on red (resp. black) in the known urn over a bet on red (resp. black) in the unknown urn, i.e.,  $e_1 \succ e_3$  (resp.  $e_2 \succ e_4$ ).

<sup>5</sup>In literature, this urn is often termed 'ambiguous'. However, due to the absence of unanimity with regards to what is called "ambiguity", we prefer to focus on ambiguity attitudes as a result of the presence of different sources of uncertainties. For instance, it is difficult to refer to this urn as ambiguous because it is not perceived as different from the other by an ambiguity neutral DM.

The preference  $e_3 \sim e_4$  can be written  $[100 \text{ on } R_U; 0 \text{ on } B_U] \sim [0 \text{ on } R_U; 100 \text{ on } B_U]$ . Under subjective expected utility, this indifference implies the following equality:  $P(R_U)u(100) + P(B_U)u(0) = P(B_U)u(100) + P(R_U)u(0)$  where  $u$  is the utility of the consequences and  $P(\cdot)$  the subjective probability measure. With the standard normalization conditions  $u(100) = 1$  and  $u(0) = 0$  it follows that  $P(R_U) = P(B_U)$ : the DM reveals that events  $R_U$  and  $B_U$  are equally likely.

However, when considering cross-choice between bets on the two urns, the preferences  $e_1 \succ e_3$  and  $e_2 \succ e_4$  rewritten as follows:  $[100 \text{ on } R_K; 0 \text{ on } B_K] \succ [100 \text{ on } R_U; 0 \text{ on } B_U]$  and  $[100 \text{ on } B_K; 0 \text{ on } R_K] \succ [100 \text{ on } B_U; 0 \text{ on } R_U]$  respectively implies under SEU,  $\frac{1}{2} = P(R_K) > P(R_U)$  and  $\frac{1}{2} = P(B_K) > P(B_U)$ . These inequalities reveal that the subjective probability of drawing a red ball in the unknown urn is less than the subjective probability of drawing a red ball in the known urn, and, at the same time, that the subjective probability of drawing a black ball in the unknown urn is less than the subjective probability of drawing a black ball in the known urn. The paradox arises because the sum in terms of subjective probability of events  $B_U$  and  $R_U$  is less than one ( $1 > P(R_U) + P(B_U)$ ), although  $B_U$  and  $R_U$  are complementary events.

The preference for bets implying known probabilities over bets implying unknown probabilities was termed "ambiguity aversion" by Ellsberg. Ambiguity averse preferences ( $e_1 \succ e_3$  and  $e_2 \succ e_4$ ) violate additivity in probability, hence SEU and PS.

Becker and Brownson (1964), Slovic and Tversky (1974) and MacCrimmon and Larsson (1979) find strong support for ambiguity avoidance. Ambiguity avoidance is also confirmed for sophisticated subjects (Hogarth and Kunreuther, 1989) as well as in experimental market settings (Sarin and Weber, 1993). It is worth noticing that ambiguity loving, as well as ambiguity neutral, behavior exists, although minority. Moreover, ambiguity attitudes are found to depend on the likelihood of events; DMs tend to be ambiguity lovers for events associated

with small likelihood levels and ambiguity averse for events associated with middle and high levels of likelihood (Hogarth and Einhorn, 1990; Abdellaoui et al., 2009a).

### 2.2.2 Common interpretations

The diversity of interpretations with regards to the existence of ambiguity attitudes may be viewed as reflecting the heterogeneity of its causes. It also corresponds to a variety of ways to incorporate ambiguity attitudes in the modeling.

#### Interpretation in terms of missing information

This interpretation explains ambiguity attitudes in terms of a difference in completeness of the information on which subjective probabilities are based. Frish and Baron (1988) highlight that the information contained in the unknown urn *could* be known. Hence, ambiguity aversion is simply due to the awareness that there is some missing information that would be relevant for the decision. The underlying dynamic approach lies in the previous 'would'. Indeed, contrary to the objective, subjective probability is not immune to revision and could be reformulated on the basis on new information.

The theoretical limitation pointed out by this interpretation is that subjective probabilities, as defined in SEU, do not incorporate the amount of evidence that leads to their formulation. As noticed by Peirce (1932):

"To express the proper state of belief, not one number but two are requisite, the first depending on the inferred probability, the second on the amount of knowledge on which that probability is based."

Hence, as a Bayesian theory of choice, subjective expected utility does not take the weight of evidence into account (Cohen, 1977) and only considers the subjective probability given what it is known.

### Interpretation in terms of a two-stage representation of uncertainty

The two-stage interpretation of ambiguity attitudes is based on the assumption that ambiguity can be cognitively assimilated to compound risk. Compound risk designates situations in which the consequences of a lottery are themselves lotteries. Segal (1987) defined an ambiguous lottery in the following way:

"The ambiguous lottery [ $x$  on  $E$ ; 0 on non  $E$ ] (ambiguous in the sense that the individual does not know the probability of  $E$ ) should be considered as a two-stage lottery, where the first, imaginary stage, is over the possible values of the probability of  $E$ ."

Ambiguity attitudes stem from the non reduction of compound lotteries principle (Segal, 1987; Klibanoff, Marinacci and Mukerji, 2005; Halevy and Ozdenoren, 2008). The *reduction of compound lottery principle* states that a two-stage lottery and the single-stage lottery obtained by the former through the conventional laws of probability calculus are equal in preference. Intuitively, the reduction of the two stages may not hold since a different source of uncertainty is at play at each stage, as will be discussed below.

As an illustration, let us consider a  $(r+b)$ -ball urn which contains  $r$  red balls and  $b$  black balls and define  $\text{urn}(r, b)$  as the lottery that gives the payoff 100 if a red ball is drawn (with probability  $r/(r+b)$ ) and nothing otherwise (with probability  $b/(r+b)$ ). Figure 2.2. describes different two-stage representations of ambiguity.

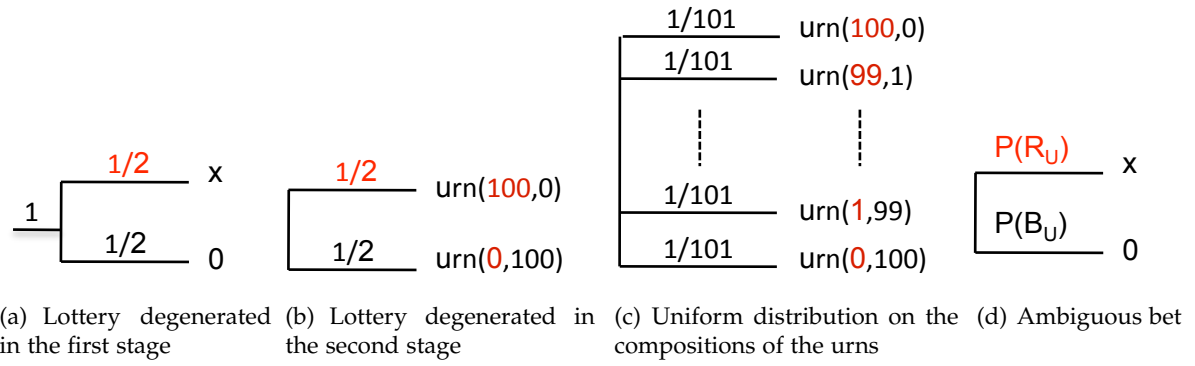


Figure 2.2: Examples of two-stage representations of ambiguity

Lottery (a) is a compound lottery which gives with certainty the lottery with a fifty-fifty chance of getting a red ball; because all the uncertainty is resolved in the second stage, this lottery is said to be degenerated in the first stage. Lottery (b) is a compound lottery that gives a full red urn or a full black urn with fifty-fifty chance; because all the uncertainty is resolved in the first stage, this lottery is said to be degenerated in the second stage. Lottery (c) is a compound lottery that assigns a uniform distribution over all the possible compositions of the unknown urn. The lotteries (a), (b) and (c) should be equivalent in terms of preference for an DM who respects the reduction of compound lottery principle.

(d) represents the ambiguous bet without any probability of events specified. A DM who identifies ambiguity with compound risk, and who is furthermore ambiguity neutral, should consider lotteries (a), (b) and (c) equivalent to the ambiguous lottery (d) in terms of preference.

DMs who do not reduce compound lotteries exhibit ambiguity attitudes. For instance, ambiguity aversion may result from an underevaluation of the second stage in lottery (c) leading to reject it in favor of its equivalent simple risk lottery  $(x, 1/2; 0, 1/2)$ .

When considering the potential complexity of the reduction operation, failures in reduction are possibly more frequent for more complex compound lotteries. Hence, a DM who does not reduce compound lottery may exhibit an increasing ambiguity aversion while he perceives a higher dispersion of outcomes/probabilities in his two-stage representation of



ambiguity. Conversely, the desirability of a two-stage lottery increases as the two stages become more degenerate. For instance (a) and (b) may be associated to less ambiguity aversion than (c).

However, it is worth noticing that the first stage, because "imaginary" (it could be for instance (a) for one DM and (b) for another), is not necessarily a uniform probability distribution on all the eventualities of the second stage, as a Bayesian DM would have imagined (c); hence, the first stage is of subjective nature while the second stays objective. Consequently, ambiguity attitudes stem more from a non reduction between the two sources of uncertainty than from a mere violation of the reduction of compound lotteries principle. DMs might reduce compound objective lotteries but they also may not want to reduce a compound lottery that entails an objective and a subjective part.

Halevy (2007) finds that individuals who correctly reduce compound lotteries are neutral to ambiguity and that failures in reduction are associated with non neutral ambiguity attitudes. In Chapter 4, we proceed to a further examination of this link. We confirm the link between attitude toward ambiguity and attitude toward compound risk, but observe that even DMs who respect the reduction axiom tend to exhibit ambiguity attitudes. Hence, ambiguity attitudes cannot be entirely explained by the inability of DMs to reduce compound lotteries.

### **Uniform sources interpretation**

The interpretation based on uniform sources argues that ambiguity attitudes may be consistent with PS in the two-urn paradox. As argued by Keynes (1921), who considered a similar Ellsberg two-urn problem few years ago, DMs a priori believe that the proportion between black and red in each urn are the same.

"In the first case we know that the urn contains black and red balls in equal proportions; in the second case the proportion of each color is unknown, and each

ball is as likely to be black as red. It is evident that *in either case* the probability of drawing a red ball is  $1/2$ , but that the weight of the argument in favor of this conclusion is greater in the first case.”

The observation that DMs are willing to exchange bets on different colors within a single urn, but reluctant to exchange bets on the same color between the two urns, suggests that events pertaining to the unknown urn ( $R_U$  and  $B_U$ ) share different features compared to events belonging to the known urn ( $R_K$  and  $B_K$ ). Consequently, there is no objective reason for comparing attitudes that rely on different sources as emphasized by Fellner (1961) :

“I will suggest, as does Ellsberg, that subjective probability judgements relating to various processes are not strictly comparable.”

Intuitively, each urn constitutes a source of uncertainty. A source is defined as *uniform* if probabilistic sophistication entails a uniform degree of ambiguity within the source (Wakker, 2008). Probabilistic sophistication based on the concept of exchangeability can be defined within the unknown urn. Exchangeability has been shown to be (together with two other technical conditions) sufficient for probabilistic sophistication (Chew and Sagi, 2006). Two disjoint events  $E_1$  and  $E_2$  are exchangeable if the preference does not change when permuting the payoffs between these two events. A partition of exchangeable events results in equal likelihood. Exchangeability in Ellsberg’s two-urn paradox corresponds to  $[x \text{ on } R_U, y \text{ on } B_U] \sim [y \text{ on } R_U, x \text{ on } B_U]$  and results in equally likely events  $P(R_U) = P(B_U) = 1/2$ . Then, source functions capture ambiguity attitudes with respect to these subjective probabilities revealed in the ambiguous urn. Chapter 3 details this approach.

Ellsberg type preferences no longer constitute a violation of probabilistic sophistication. Let us recall that Machina and Schmeidler (1992)’s PS implies that a DM ought to be indifferent between two bets that imply the same probability distribution. With the generalization

of probabilistic sophistication (its restriction to each source of uncertainty) provided by Abdellaoui et al. (2009a), a DM ought to be indifferent between two bets that imply the same probability distribution only if these bets are restricted on a single and uniform source of uncertainty. Hence, the so-called *within-source probabilistic sophistication* requires that probabilistic sophistication holds within a well-defined source but not necessarily between different sources of uncertainty.

The distinction between sources was recently supported by neurological studies that show that risk and ambiguity pertain to two distinct neuronal processes. Huettel et al. (2006) provided fMRI evidence and noticed that:

"This novel double dissociation indicates that decision making under ambiguity does not represent a special, more complex case of risky decision making; instead, these two forms of uncertainty are supported by distinct (neuronal) mechanisms."

### 2.2.3 Psychological causes of ambiguity aversion

Psychologists have shown that ambiguity attitudes may have subtle psychological causes. Notably, ambiguity attitudes have been found to vary with (i) social factors: exposure of one's choice to others, the impact of others' comparative knowledge and (ii) the way the decision problem is framed: the lack of information is rendered more obvious by a direct or a sequential comparison with the risky context.

#### **Social factors**

*The competence hypothesis.* Heath and Tversky (1991) showed that individuals prefer to bet in a context in which they consider themselves knowledgeable or competent than in a context in which they feel ignorant or uninformed. Hence, ambiguity aversion increases with the perception that others are more competent or more knowledgeable. Information about the

composition of the ambiguous urn is knowable but unknown to DMs; they consequently feel less knowledgeable and less competent, which reduces the attractiveness of the ambiguous bet. Conversely, people are more willing to bet on their own judgment over an equiprobable chance event when they consider themselves as knowledgeable.

***Unknown vs Unknowable uncertainties.*** Chow and Sarin (2002) suggest that DMs do not treat all forms of uncertainty in the same way. They establish a distinction between unknown uncertainty and unknowable uncertainty. A DM faces *unknown uncertainty* when information is available to someone else and *unknowable uncertainty* when information is available to nobody (for example, an unopened bag of candies). The authors conclude from their experiment that "People are likely to prefer, *ceteris paribus*, the known uncertainty to the unknown uncertainty with the unknowable uncertainty somewhere in-between". In the unknown case, one tends to feel ignorant as compared to others, and therefore less confident in one's choice. By contrast, in the unknowable case, one does not feel a particular information disadvantage. In this respect, Chow and Sarin's approach is consistent with the comparative ignorance hypothesis. Thus, the Ellsberg context can be viewed as the better case for obtaining ambiguity aversion since the experimenter has private information (she filled the urn).

***Fear of negative evaluation by others.*** Issues of blame, responsibility and regret are particularly salient under ambiguity. Such effects arise because information becomes available after a decision is made: an individual is often evaluated by others as if she should have known the information, even if it was not actually available at the time of the decision (Baron and Hershey, 1988). As an application of the competence hypothesis, Curley, Yates and Abrams (1986) find that increasing the number of people watching a decision enhances ambiguity aversion. The idea is as follows: if an individual chooses an ambiguous option and receives a bad outcome, then she fears criticism from others. Such criticism is easier to counter after a risky choice, since in that case a bad outcome is more easily explained by bad luck. Trautman

(2009) finds that a reduction in social evaluation reduces ambiguity aversion, and that ambiguity aversion disappears when choices become private information, i.e., when the possibility of blame is eliminated.

### **Framing effects**

*The comparative ignorance hypothesis.* Fox and Tversky (1995) conducted a series of studies to investigate the hypothesis that ambiguity aversion holds in a comparative context, but is reduced or eliminated in a noncomparative context. A subject faces a comparative condition when she evaluates *both* clear and vague prospects; she faces a non comparative condition when she evaluates only one of these prospects in isolation. More specifically, the authors replicate Ellsberg two-urn's example in a both within- and between-subject design. They find that subjects are willing to pay significantly more for the clear bet than for the vague bet in a comparative context, while there is no trace of ambiguity aversion (subjects appear to pay slightly less for the clear bet than for the vague bet) in the noncomparative condition. They conclude that the contrast between states of knowledge is the predominant source of ambiguity aversion, hence that the Ellsberg phenomenon is intrinsically a comparative effect. However, Chow and Sarin (2001) replicate Ellsberg's two and three-color examples and show that ambiguity aversion persists in a between-subject comparison (even if the difference is smaller).

*Order effects.* Fox and Weber (2002) replicate Fox and Tversky (1995)'s design by focusing on the order in which subjects face the two urns. If order effects are at play, the decision context can no longer be considered as a fully comparative one. The urn presented first has to be considered as non comparative, while the second urn becomes mechanically comparative with regard to the first one. Hence, a context which is initially non comparative may become comparative. Fox and Weber (2002) find that the ambiguous urn receives a better evaluation

when evaluated first (thus in a non comparative condition) than when evaluated after the risky one. A contrario, the risky urn receives a better evaluation when evaluated after the ambiguous one (than when evaluated first) because it is likely to benefit from a comparative effect.

However, as pointed out by Fox and Tversky (1995), the comparative nature of an experiment cannot be reduced to its pure design:

"There is no guarantee that subjects in a comparative condition actually perform the suggested comparison, or that subjects in a non comparative condition did not independently generate a comparison."

Thus, the comparative/non comparative nature of ambiguity attitudes refers more to a cognitive disposal of the subject than to a manipulation of the experimental framework. In this respect, comparative ignorance and subsequent ambiguity aversion could be due to two kinds of framing effects: an objective framing effect meaning that the comparison is generated and explicit in the experimental design; and a cognitive framing effect meaning that the comparison is cognitively and implicitly performed by the subject. Discriminating between these two kinds of framing is impossible on the sole basis of observed choices.

### **2.3 Mixed sources of uncertainty**

For normative reasons, it would be desirable to admit that all probabilities are subjective. First, it would make it possible to combine information from a variety of different sources. Second, a single decision rule could be applied in all situations. However, the one-urn Ellsberg and Machina paradoxes show that such combinations of sources of uncertainty, because mixing uncertainties of a different nature, tend to generate ambiguity attitudes that are inconsistent with two standard decision rules, respectively Savage and Choquet expected utility.

### 2.3.1 Ellsberg one-urn paradox

The three-color paradox (Ellsberg, 1961) represents the first decision problem in which objective and subjective uncertainty are mixed. The paradox starts from a 90-ball urn, in which 30 are known to be red and the remaining 60 are distributed between yellow and black through an unknown process. A ball is drawn from the urn represented in Figure 2.3.

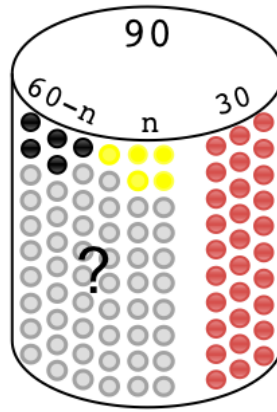


Figure 2.3: Ellsberg one-urn paradox

The nature of mixed uncertainty can be grasped through the following observation. The urn entails an hybrid partition {red, black, yellow}. Indeed, the event {red} involves only objective uncertainty while events {yellow} and {black} involve both objective and subjective uncertainty because they are each a subevent of an objective event {black, yellow} the probability of which is  $2/3$ . These subevents are referred to by Machina (2009a) as hybrid events.

The paradoxes presented in this section are essentially due to the presence of such hybrid events that turn out to be non hybrid when grouped together. Indeed, the preference of DMs for non hybrid events is in conflict with the normative requirement that a transformation of a hybrid into a non hybrid event through outcome-manipulations should not affect the decision.

Bets are described in Table 2.2. The DM's choices involve two pairs of bets  $f_1$  vs  $f_2$  and  $f_3$  vs  $f_4$ . Acts  $f_3$  and  $f_4$  are respectively obtained from acts  $f_1$  and  $f_2$  by simply replacing

the common consequence 0 by 100 on the event {yellow}. The nature - hybrid or not - of the bets depends on the nature - hybrid or not - of the partition on which they are built. More specifically, the nature of the partition depends on the way payoffs are distributed under their events within each bet. Hence,  $f_2$  and  $f_3$  are hybrid bets.  $f_1$  and  $f_4$  are known bets (lotteries) because they are based on the non hybrid union {black, yellow} of two hybrid events {black} and {yellow}.

Acts	30 balls	60 balls	
	red	black	yellow
$f_1$	100	0	0
$f_2$	0	100	0
$f'_2$	0	0	100
$f_3$	100	0	100
$f_4$	0	100	100

Table 2.2: Ellsberg bets

Ellsberg preferences  $f_1 \succ f_2 \sim f'_2$  and  $f_4 \succ f_3$  are, as in the previous section, explained by the preference for known over unknown probabilities or, alternatively, by the preference for non hybrid bets over hybrid ones. We can show that, in this paradox, ambiguity averse preferences violate PS and SEU.

The preference for  $f_1$  over  $f_2$  reveals subjective probabilities  $P(\text{red}) > P(\text{black})$  while the preference for  $f_4$  over  $f_3$  reveals subjective probability  $P(\text{black}) > P(\text{red})$ , which violates probabilistic sophistication. Moreover, bets  $f_3$  and  $f_4$  are respectively obtained from bets  $f_1$  and  $f_2$  by replacing the common outcome 0 by the common outcome 100 on the event {yellow}. Consequently, preferences  $f_1 \succ f_2$  and  $f_4 \succ f_3$  imply respectively  $P(\text{red}) > P(\text{black})$  and  $P(\text{red} \cup \text{yellow}) < P(\text{black} \cup \text{yellow})$  which contradict each other; indeed, the sure-thing principle (axiom P2 in chapter 1) - which assumes that DMs' preferences should not be sensitive to a



change of a common outcome - is violated; hence SEU.

However, such preferences are consistent with CEU which was precisely built to account for possible ambiguity averse preferences. With the normalization condition  $u(0)=0$  and  $u(100)=1$ , CEU implies:

$$f_1 \succ f_2 \Rightarrow \nu(\text{red}) > \nu(\text{black}) \text{ and}$$

$$f_4 \succ f_3 \Rightarrow \nu(\text{red} \cup \text{yellow}) < \nu(\text{black} \cup \text{yellow})$$

with  $\nu$  being a non additive measure. It follows that these inequalities are fully consistent with CEU.

### The second-order uncertainty aversion interpretation

Ellsberg preferences can be explained using a two-stage interpretation as in the two-color paradox. In this case, however, the source of uncertainty corresponding to each stage can be clearly identified. Consider the following simplified version of Ellsberg three-color paradox proposed by Ergin and Gul (2009). It involves a three-ball urn, that contains a red ball and each of the two remaining balls can be either yellow or black. The four possible compositions of the urn are given in Table 2.3:

Y	Y	B	B	ball 1
Y	B	Y	B	ball 2
R	R	R	R	ball 3
<hr/>				
yy	yb	by	bb	

Table 2.3: Three-ball urn

The representation in Table 2.3 shows that two kinds of uncertainty are combined. Row-uncertainty (called issue *a*) concerns the identity of the drawn ball (numbered by 1, 2 and 3). Column-uncertainty (issue *b*) concerns the composition of the urn, which depends on the color of balls 1 and 2. If row and column uncertainties are independent, and if the DM

considers each column/row as equally likely, then ambiguity aversion stems from a two-stage representation of the decision problem. If the column-uncertainty is resolved before the row-uncertainty, a bet on {red} is a degenerate lottery which gives the payoff 100 with probability one-third  $((100, 1/3); 1)$ , so it reduces to a bet on which ball will be drawn. On the other hand, a bet on {yellow} (or {black}) becomes a bet on (i) which ball is drawn and (ii) which is the number of the yellow (or black) ball in the urn; it can be represented by the compound lottery  $((100, 0), 1/4; (100, 1/3), 1/2; (100, 2/3), 1/4)$ . The DM who does not reduce uncertainty, does not equate his chance of winning for this compound lottery to  $1/3$ ; she is averse to the compound lottery and has a preference for a one-third probability of winning for sure, i.e., for a bet on {red}.

Hence, the DM is likely to prefer lotteries that depend only on issue  $a$  to equivalent lotteries that depend on both issues  $a$  and  $b$ . To some extent, she may prefer decision settings that involve a small number of sources of uncertainty. In this framework, ambiguity aversion is a consequence of greater aversion associated with column-uncertainty (also referred to as second-order risk).

### Separating sources in mix decision problems

Machina (2009a) proves that a proper representation of Ellsberg one-urn paradox based on a separation between sources of uncertainties allows us to reconcile Ellsberg's one-urn paradox with SEU, but not with PS, surprisingly <sup>6</sup>.

Machina (2009a)'s method consists in a diagramatic approach that 'orthogonalize' the two sources of uncertainty contained in the Ellsberg urn. The first step consists in identifying

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<sup>6</sup>Even if they differ in the method adopted, the orthogonalized method proposed by Machina (2009a) shares conceptual similarities with the uniform source method presented in the previous section. The examination of the Ellsberg two-urn paradox with the uniform source method has allowed to reconcile it with PS whenever the different sources of uncertainty are properly identified and separated. It is worth noticing that the uniform source method cannot be easily implemented for Ellsberg one-urn design since it recovers mixed sources of uncertainty. Hence, Machina's proposal has to be considered as an interesting alternative.

these two independent sources of uncertainty and the second step in separating them. The source of subjective uncertainty is the uncertainty regarding the composition of the urn, i.e., the number of balls of each color. The source of objective uncertainty refers to the identity of the ball that is drawn<sup>7</sup>. The partition of Ellsberg's urn (adapted, with 2 red balls and 4 black and yellow balls in unknown proportion) can be represented as in Table 2.4:

6	Y	Y	Y	Y	B
5	Y	Y	Y	B	B
4	Y	Y	B	B	B
3	Y	B	B	B	B
2	R	R	R	R	R
1	R	R	R	R	R
	0	1	2	3	4

Table 2.4: Orthogonalized representation of the three-color partitions

The columns describe the number of black balls in the urn (0 to 4). The rows designate each of the six balls present in the urn. Consequently, the column perspective gives bets represented as subjective acts with objective lotteries as consequences while the row perspective gives bets represented as objective lotteries with subjective acts over  $\{E_0, \dots, E_4\}$  as consequences.

The orthogonalization of the bet  $f_1$  (see Table 2.5, left matrix) should be read as follows. Intersection between column and row gives the payoff received, contingent on the number of black balls present in the urn and on the ball considered. Balls 1 and 2 are basically the red balls. Hence, whatever the number of black balls, 100 can be won if a red ball is drawn (ball 1 or 2). Note that numbering the balls does not imply any order: red balls could be alternatively designated by rows 3 and 4 as represented in Table 2.7, left matrix, or by rows 5 and 6 as represented in Table 2.7, right matrix. An advantage of such a static representation is that it excludes dynamic effects (like those highlighted by multiple-stage approaches) from the analysis.

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<sup>7</sup>The sources thus identified are similar to Ergin and Gul (2009)'s.

6	0	0	0	0	0	6	0	0	0	0	100
5	0	0	0	0	0	5	0	0	0	100	100
4	0	0	0	0	0	4	0	0	100	100	100
3	0	0	0	0	0	3	0	100	100	100	100
2	100	100	100	100	100	2	0	0	0	0	0
1	100	100	100	100	100	1	0	0	0	0	0
$f_1$	0	1	2	3	4	$f_2$	0	1	2	3	4

6	100	100	100	100	0	6	100	100	100	100	0
5	100	100	100	0	0	5	100	100	100	0	0
4	100	100	0	0	0	4	100	100	0	0	0
3	100	0	0	0	0	3	100	0	0	0	0
2	0	0	0	0	0	2	0	0	0	0	0
1	0	0	0	0	0	1	0	0	0	0	0
$f'_2$	0	1	2	3	4						

 Table 2.5: Orthogonalized representation of acts  $f_1$ ,  $f_2$  and  $f'_2$ 

6	100	100	100	100	0	6	100	100	100	100	100
5	100	100	100	0	0	5	100	100	100	100	100
4	100	100	0	0	0	4	100	100	100	100	100
3	100	0	0	0	0	3	100	100	100	100	100
2	100	100	100	100	100	2	0	0	0	0	0
1	100	100	100	100	100	1	0	0	0	0	0
$f_3$	0	1	2	3	4	$f_4$	0	1	2	3	4

 Table 2.6: Orthogonalized representation of acts  $f_3$  and  $f_4$ 

6	0	0	0	0	0	6	0	0	0	0	0
5	0	0	0	0	0	5	0	0	0	0	0
4	100	100	100	100	100	4	100	100	100	100	100
3	100	100	100	100	100	3	100	100	100	100	100
2	0	0	0	0	0	2	100	100	100	100	100
1	0	0	0	0	0	1	100	100	100	100	100
$f_1^R$	0	1	2	3	4	$f_4^R$	0	1	2	3	4

 Table 2.7: Rearranged acts  $f_1^R$  and  $f_4^R$  from  $f_1$  and  $f_4$ 

When restricting attention on the sole subjective source of uncertainty (columns), Ellsberg preference for  $f_1$  over  $f'_2$  does not imply a precise order between  $f_3$  and  $f_4$ . Indeed,  $f_1 = [(100, \frac{2}{6}) \text{ on } E_0; \dots; (100, \frac{2}{6}) \text{ on } E_4]$  and  $f_2 = [(100, 1) \text{ on } E_0; 100, \frac{1}{6}) \text{ on } E_1; (100, \frac{2}{6}) \text{ on } E_2; (100, \frac{3}{6}) \text{ on } E_3; (100, \frac{4}{6}) \text{ on } E_4]$ .  $f_1$  and  $f_2$  share the common lottery-outcome  $(100, \frac{2}{6})$  on the event  $E_2$ ; it is replaced by the common lottery-outcome  $(100, \frac{4}{6})$  in  $f_3$  and  $f_4$ :  $f_3 = [(100, 1) \text{ on } E_0; 100, \frac{5}{6})$

on  $E_1(100, \frac{4}{6})$  on  $E_2(100, \frac{3}{6})$  on  $E_3(100, \frac{2}{6})$  on  $E_4]$  and  $f_4 = [(100, \frac{4}{6})$  on  $E_0; \dots; (100, \frac{4}{6})$  on  $E_4]$ . However, such a change does not contradict the sure-thing principle since it would have imposed that the two pairs of acts differ only by this replacement. A representation of Ellsberg acts in a two-dimensional and additive space renders non additive consideration ad hoc<sup>8</sup>; the paradox as well.

When looking at the objective source of uncertainty (rows), acts  $f_4$  and  $f_3$  are obtained from  $f'_2$  and  $f_1$  by replacing the common  $1/3$  probability of act  $[0 \text{ if } E_0, \dots, 0 \text{ if } E_4]$  by the common  $1/3$  probability of act  $[100 \text{ if } E_0, \dots, 100 \text{ if } E_4]$ . Hence, the independence axiom is consistent and implies the Ellsberg preferences ( $f_1 \succ f'_2$  and  $f_4 \succ f_3$ ). Similarly, ambiguity loving ( $f'_2 \succ f_1$  and  $f_3 \succ f_4$ ) and ambiguity neutrality ( $f_1 \sim f'_2 \sim f_3 \sim f_4$ ) are consistent with the independence axiom.

However, when considering the mix objective/subjective uncertainty we observe that PS implies  $f_1 \sim f'_2 \sim f_2$  and  $f_3 \sim f_4$  because the probability of outcomes is the only determinant of preferences between bets. In this way, PS is always violated by Ellsberg preferences.

Machina (2009a) provides the extension of the above results to the 90-ball urn. Results regarding the existence of a paradox depend on the source on which it is focused. Ellsberg preferences are found to be completely consistent with the sure-thing principle and even implied by the independence axiom when separating the sources of objective and subjective uncertainty; because each of these two properties can be associated with only one dimension in the diagram, they no longer appear as a limitation of Ellsberg's paradox. However, PS has to be studied by preserving the two dimensions ; because PS considers mixed sources, it remains violated.

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<sup>8</sup>By representing Ellsberg acts as the product of two sources of uncertainty, Machina considers a larger space than the initial Ellsberg one; separation of sources restores additivity in each dimension in some sense; Gilboa and Schmeidler (1994) shows that functions on a space bearing a non additive measure turn out to be additive when considering larger space.

