# Some applications of differential-difference algebra to creative telescoping 

Shaoshi Chen

## To cite this version:

Shaoshi Chen. Some applications of differential-difference algebra to creative telescoping. Symbolic Computation [cs.SC]. Ecole Polytechnique X, 2011. English. NNT : . pastel-00576861

## HAL Id: pastel-00576861 <br> https://pastel.hal.science/pastel-00576861

Submitted on 15 Mar 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## THĖSE

## présentée à

l'École Polytechnique

Pour obtenir le titre de Docteur en Sciences Spécialités: Informatique

Shaoshi ChEN

## Quelques applications de l'algèbre différentielle et aux différences pour le télescopage créatif

Soutenue le 16 février 2011 devant le jury composé de :

| William Y. C. ChEn, | Rapporteur, |
| ---: | :--- |
| Frédéric ChYZAK, | Directeur de thèse, |
| Joris VAN DER HOEVEN, | Examinateur, |
| Manuel KAUERS, | Rapporteur, |
| Ziming LI, | Directeur de thèse, |
| François MORAIN, | Examinateur, |
| Marko PETKOVŠEK, | Rapporteur, |
| Felix UlMER, | Examinateur. |

## Doctoral Thesis

presented to<br>l'École Polytechnique

For the French degree of Doctor in Sciences
Specialty: Computer Science

Shaoshi Chen

## Some applications of differential-difference algebra to creative telescoping

Defended on February 16, 2011 in front of the committee consisting of:

| William Y. C. Chen, | Reviewer, |
| ---: | :--- |
| Frédéric ChYZAK, | Thesis advisor, |
| Joris VAN DER HoEvEn, | Examiner, |
| Manuel KAUERS, | Reviewer, |
| Ziming Li, | Thesis advisor, |
| François MORAIN, | Examiner, |
| Marko PETKOVŠEK, | Reviewer, |
| Felix ULMER, | Examiner. |


#### Abstract

Since the 1990's, Zeilberger's method of creative telescoping has played an important role in the automatic verification of special-function identities. The long-term goal initiated in this work is to obtain fast algorithms and implementations for definite integration and summation in the framework of this method. Our contributions include new practical algorithms, complexity analyses of algorithms, and theoretical criteria for the termination of existing algorithms.

On the practical side, we present a new algorithm for computing minimal telescopers for bivariate rational functions. This algorithm is based on Hermite reduction. We also improve the classical Almkvist and Zeilberger's algorithm for rational-function inputs. The Hermitereduction based algorithm and improved Almkvist and Zeilberger's algorithm are analyzed in terms of field operations. Both complexity analysis and experimental results show that our algorithms are superior to other known ones for rational-function inputs.

On the theoretic side, we present a structure theorem for multivariate hyperexponentialhypergeometric functions. This theorem is based on (multivariate) Christopher's theorem for hyperexponential functions, the Ore-Sato theorem for hypergeometric terms, and our generalization of a recent result by Feng, Singer, and Wu on compatible bivariate continuous-discrete rational functions. The structure theorem allows us to decompose a hyperexponential-hypergeometric function as a product of a rational function, several exponential and power functions, and factorial terms. Furthermore, we derive two criteria for the existence of telescopers for bivariate hyperexponential-hypergeometric functions: one is with respect to the continuous variable, and the other with respect to the discrete one. The two criteria solve the termination problem of the continuous-discrete analogue of Zeilberger's algorithms.


## Acknowledgements

I would like to express my gratitude to my two co-supervisors: Frédéric Chyzak and Ziming Li, for their academic guidance and constant supports. Thanks go to Frédéric Chyzak for his motivated lessons on mathematics, English, and French with great patience. Thanks are due to Ziming Li for his long-term seminar in which I could report my understanding and misunderstanding of each sentence of Bronstein's book Symbolic Integration I.

Special thanks go to Alin Bostan from whom I learnt the precise knowledge on complexity analysis. He spent a lot of time in reading the draft of my thesis and offered me precise and constructive comments. Without him, I could not set up the "teagroup" in the Algorithms team. I am deeply grateful to Manuel Kauers for his constant encouragement and for vivid talks. He also impressed me with his wide interest and strong ability in communicating with others.

I also want to thank Philippe Flajolet and Bruno Salvy for running the Algorithms team in an atmosphere of freedom. I thank Philippe Flajolet for all the croissants. Bruno Salvy is always willing to listen to my questions and present me great answers, which has motivated my further work. I really appreciate all of my colleagues of the Algorithms team for their assistances, particularly, our secretary Virginie Collette for helping me with dealing with all kinds of problems in my daily life, Marc Mezzarobba for answering my questions on Linux, and Alexandre Benoit for moving my books from Roquencourt to Orsay and back.

I appreciate many members of the Key Laboratory of Mathematics Mechanization for their help, particularly, Wentsün Wu for his great influence on mathematics-mechanization, Lihong Zhi for making the meeting between my wife and me possible, and Xiao-shan Gao for recommending me to join the PhD Research Program between Algorithms and KLMM. I thank Ruyong Feng, Guofeng Fu, Chunming Yuan and many other "lab-mates" for their assistances.

I thank Peter Paule for inviting me to visit RISC-Linz where I presented my first seminar talk and made a lot of new friends. My thanks also go to Moulay Barkatou for inviting my wife and me to visit his university where I gave my first conference talk. I am very grateful to Bill Chen and Qinghu Hou for offering me many opportunities to present my work in Nankai University. My thanks also goes to Zhengyi Lu for many encouragements during my PhD study.

I thank George Andrews for sending me Payne's thesis as well as Garth Payne for the updated version. Thanks go to Doron Zeilberger for a motivated discussion about his method and for his
kind recommendation letter for my postdoctoral application. I am also very grateful to Michael Singer for offering me a postdoctoral position in the North Carolina State University.

I would like to express my gratitude to Bill Chen, Manuel Kauers, and Marko Petkovšek for agreeing to be the reviewers of this thesis and their valuable and constructive comments. My thanks also go to Joris van der Hoeven, François Morain, and Felix Ulmer for agreeing to serve as the examiners of this thesis.

Last but not least, I would like to thank my parents for their education and unconditional support, my uncles and their families in Paris for the hospitality, and my wife for her love, understanding and appreciation of my work.

This work was partially supported by INRIA Paris-Rocquencourt and KLMM. Two grants of the National Natural Science Foundation of China (no. 60821002/F02 and no. 76596100) financially supported my graduate study in Beijing and several international trips.

## Contents

1 Introduction ..... 9
1.1 Motivation ..... 10
1.2 Main results ..... 12
1.3 Outline ..... 13
1.4 Notation ..... 14
2 Preliminaries ..... 15
2.1 Differential and difference rings ..... 15
2.2 Ore polynomials ..... 16
2.3 Integration of rational functions ..... 17
2.3.1 Hermite reduction ..... 18
2.3.2 Ostrogradsky and Horowitz's method ..... 21
2.3.3 Residues and Rothstein-Trager resultants ..... 22
2.4 Background on complexity ..... 24
3 Hermite Reduction for Rational-Function Telescoping ..... 27
3.1 Introduction ..... 27
3.2 Notation and hypotheses ..... 29
3.3 Hermite reduction for bivariate rational functions ..... 30
3.3.1 Output size estimates ..... 30
3.3.2 Algorithm by evaluation and interpolation ..... 32
3.4 Minimal telescopers for bivariate rational functions ..... 34
3.4.1 Hermite-reduction based method ..... 34
3.4.2 Improved Almkvist and Zeilberger's method ..... 39
3.5 Non-minimal telescopers for bivariate rational functions ..... 43
3.6 Implementation and experiments ..... 45
3.6.1 Implementation and examples ..... 45
3.6.2 Experimental results ..... 47
4 Structure of Multivariate Hyperexponential-Hypergeometric Functions ..... 51
4.1 Introduction ..... 51
4.2 Algebraic setting ..... 52
4.2.1 Hyperexponential-hypergeometric functions ..... 52
4.2.2 First-order fully integrable systems ..... 54
4.3 Rational normal forms ..... 55
4.3.1 Differential and shift rational normal forms ..... 56
4.3.2 $\mathbb{Y}$-rational normal forms ..... 58
4.4 Structure of compatible rational functions ..... 63
4.4.1 The Ore-Sato theorem ..... 64
4.4.2 Alternative proof of multivariate Christopher's theorem ..... 65
4.4.3 Multivariate extension of Feng-Singer-Wu's lemma ..... 68
4.5 Multiplicative structure ..... 72
5 Existence of Telescopers for Hyperexponential-Hypergeometric Functions ..... 75
5.1 Introduction ..... 75
5.2 Preliminaries ..... 77
5.2.1 Ring of sequences ..... 77
5.2.2 Ring of differential-recurrence operators ..... 81
5.2.3 Split polynomials ..... 81
5.3 Existence problems ..... 82
5.4 Standard representations ..... 85
5.5 Two additive decompositions ..... 87
5.5.1 Additive decomposition with respect to $x$ ..... 87
5.5.2 Additive decomposition with respect to $n$ ..... 94
5.5.3 Applying operators to additive decompositions ..... 98
5.6 Two existence criteria ..... 100
5.6.1 Existence of telescopers implies properness ..... 100
5.6.2 Properness implies the existence of telescopers ..... 103
5.7 Algorithms and examples ..... 108
5.7.1 Algorithms ..... 108
5.7.2 Examples ..... 109
6 Conclusion and Perspectives ..... 111
6.1 Extensions of the Hermite-reduction based method ..... 111
6.2 Existence criteria for telescopers in the $q$-setting ..... 113
6.3 Wilf and Zeilberger's conjecture ..... 113

## Chapter 1

## Introduction

As most special-function integrals and sums cannot be expressed in closed form, their evaluation cannot be based on table look-ups only. Even when closed forms are available, they may prove to be intractable in further manipulations. In both cases, the difficulty can be mitigated by representing functions by annihilating differential and difference operators. This motivated Zeilberger to introduce a method now known as creative telescoping [95], which applies to a large class of special functions. Zeilberger's method has been used extensively in the automatic proofs of special-function identities.

Let us illustrate the basic idea of Zeilberger's method by proving the identity

$$
\begin{equation*}
\sum_{n=0}^{+\infty} f(x, n)=\frac{1}{\sqrt{1-4 x}}, \quad \text { where } f(x, n)=\binom{2 n}{n} x^{n} \text { and } x \in[0,1 / 4) . \tag{1.1}
\end{equation*}
$$

Let $D_{x}$ and $\Delta_{n}$ denote the derivation with respect to $x$, and the difference operator with respect to $n$, respectively. The first step is to find a differential equation satisfied by the sum on the left-hand side of (1.1). To this end, we try to find a nonzero operator $L \in \mathbb{Q}(x)\left\langle D_{x}\right\rangle$ such that

$$
\begin{equation*}
L\left(x, D_{x}\right)(f)=\Delta_{n}(g), \tag{1.2}
\end{equation*}
$$

where $g$ is a function in $x$ and $n$. Moreover, the method requires that the ratio between $g$ and $f$ is a rational function. For this specific example, we have

$$
L=2-(1-4 x) D_{x} \quad \text { and } \quad g=\frac{n}{x} \cdot f=n x^{n-1}\binom{2 n}{n} .
$$

Note that taking sums with respect to $n$ commutes with the application of $L\left(x, D_{x}\right)$ and, additionally, that

$$
\sum_{n=0}^{+\infty} \Delta_{n}(g)=0
$$

This implies that the sum on the left-hand side of (1.1) satisfies

$$
2 y(x)-(1-4 x) \frac{d y(x)}{d x}=0 .
$$

It is easy to verify that the function on the right-hand side of (1.1) also satisfies the above differential equation. Moreover, the identity holds when $x=0$. By Cauchy's theorem, the identity (1.1) holds for all well-defined values $x \in[0,1 / 4)$. The operator $L$ in (1.2) is called a telescoper for $f$ with respect to $n$, and $g$ the certificate of $L$ for $f$.

This thesis focuses on the bivariate case, in which there are four kinds of telescoping problems related to different integration and summation problems.

|  | $\int / \sum$ problems | Telescoping equations |
| :--- | :---: | :--- |
| (P1) | $\int_{a}^{b} f(x, y) d y$ | $L\left(x, D_{x}\right)(f)=D_{y}(g)$ |
| (P2) | $\int_{a}^{b} f(n, y) d y$ | $L\left(n, S_{n}\right)(f)=D_{y}(g)$ |
| (P3) | $\sum_{m=a}^{b} f(m, x)$ | $L\left(x, D_{x}\right)(f)=\Delta_{m}(g)$ |
| (P4) | $\sum_{m=a}^{b} f(n, m)$ | $L\left(n, S_{n}\right)(f)=\Delta_{m}(g)$ |

Table 1.1: Four telescoping problems

### 1.1 Motivation

Since the 1990's, Zeilberger's method has been extensively studied in the literature [95, 96, 15, $92,63,6,7,65,28,16]$. The main focus of those studies is on the efficiency and termination of creative-telescoping algorithms.

Related to efficiency, progress and improvements have since been made by Abramov [7], Apagodu [16], Chyzak [35, 33], Le [63, 46, 65], Takayama [85], etc. However, very little is known about the complexity of creative-telescoping algorithms. We believe that complexity analysis helps us understand algorithms, and indicates good ways of improving and implementing them.

This is one of the motivations for Chapter 3. The existing algorithms bind the construction of telescopers with certificates. In certain applications, only telescopers are needed. For bivariate rational-function inputs, our complexity analysis shows that the arithmetic size of certificates is asymptotically larger than that of telescopers. So the complexity could be lower if one can avoid the calculations for certificates in certain applications. This motivates us to find a way in which one could choose to compute or not to compute the certificates according to the user requirements.

By the functionality of Zeilberger-style algorithms, their termination is equivalent to the existence of telescopers. In his pioneering work [95], Zeilberger has proved that telescopers exist for holonomic functions. In particular, Wilf and Zeilberger [92] have presented an elementary and constructive proof of the existence of telescopers for proper hyperexponential-hypergeometric functions by basing on ideas of Fasenmyer [40] and Verbaeten [89]. By a proper hyperexponentialhypergeometric function, we mean a function that can be written as a product of a polynomial, exponential functions, power functions, and factorial terms. Properness can be detected from the certificates of a hyperexponential-hypergeometric function. Moreover, Wilf and Zeilberger conjectured in [92, page 585] that a hyperexponential-hypergeometric function is holonomic if and only if it is proper. If this conjecture were verified, then one could algorithmically detect the holonomicity of hyperexponential-hypergeometric functions. In the case of several discrete variables, a slightly modified version of the conjecture has been proved independently by Payne in his Ph.D. thesis [73] and by Abramov and Petkovšek [14]. In particular, the case of two variables has also been shown by Hou [54, 55] and by Abramov and Petkovšek [12]. However, holonomicity is only a sufficient condition for the existence of telescopers. In fact, Chyzak, Kauers, and Salvy [34] have listed certain classes of functions that are not holonomic but still have telescopers. Therefore, the challenge is to find theoretical criteria that enable us to algorithmically detect the existence of telescopers.

In view of the theoretical difficulty, special attention has been focused mainly on the subclass of hyperexponential-hypergeometric functions. In the continuous case, the works by Bernstein [17] and Lipshitz [67] show that every hyperexponential function has a telescoper. This implies that Zeilberger's algorithm always succeeds on those inputs. However, the situation in other cases turns out to be more involved. In the discrete case and its $q$-analogue, the first com-
plete solution to the termination problem is Abramov and Le's criterion [64, 9], which decides whether telescopers exist for a given bivariate rational function in the discrete variables $m$ and $n$. According to their criterion, the rational function

$$
f=\frac{1}{m^{2}+n^{2}}
$$

has no telescoper. Soon, the criterion was extended to the general case of bivariate hypergeometric terms by Abramov [5, 6]. Basically, Abramov proves that a hypergeometric term can be written as a sum of a hypergeometric-summable term and a proper one if it has a telescoper [6, Theorem 10]. Similar results have been obtained in the general $q$-shift case by Chen, Hou and Mu [28]. These results are fundamental for predicting the termination of Zeilberger's algorithm.

A continuous-discrete analogue of Zeilberger's algorithm was presented by Almkvist and Zeilberger [15]. This analogue has been shown to be very useful in the study of orthogonal polynomials [60, Chapters 10-13]. In this setting, not all hyperexponential-hypergeometric functions have telescopers. For example, we will show in this thesis that the rather simple rational function in the continuous variable $x$ and the discrete variable $n$

$$
f=\frac{1}{x+n}
$$

has no telescoper with respect to either the continuous variable $x$ or the discrete variable $n$. Therefore, an existence criterion is also needed in this mixed setting.

### 1.2 Main results

For a bivariate rational function, we present a new algorithm to compute its minimal telescoper, which is based on Hermite reduction. We also obtain some improvements over the classical algorithm by Almkvist and Zeilberger by extending the idea of Geddes and Le to general rationalfunction inputs. Moreover, we give the first proof of a polynomial complexity (in terms of field operations) for creative telescoping on this specific class of inputs. Our algorithms are proved to be faster concerning both theoretical complexity and actual performance.

Motivated by a conjecture of Wilf and Zeilberger [92, page 585], we study the possible form of a multivariate hyperexponential-hypergeometric function. For such a function, we prove a structure theorem for its certificates, which generalizes a recent result by Feng, Singer, and

Wu [42, Proposition 5]. Combining our result with the result on multivariate hyperexponential functions in [29, 97] and the Ore-Sato theorem on multivariate hypergeometric terms, we obtain a structure theorem for multivariate hyperexponential-hypergeometric functions, which says that a multivariate hyperexponential-hypergeometric function can be written as a product of a rational function, an exponential function, power functions and factorial terms.

With the help of the result by Feng, Singer, and Wu [42, Proposition 5], we derive two criteria for the existence of telescopers for bivariate hyperexponential-hypergeometric functions. We show that a hyperexponential-hypergeometric function can be written as a sum of a hypergeometricsummable, resp. hyperexponential-integrable, function and a proper one if it has a telescoper with respect to the discrete, resp. continuous, variable. Our criteria are based on standard representations and the two adapted additive decompositions. With them, we can decide in advance the termination of the continuous-discrete analogue of Zeilberger's algorithm for bivariate hyperexponential-hypergeometric functions.

### 1.3 Outline

In this section, we provide the reader with an outline of this thesis.
Chapter 2. We recall basic notation and facts on differential rings, difference rings and Ore polynomials. Our new algorithm for the construction of minimal telescopers will be based on the Hermite reduction for the integration of the rational function. So we review the classical algorithms for rational-function integration, including Hermite reduction, Ostrogradsky-Horowitz's method, and Rothstein-Trager's algorithm. We also summarize some complexity results for later use.

Chapter 3. We focus on deriving a fast algorithm for computing minimal telescopers for bivariate rational functions. First, we present an optimal algorithm for the Hermite reduction over $k(x)$ by fast evaluation-interpolation strategy. Second, we present a new algorithm for computing the minimal telescoper of a bivariate rational function, based on a bivariate extension of Hermite reduction. Moreover, some improvements over the classical method by Almkvist and Zeilberger [15] are achieved in this chapter. We also analyze the arithmetic complexity of those algorithms. Third, we adapt and slightly extend the arguments by Lipshitz [67] and by

Bostan and others [20] to derive smaller total degree sizes of telescopers. At last, we describe our implementation and show experimental results.

Chapter 4. We first present an algebraic setting for hyperexponential-hypergeometric functions. After that, we review various normal forms of rational functions and introduce a new kind of rational normal forms, which enables us to extend the existing results to multivariate continuousdiscrete setting. Our main result is Theorem 4.4.6 on the structure of certificates of a multivariate hyperexponential-hypergeometric function. This result is a generalization of a result by Feng, Singer, and Wu [42, Proposition 5]. At last, we describe a multiplicative form of multivariate hyperexponential-hypergeometric functions.

Chapter 5. We study the existence of telescopers for hyperexponential-hypergeometric functions of one continuous variable and one discrete variable. First, we review the construction of a ring of sequences from [41, 43], which allows us to study the existence problem in an algebraic manner. After that, we introduce standard representations for bivariate hyperexponential-hypergeometric functions and then adapt two additive decompositions [13, 47] to bivariate hyperexponentialhypergeometric functions described by their standard representations. At last, we describe our existence criteria and its algorithmic description with examples.

Chapter 6. We present some conclusions and propose some topics for future work.

### 1.4 Notation

We shall use the letters $\mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{Z}$ to denote the set of complex numbers, non-negative integers, rational numbers, real numbers, and integers, respectively. For a field $F$, we denote its algebraic closure by $\bar{F}$.

## Chapter 2

## Preliminaries

In this chapter, we recall basic notation and facts on differential and difference rings and Ore polynomials from $[56,21,70]$. Moreover, we review classical techniques for rational-function integration and background on complexity analysis.

### 2.1 Differential and difference rings

Let $R$ be a commutative ring. A map $\delta: R \rightarrow R$ is said to be a derivation on $R$ if

$$
\delta(a+b)=\delta(a)+\delta(b), \quad \delta(a b)=\delta(a) b+a \delta(b) \quad \text { for all } a, b \in R
$$

The pair $(R, \delta)$ is called a differential ring. Moreover, if $R$ is a field, $R$ is called a differential field. An element $c \in R$ is called a constant with respect to $\delta$ if $\delta(c)=0$. All constants with respect to $\delta$ form a subring of $R$, denoted by $C_{\delta, R}$. If $R$ is a field, $C_{\delta, R}$ is also a field. Some basic facts concerning derivations are collected in the next lemma whose proof can be found in any book about differential algebra, e.g. [21]

Lemma 2.1.1. Let $(R, \delta)$ be a differential ring. For $a, b \in R$ and $n \in \mathbb{N}$, we have
(i) $\delta(1)=0$;
(ii) $\delta\left(a^{n}\right)=n a^{n-1} \delta(a)$;
(iii) If $b$ is invertible, then

$$
\delta\left(\frac{a}{b}\right)=\frac{\delta(a) b-a \delta(b)}{b^{2}}
$$

(iv) Logarithmic derivative identity: if $a$ and $b$ are invertible, then

$$
\frac{\delta\left(a^{m} b^{n}\right)}{a^{m} b^{n}}=m \frac{\delta(a)}{a}+n \frac{\delta(b)}{b} \quad \text { for all } m, n \in \mathbb{Z}
$$

Let $\sigma: R \rightarrow R$ be a monomorphism on $R$. Then the pair $(R, \sigma)$ is called a difference ring. Moreover, if $R$ is a field, $R$ is called a difference field. An element $c \in R$ is called a constant with respect to $\sigma$ if $\sigma(c)=c$. All constants with respect to $\sigma$ form a subring of $R$, denoted by $C_{\sigma, R}$. If $R$ is a field, $C_{\sigma, R}$ is also a field. The triple $(R, \delta, \sigma)$ is called a differential-difference ring. $(R, \delta, \sigma)$ is said to be orthogonal [62] if the derivation $\delta$ commutes with the monomorphism $\sigma$. Unless otherwise specified, all fields in the thesis are of characteristic zero and all differentialdifference rings are orthogonal.

### 2.2 Ore polynomials

Ore polynomials are a common abstraction for linear differential operators, linear difference operators, shifts, and $q$-shifts $[70,23]$. Let $(F, \delta, \sigma)$ be a differential-difference field. The ring of Ore polynomials $F[x ; \delta, \sigma]$ over $F$ is the univariate polynomial ring with indeterminate $x$, addition defined as usual, and multiplication being associative, distributive, and satisfying the commutation rule:

$$
\begin{equation*}
x f=\sigma(f) x+\delta(f), \quad \text { for any } f \in F . \tag{2.1}
\end{equation*}
$$

By the commutation rule (2.1), the product of two polynomials in $F[x ; \delta, \sigma]$ can be decided by association and distribution.

Example 2.2.1. Let $F(t)$ be the univariate rational-function field in $t$ over $F$.
(i) The usual polynomial ring $F(t)[x]$ on which $\delta$ is the zero map and $\sigma$ is the identity map.
(ii) The ring of differential operators $F(t)[x ; d / d t, 1]$, where 1 is the identical map.
(iii) The ring of recurrence operators $F(t)[x ; 0, \sigma]$, where $\sigma(f(t))=f(t+1)$ for any $f \in F(t)$.
(iv) The ring of difference operators $F(t)[x ; \Delta, \sigma]$, where $\Delta=\sigma-1$.

In this thesis, we also denote the ring of differential operators over $F(t)$ by $F(t)\left\langle D_{t}\right\rangle$ and the ring of difference operators over $F(t)$ by $F(t)\left\langle S_{t}\right\rangle$.

If $\sigma$ is an automorphism, we can perform both left and right Euclidean division on the ring $F[x ; \delta, \sigma][70]$, that is, for any $f, g \in F[x ; \delta, \sigma]$, there exist $q, r \in F[x ; \delta, \sigma]$ such that $f=q g+r$ or $f=g q+r$, where $\operatorname{deg}(r)<\operatorname{deg}(g)$. This implies that $F[x ; \delta, \sigma]$ is a left and right principal ideal domain. A multivariate extension of Ore polynomials is studied in [35, 32, 93].

### 2.3 Integration of rational functions

Symbolic evaluation of integrals is an active research domain of computer algebra, which revives the study of the integration problems from the algorithmic point of view. The first landmark algorithm, designed by Risch [77, 78] in the late 1960's, can decide whether indefinite integrals of elementary functions are elementary or not. State of the art about integrating transcendental functions has been presented in Bronstein's book [21]. For later use, we review some basic methods for rational-function integration. For more intensive presentations, see the books [21, Chapter 2], [45, Chapter 11], and [90, Chapter 22].

In this section, let $F$ be a field of characteristic zero and $F(x)$ be the rational-function field in $x$ over $F$. On the field $F(x)$, the derivation $\delta$ is defined by setting $\delta(x)=1$ and $\delta(c)=0$ for all $c \in F$. In most undergraduate calculus textbooks, we can find that any rational function $f \in$ $\mathbb{C}(x)$ over the field $\mathbb{C}$ of complex numbers has an elementary integral of the form

$$
\begin{equation*}
\int f d x=g+\sum_{i=1}^{n} \beta_{i} \log \left(x-\alpha_{i}\right) \tag{2.2}
\end{equation*}
$$

where $g \in \mathbb{C}(x)$ and $\alpha_{i}, \beta_{i} \in \mathbb{C}$ for $i=1, \ldots, n$. Here, we call $g$ the rational part of the integral and the sum of logarithms the logarithmic part of the integral. In this analytic setting when the ground field is the algebraically closed field $\mathbb{C}$, the problem of rational function integration is well-understood. However, the algorithmic question becomes more involved when the ground field is not algebraically closed. Over an arbitrary field $F$, one may need to compute objects over an algebraic extension of $F$. In this case, the method in calculus books has many practical difficulties. For any rational function $f \in F(x)$, it has been shown by Hermite [52] that the rational part of the integral $\int f d x$ still lies in $F(x)$ and he presented a method to compute this part without introducing any algebraic extension of $F$. His method is named Hermite reduction in [21, 45, 90]. For the logarithmic part, in general it is required to introduce algebraic extensions of $F$ and then the question is how to compute the minimal algebraic extension of $F$ that is
sufficient to express the integral. This question has been independently solved by Trager [87] and Rothstein [80].

### 2.3.1 Hermite reduction

Hermite reduction is a practical method for computing the rational part, which uses the method of integration by parts in order to reduce the denominator of the integrand to a square-free polynomial. More explicitly, Hermite reduction decomposes $f \in F(x)$ into

$$
\begin{equation*}
f=\delta(g)+\frac{a}{b}, \tag{2.3}
\end{equation*}
$$

where $g \in F(x)$ and $a, b \in F[x]$ with $\operatorname{deg}(a)<\operatorname{deg}(b)$ and $b$ squarefree. Such a pair $(g, a / b)$ in (2.3) is called an additive decomposition for $f$ (with respect to $x$ ). The rational functions $g$ and $a / b$ are called the rational and logarithmic parts of $f$, respectively.

We summarize the idea of Hermite reduction following the treatment in [21, Chapter 2.2]. Write the integrand as $f=A / D$ with $A, D \in F[x]$ and $\operatorname{gcd}(A, D)=1$. Let $D=D_{1} D_{2}^{2} \ldots D_{n}^{n}$ be the squarefree factorization of $D$. By computing the partial fraction decomposition for $f$, we have

$$
\begin{equation*}
f=P+\sum_{i=1}^{n} \frac{A_{i}}{D_{i}^{i}} \tag{2.4}
\end{equation*}
$$

where $P$ and the $A_{i}$ 's are in $F[x]$ and $\operatorname{deg}\left(A_{i}\right)<\operatorname{deg}\left(D_{i}^{i}\right)$ for each $i$. By the linearity of $\delta$, it is sufficient to consider the same decomposition problem for any fraction of the form

$$
\frac{P}{Q^{m}} \in F[x], \quad \text { with } m>1, \operatorname{deg}(P)<m \operatorname{deg}(Q) \text { and } Q \text { squarefree. }
$$

The square-freeness of $Q$ implies the existence of a Bézout relation

$$
\begin{equation*}
S \delta(Q)+T Q=1 \tag{2.5}
\end{equation*}
$$

where $S, T \in F[x]$ can be obtained by the extended Euclidean algorithm. Furthermore, upon multiplication and division with remainder, we get

$$
\begin{equation*}
P=S P \delta(Q)+T P Q=(G Q+B) \delta(Q)+T P Q=B \delta(Q)+C Q \tag{2.6}
\end{equation*}
$$

where $C=G \delta(Q)+T P$ with $\operatorname{deg}(C)<(m-1) \operatorname{deg}(Q)$. Now, integration by parts yields

$$
\begin{equation*}
\frac{P}{Q^{m}}=\frac{B \delta(Q)+C Q}{Q^{m}}=\delta\left(\frac{-(m-1)^{-1} B}{Q^{m-1}}\right)+\frac{C+(m-1)^{-1} \delta(B)}{Q^{m-1}} \tag{2.7}
\end{equation*}
$$

where $\operatorname{deg}\left(C+(m-1)^{-1} \delta(B)\right)<(m-1) \operatorname{deg}(Q)$ since $\operatorname{deg}(B)<\operatorname{deg}(Q)$ and $\operatorname{deg}(C)<(m-$ $1) \operatorname{deg}(Q)$. This process is repeated until the denominator is square-free.

In 1975, Mack [69] introduced a variant of Hermite reduction that requires neither partial fraction decomposition nor squarefree factorization, but only extended GCD computation.

The following lemma due to Ostrogradsky [71] shows the uniqueness (up to an additive constant) of additive decompositions. For proofs, one can follow the argument in [45, Theorem 11.4].

Lemma 2.3.1. Let $f=a / b$ be a nonzero rational function in $F(x)$ such that $a, b \in F[x]$, $\operatorname{gcd}(a, b)=1, \operatorname{deg}(a)<\operatorname{deg}(b)$, and $b$ is squarefree. Then there is no $g \in F(x)$ such that $f=\delta(g)$.

Corollary 2.3.2. Let $f$ be a rational function in $F(x)$. Then the pair ( $g, a / b$ ) satisfying (2.3) for $f$ is unique up to adding a constant to $g$.

Proof. Assume that $\left(g_{1}, a_{1} / b_{1}\right)$ and $\left(g_{2}, a_{2} / b_{2}\right)$ are two additive decompositions for $f$ with $a_{1} / b_{1} \neq$ $a_{2} / b_{2}$. The difference of logarithmic parts can be written as

$$
0 \neq a_{1} / b_{1}-a_{2} / b_{2}=\frac{A}{B}, \quad \text { where } A, B \in F[x], \operatorname{gcd}(A, B)=1, \text { and } \operatorname{deg}(A)<\operatorname{deg}(B) .
$$

Since both $b_{1}$ and $b_{2}$ are squarefree and $B$ divides $\operatorname{lcm}\left(b_{1}, b_{2}\right), B$ is also squarefree. However,

$$
\frac{A}{B}=\delta\left(g_{2}-g_{1}\right),
$$

which contradicts Lemma 2.3.1. So $a_{1} / b_{1}$ equals $a_{2} / b_{2}$ and then $g_{1}$ and $g_{2}$ differ by an additive constant.

For later use, we recall a fact on the logarithmic derivatives of rational functions.

Lemma 2.3.3. Let $f=a / b \in F(x)$ be such that $a, b \in F[x]$ and $\operatorname{gcd}(a, b)=1$, then

$$
\frac{\delta f}{f}=\frac{p}{a^{*} b^{*}},
$$

where $a^{*}$ and $b^{*}$ are, respectively, the squarefree parts of $a$ and $b$, and $p \in F[x]$ with $\operatorname{deg}(p)<$ $\operatorname{deg}\left(a^{*} b^{*}\right)$ and $\operatorname{gcd}\left(p, a^{*} b^{*}\right)=1$.

Proof. Let $a=a_{1} a_{2}^{2} \cdots a_{m}^{m}$ and $b=b_{1} b_{2}^{2} \cdots b_{n}^{n}$ be the squarefree factorizations of $a$ and $b$, respectively. Then $a^{*}=a_{1} a_{2} \cdots a_{m}$ and $b^{*}=b_{1} b_{2} \cdots b_{n}$. By Lemma 2.1.1 (iv), we have

$$
\frac{\delta f}{f}=\frac{p}{a^{*} b^{*}}, \quad \text { where } \quad p=b^{*} \sum_{i=1}^{m} \frac{i \delta\left(a_{i}\right) a^{*}}{a_{i}}-a^{*} \sum_{j=1}^{n} \frac{j \delta\left(b_{j}\right) b^{*}}{b_{i}} \in F[x] .
$$

It is easy to see that $\operatorname{deg}(p)<\operatorname{deg}\left(a^{*} b^{*}\right)$. Since the $a_{i}$ 's and $b_{j}$ 's are squarefree and pairwise coprime, we have

$$
\operatorname{gcd}\left(a^{*}, \sum_{i=1}^{m} \frac{i \delta\left(a_{i}\right) a^{*}}{a_{i}}\right)=\operatorname{gcd}\left(b^{*}, \sum_{j=1}^{n} \frac{j \delta\left(b_{j}\right) b^{*}}{b_{i}}\right)=1,
$$

which further implies $\operatorname{gcd}\left(p, a^{*} b^{*}\right)=1$. This completes the proof.
The following lemma will be useful to verify uniqueness properties, in particular, in the proofs of Theorem 4.4.4 and Lemma 4.3.10.

Lemma 2.3.4. Assume that $f$ is a rational function in $F(x), p_{1}, \ldots, p_{n}$ are pairwise coprime polynomials in $F[x]$, and $c_{1}, \ldots, c_{n}$ are constants in $F$. If

$$
\delta(f)=\sum_{i=1}^{n} c_{i} \frac{\delta\left(p_{i}\right)}{p_{i}},
$$

then $f \in F$ and either $c_{i}=0$ or $p_{i} \in F$ for all $i$ such that $1 \leq i \leq n$.
Proof. Let $p_{i}^{*}$ be the squarefree part of $p_{i}$ for all $i$ such that $1 \leq i \leq n$. By Lemma 2.3.3, we have

$$
\sum_{i=1}^{n} c_{i} \frac{\delta\left(p_{i}\right)}{p_{i}}=\sum_{i=1}^{n} c_{i} \frac{q_{i}}{p_{i}^{*}}=: \frac{a}{b},
$$

where $a, b \in F[x]$ with $\operatorname{gcd}(a, b)=1$ and the $q_{i}$ 's are polynomials in $F[x]$ with $\operatorname{gcd}\left(q_{i}, p_{i}^{*}\right)=$ 1 and $\operatorname{deg}\left(q_{i}\right)<\operatorname{deg}\left(p_{i}^{*}\right)$. Since the $p_{i}$ 's are pairwise coprime, so are the $p_{i}^{*}$ 's. Hence, the denominator $b$ is squarefree and $\operatorname{deg}(a)<\operatorname{deg}(b)$. By Lemma 2.3.1, $a / b$ must be equal to zero and then $f \in F$. By the uniqueness of squarefree partial fraction decomposition, all the fractions $c_{i} q_{i} / p_{i}^{*}$ are equal to zero. This implies that either $c_{i}=0$ or $p_{i} \in F$ for each $i$ such that $1 \leq i \leq n$.

Corollary 2.3.5. Let $c_{1}, \ldots, c_{n} \in F$ be linearly independent over $\mathbb{Z}$. If there exist rational functions $f_{1}, \ldots, f_{n} \in F(x)$ such that

$$
\sum_{i=1}^{n} c_{i} \frac{\delta\left(f_{i}\right)}{f_{i}}=0
$$

then $f_{1}, \ldots, f_{n}$ belong to $F$.

Proof. Since every element of $F$ is a constant with respect to $\delta$, we may suppose that none of $f_{1}, \ldots, f_{n}$ belongs to $F$, and look for a contradiction. Under this assumption, the sum can be rewritten as

$$
\sum_{j=1}^{m} \bar{c}_{j} \frac{\delta\left(p_{j}\right)}{p_{j}}=0
$$

where $\bar{c}_{j}$ is a $\mathbb{Z}$-linear combination of $c_{1}, \ldots, c_{n}$, and the $p_{j}$ 's are nontrivial distinct irreducible factors of the denominators of the $\delta\left(f_{i}\right) / f_{i}$. Since every $\bar{c}_{j}$ is nonzero, Lemma 2.3.4 implies that every $p_{j}$ is in $F$, which is a contradiction.

### 2.3.2 Ostrogradsky and Horowitz's method

Ostrogradsky and Horowitz's method [71,53] computes the additive decomposition of a rational function by solving a linear system. Although this method has asymptotically higher complexity than that of Hermite reduction [90, Section 22.2], it is useful for our complexity analyses in the sequel.

Let $f=P / Q \in F(x)$ be such that $P, Q \in F[x]$ and $\operatorname{gcd}(P, Q)=1$. After reading out the polynomial part of $f$, we further assume that $\operatorname{deg}(P)<\operatorname{deg}(Q)$. Let $Q^{*}$ be the squarefree part of $Q$ and $Q^{-}=Q / Q^{*}$. Denote $d_{x}^{*}=\operatorname{deg}_{x}\left(Q^{*}\right)$ and $d_{x}^{-}=\operatorname{deg}_{x}\left(Q^{-}\right)$. According to the functionality of Hermite reduction, one can find two unique polynomials $A$ and $a$ in $F[x]$ with $\operatorname{deg}_{x} A<\operatorname{deg}\left(Q^{-}\right)$ and $\operatorname{deg}_{x} a<\operatorname{deg}\left(Q^{*}\right)$ such that

$$
\begin{equation*}
\frac{P}{Q}=\delta\left(\frac{A}{Q^{-}}\right)+\frac{a}{Q^{*}} \tag{2.8}
\end{equation*}
$$

Note that $A$ and $a$ satisfy (2.8) if and only if they satisfy the equation

$$
\begin{equation*}
P=\delta(A) Q^{*}-A \widetilde{Q}+a Q^{-} \tag{2.9}
\end{equation*}
$$

where $\widetilde{Q}=Q^{*} \delta\left(Q^{-}\right) / Q^{-}$is a polynomial in $F[x]$ of degree $\operatorname{deg}\left(Q^{*}\right)-1$. Now, equation (2.9) can be solved by the method of undetermined coefficients. Write $P=\sum_{l=0}^{d_{x}^{*}+d_{x}^{-}-1} P_{l} x^{l}$ and set $A=\sum_{i=0}^{d_{x}^{-}-1} A_{i} x^{i}, a=\sum_{j=0}^{d_{x}^{*}-1} a_{j} x^{j}$ with undetermined coefficients $A_{i}$ and $a_{j}$. Then (2.9) holds if and only if

$$
\begin{equation*}
\left(A_{d_{x}^{-}-1}, \ldots, A_{0}, a_{d_{x}^{*}-1}, \ldots, a_{0}\right) M=\left(P_{d_{x}^{*}+d_{x}^{-}-1}, \ldots, P_{0}\right), \tag{2.10}
\end{equation*}
$$

where $M$ is a $\operatorname{deg}(Q) \times \operatorname{deg}(Q)$ matrix over $F$ obtained by equating the likewise powers of $x$
in (2.9). By the uniqueness of $A / Q^{-}$and $a / Q^{*}$, the system (2.10) has a unique solution, which leads to the following lemma.

Lemma 2.3.6. The matrix $M$ in (2.10) is invertible over $F$.
We call the linear system (2.10) the Ostrogradsky-Horowitz system and $M$ the OstrogradskyHorowitz matrix associated with $Q$. Note that $M$ is uniquely determined by $Q$. So we denote this matrix by $\mathcal{M}(Q)$.

### 2.3.3 Residues and Rothstein-Trager resultants

After additive decomposition, the integration problem of rational functions is reduced to computing the integrals of the form

$$
\begin{equation*}
\int \frac{a}{b} d x, \quad \text { where } a, b \in F[x] \text { with } \operatorname{gcd}(a, b)=1, \operatorname{deg}(a)<\operatorname{deg}(b) \text { and } b \text { squarefree. } \tag{2.11}
\end{equation*}
$$

Over the algebraic closure $\bar{F}$ of $F$, the rational function $a / b$ above decomposes into

$$
\frac{a}{b}=\sum_{i=1}^{n} \frac{\beta_{i}}{x-\alpha_{i}}, \quad \text { where } \alpha_{i}, \beta_{i} \in \bar{F} \text { and } b\left(\alpha_{i}\right)=0 \text { for } 1 \leq i \leq n .
$$

Consequently, the integral of $a / b$ can be expressed by

$$
\int \frac{a}{b} d x=\sum_{i=1}^{n} \beta_{i} \log \left(x-\alpha_{i}\right)
$$

By convention, the value $\beta_{i}$ is called the residue of $a / b$ at the point $x=\alpha_{i}$. According to the Lagrange formula ([44, page 38] or [45, Exercise 11.8]), the residue of $a / b$ at $\alpha_{i}$ is

$$
\begin{equation*}
\beta_{i}=\frac{a}{\delta(b)}\left(\alpha_{i}\right) \in F\left(\alpha_{i}\right) \tag{2.12}
\end{equation*}
$$

So we can always express the integrals in (2.11) over the splitting field of the denominator $b$. However, it is not necessary to compute splitting fields for obtaining the integrals. For instance, the integral below can be expressed without any algebraic extension:

$$
\int \frac{2 x}{x^{2}-2} d x=\log (x+\sqrt{2})+\log (x-\sqrt{2})=\log \left(x^{2}-2\right)
$$

In fact, Rothstein [80] and Trager [87] have shown that the minimal algebraic extension of $F$ for expressing the integrals is the splitting field of the following polynomial

$$
\begin{equation*}
R(z)=\operatorname{resultant}_{x}(b, a-z \cdot \delta(b)) \in F[z] \tag{2.13}
\end{equation*}
$$

In the literature, $R(z)$ above is called the Rothstein-Trager resultant of $a / b$ with respect to $x$, denoted henceforth by $\operatorname{RT}_{x}(a / b)$. By the formula (2.12), all the residues of $a / b$ at its poles are the roots of $R(z)$. Moreover, we have the following lemma, which appears implicitly in the literature [45, 21, 90].

