Homogenization of singular convection-diffusion equations and indefinite spectral problems

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THÈSE

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Contents

Abstract

The thesis is devoted to the homogenization of singular convection-diffusion equations and spectral problems with sign-changing density function. It consists of two parts. The first one contains both qualitative and asymptotic results for solutions of stationary and non-stationary convection-diffusion equations in bounded or unbounded domains. Among the studied problems are qualitative problem for a convection-diffusion equation in a semi-infinite cylinder, homogenization of convection-diffusion models in thin cylinders and asymptotic problems for non-stationary convection-diffusion equations with large convection term in bounded domains.

The second part of the thesis deals with the homogenization of elliptic spectral problems with sign-changing density function. We show that the asymptotic behaviour of the spectrum depends crucially on whether the density average over the period is zero or not, and construct the asymptotics of the spectrum in both these cases.

Résumé

Le but de la thèse est d'étudier l'homogénéisation d'équations de convection-diffusion singulières et de problèmes spectraux à poids indéfini. La thèse se compose de deux parties. La première partie contient des résultats qualitatifs et asymptotiques pour les solutions d'équations de type convection-diffusion stationnaires et instationnaires, qui sont définies dans des domaines bornés ou nonbornés. Les problèmes examinés comprennent des études qualitatives pour une équation elliptique avec des termes du premier ordre dans un cylindre semi-infini, l'homogénéisation de modèles de convection-diffusion dans des cylindres minces et une analyse asymptotique d'équations de convection-diffusion instationnaires avec un grand terme du premier ordre, posées dans un domaine borné.

La deuxième partie de la thèse porte sur l'homogénéisation de problèmes spectraux à poids indéfini, pouvant changer de signe. On montre que le comportement asymptotique dépend essentiellement de la moyenne du poids, notamment si la moyenne est nulle ou non nulle. On construit alors le développement asymptotique du spectre dans les deux cas.

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Introduction

The thesis consists of several closely related papers which can be divided into two groups. The first one contains both qualitative and asymptotic results for solutions of stationary and non-stationary convection-diffusion equations with periodic coefficients, in bounded or unbounded domains. This part consists of five papers:

- **A** I.Pankratova, A. Piatnitski, *On the behaviour at infinity of solutions to stationary convection-diffusion equation in a cylinder*, DCDS-B, 11 (4) (2009).
- **B** G.Panasenko, I.Pankratova, A.Piatnitski, *Homogenization of a Convection-Diffusion Equation in a Thin Rod Structure*. (English). Integral Methods in Science and Engineering. Vol.1. Analytic Methods, 279-290, Birkhauser (2010).
- **C** I.Pankratova, A.Piatnitski, *Homogenization of convection-diffusion equation in infinite cylinder* (accepted for publication in Networks and Heterogeneous Media)
- **D** G.Allaire, I.Pankratova, A.Piatnitski, *Homogenization and concentration for a diffusion equation with large convection in a bounded domain*
- **E** G.Allaire, I.Pankratova, A.Piatnitski, *Homogenization of a non-stationary convection-diffusion equation in a thin rod and in a layer*

The second group of problems addressed in the thesis is concerned with the homogenization of spectral problems with a sigh-changing density function. The results presented are the author's contribution to the following papers:

- **F** S.Nazarov, I.Pankratova, A.Piatnitski, Homogenization of spectral problem for periodic elliptic operators with sign-changing weight function, Narvik University College, R&D Report, No. 4/2008, ISSN 1890-923X (accepted for publication in Archive for Rational Mechanics and Analysis).
- **G** I.Pankratova, Spectral problem for a locally periodic elliptic operator with sign-changing weight function, Narvik University College, R&D Report, No. 9/2009, ISSN 1890-923X.

In paper **A** (see Chapter 1) we develop the qualitative theory for an elliptic equation in a semi-infinite cylinder. Paper **B** (see Chapter 2) is devoted to homogenization problems in a thin heterogeneous rod. Papers **C**, **D** and **E** deal with the non-stationary convection-diffusion equations with large convection term. The results obtained in Paper **D** are also illustrated by numerical experiments. Papers **F** and **G** are devoted to the homogenization of spectral problems for elliptic operators with sign-changing density. In Paper **F** we consider a spectral problem for an elliptic operator with periodic rapidly oscillating coefficients defined in a generic bounded domain, while Paper **G** concerns with operators defined in a thin cylinder having a locally periodic microstructure.