### 2.3.2 The reflection paradox

Machina (2009b) proposes an urn that mixes one objective with two subjective sources of uncertainty but in a fully symmetric way. He shows that some natural symmetry requirement (present in both the urn and the bets) that correspond to ambiguity averse preferences are not consistent with the predictions of CEU.

A 100-ball urn contains 50 balls that are known to be green ( $n$ ) or black ( $50-n$ ) and 50 balls that are known to be yellow ( $m$ ) or red ( $50-m$ ). This urn contains two sources of uncertainty, each bearing informational symmetry: first, objective informational symmetry because it gives an equiprobable lottery on the two non hybrid groups of subevents {black, green} and {yellow, red}; second, subjective informational symmetry since there is no information regarding the proportion of the two colors within each group.

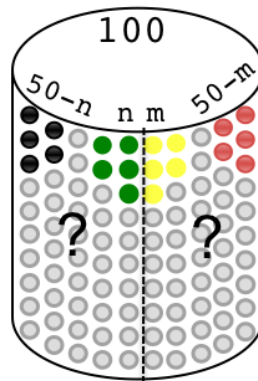


Figure 2.4: Machina's urn

The symmetry is also salient when looking at the bets described in Table 2.8. Bets  $g_4$  ( $g_3$ ) are an informationally symmetric reflection of bets  $g_1$  ( $g_2$ ). Hence, it seems natural to be indifferent between  $g_1$  and  $g_4$  and between  $g_2$  and  $g_3$ . Ambiguity averse preferences are  $g_2 \succ g_1$  and  $g_3 \succ g_4$ .

Acts	50 balls		50 balls	
	red	yellow	black	green
$g_1$	4000	8000	4000	0
$g_2$	4000	4000	8000	0
$g_3$	0	8000	4000	4000
$g_4$	0	4000	8000	4000

Table 2.8: The Machina bets

A fine analysis distinguishes between two criteria that lead to define preferences as ambiguity averse (Machina, 2009b). An ambiguity averse DM as usual prefers bets that involve known probabilities, i.e.,  $g_2$  and  $g_3$ . Conversely, an ambiguity averse DM may prefer to minimize his exposure to ambiguity by choosing bets that minimize the concentration of missing information, i.e,  $g_1$  and  $g_4$ ; in other words, he prefer bets that spread ambiguity over a greater number of hybrid events. These two kinds of ambiguity averse preferences correspond to the two pattern of informationally symmetric preferences. Ambiguity averse preferences will be found to be the most frequent pattern of choice in Chapter 5. Machina shows that ambiguity averse preferences are not compatible with CEU except if indifference holds between all the bets.

To see why this reflection poses a problem for CEU, we will assume that utility of 0 is 0, and consider the ambiguity averse pair of choices  $g_1 \succ g_2$  and  $g_4 \succ g_3$  . Substitution of CEU (Chapter 1, eq: 1.3) gives:

$$g_1 \succ g_2 \Rightarrow \nu(\text{yellow})u(8000) + [\nu(\text{yellow} \cup \text{black} \cup \text{red}) - \nu(\text{yellow})]u(4000) > \\ \nu(\text{black})u(8000) + [\nu(\text{yellow} \cup \text{black} \cup \text{red}) - \nu(\text{black})]u(4000)$$

and

$$g_4 \succ g_3 \Rightarrow \nu(\text{black})u(8000) + [\nu(\text{yellow} \cup \text{black} \cup \text{green}) - \nu(\text{black})]u(4000) >$$

$$v(\text{yellow})u(8000) + [v(\text{yellow} \cup \text{black} \cup \text{green}) - v(\text{yellow})]u(4000)$$

Consequently,

$$g_1 \succ g_2 \Rightarrow v(\text{yellow}) > v(\text{black}) \quad (1)$$

$$g_4 \succ g_3 \Rightarrow v(\text{black}) > v(\text{yellow}) \quad (2)$$

Because the revealed beliefs (1) and (2) are contradictory, CEU is violated. A CEU DM who choose  $g_1$  (resp.  $g_2$ ) in the first pair of bets should choose  $g_3$  ( $g_4$ ) in the second one.

## 2.4 Conclusion

The question we addressed in the first section is whether two natural sources of uncertainty are treated differently by DMs. We concluded that although separate, the simultaneous presence of sources of uncertainties of a different nature generates ambiguity attitudes, phenomenon that contradicts the classic modeling of subjective uncertainty, SEU and PS.

The second section has focused on ambiguity attitudes as resulting from a combination of two sources of uncertainty. When sources are mixed, ambiguity attitudes stem essentially from manipulations of consequences that change the nature of the bets. In Ellsberg's one-urn paradox, two outcomes are sufficient to create preference reversals under SEU. In Machina's one-urn paradox, three outcomes and a further source of subjective uncertainty are needed to create preference reversals under CEU. These observations point out that ever more complex decision contexts are needed to challenge normative models of decision making under uncertainty. The complexity resides in an increasing number of outcomes and sources, and in rendering their combination more fallacious. However, such paradoxes are useful to question the relevance of specific conditions that are hidden by *prima facie* innocuous functional forms.



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## Part II

# ELLSBERG PARADOX: TWO EXPERIMENTAL APPROACHES

## Chapter 3

# The Source of Uncertainty Approach

In 1961, Ellsberg's thought experiments suggested that the individual behavior could deviate from probabilistic beliefs and consequently from Subjective Expected Utility (Savage 1954). In this chapter, we provide an experimental study that reconciles Ellsberg's two-urn problem with a local form of probabilistic sophistication. The approach adopted here is based on sources of uncertainty. Sources of uncertainty are groups of events that are generated by the same mechanism of uncertainty, which implies that they share similar characteristics. Using Chew and Sagi (2008)'s concept of exchangeability, we can define choice-based probabilities locally (within particular sources) even though probabilistic sophistication does not hold globally (between sources). We elicit source functions that convert subjective choice-based probabilities into willingness to bet for each urn.

Our data establish that : (i) attitudes towards outcomes (utilities) do not depend on the urn; (ii) subjects do not depart from (local) probabilistic sophistication when facing Ellsberg urns; (iii) subjects exhibit distinct attitudes towards probability in each urn, more precisely, they are less sensitive to likelihood and more pessimistic in the ambiguous urn than in the risky one. Together, these results support the source of uncertainty approach for Ellsberg's two-urn problem.

### 3.1 Introduction

The simplest statement of ambiguity aversion as a challenge to Subjective Expected Utility (SEU: Savage, 1954), the standard model of decision under uncertainty in economics, was provided by Ellsberg (1961). Using thought experiments, he convincingly shows in his *two-color* famous example that, due to ambiguity aversion, most people prefer to bet on the color of a ball drawn from a 50-50 balls (two-color) urn, called the known urn, rather than on the color of a ball drawn from a similar urn containing 100 balls with unknown composition, the unknown urn. The preference suggested by Ellsberg are not consistent with probabilistic sophistication and consequently with SEU. Moreover, while Ellsberg's conclusion deeply influenced the subsequent theoretical investigations on ambiguity and generalizations of SEU, its experimental confirmation stays fundamentally qualitative: merely stating the presence or absence of ambiguity aversion in  $n$ -color experiments ( $n \geq 2$ ).

This chapter proposes to go one step further in the direction of a tractable analysis of attitudes towards uncertainty. We implement an experiment in which uncertainty attitudes are quantitatively analyzed by means of source functions defined on the probability interval. The analysis combines two concepts, the sources of uncertainty (Tversky and Fox, 1995; Tversky and Wakker, 1995) and exchangeability (Chew and Sagi, 2006).

Chew and Sagi (2006) define two events as exchangeable if the DM is always indifferent to permuting their corresponding outcomes. The notion of equal likelihood in the Savagean set-up expresses a particular case of exchangeability. Taking into account the above definition, a thought experiment can easily show that permuting the payoffs assigned to two given colors from the  $n$ -color unknown urn will likely keep the DM's preference unchanged. Such a conclusion is trivially true for gambles generated from the known urn. Chew and Sagi (2006)'s exchangeability allows us to reveal the existence of a uniform subjective probability



distribution on the unknown urn.

Once the unknown urn endowed with a (subjective) probability measure, we can see the willingness to bet assigned to a specific event as an urn-dependent transformation of the probability of that event. More precisely, each urn can be viewed as a source of uncertainty. Sources of uncertainty are groups of events that are generated by the same mechanism of uncertainty, which implies that they share similar characteristics. We can define choice-based probabilities locally (within particular sources) even though Machina and Schmeidler's (1992) probabilistic sophistication does not hold globally (between sources). We use three components to describe decision under uncertainty: (i) utility of outcomes; (ii) choice-based probabilities for each source of uncertainty; (iii) the source functions.

Hence, a mere comparison of the source functions will reveal relative ambiguity aversion/seeking across urns. For instance, in the two-color Ellsberg's example, the preference to bet on a specific color in the known urn corresponds to a higher transformation of the probability of  $1/2$  in this urn. The relative behavior of such transformation functions on the unit interval allows for a richer spectrum of attitudes towards uncertainty in the two urns. Needless to insist that such a set-up allows to investigate ambiguity aversion/seeking as probability-dependent. Intuitively, the DM is expected to clearly prefer betting on an unambiguous event when its probability is in the second half of the unit interval than if it is in the first half.

The model we used allows for a more sophisticated treatment of probabilities in each urn. An elevated source function will be shown to reflect DM's *optimism*. Hence, *pessimism* will result in a less elevated source function. If ambiguity aversion dominates behavior whatever the considered probability, we can expect the known urn source function to be globally more elevated than for the unknown urn. The local slope of the source function reflects the DM's capacity to *discriminate* between probabilities (in that region of the unit interval). Intuitively, a

global predominance of ambiguity in the unknown urn should result in a flatter corresponding source function. In other terms, an increase in probability should have less impact in the unknown urn than in the known one.

Section 3.2 provides the theoretical framework. Section 3.3 describes the experiment and Section 3.4 gives the results. Section 3.5 discusses.

## 3.2 Framework

### 3.2.1 Notations

$S$  is the *state space*. The outcome set is  $\mathbb{R}^+$ , designating nonnegative amounts of money. An *act*  $xEy$  gives outcome  $x$  if event  $E$  occurs and  $y$  otherwise.  $\succsim$  denotes the *preference relation* of a decision maker (henceforth, DM) over the set of acts ( $\sim$  and  $\succ$  are defined as usual). *Subjective expected utility* (SEU) holds if there exist a utility function  $u$  from  $\mathbb{R}^+$  to  $\mathbb{R}$  and a probability measure  $P$  on  $S$  such that  $xEy \mapsto P(E)u(x) + (1 - P(E))u(y)$ .

### 3.2.2 Ellsberg paradox

The Ellsberg paradox is an intuitive experiment that shows how DM's choices may contradict subjective expected utility. A DM is presented with two urns: a *known* urn ( $K$ ) contains 50 red balls and 50 black balls and an *unknown* urn ( $U$ ), which will also be called *ambiguous*, contains 100 balls red or black in unknown proportions. A ball is randomly drawn from each urn. We designate by  $R_S$  ( $B_S$ ) events "drawing a red (black) ball in urn  $S$ ". A DM who is asking to bet on a color typically exhibits the following preferences:

$$\begin{aligned} xR_K0 &\succ xR_U0 \stackrel{SEU}{\Rightarrow} P(R_K) > P(R_U) \\ xB_K0 &\succ xB_U0 \stackrel{SEU}{\Rightarrow} P(B_K) > P(B_U) \end{aligned}$$

Most people would rather bet on red balls in the known urn than in the unknown urn, and thus under SEU, they would reveal that they have a greater subjective probability of drawing a red ball in the known urn than in the unknown urn. A similar result is obtained for black balls. It thus induces that subjective probabilities of drawing a red ball or a black ball in the unknown urn are both less than one-half, which violates the additivity property of probability measure. Many empirical studies provide evidence of people's reluctance to bet on ambiguous events in the two-urn version of the Ellsberg paradox (Raiffa, 1961; Becker and Brownson, 1964; Yates and Zukowski, 1976; Kahn and Sarin, 1988; Curley and Yates, 1989; Eisenberger and Weber, 1995). A common interpretation of this paradox is that people may fail to assign subjective (additive) probability to events and therefore, deviate from probabilistic beliefs.

### 3.2.3 A general biseparable model

Ghirardato and Marinacci (2001) showed that a biseparable model agrees with a wide class of usual models, including Schmeidler's (1989) Choquet Expected Utility, Gilboa and Schmeidler's (1989) Multiple priors, Kahneman and Tversky's (1979, 1992) Prospect theory when restricted to positive binary acts. For an act  $xEy$  such that  $y \leq x$ , this model is given by:

$$xEy \mapsto W(E).u(x) + (1 - W(E).u(y))$$

$W$  denotes the *weighting function* that may be non additive, and  $u$  the utility function.

This model generalizes SEU and encompasses the Ellsberg paradox through the non additivity of the weighting function. Indeed, the Ellsberg paradox only means that  $W(R_U) + W(B_U) \leq W(R_K) + W(B_K)$ . However, we do not know yet whether the weighting function is not additive because of the non additivity of beliefs or because of a non additivity of the attitude component. The next subsection introduces a tool to discriminate between these two possible explanations.

### 3.2.4 Probabilistic sophistication

Decision makers are said to be probabilistic sophisticated if their choices are consistent with probabilistic beliefs. Formally, there exists a probability measure  $P$  such that for each prospect  $(x_1, E_1; \dots; x_n, E_n)$  the only relevant aspect regarding preferences is the probability distribution  $(x_1, p_1; \dots; x_n, p_n)$  that it generates over the outcomes, where  $p_j = P(E_j)$  for all  $j$  (Machina and Schmeidler, 1992). It follows that two different prospects generating the same probability distribution over the outcomes are equivalent in terms of preferences. A key step of our approach is to make probabilistic sophistication in the ambiguous urn observable. Chew and Sagi (2006) give particularly weak preference conditions for probabilistic sophistication based on Ramsey (1926) and de Finetti's (1937) exchangeability. Two events  $E, F$  are said exchangeable if the preference value does not change when permuting payoffs between these events. Consequently, a DM who exhibits indifference  $xEy \sim xFy$  (no matter what the outcomes  $x$  and  $y$  are) reveals that events  $E$  and  $F$  are subjectively equally likely. Chew and Sagi (2008) use this tool to define small worlds, i.e., domains of events such that an individual is probabilistically sophisticated within each domain. In the present chapter, we restrict probabilistic sophistication to two sets of events, namely, the known and the unknown urns: DMs' choices among bets in a specific urn should be consistent with probability sophistication while it can be violated when comparing bets between urns. Hence, probabilistic sophistication holds locally, within sources, but not necessarily globally, between sources. The so-called within-source probabilistic sophistication (Abdellaoui et al., 2009) generalizes Machina and Schmeidler's (1992) probabilistic sophistication by restricting it on well-defined sources.

### 3.2.5 Source functions

In each urn, the above mentioned restricted probabilistic sophistication can be combined with the biseparable model in order to decompose the attitude towards uncertainty into a

belief component and an attitude component. We will elicit the following model in each urn:

$$xEy \mapsto w(P[E]).u(x) + (1 - w(P[E])).u(y) \text{ with } (y \leq x)$$

In this representation,  $w(P[E])$  and  $(1 - w(P[E]))$  are *decision weights*.  $w$  is a *source function* that transforms probabilities into decision weights and the subjective probability of event  $E$ ,  $P[E]$  is

- a choice-based probability derived from exchangeability in the ambiguous urn,
- equal to the objective probability of winning outcome  $x$  in the risky urn.

The comparison of beliefs, source functions and utility functions across urns will enable us to better understand the Ellsberg Paradox. As an illustration, imagine that a DM has the same utility function for risk and ambiguity and also the same beliefs ( $P(R_U) = P(B_U) = P(R_K) = P(B_K) = \frac{1}{2}$ ), but he envisages the two urns as two distinct sources of uncertainty, i.e., source functions for risk and ambiguity differ. Then if she exhibits the following Ellsbergian preferences:

$$xR_K0 \succ xR_U0$$

$$xB_K0 \succ xB_U0,$$

she only reveals that she does not treat the probability one-half in the same way: the source function  $w_K(\frac{1}{2})$  for the known urn is higher than the source function  $w_U(\frac{1}{2})$  for the unknown urn. Because the non additivity resides in the attitude and not in subjective beliefs, Ellsbergian preferences is in this way reconciled with (local) probabilistic sophistication.

#### 3.2.6 A focus on source functions

Figure 3.1 below displays some typical features of the expected source functions. (a) represents the common findings about attitude towards risk, i.e., an inverse-S shaped function:

small probabilities are overweighed while medium and large probabilities are underweighting. This pattern comes from the combination of a low *likelihood sensitivity* (a weak discriminability between likelihood levels for intermediate probabilities) and *pessimism* (prospects are perceived as weakly attractive).

If ambiguity generates a lower likelihood sensitivity, then the source function under ambiguity should be more inverse-S shaped than under risk (b): discriminating between likelihood levels is indeed harder when probabilities are unknown than where they are known. In addition, DMs are often *ambiguity averse*: they feel more pessimistic when probabilities are unknown or equivalently, ambiguous prospects appear less attractive (c). (d) represents what could be expected as a result of the aforementioned effects.

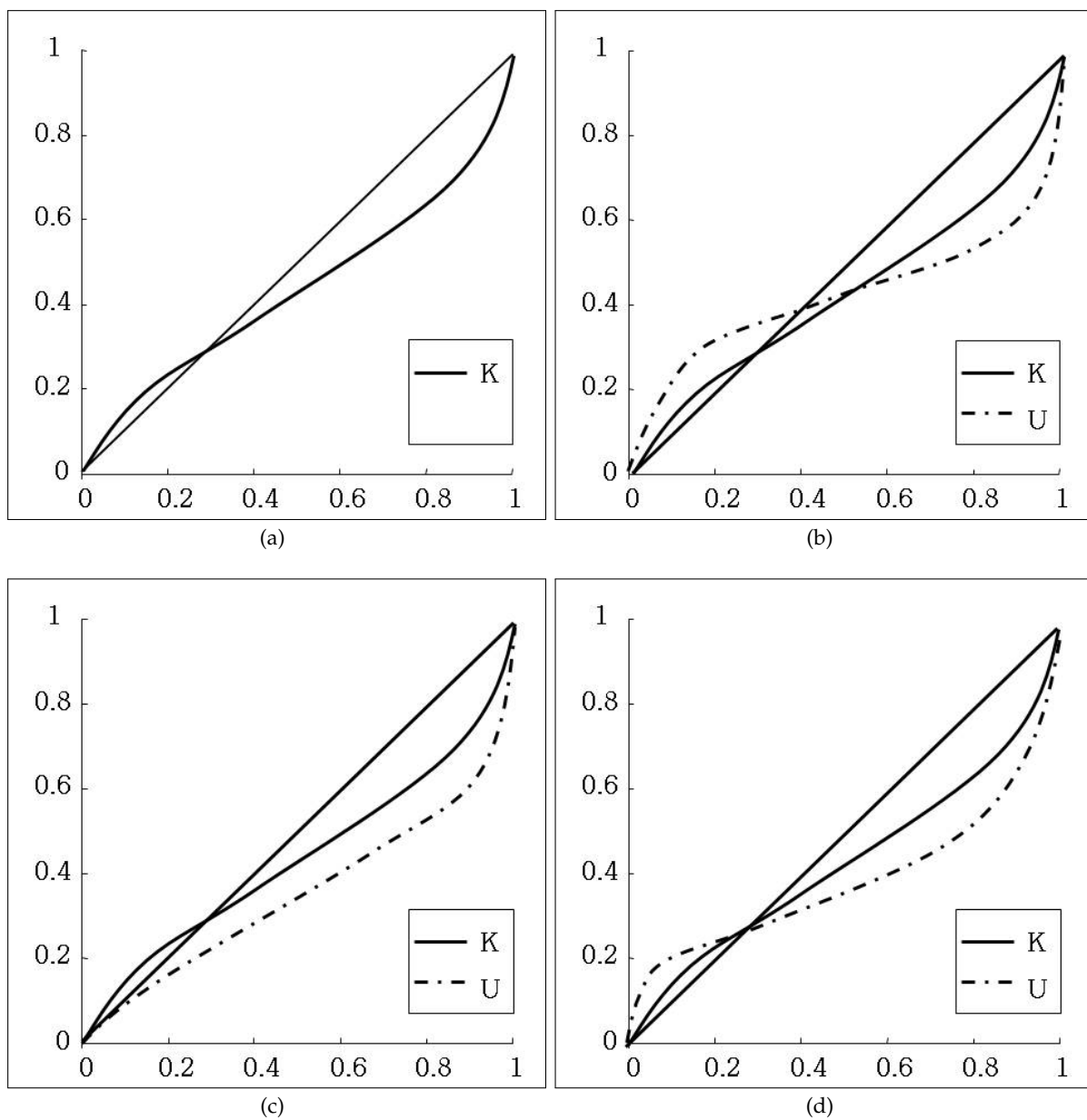


Figure 3.1: Shape of the weighting functions under risk and ambiguity

The next section describes the design and the results of the experiment which measure the model presented above. It empirically provides a disentanglement between utilities, beliefs and source-attitudes for the Ellsberg paradox.

### 3.3 Experiment

#### 3.3.1 Experimental design

##### Participants

Sixty-seven students (20 females, 47 males) at two French engineering schools (Ecole des Travaux Publics and Ecole Nationale Supérieure d'Arts et Métiers) participated in this experiment. The sample was recruited through posters and internet-based registration. They were all acquainted with probability theory but most of them had never heard of decision theory. They were told that they could win up to €25 for their participation.

##### Two decision contexts

Subjects faced two Ellsberg-like urns (see Figure 3.7 in Appendix). More precisely, they were successively confronted with the known urn and the unknown urn. In the known urn, subjects were told, and could observe, that it contained eight balls, each one a different color (red, blue, yellow, black, green, purple, brown, cyan). In the unknown urn, they could observe that there were also eight balls but they could not distinguish their colors. They knew that only the above-mentioned colors were available. Neither the subject nor the experimenter knew the true composition of the unknown urn, and it was emphasized that a new unknown urn was generated for each participant. This avoided communication between subjects concerning the composition of the unknown urn. For both urns, each ball was equally likely to be drawn. Hereafter, elementary events are denoted  $R_S, B_S, Y_S, K_S, G_S, P_S, N_S, C_S$  for  $S \in \{K, U\}$ .

##### Measuring indifferences

The experiment consisted of individual interviews using a computer. Subjects had to make choices, which were entered by the experimenter. Approximately five minutes were de-



voted to training and explanations. During the training, the subject faced the two different choice contexts. Subjects faced a twenty-six series of choice tasks; one series involved a choice between a prospect and an ascending range of sure payments. Our elicitation method is very similar to the iterative multiple price list procedure proposed by Andersen et al. (2006), i.e., a second list refines the choice where subjects switched in the first list (see Figure 3.8 in the Appendix). We introduced a third step in the choice list procedure, corresponding to the choice list that would have been generated by refining every possible switching point from the first list. In the first step we had divided the range into 5 categories and in the second step into 10 categories. Hence, in the third step we divided the domain into 50 categories. The list had been pre-filled based on the answers given and assuming monotonicity, and was presented to the subject for validation. The program also allowed respondents to backtrack if they felt regret in a previous series of choices. The experiment took about 20 minutes per subject.

The procedure allows us to finely determine a certainty equivalent for each prospect that is, the average between the lowest value the subject had rejected and the highest value he accepted. With the three-step process, we obtain certainty equivalents with a precision of 1% of the distance between the highest and the lowest outcomes of the prospect.

#### **Order treatments**

Two treatments were implemented: in the *KU treatment*, subjects faced and completed the choice concerning the known urn and then dealt with the unknown urn; in the *UK treatment*, subjects began with the unknown urn. It is worth noticing that subjects were aware of the existence of the two kinds of urn from beginning and that we deliberately put them in a comparative context.

#### Incentive mechanism

At the beginning of the experiment, subjects were told that one of their choices would be randomly drawn from the third list and played for real. This way, the system is incentive compatible and not subject to strategic behavior. If, in the randomly drawn choice, a subject had preferred the sure amount, then they received the corresponding payoff. On the contrary, if the subject preferred to play the prospect, then this prospect was played for real in the corresponding urn (specific software was created for applying the incentive mechanism).

To make things clearer for the subject, this mechanism was precisely explained and fictively applied at the end of the training session (but the payment was fictive) to make them conscious it was in their interest to choose what they really preferred. The completed choice, which was played for real, was always drawn from the series associated with the risky urn, in order to give no information about the composition of the unknown urn.

#### 3.3.2 Elicitation technique

##### Testing exchangeability

Chew and Sagi (2006) propose exchangeability as a tool to derive probabilistic sophistication within a specific source of uncertainty. Two events are said to be exchangeable when permuting the payoff of these two events does not change the preference value of the prospect for the subject. We implement this idea to test if the elementary events  $(R_U, B_U, Y_U, K_U, G_U, P_U, N_U, C_U)$  are exchangeable. Similar tests are also implemented on two-fold and four-fold unions of elementary events. Intuitively, if two elementary events (or two unions of two or four elementary events) are exchangeable, then they should be revealed equally likely. Furthermore, if  $n$  exchangeable events partition the state space, then their probability must be  $1/n$ .

#### Elicitation of the utility function

To elicit the utility function we apply the method proposed by Abdellaoui, Bleichrodt and L'Haridon (2008) to estimate the power functional form

$$u(x) = \left(\frac{x}{25}\right)^\alpha,$$

where 25 is the maximum outcome under consideration. Their method consists in eliciting the certainty equivalents of binary prospects with a fix probability (or event) and changing outcomes. Then a non-linear estimation is performed in order to get two parameters: the power  $\alpha$  and the decision weights associated with the probability (event).

#### Decision weights

We elicited subjects' certainty equivalents for prospects that yield €25 if the union of  $i$  given elementary events occur and nothing otherwise ( $i$  from 1 to 7). Such unions of events are obviously associated to objective probability  $i/8$  in the risky urn, and to subjective probability  $i/8$  in the ambiguous urn according to exchangeability. With  $u$  normalized such that  $u(0) = 0$  and  $u(25) = 1$ , the model implies that  $w(i/8) = x^\alpha$  where  $x$  is the corresponding certainty equivalent.

#### Parametric fitting of the source functions

We use the parametric specification of Goldstein and Einhorn (1987) to infer the shape of the source functions at an individual level. The general specification is as follow:

$$w(p) = \frac{(\delta p^\gamma)}{(\delta p^\gamma + (1 - p)^\gamma)}.$$

This specification is useful because each parameter precisely captures one aspect of the shape of the transformation function.  $\delta$  catches the elevation and thus refers to subjects'

pessimism, while  $\gamma$  is the curvature coefficient and reveals the DM's likelihood sensitivity. If urn K is more attractive than urn U,  $\delta$  should be higher in K than in U. Similarly, if individuals have more ability to discriminate in K than in U,  $\gamma$  should also be higher in K than in U. Thus, we can obtain a precise quantitative measurement of subjects' behavior in the two contexts of choice.

#### Indexes

To finely analyze ambiguity attitudes, we use linear indexes of pessimism and likelihood sensitivity. These indexes are inspired by Kilka and Weber (2001) and Fox and Tversky (1995) and consist in a linear regression of the transformation function on the (0,1) interval; it is defined as follows :

$$p \mapsto c + s.p$$

with  $c$  the intercept and  $s$  the slope. Then we can posit as in Abdellaoui et al. (2009):

$$a = 1 - s$$

is the sensitivity index and

$$b = 1 - s - 2c$$

is the pessimism index. These indexes can be viewed as what would be induced by the capacities being neoadditive (Chateauneuf, Eichberger and Grant, 2007).

## 3.4 Results

### 3.4.1 Exchangeability

One of the main novelties of this research is that it does not exclude the existence of subjective probabilities, even if the Ellsberg paradox holds. This is why three tests of exchangeability

were performed, in order to know whether or not the participants did act consistently with respect to the most likely subjective probability distribution, i.e., a uniform distribution. We saw that in a given context, participants should be indifferent to bet on any equally likely event. In other words, their certainty equivalents should not differ whatever the color they have to bet on.

We asked several series of questions to test exchangeability and to check that individuals were indifferent to the color of the ball. The first set of three certainty equivalents concerns elementary events ( $R_U, B_U, Y_U, K_U, G_U, P_U, N_U, C_U$ ). More precisely, certainty equivalents for three different events were elicited for each subject. An ANOVA for repeated measure shows that the hypothesis that the colors are revealed equally likely cannot be rejected ( $p=0.335$ ). Then, four certainty equivalents dealt with events two-fold unions of elementary events. Each subject faced the same four couples of colors. An ANOVA for repeated measure again shows that we cannot reject that the couple of colors are equally likely ( $p=0.245$ ). Last the certainty equivalents of two events but now referring to four colors were elicited. The hypothesis that these two events are equally likely is not rejected according to a paired t-test ( $p=0.824$ ).

Hereafter, a uniform subjective probability distribution is therefore assumed, because none of the previous tests was able to reject this hypothesis. However, since it constitutes one of the main assumptions of the remainder of the study, we carefully studied the data at the individual level. Subject 52, who exhibited strong variations in her certainty equivalents for the different colors, is now removed from the sample. Throughout the five next subsections, the mean certainty equivalent is used in the estimation, i.e., when we will need a certainty equivalent associated with one elementary event in urn  $U$ , the average of the three elicited ones will be considered.

### 3.4.2 Utility

The utility function is elicited through the semi-parametric method introduced by Abdellaoui, Bleichrodt and L'Haridon (2008). We elicited certainty equivalents in each urn for seven lotteries, whose were based on the same four-fold unions of elementary events (i.e., subjective probability equal to  $1/2$ ) but with different outcomes. Then, a nonlinear least-square estimation enables us to get the power of the utility function and the decision weight associated to the event, for each urn<sup>1</sup>.

First of all, some outliers make the means (1.429 in K and 1.960 in U) difficult to interpret and strongly different from the median (1.048 in K and 1.131 in U), on which we are thus going to focus. Two main features are noteworthy: the median is not significantly different from 1 for both urns according to sign-test (p-values are equal to 0.003 and 0.017 in K and U respectively). This is totally consistent with the theoretical arguments of Rabin (2000) and Safra and Segal (2006) for such small amounts. Moreover, a sign-test shows that the utility does not significantly differ across urns ( $p=0.712$ ) (but the decision weights are (as we will see in the next subsection). This result means that ambiguity does not seem to influence attitude towards outcomes, but mainly attitudes towards probability. Because we cannot reject that the two utilities are the same, we will see that ambiguity attitude is fully capture by the decision weights. Hence, we will hereafter interpret the differences among decision weights as differences in ambiguity attitude.

### 3.4.3 Sources functions

One interesting property of our method is that sources functions can be directly observed and compared. Events referring to different numbers of colors enabled us to vary the probability level (from  $1/8$  to  $7/8$ ). Applying the utility functions on the associated certainty equivalents

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<sup>1</sup>If the algorithm (Levenberg-Marquadt) does not converge, we estimate the power under expected utility i.e., fixing the decision weights to  $1/2$ . It happened in 4 cases, among the 132 estimations.

gave us the source functions, displayed in table 3.1.

$w$		Median	Mean	t-tests ( $w(p) = p$ )	t-tests ( $w_U = w_K$ )
1/8	K	0.193	0.255	<b>0.000</b>	0.235
	U	0.193	0.220	<b>0.000</b>	
2/8	K	0.305	0.336	<b>0.001</b>	0.203
	U	0.270	0.297	0.080	
3/8	K	0.439	0.438	<b>0.013</b>	0.164
	U	0.397	0.399	0.360	
4/8	K	0.499	0.507	0.744	<b>0.050</b>
	U	0.489	0.461	0.107	
5/8	K	0.640	0.635	0.661	<b>0.003</b>
	U	0.582	0.560	<b>0.014</b>	
6/8	K	0.749	0.741	0.661	<b>0.000</b>
	U	0.684	0.642	<b>0.000</b>	
7/8	K	0.944	0.881	0.722	<b>0.000</b>
	U	0.818	0.758	<b>0.000</b>	

Table 3.1: Source functions

First of all, some source functions are significantly different from the identity function (at 1/8, 2/8 and 3/8 for urn K, and at 1/8, 5/8, 6/8 and 7/8 for urn U). We can conclude that SEU cannot accomodate the data, which is consistent with usual findings (e.g. Hey, Lotito and Maffioletti, 2007). Figure 3.2 displays the mean source functions (mean decision weights).