Lemma 2.3.7. Let $a, b \in F[x]$ be such that $\operatorname{deg}(a)<\operatorname{deg}(b), \operatorname{gcd}(a, b)=1$ and $b$ is squarefree in $F[x]$. Let $R(z)$ be the Rothstein-Trager resultant of $a / b$ with respect to $x$. Then we have
(i) all roots of $R(z)$ are nonzero;
(ii) if $\alpha_{1}$ and $\alpha_{2}$ are two distinct roots of $R(z)$, then $p_{1}$ and $p_{2}$ are coprime over $\bar{F}$, where

$$
p_{i}=\operatorname{gcd}\left(b, a-\alpha_{i} \delta(b)\right) \in F\left(\alpha_{i}\right)[x], \quad \text { for } i=1,2 .
$$

Proof. The first assertion follows from the fact that $\operatorname{gcd}(a, b)=1$. For the second one, we suppose $p_{1}$ and $p_{2}$ are not coprime over $\bar{F}$. Then there exists $\beta \in \bar{F}$ such that $p_{1}(\beta)=p_{2}(\beta)=0$. By the definition of $p_{1}$ and $p_{2}, b(\beta)=0$ and $\left(a-\alpha_{i} \delta(b)\right)(\beta)=0$ for $i=1,2$. Since $b$ is squarefree in $F[x], \delta(b)(\beta) \neq 0$, which implies

$$
\alpha_{1}=\alpha_{2}=\frac{a}{\delta b}(\beta),
$$

which is a contradiction with $\alpha_{1} \neq \alpha_{2}$.
The next theorem is fundamental to design the algorithm for computing the logarithmic part of the integral of a rational function. For proofs, see [21, Theorem 2.4.1] or [90, Theorem 22.8].

Theorem 2.3.8 (Rothstein and Trager's theorem). Let $a, b \in F[x]$ be such that $\operatorname{deg}(a)<\operatorname{deg}(b)$, $\operatorname{gcd}(a, b)=1$ and $b$ is squarefree in $F[x]$. Let $R(z)$ be the Rothstein-Trager resultant of $a / b$ with respect to $x$. Then the integral of rational function $a / b$ can be expressed as

$$
\int \frac{a}{b} d x=\sum_{i=1}^{n} c_{i} \log \left(g_{i}\right),
$$

where $c_{1}, \ldots c_{n} \in \bar{F}$ are the distinct roots of $R(z)$ and $g_{i}=\operatorname{gcd}\left(b, a-c_{i} \delta(b)\right) \in F\left(c_{i}\right)[x]$ for all $1 \leq i \leq n$. Moreover, if $E$ is an algebraic extension of $F$ such that we have

$$
\int \frac{a}{b} d x=\sum_{i=1}^{m} \tilde{c}_{i} \log \left(\tilde{g}_{i}\right),
$$

where $\tilde{c}_{1}, \ldots, \tilde{c}_{m} \in E \backslash\{0\}$ and $\tilde{g}_{1}, \ldots, \tilde{g}_{m} \in E[x] \backslash E$ are monic, squarefree, and pairwise coprime, then $E$ contains all the roots of $R(z)$.

We conclude this section by the following theorem on the integration of rational functions.

Theorem 2.3.9. Let $f$ be a nonzero element in $F(x)$. Then there exist $g \in F(x)$, nonzero elements $c_{j} \in \bar{F}$ and $p_{j} \in F\left(c_{j}\right)[x] \backslash F\left(c_{j}\right)$ for $j=1, \ldots, n$ such that

$$
f=\delta(g)+\sum_{j=1}^{n} c_{j} \frac{\delta\left(p_{j}\right)}{p_{j}}
$$

Moreover, $p_{j}$ and $p_{j^{*}}$ are coprime over $\bar{F}$ whenever $j \neq j^{*}$.
Proof. By Hermite reduction, there exist $f \in F(x), a, b \in F[x]$ with $\operatorname{deg} a<\operatorname{deg} b, \operatorname{gcd}(a, b)=1$ and $b$ being squarefree such that

$$
f=\delta(g)+\frac{a}{b} .
$$

If $a$ is zero, then there is nothing to prove. Assume that $a$ is nonzero and let $\Lambda$ be the set of distinct roots of the Rothstein-Trager resultant $R(z)$ of $a / b$. By Lemma 2.3.7 (i), all elements in $\Lambda$ are nonzero. By Theorem 2.3.8,

$$
f=\delta(g)+\sum_{\lambda \in \Lambda} \lambda \frac{\delta\left(p_{\lambda}\right)}{p_{\lambda}}
$$

where $p_{\lambda}=\operatorname{gcd}(b, a-\lambda \delta b) \in F(\lambda)[x] \backslash F(\lambda)$. By Lemma $2.3 .7(i i), p_{\lambda}$ and $p_{\mu}$ are coprime if $\lambda$ and $\mu$ are two distinct elements of $\Lambda$. This completes the proof.

### 2.4 Background on complexity

In this section, we recall basic notation and complexity results from [90]. These results are useful to analyze algorithms in Chapter 3. The complexity results are expressed using the "big Oh" notation [90, Section 25.7]. For instance, we say that the Karatsuba algorithm for multiplying two polynomials in $\mathbb{Q}[x]$ of degree at most $n$ takes $\mathcal{O}\left(n^{\log _{2} 3}\right)$ arithmetic operations in $\mathbb{Q}$. Furthermore, we use the notation $\tilde{\mathcal{O}}(\cdot)$ to indicate cost estimates with hidden logarithmic factors [90, Definition 25.8].

Let $k$ be a field of characteristic zero. Unless otherwise specified, all complexity estimates are given in terms of arithmetic operations in $k$, which we denote by "ops". Let $k[x]_{\leq d}^{m \times n}$ be the set of $m \times n$ matrices with coefficients in $k[x]$ of degree at most $d$. Let $\omega \in[2,3]$ be a feasible exponent of matrix multiplication, so that two matrices from $k^{n \times n}$ can be multiplied using $\mathcal{O}\left(n^{\omega}\right)$ ops.

Facts 2.4.1 and 2.4.2 below show the complexity of multi-point evaluation, rational interpolation, and algebraic operations on polynomial matrices using fast arithmetic. For proofs, see [90, Corollaries 10.8, 5.18, 11.6] and [84, Theorem 7.3].

Fact 2.4.1. For a polynomial $p \in k[x]$ of degree less than $n$, pairwise distinct $u_{0}, \ldots, u_{n-1}$ in $k$, and $v_{0}, \ldots, v_{n-1} \in k$, we have:
(i) Fast evaluation: evaluating $p$ at the $u_{i}$ 's takes $\tilde{\mathcal{O}}(n)$ ops.
(ii) Fast interpolation: for $m \in\{1, \ldots, n\}$, constructing $f=s / t \in k(x)$ with $\operatorname{deg}_{x}(s)<m$ and $\operatorname{deg}_{x}(t) \leq n-m$ such that $t\left(u_{i}\right) \neq 0$ and $f\left(u_{i}\right)=v_{i}$ for $0 \leq i \leq n-1$ takes $\tilde{\mathcal{O}}(n)$ ops.

Fact 2.4.2. For $M$ in $k[x]_{\leq d}^{m \times n}$ with $d>0$, we have:
(i) If $M=\left(\begin{array}{ll}M_{1} & M_{2}\end{array}\right)$ is an invertible $n \times n$ matrix with $M_{i} \in k[x]_{\leq d_{i}}^{n \times n_{i}}$, where $i=1,2$ and $n_{1}+n_{2}=n$, then the degree of $\operatorname{det}(M)$ is at most $n_{1} d_{1}+n_{2} d_{2}$.
(ii) If $M=\left(\begin{array}{ll}M_{1} & M_{2}\end{array}\right)$ is not of full rank and with $M_{i} \in k[x]_{\leq d_{i}}^{m \times n_{i}}$, where $i=1,2$ and $n_{1}+n_{2}=$ $n$, then there exists a nonzero $u \in k[x]^{n}$ with coefficients of degree at most $n_{1} d_{1}+n_{2} d_{2}$ such that $M u=0$.
(iii) The rank $r$ and a basis of the null space of $M$ can be computed using $\tilde{\mathcal{O}}\left(n m r^{\omega-2} d\right)$ ops.

The complexity in terms of arithmetic operations in $k$ of Hermite reduction has been analyzed by Yun in [94] and also [90, Theorem 22.7].

Lemma 2.4.3 (Yun, 1977). Let $f$ be a nonzero rational function in $k(x)$ of degree at most $n$ in $x$, then Hermite reduction on $f$ can be performed using $\tilde{\mathcal{O}}(n)$ operations in $k$.

## Chapter 3

## Hermite Reduction for Rational-Function Telescoping

### 3.1 Introduction

Although creative-telescoping algorithms have been now used extensively in modern computer algebra, very little is known about their complexity: some related results seem to be the complexity analyses of an algorithm for hyperexponential indefinite integration in [49], of algorithms for rational and hypergeometric summation in [50, 19], and of an algorithm by Takayama for finding a recurrence for hypergeometric series of a special form in [86]. In order to get complexity estimates, we simplify the problem by restricting to a smaller class of inputs, namely that of bivariate rational functions. Therefore, our goal reads as follows.

Problem 1. Given a rational function $f \in k(x, y) \backslash\{0\}$, find a nonzero operator $L$ in $k(x)\left\langle D_{x}\right\rangle$ and a rational function $g$ in $k(x, y)$ such that

$$
\begin{equation*}
L\left(x, D_{x}\right)(f)=D_{y}(g) . \tag{3.1}
\end{equation*}
$$

Here, $L$ is called a telescoper for $f$ with respect to $y$ and $g$ the certificate of $L$ for $f$. Since $D_{y}$ commutes with any element in $k(x)\left\langle D_{x}\right\rangle$, we can always choose telescopers with polynomial coefficients in $k[x]$.

By considering this constrained class of inputs, we are indeed able to blend the general method of creative telescoping with the well-known Hermite reduction [52]. Although restricted,
this class already has many applications. In 1827, Abel observed that an algebraic function satisfied a linear differential equation [1]. The annihilating differential equations are important in the study of algebraic functions [37, 30, 31]. In [20], it was shown that differential equations for algebraic functions can be computed via rational-function telescoping. In combinatorics, many nontrivial problems are encoded as diagonals of rational formal power series [83, 74]. Differential equations for diagonals of a bivariate rational function can also be constructed via rationalfunction telescoping.

Essentially two algorithms for computing telescopers of minimal order can be found in the literature: the classical way [15] is to apply a differential analogue of Gosper's indefinite summation algorithm [51], which reduces the problem to solving an auxiliary linear differential equation for polynomial solutions. An algorithm developed later in [46] (see also [63]) performs Hermite reduction on $f$ to get an additive decomposition of the form

$$
f=D_{y}(g)+\sum_{i=1}^{m} \frac{u_{i}}{v_{i}}
$$

where the $u_{i}$ and $v_{i}$ are in $k(x)[y]$ and the $v_{i}$ are squarefree. Then, the algorithm in [15] is applied to each $u_{i} / v_{i}$ to get a minimal telescoper $L_{i}$ for it. The least common left multiple of the $L_{i}$ 's is proved to be a minimal telescoper for $f$. This algorithm performs well only for specific inputs (both in practice and from the complexity viewpoint), but it inspired our improvements over Almkvist and Zeilberger's method.

Our contribution. For bivariate rational functions, we present a new and provably faster algorithm for computing minimal telescopers, which is based on Hermite reduction. Over the classical method by Almkvist and Zeilberger, we make some improvements avoiding unnecessary resultant calculations and integer-root finding. We derive complexity estimates of all algorithms described in Figure 3.1, showing that our approach is faster. These complexity results give the first proof of a polynomial complexity (in terms of field operations) for creative telescoping on bivariate rational-function inputs. In the computer algebra system Maple, we have implemented our algorithms for rational-function telescoping, which has been integrated into the existing Maple library Algolib 13.0 (for Maple 13). Our experimental results show that the Hermitereduction based algorithm outperforms all other known algorithms concerning both the worstcase complexity and the actual timings of implementations.

| Algorithms | $\operatorname{deg}_{D_{x}}(L)$ | $\operatorname{deg}_{x}(L)$ | $\operatorname{deg}_{x}(g)$ | $\operatorname{deg}_{y}(g)$ | Complexity |
| :--- | :--- | :--- | :--- | :--- | :--- |
| HermiteTelescoping | $\leq d_{y}$ | $\mathcal{O}\left(d_{x} d_{y}^{2}\right)$ | $\mathcal{O}\left(d_{x} d_{y}^{2}\right)$ | $\mathcal{O}\left(d_{y}^{2}\right)$ | $\tilde{\mathcal{O}}\left(d_{x} d_{y}^{\omega+3}\right)$ |
| RationalAZ | $\leq d_{y}$ | $\mathcal{O}\left(d_{x} d_{y}^{2}\right)$ | $\mathcal{O}\left(d_{x} d_{y}^{2}\right)$ | $\mathcal{O}\left(d_{y}^{2}\right)$ | $\tilde{\mathcal{O}}\left(d_{x} d_{y}^{2 \omega+2}\right)$ |

Table 3.1: Complexity for rational-function telescoping

The rest of this chapter is organized as follows. In Section 3.2, we recall notation about bivariate polynomials and set some hypotheses used in this chapter. We study Hermite reduction over $k(x)$ in Section 3.3, proving tight output degree bounds and an optimal algorithm via fast evaluation and interpolation. We present a Hermite-reduction based algorithm for minimal telescopers and some improvements over the classical method in Section 3.4. In Section 3.5, we show the existence of telescopers and estimate their order bounds. At last, we describe in Section 3.6 an implementation and show some experimental results.

The work in this chapter is published in [18], which is a joint work with Alin Bostan, Frédéric Chyzak and Ziming Li.

### 3.2 Notation and hypotheses

In this section, we review some basic notation and introduce some hypotheses for later use.
Let $Q$ be a bivariate polynomial in $k[x, y] \backslash k[x]$. The squarefree factorisation of $Q$ with respect to $y$ is the unique product $q Q_{1} Q_{2}^{2} \cdots Q_{m}^{m}$ where $q \in k[x]$ and $Q_{i} \in k[x, y]$ satisfying $\operatorname{deg}_{y}\left(Q_{m}\right)>0$ and such that the $Q_{i}$ 's are primitive, squarefree, and pairwise coprime. The squarefree part $Q^{*}$ of $Q$ with respect to $y$ is the product $Q_{1} Q_{2} \cdots Q_{m}$. Let $Q^{-}$denote the polynomial $Q / Q^{*}$, and $\operatorname{lc}_{y}(Q)$ the leading coefficient of $Q$ with respect to $y$. The following two formulas about $Q, Q^{*}$, and $Q^{-}$can be proved by mere calculations.

Fact 3.2.1. Let $\hat{Q}_{i}$ denote $Q^{*} / Q_{i}$. Then we have
(i) $Q^{*} D_{y}\left(Q^{-}\right) / Q^{-}=\sum_{i=1}^{m}(i-1) \hat{Q}_{i} D_{y}\left(Q_{i}\right) \in k[x, y]$;
(ii) $D_{y}(Q) / Q^{-}=\sum_{i=1}^{m} i \hat{Q}_{i} D_{y}\left(Q_{i}\right) \in k[x, y]$.

Let $f=P / Q$ be a nonzero element in $k(x, y)$, where $P, Q$ are two coprime polynomials in $k[x, y]$. The degree of $f$ in $x$ is defined to be $\max \left\{\operatorname{deg}_{x}(P), \operatorname{deg}_{x}(Q)\right\}$, and denoted by $\operatorname{deg}_{x}(f)$. The degree of $f$ in $y$ is defined similarly. The bidegree of $f$ is the pair $\left(\operatorname{deg}_{x}(f), \operatorname{deg}_{y}(f)\right)$, which is denoted by bideg $(f)$. The bidegree of $f$ is said to be bounded (above) by $(\alpha, \beta)$, written $\operatorname{bideg}(f) \leq(\alpha, \beta)$, when $\operatorname{deg}_{x}(f) \leq \alpha$ and $\operatorname{deg}_{y}(f) \leq \beta$.

We say that $f=P / Q$ is proper if the degree of $P$ in $y$ is less than that of $Q$. For creative telescoping, we may always assume w.l.o.g. that $f=P / Q$ is proper. If not, rewrite $f=D_{y}(p)+\bar{f}$ with $p \in k(x)[y]$ and $\bar{f}$ proper. A telescoper $L$ for $\bar{f}$ with certificate $\bar{g}$ is a telescoper for $f$ with certificate $L(p)+\bar{g}$. So we introduce the following hypothesis.

Hypothesis (H) From now on, $P$ and $Q$ are assumed to be nonzero polynomials in $k[x, y]$ such that $\operatorname{deg}_{y}(P)<\operatorname{deg}_{y}(Q), \operatorname{gcd}(P, Q)=1$, and $Q$ is primitive with respect to $y$.

Notation From now on, we write $\left(d_{x}, d_{y}\right),\left(d_{x}^{*}, d_{y}^{*}\right)$, and $\left(d_{x}^{-}, d_{y}^{-}\right)$for the bidegrees of $Q, Q^{*}$, and $Q^{-}$, respectively.

Sometimes, we use the following hypothesis in order to make estimates concise.
Hypothesis (H') Hypothesis (H) and $\operatorname{deg}_{x}(P) \leq d_{x}$.

### 3.3 Hermite reduction for bivariate rational functions

In this section, we will apply Hermite reduction to a bivariate rational function of $k(x, y)$ with respect to $y$. We present a quasi-optimal algorithm to perform the Hermite reduction by using a fast evaluation-interpolation approach.

### 3.3.1 Output size estimates

In order to use the evaluation-interpolation strategy, we first derive an upper bound on the degrees of the outputs of the Hermite reduction on a rational function. To this end, we will use the linear system introduced in Ostrogradsky and Horowitz's method.

After reading out the polynomial part, we may assume that $f=P / Q \in k(x, y)$ with $P$ and $Q$ in $k[x, y]$ such that $\operatorname{gcd}(P, Q)=1$ and $\operatorname{deg}_{y}(P)<\operatorname{deg}_{y}(Q)$. Let $D_{x}$ and $D_{y}$ denote the usual derivations $\partial / \partial x$ and $\partial / \partial y$ on $k(x, y)$, respectively. Recall that $Q^{*}$ denotes the squarefree part
of $Q$ with respect to $y$ and $Q^{-}=Q / Q^{*}$. Set

$$
d_{x}^{*}=\operatorname{deg}_{x}\left(Q^{*}\right), d_{y}^{*}=\operatorname{deg}_{y}\left(Q^{*}\right), d_{x}^{-}=\operatorname{deg}_{x}\left(Q^{-}\right), \text {and } d_{y}^{-}=\operatorname{deg}_{y}\left(Q^{-}\right)
$$

According to Ostrogradsky and Horowitz's method, we make the ansatz

$$
\begin{equation*}
\frac{P}{Q}=D_{y}\left(\frac{A}{Q^{-}}\right)+\frac{a}{Q^{*}}, \tag{3.2}
\end{equation*}
$$

where $A, a \in k(x)[y]$ with $\operatorname{deg}_{y}(A)<d_{y}^{-}$and $\operatorname{deg}_{y}(a)<d_{y}^{*}$. In order to bound the bidegrees of $A$ and $a$, we reformulate (3.2) into the equivalent form

$$
\begin{equation*}
P=Q^{*} D_{y}(A)-\left(\frac{Q^{*} D_{y}\left(Q^{-}\right)}{Q^{-}}\right) A+Q^{-} a, \tag{3.3}
\end{equation*}
$$

where $\widetilde{Q}=Q^{*} D_{y}\left(Q^{-}\right) / Q^{-}$is a polynomial in $k[x, y]$ of bidegree at most $\left(d_{x}^{*}, d_{y}^{*}-1\right)$ by Fact 3.2.1. Viewing $A$ and $a$ as polynomials in $k(x)[y]$ with undetermined coefficients, we form the following Ostrogradsky-Horowitz system,

$$
\begin{equation*}
M\binom{\hat{A}}{\hat{a}}=\hat{P} \tag{3.4}
\end{equation*}
$$

where $M$ is the Ostrogradsky-Horowitz matrix associated to $Q$ and $\hat{A}, \hat{a}$, and $\hat{P}$ are the coefficient vectors of $A, a$, and $P$ with sizes $d_{y}^{-}, d_{y}^{*}$, and $d_{y}$, respectively. From equation (3.3), the matrix $M$ is of the form

$$
\left(\begin{array}{ll}
M_{1} & M_{2}
\end{array}\right), \quad \text { where } M_{1} \in k[x]_{\leq d_{x}^{*}}^{d_{y} \times d_{y}^{-}} \text {and } M_{2} \in k[x]_{\leq d_{x}^{-}}^{d_{y} \times d_{y}^{*}} .
$$

Let $\Delta$ be the determinant of $M$, so that $\operatorname{deg}_{x}(\Delta) \leq \mu:=d_{x}^{*} d_{y}^{-}+d_{x}^{-} d_{y}^{*}$ by Fact 2.4.2(i). For later use, we also define $\Delta^{\prime}$ as the determinant of $\mathcal{M}\left(Q^{* 2}\right)$, so that $\operatorname{deg}_{x}\left(\Delta^{\prime}\right) \leq \mu^{\prime}:=2 d_{x}^{*} d_{y}^{*}$ by Fact 2.4.2 $(i)$ and since $\left(Q^{* 2}\right)^{-}=Q^{*}$.

Lemma 3.3.1. Let $f=P / Q \in k(x, y)$ with $P, Q \in k[x, y], \operatorname{gcd}(P, Q)=1$ and $\operatorname{deg}_{y}(P)<$ $\operatorname{deg}_{y}(Q)$. Then there exist $B, b \in k[x, y]$ with $\operatorname{deg}_{y}(B)<d_{y}^{-}$and $\operatorname{deg}_{y}(b)<d_{y}^{*}$, and such that:
(i) $f=D_{y}\left(\frac{B}{\Delta Q^{-}}\right)+\frac{b}{\Delta Q^{*}}$;
(ii) $\operatorname{deg}_{x}(B) \leq \mu-d_{x}^{*}+\operatorname{deg}_{x}(P)$ and $\operatorname{deg}_{x}(b) \leq \mu-d_{x}^{-}+\operatorname{deg}_{x}(P)$.

Proof. Applying Cramer's rule to (3.4) leads to the first assertion. The second assertion follows by determinant expansions.

## Algorithm HermiteEvallnterp

Input: $P, Q \in k[x, y]$ satisfying Hypothesis (H).
Output: $(A, a) \in k(x)[y]^{2}$ solving (3.2).

1. Compute $Q^{-}:=\operatorname{gcd}\left(Q, D_{y}(Q)\right)$ and $Q^{*}:=Q / Q^{-}$;
2. Set $\lambda:=2\left(d_{x}^{*} d_{y}^{-}+d_{y}^{*} d_{x}^{-}\right)+\operatorname{deg}_{x}(P)-\min \left\{d_{x}^{-}, d_{x}^{*}\right\}$;
3. Set $S$ to the set of $\lambda+1$ smallest nonnegative integers that are lucky for $Q$;
4. For each $x_{0} \in S$, compute $\left(A_{0}, a_{0}\right) \in k[y]^{2}$ such that

$$
\frac{P\left(x_{0}, y\right)}{Q\left(x_{0}, y\right)}=D_{y}\left(\frac{A_{0}}{Q^{-}\left(x_{0}, y\right)}\right)+\frac{a_{0}}{Q^{*}\left(x_{0}, y\right)}
$$

using Hermite reduction over $k$;
5. Compute $(A, a) \in k(x)[y]$ by rational interpolation and return this pair.

Figure 3.1: Hermite reduction over $k(x)$ via evaluation and interpolation.
In what follows, we shall encounter proper rational functions with denominator $Q$ satisfy$\operatorname{ing} Q=Q^{* 2}$. The following lemma is an easy corollary of Lemma 3.3.1 for such functions.

Corollary 3.3.2. Assume that $Q=Q^{* 2}$ additionally in Lemma 3.3.1. Then there exist $B, b \in$ $k[x, y]$ with $\operatorname{deg}_{y}(B)$ and $\operatorname{deg}_{y}(b)$ less than $d_{y}^{*}$, and such that
(i) $\frac{P}{Q^{* 2}}=D_{y}\left(\frac{B}{\Delta^{\prime} Q^{*}}\right)+\frac{b}{\delta^{\prime} Q^{*}}$;
(ii) $\operatorname{deg}_{x}(B)$ and $\operatorname{deg}_{x}(b)$ are bounded by $\mu^{\prime}-d_{x}^{*}+\operatorname{deg}_{x}(P)$.

### 3.3.2 Algorithm by evaluation and interpolation

We observe that an asymptotically optimal complexity can be achieved by evaluation and interpolation at each step of Hermite reduction over $k(x)$. This inspires us to adapt Gerhard's modular method $[48,49]$ to $k(x, y)$. For simplicity, we further assume that $Q \in k[x, y]$ is nonzero and primitive over $k[x]$.

Definition 3.3.3. An element $x_{0} \in k$ is said to be lucky if $\operatorname{lc}_{y}(Q)\left(x_{0}\right) \neq 0$ and the degree in $y$ of $\operatorname{gcd}\left(Q\left(x_{0}, y\right), D_{y} Q\left(x_{0}, y\right)\right)$ is equal to $\operatorname{deg}_{y}\left(Q^{-}\right)$.

Lemma 3.3.4. There are at most $d_{x}\left(2 d_{y}^{*}-1\right)$ unlucky points.
Proof. Let $\sigma \in k[x]$ be the $d_{y}^{-}$th subresultant with respect to $y$ of $Q$ and $D_{y}(Q)$. By Corollary 5.5 in [49], all unlucky points are in the set $U=\left\{x_{0} \in k \mid \sigma\left(x_{0}\right)=0\right\}$. By Corollary 3.2 (ii) in [49], $\operatorname{deg}_{x}(\sigma) \leq d_{x}\left(2 d_{y}^{*}-1\right)$.

Lemma 3.3.5. Let $B, b$, and $\Delta$ be the same as in Lemma 3.3.1, and let $x_{0} \in k$ be lucky. Then $\Delta\left(x_{0}\right) \neq 0$ and $\left(B\left(x_{0}, y\right), b\left(x_{0}, y\right)\right)$ is the unique pair such that

$$
\begin{equation*}
\frac{P\left(x_{0}, y\right)}{Q\left(x_{0}, y\right)}=D_{y}\left(\frac{B\left(x_{0}, y\right)}{\Delta\left(x_{0}\right) Q^{-}\left(x_{0}, y\right)}\right)+\frac{b\left(x_{0}, y\right)}{\Delta\left(x_{0}\right) Q^{*}\left(x_{0}, y\right)} \tag{3.5}
\end{equation*}
$$

Proof. By the luckiness of $x_{0}, \operatorname{deg}_{y}\left(Q\left(x_{0}, y\right)\right)=d_{y}$ and $Q\left(x_{0}, y\right)^{-}=Q^{-}\left(x_{0}, y\right)$. Then we have $Q\left(x_{0}, y\right)^{*}=Q^{*}\left(x_{0}, y\right)$. This implies $\mathcal{M}(Q)\left(x_{0}, y\right)=\mathcal{M}\left(Q\left(x_{0}, y\right)\right)$, which, by Lemma 2.3.6, is invertible over $k(x)$. Hence $\Delta\left(x_{0}\right) \neq 0$, and the evaluation at $x=x_{0}$ of the equality in Lemma 3.3.1 $(i)$ is well-defined. Thus, $\left(B\left(x_{0}, y\right), b\left(x_{0}, y\right)\right)$ is a solution of (3.5). Uniqueness follows from Corollary 2.3.2.

Theorem 3.3.6. Algorithm HermiteEvallnterp in Figure 3.1 is correct and takes $\tilde{\mathcal{O}}\left(d_{x} d_{y}^{2}+\right.$ $\left.\operatorname{deg}_{x}(P) d_{y}\right)$ operations in $k$.

Proof. Set $\nu$ to $d_{x}\left(2 d_{y}^{*}-1\right)$. Lemma 3.3.4 implies that the $\lambda+1$ lucky points found in Step 3 are all less than $\lambda+\nu+1$. By Corollary 2.3.2 and 3.3.1 $(i), A=B / \Delta$ and $a=b / \Delta$. By Lemma 3.3.5, $A_{0}=B\left(x_{0}, y\right) / \Delta\left(x_{0}\right)$ and $a_{0}=b\left(x_{0}, y\right) / \Delta\left(x_{0}\right)$. By Lemma 3.3.1 (ii) and since $\operatorname{deg}_{x}(\Delta) \leq \mu$, it suffices to rationally interpolate $A$ and $a$ from values at $\lambda+1$ lucky points. This shows the correctness. The dominant computation in Step 1 is the gcd, which takes $\tilde{\mathcal{O}}\left(d_{x} d_{y}\right)$ ops by Corollary 11.9 in [90]. For each integer $i \leq \lambda+\nu$, testing luckiness amounts to evaluations at $x_{0}$ and computing $\operatorname{gcd}\left(Q\left(x_{0}, y\right), D_{y}\left(Q\left(x_{0}, y\right)\right)\right.$, which takes $\tilde{\mathcal{O}}\left(d_{y}\right)$ ops by Fact 2.4.1 (i) and Corollary 11.6 in [90]. Then, generating $S$ in Step 3 costs $\tilde{\mathcal{O}}\left((\lambda+\nu+1) d_{y}\right)$ ops. By Fact 2.4.1 $(i)$, evaluations in Step 4 take $\tilde{\mathcal{O}}\left((\lambda+1) d_{y}\right)$ ops. For each $x_{0} \in S$, the cost of the Hermite reduction in Step 4 is $\tilde{\mathcal{O}}\left(d_{y}\right)$ ops by Lemma 2.4.3. Thus, the total cost of Step 4 is $\tilde{\mathcal{O}}\left((\lambda+1) d_{y}\right)$ ops. By Fact 2.4.1(ii), Step 5 takes $\tilde{\mathcal{O}}\left((\lambda+1) d_{y}\right)$ ops. Since $\lambda \leq 2 d_{x} d_{y}+\operatorname{deg}_{x}(P)$ and $\nu \leq 2 d_{x} d_{y}$, the total cost is as announced.

Remark 3.3.7. As the generic output size of Hermite reduction is proportional to $\lambda d_{y}$, which is equal to $\mathcal{O}\left(\left(d_{x} d_{y}+\operatorname{deg}_{x}(P)\right) d_{y}\right)$, Algorithm HermiteEvallnterp has quasi-optimal complexity.

### 3.4 Minimal telescopers for bivariate rational functions

Since the 1990's, the main emphasis in existing works [95, 15, 46, 63] has been on finding telescopers of order minimal over all telescopers for $f$, which are called minimal telescopers. For a rational function $f \in k(x, y)$, all its telescopers form a left ideal $I_{f}$ in the left principal ideal domain $k(x)\left\langle D_{x}\right\rangle$. So a minimal telescoper is a generator of $I_{f}$ and two minimal telescopers differ by a multiplicative left factor in $k(x)$. Therefore, a rational function has a unique monic minimal telescoper. In this section, we present a new method for computing minimal telescopers for rational functions in $k(x, y)$, which is based on Hermite reduction. Also, we make some improvements over the classical method by Almkvist and Zeilberger. We will show that the arithmetic complexity of Hermite reduction approach is lower than that of the classical one.

### 3.4.1 Hermite-reduction based method

In some applications, we only need to compute telescopers without certificates. In the next section, we will show that the arithmetic size of certificates is asymptotically larger than that of telescopers. This motivates us to find a way in which one could choose to compute or not to compute the certificates according to the user requirements. To this end, we design a new algorithm, presented in Figure 3.2, to compute minimal telescopers for rational functions by basing on Hermite reduction.

For $f=P / Q \in k(x, y)$ and $i \in \mathbb{N}$, Hermite reduction decomposes $D_{x}^{i}(f)$ into

$$
\begin{equation*}
D_{x}^{i}(f)=D_{y}\left(g_{i}\right)+r_{i}, \tag{3.6}
\end{equation*}
$$

where $g_{i}, r_{i} \in k(x, y)$ are proper. Since the squarefree part of the denominator of $D_{x}^{i}(f)$ divides $Q^{*}$, so does the denominator of $r_{i}$. The following lemma shows that (3.6) recombines into telescopers and certificates; next, Lemma 3.4.2 implies that the first pair obtained in this way by Algorithm HermiteTelescoping in Figure 3.2 yields a minimal telescoper.

Lemma 3.4.1. The rational functions $r_{0}, \ldots, r_{d_{y}^{*}}$ are linearly dependent over $k(x)$.

Proof. The constraints on $r_{i}$ imply $\operatorname{deg}_{y}\left(r_{i} Q^{*}\right)<d_{y}^{*}$ for all $i \in \mathbb{N}$, from which follows the existence of a nontrivial linear dependence among the $r_{i}$ 's over $k(x)$.

## Algorithm HermiteTelescoping

Input: $f=P / Q \in k(x, y)$ satisfying Hypothesis (H).
Output: A minimal telescoper $L \in k[x]\left\langle D_{x}\right\rangle$ with certificate $g \in k(x, y)$.

1. Apply HermiteEvallnterp to $f$ to get $\left(g_{0}, a_{0}\right)$ such that $f=D_{y}\left(g_{0}\right)+a_{0} / Q^{*}$. If $a_{0}=0$, return $\left(1, g_{0}\right)$.
2. For $i$ from 1 to $\operatorname{deg}_{y}\left(Q^{*}\right)$ do
(a) Apply HermiteEvallnterp to $-a_{i-1} D_{x}\left(Q^{*}\right) / Q^{* 2}$ to express it as $D_{y}\left(\tilde{g}_{i}\right)+\tilde{a}_{i} / Q^{*}$.
(b) Set $g_{i}=D_{x}\left(g_{i-1}\right)+\tilde{g}_{i}$ and $a_{i}=D_{x}\left(a_{i-1}\right)+\tilde{a}_{i}$.
(c) Solve $\sum_{j=0}^{i} \eta_{j} a_{j}=0$ for $\eta_{j} \in k(x)$ using [84, Algorithm Nullspace].

If there exists a nontrivial solution, then set $(L, g):=\left(\sum_{j=0}^{i} \eta_{j} D_{x}^{j}, \sum_{j=0}^{i} \eta_{j} g_{j}\right)$, and break.
3. Compute the content $c$ of $L$ and return $\left(c^{-1} L, c^{-1} g\right)$.

Figure 3.2: Creative telescoping by Hermite reduction

Lemma 3.4.2. An integer $\rho$ is minimal such that $\sum_{i=0}^{\rho} \eta_{i} r_{i}=0$ for $\eta_{0}, \ldots, \eta_{\rho} \in k(x)$ not all zero if and only if $\sum_{i=0}^{\rho} \eta_{i} D_{x}^{i}$ is a minimal telescoper for $f$ with certificate $\sum_{i=0}^{\rho} \eta_{i} g_{i}$.

Proof. Multiplying (3.6) by $\eta_{i}$ before summing yields

$$
L(f)=D_{y}\left(\sum_{i=0}^{\rho} \eta_{i} g_{i}\right)+\sum_{i=0}^{\rho} \eta_{i} r_{i} \quad \text { for } \quad L:=\sum_{i=0}^{\rho} \eta_{i} D_{x}^{i},
$$

where the sum $\sum_{i=0}^{\rho} \eta_{i} r_{i}$ is a proper fraction in $y$ with a squarefree denominator with respect to $y$. Thus, by Corollary 2.3.2, $L$ is a telescoper of order $\rho$ for $f$ with certificate $\sum_{i=0}^{\rho} \eta_{i} g_{i}$ if and only if $\sum_{i=0}^{\rho} \eta_{i} r_{i}=0$ with $\eta_{\rho} \neq 0$. The lemma follows.

Remark 3.4.3. In the algorithm HermiteTelescoping described in Figure 3.2, one can choose not to compute the certificate $g$, because the minimal telescoper and this corresponding certificate are computed separately.

## Order bounds for minimal telescopers

Lemmas 3.4.1 and 3.4.2 combine into a tight upper bound on the order of minimal telescopers for $f$.

Corollary 3.4.4. Minimal telescopers have order at most $d_{y}^{*}$.
We also derive a lower bound on the order of the minimal telescoper, to be used as an optimisation trick in our implementation. First, we choose a lucky element $x_{0} \in k$, and then apply Hermite reduction in $k(y)$ to $D_{x}^{i}(f)\left(x_{0}, y\right)$, we get

$$
\begin{equation*}
D_{x}^{i}(f)\left(x_{0}, y\right)=D_{y}\left(g_{0, i}\right)+r_{0, i}, \tag{3.7}
\end{equation*}
$$

where $g_{0, i}, r_{0, i} \in k(y)$ are proper and the denominator of $r_{0, i}$ divides $Q^{*}\left(x_{0}, y\right)$. Let $\rho_{0}$ be the smallest integer such that $r_{0,0}, \ldots, r_{0, \rho_{0}}$ are linearly dependent over $k$.

Lemma 3.4.5. A minimal telescoper has order at least $\rho_{0}$.
Proof. We first claim that $r_{0, i}=r_{i}\left(x_{0}, y\right)$, for $r_{i}$ as in (3.6). Note that the squarefree part with respect to $y$ of the denominator of $D_{x}^{i}(f)$ divides $Q^{*}$ for all $i \in \mathbb{N}$. By Corollary 5.5 in [49], $x_{0}$ is lucky for the denominator of $D_{x}^{i}(f)$ for all $i \in \mathbb{N}$. Then the claim on $r_{0, i}$ follows from Lemma 3.3.5 applied to $D_{x}^{i}(f)$. Let $\rho$ be the minimal order of a telescoper, then $r_{0}, \ldots, r_{\rho}$ are linearly dependent over $k(x)$ by Lemma 3.4.2. Thus the rational functions $r_{0,0}, \ldots, r_{0, \rho}$ are linearly dependent over $k$, which implies $\rho_{0} \leq \rho$.

## Degree bounds for minimal telescopers

To derive degree bounds for $g_{i}$ and $r_{i}$ in (3.6), let $\Delta, \Delta^{\prime}, \mu$, and $\mu^{\prime}$ be defined as in the paragraph before Lemma 3.3.1, and set $\mu^{\prime \prime}=\mu+\mu^{\prime}-1$.

Lemma 3.4.6. Let $W$ be in $k[x, y]$ with $\operatorname{deg}_{y}(W)<d_{y}^{*}$. Then, for all $i \in \mathbb{N}$, there exist $B, b \in$ $k[x, y]$ with both bideg $(B)$ and bideg $(b)$ bounded by $\left(\operatorname{deg}_{x}(W)+\mu^{\prime \prime}, d_{y}^{*}-1\right)$, such that

$$
D_{x}\left(\frac{W}{\Delta^{i+1} \Delta^{\prime i} Q^{*}}\right)=D_{y}\left(\frac{B}{\Delta^{i+2} \Delta^{i+1} Q^{*}}\right)+\frac{b}{\Delta^{i+2} \Delta^{i+1} Q^{*}} .
$$

Proof. A straightforward calculation leads to

$$
D_{x}\left(\frac{W}{\Delta^{i+1} \Delta^{\prime i} Q^{*}}\right)=\frac{\tilde{W}}{\Delta^{i+2} \Delta^{\prime i+1} Q^{*}}-\frac{1}{\Delta^{i+1} \Delta^{\prime \prime}} \frac{W D_{x}\left(Q^{*}\right)}{Q^{* 2}},
$$

where $\operatorname{bideg}(\tilde{W}) \leq\left(\operatorname{deg}_{x}(W)+\mu^{\prime \prime}, d_{y}^{*}-1\right)$. By Corollary 3.3.2, there exist $\tilde{B}, \tilde{b} \in k[x, y]$ such that

$$
\frac{1}{\Delta^{i+1} \Delta^{\prime}} \frac{W D_{x}\left(Q^{*}\right)}{Q^{* 2}}=\frac{1}{\Delta^{i+2} \Delta^{i+1}}\left(D_{y}\left(\frac{\Delta \tilde{B}}{Q^{*}}\right)+\frac{\Delta \tilde{b}}{Q^{*}}\right)
$$

where $\operatorname{bideg}(\tilde{B})$ and $\operatorname{bideg}(\tilde{b})$ are bounded by $\left(\operatorname{deg}_{x}(W)+\mu^{\prime}-1, d_{y}^{*}-1\right)$. The proof is completed by setting $(B, b)=(-\Delta \tilde{B}, \tilde{W}-\Delta \tilde{b})$.

