



# Non-perturbative Effects in String Theory

Cezar Condeescu

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# Non-perturbative Effects in String Theory

## Effets Non-perturbatifs en Théorie des Cordes

Cezar Condeescu<sup>+</sup>

<sup>+</sup> *Centre de Physique Théorique, Ecole Polytechnique and CNRS,  
F-91128 Palaiseau, France.*

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### Jury

Constantin Bachas	Rapporteur
Ralph Blumenhagen	Rapporteur
Emilian Dudas	Directeur de thèse
Elias Kiritsis	Examineur
Augusto Sagnotti	Examineur



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# 1 Introduction

All known interactions except gravity are described at the quantum level by the phenomenological Standard Model of particle physics depending on 18 arbitrary parameters with gauge group

$$SU(3)_C \times SU(2)_L \times U(1)_Y \quad (1)$$

Although from an experimental point of view there is no data in contradiction with the Standard Model, from a theoretical point of view it is not satisfactory as a fundamental theory of Nature. Reasons to go beyond the Standard Model include explaining the values of the 18 free parameters or incorporating gravity as a quantum theory. Such problems can be addressed in the context of string theory and this provides a strong motivation for studying it. Additionally there is a correspondence between gauge theories on one side and string theory or gravity on the other side which allows one to study certain strongly coupled gauge theories by making use of gravity or string theory.

The basic idea of string theory is to replace elementary particles with quantum relativistic strings. Different modes of vibration of the elementary string will represent different particles (in the long wavelength limit). In principle one can consider two possibilities: open and closed strings. In fact there are several consistent string theories that one can define. The list contains the following: Heterotic with  $E_8 \times E_8$  or  $SO(32)$  gauge group, Type I  $SO(32)$  and Type II (A and B) theories. The Heterotic theories are theories with closed strings whereas Type I and Type II contain both closed and open strings. Let us mention here a few important features of string theory which make it a good candidate for a theory of all interactions. Closed strings have a spin 2 massless mode which can be identified with the graviton, moreover the string perturbation theory is UV-finite thus making string theory a consistent theory of quantum gravity. Standard model-like interactions can arise from string theory. Ideas like grand unification and supersymmetry can (naturally) be incorporated in string theory. Indeed, the low energy effective field theory arising from strings is a supergravity theory. Another important feature of string theories is that they are well-defined only in ten space-time dimensions, hence they predict six extra dimensions. In order for string theory to make sense as a fundamental (or effective) theory of Nature the extra dimensions have to be small (usually compact). There are indeed solutions which allow for a spacetime of the form

$$\text{SpaceTime} = \mathbb{R}^{1,3} \times K^6 \quad (2)$$

with  $\mathbb{R}^{1,3}$  being the 4d Minkowski space and  $K^6$  a compact manifold (orbifold). A

central problem in string theory is that of compactification. Although the 10d theories depend only on one scale parameter, namely the elementary string length  $l_s$ , in connecting with 4d physics one doesn't have (it is not known) a theoretical guiding principle to choose a particular compactification. This makes the task of string phenomenology a very difficult one (at least in the absence of relevant experimental data) and a very rich one in the same time. On the other hand, compactifications allowed us to find various connections (dualities) between the five basic 10d string theories pointing to the existence of a unique theory, called  $M$ -theory, which in the low energy limit should be described by 11d supergravity.

From a historic point of view the Heterotic theory was the first to be extensively studied due to its phenomenological potential in realizing  $\mathcal{N} = 1$  supersymmetric GUTs (Grand Unified Theories) by compactifying on certain Calabi-Yau manifolds. However after the discovery of the importance of  $D$ -branes [1] there was a lot of activity on the side of Type I/II theories. Type II theories with  $D$ -branes (and hence an open string sector) can have Standard Model like gauge symmetries since  $D$ -branes are connected to gauge theories. In fact Type II theories are intimately connected to Type I by means of  $T$ -duality and orientifolds.

The present work focuses on compactifications of Type I/II string theory on toroidal orbifolds/orientifolds. Toroidal orbifolds are interesting compactifications as they can realize certain singular limits of Calabi-Yau manifolds and on the other hand they admit an exact CFT description making them a very good arena for testing ideas in string theory. There are various tools (ingredients) available for building models in this context, namely  $D$ -branes, orientifolds, background magnetic fields (intersecting branes), fluxes and brane instantons. Of particular phenomenological importance is the presence of background magnetic fields (or equivalently magnetized/intersecting branes) because they allow solutions with chiral fermions. This is due to the fact that the Dirac operator with a non-zero magnetic field can have a non-zero index as well. The gauge group in this type of models is generically a product of unitary, orthogonal or symplectic factors. It is in principle possible to build (Minimal Supersymmetric) Standard Model like solutions from magnetized/intersecting branes. On the other hand there is no natural way of obtaining unification of coupling constants in such a scenario since every stack of branes comes with its own gauge bundle and coupling constant and *a priori* there is no reason for them to be equal. However this can happen if the volumes of the cycles wrapped by the branes are equal.

There has been recently activity in non-perturbative effects generated by brane instantons. These are branes that are entirely localized in the internal compact space  $K^6$ . They can realize gauge instanton effects in the context of string theory but also give rise to qualitatively different effects called stringy instanton effects. Their applications include moduli stabilization, generation of hierarchically small masses, generation of some perturbatively forbidden couplings, brane inflation, gauge symmetry breaking or supersymmetry breaking which can be used for various phenomenological applications.

The organization of this manuscript is as follows. In Chapter 2 we present the basic (lightcone) quantization of the bosonic and supersymmetric strings followed by an elementary discussion of  $D$ -branes  $O$ -planes and one-loop amplitudes for Type I/II theories. Chapter 3 is devoted to orientifold compactifications. We present two examples, namely  $\mathbb{T}^4/\mathbb{Z}_2$  and  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ . In Chapter 4 we consider compactifications of Type I with magnetized branes and the corresponding Type IIA T-dual picture of intersecting branes. Chapter 5 reviews the computation of Yukawa couplings in (toroidal) models with magnetized branes and continuous Wilson lines and show how to extract a holomorphic superpotential. One of the most important problems in both field theory and string theory is supersymmetry breaking. We describe in Chapter 6 a few mechanism known in string theory for breaking supersymmetry. Finally Chapters 7 and 8 are devoted to non-perturbative effects generated by Euclidean brane instantons. We describe briefly the instanton calculus and ADHM construction in field theory. Brane instantons which can generate corrections to the superpotential are considered next. We illustrate how they can generate linear, mass or Yukawa couplings in the superpotential and the potential for phenomenological applications. The last chapter is devoted to the study of models with Polonyi like superpotentials which are generated by stringy instantons. Two appendices collect the definitions of the characters of the  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  orientifold (with discrete torsion) and the corresponding partition functions in the presence of magnetized  $D9$  branes.

## 2 String Theory and D-branes

### 2.1 The Bosonic String

#### 2.1.1 Classical Theory

The starting point of string theory is the classical action of a relativistic string propagating in Minkowski space-time. The surface spanned by the string is called worldsheet and we shall denote it  $\Sigma$ . The worldsheet is embedded in a target space  $M$  (our space-time). As a starting point we take  $M$  to be  $D$  dimensional Minkowski space-time. Generalizations exist and they go by the name of non-linear sigma models.

Let  $X : \Sigma \rightarrow M$  be the embedding of the string worldsheet in Minkowski space. In local coordinates we are given  $D$  functions  $X^M(\tau, \sigma)$ , with  $\tau, \sigma$  being coordinates on the worldsheet. The classical action that we start with is the Polyakov action

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^M(\tau, \sigma) \partial_b X^N(\tau, \sigma) \eta_{MN} \quad (3)$$

where  $\gamma_{ab}$  is the intrinsic metric on the worldsheet and  $\eta_{MN}$  is the usual metric in Minkowski space-time. The constant  $T$  is called the string tension. Alternatively, one can work with the string length  $l_s$  or the Regge slope  $\alpha'$ . The connection between them is given by the following relation

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2} \quad (4)$$

There exists naturally on  $\Sigma$  an induced metric from  $M$ . This is the pullback of the metric on  $M$ .

$$g = X^*(\eta) \quad (5)$$

In local coordinates this reads

$$g_{ab} = \eta_{MN} \partial_a X^M \partial_b X^N \quad (6)$$

One can immediately argue that the most natural action for the string worldsheet would be its area in the target space

$$\text{Area}(X(\Sigma)) = \int d^2\sigma \sqrt{-g} \quad (7)$$

where by  $g$  we have denoted the determinant of  $g_{ab}$ . This action is non-linear in  $X$  and hard to quantize. Notice however that the equation of motion for the intrinsic metric  $\gamma$  in eq. (3)

$$\partial_a X^M \partial_b X^N \eta_{MN} - \frac{1}{2} \gamma_{ab} \partial^c X^M \partial_c X^N \eta_{MN} = 0 \quad (8)$$

implies that the intrinsic metric is proportional (with a conformal factor) to the induced metric (classically) and correspondingly the Polyakov action is proportional to the area. The condition above is nothing else than the annulation of the energy momentum tensor

$$T_{ab} = 0 \quad (9)$$

since the variation of the action with respect to the metric yields the expression of  $T_{ab}$ . We will proceed with the Polyakov action and regard the above equation as a constraint to be imposed. Now the equation of motion for  $X$  is

$$\partial_a (\sqrt{-\gamma} \gamma^{ab} \partial_b X^M(\tau, \sigma)) = 0 \quad (10)$$

The symmetries of the action are

- D-dimensional Poincaré invariance
- 2-dimensional local reparametrization invariance
- 2-dimensional Weyl invariance.

Notice that making use of local reparametrization invariance one can put the metric into a form such that it is conformally flat.

$$\gamma_{ab} = e^{2\omega} \eta_{ab} \quad (11)$$

The condition above is called the conformal gauge. The conformal factor  $e^{2\omega}$  cancels when replaced into the Polyakov action. We obtain

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma (-\partial_{\tau} X^M \partial_{\tau} X_M + \partial_{\sigma} X^M \partial_{\sigma} X_M) \quad (12)$$

and the equation of motion for  $X$  becomes the usual wave equation

$$(\partial_{\tau}^2 - \partial_{\sigma}^2) X^M(\tau, \sigma) = 0 \quad (13)$$

We use coordinates  $\tau, \sigma$  for the following form of the metric

$$ds^2 = -d\tau \otimes d\tau + d\sigma \otimes d\sigma \quad (14)$$

It is convenient to introduce light-cone coordinates

$$\begin{aligned} \sigma^+ &= \tau + \sigma \\ \sigma^- &= \tau - \sigma \end{aligned} \quad (15)$$

Then the components of the metric in the new coordinates is

$$ds^2 = -2d\sigma^+ \otimes d\sigma^- \quad (16)$$

The energy-momentum constraints in the new coordinates are

$$\begin{aligned} T_{++} &= \partial_+ X^M \partial_+ X_M = 0 \\ T_{--} &= \partial_- X^M \partial_- X_M = 0 \\ T_{+-} &= T_{-+} = 0 \text{ (automatically)} \end{aligned} \quad (17)$$

The conservation of the energy-momentum tensor  $\partial^a T_{ab} = 0$  becomes

$$\partial_- T_{++} = \partial_+ T_{--} = 0 \quad (18)$$

hence

$$T_{++} = T_{++}(\sigma^+) \quad T_{--} = T_{--}(\sigma^-) \quad (19)$$

Notice that we have an infinite number of conserved charges. Indeed, let  $f(\sigma^+)$  be an arbitrary function then we obviously have  $\partial_-(f(\sigma^+)T_{++}) = 0$  and the corresponding conserved charge is

$$Q_f = \int_0^{\bar{\sigma}} d\sigma f(\sigma^+) T_{++}(\sigma^+) \quad (20)$$

In the light-cone coordinates we have the following equation of motion for the world-sheet embedding  $X$

$$\partial_+ \partial_- X^M(\sigma^+, \sigma^-) = 0 \quad (21)$$

The general solution to this equation is of the following form

$$X^M(\sigma^+, \sigma^-) = X_L^M(\sigma^+) + X_R^M(\sigma^-) \quad (22)$$

where  $X_L$  and  $X_R$  are arbitrary functions. Now let us consider possible boundary conditions. At this point of the discussion we have two possibilities

- closed strings corresponding to periodic boundary conditions

$$X^M(\tau, \sigma + 2\pi) = X^M(\tau, \sigma) \quad (23)$$

For convenience we have reverted to the coordinates  $\tau, \sigma$ .

- open strings corresponding to Neumann boundary conditions

$$\partial_\sigma X^M(\tau, \sigma = 0, \pi) = 0 \quad (24)$$

These boundary conditions describe open strings with endpoints moving freely in D-dimensional spacetime. In addition one can consider Dirichlet boundary conditions. They describe open strings with endpoints that are fixed. They are important for describing D-branes and we will consider them later on. Let us first consider the case of closed strings. The periodic boundary conditions above imply that we have the following mode expansion

$$\begin{aligned} X_L^M(\tau + \sigma) &= \frac{1}{2}x^M + \frac{1}{2}l_s^2 p^M(\tau + \sigma) + \frac{i}{\sqrt{2}}l_s \sum_{n \neq 0} \frac{\tilde{\alpha}_n^M}{n} e^{-in(\tau + \sigma)} \\ X_R^M(\tau - \sigma) &= \frac{1}{2}x^M + \frac{1}{2}l_s^2 p^M(\tau - \sigma) + \frac{i}{\sqrt{2}}l_s \sum_{n \neq 0} \frac{\alpha_n^M}{n} e^{-in(\tau - \sigma)} \end{aligned} \quad (25)$$

The parameters  $x^M$  and  $p^M$  have the interpretation of center of mass coordinate and momentum respectively. Hence the solution is

$$X^M(\tau, \sigma) = x^M + l_s^2 p^M \tau + \frac{i}{\sqrt{2}}l_s \sum_{n \neq 0} \frac{1}{n} e^{-in\tau} (\alpha_n^M e^{in\sigma} + \tilde{\alpha}_n^M e^{-in\sigma}) \quad (26)$$

For the case of open strings we have two conditions, one for  $\sigma = 0$  and one for  $\sigma = \pi$ . The first one implies that  $X_L^M(\tau) = X_R^M(\tau)$  up to a constant that we can take to be zero. The second condition implies that  $X_L^M$  is a periodic function of its argument up to a linear polynomial function. Hence one obtains the following expansion

$$X^M(\tau, \sigma) = x_0^M + 2l_s^2 p^M \tau + i\sqrt{2}l_s \sum_{n \neq 0} \frac{\alpha_n^M}{n} e^{-in\tau} \cos(n\sigma) \quad (27)$$

Before proceeding to quantization let us write here the hamiltonian of the system.

$$H = \int_0^{\bar{\sigma}} d\sigma (P^M \dot{X}_M - L) \quad (28)$$

In terms of oscillators we get ( $\bar{\sigma} = 2\pi$  for closed strings and  $\bar{\sigma} = \pi$  for open strings)

$$\begin{aligned} H &= \frac{1}{2} \sum_{n \in \mathbb{Z}} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \quad \text{closed strings} \\ H &= \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_n \quad \text{open strings} \end{aligned} \quad (29)$$

where we have defined zero mode oscillators by

$$\begin{aligned} \alpha_0^M &= \tilde{\alpha}_0^M = \frac{l_s}{\sqrt{2}} p^M \quad \text{closed strings} \\ \alpha_0^M &= l_s \sqrt{2} p^M \quad \text{open strings} \end{aligned} \quad (30)$$

We define the Virasoro operators for the closed string as the charges at  $\tau = 0$  corresponding to the base of functions  $f_m(\sigma^\pm) = e^{im\sigma^\pm}$

$$\begin{aligned} L_m &= 2T \int_0^{2\pi} d\sigma e^{im(\tau-\sigma)} T_{--} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \\ \bar{L}_m &= 2T \int_0^{2\pi} d\sigma e^{im(\tau+\sigma)} T_{++} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \end{aligned} \quad (31)$$

They generate two copies of the so-called Virasoro algebra. In the case of the open string one obtains only one set of operators defined as follows

$$L_m = 2T \int_0^\pi d\sigma (e^{im(\tau+\sigma)} T_{++} + e^{im(\tau-\sigma)} T_{--}) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \quad (32)$$

Notice that we have the following relation between the hamiltonian and the Virasoro operators

$$\begin{aligned} H &= L_0 + \bar{L}_0 && \text{closed strings} \\ H &= L_0 && \text{open strings} \end{aligned} \quad (33)$$

We can implement the constraints  $T_{++} = T_{--} = 0$  as

$$L_m = \bar{L}_m = 0 \quad (34)$$

### 2.1.2 Quantization

For quantization we use the usual correspondence between Poisson brackets and commutators

$$[\cdot, \cdot]_{P.B.} \rightarrow \frac{1}{i} [\cdot, \cdot] \quad (35)$$

One obtains

$$\begin{aligned} [X^M(\tau, \sigma), X^N(\tau, \sigma')] &= [\dot{X}^M(\tau, \sigma), \dot{X}^N(\tau, \sigma')] = 0 \\ [X^M(\tau, \sigma), \dot{X}^N(\tau, \sigma')] &= \frac{i}{T} \delta(\sigma - \sigma') \eta^{MN} \end{aligned} \quad (36)$$

and in terms of the oscillators

$$\begin{aligned} [x^M, p^N] &= i\eta^{MN} \\ [\alpha_m^M, \alpha_n^N] &= [\tilde{\alpha}_m^M, \tilde{\alpha}_n^N] = m\delta_{m+n,0} \eta^{MN} \\ [\alpha_m^M, \tilde{\alpha}_n^N] &= 0 \end{aligned} \quad (37)$$

Notice that for  $m \neq 0$  we can rescale the operators and define  $a_m^M = \frac{1}{\sqrt{m}}\alpha_m^M$  and  $a_m^{\dagger M} = \frac{1}{\sqrt{m}}\alpha_{-m}^M$  (for  $m > 0$ ), thus obtaining the usual harmonic oscillator commutation relations. Hence we can define the ground state as the state that is annihilated by all lowering operators. Additionally we choose it to be an eigenstate of the center of mass momentum.

$$\alpha_m |0, p^M\rangle = 0 \quad m > 0 \quad (38)$$

and

$$\hat{p}^M |0, p^M\rangle = p^M |0, p^M\rangle \quad (39)$$

Notice that due to the Minkowski metric with  $\eta^{00} = -1$  we could in principle have states with negative norm.

**Theorem** (no-ghost).

Imposing the constraints on the states decouples the ghosts in 26 dimensions ( $D = 26$ ) if the normal ordering constant  $a = -1$ .

The normal ordering constant appears in the definition of  $L_0$  in the quantum theory. Indeed, we define the Virasoro operators by making use of the normal ordering

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{m-n} \cdot \alpha_n : + a \delta_{m,0} \quad (40)$$

With this definition and the commutation relations above one obtains the following Virasoro algebra

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} \quad (41)$$

where  $c = D$  is called the central charge. Each boson on the worldsheet contributes with one unit. The constraints  $L_m = 0$  cannot be imposed on all the Hilbert space. We can only ask them to be satisfied on the physical Hilbert space

$$\begin{aligned} L_m |\text{phys}\rangle &= 0 \quad m > 0 \\ L_0 |\text{phys}\rangle &= 0 \end{aligned} \quad (42)$$

Notice that  $L_m^\dagger = L_{-m}$  implies

$$\langle \text{phys}' | L_m | \text{phys} \rangle = 0 \quad \text{for all } m \neq 0 \quad (43)$$

For the closed string we need to impose similar constraints for  $\bar{L}_m$  and in addition the following constraint

$$(L_0 - \bar{L}_0) | \text{phys} \rangle = 0 \quad (44)$$

which is the generator of rigid  $\sigma$  translations. Using the equation  $p^M p_M = -m^2$  together with the constraint for  $L_0$  one obtains

$$m^2 = \frac{1}{l_s^2} \left( \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + a \right) \quad (45)$$

### 2.1.3 Lightcone Gauge and the String Spectrum

Let us notice that after imposing the condition  $\gamma_{ab} = e^{2\omega} \eta_{ab}$  one is still left with a residual symmetry, namely reparametrizations

$$\begin{aligned} \sigma^+ &\rightarrow f(\sigma^+) \\ \sigma^- &\rightarrow \tilde{f}(\sigma^-) \end{aligned} \quad (46)$$

In the case of the open string the two functions  $f$  and  $\tilde{f}$  have to be equal. One can use this freedom to further impose the following condition

$$X^+(\tau, \sigma) = x^+ + l_s^2 p^+ \tau \quad (47)$$

where we have defined lightcone coordinates

$$\begin{aligned} X^+ &= X^0 + X^1 \\ X^- &= X^0 - X^1 \end{aligned} \quad (48)$$

This amounts to put  $\alpha_m^+ = 0$ . Then by using the constraints coming from the energy-momentum tensor one can express  $\alpha_m^-$  in terms of the transverse oscillators  $\{\alpha_m^i\}_{i=2,\dots,D-1}$ .

$$\alpha_m^- = \frac{1}{l_s p^+ \sqrt{2}} \left( \sum_{n \in \mathbb{Z}} : \alpha_{m-n}^i \alpha_n^i : + 2a \delta_{m,0} \right) \quad (49)$$

$$\tilde{\alpha}_m^- = \frac{1}{l_s p^+ \sqrt{2}} \left( \sum_{n \in \mathbb{Z}} : \tilde{\alpha}_{m-n}^i \tilde{\alpha}_n^i : + 2a \delta_{m,0} \right) \quad (50)$$

for the closed string and

$$\alpha_m^- = \frac{1}{2l_s p^+ \sqrt{2}} \left( \sum_{n \in \mathbb{Z}} : \alpha_{m-n}^i \alpha_n^i : + 2a \delta_{m,0} \right) \quad (51)$$

for the open strings. Hence in the light cone gauge the physical degrees of freedom are given by the transverse coordinates.

Expressing the mass operator in terms of transverse oscillator one obtains

$$\begin{aligned} m^2 &= \frac{2}{l_s^2} \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + 2a) & (\text{closed string}) \\ m^2 &= \frac{1}{l_s^2} \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + a) & (\text{open string}) \end{aligned} \tag{52}$$

- Open string spectrum

The ground state is  $|0, p^i\rangle$ . It is a tachyonic state since its mass is given by

$$m^2 = \frac{a}{l_s^2} \tag{53}$$

The first excited state is  $\alpha_{-n}^i |0, p^j\rangle$ . This is a  $D - 2$  dimensional vector. D-dimensional Lorentz invariance requires physical states to fall into representations of the little group of  $SO(1, D - 1)$  which is  $SO(D - 2)$  for massless particles. Hence the state under discussion has to be massless. Indeed this happens if

$$a = -1 \tag{54}$$

as stated by the no-ghost theorem. One can check, level by level that we recover representations of the little group. The next excited states will be massive with masses inversely proportional to  $l_s$ . The elementary string length is supposedly very small otherwise we would have seen already strings in the accelerators, therefore the masses are very large. In the first approximation, when considering the effective low-energy theory one can work only with the massless states, provided we can find a way to kill the tachyonic state. We will see that we can find a consistent projection which will kill the tachyon in the context of superstrings.

- Closed string spectrum

Again the ground state is a tachyon  $|0, p\rangle$ . Recall that in the case of the closed string we have to impose an additional condition on the physical state (also called level matching), namely  $(L_0 - \bar{L}_0)|\text{phys}\rangle = 0$ . The first excited state is

$$\alpha_{-1}^i \otimes \tilde{\alpha}_{-1}^j |0, p\rangle \tag{55}$$

This is a reducible representation which decomposes into a symmetric traceless tensor, an antisymmetric tensor and a scalar (the trace). They are denoted with the following letters:  $G_{ij}, B_{ij}, \Phi$ . The symmetric tensor  $G_{ij}$  is identified with the graviton. The

antisymmetric tensor is called  $B$ -field and  $\Phi$  is called the dilaton. They are of course massless. Notice that due to the level matching condition we cannot consider physical a state of the form  $\alpha_{-n}^i |0, p\rangle$ . Similar selection rules apply at all levels. The next states are again massive with masses inversely proportional to  $l_s$ . Let us summarize in the end the basic features of the bosonic string

- It is well defined only in 26 dimensions
- It contains tachyons
- The closed string contains a state that can be identified with the graviton, hence we have on our hands a possible theory of quantum gravity
- There are no fermions in the theory.

We will show in the next section how to add fermions to the theory. This will allow us to project out the tachyon as well.

## 2.2 Superstrings

In order to get fermions in the theory we will add fermionic coordinates for the world-sheet.

$$\Psi_\alpha : \Sigma \rightarrow M \quad (56)$$

with  $\alpha$  being a 2-dimensional spinor index. One is thus led naturally to consider a supersymmetric version of the Polyakov action.

$$\begin{aligned} S = & -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-\gamma} \left[ \gamma^{ab} \partial_a X^M(\tau, \sigma) \partial_b X^N(\tau, \sigma) \eta_{MN} + i \bar{\psi}^M(\tau, \sigma) \rho^a \partial_a \psi^N(\tau, \sigma) \eta_{MN} \right. \\ & \left. + i \bar{\chi}_a \rho^b \rho^a \psi^M \left( \partial_b X^N - \frac{i}{4} \bar{\chi}_b \psi^N \right) \eta_{MN} \right] \end{aligned} \quad (57)$$

In the formula above we have introduced a gravitino field  $\chi_a$  as expected in local supersymmetric action and the two-dimensional Dirac matrices

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (58)$$

We are going to work in the superconformal gauge

$$\begin{aligned} \gamma_{ab} &= e^{\omega(\tau, \sigma)} \eta_{ab} \\ \chi_a &= \rho_a \chi(\tau, \sigma) \end{aligned} \quad (59)$$

where  $\chi(\tau, \sigma)$  is a Weyl fermion that factors out from the action. Indeed, the action in the superconformal gauge is

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma \left( \eta^{ab} \partial_a X^M \partial_b X_M + 2i\bar{\psi}^M \rho^a \partial_a \Psi_M \right) \quad (60)$$

The equations of motion following from this action are

$$\begin{aligned} (\partial_{\tau}^2 - \partial_{\sigma}^2) X^M(\tau, \sigma) &= 0 \\ \rho^a \partial_a \psi^M(\tau, \sigma) &= 0 \end{aligned} \quad (61)$$

In addition to these we have to impose as constraints the equations of motion for the metric and the gravitino

$$T_{ab} = \partial_a X^M \partial_b X_M - \frac{1}{2} \eta_{ab} \partial^c X^M \partial_c X_M + \frac{i}{2} \bar{\psi}^M (\rho_a \partial_b + \rho_b \partial_a) \psi_M = 0 \quad (62)$$

$$J_a = \frac{1}{2} \rho^b \rho_a \psi^M \partial_b X_M = 0 \quad (63)$$

The conservation of the energy momentum tensor and of the supercurrent, lead as in the bosonic string to an infinite number of conserved charges. It is again convenient to make use of lightcone coordinates  $\sigma^+, \sigma^-$ . The action becomes

$$S = T \int d\sigma^+ d\sigma^- \left[ \partial_+ X^M \partial_- X_M + i(\psi_+^M \partial_- \psi_{+M} + \psi_-^M \partial_+ \psi_{-M}) \right] \quad (64)$$

with the following equations of motion

$$\partial_+ \partial_- X^M = 0 \quad (65)$$

$$\partial_- \psi_+^M = \partial_+ \psi_-^M = 0 \quad (66)$$

The bosonic coordinates are treated in a similar way as for the bosonic string. We will focus on the fermionic ones. Let us consider first the open string boundary conditions. Because of dealing with fermions there are two possibilities denoted by (R) and (NS):

$$\begin{aligned} \text{(R)} \quad \psi_+^M(\tau, 0) &= \psi_-^M(\tau, 0) & \psi_+^M(\tau, \pi) &= \psi_-^M(\tau, \pi) \\ \text{(NS)} \quad \psi_+^M(\tau, 0) &= -\psi_-^M(\tau, 0) & \psi_+^M(\tau, \pi) &= \psi_-^M(\tau, \pi) \end{aligned} \quad (67)$$

In order to work with periodic and antiperiodic boundary conditions (as it is the case for the closed strings) we do what is called the “doubling trick”, i.e. we extend the range of definition of  $\psi_+$  from  $0 \leq \sigma \leq \pi$  to  $0 \leq \sigma \leq 2\pi$  by defining

$$\begin{aligned} \psi_+^M(\tau, \sigma) &= \psi^M(\tau, \sigma) \\ \psi_-^M(\tau, \sigma) &= \psi^M(\tau, 2\pi - \sigma) \end{aligned} \quad (68)$$

Then we can work with the same type of boundary conditions both in the open and closed string. These are

- periodic or Ramond (R)  $\psi_{\pm}(\tau, \sigma + 2\pi) = +\psi_{\pm}(\tau, \sigma)$
- anti-periodic or Neveu-Schwarz (NS)  $\psi_{\pm}(\tau, \sigma + 2\pi) = -\psi_{\pm}(\tau, \sigma)$

The conditions for  $\psi_+$  and  $\psi_-$  can be chosen independently in the case of the closed string. This leads to four possible sectors: NS-NS, NS-R, R-NS and R-R. In the case of the open string the left and right moving sectors are identified and we end up with only two sectors: NS and R respectively. In the lightcone coordinates the energy momentum and supercurrent become

$$\begin{aligned} T_{++} &= \partial_+ X^M \partial_+ X_M + i\psi_+^M \partial_+ \psi_{+M} \\ T_{--} &= \partial_- X^M \partial_- X_M + i\psi_-^M \partial_- \psi_{-M} \\ T_{+-} &= T_{-+} = 0 \end{aligned} \tag{69}$$

$$\begin{aligned} J_+ &= \psi_+^M \partial_+ X_M \\ J_- &= \psi_-^M \partial_- X_M \end{aligned} \tag{70}$$

and with the corresponding conservation laws

$$\partial_- T_{++} = \partial_+ T_{--} = 0 \tag{71}$$

$$\partial_- J_+ = \partial_+ J_- = 0 \tag{72}$$

Now let us turn to oscillator expansions. For the case of (R) boundary conditions we have

$$\begin{aligned} \psi_+^M(\tau, \sigma) &= \sum_{r \in \mathbb{Z}} \bar{b}_r^M e^{-ir(\tau+\sigma)} \\ \psi_-^M(\tau, \sigma) &= \sum_{r \in \mathbb{Z}} b_r^M e^{-ir(\tau-\sigma)} \end{aligned} \tag{73}$$

and for the case of (NS) boundary conditions

$$\begin{aligned} \psi_+^M(\tau, \sigma) &= \sum_{r \in \mathbb{Z}+1/2} \bar{b}_r^M e^{-ir(\tau+\sigma)} \\ \psi_-^M(\tau, \sigma) &= \sum_{r \in \mathbb{Z}+1/2} b_r^M e^{-ir(\tau-\sigma)} \end{aligned} \tag{74}$$

In order to impose the energy momentum and supercurrent constraints in the quantum theory we define super Virasoro operators

$$\begin{aligned} L_m &= 2T \int_0^{2\pi} d\sigma e^{im(\tau-\sigma)} T_{--} \\ G_r &= T \int_0^{2\pi} d\sigma e^{ir(\tau-\sigma)} J_- \end{aligned} \quad (75)$$

and similar formulas for  $\bar{L}_m$  and  $\bar{G}_r$ . In terms of oscillators we obtain

$$\begin{aligned} L_m &= \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n + \frac{1}{2} \sum_r \left( r - \frac{m}{2} \right) b_{m-r} \cdot b_r \\ G_r &= \sum_r \alpha_n \cdot b_{r-n} \end{aligned} \quad (76)$$

They generate a supersymmetric extension of the Virasoro algebra.

### 2.2.1 Quantization

The prescription for quantization of the worldsheet fermions is to replace Dirac brackets with anticommutators

$$\{\cdot, \cdot\}_{D.B.} \rightarrow \frac{1}{i} \{\cdot, \cdot\} \quad (77)$$

An elementary calculation leads to the following result

$$\begin{aligned} \{\psi_+^M(\tau, \sigma), \psi_+^N(\tau, \sigma')\} &= \{\psi_-^M(\tau, \sigma), \psi_-^N(\tau, \sigma')\} = \frac{1}{2T} \eta^{MN} \delta(\sigma - \sigma') \\ \{\psi_+^M(\tau, \sigma), \psi_-^N(\tau, \sigma')\} &= 0 \end{aligned} \quad (78)$$

and in terms of the  $b$ -oscillators

$$\{b_r^M, b_s^N\} = \eta^{MN} \delta_{r+s,0} \quad (79)$$

We now turn to the super Virasoro operators in the quantum theory. As before we have a normal ordering constant  $a$ .

$$\begin{aligned} L_m &= \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{m-n} \cdot \alpha_n : + \frac{1}{2} \sum_r \left( r - \frac{m}{2} \right) : b_{m-r} \cdot b_r : + a \delta_{m,0} \\ G_r &= \sum_r \alpha_n \cdot b_{r-n} \end{aligned} \quad (80)$$

With these definitions one show prove that the super Virasoro algebra is

$$\begin{aligned}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
\{L_m, G_r\} &= \left(\frac{m}{2} - r\right) G_{m+r} \\
[G_r, G_s] &= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s,0}
\end{aligned} \tag{81}$$

where now the central charge is given by

$$c = D + \frac{D}{2} \tag{82}$$

There is again a no-ghost theorem which tells us that  $D = 10$  and  $a=0$  for the (R) sector and  $a = -1/2$  in the (NS) sector. The physical states in the Hilbert space are determined by the conditions

$$\begin{aligned}
L_m|\text{phys}\rangle &= 0 & m \geq 0 \\
G_r|\text{phys}\rangle &= 0 & r > 0
\end{aligned} \tag{83}$$

for both the (NS) and (R) sectors. In the case of the closed fermionic string we will impose the following condition as well

$$(L_0 - \bar{L}_0)|\text{phys}\rangle = 0 \tag{84}$$

familiar from the case of the bosonic string. Before proceeding to light cone quantization let us introduce the level number operator

$$N = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r \in \mathbb{N} + \varphi}^{\infty} r b_{-r} \cdot b_r \tag{85}$$

where  $\varphi = 0$  for the (R) sector and  $\varphi = -1/2$  for the (NS)sector.

### 2.2.2 Light Cone Gauge and Spectrum

As in the case of the bosonic string one can solve the constraints in the light cone gauge. In the case of the fermionic superstring this is given by

$$\begin{aligned}
X^+ &= x^+ + l_s^2 p^+ \tau \\
\psi^+ &= 0
\end{aligned} \tag{86}$$

or equivalently  $\alpha_m^+ = 0$  for all  $m \neq 0$  and  $b_r^+ = 0$  for all  $r \in Z$ . Then one can solve the constraints and express  $\alpha_m^-$  and  $b_r^-$  in terms of the transverse oscillators

$$\begin{aligned}\alpha_m^- &= \frac{1}{2p^+} \left[ \sum_{n \in Z} : \alpha_n^i \alpha_{m-n}^i : + \sum_{r \in Z} \left( \frac{m}{2} - r \right) : b_r^i b_{m-r}^i : - 2a\delta_{m,0} \right] \\ b_r^- &= \frac{1}{p^+} \sum_{q \in Z} \alpha_{r-q}^i b_q^i\end{aligned}\tag{87}$$

Then one can find the physical spectrum by using the transverse oscillators and imposing the level matching condition for the closed string. Let us introduce the mass operator

$$m^2 = m_L^2 + m_R^2\tag{88}$$

with

$$m_R^2 = \frac{2}{l_s^2} \left( \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1}^{\infty} r b_{-r}^i b_r^i - a \right)\tag{89}$$

Then the level matching condition for physical states becomes

$$m_L^2 = m_R^2\tag{90}$$

Now let us examine the ground state in the (NS) and (R) sectors for the open string or equivalently for the left moving sector of the closed string. In both sectors the ground state is defined by

$$\alpha_m^i |0, p\rangle = b_r^i |0, p\rangle = 0 \quad \text{for all } m, r > 0\tag{91}$$

In the (NS) sector the ground state is unique and is the usual tachyon. In the (R) sector we still have the  $b_0^i$  zero modes which satisfy a Clifford algebra

$$\{b_0^M, b_0^N\} = \eta^{MN}\tag{92}$$

hence the ground state in the (R) sector will be a Majorana-Weyl 10 dimensional spinor. The number of degrees of freedom of such a spinor is indeed 8 and the (R) ground state is massless. There are however two choices of chirality for the (R) ground state. For building consistent superstring theories one needs to add an additional projector. We motivate this at this stage by the need to get rid of the tachyon. Let us introduce the GSO projector

$$P_{GSO} = \frac{1 - (-1)^F}{2}\tag{93}$$

where  $F$  is the worldsheet fermion number operator on the left moving sector. In terms of the oscillator it is given by

$$F = \sum_{r>0} b_{-r}^i b_r^i\tag{94}$$

Notice that the tachyon belongs to the kernel of the GSO projector

$$|0, p\rangle \in \text{Ker} P_{GSO} \quad (95)$$

since  $(-1)^F |0, p\rangle = +|0, p\rangle$ . In the (R) sector we use a generalized chirality operator instead of  $(-1)^F$ , that is

$$(-1)^F \rightarrow \Gamma_9 (-1)^F \quad (96)$$

where  $\Gamma_9$  is the chirality operator. There are several 10 dimensional superstring theories that can be built and we examine their (massless) spectra in the following.

### Type I

Type I string theory [2],[3] is a theory with open and closed strings with  $\mathcal{N} = 1$  supersymmetry in 10d. We have two sectors (R) and (NS). We have two choices for the GSO projection depending on the choice of chirality. Notice that the two choices are equivalent from the point of view of open strings. The massless spectrum (ground state) consists of

**(R)** one chiral spinor  $b_0^i |0, p\rangle$  with **8** real components

**(NS)** one vector field  $b_{-1/2}^i |0, p\rangle$  with **8** components

Thus in the low energy effective theory we have a vector multiplet. When we discuss Chan-Paton labels we will show how one can get non-abelian gauge interactions in open string theory.

### Type II

Type II string theory is a theory of closed and open strings with  $\mathcal{N} = 2$  supersymmetry in 10d. We have two choices, either use the same GSO projection in both the left- and right- moving sectors or use different projections. The first choice yields Type IIB theory and the second yields Type IIA theory. We have four different sectors (NS-NS) and (R-R) with spacetime bosons and (NS-R) and (R-NS) with spacetime fermions.

#### Type IIA massless spectrum

**(NS-NS)** We have the following states

$$b_{-1/2}^i \otimes \bar{b}_{-1/2}^j |0\rangle_{NS} \quad (97)$$

which decompose into representations of the transverse (little) group  $SO(8)$

- a scalar  $\phi$  called dilaton with **1** degrees of freedom
- an antisymmetric tensor field  $B^{ij} = -B^{ji}$  with **28** degrees of freedom
- a traceless symmetric tensor field  $G^{ij} = G^{ji}$ , the graviton, with **35** degrees of freedom

**(NS-R)** We have the following states

$$b_{-1/2}^i \otimes \bar{b}_0^j |0\rangle_{NS} \quad (98)$$

which decompose into

- a Weyl-Majorana spin 1/2 spinor called the dilatino with **8** degrees of freedom
- a spin 3/2 field, the gravitino with **56** degrees of freedom

**(R-NS)** We have the following states

$$b_0^i \otimes \bar{b}_{-1/2}^j |0\rangle_{NS} \quad (99)$$

which decompose into

- a Weyl-Majorana spin 1/2 spinor called the dilatino with **8** degrees of freedom
- a spin 3/2 field, the gravitino with **56** degrees of freedom

The chiralities are opposed to the ones in the (NS-R) since in Type IIA we use different GSO projections in the left-moving and right-moving sectors.