More precisely, Paper **A** concerns the study of the behaviour at infinity of solutions to second order elliptic equations with first order terms, stated in a half-cylinder. The coefficients of the equation are assumed to be measurable and bounded; Neumann boundary condition is imposed on the lateral boundary of the cylinder, while on the base we assign the Dirichlet boundary condition. Under the assumption that the coefficients of the equation stabilize to a periodic regime exponentially with large axial distance, and the functions on the right-hand side decay exponentially at infinity, we prove the existence and the uniqueness of a bounded (in the proper sense) solution and its stabilization to a constant at the exponential rate. Also we provide a necessary and sufficient condition for the uniqueness of a bounded solution. Our approach is partially based on the results from local qualitative elliptic theory, such as Harnack's inequality, Nash and De Giorgi estimates, the maximum principle, positive operator theory and a number of nontrivial a priori estimates. The problems of this type appear while constructing the asymptotic expansions of solutions to equations describing different phenomena in highly inhomogeneous media. For instance, the results obtained in Paper **A** are used in Paper **B** to construct boundary layer correctors.

The question of asymptotic behaviour at infinity for solutions of elliptic equations and systems of equations have been addressed in many mathematical works. We mention just those of them which are closely related to the problem under consideration. Elliptic equations in divergence form defined in unbounded cylinders were studied in [**27**], [**51**], [**2**]. In [**46**], [**47**] the authors has considered a boundary value problem for a second order elliptic equation with first order terms on a half-cylinder with periodic boundary conditions on the lateral boundary. Notice that, in contrast with the operators in divergence form, the presence of first order terms changes crucially the qualitative picture. In particular, a bounded solution need not be unique. The papers [**17**], [**18**], [**21**] are devoted to studying the specific classes of semi-linear elliptic equations in a half-cylinder. Many works deal with the Phragmén–Lindelöf type results for elliptic systems and elasticity problems, in particular. Among them we mark out [**22**], [**39**], [**26**], [**33**]. For more detailed comments on the existing literature see Introduction in Paper **A**.

Nowadays the homogenization theory is a extensively developed topic. Starting from the early 1960s it has been attracting much attention of mathematicians. There is a wide monographic literature devoted to this subject, we point to [**1**], [**6**], [**7**], [**8**], [**13**], [**14**], [**15**], [**30**], [**28**], [**31**], [**40**], [**43**], [**45**], [**48**], [**49**], [**50**], [**56**].

The development of the homogenization theory has been aimed at creating the rigorous mathematical description of the highly inhomogeneous media and, in particular, composite materials and porous media. Equations, describing various physical processes in such media, have rapidly oscillating coefficients because of the strong heterogeneity of the materials. Due to these oscillations, direct analytical and numerical methods of solving boundary value problems for such equations become extremely difficult and often practically unrealizable. However, in many cases it is possible to derive homogenized (effective) equations which provide a good approximation for the behaviour of the original processes. As a result, the characteristics of the original, highly heterogeneous material are well-approximated by those of the effective locally homogeneous material.

Questions of the asymptotic behaviour of thin structures have been considered in many works, both in mathematical and physical literature. In most cases, the asymptotic analysis is aimed at the dimension reduction, or, in other words, reducing the original problem to a problem in a domain of a smaller dimension. For instance, an equation stated in a thin three-dimensional plate is substituted by an equation in a two-dimensional domain; a problem describing some rod structure is reduced to an ordinary differential equation, an so on. From the point of view of the numerical analysis, the reduced problem turns out to be much simpler. Indeed, reducing the number of variables diminishes the amount of computations. Naturally, dimension reduction demands rigorous mathematical justification.

Paper **B** is devoted to the homogenization of a stationary convection-diffusion equation in a thin cylinder being a union of two nonintersecting rods with a junction at the origin. It is assumed that each of these cylinders has a periodic microstructure, and that the microstructure period is of the same order as the cylinder diameter. Under some natural assumptions on the data we construct and justify the asymptotic expansion of a solution which consists of the interior asymptotic expansion and the boundary layer correctors, arising both in the vicinity of the rod ends and the vicinity of the junction. In contrast to the divergence form operators, in the case of convection-diffusion equation the asymptotic behaviour of solutions depend crucially on the direction of the so-called effective convection. In the thesis we only consider the case when in each of the two cylinders (being the constituents of the rod) the effective convection is directed from the end of the cylinder towards the junction.