Lemma 3.4.7. For $i \in \mathbb{N}$, there exist $B_{i}, b_{i} \in k[x, y]$ such that

$$
\begin{equation*}
D_{x}^{i}(f)=D_{y}\left(\frac{B_{i}}{\Delta^{i+1} \Delta^{\prime i} Q^{* i} Q^{-}}\right)+\frac{b_{i}}{\Delta^{i+1} \Delta^{\prime i} Q^{*}} . \tag{3.8}
\end{equation*}
$$

Moreover, $\operatorname{bideg}\left(B_{i}\right) \leq\left(\operatorname{deg}_{x}(P)+\mu+i \mu^{\prime \prime}+(i-1) d_{x}^{*}, i d_{y}^{*}+d_{y}^{-}-1\right)$ and $\operatorname{bideg}\left(b_{i}\right) \leq\left(\operatorname{deg}_{x}(P)+\right.$ $\left.\mu+i \mu^{\prime \prime}-d_{x}^{-}, d_{y}^{*}-1\right)$.

Proof. We proceed by induction on $i$. For $i=0$, the claim follows from Lemma 3.3.1. Assume that $i>0$ and that the claim holds for the values less than $i$. For brevity, we set $\gamma=\operatorname{deg}_{x}(P)+\mu$, $F_{i-1}=B_{i-1} /\left(\Delta^{i} \Delta^{\prime i-1} Q^{* i-1} Q^{-}\right)$, and $G_{i-1}=b_{i-1} /\left(\Delta^{i} \Delta^{\prime i-1} Q^{*}\right)$. The induction hypothesis implies

$$
D_{x}^{i}(f)=D_{y} D_{x}\left(F_{i-1}\right)+D_{x}\left(G_{i-1}\right)
$$

with bidegree bounds on $B_{i-1}$ and $b_{i-1}$. Fact 3.2.1(i) implies that $\tilde{Q}:=Q^{*} D_{x}\left(Q^{-}\right) / Q^{-}$is in $k[x, y]$, with $\operatorname{bideg}(\tilde{Q}) \leq\left(d_{x}^{*}-1, d_{y}^{*}\right)$. Hence $D_{x}\left(1 / Q^{-}\right)=-\tilde{Q} / Q$. This observation and an easy calculation imply that

$$
D_{x}\left(F_{i-1}\right)=\frac{\tilde{B}_{i-1}}{\Delta^{i+1} \Delta^{i} Q^{* i} Q^{-}}
$$

where $\tilde{B}_{i-1} \in k[x, y]$ and $\operatorname{deg}_{x}\left(\tilde{B}_{i-1}\right) \leq \operatorname{deg}_{x}\left(B_{i-1}\right)+\mu^{\prime \prime}+d_{x}^{*}$. Furthermore, by Lemma 3.4.6 there are $\bar{B}_{i}, \bar{b}_{i} \in k[x, y]$ with bidegrees at $\operatorname{most}\left(\operatorname{deg}_{x}\left(b_{i-1}\right)+\mu^{\prime \prime}, d_{y}^{*}-1\right)$, such that

$$
D_{x}\left(G_{i-1}\right)=D_{y}\left(\frac{\bar{B}_{i}}{\Delta^{i+1} \Delta^{\prime i} Q^{*}}\right)+\frac{\bar{b}_{i}}{\Delta^{i+1} \Delta^{\prime i} Q^{*}} .
$$

Setting $B_{i}=\tilde{B}_{i-1}+\bar{B}_{i} Q^{* i-1} Q^{-}$and $b_{i}=\bar{b}_{i}$, we arrive at (3.8). It remains to verify the degree bounds. The induction hypothesis implies that $\operatorname{both}^{\operatorname{deg}}\left(\bar{B}_{i}\right)$ and $\operatorname{deg}_{x}\left(b_{i}\right)$ are bounded by $\gamma+$ $i \mu^{\prime \prime}-d_{x}^{-}$. It follows that $\operatorname{deg}_{x}\left(\bar{B}_{i} Q^{* i-1} Q^{-}\right)$is bounded by $\gamma+i \mu^{\prime \prime}+(i-1) d_{x}^{*}$. Similarly, $\operatorname{deg}_{x}\left(\tilde{B}_{i-1}\right)$ is bounded by $\gamma+i \mu^{\prime \prime}+(i-1) d_{x}^{*}$, and so is $\operatorname{deg}_{x}\left(B_{i}\right)$. The bounds on degrees in $y$ are obvious.

We next derive degree bounds for the minimal telescopers obtained at an intermediate stage of HermiteTelescoping; refined bounds on the output will be given by Theorem 3.4.16.

Lemma 3.4.8. Under ( $H^{\prime}$ ), Step 2(c) of Algorithm HermiteTelescoping computes a minimal telescoper $L \in k[x]\left\langle D_{x}\right\rangle$ with order $\rho$ and a certificate $g \in k(x, y)$ for $P / Q$ with $\operatorname{deg}_{x}(L) \in$ $\mathcal{O}\left(d_{x} d_{y} \rho^{2}\right)$ and $\operatorname{bideg}(g) \in \mathcal{O}\left(d_{x} d_{y} \rho^{2}\right) \times \mathcal{O}\left(d_{y} \rho\right)$.

Proof. By Lemma 3.4.2, we exhibit a minimal telescoper by considering the first nontrivial linear dependence among the $a_{i}$ 's in (3.8). Let $M$ be the coefficient matrix of the system in $\left(\eta_{i}\right)$ obtained from $\sum_{i=0}^{\rho} \eta_{i} a_{i}=0$. By Lemma 3.4.7, $M$ is of size at most $(\rho+1) \times d_{y}^{*}$ and with coefficients of degree at most $\sigma:=d_{x}+\mu+\rho \mu^{\prime \prime}-d_{x}^{-}$in $x$. Hence, there exists a solution $\left(\eta_{0}, \ldots, \eta_{\rho}\right) \in k[x]^{\rho+1}$ of degree at most $\sigma \rho$ in $x$ by Fact 2.4.2(ii). Since $\mu, \mu^{\prime \prime} \in \mathcal{O}\left(d_{x} d_{y}\right)$ and $d_{y}^{*} \leq d_{y}$, the degree estimates of $L$ and $g$ are as announced.

## Complexity estimates

We proceed to analyse the complexity of the algorithm in Figure 3.2.
Theorem 3.4.9. Under Hyp. ( $H^{\prime}$ ), Algorithm HermiteTelescoping in Figure 3.2 is correct and takes $\tilde{\mathcal{O}}\left(\rho^{\omega+1} d_{x} d_{y}^{2}\right)$ ops, where $\rho$ is the order of the minimal telescoper.

Proof. The formulas in Step 2(a) create the loop invariant $D_{x}^{i}(f)=D_{y}\left(g_{i}\right)+a_{i} / Q^{*}$. Correctness then follows from Lemmas 3.4.1 and 3.4.2. Step 1 takes $\tilde{\mathcal{O}}\left(d_{x} d_{y}^{2}\right)$ ops by Theorem 3.3.6 under (H'). By Lemma 3.4.7, $\operatorname{deg}_{x}\left(-a_{i-1} D_{x}\left(Q^{*}\right)\right) \in \mathcal{O}\left(i d_{x} d_{y}\right)$. So the cost for performing Hermite reduction on $-a_{i-1} D_{x}\left(Q^{*}\right) / Q^{* 2}$ in Step 2(a) is $\tilde{\mathcal{O}}\left(i d_{x} d_{y}^{2}\right)$ ops by Theorem 3.3.6. The bidegrees of $g_{i}$ and $a_{i}$ in Step 2(b) are in $\mathcal{O}\left(i d_{x} d_{y}\right) \times \mathcal{O}\left(i d_{y}\right)$ by Lemma 3.4.7. Since adding and differentiating have linear complexity, Step 2(b) takes $\tilde{\mathcal{O}}\left(i^{2} d_{x} d_{y}^{2}\right)$ ops. For each $i$, the coefficient matrix of $\sum_{j=0}^{i} \eta_{j} a_{j}=0$ in Step 2(c) is of size at most $(i+1) \times d_{y}^{*}$ and with coefficients of degree at most $\operatorname{deg}_{x}\left(a_{i}\right) \in \mathcal{O}\left(i d_{x} d_{y}\right)$. Moreover, the rank of this matrix is either $i$ or $i+1$. Then, Step 2(c) takes $\tilde{\mathcal{O}}\left(i^{\omega} d_{x} d_{y}^{2}\right)$ ops by Fact 2.4.2(iii). Computing the content and divisions in Step 3 has complexity $\tilde{\mathcal{O}}\left(d_{x} d_{y} \rho^{3}\right)$. If the algorithm returns when $i=\rho$, then the total cost is in

$$
\begin{equation*}
\sum_{i=0}^{\rho} \tilde{\mathcal{O}}\left(i^{2} d_{x} d_{y}^{2}\right)+\sum_{i=1}^{\rho} \tilde{\mathcal{O}}\left(i^{\omega} d_{x} d_{y}^{2}\right) \subset \tilde{\mathcal{O}}\left(\rho^{\omega+1} d_{x} d_{y}^{2}\right) \mathrm{ops} \tag{3.9}
\end{equation*}
$$

which is as announced.

An optimisation, based on Lemma 3.4.5, consists in guessing the order $\rho$ so as to perform Step 2(c) a few times only: As a preprocessing step, choose $x_{0} \in k$ lucky for $Q$, then detect linear dependence of $\left\{r_{0,0}, \ldots, r_{0, j}\right\}$ in (3.7). The minimal $j$ with dependence is a lower bound $\rho_{0}$ on $\rho$. So Step 2(c) is then performed only when $i \geq \rho_{0}$. In practice, the lower bound $\rho_{0}$ computed in this way almost always coincides with the actual order $\rho$. So normalising the $g_{i}$ 's becomes the dominant step, as observed in experiments. We analyse this optimisation by first estimating the cost for computing $\rho_{0}$.

Lemma 3.4.10. Under Hypothesis ( $H^{\prime}$ ), computing a lower order bound $\rho_{0}$ for minimal telescopers takes $\tilde{\mathcal{O}}\left(d_{x} d_{y} \rho_{0}^{3}\right)$ ops.

Proof. Since differentiating has linear complexity, the derivative $D_{x}^{i}(f)$ takes $\tilde{\mathcal{O}}\left(i^{2} d_{x} d_{y}\right)$ ops. By Fact 2.4.1 $(i)$, the evaluation $D_{x}^{i}(f)\left(x_{0}, y\right)$ takes as much. The cost of Hermite reduction on $D_{x}^{i}(f)\left(x_{0}, y\right)$ is $\tilde{\mathcal{O}}\left(i d_{y}\right)$ ops by Lemma 2.4.3. By Fact 2.4.2(iii) with $d=1$, computing the rank of the coefficient matrix of $\sum_{j=0}^{i} \eta_{j} r_{0, j}$, with $r_{0, j}$ as in (3.7), takes $\tilde{\mathcal{O}}\left(d_{y} i^{\omega-1}\right)$ ops. Thus, the cost for computing a lower bound on $\rho_{0}$ is $\sum_{i=0}^{\rho_{0}} \tilde{\mathcal{O}}\left(i^{2} d_{x} d_{y}\right) \in \tilde{\mathcal{O}}\left(d_{x} d_{y} \rho_{0}^{3}\right)$ ops.

Corollary 3.4.11. Assume that $\rho_{0}=\rho-\mathcal{O}(1)$. Then the previous optimisation of HermiteTelescoping takes $\tilde{\mathcal{O}}\left(\rho^{3} d_{x} d_{y}^{2}\right)$ ops.

Proof. In view of Lemma 3.4.10, the estimate (3.9) becomes

$$
\tilde{\mathcal{O}}\left(d_{x} d_{y} \rho_{0}^{3}\right)+\sum_{i=0}^{\rho} \tilde{\mathcal{O}}\left(i^{2} d_{x} d_{y}^{2}\right)+\sum_{i=\rho_{0}}^{\rho} \tilde{\mathcal{O}}\left(i^{\omega} d_{x} d_{y}^{2}\right),
$$

which is $\tilde{\mathcal{O}}\left(\rho^{3} d_{x} d_{y}^{2}\right)+\tilde{\mathcal{O}}\left(\left(\rho-\rho_{0}\right) \rho^{\omega} d_{x} d_{y}^{2}\right)$ ops, whence the result.

### 3.4.2 Improved Almkvist and Zeilberger's method

We analyse the complexity of Almkvist and Zeilberger's algorithm [15] when restricted to bivariate rational functions. In order to get a telescoper whose order $\rho$ is minimal, the resulting algorithm, denoted RationalAZ, solves the telescoping equation

$$
\sum_{i=0}^{\rho} \eta_{i} D_{x}^{i}(f)=D_{y}(g)
$$

for increasing, prescribed values of $\rho$ until it gets a solution $\left(\eta_{0}, \ldots, \eta_{\rho}, g\right) \in k(x)^{\rho+1} \times k(x, y)$ with the $\eta_{i}$ 's not all zero. For the analysis, we start by studying the parametrisation of the differential Gosper algorithm of [15] under the same restriction to $k(x, y)$.

Definition 3.4.12 ([49]). Let $K$ be a field and $a, b \in K[y]$ be nonzero polynomials. A triple $(p, q, r) \in K[y]^{3}$ is said to be a differential Gosper form of the rational function $a / b$ if

$$
\frac{a}{b}=\frac{D_{y}(p)}{p}+\frac{q}{r} \quad \text { and } \quad \operatorname{gcd}\left(r, q-\tau D_{y}(r)\right)=1 \quad \text { for all } \tau \in \mathbb{N}
$$

For hyperexponential $f$, a key step in [15] is to compute a differential Gosper form of the logarithmic derivative of $F=\sum_{i=0}^{\rho} \eta_{i} D_{x}^{i}(f)$, where the $\eta_{i}$ 's are undetermined from $k(x)$. In the analogue RationalAZ, this form is predicted by Lemma 3.4.13 below, which is a technical generalisation of a result by Le [63] on $F$ when $f$ has a squarefree denominator.

For $f=P / Q$, write $Q=t(y) T(x, y)$, splitting content and primitive part with respect to $x$. By an easy induction, $D_{x}^{i}(f)=N_{i} /\left(Q T^{* i}\right)$ for $N_{i} \in k[x, y]$. For this section, set

$$
F=\sum_{i=0}^{\rho} \eta_{i} D_{x}^{i}(f), \quad N=\sum_{i=0}^{\rho} \eta_{i} N_{i} T^{* \rho-i}, \quad \text { and } \quad H=-D_{y}(Q) / Q^{-}-\rho t^{*} D_{y}\left(T^{*}\right)
$$

Lemma 3.4.13. If $F$ is nonzero, the triple $\left(N, H, Q^{*}\right)$ is a differential Gosper form of $D_{y}(F) / F$.

Proof. First, observe $F=N /\left(Q T^{* \rho}\right)$ and $Q^{*}=t^{*} T^{*}$. Next, $D_{y}(F) / F=D_{y}(N) / N-D_{y}(Q) / Q-$ $\rho D_{y}\left(T^{*}\right) / T^{*}$ is $D_{y}(N) / N+H / Q^{*}$. It remains to prove $\operatorname{gcd}\left(Q^{*}, H-\tau D_{y}\left(Q^{*}\right)\right)=1$, for any $\tau \in \mathbb{N}$. Recall that the squarefree part $Q^{*}$ of $Q$ is the product $Q_{1} Q_{2} \cdots Q_{m}$ and that $\hat{Q}_{i}$ denotes $Q^{*} / Q_{i}$. By Fact 3.2.1(ii),

$$
Z:=H-\tau D_{y}\left(Q^{*}\right)=-\rho t^{*} D_{y}\left(T^{*}\right)-\sum_{i=1}^{m}(i+\tau) \hat{Q}_{i} D_{y}\left(Q_{i}\right)
$$

If $Q_{j}$ divides $t^{*}, Z$ reduces to $-(j+\tau) \hat{Q}_{j} D_{y}\left(Q_{j}\right)$ modulo $Q_{j}$. If not, it reduces to $-(j+$ $\tau) \hat{Q}_{j} D_{y}\left(Q_{j}\right)-\rho t^{*}\left(D_{y}\left(Q_{j}\right) T^{*} / Q_{j}\right)$, which rewrites to $-(j+\tau+\rho) \hat{Q}_{j} D_{y}\left(Q_{j}\right)$ modulo $Q_{j}$. In both cases, $Z$ is coprime with $Q^{*}$, as $j>0, \tau \geq 0$, and $\rho \geq 0$.

By induction, we have $\operatorname{bideg}\left(N_{i}\right) \leq\left(\operatorname{deg}_{x}(P)+i \operatorname{deg}_{x}\left(T^{*}\right)-i, d_{y}+i \operatorname{deg}_{y}\left(T^{*}\right)-1\right)$, so that $\operatorname{bideg}(N) \leq\left(\operatorname{deg}_{x}(P)+\rho \operatorname{deg}_{x}\left(T^{*}\right)-\rho, d_{y}+\rho \operatorname{deg}_{y}\left(T^{*}\right)-1\right)$.

## Algorithm RationalAZ

Input: $f=P / Q \in k(x, y)$ satisfying Hypothesis (H).
Output: A minimal telescoper $L \in k[x]\left\langle D_{x}\right\rangle$ with certificate $g \in k(x, y)$.

1. Compute $Q^{-}=\operatorname{gcd}\left(Q, D_{y}(Q)\right), Q^{*}=Q / Q^{-}$, and $T, T^{*}$ primitive parts of $Q, Q^{*}$ with respect to $x$, respectively;
2. Set $(\tilde{N}, N, \beta, H)$ to $\left(P, P, d_{y}^{-},-Q^{*} D_{y}(Q) / Q\right)$;
3. For $\ell=0,1, \ldots$ do
(a) Set $z$ to $\sum_{j=0}^{\beta} z_{j} y^{j}$, extract the linear system $\mathcal{M}\left(\begin{array}{ll}\eta_{i} & z_{j}\end{array}\right)^{T}=0$ from (3.10) (for $\rho=\ell$ ) and compute a basis $S$ of the null space of $\mathcal{M}$ by [84].
(b) If $S$ contains a solution $\left(\eta_{0}, \ldots, \eta_{\ell}, s\right)$ such that $\eta_{0}, \ldots, \eta_{\ell}$ are not all nonzero, then set $(L, g):=\left(\sum_{i=0}^{\ell} \eta_{i} D_{x}^{i}, s /\left(Q^{-} T^{* \ell}\right)\right)$, and go to Step 4;
(c) Update $\tilde{N}:=D_{x}(\tilde{N}) T^{*}-\tilde{N}\left(T^{*} D_{x}(T) / T+i D_{x}\left(T^{*}\right)\right), \quad N:=N T^{*}+\eta_{\ell+1} \tilde{N}$, $\beta:=\beta+\operatorname{deg}_{y}\left(T^{*}\right)$, and $H:=H-t^{*} D_{y}\left(T^{*}\right)$.
4. Compute the content $c$ of $L$ and return $\left(c^{-1} L, c^{-1} g\right)$.

## Figure 3.3: Improved Almkvist-Zeilberger algorithm

The next step in RationalAZ is, for fixed $\rho$, to reduce (3.1) by the change of unknown $g=$ $z /\left(Q^{-} T^{* \rho}\right)$, so as to determine all $\left(\eta_{i}\right) \in k(x)^{\rho+1}$ for which the differential equation in $z$

$$
\begin{equation*}
\sum_{i=0}^{\rho} \eta_{i} N_{i} T^{* \rho-i}=Q^{*} D_{y}(z)+\left(D_{y}\left(Q^{*}\right)+H\right) z \tag{3.10}
\end{equation*}
$$

has a polynomial solution in $k(x)[y]$. For later use, we recall the following consequence of Corollary 9.6 in [49].

Lemma 3.4.14. Let $a, b \in K[y]$ be such that $\beta=-\mathrm{lc}_{y}(b) / \mathrm{lc}_{y}(a)$ is a nonnegative integer and $\operatorname{deg}_{y}(b)=\operatorname{deg}_{y}(a)-1$. Let $c \in K[y]$ be such that $\beta \geq \operatorname{deg}_{y}(c)-\operatorname{deg}_{y}(a)+1$. If $u$ is a polynomial solution of $a D_{y}(z)+b z=c$, then $\operatorname{deg}_{y}(u) \leq \beta$.

The following lemma generalizes Lemma 2 in [63] to present a degree bound for $z$.

Lemma 3.4.15. If $u \in k(x)[y]$ is a solution of (3.10) for $\left(\eta_{i}\right) \in k(x)^{\rho+1}$, then $\operatorname{deg}_{y}(u)$ is bounded by $\beta=d_{y}^{-}+\rho \operatorname{deg}_{y}\left(T^{*}\right)$.

Proof. Let $a=Q^{*}$ and $b=D_{y}\left(Q^{*}\right)+H$. By the definition of $H, b=-Q^{*} D_{y}\left(Q^{-}\right) / Q^{-}-$ $\rho t^{*} D_{y}\left(T^{*}\right)$. Fact 3.2.1(i) implies that $\mathrm{lc}_{y}(b)=-\left(d_{y}^{-}+\rho \operatorname{deg}_{y}\left(T^{*}\right)\right) \mathrm{lc}_{y}(a)$. Therefore,

$$
\beta=-\mathrm{lc}_{y}(b) / \mathrm{lc}_{y}(a)=d_{y}^{-}+\rho \operatorname{deg}_{y}\left(T^{*}\right)
$$

As $\operatorname{deg}_{y}(N)<d_{y}+\rho \operatorname{deg}_{y}\left(T^{*}\right)$ and $d_{y}=d_{y}^{*}+d_{y}^{-}, \beta \geq \operatorname{deg}_{y}(N)-d_{y}^{*}+1$. The lemma holds by Lemma 3.4.14.

We end the present section using the approach of Almkvist and Zeilberger to provide tight degree bounds on the outputs from Algorithms HermiteTelescoping and RationalAZ.

Theorem 3.4.16. Under Hypothesis ( $H^{\prime}$ ), there exists a minimal telescoper $L \in k[x]\left\langle D_{x}\right\rangle$ with certificate $g \in k(x, y)$ with $\operatorname{deg}_{x}(L) \in \mathcal{O}\left(d_{x} d_{y} d_{y}^{*}\right)$ and $\operatorname{bideg}(g) \in \mathcal{O}\left(d_{x} d_{y} d_{y}^{*}\right) \times \mathcal{O}\left(d_{y} d_{y}^{*}\right)$.

Proof. By Corollary 3.4.4, there exists a smallest $\rho \in \mathbb{N}$ at $\operatorname{most} d_{y}^{*}$, for which (3.1) has a solution with the $\eta_{i}$ 's not all zero. For this $\rho$, we estimate the size of the polynomial matrix $\mathcal{M}$ derived from (3.10) by undetermined coefficients. By the remark on $N$ after Lemma 3.4.13, we have $\operatorname{bideg}(N) \leq\left(n_{x}, n_{y}\right)$ where $n_{x}:=d_{x}+\rho \operatorname{deg}_{x}\left(T^{*}\right)-\rho \in \mathcal{O}\left(\rho d_{x}\right)$ and $n_{y}:=d_{y}+\rho \operatorname{deg}_{y}\left(T^{*}\right)-1 \in$ $\mathcal{O}\left(\rho d_{y}\right)$. The matrix $\mathcal{M}$ contains two blocks $\mathcal{M}_{1} \in k[x]_{\leq n_{x}}^{\left(n_{y}+1\right) \times(\rho+1)}$ and $\mathcal{M}_{2} \in k[x]_{\leq d_{x}}^{\left(n_{y}+1\right) \times(\beta+1)}$, where $\beta \in \mathcal{O}\left(\rho d_{y}\right)$ is the same as in Lemma 3.4.15. By the minimality of $\rho$, the dimension of the null space of $\mathcal{M}$ is 1 . So there exists $u \in k[x]^{n_{y}+1}$ with coefficients of degree at most $n_{x}(\rho+1)+d_{x}(\beta+1) \in \mathcal{O}\left(d_{x} d_{y} d_{y}^{*}\right)$ in $x$ such that $\mathcal{M}(\eta z)^{T}=0$, which implies degree bounds in $x$ for $L$ and $g$. The degree bound in $y$ for $g$ is obvious.

We now analyse the complexity of the algorithm in Fig. 3.3.
Theorem 3.4.17. Under Hypothesis ( $H^{\prime}$ ), Algorithm RationalAZ in Figure 3.3 works correctly and takes $\tilde{\mathcal{O}}\left(d_{x} d_{y}^{\omega} \rho^{\omega+2}\right)$ ops, where $\rho$ is the order of the minimal telescoper.

Proof. By the existence of a telescoper, Corollary 3.4.4, and Lemma 3.4.15, the algorithm always terminates and returns a minimal telescoper $L$, of order $\rho$ at most $d_{y}^{*}$. Gcd computations dominate the cost of Steps 1 and 2, which take $\tilde{\mathcal{O}}\left(d_{x} d_{y}^{2}\right)$ ops. For each $\ell \in \mathbb{N}$, the dominating
cost in Step 3 is computing the null space of $\mathcal{M}$. Let $n_{y}=d_{y}+\ell \operatorname{deg}_{y}\left(T^{*}\right)-1 \in \mathcal{O}\left(\ell d_{y}\right)$ and $n_{x}=d_{x}+\ell \operatorname{deg}_{x}\left(T^{*}\right) \in \mathcal{O}\left(\ell d_{x}\right)$. By the same argument as in the proof of Theorem 3.4.16, the matrix $\mathcal{M}$ is of size at most $\left(n_{y}+1\right) \times(\ell+\beta+2)$ and with coefficients of degree at most $n_{x}$. Let $r$ be the rank of $\mathcal{M}$, which is either $\ell+\beta+2$ or $\ell+\beta+1$ by construction. Thus, a basis of the null space of $\mathcal{M}$ can be computed within $\tilde{\mathcal{O}}\left(n_{x}\left(n_{y}+1\right)(\ell+\beta+2) r^{\omega-2}\right)$ ops by Fact 2.4.2(iii). Since $\beta \in \mathcal{O}\left(\ell d_{y}\right), \tilde{\mathcal{O}}\left(n_{x}\left(n_{y}+1\right)(\ell+\beta+2) r^{\omega-2}\right)$ is included in $\tilde{\mathcal{O}}\left(d_{x} d_{y}^{\omega} \ell^{\omega+1}\right)$. Since Step 3 terminates at $\ell=\rho$, the total cost of the algorithm is $\sum_{\ell=0}^{\rho} d_{x} d_{y}^{\omega} \ell^{\omega+1}$ ops. This is within the announced complexity, $\tilde{\mathcal{O}}\left(d_{x} d_{y}^{\omega} \rho^{\omega+2}\right)$ ops.

Corollary 3.4.18. Algorithms HermiteTelescoping and RationalAZ in Figures 3.2 and 3.3 both output the primitive minimal telescoper $L$ together with its certificate $g$, which satisfy $\operatorname{deg}_{D_{x}}(L) \leq$ $d_{y}^{*}, \operatorname{deg}_{x}(L)$ and $\operatorname{deg}_{x}(g) \in \mathcal{O}\left(d_{x} d_{y} d_{y}^{*}\right)$, and $\operatorname{deg}_{y}(g) \in \mathcal{O}\left(d_{y} d_{y}^{*}\right)$.

Proof. Both algorithms output the primitive minimal telescoper, as they compute a minimal telescoper at an intermediate step, and owing to their last step of content removal. Bounds follow from Corollary 3.4.4 and Theorem 3.4.16.

### 3.5 Non-minimal telescopers for bivariate rational functions

In this section, we trade the minimality of telescopers for smaller total output sizes. To this end, we adapt and slightly extend the arguments in [67] and [20, Section 3].

Let $f=P / Q$ be a nonzero element in $k(x, y)$ such that $P, Q \in k[x, y]$ and $\operatorname{gcd}(P, Q)=1$. Denote $d_{x}=\max \left\{\operatorname{deg}_{x}(P), \operatorname{deg}_{x}(Q)\right\}$ and $d_{y}=\max \left\{\operatorname{deg}_{y}(P), \operatorname{deg}_{y}(Q)\right\}$. Our next goal is to find a nonzero operator $A\left(x, D_{x}, D_{y}\right)$ in $k[x]\left\langle D_{x}, D_{y}\right\rangle$ such that $A(f)=0$. To this end, we consider the $k$-vector space $\mathbb{W}_{N}$ generated by the set $\left\{x^{i} D_{x}^{j} D_{y}^{\ell} \mid i+j+\ell \leq N\right\}$ over $k$. By an easy combinatorial counting, the dimension of $\mathbb{W}_{N}$ is $\binom{N+3}{3}$ over $k$. Furthermore, for any $(i, j, \ell) \in \mathbb{N}^{3}$, a direct calculation yields

$$
\begin{equation*}
x^{i} D_{x}^{j} D_{y}^{\ell}(f)=\frac{P_{i, j, \ell}}{Q^{i+j+\ell+1}}, \tag{3.11}
\end{equation*}
$$

where $P_{i, j, \ell} \in k[x, y]$ with $\operatorname{deg}_{x}\left(P_{i, j, \ell}\right) \leq(i+j+\ell+1) d_{x}+i$ and $\operatorname{deg}_{y}\left(P_{i, j, \ell}\right) \leq(i+j+\ell+1) d_{y}$. So the set $\mathbb{W}_{N}(f)$ is included in the set

$$
\mathbb{V}_{N}=\operatorname{span}_{k}\left\{\left.\frac{x^{i} y^{j}}{Q^{N+1}} \right\rvert\, i \leq(N+1) d_{x}+N, j \leq(N+1) d_{y}\right\}
$$

where the dimension of $\mathbb{V}_{N}$ is $(N+1)\left(d_{x}+1\right)\left((N+1) d_{y}+1\right)$ over $k$. Define a linear map $\phi$ : $\mathbb{W}_{N} \rightarrow \mathbb{V}_{N}$ by $\phi(L)=L(f)$ for any $L \in \mathbb{W}_{N}$. Choosing $N=6\left(d_{x}+1\right)\left(d_{y}+1\right)$ yields the inequality

$$
\binom{N+3}{3}>(N+1)\left(d_{x}+1\right)\left((N+1) d_{y}+1\right)
$$

which implies the kernel of $\phi$ is nontrivial whenever $N \geq 6\left(d_{x}+1\right)\left(d_{y}+1\right)$. Therefore, there exists a nonzero operator $A$ in $k[x]\left\langle D_{x}, D_{y}\right\rangle$ with total degree at most $6\left(d_{x}+1\right)\left(d_{y}+1\right)$ in $x, D_{x}$, and $D_{y}$ that annihilates $f$. Moreover, $A$ can be found by solving a linear system of size $\mathcal{O}\left(\left(d_{x} d_{y}\right)^{3}\right)$ over $k$. The following lemma shows that one can construct a telescoper for $f$ from any $y$-free annihilator $A\left(x, D_{x}, D_{y}\right)$ of $f$.

Lemma 3.5.1. Let $f$ be a nonzero rational function in $k(x, y)$ and $A \in k[x]\left\langle D_{x}, D_{y}\right\rangle$ be a nonzero operator such that $A(f)=0$. Then there exists a nonzero operator $L\left(x, D_{x}\right)$ in $k[x]\left\langle D_{x}\right\rangle$ such that $L(f)=D_{y}(g)$ for some $g \in k(x, y)$.

Proof. Since $D_{y}$ commutes with $x$ and $D_{x}$, we can write $A=D_{y}^{m}\left(L\left(x, D_{x}\right)+D_{y} M\right)$, where $m \in \mathbb{N}$, $M \in k[x]\left\langle D_{x}, D_{y}\right\rangle$, and $L$ is a nonzero operator in $k[x]\left\langle D_{x}\right\rangle$. By a differential extension of a "non-commutative" trick, used by Wegschaider in [91, Theorem 3.2], there exist $w \in k[y]$ and nonzero $r \in k$ such that

$$
\begin{equation*}
w D_{y}^{m}=D_{y} Q+r, \tag{3.12}
\end{equation*}
$$

where $Q \in k[y]\left\langle D_{y}\right\rangle$. In particular, $r=(-1)^{m} m!\neq 0$ if we take $w=y^{m}$. Using the fact $r D_{y}=$ $D_{y} r$ and (3.12), we find

$$
\frac{y^{m}}{(-1)^{m} m!} A=L+D_{y} G, \quad \text { where } G \in k[x, y]\left\langle D_{x}, D_{y}\right\rangle .
$$

Since $A(f)=0, L(f)=D_{y}(-G(f))$. Note that $g=-G(f)$ is still in $k(x, y)$. This completes the proof.

From Lipshitz's argument, we see that there exists a telescoper of order in $\mathcal{O}\left(d_{x} d_{y}\right)$. Now, we slightly extend the argument in [20, Section 3] to show that a telescoper of order linear in $d_{y}$ exists. Instead of taking total degree, set $\mathbb{W}_{N_{x}, N_{\partial}}$ to the $k$-vector space

$$
\mathbb{W}_{N_{x}, N_{\partial}}=\operatorname{span}_{k}\left\{x^{i} D_{x}^{j} D_{y}^{\ell} \mid i \leq N_{x}, j+\ell \leq N_{\partial}\right\} .
$$

where the dimension of $\mathbb{W}_{N_{x}, N_{\partial}}$ is $\left(N_{x}+1\right)\binom{N_{\partial}+2}{2}$ over $k$. By (3.11), the set $\mathbb{W}_{N_{x}, N_{\partial}}(f)$ is included in the set

$$
\mathbb{V}_{N_{x}, N_{\partial}}=\operatorname{span}_{k}\left\{\left.\frac{x^{i} y^{j}}{Q^{N+1}} \right\rvert\, i \leq\left(N_{\partial}+1\right) d_{x}+N_{x}, j \leq\left(N_{\partial}+1\right) d_{y}\right\} .
$$

where the dimension of $\mathbb{V}_{N_{x}, N_{\partial}}$ is $\left(\left(N_{\partial}+1\right) d_{x}+N_{x}+1\right)\left(\left(N_{\partial}+1\right) d_{y}+1\right)$ over $k$. Choosing $N_{x}=$ $3 d_{x} d_{y}$ and $N_{\partial}=6 d_{y}$ yields the inequality

$$
\left(N_{x}+1\right)\binom{N_{\partial}+2}{2}>\left(\left(N_{\partial}+1\right) d_{x}+N_{x}+1\right)\left(\left(N_{\partial}+1\right) d_{y}+1\right) .
$$

Therefore, there exists a nonzero operator $A$ in $k[x]\left\langle D_{x}, D_{y}\right\rangle$ with degree at most $3 d_{x} d_{y}$ in $x$ and total degree at most $6 d_{y}$ in $D_{x}$ and $D_{y}$ that annihilates $f$. Again, $A$ can be found by solving a linear system, but of smaller size $\mathcal{O}\left(d_{x} d_{y}^{3}\right)$ over $k$. By the construction in Lemma 3.5.1, the order of $L$ is bounded by $6 d_{y}$, which is linear in $d_{y}$.

Remark 3.5.2. The bound $6 d_{y}$ has been shown in [20] for rational functions $y D_{y}(Q) / Q$ with $Q \in$ $k[x, y]$. Apagodu and Zeilberger [16] obtain a similar bound for a class of nonrational hyperexponential functions, but their proof does not seem to apply to rational functions, as it heavily relies on the presence of a nontrivial exponential part.

### 3.6 Implementation and experiments

We describe in this section an implementation of algorithms HermiteTelecoping and RationalAZ, and compare ours with Maple's routine.

### 3.6.1 Implementation and examples

The evaluation-interpolation algorithm HermiteEvallnterp for Hermite reduction in Figure 3.1 does not perform well, mainly because Maple's rational interpolation routines are far too slow. We thus implemented Algorithm HermiteReduce (original version) in [21, Section 2.2] (carefully avoiding redundant extended gcd calculations), and noted that it performs better. We then implemented a variant of Algorithm HermiteTelescoping in Figure 3.2, using HermiteReduce in place of HermiteEvallnterp, and including the optimisation at the end of $\S 3.4 .1$, refined by additional modular calculations. The improved Almkvist-Zeilberger's algorithm RationalAZ is also implemented. Those functions form the module RationalCT.

```
> eval(RationalCT);
```

module()
export SquareFreeParFrac, HermiteReduce, HermiteTelescoping, RationalAZ;
option package;
description "Computing the minimal telescoper for a rational function";
end module
The function SquareFreeParFrac is used to compute the partial fraction decomposition of a rational function with respect to the squarefree factorization of its denominator.
$>\mathrm{f}:=1 /\left(\mathrm{x}^{\wedge} 3+5 * \mathrm{x}^{\wedge} 2+8 * \mathrm{x}+4\right)$;

$$
f:=\frac{1}{x^{3}+5 x^{2}+8 x+4}
$$

$>$ SquareFreeParFrac(f, x, 'pfd');

$$
1,[x, 0,[[x+2,[1,-1],[1,-1]],[x+1,[1,1]]]]
$$

$>\mathrm{pf}$;

$$
-\frac{1}{x+2}-\frac{1}{(x+2)^{2}}+\frac{1}{x+1}
$$

The function HermiteReduce returns the additive decomposition of a rational function with respect to the specific variable.
$>f:=1 /(-y+y \wedge 2+x)^{\wedge} 2 ;$

$$
f:=\frac{1}{\left(-y+y^{2}+x\right)^{2}}
$$

$>$ HermiteReduce(f, y);

$$
\left[\frac{-1+2 y}{(4 x-1)\left(-y+y^{2}+x\right)}, \frac{2}{(4 x-1)\left(-y+y^{2}+x\right)}\right]
$$

In Maple, the function for computing minimal telescopers is DEtools[Zeilberger], which works for any hyperexponential inputs.
$>f:=1 /\left(-y+y^{\wedge} 2+x\right)$;

$$
f:=\frac{1}{-y+y^{2}+x}
$$

> DEtools[Zeilberger](f, x, y, Dx);

$$
\left[2+D x(4 x-1),-\frac{-1+2 y}{-y+y^{2}+x}\right]
$$

Our implementation of the algorithms HermiteTelescoping and RationalAZ is as follows.

$$
\begin{aligned}
& >\text { HermiteTelescoping }(\mathrm{f}, \mathrm{x}, \mathrm{y}, \mathrm{Dx}) ; \\
& \qquad\left[2+D x(4 x-1),-\frac{-1+2 y}{-y+y^{2}+x}\right] \\
& >\text { HermiteTelescoping(f, x, y, Dx, 'No_Certificate'); } \\
& \qquad 2+D x(4 x-1) \\
& >\text { RationalAZ(f, x, y, Dx); } \\
& \qquad\left[2+D x(4 x-1),-\frac{-1+2 y}{-y+y^{2}+x}\right]
\end{aligned}
$$

### 3.6.2 Experimental results

Now, we show some timings by our implementation and others.

## Random rational functions

We use the function randpoly in Maple to generate the numerator $P$ and the denominator $Q$ of a testing rational function with $P$ and $Q$ having the same degree in $x$ and $y$. We test the following set of rational functions:

$$
f=\frac{P}{Q}, \quad d=\operatorname{deg}_{x}(P)=\operatorname{deg}_{y}(P) \in\{1,2, \ldots, 7\}
$$

For brevity, we denote

- AZ: the function DEtools[Zeilberger] in Maple 13;
- Hermite: Hermite-reduction based method;
- RatAZ: improved Almkvist-Zeilberger algorithm.

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| AZ | 0.054 | 0.158 | 2.731 | 64.75 | 619.0 | $>\mathrm{hr}$ | $>\mathrm{hr}$ |
| RatAZ | 0.019 | 0.059 | 0.402 | 4.461 | 34.13 | 220.5 | 792.1 |
| Hermite | 0.016 | 0.057 | 0.398 | 2.664 | 18.80 | 106.2 | 422.5 |

Table 3.2: Timings on random examples (in seconds)

## Differential equations for algebraic functions

Let $\alpha(x)$ be a univariate algebraic function over $k(x)$ with minimal polynomial $P$ in $k[x, y]$. The following lemma shows that one can compute differential equations for algebraic functions via rational-function telescoping. For the proof, see [20, Proposition 2].

Lemma 3.6.1. If $L$ is a telescoper for $y D_{y}(P) / P$ with respect to $y$, then $L(\alpha(x))=0$.

In the next table, we show the timings on a set of polynomials in $\mathbb{Z}[x, y]$, which are also generated by the function randpoly. Let $d$ denote the total degree of a polynomial in $x$ and $y$. The column gfun (Maple 13) shows the timing by the function algeqtodiffeq in Maple package gfun and gfun (Algo) by the one in the updated Algolib 13.0 of INRIA Algorithms project.

| $d$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| RatAZ | 0.30 | 1.05 | 4.90 | 21.6 | 69.5 | 237. | 846. |
| Hermite | 0.21 | 0.94 | 4.53 | 20.5 | 84.7 | 231. | 864. |
| gfun (Maple 13) | 0.14 | 0.75 | 6.92 | 79.6 | 1661 | $>\mathrm{hr}>\mathrm{hr}$ |  |
| gfun (Algo) | 0.10 | 0.46 | 2.44 | 12.2 | 52.7 | 157. | 464. |

Table 3.3: Timings on computing differential equations for algebraic functions (in seconds)

## Differential equations for diagonals

For a bivariate rational power series

$$
f=\sum_{i, j \geq 0} f_{i, j} x^{i} y^{j} \in k(x, y) \cap k[[x, y]],
$$

define its diagonal by

$$
\operatorname{diag}(f):=\sum_{i=0}^{\infty} f_{i, i} x^{i}
$$

The following lemma shows that one can compute differential equations for diagonals of bivariate rational functions via rational-function telescoping. For the proof, see [67].

Lemma 3.6.2. If $L$ is a telescoper for $f(y, x / y) / y$ with respect to $y$, then $L(\operatorname{diag}(f))=0$.