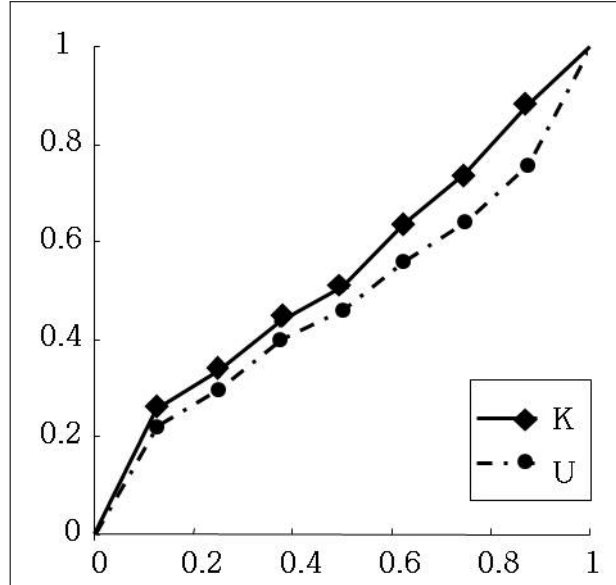


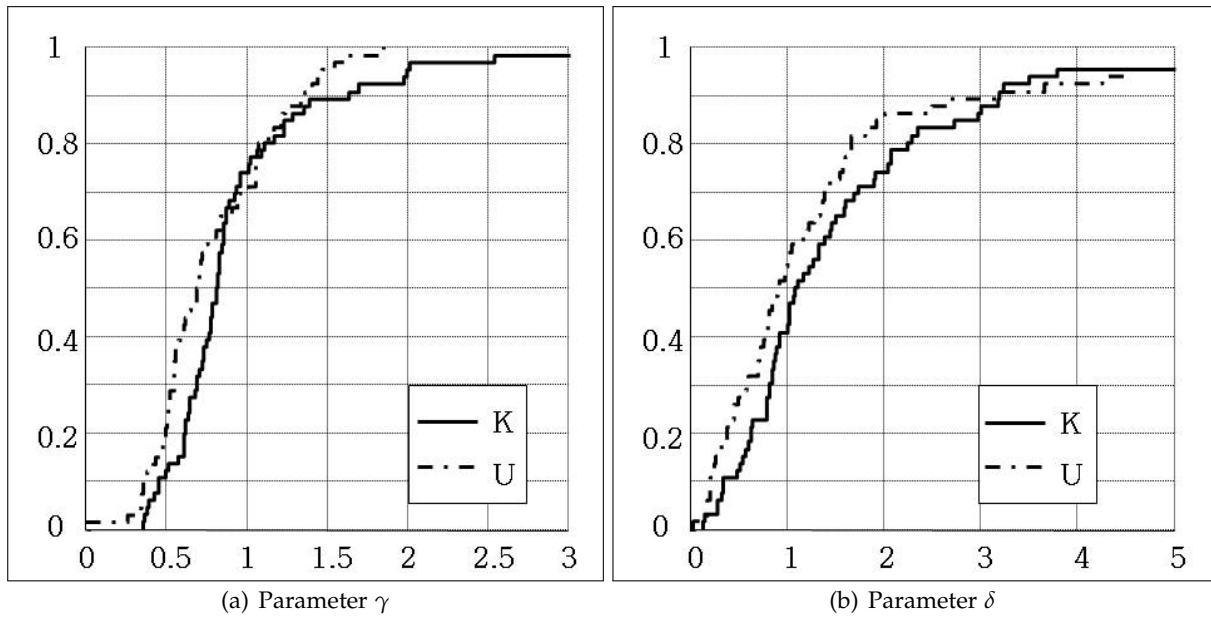
Figure 3.2: Mean source functions

We can see that the curves tend to be inverse S-shaped, but that the inverse S is more pronounced for U than for K. Furthermore, it is worth noting that the values of the source function at high probabilities ( $\geq 0.5$ ) are significantly lower in urn U than in urn K. This means that ambiguity attitude depends on the probability: the higher the probability is, the higher the difference between the source functions is and therefore the more ambiguity averse the subjects were.

#### 3.4.4 Parametric fitting of source functions

A nonlinear estimation based on decision weights gives us the parameters of the source functions. As expected, the results about  $\gamma$  (the curvature) confirm that the curves are inverse-S shaped, but according to t-tests, this is not significant in the known urn (mean=0.932, median=0.815,  $p=0.284$ ) while the curvature parameter is significantly different from 1 in the unknown urn (mean=0.785, median=0.696,  $p=0.000$ ). Moreover, the source function is more inverse-S shaped under ambiguity than under risk (a paired t-test gives  $p=0.029$ ): the ambiguous urn decreases the participants' ability to discriminate between likelihood levels.



Figure 3.3: Cumulative Distributions of parameters  $\gamma$  and  $\delta$ 

Like the utility parameter, the elevation parameter is strongly sensitive to the presence of outliers and has a minimum equal to 0 but has no upper-boundary. That is why the means (1.901 and 1.488 in K and U respectively) differ from the median (1.092 and 0.909 in K and U respectively). Moreover, large variances make t-tests non significant. Again, we are going to focus on the medians, which are not influenced by the presence of outliers. Sign-tests highlight that  $\delta$  is never significantly different from 1 ( $p=0.175$  and  $p=0.268$  in U and K respectively) and is not influenced by the urns ( $p=0.175$ ). However, Figure 3.3.b is consistent with more pessimism under ambiguity than under risk. Figure 3.4 displays the source functions based on the median parameters.

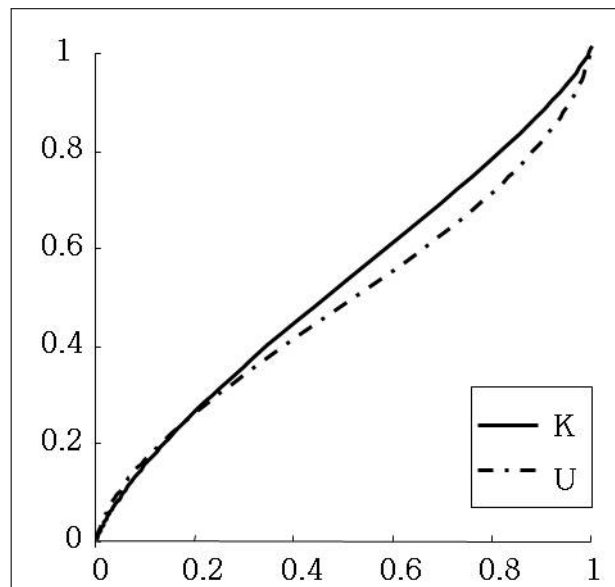
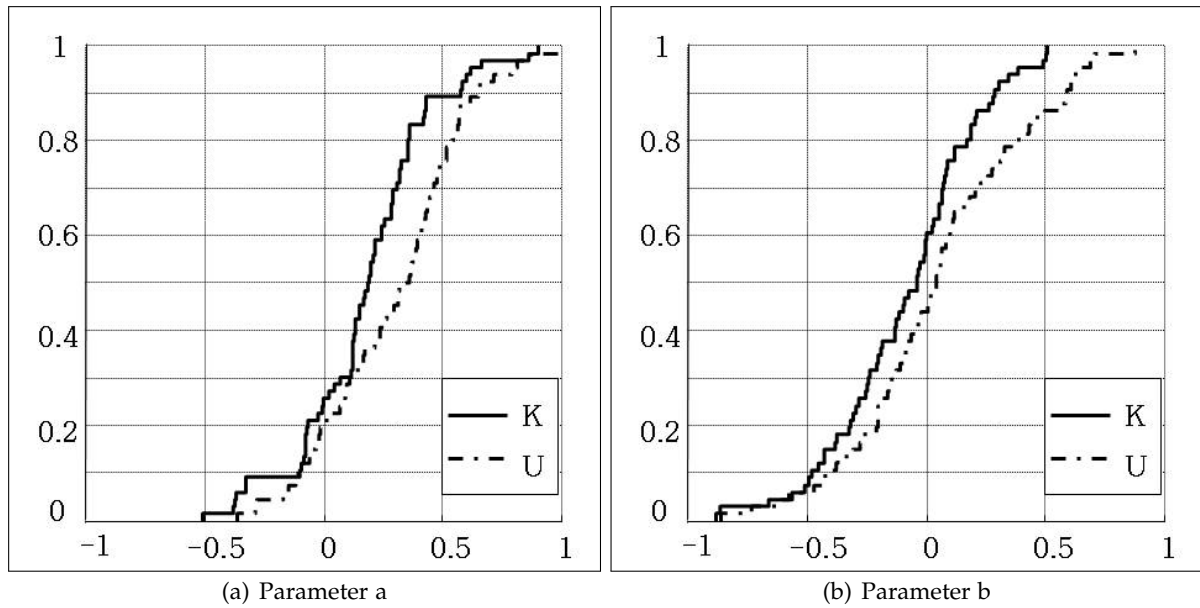


Figure 3.4: Median Weighting Function

### 3.4.5 Indexes

The sensitivity index is based on the slope of a straight line that goes through the elicited decision weights. If the source function is the identity function, then the index is equal to 0. It is positive for an inverse-S shaped source function. The sensitivity indexes are significantly positive for both curves (mean values are 0.176 for K and 0.296 for U and p-values of t-tests are 0.000 for both) and they are significantly different according to a paired t-test ( $p=0.005$ ). Figure 3.5.a displays the cumulative distribution functions of the sensitivity indexes and clearly shows these features.

Figure 3.5: Cumulative Distributions of parameters  $a$  and  $b$ 

The pessimism index (the elevation of the straight line) is positive when participants exhibit pessimism, and negative if participants exhibit optimism. Paired t-tests indicate that optimism is significant in urn K (the mean is -0.083 and the p-value is 0.029) but that there is a non-significant pessimism in U (the mean is 0.047 and the p-value 0.308). Above all, a paired t-test tells us that the pessimism index does depend on the urn ( $p=0.005$ ). Figure 3.5.b clearly represents this phenomenon through the cumulative distribution function of the pessimism indexes.

### 3.4.6 Heterogeneity

Until now, we have focused on mean and median results. Nevertheless, there is a wide range of different behaviors in our sample and this should not be neglected. The following figure displays the source functions of 9 subjects we selected among those whose data do not exhibit too much noise. The decision weights are represented by diamonds and circles, and the estimated source functions by a continuous line and a 'dash-dot' line, for K and U respectively. In the Appendix, Table 3.2 reports the parameters of the source functions.

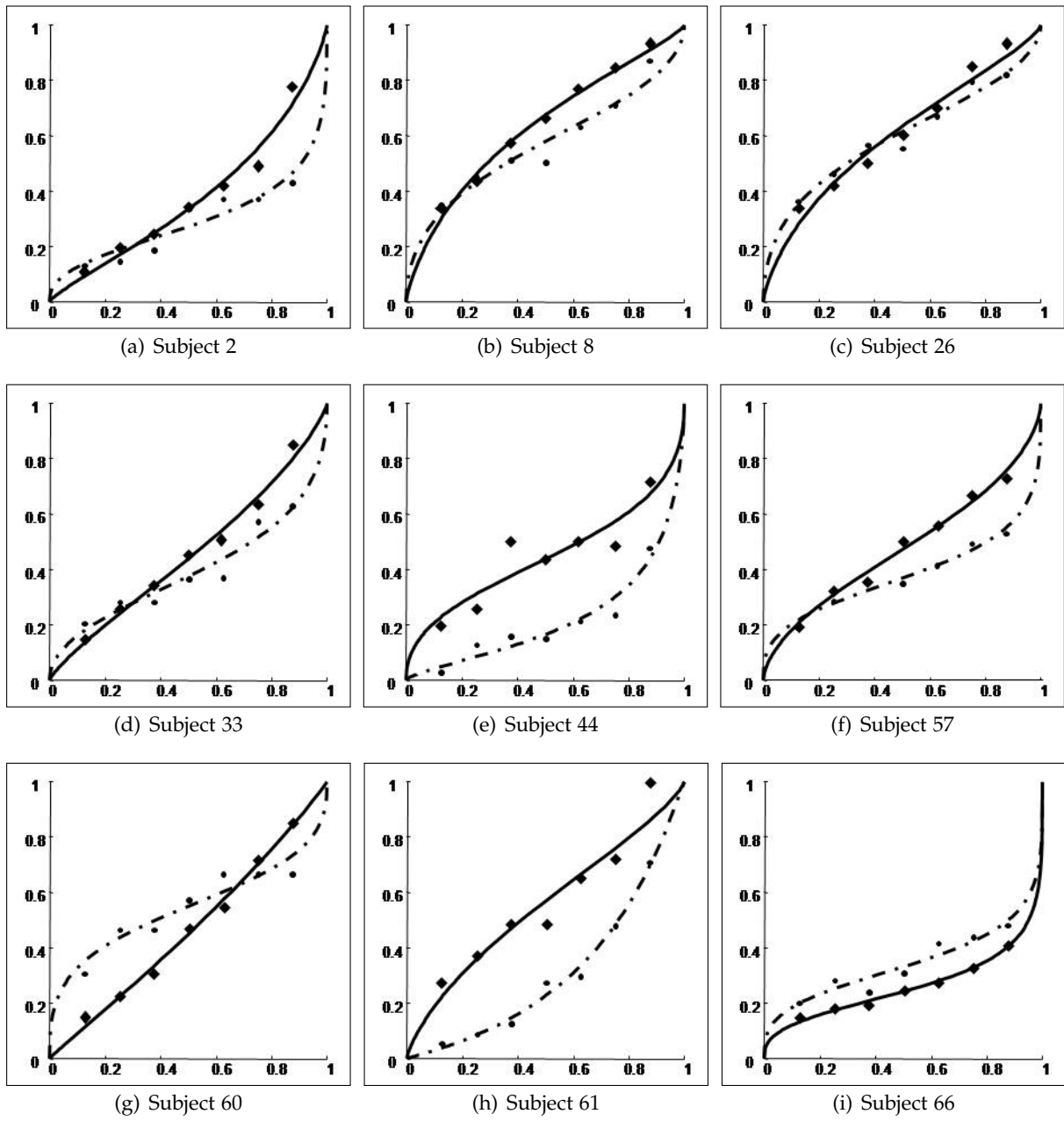


Figure 3.6: Interindividual Heterogeneity

Subjects 2, 33 and 57's curves display the common features, i.e., an inverse-S shaped source function in urn K and both more pessimism and less discriminability in U than in K. Some subjects (8, 26 and 61) are globally optimistic under risk, exhibiting a convex source function in K. Nevertheless their attitudes under ambiguity differ: subject 26's likelihood sensitivity decreases, while subject 61 becomes strongly pessimistic (concave source function).

Both phenomena are present in subject 8's source function under ambiguity. Subject 44, whose attitude under risk is mostly characterized by a low discriminability, is ambiguity averse, his source function being shifted downward by ambiguity. On the contrary, subject 66 is ambiguity seeking; the source function in urn U is above the source function in K (and displays the same curvature). Subject 60 is also mainly an ambiguity seeker but he becomes ambiguity averse for very high probabilities: ambiguity seems to bring a lower discriminability but also more optimism to this subject.

## 3.5 Discussion and further results

### 3.5.1 Qualitative features and statistical methods

Ambiguity is often (and maybe too often) associated with ambiguity aversion. This latter is even used as an axiom in models (Gilboa and Schmeidler, 1989). However, according to our results, ambiguity aversion depends on the probability level. This must be related to the fourfold pattern of risk attitudes (Tversky and Kahneman, 1992), i.e., risk seeking for small probabilities of gains and large probabilities of losses and risk aversion for large probabilities of gains and small probabilities of losses. The fact that ambiguity aversion might depend on the probability level was already highlighted by Hogarth and Einhorn (1990).

These results were obtained using nonlinear least-squares regressions at the individual level. However, we can confirm them through a global maximum likelihood analysis: in this new analysis, we use the 2212 elicited certainty equivalents to get the values of the parameters of a representative agent. The error term includes both the between-subject heterogeneity and the individual errors. Hence, we correct for the error being potentially correlated by clustering them (through the specific option *cluster* of Stata; see Harrison, 2007). Each parameter ( $\alpha$ ,  $\gamma$  and  $\delta$ ) is estimated and decomposed into a constant, plus a term that takes into account

the impact of the urn. The key features are the following (see Table 3.3 in the Appendix): the utility function tends to be convex. This comes from the fact that a lot of subjects were not interested in winning less than €5. We can also remark that the curvature coefficient is significantly different from 1, which shows that this main property of non-expected utility model plays a major role in explaining our data. Moreover, ambiguity has a significant impact only on this curvature coefficient and the variance of errors. Ambiguity decreases participants' discriminability and increases the heterogeneity between subjects and/or the individual errors. Moreover, Table 3.4 (in the Appendix) reports the results of a similar nonlinear maximum likelihood estimation, in which the Goldstein and Einhorn source function is replaced by a straight line, whose parameters are the two (sensitivity and pessimism) indexes. The power of the utility function is not significantly different from 1 and the representative agent exhibits significant insensitivity and optimism. Furthermore, the urn still has no impact on the utility but on all the other parameters: more insensitivity, more pessimism, more errors. Last, we must note that the log pseudolikelihood of this estimation is higher than the previous one and that the results totally confirm what we found in the previous section.

We can conclude that likelihood effects are particularly robust, whatever the analysis we conduct on the data: ambiguity has an impact on the sensitivity parameter, which is the parameter that makes attitude depend on probability. This is how probability matters.

#### **3.5.2 Order effects and the comparative ignorance hypothesis**

To control for order effects, we introduced two treatments (KU and UK), switching the first urn the participants had to deal with. Table 3.5 in the Appendix reports the p-values of t-test between two samples: the 31 subjects that faced treatment KU and the 35 subjects having answered to treatment UK. It is noteworthy that none of them is significant. However, if we conducted the same analysis as in section 3.4 but within each treatment, we would see that

the sensitivity index differs across urns only for treatment UK ( $p$ -value=0.023) but that the pessimism index differs across urns only for treatment KU ( $p$ -value=0.012),  $\alpha$ ,  $\gamma$  and  $\delta$  never significantly differing across urns for both treatments. This last result can be explained by the fact that assessing certainty equivalents in urn K and then in urn U increases the participants' ambiguity aversion (even if they knew since the training session that the two urns existed).

According to the comparative ignorance hypothesis, ambiguity aversion is produced by comparison with less ambiguous events. Conversely, the discrepancy between evaluation disappears (Fox and Tversky, 1995), or at least is reduced (Chow and Sarin, 2001), when a person evaluates only one of the urn in isolation. In our experiment, subjects have been put in a comparative context since they had to evaluate both the known and the unknown urn and because they were informed of the existence of the two urns since the training phase. Our results show that in our particular study, the comparative context does not necessary bring ambiguity aversion. Ambiguity aversion only appears when the subject evaluates the known urn before the unknown. Such an order effect has already been highlighted by Fox and Weber (2002). An interpretation of this asymmetry of reaction in a comparative context is that the risky urn can be viewed as a kind of informational reference point and that, to a certain extent, ambiguity aversion could be related to aversion to a loss of information.

#### **3.5.3 Ambiguity and asset prices**

According to our results, ambiguity aversion is tightly associated with the likelihood of events. This result can have concrete economic implications in particular on asset prices and portfolio holdings in financial market.

The literature on asset pricing shows that ambiguity aversion leads to a reluctance to hold an ambiguous portfolio. Specifically, Dow and Werlang (1992) obtain that for an ambiguity averse (MEU-)investor would be willing to buy an ambiguous asset at a strictly lower price

than (s)he would be willing to sell, meaning that there exists an interval of prices within which the investor would neither buy nor sell the asset. This implies portfolio inertia. Mukerji and Tallon (2004) generalizes this result, without assuming CEU or MEU preferences.

The shape of source functions we obtained could be used to shed a new light on investment decision. More specifically, we could reaffine usual pricing predictions. A main implication of our results is that ambiguity generates aversion but has also an impact on discriminability, i.e., the curvature of the source function. Some very recent results show that curvature of the source function may induce that the skewness of an asset can be priced. These results are derived for cumulative prospect theory, a model that is equivalent to our model when restricted to binary positive prospects. According to De Giorgi, Hens, and Levy (2003), pricing of (normally distributed) asset prices through the CAPM still holds when investors are CPT maximizer.

However, Barberis and Huang (2008) prove that a positively skewed asset (like a simple lottery ticket with a low probability of winning a high amount) can be overpriced, a CPT-maximizer investor overweighting small probabilities of large gains. For instance, Spalt (2007) applied this result to stock options (an option is a typical skewed asset) and concludes that employees can overvalue stock options and a risk neutral firm can take advantage of this psychological bias. As a consequence, a higher curvature under ambiguity than under risk makes positively skewed ambiguous assets more attractive than unambiguous ones.

In other words, the depreciation of an ambiguous asset implied by ambiguity aversion can be partly compensated by the value associated to its skewness. Moreover, specific investors that are ambiguity seeker for small probabilities (for instance subjects 2, 8 or 33; see subsection 4.6) may strictly prefer an ambiguous positively skewed asset to an unambiguous one (with the same distribution), whereas a uniformly ambiguity averse investor (e.g. subjects 44 and 61) will always prefer a risky asset to a ambiguous one (with the same outcome distribution).



#### 3.5.4 Sources of uncertainty

As advanced by Tversky and Kahneman (1992) (see also Tversky and Wakker, 1995) and empirically highlighted by numerous studies (Abdellaoui et al., 2009; Dolan and Jones, 2004; Fox and Tversky, 1995, 1998; Hsu et al., 2006; Kilka and Weber, 2001; Tversky and Fox, 1995), preferences among prospects do not only depend on the probabilities of events but also on their sources, e.g. investors disliking some foreign stocks relative to some domestic stocks (Fox and Tversky, 1995). But source dependency does not only mean that a source is more attractive than the other. It may also implies that likelihood sensitivity depends on the source. Ambiguity typically decreases DM's discriminability between likelihood levels. Our results are totally consistent with this literature, except that only two extreme cases (risk and ambiguity) were considered here. Abdellaoui et al. (2009) extend the source of uncertainty approach to natural uncertainties: sources are CAC 40, the temperature in Paris, and the temperature in a random foreign country. For each of these sources, they endogeneously decompose the state space in order to obtain a uniform subjective probability measure. Then, they elicit sources functions for each natural uncertainty.

#### 3.5.5 Ellsberg one-urn paradox

The Ellsberg one-urn paradox is the replication of a similar phenomenon to the two-urn problem: an urn contains 90 balls. 30 balls are red, and the remaining 60 are yellow or black. Most people would rather bet on red balls (probability  $1/3$ ) than on black balls (unknown probability) but rather bet on black or yellow balls (probability  $2/3$ ) than on red or yellow balls (unknown probability). Such preferences reveal that black balls are less likely than red balls but more likely if both are associated with yellow balls. One could think that in this urn, the three events are not exchangeable and thus, that no probabilistic belief exists. However, Chew and Sagi (2008) argue that this urn is the mixture of two sources: the known source (red

or not red) and the unknown one (black or yellow). In the unknown source, the two events are likely to be exchangeable, and thus must have the same probability. As a consequence, the three colors (red, black and yellow) are equally likely, but source functions can explain the paradox, as they did for the two-urn paradox. Applying our method to this interpretation is left for future research.

#### **3.5.6 Conclusion**

In this study, we experimentally reconciled Ellsberg two-urn's paradox with local probabilistic sophistication by eliciting source functions that depict behavior towards different sources of uncertainty. We find that the attitude towards outcome (captured by utility functions) does not differ between urns and that probabilistic sophistication holds within both urns (trivially for the known urn); hence, the source function entirely captures the attitude towards a well-defined type of uncertainty. Moreover, each source of uncertainty influences subjects' likelihood sensitivity and prospects' attractiveness differently. This observation leads us to conclude that Ellsberg two-urn's paradox is due to a difference in attitudes towards distinct sources of uncertainty and not to a violation of probabilistic beliefs as widely assumed by literature.

## Appendix

	$\gamma$		$\delta$	
	K	U	K	U
Subject 2	0.826	0.436	0.501	0.372
Subject 8	0.814	0.550	2.073	1.376
Subject 26	0.773	0.556	1.732	1.598
Subject 33	0.883	0.520	0.783	0.598
Subject 44	0.498	0.703	0.777	0.197
Subject 57	0.640	0.382	0.891	0.582
Subject 60	0.962	0.418	0.817	1.221
Subject 61	0.789	1.053	1.323	0.292
Subject 66	0.385	0.356	0.319	0.495

Table 3.2: Individual parameters

		Coef.	Robust Std. Err.	z	$P >  z $	95% Conf.	Interval
$\alpha$	constant	1.237	0.112	11.01	0.000	<b>1.017</b>	<b>1.458</b>
	urn	-0.0763	0.094	-0.81	0.418	-0.261	0.108
$\delta$	constant	1.027	0.147	7.00	0.000	0.740	1.315
	urn	-0.072	0.140	-0.51	0.607	-0.347	0.203
$\gamma$	constant	0.800	0.037	21.39	0.000	<b>0.727</b>	<b>0.873</b>
	urn	-0.176	0.035	-5.06	<b>0.000</b>	-0.244	-0.108
$\sigma$	constant	0.119	0.007	17.29	0.000	0.106	0.133
	urn	0.039	0.011	3.50	<b>0.000</b>	0.017	0.060

Log pseudolikelihood = 1137.970

The line 'urn' of each parameter is the impact of urn U with respect to urn K, i.e., the impact of ambiguity.  $\sigma$  represents the standard deviation parameter of the errors.

Table 3.3: Representative agent with the Goldstein and Einhorn parametric function

### 3. APPENDIX

		Coef.	Robust Std. Err.	z	$P >  z $	95% Conf.	Interval
$\alpha$	constant	1.065	0.084	12.72	0.000	0.901	1.229
	urn	0.067	0.089	0.75	0.453	-0.108	0.241
<b>a</b>	constant	0.178	0.029	6.21	<b>0.000</b>	0.122	0.235
	urn	0.126	0.032	3.89	<b>0.000</b>	0.062	0.189
<b>b</b>	constant	-0.098	0.042	-2.33	<b>0.020</b>	-0.181	-0.016
	urn	0.092	0.044	2.11	<b>0.035</b>	0.007	0.178
$\sigma$	constant	0.119	0.007	17.09	0.000	0.105	0.133
	urn	0.039	0.011	3.51	<b>0.000</b>	0.017	0.061

Log pseudolikelihood = 1140.837

Table 3.4: Representative agent with a linear source function

	$\alpha$		$a$		$b$		$\gamma$		$\delta$	
	K	U	K	U	K	U	K	U	K	U
t-test KU=UK	0.789	0.076	0.820	0.306	0.455	0.898	0.676	0.479	0.559	0.732

Table 3.5: Impact of order treatment

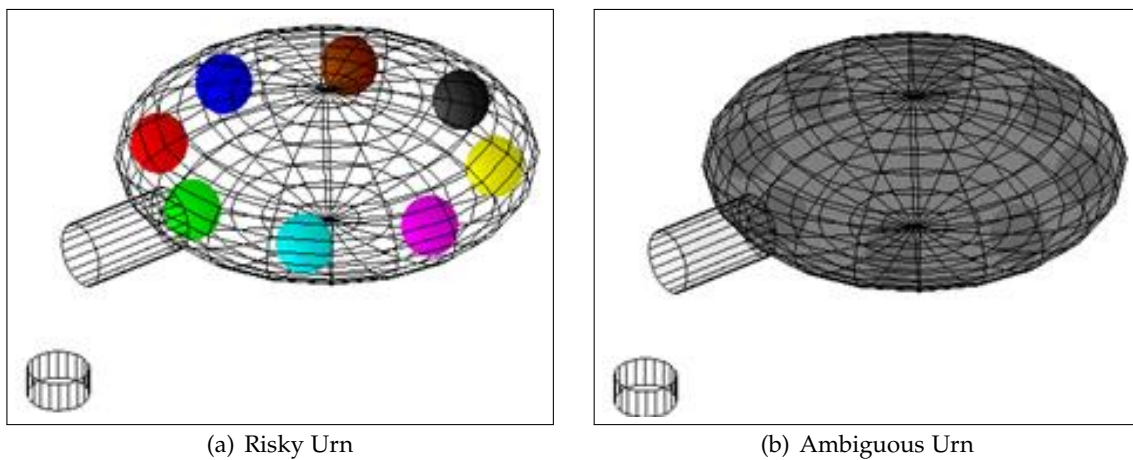
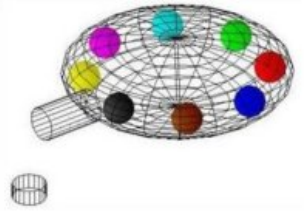


Figure 3.7: Ellsbergian urns

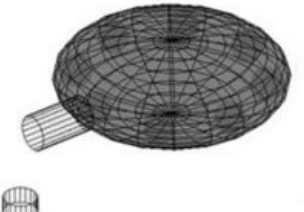
### 3. APPENDIX

**Which option do you choose?**

Option 1 Play the lottery below	1	2	Option 2 Receive this amount for sure
 <p style="text-align: center;">Win €25 if <span style="background-color: black; color: black;">■</span></p> <p>Win €0 if <span style="color: green;">■</span> or <span style="color: magenta;">■</span> or <span style="color: red;">■</span> or <span style="color: yellow;">■</span> or <span style="color: blue;">■</span> or <span style="color: cyan;">■</span> or <span style="color: brown;">■</span></p>	<input type="radio"/>	<input type="radio"/>	0 €
	<input type="radio"/>	<input type="radio"/>	5 €
	<input type="radio"/>	<input type="radio"/>	10 €
	<input type="radio"/>	<input type="radio"/>	15 €
	<input type="radio"/>	<input type="radio"/>	20 €
	<input type="radio"/>	<input type="radio"/>	25 €

(a) Risk (first step)

**Which option do you choose?**

Option 1 Play the lottery below	1	2	Option 2 Receive this amount for sure
 <p style="text-align: center;">Win €25 if <span style="background-color: black; color: black;">■</span></p> <p>Win €0 if <span style="color: green;">■</span> or <span style="color: magenta;">■</span> or <span style="color: red;">■</span> or <span style="color: yellow;">■</span> or <span style="color: blue;">■</span> or <span style="color: cyan;">■</span> or <span style="color: brown;">■</span></p>	<input type="radio"/>	<input type="radio"/>	0 €
	<input type="radio"/>	<input type="radio"/>	5 €
	<input type="radio"/>	<input type="radio"/>	10 €
	<input type="radio"/>	<input type="radio"/>	15 €
	<input type="radio"/>	<input type="radio"/>	20 €
	<input type="radio"/>	<input type="radio"/>	25 €

(b) Ambiguity (first step)

Figure 3.8: Screenshots

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## Chapter 4

# The Compound Risk Approach

Segal (1987) suggests that Ellsberg's two-urn paradox may be explained in terms of decision makers mentally representing the decision problem as a two-stage lottery and applying a non expected utility rule. Recently, Halevy (2007) suggested that attitudes towards compound risk and attitude towards ambiguity are tightly associated and provided experimental evidence that individuals who reduce compound risk to simple risk are ambiguity neutral.

We further investigate Halevy's findings by conducting an experiment that measures individual behavior under simple risk, compound risk and ambiguity for different likelihood levels. Our results confirm the link between ambiguity and compound risk attitudes but undermine the equivalence between reduction of compound lottery and ambiguity neutrality. Indeed, subjects who reduce compound risk are prone to ambiguity aversion. This tends to support only partially the hypothesis of a complete identification between ambiguity and compound risk.