**(RR)** We have the following states

$$b_0^i \otimes \bar{b}_0^j |0\rangle_{NS} \quad (100)$$

which decompose into • a 1-form (vector) field  $C^i$  with **8** components • a 3-form field (antisymmetric tensor of rank three)  $C^{ijk}$  with **56** components

### **Type IIB massless spectrum**

**(NS-NS)** We have the following states

$$b_{-1/2}^i \otimes \bar{b}_{-1/2}^j |0\rangle_{NS} \quad (101)$$

which are the same as for Type IIA.

**(NS-R)** We have the following states

$$b_{-1/2}^i \otimes \bar{b}_0^j |0\rangle_{NS} \quad (102)$$

which decompose into

- a Weyl-Majorana spin 1/2 spinor called the dilatino with **8** degrees of freedom
- a spin 3/2 field, the gravitino with **56** degrees of freedom

**(R-NS)** We have the following states

$$b_0^i \otimes \bar{b}_{-1/2}^j |0\rangle_{NS} \quad (103)$$

which decompose into

- a Weyl-Majorana spin 1/2 spinor called the dilatino with **8** degrees of freedom

- a spin 3/2 field, the gravitino with **56** degrees of freedom

The chiralities are the same to the ones in the (NS-R) since in Type IIB we use the same GSO projections in the left-moving and right-moving sectors.

**(RR)** We have the following states

$$b_0^i \otimes \bar{b}_0^j |0\rangle_{NS} \quad (104)$$

which decompose into

- a 0-form (scalar) field  $C_0$ , hence **1** degree of freedom
- a 2-form field (antisymmetric tensor of rank 2)  $C^{ij}$  with **28** components
- a self-dual 4-form field (antisymmetric tensor of rank 4)  $C^{ijkl}$  with **35** components

### Heterotic

Although throughout this thesis we are concerned with Type I and/or Type II string theory we mention here for completeness the Heterotic string theory [4],[5], which historically was the first one to be considered with phenomenological potential due to its gauge symmetry  $E_8 \times E_8$  which allowed the realization of certain supersymmetric GUT (Grand Unified Theory) scenarios in string theory by compactifying on certain Calabi-Yau manifolds [6]. The Heterotic string is a hybrid between the 26 dimensional bosonic string and the 10 dimensional fermionic string. More precisely the left-moving sector is the 26-dimensional bosonic string compactified on a self-dual 16-torus, thus yielding 10 physical dimensions. The right-moving sector is the usual fermionic string. We will distinguish between the oscillators coming from the 16-dimensional lattice and non-compactified dimensions. We have only two sectors (NS) and (R).

**Heterotic** massless spectrum

From the 10 non-compact dimensions we have

**(NS)** We have the following states

$$\alpha_{-1}^i \otimes \bar{b}_{-1/2}^j |0\rangle \quad (105)$$

which again decompose into

- a scalar  $\Phi$ , an antisymmetric tensor  $B^{ij}$  and a traceless symmetric tensor  $G^{ij}$ .

**(R)** We have the following states

$$\tilde{\alpha}_{-1}^i \bar{b}_0^j |0\rangle \quad (106)$$

which decompose into

- a Weyl-Majorana 1/2 spinor and a gravitino 3/2 spin field.

The degrees of freedom are the same as in the case of Type II string theories. From the 16-dimensional lattice we get the states responsible for the  $E_8 \times E_8$  or  $SO(32)$  gauge

symmetry of the theory.

**(NS)** We have the following states

$$\begin{aligned} \alpha_{-1}^I \otimes b_{-1/2}^i |0\rangle \\ \bar{b}_{-1/2}^i |p_L^2 = 2\rangle \end{aligned} \tag{107}$$

where the index  $I$  runs over the compactified dimensions. Together they form the **8** components of the gauge field in the adjoint representation **496** of either  $E_8 \times E_8$  or  $SO(32)$ .

In the **(R)** sector one gets the corresponding superpartner, that is the following states **(R)**

$$\begin{aligned} \alpha_{-1}^I \otimes \bar{b}_0^i |0\rangle \\ \bar{b}_0^i |p_L^2 = 2\rangle \end{aligned} \tag{108}$$

With this we end the discussion of the five basic ten dimensional superstring theories. In the next section we introduce D-branes which will be intimately related to the **(RR)** forms in Type I/II string theories and to gauge symmetries by considering Chan-Paton labels or equivalently coincident D-branes.

## 2.3 D-branes

Let  $M$  be the target space of a string theory which contains open strings. Then a  $Dp$  brane is a  $p + 1$  dimensional submanifold (a divisor) of  $M$  such that endpoints of open strings are restricted to move only in this subspace. Let us take the example of the open bosonic string, then a  $Dp$  brane will mean an open string  $X^M(\tau, \sigma)$  with Neumann boundary conditions in  $p + 1$  directions and Dirichlet boundary conditions for the rest of  $D - p - 1$  directions. In this language the open bosonic gauge sector can be thought as a  $D25$  branes and the Type I gauge sector as a  $D9$  brane. The boundary conditions for open strings corresponding to  $Dp$  branes are

$$\begin{aligned} (DD) \quad & \partial_\tau X^I(\tau, \sigma) |_{\sigma=0,\pi} = 0 \text{ for } I = p + 1, \dots, D - 1 \\ (NN) \quad & \partial_\sigma X^\mu(\tau, \sigma) |_{\sigma=0,\pi} = 0 \text{ for } \mu = 0, \dots, p \end{aligned} \tag{109}$$

Notice that we have assumed that the  $Dp$  brane always wraps the target space time direction  $X^0$ . In the chapter about instantons we will discuss branes which have Dirichlet boundary conditions in 4 dimensional spacetime, also dubbed Euclidean branes. They realize instanton effects in the context of string theory. Notice that **(DD)** boundary conditions in non-compact dimensions require the end of the strings to be fixed in those

directions. Indeed, in the case of (DD) boundary conditions one has the following oscillator expansion

$$X^I(\tau, \sigma) = x^I + 2l_s^2 p^I \sigma - l_s \sqrt{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\alpha_n}{n} e^{-in\tau} \sin n\sigma \quad (110)$$

and the endpoints are fixed

$$X^I(\tau, 0) = x^I \quad X^I(\tau, \pi) = x^I + 2\pi l_s^2 p^I \quad (111)$$

In a theory where Dp-branes of different dimensionality are present one can have also Neumann-Dirichlet (ND) boundary conditions for open strings stretched between such branes.

$$(ND) \quad \partial_\tau X(\tau, \sigma) |_{\sigma=0} = 0 \quad \partial_\sigma X(\tau, \sigma) |_{\sigma=\pi} = 0 \quad (112)$$

One obtains the following oscillator expansion for (ND) strings

$$X(\tau, \sigma) = x - l_s \sqrt{2} \sum_{n \in \mathbb{Z} + 1/2} \frac{\alpha_n}{n} e^{-in\tau} \sin n\sigma \quad (113)$$

In this case one end is fixed and the other is free to move.

For the fermionic coordinates the oscillator expansion for (DD) is similar as in the case of (NN) whereas for (ND) boundary conditions the role of the (R) and (NS) expansions are inverted.

### 2.3.1 Toroidal Compactification and T-duality

A very important subject in string theory is compactification of extra dimensions. The case that we want to analyze is a toroidal compactification. Let us take the simplest case, that is one dimension compactified on a circle (1 torus) of radius  $R$ . Let us denote the bosonic coordinate in the compactified direction by  $X$ , then we have to impose the following periodicity condition for the closed string coordinates

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + 2\pi w R \quad (114)$$

The integer  $w$  is called winding number and it counts the number of times the string has wrapped the circle. It is convenient to write the oscillator expansion in the following way

$$\begin{aligned} X_L(\tau, \sigma) &= \frac{1}{2}x + \frac{1}{2}l_s^2 p_L(\tau + \sigma) + l_s \frac{i}{\sqrt{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \tilde{\alpha}_n e^{-2in(\tau + \sigma)} \\ X_R(\tau, \sigma) &= \frac{1}{2}x + \frac{1}{2}l_s^2 p_R(\tau - \sigma) + l_s \frac{i}{\sqrt{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \alpha_n e^{-2in(\tau - \sigma)} \end{aligned} \quad (115)$$

Then imposing the periodicity condition (114) one obtains that we must have

$$\frac{1}{2}(p_L - p_R) = \frac{wR}{l_s^2} \quad (116)$$

Recall that on a compact space one has the usual quantization condition for the total momentum of the string, that is

$$\frac{p_L + p_R}{2} = \frac{m}{R} \quad (117)$$

where the integer  $m$  is called momentum number. Solving for  $p_L$  and  $p_R$  one gets

$$\begin{aligned} p_L &= \frac{m}{R} + \frac{wR}{l_s^2} \\ p_R &= \frac{m}{R} - \frac{wR}{l_s^2} \end{aligned} \quad (118)$$

Thus for the closed string in a compact space one gets the usual Kaluza-Klein states familiar from field theory indexed by  $m$  but in addition there are also new winding states.

Consider the following coordinate  $X'$

$$X'(\tau, \sigma) = X_L(\tau, \sigma) - X_R(\tau, \sigma) \quad (119)$$

By making use of eqs. (115), (118) one immediately sees that it defines again a string theory on circle of radius

$$R' = \frac{l_s^2}{R} \quad (120)$$

where the winding and the momenta numbers are interchanged

$$m \leftrightarrow w \quad (121)$$

Such a transformation is called T-duality and it is a symmetry of the bosonic closed string. This is no longer the case when one considers open strings or superstrings.

We can describe (NN) open strings by starting with the expansion (115) and imposing

$$p_L = p_R = 2p \quad \alpha_n = \tilde{\alpha}_n \quad (122)$$

where  $p$  is the center of mass momentum of the open string. Let us make the following replacement

$$x \rightarrow x + C \text{ for } X_L \text{ and } x \rightarrow x - C \text{ for } X_R \quad (123)$$

which doesn't affect the solution. The first observation is that because of the condition  $p_L = p_R$  the winding number has to be zero

$$w = 0 \quad (124)$$

thus one only gets Kaluza-Klein states for the zero-mode

$$X(\tau, \sigma) = x + 2l_s^2 \frac{m}{R} \tau + l_s \sqrt{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\alpha_n}{n} e^{-in\tau} \cos n\sigma \quad (125)$$

On the other hand, if one considers the T-dual coordinate  $X' = X_L - X_R$ , then

$$X'(\tau, \sigma) = 2C + 2l_s^2 \frac{m}{R} \sigma + l_s \sqrt{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \alpha_n e^{-in\tau} \sin n\sigma \quad (126)$$

Notice that the dual open string has now fixed endpoints in the direction of  $X'$ . Thus T-duality for open strings replaces (NN) boundary conditions with (DD). Using the T-dual radius  $R' = l_s^2/R$  and replacing  $m \leftrightarrow w$  one gets

$$X'(\tau, \sigma) = x' + 2wR'\sigma + l_s \sqrt{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \alpha_n e^{-in\tau} \sin n\sigma \quad (127)$$

where we have introduced  $x' = 2C$ .

Now let us comment briefly on the case of superstrings. For the fermionic coordinates T-duality acts like a reversal of chirality in the right-moving sector. Then Type IIA string theory compactified on a circle of radius  $R$  is T-dual to Type IIB string theory compactified on a circle of radius  $R' = l_s^2/R$ . A similar duality relates the  $SO(32)$  and  $E_8 \times E_8$  heterotic strings.

T-duality can be extended to more coordinates if one compactifies more than one dimension. Then we have that an even number of T-dualities takes Type IIA(B) to itself whereas an odd number of T-dualities takes IIA(B) to IIB(A).

### 2.3.2 Chan-Paton Labels

In the case of open strings one has the option of adding a non-dynamical quantum number at string endpoints. These are the so-called Chan-Paton labels. A state  $|k\rangle$  in the Hilbert space becomes  $|k; ij\rangle$ . The labels  $i, j$  are conserved by definition through interactions. Let  $|k; a\rangle$  be a general state, then we have the following decomposition

$$|k; a\rangle = \sum_{i,j=1}^N \lambda_{ij}^a |k; ij\rangle \quad (128)$$

One can easily see that a string diagram with open strings is proportional to the trace of a product of  $\lambda^a$  matrices. Performing the following unitary transformation

$$\lambda^a \rightarrow U \lambda^a U^\dagger \quad (129)$$

leaves all string amplitude invariant. Moreover one can show that the vertex operator on the gauge boson transforms in the adjoint of this unitary symmetry. This is sufficient to imply that the global symmetry on the worldsheet given by the Chan-Paton labels translates into a local gauge symmetry into the target space.

### 2.3.3 Unoriented Strings and CP labels

A parity operator (or involution)  $\Omega$  is an operator which satisfies

$$\Omega^2 = I \quad (130)$$

We can define such an operator on the string worldsheet by the following action

$$\begin{aligned} \Omega : (\tau, \sigma) &\rightarrow (\tau, \pi - \sigma) && \text{open strings} \\ \Omega : (\tau, \sigma) &\rightarrow (\tau, 2\pi - \sigma) && \text{closed strings} \end{aligned} \quad (131)$$

which basically amounts to reversing the orientation of the open string. Such a transformation for the worldsheet embedding coordinates  $X^M$  can be implemented by the following action on the oscillators

$$\begin{aligned} \Omega \alpha_n^M \Omega^{-1} &= \pm (-1)^n \alpha_n^M && \text{open strings} \\ \Omega \alpha_n^M \Omega^{-1} &= \tilde{\alpha}_n^M && \text{closed strings} \end{aligned} \quad (132)$$

where the sign  $\pm$  in the first equation refers to (NN) and (DD) strings respectively. By using the oscillator expansion one can easily see that this is indeed compatible with the action on the worldsheet embeddings  $X^M$

$$\begin{aligned} \Omega X^M \Omega^{-1}(\tau, \sigma) &= X^M(\tau, \pi - \sigma) && \text{open strings} \\ \Omega X^M \Omega^{-1}(\tau, \sigma) &= X^M(\tau, 2\pi - \sigma) && \text{closed strings} \end{aligned} \quad (133)$$

Similarly for the fermionic coordinates one has

$$\begin{aligned} \Omega \psi_+(\tau, \sigma) \Omega^{-1} &= \psi_-(\tau, 2\pi - \sigma) && \text{closed strings} \\ \Omega \psi_+(\tau, \sigma) \Omega^{-1} &= \psi_-(\tau, \pi - \sigma) && \text{open strings} \end{aligned} \quad (134)$$

which yields the following action on the oscillators

$$\begin{aligned} \text{(R)} \quad & \Omega b_r^M \Omega^{-1} = \bar{b}_r^M \\ \text{(NS)} \quad & \Omega b_r^M \Omega^{-1} = -\bar{b}_r^M \end{aligned} \quad (135)$$

for closed strings and

$$\Omega b_r^M \Omega^{-1} = e^{i\pi r} b_r^M \quad (136)$$

for open strings, where  $r \in \mathbb{Z}$  for the (R) sector and  $r \in \mathbb{Z} + 1/2$  for the (NS) sector. By introducing the operator

$$P(\Omega) = \frac{1 + \Omega}{2} \quad (137)$$

one can project on the states that are even under  $\Omega$ . In other words the worldsheet parity operator is a (global) symmetry of string theory and we can gauge it. The resulting theory will be a theory where we consider unoriented worldsheets as well. In the case of the bosonic string without CP labels one projects out the photon at the massless level. Indeed the vacuum is even under the worldsheet parity  $\Omega|k\rangle = |k\rangle$  and thus for the photon we have

$$\Omega \alpha_{-1}^M |k\rangle = -\alpha_{-1}^M |k\rangle \quad (138)$$

In the case with CP labels we can keep the gauge fields into the theory. The reason is that the action of the parity can be generalized to

$$\Omega \lambda_{ij} |k, ij\rangle = \lambda'_{ij} |k, ji\rangle \quad (139)$$

where in order for this transformation to be a symmetry of the string amplitudes one must have

$$\lambda' = M \lambda^T M^{-1} \quad (140)$$

Imposing that  $\Omega^2 = 1$  we obtain that

$$M M^{-1T} \lambda M^T M^{-1} = \lambda \quad (141)$$

This condition can be rewritten as  $[M M^{-1T}, \lambda] = 0$ . Now using the fact that the matrices  $\lambda$  form a complete set one obtains that

$$M M^{-1T} \sim I \quad (142)$$

Up to normalization we have two possibilities  $M M^{-1T} = \pm I$  and the corresponding solutions are

$$M = \pm M^T \quad (143)$$

that is  $M$  has to be either symmetric or antisymmetric. By a suitable choice of basis we can restrict to the two cases. The first is

$$M = I \quad (144)$$

In order for the state  $\lambda_{ij}\alpha_{-1}^M|k, ij\rangle$  to survive in the spectrum we must have  $\lambda = -\lambda^T$  and thus in this case the gauge group is  $SO(N)$ . The second case is

$$M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (145)$$

Thus for the vector we have the condition  $\lambda = M\lambda^T M$  which is the definition of  $USp(N)$ .

#### 2.3.4 D-brane Couplings and Tadpole Conditions

Let us consider a  $Dp$ -brane, that is a  $p + 1$  dimensional submanifold  $\Sigma_{p+1}$  of the ten dimensional target space  $M$ . We have an embedding of the worldvolume of the  $Dp$ -brane into the target space

$$X : \Sigma_{p+1} \rightarrow M \quad (146)$$

Let us introduce local coordinates  $(\xi^a)_{a=0,\dots,p}$  on  $\Sigma_{p+1}$ , then locally one works with the functions  $X^M(\xi^a)$  with  $M = 0, \dots, 9$ . We denote by  $g$  the metric tensor of the target space  $M$ . The induced metric on  $\Sigma_{p+1}$  is

$$X^*(g)_{ab} = \partial_a X^M \partial_b X^N g_{MN} \equiv \hat{g}_{ab} \quad (147)$$

The bosonic degrees of freedom of the  $Dp$ -brane are the massless excitations of the open strings attached to the brane. They consist of a gauge field  $A_a(\xi)$  with field strength  $F_{ab}$  and the transverse coordinates  $X^i(\xi)$ . In addition to these we consider the closed string backgrounds. For Type II theories they consist of the metric  $g$ , the 2-form field  $B_{MN}$  and the dilaton  $\Phi$  from the NSNS sector and the  $q$ -form fields  $C_q$  from the RR sector. In the case of Type IIA one has  $q = 1, 3, 5, 7, 9$  and for Type IIB  $q = 0, 2, 4, 6, 8, 10$ . For Type I theory, the projection corresponding to  $\Omega$  leaves only  $g$  and  $\Phi$  in the NSNS sector and  $C_2, C_6, C_{10}$  in the RR sector. The RR  $q$ -form fields are not all independent as they are required to satisfy a duality condition. Introducing the field strength  $F_{q+1} = dC_q$  then we have

$$F_{q+1} = *F_{9-q} \quad (148)$$

The effective action of a  $Dp$ -brane is given by the Dirac-Born-Infeld-Chern-Simons (DBI-CS) action

$$\begin{aligned}
S_{Dp} = & -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi e^{-X^*(\Phi)} \sqrt{-\det(X^*(g) + X^*(B) + 2\pi l_s^2 F)} \\
& - Q_p \int_{\Sigma_{p+1}} \text{ch}(2\pi l_s^2 F) \wedge \sqrt{\frac{\hat{A}(\mathcal{R}_T)}{\hat{A}(\mathcal{R}_N)}} \wedge \bigoplus_q X^*(C_q)
\end{aligned} \tag{149}$$

where  $\text{ch}(2\pi l_s^2 F)$  denotes the Chern character of the bundle associated to the  $Dp$ -brane. The indices  $T$  and  $R$  on  $\mathcal{R}$  refer to the curvature from the tangent and normal bundle of  $\Sigma_{p+1}$ . We will not need in our considerations the definition of the A-roof genus so we will ignore this factor. The formula above is true for one  $Dp$ -brane, that is for an abelian gauge field  $F$ . The couplings  $T_p, Q_p$  represent the tension and the RR charge respectively. For BPS branes the two are equal  $T_p = Q_p$ , that is the force due to the exchange of NSNS modes between two static parallel  $Dp$  branes is canceled by the force due to the exchange of RR fields. From the CS part of the action one can see that the RR fields present in the theory select what type of  $Dp$ -branes one can couple to a given string theory. Thus one has

$$\begin{aligned}
\text{IIA} : & D0, D2, D4, D6, D8 \\
\text{IIB} : & D(-1), D1, D3, D5, D7, D9 \\
\text{I} : & D1, D5, D9
\end{aligned} \tag{150}$$

There is a consistency condition that one has to impose, namely the RR charge cancellation. Consider a  $(q+1)$ -form field with field strength  $F_{q+2} = dC_{q+1}$ . We can write the relevant part of the action in the following form

$$S = -\frac{1}{4} \int d^{10}x \sqrt{-g} |F_{q+2}|^2 - Q_p \sum_a \int C_{q+1} \wedge \pi_{9-q}^a \tag{151}$$

where the forms  $\pi_{9-q}^a$  are closed. The equations of motion following from the action above can be written as

$$d * dC_{q+1} = 2Q_p \sum_a \pi_{9-q}^a \tag{152}$$

We will generically consider the target space to be the product of Minkowski 4d space-time with a 6d internal compact manifold  $K$ . The  $Dp$  branes that we consider will wrap 4d spacetime and cycles in the internal space  $K$ . One can integrate the equation of motion over any  $(9-q)$ -dimensional compact submanifold of  $K$ . The result depends only on the homology class of the submanifold. Introducing the cohomology class

$[\Pi_a] = [\pi_{9-q}]$ , one obtains a set of consistency conditions by integrating the equation of motion above which can be written as

$$\sum_a [\Pi_a] = 0 \quad (153)$$

which basically expresses the conservation of charge. For example consider Type II A string theory with intersecting  $D6$  branes (see Chapter 4) that wrap a 3-cycle in the internal space  $K$ . Poincaré duality states that there is an isomorphism  $H_3(K, \mathbb{Z}) \cong H^3(K, \mathbb{Z})$ . Thus to any homology 3-cycle wrapped by the  $D6$  brane  $a$  (that we integrate over) there is a unique cohomology class defined by the 3-form  $\pi_3^a$ . Alternatively, one can consider the T-dual picture of Type IIB with magnetized  $D9$  branes. We consider the case of a toroidal compact space  $K = \mathbb{T}^6$  with magnetic fields  $H_1^{(a)}, H_2^{(a)}, H_3^{(a)}$  along each two-torus, then one has a coupling to a 6-form  $C_6$  given by

$$\int_{10} C_6 \wedge \pi_4^a \quad (154)$$

where now the forms  $\pi_4^a$  are given by

$$\pi_4^a = H_1^{(a)} \wedge H_2^{(a)} + H_1^{(a)} \wedge H_3^{(a)} + H_2^{(a)} \wedge H_3^{(a)} \quad (155)$$

arising from the expansion of the Chern character with  $F$  containing the three magnetic fields  $H_i^{(a)}$ .

Satisfying the charge neutrality condition in eq. (153) only with  $D$  branes, generally leads to the necessity to introduce both branes and anti-branes which break in principle supersymmetry. Supersymmetric solutions can be found if one cancels the D-brane charges by introducing O-planes. They arise in theories which contain the orientifold projection  $\Omega$ . O-planes are defined to be the fixed point locus of  $\Omega k$ , where  $k$  is an element of an orbifold group (see Chapter 3). The effective action for O-planes is formally the same as for  $D$  branes in the limit that the fields living on the branes are put to zero. Thus they are non-dynamical objects.

$$S_{Op} = -T_{Op} \int_{\Sigma_{p+1}} d^{p+1} \xi e^{-\Phi} \sqrt{-\det g} - Q_{Op} \int_{\Sigma_{p+1}} \sqrt{\frac{L(\mathcal{R}_T/4)}{L(\mathcal{R}_N/4)}} \wedge \bigoplus_q C_q \quad (156)$$

where  $L$  is the Hirzebruch polynomial. We have omitted writing explicitly the pull-back  $X^*$  for the quantities that are defined on the world-volume of the O-plane. Similar considerations as in the case of  $Dp$ -branes lead us to consider the following coupling to the RR forms

$$- Q_{Op} \int C_{q+1} \wedge \pi_{9-q}^{Op} \quad (157)$$

which now changes the neutrality condition to

$$\sum_a [\Pi_a] + Q_{Op} [\Pi_{Op}] = 0 \quad (158)$$

where we have introduced the cohomology classes  $[\Pi_{Op}]$  corresponding to the  $9 - q$  forms  $\pi_{9-q}^{Op}$ . As we will see later on, in 10d Type I string theory one has an  $O9$  plane corresponding to the fixed locus of  $\Omega$ . Its tension and charge are  $T_{O9} = Q_{O9} = -32$ . Thus charge neutrality condition requires the introduction of 32  $D9$  branes. The values of the tension and charge are determined by an explicit calculation of the one-loop string amplitudes (see Section 2.5.2).

## 2.4 String Interactions

### 2.4.1 Vertex Operators

Let us introduce complex coordinates on the worldsheet. We first perform a Wick rotation  $\tau \rightarrow -i\tau$  to the Euclidean theory with metric  $ds^2 = d\tau \otimes d\tau + d\sigma \otimes d\sigma$ . Then we introduce the coordinates

$$z = e^{\tau - i\sigma} \quad \bar{z} = e^{\tau + i\sigma} \quad (159)$$

The metric becomes

$$ds^2 = \frac{1}{|z|^2} dz \otimes d\bar{z} \quad (160)$$

In this system of coordinates the oscillator expansions become

$$X^M(z, \bar{z}) = x^M - i \frac{l_s}{\sqrt{2}} \alpha_0^M \log z \bar{z} + i \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \alpha_n^M (z^{-n} + \bar{z}^{-n}) \quad (161)$$

in the case of the open string, and

$$\begin{aligned} X^M(z, \bar{z}) &= X_L^M(z) + X_R^M(\bar{z}) \\ X_L^M(z) &= \frac{x^M}{2} - i \frac{l_s}{\sqrt{2}} \alpha_0^M \log z + i \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \alpha_n^M z^{-n} \\ X_R^M(\bar{z}) &= \frac{x^M}{2} - i \frac{l_s}{\sqrt{2}} \tilde{\alpha}_0^M \log \bar{z} + i \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \tilde{\alpha}_n^M \bar{z}^{-n} \end{aligned} \quad (162)$$

is the case of closed strings. By taking derivatives with respect to  $z$  and  $\bar{z}$  respectively and inverting the resulting relations one obtains

$$\alpha_{-n}^M = \frac{\sqrt{2}}{l_s} \oint \frac{dz}{2\pi} z^{-n} \partial X^M(z) \quad \tilde{\alpha}_{-n}^M = \frac{\sqrt{2}}{l_s} \oint \frac{d\bar{z}}{2\pi} \bar{z}^{-n} \bar{\partial} X^M(\bar{z}) \quad (163)$$

This suggests that there is an equivalence between a state, let's say  $\alpha_{-1}|0, k\rangle$  and the contour integral around the insertion of a pointlike operator at the point  $z$  in the complex plane. In string perturbation theory, ingoing and outgoing strings will be represented by vertex operators. The S-matrix is defined through the Polyakov expansion

$$A_n = \sum_{\chi} g_s^{-\chi} \int_{M_{\Sigma}} e^{-S_P} V_1 \dots V_n \quad (164)$$

where  $g_s$  is the string coupling constant which is determined by the vacuum expectation value of the dilaton field  $g_s = e^{\langle\phi\rangle}$ . This arises in perturbation theory by adding to the Polyakov action an Einstein-Hilbert term

$$S_{\text{EH}} = \frac{\langle\phi\rangle}{4\pi} \int_{\Sigma} d^2\sigma R \quad (165)$$

which in two dimensions is a topological invariant equal to the Euler characteristic of the worldsheet  $\Sigma$ . One can show that the Euler characteristic has the following expression

$$\chi = 2 - 2h - b - c \quad (166)$$

where  $h$  is the number of handles of the surface (also called the genus),  $b$  is the number of boundaries and  $c$  is the number of crosscaps. By adding handles, boundaries and/or crosscaps to the sphere one can build any two-dimensional surface. Notice that each order in string perturbation theory is determined by  $\chi$ . At a given order one has to sum over all surfaces with given Euler characteristic. The integral is over the so-called moduli space of the surface  $M_{\Sigma}$ . In the expansion above we have assumed a theory with open and closed strings both oriented and unoriented. If one consider a pure oriented closed string theory the Euler characteristic becomes

$$\chi = 2 - 2h \quad (167)$$

and one is summing in perturbation theory over the genus of the surface.

## 2.5 Vacuum Amplitudes

### 2.5.1 Geometry of the $\chi = 0$ Surfaces

In this section we are interested in the one-loop amplitudes in string theory. They correspond to world-sheets  $\Sigma$  with zero Euler characteristic. The equation  $\chi = 0$  has

four distinct solutions

<i>Torus</i>	$h = 1, b = 0, c = 0$	(168)
<i>Klein bottle</i>	$h = 0, b = 0, c = 2$	
<i>Annulus</i>	$h = 0, b = 2, c = 0$	
<i>Möbius strip</i>	$h = 0, b = 1, c = 1$	

Let us start with the torus. Topologically the torus is a product of two circles  $\mathbb{T}^2 = S^1 \times S^1$ . It can be constructed from a parallelogram with its parallel edges identified in the same sense. Thus one has a description in terms of the quotient of the plane  $\mathbb{R}^2$  by a lattice  $\Lambda_2$  generated by two vectors  $a, b$  which can be taken to be of length 1 and  $|\tau|$  respectively. The parameter  $\tau \in \mathbb{C}$ , with  $\text{Im}\tau > 0$  is the Teichmüller parameter of

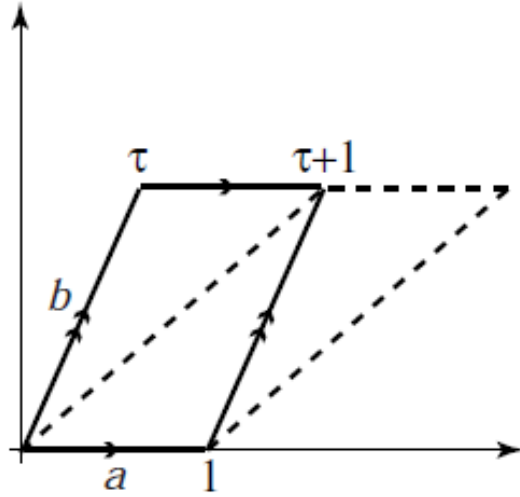


Figure 1: Fundamental Cell of the Torus

the surface and it describes the complex structure of the torus. Notice however that not all the upper half-plane describe inequivalent complex structures. Two tori for which the corresponding Teichmüller parameters  $\tau$  and  $\tau'$  are related by a  $PSL(2, \mathbb{Z})$  transformation

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \text{with } ad - bc = 1 \quad a, b, c, d \in \mathbb{Z} \quad (169)$$

have the same complex structure. Transformations of the form above are called modular transformations. The group of modular transformations is generated by the following

two transformations

$$\begin{aligned} T : \tau &\rightarrow \tau + 1 \\ S : \tau &\rightarrow -\frac{1}{\tau} \end{aligned} \tag{170}$$

In string theory the torus amplitude describes a closed string propagating at one loop. The world-sheet proper time elapsed while the string goes around the loop is  $\tau_2 \equiv \text{Im}\tau$ . From our discussion above it follows that the torus amplitude has to be modular invariant in order to be well defined. Due to this property one has actually an infinite number of choices for the proper time.

Now let us discuss the Klein bottle  $\mathcal{K}$ . It is an unoriented surface since it contains two crosscaps. It can be constructed from a right angle parallelogram for which two edges are identified in the same sense and the other two in the opposite sense. The Klein bottle has the peculiar property of not being embeddable in  $\mathbb{R}^3$ , so it is difficult to visualize. There are two canonical choices for proper time which are related by an  $S$  transformation. We will refer to them as direct and transverse channel respectively. The direct channel describes a closed string sweeping the surface, whereas the transverse channel describes a closed string propagating between two crosscaps. In string theory the crosscaps will be related to spacetime non-dynamical objects called O-planes. In describing the Klein bottle we will use the doubly covering torus which has modular parameter

$$\tau = 2i\tau_2 \tag{171}$$

and it is purely imaginary.

The annulus  $\mathcal{A}$  can be constructed again from a right angle parallelogram with two parallel edges identified in the same sense (preserving the orientation). In the direct channel it describes an open string propagating in a loop, whereas in the transverse channel it describes a closed string propagating between two boundaries which in string theory are interpreted as D-branes. Its doubly covering torus has purely imaginary modular parameter

$$\tau = \frac{1}{2}i\tau_2 \tag{172}$$

Finally, the Möbius strip  $\mathcal{M}$  is an unoriented surface which can be constructed in a similar way as the annulus with the difference of identifying the edges in the opposite sense, thus obtaining an unoriented surface. In the direct channel it describes an open string sweeping the strip, whereas in the transverse channel it describes a closed string propagating between a boundary (D-brane) and a crosscap (O-plane). Its doubly covering torus has a complex modular parameter

$$\tau = \frac{1}{2} + i\frac{\tau_2}{2} \tag{173}$$

The transverse channel proper time is no longer related to  $\frac{1}{2} + i\frac{\tau_2}{2}$  by an  $S$  transformation. For the Möbius one has to use the following transformation

$$P : \frac{1}{2} + i\frac{\tau_2}{2} \rightarrow \frac{1}{2} + i\frac{1}{2\tau_2} \quad (174)$$

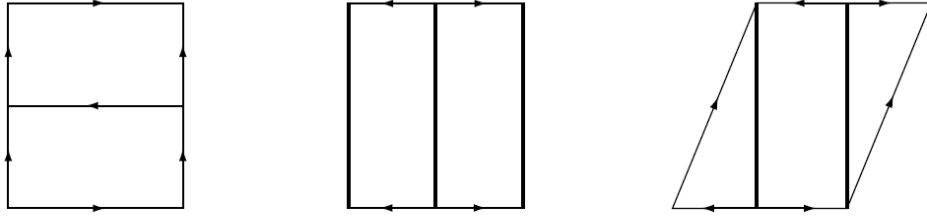


Figure 2: Double Cover Torus of the Klein bottle, Annulus and Möbius strip

### 2.5.2 Partition Functions

Let us recall the formula from field theory for the one-loop vacuum amplitude that we denote by  $\Gamma$ .

$$\Gamma = -\frac{V}{2(4\pi)^{D/2}} \int_{\epsilon}^{\infty} \frac{dt}{t^{D/2+1}} \text{Str} \left( e^{-tM^2} \right) \quad (175)$$

where  $t$  is the Schwinger parameter,  $M^2$  is the mass operator and Str denotes the fact that fermions come with an extra minus sign with respect to the bosons. In the case of strings  $t$  this will be replaced by the modular parameter or Teichmüller parameter of the corresponding worldsheet. For example a closed oriented string propagating in a loop generates a torus. The corresponding amplitude will include an integration on the fundamental region of the torus of the mass operator which we express in terms of  $L_0$  and  $\bar{L}_0$ . One then has

$$\Gamma = \frac{1}{(8\pi^2 l_s^2)^{D/2}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{D/2-1}} \text{tr} q^{L_0} \bar{q}^{\bar{L}_0} \quad (176)$$

We will neglect in what follows the numerical factor in front of the integral. The term  $\tau_2^{D/2-1}$  comes from the integration over the center of mass momentum of the string  $p^M$ ,

actually the transverse part in the light cone gauge

$$l_s^{D-2} \int d^{D-2} p e^{-\pi l_s^2 \tau_2 p^2} \quad (177)$$

The parameters  $q$  and  $\bar{q}$  are defined as

$$q = e^{2\pi i \tau} \quad \bar{q} = e^{-2\pi i \bar{\tau}} \quad (178)$$

Let us compute the torus amplitude for the Type II superstring. Introducing the harmonic oscillators  $a_n = \alpha_n / \sqrt{n}$  and  $a_n^\dagger = \alpha_{-n} / \sqrt{n}$  where  $n > 0$  we have that

$$\text{Tr } q^{\sum_{n=1}^{\infty} \alpha_{-n} \alpha_n} = \prod_{n=1}^{\infty} \text{Tr } q^{n a_n^\dagger a_n} = \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{nk} = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)} \quad (179)$$

For the fermionic oscillators one has

$$\text{Tr } q^{\frac{1}{2} \sum_{r \in Z+\phi} r b_{-r} b_r} = \prod_{r=1-\phi}^{\infty} \text{Tr } q^{r b_{-r} b_r} = \prod_{r=1-\phi}^{\infty} \sum_{k=0,1} q^{rk} = \prod_{r=1}^{\infty} (1 + q^{r-\phi}) \quad (180)$$

which holds in the (NS) sector for  $\phi = 1/2$ . In the (R) sector  $\phi = 0$  and there is an additional multiplicity arising from the zero modes  $b_0^i$ . Indeed, computing the trace in the (R) sector one has

$$\text{tr}_R (q^{L_0}) = 16 \frac{\prod_{m=1}^{\infty} (1 + q^m)^8}{\prod_{m=1}^{\infty} (1 - q^m)^8} \quad (181)$$

The coefficient 16 reflects the degeneracy of the (R) vacuum which carries a representation of the  $SO(8)$  Clifford algebra

$$\{b_0^i, b_0^j\} = \delta^{ij} \quad (182)$$

By using the above formulas and inserting in the trace the GSO projector one obtains

$$\text{tr}_{\text{NS}} \left( \frac{1 - (-1)^F}{2} q^{L_0} \right) = \frac{\prod_{m=1}^{\infty} (1 + q^{m+1/2})^8 - \prod_{m=1}^{\infty} (1 + q^m)^8}{q^{1/2} \prod_{m=1}^{\infty} (1 - q^m)^8} \quad (183)$$

The factor  $q^{1/2}$  in the denominator signals the fact that the vacuum energy in the (NS) sector is  $a = -1/2$ . Let us introduce at this point the Jacobi theta functions. They are defined by the following series

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\nu, \tau) = \sum_{n \in \mathbb{Z}} q^{i\pi\tau(n+\alpha)^2} e^{2\pi i(\beta+\nu)(n+\alpha)} \quad (184)$$

where  $\alpha, \beta \in R$ . One can easily prove the following periodicity properties.

$$\vartheta \begin{bmatrix} \alpha + 1 \\ \beta \end{bmatrix} (\nu, \tau) = \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\nu, \tau), \quad \vartheta \begin{bmatrix} \alpha \\ \beta + 1 \end{bmatrix} (\nu, \tau) = e^{2\pi i \alpha} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\nu, \tau) \quad (185)$$

The periodicity properties motivate the following definitions

$$\vartheta_1 = \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \quad \vartheta_2 = \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \quad \vartheta_3 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \vartheta_4 = \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \quad (186)$$