First, we compare the various algorithms on an example borrowed from [74]:

$$
f=\frac{1}{1-x-y-x y\left(1-x^{d}\right)}, \quad \text { where } d \in \mathbb{N} .
$$

| $d$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| AZ | 3.53 | 6.33 | 13.6 | 38.5 | 68.1 | 145 | 263. | 368. |
| RatAZ | 5.27 | 4.63 | 8.72 | 16.9 | 36.1 | 55.4 | 99.4 | 352. |
| Hermite | 2.33 | 4.52 | 8.71 | 18.6 | 36.1 | 65.3 | 121. | 169. |

Table 3.4: Timing on Pemantle and Wilson's example (in seconds)

The next test is on an example from Stanley's book [83] related to the plane walks. Let $S_{d}=$ $\left\{(i, j) \in \mathbb{N}^{2} \mid i+j=d\right\}$, consider

$$
f(x, y, d)=\frac{1}{1-\sum_{(i, j) \in S_{d}} x^{i} y^{j}}, \quad \text { where } 11 \leq d \leq 20 .
$$

From the table 3.5, we observed that the order of minimal telescopers is $d / 2$ when $d$ is even.

| $d$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| AZ | 48.7 | 5.72 | 144. | 12.4 | 400. | 23.9 | 1016 | 46.7 | $>$ hr. | 81.2 |
| RatAZ | 43.8 | 5.61 | 129. | 11.8 | 269. | 27.9 | 663.4 | 45.8 | 2976 | 88.4 |
| Hermite | 11.7 | 2.55 | 31.9 | 5.71 | 91.3 | 12.8 | 227.8 | 21.1 | 617.9 | 40.3 |
| Order | 11 | 6 | 13 | 7 | 15 | 8 | 17 | 9 | 19 | 10 |

Table 3.5: Timings on the plane walk examples (in seconds)

We conjecture that this pattern is true for all even $d$ and expect a combinatorial explanation.

## Chapter 4

## Structure of Multivariate Hyperexponential-Hypergeometric Functions

### 4.1 Introduction

Multivariate hyperexponential-hypergeometric functions are a generalization of usual exponential functions and hypergeometric terms. In their paper [92], Wilf and Zeilberger observed that a large class of identities on special functions are related to the integrals or sums of proper hyperexponential-hypergeometric functions. Moreover, they have shown that elegant and computer-constructible proofs for those identities can be obtained efficiently via Zeilberger's method of creative telescoping. Hyperexponential-hypergeometric functions also play an important role in factoring modules over Laurent-Ore algebras [93].

Motivation. It is useful to decompose a hyperexponential-hypergeometric function as a product of "simpler" functions of the same kind. For example, Christopher [29] decomposes a hyperexponential function and uses its multiplicative form to compute Liouvillian first integrals [29]. The Ore-Sato theorem [70, 81], which describes the multiplicative structure of hypergeometric terms, is rediscovered and used by Payne [73], Hou [54], Abramov, and Petkovšek [14] to prove that Wilf and Zeilberger's conjecture holds for hypergeometric terms. Feng, Singer, and Wu
present a multiplicative form for bivariate hyperexponential-hypergeometric function to study Liouvillian solutions of linear difference-differential equations [43, 42]. We will decompose a multivariate hyperexponential-hypergeometric function in the hope of studying the general Wilf and Zeilberger's conjecture.

Our contribution. We decompose a multivariate hyperexponential-hypergeometric function into a product of a rational function, a hyperexponential function, a hypergeometric term and some "simpler" functions. Our result is a generalization of a result by Feng, Singer and Wu [42, Proposition 5]. Combining this result with multivariate extension of Christopher's theorem [29, 97] and the Ore-Sato theorem, we present a structure theorem for multivariate hyperexponentialhypergeometric functions.

The rest of this chapter is organized as follows. We describe in Section 4.2 an algebraic setting for hyperexponential-hypergeometric functions. Two rational normal forms are reviewed and a new one is introduced in Section 4.3. We describe a structure of compatible rational functions in Section 4.4. Based on this structure, we obtain a multiplicative form of hyperexponentialhypergeometric functions in Section 4.5.

An earlier version of this chapter can be found in [27], which is a joint work (in progress) with Ziming Li.

### 4.2 Algebraic setting

We describe an algebraic setting for hyperexponential-hypergeometric functions, which is introduced and used in [62, 22, 93, 66]. We will regard hyperexponential-hypergeometric functions as elements in some differential-difference extensions over the field of multivariate rational functions.

### 4.2.1 Hyperexponential-hypergeometric functions

Let $A$ be a commutative ring. Recall that a derivation $\delta$ on $A$ is an additive map from $A$ to itself satisfying the Leibniz rule

$$
\delta(a b)=a \delta(b)+\delta(a) b \quad \text { for all } a, b \in A
$$

Let $\Delta$ be a finite set of derivations and automorphisms from $A$ to itself. The pair $(A, \Delta)$ is called a differential-difference ring, or $\Delta$-ring for short. If the maps in $\Delta$ commute pairwise, then $(A, \Delta)$ is said to be orthogonal [62].

Let $k$ be a field of characteristic zero. For brevity, we let $\mathbf{x}$ stand for the continuous variables $\left(x_{1}, \ldots, x_{m}\right)$ and $\mathbf{y}$ for the discrete variables $\left(y_{1}, \ldots, y_{n}\right)$. Let $k(\mathbf{x}, \mathbf{y})$ be the field of rational functions in $x_{1}, \ldots, x_{m}$ and $y_{1}, \ldots, y_{n}$ over $k$. On the field $k(\mathbf{x}, \mathbf{y})$, the derivations $\delta_{i}(1 \leq i \leq m)$ and shift operators $\sigma_{j}(1 \leq j \leq n)$ are defined for all $f \in k(\mathbf{x}, \mathbf{y})$ by

$$
\delta_{i}(f)=\frac{\partial f}{\partial x_{i}} \quad \text { and } \quad \sigma_{j}(f)=f\left(\mathbf{x}, y_{1}, \ldots, y_{j-1}, y_{j}+1, y_{j+1}, \ldots, y_{n}\right)
$$

Put $\Delta=\left\{\delta_{1}, \ldots, \delta_{m}, \sigma_{1}, \ldots, \sigma_{n}\right\}$ and note that the elements of $\Delta$ commute pairwise over $k(\mathbf{x}, \mathbf{y})$. So the pair $(k(\mathbf{x}, \mathbf{y}), \Delta)$ is a $\Delta$-field.

A ring $\mathcal{R}$ is called a $\Delta$-extension of the field $k(\mathbf{x}, \mathbf{y})$ if $\mathcal{R}$ contains $k(\mathbf{x}, \mathbf{y})$, all derivations can be extended to $\mathcal{R}$, all the shift operators can be extended to monomorphisms to $\mathcal{R}$, and the extended operators also commute pairwise over $\mathcal{R}$. The set of all extended derivations and shift operators is still denoted by $\Delta$. The ring $\mathcal{R}$ is said to be simple if it contains no ideal closed under all maps in $\Delta$ except the zero ideal and the whole ring. An element $c$ of $\mathcal{R}$ is called a constant with respect to a derivation $\delta \in \Delta$ if $\delta(c)=0$ and a constant with respect to a shift operator $\sigma \in \Delta$ if $\sigma(c)=c$. An element $c$ of $\mathcal{R}$ is called a constant if it is a constant with respect to all operators in $\Delta$. The set of all constants of $\mathcal{R}$, denoted by $C_{\mathcal{R}}$, is a subring of $\mathcal{R}$. The ring $C_{\mathcal{R}}$ is a subfield if $\mathcal{R}$ is a field.

Definition 4.2.1. Let $\mathcal{R}$ be a $\Delta$-extension of $k(\mathbf{x}, \mathbf{y})$. A nonzero element $h$ of $\mathcal{R}$ is said to be hyperexponential-hypergeometric over $k(\mathbf{x}, \mathbf{y})$ if there exist $a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n} \in k(\mathbf{x}, \mathbf{y})$ such that $b_{1} \cdots b_{n} \neq 0$ and

$$
\begin{equation*}
\delta_{1}(h)=a_{1} h, \ldots, \delta_{m}(h)=a_{m} h, \quad \text { and } \quad \sigma_{1}(h)=b_{1} h, \ldots, \sigma_{n}(h)=b_{n} h . \tag{4.1}
\end{equation*}
$$

The rational functions $a_{i}$ and $b_{j}$ above are called the certificates of $h$ with respect to $x_{i}$ and $y_{j}$, respectively.

According to the definition above, it is easy to see that derivatives, shifts, and products of hyperexponential-hypergeometric functions are also hyperexponential-hypergeometric. If $\mathcal{R}$ is simple, then any hyperexponential-hypergeometric element in $\mathcal{R}$ is invertible [66, Lemma 2.1].

Since all elements in $\Delta$ commute pairwise, the certificates $a_{i}$ 's and $b_{j}$ 's in Definition 4.2 .1 satisfy three sets of integrability conditions:

$$
\begin{align*}
\delta_{i}\left(a_{j}\right)=\delta_{j}\left(a_{i}\right) & \text { for } 1 \leq i<j \leq m,  \tag{4.2}\\
\sigma_{i}\left(b_{j}\right) b_{i}=\sigma_{j}\left(b_{i}\right) b_{j} & \text { for } 1 \leq i<j \leq n,  \tag{4.3}\\
\frac{\delta_{i}\left(b_{j}\right)}{b_{j}}=\sigma_{j}\left(a_{i}\right)-a_{i} & \text { for } 1 \leq i \leq m \text { and } 1 \leq j \leq n . \tag{4.4}
\end{align*}
$$

In the discrete case, rational functions $b_{1}, \ldots, b_{n} \in k(\mathbf{y})$ in [14] are said to be compatible if they satisfy the integrability conditions (4.3). We follow this and call $a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}$ in $k(\mathbf{x}, \mathbf{y})$ compatible if they satisfy the integrability conditions (4.2), (4.3), and (4.4). As opposed to the treatment in [14], we regard the discrete variables $y_{1}, \ldots, y_{n}$ as indeterminates, and, thus, the function $h=\left|y_{1}-y_{2}\right|$ is excluded in the thesis.

### 4.2.2 First-order fully integrable systems

A $\Delta$-extension of $k(\mathbf{x}, \mathbf{y})$ may not be an integral domain in general. For this reason, we need to construct a simple $\Delta$-ring, in which any hyperexponential-hypergeometric element is invertible. We specialize the construction in [22, Section 3] for finitely many first-order systems.

Definition 4.2.2. Let $a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}$ be rational functions in $k(\mathbf{x}, \mathbf{y})$. A first-order system

$$
\begin{equation*}
\delta_{1}(z)=a_{1} z, \ldots, \delta_{m}(z)=a_{m} z, \quad \sigma_{1}(z)=b_{1} z, \ldots, \sigma_{n}(z)=b_{n} z \tag{4.5}
\end{equation*}
$$

is said to be fully integrable over $k(\mathbf{x}, \mathbf{y})$ if $b_{1} \cdots b_{n} \neq 0$ and $a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}$ are compatible.
According to Theorem 2 in [22], given a finite number of first-order fully integrable systems, there exists a simple differential-difference extension $\mathcal{R}$ of $k(x, n)$ such that $\mathcal{R}$ contains a nonzero solution of each system. Moreover, the subring of constants in $\mathcal{R}$ is equal to $k$ if $k$ is algebraically closed. Let $\mathcal{H}(\mathbf{a}, \mathbf{b})$ denote the solution space of the first-order fully integrable system (4.5) in such an extension $\mathcal{R}$. By Theorem 2 in [22], $\mathcal{H}(\mathbf{a}, \mathbf{b})$ is one-dimensional over $k$ if $k$ is algebraically closed. In the rest of this chapter, hyperexponential-hypergeometric functions will be regarded as elements in such an extension $\mathcal{R}$. Therefore, it is legitimate to add, multiply and invert hyperexponential-hypergeometric functions. Following Definition 2 in [14], two hyperexponential-hypergeometric functions are said to be conjugate if they have the same

| Function | Expression | Conjugates given by $\mathcal{H}(\mathbf{a}, \mathbf{b})$-Notation |
| :--- | :--- | :---: |
| exponential | $\exp (f)$ | $\mathcal{H}\left(\delta_{1}(f), \ldots, \delta_{m}(f), 1, \ldots, 1\right)$ |
| constant power | $\beta^{\lambda}$ | $\mathcal{H}\left(\lambda \frac{\delta_{1} \beta}{\beta}, \ldots, \lambda \frac{\delta_{m} \beta}{\beta}, 1, \ldots, 1\right)$ |
| symbolic power | $\beta^{y_{j}}$ | $\mathcal{H}\left(y_{j} \frac{\delta_{1} \beta}{\beta}, \ldots, y_{j} \frac{\delta_{m} \beta}{\beta}, 1, \ldots, 1, \beta, 1, \ldots, 1\right)$ |
| factorial | $(\lambda)_{\mathbf{e} \cdot \mathbf{y}}$ | $\mathcal{H}\left(0, \ldots, 0, \prod_{\ell=0}^{e_{1}-1}(\mathbf{e} \cdot \mathbf{y}+\lambda+\ell), \ldots, \prod_{\ell=0}^{e_{n}-1}(\mathbf{e} \cdot \mathbf{y}+\lambda+\ell)\right)$ |

(In the table above, $f, \beta \in k(\mathbf{x}), \lambda \in k$, and $\mathbf{e} \in \mathbb{Z}^{n}$.)

Table 4.1: Familiar functions and their $\mathcal{H}(\mathbf{a}, \mathbf{b})$-representations
certificates. Two conjugate functions can only differ by a nonzero multiplicative element in $k$ if $k$ is algebraically closed.

The following lemma shows some basic properties of the space $\mathcal{H}(\mathbf{a}, \mathbf{b})$. They are derived directly from the definition.

Lemma 4.2.3. (i) For any hyperexponential-hypergeometric function $g$ over $k(\mathbf{x}, \mathbf{y})$, we have

$$
g \mathcal{H}\left(a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}\right)=\mathcal{H}\left(a_{1}+\frac{\delta_{1}(g)}{g}, \ldots, a_{m}+\frac{\delta_{m}(g)}{g}, b_{1} \frac{\sigma_{1}(g)}{g}, \ldots, b_{n} \frac{\sigma_{n}(g)}{g}\right) .
$$

(ii) $\mathcal{H}(\mathbf{a}, \mathbf{b}) \mathcal{H}(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})=\mathcal{H}(\mathbf{a}+\tilde{\mathbf{a}}, \mathbf{b} \tilde{\mathbf{b}})$, where $\mathbf{a}+\tilde{\mathbf{a}}$ and $\mathbf{b} \tilde{\mathbf{b}}$ are defined termwise.
(iii) $\delta_{i}(\mathcal{H}(\mathbf{a}, \mathbf{b}))=a_{i} \mathcal{H}(\mathbf{a}, \mathbf{b})$ for $1 \leq i \leq m$ and $\sigma_{j}(\mathcal{H}(\mathbf{a}, \mathbf{b}))=b_{j} \mathcal{H}(\mathbf{a}, \mathbf{b})$ for $1 \leq j \leq n$.

Some examples for hyperexponential-hypergeometric functions are listed in Table 4.1.

### 4.3 Rational normal forms

In this section, we review two normal forms for univariate rational functions in [13, 10, 47]. Those normal forms have played an important role in the minimal decompositions of hyperexponential and hypergeometric functions [47, 12, 13, 10]. In order to study the compatible multivariate rational functions in the next sections, we introduce a new kind of rational normal forms for the continuous-discrete rational functions in $k(\mathbf{x}, \mathbf{y})$.

### 4.3.1 Differential and shift rational normal forms

Let $F$ be a field of characteristic zero. The field $F(z)$ of univariate rational functions is equipped with both differential and difference structures by

$$
\delta(f(z))=\frac{d(f(z))}{d z} \quad \text { and } \quad \sigma(f(z))=f(z+1) \quad \text { for all } f \in F(z) .
$$

A polynomial $P \in F[z]$ is said to be squarefree with respect to $z$ over $F$ if $\operatorname{gcd}(P, \delta(P))=1$. It is said to be shift-free if $\operatorname{gcd}\left(P, \sigma^{i}(P)\right)=1$ for all $i \in \mathbb{Z} \backslash\{0\}$. In other words, any two roots of a shift-free polynomial have a non-integer distance. In the following, we collect basic facts concerning rational functions whose denominators are squarefree or shift-free, which has appeared implicitly in the literature [2, 71, 52, 53].

Lemma 4.3.1. Let $f=P / Q$ be in $F(z)$ with $\operatorname{gcd}(P, Q)=1$ and $\operatorname{deg}(P)<\operatorname{deg}(Q)$. Then
(i) If $Q$ is squarefree and $f=\delta(g)$ for some $g \in F(z)$, then $f=0$.
(ii) If $Q$ is shift-free and $f=\sigma(g)-g$ for some $g \in F(z)$, then $f=0$.

Definition 4.3.2 (Differential-reduced, shift-reduced). A rational function $f=P / Q \in F(z)$ with $\operatorname{gcd}(P, Q)=1$ is said to be differential-reduced with respect to $z$ over $F$ if

$$
\operatorname{gcd}(Q, P-i \delta(Q))=1, \quad \text { for all } i \in \mathbb{Z}
$$

It is said to be shift-reduced with respect to $z$ over $F$ if

$$
\operatorname{gcd}\left(P, \sigma^{i}(Q)\right)=1, \quad \text { for all } i \in \mathbb{Z}
$$

The following lemmas show basic properties of differential-reduced and shift-reduced rational functions, respectively. Those results are seemingly classical in [51, 15, 47, 13], but we still present their proofs for completeness.

Lemma 4.3.3. Let $f=P / Q \in F(z)$ be differential-reduced with respect to $z$, where $P, Q \in F[z]$ and $\operatorname{gcd}(P, Q)=1$. If $f=\delta(g) / g$ for some $g \in F(z)$, then $g \in F$ and $f=0$.

Proof. Suppose that $g \in F(z) \backslash F$. Then

$$
f=\frac{\delta(g)}{g}=\sum_{i=1}^{s} \frac{m_{i}}{z-\alpha_{i}}, \quad \text { where } m_{i} \in \mathbb{Z} \text { and } \alpha_{i} \in \bar{F} \text { for all } i \text { with } 1 \leq i \leq s
$$

By Theorem 2.3.8, the $m_{i}$ 's are roots of $\operatorname{RT}_{z}(f)$. $\operatorname{Sog} \operatorname{gcd}\left(Q, P-m_{i} \delta(Q)\right) \neq 1$, which contradicts the assumption that $f$ is differential-reduced. So $g \in F$ and therefore $f=0$.

Lemma 4.3.4. Let $f \in F(z)$ be shift-reduced with respect to $z$ over $F$. If $f=\sigma(g) / g$ for some $g \in F(z)$, then $g \in F$ and $f=1$.

Proof. Let $g=a / b$ with $a, b \in F[z]$ and $\operatorname{gcd}(a, b)=1$. Suppose that $g$ is not in $F$. Then either $a$ or $b$ has a positive degree. Assume that the degree of $a$ is positive. Then there exists an element $\alpha \in \bar{F}$ such that $\alpha$ is a root of $a$, but $\alpha-1$ is not. Moreover, there exists a nonnegative integer $\ell$ such that $\alpha+\ell$ is a root of $a$ but $\alpha+\ell+1$ is not.

Set $\beta=\alpha+\ell$. Since $f=\sigma(g) / g$,

$$
f=\frac{\sigma(a) b}{a \sigma(b)}
$$

We have that $\alpha-1$ is a root of $\sigma(a)$, but not a root of $a \sigma(b)$ by the definition of $\alpha$ and since $\operatorname{gcd}(\sigma(a), \sigma(b))=1$. Similarly, $\beta$ is a root of $a$, but not a root of $\sigma(a) b$ by the definition of $\beta$ and since $\operatorname{gcd}(a, b)=1$. It follows that $\alpha-1$ is a root of the numerator of $f$, while $\beta$ is a root of the denominator of $f$. Since $\beta-(\alpha-1)$ is an integer, $f$ is not shift-reduced, a contradiction. Hence, $a$ belongs to $F$. In the same vein, $b$ belongs to $F$.

Definition 4.3.5 (Differential rational normal form). For $f \in F(z)$, call a pair $(K, S) \in F(z)^{2}$ a differential rational normal form (abbreviated as DRNF) of $f$ if $f=K+\delta(S) / S$ and $K$ is differential-reduced with respect to $z$ over $F$. If, in addition, the denominators of $K$ and $S$ are coprime, then the pair $(K, S)$ is said to be a strict DRNF of $f$.

Definition 4.3.6 (Shift rational normal form). For $f \in F(z)$, call a pair $(K, S) \in F(z) \times F(z)$ a shift rational normal form (abbreviated as SRNF) of $f$ if $f=K \cdot \sigma(S) / S$ and $K$ is shift-reduced with respect to $z$ over $F$. Let $k_{1}=\operatorname{num}(K), k_{2}=\operatorname{den}(K), s_{1}=\operatorname{num}(S)$ and $s_{2}=\operatorname{den}(S)$. If, furthermore,

$$
\operatorname{gcd}\left(k_{1}, \sigma\left(s_{2}\right) s_{1}\right)=\operatorname{gcd}\left(k_{2}, \sigma\left(s_{1}\right) s_{2}\right)=1
$$

then the pair $(K, S)$ is said to be a strict SRNF of $f$.

Both strict DRNF's and strict SRNF's exist and can be computed efficiently [47, 13]. Let $h(z)$ be a hyperexponential or hypergeometric function over $F(z)$ with certificate $f$ in $F(z)$. Any differential or shift rational normal form $(K, S)$ of $f$ leads to a multiplicative factorization of $h$ in the form

$$
h(z)=S h^{\prime}, \quad \text { where the certificate of } h^{\prime} \text { is } K
$$

From this factorization, one can perform two kinds of minimal decompositions on $h$, which will be reviewed in the next chapter.

### 4.3.2 Y-rational normal forms

We introduce a new normal form for rational functions in $k(\mathbf{x}, \mathbf{y})$, which will help us derive the structure of certificates in the continuous-discrete setting.

For brevity, we set $F_{i}=k\left(x_{1}, \ldots, x_{i-1}, x_{i+1}, \ldots x_{m}, \mathbf{y}\right)$ for all $i \in\{1, \ldots, m\}$, and abbreviate "irreducible partial fraction decomposition" as IPFD.

For a nonzero element $r \in k(\mathbf{x}, \mathbf{y})$, its IPFD with respect to $x_{i}$ can be written as

$$
r=p_{0}+\sum_{i=1}^{\ell} \sum_{j=1}^{n_{i}} \frac{p_{i j}}{q_{i}^{j}}
$$

where $p_{0}, p_{i j}, q_{i}$ are in $F_{i}\left[x_{i}\right]$ such that $\operatorname{deg}_{x_{i}}\left(p_{i j}\right)<\operatorname{deg}_{x_{i}}\left(q_{i}\right)$ and $q_{i}$ is monic and irreducible.
Let $\mathbb{Y}$ be the additive group generated by $1, y_{1}, \ldots, y_{n}$. Let $r$ and its IPFD with respect to $x_{i}$ be given above. We define $r_{S}$ to be the sum of all fractions of the form $p_{i, 1} / q_{i}$ with the following two properties
(i) $p_{i, 1}=y \delta_{i}\left(q_{i}\right)$ for some nonzero element $y \in \mathbb{Y}$, and
(ii) $n_{i}=1$, that is, $q_{i}^{2}$ does not divide the denominator of $r$.

Furthermore, set $r_{K}=r-r_{S}$. We call $r_{K}$ the $\mathbb{Y}$-kernel, and $r_{S}$ the $\mathbb{Y}$-shell of $r$ with respect to $x_{i}$. For completeness, the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of zero are both defined to be zero.

Note that a $\mathbb{Y}$-shell with respect to $x_{i}$ can be written as

$$
r_{S}=\frac{\delta_{i}(f)}{f}+\sum_{j=1}^{n} y_{j} \frac{\delta_{i}\left(u_{j}\right)}{u_{j}}
$$

for some $f, u_{j} \in k(\mathbf{x}, \mathbf{y})$. Moreover, the denominators of the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of a nonzero rational function are coprime.

Lemma 4.3.7. Let $r$ be a nonzero rational function in $k(\mathbf{x}, \mathbf{y})$ with $\mathbb{Y}$-kernel $r_{K}$ and $\mathbb{Y}$-shell $r_{S}$ with respect to $x_{i}$. Let $g=y \delta_{i}(q) / q$, where $y$ is a nonzero element of $\mathbb{Y}$, and $q$ is a monic and irreducible polynomial in $F_{i}\left[x_{i}\right]$ with positive degree.
(i) If $q$ is a divisor of the denominator of $r_{K}$, then the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of $r+g$ are $r_{K}+g$ and $r_{S}$, respectively.
(ii) If $q$ is not a divisor of the denominator of $r_{K}$, then the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of $r+g$ are $r_{K}$ and $r_{S}+g$, respectively.

Proof. Since the denominator $q$ of $g$ is irreducible, it suffices to look at $q$-expansions of $r$ instead of its IPFD. The $q$-adic expansion of $r$ with respect to $x_{i}$ can be written as:

$$
\begin{equation*}
r=\frac{p_{\ell}}{q^{\ell}}+\cdots+\frac{p_{2}}{q^{2}}+\frac{p_{1}}{q}+\text { higher terms }, \tag{4.6}
\end{equation*}
$$

where $p_{\ell}, \ldots, p_{2}, p_{1} \in F_{i}\left[x_{i}\right]$ with degrees less than the degree of $q$ in $x_{i}$. It follows that the $q$-adic expansion of $r+g$ with respect to $x_{i}$ is

$$
\begin{equation*}
r+g=\frac{p_{\ell}}{q^{\ell}}+\cdots+\frac{p_{2}}{q^{2}}+\frac{p_{1}+y \delta(q)}{q}+\text { higher terms } \tag{4.7}
\end{equation*}
$$

Assume that $q$ is a divisor of the denominator of $r_{K}$. Then one of $p_{1}, p_{2}, \ldots, p_{\ell}$ is nonzero. If one of $p_{2}, \ldots, p_{\ell}$ is nonzero, then there is no fraction with denominator $q$ appearing in the IPFD of the $\mathbb{Y}$-shell of $r+g$. The first assertion holds. If $p_{2}, \ldots, p_{\ell}$ are all equal to zero, then $p_{1}$ is nonzero. By the definition of $\mathbb{Y}$-shells, $p_{1}$ is not a product of $\delta(q)$ and a nonzero element in $\mathbb{Y}$, and nor is $p_{1}+y \delta_{i}(q)$. Hence, the fraction $\left(p_{1}+y \delta_{i}(q)\right) / q$ appears in the IPFD of the $\mathbb{Y}$-kernel of $r+g$. The first assertion holds again.

Assume that $q$ is not a divisor of the denominator of $r_{K}$. Then $p_{1}$ in (4.6) is of the form $y^{\prime} \delta_{i}(q)$ for some $y^{\prime} \in \mathbb{Y}$ (noting that $y^{\prime}$ may possibly be zero). Moreover, all $p_{2}, \ldots, p_{\ell}$ are equal to zero. Hence, (4.7) becomes

$$
r+g=\left(y^{\prime}+y\right) \frac{\delta_{i}(q)}{q}+\text { higher terms }
$$

It follows that the $\mathbb{Y}$-shell of $r+g$ is equal to $r_{S}+g$.

We now define five additive subgroups of $k(\mathbf{x}, \mathbf{y})$ in order to suppress complicated expressions that would appear. Let

$$
\begin{gathered}
L_{i}=\left\{\left.\frac{\delta_{i}(f)}{f} \right\rvert\, f \in k(\mathbf{x}, \mathbf{y}), f \neq 0\right\}, \\
M_{i}=\left\{\left.\sum_{\ell=1}^{n} y_{\ell} \frac{\delta_{i}\left(u_{\ell}\right)}{u_{\ell}} \right\rvert\, u_{\ell} \in k(\mathbf{x}), u_{\ell} \neq 0\right\},
\end{gathered}
$$

and

$$
M_{i, j}=\left\{\left.\sum_{\ell=1}^{j-1} y_{\ell} \frac{\delta_{i}\left(v_{\ell}\right)}{v_{\ell}}+\sum_{\ell=j+1}^{n} y_{\ell} \frac{\delta_{i}\left(v_{\ell}\right)}{v_{\ell}} \right\rvert\, v_{\ell} \in k\left(\mathbf{x}, y_{j}\right), v_{\ell} \neq 0\right\} .
$$

Moreover, let $N_{i}=L_{i}+M_{i}+k(\mathbf{x})$ and $N_{i, j}=L_{i}+M_{i, j}+k\left(\mathbf{x}, y_{j}\right)$.
Remark 4.3.8. Let $G$ be one of the subgroups $L_{i}, M_{i}, N_{i}, M_{i, j}, N_{i, j}$. If r belongs to $G$, so does every fraction appearing in the IPFD of $r$ with respect to $x_{i}$.

The next lemma is rather technical. It tells us how the $\mathbb{Y}$-shell and $\mathbb{Y}$-kernel of an element in $N_{i, j}$ look like.

Lemma 4.3.9. Let $i \in\{1, \ldots, m\}$ and $j \in\{1, \ldots, n\}$. Let a be a nonzero element of $N_{i, j}$ with $\mathbb{Y}$-kernel $a_{K}$ and $\mathbb{Y}$-shell $a_{S}$ with respect to $x_{i}$. Then there exists $u \in k\left(\mathbf{x}, y_{j}\right)$ such that

$$
a_{K} \equiv 0 \quad \bmod M_{i, j}+k\left(\mathbf{x}, y_{j}\right) \quad \text { and } \quad a_{S} \equiv y_{j} \frac{\delta_{i}(u)}{u} \quad \bmod L_{i}+M_{i, j} .
$$

Proof. Without loss of generality, assume that $i=1$ and $j=n$. All the IPFD's, $\mathbb{Y}$-kernels and $\mathbb{Y}$-shells in the proof are with respect to $x_{1}$. Our goal is to show that

$$
\begin{equation*}
a_{K} \equiv 0 \quad \bmod M_{1, n}+k\left(\mathbf{x}, y_{n}\right) \quad \text { and } \quad a_{S} \equiv y_{n} \frac{\delta_{1}(u)}{u} \quad \bmod L_{1}+M_{1, n} \tag{4.8}
\end{equation*}
$$

for some $u \in k\left(\mathbf{x}, y_{n}\right)$.
Since $a \in N_{1, n}$, there exists $r \in k\left(\mathbf{x}, y_{n}\right)$ such that

$$
a \equiv r \quad \bmod L_{1}+M_{1, n} .
$$

Assume that $r_{K}$ and $r_{S}$ are the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of $r$, respectively. Then

$$
r_{K} \in k\left(\mathbf{x}, y_{n}\right) \quad \text { and } \quad r_{S}=\frac{\delta_{1}(f)}{f}+y_{n} \frac{\delta_{1}(u)}{u} .
$$

where $f, u \in k\left(\mathbf{x}, y_{n}\right)$. Using congruences, we have

$$
\begin{equation*}
r_{K} \equiv 0 \quad \bmod M_{1, n}+k\left(\mathbf{x}, y_{n}\right) \quad \text { and } \quad r_{S} \equiv y_{n} \frac{\delta_{1}(u)}{u} \quad \bmod L_{1}+M_{1, n} \tag{4.9}
\end{equation*}
$$

To proceed, we denote by $p$ the denominator of $r_{K}$, which is viewed as a polynomial in $x_{1}$ over $k\left(x_{2}, \ldots, x_{m}, y_{n}\right)$. Let $b=a-r$, which is in $L_{1}+M_{1, n}$. If $b=0$, then there is nothing to prove. Assume that $b \neq 0$, and that $g$ is a fraction appearing in the IPFD of $b$. By the definitions
of $L_{1}$ and $M_{1, n}$, we have that $g=y \delta_{1}(q) / q$, where $y$ is a nonzero element in $\mathbb{Y}$ free of $y_{n}$, and $q$ is a monic and irreducible polynomial in $x_{1}$ over $k\left(x_{2}, \ldots, x_{m}, \mathbf{y}\right)$. We now make a case distinction.

Case 1. If $q$ is a divisor of $p$, then we set $r_{K}^{\prime}=r_{K}+g$ and $r_{S}^{\prime}=r_{S}$.
Case 2. If $q$ is not a divisor of $p$, then we set $r_{K}^{\prime}=r_{K}$ and $r_{S}^{\prime}=r_{S}+g$.
Let $r^{\prime}=r_{K}^{\prime}+r_{S}^{\prime}$. By Lemma 4.3.7, $r_{K}^{\prime}$ and $r_{S}^{\prime}$ are the $\mathbb{Y}$-kernel and $\mathbb{Y}$-shell of $r^{\prime}$, respectively. In Case $1, q$ is a factor of $p$. So $q$ is in $k\left(\mathbf{x}, y_{n}\right)$. Accordingly, $g$ is in $M_{1, n}+k\left(\mathbf{x}, y_{n}\right)$. It follows from (4.9) that

$$
r_{K}^{\prime} \equiv 0 \quad \bmod M_{1, n}+k\left(\mathbf{x}, y_{n}\right) \quad \text { and } \quad r_{S}^{\prime} \equiv y_{n} \frac{\delta_{1}(u)}{u} \quad \bmod L_{1}+M_{1, n}
$$

By Remark 4.3.8, $g$ is in $L_{1}+M_{1, n}$, which, together with (4.9), implies that the above two congruences also hold in Case 2.

Since $a-r^{\prime}=a-r-g$ and $g \in L_{1}+M_{1, n}$, we have

$$
a \equiv r^{\prime} \quad \bmod L_{1}+M_{1, n}
$$

Let $b^{\prime}=a-r^{\prime}$. Then the number of fractions in the IPFD of $b^{\prime}$ is less than that in the IPFD of $b$. Repeating the same argument on $b^{\prime}$, we see that (4.8) follows from an easy induction.

The lemma below is useful to generalize Proposition 5 in [42].

Lemma 4.3.10. Let $i \in\{1, \ldots, m\}$ and $p, q \in\{1, \ldots, n\}$ with $p \neq q$. Then

$$
N_{i, p} \cap N_{i, q}=N_{i}
$$

Proof. Without loss of generality, we assume that $i=1, p=1, q=n$ and $n>1$. Since

$$
M_{1} \subset M_{1,1}+k\left(\mathbf{x}, y_{1}\right) \quad \text { and } \quad M_{1} \subset M_{1, n}+k\left(\mathbf{x}, y_{n}\right)
$$

we have $N_{1}$ is included in the intersection $N_{1,1}$ and $N_{1, n}$ by definition. It suffices to show the opposite inclusion. Assume that $a$ is a nonzero element in the intersection of $N_{1,1}$ and $N_{1, n}$. Let $a_{S}$ and $a_{K}$ be, respectively, the $\mathbb{Y}$-shell and $\mathbb{Y}$-kernel of $a$ with respect to $x_{1}$. By Lemma 4.3.9, the $\mathbb{Y}$-shell of $a$ with respect to $x_{1}$ is of the form

$$
a_{S}=\frac{\delta_{1}\left(f_{1}\right)}{f_{1}}+\sum_{j=1}^{n} y_{j} \frac{\delta_{1}\left(v_{j}\right)}{v_{j}}=\frac{\delta_{1}\left(f_{1}^{\prime}\right)}{f_{1}^{\prime}}+\sum_{j=1}^{n} y_{j} \frac{\delta_{1}\left(v_{j}^{\prime}\right)}{v_{j}^{\prime}}
$$

where $f_{1}, f_{1}^{\prime} \in k(\mathbf{x}, \mathbf{y}), v_{j} \in k\left(\mathbf{x}, y_{1}\right)$ and $v_{j}^{\prime} \in k\left(\mathbf{x}, y_{n}\right)$ for all $j$ with $1 \leq j \leq n$. Since the elements $1, y_{1}, \ldots, y_{n}$ are $\mathbb{Z}$-linearly independent constants with respect to $\delta_{1}$, Corollary 2.3.5 implies that

$$
\frac{\delta_{1}\left(f_{1}\right)}{f_{1}}=\frac{\delta_{1}\left(f_{1}^{\prime}\right)}{f_{1}^{\prime}} \quad \text { and } \quad \frac{\delta_{1}\left(v_{j}\right)}{v_{j}}=\frac{\delta_{1}\left(v_{j}^{\prime}\right)}{v_{j}^{\prime}}
$$

for all $j$ with $1 \leq j \leq n$. Since the differential equation

$$
\delta_{1}(z)=\frac{\delta_{1}\left(v_{j}^{\prime}\right)}{v_{j}^{\prime}} z
$$

has a solution $v_{j} \in k\left(\mathbf{x}, y_{1}\right)$ and its coefficients belong to $k\left(\mathbf{x}, y_{n}\right)$, it must have a solution $u_{j}$ in $k(\mathbf{x})$. Hence,

$$
a_{S}=\frac{\delta_{1}\left(f_{1}\right)}{f_{1}}+\sum_{j=1}^{n} y_{j} \frac{\delta_{1}\left(u_{j}\right)}{u_{j}} \in N_{1} .
$$

It remains to show that the $\mathbb{Y}$-kernel of $a$ with respect to $x_{1}$ belongs to $N_{1}$. By Lemma 4.3.9, the $\mathbb{Y}$-kernel of $a$ is of the form

$$
a_{K}=y_{n} \frac{\delta_{1}\left(w_{n}\right)}{w_{n}}+\sum_{\ell=2}^{n-1} y_{\ell} \frac{\delta_{1}\left(w_{\ell}\right)}{w_{\ell}}+s=y_{1} \frac{\delta_{1}\left(w_{1}^{\prime}\right)}{w_{1}^{\prime}}+\sum_{\ell=2}^{n-1} y_{\ell} \frac{\delta_{1}\left(w_{\ell}^{\prime}\right)}{w_{\ell}^{\prime}}+s^{\prime}
$$

where $w_{2}, \ldots, w_{n}, s \in k\left(\mathbf{x}, y_{1}\right)$ and $w_{1}^{\prime}, \ldots, w_{n-1}^{\prime}, s^{\prime} \in k\left(\mathbf{x}, y_{n}\right)$. Note that $a_{K}$ is a polynomial in $y_{1}$ over $k\left(\mathbf{x}, y_{2}, \ldots, y_{n}\right)$ whose degree (in $\left.y_{1}\right)$ is less than or equal to 1 by the second equality given above. So $w_{2}, \ldots, w_{n}$ in the first equality can be taken as elements in $k(\mathbf{x})$. For, otherwise, the denominator of $a_{K}$ would involve $y_{1}$, a contradiction. It follows that

$$
s=c y_{1}+d
$$

where $c, d \in k(\mathbf{x})$. In the same vein, we have

$$
a_{K}=y_{1} \frac{\delta_{1}\left(w_{1}^{\prime}\right)}{w_{1}^{\prime}}+\sum_{\ell=2}^{n-1} y_{\ell} \frac{\delta_{1}\left(w_{\ell}^{\prime}\right)}{w_{\ell}^{\prime}}+c^{\prime} y_{n}+d^{\prime}
$$

where $w_{1}^{\prime}, \ldots, w_{n-1}^{\prime}$, and $c^{\prime}, d^{\prime}$ are chosen to be in $k(\mathbf{x})$. It follows that $c=\delta_{1}\left(w_{1}^{\prime}\right) / w_{1}^{\prime}$. Consequently,

$$
a_{K}=y_{1} \frac{\delta_{1}\left(w_{1}^{\prime}\right)}{w_{1}^{\prime}}+\sum_{\ell=2}^{n} y_{\ell} \frac{\delta_{1}\left(w_{\ell}\right)}{w_{\ell}}+d
$$

which belongs to $N_{1}$. We have proved that both $a_{S}$ and $a_{K}$ are in $N_{1}$, and so is $a$.
The lemma below can be viewed as a discrete analogue of Lemma 4.3.10.

Lemma 4.3.11. Let $p, q \in\{1, \ldots, m\}$ with $p \neq q$, and $j \in\{1, \ldots, n\}$. If a nonzero rational function $b \in k(\mathbf{x}, \mathbf{y})$ can be written as

$$
\begin{equation*}
b=\frac{\sigma_{j}\left(f_{p}\right)}{f_{p}} \beta_{p} \alpha_{p}=\frac{\sigma_{j}\left(f_{q}\right)}{f_{q}} \beta_{q} \alpha_{q} \tag{4.10}
\end{equation*}
$$

for some $f_{p}, f_{q} \in k(\mathbf{x}, \mathbf{y}), \beta_{p}, \beta_{q} \in k(\mathbf{x}), \alpha_{p} \in k\left(x_{p}, \mathbf{y}\right)$ and $\alpha_{q} \in k\left(x_{q}, \mathbf{y}\right)$, then there exist $f \in$ $k(\mathbf{x}, \mathbf{y}), \alpha \in k(\mathbf{y})$, and $\beta \in k(\mathbf{x})$ such that

$$
b=\frac{\sigma_{j}(f)}{f} \beta \alpha .
$$

Proof. Without loss of generality, we may assume that both $\alpha_{p}$ and $\alpha_{q}$ are shift-reduced with respect to $y_{j}$. Suppose that

$$
\begin{equation*}
\frac{\alpha_{p}}{\alpha_{q}}=\frac{\sigma_{j}(g)}{g} \frac{1}{u}, \tag{4.11}
\end{equation*}
$$

in which $(g, 1 / u)$ is a shift rational normal form of $\alpha_{p} / \alpha_{q}$ with respect to $y_{j}$. Then $g$ belongs to $k(\mathbf{y})$, because $x_{p}$ and $x_{q}$ are two distinct indeterminates, and $\alpha_{p}, \alpha_{q}$ are shift-reduced. It follows from (4.10) and (4.11) that

$$
\frac{\sigma_{j}(w)}{w}=\frac{\beta_{q}}{\beta_{p}} u
$$

where $w=g f_{p} / f_{q}$. Since $u$ is shift-reduced with respect to $y_{j}$, and $\beta_{p}, \beta_{q} \in k(\mathbf{x})$, the above equation and Lemma 4.3.4 imply that $\sigma_{j}(w)=w$, and

$$
\frac{\beta_{q}}{\beta_{p}} u=1,
$$

and, hence, $u \in k(\mathbf{x})$. By (4.11), $\alpha_{p} \in k\left(x_{p}, \mathbf{y}\right)$ is a product of an element in $k(\mathbf{y})$, an element in $k(\mathbf{x})$, and $\alpha_{q}$ in $k\left(x_{q}, \mathbf{y}\right)$. By the uniqueness of the factorization of rational functions, $\alpha_{p}$ can be written as a product $c d$ for some $c \in k\left(x_{p}\right)$ and $d \in k(\mathbf{y})$. Setting $f=f_{p}, \beta=\beta_{p} c$, and $\alpha_{p}=d$ yields the lemma.