The obtained results rely on the nontrivial analysis of a convection-diffusion equation in semi-infinite and infinite cylinders. The case of semi-infinite cylinder has been studied in Paper **A**, while the case of the infinite cylinder is addressed in Section 6, Paper **B**.

The problems stated in half-infinite cylindrical domains for the elasticity system have been intensively studied in the existing literature. We quote here the works [**24**], [**25**], [**32**], [**34**], [**35**], [**53**], [**52**], [**55**]. The contact problem of two heterogeneous bars was considered in [**41**], [**42**], [**44**]. Elliptic equations in divergence form have been addressed, for example, in [**6**] and [**43**].

Papers **C**, **D** and **E** focus on the non-stationary convection-diffusion equations with large convection term. In paper **C** the asymptotic behaviour of a solution to a parabolic equation defined in a thin infinite cylinder is studied. Namely, we consider the following operator:

$$\partial_t u - \operatorname{div}\big(a(\frac{x}{\varepsilon})\nabla u\big) + \frac{1}{\varepsilon}\big(b(\frac{x}{\varepsilon}), \nabla u\big)$$

with a small parameter ε. Assuming that the coefficients of the operator are periodic in the axial direction of the cylinder, and imposing homogeneous Dirichlet boundary condition on the lateral boundary, we study the asymptotic behaviour of the solution u^ε, as $\varepsilon \to 0$. Similar equation defined in the whole space has been considered in [**16**] and [**5**]. In contrast with the case of solenoidal vector-field $b(y)$ with zero mean value, for general periodic function $b(y)$ one cannot expect the convergence of the sequence of solutions u^ε in the fixed spatial reference frame. It has been shown in [**16**], [**5**] that the convergence in moving coordinates $(t, x + \bar{b}t/\varepsilon)$ takes place with the constant vector $\bar{b}$ called the effective drift or effective convection (see also [**47**]). In other words, the translated sequence $u^\varepsilon(t, x + \bar{b}t/\varepsilon)$ converges to the solution of a homogenized parabolic equation.

In the case of thin infinite cylinder, considered in Paper **C**, the behaviour of u^ε is governed by the axial component of the effective convection. It is shown that homogenization result holds in moving coordinates, and that the solution admits an asymptotic expansion which consists of the interior expansion being regular in time, and an initial layer. The estimates for the rate of convergence are obtained.

Obviously, the convergence in moving coordinates is not well-defined in a bounded domain. The goal of Papers **D** and **E** is to study the asymptotics of a solution to a non-stationary convection-diffusion equation in a bounded domain. The case of general domain is considered in **D**. while in **E** we study the equation stated either in a thin rod or in a layer. The special structure of the domain allows us to weaken the assumptions imposed on the initial function and construct higher order terms of the asymptotics.

More precisely, in Paper **D** we study the following initial boundary value problem in a bounded domain Ω:

$$\begin{cases} \partial_t u^\varepsilon - \operatorname{div}\big(a(\frac{x}{\varepsilon})\nabla u^\varepsilon\big) + \frac{1}{\varepsilon}\big(b(\frac{x}{\varepsilon}), \nabla u^\varepsilon\big) = 0, & \text{in } (0,T)\times\Omega, \\ u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times\partial\Omega, \\ u^\varepsilon(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

We assume that the initial function u_0 has a compact support in Ω. Intuitively, it is clear that in a bounded domain the initial profile should move rapidly in the direction of the effective drift $\bar{b}$ until it reaches the boundary, and then dissipate due to the homogeneous Dirichlet boundary condition, as t grows. Since the convection term is large, the dissipation increases, as $\varepsilon \to 0$, so that the solution vanishes at any positive time $t > 0$. In order to study the behaviour of this solution for $t \gg \varepsilon$, one should determine the rate of decay and study the rescaled solution. It turns out that both the rate of decay and the behaviour of the rescaled solution can be described in terms of the first eigenpair of the cell eigenproblem with an optimal exponential parameter called Θ (see [**10**]). We prove that, $u^\varepsilon(t,x)$ concentrates in the neighbourhood of a point $\xi \in \partial\Omega$ which depends on Θ. It is interesting to notice that in general, the point of concentration, for large t, does not coincide with the point of intersection of the effective drift $\bar{b}$ with $\partial\Omega$, which is also illustrated by the numerical example given in the paper.