To relate the theta functions to our vacuum amplitude we need the product formulae

$$\begin{aligned} \vartheta_1(\nu, \tau) &= 2q^{1/8} \sin \pi \nu \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2\pi i \nu})(1 - q^n e^{-2\pi i \nu}) \\ \vartheta_2(\nu, \tau) &= 2q^{1/8} \cos \pi \nu \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n e^{2\pi i \nu})(1 + q^n e^{-2\pi i \nu}) \\ \vartheta_3(\nu, \tau) &= \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1/2} e^{2\pi i \nu})(1 + q^{n-1/2} e^{-2\pi i \nu}) \\ \vartheta_4(\nu, \tau) &= \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1/2} e^{2\pi i \nu})(1 - q^{n-1/2} e^{-2\pi i \nu}) \end{aligned} \quad (187)$$

Introducing also the Dedekind  $\eta$ -function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (188)$$

we can write the vacuum amplitudes in terms of theta functions

$$\begin{aligned} \text{tr}_R q^{L_0} &= \frac{\vartheta_2^4(0, \tau)}{\eta^{12}(\tau)} \\ \text{tr}_{NS} \left( \frac{1 - (-1)^F}{2} q^{L_0} \right) &= \frac{\vartheta_3^4(0, \tau) - \vartheta_4^4(0, \tau)}{\eta^{12}(\tau)} \end{aligned} \quad (189)$$

Using the above results one can then write the torus amplitude of the Type IIB string theory

$$\mathcal{T}_{\text{IIB}} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} \frac{1}{\tau_2^4 |\eta(\tau)|^{16}} \left| \frac{\vartheta_3^4(0, \tau) - \vartheta_4^4(0, \tau) - \vartheta_2^4(0, \tau) - \vartheta_1^4(0, \tau)}{\eta^4(\tau)} \right|^2 \quad (190)$$

A few remarks are in order at this point. The contribution from the 8 transverse bosonic coordinates to the partition function is  $\frac{1}{\tau_2^4 \eta(\tau)^8 \eta(\bar{\tau})^8}$  thus one real boson contributes

$$\frac{1}{\sqrt{\tau_2} \eta(\tau) \eta(\bar{\tau})} \quad (191)$$

For one real fermionic coordinate there are four possibilities which can be written as

$$\sqrt{\frac{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}}{\eta(\tau)}} \quad \alpha, \beta = 0, 1/2 \quad (192)$$

The four terms that we get in the partition function correspond to the four different spin structures that one can write on the torus. The torus has two non-contractible cycles. Around each of these cycles a fermion can have either periodic or antiperiodic boundary conditions. Indeed, let  $\psi(\sigma^1, \sigma^2)$  be a spinor defined on a two torus. Then one can write the following boundary conditions

$$\begin{aligned} \psi(\sigma^1 + 1, \sigma^2) &= \pm \psi(\sigma^1, \sigma^2) \\ \psi(\sigma^1, \sigma^2 + 1) &= \pm \psi(\sigma^1, \sigma^2) \end{aligned} \quad (193)$$

where for simplicity we have taken the periodicities of the torus to be equal to 1. There are four possibilities in defining a spinor on a torus called spin structures. We denote them by  $(+, +), (+, -), (-, +)$  and  $(-, -)$  and they correspond to the signs in eq. (193). One can write down more general boundary conditions by introducing complex spinors

$$\begin{aligned} \psi(\sigma^1 + 1, \sigma^2) &= -e^{2\pi i \theta} \psi(\sigma^1, \sigma^2) \\ \psi(\sigma^1, \sigma^2 + 1) &= -e^{2\pi i \phi} \psi(\sigma^1, \sigma^2) \end{aligned} \quad (194)$$

The contribution to the partition function of a complex fermion with the boundary conditions given in eq. (194) is

$$\frac{\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (0, \tau)}{\eta(\tau)} \quad (195)$$

In view of this, the four spin structures contributions are

$$\begin{aligned} (+, +) \quad \theta = 1/2, \phi = 1/2 &\rightarrow \frac{\vartheta_1^4}{\eta^4} \\ (+, -) \quad \theta = 1/2, \phi = 0 &\rightarrow \frac{\vartheta_2^4}{\eta^4} \\ (-, -) \quad \theta = 0, \phi = 0 &\rightarrow \frac{\vartheta_3^4}{\eta^4} \\ (-, +) \quad \theta = 0, \phi = 1/2 &\rightarrow \frac{\vartheta_4^4}{\eta^4} \end{aligned} \quad (196)$$

By introducing the phases

$$c_{\alpha\beta} = (-1)^{2(\alpha+\beta+2\alpha\beta)} \quad (197)$$

we can write the torus amplitude in the following form

$$\mathcal{T}_{\text{IIB}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^4 |\eta(\tau)|^{16}} \left| \sum_{\alpha, \beta \in \{0, 1/2\}} (-1)^{2(\alpha+\beta+2\alpha\beta)} \frac{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^4(0; \tau)}{\eta^4(\tau)} \right|^2 \quad (198)$$

The relative signs between the contributions from different spin structures, that is the coefficients  $c_{\alpha\beta}$  are determined by the requirement of modular invariance of  $\mathcal{T}$ . Indeed, one can show that the partition function above is modular invariant by using the transformation properties of the theta functions

$$\begin{aligned} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu, \tau + 1) &= e^{-i\pi\alpha(\alpha-1)} \vartheta \begin{bmatrix} \alpha \\ \alpha + \beta - 1/2 \end{bmatrix}(\nu, \tau) \\ \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right) &= \sqrt{-i\tau} e^{2\pi i\alpha\beta + i\pi\frac{\nu^2}{\tau}} \vartheta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}(\nu, \tau) \end{aligned} \quad (199)$$

Another important property of the torus amplitude is its ultraviolet finiteness. The ultraviolet limit corresponds to the region where the modular parameter  $\tau_2$  approaches zero. Notice however that by using a modular transformation one can choose the fundamental domain of the torus  $\mathcal{F}$  to be the following strip

$$\mathcal{F} = \{\tau \in \mathbb{C}, \text{Im}\tau \geq 0, -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, |\tau| \geq 1\} \quad (200)$$

which excludes the region around the origin.

A convenient way to encode the one-loop amplitudes is by making use of the characters of the  $\widehat{so(2n)}$  algebra (i.e. the affine extension of the Lie algebra  $so(2n)$ ).

$$\begin{aligned} O_{2n} &= \frac{\vartheta_3^n + \vartheta_4^n}{2\eta^n} & V_{2n} &= \frac{\vartheta_3^n - \vartheta_4^n}{2\eta^n} \\ S_{2n} &= \frac{\vartheta_2^n + i^{-n}\vartheta_1^n}{2\eta^n} & C_{2n} &= \frac{\vartheta_2^n - i^{-n}\vartheta_1^n}{2\eta^n} \end{aligned} \quad (201)$$

The interesting case for 10d superstrings is  $n = 4$ , that is the affine algebra  $\widehat{so(8)}$ . It has four conjugacy classes of representations and at level one has four integrable representations. They correspond to four sublattices of the weight lattice and include the vector, the scalar and two 8-dimensional spinors. In terms of these, the Type IIB torus amplitude can be written as

$$\mathcal{T}_{\text{IIB}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^4 |\eta(\tau)|^{16}} |V_8 - S_8|^2 \quad (202)$$

The massless spectrum can now be easily derived from the partition function above. Indeed we obtain the four sectors of the Type IIB string theory

$$\begin{aligned}
(\text{NSNS}) \quad & V_8 \times \bar{V}_8 \rightarrow (G_{MN}, B_{MN}, \Phi) \\
(\text{NSR}) \quad & V_8 \times \bar{S}_8 \rightarrow \left( \tilde{\Phi}_\alpha, \tilde{G}_M^\alpha \right)_1 \\
(\text{RNS}) \quad & S_8 \times \bar{V}_8 \rightarrow \left( \tilde{\Phi}_\alpha, \tilde{G}_M^\alpha \right)_2 \\
(\text{RR}) \quad & S_8 \times \bar{S}_8 \rightarrow (C_0, C_2, C_4)
\end{aligned} \tag{203}$$

The massless spectrum is that of gravity with  $\mathcal{N} = 2$  supersymmetry. Particularly useful for checking modular invariance or for passing from the direct to the transverse channel is the representation of the modular transformations  $S, T$  on the  $\widehat{so(2n)}$  characters  $O_{2n}, V_{2n}, S_{2n}, C_{2n}$ .

$$T = e^{-in\frac{\pi}{12}} \text{diag} \left( 1, -1, e^{in\frac{\pi}{4}}, e^{-in\frac{\pi}{4}} \right) \tag{204}$$

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-n} & -i^{-n} \\ 1 & -1 & -i^{-n} & i^{-n} \end{pmatrix} \tag{205}$$

For  $\widehat{so(8)}$ , for the characters divided by the contribution of the bosonic coordinates

$$\frac{O_8}{\tau_2^4 \eta^8} \quad \frac{V_8}{\tau_2^4 \eta^8} \quad \frac{S_8}{\tau_2^4 \eta^8} \quad \frac{C_8}{\tau_2^4 \eta^8} \tag{206}$$

the transformation  $T$  takes the simpler form

$$T = \text{diag} (-1, 1, 1, 1) \tag{207}$$

Remember that for the Möbius amplitude in order to pass from the direct channel to the transverse channel one needs to perform a  $P$  transformation which in terms of  $S$  and  $T$  in eqs. (204),(205) reads

$$P = T^{1/2} S T^2 S T^{1/2} \tag{208}$$

In terms of the basis (206)  $P$  has the same form as  $T$

$$P = (-1, 1, 1, 1) \tag{209}$$

More general, for the  $\widehat{so(2n)}$  characters one has the transformation

$$P = \begin{pmatrix} \cos \frac{n\pi}{4} & \sin \frac{n\pi}{4} & 0 & 0 \\ \sin \frac{n\pi}{4} & -\cos \frac{n\pi}{4} & 0 & 0 \\ 0 & 0 & e^{-in\frac{\pi}{4}} \cos \frac{n\pi}{4} & ie^{-in\frac{\pi}{4}} \sin \frac{n\pi}{4} \\ 0 & 0 & ie^{-in\frac{\pi}{4}} \sin \frac{n\pi}{4} & e^{-in\frac{\pi}{4}} \cos \frac{n\pi}{4} \end{pmatrix} \quad (210)$$

So far we have discussed in detail the ten dimensional Type IIB theory. Remember that the difference between Type IIB and Type IIA was the chirality of the gravitinos. In order to obtain the torus one just has to replace  $S_8$  with  $C_8$  in let's say the left moving part. Thus one has

$$\mathcal{T}_{\text{IIA}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^4 |\eta(\tau)|^{16}} (V_8 - C_8)(\bar{V}_8 - \bar{S}_8) \quad (211)$$

Next we discuss the amplitudes of Type I string theory in ten dimensions (for a review on open strings see [7]). It contains closed and open strings on both oriented and unoriented surfaces. There are four vacuum amplitudes at one-loop, namely the torus  $\mathcal{T}$ , the Klein bottle  $\mathcal{K}$ , the annulus  $\mathcal{A}$  and the Möbius strip  $\mathcal{M}$ . The torus  $\mathcal{T}$  will have the same form as for Type IIB. The Klein bottle amplitude can be computed by inserting into the trace the world-sheet involution  $\Omega$ .

$$\mathcal{K} = \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \frac{1}{\tau_2^{D/2-1}} \text{Tr } q^{L_0} \bar{q}^{\bar{L}_0} \Omega \quad (212)$$

Then one obtains

$$\mathcal{K} = \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \frac{(V_8 - S_8)(2i\tau_2)}{\eta^8(2i\tau_2)} \quad (213)$$

where  $2i\tau_2$  is the Teichmüller parameter of the doubly covering torus of the Klein bottle. Before analyzing the transverse channel let us write down the open string amplitudes  $\mathcal{A}$ ,  $\mathcal{M}$  as well. They can be computed starting with the following formulas

$$\begin{aligned} \mathcal{A} &= \frac{N^2}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \frac{1}{\tau_2^{D/2-1}} \text{Tr } q^{L_0} \\ \mathcal{M} &= \frac{\epsilon N}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \frac{1}{\tau_2^{D/2-1}} \text{Tr } q^{L_0} \Omega \end{aligned} \quad (214)$$

The Teichmüller parameters of the doubly covering torii of the two surfaces are  $\frac{i\tau_2}{2}$  and  $\frac{i\tau_2}{2} + \frac{1}{2}$ . The factors  $N^2$  and  $N$  appearing into the amplitudes above take into account the multiplicities arising from considering Chan-Paton labels. The sign  $\epsilon$  is related to

the orientifold tension and RR charge. It will be fixed by demanding cancelation of tadpoles. Computing the amplitudes in the open string sector, one obtains

$$\begin{aligned}\mathcal{A} &= \frac{N^2}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \frac{(V_8 - S_8)(\frac{1}{2}i\tau_2)}{\eta^8(\frac{1}{2}i\tau_2)} \\ \mathcal{M} &= \frac{\epsilon N}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \frac{(\hat{V}_8 - \hat{S}_8)(\frac{1}{2}i\tau_2 + \frac{1}{2})}{\hat{\eta}^8(\frac{1}{2}i\tau_2 + \frac{1}{2})}\end{aligned}\tag{215}$$

For the Möbius amplitude we have introduced hatted characters. In conformal field theory a character  $\chi_i$  encodes the content of a Verma module. In general it is characterized by a primary operator of conformal weight  $h_i$  and the corresponding descendants. It can be expanded as

$$\chi_i(q) = q^{h_i - \frac{c}{24}} \sum_k d_k q^k \tag{216}$$

The hatted characters differ from the definition above by a phase factor  $e^{-i\pi(h_i - \frac{c}{24})}$ . Thus one has

$$\hat{\chi}_i\left(i\tau_2 + \frac{1}{2}\right) = q^{h_i - \frac{c}{24}} \sum_k (-1)^k d_{(i)k} q^k \tag{217}$$

with  $q = e^{-2\pi\tau_2}$ .

With the normalization that we have chosen the total vacuum amplitude is given by

$$\frac{1}{2}\mathcal{T} + \mathcal{K} + \mathcal{A} + \mathcal{M} \tag{218}$$

In eq. (215) the sign  $\epsilon$  determines whether the vector from the massless open string spectrum is in the antisymmetric or symmetric representation of the CP gauge group

$$\frac{N(N + \epsilon)}{2} \tag{219}$$

The case  $\epsilon = 1$  yields an  $USp(N)$  gauge symmetry, whereas the case  $\epsilon = -1$  yields  $SO(32)$ . Only the  $SO(32)$  satisfies both the NSNS and RR tadpole conditions. The tadpoles arise because the amplitudes  $\mathcal{K}$ ,  $\mathcal{A}$ ,  $\mathcal{M}$  have ultraviolet ( $\tau_2 \rightarrow 0$ ) divergences. Unlike for the torus, now the integration domain is from 0 to  $\infty$  thus leading to ultraviolet divergencies. It is convenient to analyze the divergence in the transverse channel where they arise from the exchange of massless modes. For the Klein bottle, the transverse time is

$$2\tau_2 \rightarrow l = \frac{1}{2\tau_2} \tag{220}$$

and it represents a tube terminating on two crosscaps. In spacetime, crosscaps define non-dynamical objects called orientifold planes.

For the annulus the transverse time is

$$\frac{1}{2}\tau_2 \rightarrow l = \frac{2}{\tau_2} \quad (221)$$

and in this channel it represents a closed string exchanged by the boundaries. In spacetime we interpret the boundaries as extended objects called D-branes.

Finally the Möbius amplitude in the transverse channel represents the exchange of a closed string between a D-brane and an O-plane. The transverse time in this case is given by the  $P$  transformation

$$\frac{1}{2}i\tau_2 + \frac{1}{2} \rightarrow \frac{i}{2\tau_2} + \frac{1}{2} = il + \frac{1}{2} \quad (222)$$

Extracting the singular part from the transverse channel amplitudes

$$\begin{aligned} \tilde{\mathcal{K}} &= \frac{2^5}{2} \int_0^\infty dl \frac{(V_8 - S_8)(il)}{\eta^8(il)} \\ \tilde{\mathcal{A}} &= \frac{2^{-5}N^2}{2} \int_0^\infty dl \frac{(V_8 - S_8)(il)}{\eta^8(il)} \\ \tilde{\mathcal{M}} &= 2\frac{\epsilon N}{2} \int_0^\infty dl \frac{(\hat{V}_8 - \hat{S}_8)(il + \frac{1}{2})}{\hat{\eta}^8(il + \frac{1}{2})} \end{aligned} \quad (223)$$

one finds the following tadpole condition

$$\frac{2^5}{2} + \frac{2^{-5}N^2}{2} + 2\frac{\epsilon N}{2} = \frac{2^{-5}}{2}(N + 32\epsilon)^2 = 0 \quad (224)$$

which determines  $\epsilon = -1$  and thus  $SO(32)$  as gauge group. Type I string theory contains  $D9$  branes and  $O9$  planes. In general  $Dp$ -branes and  $Op$ -planes have tensions  $T_p$  and charges  $Q_p$  under the RR potentials  $C_{p+1}$  or in other words they act as sources for the RR forms.  $Dp$  branes have positive tension and RR charge. One can have  $Op$ -planes with negative tension and negative RR charge that we denote by  $Op_-$  and  $Op$ -planes with positive tension and RR charge denoted by  $Op_+$ . In addition one can define anti- $Dp$ -branes and anti  $Op$ -planes which have opposite RR charge. In ten dimensional Type I string theory we have 32  $D9$ -branes which cancel the charge of the orientifold plane  $O9_-$ . In general there are two types of tadpole conditions (NSNS) and (RR) which are qualitatively different. (NSNS) conditions ensure that there are no dilaton tadpoles whereas the (RR) is the condition of conservation of charge. In the case of supersymmetric models they are the same.

### 2.5.3 Compactified Dimensions

The partition functions in the previous section assume ten dimensional Minkowski spacetime. To connect string theory with four dimensional physics one needs to compactify the extra dimensions. The case that we consider in this section is toroidal compactification. Later on we will approach the subject of orientifold and magnetized/intersecting branes. In section 2.3.1 we have discussed how the oscillator expansions are modified when strings are compactified on a circle (or a torus). Let us consider the contribution to the torus partition function of a bosonic coordinate compactified on a circle. The oscillators give the same contribution whereas from the zero modes we have to sum over the winding and momentum numbers  $w, m$ . Thus the contribution of a compact boson to the torus amplitude is

$$\Lambda_{m,w} = \sum_{m,w} \frac{q'^2_s \frac{p_L^2}{4} \bar{q}'^2_s \frac{p_R^2}{4}}{\eta \bar{\eta}} \quad (225)$$

where  $p_{L,R}$  have been defined in eq. (118). In the case of (NN) open strings one sums only over momentum numbers. Thus the contribution of one real boson to the annulus partition function is

$$P_m = \sum_m \frac{q'^2_s \frac{m^2}{2R^2}}{\eta} \quad (226)$$

It is convenient to introduce the corresponding sum over the winding numbers

$$W_w = \sum_w \frac{q^{\frac{w^2 R^2}{4l_s^2}}}{\eta} \quad (227)$$

The formulas above can be generalized for higher dimensional tori. Let us consider a d-dimensional torus  $\mathbb{T}^d$  with metric tensor  $g$  then  $p_{L,R}, m$  and  $w$  are d-dimensional vectors. One then just needs to do the replacements

$$\begin{aligned} p_{L,R}^2 &\rightarrow p_{L,R}^T g^{-1} p_{L,R} \\ m^2 &\rightarrow m^T g^{-1} m \\ w^2 &\rightarrow w^T g^{-1} w \end{aligned} \quad (228)$$

Let us consider the particular case of a factorizable six-torus  $\mathbb{T}^6 = \bigotimes_{i=1}^3 \mathbb{T}_i^2$  that is we take spacetime to be of the form  $\mathbb{R}^{1,3} \times \mathbb{T}^6$ . From a 4d point of view the components of the metric of the internal space are scalar fields which are called moduli. The metric in each two-torus  $\mathbb{T}_i^2$  has three independent degrees of freedom which are described in

terms of a complex scalar  $U_i$  called complex structure (also denoted by  $\tau_i$ ) and a real scalar  $\text{Im}T_i$  called Kähler modulus. In terms of these the metric tensor is written as

$$g_{ab}^{(i)} = \frac{\text{Im}T}{\text{Im}U} \begin{pmatrix} 1 & \text{Re}U \\ \text{Re}U & |U|^2 \end{pmatrix} \quad (229)$$

The real part of the modulus is given by 2-form field  $B_{ab}^{(i)}$  from the NSNS sector with legs on the  $i$ -th two-torus

$$B_{ab}^{(i)} = \begin{pmatrix} 0 & \text{Re}T \\ \text{Re}T & 0 \end{pmatrix} \quad (230)$$

In terms of these moduli fields the 4d effective supergravity is written in standard form. To conclude this section we write down the partition functions of Type I string theory compactified on a circle

$$\begin{aligned} \mathcal{T} &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{7/2}} \frac{1}{|\eta|^{14}} |V_8 - S_8|^2 \Lambda_{m,w} \\ \mathcal{K} &= \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{1}{\eta^7} (V_8 - S_8) (2i\tau_2) P_m \end{aligned} \quad (231)$$

and for the open string sector

$$\begin{aligned} \mathcal{A} &= \frac{N^2}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{1}{\eta^7} (V_8 - S_8) \left(\frac{1}{2}i\tau_2\right) P_m \\ \mathcal{M} &= -\frac{N}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{1}{\hat{\eta}^7} (\hat{V}_8 - \hat{S}_8) \left(\frac{1}{2}i\tau_2 + \frac{1}{2}\right) P_m \end{aligned} \quad (232)$$

Notice that the fermions give the same contribution to the partition function and that the gauge group remains  $SO(32)$  as in the ten dimensional case. In the following chapters we will consider more sophisticated compactifications. Let us mention a few references in the end. There are of course the classical textbooks on string theory [8],[9],[10],[11] and the more recent [12]. Particularly useful for the quantization of strings was [13]. For the section on D-branes we mention [14], [15].

## 3 Orientifolds

### 3.1 Orbifolds

Let  $\mathcal{M}$  be a differentiable manifold and  $\Gamma$  a finite group endowed with an (left) action on  $M$

$$\Gamma \times M \rightarrow M \quad (233)$$

Then an orbifold is the space of orbits  $\mathcal{O} \equiv M/\Gamma$ , that is we identify any two points  $x, y \in M$  if there exists an element  $g \in \Gamma$  such that  $gx = y$ . Notice that the quotient space is not in general a smooth manifold. This is due to the fact that there may be fixed points of the action of  $\Gamma$ , i.e. points in  $M$  that satisfy  $gx = x$  for all  $g \in \Gamma$ . The orbifold space will have in general singularities corresponding to these points. An action of an orbifold group  $\Gamma$  is called free if there are no fixed points. In this case one obtains a smooth manifold (if the parent space was a smooth manifold).

An alternative definition of an orbifold is a space  $\mathcal{O}$  which is locally diffeomorphic to  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ ) except for a finite number of points.

In string theory we will consider target spaces with the geometry of an orbifold. There will be an induced action on the string coordinates and we will consider the quotient theory. Even though the target space has singularities the corresponding world-sheet CFTs [16], [17] are smooth and well defined everywhere. In addition one gets new twisted sectors of the Hilbert space that we denote by  $\mathcal{H}_h$  for each  $h \in \Gamma$ . This is true for abelian groups. The states in the twisted sectors are localized at the orbifold singularities. The reason why this happens is that in the orbifold theory  $M/\Gamma$  there are new strings that become closed due to the orbifold identifications. Indeed, for the closed bosonic string coordinates one can write the following periodicity conditions

$$X^M(\tau, \sigma + 2\pi) = {}^h X^M(\tau, \sigma) \quad (234)$$

where we have denoted by  ${}^h X^M$  the result of the action of the element  $h \in \Gamma$ . In addition one needs to project each (twisted) sector onto the orbifold invariant states. This is achieved by the use of the following projector

$$P(\Gamma) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g \quad (235)$$

where we have denoted by  $|\Gamma|$  the set cardinal (or volume) of the group  $\Gamma$ . The total partition function of the orbifold theory will be given by

$$\mathcal{T}_{M/\Gamma} = \frac{1}{|\Gamma|} \sum_{g, h \in \Gamma} \text{Tr}_{\mathcal{H}_h} \left( q^{L_0} \bar{q}^{\bar{L}_0} g \right) \quad (236)$$

From the CFT point of view the twisted sectors arise because of modular invariance. However this requirement doesn't always fix uniquely the partition function. Indeed consider the following partition function

$$\mathcal{T}'_{M/\Gamma} = \frac{1}{|\Gamma|} \sum_{g,h \in \Gamma} \epsilon(g,h) Z(g,h) \quad (237)$$

where we have denoted by  $Z(g,h)$  the trace over the twisted sector  $\mathcal{H}_h$  projected by the element  $g$ . If the phases  $\epsilon(g,h)$  satisfy the following conditions

$$\begin{aligned} \epsilon(h_1 h_2, g) &= \epsilon(h_1, g) \epsilon(h_2, g) \\ \epsilon(h, g) &= \epsilon(g, h)^{-1} \end{aligned} \quad (238)$$

then  $\mathcal{T}'$  is also modular invariant (provided  $\mathcal{T}$  was modular invariant). One can show that the inequivalent discrete torsion theories are classified by the second order cohomology group of  $\Gamma$  with values in  $U(1)$  which is denoted by  $H^2(\Gamma, U(1))$  (see [18]). It consists of cocycles

$$c : \Gamma \times \Gamma \rightarrow U(1) \quad (239)$$

such that

$$c(g_1, g_2 g_3) c(g_2, g_3) = c(g_1 g_2, g_3) c(g_1, g_2) \quad (240)$$

for all  $g_1, g_2, g_3 \in \Gamma$ . Furthermore, we identify two cocycles that differ by a coboundary

$$c'(g, h) = \frac{c_g c_h}{c_{gh}} c(g, h) \quad (241)$$

with  $c_g \in U(1)$  for all  $g \in \Gamma$ . These conditions imply indeed (are equivalent to) the conditions for modular invariance of the partition function ([19], [20]). In the theories with discrete torsion the orbifold projection into the twisted sectors is modified to

$$P(\Gamma) |_{\mathcal{H}_h} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \epsilon(g, h) g |_{\mathcal{H}_h} \quad (242)$$

In order to connect string theory with 4d physics one needs to compactify the extra dimensions. Thus one is lead to consider target spaces of the following form

$$R^{1,3} \times K_6 \quad (243)$$

where  $R^{1,3}$  is the usual Minkowski space and  $K_6$  is a compact six dimensional manifold or orbifold. The spaces  $K_6$  often encountered in string theory are Calabi-Yau threefolds and toroidal orbifolds. If one starts with a theory with  $\mathcal{N} = 1$  supersymmetry in ten dimensions, then by dimensional reduction one obtains a theory with  $\mathcal{N} = 4$

supersymmetry in four dimensions. Indeed, let us consider a toroidal compactification. A massless Majorana-Weyl spinor is an  $\mathbf{8}$  representation of the little group  $SO(8)$ . In reducing to four dimensions we have to decompose this representation under the representation of  $SO(2) \times SO(6)$ .

$$\mathbf{8} = (\mathbf{2}, \mathbf{4}) \quad (244)$$

thus one obtains four supercharges in 4d. By using an orbifold one can project out some of the supersymmetry generators. The requirement is that the orbifold space to have  $SU(3)$  holonomy in order to obtain  $\mathcal{N} = 1$  supersymmetry. If one decomposes  $SO(6)$  further to its  $SU(3)$  subgroup then

$$(\mathbf{2}, \mathbf{4}) = (\mathbf{2}, \mathbf{1}) + (\mathbf{2}, \mathbf{3}) \quad (245)$$

On a space of  $SU(3)$  holonomy only the singlet spinor survives and one obtains a theory with  $\mathcal{N} = 1$  supersymmetry. Calabi-Yau manifolds are spaces with  $SU(3)$  holonomy. We will focus on toroidal orbifolds. Let us take our compact space to be

$$K_6 = T^6 / \Gamma \quad (246)$$

where  $\Gamma$  is a finite subgroup of the group of rotations in six dimensions  $SO(6)$  or of its subgroup  $SU(3)$ . By introducing complex bosonic coordinates

$$Z^1 = \frac{1}{\sqrt{2}}(X^4 + iX^5) \quad Z^2 = \frac{1}{\sqrt{2}}(X^6 + iX^7) \quad Z^3 = \frac{1}{\sqrt{2}}(X^8 + iX^9) \quad (247)$$

we consider the simplified case of an element  $\theta \in \Gamma$  of the form

$$\theta(Z^1, Z^2, Z^3) = (e^{2\pi i v_1} Z^1, e^{2\pi i v_2} Z^2, e^{2\pi i v_3} Z^3) \quad (248)$$

The vector  $\mathbf{v} = (v_1, v_2, v_3)$  is called the twist vector. Now let us consider the action of  $\theta$  on a ten dimensional Majorana-Weyl spinor  $|s_1, s_2, s_3, s_4\rangle$  where  $s_i = \pm 1/2$  are the helicities.

$$\theta|s_1, s_2, s_3, s_4\rangle = e^{2\pi i(v_1 s_2 + v_2 s_3 + v_3 s_4)}|s_1, s_2, s_3, s_4\rangle = e^{\pi i(\pm v_1 \pm v_2 \pm v_3)}|s_1, s_2, s_3, s_4\rangle \quad (249)$$

Imposing

$$v_1 \pm v_2 \pm v_3 = 0 \quad (250)$$

for a certain fixed sign choice, and all  $v_i \neq 0$  then the holonomy of the orbifold will be  $SU(3)$  and thus one obtains a theory with  $\mathcal{N} = 1$  supersymmetry. Extended  $\mathcal{N} = 2$  supersymmetry is also possible if, for example  $v_3 = 0$  and  $v_1 + v_2 = 0$ . The most

common examples of orbifold groups in string theory are the  $\mathbb{Z}_N$  and  $\mathbb{Z}_N \times \mathbb{Z}_M$ . Notice that  $\mathbb{Z}_N$  is the group generated by one element  $\theta$  with the constraint  $\theta^N = 1$ . Then its elements are

$$\{1, \theta, \dots, \theta^{N-1}\} \quad (251)$$

and the orbifold projector can be written as

$$P(\mathbb{Z}_N) = \frac{1}{N}(1 + \theta + \theta^2 + \dots + \theta^{N-1}) \quad (252)$$

In the discussion above about supersymmetry we have implicitly assumed  $\mathcal{N} = 1$  in ten dimensions which for the Type I or Type II theory involves the  $\Omega$  projection in addition to the orbifold projection. The combination of the two is called an orientifold and it is the subject of a separate section. Before that let us consider the simplest orbifold of the bosonic string, namely  $S^1/\mathbb{Z}_2$ . It consists of two elements  $o, g$  where  $o$  is the identity and  $g$  satisfies the constraint  $g^2 = o$ . We consider bosonic string theory compactified on a circle of radius  $R$ . We denote the corresponding coordinate by  $X^{25}$ . The orbifold group acts by

$$X^{25} \rightarrow -X^{25} \quad (253)$$

There are two fixed points for the action of this orbifold  $X^{25} = 0$  and  $X^{25} = \pi R$ . There is one twisted sector corresponding to the element  $g \in \mathbb{Z}_2$ . Notice that in this case there is no discrete torsion since  $H^2(\mathbb{Z}_2, U(1)) = \{0\}$ . The unique modular invariant partition function of the theory is

$$\mathcal{T} = \frac{1}{2} \left[ \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} q^{l_s^2 p_L^2 / 4} \bar{q}^{l_s^2 p_R^2 / 4} + \left| \frac{2\eta}{\vartheta_2} \right| + 2 \left| \frac{\eta}{\vartheta_4} \right| + 2 \left| \frac{\eta}{\vartheta_3} \right| \right] \quad (254)$$

where we have written only the contribution from the compact dimension. Notice in the twisted part of the partition function (the last two terms) there is a factor of 2 counting the number of fixed points of the orbifold.

### 3.2 Orientifolds

As advertised earlier, more general constructions than orbifolds are possible, namely orientifolds. In this case aside from projecting by an orbifold group  $\Gamma$  we project also the orientation reversing parity operator  $\Omega$ . Thus one introduces the projector  $P(\Omega)$  when performing the traces.

$$P(\Omega) = \frac{1}{2}(1 + \Omega) \quad (255)$$

If the orbifold group has elements  $\Gamma = \{1, \theta^k\}_{k=1, \dots, |\Gamma|-1}$  then the total orientifold group is

$$\{1, \Omega, \theta^k, \Omega\theta^k\} \quad (256)$$

### 3.2.1 The $\mathbb{T}^4/\mathbb{Z}_2$ orientifold

Let us consider a six dimensional compactification. The action of the orbifold generator  $g \in \mathbb{Z}_2$  on the bosonic coordinates is

$$g : (X^6, X^7, X^8, X^9) \rightarrow (-X^6, -X^7, -X^8, -X^9) \quad (257)$$

We place ourselves in the context of Type I string theory. The locus of points invariant under  $\Omega$  defines an  $O9$ -plane. Its charge under the RR-forms has to be canceled by introducing the corresponding number of  $D9$  branes. In addition to  $O9$  planes one has  $O5$  planes which are the locus of fixed points under the orientifold operation  $\Omega g$ . The  $O5$  planes are extended in the six non-compact dimensions and are pointlike in the compact  $\mathbb{T}^4$ , placed at the fixed points of  $\Omega g$ . To cancel the corresponding RR charge one needs to introduce  $D5$  branes.

One needs to provide further the action of  $g$  on the fermions. For this, let us decompose the  $SO(8)$  characters under the  $SO(4) \times SO(4)$

$$\begin{aligned} V_8 &= V_4 O_4 + O_4 V_4 & O_8 &= O_4 O_4 + V_4 V_4 \\ S_8 &= S_4 S_4 + C_4 C_4 & C_8 &= S_4 C_4 + C_4 S_4 \end{aligned} \quad (258)$$

One acts in the following way on the internal  $SO(4)$  characters

$$g \begin{pmatrix} O_4 \\ V_4 \\ S_4 \\ C_4 \end{pmatrix} = \begin{pmatrix} O_4 \\ -V_4 \\ -S_4 \\ C_4 \end{pmatrix} \quad (259)$$

It is convenient to introduce characters that are eigenvectors of the  $\mathbb{Z}_2$  generator  $g$ .

$$\begin{aligned} Q_o &= V_4 O_4 - C_4 C_4 & Q_v &= O_4 V_4 - S_4 S_4 \\ Q_s &= O_4 C_4 - S_4 O_4 & Q_c &= V_4 S_4 - C_4 V_4 \end{aligned} \quad (260)$$

In terms of these one can write the modular invariant torus amplitude

$$\mathcal{T} = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^4} \left[ |Q_o + Q_v|^2 \Lambda_{m,n} + |Q_o - Q_v|^2 \left| \frac{2\eta}{\vartheta_2} \right|^4 + 16 |Q_s + Q_c|^2 \left| \frac{\eta}{\vartheta_4} \right|^4 + 16 |Q_s - Q_c|^2 \left| \frac{\eta}{\vartheta_3} \right|^4 \right] \frac{1}{|\eta|^8} \quad (261)$$

One can extract the massless spectrum from the above partition function. It consists of the  $\mathcal{N} = (2, 0)$  gravitational multiplet and 21 tensor multiplets. Recall that the content of the  $\mathcal{N} = (2, 0)$  gravitational multiplet contains the metric, five self-dual two-forms and two left-handed gravitinos. Similarly, the tensor multiplet contains an antiself-dual two-form, five scalars and two right-handed spinors.