### 4.1 Introduction

Ellsberg (1961) demonstrates that not all uncertainties (and especially what he terms 'ambiguity') can be reduced to risk and contradicts at the same time subjective expected utility (Savage, 1954) which assumes that all uncertainties are substitutable. Segal (1987) proposes an interpretation of the Ellsberg two-urn paradox where Ellsberg type behavior may be explained in terms of decision makers (henceforth, DMs) mentally representing the decision problem as a two-stage lottery and applying a non expected utility rule. More precisely, an ambiguous lottery should be considered as a two-stage lottery where the first stage deals with all the possible values of the probability of the event under consideration and does not impose the reduction of compound lottery principle between the two stages. Segal's approach has been revived in the modeling of uncertainty. Klibanoff, Marinacci and Mukerji (2005) describe uncertainty in two stages. The first (subjective) stage depicts the beliefs over the possible resolutions of the uncertainty in the second stage. Ambiguity aversion is captured through a function that underweights the second stage expected utilities calculated with respect to each possible probability measure. In a different framework, Ergin and Gul (2009) similarly explain the Ellsberg paradox as a consequence of aversion to second-order risk which corresponds to the possible compositions of Ellsberg's urn.

In a recent article, Halevy (2007) confirms Segal's intuition and empirically establishes the association between ambiguity attitudes and attitudes towards compound risk. More precisely, he finds that subjects who reduce a compound lottery resulting in one-half a chance of winning are ambiguity neutral (they give the same certainty equivalent for the bet on the compound lottery, for the bet on Ellsberg's risky urn, and for the bet on Ellsberg's ambiguous urn); conversely, subjects who do not reduce compound lotteries are found to exhibit ambiguity attitudes. Hence, Halevy postulates equivalence between attitudes towards compound

risk and ambiguity in noting:

(...) subjects who reduced compound lotteries were almost always ambiguity neutral, and most subjects who were ambiguity neutral reduced compound lotteries appropriately .

Following Halevy (2007), axiomatizations have been developed that explicitly relate attitudes towards ambiguity and compound lotteries. Seo (2008) provides an axiomatization in which a DM who reduces compound lotteries does not exhibit Ellsberg-type behavior. Similarly, Halevy and Ozdenoren (2008) give the condition to obtain second-order probabilistic sophistication that allows for non neutral behavior towards ambiguity. The reduction of compound lottery axiom is needed to obtain basic probabilistic sophistication and ambiguity neutrality.

In an Ellsberg-like framework we further investigate the link between ambiguity and two-stage lotteries. A first range of finding deals with reduction of compound lotteries *per se* with no respect to ambiguity. We confirm that failures in reduction are the prevailing behavior. Moreover, our data support time neutrality (Segal, 1990) and attitude towards compound risk is found to be likelihood-dependent.

A second range of findings confirms the relationship between attitude towards ambiguity and attitude towards compound risk. However, our data show that ambiguity attitudes cannot exclusively be associated with non reduction of two-stages lotteries. While almost no subject exhibits both ambiguity neutrality and reduction conformity, subjects who reduce compound lottery tend to be ambiguity averse. This result suggests that ambiguity attitudes cannot entirely be attributed to compound risk attitudes. However, when focusing on subjects who are not ambiguity neutral, our data do not allow to distinguish between subjects that do not adopt a representation of in terms of compound risk and those who do but fail to reduce compound risk correctly.

Section 4.2 details the experimental protocol. Section 4.3 focuses on the findings regarding the reduction of compound lotteries. Section 4.4 discusses preliminary findings on the relationship between ambiguity and compound risk attitudes. Section 4.5 concludes.

## 4.2 Experiment

### 4.2.1 Uncertainties

Subjects faced three types of uncertainties, namely, simple risk, compound risk and ambiguity, each represented by Ellsberg-like urns. The common feature to these three types of uncertainty is that each urn contained twelve balls; each ball was equally likely to be drawn and each ball could be of one of the twelve colors available.

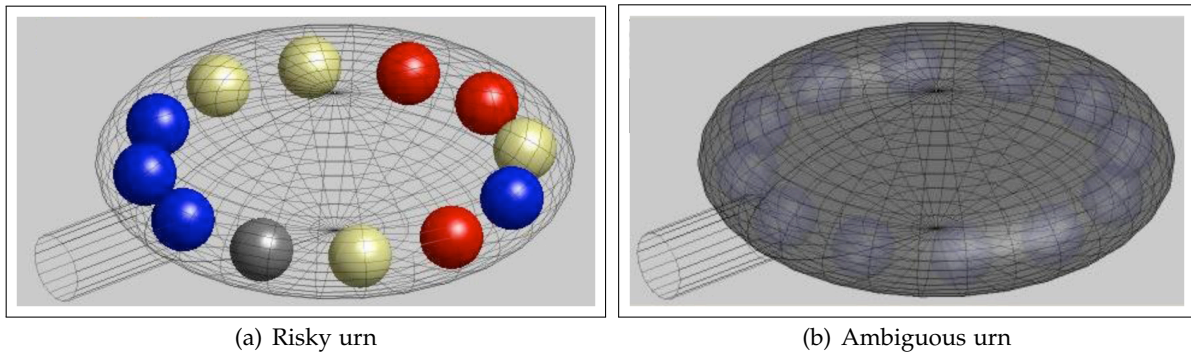


Figure 4.1: Ellsberg-like urns

For the simple (compound) risk, the subjects visualized the color of the balls in one (two successive) urn(s) and thus could infer probabilistic information. For ambiguity, the composition of the urn was hidden. Figure 4.1 represents the risky (a) and ambiguous (b) urns which were used in the software in order to describe the informational context of each choice situation. However, physical urns were used to concretize these uncertainties during the explanatory phase and for implementing the incentive mechanism.

### 4.2.2 Mechanisms to create ambiguity

The choice of a mechanism to create lab-ambiguity is controversial. Many previous empirical studies approximated ambiguity by second-order probabilities (Chow and Sarin 2002; Yates and Zukowski, 1976). A more sophisticated approach (Hey, Lotito and Maffioletti, 2008) uses a transparent box containing balls that are put into continuous motion by a current of air. The so-called British Bingo Blower allows simulating different degrees of ambiguity by varying both the number of balls put in the Blower and their composition. This approach is limited by the fact that it does not allow full ambiguity because it assimilates ambiguity and imprecision.

Our mechanism to create ambiguity is quite different since we want to keep the number of balls constant in the ambiguous urn. To this end, ambiguity was represented by a meta-ambiguous urn for which neither the number of balls nor the composition was specified to the subject. Naturally, this meta-urn was endowed with a uniform distribution (144 balls, 12 balls of each color). At the beginning of the experiment, each subject creates his own ambiguous urn by drawing twelve balls from the meta-urn. Each ball of the meta-urn was marked with one of the twelve colors beforehand, but this mark was hidden so that each ball in itself bore ambiguity. With such a two-stage generation of ambiguity, it becomes difficult to infer any probability distribution on the final individual urn. Note that it forces subjects to adopt a two-stage vision of ambiguity but does not guarantee that subjects operated such a representation. Since our purpose is to study the relationship between ambiguity and two-stage risk, one cannot argue that the two-stage feature of ambiguity was undermined by the design. Moreover, with this procedure, our final ambiguous urn is not far from what is called unknowable uncertainty (Chow and Sarin, 2002); neither the subject nor the experimenter knew the composition of the final 12-ball ambiguous urn. The advantage of such a procedure is that it avoids artificial ambiguity effects such as comparative ignorance effects (Fox and Tversky, 1995).

### 4.2.3 Sample

62 subjects were recruited from HEC (Management School,  $n=14$ ) and Ecole Centrale (Engineering School,  $n=48$ ) Paris. Subjects were all well acquainted with probability theory but had no knowledge in decision theory. The experiment consisted of individual and computer-based interviews using specific software built for the experiment. Choices were entered by the experimenter in order to avoid some "click mania". Subjects who asked to were allowed to calculate with pencil and paper (only two subjects did).

### 4.2.4 Procedure

Subjects faced three parts, successively simple risk, ambiguity, and compound risk (see respectively Figure 4.3, 4.4 and 4.6 in the Appendix). The compound risk part consisted in two subparts: lotteries with relatively low probabilities in the first stage; and (equivalent) lotteries with relatively high probability in the first stage. These two subparts were permuted between groups ( $n=31$  each) to control for possible order effects. A training question was asked before each part to check whether subjects had a correct understanding of the design and of the type of uncertainty faced.

On the overall experiment, subjects faced forty-one choice tasks. For each task, subjects had to choose between receiving a sure amount or playing the urn. The maximum amount was €50 and €0 was the minimum. We conducted likelihood treatments; for each type of uncertainty, certainty equivalents were elicited for 7 likelihood levels ( $1/12$ ;  $2/12$ ;  $4/12$ ;  $6/12$ ;  $8/12$ ;  $10/12$ ;  $11/12$ ). Likelihood treatments are described in Table 4.1.



## 4.2. EXPERIMENT

Uncertainty	Risk	Comp. risk (1)	Comp. risk (2)	Ambiguity
<b>1/12</b>	$(\frac{1}{12}, 50; 0)$	$(\frac{2}{12} \times \frac{6}{12}, 50; 0)$	$(\frac{6}{12} \times \frac{2}{12}, 50; 0)$	(1 color, 50; 0)
<b>1/6</b>	$(\frac{2}{12}, 50; 0)$	$(\frac{4}{12} \times \frac{6}{12}, 50; 0)$	$(\frac{6}{12} \times \frac{4}{12}, 50; 0)$	(2 colors, 50; 0)
<b>1/3</b>	$(\frac{4}{12}, 50; 0)$	$(\frac{6}{12} \times \frac{8}{12}, 50; 0)$	$(\frac{8}{12} \times \frac{6}{12}, 0; 50)$	(4 colors, 50; 0)
<b>1/2</b>	$(\frac{6}{12}, 50; 0)$	$(\frac{8}{12} \times \frac{9}{12}, 50; 0)$	$(\frac{9}{12} \times \frac{8}{12}, 50; 0)$	(6 colors, 50; 0)
<b>2/3</b>	$(\frac{8}{12}, 50; 0)$	$(\frac{4}{12} \times \frac{6}{12}, 50; 0)$	$(\frac{6}{12} \times \frac{4}{12}, 50; 0)$	(8 colors, 50; 0)
<b>5/6</b>	$(\frac{10}{12}, 50; 0)$	$(\frac{6}{12} \times \frac{8}{12}, 0; 50)$	$(\frac{8}{12} \times \frac{6}{12}, 0; 50)$	(10 colors, 50; 0)
<b>11/12</b>	$(\frac{11}{12}, 50; 0)$	$(\frac{6}{12} \times \frac{2}{12}, 0; 50)$	$(\frac{2}{12} \times \frac{6}{12}, 0; 50)$	(11 colors, 50; 0)

Table 4.1: Likelihood Treatments

The elicitation method is very similar to the iterative multiple price list procedure of Andersen et al. (2006). One series involved a choice between a lottery and an ascending range of sure payments. Each series contained three steps. The first step consisted in six choices between the lottery and a sure payment; sure payments were equally spaced between the minimum and the maximum amount of the corresponding lottery (see Figure 4.4 (a) in the Appendix). The second step consisted in a new set of eleven choices, spanning the narrower range between the lowest sure payment that the respondent had rejected and the highest sure payment he had accepted in the previous step (Figure 4.4 (b)). The third step summed up all the 51 subject's choices for the series just completed, and was submitted for final validation (Figure 4.5). While the software forced consistency in the first two steps, violations of monotonicity were allowed in step three. Hence, subjects were not allowed to

choose €10 for sure rather than the lottery and then choose the same lottery rather than €20 in the first two steps; but this was authorized in the validation step. None of the subjects acted this way. The program also allowed respondents to backtrack if they felt regret in a previous series of choices.

### 4.2.5 Incentives

The efficiency of the incentive mechanisms is quite a controversial issue, especially in the presence of compound lotteries (Holt, 1986; for a large discussion see Wakker's "Message to referees who want to embark on yet another discussion of the random-lottery incentive system for individual choice"). Moreover, there is no information about the efficiency of incentives in the presence of ambiguity. Therefore, two incentive treatments were implemented: The Random Lottery Incentive (RLI) mechanism and Flat Payment procedure (FP). The RLI was applied on a group of 30 subjects. Subjects performed a series of choices tasks knowing that one of their choices would be randomly drawn at the end of the session and played for real. If the choice implied (compound) risk, the (two) corresponding urn(s) was (were) created and the subject picked the ball in one (two successive) bag(s). In the case of the ambiguous urn, they drew the ball from the urn they generated at the beginning of the experiment. Subject could win up to 50 euros. The FP procedure consists in paying subjects enough to guarantee their extrinsic motivation in responding to the choice task. The 32 subjects who belonged to flat payment group each received 15 euros for their participation. The total amount spent for the payment was 1265 euros.

## 4.3 Results

### 4.3.1 Failures in reduction of compound lotteries

One of the main assumptions of most economic models is that the nature and the complexity of a lottery should not affect its evaluation. Especially, models implicitly assume the *reduction of compound lotteries* principle that requires a DM to be indifferent between the extensive and the reduced form of a compound lottery. However, a large body of empirical literature demonstrates that individuals are widely sensitive to the structure of compound lotteries; notably, the number and the order of stages, and the span of the tree significantly affect the behavior.

Most empirical studies proceed through direct comparison between the reduced and the extensive form of the lottery (except Friedman, 2005). Our experiment proceeds differently; preferences are captured by asking subjects for their certainty equivalents for simple risk and the two forms of compound risks, separately. Table 4.2 tests the equality of certainty equivalents for both forms of compound lotteries (KK1 designates the compound risk (1) with low probability in first stage and KK2 the compound risk (2) with high probability in first stage) where the compound risk (1) is equivalent to (2) when permuting the first and the second stage.

Probabilities	1/12	2/12	4/12	6/12	8/12	10/12	11/12
<b>t-test KK1=KK2</b>	0.2173	0.2534	0.0687	0.1835	0.2958	0.7286	0.1300
[95% Confidence Int.]	[0.3215, 1.3860]	[-0.4026, 1.4993]	[0.0969, 2.5485]	[-0.5727, 2.9275]	[-0.6360, 2.0553]	[-1.0694, 1.5211]	[-0.1953, .4857]

Table 4.2: Two measures of reduction of compound lotteries

#### **Time neutrality**

Table 4.2 shows that, in the aggregate, exchanging first and second stage in the compound lottery makes no difference. This corresponds to the property called time neutrality in Segal (1990). Our data suggest that in the aggregate, time neutrality - which states that the DM is indifferent to the timing of the resolution of the uncertainty - is satisfied. This result is consistent with Friedman (2005) who does not find order effects due to the permutation of the first and the second stages. Time non-neutrality is often considered as the reason for DMs not to be indifferent between one- and two-stage lotteries (Segal, 1990). Note that our sample exhibits time neutrality and does not reduce compound lotteries, suggesting that sensitivity to timing of the resolution of uncertainty is a minor factor for the observed deviations of the reduction principle.

Moreover, we do not find any significant order effect between the group which begins by evaluating the compound lotteries with (relatively) high first stage probability (compound risk (2):  $n=31$ ) and the one which begins with the compound lotteries with (relatively) low first stage probability (compound risk (1):  $n=31$ ).

#### **Reduction and likelihood treatments**

Table 4.2 shows that our data are consistent with earlier findings of widespread non reduction of compound lotteries in the economic and psychological literature (Bar-Hillel, 1973; Kahneman and Tversky, 1979; Conlisk, 1989; Bernasconi and Loomes, 1992; Friedman, 2005). Table 4.3 gives the number of subjects who gave the same certainty equivalent for the two sorts of compound risk; it suggests that reductions are more frequent for small and very high probability levels:

Likelihood treatment	1/12	2/12	4/12	6/12	8/12	10/12	11/12
Count KK1=KK2	26	23	23	21	13	18	20

Table 4.3: Reduction between compound risks and likelihood treatments

### Individual level

We can also look at the individual subject level to see if there is substantial heterogeneity in behavior towards the two stages of the compound lottery. An exact classification requiring unanimity (the reduction holds at each of the seven probability levels) shows that none of the subjects identifies as similar two compound identical lotteries. A classification authorizing an error equal to 5 % of the expected value of the compound lottery is more permissive and reveals only a few subjects who reduce the two compound identical lotteries. A classification based on majority (the reduction holds for at least 4 over 7 likelihood levels) increases the number of subjects who reduce similar compound lotteries.

Criteria	KK1=KK2*	KK1>KK2	KK1<KK2	Mixed
7 points (unanimity)	0	2	4	56
Error +/-2,5 (i.e., 5%EV)	4	21	14	23
At least 4 points over 7	14	16	8	24

\*KK1(2) = compound lottery with relatively low probability in the first (second) stage

Table 4.4: Attitude towards two types of compound risk

Empirical studies show that for positive outcomes, subjects prefer lotteries with high probability in first-stage (Bar-Hillel, 1973; Kahneman and Tversky, 1979). However, all previous studies implied direct comparison between the simple lottery and its extensive form. Our data contradict the existing findings since subjects exhibit certainty equivalents higher for the compound lottery with relatively low probability in first stage (according to error-permissive

and majority classifications).

### Simple risk and mean compound risk

The two measures of compound risk (1) and (2) are not significantly different (Table 4.2); in the following, we take the mean of the two to simplify the analysis.

Table 4.5 below gives the certainty equivalents for simple risk and the mean certainty equivalent for compound risk. It shows that subjects tend to overweight compound lottery to their reduced equivalents for probabilities less or equal to 8/12. For higher probabilities, subjects instead tend to underweight the compound lottery to its simple form. This result refines previous findings that subjects overweight compound lotteries whatever the probability level.

Probability level	1/12	2/12	4/12	6/12	8/12	10/12	11/12
Expected value	4,16	8,33	16,66	25	33,33	41,65	45,81
Simple risk	7.41 (4.52)	11.12 (5.11)	18.03 (5.84)	24.87 (5.31)	31.35 (6.16)	41.09 (4.94)	45.27 (3.41)
Compound risk	8.37 (4.39)	13.32 (7.35)	18.74 (5.95)	25.50 (6.50)	32.01 (6.08)	37.41 (6.08)	42.16 (5.58)

Table 4.5: Mean (Standard Deviation) certainty equivalent for simple and mean compound risk

### 4.3.2 Exchangeability

Exchangeability verifies that a subject reveals by his choices a uniform probability distribution in the ambiguous urn (the method is detailed in Abdellaoui et al., 2009). The subject is asked to give the certainty equivalent  $E_j$  for a bet involving a winning of a single color  $j$  in the ambiguous urn against the other eleven  $-j$ , for three different  $j$ . Exchangeability was tested over all the sample since the combination of the three colors was unique for each subject.

Exchangeability, if checked (i.e., if a subject gives the same certainty equivalent for the three colors) has two properties: first, it reveals that the subject is indifferent to the color of the ball, avoiding any paranoiac behavior regarding the composition of the urn; second, it allows to implicitly increment the likelihood level by adding a further color in a bet (adding a color increases the number of favorable cases over the number of possible cases).

The minimum certainty equivalent assigned for one color is 0.5 and the maximum is 30.5. Among 62 subjects, only 7 gave different certainty equivalents for at least two of the three different colors they faced. The maximum standard deviation is 5.7735.

An ANOVA for repeated measure shows that the hypothesis that the colors are revealed equally likely cannot be rejected within subject ( $p=0.6123$ ). Consequently, exchangeability is not rejected and the ambiguous urn can be endowed with a uniform subjective probability distribution. Consequently, likelihood treatment can be implemented even under ambiguity.

#### **4.4 Attitude towards ambiguity and attitude towards compound risk**

Since Segal (1987)'s intuition, very few papers have explored the link between attitude towards ambiguity and attitude towards compound risk. We notice two main (and contradictory) empirical contributions dealing with this relationship. Bernasconi and Loomes (1992) test Segal's hypothesis and conclude:

"the Segal's resolution of the Ellsberg's paradox is, at best, only a partial explanation. [...] Our findings that 'ambiguous lotteries' in the sense of Ellsberg cannot be fully characterized by 'distributed lotteries' as suggested by Segal also undermine the possibility of viewing ambiguity aversion and risk aversion as 'two sides of the same coin'."

More recently Halevy (2007) established empirically that attitude towards ambiguity is related to attitude towards compound risk. In fact, Halevy goes even further and claims that the lack of neutrality towards compound risk, i.e., non reduction of compound lotteries, is necessary for non neutral attitude towards ambiguity.

“The results suggest that failure to reduce compound (objective) lotteries is the underlying factor of the Ellsberg paradox, and call upon decision theory to uncover the theoretical relationship between ambiguity aversion and different forms in which reduction may fail.”

Our experimental evidence confirms the presence of a relationship between attitude towards ambiguity and towards compound risk. However, we do not find that neutrality towards compound risk is strongly associated with neutrality towards ambiguity. Thus, our data suggests that the conclusion in the quote from Halevy above is premature. As we discuss in the next section, we also find that both of these attitudes are strongly influenced by the event on which the good outcome is realized: larger events in the sense of set containment make both ambiguity and compound risk less attractive.

##### 4.4.1 Contradictory findings for probability one-half

We begin with a simple contingency table relating neutrality/non neutrality towards ambiguity and compound risk as revealed by subjects' choices in our treatment giving the good outcome with the probability one-half (under ambiguity, half the colors win). Recall in this treatment we observe a certainty equivalent for the simple lottery  $(50, 1/2; 0)$ , the compound lottery  $((50, 3/4), 2/3; 0)$ , the compound lottery  $((50, 2/3), 3/4; 0)$ , and the ambiguous bet  $(50 \text{ if } 6 \text{ colors; } 0 \text{ otherwise})$ . Reduction of (neutrality towards) compound risk is satisfied if the certainty equivalents for the simple lottery and the two compound lotteries are equal. Ambiguity neutrality is satisfied if the certainty equivalent for the simple risk lottery (de-



#### 4.4. ATTITUDE TOWARDS AMBIGUITY AND ATTITUDE TOWARDS COMPOUND RISK

noted by K) and for the ambiguous bet (denoted by U) are equal. In Table 4.6, we choose to report the probability one-half (under ambiguity, half the colors) treatment first because the simple lottery, compound lotteries and the ambiguous bet studied in Halevy (2007) also have probability one-half (under ambiguity, half the colors).

			Ambiguity (Halevy 2007)			Ambiguity (our data if K=KK)			Ambiguity (our data if K=KK1=KK2)		
			Non neutral	Neutral	Total	Non neutral	Neutral	Total	Non neutral	Neutral	Total
Compound lotteries	Reduction	Count	1	22	23	8	1	9	6	2	8
		Expected	18.5	4.5		7.55	1.45		6.96	1.29	
	No reduction	Count	113	6	119	44	9	53	48	8	56
		Expected	95.5	23.5		44.45	8.55		48.77	9.03	
	Total		114	28	142	52	10	62	54	10	62
	Fisher's statistic		0.0000			1			0.5998		

Table 4.6: Contingency table relating compound risk and ambiguity attitudes for probability one-half

Our results suggest that the relationship (if any) between attitude towards compound risk and ambiguity is more complex than simple identification. To further explore this relationship we use regression analysis to relate the ambiguity premium to the compound risk premium. We begin by defining the ambiguity premium as the certainty equivalent for the corresponding simple lottery minus the certainty equivalent for the ambiguous bet. Similarly, we define the compound risk premium as the certainty equivalent for the corresponding simple lottery minus the average certainty equivalent across the two compound risks. The question of the link between reduction of compound lotteries and attitudes towards ambiguity has never been studied letting varying the probability level. The regression below pool observations across all likelihood treatments and then consider each likelihood treatment separately.

### 4.4.2 Regressions

As a first step to study the link between the ambiguity premium ( $U - K$ ) and the compound risk premium ( $KK - K$ ), we pool all the observations and use dummies for likelihood treatments. We estimate the following model:  $U - K = a(KK - K) + \frac{t_2}{12} \frac{d_2}{12} + \frac{t_4}{12} \frac{d_4}{12} + \frac{t_6}{12} \frac{d_6}{12} + \frac{t_8}{12} \frac{d_8}{12} + \frac{t_{10}}{12} \frac{d_{10}}{12} + \frac{t_{11}}{12} \frac{d_{11}}{12} + c + e$  where  $d_i/12$  is the dummy for the likelihood treatment  $i/12$ ,  $c$  is the intercept and  $e$  the error term.

U-K	KK-K	d2	d4	d6	d8	d10	d11	Intercept
Coef.	0.80	-0.26	-0.13	-0.22	-0.00	0.23	0.05	-1.23
Std. Err.	0.2199	0.2854	0.2696	0.2812	0.2723	0.2650	0.2898	0.4028
P> t	0.000	0.350	0.628	0.420	0.971	0.383	0.865	0.002

R-squared = 0.2618; Adj R-squared = 0.2497

Table 4.7: Pooled regression of the compound risk premium on the ambiguity premium

Wald test accepts the equality of all the coefficients ( $F(6, 426) = 1.10$ ,  $\text{Prob} > F = 0.3609$ ). One can see that the relationship between the two premia is highly significant ( $p=0.000$ ,  $\text{coef}=0.8080$ ), though quite partial ( $R^2=0.2618$ ). Examination of the scatter plot shows that linear regression is not a bad approximation of this relationship.

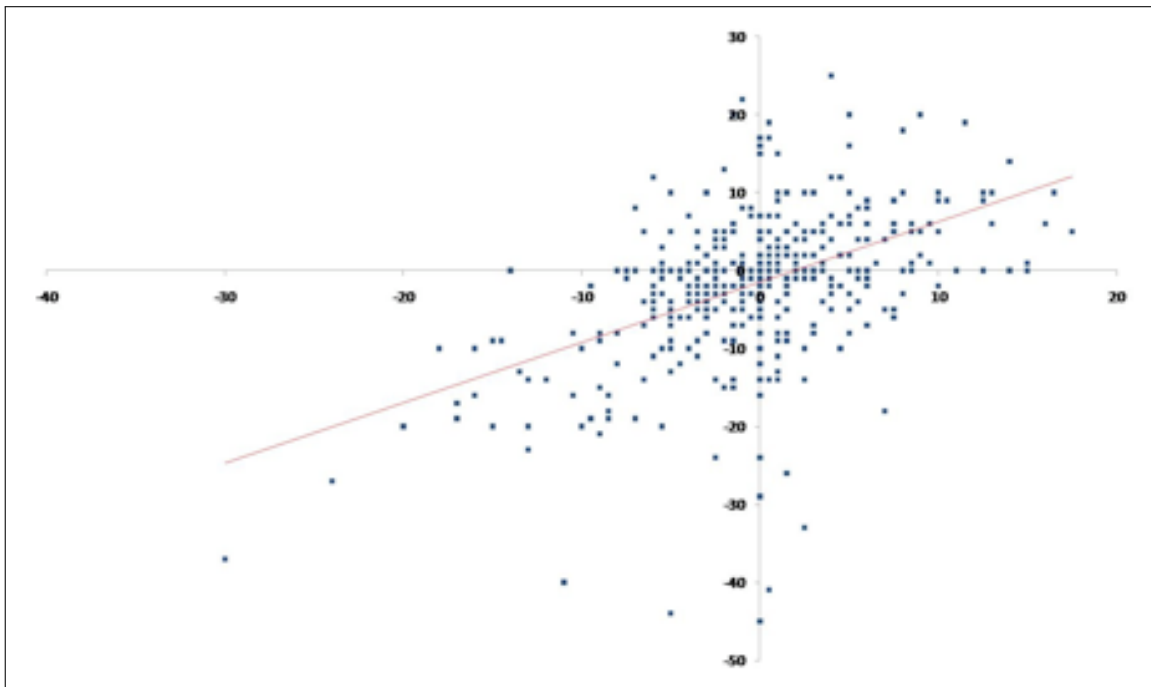


Figure 4.2: Scatter plot: Ambiguity premium as a function of compound risk premium

Further note that the intercept in this relationship is significantly different from zero ( $p=0.0022$ ), demonstrating that if subjects reduce compound risk the best prediction is they are non neutral towards ambiguity. More specifically, since the coefficient estimate is significantly negative ( $p=0.0011$ ), they will tend to be ambiguity averse. This finding, together with the modest explanatory power of the regression is our basis for questioning the concluding quote in Halevy (2007).

We then turn to an aspect of both these attitudes that could not be examined by earlier works: the effects of different likelihood treatments.

#### 4.4.3 The impact of likelihood on attitudes towards ambiguity, risk and compound risk

Possibly the most striking aspect revealed by our data is the strong and systematic effect of likelihood of the good outcome on attitudes towards all three types of uncertainties. The

effects we find are to our knowledge entirely novel with regard to compound risk and are consistent with the small body of existing literature examining this for ambiguity. For simple risk, this aspect of our findings is consistent with a large body of previous work. Our basic finding is that increases in likelihood of the good outcome (under ambiguity, making the winning event larger) make both the ambiguity and compound risk premia increase while also increasing the simple risk premium. This highlights the importance of controlling for likelihood when measuring attitudes towards uncertainty. It also suggests that any descriptive model intended to apply to the full range of uncertain situations must allow attitudes to change with likelihoods.

Table 4.8 displays the median and mean certainty equivalent for the simple risk (K), the mean compound risk (KK) and ambiguity (U). It shows that the standard deviation is always higher for ambiguity compared with the two sorts of risk, and that it increases with the size of the winning event.

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Likelihood	Expected Value	Urn	Median	Mean (SD)
1/12 (1/2.2/12)	4.1667	K	6.50	7.41 (4.5206)
		KK	7.50	8.37 (4.3998)
		U	9.50	10.16 (6.9048)
2/12 (1/2.4/12)	8.3333	K	10.00	11.12 (5.1158)
		KK	12.00	12.27 (5.2936)
		U	12.00	13.32 (7.3586)
4/12 (1/2.2/3)	16.6667	K	16.50	18.03 (5.8442)
		KK	17.00	18.74 (5.9511)
		U	16.50	17.64 (7.7773)
6/12 (2/3.3/4)	25	K	24.50	24.87 (5.3139)
		KK	25.50	25.50 (6.5082)
		U	24.50	23.22 (8.4663)
8/12 (1/2+1/3.1/2)*	33.3333	K	32.50	31.35 (6.1693)
		KK	33.00	32.01 (6.0861)
		U	30.50	28.40 (9.5946)
10/12 (1/2+2/3.1/2)	41.6667	K	40.50	41.09 (4.9405)
		KK	38.50	37.41 (6.0842)
		U	37.00	34.06 (10.8682)
11/12 (1/2+5/6.1/2)	45.8333	K	45.50	45.27 (3.4185)
		KK	44.00	42.16 (5.5845)
		U	44.00	39.83 (10.5845)

\*: This notation means that at the first stage with probability one-half you win and with probability one-third you continue to the second stage and with probability one-sixth you lose. In the second stage you win with probability one-half.

Table 4.8: Impact of likelihood level on certainty equivalent for simple risk, compound risk and ambiguity

Table 4.9 describes the average premia. An ANOVA with repeated measures rejects the equality of means across likelihood treatments for each premium ( $p=0.000$  for each premium). We observe that, in average, subjects tend to be simple risk seeking for probability less than  $1/3$  and risk lover for higher probability levels. They tend to be compound risk seeking except

## 4.5. SUMMARY AND CONCLUSION

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for very high probability (10/12 and 11/12). Ambiguity loving is predominant for very small winning events (one ball wins; two ball win) then subjects turn to be averse for the remaining likelihood levels.