Without the assumption that u_0 has a compact support in Ω, one faces the necessity to construct boundary layer correctors in the neighbourhood of $\partial\Omega$. It is well known that this problem cannot be solved in the case of general bounded domain. However, it is getting feasible in some special cases when the periodic structure agrees with the geometry of the boundary of Ω. In Paper **G** we consider two such cases. Namely, we study a convection-diffusion models in a thin rod and in a layer in $\mathbb{R}^d$. In the case of a thin rod we impose homogeneous Neumann boundary conditions on the lateral boundary of the rod and homogeneous Dirichlet boundary conditions on its bases. As was noticed above, the solution vanishes for time $t \gg \varepsilon$. We determine the rate of vanishing of the solution and describe the evolution of its profile. If the effective axial drift is not zero (we study only this case), the rescaled solution concentrates in the vicinity of one of the rod ends, and the choice of the end depends on the sign of the effective convection. In order to characterize the rate of decay we introduce a 1-parameter family of auxiliary cell spectral problems (see [**7**], [**10**]). The asymptotic behaviour of the solution is then governed by the first eigenpair of the said spectral problem. Among the technical tools used in the paper, are factorization principle (see [**23**], [**54**], [**56**], [**10**], [**11**]), dimension reduction arguments and qualitative results required for constructing boundary layer correctors.

In the case of a layer addressed in Section 3 of Paper **G**, in addition to the factorization principle, we also have to introduce moving coordinates. More precisely, we use a parameterized cell spectral problem and factorization principle to suppress the component of the effective drift which is perpendicular to the layer boundary. While, due to the presence of the longitudinal components of the effective convection we have to introduce moving coordinates.

The second part of the thesis (Papers **F** and **G**) deals with the homogenization of a special class of spectral problems. Various homogenization problems in spectral theory have been extensively explored since 1970s. At present it is a well-developed field comprising many efficient methods and approaches.

Paper **F** focuses on the homogenization of a Dirichlet spectral problem for an elliptic equation in divergence form stated in a regular domain Ω. Namely, the problem under consideration has the form

$$\begin{aligned} -\mathrm{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla u^\varepsilon(x)\Big) &= \lambda\,\rho\big(\frac{x}{\varepsilon}\big)\,u^\varepsilon(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{0.1}$$

with ε being a small positive parameter. It is assumed that the coefficients of the equation $a_{ij}(y)$ and the spectral density function $\rho(y)$ are periodic with respect to y, and that $\rho(y)$ changes sign. It is shown that the spectrum of problem (0.1) consists of two infinite sequences tending to $\pm\infty$. The asymptotic behaviour of the eigenpairs depends essentially on whether the average of ρ over the period is equal to zero or not. When the average of ρ

is zero, the effective spectral problem is that for a quadratic operator pencil, and the eigenvalues are of order ε^{-1}. In the second case the positive eigenvalues and the corresponding eigenfunctions show the same regular behaviour as in the case of point-wise positive spectral density. In both cases we justify the convergence of spectra and present the estimates for the rate of convergence.

The main peculiarity of the problem considered in Paper **F** is the fact that the spectral density function is sign-changing. Previously, a spectral problem with sign-changing density for the Laplace operator has been considered in [**36**]; in this work the limit behaviour of spectrum has been studied under the assumption that the density consists of a fixed positive part and asymptotically vanishing negative part. The results obtained in Paper **F** have been generalized to spectral problems for elliptic systems in [**37**].

The homogenization of spectral problems in the case of point-wise positive weight ρ is well-studied nowadays. Such problems have been first considered in [**19**], [**20**] and then in many other papers. The homogenization of spectral problems in perforated domains has been studied in [**54**] followed by many other works on the subject. The limit behaviour of spectrum of elasticity system in perforated domain has been addressed in [**40**]. In [**12**] the authors have generalized the results obtained in [**40**] by making weaker the assumptions on the regularity of the inclusions and external forces. The spectral problems for locally periodic symmetric second order elliptic operators with large potential have been studied in [**3**]. The work [**4**] dealt with the asymptotic behaviour of spectrum for a periodic symmetric elliptic system with large potential.