Now let us consider the  $\Omega$  projected closed string amplitude.

$$\mathcal{K} = \frac{1}{4} \int_0^\infty \frac{d\tau_2}{\tau_2^4} \left[ (Q_o + Q_v)(P_m + W_n) + 2 \times 16(Q_s + Q_c) \left( \frac{\eta}{\vartheta_4} \right)^2 \right] \quad (262)$$

The factor  $1/4$  in front of the Klein bottle amplitude comes from the fact that we are inserting two projectors in the trace for the total vacuum amplitude

$$P(\Omega) = \frac{1}{2}(1 + \Omega) \quad P(\mathbb{Z}_2) = \frac{1}{2}(1 + g) \quad (263)$$

The massless spectrum consists of the  $\mathcal{N} = (1, 0)$  gravitational multiplet (metric, self-dual two form and left-handed gravitino), one tensor multiplet (antiself-dual two form, scalar, right-handed spinor) and 20 hypermultiplets (4 scalars, right-handed spinor). In order to see explicitly the presence of  $O5$  planes in addition to  $O9$  planes it is instructive to write the Klein bottle amplitude in the transverse channel. By introducing the internal volume of the four-torus

$$v_4 = \frac{1}{l_s^4} \sqrt{\det g} \quad (264)$$

one obtains

$$\tilde{\mathcal{K}} = \frac{2^5}{4} \int_0^\infty dl \left[ (Q_o + Q_v) \left( v_4 W_n + \frac{1}{v_4} P_m \right) + 2(Q_o - Q_v) \left( \frac{2\eta}{\vartheta_2} \right)^2 \right] \frac{1}{\eta^4} \quad (265)$$

The massless tadpole contribution is obtained by considering  $\tilde{\mathcal{K}}$  at the origin of the lattice (winding and momenta) sums.

$$\tilde{\mathcal{K}}_0 \sim \frac{2^5}{4} \left[ Q_o \left( \sqrt{v_4} + \frac{1}{\sqrt{v_4}} \right)^2 + Q_v \left( \sqrt{v_4} - \frac{1}{\sqrt{v_4}} \right)^2 \right] \quad (266)$$

From this expression one sees that in addition to  $O9$  planes there are  $O5_-$  planes with standard negative tension and RR charge. The corresponding tadpoles require the presence of  $D9$  and  $D5$  branes. Let us denote the CP multiplicity of the  $D9$  branes by  $N$  and the one of  $D5$  branes by  $D$ . There is an action of the orbifold generator  $g$  on

the CP labels which we will denote by  $R_N$  and  $R_D$  respectively. With these one can write the annulus amplitude

$$\begin{aligned} \mathcal{A} = & \frac{1}{4} \int_0^\infty \frac{d\tau_2}{\tau_2^4} \left[ (Q_o + Q_v)(N^2 P_m + D^2 W_n) + (R_N^2 + R_D^2)(Q_o - Q_v) \left( \frac{2\eta}{\vartheta_2} \right)^2 \right. \\ & \left. + 2ND(Q_s + Q_c) \left( \frac{\eta}{\vartheta_4} \right)^2 + 2R_N R_D(Q_s - Q_c) \left( \frac{\eta}{\vartheta_3} \right)^2 \right] \frac{1}{\eta^4} \end{aligned} \quad (267)$$

It is instructive to comment on the various contributions to this amplitude. The overall  $1/\tau_2^2 \eta^4$  comes from the non-compact bosons which always have (NN) boundary condition. The term  $N^2 P_m$  comes from compact bosons with (NN) boundary conditions corresponding to the  $D9$  branes, whereas  $D^2 W_n$  comes from compact bosons with (DD) boundary conditions corresponding to the  $D5$  branes. The contribution  $2ND(\eta/\vartheta_4)^2$  comes from open strings stretched between  $D9$  and  $D5$  branes in the compact four-torus and thus have (ND) boundary conditions. The contribution from the fermions is affected accordingly leading to  $Q_s + Q_c$ . The rest of the terms represent the action of the orbifold generator  $g$ . Let us consider the annulus in the transverse channel

$$\begin{aligned} \tilde{\mathcal{A}} = & \frac{2^{-5}}{4} \int_0^\infty dl \left[ (Q_o + Q_v) \left( N^2 v_4 W_n + \frac{D^2}{v_4} P_m \right) + 2ND(Q_o - Q_v) \left( \frac{2\eta}{\vartheta_2} \right)^2 \right. \\ & \left. + 16(R_N^2 + R_D^2)(Q_s + Q_c) \left( \frac{\eta}{\vartheta_4} \right)^2 - 2 \times 4R_N R_D(Q_s - Q_c) \left( \frac{\eta}{\vartheta_3} \right)^2 \right] \frac{1}{\eta^4} \end{aligned} \quad (268)$$

The contribution to the tadpoles from the massless part of the spectrum is

$$\begin{aligned} \tilde{\mathcal{A}}_0 \sim & \frac{2^{-5}}{4} \left[ Q_o \left( N\sqrt{v_4} + \frac{D}{\sqrt{v_4}} \right)^2 + Q_v \left( N\sqrt{v_4} - \frac{D}{\sqrt{v_4}} \right)^2 \right] \\ & + Q_s[15R_N^2 + (R_N - 4R_D)^2] + Q_c[15R_N^2 + (R_N + 4R_D)^2] \end{aligned} \quad (269)$$

One can immediately see from the untwisted sector that the CP multiplicities  $N$  and  $D$  determine the number of  $D9$  and  $D5$  branes. Their contribution matches the contributions from the Klein bottle amplitude. The twisted part of the formula above encodes the distribution of the D-branes. The  $D9$  branes see all 16 fixed points whereas the  $D5$  branes are all placed at the same fixed point. Now let us write down the Möbius amplitude as well

$$\mathcal{M} = -\frac{1}{4} \int_0^\infty \frac{d\tau_2}{\tau_2^4} \left[ (\hat{Q}_o + \hat{Q}_v)(N\hat{P}_m + D\hat{W}_n) - (N + D)(\hat{Q}_o - \hat{Q}_v) \left( \frac{2\hat{\eta}}{\hat{\vartheta}_2} \right)^2 \right] \frac{1}{\hat{\eta}^4} \quad (270)$$

There is immediately one consequence that can be extracted from the amplitude above. At the massless level there is no gauge boson propagating in the Möbius. As a general rule this implies that the gauge group is unitary and the correct parametrization of the CP multiplicity is

$$N = p + \bar{p} \quad D = d + \bar{d} \quad (271)$$

and thus the gauge group is of the form

$$U(p) \times U(d) \quad (272)$$

We still have to specify the action of  $g$  on the CP factors  $N, D$ . This will be determined by the (twisted) tadpole conditions. In order to obtain the tadpole conditions we write the transverse channel amplitude

$$\tilde{\mathcal{M}} = -\frac{2}{4} \int_0^\infty dl \left[ (\hat{Q}_o + \hat{Q}_v) \left( N v_4 W_n + \frac{D}{v_4} P_m \right) + (N + D)(Q_o - Q_v) \left( \frac{2\hat{\eta}}{\hat{v}_2} \right)^2 \right] \quad (273)$$

which yields the following expression at the massless level

$$\tilde{\mathcal{M}}_0 \sim -\frac{2}{4} \left[ (\hat{Q}_o \left( N\sqrt{v_4} + \frac{D}{\sqrt{v_4}} \right) \left( \sqrt{v_4} + \frac{1}{\sqrt{v_4}} \right) + \hat{Q}_v \left( N\sqrt{v_4} - \frac{D}{\sqrt{v_4}} \right) \left( \sqrt{v_4} - \frac{1}{\sqrt{v_4}} \right) \right] \quad (274)$$

By comparing eqns. (266), (269), (274) one can see that we have the following relation

$$\tilde{\mathcal{M}}_0 = \sqrt{\tilde{\mathcal{K}}_0} \sqrt{\tilde{\mathcal{A}}_0} \quad (275)$$

which holds state by state, and can be used in principle to determine the Möbius amplitude from the annulus and the Klein. Imposing the tadpole cancelation condition

$$\tilde{\mathcal{K}}_0 + \tilde{\mathcal{A}}_0 + \tilde{\mathcal{M}}_0 = 0 \quad (276)$$

one obtains that the untwisted tadpole conditions are

$$N = 32 \quad D = 32 \quad (277)$$

and the twisted ones

$$R_N = R_D = 0 \quad (278)$$

In view of our parametrization of the CP factors in eq. (271) we must have

$$\begin{aligned} p &= d = 16 \\ R_N &= i(p - \bar{p}) \\ R_D &= i(d - \bar{d}) \end{aligned} \quad (279)$$

thus the gauge group of this model is  $U(16)_9 \times U(16)_5$  and the open string part of the massless spectrum consists of hypermultiplets in the following representations  $(120 + \overline{120}, 1)$ ,  $(1, 120 + \overline{120})$  and  $(16, \overline{16})$ . In addition there are of course the corresponding gauge multiplets for the two factors of the gauge group.

In the model that we have presented above the  $D5$  branes were all sitting at the same fixed point. There exists a generalization of this in the sense that one can consider arbitrary positions for the  $D5$  branes. This is equivalent to saying that we add Wilson lines to the model. One has to distinguish between two cases. When a  $D5$  brane sits at a fixed point the gauge group is unitary. On the contrary, when a  $D5$  brane sits in the bulk the gauge group is symplectic. The parametrization of CP labels for the model with Wilson lines is

$$\begin{aligned} N &= p + \bar{p} & R_N &= i(p - \bar{p}) \\ D_i &= d_i + \bar{d}_i & R_D &= i(d_i - \bar{d}_i) & i &= 1, \dots, 16 \\ D_k &= 2d_k & k &= 1, \dots, p \end{aligned} \quad (280)$$

where the index  $i$  goes over the orbifold fixed points and the index  $k$  goes over the branes in the bulk. Notice that for a brane in the bulk the action of the orbifold will create an image brane. This is the reason behind the rescaling of the corresponding CP factor above. The gauge group will be broken to

$$U(n) \times \prod_{i=1}^{16} U(d_i) \times \prod_{k=1}^p USp(d_k) \quad (281)$$

depending on the distribution of the  $D5$ -branes. Any configuration has to satisfy the untwisted tadpole conditions

$$n = 16 \quad \sum_{i=1}^{16} d_i + \sum_{k=1}^p d_k = 16 \quad (282)$$

In our construction of the  $T^4/Z_2$  orientifold we have introduced standard  $O5$  orientifold planes with negative tension and RR charge. It is possible to consider a different model with exotic  $O5$  planes, that is with positive tension and RR charge. In this case one is forced to introduce anti  $D5$  branes to satisfy tadpoles. Such a scenario leads to brane supersymmetry breaking which will be discussed in another chapter.

### 3.2.2 The $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold

We start with a factorizable six torus  $\mathbb{T}_1^2 \times \mathbb{T}_2^2 \times \mathbb{T}_3^2$  and we introduce complex coordinates  $(Z^1, Z^2, Z^3)$  corresponding to each two-torus

$$Z^1 = \frac{1}{\sqrt{2}}(X^4 + iX^5) \quad Z^2 = \frac{1}{\sqrt{2}}(X^6 + iX^7) \quad Z^3 = \frac{1}{\sqrt{2}}(X^8 + iX^9) \quad (283)$$

In terms of these the action of the three generators  $g, f, h$  of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  can be written as

$$g : (+, -, -) \quad f : (-, +, -) \quad h : (-, -, +) \quad (284)$$

where a sign  $\pm$  in the  $i$ -th position means that  $Z_i$  is mapped into  $\pm Z_i$  by the orbifold generator. There is of course an identity element that we denote by  $o$ .

The  $\mathbb{Z}_2 \times \mathbb{Z}_2$  is the simplest orbifold group that has discrete torsion. One can show that we have

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \quad (285)$$

thus one has two disconnected modular orbits which can be assembled together with one of the two signs  $\epsilon = \epsilon(g, h) = \pm 1$ . This is visible if one writes down the torus amplitude

$$\begin{aligned} \mathcal{T} = & \frac{1}{4} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3} \left[ |T_{oo}|^2 \Lambda_1 \Lambda_2 \Lambda_3 + |T_{og}|^2 \Lambda_1 \left| \frac{2\eta}{\vartheta_2} \right|^4 + |T_{of}|^2 \Lambda_2 \left| \frac{2\eta}{\vartheta_2} \right|^4 + |T_{oh}|^2 \Lambda_3 \left| \frac{2\eta}{\vartheta_2} \right|^4 \right. \\ & + |T_{go}|^2 \Lambda_1 \left| \frac{2\eta}{\vartheta_4} \right|^4 + |T_{gg}|^2 \Lambda_1 \left| \frac{2\eta}{\vartheta_3} \right|^4 + |T_{fo}|^2 \Lambda_2 \left| \frac{2\eta}{\vartheta_4} \right|^4 + |T_{ff}|^2 \Lambda_2 \left| \frac{2\eta}{\vartheta_3} \right|^4 \\ & + |T_{ho}|^2 \Lambda_3 \left| \frac{2\eta}{\vartheta_4} \right|^4 + |T_{hh}|^2 \Lambda_3 \left| \frac{2\eta}{\vartheta_3} \right|^4 \\ & \left. + \epsilon (|T_{gh}|^2 + |T_{gf}|^2 + |T_{fg}|^2 + |T_{fh}|^2 + |T_{hg}|^2 + |T_{hf}|^2) \left| \frac{8\eta^3}{\vartheta_2 \vartheta_3 \vartheta_4} \right|^2 \right] \frac{1}{|\eta|^4} \end{aligned} \quad (286)$$

where we have denoted by  $\Lambda_i$  the lattice sum  $\Lambda_{m_i, n_i}$  in the  $i$ -th torus. Furthermore,  $T_{kl}$  denotes the contribution to the partition function from the  $k$ -twisted sector projected by the  $l$  generator with  $k, l$  running over all elements of the orbifold  $\{o, g, f, h\}$ . For example  $T_{oo} = V_8 - S_8$  is the familiar ten dimensional Type IIB. Using the decomposition of the characters under

$$SO(8) \rightarrow SO(2) \times SO(2) \times SO(2) \times SO(2) \quad (287)$$

appropriate for a compactification on a factorizable six-torus and introducing characters  $\tau_{kl}$  that are eigenvectors of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  (see eq. (557) in Appendix A for their definition),

one can write

$$\begin{aligned}
T_{ko} &= \tau_{ko} + \tau_{kg} + \tau_{kh} + \tau_{kf} & T_{kg} &= \tau_{ko} + \tau_{kg} - \tau_{kh} - \tau_{kf} \\
T_{kh} &= \tau_{ko} - \tau_{kg} + \tau_{kh} - \tau_{kf} & T_{kf} &= \tau_{ko} - \tau_{kg} - \tau_{kh} + \tau_{kf}
\end{aligned} \tag{288}$$

Explicitly, we can write down the induced action of the orbifold generators on the familiar  $SO(2)$  characters in each torus.

Torus	$T_1^2$	$T_2^2$	$T_3^2$
Generator/Characters	$(O_2, V_2, S_2, C_2)$	$(O_2, V_2, S_2, C_2)$	$(O_2, V_2, S_2, C_2)$
$g$	$(1, 1, 1, 1)$	$(1, -1, -i, i)$	$(1, -1, i, -i)$
$f$	$(1, -1, -i, i)$	$(1, 1, 1, 1)$	$(1, -1, i, -i)$
$h$	$(1, -1, -i, i)$	$(1, -1, i, -i)$	$(1, 1, 1, 1)$

The massless spectrum is different for the two choices  $\epsilon = \pm 1$ . Thus for the model with discrete torsion  $\epsilon = -1$  one obtains  $\mathcal{N} = 2$  supergravity with 52 hypermultiplets and 3 vector multiplets whereas for the model without discrete torsion  $\epsilon = 1$  one has again  $\mathcal{N} = 2$  with 4 hypermultiplets and 51 vector multiplets. They describe the orbifold limits of Calabi-Yau compactifications with Hodge numbers  $(51, 3)$  and  $(3, 51)$  respectively. We now implement the  $\Omega$  projection in the two cases. One introduces additional signs  $\epsilon_g, \epsilon_f, \epsilon_h$  corresponding to the tension and RR charges of the three  $O5$  planes introduced by the orientifold group generators  $\Omega g, \Omega f, \Omega h$ . The corresponding signs have to satisfy the following constraint

$$\epsilon_g \epsilon_f \epsilon_h = \epsilon \tag{289}$$

The Klein amplitude is

$$\begin{aligned}
\mathcal{K} &= \frac{1}{8} \int_0^\infty \frac{d\tau_2}{\tau_2^3} [(P_1 P_2 P_3 + P_1 W_2 W_3 + W_1 P_2 W_3 + W_1 W_2 P_3) T_{oo} \\
&\quad + 2 \times 16 [\epsilon_g (P_1 + \epsilon W_1) T_{go} + \epsilon_f (P_2 + \epsilon W_2) T_{fo} + \epsilon_h (P_3 + \epsilon W_3) T_{ho}] \left( \frac{\eta}{\vartheta_4} \right)^2] \frac{1}{\eta^2}
\end{aligned} \tag{290}$$

with  $P_i = P_{m_i}$  and  $W_i = W_{n_i}$  denoting the momentum and winding number sums in the corresponding tori. By methods that we have illustrated one can write the transverse

channel amplitude. We reproduce here only the massless part of it

$$\begin{aligned}
\tilde{\mathcal{K}}_0 \sim \frac{2^5}{8} & \left[ \left( \sqrt{v_1 v_2 v_3} + \epsilon_g \frac{v_1}{v_2 v_3} + \epsilon_f \frac{v_2}{v_1 v_3} + \epsilon_h \frac{v_3}{v_1 v_2} \right)^2 \tau_{oo} \right. \\
& + \left( \sqrt{v_1 v_2 v_3} + \epsilon_g \frac{v_1}{v_2 v_3} - \epsilon_f \frac{v_2}{v_1 v_3} - \epsilon_h \frac{v_3}{v_1 v_2} \right)^2 \tau_{og} \\
& + \left( \sqrt{v_1 v_2 v_3} - \epsilon_g \frac{v_1}{v_2 v_3} + \epsilon_f \frac{v_2}{v_1 v_3} - \epsilon_h \frac{v_3}{v_1 v_2} \right)^2 \tau_{of} \\
& \left. + \left( \sqrt{v_1 v_2 v_3} - \epsilon_g \frac{v_1}{v_2 v_3} - \epsilon_f \frac{v_2}{v_1 v_3} + \epsilon_h \frac{v_3}{v_1 v_2} \right)^2 \tau_{oh} \right]
\end{aligned} \tag{291}$$

One can see from the formula above that reversing the sign  $\epsilon_k$  reverses the charge and tension of the corresponding  $O5$  plane. There are four possibilities consistent with the condition (289):

$(\epsilon_g, \epsilon_f, \epsilon_h) = (+, +, +)$  gives a model with 48 chiral multiplets from the twisted sectors, the choice  $(+, -, -)$  contains 16 chiral multiplets and 32 vector multiplets from the twisted sectors. The two choices with discrete torsion  $(+, +, -)$  and  $(-, -, -)$  yield the same massless spectra with 48 chiral multiplets from the twisted sectors.

Notice that in the case with discrete torsion one must have at least one exotic orientifold plane, and thus cancelation of tadpoles will require introduction of anti  $D5$  branes thus leading to a scenario with brane supersymmetry breaking. An interesting case is to consider the discrete torsion model with magnetized  $D9$  branes. In this case it is possible to satisfy tadpoles without introducing  $\overline{D5}$  branes and thus preserving supersymmetry in the case  $(+, +, -)$ . We will discuss this in the next chapter where we introduce magnetizations on the D-branes.

## 4 Internal Magnetic Fields and Intersecting Branes

### 4.1 Non-linear sigma model

The non-linear sigma model is a generalization of the actions that we have considered so far for strings. In principle one considers strings in a non-trivial background. For example let us consider the bosonic string. We have seen that the massless spectrum of the closed bosonic string consists of three fields: the graviton  $G_{MN}$ , the antisymmetric field  $B_{MN}$  and the dilaton  $\Phi$ . One can write down an action for the bosonic string using the aforementioned background.

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-\gamma} [G_{MN}(X) \gamma^{ab} \partial_a X^M \partial_b X^N + B_{MN}(X) \epsilon^{ab} \partial_a X^M \partial_b X^N + \alpha' \Phi R] \quad (292)$$

For general backgrounds it is not known how to quantize this action. Weyl invariance requires that all beta functions of the theory vanish. At the first order in  $\alpha'$  one obtains

$$\begin{aligned} \beta_{MN}^G &= \alpha' R_{MN} + 2\alpha' \nabla_M \nabla_N \Phi - \frac{\alpha'}{4} H_{MPQ} H_N^{PQ} + O(\alpha'^2) \\ \beta_{MN}^B &= -\frac{\alpha'}{2} \nabla^P H_{PMN} + \alpha' \nabla^P \Phi H_{PMN} + O(\alpha'^2) \\ \beta^\Phi &= \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_P \Phi \nabla^P \Phi - \frac{\alpha'}{24} H_{MNP} H^{MNP} + O(\alpha'^2) \end{aligned} \quad (293)$$

where  $R_{MN}$  is the target space Ricci tensor and  $H_{MNP}$  is the field strength of the  $B$ -field

$$H = dB \quad (294)$$

An important observation is that the equation  $\beta_{MN}^G$  resembles the Einstein's equations with  $B_{MN}$  and  $\Phi$  fields as sources. The second equation  $\beta_{MN}^B = 0$  is generalization of Maxwell equation determining the divergence of the field strength tensor.

One can add a boundary term to the action of the non-linear sigma model.

$$S_{\text{boundary}} = -q_L \int d\tau A_M(X) \partial_\tau X^M |_{\sigma=0} - q_R \int d\tau A_M(X) \partial_\tau X^M |_{\sigma=\pi} \quad (295)$$

This will be important in the context of magnetized/intersecting branes that we will discuss later on.

## 4.2 Magnetized Branes

Let us consider the open bosonic string in the presence of a constant magnetic field  $F_{MN}$ . The vector potential can then be chosen of the following form

$$A_M = -\frac{1}{2}F_{MN}X^N \quad (296)$$

One starts with the following world-sheet action [21]

$$S = -\frac{1}{4\pi l_s^2} \int d\tau d\sigma \partial_a X^M \partial^a X_M - q_L \int d\tau A_M \partial_\tau X^M |_{\sigma=0} - q_R \int d\tau A_M \partial_\tau X^M |_{\sigma=\pi} \quad (297)$$

written in the conformal gauge. Variation of this action leads to the following PDE (partial differential equations) problem

$$\begin{aligned} (\partial_\tau^2 - \partial_\sigma^2)X^M &= 0 \\ \frac{1}{2\pi l_s^2} \partial_\sigma X^M - q_L F_N^M \partial_\tau X^N &= 0, \quad \text{for } \sigma = 0 \\ \frac{1}{2\pi l_s^2} \partial_\sigma X^M + q_R F_N^M \partial_\tau X^N &= 0, \quad \text{for } \sigma = \pi \end{aligned} \quad (298)$$

which is nothing else than the wave equation with general boundary conditions given by a linear combination of Neumann and Dirichlet. This is a problem that can be solved exactly for a constant magnetic field.  $F_{MN}$  is an antisymmetric matrix of even dimension (both the bosonic and the superstring live in an even number of dimensions 26, and 10 respectively) and by a change of basis can be put into the following standard form

$$F_{MN} = \begin{pmatrix} 0 & H_1 & 0 & \cdot & \cdot & 0 \\ -H_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & 0 & H_{D/2} & \\ 0 & 0 & & -H_{D/2} & 0 & \end{pmatrix} \quad (299)$$

thus one can reduce the problem to a plane  $(X^1, X^2)$ . Let us denote  $F_{12} = H$ . It is convenient to introduce complex coordinates

$$Z = \frac{1}{\sqrt{2}}(X^1 + iX^2) \quad \bar{Z} = \frac{1}{\sqrt{2}}(X^1 - iX^2) \quad (300)$$

In terms of these the action can be written as

$$S = \frac{1}{2\pi l_s^2} \int d\tau d\sigma \partial_a \bar{Z} \partial^a Z + i q_L H \int d\tau \bar{Z} \partial_\tau Z \big|_{\sigma=0} + i q_R H \int d\tau \bar{Z} \partial_\tau Z \big|_{\sigma=\pi} \quad (301)$$

By introducing the following quantities

$$\alpha = 2\pi l_s^2 q_L H \quad \beta = 2\pi l_s^2 q_R H \quad (302)$$

one can write the boundary conditions as

$$\begin{aligned} \partial_\sigma Z + i\alpha \partial_\tau Z &= 0 & \sigma &= 0 \\ \partial_\sigma Z - i\beta \partial_\tau Z &= 0 & \sigma &= \pi \end{aligned} \quad (303)$$

One distinguishes two cases depending on the total charge of the string  $Q = q_L + q_R$ .

### Case I. $Q \neq 0$

In this case the frequencies of the oscillator modes are shifted by

$$\zeta = \frac{1}{\pi}(\gamma + \gamma') \quad (304)$$

where  $\gamma = \tan^{-1}(\alpha)$  and  $\gamma' = \tan^{-1}(\beta)$ . The solution can be written as follows

$$Z(\tau, \sigma) = z + i l_s \sqrt{2} \left[ \sum_{n=1}^{\infty} a_n \phi_n(\tau, \sigma) - \sum_{m=0}^{\infty} \tilde{a}_m^\dagger \phi_{-m}(\tau, \sigma) \right] \quad (305)$$

where the functions  $\psi_n$  are defined as

$$\phi_n(\tau, \sigma) = \frac{1}{\sqrt{|n - \zeta|}} \cos[(n - \zeta)\sigma + \gamma] e^{-i(n - \zeta)\tau} \quad (306)$$

Quantization of this system leads to the usual commutations relations for the Fourier coefficients  $a_m$  and  $b_m^\dagger$ .

$$\begin{aligned} [a_n, a_m^\dagger] &= [\tilde{a}_n, \tilde{a}_m^\dagger] = \delta_{mn} \\ [a_n, a_m] &= [a_n^\dagger, a_m^\dagger] = [\tilde{a}_n, \tilde{a}_m] = [\tilde{a}_n^\dagger, \tilde{a}_m^\dagger] = 0 \end{aligned} \quad (307)$$

For the zero modes one obtains

$$[z, \bar{z}] = \frac{2\pi l_s^2}{\alpha + \beta} \quad (308)$$

which results in the familiar spectrum of Landau levels.

One can compute the Virasoro operators by the usual methods that we have illustrated. We reproduce here  $L_0$  since this is the one necessary for computing partition functions

$$L_0 = \sum_{m=1}^{\infty} (m - \zeta) a_m^\dagger a_m + \sum_{m=0}^{\infty} (m + \zeta) \tilde{a}_m^\dagger \tilde{a}_m \quad (309)$$

An important consequence of the presence of boundary magnetic fields is the shift of the masses. Take for example the first excited level

$$a_1^\dagger|z\rangle \text{ and } \tilde{a}_1^\dagger|z\rangle \quad (310)$$

which was massless in the case of zero magnetic field. Their masses are

$$\begin{aligned} M^2(a_1^\dagger|z\rangle) &= -\frac{1}{2}\zeta(1+\zeta) \\ M^2(\tilde{a}_1^\dagger|z\rangle) &= +\frac{1}{2}\zeta(1+\zeta) \end{aligned} \quad (311)$$

thus one becomes tachyonic and the other becomes massive.

By making use of eq. (309) one can compute the contribution to the partition function (annulus) of a complex magnetized boson

$$\mathcal{A} \sim \frac{i\eta}{q^{\frac{1}{2}\zeta^2}\vartheta_1(\zeta\tau, \tau)} \quad (312)$$

which holds both in the compact (toroidal) and non-compact case. The reason is that in the expansion in eq. (305) the quantized momentum zero-mode does not appear. Now let us discuss the case of the neutral strings.

#### Case II. $Q = 0$

The frequencies of the oscillators are no longer shifted and the boundary conditions allow the presence of new zero modes

$$Z(\tau, \sigma) = \frac{z + \bar{p}[\tau - i\alpha(\sigma - \pi/2)]}{\sqrt{1 + \alpha^2}} + il_s\sqrt{2}\sum_{n=1}^{\infty}[a_n\phi_n(\tau, \sigma) - \tilde{a}_n^\dagger\phi_{-n}(\tau, \sigma)] \quad (313)$$

Canonical quantization leads again to the usual commutation relations for the oscillators  $a_m$  and  $b_m^\dagger$  whereas for the zero modes we have

$$[x_i, x_j] = 0 \quad [p_i, p_j] = 0 \quad [x_i, p_j] = i\delta_{ij} \quad (314)$$

The contribution to the annulus of a magnetized neutral string is

$$\mathcal{A} \sim (1 + (2\pi l_s^2 q H)^2) \frac{1}{\tau_2 \eta^2} \quad (315)$$

where we have introduced  $q = q_L = -q_R$ . Notice that in the compact case the momentum is quantized as

$$p = \frac{m}{R\sqrt{1 + \alpha^2}} \quad (316)$$

We will denote by  $\tilde{P}_m$  the corresponding sum over “boosted” momenta.

A supersymmetric extension of the above construction exists (see [22]). One has to add the fermionic part

$$S = \frac{i}{4\pi l_s^2} \int d\tau d\sigma \bar{\psi} \rho^a \partial_a \psi + \frac{i}{4} q_L \int d\tau F_{MN} \bar{\psi}^M \rho^0 \psi^N |_{\sigma=0} + \frac{i}{4} q_R \int d\tau F_{MN} \bar{\psi}^M \rho^0 \psi^N |_{\sigma=\pi} \quad (317)$$

Restricting as before to a plane  $(X^1, X^2)$  or a two torus, we introduce the superpartners of the complex bosonic coordinate  $Z$

$$\psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2) \quad (318)$$

The variation of the action leads to the following boundary conditions for the fermionic coordinate

$$\begin{aligned} \text{(R)} + \text{(NS)} \quad & (1 - i\alpha)\psi_R = (1 + i\alpha)\psi_L \quad \text{for } \sigma = 0 \\ \text{(NS)} \quad & (1 + i\beta)\psi_R = -(1 - i\beta)\psi_L \quad \text{for } \sigma = \pi \\ \text{(R)} \quad & (1 + i\beta)\psi_R = +(1 - i\beta)\psi_L \quad \text{for } \sigma = \pi \end{aligned} \quad (319)$$

The oscillator expansion that we obtain is

$$\begin{aligned} \psi &= \sum_{n=1}^{\infty} b_n \psi_{n,L}(\tau, \sigma) + \sum_{n=0}^{\infty} \tilde{b}_n^\dagger \psi_{-n,R}(\tau, \sigma) \quad \text{(R)} \\ \psi &= \sum_{r=1/2}^{\infty} \left[ b_r \psi_{r,L}(\tau, \sigma) + \tilde{b}_r^\dagger \psi_{-r,R}(\tau, \sigma) \right] \quad \text{(NS)} \end{aligned} \quad (320)$$

with the functions  $\psi_n$  given by

$$\psi_{n,L/R}(\tau, \sigma) = \frac{1}{\sqrt{2}} e^{-i(n-\zeta)(\tau \mp \sigma) \pm i\gamma} \quad (321)$$

where the sign choices refer to the left-moving and right-moving sectors respectively. Canonical quantization leads to the usual oscillator algebra

$$\begin{aligned} \{b_n, b_m^\dagger\} &= \{\tilde{b}_n, \tilde{b}_m^\dagger\} = \delta_{mn} \\ \{b_n, b_m\} &= \{\tilde{b}_n, \tilde{b}_m\} = \{b_n^\dagger, b_m^\dagger\} = \{\tilde{b}_n^\dagger, \tilde{b}_m^\dagger\} = 0 \end{aligned} \quad (322)$$

There is a neat formula, derived in [23], which encodes the shifts of all the string excitations due to the boundary magnetic field

$$\delta M^2 = (2n + 1)|\zeta| + 2\zeta \Sigma_{12} \quad (323)$$

where  $\Sigma_{12}$  is the spin operator in the directions of the plane or torus  $(X^1, X^2)$ . Now let us discuss the contribution to the partition function of magnetized fermions. Due to the shift  $\zeta$  of the oscillator modes one obtains a contributions of the following form

$$q^{\frac{1}{2}\zeta^2} \frac{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\zeta\tau, \tau)}{\eta(\tau)} \quad (324)$$

valid for both the compact and non-compact case. Notice that the factor  $q^{\frac{1}{2}\zeta^2}$  cancel between the bosonic and fermionic coordinates as required by world-sheet supersymmetry and we will ignore it in the following. It is convenient to introduce  $\widehat{so(2n)}$  characters with non-vanishing argument  $\zeta$ :

$$\begin{aligned} O_{2n}(\zeta) &= \frac{1}{2\eta^n(\tau)} [\vartheta_3^n(\zeta, \tau) + \vartheta_4^n(\zeta, \tau)] \\ V_{2n}(\zeta) &= \frac{1}{2\eta^n(\tau)} [\vartheta_3^n(\zeta, \tau) - \vartheta_4^n(\zeta, \tau)] \\ S_{2n}(\zeta) &= \frac{1}{2\eta^n(\tau)} [\vartheta_2^n(\zeta, \tau) + i^{-n}\vartheta_1^n(\zeta, \tau)] \\ O_{2n}(\zeta) &= \frac{1}{2\eta^n(\tau)} [\vartheta_2^n(\zeta, \tau) - i^{-n}\vartheta_1^n(\zeta, \tau)] \end{aligned} \quad (325)$$

We will express in terms of these the corresponding partition functions. An important property of compactifications with internal magnetic fields is that they allow the presence of chiral fermions which is a desirable feature from phenomenological point of view. One can see this by examining eq. (323). Consider the characters  $S_2(\zeta)$  and  $C_2(\zeta)$  in the corresponding plane (or two-torus) then the mass shift due to the internal magnetic field is  $+\zeta$  and  $-\zeta$  respectively. On the other hand from the bosonic coordinates contribution, i.e.  $i\eta/\vartheta_1(\zeta\tau)$ , the shift (at the ground level  $n = 0$ ) is  $|\zeta|$ . Hence only one of the two fermions  $S_2(\zeta\tau)$  or  $C_2(\zeta\tau)$  will stay massless for non-zero magnetization. This considerations hold at every mass level.

One can easily generalize the eq. (323) to a factorizable six-torus  $\mathbb{T}^6 = \bigotimes_{i=1}^3 \mathbb{T}_i^2$  with magnetic fields in each factor  $H_i$ . Then one has

$$\delta M^2 = \sum_{i=1}^3 (2n_i + 1) |\zeta_i| + 2\zeta_i \Sigma_i \quad (326)$$

### 4.2.1 Toroidal Compactification

Let us consider Type I string theory compactified on a (factorizable) six-torus

$$M = \mathbb{R}^{1,3} \otimes \mathbb{T}^6 \quad (327)$$

with the  $\mathbb{T}^6$  decomposable into three 2-tori  $\mathbb{T}_1^2 \otimes \mathbb{T}_2^2 \otimes \mathbb{T}_3^2$ . For each stack  $a$  of  $D9$ -branes we introduce  $U(1)$  background gauge fields:

$$H_1^{(a)}, H_2^{(a)}, H_3^{(a)} \quad (328)$$

in each of the three 2-tori  $\mathbb{T}_i^2$ . A Dirac quantization condition requires the magnetic fields to have the following form

$$H_i^{(a)} = \frac{m_i^a}{n_i^a v_i} \quad (329)$$

with  $m_i^a, n_i^a \in \mathbb{Z}$ . We define the “angles”  $\zeta_i^a$  by

$$\tan \zeta_i^a = H_i^{(a)} \quad (330)$$

The quantities  $\zeta_i^a$  will indeed have the interpretation of angles in the T-dual picture of Type IIA theory with intersecting  $D6$  branes that we will discuss later.

The closed string amplitudes are not affected by the magnetic fields. It is convenient to introduce the notion of an image brane  $a'$  under the orientifold  $\Omega$ . The corresponding magnetic field is obtained by the mapping

$$m_i^a \rightarrow -m_i^a \quad (331)$$

The CP factors will be parametrized as

$$N_a = p_a + \bar{p}_a \quad (332)$$

anticipating the fact that the gauge group will be unitary  $U(p_a)$ . The action of  $\Omega$  on the CP labels is  $p_a \rightarrow \bar{p}_a$ .

For the annulus one obtains

$$\begin{aligned} \mathcal{A} = & \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \left\{ \sum_a (p_a \bar{p}_a + \bar{p}_a p_a) T_{oo}(0) \tilde{P}_1 \tilde{P}_2 \tilde{P}_3 \right. \\ & + \sum_a I_{aa'} [p_a^2 T_{oo}(2\zeta_i^a \tau) + \bar{p}_a^2 T_{oo}(-2\zeta_i^a \tau)] \prod_{i=1}^3 \frac{i\eta}{\vartheta_1(2\zeta_i^a \tau)} \\ & + \sum_{a < b} I_{ab} [p_a \bar{p}_b T_{oo}(\zeta_i^{ab} \tau) + \bar{p}_a p_b T_{oo}(-\zeta_i^{ab} \tau)] \prod_{i=1}^3 \frac{i\eta}{\vartheta_1(\zeta_i^{ab} \tau)} \\ & \left. + \sum_{a < b} I_{ab'} [p_a p_b T_{oo}(\zeta_i^{ab'} \tau) + \bar{p}_a \bar{p}_b T_{oo}(-\zeta_i^{ab'} \tau)] \prod_{i=1}^3 \frac{i\eta}{\vartheta_1(\zeta_i^{ab'} \tau)} \right\} \frac{1}{\eta^2} \end{aligned} \quad (333)$$

where  $T_{oo}$  has been defined in eq. (288). The notation  $T_{oo}(2\zeta_i^a\tau)$  means that the corresponding  $so(2)$  characters have non-vanishing arguments (see eq. (325))  $2\zeta_1^a\tau, 2\zeta_2^a\tau, 2\zeta_3^a\tau$  in the corresponding tori. We have introduced the following notations

$$\begin{aligned}\zeta_i^{ab} &= \zeta_i^a - \zeta_i^b \\ I_{ab} &= \prod_{i=1}^3 (m_i^a n_i^b - n_i^a m_i^b)\end{aligned}\tag{334}$$

The numbers  $I_{ab}$  will also have a geometrical interpretation in the Type IIA picture as intersection numbers between the cycles wrapped by the branes  $a$  and  $b$ . Notice that  $\Omega$  also takes  $\zeta_i^a$  into  $-\zeta_i^a$  thus making the combinations appearing in  $\mathcal{A}$  invariant under it (as it should be). Now let us write the Möbius amplitude

$$\mathcal{M} = -\frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^3} \sum_a \left\{ \prod_{i=1}^3 (2m_i^a) \left[ p_a \hat{T}_{oo}(2\zeta_i^a\tau) + \bar{p}_a \hat{T}_{oo}(-2\zeta_i^a\tau) \right] \prod_{i=1}^3 \frac{i\hat{\eta}}{\hat{\vartheta}_1(2\zeta_i^a\tau)} \right\} \frac{1}{\hat{\eta}^2}\tag{335}$$

One important consequence of the above formula is that due to the presence of the magnetic fields the character corresponding to the vector boson

$$V_2 O_2(2\zeta_1^a\tau) O_2(2\zeta_2^a\tau) O_2(2\zeta_3^a\tau)\tag{336}$$

becomes massive in the Möbius because the contribution from the internal bosons is

$$\prod_{i=1}^3 \frac{i\hat{\eta}}{\hat{\vartheta}_1(2\zeta_i^a\tau)}\tag{337}$$

The theory is not supersymmetric for any choice of magnetic fields. In order to have one supersymmetry preserved in 4d one needs the following condition

$$\sum_{i=1}^3 H_i^{(a)} = \prod_{i=1}^3 H_i^{(a)} \quad \text{for all } a\tag{338}$$

This ensures that the stack  $a$  preserves  $\mathcal{N} = 1$  supersymmetry. In terms of the angles  $\zeta_i^a$  this can be written as

$$\sum_{i=1}^3 \zeta_i^a = 0\tag{339}$$

In order for all stack of branes to preserve the same supersymmetry, the magnetic fields have to satisfy the following constraint

$$H_1^{(a)} H_2^{(a)} + H_1^{(a)} H_3^{(a)} + H_2^{(a)} H_3^{(a)} \leq 1 \quad \text{for all } a\tag{340}$$

To build consistent models one has to satisfy tadpole conditions, which in this case are

$$\sum_a (p_a + \bar{p}_a) n_1^a n_2^a n_3^a = 32 \quad (341)$$

The gauge group is of the form

$$G_{CP} = \prod_a U(p_a) \quad (342)$$

Given that the magnetic fields preserve supersymmetry then the massless spectrum of the model consists of chiral multiplets with multiplicities and representations given in Table 1.

Multiplicity	Representation	Index
3	$p_a \bar{p}_a$	$\forall a$
$\frac{1}{2} (I_{aa'} + I_{aO})$	$\frac{p_a(p_a-1)}{2}$	$\forall a$
$\frac{1}{2} (I_{aa'} - I_{aO})$	$\frac{p_a(p_a+1)}{2}$	$\forall a$
$I_{ab}$	$(p_a, \bar{p}_b)$	$a < b$
$I_{ab'}$	$(p_a, p_b)$	$a < b$

Table 1: Massless spectrum of the magnetized  $\mathbb{T}^6$  compactification of Type I string theory

In addition there are the gauge multiplets corresponding to each factor in the gauge group. An important property from phenomenological point of view of the spectrum above is chirality and replication of chiral fermions. Thus the number of families is determined by the intersection numbers  $I_{aa'}, I_{ab}, I_{aO}$ . In the table above we have introduced the intersection with the orientifold plane defines as

$$I_{aO} = 8m_1^a m_2^a m_3^a \quad (343)$$

In this section we have assumed that all stacks of branes have non-zero magnetizations. Of course one can have also non-magnetized D9 branes. Their corresponding gauge group will be orthogonal.