### 4.4 Structure of compatible rational functions

The goal of this section is to show the following theorem, which describes the structure of compatible rational functions. To this end, let us recall a useful notation from Payne's thesis [73].

For any integers $a, b \in \mathbb{Z}$ and a sequence of expressions $A_{i}$, define

$$
\prod_{i}^{b} A_{i}= \begin{cases}\prod_{i=a}^{b-1} A_{i}, & \text { if } b>a \\ 1, & \text { if } b=a \\ 1 / \prod_{i=b}^{a-1} A_{i}, & \text { if } b<a\end{cases}
$$

Theorem 4.4.1. Assume that $a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n} \in k(\mathbf{x}, \mathbf{y})$ are $m+n$ rational functions such that $b_{1} \cdots b_{n} \neq 0$, and that all the equalities in (4.2), (4.3) and (4.4) hold. Then there exist $f \in$ $k(\mathbf{x}, \mathbf{y})$, univariate rational functions $r_{v} \in k(z)$ for each $v$ in a finite set $V \subset \mathbb{Z}^{n}, c_{1}, \ldots, c_{L} \in \bar{k}$, $g_{0}, \beta_{1}, \ldots, \beta_{n} \in k(\mathbf{x})$, and $g_{1}, \ldots, g_{L} \in \bar{k}(\mathbf{x})$ such that

$$
\begin{equation*}
a_{i}=\delta_{i}\left(g_{0}\right)+\frac{\delta_{i}(f)}{f}+\sum_{\ell=1}^{L} c_{\ell} \frac{\delta_{i}\left(g_{\ell}\right)}{g_{\ell}}+\sum_{j=1}^{n} y_{j} \frac{\delta_{i}\left(\beta_{j}\right)}{\beta_{j}} \quad \text { for all } i \text { with } 1 \leq i \leq m \tag{4.12}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{j}=\frac{\sigma_{j}(f)}{f} \beta_{j} \prod_{v \in V} \prod_{p}^{v_{j}} r_{v}(\mathbf{y} \cdot v+p) \quad \text { for all } j \text { with } 1 \leq j \leq n . \tag{4.13}
\end{equation*}
$$

where $\mathbf{y} \cdot v$ denotes the inner product $y_{1} v_{1}+\ldots+y_{n} v_{n}$.

Before proving this theorem, we review two special cases in Sections 4.4.1 and 4.4.2.

### 4.4.1 The Ore-Sato theorem

The structure of rational solutions of the recurrence equation

$$
R_{1}(m, n+1) R_{2}(m, n)=R_{1}(m, n) R_{2}(m+1, n)
$$

has been described by Ore [70]. The multivariate extension of Ore's theorem was obtained by Sato [81] in the 1960s when he developed the theory of prehomogeneous vector spaces. In the process of proving the discrete case of Wilf and Zeilberger's conjecture on holonomic hypergeometric terms, the Ore-Sato theorem was discovered and proved again by Payne in his Ph.D. thesis [73] and independently by Abramov and Petkovšek [14]. In particular, the case of two variables has also been shown by Hou [54, 55] and by Abramov and Petkovšek [12]. Their results reveal a multiplicative structure of the certificates $b_{1}, \ldots, b_{n}$ of a hyperexponential-hypergeometric function from its integrability conditions (4.3). The following presentation of the Ore-Sato theorem is taken from Payne's thesis [73, Theorem 2.8.4].

Theorem 4.4.2 (Ore-Sato theorem). Let $b_{1}, \ldots, b_{n} \in k(\mathbf{y})$ be nonzero rational functions such that

$$
b_{i} \sigma_{i}\left(b_{j}\right)=b_{j} \sigma_{j}\left(b_{i}\right), \quad \text { for } 1 \leq i<j \leq n
$$

Then there exist a rational function $f \in k(\mathbf{y})$, a finite set $V \subset \mathbb{Z}^{n}$, and univariate rational functions $r_{v}$ in $k(z)$ for each $v \in V$ such that for all $j$ with $1 \leq j \leq n$,

$$
b_{j}=\frac{\sigma_{j}(f)}{f} \prod_{v \in V} \prod_{p}^{v_{j}} r_{v}(\mathbf{y} \cdot v+p)
$$

where $\mathbf{y} \cdot v$ denotes the inner product $y_{1} v_{1}+\ldots+y_{n} v_{n}$.

### 4.4.2 Alternative proof of multivariate Christopher's theorem

A multivariate hyperexponential function $h(\mathbf{x})$ is a nonzero solution of the first-order fully integrable system

$$
\delta_{1}(z)=a_{1} z, \quad \ldots, \quad \delta_{m}(z)=a_{m} z
$$

where $a_{1}, \ldots, a_{m}$ are compatible rational functions in $k(\mathbf{x})$, i.e., they satisfy the integrability conditions (4.3). In his refinement of Singer's theorem [82] on Liouvillian first integrals, Christopher [29] has described a possible form for those compatible $a_{i}$ 's in the bivariate case.

Theorem 4.4.3 (Christopher, 1999). Any bivariate hyperexponential function $h\left(x_{1}, x_{2}\right)$ over the field $\mathbb{C}\left(x_{1}, x_{2}\right)$ can be written as

$$
\exp (f) \prod_{\ell=1}^{L} g_{\ell}^{c_{\ell}}, \quad \text { where } f, g_{\ell} \in \mathbb{C}\left(x_{1}, x_{2}\right) \text { and } c_{\ell} \in \mathbb{C}
$$

Consequently, the two certificates $a_{1}$ and $a_{2}$ of $h\left(x_{1}, x_{2}\right)$ can be written as

$$
a_{i}=\delta_{i}(f)+\sum_{\ell=1}^{L} c_{\ell} \frac{\delta_{i}\left(g_{\ell}\right)}{g_{\ell}}, \quad \text { for } i=1,2 .
$$

Zoladek [97] has shown the multivariate extension of Christopher's theorem by using a result in Cerveau and Mattei's book [25]. We offer an alternative proof involving only rational functions.

Theorem 4.4.4 (Multivariate Christopher's theorem). Let $a_{1}, \ldots, a_{m} \in k(\mathbf{x})$ be rational functions such that

$$
\delta_{i}\left(a_{j}\right)=\delta_{j}\left(a_{i}\right), \quad \text { for } 1 \leq i<j \leq m .
$$

Then there exist $f \in k(\mathbf{x})$, nonzero elements $c_{\ell} \in \bar{k}$, and $g_{\ell} \in k\left(c_{\ell}\right)(\mathbf{x})$ for $1 \leq \ell \leq n$ such that

$$
a_{i}=\delta_{i}(f)+\sum_{\ell=1}^{n} c_{\ell} \frac{\delta_{i}\left(g_{\ell}\right)}{g_{\ell}}, \quad \text { for } i=1, \ldots, m .
$$

The proof proceeds by induction on $m$. To this end, we first present a lemma for the induction step. In the proof of the lemma below, we use the formula:

$$
\begin{equation*}
\delta_{i}\left(\frac{\delta_{j}(f)}{f}\right)=\delta_{j}\left(\frac{\delta_{i}(f)}{f}\right) \quad \text { for all nonzero } f \in k(\mathbf{x}) \text { and } 1 \leq i<j \leq m \tag{4.14}
\end{equation*}
$$

Lemma 4.4.5. Let $K$ denote $k\left(x_{2}, x_{3}, \ldots, x_{m}\right)$ and $a_{1}, \ldots, a_{m} \in k(\mathbf{x})$ be rational functions such that

$$
\delta_{i}\left(a_{j}\right)=\delta_{j}\left(a_{i}\right) \quad \text { for all } i, j \text { with } 1 \leq i<j \leq m .
$$

Then there exist $f \in k(\mathbf{x}), A_{\ell} \in K$ with $\ell=2, \ldots, m$, nonzero elements $c_{j} \in \bar{k}$ and $p_{j} \in$ $K\left(c_{j}\right)\left[x_{1}\right] \backslash K\left(c_{j}\right)$ with $1 \leq j \leq n$ for some finite $n \in \mathbb{N}$ such that $\delta_{i}\left(A_{j}\right)=\delta_{j}\left(A_{i}\right)$ for all $i, j$ with $2 \leq i<j \leq m$, and

$$
\begin{aligned}
& a_{1}=\delta_{1}(f)+\sum_{j=1}^{n} c_{j} \frac{\delta_{1}\left(p_{j}\right)}{p_{j}} \\
& a_{\ell}=\delta_{\ell}(f)+\sum_{j=1}^{n} c_{j} \frac{\delta_{\ell}\left(p_{j}\right)}{p_{j}}+A_{\ell} \quad \text { for } \ell=2, \ldots, m
\end{aligned}
$$

Moreover, the $p_{j}$ 's are pairwise coprime polynomials over $K\left(c_{1}, \ldots, c_{n}\right)$.
Proof. Lemma 2.3.9 asserts that the theorem holds for $m=1$. Applying Lemma 2.3.9 to $a_{1} \in$ $K\left(x_{1}\right)$ yields that there exist $f \in k(\mathbf{x})$, nonzero elements $c_{j} \in \bar{K}$ and $p_{j} \in K\left(c_{j}\right)\left[x_{1}\right] \backslash K\left(c_{j}\right)$ with $1 \leq j \leq n$ for some $n \in \mathbb{N}$ such that

$$
\begin{equation*}
a_{1}=\delta_{1}(f)+\sum_{j=1}^{n} c_{j} \frac{\delta_{1}\left(p_{j}\right)}{p_{j}} \tag{4.15}
\end{equation*}
$$

Moreover, the $c_{j}$ 's are the distinct roots of the Rothstein-Trager resultant of the logarithmic part $A / D$ of $a_{1}$ with respect to $x_{1}$ and $p_{j}=\operatorname{gcd}\left(D, A-c_{j} \delta_{1}(D)\right)$. So the $p_{j}$ 's are pairwise coprime polynomials over $K\left(c_{1}, \ldots, c_{n}\right)$ by Lemma 2.3.7.

We show that all the $c_{j}$ 's are in $\bar{k}$. For all $\ell$ with $2 \leq \ell \leq m$, the commutative formula (4.14) implies

$$
\delta_{\ell}\left(a_{1}\right)=\delta_{1}\left(\delta_{\ell}(f)\right)+\delta_{1}\left(\sum_{j=1}^{n} c_{j} \frac{\delta_{\ell}\left(p_{j}\right)}{p_{j}}\right)+\sum_{j=1}^{n} \delta_{\ell}\left(c_{j}\right) \frac{\delta_{1}\left(p_{j}\right)}{p_{j}} .
$$

Now, it follows from the integrability condition $\delta_{\ell}\left(a_{1}\right)=\delta_{1}\left(a_{\ell}\right)$ that

$$
\delta_{1}\left(a_{\ell}-\delta_{\ell}(f)-\sum_{j=1}^{n} c_{j} \frac{\delta_{\ell}\left(p_{j}\right)}{p_{j}}\right)=\sum_{j=1}^{n} \delta_{\ell}\left(c_{j}\right) \frac{\delta_{1}\left(p_{j}\right)}{p_{j}} .
$$

Since $p_{j} \in K\left(c_{j}\right)\left[x_{1}\right] \backslash K\left(c_{j}\right)$, we have $\delta_{1}\left(p_{j}\right) \neq 0$. By Lemma 2.3.4, for all $\ell$ with $2 \leq \ell \leq m$, we have $\delta_{\ell}\left(c_{j}\right)=0$ and

$$
a_{\ell}=\delta_{\ell}(f)+\sum_{j=1}^{n} c_{j} \frac{\delta_{\ell}\left(p_{j}\right)}{p_{j}}+A_{\ell}, \quad \text { for some } A_{\ell} \in K\left(c_{1}, \ldots, c_{n}\right) .
$$

So the $c_{j}$ 's are in $\bar{k}$ by Lemma 3.3.2 in [21]. Moreover, for all $\ell$ with $2 \leq \ell \leq m$, the sum

$$
\sum_{j=1}^{n} c_{j} \frac{\delta_{\ell}\left(p_{j}\right)}{p_{j}}
$$

is in $k(\mathbf{x})$, because the $c_{j}$ 's are distinct roots of some Rothstein-Trager resultant with coefficients in $k\left(x_{2}, \ldots, x_{m}\right)$ and the sum above is invariant under every permutation of the $c_{j}$ 's. This implies the $A_{\ell}$ 's are in $K$. The integrability conditions $\delta_{i}\left(a_{j}\right)=\delta_{j}\left(a_{i}\right)$ imply that $\delta_{i}\left(A_{j}\right)=\delta_{j}\left(A_{i}\right)$ for all $i, j$ with $2 \leq i<j \leq m$. This completes the proof.

Now, we present the proof of Theorem 4.4.4.

Proof. We proceed by induction on $m$. Lemma 2.3.9 asserts the theorem holds for the base case $m=1$. We assume that $m \geq 2$ and that the theorem holds for $m-1$. Let $K$ denote $k\left(x_{2}, x_{3}, \ldots, x_{m}\right)$. By Lemma 4.4.5, there exist $f \in k(\mathbf{x}), A_{\ell} \in K$ for $\ell=2, \ldots, m$, nonzero elements $c_{j} \in \bar{k}$ and $p_{j} \in K\left(c_{j}\right)\left[x_{1}\right] \backslash K\left(c_{j}\right)$ with $1 \leq j \leq n$ for some finite $n \in \mathbb{N}$ such that $\delta_{i}\left(A_{j}\right)=\delta_{j}\left(A_{i}\right)$, for all $i, j$ with $2 \leq i<j \leq m$, and

$$
\begin{aligned}
& a_{1}=\delta_{1}(f)+\sum_{j=1}^{n} c_{j} \frac{\delta_{1}\left(p_{j}\right)}{p_{j}}, \\
& a_{\ell}=\delta_{\ell}(f)+\sum_{j=1}^{n} c_{j} \frac{\delta_{\ell}\left(p_{j}\right)}{p_{j}}+A_{\ell}, \quad \text { for } \ell=2, \ldots, m .
\end{aligned}
$$

By the induction hypothesis, for the $m-1$ rational functions $A_{\ell}$, there exist $\bar{f} \in K$, nonzero elements $\bar{c}_{j} \in \bar{k}$ and $\bar{p}_{j} \in K\left(c_{j}\right)$ for $j=1, \ldots, \bar{n}$ such that

$$
A_{\ell}=\delta_{\ell}(\bar{f})+\sum_{j=1}^{\bar{n}} \bar{c}_{j} \frac{\delta_{\ell}\left(\bar{p}_{j}\right)}{\bar{p}_{j}}, \quad \text { for all } \ell \text { with } 2 \leq \ell \leq m .
$$

Since the $\bar{f}$ and $\bar{p}_{j}$ 's are free of $x_{1}$, we have

$$
a_{i}=\delta_{i}(f+\bar{f})+\sum_{j=1}^{n} c_{j} \frac{\delta_{i}\left(p_{j}\right)}{p_{j}}+\sum_{j=1}^{\bar{n}} \bar{c}_{j} \frac{\delta_{i}\left(\bar{p}_{j}\right)}{\bar{p}_{j}}, \quad \text { for all } i \text { with } 1 \leq i \leq m
$$

This completes the proof.

### 4.4.3 Multivariate extension of Feng-Singer-Wu's lemma

In the bivariate continuous-discrete case, Feng, Singer, and Wu [42, Proposition 5] have shown that there exist $f \in k\left(x_{1}, y_{1}\right), \alpha \in k\left(y_{1}\right)$, and $\beta, \gamma \in k\left(x_{1}\right)$ such that

$$
a_{1}=\frac{\delta_{1}(f)}{f}+y_{1} \frac{\delta_{1}(\beta)}{\beta}+\gamma \quad \text { and } \quad b_{1}=\frac{\sigma_{1}(f)}{f} \beta \alpha .
$$

These two equalities are used to develop algorithms for computing Liouvillian solutions of primeorder linear difference-differential equations in [42]. We generalize Feng-Singer-Wu's result to the multivariate case.

Theorem 4.4.6. Assume that $a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}$ are in $k(\mathbf{x}, \mathbf{y})$ with $b_{1} \cdots b_{n} \neq 0$. If all the equalities in (4.2), (4.3) and (4.4) hold, then there exist $f \in k(\mathbf{x}, \mathbf{y}), \beta_{1}, \ldots, \beta_{n} \in k(\mathbf{x})$, $\gamma_{1}, \ldots, \gamma_{m} \in k(\mathbf{x})$ and $\alpha_{1}, \ldots, \alpha_{n} \in k(\mathbf{y})$ such that

$$
\begin{equation*}
a_{i}=\frac{\delta_{i}(f)}{f}+\sum_{j=1}^{n} y_{j} \frac{\delta_{i}\left(\beta_{j}\right)}{\beta_{j}}+\gamma_{i} \quad \text { for all } i \text { with } 1 \leq i \leq m \tag{4.16}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{j}=\frac{\sigma_{j}(f)}{f} \beta_{j} \alpha_{j} \quad \text { for all } j \text { with } 1 \leq j \leq n \tag{4.17}
\end{equation*}
$$

Moreover, all the equalities in (4.2), (4.3) and (4.4) remain valid if $a_{i}$ is replaced by $\gamma_{i}$ and $b_{j}$ is replaced by $\alpha_{j}$ for all $i$ with $1 \leq i \leq m$ and $j$ with $1 \leq j \leq n$.

Before presenting a proof of the theorem, we look at a few special cases. If $n=0$, then we may set $f=1$ and $\gamma_{i}=a_{i}$ for all $i$ with $1 \leq i \leq m$. Similarly, if $m=0$, then we set $f=1$, $\beta_{j}=1$ and $b_{j}=\alpha_{j}$ for all $j$ with $1 \leq j \leq n$. Hence, the theorem holds if either $n=0$ or $m=0$. If $m=n=1$, then the theorem holds by Proposition 5 in [42]. Note that $k$ is assumed to be algebraically closed in [42]. But the proposition holds without this assumption when one reads their proof carefully. Hence, it suffices to show the theorem when $m>1$ and $n \geq 1$ or when $m \geq 1$ and $n>1$.

The proof proceeds by induction on $m$ and $n$. To this end, we first show that Theorem 4.4.6 holds for $m=1$ and $n$ arbitrary.

Lemma 4.4.7. Let $a, b_{1}, \ldots, b_{n}$ be rational functions in $k(x, \mathbf{y})$ such that $b_{1}, \ldots, b_{n}$ satisfy the integrability condition (4.3), $b_{1} \cdots b_{n} \neq 0$ and

$$
\begin{equation*}
\frac{\delta\left(b_{j}\right)}{b_{j}}=\sigma_{j}(a)-a \quad \text { for all } j \text { with } 1 \leq j \leq n \tag{4.18}
\end{equation*}
$$

where $\delta=d / d x$. Then there exist $f \in k(x, \mathbf{y}), \alpha_{1}, \ldots, \alpha_{n} \in k(\mathbf{y}), \beta_{1}, \ldots, \beta_{n} \in k(x)$, and $\gamma \in k(x)$ such that

$$
\begin{equation*}
a=\frac{\delta(f)}{f}+\sum_{j=1}^{n} y_{j} \frac{\delta\left(\beta_{j}\right)}{\beta_{j}}+\gamma \quad \text { and } \quad b_{j}=\frac{\sigma_{j}(f)}{f} \beta_{j} \alpha_{j} \quad \text { for } 1 \leq j \leq n \tag{4.19}
\end{equation*}
$$

Moreover, the $\alpha_{i}$ 's also satisfy (4.3).
Proof. We proceed by induction on $n$. If $n=1$, then the lemma follows from Proposition 5 in [42]. Assume that $n$ is greater than 1, and the claim holds for the values lower than $n$. Re$\operatorname{gard} a, b_{1}, \ldots, b_{n}$ as rational functions in $x, y_{1}, \ldots, y_{n-1}$ over $k\left(y_{n}\right)$. By the induction hypothesis, there exist $\tilde{f}$ in $k(x, \mathbf{y}), \tilde{\beta}_{1}, \ldots, \tilde{\beta}_{n-1}$ in $k\left(x, y_{n}\right)$ and $\tilde{\gamma}$ in $k\left(x, y_{n}\right)$ such that

$$
a=\frac{\delta(\tilde{f})}{\tilde{f}}+y_{1} \frac{\delta\left(\tilde{\beta}_{1}\right)}{\tilde{\beta}_{1}}+\cdots+y_{n-1} \frac{\delta\left(\tilde{\beta}_{n-1}\right)}{\tilde{\beta}_{n-1}}+\tilde{\gamma}
$$

Similarly, regarding $a, b_{1}, \ldots, b_{n}$ as rational functions in $x, y_{2}, \ldots, y_{n}$ over $k\left(y_{1}\right)$, we have

$$
a=\frac{\delta\left(f^{\prime}\right)}{f^{\prime}}+y_{2} \frac{\delta\left(\beta_{2}^{\prime}\right)}{\beta_{2}^{\prime}}+\cdots+y_{n} \frac{\delta\left(\beta_{n}^{\prime}\right)}{\beta_{n}^{\prime}}+\gamma^{\prime}
$$

where $f^{\prime}$ is in $k(x, \mathbf{y}), \beta_{2}^{\prime}, \ldots, \beta_{n}^{\prime}$ are in $k\left(x, y_{1}\right)$, and $\gamma^{\prime}$ is in $k\left(x, y_{1}\right)$. It follows from Lemma 4.3.10 that there exist $f \in k(x, \mathbf{y})$ and $\beta_{1}, \ldots, \beta_{n}, \gamma \in k(x)$ such that

$$
\begin{equation*}
a=\frac{\delta(f)}{f}+\sum_{j=1}^{n} y_{j} \frac{\delta\left(\beta_{j}\right)}{\beta_{j}}+\gamma \tag{4.20}
\end{equation*}
$$

Assume that, for $j$ with $1 \leq j \leq n$,

$$
\begin{equation*}
b_{j}=\frac{\sigma_{j}(f)}{f} \beta_{j} \alpha_{j} \tag{4.21}
\end{equation*}
$$

for some $\alpha_{j} \in k(x, \mathbf{y})$. By (4.20),

$$
\sigma_{j}(a)-a=\frac{\sigma_{j}(\delta(f))}{\sigma_{j}(f)}-\frac{\delta(f)}{f}+\frac{\delta_{j}(\beta)}{\beta}
$$

and, by (4.21),

$$
\frac{\delta\left(b_{j}\right)}{b_{j}}=\frac{\delta\left(\sigma_{j}(f)\right)}{\sigma_{j}(f)}-\frac{\delta(f)}{f}+\frac{\delta\left(\beta_{j}\right)}{\beta_{j}}+\frac{\delta\left(\alpha_{j}\right)}{\alpha_{j}} .
$$

These two equalities, the integrability condition (4.18), and the commutativity of $\delta$ and $\sigma_{j}$ imply

$$
\frac{\delta\left(\alpha_{j}\right)}{\alpha_{j}}=0
$$

Consequently, $\alpha_{j}$ belongs to $k(\mathbf{y})$. By (4.3), the $\alpha_{j}$ 's in the above proof satisfy

$$
\frac{\sigma_{q}\left(\alpha_{j}\right)}{\alpha_{j}}=\frac{\sigma_{j}\left(\alpha_{q}\right)}{\alpha_{q}}
$$

for all $j, q$ with $1 \leq j<q \leq n$. This completes the proof.

Now, we present a proof of Theorem 4.4.6.
Proof. We proceed by induction on $m$ with Lemma 4.4.7 as our induction base. Let $n$ be a fixed positive integer, and assume that the theorem holds for the values lower than $m$. Consider the continuous variables $x_{1}, \ldots, x_{m-1}$ and the discrete ones $y_{1}, \ldots, y_{n}$ over $k\left(x_{m}\right)$. We have, for all $j$ with $1 \leq j \leq n$, there exist $\tilde{f}_{j} \in k(\mathbf{x}, \mathbf{y}), \tilde{\beta}_{j} \in k(\mathbf{x})$ and $\tilde{\alpha}_{j} \in k\left(x_{m}, \mathbf{y}\right)$ such that

$$
b_{j}=\frac{\sigma_{j}\left(\tilde{f}_{j}\right)}{\tilde{f}_{j}} \tilde{\beta}_{j} \tilde{\alpha}_{j}
$$

Consider the continuous variables $x_{2}, \ldots, x_{m}$ and the discrete ones $y_{1}, \ldots, y_{n}$ over $k\left(x_{1}\right)$. We have that for all $j$ with $1 \leq j \leq n$, there exist $\hat{f}_{j} \in k(\mathbf{x}, \mathbf{y}), \hat{\beta}_{j} \in k(\mathbf{x})$ and $\hat{\alpha}_{j} \in k\left(x_{1}, \mathbf{y}\right)$ such that

$$
b_{j}=\frac{\sigma_{j}\left(\hat{f}_{j}\right)}{\hat{f}_{j}} \hat{\beta}_{j} \hat{\alpha}_{j}
$$

By the above two equalities and Lemma 4.3.11, there exist $f_{j}^{\prime} \in k(\mathbf{x}, \mathbf{y}), \beta_{j}^{\prime} \in k(\mathbf{x}), \alpha_{j}^{\prime} \in k(\mathbf{y})$ such that

$$
\begin{equation*}
b_{j}=\frac{\sigma_{j}\left(f_{j}^{\prime}\right)}{f_{j}^{\prime}} \beta_{j}^{\prime} \alpha_{j}^{\prime} \quad \text { for all } j \text { with } 1 \leq j \leq n \tag{4.22}
\end{equation*}
$$

Now, we claim that there exist $f \in k(\mathbf{x}, \mathbf{y}), \beta_{1}, \ldots, \beta_{n} \in k(\mathbf{x})$ and $\alpha_{1}, \ldots, \alpha_{n} \in k(\mathbf{y})$ such that

$$
b_{j}=\frac{\sigma_{j}(f)}{f} \beta_{j} \alpha_{j} \quad \text { for all } j \text { with } 1 \leq j \leq n
$$

We prove the claim by induction on $n$. If $n=1$, then the claim is true by (4.22). Assume that the claim holds for $\ell<n$, and that

$$
\begin{equation*}
b_{\ell+1}=\frac{\sigma_{\ell+1}(f)}{f} u \tag{4.23}
\end{equation*}
$$

for some $u \in k(\mathbf{x}, \mathbf{y})$. By (4.3),

$$
\sigma_{1}(u)=\frac{\sigma_{\ell+1}\left(\alpha_{1}\right)}{\alpha_{1}} u, \ldots, \sigma_{\ell}(u)=\frac{\sigma_{\ell+1}\left(\alpha_{\ell}\right)}{\alpha_{\ell}} u
$$

Since $\alpha_{1}, \ldots, \alpha_{\ell} \in k(\mathbf{y})$,

$$
u=v w,
$$

where $v \in k\left(\mathbf{x}, y_{\ell+1}, \ldots, y_{n}\right)$ and $w \in k(\mathbf{y})$. By (4.22) and (4.23),

$$
\begin{equation*}
\frac{\sigma_{\ell+1}(f)}{f} v w=\frac{\sigma_{\ell+1}\left(f_{\ell+1}^{\prime}\right)}{f_{\ell+1}^{\prime}} \beta_{\ell+1}^{\prime} \alpha_{\ell+1}^{\prime} . \tag{4.24}
\end{equation*}
$$

From the equality (4.24), it follows that

$$
\begin{equation*}
v=\frac{\sigma_{\ell+1}(p)}{p} \beta_{\ell+1}^{\prime} q, \tag{4.25}
\end{equation*}
$$

where $p=f^{\prime} / f$ and $q=\alpha_{\ell+1}^{\prime} / w$. Since $k$ is of characteristic zero, there exist $c_{1}, \ldots, c_{\ell} \in k$ such that the denominators and numerators of $p$ and $q$ evaluated at $y_{1}=c_{1}, \ldots, y_{\ell}=c_{\ell}$ are all nonzero. Substituting $c_{1}, \ldots, c_{\ell}$ for $y_{1}, \ldots, y_{\ell}$, respectively, into (4.25), we see that

$$
v=\frac{\sigma_{\ell+1}(g)}{g} \beta_{\ell+1}^{\prime} r,
$$

where $g \in k\left(\mathbf{x}, y_{\ell+1}, \ldots, y_{n}\right)$ and $r \in k\left(y_{\ell+1}, \ldots, y_{n}\right)$. Setting $\beta_{\ell+1}=\beta_{\ell+1}^{\prime}$ and $\alpha_{\ell+1}=r w$, we see that

$$
b_{\ell+1}=\frac{\sigma_{\ell+1}(f g)}{f g} \beta_{\ell+1} \alpha_{\ell+1} .
$$

Moreover, since $g$ is a constant with respect to $\sigma_{1}, \ldots, \sigma_{\ell}$,

$$
b_{j}=\frac{\sigma_{j}(f g)}{f g} \beta_{j} \alpha_{j} \quad \text { for all } j \text { with } 1 \leq j \leq \ell
$$

The claim holds for $\ell+1$ when we replace $f$ by $f g$.
It remains to obtain the form for the $a_{i}$ 's. Assume that, for all $i$ with $1 \leq i \leq m$,

$$
a_{i}=\frac{\delta_{i}(f)}{f}+\sum_{j=1}^{m} y_{i} \frac{\delta_{i}\left(\beta_{j}\right)}{\beta_{j}}+\gamma_{i}
$$

where $\gamma_{i} \in k(\mathbf{x}, \mathbf{y})$ is to be determined. Using the integrability conditions in (4.4), and performing a calculation similar to that below (4.21) in the proof of Lemma 4.4.7, we find that

$$
\sigma_{j}\left(\gamma_{i}\right)=\gamma_{i} \quad \text { for all } j \text { with } 1 \leq j \leq n,
$$

which implies that $\gamma_{i}$ belongs to $k(\mathbf{x})$. Moreover, for all $1 \leq i<p \leq m$,

$$
\delta_{i}\left(\gamma_{p}\right)=\delta_{p}\left(\gamma_{i}\right)
$$

By the integrability conditions (4.3)

$$
\frac{\sigma_{j}\left(\alpha_{q}\right)}{\alpha_{q}}=\frac{\sigma_{q}\left(\alpha_{j}\right)}{\alpha_{j}}
$$

for all $j, q$ with $1 \leq j<q \leq n$. This completes the proof of Theorem 4.4.6.
Remark 4.4.8. Applying Theorem 4.4.2 and Theorem 4.4.4 to the $\gamma_{i}$ 's and $\alpha_{j}$ 's in Theorem 4.4.6, respectively, we readily see that the structure Theorem 4.4.1 holds.

### 4.5 Multiplicative structure

In this section, we assume that $k$ is algebraically closed. We derive a multiplicative form of a hyperexponential-hypergeometric function from the structure of its certificates.

Let $h(\mathbf{x}, \mathbf{y})$ be a hyperexponential-hypergeometric function over $k(\mathbf{x}, \mathbf{y})$ with $m+n$ certificates $a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}$ such that the product $b_{1} \cdots b_{n}$ is nonzero. Then (4.2), (4.3) and (4.4) hold. By Theorem 4.4.6, there exist $f \in k(\mathbf{x}, \mathbf{y}), \beta_{1}, \ldots, \beta_{n} \in k(\mathbf{x}) \backslash\{0\}, \gamma_{1}, \ldots, \gamma_{m} \in k(\mathbf{x})$ and $\alpha_{1}, \ldots, \alpha_{n} \in k(\mathbf{y})$ such that

$$
a_{i}=\frac{\delta_{i}(f)}{f}+\sum_{j=1}^{n} y_{j} \frac{\delta_{i}\left(\beta_{j}\right)}{\beta_{j}}+\gamma_{i}
$$

for all $i$ with $1 \leq i \leq m$, and

$$
b_{j}=\frac{\sigma_{j}(f)}{f} \beta_{j} \alpha_{j}
$$

for all $j$ with $1 \leq j \leq n$. Moreover,

$$
\gamma_{1}, \ldots, \gamma_{m}, \alpha_{1}, \ldots, \alpha_{n}
$$

satisfy the integrability conditions (4.2), (4.3) and (4.4). By Theorem 2 in [22], there exists a simple differential-difference extension $R$ of $k(\mathbf{x}, \mathbf{y})$ containing functions $h, \beta^{y_{i}}$, and any $h^{\prime} \in$ $\mathcal{H}\left(\gamma_{1}, \ldots, \gamma_{m}, \alpha_{1}, \ldots, \alpha_{n}\right)$. Every hyperexponential-hypergeometric element in $R$ is invertible, since $R$ is simple. Moreover, two hyperexponential-hypergeometric elements having the same certificates differ by a multiplicative constant in $k$. It is straightforward to verify that $h$ and $f \beta_{1}^{y_{1}} \cdots \beta_{n}^{y_{n}} h^{\prime}$ have the same certificates. Therefore,

$$
h=c f \beta_{1}^{y_{1}} \cdots \beta_{n}^{y_{n}} h^{\prime},
$$

where $c \in k$ and the certificates of $h^{\prime}$ are $\gamma_{1}, \ldots, \gamma_{m}, \alpha_{1}, \ldots, \alpha_{n}$. Note that the $\gamma_{i}$ 's are functions in $\mathbf{x}$, and the $\alpha_{j}$ 's are functions in $\mathbf{y}$. Then we have the factorization

$$
\mathcal{H}\left(\gamma_{1}, \ldots, \gamma_{m}, \alpha_{1}, \ldots, \alpha_{n}\right)=\mathcal{H}\left(\gamma_{1}, \ldots, \gamma_{m}, 1, \ldots, 1\right) \mathcal{H}\left(0, \ldots, 0, \alpha_{1}, \ldots, \alpha_{n}\right)
$$

By Theorem 4.4.4, any element $h_{e} \in \mathcal{H}\left(\gamma_{1}, \ldots, \gamma_{m}, 1, \ldots, 1\right)$ is conjugate to

$$
\exp \left(g_{0}\right) \prod_{\ell=1}^{L} g_{\ell}^{c_{\ell}}
$$

where $c_{1}, \ldots, c_{L} \in k$ and $g_{0}, g_{1}, \ldots, g_{L} \in k(\mathbf{x})$. By Corollary 3.7.3 in [73] or Corollary 4 in [14], any element $h_{g} \in \mathcal{H}\left(0, \ldots, 0, \alpha_{1}, \ldots, \alpha_{n}\right)$ is conjugate to

$$
\tilde{f}(\mathbf{y}) T(\mathbf{y})
$$

where $\tilde{f} \in k(\mathbf{y})$ and $T(\mathbf{y})$ is a factorial term in $\mathbf{y}$ (see [73, Definition 3.5.1] or [14, Definition 5]). We summarize the above discussion by the following theorem.

Theorem 4.5.1. If $k$ is algebraically closed, then any hyperexponential-hypergeometric function $h$ over $k(\mathbf{x}, \mathbf{y})$ is conjugate to a function

$$
\begin{equation*}
r(\mathbf{x}, \mathbf{y}) \exp \left(g_{0}(\mathbf{x})\right) \prod_{\ell=1}^{L} g_{\ell}(\mathbf{x})^{c_{\ell}} \prod_{j=1}^{n} \beta_{j}(\mathbf{x})^{y_{j}} T(\mathbf{y}) \tag{4.26}
\end{equation*}
$$

where $r \in k(\mathbf{x}, \mathbf{y}), g_{0}, g_{1}, \ldots, g_{L}, \beta_{1}, \ldots, \beta_{n} \in k(\mathbf{x}), c_{l}, \ldots, c_{L} \in k$, and $T(\mathbf{y})$ is a factorial term.
Example 4.5.2 (Jacobi polynomials).

$$
P_{n}^{\left(m_{1}, m_{2}\right)}(x)=\sum_{\ell} h\left(x, \ell, n, m_{1}, m_{2}\right),
$$

where $h$ is a hyperexponential-hypergeometric function over $k\left(x, \ell, n, m_{1}, m_{2}\right)$ with certificates:

$$
\begin{aligned}
a & =\frac{2 \ell+n x-n}{x^{2}-1}, \quad b_{1}=\frac{(x-1)(n-\ell)\left(n+m_{2}-\ell\right)}{\left(m_{1}+\ell+1\right)(\ell+1)(x+1)}, \\
b_{2} & =\frac{(x+1)\left(n+m_{2}+1\right)\left(n+m_{1}+1\right)}{2(n-\ell+1)\left(n+m_{2}-\ell+1\right)}, \quad b_{3}=\frac{n+m_{1}+1}{m_{1}+\ell+1}, \quad b_{4}=\frac{n+m_{2}+1}{n+m_{2}-\ell+1}
\end{aligned}
$$

According to Theorem 4.4.6, we write $\left(a, b_{1}, b_{2}, b_{3}\right)$ as

$$
\begin{aligned}
a & =\ell \frac{\delta\left(\frac{x-1}{x+1}\right)}{\frac{x-1}{x+1}}+n \frac{\delta(x+1)}{x+1}, \quad b_{1}=\frac{x-1}{x+1} \cdot \frac{(n-\ell)\left(n+m_{2}-\ell\right)}{\left(m_{1}+\ell+1\right)(\ell+1)}, \\
b_{2} & =(x+1) \cdot \frac{1}{2} \cdot \frac{\left(n+m_{2}+1\right)\left(n+m_{1}+1\right)}{(n-\ell+1)\left(n+m_{2}-\ell+1\right)}, \quad b_{3}=\frac{n+m_{1}+1}{m_{1}+\ell+1}, \quad b_{4}=\frac{n+m_{2}+1}{n+m_{2}-\ell+1}
\end{aligned}
$$

Then $h$ is a hyperexponential-hypergeometric function with a multiplicative form

$$
h=\left(\frac{x-1}{x+1}\right)^{\ell}(x+1)^{n} h^{\prime}, \quad \text { where } \quad h^{\prime}=\frac{1}{2^{n}}\binom{n+m_{1}}{n-\ell}\binom{n+m_{2}}{\ell} .
$$

Definition 4.5.3 (Properness). A hyperexponential-hypergeometric function over $k(\mathbf{x}, \mathbf{y})$ is said to be proper if it is conjugate to a function

$$
\begin{equation*}
p(\mathbf{x}, \mathbf{y}) \exp \left(g_{0}(\mathbf{x})\right) \prod_{\ell=1}^{L} g_{\ell}(\mathbf{x})^{c_{\ell}} \prod_{j=1}^{n} \beta_{j}(\mathbf{x})^{y_{j}} T(\mathbf{y}) \tag{4.27}
\end{equation*}
$$

where $p$ is a polynomial in $k[\mathbf{x}, \mathbf{y}], g_{0}, g_{1}, \ldots, g_{L}, \beta_{1}, \ldots, \beta_{n} \in k(\mathbf{x}), c_{1}, \ldots, c_{L} \in k$, and $T(\mathbf{y})$ is a factorial term over $k(\mathbf{y})$.

By Definition 4.5.3 and Theorem 4.4.4, we obtain the following result.

Corollary 4.5.4. Any multivariate hyperexponential function $H(\mathbf{x})$ over $k(\mathbf{x})$ is proper.
In their paper [92], Wilf and Zeilberger have listed several familiar special functions which are proper hyperexponential-hypergeometric functions.

## Chapter 5

## Existence of Telescopers for Hyperexponential-Hypergeometric Functions

### 5.1 Introduction

The termination problem of Zeilberger's algorithms for creative telescoping has been extensively studied in the last two decades. The algorithms terminate if the existence of telescopers is guaranteed. Zeilberger [95] has shown that holonomicity guarantees the success of his algorithms. In particular, Wilf and Zeilberger have provided an elementary proof, based on the ideas of Fasenmyer [40] and Verbaeten [89], that telescopers always exist for proper hyperexponentialhypergeometric functions [95, 92]. However, holonomicity is only a sufficient condition, i.e., there are cases in which the input functions are not holonomic but Zeilberger's algorithm still terminates [34]. Therefore, one challenge is to find theoretical criteria which enable us to algorithmically detect the existence of telescopers.

In view of the theoretical difficulty, special attention has been focused on the subclass of hyperexponential-hypergeometric functions. In the continuous case, the work by Bernstein [17] and Lipshitz [67] shows that every hyperexponential function has a telescoper. This implies that Zeilberger's algorithm always succeeds on those inputs. However, the situation in other cases
turns out to be more involved. In the discrete case and its $q$-analogue, the first complete solution to the termination problem is Abramov and Le's criterion [64, 9], which decides whether telescopers exist for a given bivariate rational function in the discrete variables $m$ and $n$. According to their criterion, the rational function

$$
f=\frac{1}{m^{2}+n^{2}}
$$

has no telescoper. Soon, the criterion was extended to the general case of bivariate hypergeometric terms by Abramov [5, 6]. Basically, Abramov proves that a hypergeometric term can be written as a sum of a hypergeometric-summable term and a proper one if it has a telescoper [6, Theorem 10]. Similar results have been obtained in the general $q$-shift case by Chen, Hou and Mu [28]. These results are fundamental for detecting the termination of Zeilberger's algorithm.

The continuous-discrete analogue of Zeilberger's algorithm is presented by Almkvist and Zeilberger [15]. This analogue has been shown to be useful in the study of orthogonal polynomials [60, Chapters 10-13].

Motivation. In the bivariate mixed case, not all hyperexponential-hypergeometric functions have telescopers. For example, we will show in this chapter that the simple rational function

$$
f=\frac{1}{x+n}
$$

has no telescoper with respect to either the continuous variable $x$ or the discrete variable $n$. Therefore, an Abramov-like criterion is also needed in this mixed case.

Our contribution. The main contribution in this chapter is two criteria for the existence of telescopers for bivariate hyperexponential-hypergeometric functions. Our criteria are of the same type as Abramov's, which says that a hyperexponential-hypergeometric function can be written as a sum of a hypergeometric-summable (integrable) function and a proper one if it has a telescoper with respect to the discrete (continuous) variable. The key ingredients of our proof are: standard representations of bivariate hyperexponential-hypergeometric functions in [42], and two additive decompositions: one is the Abramov-Petkovšek algorithm [11, 13] with respect to the discrete variable; and the other is the Hermite-like reduction [47] with respect to the continuous one.