The problem addressed in Paper **F** might have interesting applications in the modern theory of metamaterials, that is artificial composite materials designed to produce a desired electromagnetic behavior with significantly enhanced performance over ”natural” structures. For example, when the world is observed through conventional lenses, the sharpness of the image is determined by and limited to the wavelength of light. Metamaterials with negative refractive index are aimed at creation of ”perfect” lenses, that is lenses with capabilities beyond conventional (positive index) ones. It is observed that the double negative media (i.e. metamaterials having both negative permittivity and permeability) could lead to a negative index of refraction. The so-called single negative metamaterials, where either permittivity or permeability are negative, is the next example. For instance, many plasmas, as well as some metals (gold and silver), possess negative permittivity while the permeability is positive. Double positive materials do occur in the nature (dielectrics). Tunable negative index metamaterials became very popular due to the backward wave propagation and subwavelength resolution. Such materials have applications not only in ”superlenses”, but also in miniature metamaterial antennas construction.

It is interesting to note that equation similar to (0.1) can be derived from Maxwell's equations. Indeed, consider the equation for the vector of the electrical field in a bounded 3-dimensional domain

$$\mathrm{rot}\Big(\frac{1}{\widetilde{\mu}(x)}\mathrm{rot}\vec{E}(t,x)\Big) = -\widetilde{\varepsilon}(x)\frac{\partial^2 \vec{E}}{\partial t^2}(t,x), \quad (0,T)\times\Omega.$$

Here $\vec{E}$ is the vector of the electrical field, $\widetilde{\varepsilon}(x)$ and $\widetilde{\mu}(x)$ are the permittivity and permeability, respectively. Let us consider the case of E-polarized plane wave

$$\vec{E} = \{E(x), 0, 0\}e^{i\omega t}, \ E(x) = E(x_2, x_3).$$

Substituting this representation into the Maxwell equations and assuming that $\widetilde{\mu}(x) = \mu(x_2, x_3)$, $\widetilde{\varepsilon}(x) = \varepsilon(x_2, x_3)$ one has

$$-\frac{\partial}{\partial x_2}\Big(\frac{1}{\widetilde{\mu}(x)}\frac{\partial E}{\partial x_2}\Big) - \frac{\partial}{\partial x_3}\Big(\frac{1}{\widetilde{\mu}(x)}\frac{\partial E}{\partial x_3}\Big) = \omega^2\widetilde{\varepsilon}(x)E.$$

Comparing the obtained equation with (0.1) we see that the possibility for ρ to change its sign is equivalent to the sign-changing $\widetilde{\varepsilon}(x)$. In particular, the negative index materials correspond to the case when both λ and ρ in (0.1) are negative; the single negative metamaterials - to the case of positive λ and $\rho < 0$ or $\lambda < 0$ and $\rho > 0$. These simple arguments give an idea that the problem studied in Paper **F** might have applications in the theory of metamaterials.

In Paper **G** a homogenization problem for a second-order self-adjoint operator, stated in a thin cylinder, is considered. The homogeneous Neumann boundary condition is set on the lateral boundary, while on the bases of the cylinder the homogeneous Dirichlet boundary conditions are imposed. As in Paper **F**, the spectral density function is assumed to change sign. Both the coefficients of the equation and the spectral density function are supposed to be locally periodic in the axial direction of the cylinder. The asymptotic behaviour of the spectrum depends on whether the average of the spectral density function over the period is zero or not.

For the density function having positive average the effective spectral problem happens to be a Sturm-Liouville problem. In this case the convergence of the positive part of the spectrum is justified using the theory of convergence in variable spaces with singular measures.

In the case of zero average weight function the limit spectral problem is that for a quadratic operator pencil. To study this operator pencil we apply the results from [**29**] combined with usual arguments used when studying Sturm-Liouville problems. It should be noted that in contrast with [**38**], the presence of slow variable in the coefficients makes the limit operator pencil nontrivial, so that it can not be reduced to the standard Sturm-Liouville problem.