### 4.3 Branes at Angles

Let us start by considering the boundary conditions in eq. (298) and let us choose a magnetic field as in eq. (299) such that we can reduce to a plane (or a two-torus) of

coordinates  $(X^1, X^2)$  with  $F_{12} = H$ . Then the boundary conditions in this plane are

$$\begin{aligned}\partial_\sigma X^1 - 2\pi l_s^2 q_L H \partial_\tau X^2 &= 0, \\ \partial_\sigma X^2 + 2\pi l_s^2 q_L H \partial_\tau X^1 &= 0\end{aligned}\quad \text{for } \sigma = 0 \quad (344)$$

and

$$\begin{aligned}\partial_\sigma X^1 + 2\pi l_s^2 q_R H \partial_\tau X^2 &= 0, \\ \partial_\sigma X^2 - 2\pi l_s^2 q_R H \partial_\tau X^1 &= 0\end{aligned}\quad \text{for } \sigma = \pi \quad (345)$$

Let us perform a T-duality in the  $X^2$  direction. This amounts to the exchange  $\partial_\tau \leftrightarrow \partial_\sigma$  in the corresponding coordinate

$$\begin{aligned}\partial_\tau X^2 &= \partial_\sigma Y^2 \\ \partial_\sigma X^2 &= \partial_\tau Y^2\end{aligned}\quad (346)$$

where we have denotes by  $Y^2$  the T-dual coordinate of  $X^2$ . The equations above can be written compactly as  $\partial_a X^2 = \epsilon_{ab} \partial^a Y^2$ . In terms of the new coordinate the boundary conditions become

$$\begin{aligned}\partial_\sigma (X^1 + \tan \theta_L Y^2) &= 0 \\ \partial_\tau (Y^2 - \tan \theta_L X^1) &= 0\end{aligned}\quad \text{for } \sigma = 0 \quad (347)$$

and a similar equation for  $\theta_R$  at  $\sigma = \pi$ . The T-dual boundary conditions now describe a string stretched between branes rotated by  $\theta_L$  and  $\theta_R$  respectively. The relation between the angles and the magnetizations is given by the following relation

$$\tan \theta_{L,R} = 2\pi l_s^2 q_{L,R} H \quad (348)$$

If before one had branes that were extended in the whole plane, now one has branes of one dimension lower thus extended in one direction. Their worldvolume is a 1-cycle inside  $\mathbb{T}^2$ . The torus  $\mathbb{T}^2$  can be described as the quotient of the complex plane  $\mathbb{C}$  by the two dimensional lattice  $\Lambda^{(2)}$  generated by the vectors  $\vec{a}, \vec{b}$ . Let  $\Pi$  be an arbitrary one-cycle, then we denote by  $[\Pi]$  the corresponding homology class, that is

$$[\Pi] \in H_1(\mathbb{T}^2, \mathbb{Z}) \quad (349)$$

The homology classes of the 1-cycles  $a, b$  form a basis for  $H_1(\mathbb{T}^2, \mathbb{Z})$ , thus any 1-cycle  $\Pi_\alpha$  can be expressed as

$$[\Pi_\alpha] = m^\alpha [a] + n^\alpha [b] \quad (350)$$

with  $m^\alpha, n^\alpha \in \mathbb{Z}$ . The integers  $m_\alpha, n_\alpha$  are called the wrapping numbers of the 1-cycle (brane)  $\Pi_\alpha$ . On the space of 1-cycles one can define a pairing  $\circ$  called intersection number. It is topological invariant and thus is well defined on the homology classes

$$\circ : H_1(\mathbb{T}^2, \mathbb{Z}) \times H_1(\mathbb{T}^2, \mathbb{Z}) \rightarrow \mathbb{Z} \quad (351)$$

In order to take into account the orientation of the 1-cycles the pairing  $\circ$  is antisymmetric. Hence for the elementary cycles  $a, b$  one has

$$\begin{aligned} [a] \circ [b] &= -[b] \circ [a] = 1 \\ [a] \circ [a] &= [b] \circ [b] = 0 \end{aligned} \quad (352)$$

Let  $\Pi_\alpha, \Pi_\beta$  be two arbitrary 1-cycles (inside  $\mathbb{T}^2$ ) then the corresponding intersection number can be expressed as

$$[\Pi_\alpha] \circ [\Pi_\beta] = m^\alpha n^\beta - n^\alpha m^\beta = I_{\alpha\beta} \quad (353)$$

The 1-cycle wrapped by the image brane  $\alpha'$  can be expressed as

$$[\Pi_{\alpha'}] = -m^\alpha [a] + n^\alpha [b] \quad (354)$$

which is in accord with the action of  $\Omega$  on the wrapping numbers in eq. (331). Finally, if we start with an O-plane extended in the whole two-torus  $\mathbb{T}^2$ , then after T-duality we are left with a 1-cycle  $\Pi_O$  given by

$$[\Pi_O] = 2[b] \quad (355)$$

Finally, let us denote by  $R_1, R_2$  the two radii of the circles that generate the T-dual torus  $\mathbb{T}^2$ . Then the tangent of the angle  $\theta^{(a)}$  corresponding to a rotated brane  $a$  will obey a similar quantization condition as in eq. (329) where one has to make the substitution  $R_2 \rightarrow 1/R_2$ , thus obtaining

$$\tan \theta^{(a)} = \frac{m^a R_2}{n^a R_1} \quad (356)$$

Notice that the angles  $\theta^{(a)}$  are the same quantities as  $\zeta^a$  defined in eq. (330). One can generalize the formulas above for a six-torus  $\mathbb{T}^6$ . We do this in the next section.

#### 4.3.1 Toroidal Compactification with Intersecting Branes

Let us consider a factorizable six-torus  $\mathbb{T}^6 = \bigotimes_{i=1}^3 \mathbb{T}_i^2$ . If we start with Type I string theory with  $D9$  branes and  $O9$  planes, then by performing three T-dualities, one in each

two-torus, we obtain an orientifold of Type IIA with  $D6$  branes and  $O6$  planes which wrap a 3-cycle in the internal six-torus (in addition to 4d spacetime). We introduce a basis of 1-cycles  $\{a_i, b_i\}_{i=1,2,3}$  in each two-torus  $\mathbb{T}_i^2$ . Then a general 3-cycle  $[\Pi_\alpha]$  wrapped by a  $D6$  brane can be expressed as

$$[\Pi_\alpha] = \bigotimes_{i=1}^3 (m_i^\alpha [a_i] + n_i^\alpha [b_i]) \quad (357)$$

Now the intersection between two  $D6$  branes  $\alpha$  and  $\beta$  becomes

$$[\Pi_\alpha] \circ [\Pi_\beta] = \prod_{i=1}^3 (m_i^\alpha n_i^\beta - n_i^\alpha m_i^\beta) = \prod_{i=1}^3 I_{\alpha\beta}^i \equiv I_{\alpha\beta} \quad (358)$$

which is exactly the same quantity in the second line of eq. (334). Using this result and the definition of the 3-cycle wrapped by the  $O6$  planes

$$[\Pi_{O6}] = 8 \bigotimes_{i=1}^3 [b_i] \quad (359)$$

one can express the chiral part of the massless spectrum in Table 1 as (see for example [24])

Multiplicity	Representation	Index
$\frac{1}{2} ([\Pi_\alpha] \circ [\Pi_{\alpha'}] + [\Pi_\alpha] \circ [\Pi_{O6}])$	$\frac{p_\alpha(p_\alpha-1)}{2}$	$\forall \alpha$
$\frac{1}{2} ([\Pi_\alpha] \circ [\Pi_{\alpha'}] - [\Pi_\alpha] \circ [\Pi_{O6}])$	$\frac{p_\alpha(p_\alpha+1)}{2}$	$\forall \alpha$
$[\Pi_\alpha] \circ [\Pi_\beta]$	$(p_\alpha, \bar{p}_\beta)$	$\alpha < \beta$
$[\Pi_\alpha] \circ [\Pi_{\beta'}]$	$(p_\alpha, p_\beta)$	$\alpha < \beta$

Table 2: Chiral spectrum of a toroidal compactification with intersecting  $D6$  branes in Type IIA.

The formulas in Table 2 can be used to compute the chiral massless spectrum in more general compactifications like Calabi-Yau's or toroidal orientifolds. Notice that the intersection numbers determine only the chiral part of the spectrum. They correspond to the index of a Dirac operator.

$$index(\not{D}) = n_+ - n_- = I_{\alpha\beta} \quad (360)$$

In order to get the full spectrum one needs to make the decomposition

$$[\Pi_\alpha] \circ [\Pi_\beta] = [\Pi_\alpha \cup \Pi_\beta]^+ - [\Pi_\alpha \cup \Pi_\beta]^- \quad (361)$$

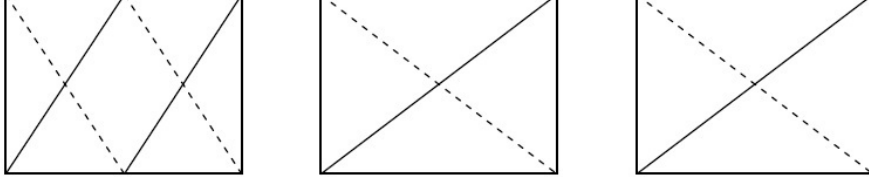


Figure 3: Rotated brane with wrapping numbers  $(2, 1) \otimes (1, 1) \otimes (1, 1)$  is represented with a continuous line and the corresponding image with a dashed line

The RR tadpole conditions in eq. (341) can now be written in terms of homology cycles as

$$\sum_{\alpha} p_{\alpha} ([\Pi_{\alpha} + \Pi_{\alpha'}]) = 4[\Pi_{O6}] \quad (362)$$

One advantage of the intersecting brane picture is the geometric visualization of the  $D6$  brane cycles and their intersections. For example take a  $D6$  brane with wrapping numbers  $\alpha : (2, 1) \otimes (1, 1) \otimes (1, 1)$  then one has the geometric representation in Figure 3.

#### 4.4 $T^4/\mathbb{Z}_2$ with Magnetized/Intersecting Branes

Let us discuss in this section the compactification of Type I string theory on  $T^4/\mathbb{Z}_2$  where the  $D9$  branes are magnetized [25]. Alternatively, one can talk about intersecting  $D7$  branes. Because of the  $\mathbb{Z}_2$  orientifold one has  $O9/O5$  planes. Due to the presence of magnetizations one can satisfy tadpoles without introducing  $D5$  branes. Only the open string amplitudes are affected by the presence of the magnetizations

$$H_i^{(\alpha)} = \frac{m_i^{\alpha}}{n_i^{\alpha} v_i} \quad i = 1, 2 \quad (363)$$

With the following parametrization of the CP factors

$$\begin{aligned} N_{\alpha} &= p_{\alpha} + \bar{p}_{\alpha} & R_{N_{\alpha}} &= i(p_{\alpha} - \bar{p}_{\alpha}) \\ D &= d + \bar{d} & R_D &= i(d - \bar{d}) \end{aligned} \quad (364)$$

where  $N_\alpha$  denotes the CP label for magnetized branes and  $D$  the one for  $D5$  branes, one can write down the following amplitudes

$$\begin{aligned}
\mathcal{A} = & \frac{1}{4} \left\{ (Q_o + Q_v)(0; 0) \left[ 2 \sum_a p_a \bar{p}_a \tilde{P}_1 \tilde{P}_2 + (d + \bar{d})^2 W_1 W_2 \right] \right. \\
& + \sum_a I_{aa'} \left[ p_a^2 (Q_o + Q_v)(2z_1^a \tau; 2z_2^a \tau) + \bar{p}_a^2 (Q_o + Q_v)(-2z_1^a \tau; -2z_2^a \tau) \right] \prod_{i=1}^2 \frac{i\eta}{\theta_1(2z_i^a \tau)} \\
& + \sum_{a < b} 2I_{ab} \left[ p_a \bar{p}_b (Q_o + Q_v)(z_1^{ab} \tau; z_2^{ab} \tau) + \bar{p}_a p_b (Q_o + Q_v)(-z_1^{ab} \tau; -z_2^{ab} \tau) \right] \prod_{i=1}^2 \frac{i\eta}{\theta_1(z_i^{ab} \tau)} \\
& + \sum_{a < b} 2I_{ab'} \left[ p_a p_b (Q_o + Q_v)(z_1^{ab'} \tau; z_2^{ab'} \tau) + \bar{p}_a \bar{p}_b (Q_o + Q_v)(-z_1^{ab'} \tau; -z_2^{ab'} \tau) \right] \prod_{i=1}^2 \frac{i\eta}{\theta_1(z_i^{ab'} \tau)} \\
& + \sum_a 2I_{a5} \left[ p_a (d + \bar{d})(Q_s + Q_c)(z_1^a \tau; z_2^a \tau) + \bar{p}_a (d + \bar{d})(Q_s + Q_c)(-z_1^a \tau; -z_2^a \tau) \right] \prod_{i=1}^2 \frac{\eta}{\theta_4(z_i^a \tau)} \\
& + (Q_o - Q_v)(0; 0) \left[ 2 \sum_a p_a \bar{p}_a - (d - \bar{d})^2 \right] \left( \frac{2\eta}{\theta_2(0)} \right)^2 \\
& - \left[ p_a^2 (Q_o - Q_v)(2z_1^a \tau; 2z_2^a \tau) + \bar{p}_a^2 (Q_o - Q_v)(-2z_1^a \tau; -2z_2^a \tau) \right] \prod_{i=1}^2 \frac{2\eta}{\theta_2(2z_i^a \tau)} \\
& + \sum_{a < b} 2S_{ab} \left[ p_a \bar{p}_b (Q_o - Q_v)(z_1^{ab} \tau; z_2^{ab} \tau) + \bar{p}_a p_b (Q_o - Q_v)(-z_1^{ab} \tau; -z_2^{ab} \tau) \right] \prod_{i=1}^2 \frac{\eta}{\theta_2(z_i^{ab} \tau)} \\
& - \sum_{a < b} 2S_{ab} \left[ p_a p_b (Q_o - Q_v)(z_1^{ab'} \tau; z_2^{ab'} \tau) + \bar{p}_a \bar{p}_b (Q_o - Q_v)(-z_1^{ab'} \tau; -z_2^{ab'} \tau) \right] \prod_{i=1}^2 \frac{\eta}{\theta_2(z_i^{ab'} \tau)} \\
& + \sum_a 2S_{a5} \left[ p_a (d - \bar{d})(Q_s - Q_c)(z_1^a \tau; z_2^a \tau) + \bar{p}_a (d - \bar{d})(Q_s - Q_c)(-z_1^a \tau; -z_2^a \tau) \right] \prod_{i=1}^2 \frac{\eta}{\theta_3(z_i^a \tau)} \left. \right\} \\
& \hspace{15em} (365)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M} = & -\frac{1}{4} \left\{ (\hat{Q}_o + \hat{Q}_v)(0;0)(d + \bar{d})W_1W_2 \right. \\
& + \prod_{i=1}^2 (2m_i^{(a)}) \left[ p_a(\hat{Q}_o + \hat{Q}_v)(2z_1^a\tau; 2z_2^a\tau) + \bar{p}_a(\hat{Q}_o + \hat{Q}_v)(-2z_1^a\tau; -2z_2^a\tau) \right] \prod_{i=1}^2 \frac{i\hat{\eta}}{\hat{\theta}_1(2z_i^a\tau)} \\
& - (d + \bar{d})(\hat{Q}_o - \hat{Q}_v)(0;0) \left( \frac{2\eta}{\theta_2(0)} \right)^2 \\
& \left. - \left[ p_a(\hat{Q}_o - \hat{Q}_v)(2z_1^a\tau; 2z_2^a\tau) + \bar{p}_a(\hat{Q}_o - \hat{Q}_v)(-2z_1^a\tau; -2z_2^a\tau) \right] \prod_{i=1}^2 \frac{2n_i^{(a)}\hat{\eta}}{\hat{\theta}_2(2z_i^a\tau)} \right\}
\end{aligned} \tag{366}$$

From the equations above one finds the following untwisted tadpole conditions

$$\begin{aligned}
\sum_a (p_a + \bar{p}_a) n_1^a n_2^a &= 32 \\
\sum_a (p_a + \bar{p}_a) m_1^a m_2^a + d + \bar{d} &= 32
\end{aligned} \tag{367}$$

In addition one has the twisted tadpole conditions

$$R_{N_a} = R_D = 0 \tag{368}$$

The gauge group is a product of unitary factors

$$G_{CP} = \prod_a U(p_a) \times U(d) \tag{369}$$

The magnetized characters in the vacuum amplitudes have the following expressions

$$\begin{aligned}
Q_o(\nu_1; \nu_2) &= V_4 [O_2(\nu_1)O_2(\nu_2) + V_2(\nu_1)V_2(\nu_2)] - C_4 [S_2(\nu_1)C_2(\nu_2) + C_2(\nu_1)S_2(\nu_2)] \\
Q_v(\nu_1; \nu_2) &= O_4 [V_2(\nu_1)O_2(\nu_2) + O_2(\nu_1)V_2(\nu_2)] - S_4 [S_2(\nu_1)S_2(\nu_2) + C_2(\nu_1)C_2(\nu_2)] \\
Q_s(\nu_1; \nu_2) &= O_4 [S_2(\nu_1)C_2(\nu_2) + C_2(\nu_1)S_2(\nu_2)] - S_4 [O_2(\nu_1)O_2(\nu_2) + V_2(\nu_1)V_2(\nu_2)] \\
Q_c(\nu_1; \nu_2) &= V_4 [S_2(\nu_1)S_2(\nu_2) + C_2(\nu_1)C_2(\nu_2)] - C_4 [V_2(\nu_1)O_2(\nu_2) + O_2(\nu_1)V_2(\nu_2)]
\end{aligned} \tag{370}$$

The condition for supersymmetry in this case is

$$H_1^{(a)} = H_2^{(a)} \quad \text{or equivalently} \quad z_1^a = z_2^a \tag{371}$$

The massless spectrum can be computed by the methods that we have illustrated and it consists of hypermultiplets with multiplicities and representations given in Table 3.

Multiplicity	Representation	Relevant Indices
$\frac{1}{4}(I_{aa'} + I_{aO} + 4)$	$\left(\frac{p_a(p_a-1)}{2}, 1\right)$	$\forall a$
$\frac{1}{4}(I_{aa'} - I_{aO} + 4)$	$\left(\frac{p_a(p_a+1)}{2}, 1\right)$	$\forall a$
$\frac{1}{2}(I_{ab} - S_{ab})$	$(p_a, \bar{p}_b)$	$a < b$
$\frac{1}{2}(I_{ab'} + S_{ab})$	$(p_a, p_b)$	$a < b$
$\frac{1}{2}(I_{a5} - S_{a5})$	$(p_a, d)$	$\forall a$
$\frac{1}{2}(I_{a5} + S_{a5})$	$(p_a, \bar{d})$	$\forall a$
1	$\left(1, \frac{d(d-1)}{2} + \frac{\bar{d}(\bar{d}-1)}{2}\right)$	

Table 3: Representations and multiplicities of charged hypermultiplets on a  $\mathbb{T}^4/\mathbb{Z}_2$  orbifold in the presence of magnetic backgrounds.

The numbers  $I_{ab}$  and  $I_{aO}$  have the following definitions in the case of  $\mathbb{T}^4/\mathbb{Z}_2$

$$\begin{aligned}
I_{ab} &= \prod_{i=1}^2 (m_i^a n_i^b - n_i^a m_i^b) \\
I_{aO} &= 4 \left( \prod_{i=1}^2 m_i^a + \prod_{i=1}^2 n_i^a \right)
\end{aligned} \tag{372}$$

By performing two T-dualities, one in each two-torus, one is left with intersecting  $D7$  branes in Type IIB with orientifold  $\Omega' = \Omega I_2$ , where  $I_2$  reverses the sign of the coordinates that have been T-dualized. In order to describe the cycles wrapped by the branes we introduce the homology basis of the covering space  $\mathbb{T}_1^2 \times \mathbb{T}_2^2$

$$[a_i], [b_i] \quad \text{with } i = 1, 2 \tag{373}$$

such that any factorizable two-cycle of  $\mathbb{T}^4$  can be written as

$$\Pi_a^{\mathbb{T}^4} = \bigotimes_{i=1}^2 (m_i^a [a_i] + n_i^a [b_i]) \tag{374}$$

Since we are working with an orbifold space  $\mathbb{T}^4/\mathbb{Z}_2$  one introduces bulk 2-cycles  $\Pi_a^B$ . In general for an orbifold group  $\Gamma$  and a covering space  $\mathcal{M}$  one has the following definition for the bulk cycles

$$\Pi_a^B = \sum_{k \in \Gamma} k \Pi_a^{\mathcal{M}} \tag{375}$$

The intersection form of two bulk cycles has the following expression [26]

$$[\Pi_a^B] \circ [\Pi_a^B] = \frac{1}{|\Gamma|} \left[ \sum_{k \in \Gamma} k \Pi_a^{\mathcal{M}} \right] \circ \left[ \sum_{k \in \Gamma} k \Pi_b^{\mathcal{M}} \right] \quad (376)$$

where we have denoted by  $|\Gamma|$  the number of elements of the orbifold group  $\Gamma$ . In our case we have  $\Gamma = \mathbb{Z}_2$  and one can identify  $\Pi_a^{\mathbb{T}^4}$  with  $g \Pi_a^{\mathbb{T}^4}$ . Then we write that

$$[\Pi_a^B] = 2[\Pi_a^{\mathbb{T}^4}] \quad (377)$$

thus obtaining the following intersections between the bulk 2-cycles

$$[\Pi_a^B] \circ [\Pi_b^B] = 2[\Pi_a^{\mathbb{T}^4}] \circ [\Pi_b^{\mathbb{T}^4}] = 2I_{ab} \quad (378)$$

In addition to the bulk 2-cycles, for each fixed point  $l \in F_g$  we introduce a collapsed 2-cycle  $[e_l^g]$ . We have denoted by  $F_g$  the set of fixed points of the orbifold generator  $g \in \mathbb{Z}_2$ . The intersection numbers of the collapsed cycles are [27]

$$[e_l^g] \circ [e_k^g] = -2\delta_{lk} \quad (379)$$

Now consider a fractional brane  $a$ , that is a brane that passes through fixed points of  $\mathbb{Z}_2$  then we can write

$$\Pi_a^F = \frac{1}{2}\Pi_a^B + \frac{1}{2} \sum_{l \in F_g} \epsilon_l^{a,g} e_l^g \quad (380)$$

where the factor  $1/2$  reflects the fact that one needs two fractional branes in order to get a brane in the bulk. The quantity  $\epsilon_l^g$  is equal to one if the brane  $a$  passes through the fixed point  $l$  and it is equal to zero otherwise. With the formula above one has the calculation

$$\begin{aligned} [\Pi_a^F] \circ [\Pi_b^F] &= \frac{1}{4} \left( [\Pi_a^B] \circ [\Pi_b^B] + \sum_{l,k} \epsilon_l^{a,g} \epsilon_k^{b,g} [e_l] \circ [e_k] \right) \\ &= \frac{1}{2} \left( I_{ab} - \sum_{l,k} \epsilon_l^{a,g} \epsilon_k^{b,g} \delta_{lk} \right) = \frac{1}{2} (I_{ab} - S_{ab}) \end{aligned} \quad (381)$$

where now the interpretation of  $S_{ab} := \sum_l \epsilon_l^{a,g} \epsilon_l^{b,g}$  is explicit as the number of fixed points of the  $\mathbb{Z}_2$  orbifold that both branes  $a$  and  $b$  intersect. In order to reproduce the spectrum in Table 3 one needs the 2-cycle of the image brane  $a'$

$$\Pi_{a'}^F = \frac{1}{2}\Pi_{a'}^B - \frac{1}{2} \sum_{l \in F_g} \epsilon_l^{a,g} e_l^g \quad (382)$$

and of the orientifold planes corresponding to fixed points of  $\Omega'$  and  $\Omega'g$

$$[\Pi_{O7}] = 2 \left( \bigotimes_{i=1}^2 [b_i] + \bigotimes_{i=1}^2 [a_i] \right) \quad (383)$$

Notice that a  $D5$  brane in the Type I magnetized pictures is described in the T-dual picture by a  $D7$  brane with the following wrapping numbers

$$D5 \rightarrow D7 : (1, 0) \otimes (1, 0) \quad (384)$$

corresponding to an angle of  $\pi/2$  or infinite magnetic field. Now one can easily check that

$$[\Pi_a^F] \circ [\Pi_{O7}] = 2 \left( \prod_{i=1}^2 m_i^a + \prod_{i=1}^2 n_i^a \right) = \frac{1}{2} I_{aO} \quad (385)$$

and put the chiral spectrum in the form displayed in Table 2 with  $\Pi_a \rightarrow \Pi_a^F$  and  $\Pi_{O6} \rightarrow \Pi_{O7}$ .

#### 4.5 $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with Magnetized/Intersecting Branes and Discrete Torsion

Now let us consider the magnetic deformation of the  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  orientifold with discrete torsion [28] introduced in Section 3.2.2. Our choice of discrete torsion is

$$(\epsilon_g, \epsilon_f, \epsilon_h) = (+, +, -) \quad (386)$$

that is we have two standard  $O5$  planes denoted by  $O5_{1-}, O5_{2-}$  corresponding to the fixed point locus of  $\Omega g$  and  $\Omega f$  and one exotic  $O$ -plane denoted by  $O5_{3+}$  corresponding to the fixed point locus of  $\Omega h$ . It is convenient to introduce two “families” of  $D9$  branes labeled by the CP labels  $p_a, q_\alpha$  and their complex conjugates. The action of the orbifold generators on these is given by

$$\begin{aligned} N_{a,o} &= p_a + \bar{p}_a, & N_{\alpha,o} &= q_\alpha + \bar{q}_\alpha, \\ N_{a,g} &= i(p_a - \bar{p}_a), & N_{\alpha,g} &= i(q_\alpha - \bar{q}_\alpha), \\ N_{a,f} &= i(p_a - \bar{p}_a), & N_{\alpha,f} &= -i(q_\alpha - \bar{q}_\alpha), \\ N_{a,h} &= p_a + \bar{p}_a, & N_{\alpha,h} &= -q_\alpha - \bar{q}_\alpha. \end{aligned} \quad (387)$$

The gauge group is a product of unitary factors

$$G_{CP} = \prod_a U(p_a) \times \prod_\alpha U(q_\alpha) \quad (388)$$

One has solutions with  $\mathcal{N} = 1$  supersymmetry in 4d if the conditions in eqs. (551), (340) are satisfied by the internal magnetic fields. Consistent models have to satisfy tadpole conditions. They are

$$\begin{aligned}
\sum_a p_a n_1^a n_2^a n_3^a + \sum_\alpha q_\alpha n_1^\alpha n_2^\alpha n_3^\alpha &= 16, \\
\sum_a p_a n_1^a m_2^a m_3^a + \sum_\alpha q_\alpha n_1^\alpha m_2^\alpha m_3^\alpha &= -16 \epsilon_g, \\
\sum_a p_a m_1^a n_2^a m_3^a + \sum_\alpha q_\alpha m_1^\alpha n_2^\alpha m_3^\alpha &= -16 \epsilon_f, \\
\sum_a p_a m_1^a m_2^a n_3^a + \sum_\alpha q_\alpha m_1^\alpha m_2^\alpha n_3^\alpha &= -16 \epsilon_h,
\end{aligned} \tag{389}$$

and from the twisted sectors

$$\begin{aligned}
\sum_a p_a m_1^a \epsilon_l^{a,g} + \sum_\alpha q_\alpha m_1^\alpha \epsilon_l^{\alpha,g} &= 0, \\
\sum_a p_a m_2^a \epsilon_l^{a,f} - \sum_\alpha q_\alpha m_2^\alpha \epsilon_l^{\alpha,f} &= 0, \\
\sum_a p_a n_3^a \epsilon_l^{a,h} - \sum_\alpha q_\alpha n_3^\alpha \epsilon_l^{\alpha,h} &= 0,
\end{aligned} \tag{390}$$

where  $\epsilon_l^{a,k}$  is equal to one if the brane  $a$  passes through the fixed point  $l$  of the  $k$ -th orbifold generator. Notice that for each generator of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  there are 16 fixed points, thus  $l = 1, \dots, 16$ . Due to the presence of the  $\epsilon_l^{a,k}$  the twisted tadpole conditions are satisfied fixed point per fixed point. The tadpole conditions (389),(390) can be derived from the one-loop string amplitudes reproduced in Appendix B. See also [28] and [49]. Notice from the tadpole conditions (or alternatively by making use of the  $DBI - CS$  action) that magnetized  $D9$  branes have induced  $D5$  charges given by

$$\begin{aligned}
D5_1 &: Q_9 n_1^a m_2^a m_3^a \\
D5_2 &: Q_9 m_1^a n_2^a m_3^a \\
D5_3 &: Q_9 m_1^a m_2^a n_3^a
\end{aligned} \tag{391}$$

In order to satisfy the tadpole for an exotic  $O$ -plane one needs some of these induced charges and tensions to be negative. Suppose that  $n_i^a$  are positive, then from the supersymmetry condition it follows that we have two possibilities, either  $m_i^a, m_j^a > 0, m_k^a < 0$  or  $m_i^a, m_j^a < 0, m_k^a < 0$  with  $i \neq j \neq k = 1, 2, 3$ . In both cases one can see that one of the induced  $D5$  brane charges will be negative thus allowing for

supersymmetric solutions also in the case when discrete torsion is present. Recall that in a supersymmetric model the tension and charge of a given  $D$ -brane are equal. The tension of a magnetized  $D9$  brane is given by

$$T = |n_1 n_2 n_3| T_9 \sqrt{(1 + H_1^2)(1 + H_2^2)(1 + H_3^2)} \quad (392)$$

By making use of the supersymmetry condition  $H_1 + H_2 + H_3 = H_1 H_2 H_3$  one can show that the “induced tensions” are

$$T = |n_1 n_2 n_3 T_9 - T_5 (n_1 m_2 m_3 + m_1 n_2 m_3 + m_1 m_2 n_3) \text{sgn}(n_1 n_2 n_3)| \quad (393)$$

matching precisely the RR charges in eq. (391). In addition to tadpole conditions there are K-theory constraints coming from the fact the RR charges are classified by the K-theory groups [29] (and not the homology groups). The explicit form of these constraints in our case can be found in [30]. The massless spectrum is given in Table 4.

The intersection numbers for the  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  are defined as

$$\begin{aligned} I_{ab} &= \prod_{i=1}^3 (m_i^a n_i^b - n_i^a m_i^b) \\ I_{aO} &= 8(m_1^a m_2^a m_3^a - \epsilon_g m_1^a n_2^a n_3^a - \epsilon_f n_1^a m_2^a n_3^a - \epsilon_h n_1^a n_2^a m_3^a) \end{aligned} \quad (394)$$

Again one can consider the T-dual picture of intersecting  $D6$  branes in Type IIA. This is obtained by performing three T-dualities, one in each factor of the three two-tori. The cycles that can be wrapped by the  $D6$  branes are the bulk cycles inherited from the covering torus  $\mathbb{T}^6$

$$[\Pi_a^B] = 4[\Pi_a^{\mathbb{T}^6}] \quad (395)$$

with the following intersection

$$[\Pi_a^B] \circ [\Pi_b^B] = 4I_{ab} \quad (396)$$

In addition to the cycles inherited from the covering space one has extra 32 special cycles for each element of the orbifold group  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , thus a total of 96 cycles. A basis is given by the 3-cycles

$$[\alpha_{l,m}^k] = 2[e_l^k] \otimes [a_{i_k}] \quad [\alpha_{l,n}^k] = 2[e_l^k] \otimes [b_{i_k}] \quad (397)$$

where  $k = g, f, h$  are the orbifold generators,  $i_k = 1, 2, 3$  labels the two-torus  $\mathbb{T}_{i_k}^2$  on which the orbifold generator  $k$  acts trivially and  $l \in F_k$  runs over the fixed points under the action of  $k$ . As in the case of the  $\mathbb{Z}_2$  orbifold we have  $[e_l^k] \circ [e_r^p] = -2\delta_{lr}\delta^{kp}$ , and

Multiplicity	Representation	Relevant Indices
$\frac{1}{8}(I_{aa^*} + I_{aO} - 4I_{aa^*}^1 - 4I_{aa^*}^2 + 4I_{aa^*}^3)$	$\left(\frac{p_a(p_a-1)}{2}, 1\right)$	$\forall a$
$\frac{1}{8}(I_{\alpha\alpha^*} + I_{\alpha O} - 4I_{\alpha\alpha^*}^1 - 4I_{\alpha\alpha^*}^2 + 4I_{\alpha\alpha^*}^3)$	$\left(1, \frac{q_\alpha(q_\alpha-1)}{2}\right)$	$\forall \alpha$
$\frac{1}{8}(I_{aa^*} - I_{aO} - 4I_{aa^*}^1 - 4I_{aa^*}^2 + 4I_{aa^*}^3)$	$\left(\frac{p_a(p_a+1)}{2}, 1\right)$	$\forall a$
$\frac{1}{8}(I_{\alpha\alpha^*} - I_{\alpha O} - 4I_{\alpha\alpha^*}^1 - 4I_{\alpha\alpha^*}^2 + 4I_{\alpha\alpha^*}^3)$	$\left(1, \frac{q_\alpha(q_\alpha+1)}{2}\right)$	$\forall \alpha$
$\frac{1}{4}(I_{aa^*} - S_g^{a\alpha} I_{aa^*}^1 + S_f^{a\alpha} I_{aa^*}^2 - S_h^{a\alpha} I_{aa^*}^3)$	$(p_a, q_\alpha)$	$\forall a, \forall \alpha$
$\frac{1}{4}(I_{a\alpha} + S_g^{a\alpha} I_{a\alpha}^1 - S_f^{a\alpha} I_{a\alpha}^2 - S_h^{a\alpha} I_{a\alpha}^3)$	$(p_a, \bar{q}_\alpha)$	$\forall a, \forall \alpha$
$\frac{1}{4}(I_{ab^*} - S_g^{ab} I_{ab^*}^1 - S_f^{ab} I_{ab^*}^2 + S_h^{ab} I_{ab^*}^3)$	$(p_a, p_b)$	$a < b$
$\frac{1}{4}(I_{ab} + S_g^{ab} I_{ab}^1 + S_f^{ab} I_{ab}^2 + S_h^{ab} I_{ab}^3)$	$(p_a, \bar{p}_b)$	$a < b$
$\frac{1}{4}(I_{\alpha\beta^*} - S_g^{\alpha\beta} I_{\alpha\beta^*}^1 - S_f^{\alpha\beta} I_{\alpha\beta^*}^2 + S_h^{\alpha\beta} I_{\alpha\beta^*}^3)$	$(q_\alpha, q_\beta)$	$\alpha < \beta$
$\frac{1}{4}(I_{\alpha\beta} + S_g^{\alpha\beta} I_{\alpha\beta}^1 + S_f^{\alpha\beta} I_{\alpha\beta}^2 + S_h^{\alpha\beta} I_{\alpha\beta}^3)$	$(q_\alpha, \bar{q}_\beta)$	$\alpha < \beta$
1	$(p_a, \bar{q}_\alpha) + (\bar{p}_a, q_\alpha)$	if $(m_a^i, n_a^i) = (m_\alpha^i, n_\alpha^i) \forall i$
1	$(p_a, q_\alpha) + (\bar{p}_a, \bar{q}_\alpha)$	if $(m_a^i, n_a^i) = (-m_\alpha^i, n_\alpha^i) \forall i$

Table 4: Representations and multiplicities of charged chiral superfields on a  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  orbifold with discrete torsion, in the presence of magnetic backgrounds.

zero intersection with the bulk cycles.