The rest of this chapter is organized as follows. We describe an algebraic setting for bivariate hyperexponential-hypergeometric functions in Section 5.2. We discuss on the existence of telescopers in Section 5.3. A standard representation of hyperexponential-hypergeometric functions is introduced in Section 5.4. The Abramov-Petkovšek decomposition and the Hermite-like reduction are adapted to the continuous-discrete setting in Section 5.5. Our criteria for the existence of telescopers are presented in Section 5.6.

An earlier version of this chapter can be found in [26], which is a joint work (in progress) with Frédéric Chyzak, Ruyong Feng, and Ziming Li.

### 5.2 Preliminaries

In his papers [77, 78], Risch based his algorithm on Ritt's theory of differential fields and their extensions. From then on, differential algebra $[79,56,61]$ has been the algebraic foundation of symbolic integration [21]. The analog investigation [57, 58] has been carried out for summation problems via difference algebra [36]. In this section, we describe an algebraic setting for hyperexponential-hypergeometric functions, which permits us to deal with the telescoping problem in an algebraic way.

In the rest of this chapter, let $k$ be an algebraically closed field of characteristic zero, and $k(x, n)$ be the field of rational functions in $x$ and $n$ over $k$. We view $x$ as the continuous variable and $n$ as the discrete variable. On the field $k(x, n)$, the derivation $\delta$ and the shift operator $\sigma$ are defined as

$$
\delta(f(x, n))=\frac{\partial f}{\partial x} \quad \text { and } \quad \sigma(f(x, n))=f(x, n+1)
$$

for all $f \in k(x, n)$, respectively.

### 5.2.1 Ring of sequences

We recall the ring of sequences defined in $[41,43]$ and regard a hyperexponential-hypergeometric function as an element in that ring.

Let $\mathcal{K}$ be the universal Picard-Vessiot field [88, Chapter 10] of the field $k(x)$. Then $(\mathcal{K}, \delta)$ is a differential field extension of $(k(x), \delta)$. Furthermore, the constants in $\mathcal{K}$ with respect to $\delta$ are
exactly the elements of $k$, since $k$ is algebraically closed. Let $\mathcal{K}^{\mathbb{N}}$ be the ring of infinite sequences over $\mathcal{K}$, where the addition and multiplication of sequences are defined termwise. For a sequence

$$
s=\left(s_{0}, s_{1}, s_{2}, \ldots\right)
$$

in $\mathcal{K}^{\mathbb{N}}$, define

$$
\delta^{\prime}(s)=\left(\delta\left(s_{0}\right), \delta\left(s_{1}\right), \delta\left(s_{2}\right), \ldots\right) .
$$

Then $\delta^{\prime}$ is a derivation operator on $\mathcal{K}^{\mathbb{N}}$. Let $I \subset \mathcal{K}^{\mathbb{N}}$ be the ideal consisting of sequences with finitely many nonzero terms. Since $I$ is closed under $\delta^{\prime}$, the quotient ring $\mathcal{S}=\mathcal{K}^{\mathbb{N}} / I$ is a differential ring whose derivation $\bar{\delta}$ is induced by $\delta^{\prime}$. In other words:

$$
\bar{\delta}(s+I)=\delta^{\prime}(s)+I
$$

We define a map $\bar{\sigma}: \mathcal{S} \rightarrow \mathcal{S}$ by sending

$$
s+I \mapsto\left(s_{1}, s_{2}, s_{3}, \ldots\right)+I .
$$

It is easy to verify that $\bar{\sigma}$ is a well-defined automorphism, and $\bar{\delta}$ and $\bar{\sigma}$ commute with each other.
We now embed the field $k(x, n)$ of rational functions into $\mathcal{S}$. For an element $f \in k(x, n)$, there exists $m \in \mathbb{N}$ such that $f(x, i)$ is well-defined when $i \geq m$. Define a map $\phi: k(x, n) \rightarrow \mathcal{S}$ by sending

$$
f(x, n) \mapsto(\underbrace{0, \ldots, 0}_{m}, f(x, m), f(x, m+1), f(x, m+2), \ldots)+I .
$$

By the definition of $I, \phi$ is a well-defined monomorphism such that $\phi \circ \delta=\bar{\delta} \circ \phi$ and $\phi \circ \sigma=\bar{\sigma} \circ \phi$. Let us identify $k(x, n)$ with $\phi(k(x, n))$. Then $\bar{\delta}$ and $\bar{\sigma}$ can be viewed as extensions of $\delta$ and $\sigma$, respectively. Hence, we may also identify $\bar{\delta}$ with $\delta$, and $\bar{\sigma}$ with $\sigma$. Consequently, $\mathcal{S}$ is a differentialdifference ring extension of $k(x, n)$, and $k$ is the set of constants in $\mathcal{S}$.

Lemma 5.2.1. Let $a$ and $b$ be in $k(x, n)$ with $b \neq 0$ and

$$
\begin{equation*}
\frac{\delta(b)}{b}=\sigma(a)-a . \tag{5.1}
\end{equation*}
$$

Then there exists an invertible hyperexponential-hypergeometric function in $\mathcal{S}$ such that its $x$ certificate is equal to a and its $n$-certificate is equal to $b$.

Proof. View $a$ and $b$ as rational functions in $n$. Let $N$ be a sufficiently large integer such that for any $j \geq N, a(x, j)$ and $b(x, j)$ are well-defined and $b(x, j) \neq 0$. There exists a nonzero element $v \in \mathcal{K}$ such that

$$
\delta(v)=a(x, N) v .
$$

Let $h_{i}=0$ for all $i$ with $0 \leq i \leq N-1, h_{N}=v$, and $h_{i+1}=b(x, i) h_{i}$ for all $i>N$. We claim that

$$
h=\left(h_{0}, h_{1}, \ldots, h_{N}, h_{N+1}, \ldots\right)+I
$$

is an invertible element. Furthermore, its $x$-certificate and $n$-certificate are $a$ and $b$, respectively.
Invertibility follows from the fact $h_{i} \neq 0$ for all $i \geq N$. We now verify that $\sigma(h)=b h$ by the following calculation:

$$
\begin{aligned}
\sigma(h) & =(\underbrace{0, \ldots, 0}_{N-1}, h_{N}, h_{N+1}, h_{N+2}, \cdots)+I \\
& =(\underbrace{0, \ldots, 0}_{N-1}, h_{N}, b(x, N) h_{N}, b(x, N+1) h_{N+1}, \cdots)+I \\
& =(\underbrace{0, \ldots, 1}_{N}, b(x, N), b(x, N+1), \cdots)(\underbrace{0, \ldots, 0}_{N-1}, h_{N}, h_{N}, h_{N+1}, \cdots)+I \\
& =b h .
\end{aligned}
$$

Next, we verify that $\delta\left(h_{i}\right)=a(x, i) h_{i}$ for $i \geq N$, which implies $\delta(h)=a h$. When $i=N$, the equality holds by the definition of $h_{N}$. Assume that the equality holds for all $i$ with $N \leq i \leq \ell$. By (5.1), we have

$$
\delta(b(x, \ell))=a(x, \ell+1) b(x, \ell)-a(x, \ell) b(x, \ell) .
$$

It follows that

$$
\begin{aligned}
\delta\left(h_{\ell+1}\right) & =\delta\left(b(x, \ell) h_{\ell}\right)=\delta(b(x, \ell)) h_{\ell}+b(x, \ell) \delta\left(h_{\ell}\right) \\
& =(a(x, \ell+1) b(x, \ell)-a(x, \ell) b(x, \ell)) h_{\ell}+b(x, \ell) a(x, \ell) h_{\ell} \\
& =a(x, \ell+1) b(x, \ell) h_{\ell}=a(x, \ell+1) h_{\ell+1} .
\end{aligned}
$$

The claim is proved for $\ell+1$, and so is the lemma by induction.

Example 5.2.2. Assume that $\beta(x)$ is a nonzero element of $k(x)$. Then the pair $\left(n \frac{\delta(\beta)}{\beta}, \beta\right)$ satisfies (5.1). It follows that

$$
\left(1, \beta, \beta^{2}, \ldots\right)+I
$$

is a hyperexponential-hypergeometric function in $\mathcal{S}$ with certificates $n \delta(\beta) / \beta$ and $\beta$. For later convenience, this element is denoted as $\beta^{n}$.

Let $a$ and $b$ be the same as in Lemma 5.2.1, and assume that (5.1) holds. Recall that

$$
\mathcal{H}(a, b) \triangleq\{h \in \mathcal{S} \mid \delta(h)=a h \text { and } \sigma(h)=b h\} .
$$

Proposition 5.2.3. The set $\mathcal{H}(a, b)$ is a one-dimensional vector space over $k$, in which every nonzero element is invertible in $\mathcal{S}$.

Proof. By Lemma 5.2.1, $\mathcal{H}(a, b)$ contains an invertible element, say $h$. Let $g$ be another element of $\mathcal{H}(a, b)$. Then $\delta\left(g h^{-1}\right)=0$ and $\sigma\left(g h^{-1}\right)=g h^{-1}$ by a straightforward calculation. Hence, $g h^{-1}$ is a constant. The lemma follows from the fact that $k$ is the set of constants in $\mathcal{S}$.

This proposition implies that every first-order fully integrable system over $k(x, n)$ has a onedimensional solution space over $k$. For later use, we specialize the three formulas in Lemma 4.2.3 for $\mathcal{H}(a, b)$.

Corollary 5.2.4. (i) For any hyperexponential-hypergeometric function $g \in \mathcal{S}$, we have

$$
g \mathcal{H}(a, b)=\mathcal{H}\left(a+\frac{\delta(g)}{g}, b \frac{\sigma(g)}{g}\right) ;
$$

(ii) For all $a, b, a^{\prime}, b^{\prime} \in k(x, n)$ with $b b^{\prime} \neq 0$ such that both $(a, b)$ and ( $\left.a^{\prime}, b^{\prime}\right)$ satisfy (5.1), we have

$$
\mathcal{H}(a, b) \mathcal{H}\left(a^{\prime}, b^{\prime}\right)=\mathcal{H}\left(a+a^{\prime}, b b^{\prime}\right) ;
$$

(iii) $\delta(\mathcal{H}(a, b))=a \mathcal{H}(a, b)$ and $\sigma(\mathcal{H}(a, b))=b \mathcal{H}(a, b)$.

From now on, we assume that all hyperexponential-hypergeometric functions belong to $\mathcal{S}$, unless otherwise stated.

### 5.2.2 Ring of differential-recurrence operators

Let $k(x, n)\left\langle D_{x}, S_{n}\right\rangle$ be the ring of differential-recurrence operators whose commutation rules are

$$
S_{n} D_{x}=D_{x} S_{n}, \quad D_{x} f=f D_{x}+\delta(f), \quad \text { and } \quad S_{n} f=\sigma(f) S_{n},
$$

for all $f \in k(x, n)$. In this ring, we denote $\Delta_{n}$ the forward difference operator $S_{n}-1$. For every $s \in \mathcal{S}$, define two actions: $D_{x}(s)=\delta(s)$ and $S_{n}(s)=\sigma(s)$. Then $\mathcal{S}$ becomes a left module over $k(x, n)\left\langle D_{x}, S_{n}\right\rangle$. Assume that $h$ is a hyperexponential-hypergeometric function with $x$-certificate $a$ and $n$-certificate $b$. Then the integrability condition (5.1) can be rewritten as $S_{n} D_{x}(h)=D_{x} S_{n}(h)$.

Two hyperexponential-hypergeometric functions $h_{1}$ and $h_{2}$ are said to be similar if the ratio $h_{1} / h_{2}$ is in $k(x, n)$. It is easy to verify that similarity is an equivalence relation among hyperexponential-hypergeometric functions.

Lemma 5.2.5. Let $h_{1}$ and $h_{2}$ be two hyperexponential-hypergeometric functions over $k(x, n)$. If $h_{1}$ and $h_{2}$ are similar, then
(i) $h_{1}+h_{2}$ is either equal to zero or similar to $h_{1}$;
(ii) $L\left(h_{1}\right)$ is either equal to zero or similar to $h_{1}$ for any $L \in k(x, n)\left\langle D_{x}, S_{n}\right\rangle$.

Proof. Let $r \in k(x, n)$ be the nonzero ratio $h_{1} / h_{2}$. Then the first assertion follows from the equality $h_{1}+h_{2}=(1+1 / r) h_{1}$. Since $h_{1}$ is hyperexponential-hypergeometric over $k(x, n)$, successive derivatives and shifts of $h_{1}$ are all similar to $h_{1}$. By the first assertion, the second one follows.

### 5.2.3 Split polynomials

For a nonzero element $f \in k(x, n)$, the denominator and numerator of $f$ are denoted by $\operatorname{den}(f)$ and num $(f)$, respectively. $\operatorname{den}(f)$ and $\operatorname{num}(f)$ are two coprime polynomials in $k[x, n]$. We introduce the definition below to describe a standard form of rational functions in $k(x, n)$.

Definition 5.2.6. A polynomial $p \in k[x, n]$ is said to be split if it is of the form $p_{1}(x) p_{2}(n)$ with $p_{1} \in k[x]$ and $p_{2} \in k[n]$.

A rational function $f$ can be decomposed as $f_{1}(x) f_{2}(n) f_{3}(x, n)$, where $f_{1} \in k(x), f_{2} \in k(n)$ and both $\operatorname{num}\left(f_{3}\right)$ and $\operatorname{den}\left(f_{3}\right)$ have no split factors. The product $f_{1} f_{2}$ is called the split part of $f$.

Algorithms for computing rational solutions of linear differential and difference equations with polynomial coefficients have been presented by Abramov [3, 8, 4]. The lemma below concerns the shape of denominators of rational solutions of linear differential and difference equations with the leading coefficients split.

Lemma 5.2.7. Let $L$ be an operator either in $k[x, n]\left\langle D_{x}\right\rangle$ or in $k[x, n]\left\langle S_{n}\right\rangle$ and let $p$ be a polynomial in $k[x, n]$. If the leading coefficient of $L$ is split, so is the denominator of rational solutions of $L(y(x, n))=p$.

Proof. Let $f \in k(x, n)$ be any rational solution of $L(y(x, n))=p$ and $q \in k[x, n]$ be the leading coefficient of $L$. If $L$ is in $k[x, n]\left\langle D_{x}\right\rangle$, the splitness of $\operatorname{den}(f)$ follows from the well-known fact that any root of $\operatorname{den}(f)$ in $\overline{k(n)}$ is also a root of $q$, see [3]. If $L$ is in $k[x, n]\left\langle S_{n}\right\rangle$, the denominator of $f$ is a divisor of the product $Q(x, n):=q(x, n) q(x, n-1) \cdots q(x, n-d)$ for some finite $d \in \mathbb{N}$ according to Abramov's algorithm for computing a denominator bound in [4]. If $q$ is split, so is the product $Q$. Therefore, $\operatorname{den}(f)$ is split in both cases.

### 5.3 Existence problems

The fact that differential and difference cases can be treated in similar-looking ways has been observed for a long time in mathematics. Once a result is obtained in one side, it can often be analogized in the other one. In the sequel, our presentation will reflect the symmetry between the two counterparts of differential and difference variables.

The method of differencing under the integral sign was first formulated by Almkvist and Zeilberger in [15], which can be used to find a linear recurrence equation for the integral (if exists)

$$
H(n):=\int_{0}^{+\infty} h(x, n) d x
$$

where $h(x, n)$ is a hyperexponential-hypergeometric function over $k(x, n)$. Here, we suppose integrals are well-defined over $k$, say $\mathbb{C}$. The key step of Almkvist and Zeilberger's algorithm
tries to find a nonzero linear recurrence operator $L\left(n, S_{n}\right)$ in $k(n)\left\langle S_{n}\right\rangle$ such that

$$
\begin{equation*}
L\left(n, S_{n}\right)(h)=D_{x}(g), \tag{5.2}
\end{equation*}
$$

for some hyperexponential-hypergeometric function $g$ over $k(x, n)$.

Definition 5.3.1. Let $h$ be a hyperexponential-hypergeometric function over $k(x, n)$. A nonzero linear recurrence operator $L\left(n, S_{n}\right) \in k(n)\left\langle S_{n}\right\rangle$ is called a telescoper for $h$ with respect to $x$ if there exists another hyperexponential-hypergeometric function g such that (5.2) holds.

After that, applying the integral sign to both sides of (5.2) yields

$$
L\left(n, S_{n}\right)(H(n))=g(+\infty, n)-g(0, n) .
$$

This further implies that $L\left(n, S_{n}\right)$ is indeed the recurrence relation satisfied by $H(n)$ under certain nice boundary condition, say $g(+\infty, n)=g(0, n)$. For example, consider the integral

$$
A(n)=\int_{0}^{+\infty} x^{n-1} \exp (-x) d x
$$

Almkvist and Zeilberger's algorithm returns a pair $(L, g)$ with

$$
L=S_{n}-n \quad \text { and } \quad g=-x^{n} \exp (-x) .
$$

Note that $g(+\infty, n)=g(0, n)=0$, which implies that $L(A(n))=A(n+1)-n A(n)=0$. So we recognize that $A(n)=(n-1)$ ! since the value $A(1)$ is equal to 1 .

The differential analogue of Almkvist and Zeilberger's algorithm tries to construct a linear differential equation $L\left(x, D_{x}\right) \in k(x)\left\langle D_{x}\right\rangle$ such that

$$
\begin{equation*}
L\left(x, D_{x}\right)(h)=\Delta_{n}(g), \tag{5.3}
\end{equation*}
$$

for some hyperexponential-hypergeometric function $g$ over $k(x, n)$.

Definition 5.3.2. Let $h$ be a hyperexponential-hypergeometric function over $k(x, n)$. A nonzero linear differential operator $L\left(x, D_{x}\right) \in k(x)\left\langle D_{x}\right\rangle$ is called a telescoper for $h$ with respect to $n$ if there exists another hyperexponential-hypergeometric function g such that (5.3) holds.

Both the $g$ 's in (5.2) and (5.3) are either zero or similar to $h$ by Lemma 5.2.5. This analogue can be used to construct a linear differential equation for the sum

$$
H(x):=\sum_{n=0}^{+\infty} h(x, n) .
$$

For more interesting examples, see the appendix of [15] or Koepf's book [60, Chapters 10-13].
As in the discrete case, not all hyperexponential-hypergeometric functions have telescopers with respect to $x$ or $n$. That is, Almkvist and Zeilberger's algorithm does not terminate on all hyperexponential-hypergeometric inputs.

Example 5.3.3. The non-existence phenomenon is well illustrated by the rational function

$$
f=\frac{1}{x+n} .
$$

Let us show that $f$ has no telescoper with respect to either $x$ or $n$.

Differential case: Suppose that $f$ has a telescoper with respect to $x$, i.e., there exists a nonzero operator $L$ in $k(n)\left\langle S_{n}\right\rangle$ such that $L(f)=D_{x}(g)$ for some $g \in k(x, n)$. Write $L=\sum_{i=0}^{\rho} \ell_{i}(n) S_{n}^{i}$ with $\rho \in \mathbb{N}$ and $\ell_{i} \in k(n)$. Then we have

$$
L(f)=\sum_{i=0}^{\rho} \ell_{i}(n) S_{n}^{i}\left(\frac{1}{x+n}\right)=\sum_{i=0}^{\rho}\left(\frac{\ell_{i}(n)}{x+n+i}\right)=\frac{A}{D}
$$

where $D$ divides the product $(x+n)(x+n+1) \cdots(x+n+\rho)$, which is squarefree with respect to $x$, and $A \in k(n)[x]$ with $\operatorname{deg}_{x}(A)<\operatorname{deg}_{x}(D)$ and $\operatorname{gcd}(A, D)=1$ as polynomials in $x$. Since $A / D=D_{x}(g), A=0$ by Lemma 4.3.1 (i). So $f$ is a rational solution of the linear difference equation $L(z)=0$. By Lemma 5.2.7, the denominator of $f$ should be split in $k[x, n]$ since the leading coefficient of $L$ is free of $x$, but $x+n$ is not split, a contradiction.

Difference case: Suppose that $f$ has a telescoper with respect to $n$, i.e., there exists a nonzero operator $L$ in $k(x)\left\langle D_{x}\right\rangle$ such that $L(f)=\Delta_{n}(g)$ for some $g \in k(x, n)$. Write $L=\sum_{i=0}^{\rho} \ell_{i}(x) D_{x}^{i}$ with $\rho \in \mathbb{N}$ and $\ell_{i} \in k(x)$. Then we have

$$
L(f)=\sum_{i=0}^{\rho} \ell_{i}(x) D_{x}^{i}\left(\frac{1}{x+n}\right)=\sum_{i=0}^{\rho}\left(\frac{(-1)^{i} \ell_{i}(x)}{(x+n)^{i+1}}\right)=\frac{A}{D}
$$

where $D$ is a factor of $(x+n)^{\rho+1}$, which is shift-free with respect to $n$, and $A \in k(x)[n]$ with $\operatorname{deg}_{n}(A)<\operatorname{deg}_{n}(D)$ and $\operatorname{gcd}(A, D)=1$ as polynomials in $n$. Since $A / D=\Delta_{n}(g), A=0$ by Lemma 4.3.1 (ii). So $f$ is a rational solution of the linear differential equation $L(z)=0$. By Lemma 5.2.7, the denominator of $f$ should be split in $k[x, n]$ since the leading coefficient of $L$ is free of $n$, but $x+n$ is not split, a contradiction.

In this chapter, we solve the following problem, which is equivalent to the termination problem of Almkvist and Zeilberger's algorithm.

Problem 5.3.4. Given a hyperexponential-hypergeometric function $h(x, n)$ over $k(x, n)$, decide whether $h$ has a telescoper with respect to $x$ or $n$.

### 5.4 Standard representations

We derive a standard representation of hyperexponential-hypergeometric functions, based on [42, Proposition 5], i.e., the bivariate case of Theorem 4.4.6.

Lemma 5.4.1. Let $a$ and $b$ be two nonzero rational functions in $k(x, n)$ such that the integrability condition (5.1) holds. Then there exist $f \in k(x, n)$ whose split part is trivial, $\alpha \in k(n)$, and $\beta, \gamma \in$ $k(x)$ such that

$$
\begin{equation*}
a=\frac{\delta(f)}{f}+\frac{\delta(\beta(x))}{\beta(x)} n+\gamma(x) \quad \text { and } \quad b=\frac{\sigma(f)}{f} \alpha(n) \beta(x) . \tag{5.4}
\end{equation*}
$$

Proof. By Proposition 5 in [42], there exist $f \in k(x, n), \alpha \in k(n)$, and $\beta, \gamma \in k(x)$ such that (5.4) holds. Write $f$ as $f_{1} f_{2} f_{3}$ where $f_{1} \in k(x), f_{2} \in k(n)$, and $f_{3} \in k(x, n)$ whose split part is trivial. Replacing $f$ by $f_{3}, \gamma$ by $\gamma+\delta\left(f_{1}\right) / f_{1}$, and $\alpha$ by $\alpha \sigma\left(f_{2}\right) / f_{2}$ in (5.4) yields the lemma.

By Lemma 5.4.1, any hyperexponential-hypergeometric function can be written in a unique multiplicative form.

Proposition 5.4.2. Let $h$ be a hyperexponential-hypergeometric function. Then there exist $f \in$ $k(x, n)$ with trivial split part, $\alpha \in k(n)$ with monic numerator and denominator, and $\beta, \gamma \in k(x)$ such that

$$
\begin{equation*}
h \in f(x, n) \beta(x)^{n} \mathcal{H}(\gamma(x), \alpha(n)) . \tag{5.5}
\end{equation*}
$$

Moreover, if

$$
h \in f^{\prime}(x, n) \beta^{\prime}(x)^{n} \mathcal{H}\left(\gamma^{\prime}(x), \alpha^{\prime}(n)\right)
$$

where $f^{\prime} \in k(x, n)$ with trivial split part, $\alpha^{\prime} \in k(n)$ with monic numerator and denominator, and $\beta^{\prime}, \gamma^{\prime} \in k(x)$, then $f / f^{\prime} \in k, \alpha=\alpha^{\prime}, \beta=\beta^{\prime}$, and $\gamma=\gamma^{\prime}$.

Proof. Let $a$ and $b$ be the $x$-certificate and $n$-certificate of $h$, respectively. By Lemma 5.4.1, there exist $f \in k(x, n)$ with trivial split part, $\alpha \in k(n)$, and $\beta, \gamma \in k(x)$ such that (5.4). By a straightforward calculation, $h$ and a nonzero element of $f \beta^{n} \mathcal{H}(\gamma, \alpha)$ have the same certificates. Hence, (5.5) holds by Proposition 5.2.3. Assume that $h$ belongs to $f^{\prime}(x, n) \beta^{\prime}(x)^{n} \mathcal{H}\left(\gamma^{\prime}(x), \alpha^{\prime}(n)\right)$, as given in the statement of the proposition. Then the $x$-certificate and $n$-certificate of every nonzero element in $f^{\prime}(x, n) \beta^{\prime}(x)^{n} \mathcal{H}\left(\gamma^{\prime}(x), \alpha^{\prime}(n)\right)$ are $a$ and $b$, respectively. It follows that

$$
\begin{equation*}
\frac{\delta(f)}{f}+\frac{\delta(\beta(x))}{\beta(x)} n+\gamma(x)=\frac{\delta\left(f^{\prime}\right)}{f^{\prime}}+\frac{\delta\left(\beta^{\prime}(x)\right)}{\beta^{\prime}(x)} n+\gamma^{\prime}(x) \tag{5.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\sigma(f)}{f} \alpha(n) \beta(x)=\frac{\sigma\left(f^{\prime}\right)}{f^{\prime}} \alpha^{\prime}(n) \beta^{\prime}(x) \tag{5.7}
\end{equation*}
$$

Consequently,

$$
\frac{\delta(f)}{f}=\frac{\delta\left(f^{\prime}\right)}{f^{\prime}}, \quad \frac{\delta(\beta(x))}{\beta(x)}=\frac{\delta\left(\beta^{\prime}(x)\right)}{\beta^{\prime}(x)}, \quad \text { and } \gamma(x)=\gamma^{\prime}(x)
$$

by (5.6), the uniqueness of the partial fraction decomposition of $a$, and the assumption that both $f$ and $f^{\prime}$ have trivial split parts. Hence, $f=c f^{\prime}$ and $\beta=c^{\prime} \beta^{\prime}$ for some $c, c^{\prime} \in k(n)$. Again, by the assumption on $f$ and $f^{\prime}, c$ belongs to $k$. Since $\beta$ and $\beta^{\prime}$ are free of $n$, so is $c^{\prime}$. It follows from (5.7) that $\alpha(n)=c^{\prime} \alpha(n)$. Thus, $c^{\prime}$ is equal to 1 since the numerators and denominators of $\alpha(n)$ and $\alpha^{\prime}(n)$ are assumed to be monic.

Definition 5.4.3. Let $h(x, n) \in \mathcal{H}(a, b)$ be a hyperexponential-hypergeometric function. We say that a quadruple $(f(x, n), \alpha(n), \beta(x), \gamma(x)) \in k(x, n)^{4}$ is the standard representation of $h$ if $f \in$ $k(x, n)$ with trivial split part, $\alpha \in k(n)$ with monic numerator and denominator, $\beta, \gamma \in k(x)$, and (5.5) holds.

By the definition in [92] or Definition 4.5.3 in Chapter 4, a bivariate hyperexponentialhypergeometric function is proper if its standard representation is

$$
\begin{equation*}
(p(x, n), \alpha(n), \beta(x), \gamma(x)) \tag{5.8}
\end{equation*}
$$

where $p \in k[x, n], \alpha \in k(n)$, and $\beta, \gamma \in k(x)$.
Proposition 5.4.4. Let $h \in f(x, n) \beta(x)^{n} \mathcal{H}(\gamma(x), \alpha(n))$ with $f \in k(x, n), \alpha \in k(n)$, and $\beta, \gamma \in$ $k(x)$. If the denominator of $f$ is split, then $h$ is proper.

Proof. Write $f=p / q$ with $p, q \in k[x, n]$ and $\operatorname{gcd}(p, q)=1$. Since $q$ is split, $q=A(x) B(n)$ for some $A \in k[x]$ and $B \in k[n]$. By Corollary 5.2.4 (i), we have

$$
h \in p(x, n) \beta(x)^{n} \mathcal{H}\left(\gamma(x)-\frac{\delta(A(x))}{A(x)}, \alpha(n) \frac{B(n)}{\sigma(B(n))}\right) .
$$

Therefore, $h$ is proper by definition.

### 5.5 Two additive decompositions

For a bivariate hyperexponential-hypergeometric function $h(x, n)$ in the continuous variable $x$ and the discrete one $n$, we can perform two kinds of complete additive decompositions: one is with respect to the continuous variable, obtaining $h=D_{x}\left(h_{1}\right)+h_{2}$, and the other is with respect to the discrete variable, obtaining $h=\Delta_{n}\left(h_{1}\right)+h_{2}$. The $h_{1}$ 's and $h_{2}$ 's above are hyperexponentialhypergeometric functions with $h_{2}$ minimal in some sense. Algorithms for complete additive decompositions generalize the capability of Gosper's algorithm [51] and its differential analogue [15]. That is, it returns $h_{2}=0$ if $h=D_{x}(g)$ or $h=\Delta_{n}(g)$ for some $g$ hyperexponential-hypergeometric. In this section, we adapt the two additive decompositions to hyperexponential-hypergeometric functions described by their standard representations.

Notation. In what follows, let $F$ be either $k(n)$ if the decomposition is with respect to $x$ or $k(x)$ if it is with respect to $n$.

### 5.5.1 Additive decomposition with respect to $x$

In Chapter 3, Hermite reduction or Ostrogradsky and Horowitz's method decomposes a rational function $f \in F(x)$ into $f=D_{x}(g)+r$, where $r=a / b$ is minimal in the sense that $\operatorname{deg}_{x}(a)<$ $\operatorname{deg}_{x}(b)$ and $b$ is squarefree, that is, the factors of $b$ have minimal multiplicity. Hermite reduction was later extended by Davenport [39] and by Geddes, Le and Li [47] with two different motivations: one is to solve the Risch differential equation

$$
D_{x}(y)+f y=g, \quad \text { where } f \text { and } g \text { are in } F(x)
$$

and the other is to solve the decomposition problem of hyperexponential functions. We will use the latter to compute additive decompositions of hyperexponential-hypergeometric functions with respect to $x$. To this end, we first recall some terminology from [47].

Definition 5.5.1 (GLL triple). Let $u, v \in F[x]$ and $w \in F(x)$. We call $(u, v, w)$ a Geddes $-\mathrm{Le}-\mathrm{Li}$ triple, in short GLL triple, if the following conditions are satisfied:
(i) $\operatorname{gcd}(u, v)=1$;
(ii) $u$ is squarefree with respect to $x$;
(iii) $w$ is differential-reduced with respect to $x$;
$(i v) \operatorname{gcd}(u, \operatorname{den}(w))=1$.

Definition 5.5.2 (Hyperexponential-integrable). A hyperexponential function $h$ over $F(x)$ is said to be hyperexponential-integrable if there exists a hyperexponential function $g$ such that $h=$ $D_{x}(g)$.

Note that the function $g$ in the above definition is similar to $h$ if it exists.

Definition 5.5.3 (Additive decomposition with respect to $x)$. Let $h$ be in $\mathcal{H}(a, b)$ with $a, b \in$ $F(x)$. An additive decomposition of $h$ with respect to $x$ is of the form

$$
h=D_{x}\left(h_{1}\right)+h_{2},
$$

where $h_{1}$ is hyperexponential-hypergeometric over $F(x)$, and either $h_{2}$ is zero or

$$
h_{2} \in \frac{v}{u} \cdot \mathcal{H}\left(a_{2}, b_{2}\right), \quad \text { where } b_{2} \in F(x) \text { and }\left(u, v, a_{2}\right) \text { is a GLL triple. }
$$

Such a decomposition is said to be complete if $h_{2}=0$ whenever $h$ is hyperexponential-integrable.

Remark 5.5.4. In the sequel we never use complete additive decompositions. In fact, the $h_{1}$ and $h_{2}$ above are similar to $h$ by the same argument as in [75, Proposition 5.6.2].

## Geddes-Le-Li's algorithm

We recall the algorithm from [47] to perform additive decompositions with respect to $x$. View a hyperexponential-hypergeometric function $h \in \mathcal{H}(a, b)$ as a hyperexponential function with $x$ certificate $a \in F(x)$. Let $(K, S) \in F(x) \times F(x)$ be a strict DRNF of $a$, that is, $K$ is differentialreduced with respect to $x$ and

$$
a=K+\frac{\delta(S)}{S}, \quad \text { where } \operatorname{gcd}(\operatorname{den}(K), \operatorname{den}(S))=1
$$

Immediately, the additive form above for $a$ leads to a multiplicative form for $h$, that is,

$$
h=S \cdot \tilde{h}, \quad \text { where } \tilde{h} \in \mathcal{H}\left(K, b \frac{S}{\sigma(S)}\right) .
$$

Now comes the Hermite-like reduction: Theorem 2 in [47] shows that there exists $S_{1} \in F(x)$ such that

$$
\begin{equation*}
S=\delta\left(S_{1}\right)+K S_{1}+\frac{v}{u \cdot \operatorname{den}(K)^{i}}, \tag{5.9}
\end{equation*}
$$

where $i \in\{0,1\}$ and $(u, v, K)$ is a GLL triple. Since $D_{x}(\tilde{h})=K \tilde{h}$, the decomposition for $S$ in (5.9) further implies that

$$
h=D_{x}\left(S_{1} \tilde{h}\right)+\frac{v}{u \cdot \operatorname{den}(K)^{i}} \tilde{h},
$$

In order to get a complete additive decomposition for $h$, we still need to decide whether

$$
\begin{equation*}
h_{2}=\frac{v}{u \cdot \operatorname{den}(K)^{i}} \tilde{h} \tag{5.10}
\end{equation*}
$$

is hyperexponential-integrable or not. A necessary condition is as follows.

Lemma 5.5.5 (Geddes-Le-Li, 2004). Let $h$ be a hyperexponential function over $F(x)$. If $h$ is hyperexponential-integrable, and

$$
\frac{\delta(h)}{h}=w+\frac{\delta(v / u)}{v / u},
$$

where $v, u \in F[x], w \in F(x)$, and $(u, v, w)$ is a GLL triple, then $u$ is in $F$.

Proof. For the proof, see [47, Theorem 4].

## Additive decompositions with respect to $x$ from standard representations

Let $(f(x, n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of $h \in \mathcal{H}(a, b)$, that is,

$$
h \in f(x, n) \beta(x)^{n} \mathcal{H}(\gamma(x), \alpha(n)),
$$

where $f \in k(x, n)$ with trivial split part, $\alpha \in k(n)$ with monic numerator and denominator, and $\beta, \gamma \in k(x)$. We show how to obtain an additive decomposition of $h$ with respect to $x$ from its standard representation.

First, we compute a strict DRNF for $a$ from the standard representation of $h$. The following lemma helps us decide whether certain rational functions are differential-reduced.

Lemma 5.5.6. Let

$$
g=\frac{p}{q}+n \frac{s}{t}+\frac{u}{v},
$$

where $p, q, s, t, u, v \in k[x]$ with $\operatorname{gcd}(p, q)=\operatorname{gcd}(s, t)=\operatorname{gcd}(u, v)=1$. Assume that $v \mid t$. Then
(i) the denominator of $g$ is the least common multiple of $q$ and $t$;
(ii) if $p / q$ is differential-reduced, so is $g$;
(iii) if $p / q+n s / t$ is differential-reduced, so is $g$.

Proof. Let $M=\operatorname{lcm}(q, t), w_{1}=M / q$ and $w_{2}=M / t$. We first show the following claim.
Claim. If $w$ is a nontrivial irreducible factor of

$$
\operatorname{gcd}\left(M, s w_{2} n+r\right), \quad \text { where } r \text { is an element in } k[x],
$$

then $w$ divides $w_{2}, r$ and $q$, but does not divide $w_{1}$.
Proof of the Claim. Since $n$ is an indeterminate over $k, w$ is a factor of $s w_{2}$. Suppose that $w$ is not a factor of $w_{2}$. Then it is a factor of $s$. Since $w$ is a factor of $M$, it is a factor of $w_{2} t$, and, thus, a factor of $t$, a contradiction to the assumption that $\operatorname{gcd}(s, t)=1$. We have concluded that $w$ is a factor of $w_{2}$. Consequently, it is a factor of $r$ because $n$ is an indeterminate, but it is not a factor of $w_{1}$ because $\operatorname{gcd}\left(w_{1}, w_{2}\right)=1$. Since $w$ divides $q w_{1}$, it divides $q$. The claim is proved.

To prove the first assertion, we let $w_{3}=M / v$, which is a multiple of $w_{2}$ since $t$ is a multiple of $v$. Put $W=p w_{1}+s w_{2} n+u w_{3}$. We have $g=W / M$. It remains to show that $\operatorname{gcd}(M, W)=1$. Set $r=p w_{1}+u w_{3}$ and assume that $w$ is an irreducible factor of $\operatorname{gcd}(M, W)$. By the claim, $w$ divides $r$ and $w_{2}$, hence, it divides $p w_{1}$. The claim then implies that $w$ divides $p$. But $w$ also divides $q$ by the same claim. Thus, $w \in k$ since $\operatorname{gcd}(p, q)=1$.

To prove the second assertion, we only need to show that $\operatorname{gcd}(q, p-\ell \delta(q))=1$ implies that $\operatorname{gcd}(M, W-\ell \delta(M))=1$. Setting $r=-p w_{1}-u w_{2}-\ell \delta(M)$, we have

$$
\operatorname{gcd}(M, W-\ell \delta(M))=\operatorname{gcd}\left(M, s w_{2} n+r\right) .
$$

Assume that $w$ is an irreducible factor of $\operatorname{gcd}(M, W-\ell \delta(M))$. By the claim, $w$ divides $r$ and $w_{2}$, therefore, it divides $p w_{1}-\ell \delta(M)$, which is equal to $p w_{1}-\ell w_{1} \delta(q)-\ell \delta\left(w_{1}\right) q$. Since $w$ divides $q$, it divides $(p-\ell \delta(q)) w_{1}$. Again, by the claim, $w$ divides $p-\ell \delta(q)$. On the other hand, $w$ divides $q$ because $w \mid M$ and $w \nmid w_{1}$. Hence, $w$ is a factor of $\operatorname{gcd}(q, p-\ell \delta(q))$, which is equal to 1 , because $p / q$ is differential-reduced. Consequently, $w$ is in $k$.

To prove the third assertion, it suffices to show that $\operatorname{gcd}\left(M, p w_{1}+s w_{2} n-\ell \delta(M)\right)=1$ implies $\operatorname{gcd}(M, W-\ell \delta(M))=1$. Let $r$ and $w$ be defined as above. By the claim, $w$ divides $w_{3}$ since $w_{2}$ divides $w_{3}$. It follows from the definition of $r$ that $w$ divides $p w_{1}+s w_{2} n-\ell \delta(M)$, and, hence, $w$ is in $k$.

For later use, we recall the definition of pumps, given in [13, page 533].

Definition 5.5.7 (Pump). Let $p, q \in F[x]$ be such that $p \mid q$. A factor $\widetilde{p}$ of $q$ is said to be the pump of $p$ in $q$ if $\operatorname{gcd}(q / \tilde{p}, \widetilde{p})=1, \quad p \mid \widetilde{p}$, and any irreducible factor of $\widetilde{p}$ divides $p$.

Example 5.5.8. Let $p_{1}, p_{2}, p_{3}$ be in $F[x]$ with positive degrees. Assume that they are pairwise coprime and $p=p_{1} p_{2}^{2} p_{3}^{3}$. Then the pump of $p_{1} p_{3}^{2}$ in $p$ is equal to $p_{1} p_{3}^{3}$.

In other words, a pump of a factor $p$ of $q$ has the same squarefree part as $p$ 's, but each of its irreducible factors has the maximal multiplicity in $q$. For a rational function $f \in k(x, n)$, denote by $f^{*}$ the squarefree part of $\operatorname{num}(f) \cdot \operatorname{den}(f)$. Now, we show how to compute a strict DRNF of $a$ from that of $\gamma$.

Lemma 5.5.9. Let $(f(x, n), \alpha(n), \beta(x), \gamma(x))$ be a standard representation of a nonzero element in $\mathcal{H}(a, b)$. Let $(\widetilde{K}, \widetilde{S})$ be a strict DRNF of $\gamma$. Let $u=\operatorname{gcd}\left(\operatorname{den}(\widetilde{S}), \beta^{*}\right)$ and $T$ the pump of $u$ in $\operatorname{den}(\widetilde{S})$. If

$$
\begin{equation*}
K=\widetilde{K}+n \frac{\delta(\beta)}{\beta}-\frac{\delta(T)}{T} \quad \text { and } \quad S=f \widetilde{S} T \tag{5.11}
\end{equation*}
$$

then $(K, S)$ is a strict DRNF of a with respect to $x$.
Proof. Let $(\tilde{K}, \tilde{S})$ be a strict DRNF of $\gamma(x)$. Then by definition, we have

$$
a=\frac{\delta(f)}{f}+n \frac{\delta(\beta)}{\beta}+\gamma=\frac{\delta(f)}{f}+n \frac{\delta(\beta)}{\beta}+\frac{\delta(\tilde{S})}{\tilde{S}}+\tilde{K} .
$$

By Lemma 2.3.3, the denominators of $\delta(\beta) / \beta$ and $\delta(T) / T$ are $\beta^{*}$ and $T^{*}$, respectively. By the definition of pump and $u \mid \beta^{*}, u$ is the squarefree part of $T$, which implies $T^{*} \mid \beta^{*}$. Hence, $K$ is differential-reduced by Lemma 5.5.6 (ii).