In contrast with the problems investigated in [**37**] and [**38**], for the model considered in the Paper **G** the limit spectral problem is one-dimensional, so that dimension reduction arguments are to be used. We combine the asymptotic expansion technique with the singular measure approach developed in [**57**] and [**9**]. The fact that the considered operator is defined in a thin cylinder allows us to construct boundary layer correctors in the neighbourhood of the cylinder bases and, as a result, improve essentially the asymptotics. As a matter of fact, if the coefficients are sufficiently regular, then arbitrary many terms of the asymptotic expansion can be constructed. The existence of exponentially decaying boundary layer correctors in assured by the results obtained in Paper **A**.

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Paper A

PAPER A

On the behaviour at infinity of solutions to stationary convection-diffusion equation in a cylinder

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ABSTRACT. The work focuses on the behaviour at infinity of solutions to second order elliptic equation with first order terms in a semi-infinite cylinder. Neumann's boundary condition is imposed on the lateral boundary of the cylinder and Dirichlet condition on its base. Under the assumption that the coefficients stabilize to a periodic regime, we prove the existence of a bounded solution, its stabilization to a constant, and provide necessary and sufficient condition for the uniqueness.

1. Introduction

This work deals with the behaviour at infinity of solutions to stationary convection-diffusion equations defined in a semi-infinite cylinder. We assume that Neumann boundary condition is imposed on the lateral boundary of the cylinder, and that the coefficients of the equation are periodic along the cylinder axis or stabilize at the exponential rate to a periodic regime for asymptotically large axial distance. Under these assumptions we study the existence and uniqueness of a bounded solution, and its stabilization to a constant at infinity.

The question of validity of the Saint-Venant and Phragmén–Lindelöf principles, as well as other questions related to the behaviour at infinity of solutions to elliptic equations and systems of equations, received a lot of attention of mechanicians and mathematicians starting from the beginning of 20th century.

A number of rigorous mathematical works are devoted to this subject. Dirichlet and Neumann boundary value problems in a cylindrical domains for second order linear elliptic equations in divergence form were studied by many authors. Early contributions include [**10**], [**6**] and [**7**] which contain results like Saint-Venant's principle for special classes of Neumann problems. As to the later works on this topic, we mention just some of them closely related to the present paper.

In [**14**] an equation in divergence form in a half-cylinder with periodic coefficients on all variables except for one was considered, the exponential stabilization to a constant was proved. The periodic boundary conditions were imposed on the lateral boundary of the cylinder. The technique used in this work relies on specific geometrical methods. As was communicated to the authors ([**22**]), the question of the existence of a solution that converges exponentially to a constant with large axial distance was studied in 1976 for an equation in divergence form under some natural assumptions on the right-hand side. The method relied on a variation of the Lax-Milgram lemma. This result has partially been written in [**15**]. Another proof of the exponential decay of the solution to the same equation was given in [**1**]and it is valid also for non-flat base of the cylinder.

A boundary value problem for a second order elliptic equation with first order terms on a half-cylinder with periodic boundary conditions on the lateral boundary of the cylinder was studied in [**19**] and [**20**]. In these works, under the assumption of C^2 regularity and periodicity of the coefficients, the existence of a bounded solution and its exponential stabilization to a constant at infinity was proved by means of diffusion processes techniques. Moreover, the necessary and sufficient conditions for the uniqueness of a bounded solution were given. In [**20**] the obtained results were applied to a homogenization problem for singularly perturbed operators defined in a layer. Also, without the assumption on periodicity in axial variable the following conditional result was obtained in [**20**]: if the adjoint problem has a bounded uniformly positive solution, then the effective axial drift can be defined and the results proved in periodic case, remain valid.

In the present paper we study operators with measurable coefficients and assume only Lipschitz continuity of the boundary of the cross-section. In this case the usage of probabilistic techniques is getting embarrassing and sometimes impossible, especially if the boundary condition is not homogeneous. Our approach relies on the various results from local qualitative elliptic theory, such as Harnack's inequality, Nash and De Giorgi estimates, the maximum principle, positive operators theory and a number of non-trivial a priori estimates which include as a weight the function forming the kernel of the adjoint periodic operator. We consider here not only operators with periodic coefficients, but also with coefficients which stabilize to periodic regime at infinity. Another issue addressed in the paper is the generalization of the existence, uniqueness and stabilization results to the case of nonhomogeneous equations with H^{-1} function on the right-hand side. It should be noted that obtaining the a priori estimates in this case is getting more complicated than in the case of data from L^2.