The general 3-cycle that can be wrapped by a fractional  $D6$  brane can be written as

$$\Pi_a^F = \frac{1}{4}\Pi_a^B + \frac{1}{4}\left(\sum_{l \in F_g} \epsilon_l^{a,g} \Pi_{l,a}^g\right) + \frac{1}{4}\left(\sum_{l \in F_f} \epsilon_l^{a,f} \Pi_{l,a}^f\right) + \frac{1}{4}\left(\sum_{l \in F_h} \epsilon_l^{a,h} \Pi_{l,a}^h\right) \quad (398)$$

where we have introduced the 3-cycles  $\Pi_{l,a}^k$  defined as

$$[\Pi_{l,a}^k] = m_{i_k}^a [\alpha_{l,m}^k] + n_{i_k}^a [\alpha_{l,n}^k] \quad (399)$$

One can easily show that their intersection is of the following form

$$[\Pi_{l,a}^k] \circ [\Pi_{r,b}^p] = 4\delta_{lr}\delta^{kp} I_{ab}^{i_k} \quad (400)$$

where  $I_{ab}^{i_k} = m_{i_k}^a n_{i_k}^b - n_{i_k}^a m_{i_k}^b$  is the intersection in the  $i_k$ -th torus. In the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  orientifold one has four types of  $O6$  planes corresponding to the fixed point locus of

the projections  $\Omega, \Omega g, \Omega f, \Omega h$ . The 3-cycle wrapped by the  $O6$ -planes can be written as

$$\Pi_{O6} = 2 \left( \bigotimes_{i=1}^3 [b_i] - \epsilon_g [b_1] \otimes [a_2] \otimes [a_3] - \epsilon_f [a_1] \otimes [b_2] \otimes [a_3] - \epsilon_h [a_1] \otimes [a_2] \otimes [b_3] \right) \quad (401)$$

Now let us specify the action of  $\Omega$  on the collapsed cycles  $\alpha_{l,m}^k, \alpha_{l,n}^k$ . It is given by

$$\Omega : \alpha_{l,m}^k \rightarrow -\epsilon_k \alpha_{l,m}^k \quad \alpha_{l,n}^k \rightarrow \epsilon_k \alpha_{l,n}^k \quad (402)$$

By using the formulas in Table 2 one can again reproduce the chiral spectrum of the magnetized  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  with discrete torsion. Recall that we have introduced two families of branes  $p_a$  and  $q_\alpha$ . The general form of the 3-cycles wrapped by the  $p_a$  branes is given in eq. (403) whereas a 3-cycle wrapped by the  $q_\alpha$  branes can be written as

$$\Pi_\alpha^F = \frac{1}{4} \Pi_\alpha^B + \frac{1}{4} \left( \sum_{l \in F_g} \epsilon_l^{\alpha,g} \Pi_{l,\alpha}^g \right) - \frac{1}{4} \left( \sum_{l \in F_f} \epsilon_l^{\alpha,f} \Pi_{l,\alpha}^f \right) - \frac{1}{4} \left( \sum_{l \in F_h} \epsilon_l^{\alpha,h} \Pi_{l,\alpha}^h \right) \quad (403)$$

## 5 Yukawa Couplings and Wilson Lines in Magnetized Branes Models

In this section we want to sketch the results in [31] where an explicit expression for the Yukawa couplings in models compactified on magnetized tori have been computed. An alternative calculation is that of T-dual intersecting branes in Type IIA (see [32]). The strategy of the calculation is to make use of  $\mathcal{N} = 1$  Super Yang-Mills theory in 10d and compute the effective 4d Yukawa couplings. The starting point is the  $D = 10$  action

$$S = \int d^{10}x \left[ -\frac{1}{2g^2} \text{Tr } F_{MN} F^{MN} + \frac{i}{2g^2} \text{Tr } \bar{\lambda} \Gamma_M D^M \lambda \right] \quad (404)$$

where both  $F_{MN}$  and  $\lambda$  are in the adjoint representation of the gauge group  $G$ .

$$\begin{aligned} F_{MN} &= \partial_M A_N - \partial_N A_M - i[A_M, A_N] \\ D_M \lambda &= \partial_M \lambda - i[A_M, \lambda] \end{aligned} \quad (405)$$

One compactifies six dimensions such that

$$\text{Space} - \text{Time} = \mathbb{R}^{3,1} \times \mathcal{M}_6 \quad (406)$$

It is convenient to introduce the following decomposition of the fields

$$\begin{aligned} \lambda(x^\mu, y^m) &= \sum_n \chi_n(x^\mu) \otimes \psi_n(y^m) \\ A_M(x^\mu, y^m) &= \sum_n \varphi_{n,M}(x^\mu) \otimes \phi_{n,M}(y^m) \end{aligned} \quad (407)$$

One can choose the internal wavefunctions to be eigenvalues of the corresponding internal wave equation

$$\begin{aligned} i\Gamma^m D_m \psi_n &= m_n \psi_n \\ \Delta_6 \phi_{n,M} &= M_{n,M}^2 \phi_{n,M} \end{aligned} \quad (408)$$

We consider magnetized compactifications that is  $\langle A_m(y) \rangle \neq 0$  such that we have a non-zero flux  $\langle F_{mn} \rangle \neq 0$  in the internal manifold. As we have seen in the chapter about magnetized branes such a theory can have properties like chirality and replication of chiral fermions, very desirable from the phenomenological point of view. From a field theory point of view this is due to the fact that in the presence of magnetic fields the index of the Dirac operator can be non-zero. Indeed, suppose that we have fermions in the representation  $Q$  of the gauge group then the index has the following expression

$$\text{index}_Q \not{D}_6 = \frac{1}{48(2\pi)^3} \int_{\mathcal{M}_6} \left[ \text{Tr}_Q(F \wedge F \wedge F) - \frac{1}{8} \text{Tr}_Q(F) \wedge \text{Tr}(R \wedge R) \right] \quad (409)$$

In addition the original gauge group  $G$  will be broken to a subgroup  $H$ . In order to obtain a canonical kinetic term in four dimensions one normalizes the internal wavefunctions such that

$$\int_{\mathcal{M}_6} d^6 y \psi_j(y)^\dagger \psi_k(y) = \delta_{jk} \quad (410)$$

and a similar relation for  $\phi_{n,M}$ . One obtains a Yukawa coupling in 4d from dimensional reduction of the term  $A \cdot \lambda \cdot \lambda$ . Its expression is obtained by integrating over the internal manifold  $\mathcal{M}_6$

$$Y_{ijk} = \int_{\mathcal{M}_6} d^6 y \psi_i^{a\dagger} \Gamma^m \psi_j^b \phi_{k,m}^c f_{abc} \quad (411)$$

The case that is of interest for us is a toroidal compactification, that is we take  $\mathcal{M}_6 = \mathbb{T}^6$  which for simplicity we take it to be factorizable. We turn on a magnetic flux of the following form

$$F = \sum_{r=1}^3 \frac{i\pi}{\text{Im}\tau_r} \begin{pmatrix} \frac{m^a}{n^a} \mathbb{I}_{p_a} & & \\ & \frac{m^b}{n^b} \mathbb{I}_{p_b} & \\ & & \frac{m^c}{n^c} \mathbb{I}_{p_c} \end{pmatrix} dz^r \wedge \bar{z}^r \quad (412)$$

where, let's say,  $\mathbb{I}_{p_a}$  denotes the  $p_a \times p_a$  identity matrix and  $m^a, n^a \in \mathbb{Z}$  such that  $\text{g.c.d.}(m^a, n^a) = 1$ . We have denoted by  $\tau^r$  the complex structure modulus of the  $r$ -th torus. Complex coordinates are  $dz^r = dx^r + \tau_r dy^r$ . In string theory such a situation corresponds to three stacks of branes  $a, b, c$  with magnetizations

$$H^\alpha = \frac{m^\alpha}{n^{\alpha v}} \quad \alpha = a, b, c \quad (413)$$

and gauge group

$$U(p_a) \times U(p_b) \times U(p_c) \quad (414)$$

In addition we consider configurations of continuous Wilson lines which do not break the gauge symmetry

$$A_{\text{W.L.}} = \sum_{r=1}^3 \frac{i\pi}{\tau_r} \begin{pmatrix} \text{Im}(\xi_a^r d\bar{z}^r) & & \\ & \text{Im}(\xi_b^r d\bar{z}^r) & \\ & & \text{Im}(\xi_c^r d\bar{z}^r) \end{pmatrix} \quad (415)$$

where  $\xi_{\alpha,x}^r, \xi_{\alpha,y}^r \in [0, 1/n_\alpha^r)$  are the real Wilson lines along the  $r$ -th two torus  $\mathbb{T}_r^2$  and we define complex Wilson lines by

$$\xi_\alpha^r = \xi_{\alpha,x}^r + \tau_r \xi_{\alpha,y}^r \quad (416)$$

In a T-dual picture with intersecting  $D6$  branes the Wilson lines correspond to positions of branes in the corresponding two tori. As we have seen in section 4.2.1 one has  $|I_{\alpha\beta}|$

chiral superfields  $\Phi_{\alpha\beta}^{\vec{i}}$  transforming in the bifundamental representation  $p_\alpha, \bar{p}_\beta$  of the gauge group, where  $\alpha, \beta = a, b, c$ . The multi-index  $\vec{i} = (i_1, i_2, i_3)$  can have the following values  $i_r = 0, \dots, |I_{\alpha\beta}^r| - 1$ . We are interested in the Yukawa coupling corresponding to the following term in the 4d lagrangian

$$Y_{ijk} \Phi_{ab}^{\vec{i}} \Phi_{bc}^{\vec{j}} \Phi_{ca}^{\vec{k}} \quad (417)$$

In the T-dual picture of Type IIA intersecting  $D6$  branes one has the following geometric interpretation of the Yukawa coupling as the area of the triangle formed by the branes  $a, b, c$  (in each two-torus). At the endpoints of the triangle one has the fields  $\Phi_{ab}, \Phi_{bc}, \Phi_{ca}$  living at the intersection of the D-branes. This is illustrated in Figure 4.

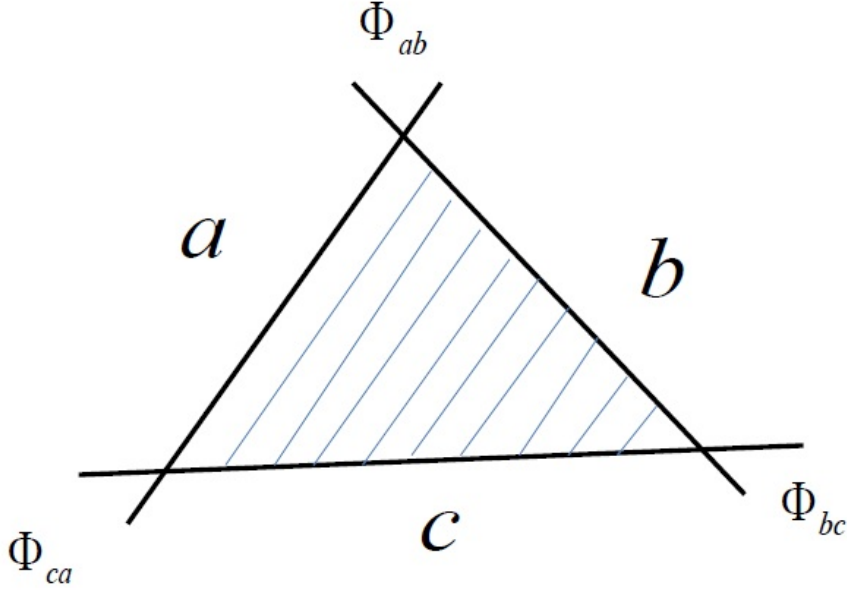


Figure 4: Yukawa couplings in intersecting branes

The calculation in [31] shows that the Yukawa coupling has the following expression

in the case of a toroidal compactification

$$Y_{ijk} = g \prod_{r=1}^3 \left( \frac{2 \text{Im } \tau_r}{\mathcal{A}^2} \right)^{1/4} \mathcal{N}_{\mathcal{I}_{ab}^r} \mathcal{N}_{\mathcal{I}_{bc}^r} \mathcal{N}_{\mathcal{I}_{ca}^r} e^{i\pi \left( \frac{\xi_{ab}^r \text{Im } \xi_{ab}^r}{\mathcal{I}_{ab}^r} + \frac{\xi_{bc}^r \text{Im } \xi_{bc}^r}{\mathcal{I}_{bc}^r} + \frac{\xi_{ca}^r \text{Im } \xi_{ca}^r}{\mathcal{I}_{ca}^r} \right) / \text{Im } \tau_r} \times \vartheta \left[ \begin{matrix} \delta_{ijk}^r \\ 0 \end{matrix} \right] (\xi^r; \tau^r | I_{ab}^r I_{bc}^r I_{ca}^r |) \quad (418)$$

where  $g$  is the 10d gauge coupling constant,  $\mathcal{A}$  is the area of the 2-torus and  $\mathcal{N}_{\mathcal{I}_{\alpha\beta}^r}$  are factors depending on the intersection numbers that will enter the Kähler metric. In addition we have the following notations

$$\begin{aligned} \mathcal{I}_{\alpha\beta}^r &\equiv \frac{I_{\alpha\beta}^r}{n_r^\alpha n_r^\beta} \\ \delta_{ijk}^r &\equiv \frac{i_r}{I_{ab}^r} + \frac{j_r}{I_{ca}^r} + \frac{k_r}{I_{bc}^r} \\ \xi_{\alpha\beta}^r &\equiv \xi_\alpha^r - \xi_\beta^r \\ \xi^r &\equiv I_{ab}^r n_r^c \xi_c^r + I_{bc}^r n_r^a \xi_a^r + I_{ca}^r n_r^b \xi_b^r \end{aligned} \quad (419)$$

We restrict ourselves to supersymmetric models and we are interested in computing the corresponding superpotential. The Yukawa coupling in eq. (542) has to fit in the general supergravity formula

$$Y_{ijk} \hat{\Phi}_{ab}^{\vec{i}} \hat{\Phi}_{bc}^{\vec{j}} \hat{\Phi}_{ca}^{\vec{k}} = (K_{ab} K_{bc} K_{ca})^{-1/2} e^{K/2} W_{ijk} \hat{\Phi}_{ab}^{\vec{i}} \hat{\Phi}_{bc}^{\vec{j}} \hat{\Phi}_{ca}^{\vec{k}} \quad (420)$$

where  $W_{ijk}$  is the coupling appearing in the superpotential,  $\hat{\Phi}_{ab}^{\vec{i}}$  represent the chiral variables in terms of which the superpotential is holomorphic,  $K$  is the Kähler potential and  $K_{ab} = \partial_{ab} \bar{\partial}_{ab} K$  are the kinetic terms in the  $ab$  sector. The  $\mathcal{N} = 1$  chiral variables  $\hat{\Phi}_{ab}^{\vec{j}}$  are defined as [33]

$$\hat{\Phi}_{\alpha\beta}^{\vec{j}} = \left( \frac{W}{\overline{W}} \right)^{\frac{1}{4}} \left( \prod_{r=1}^3 (\text{Im } \tau_r)^{\frac{1}{4}} e^{i\pi \frac{\xi_{\alpha\beta}^r \text{Im } \xi_{\alpha\beta}^r}{\mathcal{I}_{\alpha\beta}^r \text{Im } \tau_r}} \right) \Phi_{\alpha\beta}^{\vec{j}} \quad (421)$$

Then one can write the following superpotential

$$W = \hat{\Phi}_{ab}^{\vec{i}} \hat{\Phi}_{bc}^{\vec{j}} \hat{\Phi}_{ca}^{\vec{k}} \prod_{r=1}^3 \vartheta \left[ \begin{matrix} \delta_{ijk}^r \\ 0 \end{matrix} \right] (\xi^r; \tau^r | I_{ab}^r I_{bc}^r I_{ca}^r |) \quad (422)$$

## 6 Supersymmetry Breaking in String Theory

Supersymmetry breaking is an outstanding problem in both particle physics and string phenomenology. We would like to describe in this chapter a few mechanisms for breaking supersymmetry in the context of string theory, namely: magnetic fields [23], brane supersymmetry breaking [34], [35], [36], [37], Scherk-Schwarz compactification [38], [39], [40], [41] or [42], [43] on the Heterotic side and closed string fluxes [44], [45].

### 6.1 Magnetic Fields

The first mechanism for breaking supersymmetry that we consider is through magnetic fields on the worldvolume of the D-branes. For definiteness we consider Type I string theory compactified on a factorizable six-torus  $\bigotimes_{i=1}^3 \mathbb{T}_i^2$  with magnetized D-branes. For a stack  $a$  the magnetic fields are given by

$$H_i^{(a)} = \frac{m_i^a}{n_i^a v_i} \quad (423)$$

We consider three different cases.

I.  $H_1^{(a)} \neq 0$ ,  $H_{2,3}^{(a)} = 0$  that is we consider a non-zero magnetic flux only along one two-torus. The shift in the masses resulted by the introduction of the magnetic field is given by the formula

$$\delta M^2 = (2n + 1)|\zeta| + 2\zeta\Sigma_1 \quad (424)$$

We will restrict our analysis to the  $n = 0$  level, that is the massless level before introducing magnetic fields. Recall that the 10d open string massless spectrum consists of a vector boson and a Weyl fermion. Dimensional reduction yields an  $\mathcal{N} = 1$  vector multiplet and three chiral multiplets in 4d. This is encoded in the following decomposition of characters

$$V_8 - S_8 = \tau_{oo} + \tau_{og} + \tau_{of} + \tau_{oh} \quad (425)$$

where  $\tau_{kl}$  are defined in Appendix A. Let us consider the chiral multiplet contained in  $\tau_{og}$ . It contains a complex boson with internal helicities in the first torus  $\Sigma_1 = \pm 1$  and a Weyl fermion with helicities  $\Sigma_1 = \pm 1/2$ . Thus the shift in the masses due to the magnetic field is

$$\begin{aligned} \delta M^2(\text{bosons}) &= |\zeta| \pm 2\zeta \\ \delta M^2(\text{fermion}) &= |\zeta| \pm \zeta \end{aligned} \quad (426)$$

From the formula above one can see immediately that for non-zero magnetic fields  $\zeta \neq 0$  supersymmetry is broken as the fermions and bosons in the chiral multiplet have

different masses. Notice that one has a massless chiral fermion. Moreover one of the (two real) bosons develops a tachyonic mass leading to a Nielsen-Olesen instability.

II.  $H_1^{(a)} \neq 0$ ,  $H_2^{(a)} \neq 0$ ,  $H_3^{(a)} = 0$  that is we have non-zero magnetic fields along  $\mathbb{T}_1^2$  and  $\mathbb{T}_2^2$ . In this case the mass shifts due to the presence of magnetic fields is given by

$$\delta M^2 = \sum_{i=1}^2 [(2n_i + 1)|\zeta_i| + 2\zeta_i \Sigma_i] \quad (427)$$

Let us consider again the chiral multiplet contained in  $\tau_{og}$ . The corresponding shifts in masses are given by

$$\begin{aligned} \delta M^2(\text{bosons}) &= |\zeta_1| + |\zeta_2| \pm 2\zeta_1 \\ \delta M^2(\text{fermion}) &= |\zeta_1| + |\zeta_2| \pm \zeta_1 \mp \zeta_2 \end{aligned} \quad (428)$$

For arbitrary magnetic fields one can easily see that supersymmetry is broken. Notice that if  $|\zeta_2| < |\zeta_1|$  one of the bosons in  $\tau_{og}$  becomes tachyonic. However if the following relation is satisfied

$$\zeta_1 + \zeta_2 = 0 \quad (429)$$

then half of the degrees of freedom in  $\tau_{og}$  remain massless. Including the contribution from the image through  $\Omega$  one obtains a full massless chiral multiplet and a massive one as well. In fact  $\tau_{oo}$  and  $\tau_{oh}$  become massive whereas  $\tau_{og}$  and  $\tau_{of}$  give massless chiral multiplets. From a six-dimensional point of view they combine into a hypermultiplet. III.  $H_i^{(a)} \neq 0$ , for all  $i = 1, 2, 3$ . The mass formula contains now three terms, one for each two-torus  $\mathbb{T}^2$ .

$$\delta M^2 = \sum_{i=1}^3 [(2n_i + 1)|\zeta_i| + 2\zeta_i \Sigma_i] \quad (430)$$

We again consider the effect of the magnetic field on the chiral multiplet contained in  $\tau_{og}$ . According to the formula above we have

$$\begin{aligned} \delta M^2(\text{bosons}) &= |\zeta_1| + |\zeta_2| + |\zeta_3| \pm 2\zeta_1 \\ \delta M^2(\text{fermion}) &= |\zeta_1| + |\zeta_2| + |\zeta_3| \pm \zeta_1 \mp \zeta_2 \mp \zeta_3 \end{aligned} \quad (431)$$

It is again apparent that for arbitrary values of the magnetic field supersymmetry is broken. Tachyonic instabilities appear from  $\tau_{og}$  if  $|\zeta_2| + |\zeta_3| < |\zeta_1|$ . Similar arguments apply to the rest of the spectrum. One can still recover a supersymmetric spectrum if the following condition is satisfied

$$\zeta_1 + \zeta_2 + \zeta_3 = 0 \quad (432)$$

For definiteness consider the case  $\zeta_1 > 0, \zeta_2 < 0, \zeta_3 < 0$  then  $\tau_{og}$  will give rise to a massless chiral superfield in 4d (after adding the contribution from the image brane) plus a massive one. The rest of the characters  $\tau_{oo}, \tau_{of}, \tau_{oh}$  coming from the dimensional reduction of  $V_8 - S_8$  will become massive. Notice that in the case of a supersymmetric spectrum there are no tachyonic instabilities.

## 6.2 Brane Supersymmetry Breaking

The basic extended objects present in 10d Type I string theory are  $D9$  branes and  $O9$  planes characterized by their tension  $T$  and RR charge  $Q$ . The complete list of objects is given in Table 5 below

Type	$T$	$Q$
$D9$	+	+
$\overline{D9}$	+	-
$O9_-$	-	-
$O9_+$	+	+
$\overline{O9_-}$	-	+
$\overline{O9_+}$	+	-

Table 5: Tensions and RR charges of  $D9/O9$  branes/planes in Type I string theory

A sign  $\pm$  means that the tension  $T$  or RR charge  $Q$  is positive or negative. Let us denote by  $N_+$  the number of  $D9$  branes and by  $N_-$  the number of  $\overline{D9}$  branes. The open string amplitudes involving the objects in Table 5 can be written as

$$\mathcal{A} = \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} [(N_+^2 + N_-^2)(V_8 - S_8) + 2N_+N_-(O_8 - C_8)] \frac{1}{\eta^8} \quad (433)$$

$$\mathcal{M} = \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} [\epsilon_{NS}(N_+ + N_-)\hat{V}_8 - \epsilon_R(N_+ - N_-)\hat{S}_8] \frac{1}{\hat{\eta}^8} \quad (434)$$

The amplitudes above yield in general a non-supersymmetric spectrum. The NSNS and RR tadpole conditions can be extracted from the transverse channel amplitudes

$$\tilde{\mathcal{A}} = \frac{2^{-5}}{2} \int_0^\infty dl [(N_+ + N_-)^2 V_8 - (N_+ - N_-)^2 S_8] \frac{1}{\eta^8} \quad (435)$$

$$\tilde{\mathcal{M}} = \frac{2}{2} \int_0^\infty dl [\epsilon_{NS}(N_+ + N_-)\hat{V}_8 - \epsilon_R(N_+ - N_-)\hat{S}_8] \frac{1}{\hat{\eta}^8} \quad (436)$$

The sign  $\epsilon_{NS} = \pm 1$  represents the sign of the tension of the  $O9$  planes whereas  $\epsilon_R$  labels the signs of the RR charges of the  $O9$  planes. Notice that the NSNS and RR tadpoles are different for non-supersymmetric models. Indeed, in this case we have

$$\begin{aligned} \text{NSNS} \quad N_+ + N_- &= -\epsilon_{NS} 32 \\ \text{RR} \quad N_+ - N_- &= -\epsilon_R 32 \end{aligned} \tag{437}$$

Notice that the solution  $N_+ = 32$ ,  $N_- = 0$  and  $\epsilon_{NS} = \epsilon_R = -1$  yields the usual  $SO(32)$  Type I superstring. Allowing for the existence of a NSNS tadpole one has also the solutions  $\epsilon_{NS} = \epsilon_R = -1$  and  $N_+ - N_- = 32$  with gauge group  $SO(N_+) \times SO(N_-)$ . Notice that all models with both  $N_+ \neq 0$  and  $N_- \neq 0$  have tachyonic instabilities that reflect the attraction between branes and anti-branes.

The simplest example of a model that exhibits (brane) supersymmetry breaking and is free of tachyons is the so-called Sugimoto model in [46] and corresponds to the following solution

$$\epsilon_{NS} = \epsilon_R = +1 \quad N_+ = 0 \quad N_- = 32 \tag{438}$$

The model contains  $N_-$   $\overline{D9}$  branes and  $O9_+$  planes. The open string massless spectrum can be extracted from the amplitudes

$$\mathcal{A}_0 + \mathcal{M}_0 \sim \frac{N_-(N_- + 1)}{2} V_8 - \frac{N_-(N_- - 1)}{2} S_8 \tag{439}$$

and it consists of a vector boson in the adjoint (symmetric) representation of  $USp(32)$  and a Weyl fermion in the antisymmetric representation. Notice that the presence of the dilaton tadpole is incompatible with Minkowski space-time. The model admits a background with  $SO(1,8)$  symmetry with a warping of the ninth dimension (see [47], [48]).

Another solution with brane supersymmetry breaking is the  $\mathbb{T}^4/\mathbb{Z}_2$  orientifold with (non-magnetized)  $D9/\overline{D5}$  branes and  $O9_-/O5_+$  planes. The model is free of tachyons and the gauge group is

$$SO(16)_9^2 \times USp(16)_5^2 \tag{440}$$

The six-dimensional massless spectrum is given in Table 6.

The model admits magnetic deformations which are tachyon free as well if the following condition is satisfied by the magnetic fields on the  $D9$  branes

$$H_1^{(a)} + H_2^{(a)} = 0 \tag{441}$$

However the spectrum is still non-supersymmetric.

An interesting 4d model with brane supersymmetry breaking is the  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  orientifold with discrete torsion. More precisely we consider the model with  $(\epsilon_g, \epsilon_f, \epsilon_h) =$

Multiplicity	Representation	Multiplet/Field
1	$(16, 16, 1, 1)$	Hypermultiplet
1	$(1, 1, 120, 1) + (1, 1, 1, 120)$	Weyl Fermion
1	$(1, 1, 16, 16)$	Weyl Fermion
4	$(1, 1, 16, 16)$	Scalar
2	$(16, 1, 1, 16) + (1, 16, 16, 1)$	Scalar
1	$(16, 1, 16, 1) + (1, 16, 1, 16)$	Symplectic Majorana-Weyl Fermion

Table 6: Spectrum of the non-supersymmetric  $\mathbb{T}^4/\mathbb{Z}_2$  orientifold.

$(+, +, -)$ . The model contains  $O9$  planes and  $O5_i$  planes. The index  $i = 1, 2, 3$  labels the corresponding two-torus that the  $O5_i$  plane wraps. Our choice of discrete torsion implies that in the theory we have the following O-planes:  $O9_-, O5_{1-}, O5_{2-}, O5_{3+}$ . In order to satisfy tadpoles without introducing magnetizations one needs to introduce  $D9, D5_1, D5_2$  and  $\overline{D5}_3$  branes. The spectrum of this model is indeed non-supersymmetric and tachyon free. We don't reproduce it here (for details see [34]). The tadpole conditions select the following gauge group

$$U(8)_9^2 \times U(8)_{5_1}^2 \times U(8)_{5_2}^2 \times USp(8)_{5_3}^2 \quad (442)$$

where by the labels  $9, 5_i$  we have emphasized the provenience of the various factors in the gauge group. As we have seen, if one magnetizes the  $D9$  branes then supersymmetric solutions to the tadpole conditions exist for particular values of the magnetic fields.

### 6.3 Scherk-Schwarz Compactification

We consider in this section a compactification of Type IIB (or Type I) string theory that breaks supersymmetry. The space that we compactify is an orbifold  $S^1/g$  where  $g = (-1)^F \delta$ .  $F$  is the (spacetime) fermion number and  $\delta$  acts on the compact coordinate  $X^9$  in the following way

$$\delta : X^9 \rightarrow X^9 + \pi R \quad (443)$$

where  $R$  is the radius of the circle  $S^1$ . Working out the torus amplitude of the model, one obtains

$$\begin{aligned} \mathcal{T} = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{11/2}} [ & |V_8 - S_8|^2 \Lambda_{m,n} + |V_8 + S_8|^2 (-1)^m \Lambda_{m,n} \\ & + |O_8 - C_8|^2 \Lambda_{m,n+1/2} + |O_8 + C_8|^2 (-1)^m \Lambda_{m,n+1/2} ] \frac{1}{|\eta|^{14}} \end{aligned} \quad (444)$$

The first term in the torus amplitude is the usual Type IIB on a circle in eq. (231). The second term arises from the action of the orbifold generator  $g = (-)^F \delta$ . Thus  $(-1)^F$  is responsible for the flipped sign in front of  $S_8$  whereas  $\delta$  generates the factor  $(-1)^m$ . Indeed by labeling with  $|m, n\rangle$  a state with  $m, n$  as momentum and winding numbers one can easily see that

$$\delta|m, n\rangle = (-1)^m |m, n\rangle \quad (445)$$

thus leading to the following lattice sum when performing the trace over the zero-modes

$$(-1)^m \Lambda_{m,n} = \frac{1}{\eta\bar{\eta}} \sum_{m,n} (-1)^m q^{\frac{l_s^2}{4} \left( \frac{m}{R} + \frac{nR}{l_s^2} \right)^2} \bar{q}^{\frac{l_s^2}{4} \left( \frac{m}{R} - \frac{nR}{l_s^2} \right)^2} \quad (446)$$

The second line in eq. (444) is obtained by demanding the amplitude to be modular invariant. It is convenient to rewrite the torus amplitude in the following form

$$\begin{aligned} \mathcal{T} = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{11/2}} [ & (V_8 \bar{V}_8 + S_8 \bar{S}_8) \Lambda_{2m,n} + (O_8 \bar{O}_8 + C_8 \bar{C}_8) \Lambda_{2m,n+1/2} \\ & - (V_8 \bar{S}_8 + S_8 \bar{V}_8) \Lambda_{2m+1,n} - (O_8 \bar{C}_8 + C_8 \bar{O}_8) \Lambda_{2m+1,n+1/2} ] \end{aligned} \quad (447)$$

Let us perform the following rescaling of the radius  $R \rightarrow R/2$ . Then in terms of the new radius the torus amplitude becomes

$$\begin{aligned} \mathcal{T} = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{11/2}} [ & (V_8 \bar{V}_8 + S_8 \bar{S}_8) \Lambda_{m,2n} + (O_8 \bar{O}_8 + C_8 \bar{C}_8) \Lambda_{m,2n+1} \\ & - (V_8 \bar{S}_8 + S_8 \bar{V}_8) \Lambda_{m+1/2,2n} - (O_8 \bar{C}_8 + C_8 \bar{O}_8) \Lambda_{m+1/2,2n+1} ] \end{aligned} \quad (448)$$

In this form it is apparent that the Kaluza-Klein (momentum) modes of the fermions are shifted by  $1/2$  with respect to the bosons thus breaking supersymmetry. Notice that for  $R < 2l_s$  the closed string spectrum contains a tachyon. In the limit  $R \rightarrow \infty$  supersymmetry is restored as one recovers the usual 10d Type IIB theory. Now let us consider the orientifold projection of the closed string model. The Klein bottle

amplitude is the symmetric part of the torus, since  $\Omega$  identifies the left and right sectors. In particular one has  $p_L = p_R$  which implies  $n = 0$ . Thus one obtains

$$\mathcal{K} = \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{1}{\eta^7} (V_8 - S_8) P_m \quad (449)$$

In the transverse channel one has

$$\tilde{\mathcal{K}} = \frac{2^5}{4} \int_0^\infty dl \frac{1}{\eta^7} v (V_8 - S_8) W_{2n} \quad (450)$$

with  $v = R/l_s$ . Now let us introduce the transverse channel open string amplitudes, the annulus

$$\tilde{\mathcal{A}} = \frac{2^{-5}}{4} N^2 \int_0^\infty dl \frac{1}{\eta^7} v [(V_8 - S_8) W_{2n} + (O_8 - C_8) W_{2n+1}] \quad (451)$$

and the Möbius strip

$$\tilde{\mathcal{M}} = -\frac{N}{2} \int_0^\infty dl \frac{1}{\hat{\eta}^7} v \left( \hat{V}_8 W_{2n} - \hat{S}_8 (-1)^n W_n \right) \quad (452)$$

Cancellation of tadpoles requires  $N = 32$  and the gauge group is the usual  $SO(32)$ . Finally the open string amplitudes in the direct channel are given by

$$\begin{aligned} \mathcal{A} &= \frac{N^2}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{1}{\eta^7} (V_8 P_m - S_8 P_{m+1/2}) \\ \mathcal{M} &= -\frac{N}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{1}{\hat{\eta}^7} \left( \hat{V}_8 P_m - \hat{S}_8 P_{m+1/2} \right) \end{aligned} \quad (453)$$

In the open string sector one can see again that the fermions have the momentum modes shifted by  $1/2$  with respect to the bosons. One can realize such a mechanism for breaking supersymmetry in field theory. Consider a  $D$ -dimensional field theory with a scalar field  $\Phi$  and a fermion field  $\Psi$ . We compactify the theory on a circle  $S^1$  of radius  $R$ . We denote by  $x^\mu$  the coordinates transverse to  $S^1$  and by  $y$  the compact dimension. By imposing the following periodicity conditions

$$\Xi(x^\mu, y + 2\pi R) = (-1)^F \Xi(x^\mu, y) \quad (454)$$

where  $\Xi$  is either  $\Phi$  or  $\Psi$ , one has the following Kaluza-Klein expansions

$$\Phi(x^\mu, y) = \sum_{n \in \mathbb{Z}} e^{\frac{iny}{R}} \Phi_n(x^\mu) \quad (455)$$

$$\Psi(x^\mu, y) = \sum_{n \in \mathbb{Z}} e^{\frac{i(n+1/2)y}{R}} \Psi_n(x^\mu) \quad (456)$$

where we see that the Kaluza-Klein modes of the fermion field  $\Psi$  are shifted by  $1/2$  with respect to the modes of the boson  $\Phi$  and thus breaking supersymmetry in a similar way that we have seen in the string model. There are other Scherk-Schwarz type models than the one that we have considered. For details see [7].

## 6.4 Closed String Fluxes

We have considered in Chapter 4 compactifications of string theory in the presence of internal magnetic fields which coupled to the elementary strings through the action in eq. (295). In the world-sheet non-linear sigma model action (292) one can couple the 2-form field  $B_{MN}$ . Moreover in Type II theory the D-branes source the RR p-form fields  $C_p$ . Let us introduce the corresponding field strengths

$$F_{p+1} = dC_p \quad (457)$$

and for the NSNS 2-form field

$$H_3 = dB \quad (458)$$

In string theory one has the following duality condition for the  $p+1$ -form field strengths

$$*F_{p+1} = F_{9-p} \quad (459)$$

The action of a p-form field is given by

$$S = \int F_{p+1} \wedge *F_{9-p} \quad (460)$$

Flux compactification implies considering configurations with a non-zero flux of the field strength

$$\int_{\Sigma_{p+1}} F_{p+1} \neq 0 \quad (461)$$

Such a condition requires that the  $p+1$  homology group of the target space  $M$  be non-trivial

$$H_{p+1}(M, \mathbb{R}) \neq \{0\} \quad (462)$$

In particular this implies that the target space is non-trivial from a topological point of view. Thus, if one wants to preserve 4d Lorentz invariance then  $p$ -form fluxes can be turned on only in the compact space  $K$ .

$$M = \mathbb{R}^{1,3} \times K \quad (463)$$

with  $H_{p+1}(K, \mathbb{R}) \neq \{0\}$ . Introducing a basis of  $p + 1$ -cycles  $\sigma_i \in H_{p+1}(K, \mathbb{R})$  one has the following quantization condition for the field strength fluxes

$$\int_{\sigma_i} F_{p+1} = n_i \quad (464)$$

where  $i = 1, \dots, b_{p+1}$  with  $b_{p+1}$  being the  $p + 1$ -th Betti number of the compactification manifold  $K$ , i.e.  $b_{p+1} := \dim H_{p+1}(K, \mathbb{R})$ . The components of the metric on  $K$  are moduli fields from the 4d point of view. They enter in the action of a  $p$ -form field thus resulting in a scalar potential for the moduli given by

$$V = \int_K F_{p+1} \wedge *F_{5-p} \quad (465)$$

If this potential is generic enough then its minimization can lead to the stabilization of the metric moduli. Besides stabilization of moduli, turning on fluxes can lead to spontaneous supersymmetry breaking.

## 7 Instantons in String Theory

In this chapter we would like to present the results in [49] where we build some global models based on the  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  orientifold with discrete torsion where  $E1$  stringy instantons generate linear terms and mass terms in the non-perturbative superpotential. One of the models had conformal invariance (at one-loop) which is then broken by non-perturbative effects (hierarchically small mass terms). Before the results of the paper we present a general introduction to field theory instantons and stringy instantons.

### 7.1 Field Theory Instantons

The starting point of our discussion is the  $SU(N)$  (pure) Yang-Mills action in 4d Euclidean space

$$S[A] = - \int \text{Tr}_N(F \wedge *F) + i\theta \left(\frac{g}{2\pi}\right)^2 \int \text{Tr}_N(F \wedge F) \quad (466)$$

where  $F$  is the (Lie algebra valued) 2-form field strength which expressed in components reads  $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$  and  $*$  denotes the Hodge dual form  $*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ . In terms of the gauge potential 1-form  $A = A_\mu dx^\mu$  one has

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \quad (467)$$

The following quantity

$$k = - \left(\frac{g}{2\pi}\right)^2 \int \text{Tr}_N(F \wedge F) \in \mathbb{Z} \quad (468)$$

is called the topological charge (of the gauge bundle). In component notation one can write the action in the following form

$$S[A] = -\frac{1}{2} \int d^4x \text{Tr}_N(F_{\mu\nu}F^{\mu\nu}) + i\theta k \quad (469)$$

Notice that our conventions are such that the gauge field  $A_\mu$  is anti-Hermitian and so one has the following inequality

$$\int d^4x \text{Tr}_N[(F_{\mu\nu} \pm *F_{\mu\nu})(F^{\mu\nu} \pm *F^{\mu\nu})] \leq 0 \quad (470)$$

from which one can write the following bound on the real part of the action

$$-\frac{1}{2} \int d^4x \text{Tr}_N(F_{\mu\nu}F^{\mu\nu}) \geq \frac{8\pi^2}{g^2} |k| \quad (471)$$

We have equality if the gauge field is self-dual or anti-self-dual

$$*F_{\mu\nu} = \pm F_{\mu\nu} \quad (472)$$

In general instantons are defined to be finite action solutions of the classical equations of motion. In our particular case, instantons are the self-dual solutions with positive instanton number  $k > 0$  whereas anti-instantons are the anti-self-dual solutions with negative instanton number  $k < 0$ . Then the action of an instanton solution is equal to

$$S_0 = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \quad (473)$$

First let us consider the explicit solution of an  $SU(2)$  gauge theory with  $k = 1$  [50],[51]

$$A_\mu = g^{-1} \frac{2(x - X)^\nu \sigma_{\nu\mu}}{(x - X)^2 + \rho^2} \quad (474)$$

where the matrices  $\sigma_{\mu\nu}$  are self-dual

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu) \quad (475)$$

with  $\sigma_\mu$  being  $2 \times 2$  matrices which are related to the Pauli matrices  $\tau^i$ ,  $i = 1, 2, 3$  by  $\sigma_\mu = (i\vec{\tau}, 1_2)$ . The instanton solution in eq. (474) depends on five parameters: the scale size  $\rho$  and the four-vector position  $X_\mu$ . In addition to these there are three extra parameters corresponding to  $SU(2)$  global gauge transformations (which are not fixed by the usual covariant gauge fixing). In total there are eight free parameters counting the inequivalent solutions to the self-dual Yang-Mills equations (472). We call these parameters *collective coordinates* and they span the *moduli space* of instanton solutions. In general, for arbitrary topological charge  $k$ , one gets  $4kN$  collective coordinates for a  $SU(N)$  gauge theory. We denote the corresponding moduli space by  $\mathfrak{M}_k$ . An important notion is that of *zero-mode*, that is we consider small fluctuations  $A_\mu(x) + \delta A_\mu(x)$  around a self-dual solution  $A_\mu(x)$ . To linear order the fluctuations have to satisfy

$$D_\mu \delta A_\nu - D_\nu \delta A_\mu = \epsilon_{\mu\nu\rho\sigma} D^\rho \delta A^\sigma \quad (476)$$

It is convenient to use quaternionic notation for 4d Euclidean spacetime. By making use of the local isomorphism between  $SO(4)$  and  $SU(2)_L \times SU(2)_R$  given by the matrices  $\sigma_\mu$  and  $\bar{\sigma}_\mu = (-i\vec{\tau}, 1_2)$  one can write a four vector  $x_\mu$  as

$$x_{\alpha\dot{\alpha}} = x_\mu \sigma_{\alpha\dot{\alpha}}^\mu \quad \bar{x}^{\dot{\alpha}\alpha} = x^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \quad (477)$$

The same definitions apply to any quantity with a four-vector index like  $\partial_\mu$ ,  $D_\mu$ ,  $\delta A_\mu$  and so on. Notice that since we are working in Euclidean spacetime there is no difference

between quantities with indices  $\mu, \nu, \dots$  up or down since the metric is equal to  $\delta_{\mu\nu}$ . In quaternionic notation one can rewrite eq. (476) as

$$\bar{\tau}^{\dot{\alpha}}_{\dot{\beta}} \bar{D}^{\dot{\beta}\alpha} \delta A_{\alpha\dot{\alpha}} = 0 \quad (478)$$

In order to implement the fact that two solutions that differ by a local gauge transformations are equivalent one asks that the variations  $\delta A_\mu$  are orthogonal to gauge transformations  $\Omega$

$$\int d^4x \text{Tr}_N (D_\mu \Omega \delta A^\mu) = 0 \quad (479)$$

Integration by parts of the equation above and passing to quaternionic notation one can write the equivalent orthogonality condition

$$\bar{D}^{\dot{\alpha}\alpha} \delta A_{\alpha\dot{\alpha}} = 0 \quad (480)$$

One can combine the two conditions in eqs. (478) and (480)

$$\bar{D}^{\dot{\alpha}\alpha} \delta A_{\alpha\dot{\beta}} = 0 \quad (481)$$

The fluctuations  $\delta A_\mu(x)$  which satisfy the equation above represent physical fluctuations in field space which do not change the value of the action. This is no longer the case for non-zero mode fluctuations. One can make use of the functional inner product of zero-modes in order to define a metric in the moduli space  $\mathfrak{M}_k$

$$g_{mn}(X) = -2g^2 \int d^4x \text{Tr}_N (\delta_m A_\mu(x, X) \delta_n A_\nu(x, X)) \quad (482)$$

where we have denoted by  $X$  the ensemble of collective coordinates of  $\mathfrak{M}_k$ , and  $m, n = 1, \dots, 4kN = \dim \mathfrak{M}_k$ . The metric above is hyper-Kähler, meaning that the moduli space as a manifold has reduced holonomy  $USp(kN) \subset SO(4kN)$ .