It remains to show $\operatorname{gcd}(\operatorname{den}(K), \operatorname{den}(S))=1$. By the definition of pumps, $\operatorname{den}(\widetilde{S})=T w$, where $w \in k[x]$ and $\operatorname{gcd}(T, w)=1$. On the other hand,

$$
\operatorname{den}(S)=\operatorname{den}(f) \operatorname{den}(\widetilde{S} T)=\operatorname{den}(f) w,
$$

since $\operatorname{num}(f) \operatorname{den}(f)$ has only trivial factors in $k[x]$. It follows that

$$
\operatorname{gcd}(\operatorname{den}(K), \operatorname{den}(S))=\operatorname{gcd}(\operatorname{den}(K), \operatorname{den}(f) w)
$$

which is equal to $\operatorname{gcd}(\operatorname{den}(K), w)$ since $\operatorname{den}(K) \in k[x]$. So it suffices to show that $\operatorname{den}(K)$ and $w$ are coprime. By Lemma 5.5.6 $(i), \operatorname{den}(K)=\operatorname{lcm}\left(\operatorname{den}(\widetilde{K}), \beta^{*}\right)$. So it suffices to show that

$$
\operatorname{gcd}(\operatorname{den}(\widetilde{K}), w)=1 \quad \text { and } \quad \operatorname{gcd}\left(\beta^{*}, w\right)=1
$$

The first equality is immediate from the assumptions that $(\widetilde{K}, \widetilde{S})$ is a strict DRNF and that $w$ is a factor of $\operatorname{den}(\widetilde{S})$. To prove the second, we assume that $p$ is an irreducible factor of $\operatorname{gcd}\left(\beta^{*}, w\right)$. Then $p$ is a factor of $u$ by the definitions of $u$ and $w$. Hence, it is a factor of $T$. Consequently, $p$ is a common factor of $T$ and $w$. We conclude that $p$ is in $k$.

We apply Geddes-Le-Li's algorithm to a hyperexponential-hypergeometric function described by its standard representation and obtain a special form of its additive decomposition with respect to $x$.

Theorem 5.5.10. Let $(f(x, n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of a nonzero element $h$ in $\mathcal{H}(a, b)$. Then one can find a hyperexponential-hypergeometric function $h_{1}$ and $\tilde{\gamma}(x) \in$ $k(x)$ such that

$$
\begin{equation*}
h-D_{x}\left(h_{1}\right) \in \frac{v}{u} \cdot \mathcal{H}\left(\tilde{\gamma}+n \frac{\delta(\beta)}{\beta}, \beta \alpha\right) \tag{5.12}
\end{equation*}
$$

where $u, v$ are in $k(n)[x]$ and $(u, v, \tilde{\gamma}+n \delta(\beta) / \beta)$ is a GLL triple.
Proof. By Lemma 5.5.9, the $x$-certificate $a$ of $h$ has a strict DRNF $(K, S)$ of the form

$$
\left(\tilde{K}+n \cdot \frac{\delta(\beta)}{\beta}-\frac{\delta(T)}{T}, f \cdot \tilde{S} \cdot T\right)
$$

where $(\tilde{K}, \tilde{S})$ is a strict DRNF of $\gamma(x)$ and $T \in k[x]$. A direct calculation verifies that

$$
h=S \tilde{h} \quad \text { for some } \tilde{h} \in \mathcal{H}(K, \beta \alpha) .
$$

By (5.9), Hermite-like reduction rewrites $S$ into

$$
S=\delta\left(S_{1}\right)+S_{1} K+\frac{v}{u \operatorname{den}(K)^{i}},
$$

where $i \in\{0,1\}, S_{1} \in k(x, n)$ and $(u, v, K)$ is a GLL triple. It follows that

$$
h=D_{x}\left(S_{1} \tilde{h}\right)+\frac{v}{u \operatorname{den}(K)^{i}} \tilde{h} .
$$

Since $\operatorname{den}(K)$ is in $k[x]$ and by Corollary 5.2.4

$$
\begin{aligned}
\frac{v}{u \operatorname{den}(K)^{i}} \mathcal{H}(K, \beta \alpha) & =\frac{v}{u} \mathcal{H}\left(K-\frac{\delta\left(\operatorname{den}(K)^{i}\right)}{\operatorname{den}(K)^{i}}, \beta \alpha \frac{\operatorname{den}(K)^{i}}{\sigma\left(\operatorname{den}(K)^{i}\right)}\right) \\
& =\frac{v}{u} \mathcal{H}\left(K-\frac{\delta\left(\operatorname{den}(K)^{i}\right)}{\operatorname{den}(K)^{i}}, \beta \alpha\right) .
\end{aligned}
$$

Setting $h_{1}=S_{1} \tilde{h}$ and $\tilde{\gamma}=\tilde{K}-\frac{\delta(T)}{T}-\frac{\delta\left(\operatorname{den}(K)^{i}\right.}{\operatorname{den}(K)^{i}}$ yields (5.12). It remains to show that the triple $\left(u, v, \tilde{\gamma}+n \frac{\delta(\beta)}{\beta}\right)$ is a GLL triple. Note that

$$
g:=\tilde{\gamma}+n \frac{\delta(\beta)}{\beta}=K-\frac{\delta\left(\operatorname{den}(K)^{i}\right)}{\operatorname{den}(K)^{i}} .
$$

Since $K$ is differential-reduced, so is $g$ by definition. Moreover, $u$ and $\operatorname{den}(g)$ are coprime with respect to $x$, as $u$ and $\operatorname{den}(K)$ are coprime. It follows that $(u, v, g)$ is a GLL triple.

Remark 5.5.11. An additive decomposition with respect to $x$ can be computed by the algorithm ReducedHyperexp in [47].

### 5.5.2 Additive decomposition with respect to $n$

In the discrete case, the algorithms in $[2,4,76,72]$ decompose a rational function $f \in F(n)$ into $f=\Delta_{n}(g)+r$, where $r=a / b$ is minimal in the sense that $\operatorname{deg}_{n}(a)<\operatorname{deg}_{n}(b)$ and $b$ is shift-free, that is $b$ has minimal dispersion. Those decomposition algorithms are extended to the case of hypergeometric terms by Abramov and Petkovšek [11, 13]. This extension also generalizes the capability of the well-known Gosper algorithm [51], which decides whether the indefinite sum of a hypergeometric term is hypergeometric. More precisely, for a given hypergeometric term $H(n)$ over $F(n)$, Abramov and Petkovšek's algorithm computes two hypergeometric terms $H_{1}(n)$ and $H_{2}(n)$ such that $H=\Delta_{n}\left(H_{1}\right)+H_{2}$, where $H_{2}$ is minimal in some sense. In particular, $H_{2}$ is identically zero if $H(n)$ has a hypergeometric indefinite sum. Here, we will use Abramov and Petkovšek's algorithm to compute additive decompositions of hyperexponentialhypergeometric functions with respect to the discrete variable $n$. To this end, we recall some terminology from [13].

Definition 5.5.12 (AP triple). Let $u, v \in F[n]$ and $w \in F(n)$. We call ( $u, v, w)$ an AbramovPetkovšek triple, in short AP triple, if the following conditions are satisfied:
(i) $\operatorname{gcd}(u, v)=1$;
(ii) $u$ is shift-free with respect to $n$;
(iii) $w$ is shift-reduced with respect to $n$;
(iv) $\operatorname{gcd}\left(u, \sigma^{-\ell}(\operatorname{num}(w))\right)=\operatorname{gcd}\left(u, \sigma^{\ell}(\operatorname{den}(w))\right)=1$ for all $\ell \in \mathbb{N}$.

Definition 5.5.13 (Hypergeometric-summable). A hypergeometric term $h$ over $F(n)$ is said to be hypergeometric-summable if there exists a hypergeometric function $g$ such that $h=\Delta_{n}(g)$.

Note that the $g$ in the above definition is similar to $h$ if it exists.

Definition 5.5.14 (Additive decomposition with respect to $n$ ). Let $h$ be in $\mathcal{H}(a, b)$ with $a, b \in$ $F(n)$. An additive decomposition of $h$ with respect to $n$ is of the form

$$
h=\Delta_{n}\left(h_{1}\right)+h_{2},
$$

where $h_{1}$ is hyperexponential-hypergeometric over $F(n)$, and either $h_{2}$ is zero or

$$
h_{2} \in \frac{v}{u} \cdot \mathcal{H}\left(a_{2}, b_{2}\right), \quad \text { where } a_{2} \in F(n) \text { and }\left(u, v, b_{2}\right) \text { is an AP triple. }
$$

Such a decomposition is said to be complete if $h_{2}=0$ whenever $h$ is hypergeometric-summable.
Remark 5.5.15. In the sequel we never use complete additive decompositions. In fact, the $h_{1}$ and $h_{2}$ above are similar to $h$ by the same argument in [75, Proposition 5.6.2].

## Abramov and Petkovšek's algorithm

We recall the algorithm from $[11,13]$ to perform additive decomposition with respect to $n$. View a hyperexponential-hypergeometric function $h \in \mathcal{H}(a, b)$ as a hypergeometric term with $n$ certificate $b \in F(n)$. Let $(K, S) \in F(n) \times F(n)$ be a strict SRNF of $b$, that is, $K$ is shift-reduced with respect to $n$ and

$$
b=K \cdot \frac{\sigma(S)}{S}
$$

where $\operatorname{gcd}(\operatorname{num}(K), \sigma(\operatorname{den}(S)) \cdot \operatorname{num}(S))=\operatorname{gcd}(\operatorname{den}(K), \sigma(\operatorname{num}(S)) \cdot \operatorname{den}(S))=1$. Immediately, the multiplicative form above for $b$ leads to a multiplicative form for $h$, that is,

$$
h=S \cdot \tilde{h}, \quad \text { where } \tilde{h} \in \mathcal{H}\left(a-\frac{\delta(S)}{S}, K\right) .
$$

Now comes Abramov and Petkovšek's reduction: Lemma 9 in [13] implies that there exists $S_{1} \in$ $F(n)$ such that

$$
\begin{equation*}
S=\sigma\left(S_{1}\right) K-S_{1}+\frac{v}{u \cdot\left(\sigma^{-1}\left(k_{1}\right)\right)^{i} \cdot k_{2}^{j}}, \tag{5.13}
\end{equation*}
$$

where $i, j \in\{0,1\}, k_{1}$ and $k_{2}$ are the numerator and denominator of $K$, respectively, and ( $u, v, K$ ) is an AP triple. Since $S_{n}(\tilde{h})=K \tilde{h}$, the decomposition for $S$ in (5.13) further implies that

$$
h=\Delta_{n}\left(S_{1} \tilde{h}\right)+\frac{v}{u \cdot\left(\sigma^{-1}\left(k_{1}\right)\right)^{i} \cdot k_{2}^{j}} \tilde{h} .
$$

In order to get a complete additive decomposition of $h$, we still need to decide whether

$$
\begin{equation*}
h_{2}=\frac{v}{u \cdot\left(\sigma^{-1}\left(k_{1}\right)\right)^{i} \cdot k_{2}^{j}} \tilde{h} \tag{5.14}
\end{equation*}
$$

is hypergeometric-summable or not. A necessary condition is as follows.

Lemma 5.5.16 (Abramov-Petkovšek, 2002). Let $h$ be a hypergeometric function over $F(n)$. If $h$ is hypergeometric-summable, and

$$
\frac{\sigma(h)}{h}=\frac{\sigma(v / u)}{v / u} w
$$

where $v, u \in F[n], w \in F(n)$, and $(u, v, w)$ is an AP triple, then $u$ is in $F$.
Proof. For the proof, see [13, Theorem 11].

## Additive decompositions with respect to $n$ from standard representations

Let $(f(x, n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of $h \in \mathcal{H}(a, b)$. We show how to obtain an additive decomposition of $h$ with respect to $n$ from its standard representation.

The next lemma relates a strict SRNF of the $n$-certificate $b$ of $h$ to its standard representation.
Lemma 5.5.17. Let $(f(x, n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of a nonzero element in $h \in \mathcal{H}(a, b)$. If $(\tilde{K}, \tilde{S})$ is a strict SRNF of $\alpha(n)$ with respect to $n$, then $(\beta \tilde{K}, f \tilde{S})$ is a strict SRNF of the $n$-certificate $a$ of $h$.

Proof. Let $K=\beta \tilde{K}$ and $S=f \tilde{S}$. Since $(\tilde{K}, \tilde{S})$ is a strict SRNF of $\alpha(n), \alpha=\tilde{K} \sigma(\tilde{S}) / \tilde{S}$. It follows that

$$
b=\frac{\sigma(h)}{h}=\beta \alpha \frac{\sigma(f)}{f}=\beta\left(\tilde{K} \frac{\sigma(\tilde{S})}{\tilde{S}}\right) \frac{\sigma(f)}{f}=\beta \tilde{K} \cdot \frac{\sigma(f \tilde{S})}{f \tilde{S}} .
$$

The product $\beta \tilde{K}$ is shift-reduced with respect to $n$, because $\tilde{K}$ is shift-reduced and $\beta$ is in $k(x)$. Write $K=k_{1} / k_{2}$ and $S=s_{1} / s_{2}$ with $\operatorname{gcd}\left(k_{1}, k_{2}\right)=\operatorname{gcd}\left(s_{1}, s_{2}\right)=1$ in $k[x, n]$. It remains to verify the GCD condition

$$
\begin{equation*}
\operatorname{gcd}\left(k_{1}, \sigma\left(s_{2}\right) s_{1}\right)=\operatorname{gcd}\left(k_{2}, \sigma\left(s_{1}\right) s_{2}\right)=1 \tag{5.15}
\end{equation*}
$$

for $(K, S)$. Since $\beta \in k(x)$ and $\tilde{K} \in k(n)$, we have

$$
k_{1}=\operatorname{num}(\beta) \cdot \operatorname{num}(\tilde{K}) \quad \text { and } \quad k_{2}=\operatorname{den}(\beta) \cdot \operatorname{den}(\tilde{K}) .
$$

Similarly, since $\tilde{S} \in k(n)$ and $f$ has a trivial split part, we have

$$
s_{1}=\operatorname{num}(f) \cdot \operatorname{num}(\tilde{S}), \quad \text { and } \quad s_{2}=\operatorname{den}(f) \cdot \operatorname{den}(\tilde{S})
$$

Hence, the condition (5.15) holds, because ( $\tilde{K}, \tilde{S}$ ) is a strict SRNF, and every nonzero polynomial in $k[n]$ is coprime with both $\operatorname{num}(f)$ and $\operatorname{den}(f)$.

We apply Abramov-Petkovšek's algorithm to a hyperexponential-hypergeometric function described by standard representation and obtain a special form of its additive decomposition with respect to $n$.

Theorem 5.5.18. Let $(f(x, n), \alpha(n), \beta(x), \gamma(x))$ be the standard representation of a nonzero element in $h \in \mathcal{H}(a, b)$. Then one can find $\tilde{\gamma}(x) \in k(x), \tilde{\alpha} \in k(n)$, and a hyperexponentialhypergeometric function $h_{1}$ such that

$$
h-\Delta_{n}\left(h_{1}\right) \in \frac{v}{u} \cdot \mathcal{H}\left(\tilde{\gamma}+n \frac{\delta(\beta)}{\beta}, \beta \tilde{\alpha}\right)
$$

where $u, v \in k(x)[n]$ and $(u, v, \beta \tilde{\alpha})$ is an AP triple.
Proof. By Lemma 5.5.17, the $n$-certificate $b$ of $h$ has a strict SRNF $(K, S)$ of the form

$$
(\beta \tilde{K}, f \tilde{S})
$$

where $(\tilde{K}, \tilde{S})$ is a strict SRNF of $\alpha(n)$. A direct calculation verifies that

$$
h=S \tilde{h} \quad \text { for some } \tilde{h} \in \mathcal{H}\left(\gamma+n \frac{\delta(\beta)}{\beta}, K\right) .
$$

Write $K=k_{1} / k_{2}$ and $S=s_{1} / s_{2}$ with $\operatorname{gcd}\left(k_{1}, k_{2}\right)=\operatorname{gcd}\left(s_{1}, s_{2}\right)=1$ in $k[x, n]$. Now, AbramovPetkovšek's reduction [13] decomposes $S$ into

$$
S=\delta\left(S_{1}\right) K-S_{1}+\frac{v}{u\left(\sigma^{-1}\left(k_{1}\right)\right)^{i} k_{2}^{j}},
$$

where $i, j \in\{0,1\}, S_{1} \in k(x, n)$ and $(u, v, K)$ is an AP triple. It follows that

$$
h=\Delta_{n}\left(S_{1} \tilde{h}\right)+\frac{v}{u\left(\sigma^{-1}\left(k_{1}\right)\right)^{i} k_{2}^{j}} \tilde{h}
$$

Set $T=\left(\sigma^{-1}\left(k_{1}\right)\right)^{i} k_{2}^{j} \in k[x, n]$. Since $K$ is split in $k[x, n], T$ is split in $k[x, n]$. By Corollary 5.2.4, we have

$$
\frac{v}{u \cdot T} \tilde{h} \in \frac{v}{u \cdot T} \mathcal{H}\left(\gamma+n \frac{\delta(\beta)}{\beta}, K\right)=\frac{v}{u} \mathcal{H}\left(\gamma+n \frac{\delta(\beta)}{\beta}-\frac{\delta(T)}{T}, K \frac{T}{\sigma(T)}\right) .
$$

Set $h_{1}=S_{1} \tilde{h}, \tilde{\gamma}=\gamma-\frac{\delta(T)}{T}$, and $\tilde{\alpha}=\tilde{K} \frac{T}{\sigma(T)}$. Since $T$ is split, $\gamma \in k(x)$, and $\tilde{K} \in k(n)$, we have that $\tilde{\gamma} \in k(x)$ and $\tilde{\alpha} \in k(n)$ hold. It remains to show that ( $u, v, \beta \tilde{\alpha}$ ) is an AP triple.

We only need to verify the last two conditions in Definition 5.5.12. Since $i, j \in\{0,1\}, \beta \tilde{\alpha}$ is in the set

$$
\left\{\frac{k_{1}}{k_{2}}, \frac{\sigma^{-1}\left(k_{1}\right)}{k_{2}}, \frac{k_{1}}{\sigma\left(k_{2}\right)}, \frac{\sigma^{-1}\left(k_{1}\right)}{\sigma\left(k_{2}\right)}\right\} .
$$

Then $\beta \tilde{\alpha}$ is shift-reduced, because $K=k_{1} / k_{2}$ is. Note that the numerator and denominator of $\beta \tilde{\alpha}$ are just some shifts of that of $K$, respectively. Then the last gcd condition follows from the assumptions on $u$ and $k_{1}, k_{2}$.

Remark 5.5.19. An additive decomposition with respect to $n$ can be computed by the algorithm dterm in [13].

### 5.5.3 Applying operators to additive decompositions

In this section, we apply an operator in $k(x)\left\langle D_{x}\right\rangle$, resp. in $k(n)\left\langle S_{n}\right\rangle$, to an additive decomposition with respect to $n$, resp. $x$, and prove that the application results in another additive decomposition. To this end, we need a simple lemma.

Lemma 5.5.20. Let $u$ and $v$ be in $k(x)[n]$ and let $w$ be any factor of $u^{m}$ with $m \in \mathbb{N}$. Then
(i) if $u$ is shift-free with respect to $n$, so is $w$;
(ii) for all $i \in \mathbb{Z}$, if $\operatorname{gcd}\left(u, \sigma^{i}(v)\right)=1$, then $\operatorname{gcd}\left(w, \sigma^{i}(v)\right)=1$.

Proof. A polynomial in $k(x)[n]$ is shift-free with respect to $n$ if and only if the largest integer distance of its roots is equal to zero [2]. This fact implies the first assertion. For any $i \in \mathbb{Z}$, if $\operatorname{gcd}\left(u, \sigma^{i}(v)\right)=1$, then $\operatorname{gcd}\left(u^{m}, \sigma^{i}(v)\right)=1$ for all $m \in \mathbb{N}$, which implies $\operatorname{gcd}\left(w, \sigma^{i}(v)\right)=1$. The second assertion holds.

Proposition 5.5.21. With the assumptions and notation introduced in Theorem 5.5.10, assume further that $h_{2}=h-D_{x} h_{1}$, and that $L$ belongs to $k(n)\left\langle S_{n}\right\rangle$ such that $L\left(h_{2}\right)$ is nonzero. Set

$$
A_{2}=\tilde{\gamma}+n \frac{\delta(\beta)}{\beta}-\rho \frac{\delta(\operatorname{den}(\beta))}{\operatorname{den}(\beta)} \quad \text { and } \quad B_{2}=\beta \alpha
$$

where $\rho$ denotes the order of $L$. Then

$$
L\left(h_{2}\right) \in \frac{V}{U} \mathcal{H}\left(A_{2}, B_{2}\right)
$$

for some $U, V \in k(n)[x]$, and $\left(U, V, A_{2}\right)$ is a GLL triple.

Proof. Setting $a_{2}=\tilde{\gamma}+n \delta(\beta) / \beta$. we get $A_{2}=a_{2}-\rho \delta(\operatorname{den}(\beta)) / \operatorname{den}(\beta)$.
By Theorem 5.5.10, $h_{2} \in \frac{v}{u} \cdot \mathcal{H}\left(a_{2}, B_{2}\right)$. An easy calculation implies

$$
L\left(h_{2}\right)=\frac{v^{\prime}}{u^{\prime} \operatorname{den}(\beta)^{\rho}} \cdot \mathcal{H}\left(a_{2}, B_{2}\right)
$$

where $v^{\prime} \in k(n)[x]$ and $u^{\prime}$ is the least common multiple of $u, S_{n}(u), \ldots, S_{n}^{\rho}(u)$ in $k(n)[x]$. Since $u$ is squarefree with respect to $x$, so are its shifts. Consequently, $u^{\prime}$ is squarefree with respect to $x$. Set

$$
U=\operatorname{den}\left(v^{\prime} / u^{\prime}\right) \quad \text { and } \quad V=\operatorname{num}\left(v^{\prime} / u^{\prime}\right) .
$$

Then $L\left(h_{2}\right)=(V / U) \cdot \mathcal{H}\left(A_{2}, B_{2}\right)$.
It remains to show that $\left(U, V, A_{2}\right)$ is a GLL triple. First, $U$ and $V$ are coprime by their definition. Second, $U$ is squarefree with respect to $x$, as $u^{\prime}$ is squarefree with respect to $x$. To verify the third condition in Definition 5.5.1, we observe that the squarefree part of den $(\beta)$ num $(\beta)$ is the denominator of $\delta(\beta) / \beta$, and that $\operatorname{den}(\beta)$ is the denominator of $\rho \delta(\operatorname{den}(\beta)) / \operatorname{den}(\beta)$, which divides the former denominator. This observation enables us to apply Lemma 5.5.6. We conclude that $A_{2}$ is differential-reduced by Lemma 5.5 .6 (iii) and since $\tilde{\gamma}+n \frac{\delta(\beta)}{\beta}$ is differential-reduced. At last, we verify that $U$ and $\operatorname{den}\left(A_{2}\right)$ are coprime over $k(n)$. By Lemma 5.5.6 (i), $\operatorname{den}\left(a_{2}\right)=$ $\operatorname{den}\left(A_{2}\right)$. Since $\left(u, v, a_{2}\right)$ is a GLL triple, $\operatorname{gcd}\left(u, \operatorname{den}\left(a_{2}\right)\right)=1$. It follows from the fact $\operatorname{den}\left(a_{2}\right) \in$ $k[x]$ that

$$
\operatorname{gcd}\left(S_{n}^{i}(u), \operatorname{den}\left(a_{2}\right)\right)=1
$$

for all $i \in \mathbb{N}$, and, consequently, $\operatorname{gcd}\left(u^{\prime}, \operatorname{den}\left(a_{2}\right)\right)=1$. This implies that $\operatorname{gcd}\left(U, \operatorname{den}\left(a_{2}\right)\right)=1$, that is, $\operatorname{gcd}\left(U, \operatorname{den}\left(A_{2}\right)\right)=1$.

Proposition 5.5.22. With the assumptions and notation introduced in Theorem 5.5.18, assume further that $h_{2}=h-\Delta_{n}\left(h_{1}\right)$, and that $L$ belongs to $k(x)\left\langle D_{x}\right\rangle$ such that $L\left(h_{2}\right)$ is nonzero. Set

$$
A_{2}=\tilde{\gamma}+n \frac{\delta(\beta)}{\beta} \quad \text { and } \quad B_{2}=\beta \tilde{\alpha}
$$

Then

$$
L\left(x, D_{x}\right)\left(h_{2}\right) \in \frac{V}{U} \cdot \mathcal{H}\left(A_{2}, B_{2}\right)
$$

for some $U, V \in k(x)[n]$, and $\left(U, V, B_{2}\right)$ is an AP triple.

Proof. By Theorem 5.5.18,

$$
h_{2} \in \frac{v}{u} \mathcal{H}\left(A_{2}, B_{2}\right)
$$

for some $u, v \in k(x)[n]$ and $\left(u, v, B_{2}\right)$ is an AP triple. A straightforward calculation leads to

$$
L\left(x, D_{x}\right)\left(h_{2}\right) \in(V / U) \cdot \mathcal{H}\left(A_{2}, B_{2}\right)
$$

for some $U, V \in k(x)[n]$ with $\operatorname{gcd}(U, V)=1$. By Theorem 5.5.18, $\operatorname{den}\left(A_{2}\right)$ is in $k[x]$. It follows that $U$ is a factor of a power of $u$ in $k(x)[n]$. Thus, $\left(U, V, B_{2}\right)$ is an AP triple by Lemma 5.5.20 and since $\left(u, v, B_{2}\right)$ is an AP triple, as stated in Theorem 5.5.18.

### 5.6 Two existence criteria

The Fundamental Theorem in [92] shows that telescopers exist for proper hyperexponentialhypergeometric functions. However, properness is just a sufficient condition. For instance, the rational function $f=1 /(x+n)^{2}$ is not proper, but it does have a telescoper with respect to $x$ since $f=D_{x}(-1 /(x+n))$. On the other hand, Example 5.3.3 indicates that not all hyperexponential-hypergeometric functions have telescopers. In this section, we present sufficient and necessary conditions in the following theorems for the existence of telescopers.

Theorem 5.6.1. Let $h$ be a hyperexponential-hypergeometric function over $k(x, n)$ and $h=$ $D_{x}\left(h_{1}\right)+h_{2}$ be an additive decomposition of $h$ with respect to $x$. Then $h$ has a telescoper with respect to $x$ if and only if $h_{2}$ is either zero or proper.

Theorem 5.6.2. Let $h$ be a hyperexponential-hypergeometric function over $k(x, n)$ and $h=$ $\Delta_{n}\left(h_{1}\right)+h_{2}$ be an additive decomposition of $h$ with respect to $n$. Then $h$ has a telescoper with respect to $n$ if and only if $h_{2}$ is either zero or proper.

Our proofs of these two theorems will be divided into several steps.

### 5.6.1 Existence of telescopers implies properness

The lemma below shows that the existence problem for $h$ is equivalent to that for $h_{2}$. The proofs take the same argument as in [9, page 3].

Lemma 5.6.3. Let $h, h_{1}$ and $h_{2}$ be three hyperexponential-hypergeometric functions in the ring $\mathcal{S}$ of sequences.
(i) Assume that $h=D_{x}\left(h_{1}\right)+h_{2}$. Then $L$ is a telescoper for $h$ with respect to $x$ if and only if it is a telescoper for $h_{2}$ with respect to $x$.
(ii) Assume that $h=\Delta_{n}\left(h_{1}\right)+h_{2}$. Then $L$ is a telescoper for $h$ with respect to $n$ if and only if it is a telescoper for $h_{2}$ with respect to $n$.

Proof. By the commutativity between $L\left(n, S_{n}\right) \in k(n)\left\langle S_{n}\right\rangle$ and $D_{x}$, we have

$$
L\left(n, S_{n}\right)(h)=D_{x}(g) \quad \Leftrightarrow \quad L\left(n, S_{n}\right)\left(h_{2}\right)=D_{x}\left(g-L\left(h_{1}\right)\right) .
$$

This implies the first assertion. Similarly, the second one follows from the commutativity between $L\left(x, D_{x}\right) \in k(x)\left\langle D_{x}\right\rangle$ and $\Delta_{n}$.

## Proof of the necessity in Theorem 5.6.1

Proof. Assume that $h$ has a telescoper with respect to $x$, and that $h_{2}$ is nonzero. Our goal is to show that $h_{2}$ is proper. By Theorem 5.5.10,

$$
h_{2}=\frac{v}{u} H \quad \text { for some } H \in \mathcal{H}\left(a_{2}, b_{2}\right),
$$

where $b_{2}=\beta \alpha$ with $\beta \in k(x)$ and $\alpha \in k(n)$, and $\left(u, v, a_{2}\right)$ is a GLL triple. Without loss of generality, we further assume that $u$ is in $k[x, n]$. To prove that $h_{2}$ is proper, it suffices to prove that $u$ is split by Proposition 5.4.4. Let $L \in k(n)\left\langle S_{n}\right\rangle$ be a telescoper for $h$ with respect to $x$. By Lemma 5.6.3, $L$ is also a telescoper for $h_{2}$ with respect to $x$. Then

$$
\begin{equation*}
L\left(h_{2}\right)=L\left(\frac{v}{u} H\right)=D_{x}(g), \tag{5.16}
\end{equation*}
$$

for some hyperexponential-hypergeometric function $g$. Since $b_{2}$ is split, a direct calculation yields

$$
\begin{equation*}
L\left(h_{2}\right)=M\left(\frac{v}{u}\right) H \tag{5.17}
\end{equation*}
$$

for some nonzero element $M \in k(n, x)\left\langle S_{n}\right\rangle$ whose coefficients are split. On the other hand, Proposition 5.5.21 implies

$$
\begin{equation*}
L\left(h_{2}\right) \in \frac{V}{U} \mathcal{H}\left(A_{2}, B_{2}\right) \tag{5.18}
\end{equation*}
$$

where $U, V \in k(n)[x]$ and $\left(U, V, A_{2}\right)$ is a GLL triple. Since $L\left(h_{2}\right)$ is hyperexponential-integrable by (5.16), it follows from Lemma 5.5 .5 that $U$ is free of $x$.

By Proposition 5.5.21, $A_{2}=a_{2}-\rho \delta(\operatorname{den}(\beta)) / \operatorname{den}(\beta)$ and $B_{2}=b_{2}$, where $\rho$ denotes the order of $L$. It follows from (5.18) and $\operatorname{den}(\beta) \in k[x]$ that

$$
L\left(h_{2}\right) \in \frac{V}{U \operatorname{den}(\beta)^{\rho}} \mathcal{H}\left(a_{2}, b_{2}\right)
$$

which, together with (5.17) and $\mathcal{H}\left(a_{2}, b_{2}\right)=\{c H \mid c \in k\}$, implies that

$$
M\left(\frac{v}{u}\right)=\frac{c V}{U \operatorname{den}(\beta)^{\rho}} \quad \text { for some } c \in k
$$

Hence, there exist a nonzero operator $M^{\prime} \in k[x, n]\left\langle S_{n}\right\rangle$ with leading coefficient in $k[n]$ and a rational function $g \in k(x)$ such that $M^{\prime}(g v / u)$ belongs to $k[x, n]$. Consequently, $u$ is split by Lemma 5.2.7, and, hence, $h_{2}$ is proper.

## Proof of the necessity in Theorem 5.6.2

Proof. Assume that $h$ has a telescoper with respect to $n$, and that $h_{2}$ is nonzero. Our goal is to show that $h_{2}$ is proper. By Theorem 5.5.18,

$$
h_{2}=\frac{v}{u} H \quad \text { for some } H \in \mathcal{H}\left(a_{2}, b_{2}\right),
$$

where $a_{2}=\tilde{\gamma}+n \frac{\delta(\beta)}{\beta}$ for some $\tilde{\gamma} \in k(x)$, and $\left(u, v, b_{2}\right)$ is an AP triple. Without loss of generality, we further assume that $u$ is in $k[x, n]$. To prove that $h_{2}$ is proper, it suffices to prove that $u$ is split by Proposition 5.4.4.

Let $L \in k(x)\left\langle D_{x}\right\rangle$ be a telescoper for $h$ with respect to $n$. By Lemma 5.6.3, $L$ is also a telescoper for $h_{2}$ with respect to $n$. Then

$$
\begin{equation*}
L\left(h_{2}\right)=L\left(\frac{v}{u} H\right)=\Delta_{n}(g), \tag{5.19}
\end{equation*}
$$

for some hyperexponential-hypergeometric function $g$. For all $i \in \mathbb{N}$, let $f_{i}$ be the rational function in $k(x, n)$ such that

$$
D_{x}^{i}(H)=f_{i} H
$$

Then $\operatorname{den}\left(f_{i}\right)$ is in $k[x]$, because $\operatorname{den}\left(a_{2}\right)$ is in $k[x]$. It follows from Leibniz's formula that there exists $M_{i} \in k(x)[n]\left\langle D_{x}\right\rangle$ such that

$$
D_{x}^{i}\left(h_{2}\right)=M_{i}\left(\frac{v}{u}\right) H
$$

Moreover, the leading coefficient of $M_{i}$ is in $k$. Consequently, there exists $M \in k(x)[n]\left\langle D_{x}\right\rangle$ with leading coefficient in $k(x)$ such that

$$
\begin{equation*}
L\left(h_{2}\right)=M\left(\frac{v}{u}\right) H \tag{5.20}
\end{equation*}
$$

On the other hand, Proposition 5.5.22 and $\mathcal{H}\left(a_{2}, b_{2}\right)=\{c H \mid c \in k\}$ imply,

$$
\begin{equation*}
L\left(h_{2}\right)=\frac{V}{U} H . \tag{5.21}
\end{equation*}
$$

where $U$ and $V$ are in $k(x)[n]$, and $\left(U, V, b_{2}\right)$ is an AP triple. Since $L\left(h_{2}\right)$ is hypergeometricsummable by (5.19), it follows from Lemma 5.5.16 that $U$ is free of $n$.

Comparing (5.20) with (5.21), we have that $M(v / u)=V / U$, which is in $k(x)[n]$. It follows that there exists a nonzero operator $M^{\prime} \in k[x, n]\left\langle D_{x}\right\rangle$ with leading coefficient in $k[x]$ such that $M^{\prime}(v / u)$ belongs to $k[x, n]$. By Lemma 5.2.7, $u$ is split, and then $h_{2}$ is proper.

### 5.6.2 Properness implies the existence of telescopers

Wilf and Zeilberger present an elementary proof of the existence of telescopers for proper multivariate hypergeometric terms [92, Theorem 3.1], and indicate that their argument should be also valid in the continuous-discrete setting. For the sake of completeness, we elaborate on the proof of the following theorem.

Theorem 5.6.4. If $h$ is a proper hyperexponential-hypergeometric function over $k(x, n)$, then $h$ has a telescoper with respect to $n$ and a telescoper with respect to $x$.

## Preparation lemmas

Before the proof of the theorem above, we shall show some lemmas.

Lemma 5.6.5. Let $h \in \mathcal{H}(a, b)$ be a hyperexponential-hypergeometric function over $k(x, n)$.
(i) If $h$ has a telescoper with respect to $n$, so does the product $H h$ for any hyperexponentialhypergeometric function $H$ with $\Delta_{n}(H)=0$.
(ii) If $h$ has a telescoper with respect to $x$, so does the product $H h$ for any hyperexponentialhypergeometric function $H$ with $D_{x}(H)=0$.

Proof. Assume that $H$ is a nonzero element of $\mathcal{H}(A, B)$. If $\Delta_{n}(H)=0$, then $B=1$ and $a \in k(x)$ by the integrability condition $\sigma(A)-A=\delta(B) / B$. It is straightforward to verify that $H D_{x}(h)=$ $\left(D_{x}-A\right) H h$, which, together with an easy induction, implies that

$$
H D_{x}^{i}(h)=\left(D_{x}-A\right)^{i}(H h) \quad \text { for all } i \in \mathbb{N} .
$$

Let $L \in k(x)\left\langle D_{x}\right\rangle$ be a telescoper of $h$. By the equality above,

$$
H L(h)=M(H h)
$$

where $M$ is obtained by replacing $D_{x}^{i}$ in $L$ by $\left(D_{x}-A\right)^{i}$. Since there exists a hypergeometric function $g$ such that $L(h)=\Delta_{n}(g)$ and $\Delta_{n}(H)=0$, the equality above implies that

$$
M(H h)=H \Delta_{n}(g)=\Delta_{n}(H g) .
$$

The first assertion is proved.
To prove the second assertion, assume that $H$ is a nonzero element in $\mathcal{H}(A, B)$. If $D_{x}(H)$ is equal to 0 , then $A=0$ and $B \in k(n)$ by the integrability condition $\sigma(A)-A=\delta(B) / B$. Since $H S_{n}(h)=S_{n}(H h) / B$, we have that, for all $i \in \mathbb{N}$,

$$
H S_{n}^{i}(h)=B_{i} S_{n}^{i}(H h) \quad \text { for some } B_{i} \in k(n) .
$$

Let $L \in k(n)\left\langle S_{n}\right\rangle$ be a telescoper of $h$. By the equality above,

$$
H L(h)=M(H h)
$$

where $M$ is obtained by replacing $S_{n}^{i}$ in $L$ by $B_{i} S_{n}^{i}$. Since there exists a hypergeometric function $g$ such that $L(h)=D_{x}(g)$ and $D_{x}(H)=0$, the equality above implies that

$$
M(H h)=H D_{x}(g)=D_{x}(H g)
$$

The second assertion is proved.
Lemma 5.6.6. Let $h_{1}$ and $h_{2}$ be two similar hyperexponential-hypergeometric functions over the field $k(x, n)$. Let $\mathbb{V}_{n}$ be the vector space spanned by $\left\{\Delta_{n}^{i}\left(h_{j}\right) \mid i \in \mathbb{N}, j \in\{1,2\}\right\}$ over $k(x)$, and $\mathbb{V}_{x}$ be the vector space spanned by $\left\{D_{x}^{i}\left(h_{j}\right) \mid i \in \mathbb{N}, j \in\{1,2\}\right\}$ over $k(n)$.

1. If both $h_{1}$ and $h_{2}$ have telescopers with respect to $n$, so does any nonzero element in $\mathbb{V}_{n}$.
2. If both $h_{1}$ and $h_{2}$ have telescopers with respect to $x$, so does any nonzero element in $\mathbb{V}_{x}$.

Proof. Assume that there exists a nonzero $L_{j} \in k(x)\left\langle D_{x}\right\rangle$ such that $L_{j}\left(h_{j}\right)=\Delta_{n}\left(g_{j}\right)$ for some hyperexponential-hypergeometric function $g_{j}$, and for $j=1,2$. Then

$$
L_{j}\left(\Delta_{n}^{i}\left(h_{j}\right)\right)=\Delta_{n}\left(\Delta_{n}^{i}\left(g_{j}\right)\right) .
$$

Hence, every element of $\left\{\Delta_{n}^{i}\left(h_{j}\right) \mid i \in \mathbb{N}, j \in\{1,2\}\right\}$ has a telescoper. If two hypergeometric functions have telescopers with respect to $n$, say $T_{1}$ and $T_{2}$, then their sum, assumed to be nonzero, also has a telescoper with respect to $n$, which can be taken as a common left multiple of $T_{1}$ and $T_{2}$. The first assertion then follows from Lemma 5.6.5 (ii). The second assertion can be proved in the same manner.

In the proof of the Fundamental Theorem [92], Wilf and Zeilberger neglected the case that the remainder of a nonzero $m$-free operator $P\left(n, S_{n}, S_{m}\right) \in k(n)\left\langle S_{n}, S_{m}\right\rangle$ by $S_{m}-1$ may be zero. Wegschaider used a "noncommutative trick" to deal with this case. Here, we will apply Wegschaider's trick and its differential analogue to transform an $x$-free or $n$-free operator to a telescoper with respect to $n$ or $x$.

Lemma 5.6.7 (Wegschaider's trick). Let $h$ be a hypergeometric function over $k(x, n)$.
(i) If there exists a nonzero operator $A \in k(x)\left\langle S_{n}, D_{x}\right\rangle$ such that $A(h)=0$, then $h$ has a telescoper with respect to $n$.
(ii) If there exists a nonzero operator $A \in k(n)\left\langle S_{n}, D_{x}\right\rangle$ such that $A(h)=0$, then $h$ has a telescoper with respect to $x$.

Proof. Assume that there exists a nonzero operator $A$ in $k(x)\left\langle S_{n}, D_{x}\right\rangle$ such that $A(h)=0$. Since $\Delta_{n}=S_{n}-1$, we can write $A=\Delta_{n}^{m}\left(L\left(x, D_{x}\right)+\Delta_{n} M\right)$, where $m$ is in $\mathbb{N}, L$ is a nonzero operator in $k(x)\left\langle D_{x}\right\rangle$ and $M$ is in $k(x)\left\langle S_{n}, D_{x}\right\rangle$. By Wegschaider's trick in [91, Theorem 3.2], there exist $w \in k(n)$ and $r \in k$ with $r \neq 0$ such that

$$
\begin{equation*}
w \Delta_{n}^{m}=\Delta_{n} Q+r \tag{5.22}
\end{equation*}
$$

for some $Q \in k(n)\left\langle S_{n}\right\rangle$. In particular, $r=(-1)^{m} m$ ! if we take $w=n^{m}$. Using the fact $r \Delta_{n}=$ $\Delta_{n} r$ and (5.22), we find

$$
\frac{w}{r} A=L+\Delta_{n} N \quad \text { for some } N \in k(x, n)\left\langle D_{x}, S_{n}\right\rangle
$$

Hence, $L$ is a telescoper for $h$ with respect to $n$.
The second assertion can be proved in a similar way. Instead of (5.22), we need to find $w \in$ $k(x)$ and $r \in k \backslash\{0\}$ such that

$$
w D_{x}^{m}=D_{x} Q+r
$$

for some $Q \in k(x)\left\langle D_{x}\right\rangle$. In particular, $r=(-1)^{m} m$ ! if we take $w=x^{m}$.