Likelihood treatment	Average premia		
	Simple risk (EV-ECK)	Compound risk (ECK-ECKK)	Ambiguity (ECK-ECU)
1/12	-3.2527	-0.9586	-2.7419
2/12	-2.7956	-1.1452	-2.1936
4/12	-1.3656	-0.7096	0.3871
6/12	0.1290	-0.6371	1.6452
8/12	1.9785	-0.6613	2.9516
10/12	0.5698	3.6774	7.0323
11/12	0.5591	3.1129	5.4355

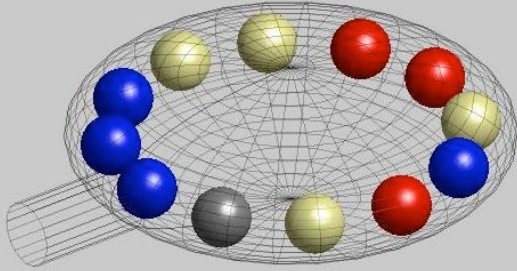
Table 4.9: Uncertainties premia

## 4.5 Summary and conclusion

We obtain no compelling evidence that we necessary need a theory that incorporates both ambiguity and reduction of compound lotteries. A descriptively valid theory should account for compound risk attitude and ambiguity attitude separately and with respect to likelihoods of events.

## Appendix

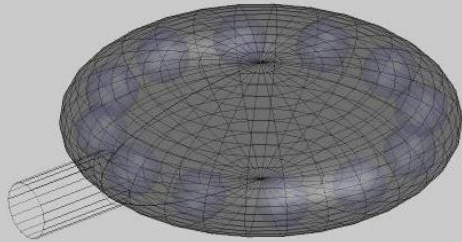
Which option do you choose ?

OPTION 1	1 2	OPTION 2
Play the lottery below		Receive this amount for sure
	<input type="radio"/> <input type="radio"/>	€ 0
	<input type="radio"/> <input type="radio"/>	€ 10
	<input type="radio"/> <input type="radio"/>	€ 20
	<input type="radio"/> <input type="radio"/>	€ 30
	<input type="radio"/> <input type="radio"/>	€ 40
	<input type="radio"/> <input type="radio"/>	€ 50

Receive € 50 if ■ or ■ or ■  
Receive € 0 if ■

Figure 4.3: Display - Risky urn (first step)

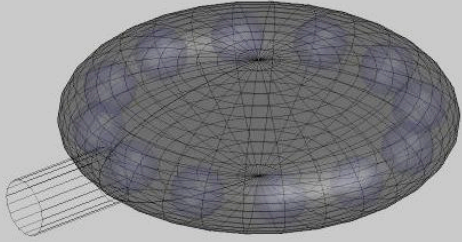
**Which option do you choose ?**

OPTION 1 <small>Play the lottery below</small>	1 2	OPTION 2 <small>Receive this amount for sure</small>
	<input type="radio"/> <input type="radio"/>	€ 0
	<input type="radio"/> <input type="radio"/>	€ 10
	<input type="radio"/> <input type="radio"/>	€ 20
	<input type="radio"/> <input type="radio"/>	€ 30
	<input type="radio"/> <input type="radio"/>	€ 40
	<input type="radio"/> <input type="radio"/>	€ 50

Receive € 50 if ■ or ■ or ■ or ■ or ■ or ■ or ■ or ■  
 Receive € 0 if ■ or ■ or ■ or ■

(a) Ambiguous urn (first step)

**Which option do you choose ?**

OPTION 1 <small>Play the lottery below</small>	1 2	OPTION 2 <small>Receive this amount for sure</small>
	<input type="radio"/> <input type="radio"/>	€ 20
	<input type="radio"/> <input type="radio"/>	€ 21
	<input type="radio"/> <input type="radio"/>	€ 22
	<input type="radio"/> <input type="radio"/>	€ 23
	<input type="radio"/> <input type="radio"/>	€ 24
	<input type="radio"/> <input type="radio"/>	€ 25
	<input type="radio"/> <input type="radio"/>	€ 26
	<input type="radio"/> <input type="radio"/>	€ 27
	<input type="radio"/> <input type="radio"/>	€ 28
	<input type="radio"/> <input type="radio"/>	€ 29
<input type="radio"/> <input type="radio"/>	€ 30	

Receive € 50 if ■ or ■ or ■ or ■ or ■ or ■ or ■ or ■  
 Receive € 0 if ■ or ■ or ■ or ■

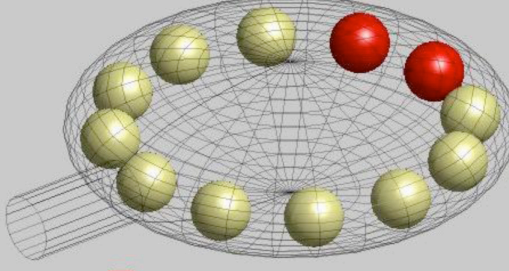


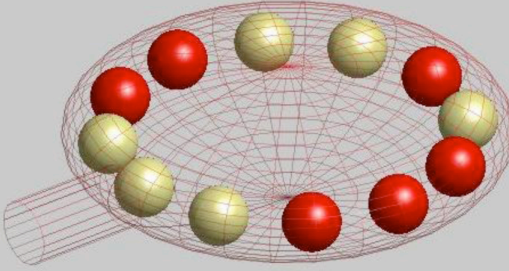


(b) Ambiguous urn (second step: refinement)

Figure 4.4: Display - Ambiguous urn





Which option do you choose ?

OPTION 1	1 2	OPTION 2
Play the lottery below		Receive this amount for sure
	<input type="radio"/> <input type="radio"/>	€ 0
Play the red lottery if  Receive € 0 if 	<input type="radio"/> <input type="radio"/>	€ 10
	<input type="radio"/> <input type="radio"/>	€ 20
	<input type="radio"/> <input type="radio"/>	€ 30
Receive € 50 if  Receive € 0 if 	<input type="radio"/> <input type="radio"/>	€ 40
	<input type="radio"/> <input type="radio"/>	€ 50

Continue

Figure 4.6: Display - Compound urns (first step)

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## **Part III**

# **MACHINA PARADOX: A CHALLENGE FOR AMBIGUITY MODELS?**

## Chapter 5

# Machina Paradox and CEU: An Empirical Evidence

In a recent paper, Machina (2009) suggested choice problems in the spirit of Ellsberg (1961) which challenge tail-separability, an implication of Choquet Expected Utility (CEU) to a similar extent as the Ellsberg paradox challenged the sure-thing principle implied by Subjective Expected Utility (SEU). We have tested choice behavior for bets on one of Machina's choice problems, the reflection example. Our results indicate that tail-separability is violated by a large majority of subjects (over 70% of the sample). These empirical findings complement the theoretical analysis of Machina (2009) and, together, they confirm the need for new approaches in the analysis of ambiguity for decision making.

## 5.1 Introduction

In the past twenty years, there has been a growing attention in decision theory and decision analysis toward ambiguity (Schmeidler, 1989; Gilboa and Schmeidler, 1989; Camerer and Weber, 1992; Fox and Tversky, 1995; Halevy, 2007). Simply stated, ambiguity may be defined as uncertainty about unknown probability. The starting point of any study on ambiguity is Ellsberg's well-known two-color example. In this thought experiment the decision maker (DM) has a choice between two bets: betting a sum of money on a red ball drawn from an urn with 50 red balls and 50 black balls or betting the same sum of money on a red ball drawn from a 100-ball urn with unknown numbers of red balls and black balls. Ellsberg predicted that most people would prefer to bet on the first urn and defined this behavior as ambiguity aversion. In such a case, the DM prefers the first urn because it provides a clear information - known probabilities - rather than a vague information - unknown probabilities - about the likelihood of receiving the sum of money.

In his 1961 article, Ellsberg also proposed the following choice problem, known as the three-color example. An urn contains 90 balls, 30 of which are red and 60 are either yellow or black in unknown proportion. One ball will be drawn at random. An act pays a particular sum of money depending on the color of the ball drawn. Table 5.1, below, presents four acts similar to those of Ellsberg (1961). We use different sums of money here in accordance with the design of our experiment. For example, act  $f_1$  has two outcomes: 50 in the event of a red ball, and 0 in the event of a yellow or black ball. Similarly,  $g_1$  pays 50 in the event of a black ball and 0 otherwise.



Acts	30 balls	60 balls	
	Red	Yellow	Black
$f_1$	50	0	0
$g_1$	0	0	50
$\hat{f}_1$	50	50	0
$\hat{g}_1$	0	50	50

Table 5.1: Ellsberg Acts

It has been widely documented (Slovic and Tversky, 1974; MacCrimmon and Larsson, 1979) that people prefer act  $f_1$  to act  $g_1$ . As an explanation similar arguments as in the two-color urn problem have been put forward: there is precise information about the likelihood of receiving 50 in act  $f_1$ , as opposed to act  $g_1$ , where the range of likelihood is between 0 and  $2/3$ . Aversion to this lack of information about the outcome 50 in act  $g_1$  has been identified as a potential cause for the exhibited preferences.

The same aversion to lack of information leads to a preference for act  $\hat{g}_1$  over  $\hat{f}_1$ , because there is a precise  $2/3$  chance of getting 50 in  $\hat{g}_1$  whereas there is imprecise probability ranging between  $1/3$  and 1 of getting 50 in  $\hat{f}_1$ . Choice situations like these, which involve acts over events that have imprecise probabilities (e.g., the events "the drawn ball is black", "the drawn ball is yellow" or "the drawn ball is black or red" and "the drawn ball is yellow or red") have been termed ambiguous.

While most people choose act  $f_1$  over  $g_1$  and also prefer act  $\hat{g}_1$  over  $\hat{f}_1$ , which has been interpreted as ambiguity aversion, there are a few people exhibiting the opposite, ambiguity seeking choice behavior (Slovic and Tversky, 1974). Both patterns of choices are in contrast to what subjective expected utility (SEU) would predict. Under SEU, preferences must be consistent in the sense that  $f_1$  is preferred to  $g_1$  if and only if  $\hat{f}_1$  is preferred to  $\hat{g}_1$ . This principle of consistency is a direct implication of the sure-thing principle (Savage, 1954), which

requires preferences to be independent of common outcomes. Hence, the choice between acts  $f_1$  and  $g_1$  should not depend on the common outcome that obtains if the event "the ball is yellow" occurs. More precisely, the sure-thing principle says that the common outcomes can be replaced by any other common outcomes without influencing the preference, hence, in this example, 0 can be replaced by 50 for the event "the ball is yellow". But notice that this transforms the choice problem  $f_1$  versus  $g_1$  into the problem  $\hat{f}_1$  versus  $\hat{g}_1$ . This way Ellsberg uncovered a major descriptive shortcoming of Savage's SEU.

Generalizations of SEU have been developed in order to tackle the issues raised by Ellsberg and more generally to take into account sensitive behavior towards ambiguity. Among the most influential of these theories, Multiple Priors (Gilboa and Schmeidler, 1989), Choquet Expected Utility (CEU: Gilboa, 1987; Schmeidler, 1989; Sarin and Wakker, 1992) and Cumulative Prospect Theory (CPT: Tversky and Kahneman, 1992; Wakker and Tversky, 1993). Although we present our results in relation to a fundamental principle underlying CEU, our findings also apply to CPT because, if consequences are all gains (or all losses), CPT agrees with CEU.

The main derivations of CEU build up the idea of rank-dependence introduced for risk by Quiggin (1982). Paraphrasing Diecidue and Wakker (2001), the intuition of rank-dependence may be expressed as the fact that "the attention paid to an event depends not only on the event but also on how good the outcome yielded by the event is in comparison to the outcomes yielded by the other events". This has two main consequences. First, attitudes toward risk are no longer modeled solely through the utility function but also through the perception of risk and uncertainty. Second, not only the likelihood of an event matters but also its ranking compared to others possible events. More accurately, non-expected utility with rank-dependence (including CEU) restricts the sure-thing principle to comonotonic acts and this can explain the pattern of preference derived from the Ellsberg paradox (Chew and Wakker, 1996). Comonotonicity may be defined as follows: if two acts have the same ranking of

events, then any change of their common outcomes that does not change the ranking of events should leave the preference between these acts unaffected. Under the comonotonic sure-thing principle, preferences must be independent of common outcomes only for comonotonic acts. In Ellsberg's three-color example, the event "the ball is yellow" is rank-ordered differently in the two proposed choices: it is of rank 0 (associated with the worst consequence) in choice between  $f_1$  and  $g_1$  and of rank 1 (associated with the best consequence) in choice between  $\hat{f}_1$  and  $\hat{g}_1$ . As a consequence, acts are not comonotonic in the Ellsberg's three-color example, and independence does not need to hold.

The comonotonic sure-thing principle is, on its own, more general than tail-separability, the condition we test experimentally. According to tail-separability if two acts have the same tail, on best or worst outcomes, then any change of their common tail should leave the preference between these acts unaffected. CEU implies both tail-separability and the comonotonic sure-thing principle. Tail-separability may be defined in two distinct ways depending on whether indifference between adjacent outcomes is allowed or not. In the former case, we have weak tail-separability, in the latter case, strong tail-separability.

Two reasons may explain CEU's success. First, CEU keeps the main structure of SEU but introduces more realistic, but still measurable, features of individual behavior. The idea of rank-dependence inherent to CEU has proven to be capable of explaining both the observed deviations from expected utility (the Allais paradox and the Ellsberg paradox). Second, CEU provides generalizations of classic results in various areas of economics such as insurance demand, portfolio choice and asset pricing, or inequality measurement (see Mukerji and Tallon, 2004 for a survey). Throughout this paper we choose to concentrate on CEU as the main and most popular non-expected utility theory but our results may be extended not only to any model that implies tail-separability but also to the main Multiple Priors models (Baillon, L'Haridon and Placido, 2009).

In his *reflection example* Machina (2009) modified the original three-color example of Ellsberg by adding a further imprecise probability event, a fourth color event. This modified Ellsberg urn contains fifty red or yellow balls in unknown proportion and fifty black or green balls in unknown proportion. An important aspect of the structure of the reflection example is *informational symmetry*: notably, there are two symmetric events with precise probabilities (events "the drawn ball is red or yellow" and "the drawn ball is black or green" are equally likely) and further, within the two events "the drawn ball is red or yellow" and "the drawn ball is black or green" the ambiguity about the distribution of colors is similar. Machina illustrated that having two informationally symmetric sources of ambiguity poses serious difficulties for tail-separability under CEU. More precisely, Machina showed that a specific replacement of common outcomes at the tails of acts with other common outcomes leads to a reflected pair of acts that were informationally symmetric to the original acts, so that a preference in the former pair of acts would be reflected in the latter pair of acts, contradicting the consistency requirement under CEU.

We present details of the reflection example in Section 5.2, where we also illustrate that CEU requires consistent choice behavior between acts and their reflected, informationally symmetric, acts. We tested these predictions in our experiment and found that more than 70% of subjects violate models implying weak tail-separability suggesting sensitivity to the informationally symmetric structure in Machina's choice problem.

The remainder of the chapter is organized as follows: Section 5.2 presents the theoretical and conceptual framework. Section 5.3 details the experimental studies. We report the findings in Section 5.4. In Section 5.5 we confront the results in respect to CEU and attitudes towards ambiguity.

## 5.2 Framework

In this section we recall the reflection example of Machina (2009). To have a clear picture of the challenge for the non-expected utility theories, including the rank-dependent theories, it is important to introduce some notation. We recall briefly the classical subjective expected utility model of Savage (1954) and present the sure-thing principle that underlies this theory before we look at variations of this property that underpin rank-dependent theories.

As in the framework of Savage, we assume a state space  $S$ , subsets of which we call events. An act  $f$  assigns to each event a consequence. For simplicity of exposition, the set of consequences is  $\mathbb{R}$ , designating money. For our purposes it will be sufficient to look at simple acts, that is, acts that have only finitely many consequences. An act, therefore, can be represented as  $f = (E_1, x_1; \dots; E_n, x_n)$  for a natural number  $n$ , with the understanding that  $x_i$  is obtained if event  $E_i$  is true. With this notation it is implicitly assumed that the collection of events  $\{E_1, \dots, E_n\}$  form an (ordered) partition of the state space  $S$ , that is, they are mutually exclusive and exhaustive.

We assume a preference  $\succsim$  over acts, denoting weak preference, and we adopt the usual notation  $\succ$  and  $\sim$  for strict preference and indifference, respectively ( $\precsim$  and  $\prec$  denote corresponding reversed preferences). Next we look at different models to evaluate acts such that the assigned values allow for a comparison of acts in agreement with the preference  $\succsim$ . That is, we consider functions  $V$  that assign to each act a real value such that  $V(f) \geq V(g)$  whenever  $f \succsim g$ , for any acts  $f, g$ .

### 5.2.1 Subjective expected utility

Subjective expected utility (SEU) holds if each act  $f = (E_1, x_1; \dots; E_n, x_n)$  is evaluated by  $\sum_{i=1}^n p(E_i)u(x_i)$ . Here  $p$  is a uniquely defined probability measure and the continuous and strictly increasing utility function  $u$ , which assigns to each consequence a real number, is

cardinal.

We write  $h_E f$  for the act that agrees with  $h$  if event  $E$  obtains and otherwise agrees with the act  $f$ . A necessary condition for SEU is that the preference satisfies the sure-thing principle:

$$h_E f \succsim h_E g \Leftrightarrow h'_E f \succsim h'_E g,$$

for all acts  $h_E f, h_E g, h'_E f$  and  $h'_E g$ . Thus, under SEU, the preference between any two acts is independent of consequence-event pairs that are common.

### 5.2.2 Choquet expected utility

Choquet Expected Utility (CEU) holds if each act  $f = (E_1, x_1; \dots; E_n, x_n)$  is evaluated by  $\sum_{i=1}^n \pi(E_j : x_j \geq x_i) u(x_i)$ . The utility under CEU is also cardinal, like under SEU. The difference between the two models consists in the weights that precede utility when evaluating an act. The weights  $p(E_i)$  under SEU are generated by a probability measure, thus, an additive measure on the state space  $S$ , while the weights  $\pi(E_j : x_j \geq x_i)$  under CEU are generated by a (possibly) non-additive measure. This capacity  $v$ , assigns weight 0 to the empty set and weight 1 for the entire state space  $S$  and is monotonic (i.e.,  $v(A \cup B) \geq v(A)$  for all events  $A, B \in S$ ). The decision weights  $\pi(E_j : x_j \geq x_i)$ ,  $i = 1, \dots, n$ , are defined as follows: Take any permutation  $\rho$  of  $\{1, \dots, n\}$  such that  $x_{\rho(1)} \geq \dots \geq x_{\rho(n)}$ . Then,  $\pi(E_j : x_j \geq x_i) = v(\cup_{x_{\rho(k)} \geq x_i} E_{\rho(k)}) - v(\cup_{x_{\rho(k)} \geq x_i, \rho(k) \neq i} E_{\rho(k)})$ ,  $i = 1, \dots, n$ .

Like the probability measure  $p$  under SEU, the capacity  $v$  is uniquely determined under CEU.

A necessary property of rank-dependent utility models is *tail-separability*. It states that, if two acts share a common tail, then this tail can be modified without altering the preference between the acts. Let formally introduce *weak* tail-separability:

$$h_E f \succsim h_E g \Leftrightarrow h'_E f \succsim h'_E g,$$

for all acts  $h_E f, h_E g$ ,  $h'_E f$  and  $h'_E g$ , such that either all outcomes that obtain under event  $E$  are ranked weakly above those of  $f$  and of  $g$  or all outcomes that obtain under event  $E$  are ranked weakly below those of  $f$  and of  $g$ . *Strong* tail-separability holds if we require the previous equivalence to hold whenever all outcomes that obtain under event  $E$  are ranked strictly above those of  $f$  and of  $g$  or all outcomes that obtain under event  $E$  are ranked strictly below those of  $f$  and of  $g$ .

The two variants of tail-separability, both being implications of the sure-thing principle, are equivalent if other standard assumptions are invoked. These assumptions are required under SEU and CEU, under CPT and under the outcome-dependent capacity model of Chew and Wakker (1996). All these models imply both forms of tail-separability.

At this stage it is important to clarify the extent to which our experimental results apply. While we are providing experimental evidence against weak tail-separability our tests do not say anything about strong tail-separability. This point will be further illustrated in subsection 5.2.3 when we review Machina's (2009) refutation example. Moreover, we like to note here that the specific test of weak separability that we focus on is not exclusively a test of CEU and the other rank-dependent theories. Because we test weak tail-separability only by looking at extreme consequences, the results we obtain also provide a test, and as we show a challenge, for other "separable" models that have been put forward in the literature on ambiguity (for a more detailed discussion see Baillon, et al. 2009). However, given the popularity of CEU for the analysis of ambiguity, we present our results in relation to (weak) tail-separability. As an illustration of the general property that we are testing, consider the following four acts (assuming that outcomes are ordered from best to worst and that  $x_{j-1} \geq y_j$  and  $z_{j-1} \geq x_j$ ):

$$h_1 = (G_1, z_1; \dots; G_{j-1}, z_{j-1}; E_j, x_j; \dots; E_n, x_n)$$

$$h_2 = (G_1, z_1; \dots; G_{j-1}, z_{j-1}; E_j, y_j; \dots; E_n, y_n)$$

$$h_3 = (G_1, x_1; \dots; G_{j-1}, x_{j-1}; E_j, x_j; \dots; E_n, x_n)$$

$$h_4 = (G_1, x_1; \dots; G_{j-1}, x_{j-1}; F_j, y_j; \dots; F_n, y_n)$$

The two acts  $h_1$  and  $h_2$  share common outcomes on the same events that give outcomes strictly better than  $x_j$  (ie:  $z_1, \dots, z_{j-1}$ ); they have a *common upper tail*  $(G_1, z_1; \dots; G_{j-1}, z_{j-1})$ . Similarly, acts  $h_3$  and  $h_4$  have a common upper tail  $(G_1, x_1; \dots; G_{j-1}, x_{j-1})$ . Due to tail-separability, a preference for  $h_1$  over  $h_2$  implies the preference for  $h_3$  over  $h_4$  (see Appendix 1 for a proof).

The two acts  $h_2$  and  $h_4$  share common outcomes on the same events that give outcomes strictly lower than  $z_{j-1}$  and  $x_{j-1}$ ; they have a *common lower tail*  $(F_j, y_j; \dots; F_n, y_n)$ . Similarly, acts  $h_1$  and  $h_3$  have a common lower tail  $(E_j, x_j; \dots; E_n, x_n)$ . The same reasoning as before applies and the tail-separability property implies  $h_1 \succsim h_2 \Leftrightarrow h_3 \succsim h_4$ . In other words, in both cases the preference is determined by the tail on which acts differ.

One should remark here that the common tail may not need to be maximal i.e., must not contain the whole sequence.  $(F_n, y_n)$  is a common lower tail for acts  $h_2$  and  $h_4$ , but also any tail formed by the partition  $\{F_n \setminus H, H\}$  of event  $F_n$ . Then the common tail  $(F_n, y_n)$  can be replaced by the common tail  $(F_n \setminus H, w_n, H, w_{n+1})$  if  $w_{n+1} \leq w_n \leq z_{j-1}$ . Such a replacement of a common tail is possible only if outcomes are not required to be strictly rank-ordered:  $x_1 \geq \dots \geq x_n$ . If outcomes are strictly rank-ordered:  $x_1 > \dots > x_n$  it is not possible to split an event further. The former defines *weak tail-separability* which is the matter of this article; the latter, *strong tail-separability* is beyond our scope.

### 5.2.3 The reflection example

Machina (2009) presented two new choice problems, namely the 50:51 example and the reflection example. In our experiment we investigate the latter through an urn containing twenty identical balls except for color. Ten of these balls are red or yellow in unknown proportion, and the remaining ten are black or green in unknown proportion. One ball is drawn at random from the urn. Acts that give different outcomes depending on the color drawn are



described in Tables 5.2 and 5.3. The choice pattern  $f_t \succ g_t$  and  $\hat{f}_t \succ \hat{g}_t$  is shortly designated by  $f_t \hat{f}_t$ . Index  $t$  refers to choices between bets of a table 5. $t$ .

Acts	10 balls		10 balls	
	$R$	$Y$	$B$	$G$
$f_2$	0	50	25	25
$g_2$	0	25	50	25
$\hat{f}_2$	25	50	25	0
$\hat{g}_2$	25	25	50	0

Table 5.2: The reflection example with Lower Tail Shifts

$\hat{f}_2(\hat{g}_2)$  is obtained from  $f_2(g_2)$  as follows. Suppose that  $g_2 \succsim f_2$  is observed (the arguments below also apply if  $f_2 \succsim g_2$  is assumed). We rewrite this to highlight the common lower tails  $(G; 25; R; 0)$  and  $(R; 0)$  of  $g_2$  and  $f_2$ .

$$\begin{aligned}
g_2 &\succsim f_2 \\
&\Leftrightarrow \\
(B, 50; Y, 25; G, 25; R, 0) &\succsim (Y, 50; B, 25; G, 25; R, 0).
\end{aligned}$$

Now we replace only the common tail  $(R, 0)$  with the common tail  $(R, 25)$  and obtain:

$$\begin{aligned}
(B, 50; Y, 25; G, 25; R, 0) &\succsim (Y, 50; B, 25; G, 25; R, 0) \\
&\Leftrightarrow \\
(B, 50; Y, 25; G, 25; R, 25) &\succsim (Y, 50; B, 25; G, 25; R, 25).
\end{aligned}$$

As remarked in the previous section, this latter replacement is allowed under the weak tail-separability. Under the strong tail-separability the common tail  $(R, 0)$  can only be replaced by a common tail  $(R, z)$  with  $z < 25$ , hence, to get the above equivalence, one would need to invoke additional preference conditions like outcome-continuity. However, notice something typical for rank-dependence with weakly rank-ordered outcomes: we can rewrite the acts in

the last indifference by interchanging the order of the events  $G$  and  $R$  without affecting the preference between those acts. This would not be possible when outcomes are required to be strictly rank-ordered. Hence, we obtain the equivalence:

$$(B, 50; Y, 25; G, 25; R, 25) \succsim (Y, 50; B, 25; G, 25; R, 25)$$

$$\Leftrightarrow$$

$$(B, 50; Y, 25; R, 25; G, 25) \succsim (Y, 50; B, 25; R, 25; G, 25).$$

where the last two acts have common lower tails  $(R, 25; G, 25)$  and  $(G, 25)$ . We, finally, replace the common tail  $(G, 25)$  with the common tail  $(G, 0)$  and obtain

$$(B, 50; Y, 25; R, 25; G, 25) \succsim (Y, 50; B, 25; R, 25; G, 25)$$

$$\Leftrightarrow$$

$$(B, 50; Y, 25; R, 25; G, 0) \succsim (Y, 50; B, 25; R, 25; G, 0).$$

where the last preference is equivalent to  $\hat{g}_2 \succsim \hat{f}_2$ .

Notice, that the exercise of replacing common lower tails with other common tails, which transforms the choice problem " $f_2$  vs.  $g_2$ " into " $\hat{f}_2$  vs.  $\hat{g}_2$ " has also lead to a replacement of known probability events with unknown probability events when going from  $f_2$  to  $\hat{f}_2$ . It has also lead to the opposite reflected replacement of unknown probability events with known probability events when going from  $g_2$  to  $\hat{g}_2$ . That is, the precise information that the likelihood of obtaining 25 in the event "the drawn ball is black or green" in act  $f_2$  has now changed into the ambiguous information that the likelihood of obtaining 25 ranges between 0 and 1 in act  $\hat{f}_2$ . Similarly, the imprecise information that the likelihood of obtaining 25 ranges between 0 and 1 in act  $g_2$  has now been changed into the precise information that the likelihood of obtaining 25 in the event "the drawn ball is red or yellow" in act  $\hat{g}_2$ .

Observe that there is informational symmetry when comparing acts  $f_2$  and  $\hat{g}_2$ : there is a 50% chance of getting 25 in each act and an imprecise probability of getting 50 or 0. Except

for the names of the corresponding events there is no informational asymmetry about the outcomes of the respective acts. Likewise, there is informational symmetry between acts  $\hat{f}_2$  and  $g_2$ : there is an imprecise probability  $p$  ranging between 0 and 1/2 of getting 0, an imprecise probability  $q$  ranging between 0 and 1/2 of getting 50, and an imprecise probability,  $1 - p - q$ , of getting 25. So, weak tail-separability at the lower tail has reflected the ambiguity that may have influenced a preference for  $f_2$  over  $g_2$  into a similar situation of ambiguity that may influence a choice of  $\hat{g}_2$  over  $\hat{f}_2$ .

To see why this reflection poses a problem for CEU assume that utility of 0 is 0, and consider the choice pattern  $f_2\hat{g}_2$  (note that a similar argument applies for  $g_2\hat{f}_2$ ). Substitution of CEU gives:

$$\begin{aligned} f_2 \succ g_2 \Rightarrow & v(Y)u(50) + [v(Y \cup B \cup G) - v(Y)]u(25) > \\ & v(B)u(50) + [v(Y \cup B \cup G) - v(B)]u(25) \end{aligned}$$

and

$$\begin{aligned} \hat{g}_2 \succ \hat{f}_2 \Rightarrow & v(B)u(50) + [v(R \cup Y \cup B) - v(B)]u(25) > \\ & v(Y)u(50) + [v(R \cup Y \cup B) - v(Y)]u(25) \end{aligned}$$

Consequently,

$$f_2 \succ g_2 \Rightarrow v(Y) > v(B) \quad (1)$$

$$\hat{g}_2 \succ \hat{f}_2 \Rightarrow v(B) > v(Y) \quad (2)$$

Because the revealed beliefs (1) and (2) are contradictory, informational symmetry leads to preferences  $f_2\hat{g}_2$  or  $g_2\hat{f}_2$  that are not compatible with weak tail-separability. A CEU DM should exhibit either  $f_2\hat{f}_2$  or  $g_2\hat{g}_2$ . A CEU DM who furthermore follows informational symmetry should exhibit  $f_2 \sim g_2$  and  $\hat{f}_2 \sim \hat{g}_2$ .

Acts	10 balls		10 balls	
	$R$	$Y$	$B$	$G$
$f_3$	50	50	25	75
$g_3$	50	25	50	75
$\hat{f}_3$	75	50	25	50
$\hat{g}_3$	75	25	50	50

Table 5.3: The reflection example with Upper Tail Shifts

Table 5.3 shows the reflection example with upper tail shifts. Acts  $\hat{f}_3$  and  $\hat{g}_3$  are obtained from events  $f_3$  and  $g_3$  by an ordered sequence of upper tail shifts.  $\hat{f}_3$  is obtained from  $f_3$  by two successive shifts. First, a shift of the payoffs in event  $G$  from €75 down to €50. Second, a shift of the payoffs in event  $R$  from €50 up to €75. The same applies for the way  $\hat{g}_3$  is obtained from  $g_3$ . As previously these shifts also create a mirror-image effect by making  $f_3$  ( $g_3$ ) symmetric with  $\hat{g}_3$  ( $\hat{f}_3$ ). As a consequence choice patterns which correspond to informational symmetry are  $f_3\hat{g}_3$  or  $g_3\hat{f}_3$  while strict choice patterns which correspond to CEU are  $f_3\hat{f}_3$  or  $g_3\hat{g}_3$ . One may note that  $f_3 \sim g_3$  and  $\hat{f}_3 \sim \hat{g}_3$  is the only choice pattern consistent both with informational symmetry and CEU.

Choice situations presented above enable us to test for preference conditions that allow discriminating between behaviors which are consistent with CEU with weak tail-separability and behaviors which follow informational symmetry. More accurately, any strict choice pattern consistent with informational symmetry violate weak tail-separability and is therefore a preference reversal under a CEU representation based on such hypothesis.

#### 5.2.4 Proper criteria to analyze ambiguity

In an earlier draft of his paper, Machina proposed three criteria to analyze ambiguity for the reflection example. Depending on the criteria retained, a DM's behavior may or may not

be compatible with weak tail-separability. In what follows, we refer to acts of Table 5.2 to highlight these aspects.