A natural question that arises is how do the instantons contribute to the functional integral of field theory (in the semi-classical limit). One can show that

$$\int [dA_\mu][db][dc] e^{S[A,b,c]} \longrightarrow \frac{e^{S_0}}{g^{4kN}} \int \prod_{m=1}^{4kN} \frac{dX^m}{\sqrt{2\pi}} \sqrt{\det g(X)} \frac{\det(-D^2)}{\det' \Delta^{(+)}} \quad (483)$$

as  $g \rightarrow 0$ . In the equation above  $b, c$  are ghost fields implementing the gauge fixing condition,  $S_0$  is the (classical) instanton action in eq. (473). The fluctuation determinant arises from integration over the non-zero mode fluctuations and one can recognize the volume form  $\omega$  of the moduli space  $\mathfrak{M}_k$

$$\int_{\mathfrak{M}_k} \omega := \int \prod_{m=1}^{4kN} \frac{dX^m}{\sqrt{2\pi}} \sqrt{\det g(X)} \quad (484)$$

The operator  $\Delta^{(+)}$  is defined by

$$\Delta^{(+)} = -D_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}\alpha} = -1_2 D_\mu D^\mu - g\sigma_{\mu\nu}F^{\mu\nu} \quad (485)$$

Before considering instantons in supersymmetric theories let us show the ADHM construction of instantons [53]. Let  $\Delta(x)$  be an  $(N+2k) \times 2k$  complex valued matrix which depends linearly on  $x_\mu$ , that is we can define matrices  $a, b \in \mathcal{M}_{(N+2k) \times 2k}(\mathbb{C})$  such that

$$\Delta_{\lambda i \dot{\alpha}}(x) = a_{\lambda i \dot{\alpha}} + b_{\lambda i}^\alpha x_{\alpha \dot{\alpha}} \quad \bar{\Delta}_i^{\dot{\alpha} \lambda}(x) = \bar{a}_i^{\dot{\alpha} \lambda} + \bar{x}^{\dot{\alpha} \alpha} \bar{b}_{i \alpha}^\lambda \quad (486)$$

where  $\bar{\Delta}(x)$  is the Hermitian conjugate matrix of  $\Delta(x)$ . The indices above can take the values  $\lambda = 1, \dots, N+2k$ ,  $i = 1, \dots, k$ , with  $k$  being the instanton number. Suppose that  $\Delta(x) : \mathbb{C}^{2k} \rightarrow \mathbb{C}^{2k+N}$  is injective then we have that  $\text{Ker}(\bar{\Delta}(x))$  is  $N$ -dimensional. Let  $(U_{\lambda u}(x))_{u=1, \dots, N}$  be an orthonormal basis of  $\text{Ker}(\bar{\Delta}(x))$  then one can write

$$\bar{\Delta}_i^{\dot{\alpha} \lambda} U_{\lambda u} = \bar{U}_u^\lambda \Delta_{\lambda i \dot{\alpha}} = 0 \quad (487)$$

$$\bar{U}_u^\lambda U_{\lambda v} = \delta_{uv} \quad (488)$$

If the following condition is satisfied

$$\bar{\Delta}_i^{\dot{\alpha} \lambda} \Delta_{\lambda j \dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}} (f^{-1})_{ij} \quad (489)$$

with  $f$  an arbitrary  $k \times k$  Hermitian matrix depending on  $x$ , then the gauge field

$$(A_\mu)_{uv} = g^{-1} \bar{U}_u^\lambda \partial_\mu U_{\lambda v} \quad (490)$$

yields a self-dual field strength which can be shown to be

$$F_{\mu\nu} = 4g^{-1} \bar{U} b \sigma_{\mu\nu} f \bar{b} U \quad (491)$$

With this method one can construct all self-dual solutions of the Yang-Mills equations. The matrices  $a, b$  parametrize the moduli space  $\mathfrak{M}_k$  but they are subject to the additional constraints in eq. (489). Since  $f_{ij}(x)$  is arbitrary, one can extract the following  $x$ -independent conditions on the matrices  $a$  and  $b$

$$\begin{aligned} \bar{a}_i^{\dot{\alpha} \lambda} a_{\lambda j \dot{\beta}} &= \left( \frac{1}{2} \bar{a} a \right)_{ij} \delta_{\dot{\beta}}^{\dot{\alpha}} \\ \bar{a}_i^{\dot{\alpha} \lambda} b_{\lambda j}^\beta &= \bar{b}_i^{\beta \lambda} a_{\lambda j}^{\dot{\alpha}} \\ \bar{b}_{\alpha i}^\lambda b_{\lambda j}^\beta &= \left( \frac{1}{2} \bar{b} b \right)_{ij} \delta_\alpha^\beta \end{aligned} \quad (492)$$

The three conditions above are called ADHM constraints. Notice that the ADHM construction is left unaffected if one performs the following  $x$ -independent transformations

$$\Delta \rightarrow \Lambda \Delta \Upsilon^{-1} \quad U \rightarrow \Lambda U \quad f \rightarrow \Upsilon f \Upsilon^\dagger \quad (493)$$

with  $\Lambda \in U(N+2k)$  and  $\Upsilon \in Gl(k, \mathbb{C})$ . Making use of this symmetry and decomposing the index  $\lambda = u + i\alpha$  one can choose the matrix  $b$  to have the following form

$$b_{\lambda j}^\beta = b_{(u+i\alpha),j}^\beta = \begin{pmatrix} 0 \\ \delta_\alpha^\beta \delta_{ij} \end{pmatrix} \quad \bar{b}_{\beta j}^\lambda = \bar{b}_{\beta j}^{(u+i\alpha)} = (0 \quad \delta_\beta^\alpha \delta_{ji}) \quad (494)$$

Notice that the form above is left invariant by a  $U(k)$  subgroup of  $U(N+2k) \times Gl(k, \mathbb{C})$  defined by the transformations of the following form

$$\Lambda = \begin{pmatrix} 1_N & 0 \\ 0 & \Xi 1_2 \end{pmatrix} \quad \Upsilon = \Xi, \quad \Xi \in U(k) \quad (495)$$

Then the ADHM constraints can be put in the following form

$$\bar{\tau}_{\dot{\beta}}^{\dot{\alpha}} (\bar{a}^{\dot{\beta}} a_{\dot{\alpha}}) = 0 \quad (496)$$

with  $(a'_\mu)^\dagger = a'_\mu$  defined by

$$a_{\lambda j \dot{\alpha}} = a_{(u+i\alpha)j \dot{\alpha}} = \begin{pmatrix} w_{uj \dot{\alpha}} \\ (a'_{\alpha \dot{\alpha}})_{ij} \end{pmatrix} \quad \bar{a}_j^{\dot{\alpha} \lambda} = \bar{a}_j^{\dot{\alpha} (u+i\alpha)} = (w_{ju}^{\dot{\alpha}} \quad (\bar{a}'^{\dot{\alpha} \alpha})_{ji}) \quad (497)$$

To summarize, the moduli space of instanton solutions  $\mathfrak{M}_k$  is described in the ADHM construction by the variables  $a_{\dot{\alpha}} = \{w_{\dot{\alpha}}, a'_\mu\}$ , with  $a'_\mu$  being Hermitian  $k \times k$  matrices, subject to the constraints in eq. (496) and quotiented by the residual symmetry group  $U(k)$  in eq. (495) which acts in the following way on  $a_{\dot{\alpha}}$

$$w_{ui \dot{\alpha}} \rightarrow w_{\dot{\alpha}} \Xi \quad a'_\mu \rightarrow \Xi^\dagger a'_\mu \Xi \quad (498)$$

One can construct an instanton calculus also in the case of supersymmetric Yang-Mills theory. Thus consider the following Euclidean space action valid for super Yang-Mills theory with  $\mathcal{N} = 1, 2, 4$  supersymmetry

$$S[A, \lambda, \bar{\lambda}, \Phi] = \int d^4x \text{Tr}_N \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{i\theta g^2}{16\pi^2} F_{\mu\nu} * F^{\mu\nu} - 2D_\mu \bar{\lambda}_A \bar{\sigma}^\mu \lambda^A + D_\mu \Phi_a D^\mu \Phi_a \right. \\ \left. - g \bar{\lambda}_A \Sigma_a^{AB} [\Phi_a, \bar{\lambda}_B] - g \lambda^A \bar{\Sigma}_{aAB} [\Phi_a, \lambda^B] - \frac{1}{2} g^2 [\Phi_a, \Phi_b]^2 \right\} \quad (499)$$

where the matrices  $\Sigma$  are associated with the  $R$ -symmetry for theories with extended supersymmetry  $\mathcal{N} = 2, 4$ . We don't need their explicit form. The fermions  $\lambda^A$  and  $\bar{\lambda}_A$  are the superpartners (gauginos) of the gauge field and the index  $A$  is an  $R$ -symmetry index  $A = 1, \dots, \mathcal{N}$ . In the theories with extended supersymmetry there are additionally the real scalars  $\Phi_a$  with  $a = 1, \dots, 2(\mathcal{N} - 1)$ . The equations of motion following from the action (499) are

$$\begin{aligned} D^\nu F_{\mu\nu} &= 2g[\Phi_a, D_\mu \Phi_a] + 2g\bar{\sigma}_\mu\{\lambda^A, \bar{\lambda}_A\} \\ D^\mu \bar{\sigma}_\mu \lambda^A &= g\Sigma_a^{AB}[\Phi_a, \bar{\lambda}_B] \\ D_\mu \sigma^\mu \bar{\lambda}_A &= g\bar{\Sigma}_{aAB}[\Phi_a, \lambda^B] \\ D^2 \Phi_a &= g^2[\Phi_b, [\Phi_b, \Phi_a]] + g\bar{\Sigma}_{aAB}\lambda^A \lambda^B + g\Sigma_a^{AB}\bar{\lambda}_A \bar{\lambda}_B \end{aligned} \tag{500}$$

In order to find the super-instanton solution of the equations above one proceeds perturbatively order by order in the coupling constant. The starting point is the self-dual solution  $A_\mu(x, X)$  with all other fields set to zero. Expanding around this solution, to linear order one has to solve the equations

$$\begin{aligned} D^\mu \bar{\sigma}_\mu \lambda^A &= 0 \\ D_\mu \sigma^\mu \bar{\lambda}_A &= 0 \\ D^2 \Phi_a &= 0 \end{aligned} \tag{501}$$

Notice that this is consistent with the equations of motion because  $\text{Ker}(D_\mu \sigma^\mu) = \{0\}$  in the instantonic background. Thus we have  $\bar{\lambda}_A = 0$ . On the other hand for  $\lambda^A$  one finds  $2kN$  solutions (zero-modes). Let us denote the corresponding collective coordinates by  $\Psi^{iA}$ . For the case of the super-instanton one can define analogous fermionic ADHM constraints in terms of the Grassmann-valued partners  $\mathcal{M} = \{\mu, \bar{\mu}, \mathcal{M}'_\alpha\}$  of the ADHM variables  $a_{\dot{\alpha}} = \{w_{\dot{\alpha}}, a'_{\dot{\alpha}}\}$ . The variables  $\mathcal{M}$  are  $(N+2k) \times k$  matrices satisfy the following constraints

$$\bar{\mathcal{M}}a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}}\mathcal{M} = \bar{\mu}w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}}\mu + [\mathcal{M}^\alpha, a'_{\alpha\dot{\alpha}}] = 0 \tag{502}$$

We don't go into details here (see [52] for a review).

Let us write the fermionic zero-modes in the form

$$\lambda_\alpha = g^{-1/2}\Lambda_\alpha(\mathcal{M}) \tag{503}$$

then the scalar product of fermionic zero-modes defines the inner product of symplectic tangent vectors on the moduli space  $\mathfrak{M}_k$

$$\Omega(\mathcal{M}, \mathcal{N}) = -4 \int d^4x \text{Tr}_N(\Lambda(\mathcal{M})\Lambda(\mathcal{N})) \tag{504}$$

Estimating the contribution of the super-instanton to the field theory functional integral one obtains

$$\int [dA][d\lambda][d\Phi][db][dc] e^{-S} \rightarrow \left(\frac{\mu}{g}\right)^{kN(4-\mathcal{N})} e^{S_0} \int_{\mathfrak{M}_k} \omega^{(\mathcal{N})} e^{-\tilde{S}} \quad (505)$$

where the  $\mathcal{N}$  supersymmetric volume form on  $\mathfrak{M}_k$  is

$$\int_{\mathfrak{M}_k} \omega^{(\mathcal{N})} := \int \frac{\det g(X)}{[\text{Pfaff}_{\frac{1}{2}} \Omega(X)]^{\mathcal{N}}} \prod_{m=1}^{4kN} \frac{dX^m}{\sqrt{2\pi}} \prod_{A=1}^{\mathcal{N}} \prod_{i=1}^{2kN} d\Psi^{iA} \quad (506)$$

where  $\tilde{S}$  is the action of the super-instanton solution at the order  $g^0$ .

In the end we just want to mention a couple of applications of instantons for spontaneous breaking of classical symmetries in QFT through dynamical generation of condensates. Gaugino condensation induced by fractional gauge instantons in supersymmetric theories lead to spontaneous breaking of  $R$ -symmetry. The 't Hooft instantons in QCD induce a chiral condensate which leads to spontaneous breaking of chiral symmetry.

## 7.2 Euclidean Brane Instantons

In string theory instanton effects are realized by branes with their worldvolume extended only in the internal space. Since such branes don't wrap any direction in 4d spacetime, thus being an instant from 4d point of view, we call them Euclidean brane instantons. We will denote with  $E_p$  a  $p + 1$ -dimensional Euclidean brane. Recall that in string theory (I and II) the dimensionality of the branes that one can have is determined by the spectrum of RR forms. In Type I we have  $C_2$  and  $C_6$  which can couple to  $E1$  and  $E5$ , that is Euclidean branes wrapping a 2-cycle of the internal space  $K$  or the whole internal space respectively. We will always consider orientifold theories. Thus, Type IIB with the standard orientifold  $\Omega$  yields Type I with  $D9/D5$  branes and  $O9/O5$  planes. There is an alternative orientifold projection  $\Omega'$  which couples Type IIB to  $D3/D7$  branes and  $O3/O7$  planes. They are related by six T-dualities in the internal space which map  $D9$  and  $D5$  branes to  $D3$  and  $D7$  branes respectively (and similarly for O-planes). For IIB/ $\Omega'$  one has  $E(-1)$  and  $E3$  instantons. Finally in Type IIA with intersecting  $D6$  branes we can have  $E0$ ,  $E2$  and  $E4$  instantons. Relevant for us will be only the  $E2$  instantons which are BPS with respect to  $D6$  branes.

Euclidean  $E(p - 4)$  branes wrapping the same internal cycle in the internal space as a  $Dp$  brane realize gauge instantons for the worldvolume theory of a  $Dp$  brane (see

for example [54],[55]). The massless spectrum of the strings stretched between the instanton brane and itself and of the strings stretched between the instanton brane and the  $Dp$  brane match precisely the zero modes found in field theory. Thus, the massless modes of the strings involving the instanton brane will be interpreted as the collective coordinate integral. Qualitatively different effects arise when one considers Euclidean  $Eq$  branes wrapping different cycles in the internal space than the  $Dp$  brane. We call such effects stringy instanton effects.

We will focus on instanton branes in Type I that can generate (directly) corrections to the holomorphic superpotential. An important feature of Euclidean branes that can contribute to the superpotential is that they have to wrap rigid cycles. This is necessary in order for the instanton to have the minimum number of (two) uncharged fermionic zero modes  $\theta_\alpha$ . In the 4d effective supergravity these modes are interpreted as the coordinates of the  $\mathcal{N} = 1$  superspace. In addition to  $\theta_\alpha$ , there are four bosonic modes  $x^\mu$  corresponding to the positions of the instanton brane in 4d spacetime. A non-perturbative superpotential is generated

$$S_W = \int d^4x d^2\theta W_{\text{np}} \quad (507)$$

The classical instanton action is given by the volume of the internal cycles wrapped by the corresponding Euclidean branes. For  $E1$  and  $E5$  instantons in the context of a factorizable toroidal orientifold compactification the relevant moduli are the complex axion-dilaton  $S$  and the Kähler moduli  $T_k$ .

$$S = C_6 + ie^{\Phi/2} \prod_{i=1}^3 \text{Vol}_r \quad T_k = C_{2,k} + ie^{-\Phi/2} \text{Vol}_k \quad (508)$$

In the expressions above,  $\Phi$  is the 10d dilaton,  $\text{Vol}_r$  is the volume of the  $r$ -th two-torus,  $C_6$  is the component of the RR 6-form along the compact space and  $C_{2,k}$  is the component of the RR 2-form along the  $k$ -th two-torus. Then the tree-level actions of the  $E1$  and  $E5$  instantons are given by

$$S_{E5} = S + M_0 \quad S_{E1_k} = T_k + M_k \quad (509)$$

where  $M_0$  and  $M_k$  are linear combinations of complex blow-up moduli with coefficients which depend on the discrete Wilson lines, position and CP factors of the instanton. Abelian anomalies in Type I are canceled through a generalized Green-Schwarz mechanism involving the RR forms  $C_2$  and  $C_6$ . One can show that under a  $U(1)_a$  gauge symmetry with gauge parameter  $\Lambda_a$  the instanton action transforms as

$$e^{-S_E} \rightarrow e^{iQ_a(E)\Lambda_a} e^{-S_E} \quad (510)$$

If  $Q_a(E) \neq 0$  then there is the interesting possibility to generate couplings which are perturbatively forbidden by the corresponding  $U(1)_a$ . The superpotential is of the following form

$$W_{\text{np}} = \prod_i \Phi_i e^{-S_E} \quad (511)$$

where  $\prod_i \Phi_i$  is a product of matter superfields which is charged under  $U(1)_a$  such that  $W_{\text{np}}$  is gauge invariant. The rules for the (superpotential) instanton calculus have been outlined in [56]. The instanton contribution to the superpotential can be expressed in the following form in terms of disk and one-loop open string amplitudes

$$\begin{aligned} \langle \Phi_{a_1 b_1} \dots \Phi_{a_M b_M} \rangle_E &= \int d^4 x d^2 \theta \sum_{\text{conf.}} \prod_a \left( \prod_i d\eta_a^i \right) \left( \prod_j d\bar{\eta}_a^j \right) \\ &\times e^{-S_E} e^{\sum_a \mathcal{A}'(E, a) + \mathcal{M}'(E, O)} \langle \Phi_{a_1, b_1} \rangle_{\eta_{a_1}, \bar{\eta}_{b_1}} \dots \langle \Phi_{a_M, b_M} \rangle_{\eta_{a_M}, \bar{\eta}_{b_M}} \end{aligned} \quad (512)$$

In the formula above one recognizes the integral over the uncharged zero-modes  $x^\mu$ ,  $\theta_\alpha$ . They correspond to the symmetries broken by the instanton brane, translation invariance in 4d spacetime and 1/2 of supersymmetry (since the instanton and the branes that we consider are 1/2 BPS). Additionally one integrates over charged zero-modes  $\eta_a^i$ ,  $\bar{\eta}_a^j$  coming from strings stretched between the instanton and the various branes (labeled by the index  $a$ ) in the compactification. The indices  $i$  and  $j$  counts the corresponding multiplicities.  $S_E$  is the classical instanton action,  $\mathcal{A}'(E, a)$  is the annulus amplitude for open strings between the instanton  $E$  and the D-brane  $a$ ,  $\mathcal{M}(E, O)$  is the Möbius amplitude of the instanton. The prime denotes the fact that one has to remove the zero-modes. The one-loop dependence of (512) is related to the 1-loop Pfaffians/determinants for quantum fluctuations around the instanton background

$$\frac{\text{Pfaff}'(\mathcal{D}_F)}{\sqrt{\det'(\mathcal{D}_B)}} = \exp \left( \sum_a \mathcal{A}'(E, a) + \mathcal{M}'(E, O) \right) \quad (513)$$

Finally, eq. (512) contains disk amplitudes  $\langle \Phi_{a_k, b_k} \rangle_{\eta_{a_k}, \bar{\eta}_{b_k}}$  with insertions of the corresponding product  $\Phi_{a_k, b_k}$  of boundary changing vertex operators. The sum over configurations goes over all the possible ways of inserting the charged fermionic zero-modes  $\eta_a^i$ ,  $\bar{\eta}_a^j$  in the disk amplitudes involving matter fields. The considerations above apply in principle to any instanton brane that have the appropriate structure of uncharged zero-modes. That's why we have not specified the dimensionality of the E-branes or of the 4d spacetime filling D-branes and O-planes. The expression in eq. (512) applies in the case of one-instanton contributions, that is one Euclidean brane. We will not

consider here multi-instanton effects corresponding to having multiple coinciding Euclidean branes.

An important remark is in order here. In general there are several instantons in the internal geometry that can contribute to the superpotential. Thus the superpotential will be of the following form

$$W = W_p + \sum_{\alpha} W_{(\alpha)} \quad (514)$$

where  $W_p$  is the perturbative superpotential,  $\alpha$  labels the various single instantons that contribute directly and  $W_{(\alpha)}$  is the corresponding non-perturbative contribution and in general it has the form in eq. (511). We will show in the next section a few applications and concrete examples of  $Ep$ -branes.

### 7.3 Applications

In the recent years there have been some activity in phenomenological applications of stringy instanton effects in D-brane models. These include neutrino Majorana masses [56], [57],  $\mu$ -terms in the MSSM [58], stabilization of moduli fields [59], fermion masses [60] or supersymmetry breaking [61].

Let us consider a concrete class of models where stringy instantons can generate non-perturbative superpotentials. We consider the familiar compactification of Type I string theory with magnetized  $D9$ -branes on a factorizable  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  with discrete torsion. Our choice of discrete torsion is  $(\epsilon_g, \epsilon_f, \epsilon_h) = (+, +, -)$ . This implies that we have an exotic (positive tension and RR charge) orientifold plane  $O5_{3+}$  wrapping the third two-torus  $\mathbb{T}_3^2$ . A list of O-planes, D- and E-branes (which are mutually BPS or 1/2 BPS) that one can introduce in such a compactification is reproduced in Table 7.

	$x$	$z_1$	$z_2$	$z_3$		$x$	$z_1$	$z_2$	$z_3$		$x$	$z_1$	$z_2$	$z_3$
$O9$	$\times$	$\times$	$\times$	$\times$	$D9$	$\times$	$\times$	$\times$	$\times$	$E5$	$\cdot$	$\times$	$\times$	$\times$
$O5_{1-}$	$\times$	$\times$	$\cdot$	$\cdot$	$D5_1$	$\times$	$\times$	$\cdot$	$\cdot$	$E1_1$	$\cdot$	$\times$	$\cdot$	$\cdot$
$O5_{2-}$	$\times$	$\cdot$	$\times$	$\cdot$	$D5_2$	$\times$	$\cdot$	$\times$	$\cdot$	$E1_2$	$\cdot$	$\cdot$	$\times$	$\cdot$
$O5_{3+}$	$\times$	$\cdot$	$\cdot$	$\times$	$D5_3$	$\times$	$\cdot$	$\cdot$	$\times$	$E1_3$	$\cdot$	$\cdot$	$\cdot$	$\times$

Table 7: Extended objects in the  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  orientifold with discrete torsion. A cross (dot) denotes that the object wraps (is localized along) the corresponding directions ( $x$  denotes the spacetime coordinate and  $z_i$  is the complex coordinate along  $\mathbb{T}_i^2$ ).

We are hunting for instantons that have only two fermionic uncharged zero-modes  $\theta_\alpha$ .

One can show that this is indeed the case for the instanton wrapping the same two-torus as the exotic O-plane (see [49], [62]). The gauge group of the aforementioned instanton  $E1_3$  is orthogonal due to the presence of the exotic O-plane and for the case of a single E-brane, that is an  $O(1)$  instanton, one obtains the desired structure of uncharged zero-modes in order to generate directly a contribution to the superpotential. We reproduce in Table 8 the CP gauge group and the minimum number of uncharged fermionic zero-modes for the  $E5$  and  $E1_k$  instantons.

	$G_{CP}$	number of uncharged fermionic zero-modes for a single brane
$E5$	$U(n)$	4
$E1_1$	$U(n)$	4
$E1_2$	$U(n)$	4
$E1_3$	$SO(n)$	2

Table 8: CP gauge group and minimum number of uncharged fermionic zero-modes for the  $E5$  and  $E1_k$  instantons.

Let us examine more closely the  $E1_3$  instantons. One can write the following wrapping (magnetization) numbers for them

$$(m_i, n_i) = (1, 0) \otimes (-1, 0) \otimes (0, 1) \quad (515)$$

The fractional  $E1_3$  instantons can be positioned at any of the 16 fixed points of  $h \in \mathbb{Z}_2 \times \mathbb{Z}_2$  situated inside  $\mathbb{T}_1^2 \times \mathbb{T}_2^2$ . In addition there are the discrete Wilson lines in the world-volume of the instanton. We parametrize these by means of three discrete complex parameters

$$\xi_{E1}^r = \epsilon^r + \tau_r \epsilon^{r+3} \quad (516)$$

where  $\epsilon^r = 0, 1/2$ . Notice that  $\xi_{E1}^r$  are not physical fields since the instanton action lacks the corresponding kinetic terms. There are four different choices of CP charges

$$\begin{aligned} D_{1;o}^1 &= k_1, & D_{2;o}^2 &= k_2, & D_{3;o}^3 &= k_3, & D_{4;o}^4 &= k_4, \\ D_{1;g}^1 &= k_1, & D_{2;g}^2 &= k_2, & D_{3;g}^3 &= -k_3, & D_{4;g}^4 &= -k_4, \\ D_{1;f}^1 &= -k_1, & D_{2;f}^2 &= k_2, & D_{3;f}^3 &= k_3, & D_{4;f}^4 &= -k_4, \\ D_{1;h}^1 &= -k_1, & D_{2;h}^2 &= k_2, & D_{3;h}^3 &= -k_3, & D_{4;h}^4 &= k_4. \end{aligned} \quad (517)$$

In addition to the uncharged zero-modes  $x^\mu, \theta_\alpha$  one has extra zero-modes  $\eta^i$  coming from strings stretched between the instanton brane and the other spacetime filling

D-branes. These modes are charged under the corresponding gauge group factors. Details on the partition functions of the instantons can be found in [49]. Recall that for magnetized  $D9$  branes we have introduced two families of branes labeled by the complex CP charges  $p_\beta$  and  $q_\beta$  (see Section 4.5). One can show that the spectrum of charged zero-modes is of the form presented in Table 9. Let us consider a couple of

CP choice	Multiplicity	Representation
1	$\frac{1}{4} (I^{ab} - S_g^{ab} I_1^{ab} + S_f^{ab} I_2^{ab} - S_h^{ab} I_3^{ab})$	$(k_1, \bar{p}_b)$
	$\frac{1}{4} (I^{a\beta} - S_g^{\alpha\beta} I_1^{a\beta} - S_f^{\alpha\beta} I_2^{a\beta} + S_h^{\alpha\beta} I_3^{a\beta})$	$(k_1, \bar{q}_\beta)$
2	$\frac{1}{4} (I^{ab} - S_g^{ab} I_1^{ab} - S_f^{ab} I_2^{ab} + S_h^{ab} I_3^{ab})$	$(k_2, \bar{p}_b)$
	$\frac{1}{4} (I^{a\beta} - S_g^{\alpha\beta} I_1^{a\beta} + S_f^{\alpha\beta} I_2^{a\beta} - S_h^{\alpha\beta} I_3^{a\beta})$	$(k_2, \bar{q}_\beta)$
3	$\frac{1}{4} (I^{ab} + S_g^{ab} I_1^{ab} - S_f^{ab} I_2^{ab} - S_h^{ab} I_3^{ab})$	$(k_3, \bar{p}_b)$
	$\frac{1}{4} (I^{a\beta} + S_g^{\alpha\beta} I_1^{a\beta} + S_f^{\alpha\beta} I_2^{a\beta} + S_h^{\alpha\beta} I_3^{a\beta})$	$(k_3, \bar{q}_\beta)$
4	$\frac{1}{4} (I^{ab} + S_g^{ab} I_1^{ab} + S_f^{ab} I_2^{ab} + S_h^{ab} I_3^{ab})$	$(k_4, \bar{p}_b)$
	$\frac{1}{4} (I^{a\beta} + S_g^{\alpha\beta} I_1^{a\beta} - S_f^{\alpha\beta} I_2^{a\beta} - S_h^{\alpha\beta} I_3^{a\beta})$	$(k_4, \bar{q}_\beta)$

Table 9: Charged zero modes for  $E1_3$  branes. The index  $a$  in the multiplicities corresponds to the  $E1_3$  instantons, with wrapping numbers  $(m_i, n_i) = (1, 0) \otimes (-1, 0) \otimes (0, 1)$ .

explicit models (presented in [28], [49]) where stringy instantons generate linear terms or mass terms in the superpotential. The first model contains four stacks of magnetized  $D9$  branes labeled with the CP charges  $p_1, p_2, q_1, q_2$  and having the following wrapping numbers

$$\begin{aligned}
p_1, q_1 &: (1, 1) \otimes (1, 1) \otimes (-1, 1) \\
p_2, q_2 &: (-1, 1) \otimes (-1, 1) \otimes (1, 1)
\end{aligned} \tag{518}$$

Notice that even though the stacks  $p_1$  and  $q_1$  have the same magnetizations they differ by the action of the orbifold group on the corresponding CP factors, hence by the twisted charges. The gauge group of the model is

$$G_{CP} = U(4)^4 \tag{519}$$

The massless spectrum contains, among other fields, 8 chiral supermultiplets in the antisymmetric representation of each  $U(4)$  factor of the gauge group. Let us denote

them with  $A_{ij}^s = -A_{ji}^s$  where  $s = 1, 2, 3, 4$  labels the four  $U(4)$  factors of the total gauge group. The charged spectrum of the  $E1_3$  branes for single instanton configurations is given in Table 10. The  $U(1)$  charges of the antisymmetric fields  $A_{ij}^s$  and of the charged

CP choice	$(k_1, k_2, k_3, k_4)$	Representation	Zero mode
1	$(1, 0, 0, 0)$	$(4, 1, 1, 1)$	$\eta_i^1$
2	$(0, 1, 0, 0)$	$(1, 4, 1, 1)$	$\eta_i^2$
3	$(0, 0, 1, 0)$	$(1, 1, \bar{4}, 1)$	$\eta_i^3$
4	$(0, 0, 0, 1)$	$(1, 1, 1, \bar{4})$	$\eta_i^4$

Table 10: Charged zero-mode structure for  $O(1)$  E1-brane instantons in the  $U(4)^4$  model. The index  $i$  runs over the (anti)fundamental representation of  $U(4)$ .

instanton zero-modes  $\eta_i^s$  are in such a way that the operator

$$\sum_{i,j=1}^4 \eta_i^s A_{ij}^s \eta_j^s \quad (520)$$

is gauge invariant. Schematically, integrating the following expression over the charged zero-modes

$$\int \left( \prod_{i=1}^4 d\eta_i^s \right) e^{-S_{E1} - \sum_{i,j} \eta_i^s A_{ij}^s \eta_j^s} \quad (521)$$

one obtains the non-perturbative superpotential

$$W_{\text{np}} = e^{-S_{E1}} \sum_{s=1}^4 \sum_{i,j,k,l=1}^4 \epsilon_{ijkl} A_{ij}^s A_{kl}^s \quad (522)$$

Let us comment on the form of the non-perturbative superpotential above. Notice that the operator  $A^s \cdot A^s$  is forbidden perturbatively since it is charged under the corresponding  $U(1)_s \subset U(4)_s$ . In general there is a coupling in front of the mass term in eq. (522) which depends on the open string moduli and of the complex structure moduli. For an explicit computation of such couplings see [30] or [57] for the case of (neutrino) Majorana masses. An important comment at this point is that any mass term like the one in eq. (522) is weighted by a factor  $e^{-S_{E1}}$  containing the classical instanton action

$$e^{-S_{E1}} = e^{-T_3} \simeq e^{-\text{Vol}} \quad (523)$$

giving naturally a hierarchically small mass. Actually the model splits into two non-interacting parts each with low-energy gauge group  $SU(4)^2$  having conformal symmetry at one-loop. This is then broken by the masses generated by stringy instanton effects [49].

The second explicit example contains four stacks of magnetized  $D9$  branes labeled by the CP charges  $p_1, p_2, q_1, q_2$ , one stack of non-magnetized  $D9$  branes labeled by  $N = n_1 + n_2$  and one stack of  $D5_1$  branes labeled by  $D = d_1 + d_2$ . The corresponding wrapping numbers are

$$\begin{aligned} p_1, q_1 &: (2, 1) \otimes (1, 1) \otimes (-1, 1) \\ p_2, q_2 &: (-2, 1) \otimes (-1, 1) \otimes (1, 1) \\ n_1, n_2 &: (0, 1) \otimes (0, 1) \times (0, 1) \\ d_1, d_2 &: (0, 1) \otimes (1, 0) \otimes (-1, 0) \end{aligned} \tag{524}$$

The gauge group of the model is

$$G_{CP} = U(2)^4 \times USp(4)^4 \tag{525}$$

We will focus our attention again on the antisymmetric fields coming from the magnetized stack of branes. One can see that the model contains 12 antisymmetric fields  $A_{ij}^s$  with respect to each unitary gauge group factor. The charged zero-mode instanton spectrum is given in Table 11.

CP choice	Representation	Zero mode
1	$(2, 1, 1, 1, 1, 1, 1, 1)$	$\eta_i^1$
2	$(1, 2, 1, 1, 1, 1, 1, 1)$	$\eta_i^2$
3	$(1, 1, \bar{2}, 1, 1, 1, 1, 1)$	$\eta_i^3$
4	$(1, 1, 1, \bar{2}, 1, 1, 1, 1)$	$\eta_i^4$

Table 11: Charged zero-mode structure for  $O(1)$  E1-brane instantons in the  $U(2)^2 \times U(2)^2 \times USp(4)^2 \times USp(4)^2$  model. The index  $i$  refers to the (anti)fundamental representation of  $U(2)$ .

As in the previous model the operator  $\sum_{i,j=1}^2 \eta_i^s A_{ij}^s \eta_j^s$  is gauge invariant. Performing the integral over the charged zero-modes of the instanton

$$\int \left( \prod_{i=1}^2 \eta_i^s \right) e^{-S_{E1} - g_s(\xi, \tau_r) \sum_{i,j} \eta_i^s A_{ij}^s \eta_j^s} \tag{526}$$

Multiplicity	Representation	Zero-mode
$[\Xi \cap \Pi_a]^+$	$(-1_E, N_a)$	$\eta_a$
$[\Xi \cap \Pi_a]^-$	$(1_E, \bar{N}_a)$	$\bar{\eta}_a$
$[\Xi \cap \Pi_{a'}]^+$	$(-1_E, \bar{N}_a)$	$\eta_{a'}$
$[\Xi \cap \Pi_{a'}]^-$	$(1_E, N_a)$	$\bar{\eta}_{a'}$

Table 12: Charged instanton zero-modes for  $E2$  branes.

one obtains the following non-perturbative superpotential

$$W_{\text{np}} = e^{-S_{E1}} \sum_{s=1}^4 g_s(\xi, \tau_r) \sum_{i,j=1}^2 \epsilon_{ij} A_{ij}^s \quad (527)$$

We have introduced explicitly the couplings  $g_s(\xi, \tau_r)$  which depend on the complex structure moduli  $\tau_r$  and the open string moduli that we have denoted generically with  $\xi$ . In Chapter 8 we will sketch the computation of these couplings. More details can be found in [30]. A superpotential of the form in eq. (527) containing a linear term in a charged superfield can have important phenomenological applications like moduli stabilization and Polonyi-like supersymmetry breaking. Intuitively speaking, a potential is generated for the moduli  $\xi$  and  $\tau_r$  appearing in the couplings  $g_s$ , which is of the following form

$$V \simeq |g_s(\xi, \tau_r)|^2 \quad (528)$$

We have shown in [30] that a supersymmetric non-perturbative vacuum exists in the model under discussion. We postpone the discussion of it to Chapter 8.

With the excuse of showing how stringy instantons can generate Majorana masses let us discuss the T-dual setting of Type IIA with intersecting  $D6$  branes. We will basically review the discussion in [56]. The instantons that are relevant for our discussion are  $E2$  branes. They wrap a 3-cycle  $\Xi$  in the internal space  $K$ . At the intersection of  $\Xi$  with 3-cycles  $\Pi_a$  wrapped by the  $D6$  branes one will have charged instanton zero-modes. Using the decomposition of the topological intersection number into positive and negative chirality

$$[\Xi] \circ [\Pi_a] = [\Xi \cap \Pi_a]^+ - [\Xi \cap \Pi_a]^- \quad (529)$$

one can write the charged zero-mode spectrum of the  $E2$  instanton in the form presented in Table 12.

The total charge of the fermionic zero-modes arising at the intersection of  $\Xi$  with  $\Pi_a$  and  $\Pi_{a'}$  is

$$Q_a(E2) = N_a [\Xi] \circ ([\Pi_a] - [\Pi_{a'}]) \quad (530)$$

In order to be concrete let us suppose we have an MSSM like model realized on the  $D6$  branes. One starts with four stacks of branes with the following gauge group [63]

$$U(3)_a \times USp(2)_b \times U(1)_c \times U(1)_d \quad (531)$$

In the formula above we have labeled the four stacks of branes with  $a, b, c$  and  $d$  respectively. The Standard Model matter fields can be realized by bifundamental representations. They are reproduced in Table 13. Notice that the MSSM hypercharge

Intersection	Field	Representation	Hypercharge Y
$a - b$	$Q_L$	$3 \times (3, 2)_{(1,0,0)}$	$1/3$
$a - c$	$(U_R)^c$	$3 \times (\bar{3}, 1)_{(-1,1,0)}$	$-4/3$
$a' - c$	$(D_R)^c$	$3 \times (\bar{3}, 1)_{(-1,-1,0)}$	$2/3$
$b - d$	$L_L$	$3 \times (1, 2)_{(0,0,-1)}$	$-1$
$c - d$	$(E_R)^c$	$3 \times (1, 1)_{(0,-1,1)}$	$2$
$c' - d$	$(N_R)^c$	$3 \times (1, 1)_{(0,1,1)}$	$0$

Table 13: MSSM from a four stack model.

is the following linear combination of the various  $U(1)$ 's present in the model

$$U(1)_Y = \frac{1}{3}U(1)_a - U(1)_c + U(1)_d \quad (532)$$

Notice that due to the extra  $U(1)$  symmetries one cannot write perturbatively a Majorana mass term for the right-handed neutrinos

$$W_m = M_m (N_R)^c (N_R)^c \quad (533)$$

However such a term can be generated by  $E2$  instanton branes if the following conditions are satisfied

$$\begin{aligned} [\Xi] \cap [\Pi_a] &= [\Xi] \cap [\Pi_{a'}] = [\Xi] \cap [\Pi_b] = [\Xi] \cap [\Pi_{b'}] = [\Xi] \cap [\Pi_c] = [\Xi] \cap [\Pi_{c'}] = 0 \\ [\Xi \cap \Pi_{c'}]^+ &= 2, \quad [\Xi \cap \Pi_{c'}]^- = 0, \quad [\Xi \cap \Pi_d]^- = 2, \quad [\Xi \cap \Pi_d]^+ = 0 \end{aligned} \quad (534)$$

The mass generated by stringy instantons will have the following form

$$M_m = x M_s e^{-\text{Vol}_{E2}} \quad (535)$$

where  $x$  contains the contributions from the Kähler potential and the one-loop determinant and  $M_s$  is the stringy mass scale. As we have commented in the case of  $E1$

instantons the volume factor will naturally create a hierarchy between the string scale  $M_s$  and the Majorana mass of the right-handed neutrinos  $M_m$  (assuming that  $x$  is of order  $O(1)$ ). Of course the discussion about the  $E2$  instanton is valid if it has the correct structure of uncharged zero-modes in order to contribute to the superpotential. Another phenomenological application of stringy instantons is generation of some perturbatively forbidden Yukawa couplings. This is the case in certain Grand Unified  $SU(5)$  models realized with intersecting/magnetized branes where the top trilinear Yukawa coupling  $\mathbf{10} \mathbf{10} \mathbf{5_H}$  is forbidden. Under certain conditions instantons can generate this coupling non-perturbatively.