## Proof of Theorem 5.6.4

Now, we present the proof of Theorem 5.6.4.

Proof. Assume that $h$ is a proper hyperexponential-hypergeometric function. Then the standard representation of $h$ is of the form

$$
(p(x, n), \beta(x), \gamma(x), \alpha(n)),
$$

where $p \in k[x, n], \beta, \gamma \in k(x)$, and $\alpha \in k(n)$. Writing $p$ as $\sum_{i=0}^{m} p_{i}(x) n^{i}$ with $p_{i} \in k[x]$ yields

$$
p \beta^{n} \mathcal{H}(\gamma, \alpha)=\sum_{i=0}^{m} p_{i} n^{i} \beta^{n} \mathcal{H}(\gamma, \alpha)=\sum_{i=0}^{m} \beta^{n} \mathcal{H}\left(\gamma+\frac{\delta\left(p_{i}\right)}{p_{i}}, \alpha(n) \frac{\sigma\left(n^{i}\right)}{n^{i}}\right) .
$$

It follows that

$$
\begin{equation*}
h=\sum_{i=0}^{m} \beta(x)^{n} G_{i} H_{i} \tag{5.23}
\end{equation*}
$$

for some $G_{i} \in \mathcal{H}\left(\gamma(x)+\frac{\delta\left(p_{i}\right)}{p_{i}}, 1\right)$ and $H_{i} \in \mathcal{H}\left(0, \alpha(n) \frac{\sigma\left(n^{i}\right)}{n^{i}}\right)$.
First, we show that $h$ has a telescoper with respect to $n$. Note that each $G_{i}$ in (5.23) has $n$-certificate equal to 1 . By Lemmas 5.6.5 and 5.6.6, it suffices to show that a hypergeometric function $\hat{h} \in \beta(x)^{n} \cdot \mathcal{H}(0, g(n))$ with $g \in k(n)$ has a telescoper with respect to $n$. Let $s=\operatorname{num}(\beta)$, $t=\operatorname{den}(\beta), a=\operatorname{num}(g)$ and $b=\operatorname{den}(g)$. Moreover, let $v=\operatorname{num}(\delta(\beta) / \beta)$ and $w=\operatorname{den}(\delta(\beta) / \beta)$. A straightforward calculation yields that the $x$-certificate and $n$-certificate of $\hat{h}$ are, respectively,

$$
\frac{n v}{w} \text { and } \frac{s a}{t b} .
$$

Note that $s, t, v, w$ are in $k[x]$, and $a, b$ in $k[n]$.
We claim that there exists a nonzero operator $A$ in $k(x)\left\langle S_{n}, D_{x}\right\rangle$ such that $A(\hat{h})=0$. The claim will be proved by a well-known argument used in [67]. The first assertion will then follow from Lemma 5.6.7.

Let $\mathcal{F}_{N}$ be the linear space spanned by $\left\{S_{n}^{i} D_{x}^{j} \mid i+j \leq N\right\}$ over $k(x)$. Let $\mu$ be the maximum of degrees of $a$ and $b$ in $n$, and let

$$
\mathcal{W}_{N}=\operatorname{span}_{k(x)}\left\{\left.\frac{n^{i} \hat{h}}{b(n+N-1) \cdots b(n)} \right\rvert\, i \leq(\mu+1) N\right\} .
$$

An easy induction on $i$ and $j$ yields

$$
S_{n}^{i} D_{x}^{j}(\hat{h})=\frac{q(n, x) \hat{h}}{b(n+i-1) \cdots b(n)}, \text { where } \operatorname{deg}_{n}(q) \leq i \mu+j \text {. }
$$

Hence, $S_{n}^{i} D_{x}^{j}(\hat{h})$ belongs to $\mathcal{W}_{N}$ if $i+j \leq N$. Accordingly, there is a $k(x)$-linear map $\phi_{N}$ from $\mathcal{F}_{N}$ to $\mathcal{W}_{N}$ that sends $L$ to $L(\hat{h})$ for all $L \in \mathcal{F}_{N}$. Since the dimension of $\mathcal{F}_{N}$ over $k(x)$ is $\binom{N+2}{2}$, while that of $\mathcal{W}_{N}$ is $(\mu+1) N+1$, the kernel of $\phi_{N}$ is nontrivial when $N$ is sufficiently large. Any nonzero element in the kernel annihilates $\hat{h}$. The claim is established.

Second, we show that $h$ has a telescoper with respect to $x$. Note that each nonzero element in $\mathcal{H}\left(0, \alpha(n) \frac{\sigma\left(n^{i}\right)}{n^{i}}\right)$ in (5.23) has $x$-certificate equal to zero. By Lemmas 5.6.5 and 5.6.6, it suffices to show that any hypergeometric function $\hat{h}(x, n)$ in the set

$$
\beta(x)^{n} \cdot \mathcal{H}(g(x), 1) \quad \text { with } g \in k(x)
$$

has a telescoper with respect to $x$. Let $s=\operatorname{num}(\beta), t=\operatorname{den}(\beta)$,

$$
v=\operatorname{num}\left(n \frac{\delta(\beta)}{\beta}+g\right) \quad \text { and } \quad w=\operatorname{den}\left(n \frac{\delta(\beta)}{\beta}+g\right) .
$$

A straightforward calculation yields that the $x$-certificate and $n$-certificate of $\hat{h}$ are, respectively,

$$
\frac{v(x, n)}{w(x)} \text { and } \frac{s(x)}{t(x)}
$$

Note that $s, t$, and $w$ are in $k[x]$ and $v$ is in $k[x, n]$ with $\operatorname{deg}_{n}(v)=1$.
We claim that there exists a nonzero operator $A$ in $k(n)\left\langle S_{n}, D_{x}\right\rangle$ such that $A(\hat{h})=0$. The second assertion then follows from Lemma 5.6.7.

Following an argument similar to the one above, we now consider a linear space $\mathcal{F}_{N}$ spanned by $\left\{S_{n}^{i} D_{x}^{j} \mid i+j \leq N\right\}$ over $k(n)$. Let $\mu$ be the maximum of the degrees in $x$ of $s, t, v$ and $w$, and let

$$
\mathcal{W}_{N}=\operatorname{span}_{k(n)}\left\{\left.\frac{x^{i} \hat{h}}{(t w)^{N}} \right\rvert\, i \leq 2 \mu N\right\} .
$$

## Algorithm ExistenceContinuous

Input: A hyperexponential-hypergeometric function $h \in \mathcal{H}(a, b)$ with $a, b \in k(x, n)$.
Output: True, if $h$ has a telescoper with respect to $x$; false, otherwise.

1. Compute a strict DRNF $(K, S)$ of $a$ with respect to $x$;
2. Apply ReduceCert in [47] to $(K, S)$ to get the decomposition

$$
S=\delta\left(S_{1}\right)+S_{1} K+\frac{v}{u \operatorname{den}(K)^{i}}
$$

3. If $u$ is split, then return true, otherwise, return false.

Figure 5.1: Algorithm for deciding the existence of telescopers with respect to $x$.

An easy induction on $i$ and $j$ yields

$$
S_{n}^{i} D_{x}^{j}(\hat{h})=\frac{q(n, x)}{t^{i} w^{j}} \hat{h}, \text { where } \operatorname{deg}_{x}(q) \leq(i+j) \mu
$$

Hence, $S_{n}^{i} D_{x}^{j}(\hat{h})$ belongs to $\mathcal{W}_{N}$ if $i+j \leq N$. Accordingly, there is a $k(n)$-linear map $\phi_{N}$ from $\mathcal{F}_{N}$ to $\mathcal{W}_{N}$ that sends $L$ to $L(\hat{h})$ for all $L \in \mathcal{F}_{N}$. Since the dimension of $\mathcal{F}_{N}$ over $k(n)$ is $\binom{N+2}{2}$, while that of $\mathcal{W}_{N}$ is $2 \mu N+1$, the kernel of $\phi_{N}$ is nontrivial when $N$ is sufficiently large. Any nonzero element in the kernel annihilates $\hat{h}$. The claim is established.

Remark 5.6.8. Theorems 5.6.1 and 5.6.2 hold by the proofs of necessity in Theorems 5.6.1 and 5.6.2, and by Theorem 5.6.4

### 5.7 Algorithms and examples

### 5.7.1 Algorithms

According to our criteria, we only use the continuous, resp. discrete, certificate for deciding the existence of telescopers with respect to the continuous, resp. discrete, variable. Algorithms in Figures 5.1 and 5.2 are based on strict DRNF and strict SRNF computation, two additive decompositions, and our criteria.

## Algorithm ExistenceDiscrete

Input: A hyperexponential-hypergeometric function $h \in \mathcal{H}(a, b)$ with $a, b \in k(x, n)$.
Output: True, if $h$ has a telescoper with respect to $n$; false, otherwise.

1. Compute a strict SRNF $(K, S)$ of $b$ with respect to $n$;
2. Apply dcert in [13] to $(K, S)$ to get the decomposition

$$
S=\sigma\left(S_{1}\right) K-S_{1}+\frac{v}{u\left(\sigma^{-1}(\operatorname{num}(K))\right)^{i} \operatorname{den}(K)^{j}}
$$

3. If $u$ is split, then return true, otherwise, return false.

Figure 5.2: Algorithm for deciding the existence of telescopers with respect to $n$.

### 5.7.2 Examples

Example 5.7.1. It is possible that a hyperexponential-hypergeometric function has a telescoper with respect to one variable but no telescoper with respect to the other variable. Consider the rational function

$$
h=\frac{1}{(x+n)^{2}} .
$$

Applying Hermite reduction to $h$ with respect to $x$ yields

$$
h=D_{x}\left(\frac{-1}{x+n}\right),
$$

which implies that 1 is a telescoper for $h$ with respect to $x$. However, $h$ has no telescoper with respect to $n$, because $(x+n)^{2}$ is shift-free with respect to $n$ and it is not split. Similarly, consider the rational function

$$
h=\frac{1}{(x+n)(x+n+1)} .
$$

Since $h=\Delta_{n}(-1 /(x+n)), 1$ is a telescoper for $h$ with respect to $n$. However, $h$ has no telescoper with respect to $x$, because $(x+n)(x+n+1)$ is squarefree with respect to $x$ and it is not split.

Example 5.7.2. As we know, the properness is only a sufficient condition for the existence of telescopers. We show a concrete example to illustrate this fact. Consider the hyperexponentialhypergeometric function

$$
h(x, n)=\frac{-x+2 n x+2 n^{2}}{(x+n)^{2} x} \cdot x^{n} \cdot e^{-x}
$$

Though $h$ is not proper, it still has a telescoper with respect to $x$ since $h$ can be decomposed into

$$
h=D_{x}\left(\frac{1}{x+n} \cdot x^{n} \cdot e^{-x}\right)+x^{n-1} \cdot e^{-x}
$$

where $x^{n-1} \cdot e^{-x}$ is proper. In fact, our criteria indicate that all nonproper examples are of this form.

## Chapter 6

## Conclusion and Perspectives

In this thesis, we have focused on the practical efficiency, theoretical complexity, and the termination of creative-telescoping algorithms. For bivariate rational functions, we have managed to blend the general method of creative telescoping with the Hermite reduction. According to our complexity analyses and experiments, the Hermite reduction based method has been proved to outperform the existing methods. In order to study the general case of Wilf and Zeilberger's conjecture, we have presented a structure theorem for multivariate hyperexponentialhypergeometric functions. With the help of standard representations and two additive decompositions, we have derived two existence criteria for deciding whether a bivariate hyperexponentialhypergeometric function has a telescoper with respect to the continuous variable or the discrete one. Our criteria solve the termination problem of Zeilberger-style algorithms for bivariate hyperexponential-hypergeometric functions. Still, there are many further studies to be carried out. In what follows, we would like to mention a few directions for future work.

### 6.1 Extensions of the Hermite-reduction based method

For the moment, the Hermite-reduction based method only works for bivariate rational-function inputs. As we have already seen in Chapter 5, Hermite reduction can be extended to hyperexponential or hypergeometric cases. So it is natural to wonder whether the approach of the Hermite-reduction based method can be used to compute the minimal telescopers for bivariate hyperexponential functions or hypergeometric terms. We have considered two possible exten-
sions: one is for hyperexponential functions, especially in simple radical extensions; and the other is for multivariate rational functions. In the multivariate rational case, the corresponding telescoping problem is as follows.

Problem 6.1.1. Given a rational function $f \in k\left(x, y_{1}, \ldots, y_{n}\right)$, construct a nonzero operator $L \in$ $k(x)\left\langle D_{x}\right\rangle$ such that

$$
L\left(x, D_{x}\right)(f)=\sum_{i=1}^{n} D_{y_{i}}\left(g_{i}\right) \quad \text { for some } g_{i} \in k\left(x, y_{1}, \ldots, y_{n}\right)
$$

In order to solve the above problem, we have tried to extend the Hermite reduction to the multivariate case. That is, we would like to decompose a multivariate rational function $f$ in $k\left(y_{1}, \ldots, y_{n}\right)$ into

$$
f=D_{y_{1}}\left(g_{1}\right)+\cdots+D_{y_{n}}\left(g_{n}\right)+r,
$$

where $r$ is minimal in some sense. For a special class of rational functions, we show this can be obtained.

Let $f=P / Q^{m}$ be a rational function in $k\left(y_{1}, \ldots, y_{n}\right)$ with $m>1, P, Q \in k\left[y_{1}, \ldots, y_{n}\right]$, and $\operatorname{gcd}(P, Q)=1$. Let $\mathcal{I}$ be the ideal generated by $Q, D_{y_{1}}(Q), \ldots, D_{y_{n}}(Q)$ in $k\left[y_{1}, \ldots, y_{n}\right]$. We observe that the ideal $\mathcal{I}$ is equal to the ring $k\left[y_{1}, \ldots, y_{n}\right]$ in certain applications, see examples gathered in [74]. In the sequel, we proceed the discussion under the above constraints on $Q$. Then we can always express $P$ in the form

$$
P=A Q+\sum_{i=1}^{n} B_{i} D_{y_{i}}(Q)
$$

where $A, B_{1}, \ldots, B_{n}$ are polynomials in $k\left(y_{1}, \ldots, y_{n}\right)$. Now, integration by parts yields

$$
\frac{P}{Q^{m}}=\frac{A Q+\sum_{i=1}^{n} B_{i} D_{y_{i}}(Q)}{Q^{m}}=\sum_{i=1}^{n} D_{y_{i}}\left(\frac{-B_{i}}{m Q^{m-1}}\right)+\frac{A+m^{-1} \sum_{i=1}^{n} D_{y_{i}}\left(B_{i}\right)}{Q^{m-1}} .
$$

Repeating the above process at most $m-1$ times, we get

$$
\begin{equation*}
\frac{P}{Q^{m}}=\sum_{i=1}^{n} D_{y_{i}}\left(g_{i}\right)+\frac{\widetilde{P}}{Q} \tag{6.1}
\end{equation*}
$$

where $g_{i} \in k\left(y_{1}, \ldots, y_{n}\right)$ and $\widetilde{P} \in k\left[y_{1}, \ldots, y_{n}\right]$. In the univariate case when $Q$ is squarefree, this is exactly Hermite reduction. When $n>1$, the essential difference from the univariate case is that one cannot make the remaining fraction $\widetilde{P} / Q$ proper with respect to all the variables. The
uniqueness of the decomposition in (6.1) is also not well studied. In the future, we would like to understand this decomposition better and expect it to be useful to solve the telescoping problem for multivariate rational functions.

### 6.2 Existence criteria for telescopers in the $q$-setting

There are nine termination problems of Zeilberger's method for bivariate hyperexponentialhypergeometric functions if telescopers could involve the $q$-shift operator, denoted by $Q_{n}$. In Table 6.1, we mark the solved instances by $\checkmark$ and unsolved by ?. In order to solve the remaining termination problems, we would like to extend Theorem 4.4.1 to the $q$-shift setting. Moreover, we expect a unified way to deal with all of the nine cases.

| $(L, g)$ | $D_{y}$ | $S_{n}-1$ | $Q_{n}-1$ |
| :---: | :---: | :---: | :---: |
| $L\left(x, D_{x}\right)$ | $\checkmark$ | $\checkmark$ | $?$ |
| $L\left(n, S_{n}\right)$ | $\checkmark$ | $\checkmark$ | $?$ |
| $L\left(n, Q_{n}\right)$ | $?$ | $?$ | $\checkmark$ |

Table 6.1: Solved and unsolved termination problems

### 6.3 Wilf and Zeilberger's conjecture

We present some remarks on the general case of Wilf and Zeilberger's conjecture. First, we recall basic notation about holonomic functions and modules following the presentation in Coutinho's book [38]. Let $\mathcal{A}_{m}$ denote the $m$-th Weyl algebra $k\left[x_{1}, \ldots, x_{m}\right]\left\langle D_{1}, \ldots, D_{m}\right\rangle$, in which we have the following multiplication rule: for all $1 \leq i, j \leq m$,

$$
x_{i} x_{j}=x_{j} x_{i}, \quad D_{i} D_{j}=D_{j} D_{i}, \quad \text { and } \quad D_{i} x_{j}-x_{j} D_{i}=\delta_{i j},
$$

where $\delta_{i j}$ is the Kronecker delta symbol: it equals one if $i=j$ and zero otherwise. A finitely generated left $\mathcal{A}_{m}$-module is holonomic if it is zero, or if it has Hilbert dimension $m$ [38, Chapter 5]. By Bernstein's inequality, any finitely generated nonzero left $\mathcal{A}_{m}$-module has dimension at least $m$. So holonomicity indicates the minimality of dimension for nonzero $\mathcal{A}_{m}$-modules.

Let $H(\mathbf{x})$ be a function in a left $\mathcal{A}_{m}$-module of functions. Then one can define the annihilating ideal of $H(\mathbf{x})$ in $\mathcal{A}_{m}$ by

$$
\operatorname{ann}_{\mathcal{A}_{m}}(H)=\left\{p \in \mathcal{A}_{m} \mid p \cdot H=0\right\} .
$$

The function $H(\mathbf{x})$ is said to be holonomic if its annihilating ideal is holonomic as a left $\mathcal{A}_{m}$ module. When the function $H(\mathbf{x})$ can be viewed as an element in a left module over the ring $k\left(x_{1}, \ldots, x_{m}\right)\left\langle D_{1}, \ldots, D_{m}\right\rangle$ of differential operators, $H(\mathbf{x})$ is said to be $D$-finite if the vector space generated by all derivatives $D_{1}^{i_{1}} \cdots D_{m}^{i_{m}}(H)$ over $k(\mathbf{x})$ is finite dimensional over $k(\mathbf{x})$ [67, 68]. In the continuous case, an elementary proof of Kashiwara's equivalence [59] between holonomicity and D-finiteness was presented by Takayama in [85]. For this reason, we choose the following definition, used in [24, page 59], for holonomicity of continuous-discrete functions.

Definition 6.3.1. A function $h(\mathbf{x}, \mathbf{y})$ is said to be holonomic if its associated generating function

$$
H(\mathbf{x}, \mathbf{z})=\sum_{y_{1}, \ldots, y_{n} \in \mathbb{N}} h(\mathbf{x}, \mathbf{y}) z_{1}^{y_{1}} \cdots z_{n}^{y_{n}}
$$

is a holonomic function in a left module over the ring $k(\mathbf{x}, \mathbf{z})\left\langle D_{x_{1}}, \ldots, D_{x_{m}}, D_{z_{1}}, \ldots, D_{z_{n}}\right\rangle$.
Now, we can describe the conjecture by Wilf and Zeilberger [92, page 585] on holonomic hyperexponential-hypergeometric functions.

Conjecture 6.3.2 (Wilf and Zeilberger). Let $h(\mathbf{x}, \mathbf{y})$ be a hyperexponential-hypergeometric function over $k(\mathbf{x}, \mathbf{y})$. Then $h$ is holonomic if and only if it is proper.

In the case of several discrete variables, a slightly modified version of the conjecture was proved independently by Payne in his Ph.D. thesis [73] and by Abramov and Petkovšek [14]. In particular, the case of two variables has also been shown by Hou [54, 55] and by Abramov and Petkovšek [12]. In the continuous case, any multivariate hyperexponential function is D-finite, and then holonomic by Kashiwara's equivalence. By Corollary 4.5.4, it is also proper. Thus, Wilf and Zeilberger's conjecture holds naturally in this case.

In the discrete case, the structure theorem on multivariate hypergeometric terms helps us reduce Wilf and Zeilberger's conjecture to rational case [73, 14], where one only need to show that any holonomic rational functions are proper. For the moment, we have not obtained any complete proof of the general continuous-discrete case of the conjecture. In the future, we would like to prove the following conjecture on holonomic rational functions in the mixed setting.

Conjecture 6.3.3. Let $f(\mathbf{x}, \mathbf{y})$ be a rational function in $k(\mathbf{x}, \mathbf{y})$. Then $f$ is holonomic if and only if the denominator of $f$ splits into the form

$$
A(\mathbf{x}) \prod_{\ell=1}^{L}\left(v_{\ell} \cdot \mathbf{y}+\lambda_{\ell}\right)
$$

where $A \in k[\mathbf{x}], \lambda_{1}, \ldots, \lambda_{L} \in \bar{k}, v_{\ell} \in \mathbb{Z}^{n}$ for all $\ell$ with $1 \leq \ell \leq L$, and the $v_{\ell} \cdot \mathbf{y}$ means the inner product of vectors $v_{\ell}$ and $\mathbf{y}$.

## Bibliography

[1] N. H. Abel. Euvres Complètes de Niels Henrik Abel. Tome II. Imprimerie de Grøndahl \& Son, Christiania, 1981. Contenant les mémoirs posthumes d'Abel. [Containing the posthumous memoirs of Abel], Edited and with notes by L. Sylow and S. Lie.
[2] S. A. Abramov. The rational component of the solution of a first order linear recurrence relation with rational right hand side. Ž. Vyčisl. Mat. i Mat. Fiz., 15(4):1035-1039, 1090, 1975.
[3] S. A. Abramov. Rational solutions of linear differential and difference equations with polynomial coefficients. USSR Comput. Math. Math. Phys., 29(6):7-12, 1989.
[4] S. A. Abramov. Rational solutions of linear difference and $q$-difference equations with polynomial coefficients. In ISSAC '95: Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation, pages 285-289, New York, NY, USA, 1995. ACM.
[5] S. A. Abramov. Applicability of Zeilberger's algorithm to hypergeometric terms. In $I S$ SAC'02: Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation, pages 1-7, New York, 2002. ACM.
[6] S. A. Abramov. When does Zeilberger's algorithm succeed? Adv. in Appl. Math., 30(3):424441, 2003.
[7] S. A. Abramov, J. J. Carette, K. O. Geddes, and H. Q. Le. Telescoping in the context of symbolic summation in Maple. J. Symbolic Comput., 38(4):1303-1326, 2004.
[8] S. A. Abramov and K. Y. Kvasenko. Fast algorithms to search for the rational solutions of linear differential equations with polynomial coefficients. In ISSAC '91: Proceedings of
the 1991 International Symposium on Symbolic and Algebraic Computation, pages 267-270, New York, NY, USA, 1991. ACM.
[9] S. A. Abramov and H. Q. Le. A criterion for the applicability of Zeilberger's algorithm to rational functions. Discrete Math., 259(1-3):1-17, 2002.
[10] S. A. Abramov, H. Q. Le, and M. Petkovšek. Rational canonical forms and efficient representations of hypergeometric terms. In ISSAC'03: Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, pages 7-14, New York, 2003. ACM.
[11] S. A. Abramov and M. Petkovšek. Minimal decomposition of indefinite hypergeometric sums. In ISSAC'01: Proceedings of the 2001 International Symposium on Symbolic and Algebraic Computation, pages 7-14 (electronic), New York, 2001. ACM.
[12] S. A. Abramov and M. Petkovšek. Proof of a conjecture of Wilf and Zeilberger, 2001. Preprints Series of the Institute of Mathematics, Physics and Mechanics 39(748)(2001), Ljubljana, March 9, 2001.
[13] S. A. Abramov and M. Petkovšek. Rational normal forms and minimal decompositions of hypergeometric terms. J. Symbolic Comput., 33(5):521-543, 2002.
[14] S. A. Abramov and M. Petkovšek. On the structure of multivariate hypergeometric terms. Adv. Appl. Math., 29(3):386-411, 2002.
[15] G. Almkvist and D. Zeilberger. The method of differentiating under the integral sign. J. Symbolic Comput., 10:571-591, 1990.
[16] M. Apagodu and D. Zeilberger. Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf- Zeilberger theory. Adv. in Appl. Math., 37(2):139-152, 2006.
[17] J. Bernšteĭn. Modules over a ring of differential operators. An investigation of the fundamental solutions of equations with constant coefficients. Funkcional. Anal. i Priložen., 5(2):1-16, 1971.
[18] A. Bostan, S. Chen, F. Chyzak, and Z. Li. Complexity of creative telescoping for bivariate rational functions. In ISSAC '10: Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation, pages 203-210, New York, NY, USA, 2010. ACM.
[19] A. Bostan, F. Chyzak, T. Cluzeau, and B. Salvy. Low complexity algorithms for linear recurrences. In J.-G. Dumas, editor, ISSAC'06: Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation, pages 31-38. ACM Press, 2006.
[20] A. Bostan, F. Chyzak, B. Salvy, G. Lecerf, and É. Schost. Differential equations for algebraic functions. In ISSAC'07: Proceedings of the 2007 International Symposium on Symbolic and Algebraic Computation, pages 25-32. ACM, New York, 2007.
[21] M. Bronstein. Symbolic Integration I: Transcendental Functions, volume 1 of Algorithms and Computation in Mathematics. Springer-Verlag, Berlin, second edition, 2005.
[22] M. Bronstein, Z. Li, and M. Wu. Picard-Vessiot extensions for linear functional systems. In ISSAC '05: Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation, pages 68-75, New York, USA, 2005. ACM.
[23] M. Bronstein and M. Petkovšek. An introduction to pseudo-linear algebra. Theoretical Computer Science, 157:3-33, 1996.
[24] P. Cartier. Démonstration "automatique" d'identités et fonctions hypergéométriques (d'après D. Zeilberger). Astérisque, (206):Exp. No. 746, 3, 41-91, 1992. Séminaire Bourbaki, Vol. 1991/92.
[25] D. Cerveau and J.-F. Mattei. Formes Intégrables Holomorphes Singulières, volume 97 of Astérisque. Société Mathématique de France, Paris, 1982. With an English summary.
[26] S. Chen, F. Chyzak, R. Feng, and Z. Li. The existence of telescopers for hyperexponentialhypergeometric functions, 2010. MM-Res. Preprints (2010) No. 29, 239-267.
[27] S. Chen and Z. Li. A multiplicative form of multivariate hyperexponential-hypergeometric functions, 2010. MM-Res. Preprints (2010) No. 29, 25-35.
[28] W. Y. C. Chen, Q.-H. Hou, and Y.-P. Mu. Applicability of the $q$-analogue of Zeilberger's algorithm. J. Symbolic Comput., 39(2):155-170, 2005.
[29] C. Christopher. Liouvillian first integrals of second order polynomial differential equations. Electron. J. Differential Equations, pages No. 49, 7 pp. (electronic), 1999.
[30] D. V. Chudnovsky and G. V. Chudnovsky. On expansion of algebraic functions in power and Puiseux series. I. J. Complexity, 2(4):271-294, 1986.
[31] D. V. Chudnovsky and G. V. Chudnovsky. On expansion of algebraic functions in power and Puiseux series. II. J. Complexity, 3(1):1-25, 1987.
[32] F. Chyzak. Fonctions Holonomes en Calcul Formel. PhD thesis, École Polytechnique, Paris, France, 1998. INRIA, TU 0531. 227 pages.
[33] F. Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. Discrete Math., 217(1-3):115-134, 2000.
[34] F. Chyzak, M. Kauers, and B. Salvy. A non-holonomic systems approach to special function identities. In ISSAC '09: Proceedings of the 2009 International Symposium on Symbolic and Algebraic Computation, pages 111-118, New York, NY, USA, 2009. ACM.
[35] F. Chyzak and B. Salvy. Non-commutative elimination in Ore algebras proves multivariate identities. J. Symbolic Comput., 26(2):187-227, 1998.
[36] R. M. Cohn. Difference Algebra. Interscience Publishers John Wiley \& Sons, New York-London-Sydeny, 1965.
[37] O. Cormier, M. F. Singer, B. M. Trager, and F. Ulmer. Linear differential operators for polynomial equations. J. Symbolic Comput., 34(5):355-398, 2002.
[38] S. C. Coutinho. A Primer of Algebraic D-modules, volume 33 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1995.
[39] J. H. Davenport. The Risch differential equation problem. SIAM J. Comput., 15(4):903-918, 1986.
[40] M. C. Fasenmyer. Some generalized hypergeometric polynomials. Bull. Amer. Math. Soc., 53:806-812, 1947.
[41] R. Feng. A note on the Galois group of linear differential-difference equations, 2010. MMRes. Preprints (2010) No. 29, 1-8.
[42] R. Feng, M. F. Singer, and M. Wu. An algorithm to compute Liouvillian solutions of prime order linear difference-differential equations. J. Symbolic Comput., 45(3):306-323, 2010.
[43] R. Feng, M. F. Singer, and M. Wu. Liouvillian solutions of linear difference-differential equations. J. Symbolic Comput., 45(3):287-305, 2010.
[44] R. V. Gamkrelidze, editor. Analysis. I, volume 13 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 1989. Integral representations and asymptotic methods, A translation of Sovremennye problemy matematiki. Fundamentalnye napravleniya, Tom 13, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1986, Translation by D. Newton, Translation edited by R. V. Gamkrelidze.
[45] K. O. Geddes, S. R. Czapor, and G. Labahn. Algorithms for Computer Algebra. Kluwer Academic Publishers, Boston, MA, 1992.
[46] K. O. Geddes and H. Q. Le. An algorithm to compute the minimal telescopers for rational functions (differential-integral case). In Mathematical Software, pages 453-463. WSP, 2002.
[47] K. O. Geddes, H. Q. Le, and Z. Li. Differential rational normal forms and a reduction algorithm for hyperexponential functions. In ISSAC'04: Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation, pages 183-190, New York, USA, 2004. ACM.
[48] J. Gerhard. Fast modular algorithms for squarefree factorization and Hermite integration. Appl. Algebra Engrg. Comm. Comput., 11(3):203-226, 2001.
[49] J. Gerhard. Modular Algorithms in Symbolic Summation and Symbolic Integration (LNCS). SpringerVerlag, 2004.
[50] J. Gerhard, M. Giesbrecht, A. Storjohann, and E. V. Zima. Shiftless decomposition and polynomial-time rational summation. In Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, ISSAC '03, pages 119-126, New York, NY, USA, 2003. ACM.
[51] R. W. Gosper, Jr. Decision procedure for indefinite hypergeometric summation. Proc. Nat. Acad. Sci. U.S.A., 75(1):40-42, 1978.
[52] C. Hermite. Sur l'intégration des fractions rationnelles. Ann. Sci. École Norm. Sup. (2), 1:215-218, 1872.
[53] E. Horowitz. Algorithms for partial fraction decomposition and rational function integration. In $S Y M S A C^{\prime} 71$, pages 441-457, New York, USA, 1971. ACM.
[54] Q.-H. Hou. Algebraic Method in Combinatorics. PhD thesis, Nankai University, Tianjin, China, 2001.
[55] Q.-H. Hou. k-free recurrences of double hypergeometric terms. Adv. in Appl. Math., 32(3):468-484, 2004.
[56] I. Kaplansky. An Introduction to Differential Algebra. Actualités Sci. Ind., No. 1251, Publ. Inst. Math. Univ. Nancago, No. 5. Hermann, Paris, 1957.
[57] M. Karr. Summation in finite terms. J. Assoc. Comput. Mach., 28(2):305-350, 1981.
[58] M. Karr. Theory of summation in finite terms. J. Symbolic Comput., 1(3):303-315, 1985.
[59] M. Kashiwara. On the holonomic systems of linear differential equations. II. Invent. Math., 49(2):121-135, 1978.
[60] W. Koepf. Hypergeometric Summation. Advanced Lectures in Mathematics. Friedr. Vieweg \& Sohn, Braunschweig, 1998. An algorithmic approach to summation and special function identities.
[61] E. R. Kolchin. Differential Algebra and Algebraic Groups. Academic Press, New York, 1973. Pure and Applied Mathematics, Vol. 54.
[62] G. Labahn and Z. Li. Hyperexponential solutions of finite-rank ideals in orthogonal Ore rings. In ISSAC'04: Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation, pages 213-220. ACM, New York, 2004.
[63] H. Q. Le. On the differential-integral analogue of Zeilberger's algorithm to rational functions. In Computer mathematics (Chiang Mai, 2000), volume 8 of Lecture Notes Ser. Comput., pages 204-213. World Sci. Publ., River Edge, NJ, 2000.
[64] H. Q. Le. On the $q$-analogue of Zeilberger's algorithm to rational functions. Program. Comput. Softw., 27(1):35-42, 2001.
[65] H. Q. Le. Algorithms for the construction of the minimal telescopers. PhD thesis, University of Waterloo, Waterloo, Ont., Canada, 2003. AAINQ83003.
[66] Z. Li and D. Zheng. Determining whether a multivariate hyperexponential function is algebraic. J. Syst. Sci. Complex., 19(3):352-364, 2006.
[67] L. M. Lipshitz. The diagonal of a D-finite power series is D-finite. J. Algebra, 113(2):373-378, 1988.
[68] L. M. Lipshitz. D-finite power series. J. Algebra, 122(2):353-373, 1989.
[69] D. Mack. On rational integration. Research Report UCP-38, 1975.
[70] O. Ore. Sur la forme des fonctions hypergéométriques de plusieurs variables. J. Math. Pures Appl. (9), 9(4):311-326, 1930.
[71] M. V. Ostrogradskiĭ. De l'intégration des fractions rationnelles. Bull. de la classe physicomathématique de l'Acad. Impériale des Sciences de Saint-Pétersbourg, 4:145-167, 286-300, 1845.
[72] P. Paule. Greatest factorial factorization and symbolic summation. J. Symbolic Comput., 20(3):235-268, 1995.
[73] G. H. Payne. Multivariate Hypergeometric Terms. PhD thesis, Pennsylvania State University, Pennsylvania, USA, 1997.
[74] R. Pemantle and M. C. Wilson. Twenty combinatorial examples of asymptotics derived from multivariate generating functions. SIAM Rev., 50(2):199-272, 2008.
[75] M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A=B$. A K Peters Ltd., Wellesley, MA, 1996. With a foreword by Donald E. Knuth.
[76] R. Pirastu and V. Strehl. Rational summation and Gosper-Petkovšek representation. J. Symbolic Comput., 20(5-6):617-635, 1995. Symbolic computation in combinatorics (Ithaca, NY, 1993).
[77] R. H. Risch. The problem of integration in finite terms. Trans. Amer. Math. Soc., 139:167189, 1969.
[78] R. H. Risch. The solution of the problem of integration in finite terms. Bull. Amer. Math. Soc., 76:605-608, 1970.
[79] J. F. Ritt. Differential Algebra. American Mathematical Society Colloquium Publications, Vol. XXXIII. American Mathematical Society, New York, N. Y., 1950.
[80] M. Rothstein. A new algorithm for integration of exponential and logarithmic functions. In Proceedings of the 1977 MACSYMA Users Conference (Berkeley, CA), pages 263-274, Washington DC, 1977. NASA.
[81] M. Sato. Theory of prehomogeneous vector spaces (algebraic part) - the English translation of Sato's lecture from Shintani's note. Nagoya Math. J., 120:1-34, 1990. Notes by Takuro Shintani, Translated from the Japanese by Masakazu Muro.
[82] M. F. Singer. Liouvillian first integrals of differential equations. Trans. Amer. Math. Soc., 333(2):673-688, 1992.
[83] R. P. Stanley. Enumerative Combinatorics. Vol. 2, volume 62 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.
[84] A. Storjohann and G. Villard. Computing the rank and a small nullspace basis of a polynomial matrix. In ISSAC'05: Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation, pages 309-316. ACM, New York, 2005.
[85] N. Takayama. An approach to the zero recognition problem by Buchberger algorithm. J. Symbolic Comput., 14(2-3):265-282, 1992.
[86] N. Takayama. An algorithm for finding recurrence relations of binomial sums and its complexity. J. Symbolic Comput., 20(5-6):637-651, 1995. Symbolic computation in combinatorics $\Delta_{1}$ (Ithaca, NY, 1993).
[87] B. M. Trager. Algebraic factoring and rational function integration. In SYMSAC’»6: Proceedings of the Third ACM Symposium on Symbolic and Algebraic Computation, pages 219226, New York, NY, USA, 1976. ACM.
[88] M. van der Put and M. F. Singer. Galois Theory of Linear Differential Equations, volume 328 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 2003.
[89] P. Verbaeten. The automatic construction of pure recurrence relations. SIGSAM Bull., 8(3):96-98, 1974.
[90] J. von zur Gathen and J. Gerhard. Modern Computer Algebra. Cambridge University Press, Cambridge, second edition, 2003.
[91] K. Wegschaider. Computer Generated Proofs of Binomial Multi-Sum Identities. Master's thesis, RISC, J. Kepler University, May 1997.
[92] H. S. Wilf and D. Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and " $q$ ") multisum/integral identities. Invent. Math., 108(3):575-633, 1992.
[93] M. Wu. On Solutions of Linear Functional Systems and Factorization of Modules over Laurent-Ore Algebras. PhD thesis, Chinese Academy of Sciences and Université de NiceSophia Antipolis, Beijing, China and Nice, France, 2005.
[94] D. Y. Y. Yun. Fast algorithm for rational function integration. In Information processing 77 (Proc. IFIP Congr., Toronto, Ont., 1977), pages 493-498. IFIP Congr. Ser., Vol. 7. North-Holland, Amsterdam, 1977.
[95] D. Zeilberger. A holonomic systems approach to special functions identities. J. Comput. Appl. Math., 32:321-368, 1990.
[96] D. Zeilberger. The method of creative telescoping. J. Symbolic Comput., 11(3):195-204, 1991.
[97] H. Zoladek. The extended monodromy group and Liouvillian first integrals. J. Dynam. Control Systems, 4(1):1-28, 1998.

## Index

Abramov-Petkovšek triple, 94
additive decomposition, 18
bidegree, 30
certificate, 10
compatible, 54
complete, 88
conjugate, 54
constant, 15
creative telescoping, 9

D-finite, 114
derivation, 15
difference field, 16
difference ring, 16
differential field, 15
differential Gosper form, 40
differential rational normal form, 57
differential ring, 15
differential-difference ring, 16
differential-reduced, 56
existence criteria, 100
fully integrable, 54

Geddes-Le-Li triple, 88
Hermite reduction, 17, 28
holonomic, 113
hyperexponential-hypergeometric, 51
hyperexponential-integrable, 88
hypergeometric-summable, 94
integrability conditions, 54
logarithmic part, 17
lucky, 32
minimal telescoper, 34
orthogonal, 16
Ostrogradsky-Horowitz system, 22
properness, 73
pump, 91
rational part, 17
residue, 22
ring of Ore polynomials, 16
Rothstein-Trager resultant, 23
shift rational normal form, 57
shift-reduced, 56
similar, 81
split, 81
squarefree, 56
squarefree factorization, 29
squarefree part, 29
standard representation, 86
strict DRNF, 57
strict SRNF, 57
telescoper, 10

Zeilberger's method, 9

## Curriculum Vitae

## Personal Data

- Name: Shaoshi Chen
- Gender: Male
- Nationality: Chinese
- Date and place of birth: July 17, 1983, Zhejiang Province, China
- Marital state: Married, no children


## Contact

- Email: schen@amss.ac.cn


## Education

2001-2005, Studies in mathematics at Jiangsu University, China

2005-2007, Studies in symbolic computation at Key Laboratory of Mathematics Mechanization, Chinese Academy of Sciences.

Supervisor: Ziming Li

2007-2010, Doctorate studies in symbolic computation at Key Laboratory of Mathematics Mechanization (Chinese Academy of Sciences, China) and Algorithms Project (INRIARocquencourt, France).

Co-supervisors: Frédéric Chyzak and Ziming Li

## Published Papers

- A. Bostan, S. Chen, F. Chyzak, and Z. Li. Complexity of creative telescoping for bivariate rational functions. In Stephen M. Watt, editor, ISSAC 2010 Proc. 2010 Internat. Symp. Symbolic Algebraic Comput., New York, N. Y., 2010. ACM Press pp. 203-210.


## Posters

- S. Chen and Z. Li. On the structure of multivariate hyperexponential-hypergeometric functions. Poster at the conference DART IV (Beijing, China), 2010.
- A. Bostan, S. Chen, F. Chyzak, and Z. Li. Rational-function telescoping: blending creative telescoping with Hermite reduction. Poster at the conference ISSAC'09 (Seoul, South Korea), 2009.
- S. Chen and Z. Li. A speed-up of the Hermite reduction for rational functions. Poster at the conference ICMM'09 (Beijing, China), 2009.


## Preprints

- S. Chen, F. Chyzak, R. Feng and Z. Li. The existence of telescopers for hyperexponentialhypergeometric functions. MM Research Preprints, KLMM, AMSS, Academia Sinica, Vol. 29, p. 239-267, July 2010
- S. Chen and Z. Li. A note on Ostrogradsky and Horowitz's method. MM Research Preprints, KLMM, AMSS, Academia Sinica, Vol. 29, p. 36-47, July 2010.
- S. Chen and Z. Li. A multiplicative form of multivariate hyperexponential-hypergeometric functions. MM Research Preprints KLMM, AMSS, Academia Sinica, Vol. 29, p. 25-35, July 2010.
- S. Chen, R. Feng, Z. Li and H. Wang. An exercise on real elementary functions in the book Symbolic Integration I (second edition). MM Research Preprints KLMM, AMSS, Academia Sinica, Vol. 26, p. 115-125, January 2008.