### Individual payoffs

Acts  $f_2$  and  $g_2$  ( $\hat{f}_2$  and  $\hat{g}_2$ ) offer €0 on the same event  $R$  ( $G$ ) and €50 on equally ambiguous event  $Y$  and  $B$ . The difference between  $f_2$  and  $g_2$  ( $\hat{f}_2$  and  $\hat{g}_2$ ) lies in the fact that  $f_2$  ( $\hat{g}_2$ ) offers the intermediary outcome €25 with probability one half while  $g_2$  ( $\hat{f}_2$ ) offers the same outcome with a probability that can range from 0 to 1. When considering individual payoffs, the main difference between  $f_2$  ( $\hat{g}_2$ ) and  $g_2$  ( $\hat{f}_2$ ) is based on the nature of the intermediary outcome. For acts  $f_2$  ( $\hat{g}_2$ ), intermediate outcome is not ambiguous while this is the case for acts  $g_2$  ( $\hat{f}_2$ ). Thus, a DM who is ambiguity-averse in terms of individual payoffs would rather choose  $f_2\hat{g}_2$ . If ambiguity is defined as *uncertainty about probability, created by missing information that is relevant and could be known* (Frisch and Baron, 1988), then the probability of winning €25 is the missing information that is relevant and could be known. In that sense ambiguity in terms of individual payoffs coincides with Camerer and Weber (1992)'s "ambiguity about probability". The difference is that ambiguity about probability is defined between urns in the Ellsberg (1961) two-color problem whereas ambiguity in terms of individual payoffs is defined within the reflection example.

### Decumulative payoff events

As before, the best outcome €50 is equally ambiguous under  $f_2$  and  $g_2$  ( $\hat{f}_2$  and  $\hat{g}_2$ ) and the worst outcome €0 is placed on the same event  $R$  ( $G$ ). A closer look at decumulative payoff events shows that  $f_2$  and  $g_2$  yield €25 or more on the same event  $Y \cup B \cup G$ . In term of decumulative payoff events, this event is equally ambiguous across  $f_2$  and  $g_2$  (and  $R \cup Y \cup B$  is also equally ambiguous between  $\hat{f}_2$  and  $\hat{g}_2$ ), the missing information being the same between

acts. As a consequence, if ambiguity is defined in terms of decumulative payoff events a DM would be indifferent between  $f_2(\hat{f}_2)$  and  $g_2(\hat{g}_2)$ . We note that CEU maximizers who follow informational symmetry fall in this category. Decision makers who are indifferent between acts exhibit consistent beliefs and hence reveal no preference reversal. If one considers that following informational symmetry is a necessary condition for a rational choice then indifference is the only behavior consistent with CEU. If individual are not sensitive to informational symmetry but CEU maximizers, they would exhibit strict preferences ( $f_2\hat{f}_2$  or  $g_2\hat{g}_2$ ).

### Exposure to ambiguity

$f_2(\hat{g}_2)$  concentrates ambiguity on the 10 yellow or red balls, whereas  $g_2(\hat{f}_2)$  concentrates this amount over the 20 balls. The missing information that is relevant to the decision is concentrated within the set of 10 yellow or red balls in  $f_2(\hat{g}_2)$  whereas it is distributed over the whole urn in  $g_2(\hat{f}_2)$ . Thus, an individual who is averse to exposure to ambiguity minimize the concentration of missing information and prefers to span ambiguity over the 20 balls rather than over only 10 balls. Then, she will choose  $g_2\hat{f}_2$ .

It worth noticing that, while the second criterion allows for behavior consistent with CEU, the first and third criteria violate weak tail-separability. If one considers that informational symmetry is inherent to the urn and this, independently of any specification of the acts, then a CEU maximizer should be indifferent between both pairs of acts and should satisfy the second criteria only.

The following subsection describes an experimental study mainly based on Machina (2009)'s proposal which aims at testing the validity of the first and third criteria. The validity of the second criteria is discussed in section 5.5.1 through a specific replication of the main experiment.

### 5.3 Experiment

Four groups of subjects (94 students, 39 females and 55 males) enrolled in economics courses at IUFM and Ecole Centrale Paris participated in this experiment. Most of the students were acquainted with probability theory but they had no explicit training in decision theory. The experiment consisted of a pencil and paper questionnaire. Subjects were presented with choice-situations described in the above three tables; each choice-situation was described as the corresponding urn with balls of different colors, and a picture of the urn was also displayed. Subjects could read the composition of the urn and were asked to choose between two options labeled A and B (See Figure 5.1 of the Appendix 2 for a typical display).

As an introduction, subjects were told there were no right or wrong answers, and they had to choose the alternative they preferred. In order to increase motivation, we introduced a random incentive mechanism similar to the one used by Camerer and Ho (1994), Harrison, Lau and Rutström (2007a), and Harrison, List and Towe (2007b). The mechanism worked as follows. In each of the four groups, one of the subjects was randomly selected from that group. Only for these subjects one of their task was selected and their choice was played for real and each selected subject could win up to €75 depending on her responses. Subjects were informed about the mechanism prior to the experiment. There was no time constraint. We controlled for order effects, permuting situations A and B (two groups: 49 students for one and 45 for the other). Moreover, for each subject, we also controlled for color effect in order to guarantee that we effectively captured a preference toward an alternative rather than a preference for a particular color. Thus, we replicated choice situations of Tables 5.2 and 5.3 by reversing payoffs between colors. This prevented subjects from thinking that the ratios of colors were chosen to bias the bets in favor of the experimenters.

In this specific experiment, indifference between options A and B was not allowed. Two

reasons justify such a protocol that forces subjects to express outright choice for one of the two options. First, using strict preference patterns, the protocol generates a sharp distinction between behavior consistent with CEU and behavior consistent with informational symmetry. By making these two behaviors mutually exclusive, we obtained a direct and clear test on the possibility - or the difficulty - to observe a paradox for CEU. Second, the absence of indifference avoids certain choice behaviors such as randomization or indecisiveness. If individuals are subject to randomization or indecisiveness, indifference may be viewed as a way to escape from the choice problem and not as an equivalence judgment between the two options. Not allowing for indifference is therefore a first step to test whether CEU could be prone to a paradox. Section 5.5.1 discusses the importance of the indifference hypothesis and presents results from a replication of the experiment on another set of 42 subjects. We found that, even if indifference is allowed, more than 90% of subjects still express strict preferences.

## 5.4 Results

### 5.4.1 Confirming Ellsberg paradox

A first result is that 65% of subjects exhibit a preference reversal against the SEU prediction. This result confirms the classic Ellsberg paradox and replicates the most commonly observed choice pattern in the three-color example. For example, Slovic and Tversky (1974) also find that 65% of subjects ( $n = 29$ ) violate SEU (percentage raises to 72% after subjects have received arguments pro and con SEU). MacCrimmon and Larsson (1979) found 79% of answers ( $n = 19$ ) inconsistent with SEU.



### 5.4.2 Informational symmetry

Table 5.4 summarizes subjects' choices between acts in Tables 5.2 and 5.3 described above. For each pair of bets, the table gives the number of subjects that chose each of the four possible patterns of choice. The table also provides the proportion of preference reversals observed under weak tail-separability and the significance of this proportion as compared to one half through the p-value of a binomial test.

Modified Ellsberg Acts	Color-treatment	$f_t \hat{f}_t$	$f_t \hat{g}_t$	$g_t \hat{g}_t$	$g_t \hat{f}_t$	% reversal (against CEU)	p-value
Table 5.2 (Lower tail Shifts)	1	11	44	15	24	72%	0.000
	2	10	43	13	28	76%	0.000
Table 5.3 (Upper tail Shifts)	1	8	47	6	33	85%	0.000
	2	4	54	7	29	88%	0.000

Note a: Treatment 2 proposes similar acts to those described in Table 5.2 but reverses payoffs between colors.

Note b: % of reversal is given by the percentage of subjects exhibiting  $f_t \hat{g}_t$  or  $g_t \hat{f}_t$ . The p-value corresponds to a binomial test of the difference between the preference reversal proportion and 0.5.

Table 5.4: Subjects' choices and Preference reversals

Informational symmetry, which is due to both the symmetric structure of the urn and the symmetry between acts, is not violated by a significant proportion of subjects. Indeed, information symmetric behavior corresponding to following patterns  $f_t \hat{g}_t$  or  $g_t \hat{f}_t$  is exhibited by 74% of subjects in the lower tail case and 86.5% in the upper tail case. A necessary condition for CEU to accommodate informational symmetry is  $f_t \sim g_t$  and  $\hat{f}_t \sim \hat{g}_t$ . The results provide evidence against weak tail-separability and highlight the relevance of informational symmetry.

### 5.4.3 A paradox for Choquet expected utility

Table 5.4 shows that violations of weak tail-separability in Machina's reflection example are greater than those observed in Ellsberg urns under SEU. All percentages of preference reversals are above 70% and all are significantly different from 0.5. In the same vein, Wu (1994) empirically finds that more than 50% of subjects violate the ordinal independence axiom under risk, and consequently, that the rank-dependent expected utility model is not sufficient to explain the observed behavior. At odds with our results, Fennema and Wakker (1996) find that only 25 % of subjects violate upper tail-separability under uncertainty. We used the Conlisk's D statistic (Conlisk, 1989) to test and compare preference reversals between upper and lower tail separability. Interestingly, we observe a greater amount of preference reversal under CEU with upper tail-separability than with lower tail-separability (Conlisk's D statistic is  $D = 2.12$ ,  $p = 0.02$  for treatment 1,  $D = 2.28$ ,  $p = 0.01$  for treatment 2). An Anova with repeated measures rejects the equality of preference reversals proportions across the four situations (p-value=0.0027).

### 5.4.4 An empirically-consistent approach for ambiguity

The experiment provides an empirical complement to the three criteria proposed by Machina to analyze ambiguity. Table 5.4 shows that the prevailing pattern of observed choice is  $f_t \hat{g}_t$  (for 50% of the sample on average), agreeing with the idea of ambiguity aversion based on individual payoffs. Moreover, about 30% of the subjects exhibit the pattern  $g_t \hat{f}_t$  which is compatible with an approach of ambiguity in terms of exposure to ambiguity. In order to evaluate to which extent violations of tail-separability are systematic rather than random, we used the Conlisk's Z statistic. This statistic tests whether the percentage of  $f_t \hat{g}_t$  is significantly different from the percentage of  $g_t \hat{f}_t$ . If all Machina's choice problems we found large values of Z ( $Z=3.55$ ,  $p=0.002$  and  $Z= 2.41$ ,  $p=0.007$  for lower tail shifts and  $Z=1.86$ ,  $p=0.03$  and  $Z=3.26$ ,

p less than 0.001 for upper tail shifts). As a consequence, one can conclude that violations of tail-separability are not only frequent but also systematic. Observed choices cannot be justified by errors made by subjects close to indifference.

#### 5.4.5 Other effects

In order to identify possible effects from order, age, gender, color and treatment group, Table 5.5 displays estimates from a panel random-effect probit regression of preference reversals for the four modified Ellsberg choice situations. We find no effect of age and gender on preference reversals and no significant order effect. Moreover, the colors used in the experiment have no effect on preference reversals. The only significant variable is group treatment suggesting that, in our sample, engineers from Ecole Centrale (N=56) are more prone to preference reversals.

Variable	Age	Gender	Order	Students type	Color-treatment
Coefficient	0.0004	0.525	0.035	0.456	0.168
t-statistic	0.00	1.66	0.13	2.97	0.97

Notes: The Log-likelihood value is -170.27; The Wald test for the null hypothesis that all coefficients are equal to zero has a chi-square value of 54.43 with 4 degrees of freedom (p-value=0.00). The fraction of the total variance due to random individual effects is estimated to be 0.446, with a standard error of 0.097.

Table 5.5: Panel random-effect probit regression of preference reversals

## 5.5 Discussion

### 5.5.1 Choquet expected utility versus informational symmetry?

Our results provide experimental evidence for the generalized Ellsberg paradox following Machina (2009). Informational symmetry is an important feature of preferences that calls for a reassessment of rank-dependence and specifically of weak tail-separability implications. The most common observed pattern of choice (47% on average) suggests that individuals are

ambiguity averse in terms of individual payoffs. We also observed a significant proportion of choices (29% on average) compatible with the hypothesis of ambiguity aversion in terms of exposure to ambiguity.

At this stage, our experiment did not account for indifference between acts. This has two main consequences. First, it does not envisage the case where a CEU subject may decide to treat the various reflected events as informationally symmetric. In such a case, according to cumulative dominance (Sarin and Wakker, 1992) she will be indifferent between each pair of acts. Second, since indifference captures ambiguity in terms of decumulative events, we obtained no information about this specific definition of ambiguity. In order to clarify this point, we run a fourth experimental session with 42 students enrolled in economic courses at Ecole Normale Supérieure Cachan. Subjects faced the same questionnaire but have the possibility to express indifference between two acts. Overall only 2 subjects appear to be indifferent between all acts (4.7%) and thus satisfy both CEU and informational symmetry. This suggests that ambiguity in terms of decumulative events is rarely found. Results on preference reversals and informational symmetry remain (79% and 69% for lower tail shifts, 79% and 83% for upper tail shifts, with a majority of subjects ambiguity averse in terms of individual payoffs). Consequently, this fourth session casts doubt on the possibility for informational symmetry to be an inherent part of CEU. Results from this fourth session confirm the preeminence of informational symmetry over CEU even when indifference is allowed. This rules out the possibility of 'informational symmetric Choquet preferences' being the most common observed pattern of choice. One may argue that our protocol involved no precise incentive to express indifference. This may appear as a drawback of our experiment. However if indifference was the dominant pattern, we should have observed random choices rather than systematic choices at the aggregate level.

### 5.5.2 Informational symmetry and editing

Many violations in decision analysis can be explained by cognitive operations which forgo the evaluation of an act. For example, Wu (1994) explains observed violations of tail-separability under risk by a combination of editing and composition rules in a two-stage procedure. At the first stage most people cancel common tails between prospects. Editing rule at hand is that people only cancel common tails that are directly apparent (unapparent common tails are said to be opaque). At the second stage, people use a composition rule that evaluate the lotteries, event by event. This composition rule used under risk is derived from original prospect theory (Kahneman and Tversky, 1979), a theory which was not designed to deal with uncertainty. In our experiment, subjects may have used such an editing operation.

Table 5.2 shows that an editing rule could be applied to cancel out events  $R$  and  $G$  across acts  $f_2$  and  $g_2$  and between acts  $\hat{f}_2$  and  $\hat{g}_2$ . With editing,  $f_2$  is equivalent to  $\hat{f}_2$  and  $g_2$  is equivalent to  $\hat{g}_2$ . Such a cancellation of common tails must yield results consistent with weak tail-separability. Indeed the editing rule proposed by Wu (1994) is more subtle: subjects cancel apparent common tails but not opaque common tails. When we write acts of Table 5.2  $f_2 = (Y, 50; B \cup G, 25; R, 0)$  and  $g_2 = (B, 50; Y \cup G, 25; R, 0)$ , we observe that the only apparent common tail is on event  $R$  while the commonality of event  $G$  is opaque. The way we framed choices to the subjects (see Figure 5.1 of the Appendix 2 for a typical display) promotes such cancelling of the common tail (Erev, Wakker and Weber, 1994, under risk, Fennema and Wakker, 1996, under uncertainty). Let  $f'_2$  and  $g'_2$  be the acts the subject considered once editing is performed. Hence,  $f'_2 = (Y, 50; B \cup G, 25)$  and  $g'_2 = (B, 50; Y \cup G, 25)$ . Thus, the choice appears as a fifty percent chance to win exactly €25 plus an extra ambiguous chance to win €50 ( $f'_2$ ) or as a fifty percent chance to win at least €25 (possibly €50) plus an extra ambiguous chance to win €25 ( $g'_2$ ).

Similarly, after an editing phase, the choice between acts  $\hat{f}_2$  and  $\hat{g}_2$  becomes choice between

$\hat{f}'_2 = (Y, 50; B \cup R, 25)$  and  $\hat{g}'_2 = (B, 50; Y \cup R, 25)$ . Thus, the choice appears also as a fifty percent chance to win exactly €25 plus an extra ambiguous chance to win €50 ( $\hat{g}'_2$ ) or a fifty percent chance to win at least €25 plus an extra ambiguous chance to win €25 ( $\hat{f}'_2$ ). Then, if people cancel only apparent common tails and not opaque tails they would identify  $f_2$  and  $\hat{g}_2$  ( $g_2$  and  $\hat{f}_2$ ) as similar. Such an editing operation *à la* Wu (1994) explains violations of weak tail-separability and is in agreement with informational symmetry. In subsection 5.2.3 we showed that acts  $\hat{f}_2$  and  $\hat{g}_2$  are obtained from  $f_2$  and  $g_2$  through two lower tail replacements from  $(R; 0)$  to  $(R; 25)$  and from  $(G; 25)$  to  $(G; 0)$  under a weak tail separability condition. This common lower tail replacements have been presented in a coalesced form to the subjects (see Appendix 2 for a display) and then could not be considered as transparent. As Birnbaum and colleagues have shown for the special case of risk (see Birnbaum 2008 for a review), presenting choice problems in coalesced form leads to more frequent violations of tail-separability. This suggests that the observed violations of weak tail-separability are mainly a consequence of violation of coalescing.

### 5.5.3 Quality of the CEU model

Most of the empirical work on non-expected utility theories under uncertainty has been carried out with only two outcomes-lotteries (see Abdellaoui, Vossman and Weber, 2005 for a review). Exceptions are MacCrimmon and Larsson (1979), Tversky and Kahneman (1992), Fennema and Wakker (1996), Wu and Gonzalez (1999a, 1999b), Hey, Lotito and Maffioletti (2007) and Diecidue, Wakker and Zeelenberg (2007). Experiments involving two outcomes are well suited to the study of determinants and shapes of decision weights under uncertainty but miss middle-ranked positions. Such positions are important for a general study of rank-dependence. Although rank-dependence has been shown to be a major descriptive improvement when precise probabilities for events are given, things are less clear-cut under

ambiguity. Fennema and Wakker (1996) test upper tail-separability with three-outcome acts in a more general setting than ours. According to their results, the CEU model does not provide any descriptive improvement over SEU. They concluded that "RDU can be of descriptive value in specific domains of decision making" i.e., if the certainty and possibility effect applies in the case of multi-outcomes gambles or within the Ellsberg paradox. Note that our experiment does not apply to risk (see L'Haridon and Placido 2008 on this topic). Under risk, RDU - and then rank-dependence - may still be descriptively superior when compared to Expected Utility. Using de Finetti's betting-odds system, Diecidue, Wakker and Zeelenberg (2007) elicited decision weights under uncertainty with three outcomes in a setting where the certainty effect applies and find evidence for rank-dependence. Using a British Bingo Blower to study individual attitudes towards ambiguity, Hey, Lotito and Maffioletti (2007) also found support in favor of CEU. The results of the present experiment raises the question of the sensitivity of the weak-tail separability to the number of outcomes and to the weight put on intermediate outcomes.

The problem investigated in this paper reveals a deeper, more fundamental question: is rank-dependence a general recipe for the study of ambiguity? In the Ellsberg paradox, subjects face a single source of ambiguity. In the reflection example tested in this paper, there are two sources of ambiguity. The informational symmetric reflections of events within acts lead to violations of comotonicity because they induce specific ambiguity attitudes due to manipulation of ambiguity. By focusing on ambiguity on intermediate events, the reflection example shifts the DM attention away from extreme events. As a consequence, the CEU model with weak tail-separability, which aims precisely at focusing on these extreme events may not predict accurately the DM's choices.

One may interpret Machina's reflection example in terms of sources of ambiguity (Siniscalchi, 2008). Unlike the Ellsberg paradox where only one source of ambiguity exists, the

reflection example offers a choice between acts with two sources of ambiguity ( $R$  vs  $Y$  and  $B$  vs  $G$  for  $g_2$  and  $\hat{f}_2$ , respectively) and acts with a unique source of ambiguity ( $R$  vs  $Y$  for  $f_2$  and  $B$  vs  $G$  for  $\hat{g}_2$ ). Our experimental results suggest that CEU keeps its descriptive value if a single source of ambiguity is considered but a more refined approaches are needed to deal with multiple sources of ambiguity and the information attached to events. The model of Siniscalchi (2008) captures complementarities among ambiguous events and is able to explain the main pattern of preference observed in this experiment.



## Appendix 1

We consider the four following acts described in the main text:

$$h_1 = (G_1, z_1; \dots; G_{j-1}, z_{j-1}; E_j, x_j; \dots; E_n, x_n)$$

$$h_2 = (G_1, z_1; \dots; G_{j-1}, z_{j-1}; F_j, y_j; \dots; F_n, y_n)$$

$$h_3 = (G_1, x_1; \dots; G_{j-1}, x_{j-1}; E_j, x_j; \dots; E_n, x_n)$$

$$h_4 = (G_1, x_1; \dots; G_{j-1}, x_{j-1}; F_j, y_j; \dots; F_n, y_n)$$

Assuming CEU preference for  $h_1$  over  $h_2$  gives:

$$h_1 \succsim h_2$$

$$\Leftrightarrow$$

$$CEU(G_1, z_1; \dots; G_{j-1}, z_{j-1}; E_j, x_j; \dots; E_n, x_n) \geq$$

$$CEU(G_1, z_1; \dots; G_{j-1}, z_{j-1}; F_j, y_j; \dots; F_n, y_n)$$

The value of the common term of  $h_1$  and  $h_2$  is:

$$\sum_{k=1}^{j-2} v(\cup G_k)[u(z_k) - u(z_{k+1})] + v(\cup G_k)[u(z_{j-1})]$$

Replacing  $z_k$  by  $x_k$  for  $k = 1 \dots i - 1$  gives:

$$\sum_{k=1}^{j-2} v(\cup G_k)[u(x_k) - u(x_{k+1})] + v(\cup G_k)[u(x_{j-1})]$$

It follows that:

$$CEU(G_1, x_1; \dots; G_{j-1}, x_{j-1}; E_j, x_j; \dots; E_n, x_n) \geq$$

$$CEU(G_1, x_1; \dots; G_{j-1}, x_{j-1}; F_j, y_j; \dots; F_n, y_n)$$

$$\Leftrightarrow$$

$$h_3 \succsim h_4$$

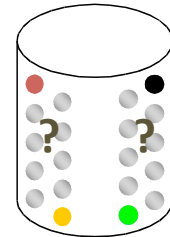
## Appendix 2

### Choice Task n°3

The following urn contains 20 balls:

-10 red or yellow balls in unknown proportion

-10 black or green balls in unknown proportion



A ball will be drawn at random within this urn

Which situation do you choose?

Situation A	Situation B
If the ball is <b>yellow</b> , you receive 50 €	If the ball is <b>black</b> , you receive 50 €
If the ball is <b>black or green</b> , you receive 25 €	If the ball is <b>green or yellow</b> , you receive 25 €
If the ball is <b>red</b> , you receive 0 €	If the ball is <b>red</b> , you receive 0 €

☐
☐

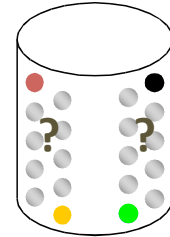
Figure 5.1: A typical display used in the experiment (indifference not allowed)

## Choice Task n°3

The following urn contains 20 balls:

-10 red or yellow balls in unknown proportion

-10 black or green balls in unknown proportion



A ball will be drawn at random within this urn

Which situation do you choose?

Situation A	Situation B
If the ball is <b>yellow</b> , you receive 50 €	If the ball is <b>black</b> , you receive 50 €
If the ball is <b>black or green</b> , you receive 25 €	If the ball is <b>green or yellow</b> , you receive 25 €
If the ball is <b>red</b> , you receive 0 €	If the ball is <b>red</b> , you receive 0 €

☐
☐
☐

Figure 5.2: A typical display used in the replication with indifference allowed

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## Chapter 6

# An Allais-like Paradox for Generalized Expected Utility Theories ?

This chapter reports the results of an experiment which aims at providing a test of ordinal independence, a necessary property of Generalized Expected Utility theories such as Rank-Dependent Expected Utility theory (RDEU). Our experiment is based on a modified version of the Allais paradox proposed by Machina (2007), which allows testing ordinal independence restricted to simple lotteries, i.e. the tail-separability property.

The results tend to support RDEU models since tail-separability is not violated by 71% of subjects while 73% violate the independence condition of classic Allais paradox. This confirms the relative theoretical soundness of RDEU models over Expected Utility model for the particular context of risk.



## 6.1 Introduction

In 1953, Allais severely questioned classical Expected Utility (vNM: von Neumann and Morgenstern, 1944) suggesting that choice behavior could not be consistent with a necessary condition of the theory, the independence axiom. With its empirical confirmations (MacCrimmon and Larson, 1979; Kahneman and Tversky, 1979), the so-called Allais paradox belongs to a broad series of systematic violations of Expected Utility (see Starmer, 2000 for a survey) which compromised Expected Utility as a descriptively valid theory of choice under risk.

Allais (1953) proposes the four lotteries described in Table 6.1.

Probabilities	0,89	0,10	0,01
$l_1$	1 000 000	1 000 000	1 000 000
$l_2$	1 000 000	5 000 000	0
$l_3$	0	1 000 000	1 000 000
$l_4$	0	5 000 000	0

Table 6.1: Allais lotteries

In the vNM theory, as well in the Savage expected utility theory (SEU: Savage, 1954), consistent preferences are those for both  $l_1$  and  $l_3$  or for both  $l_2$  and  $l_4$ . However, most of the individuals express preferences  $l_1 \succ l_2$  and  $l_4 \succ l_3$ , which means under expected utility:

$$u(1000000).0,89 + u(1000000).0,10 + u(1000000).0,01 >$$

$$u(1000000).0,89 + u(5000000).0,10 + u(0).0,01$$

$$u(0).0,89 + u(5000000).0,10 + u(0).0,01 > u(0).0,89 + u(1000000).0,10 + u(1000000).0,01$$

Under the normalization condition  $u(0) = 0$  it implies:

$$0,11.u(1000000) > u(5000000).0,10$$

$$u(5000000).0,10 > 0,11.u(1000000)$$

Together, these inequalities are contradictory. According to the independence axiom (vNM), as well as the sure thing principle (SEU), the change of the common consequence 1 000 000 to 0 from the first to the second pair of lotteries should not reverse the preference within each pair. The intuitive behavioral justification for this paradox is that the change of the common outcome make the status of the lottery  $l_1$  change from a degenerate lottery (that gives one million with certainty) to a risky lottery  $l_3$ . Hence, it loses the advantage of the certainty. Note that this modification marginally affects the original  $l_2$  since  $l_4$  remains risky.

Following Allais' intuition, alternative models were developed in order to allow non linear treatments of probabilities. Among these models, Quiggin's (1982) Rank Dependent Expected Utility (RDEU) model was the first successful, since it avoided violations of stochastic dominance through the introduction of decision weights that incorporate the relative ranking of the outcomes instead of a direct transformation of probability (for other rank-dependent generalizations, see Yaari, 1987; Segal, 1987). All these expected utility generalizations involving rank-dependence are based on a weaker form of the vNM's original independence axiom, namely comonotonic independence. The comonotonicity requirement was further weakened in the ordinal independence property (Green and Jullien, 1988; Quiggin, 1993; Wakker and Zank, 2002), also called tail-separability when restricted to simple lotteries.

Machina (2007) proposes thought experiments in the spirit of Allais and Ellsberg that points out the possible vulnerability of RDEU models through the tail-separability property. Moreover, recent results show that the tail-separability property defined over events challenges the descriptive validity of the counterpart of RDEU under uncertainty, i.e. Choquet Expected Utility (L'Haridon and Placido, 2009). In the present chapter, we provide an empirical test based on Machina's examples in order to empirically confront RDEU in the particular context of risk.

The rest of the chapter is organized as follows: In Section 6.2 we present the general

framework of the experiment. We report the findings of the experiment in Section 6.3. Section 6.4 concludes.

## 6.2 Framework of the experiment

### 6.2.1 Rank-dependent expected utility

We consider an individual who has to make a choice between three-outcome risky lotteries. We restrict the formulation of RDEU to such lotteries. Let  $L = (x_1, p_1; x_2, p_2; x_3, p_3)$  denotes the risky lottery which yields the monetary payoff  $x_i$  with probability  $p_i, i = 1, \dots, 3$ . Monetary payoffs are rank-ordered:  $x_1 \geq x_2 \geq x_3$ . The RDEU of lottery  $L$  is given by  $V_{RDEU}(L)$ :

$$V_{RDEU}(L) = w(p_1)u(x_1) + [w(p_1 + p_2) - w(p_1)]u(x_2) + [1 - w(p_1 + p_2)]u(x_3) \quad (6.1)$$

$u(\cdot)$  is a strictly increasing utility function over payoffs and  $w(\cdot)$  a strictly increasing probability weighting function from  $[0, 1]$  to  $[0, 1]$  with  $w(0) = 0$  and  $w(1) = 1$ .

A decision maker (DM) who has RDEU preferences satisfies all EU axioms except the independence axiom, which is replaced by a similar condition on rank-dependence, the ordinal independence axiom (see Quiggin, 1993 and Marley and Luce, 2005 for a review). Ordinal independence requires that, if two lotteries agree on a given segment of the cumulative distribution function, the value they take on that segment should not affect their ranking (Quiggin, 1993). In our experiment, we focus on tail-separability, a special case of ordinal independence where the common segment of the cumulative distribution function is one of both tails. Tail-separability implies that if two lotteries share a common tail then the substitution of another common tail maintains the preference order between the lotteries.

When the common tail concerns the higher payoffs, we more precisely refer to upper tail-separability. As an illustration, we consider the four following lotteries:

$$L_1 = (x_1, p_1; x_2; p_2; x_3, p_3) \text{ vs. } L_2 = (x_1, p_1; x_2; q_2; x_3, q_3)$$

$$L_3 = (x_1, p'_1; x_2; p_2; x_3, p_3) \text{ vs. } L_4 = (x_1, p'_1; x_2; q_2; x_3, q_3)$$

Lotteries  $L_1$  and  $L_2$  share the common upper tail  $(x_1, p_1)$  and lotteries  $L_3$  and  $L_4$  share the common upper tail  $(x_1, p'_1)$ . Under RDEU, the choice between  $L_1$  and  $L_2$ , ( $L_3$  and  $L_4$ ) depends on the sign of the difference  $V_{RDEU}(L_1) - V_{RDEU}(L_2)$  ( $V_{RDEU}(L_3) - V_{RDEU}(L_4)$ ) and the intensity of preference is given by the absolute amount of this difference (up to a positive affine transformation). Applying (6.1) one gets:

$$V_{RDEU}(L_1) - V_{RDEU}(L_2) = [w(p_1 + p_2) - w(p_1 + q_2)][u(x_2) - u(x_3)] \quad (6.2)$$

$$V_{RDEU}(L_3) - V_{RDEU}(L_4) = [w(p'_1 + p_2) - w(p'_1 + q_2)][u(x_2) - u(x_3)] \quad (6.3)$$

Using  $p_1 + p_2 + p_3 = p_1 + q_2 + q_3 = 1$  and  $p'_1 + p_2 + p_3 = p'_1 + q_2 + q_3 = 1$ , (2) and (3) become

$$V_{RDEU}(L_1) - V_{RDEU}(L_2) = [w(1 - p_3) - w(1 - q_3)][u(x_2) - u(x_3)] \quad (6.4)$$

$$V_{RDEU}(L_3) - V_{RDEU}(L_4) = [w(1 - p_3) - w(1 - q_3)][u(x_2) - u(x_3)] \quad (6.5)$$

As a consequence,  $V_{RDEU}(L_1) - V_{RDEU}(L_2) = V_{RDEU}(L_3) - V_{RDEU}(L_4)$ . Any shift of the common upper tail  $(x_1, p_1)$  ( $x_1, p'_1$ ) between lotteries  $L_1$  and  $L_2$  ( $L_3$  and  $L_4$ ) does not change the preference order neither the intensity of preference.