## 8 Linear Term Instabilities

In this chapter we will discuss the results in [30] where the non-perturbative superpotential with linear terms in the charged fields is computed in global models. We identify the correct way of summing over one-instanton contributions (see eq. (544)). Identification of the correct form of the superpotential relies on the results in [33] where we derive the holomorphic variables (field redefinitions by the Wilson line moduli) in models with magnetized branes and continuous Wilson lines.

Stringy instanton effects can induce dynamical condensates and spontaneous breaking of symmetries in the effective 4d theory. A situation that we are referring to is two stacks of fractional branes with opposite twisted charges passing through an orbifold singularity. The massless spectrum contains chiral multiplets  $\Phi_{\bar{a}b}$  and  $\tilde{\Phi}_{ab}$  transforming in the

$$(\text{Fund}(G_a), \overline{\text{Fund}(G_b)}) \text{ and } (\overline{\text{Fund}(G_a)}, \text{Fund}(G_b)) \quad (536)$$

representations of the CP gauge group  $G_a \times G_b$ . Since we have assumed that branes have opposite twisted charges, then the recombined stack has vanishing total twisted charge and hence it can move off the singularity. Its position modulus will be related to the condensate  $\langle \Phi_{\bar{a}b} \tilde{\Phi}_{ab} \rangle$  which parametrizes the recombination of the two stacks and the spontaneous breaking of the gauge group to a diagonal subgroup. Such a non-trivial condensate can be induced by Euclidean brane instantons. We depict this situation in Figure 5. We study in this chapter situations where there is a non-perturbatively generated linear superpotential

$$W_{\text{np}} = g(\xi, \tau_r) e^{-S_E} \Phi \quad (537)$$

in the gauge theory. The function  $g$  depends on the open string moduli denoted generically with  $\xi$  and the complex structure moduli  $\tau_r$ . A non-perturbative superpotential of the form above is generated if the following conditions are satisfied

- There is an  $O(1)$  instanton with  $\theta_\alpha$  and  $x^\mu$  as uncharged zero-modes arising from strings with both ends on the Euclidean brane;
- There is an appropriate number of (two) charged instanton zero-modes  $\eta_i$  arising from strings stretched between the brane and the instantons;
- There is an  $U(1)$  or  $U(2)$  gauge theory realized on the D-branes;

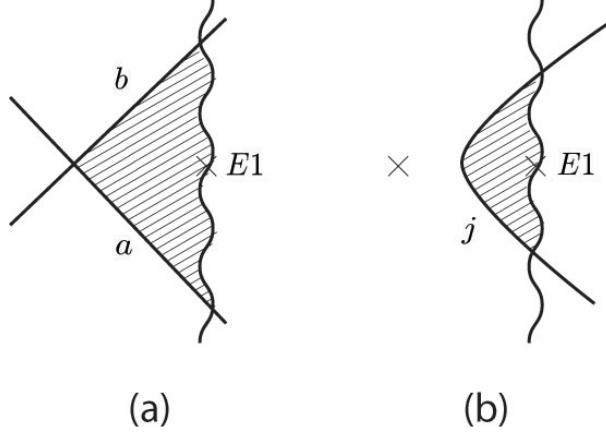


Figure 5: Non-perturbative Higgs mechanism through a dynamical condensate induced by a stringy Euclidean brane instanton. (a) Two fractional branes with opposite twisted charge are stuck in a given singularity. An instanton (represented by the wavy line) sits at a distant singularity. (b) The instanton may induce a condensate  $\langle \Phi_{ab} \tilde{\Phi}_{\bar{a}\bar{b}} \rangle$  whose VEV parameterizes the position of the recombined brane  $j$  and minimizes the area of the disc amplitude spanned by the branes and the instanton (shaded region).

- There is a gauge invariant Yukawa coupling  $\eta_1 \Phi \eta_2$ , with  $\Phi$  being a matter field charged under  $U(1)$ .

The  $F$ -term contribution of the field  $\Phi$  to the vacuum energy is of the form

$$\Delta V \simeq |g(\xi, \tau_r) e^{-S_E}|^2 \quad (538)$$

and thus one can interpret the dependence on the open string moduli as a force acting on the  $D$ -brane by the singularity where the instanton resides.

In globally consistent models there are usually several rigid instantons that can contribute to the superpotential

$$W_{\text{np}} = \sum_{\alpha} g_{\alpha}(\xi, \tau_r) e^{-S_{E\alpha}} \Phi \quad (539)$$

The non-perturbative vacuum will be found by minimizing the resulting scalar potential for the positions of the brane. The functions  $g_{\alpha}(\xi, \tau_r)$  contain additional effects related to the dynamics of the other  $D$ -branes present in the model. For instance, if a spectator brane moves towards the singularities where the instantons reside then it can change the number of zero-modes and therefore destroy the original instability.

We would like to illustrate in the following how one computes the couplings  $g_\alpha$  in the case of a toroidal compactification with an orbifold group  $\Gamma$  that contains a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  with discrete torsion required in order to have instantons that can generate corrections to the superpotential. Thus we consider Type I string theory on a factorizable  $\mathbb{T}^6/\Gamma$  with magnetized  $D9$  branes where  $E1_3$  Euclidean branes can generate linear terms in the superpotential. This will apply in particular to the model considered in Section 7.3.

Suppose that we have a chiral field  $\Phi_{ab}$  which transforms in the bifundamental representation of  $U(1)_a \times U(1)_b$  and suppose that we have two instanton fermionic zero-modes  $\eta_a$  and  $\eta_b$  charged under  $U(1)_a$  and  $U(1)_b$  respectively in such a way that the Yukawa coupling  $\eta_a \Phi_{ab} \eta_b$  is gauge invariant. Integration over the charged instanton zero-modes yields the non-perturbative superpotential

$$W_{\text{np}} = g(\xi, \tau_r) e^{-S_E} \Phi_{ab} \quad (540)$$

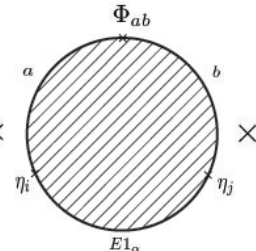
The zeroes of  $g(\xi, \tau_r)$  will determine the non-perturbative supersymmetric vacuum. There are two other situations for which one can get a linear term in the superpotential, namely for an antisymmetric chiral superfield  $A_{ij}$  of a  $U(2)$  gauge group and for a symmetric chiral field  $S_{12}$  of a  $U(1)$  gauge group. For the antisymmetric field one needs one charged fermionic zero-mode in the fundamental of the  $U(2)$ , and for the symmetric field two charged instanton zero-modes (with the same  $U(1)$  charge). Then one find non-perturbative superpotentials with linear terms

$$W_{\text{np}} = g(\xi, \tau_r) e^{-S_E} \sum_{i,j=1}^2 A_{ij} \quad (541)$$

$$W_{\text{np}} = g(\xi, \tau_r) e^{-S_E} S_{12}$$

We can make use of the D-brane instanton calculus to compute the function  $g(\xi, \tau_r)$ , or equivalently the  $F$ -term associated to the field  $\Phi_{ab}$ . This can be expressed as

$$\langle F_\Phi \rangle = \sum_\alpha \int d^4x d^2\theta d^2\eta \exp(2\pi i S_{E1_\alpha}) \times$$



$$\times \exp \left[ \sum_k \left( \text{Diagram of a cylinder with E1_alpha on the left and D9_k on the right} \right) + \left( \text{Diagram of a cylinder with E1_alpha on the left and O on the right} \right) \right]$$

where the sum in  $\alpha$  goes over all possible configurations for the position, discrete Wilson lines and CP charges of the instanton (see eqs. (516) and (517) for the  $E1_3$  instanton). In general the fields  $\Phi_{ab}$  come with a given multiplicity, hence there are several  $F$ -terms of the form shown above. In order to compute  $\langle F_\Phi \rangle$  one needs to compute the Yukawa coupling  $\eta_a \Phi_{ab} \eta_b$  and the one-loop amplitudes involving the instanton and all the branes in the compactification. In Chapter 5 we have sketched how Yukawa couplings are computed in a toroidal compactification. Making use of the result in eq. (542) we can write the following expression for the 3-point disk scattering amplitude involving two magnetized branes and a rigid  $E1$  brane with two fermionic charged instanton zero-modes

$$\mathcal{D}_{abE1}^{\vec{i}} \sim \prod_{r=1}^3 \exp \left[ \frac{\pi i \xi_{abE1}^r \text{Im } \xi_{abE1}^r}{|I_{ab}^r| \text{Im } \tau_r} \right] \vartheta \left[ \begin{matrix} \frac{i_r}{I_{ab}^r} \\ 0 \end{matrix} \right] (\xi_{abE1}^r ; \tau_r | I_{ab}^r) \quad (542)$$

We have omitted some normalization factors coming from the Kähler metric and some (undetermined) Wilson line dependent phases (for a discussion of their importance see [33]). The dependence on the continuous Wilson line moduli  $\xi_a^r, \xi_b^r$  of the branes  $a$  and  $b$  and the discrete parameters  $\xi_{E1}^r$  of the instanton is encoded in the quantity

$$\xi_{abE1}^r = I_{ab}^r \xi_{E1}^r + I_{E1a}^r \xi_b^r + I_{bE1}^r \xi_a^r \quad (543)$$

The one-loop amplitudes of the instanton have been computed in [49]. More details about the calculation of the  $F$ -term under consideration can be found in [30]. We reproduce here the final expression of the non-perturbative superpotential for  $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$

$$\begin{aligned} W_{\text{np}} = & e^{2\pi i \hat{T}_3 \hat{\Phi}_{ab}^{\vec{i}}} \left[ \eta(\tau_1)^{1+2N_{D52}} \eta(\tau_2)^{1+2N_{D51}} \eta(\tau_3)^{1+2N_{D9}} \right]^{-1} \times \\ & \times \prod_{r=1}^3 \sum_{\epsilon^r, \epsilon^{r+3}=0,1/2} \left( e^{2\pi i [N_{D9}(\epsilon^3 + \epsilon^3 \epsilon^6) + N_{D51}(\epsilon^2 + \epsilon^2 \epsilon^5) + N_{D52}(\epsilon^1 + \epsilon^1 \epsilon^4)]} e^{-i\pi |I_{ab}^r| \epsilon^r \epsilon^{r+3}} \right. \\ & \quad \left. \vartheta \left[ \begin{matrix} \frac{i_r}{I_{ab}^r} + \epsilon^{r+3} \\ I_{ab}^r \epsilon^r \end{matrix} \right] (I_{bE1}^r \xi_a^r + I_{E1a}^r \xi_b^r ; \tau^r | I_{ab}^r) \right. \\ & \quad \left. \times \prod_{K=1}^{N_{D9}} \vartheta^2 \left[ \begin{matrix} \frac{1}{2} + \epsilon^6 \\ \frac{1}{2} + \epsilon^3 \end{matrix} \right] (\xi_K^3; \tau_3) \times \prod_{Q=1}^{N_{D51}} \vartheta^2 \left[ \begin{matrix} \frac{1}{2} + \epsilon^5 \\ \frac{1}{2} + \epsilon^2 \end{matrix} \right] (\xi_Q^2; \tau_2) \times \prod_{P=1}^{N_{D52}} \vartheta^2 \left[ \begin{matrix} \frac{1}{2} + \epsilon^4 \\ \frac{1}{2} + \epsilon^1 \end{matrix} \right] (\xi_P^1; \tau_1) \right) \end{aligned} \quad (544)$$

This expression is holomorphic and periodic under shifts of the D-brane moduli and  $U(1)$  gauge transformations, and behaves as a holomorphic modular form of weight

$-1$  under  $SL(2, \mathbb{Z})$  transformations of the complex structure parameters. Of course, symmetrization with respect to all the orbifold and orientifold operations should be understood. Let us explain the notation in eq. (544).  $N_{D9}$  is the number of non-magnetized  $D9$ -branes and  $N_{D5_s}$  is the number of  $D5$ -branes wrapping the  $s$ -th torus ( $s = 1, 2$ ).  $\xi_K^3$  is the complexified continuous Wilson line modulus of the  $K$ -th  $D9$  brane along the third two-torus,  $\xi_P^1$  ( $\xi_Q^2$ ) is the complexified position modulus of the  $P$ -th  $D5_2$  brane (the  $Q$ -th  $D5_1$  brane) along the first (second) two-torus. The variables  $\hat{\Phi}_{ab}^i$  and  $\hat{T}_3$  appearing in eq. (544) have been redefined by the continuous Wilson line moduli of the branes according to eqs. (421) and (546). For a factorizable six-torus compactification with magnetized branes and continuous Wilson lines the redefinitions of the axion-dilaton  $S$  and of the Kähler moduli  $T_k$  were found to be [33]

$$\hat{S} = S + \sum_{\{I\}} \sum_{r=1}^3 c_I^r \frac{\xi_I^r \text{Im } \xi_I^r}{\text{Im } \tau_r} \quad (545)$$

$$\hat{T}_k = T_k - \sum_{\{I\}} \left[ c_I^0 \frac{\xi_I^k \text{Im } \xi_I^k}{\text{Im } \tau_k} - \sum_{p \neq q \neq k}^3 c_I^q \frac{\xi_I^p \text{Im } \xi_I^p}{\text{Im } \tau_p} \right], \quad k = 1, 2, 3 \quad (546)$$

where  $c_I^r$  are the RR charges of  $D5$  brane induced in the worldvolume of  $D9$  brane  $I$  by the magnetization

$$c_I^r = n_r^I m_j^I m_k^I \quad r \neq j \neq k = 1, 2, 3 \quad (547)$$

and  $c_I^0$  is defined to be

$$c_I^0 = n_1^I n_2^I n_3^I \quad (548)$$

Notice that the superpotential in eq. (544) contains a sum over the discrete instanton parameters (positions and Wilson line moduli) weighted by generally non-trivial phases. From a local point of view each term in this sum has two types of zeroes:

- Points in the moduli space for which the one-loop determinant vanishes due to the appearance of extra fermionic zero modes which run in the loops. From a target space point of view, the extra fermionic modes arise from strings stretching between the instanton and the non-magnetized (bulk)  $D$ -branes which become massless when the brane sits on top of the instanton.
- Points in the moduli space for which the area of the 3-point disc scattering amplitude becomes zero, leading to a vanishing Yukawa coupling between the two fermionic charged zero-modes of the instanton and the matter field. The interpretation of this process is that the  $D$ -branes move in order to reduce the area of the disc and thus minimize the vacuum energy.

The picture illustrated above is complicated by global issues. In a given compact model there are usually several locations in the internal space where instantons reside and thus, supersymmetric vacua in global models are generically the result of a combined cancelation between the different non-vanishing instanton contributions. Let us illustrate this with a simple toy model. We consider a toroidal compactification with  $D5_3$  branes equally distributed and sitting on top of the  $E1$  instantons at the singularities. In addition there are also  $N_{D5_1} = 4$  bulk  $D5_1$  branes. The induced non-perturbative superpotential is [33]

$$W_{\text{np}} = e^{2\pi i \hat{T}_3} [\eta(\tau_1)^1 \eta(\tau_2)^9 \eta(\tau_3)^1]^{-1} \sum_{\epsilon^2, \epsilon^5=0,1/2} \prod_{q=1}^4 \vartheta^2 \left[ \begin{matrix} \frac{1}{2} + \epsilon^5 \\ \frac{1}{2} + \epsilon^2 \end{matrix} \right] (\xi_q^2; \tau_2) \quad (549)$$

where  $\xi_q^2$  are the positions of the  $D5_1$ -branes on the second torus. From a local point of view, when the  $D5_1$  branes sit on top of the instantons, i.e.

$$\xi_1^2 = 0, \quad \xi_2^2 = \frac{1}{2}, \quad \xi_3^2 = \frac{\tau_2}{2}, \quad \xi_4^2 = \frac{1}{2} + \frac{\tau_2}{2}, \quad (550)$$

each term in the non-perturbative superpotential vanishes. The gauge group is  $U(1)^8$ . Outside of this locus one or more instantons give a non-zero contribution to the superpotential. Hence, from a local analysis, one would conclude the (550) is a supersymmetric minimum of the scalar potential, with no flat directions. However, from the global point of view, even if each single instanton contribution is non-zero, there can be cancelations between the different terms leading to new supersymmetric vacua. In the model at hand there is a one-parameter family of supersymmetric vacua.

$$\xi_1^2 = \frac{\rho}{2} + \frac{\rho\tau_2}{2}, \quad \xi_2^2 = \frac{1-\rho}{2} + \frac{\rho\tau_2}{2}, \quad \xi_3^2 = \frac{\rho}{2} + \frac{(1-\rho)\tau_2}{2}, \quad \xi_4^2 = \frac{1-\rho}{2} + \frac{(1-\rho)\tau_2}{2} \quad (551)$$

with  $\rho \in [0, 1)$ . We illustrate this in Figure 6. Now let us comment a little bit on the explicit model in eq. (524) with gauge group  $U(2)^4 \times USp(4)^4$  and a non-perturbative linear superpotential for the antisymmetric fields of the unitary factors. The detailed analysis can be found in [30]. Recall that the model contained bulk non-magnetized  $D9$  branes and  $D5_1$  branes. Accordingly, one finds a supersymmetric vacuum in the following situations

- The discrete Wilson lines of non-magnetized bulk  $D9$ -branes in the third 2-torus are equally distributed among the four possible choices

$$\xi_1^3 = 0, \quad \xi_2^3 = \frac{1}{2}, \quad \xi_3^3 = \frac{\tau_3}{2}, \quad \xi_4^3 = \frac{1}{2} + \frac{\tau_3}{2} \quad (552)$$

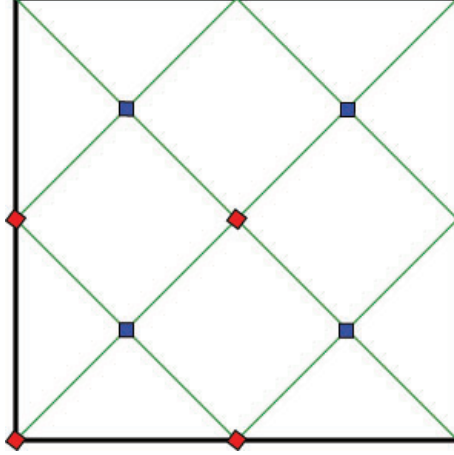


Figure 6: Non-perturbative supersymmetric flat directions for the motion of  $D5_1$ -branes along the second torus. The branes can only move along the continuous lines, where gauge group is  $USp(2)^4$ . Points with special gauge symmetry group have been also marked: (blue) squares for  $USp(4)^2$  points, and (red) rhombus for  $U(1)^8$  points. The four branes can only move in a correlated way, so that they approach an  $USp(4)^2$  point all at the same time, each one through a different direction.

- Bulk  $D5$ -branes are equally distributed among the four singularities in the second 2-torus

$$\phi_1^2 = 0, \quad \phi_2^2 = \frac{1}{2}, \quad \phi_3^2 = \frac{\tau_2}{2}, \quad \phi_4^2 = \frac{1}{2} + \frac{\tau_2}{2} \quad (553)$$

In both cases extra-fermionic zero-modes are generated. The initial gauge symmetry is spontaneously broken at the non-perturbative supersymmetric vacuum

$$U(2)^2 \times U(2)^2 \times USp(4)^2 \times USp(4)^2 \rightarrow U(2)^2 \times U(2)^2 \times U(1)^8 \times USp(4)^2 \quad (554)$$

Fluctuations of non-magnetized bulk  $D9$  branes or  $D5$  branes moduli around this vacuum induce a non-trivial scalar potential also for the Wilson line moduli of the magnetized  $D9$  branes. We plot this potential for both the bulk non-magnetized and magnetized branes in Figure 7. The four stacks of fractional magnetized  $D9$  branes have total twisted charge equal to zero and thus they can recombine and move in the bulk. The potential for moduli of the recombined brane has a minimum at

$$\xi_a^2 = \frac{1}{4} + \frac{\tau_2}{4} \pmod{\frac{1}{2}, \frac{\tau_2}{2}} \quad (555)$$

Hence, while  $\phi_1^2$  is rolling down towards the supersymmetric vacuum at the origin, the recombination moduli of the fractional magnetized branes feel a potential, inducing

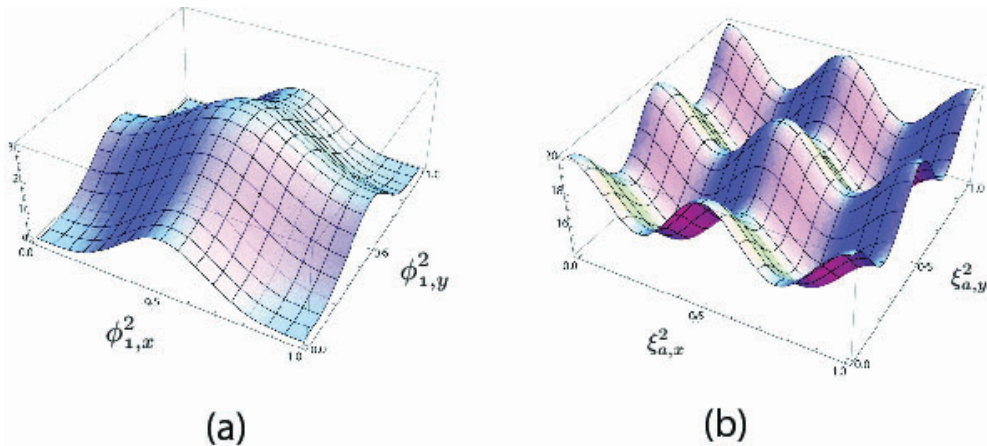


Figure 7: (a) Non-perturbative scalar potential for the complex field  $\phi_1^2$ , parameterizing the position of one of the D5-branes along the second 2-torus. Other D5-branes are equally distributed over the singularities in the second 2-torus, except for the origin. (b) Non-perturbative scalar potential for the complex Wilson line modulus of the recombined D9-brane along the second 2-torus, for an arbitrary VEV of  $\phi_1^2$ .

the spontaneous breaking of the gauge symmetry

$$U(2)^2 \times U(2)^2 \times USp(4)^2 \times USp(4)^2 \rightarrow U(2)_{\text{diag.}} \times USp(4)^2 \times USp(4)^2 \quad (556)$$

Fluctuations around the non-perturbative vacuum in this model will favor a gauge group  $U(2) \times U(1)^{16}$ .

We would like to end this section with a few remarks about applications of non-perturbative linear superpotentials. These range from moduli stabilization to phenomenological applications such as models of D-brane inflation [64], or stringy realizations of composite Higgs models. In addition, superpotentials of the form eq. (539), which have an  $R$ -symmetry, have also been recently employed to realize supersymmetry breaking a la Polonyi and gauge mediation in string theory [65]. In this sense, it is generally accepted that if the number of complex fields on which  $g_\alpha(\xi, \tau_r)$  depend is lower than the multiplicity of fields  $\Phi$ , the superpotential in eq. 539 spontaneously breaks supersymmetry [66],[67]. This is true when the functions  $g_\alpha(\xi, \tau)$  are generic enough. However, our results [30] suggest that there are often correlations between the zeroes of  $g_\alpha(\xi, \tau_r)$  allowing for supersymmetric vacua even in cases where the non-perturbative superpotential preserves an  $R$ -symmetry. The reason is that due to the particular dependence on the open string moduli  $\xi$ , the superpotential is not generic.

## A Characters of the $T^6/Z_2 \times Z_2$ orientifold

$$\begin{aligned}
\tau_{oo} &= V_2 O_2 O_2 O_2 + O_2 V_2 V_2 V_2 - S_2 S_2 S_2 S_2 - C_2 C_2 C_2 C_2 \\
\tau_{og} &= O_2 V_2 O_2 O_2 + V_2 O_2 V_2 V_2 - C_2 C_2 S_2 S_2 - S_2 S_2 C_2 C_2 \\
\tau_{oh} &= O_2 O_2 O_2 V_2 + V_2 V_2 V_2 O_2 - C_2 S_2 S_2 C_2 - S_2 C_2 C_2 S_2 \\
\tau_{of} &= O_2 O_2 V_2 O_2 + V_2 V_2 O_2 V_2 - C_2 S_2 C_2 S_2 - S_2 C_2 S_2 C_2 \\
\tau_{go} &= V_2 O_2 S_2 C_2 + O_2 V_2 C_2 S_2 - S_2 S_2 V_2 O_2 - C_2 C_2 O_2 V_2 \\
\tau_{gg} &= O_2 V_2 S_2 C_2 + V_2 O_2 C_2 S_2 - S_2 S_2 O_2 V_2 - C_2 C_2 V_2 O_2 \\
\tau_{gh} &= O_2 O_2 S_2 S_2 + V_2 V_2 C_2 C_2 - C_2 S_2 V_2 V_2 - S_2 C_2 O_2 O_2 \\
\tau_{gf} &= O_2 O_2 C_2 C_2 + V_2 V_2 S_2 S_2 - S_2 C_2 V_2 V_2 - C_2 S_2 O_2 O_2 \\
\tau_{ho} &= V_2 S_2 C_2 O_2 + O_2 C_2 S_2 V_2 - C_2 O_2 V_2 C_2 - S_2 V_2 O_2 S_2 \\
\tau_{hg} &= O_2 C_2 C_2 O_2 + V_2 S_2 S_2 V_2 - C_2 O_2 O_2 S_2 - S_2 V_2 V_2 C_2 \\
\tau_{hh} &= O_2 S_2 C_2 V_2 + V_2 C_2 S_2 O_2 - S_2 O_2 V_2 S_2 - C_2 V_2 O_2 C_2 \\
\tau_{hf} &= O_2 S_2 S_2 O_2 + V_2 C_2 C_2 V_2 - C_2 V_2 V_2 S_2 - S_2 O_2 O_2 C_2 \\
\tau_{fo} &= V_2 S_2 O_2 C_2 + O_2 C_2 V_2 S_2 - S_2 V_2 S_2 O_2 - C_2 O_2 C_2 V_2 \\
\tau_{fg} &= O_2 C_2 O_2 C_2 + V_2 S_2 V_2 S_2 - C_2 O_2 S_2 O_2 - S_2 V_2 C_2 V_2 \\
\tau_{fh} &= O_2 S_2 O_2 S_2 + V_2 C_2 V_2 C_2 - C_2 V_2 S_2 V_2 - S_2 O_2 C_2 O_2 \\
\tau_{ff} &= O_2 S_2 V_2 C_2 + V_2 C_2 O_2 S_2 - C_2 V_2 C_2 O_2 - S_2 O_2 S_2 V_2
\end{aligned} \tag{557}$$

## B Partition functions of the magnetized $T^6/Z_2 \times Z_2$ with discrete torsion

We reproduce here the general partition function of magnetized D9 branes on a  $T^6/Z_2 \times Z_2$  orbifold with discrete torsion.  $p_a$  and  $q_\alpha$  denote different stacks of branes, while the intersection numbers and  $S_i^{AB}$  are defined in eqs. The magnetic field deformation on each  $T^2$  is encoded in  $z_i^A$ , while  $z_i^{AB} = z_i^A - z_i^B$  and  $z_i^{AB'} = z_i^A + z_i^B$ .

The spectrum of open strings stretched between a  $D9^A$  brane and itself or its image

$D9^{A'}$  is encoded in the annulus amplitude

$$\begin{aligned}
\mathcal{A}^{A,A^{(v)}} = & \frac{1}{4} \int_0^\infty \frac{dt}{t^3} \left\{ p_a \bar{p}_a \left[ \tilde{P}_1 \tilde{P}_2 \tilde{P}_3 T_{oo}(0) + \left( \tilde{P}_1 T_{og}(0) + \tilde{P}_2 T_{of}(0) + \tilde{P}_3 T_{oh}(0) \right) \left( \frac{2\eta}{\theta_2} \right)^2 \right] \right. \\
& + I^{aa'} \left[ \frac{p_a^2}{2} T_{oo}(2z_i^a \tau) + \frac{\bar{p}_a^2}{2} T_{oo}(-2z_i^a \tau) \right] \prod_{i=1}^3 \frac{i\eta}{\theta_1(2z_i^a \tau)} \\
& - 4I_1^{aa'} \left[ \frac{p_a^2}{2} T_{og}(2z_i^a \tau) + \frac{\bar{p}_a^2}{2} T_{og}(-2z_i^a \tau) \right] \frac{i\eta}{\theta_1(2z_1^a \tau)} \prod_{i=2,3} \frac{\eta}{\theta_2(2z_i^a \tau)} \\
& - 4I_2^{aa'} \left[ \frac{p_a^2}{2} T_{of}(2z_i^a \tau) + \frac{\bar{p}_a^2}{2} T_{of}(-2z_i^a \tau) \right] \frac{i\eta}{\theta_1(2z_2^a \tau)} \prod_{i=1,3} \frac{\eta}{\theta_2(2z_i^a \tau)} \\
& + 4I_3^{aa'} \left[ \frac{p_a^2}{2} T_{oh}(2z_i^a \tau) + \frac{\bar{p}_a^2}{2} T_{oh}(-2z_i^a \tau) \right] \frac{i\eta}{\theta_1(2z_3^a \tau)} \prod_{i=1,2} \frac{\eta}{\theta_2(2z_i^a \tau)} \\
& \left. + (a, a' \rightarrow \alpha, \alpha') \right\} \frac{1}{\eta^2}
\end{aligned} \tag{558}$$

and in the Möbius-strip amplitude

$$\begin{aligned}
\mathcal{M} = & -\frac{1}{4} \int_0^\infty \frac{dt}{t^3} \left\{ \prod_{i=1}^3 (m_i^{(a)}) \left[ p_a \hat{T}_{oo}(2z_i^a \tau) + \bar{p}_a \hat{T}_{oo}(-2z_i^a \tau) \right] \prod_{i=1}^3 \frac{i\hat{\eta}}{\hat{\theta}_1(2z_i^a \tau)} \right. \\
& - \epsilon_1 \left[ p_a \hat{T}_{og}(2z_i^a \tau) + \bar{p}_a \hat{T}_{og}(-2z_i^a \tau) \right] \frac{im_1^{(a)} \hat{\eta}}{\hat{\theta}_1(2z_1^a \tau)} \prod_{i=2,3} \frac{n_i^{(a)} \hat{\eta}}{\hat{\theta}_2(2z_i^a \tau)} \\
& - \epsilon_2 \left[ p_a \hat{T}_{of}(2z_i^a \tau) + \bar{p}_a \hat{T}_{of}(-2z_i^a \tau) \right] \frac{im_2^{(a)} \hat{\eta}}{\hat{\theta}_1(2z_2^a \tau)} \prod_{i=1,3} \frac{n_i^{(a)} \hat{\eta}}{\hat{\theta}_2(2z_i^a \tau)} \\
& - \epsilon_3 \left[ p_a \hat{T}_{oh}(2z_i^a \tau) + \bar{p}_a \hat{T}_{oh}(-2z_i^a \tau) \right] \frac{im_3^{(a)} \hat{\eta}}{\hat{\theta}_1(2z_3^a \tau)} \prod_{i=1,2} \frac{n_i^{(a)} \hat{\eta}}{\hat{\theta}_2(2z_i^a \tau)} \\
& \left. + (a \rightarrow \alpha) \right\} \frac{1}{\eta^2}.
\end{aligned} \tag{559}$$

Here  $\tilde{P}_i$  denote “boosted” compactification lattices obtained replacing Kaluza-Klein momenta  $k_i$  in the  $i$ -th torus by  $k_i \rightarrow k_i / \sqrt{n_i^2 + (m_i^2/v_i^2)}$ .

Oriented open strings, stretched between different stacks of branes, yield the fol-

lowing contributions to the annulus amplitude

$$\begin{aligned}
\mathcal{A}^{a,b^{(\prime)}} = & \frac{1}{4} \int_0^\infty \frac{dt}{t^3} \frac{1}{\eta^2} \left\{ I^{ab} [p_a \bar{p}_b T_{oo}(z_i^{ab} \tau) + \bar{p}_a p_b T_{oo}(-z_i^{ab} \tau)] \prod_{i=1}^3 \frac{i\eta}{\theta_1(z_i^{ab} \tau)} \right. \\
& + I^{ab'} [p_a p_b T_{oo}(z_i^{ab'} \tau) + \bar{p}_a \bar{p}_b T_{oo}(-z_i^{ab'} \tau)] \prod_{i=1}^3 \frac{i\eta}{\theta_1(z_i^{ab'} \tau)} \\
& + S_g^{ab} I_1^{ab} [p_a \bar{p}_b T_{og}(z_i^{ab} \tau) + \bar{p}_a p_b T_{og}(-z_i^{ab} \tau)] \frac{i\eta}{\theta_1(z_1^{ab} \tau)} \prod_{i=2,3} \frac{\eta}{\theta_2(z_i^{ab} \tau)} \\
& - S_g^{ab} I_1^{ab'} [p_a p_b T_{og}(z_i^{ab'} \tau) + \bar{p}_a \bar{p}_b T_{og}(-z_i^{ab'} \tau)] \frac{i\eta}{\theta_1(z_1^{ab'} \tau)} \prod_{i=2,3} \frac{\eta}{\theta_2(z_i^{ab'} \tau)} \\
& + S_f^{ab} I_2^{ab} [p_a \bar{p}_b T_{of}(z_i^{ab} \tau) + \bar{p}_a p_b T_{of}(-z_i^{ab} \tau)] \frac{i\eta}{\theta_1(z_2^{ab} \tau)} \prod_{i=1,3} \frac{\eta}{\theta_2(z_i^{ab} \tau)} \\
& - S_f^{ab} I_2^{ab'} [p_a p_b T_{of}(z_i^{ab'} \tau) + \bar{p}_a \bar{p}_b T_{of}(-z_i^{ab'} \tau)] \frac{i\eta}{\theta_1(z_2^{ab'} \tau)} \prod_{i=1,3} \frac{\eta}{\theta_2(z_i^{ab'} \tau)} \\
& + S_h^{ab} I_3^{ab} [p_a \bar{p}_b T_{oh}(z_i^{ab} \tau) + \bar{p}_a p_b T_{oh}(-z_i^{ab} \tau)] \frac{i\eta}{\theta_1(z_3^{ab} \tau)} \prod_{i=1,2} \frac{\eta}{\theta_2(z_i^{ab} \tau)} \\
& \left. + S_h^{ab} I_3^{ab'} [p_a p_b T_{oh}(z_i^{ab'} \tau) + \bar{p}_a \bar{p}_b T_{oh}(-z_i^{ab'} \tau)] \frac{i\eta}{\theta_1(z_3^{ab'} \tau)} \prod_{i=1,2} \frac{\eta}{\theta_2(z_i^{ab'} \tau)} \right\}, \tag{560}
\end{aligned}$$

$$\mathcal{A}^{\alpha,\beta^{(\prime)}} = \mathcal{A}^{a,b^{(\prime)}} \quad \text{with} \quad a, b, b' \rightarrow \alpha, \beta, \beta', \tag{561}$$

and

$$\begin{aligned}
\mathcal{A}^{a,\alpha^{(r)}} = & \frac{1}{4} \int_0^\infty \frac{dt}{t^3} \frac{1}{\eta^2} \left\{ I^{a\alpha} [p_a \bar{q}_\alpha T_{oo}(z_i^{a\alpha} \tau) + \bar{p}_a q_\alpha T_{oo}(-z_i^{a\alpha} \tau)] \prod_{i=1}^3 \frac{i\eta}{\theta_1(z_i^{a\alpha} \tau)} \right. \\
& + I^{a\alpha'} [p_a q_\alpha T_{oo}(z_i^{a\alpha'} \tau) + \bar{p}_a \bar{q}_\alpha T_{oo}(-z_i^{a\alpha'} \tau)] \prod_{i=1}^3 \frac{i\eta}{\theta_1(z_i^{a\alpha'} \tau)} \\
& + S_g^{a\alpha} I_1^{a\alpha} [p_a \bar{q}_\alpha T_{og}(z_i^{a\alpha} \tau) + \bar{p}_a q_\alpha T_{og}(-z_i^{a\alpha} \tau)] \frac{i\eta}{\theta_1(z_1^{a\alpha} \tau)} \prod_{i=2,3} \frac{\eta}{\theta_2(z_i^{a\alpha} \tau)} \\
& - S_g^{a\alpha} I_1^{a\alpha'} [p_a q_\alpha T_{og}(z_i^{a\alpha'} \tau) + \bar{p}_a \bar{q}_\alpha T_{og}(-z_i^{a\alpha'} \tau)] \frac{i\eta}{\theta_1(z_1^{a\alpha'} \tau)} \prod_{i=2,3} \frac{\eta}{\theta_2(z_i^{a\alpha'} \tau)} \\
& - S_f^{a\alpha} I_2^{a\alpha} [p_a \bar{q}_\alpha T_{of}(z_i^{a\alpha} \tau) + \bar{p}_a q_\alpha T_{of}(-z_i^{a\alpha} \tau)] \frac{i\eta}{\theta_1(z_2^{a\alpha} \tau)} \prod_{i=1,3} \frac{\eta}{\theta_2(z_i^{a\alpha} \tau)} \\
& + S_f^{a\alpha} I_2^{a\alpha'} [p_a q_\alpha T_{of}(z_i^{a\alpha'} \tau) + \bar{p}_a \bar{q}_\alpha T_{of}(-z_i^{a\alpha'} \tau)] \frac{i\eta}{\theta_1(z_2^{a\alpha'} \tau)} \prod_{i=1,3} \frac{\eta}{\theta_2(z_i^{a\alpha'} \tau)} \\
& - S_h^{a\alpha} I_3^{a\alpha} [p_a \bar{q}_\alpha T_{oh}(z_i^{a\alpha} \tau) + \bar{p}_a q_\alpha T_{oh}(-z_i^{a\alpha} \tau)] \frac{i\eta}{\theta_1(z_3^{a\alpha} \tau)} \prod_{i=1,2} \frac{\eta}{\theta_2(z_i^{a\alpha} \tau)} \\
& \left. - S_h^{a\alpha} I_3^{a\alpha'} [p_a q_\alpha T_{oh}(z_i^{a\alpha'} \tau) + \bar{p}_a \bar{q}_\alpha T_{oh}(-z_i^{a\alpha'} \tau)] \frac{i\eta}{\theta_1(z_3^{a\alpha'} \tau)} \prod_{i=1,2} \frac{\eta}{\theta_2(z_i^{a\alpha'} \tau)} \right\}. \tag{562}
\end{aligned}$$

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