Similarly, when the common tail concerns the lower payoffs, we more precisely defined lower tail-separability. As an illustration, we consider the four following lotteries:

$$L_1 = (x_1, p_1; x_2; p_2; x_3, p_3) \text{ vs. } L_2 = (x_1, q_1; x_2; q_2; x_3, p_3)$$

$$L_3 = (x_1, p_1; x_2; p_2; x_3, p'_3) \text{ vs. } L_4 = (x_1, q_1; x_2; q_2; x_3, p'_3)$$

Lotteries  $L_1$  and  $L_2$  share the common lower tail  $(x_3, p_3)$  and lotteries  $L_3$  and  $L_4$  share the common lower tail  $(x_3, p'_3)$ . Applying (6.1) to binary choices between  $L_1$  and  $L_2$  and  $L_3$  and  $L_4$  gives:

$$V_{RDEU}(L_1) - V_{RDEU}(L_2) = [w(p_1) - w(q_1)][u(x_1) - u(x_2)] = V_{RDEU}(L_3) - V_{RDEU}(L_4) \quad (6.6)$$

Lower tail-separability applies since any shift of the common lower tail  $(x_3, p_3)$  does not change the preference order and the intensity of preference.

### 6.2.2 Allais-like choices

Machina (2007) proposes choices built on Allais classic paradox that may question RDEU ordinal independence axiom in the same way that the Allais paradox questioned the EU independence axiom. The choices proposed are as follows:

$$L_1 = (75, 0.05; 45, 0.90; 15, 0.05) \text{ vs. } L_2 = (75, 0.05; 60, 0.45; 15, 0.50)$$

$$L_3 = (60, 0.05; 45, 0.90; 0, 0.05) \text{ vs. } L_4 = (60, 0.50; 15, 0.45; 0, 0.05)$$

These modified Allais lotteries are used in our experiment to test ordinal independence through tail-separability. Each pair of lotteries shares both a common upper tail (a 5% chance to get the best consequence) and a common lower tail (a 5% chance to get the worst consequence). The remaining 90% are split among monetary payoffs: concentrated on the intermediary payoff 45 in  $L_1$  and  $L_3$ , split between the intermediary 60 and the worst payoff 15 in  $L_2$  and between the intermediary 15 and the best payoff 60 in  $L_4$ . Common tails become apparent if one writes:

$$L_1 = (75, 0.05; 45, 0.90; 15, 0.05) \text{ vs. } L_2 = (75, 0.05; 60, 0.45; 15, 0.45; 15, 0.05)$$

$$L_3 = (60, 0.05; 45, 0.90; 0, 0.05) \text{ vs. } L_4 = (60, 0.05; 60, 0.45; 15, 0.45; 0, 0.05)$$

Then, choice between  $L_1$  and  $L_2$ , and  $L_3$  and  $L_4$  are given by the sign of the following differences:

$$\begin{aligned} V_{RDEU}(L_1) - V_{RDEU}(L_2) &= -[w(0.50) - w(0.05)]u(60) + [w(0.95) - w(0.05)]u(45) \\ &\quad - [w(0.95) - w(0.50)]u(15) \end{aligned}$$

$$\begin{aligned} V_{RDEU}(L_3) - V_{RDEU}(L_4) &= -[w(0.50) - w(0.05)]u(60) + [w(0.95) - w(0.05)]u(45) \\ &\quad - [w(0.95) - w(0.50)]u(15) \end{aligned}$$

As a consequence:  $V_{RDEU}(L_1) - V_{RDEU}(L_2) = V_{RDEU}(L_3) - V_{RDEU}(L_4)$ . A DM who exhibits preference for  $L_3$  over  $L_4$  should also exhibit a preference for  $L_1$  over  $L_2$ . An individual may prefer  $L_1$  over  $L_2$  because the latter offer a slightly higher chance to get the worst outcome while the chance to obtain the best outcome stays unchanged. However, people may also prefer  $L_4$  over  $L_3$  because the latter offers a higher chance to get the best outcome while the chance to get the worst outcome is unchanged. An individual who exhibit theses preferences violates tail-separability and consequently RDEU.

### 6.2.3 Experiment

Ninety-four students (39 females and 55 males) took part in the experiment. Students were enrolled in economics courses at IUFM and Ecole Centrale Paris. Most of the students were acquainted with probability theory but they had never heard of decision theory. The experiment consisted of a paper-pencil questionnaire where subjects were confronted with the two pairs of binary choices presented above. Subjects were told there were neither right nor wrong answers, and they had to choose the situation they preferred, without any time constraint. We run three sessions and within each session, subjects were informed that one of them would be

randomly selected to have her choice played out for real. In order to control for order effects, we permuted situations on the questionnaire. As an introduction, subjects faced a version of the Allais paradox including monetary payoffs similar to the one used in lotteries  $L_1$  to  $L_4$ . Classic Allais choices were the following:

$$L_{A1}=(\text{€}15,1) \text{ vs. } L_{A2}=(\text{€}75,0.10;\text{€}15,0.89;\text{€}0,0.01)$$

$$L_{A3}=(\text{€}75,0.1;\text{€}0,0.90) \text{ vs. } L_{A4}=(\text{€}15,0.11;\text{€}0,0.89)$$

### 6.3 Results

Table 6.2 summarizes subjects' choices for the two choice situations designed to test tail-separability. For each pair of lotteries, the following table gives the number of subjects that chose each of the four possible patterns of choice. Overall 68% of subjects revealed choices consistent with RDEU and 32% of subjects exhibit a preference reversal under RDEU. Moreover results from the first part of the experiment on Allais paradox also plead in favor of RDEU: 73% of the subjects satisfied the Allais paradox and thus exhibit preference reversals under EU.

Choice	$L_1L_3$	$L_2L_4$	$L_1L_4$	$L_2L_3$
n	45	19	19	11

Table 6.2: Subjects' choices

At the individual level, over the 25 subjects whose answers where EU compatible in the classic Allais paradox part of the experiment, 15 also gave answers RDEU compatible (60%). Among the 69 subjects whose answers were incompatible with EU in Allais, 49 gave answers compatible with RDEU (71%) in modified Allais choices. This left 20 subjects who gave answers

incompatible with both EU and RDEU. The most common pattern of choice was therefore incompatible with EU preferences and the independence axiom but compatible with ordinal independence necessary to RDEU preferences

One should also notice that we found no significant effects from order, age, gender, and session (p-values of correlations between each variable and pattern of choices were all greater than 0.12).

## 6.4 Conclusion

Our results show that the majority of subjects exhibit a behavior that violate the independence axiom (and hence EU) but that is compatible with the ordinal independence axiom (i.e RDEU). This suggests that RDEU models are less vulnerable to independence-type violations in comparison with EU. Thus, we found no "Allais paradox" for generalized expected utility theory. Our results are consistent with existing literature. Weber and Kirsner (1997) show that the number of violation of comonotonic independence are significantly less than those for non comonotonic independence. Wu (1994) tests violations of tail-separability under risk and finds similar results in a different setting. Wu reports a 38% of within-subjects violation of upper tail-separability whereas we found only 32%. Wakker, Erev and Weber (1994) test comonotonic independence. They show that this axiom is well suited for Allais-type choices but loses in performance in more general choice contexts (in particular when the certainty effect does not apply). Our experiment reinforces such evidence.



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## Chapter 7

# Machina Paradox's Collateral Damages for Ambiguity

Machina (2009) introduced two examples that falsify Choquet expected utility, presently one of the most popular models of ambiguity. This chapter shows that Machina's examples do not only falsify the model mentioned, but also four other popular models for ambiguity of the literature, namely maxmin expected utility, variational preferences,  $\alpha$ -maxmin and the smooth model of ambiguity aversion. Thus, Machina's examples pose a challenge to most of the present field of ambiguity.

## 7.1 Introduction

Ellsberg (1961) constructed counterexamples to show the limitations of Savage's (1954) subjective expected utility (SEU). Ellsberg's examples involved a comparison between objective uncertainty (or risk), in which probabilities are clearly determined, and subjective uncertainty, in which they are not. The prevailing preference for objective over subjective uncertainty, known as ambiguity aversion, raised an important paradox for economic theory under uncertainty. Since Ellsberg's canonical work, many models have been developed to generalize SEU, so as to represent this preference for objective over subjective uncertainty.

In the same vein as Ellsberg, Machina (2009) proposed two examples that falsify one of the SEU generalizations, Schmeidler's (1989) Choquet expected utility (CEU). In this chapter, we show that the impact of Machina's examples is not restricted to the model initially targeted. His examples pose difficulties not only for CEU, but also the other four most popular and widely-used models of ambiguity-averse preferences, namely maximin expected utility, variational preferences,  $\alpha$ -maxmin and the smooth model of ambiguity aversion. Consequently, implications for economics are more profound than initially thought.

The chapter proceeds as follows: Section 7.2 formally presents four major models of ambiguity aversion. In Section 7.3 and 7.4, these models are confronted with Machina's examples. Section 7.5 reports further results from the literature and Section 7.6 concludes.

## 7.2 Four popular models of ambiguity-averse preferences

Machina (2009) proposed two examples that point out the vulnerability of CEU. In this section, we present four major models that are as well challenged by Machina's examples. A typical decision problem under uncertainty involves a state space  $S$ , which contains all possible states of nature. Only one of those states is (will be) true, but we do not know which one. By  $D$  we

denote the set of all probability measures (typically denoted  $p$ ) over  $S$ . An act is a mapping from the state space  $S$  to a set of monetary outcomes.  $U_p(f)$  refers to the expected utility of act  $f$  if the probability distribution is  $p$ . Using this notation, Gilboa and Schmeidler's (1989) Maxmin Expected Utility (MEU), also called *multiple priors*, holds if preferences can be represented by

$$MEU(f) = \min_{p \in \Delta} U_p(f), \quad (7.1)$$

where  $\Delta$  is a subset of  $D$  and is called the *set of priors*.  $\Delta$  need not be equal to  $D$ , i.e., decision makers (DMs) may think that some probability distributions in  $D$  are not possible or irrelevant. MEU is the basis of several results in economics and finance. For instance, Dow and Werlang (1992), and Epstein and Wang (1994), followed by many others, have studied the impact of multiple priors in asset pricing. Introducing multiplier preferences, Hansen and Sargent (2001) showed how applications of robust-control theory used to account for model misspecification in macroeconomic modeling are related to MEU.

Maccheroni, Marinacci and Rustichini (2006) proposed a general model, called variational preferences (VP), which captures both MEU and multiplier preferences. Under VP, preferences are represented as follows:

$$VP(f) = \min_{p \in D} \{U_p(f) + c(p)\}, \quad (7.2)$$

where  $c(p) : D \rightarrow [0, \infty]$  is an index of ambiguity aversion assigned to the probability distribution  $p$ . MEU is a special case of VP where  $c(p) = 0$  if  $p \in \Delta$  and  $c(p) = \infty$  otherwise. Hansen and Sargent's (2001) multiplier preferences correspond to a case with  $c$  a function of relative entropy.

The third model, axiomatized by Ghirardato, Maccheroni and Marinacci (2004), is called  $\alpha$ -maxmin ( $\alpha M$ ), which is a linear combination of MEU (maxmin) and its opposite (maxmax),

in which not the worst but the best expected utility is considered. This model directly extends the well-known Hurwitz criterion to ambiguity.  $\alpha M$  holds if preferences can be represented by:

$$\alpha M(f) = \alpha \min_{p \in \Delta} U_p(f) + (1 - \alpha) \max_{p \in \Delta} U_p(f). \quad (7.3)$$

In this model, the set of priors  $\Delta$  and the parameter  $\alpha$  are, respectively, interpreted as ambiguity and ambiguity attitude. Consider an  $\alpha M$  DM facing Ellsberg's three-color urn (an urn with 20 red balls and 40 balls that may be yellow or black, one ball being randomly drawn; the DM has to bet on the color of the ball) and who strictly prefers to bet on red rather than on yellow or on black. Such a DM is clearly ambiguity averse and violates SEU. It can be shown that in such a case,  $\alpha$  must be higher than 1/2.

The fourth model was introduced by Klibanoff, Marinacci and Mukerji (2005). Their approach is slightly different from the previous ones. Their "smooth model of ambiguity aversion", usually called KMM, involves a two-stage decomposition of the decision process into risk and ambiguity. Each stage is represented by an expected-utility-like functional form. Preferences are represented by

$$KMM(f) = \sum_{p \in D} \mu(p) \varphi(U_p(f)), \quad (7.4)$$

where  $\mu$  is a subjective probability measure over  $D$ , that is, the measure of the subjective relevance of  $p$  to be the 'right' probability. Ambiguity attitude is contained in  $\varphi$ . More precisely, concavity of  $\varphi$  implies ambiguity aversion. For instance, a DM who prefers to bet on red rather than on yellow or on black in Ellsberg's urn cannot have a convex  $\varphi$ . Klibanoff, Marinacci and Mukerji (2005) defined ambiguity aversion as aversion to mean preserving spreads in expected utility values (see Section 7.3 for more explanations) and their model deals with

ambiguity aversion as expected utility does with risk aversion. Hence, it is particularly convenient for applications (e.g., in macroeconomics, Hansen, 2007; in health and environmental policy, Treich, 2008; in finance, Gollier, 2006).

In the next two sections, we show precisely how Machina's examples pose difficulties for all the four models presented above.

### 7.3 The 50:51 example

The first example proposed by Machina (2009) is based on an urn with 101 balls. 50 balls are marked with either 1 or 2 and 51 balls are marked with either 3 or 4. Each ball is equally likely to be drawn.  $E_k$  denotes the event "a ball marked with an  $k$  is drawn". Table 7.1 displays the outcomes assigned to each event by four acts. These outcomes are expressed in utility units. We use Machina's flexibility of outcomes to choose outcomes that are equally-spaced on the utility scale<sup>1</sup>. This adaptation of Machina's original example enables us to derive particularly clear counter-examples for MEU,  $\alpha$ M, and VP, but is unnecessary for the KMM model. The specific numbers 0, 101, 202, 303 are proposed for convenience, to simplify some formulas (they are multiples of the number of balls in the urn). These numbers do not constitute any further restriction since under all the models we are dealing with, utility is defined up to unit and level.

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<sup>1</sup>Eliciting such outcomes is rather easy for the four models MEU,  $\alpha$ M, VP, and KMM, because they all assume expected utility under risk. A simple way to determine such outcomes is to fix two outcomes (e.g., \$0 and \$10,000) and find two certainty equivalents: first, the certainty equivalent of the lottery giving the high outcome with probability 1/3 and giving the low outcome otherwise; second, the certainty equivalent of the lottery giving the high outcome with probability 2/3 and the low outcome otherwise. Imagine that these certainty equivalents are \$1,500 and \$4,000. We can conclude that \$0, \$1,500, \$4,000 and \$10,000 are equally-spaced in terms of utility units.

Acts	50 balls		51 balls	
	$E_1$	$E_2$	$E_3$	$E_4$
$f_1$	202	202	101	101
$f_2$	202	101	202	101
$f_3$	303	202	101	0
$f_4$	303	101	202	0

Table 7.1: The 50:51 example

If a DM is sufficiently ambiguity averse, she will prefer  $f_1$  to  $f_2$ , as argued by Machina. Indeed,  $f_1$  is clearly unambiguous whereas  $f_2$  is ambiguous, but benefits from a slight advantage due to the 51<sup>th</sup> ball that may yield 202. There is thus a tradeoff between this advantage offered by  $f_2$  and the absence of ambiguity offered by  $f_1$ . Such a tradeoff is less obvious in the choice between  $f_3$  and  $f_4$ . Like  $f_2$ ,  $f_4$  benefits from the 51<sup>th</sup> ball but  $f_3$  does not offer a particular informational advantage at hand. Two normative principles are conflicting in this example. A Bayesian DM (an Expected Utility maximizer assuming a uniform distribution over the balls) should prefer  $f_2$  and  $f_4$ . And yet, a DM who values unambiguous information may prefer  $f_1$  to  $f_2$  and may be indifferent between  $f_3$  and  $f_4$ . The informational advantage of  $f_1$  can more than offset its Bayesian disadvantage with respect to  $f_2$  whereas  $f_3$  benefits from no such informational advantage that could compensate its Bayesian disadvantage with respect to  $f_4$ . This would lead to  $f_1 \succ f_2$  and  $f_3 \prec f_4$ . However, Machina showed that under CEU,  $f_1 \succ f_2$  if and only if  $f_3 \succ f_4$ . We show that  $f_1 \succ f_2$  also implies  $f_3 \succ f_4$  if the DM's preferences are represented by MEU, VP,  $\alpha$ M, or KMM with  $\varphi$  concave.

Throughout this section, any possible probability distribution over the state space is fully characterized by a pair of numbers  $(i, j)$  where  $i$  denotes the number of balls marked with a 1 and  $j$  denotes the number of balls marked with a 3. There are  $50 - i$  balls marked with a 2 and  $51 - j$  balls marked with a 4.  $D = \{0, \dots, 50\} \times \{0, \dots, 51\}$  refers to the set of all such



distributions.  $U_{(i,j)}(f)$  denotes the expected utility of act  $f$  if the distribution is characterized by  $(i, j)$ . In what follows, we often suppress  $f$  in  $U_{(i,j)}(f)$ .

First consider MEU. For  $f_2$  and  $f_3$ , increasing  $i$  or  $j$  by 1 increases  $U_{(i,j)}$  by 1. For  $f_4$ , it increases  $U_{(i,j)}$  by 2. As a consequence, a MEU DM will take into account the minimum of  $i + j$  for  $f_2$ ,  $f_3$ , and  $f_4$ . The same prior can thus be applied to evaluate the four acts, the prior having no impact on the evaluation of  $f_1$ , the DM knowing for sure the probability of each outcome. With the same prior for the four acts, we are back to SEU. Hence, MEU implies the same restriction as CEU (and SEU):  $f_1 \succ f_2$  if and only if  $f_3 \succ f_4$ .

Under  $\alpha M$ , it can easily be shown that the same result holds. As for MEU, the priors that are used to evaluate  $f_1$ ,  $f_2$ ,  $f_3$  and  $f_4$  are the same. In the 50:51 example,  $\alpha M$  corresponds to SEU with a specific probability distribution:  $\alpha$  times the distribution that minimizes  $i + j$  plus  $(1 - \alpha)$  times the distribution that maximizes  $i + j$  (over the set of priors  $\Delta$ ).

Result 1 in the Appendix establishes that an ambiguity averse DM, who prefers  $f_1$  to  $f_2$ , will violate<sup>2</sup> VP if  $f_3 \prec f_4$ . A similar result can be derived for KMM with  $\varphi$  concave. Using the functional given by (7.4), the values of the acts in the 50:51 example are:

$$\begin{aligned} KMM(f_1) &= \varphi(151), \\ KMM(f_2) &= \sum_{(i,j) \in D} \mu(i, j) \varphi(i + j + 101), \\ KMM(f_3) &= \sum_{(i,j) \in D} \mu(i, j) \varphi(i + j + 100), \text{ and} \\ KMM(f_4) &= \sum_{(i,j) \in D} \mu(i, j) \varphi(2i + 2j + 50). \end{aligned}$$

Figure 7.1 represents the impact of the concavity of  $\varphi$  on the evaluation of the acts for  $i + j < 50$  and  $i + j > 50$ . The case  $i + j = 50$  is straightforward: it implies  $\varphi(151) - \varphi(101 + i + j) = \varphi(100 + i + j) - \varphi(50 + 2i + 2j)$ . For  $i + j < 50$ ,  $U_{(i,j)}(f_1) > U_{(i,j)}(f_2) > U_{(i,j)}(f_3) > U_{(i,j)}(f_4)$ . Moreover, the difference between  $U_{(i,j)}(f_1)$  and  $U_{(i,j)}(f_2)$  on the one hand and  $U_{(i,j)}(f_3)$  and  $U_{(i,j)}(f_4)$  on the other hand is the same. Figure 7.1(a) shows how concavity of  $\varphi$

<sup>2</sup>Note that unlike under CEU, MEU, and  $\alpha M$ ,  $f_1 \prec f_2$  and  $f_3 \succ f_4$  may both hold under VP and KMM. These preferences are however not plausible under natural ambiguity aversion.

implies that  $\varphi(151) - \varphi(101 + i + j) \leq \varphi(100 + i + j) - \varphi(50 + 2i + 2j)$  for all  $(i + j) < 50$ . The same result holds if  $i + j > 50$  as can be seen in Figure 7.1(b). Under KMM, a preference for both  $f_1$  over  $f_2$  and  $f_4$  over  $f_3$  implies  $KMM(f_1) - KMM(f_2) > KMM(f_3) - KMM(f_4)$  which is not possible because  $\varphi(151) - \varphi(101 + i + j) \leq \varphi(100 + i + j) - \varphi(50 + 2i + 2j)$  for all  $(i, j)$ . This leads to a contradiction. A DM with  $\varphi$  concave cannot exhibit both  $f_1 \succ f_2$  and  $f_3 \prec f_4$ . Note that this result can easily be extended to outcomes that are not equally spaced in terms of utility unit, the proof being very similar.

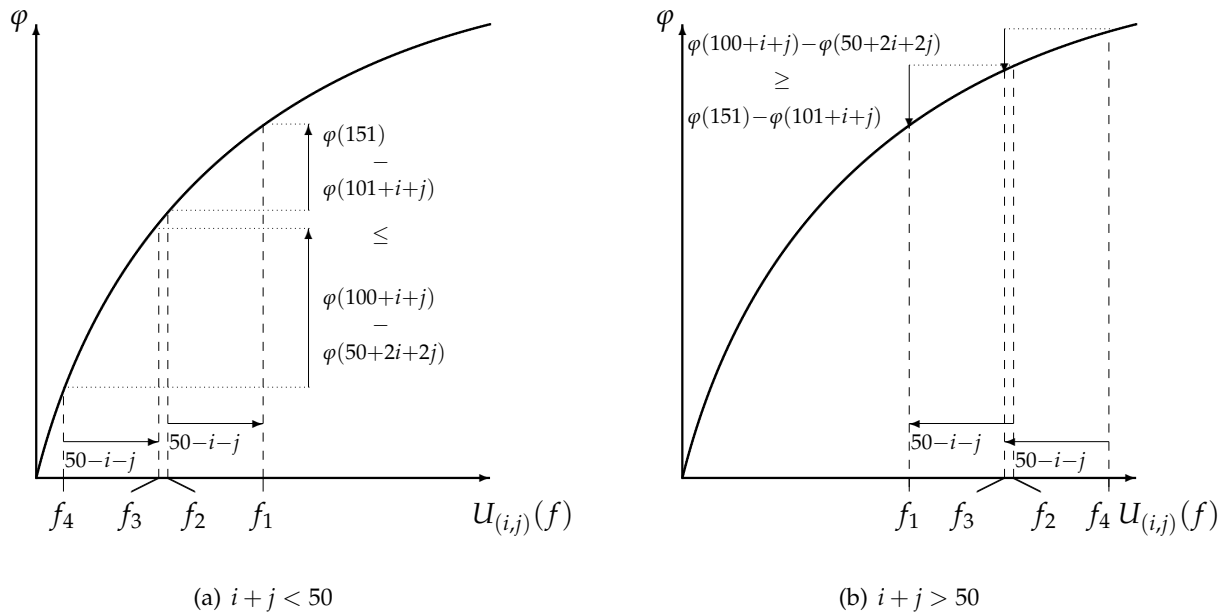


Figure 7.1: Impact of the concavity of  $\varphi$  on the evaluation of the acts

## 7.4 The reflection example

The second example proposed by Machina (2009), the *reflection example*, entails a slight modification of the previous urn; not 51 but 50 balls are marked with a 3 or a 4. Table 7.2 describes four acts assigning outcomes evaluated in terms of utility to the four events (with  $0 < \pi < 1$ ). Unlike in the previous example, the outcomes need not be equally-spaced on the utility scale.

Acts	50 balls		50 balls	
	$E_1$	$E_2$	$E_3$	$E_4$
$f_5$	$100\pi$	<b>100</b>	$100\pi$	0
$f_6$	$100\pi$	$100\pi$	<b>100</b>	0
$f_7$	0	<b>100</b>	$100\pi$	$100\pi$
$f_8$	0	$100\pi$	<b>100</b>	$100\pi$

Table 7.2: The reflection example

### 7.4.1 Decision criteria and experimental results

$E_1$  and  $E_2$  ( $E_3$  and  $E_4$ ) are *informationally symmetric*: there is no more evidence in favor of one event or the other. Moreover, the two pairs  $(E_1, E_2)$  and  $(E_3, E_4)$  are also informationally symmetric. This is why Machina (2009) argues that  $f_8$  is an (*informationally symmetric*) *left-right reflection* of  $f_5$  and  $f_7$  is a left-right reflection of  $f_6$ . As a consequence, there is no reason to prefer  $f_8$  to  $f_7$  if one prefers  $f_6$  to  $f_5$ . We will say that preferences should be *reflected*. Machina shows that under CEU,  $f_5 \prec f_6$  is equivalent to  $f_7 \prec f_8$  and thus preferences should not be reflected, unless indifference holds. Hence, CEU can only account for reflected preferences through indifference ( $f_5 \sim f_6$  and  $f_7 \sim f_8$ ). However, in an experimental study of the reflection example, L'Haridon and Placido (2009) showed that such indifferences are rejected (over 90% of the subjects expressed strict preferences when indifference was allowed) while reflected preferences hold for more than 70% of subjects.

More can be said about the pattern of preferences on this set of acts. Let us begin with a normative point of view. Bayesianism suggests indifference between the four acts, because they lead to the same expected utility (assuming a uniform prior). Decision makers may have to find other criteria unless they accept to be indifferent. We first present two arguments in favor of  $f_5 \prec f_6$  and  $f_7 \succ f_8$  and then two arguments in favor of  $f_5 \succ f_6$  and  $f_7 \prec f_8$ .

In the Ellsberg paradox<sup>3</sup> as presented in Section 7.2, Bayesianism also suggests indifference, but DMs who do not feel indifferent between betting on red or on yellow usually choose acts the outcomes of which are associated with known probabilities. This phenomenon of ambiguity aversion is argued to be a normative principle by Ellsberg (1961). In the reflection example, the outcomes of  $f_5$  and  $f_8$  are not assigned objective probabilities whereas the middle outcome ( $100\pi$ ) has a known probability ( $1/2$ ) in  $f_6$  and  $f_7$ . Consequently, in the light of Ellsberg, it seems natural that  $f_6$  and  $f_7$  are less ambiguous than  $f_5$  and  $f_8$  and hence, that they should be preferred.

Furthermore, consider an act assigning  $100\pi$  to  $E_1$ , and 100 to both  $E_2$  and  $E_3$ . Should a DM prefer to remove  $100(1 - \pi)$  from  $E_2$  (yielding  $f_6$ ) or from  $E_3$  (yielding  $f_5$ )? Subtracting it from  $E_2$  removes an exposure to ambiguity (the respective proportions of balls marked with 1 and with 2 do not matter anymore). We can call this a *categorical change*, from ambiguity to unambiguity. Subtracting the same amount from  $E_3$  only decreases a previously-existing exposure to ambiguity (a utility of 100 is exposed to the ambiguity of  $E_3/E_4$  before the change, but only  $100\pi$  after). This can be viewed as a *gradual change*. For many DMs, a categorical change brings some extra value with respect to a gradual change. As shown by economics of innovation it is often easier to reach a higher utility level by consuming a new good (categorical change) than by increasing the quantity of an already-consumed one (gradual change). Among others, Hausman (1997) and Petrin (2002) provided empirical evidence on the welfare gains of such categorical change over gradual change. Consequently,  $f_6$  (which results from the categorical change) should be preferred to  $f_5$  (and with a similar reasoning,  $f_7$  to  $f_8$ ).

However, Machina argued that both reflected preference patterns are plausible and gives an argument for  $f_5 \succ f_6$  and  $f_7 \prec f_8$ . He explained that  $f_6$  may be perceived more ambiguous than  $f_5$  because  $f_6$  has a utility of 100 riding on the subjective uncertainty implied by  $\{E_3, E_4\}$

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<sup>3</sup>Recall that this paradox involves an urn with 20 red balls and 40 balls that are either yellow or black.

whereas  $f_5$  splits this ambiguous utility between  $100(1 - \pi)$  on  $\{E_1, E_2\}$  and  $100\pi$  on  $\{E_3, E_4\}$ .

A last criterion consists in avoiding mean preserving spreads in terms of expected utility values. This is the way Klibanoff, Marinacci and Mukerji (2005) define ambiguity aversion. Assuming that all the possible probability distributions over the urn are equally likely<sup>4</sup>,  $f_6$  ( $f_7$ ) can be derived by a series of mean preserving spreads in expected utility values from  $f_5$  ( $f_8$ ) (see Result 2 in the Appendix). A DM who would apply this rule should prefer  $f_5$  to  $f_6$  and  $f_8$  to  $f_7$ .

The four normative arguments do not allow us to clearly predict what the preferences should be. However, we can still let the data speak. Up to now, the only experimental test of the reflection example we are aware of has been conducted by L'Haridon and Placido (2009). The typical preference pattern they found was  $f_5 \prec f_6$  and  $f_7 \succ f_8$  (46% of the participants), even if 28% of the subjects exhibited  $f_5 \succ f_6$  and  $f_7 \prec f_8$ . Furthermore, they replicated the Ellsberg paradox and found that  $f_5 \prec f_6$  and  $f_7 \succ f_8$  was still the most common pattern when only the subjects that are clearly ambiguity averse according to the Ellsberg paradox are considered. This confirms that ambiguity averse DMs tend to have this pattern of preferences. As a consequence, one might expect that a model of ambiguity aversion can account for  $f_5 \prec f_6$  and  $f_7 \succ f_8$ . This is what we will check for the four models under consideration in this chapter.

### 7.4.2 Analysis of the reflection example

In what follows,  $(i, j)$  denotes any possible probability distribution over the state space with  $i$  the number of balls marked with a 2 (whereas it denoted the balls marked with a 1 in the previous section) and  $j$  the number of balls marked with a 3. Therefore, there are  $50 - i$  balls

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<sup>4</sup>A sufficient condition for what follows is that  $\mu(i, j) = \mu(j, i)$  where  $\mu$  is a second-order distribution on  $D = \{0, \dots, 50\} \times \{0, \dots, 50\}$  and  $(i, j) \in D$  denotes any possible probability distribution over the state space with  $i$  the number of balls marked with a 2 and  $j$  the number of balls marked with a 3.

with a 1 and  $50 - j$  balls with a 4.  $D = \{0, \dots, 50\} \times \{0, \dots, 50\}$  is the set of all  $(i, j)$  distributions.

First consider MEU. It can be shown that MEU will minimize some linear combinations of  $i$  and  $j$  in  $f_5$  and  $f_8$  whereas it minimizes only  $j$  in  $f_6$  and only  $i$  in  $f_7$ . It is thus impossible for both  $f_6$  and  $f_7$  to be preferred to  $f_5$  and  $f_8$  respectively (see Result 3 in the Appendix). However,  $f_5 \succ f_6$  and  $f_7 \prec f_8$  may hold. MEU predicts that, if preferences are reflected, an ambiguity averse DM will prefer the acts in which none of the outcomes are associated with a known probability. It cannot represent what L'Haridon and Placido (2009) found as being the prevailing ambiguity averse preferences. VP also fails to account for these preferences. The derivation of this result follows the same steps as in the MEU case (Result 4 in the Appendix).

At odds with MEU and VP,  $\alpha M$  can explain the reflection example for any  $\alpha$  as soon as  $\Delta \neq D$  and  $\alpha \neq 1$ . If the set of priors equates the set of all possible distributions ( $\Delta = D$ ), indifference should hold between the four acts. Assume now, for instance,  $\Delta = D - \{(50, 50)\}$  and  $\alpha \neq 1$ . Note that  $i = 50$  or  $j = 50$  are still possible independently. The maximum expected utility ( $50 + 50\pi$ ) is still possible for  $f_6$  and  $f_7$  but not for  $f_5$  and  $f_8$ . Assume that  $\pi \geq 1/2$ . The valuations of the acts are:  $\alpha M(f_5) = \alpha M(f_8) = 50\pi + (1 - \alpha)(49 + \pi)$ , which is smaller than  $\alpha M(f_6) = \alpha M(f_7) = 50\pi + (1 - \alpha)50$ . Similarly assume that  $\pi < 1/2$ . In such a case:  $\alpha M(f_5) = \alpha M(f_8) = 50\pi + (1 - \alpha)(50 - \pi)$ , which is also smaller than  $\alpha M(f_6) = \alpha M(f_7) = 50\pi + (1 - \alpha)50$ . Thus,  $f_5 \prec f_6$  and  $f_7 \succ f_8$  can both hold.

However, this result may be considered counter-intuitive. One may think that the informational symmetry of the decision problem should be present in the set of priors. We will say that the set of priors *replicates* the informational symmetry of the decision problem if  $(i, j) \in \Delta$  implies  $(50 - i, j) \in \Delta$ ,  $(i, 50 - j) \in \Delta$ , and  $(j, i) \in \Delta$ . If  $\Delta$  ( $\Delta \neq D$ ) replicates the informational symmetry<sup>5</sup>,  $f_5 \prec f_6$  ( $f_7 \succ f_8$ ) implies  $\alpha < 1/2$  (see Result 5). As a consequence, either  $\Delta$  does not replicate the informational symmetry or  $\alpha < 1/2$  (or both). In other words, DMs exhibit-

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<sup>5</sup>This excludes the above example  $\Delta = D - \{(50, 50)\}$ .

ing  $f_5 \prec f_6$  and  $f_7 \succ f_8$  must change their preferences for some permutations of  $E_1$  with  $E_2$ ,  $E_3$  with  $E_4$ , or  $(E_1, E_2)$  with  $(E_3, E_4)$  (if  $\Delta$  does not replicate the informational symmetry, the numbers 1, 2, 3 and 4 must matter) or they must prefer to bet on the yellow and on the black balls rather than on the red balls in the aforementioned Ellsberg urn (otherwise,  $\alpha$  cannot be smaller than  $1/2$ ).

Finally, let us study Klibanoff, Marinacci and Mukerji's smooth model of ambiguity.  $f_5 \prec f_6$  and  $f_7 \succ f_8$  imply that a KMM DM cannot have a concave  $\varphi$  (see Result 6), no matter what  $\mu$  is, i.e., whatever a KMM DM thinks about the relevance of each probability distribution. Moreover, if this preference pattern does not depend on the outcomes under consideration,  $\varphi$  must be convex. Similarly, if the Ellsberg paradox holds whatever the color and the outcomes,  $\varphi$  must be concave. This leads to a contradiction.

It is noteworthy that the four models under consideration rule out not only a way of thinking that has been found to be dominant in the first empirical test of the reflection example but also what can be justified by some decision criteria. When choosing a model, an economist might want to have in mind the models' descriptive abilities but one may prefer a model for normative reasons. Machina's examples are an efficient tool to understand what each model implies, both from a normative and from a descriptive viewpoint.

## 7.5 The implications of Machina's examples for other models

Up to now, we have focused on four popular models of ambiguity aversion. Lehrer (2007a) analyzed the impact of the reflection example on two other models: Lehrer's (2009) concave integral for capacities and Lehrer's (2007b) expected utility maximization w.r.t partially-specified probabilities. In both cases, he found that  $f_5 \succ f_6$  and  $f_7 \prec f_8$ , but not the opposite preferences that were experimentally found. As a consequence, these two models have the

same prediction as MEU and VP for the reflection example. Lo (2007) showed similar results for Klibanoff's (2001) version of MEU thanks to an unpublished example proposed by Machina in an earlier draft. Lo's (2007) results are consistent with ours.

Siniscalchi (2008) proposed a model, called Vector Expected Utility (VEU), that is able to account for both the 50:51 and the reflection examples. His model is decomposed into an expected utility term and an adjustment term capturing attitude towards ambiguity. Complementarities among ambiguous events (in the above studied examples,  $E_1$  and  $E_2$  on the one hand,  $E_3$  and  $E_4$  on the other hand, have such complementarities) are represented through adjustment factors. The second term of the VEU model is a function defined over these adjustment factors. It is negative if Chateauneuf and Tallon's (2002) diversification axiom holds. Furthermore, it is negative and concave if Gilboa and Schmeidler's (1989) uncertainty aversion axiom holds (this axiom is necessary for VP and MEU). VEU can handle the preference patterns under focus in the present chapter but this implies that the adjustment function is negative and not concave, meaning that the diversification axiom holds but not the uncertainty aversion axiom.

## 7.6 Conclusion

This chapter shows that Machina's (2009) examples which were initially designed to challenge CEU also question a wide range of ambiguity models. Our results show that, the seminal MEU model but also two major extensions ( $\alpha$ M and VP) and the KMM model, also fail to account for some important patterns of preference implied by Machina's examples. In the 50:51 example, the preferences of ambiguity averse DMs cannot be better represented by MEU, VP,  $\alpha$ M and KMM (with  $\varphi$  concave). In the reflection example, a pattern of preferences that is especially relevant according to informational symmetry and to available experimen-



tal evidence cannot be represented by MEU, VP and KMM (with  $\varphi$  concave). It cannot be represented by  $\alpha M$  unless it satisfies some strong and counter-intuitive restrictions.

Machina's examples, together with Allais's and Ellsberg's, highlight the relevance of thought experiments for clarifying the behavior under uncertainty. Machina's (2009) examples are exciting call for future theoretical and empirical research.

## Appendix

**Result 1.** *In the 50:51 example, VP implies  $f_1 \succ f_2 \Rightarrow f_3 \succ f_4$ .*

We can define  $(i_h, j_h)$  as any element of  $\operatorname{argmin}_{(i,j) \in D} \{U_{(i,j)}(f_h) + c(i, j)\}$ . As a consequence,  $VP(f_1) = 151 + c(i_1, j_1)$ ,  $VP(f_2) = 101 + i_2 + j_2 + c(i_2, j_2)$ ,  $VP(f_3) = 100 + i_3 + j_3 + c(i_3, j_3)$  and  $VP(f_4) = 50 + 2i_4 + 2j_4 + c(i_4, j_4)$ .

First, suppose that  $f_1 \succ f_2$  and  $f_3 \prec f_4$ . Hence,  $50 + c(i_1, j_1) > i_2 + j_2 + c(i_2, j_2)$ . Replacing  $i_4$  and  $j_4$  by  $i_3$  and  $j_3$  in  $VP(f_4)$ , because this can only increase the evaluation of the act, we obtain  $i_3 + j_3 > 50$ . By definition of  $(i_1, j_1)$ ,  $c(i_1, j_1) \leq c(i_3, j_3)$ . The sum of these inequalities gives  $50 + c(i_1, j_1) < i_3 + j_3 + c(i_3, j_3)$ . As  $i_2 + j_2 + c(i_2, j_2) = i_3 + j_3 + c(i_3, j_3)$  must hold, we have  $50 + c(i_1, j_1) < i_2 + j_2 + c(i_2, j_2)$ . This leads to a contradiction.

To show that  $f_1 \prec f_2$  and  $f_3 \succ f_4$  are possible, let us give an example. Imagine that  $c(i, j) = 0$  if  $i + j > 60$ ,  $c(i, j) = 90 - 1.5(i + j)$  if  $30 < i + j \leq 60$  and  $c(i, j) = 135 - 3(i + j)$  otherwise (note that  $c$  is a convex function on the simplex generated by  $D$ ). With such a function, we must have  $i_1$  and  $j_1$  such that  $i_1 + j_1 \geq 60$ ,  $i_2 = j_2 = i_3 = j_3 = 30$  and  $i_4 = j_4 = 15$ . As a consequence,  $VP(f_1) = 151$ ,  $VP(f_2) = 161$ ,  $VP(f_3) = 160$ , and  $VP(f_4) = 155$ . Then, there exists a  $c(i, j)$  such that  $f_1 \prec f_2$  and  $f_3 \succ f_4$ .

**Result 2.** *Assuming  $\mu(i, j) = \mu(j, i) \forall (i, j) \in D$  (where  $\mu$  is a probability measure over  $D$ ),  $f_6$  ( $f_7$ ) can be derived from  $f_5$  ( $f_8$ ) by a series of mean preserving spreads in terms of expected utility values.*

Let  $\mu_{f_h}$  be the density function over the expected utility values induced by  $f_h$  and  $\mu$  ( $h \in \{5, 6\}$ ). We assume that  $\mu(i, j) = \mu(j, i)$ . Note that:

$$U_{(i,j)}(f_5) = i + (50 - i + j)\pi \quad (7.5)$$

$$U_{(i,j)}(f_6) = j + 50\pi \quad (7.6)$$

For all  $(i, j)$  such that  $i = j$ , both  $\mu_{f_5}$  and  $\mu_{f_6}$  assigns  $\mu(i, j)$  to  $i + 50\pi$ .

Let us now consider each  $(i, j) \in D$  such that  $j < i$  and its symmetric distribution  $(j, i)$  (we are thus dealing with every cases satisfying  $i \neq j$ ). If  $1/2 \leq \pi < 1$ :

$$U_{(i,j)}(f_6) < U_{(i,j)}(f_5) \leq U_{(j,i)}(f_5) < U_{(j,i)}(f_6). \quad (7.7)$$

Otherwise ( $0 < \pi < 1/2$ ):

$$U_{(i,j)}(f_6) < U_{(j,i)}(f_5) < U_{(i,j)}(f_5) < U_{(j,i)}(f_6). \quad (7.8)$$

Therefore, whatever  $\pi$  is,  $\mu_{f_5}$  assign  $\mu(i, j)$  and  $\mu(j, i)$  (which are equal, by assumption) to intermediate values while  $\mu_{f_6}$  assigns them to extreme values, moving a probability mass from the center to the tails of the distribution. Moreover, the mean expected utility has not changed because  $U_{(i,j)}(f_5) + U_{(j,i)}(f_5) = U_{(i,j)}(f_6) + U_{(j,i)}(f_6)$ . This correspond to Rothschild and Stiglitz's (1970) definition of a mean preserving spread (for discrete distribution). As a consequence,  $\mu_{f_6}$  can be derived through as many mean preserving spreads in  $\mu_{f_5}$  as there are cases  $j < i$ . The same result can be obtained for  $f_7$  and  $f_8$  by symmetry.

**Result 3.** *In the reflection example, MEU preferences  $f_5 \prec f_6$  and  $f_7 \succ f_8$  cannot both hold.*

We first define  $(i_h, j_h)$  as an element of  $\operatorname{argmin}_{(i,j) \in \Delta} U_{(i,j)}(f_h)$ . Thus we obtain  $MEU(f_5) = i_5 + (50 - i_5 + j_5)\pi$ ,  $MEU(f_6) = j_6 + 50\pi$ ,  $MEU(f_7) = i_7 + 50\pi$  and  $MEU(f_8) = j_8 + (50 - j_8 + i_8)\pi$ .  $f_5 \prec f_6$  and  $f_7 \succ f_8$  imply  $j_6 + i_7 > (i_5 + j_8)(1 - \pi) + (j_5 + i_8)\pi$ . However, the  $j_h$ s and  $i_h$ s being defined as the arguments of the minimization problems,  $j_6$  ( $i_7$ ) cannot be higher than the others  $j_h$ s ( $i_h$ s). As a consequence,  $j_6 + i_7 > (i_5 + j_8)(1 - \pi) + (j_5 + i_8)\pi$  cannot hold under MEU.

**Result 4.** *In the reflection example, VP implies that  $f_5 \prec f_6$  and  $f_7 \succ f_8$  cannot both hold.*

We can define  $(i_h, j_h)$  as any element of  $\operatorname{argmin}_{(i,j) \in D} \{U_{(i,j)}(f_h) + c(i, j)\}$ . We must have  $VP(f_5) = i_5 + (50 - i_5 + j_5)\pi + c(i_5, j_5)$ ,  $VP(f_6) = j_6 + 50\pi + c(i_6, j_6)$ ,  $VP(f_7) = i_7 + 50\pi + c(i_7, j_7)$ , and  $VP(f_8) = j_8 + (50 - j_8 + i_8)\pi + c(i_8, j_8)$ .  $f_5 \prec f_6$  and  $f_7 \succ f_8$  imply  $j_6 + i_7 + c(i_6, j_6) + c(i_7, j_7) > (i_5 + j_8)(1 - \pi) + (j_5 + i_8)\pi + c(i_5, j_5) + c(i_8, j_8)$ . The  $j_h$ s and  $i_h$ s are defined as the arguments of the minimization problems. As a consequence,  $j_6 + c(i_6, j_6) \leq j_5 + c(i_5, j_5)$ ,  $j_6 + c(i_6, j_6) \leq j_8 + c(i_8, j_8)$ ,  $i_7 + c(i_7, j_7) \leq i_5 + c(i_5, j_5)$ ,  $i_7 + c(i_7, j_7) \leq i_8 + c(i_8, j_8)$ . If we multiply the first and the fourth inequalities by  $\pi$  and the two others by  $(1 - \pi)$  and sum up them, we obtain  $j_6 + i_7 + c(i_6, j_6) + c(i_7, j_7) \leq (i_5 + j_8)(1 - \pi) + (j_5 + i_8)\pi + c(i_5, j_5) + c(i_8, j_8)$ , which contradicts the inequality implied by the preferences.

**Result 5.** *If  $\Delta$  ( $\Delta \neq D$ ) replicates the informationnal symmetry of the decision problem,  $f_5 \prec f_6$  ( $f_7 \succ f_8$ ) implies  $\alpha < 1/2$*

We say that the set of priors replicates the informationnally symmetric left-right reflection whenever  $(i, j) \in \Delta$  implies  $(50 - i, j) \in \Delta$ ,  $(i, 50 - j) \in \Delta$ , and  $(j, i) \in \Delta$ . Note that this definition also implies  $(50 - i, 50 - j) \in \Delta$ . Assume that  $(i', j') \in \operatorname{argmin}_{(i,j) \in \Delta} U_{(i,j)}(f_5)$  and  $(i'', j'') \in \operatorname{argmin}_{(i,j) \in \Delta} U_{(i,j)}(f_6)$ . As a consequence of the structure of  $\Delta$  and by reflection,  $(j', i') \in \operatorname{argmin}_{(i,j) \in \Delta} U_{(i,j)}(f_8)$  and  $(j'', i'') \in \operatorname{argmin}_{(i,j) \in \Delta} U_{(i,j)}(f_7)$ .

It also implies that  $(50 - i', 50 - j') \in \operatorname{argmax}_{(i,j) \in \Delta} U_{(i,j)}(f_5)$ ,

$(50 - i'', 50 - j'') \in \operatorname{argmax}_{(i,j) \in \Delta} U_{(i,j)}(f_6)$ ,  $(50 - j', 50 - i') \in \operatorname{argmax}_{(i,j) \in \Delta} U_{(i,j)}(f_8)$  and  $(50 - j'', 50 - i'') \in \operatorname{argmax}_{(i,j) \in \Delta} U_{(i,j)}(f_7)$ .  $f_5 \prec f_6$  (or  $f_7 \succ f_8$ ) implies:

$$\alpha (i' + (j' - i')\pi) + (1 - \alpha) (50 - i' + (i' - j')\pi) < \alpha j'' + (1 - \alpha)(50 - j'') \quad (7.9)$$

$$\alpha (i' - j'' + (j' - i')\pi) < (1 - \alpha) (i' - j'' + (j' - i')\pi) \quad (7.10)$$

By definition of  $i'$ ,  $j'$  and  $j''$  and because of the symmetry of  $\Delta$ ,  $j'' \leq j'$  and  $j'' \leq i'$ . As a consequence,  $(i' - j'' + (j' - i')\pi) \geq (i' - j'')(1 - \pi) \geq 0$ . If  $(i' - j'' + (j' - i')\pi) = 0$ , indifference

should hold. It must thus be strictly positive. Hence,  $\alpha < 1 - \alpha$  and therefore,  $\alpha < 1/2$ .

**Result 6.** *If a KMM DM has a concave  $\varphi$ ,  $f_5 \succ f_6$  and  $f_7 \succ f_8$  cannot both hold.*

In the reflection example, values of the acts are:

$$KMM(f_5) = \sum_{(i,j) \in D} \mu(i,j) \varphi(i + (50 + j - i)\pi),$$

$$KMM(f_6) = \sum_{(i,j) \in D} \mu(i,j) \varphi(j + 50\pi),$$

$$KMM(f_7) = \sum_{(i,j) \in D} \mu(i,j) \varphi(i + 50\pi), \text{ and}$$

$$KMM(f_8) = \sum_{(i,j) \in D} \mu(i,j) \varphi(j + (50 + i - j)\pi).$$

A preference for both  $f_6$  and  $f_7$  against  $f_5$  and  $f_8$  implies

$$\sum_{(i,j) \in D} \mu(i,j) [\varphi(j + (50 + j - i)\pi) - \varphi(j + 50\pi) + \varphi(i + (50 + i - j)\pi) - \varphi(i + 50\pi)] < 0.$$

However, if  $\varphi$  is concave, for all  $(i, j)$ :

$$\varphi(j + (50 + j - i)\pi) - \varphi(j + 50\pi) + \varphi(i + (50 + i - j)\pi) - \varphi(i + 50\pi) \geq 0.$$

To prove this, let us define  $a(i, j) = \varphi(j + (50 + i - j)\pi) - \varphi(j + 50\pi)$ , and

$$b(i, j) = \varphi(i + (50 + j - i)\pi) - \varphi(i + 50\pi).$$

Assume  $j \geq i$ ; hence,  $a(i, j) \leq 0$  and  $b(i, j) \geq 0$ . Note that  $j + 50\pi - (j + (50 + i - j)\pi) = i + (50 + j - i)\pi - (i + 50\pi) = (j - i)\pi > 0$ . Consequently, the same increase (i.e.,  $(j - i)\pi$ ) of the argument of  $\varphi$  is applied to two different levels:  $i + 50\pi$  and  $j + (50 + i - j)\pi$ . If  $j \geq i$ ,  $i + 50\pi \leq j + (50 + i - j)\pi$ .  $\varphi$  being increasing and concave, the impact of an increase of the arguments in terms of  $\varphi$  units should be lower for the highest argument. As a consequence,  $b(i, j) \geq -a(i, j)$ .

The opposite case  $i \geq j$  is obtained by symmetry.

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# Discussion

The discussion is built around two opened questions related to 1. the nature - rational or not - of ambiguity attitudes, and 2. the role of different actors - decision theorists, contradictors and experimentalists - with regards to improvements of the field.

## Are ambiguity attitudes rational?

The existence of ambiguity attitudes is widely supported by empirical evidence. It follows that the question of whether or not DMs *are* ambiguity neutral was quickly cut off<sup>6</sup>; but whether DMs *should* be ambiguity neutral is a still-in-debate question. It leads to discuss three hypotheses:

1. ambiguity attitudes are a convincing and consistent behavioral demonstration of the descriptive insufficiencies of Savage expected utility,
2. ambiguity attitudes result from possible cognitive limitations of DMs who act on the basis of series of heuristics,
3. ambiguity attitudes result from sensitivity to framing effects or to more generally pernicious representations of decision problems.

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<sup>6</sup>An alternative strategy exists: de Finetti (1977) banishes ambiguity by simply stating: "all probabilities are equally well known to ourselves so that ambiguity is meaningless".

While (1) leads to update what has to be taken as normative in decision models, (2) is an in-between position and envisages ambiguity attitudes as a kind of second-best in decisions and (3) leads to qualify ambiguity attitudes as irrational.

### **Is ambiguity neutrality rational?**

The nature, rational or irrational, of ambiguity attitudes, depends on the definition of rationality adopted.

#### **Model-based rationality**

First, a *model-based* definition of rationality would imply that any DM who departs from the standard SEU model by treating objective and subjective probabilities differently, or by failing in assigning a personal probability to an event, is irrational; it follows that a DM is irrational whenever he is not ambiguity neutral. To show the limitation of this approach, let us discuss whether ambiguity neutrality is necessarily rational.

A first perspective is given by Gilboa et al. (2008b). They argue that, because Savage (1954) stays mute about the origin and formation of the prior beliefs, being Savagean is not sufficient for rationality; indeed:

"Many beliefs in super natural phenomena and many superstitions are considered 'irrational' because they conflict with evidence. Yet, nothing prevents a DM from holding such beliefs and also satisfying Savage's axioms. In others words, Savage's axiomatic system restricts choices only to be internally coherent and is therefore insufficient for an intuitive definition of rationality."

Hence, being ambiguity neutral in the meaning of always having the ability to formulate a subjective additive probability on any event is not sufficient for rationality. The assumption about completeness of beliefs that underlies Savage's theory is too strong for rationality.

We conclude that ambiguity attitudes cannot be pointed out as irrational when considering *model-based rationality*. Another side of literature argues that rationality is fundamentally of subjective nature.

### **Subjective rationality**

Gilboa (2008) proposes two definitions of rationality that differ in their between or within person foundations. He defines a decision as objectively rational if one can convince any reasonable person that it is the right thing to do; a decision is subjectively rational if the DM cannot be convinced that it is a wrong thing to do. The within-person definition of rationality would imply that the DM is not willing to change his behavior after having analyzed it. Ambiguity aversion has been confirmed for sophisticated subjects (Hogarth and Kunreuther, 1989); hence, being ambiguity averse is rational in the second meaning of subjective rationality. It is worth noticing that until now, no empirical data confront the first between-person definition of subjective rationality.

### **Ambiguity aversion as heuristic**

Ambiguity aversion can be viewed as a good strategy in some decision contexts that are non common for the DM, or when better informed agents (like experts) are present for substituting to the DM when a choice has to be made. In certain cases, it is rational not to decide as argued by Gilboa, et al. (2008a):

"In the absence of information, it appears more rational to be silent than to pretend to have knowledge that one does not have."

It follows that it may be reasonable for DMs to be ambiguity averse in the sense of avoiding situations in which they feel there is not enough information to decide; similarly, delegating decisions may be a good strategy when experts can be involved.

Frish and Baron (1988) list reasons that lead to the heuristic that it is a good rule to avoid making decision under ambiguity: the possibility of the existence of better informed opponent, the existence of an hostile opponent who will bias the situation in your disadvantage, a series of identical ambiguous gambles are more risky than a series of non-ambiguous gambles in the long run, ambiguous options may incitate to wait for obtaining more information, the blame and regret arguments ... Hence, even when it results from heuristics, ambiguity aversion is a kind of second best in decisions.

### **Are ambiguity attitudes a consequence of mere framing effects<sup>7</sup>?**

Attributing ambiguity attitudes to framing have concrete normative implications: a model should account for sensitivity of individuals to the informational structure of a decision context to the extent it cannot be attributed to framing effects since the latter result from judgment errors.

We already discussed ambiguity attitudes as resulting from framing in the case of Ellsberg two-urn example (2.2.3.). More generally, Frish and Baron (1988) argue that, an ambiguous urn may be transformed in unambiguous by simple framing. Assume that the decision is one of a sequence of identical but independent decisions. Propose the subject a bet on either (red or black) color in the ambiguous urn, one hundred or one thousand times, with a different ambiguous urn used each time. In the long run, over a long series of such a bets, the information becomes irrelevant and reduces to half a chance of winning; hence, the ambiguous urn turns to be unambiguous. This is a possible frame for the classic risky urn.

The question of whether attitudes in Machina (2009) result from ambiguity or from a mere framing effect is still in debate. We have seen that Machina paradox involves a complex uncertainty mix whose association with a specific payoffs scheme leads to reflection effects.

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<sup>7</sup>Framing effects are at play if a DM changes his decision when facing two different descriptions of the same choice problem.

Machina assumed that DMs ought to be sensitive to reflection effects since they entail natural symmetry considerations. The way Machina's problem was proposed to subjects (detailed in Chapter 6) undermines such symmetry effects but promotes others, such as cancellation of common parts of the bets in the editing phase, potentially followed by violations of coalescing. However, as soon as choice problems become more complex, framing effects become more likely to appear.

### **Theories and paradoxes of ambiguity**

The theory of choice under uncertainty has evolved following a series of paradoxes highlighting the contradictory aspects of choice behavior. A paradox of behavior is originally a thought experiment whose empirical confirmation contains enough evidence to conclude to a violation of the theory of choice assumed beforehand. Hence, the role of the paradoxes is to lead to further theoretical constructs that however should not depart from minimum rationality conditions. Paradoxes shake theories but at the same time give rise to the normative and descriptive improvements required.

The evolution of the theory of individual decision making is motivated by three actors: the theorists, who define reasonable principles that a DM should follow; the contradictors, who are interesting in the failure of the theory; and the experimenters who confront behavioral evidence to theoretical predictions. Interactions between actors create fruitful complementarities but different questions can be individually addressed to each of these three actors.

### **Questions to address to the decision theorist**

What should be the limits of theoretical innovations and what are the criteria that lead a decision theorist to consider a behavioral trait as normatively desirable? What should the theorist do when the normative and descriptive purposes are in conflict? It is not worthwhile

to define a behavior that does not exist in the real life, and at the same time, it is not relevant to describe an existing behavior that cannot be seen as desirable. Applying this to the ambiguity aversion phenomenon, should it be incorporated in economic models for descriptive purpose and sometimes until to become a rule of behavior as for instance done by the uncertainty aversion axiom (MEU: Gilboa and Schmeidler, 1989). Should we say that economic models should account for ambiguity attitudes because it unquestionably constitutes a behavioral regularity of the individual decision making or should economic models stay mute arguing that ambiguity attitudes ought not to play a role in decision. Observation of violations of normative models leads to consider two solution as suggest by Gilboa (2008):

" [...] we can either bring the theory closer to reality (making the theory a better descriptive one), or bring reality closer to theory (preaching the theory as a normative one)."

### **Questions to address to the contradictor**

When should we stop looking for behavioral anomalies? The complexity of the human behavior and its dependence to the decision context makes it easy to build a tricky example that ends up generating behavior inconsistent with the theory. A theory of decision making could be falsified as soon as we can find an individual who does not conform with one of its axiom. The term 'falsification' may appear too strong especially in the social studies field. A major falsification, say, a falsification that could give birth to new theories should embedded enough empirical evidence and should be thought as representing sufficiently concrete and general context.

**Questions to address to the (lab) experimenter**

Two main questions arise when time comes to test paradoxes that incorporate ambiguity features. The first one concerns the proper and credible mechanism one should use to generate ambiguity in the laboratory. By nature, ambiguity should not be determined by a prespecified nor predictable random mechanism. Second, let us suppose that a proper way to generate lab-ambiguity has been found. Then, the question of the role and impact of the incentive mechanism on ambiguity attitudes should be addressed.

*What is the proper way to simulate ambiguity?* Properly creating ambiguity using a physical device has always been an experimental challenge. The difficulty for the experimenter is to find a transparent and credible mechanism for generating ambiguity (this question does not apply to real-life events like "the temperature in Paris tomorrow", which finds its natural resolution). It follows that nobody (neither the subject nor the experimenter) should have information on the way ambiguity is generated, i.e., on the law that governs the random process. Under risk, it is plausible that subjects do not distinguish between different ways of generating objectively known probabilities ("apart from some volatile psychological effects", see for instance Wakker, 2008). As suggested by the source approach, distinct supports of uncertainty determine distinct ambiguity attitudes.

In the literature, many empirical studies approximate ambiguity by second order probabilities (Chow and Sarin, 2002; Yates and Zukowski, 1976). A more sophisticated approach (Hey, Lotito and Maffioletti, 2007) consists in constructing a transparent box containing balls that are in continuous motion thanks to a jet of wind. Hence, the so-called British Bingo Blower allows simulating different degrees of ambiguity by varying the number of balls put in the Blower. Another approach consists in using a quantum random number generator (Abdel-laoui, L'Haridon and Nebout, 2009), the physical properties of which intrinsically guarantee the randomness of the process. All these artificial lab-mechanisms are at best a way to clear

the experimenter's conscience. Indeed, ambiguity is ultimately an affair of what the subjects actually believe and none of the previous device could pretend to control this.

*What about the efficiency of incentives in the presence of ambiguity?* Until there, experiments under ambiguity have raised the problem of how to properly generate ambiguity experimentally. By contrast, the question of whether the magnitude of the stakes affects ambiguity attitudes and, as a corollary, the question of the proper incentive mechanism has not received much attention. Obviously, the second question can be answered once the first one has received its natural response. More than risk, ambiguity raises serious problems with regard to the implementation and the efficiency of incentives. In this respect, the growing experimental literature on ambiguity appeals from an add-on of Holt and Laury (2002, 2005) contribution for risk. Thus, if the impacts of incentives on behavior under risk are well-known, a precise investigation of these effects under ambiguity seems topical.

The major limitation of this triptych is that it defines and observes the sole choice behavior, neglecting cognitive processes that lead to the decision. As a consequence, ambiguity models and their experimental measures appear sometimes speculative. Indeed, they are built on pre-suppositions on the manner individuals represent themselves a decision problem (ambiguity as a source of uncertainty; ambiguity as a set of beliefs; ambiguity as compound risk). However, the diversity of approaches that results from the lack of information regarding the true cognitive process that governs decision is virtuous since it may reflect the inter-individual heterogeneity of human decision processes. A complementary psychological perspective would permit a better investigation of foundations of all of these approaches.



### **Conclusion**

In this thesis, ambiguity has been studied in a framework where decisions are made in one-shot problems. Such approach sheds light on what are ambiguity and ambiguity attitudes. However, it stays quite restrictive; indeed, by definition and by opposition to objective probabilities, subjective beliefs have the property to evolve as soon as new information arise. Hence, ambiguity and ambiguity attitudes have to be envisaged in a dynamic framework that authorizes DMs to update her beliefs. Many theoretical studies exist (for instance: Ozdenoren and Peck, 2008; Hanany and Klibanoff, 2007); however, very few empirical studies are concerned with ambiguity in dynamic settings. It will certainly be a challenging way of research.

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## **Abstract**

### **From Ellsberg to Machina : Confronting decision models under ambiguity with experimental evidence**

How *do* decision makers act and how *should* they act when confronted with uncertainty ? Economic behavior under uncertainty is often influenced by the informational structure of the decision context. Remarkably, the joint presence (juxtaposition or combination) of two sources of uncertainty - namely risk (known probability) and ambiguity (unknown probability of events) - gives rise to behaviors that depart from standard models of decision making, subjective expected utility and its extension to non-additive probability, Choquet expected utility ; the essential behavioral component beyond paradoxes of uncertainty is (non neutral) attitude toward ambiguity. The studies reported in this thesis empirically investigate the heterogeneity of ambiguity attitudes in the light of the variability of the features of uncertainty. They deal with two main sorts of cases : the case where a decision maker faces two separate sources of uncertainty (two-color Ellsberg paradox) ; the case where a decision maker faces mixed sources of uncertainty (Machina paradox).

Key words : Ambiguity, Behavioral Decision Making, Experimental Economics, Machina Paradox, Sources of Uncertainty.

## **Résumé**

### **D'Ellsberg à Machina : les modèles de décision dans l'ambiguïté à l'épreuve de l'expérimentation**

Dans quelle mesure le comportement des décideurs se conforme-t-il aux prédictions des modèles de décision en environnement incertain ? Le comportement économique est souvent influencé par la structure informationnelle du contexte de décision. Notamment, la concomitance (juxtaposition ou combinaison) de deux sources d'incertitudes - le risque (probabilités connues) et l'ambiguïté (probabilités inconnues des événements) - donne lieu à des comportements non compatibles avec les modèles standards de théorie de la décision, le modèle d'utilité espérée subjective et son extension aux probabilités non-additives, le modèle d'utilité espérée à la Choquet ; la composante comportementale à la base des paradoxes de l'incertitude est le fait que les individus ont une attitude (non neutre) face à l'ambiguïté. Cette thèse propose différentes études empiriques visant à mettre en évidence l'hétérogénéité des attitudes face à l'ambiguïté à la lumière de la variabilité des structures d'incertitude. Ces études traitent de deux principaux cas : lorsque le décideur est confronté à deux sources séparées d'incertitude (paradoxe d'Ellsberg à deux couleurs) ; lorsque le décideur est confronté à un *mix* d'incertitudes (paradoxe de Machina).

Mots clés : Ambiguïté, Économie Expérimentale, Paradoxe de Machina, Sources d'Incertainitude, Théorie de la Décision.