Also we pay special attention to the asymptotic behaviour of the solutions defined in a growing family of finite cylinders. This gives a clear picture of how solutions defined in finite cylinders approximate the limit bounded solution. This analysis allows us, in particular, to distinguish the special case when the so-called effective axial drift is equal to zero.

In [**3**], [**4**] and [**8**] specific classes of semi-linear elliptic equations in a half-cylinder were considered. It was proved that a global solution, when it exists, decays at least exponentially with large axial distance. The technique involves the derivation of a first order differential inequality for the energy flux across a cross-section of the cylinder. With the help of this technique spatial behaviour of solutions to elliptic systems, in particular those of linearized and linear elasticity, was studied also (see, for example, [**9**]).

A priori estimates similar to Saint-Venant's principle in elasticity theory were discussed in [**17**] under some dissipativity type assumptions on the coefficients. Also in this work interesting uniqueness results in proper classes of growing functions were obtained for Dirichlet and Neumann problems for second order linear elliptic equations in unbounded domains .

In [**11**] the authors investigated elliptic systems with complex constant coefficients, assuming that a weighted Dirichlet integral is bounded. The paper deals with finite energy solutions for the system of linear elasticity.

The asymptotic behaviour at infinity of solutions to symmetric elliptic systems were treated in [**16**]. This work focused on the existence of solutions in weighted spaces with various exponentially growing or decaying weights.

In [**13**] the behaviour of solutions to nonlinear elliptic equations with a dissipative nonlinear zero order terms was studied by means of the barrier functions techniques.

The goal of this work is to study the behaviour at infinity of solutions to a linear stationary convection-diffusion equation in a semi-infinite cylinder. We impose Dirichlet boundary condition on the base of the cylinder and Neumann condition on the lateral boundary. Under the assumptions that the coefficients of the equation stabilize exponentially to a periodic regime, and the functions on right-hand side of the equation and of the boundary operator decays sufficiently fast at infinity, we prove the existence of a bounded solution and its stabilization to a constant at the exponential rate. Also we provide a necessary and sufficient condition for the uniqueness of a bounded solution. It should be noted that, in contrast with the divergence form operators, for the operators with first-order terms the question of uniqueness of a bounded solution is getting more complicated. We show that whether a solution is unique or not depends on the sign of some constant called effective axial drift (or flux), which can be determined in terms of a solution to auxiliary periodic problem for the formally adjoint operator.

The problems of this type appear while constructing the asymptotic expansions of solutions to equations describing different phenomena in highly inhomogeneous medium. For instance, such results allow one to construct boundary layer functions in various homogenization problems. Moreover, these results are of independent interest in mechanics and other applied fields and, of course, in mathematics.

The paper is organized as follows. Sections 1–6 focus on the homogeneous problem with periodic coefficients. In these sections we start with the problem setup and auxiliary results, and then proceed with the existence, the uniqueness and the stabilization to a constant of a bounded solution to the problem under consideration. In Sections 7–8 we obtain similar results for inhomogeneous problems and equations with coefficients stabilizing to a periodic regime.

2. Problem statement

Let $G = (0, \infty) \times Q$ be a semi-infinite cylinder in $\mathbb{R}^d$ with the axis directed along x_1, where Q is a bounded domain in $\mathbb{R}^{d-1}$ with a Lipschitz boundary ∂Q. The lateral boundary of G is denoted by $\Sigma = (0, +\infty) \times \partial Q$. We study the following boundary-value problem:

$$\begin{cases} -\mathrm{div}\ (a(x)\, \nabla\, u(x)) - (b(x), \nabla\, u(x)) = 0, & x \in G, \\ \dfrac{\partial u}{\partial n_a} = 0, & x \in \Sigma, \\ u(0, x') = \varphi(x'), & x' \in Q. \end{cases} \tag{2.1}$$

Here $a(x)$ is a $d \times d$ matrix and $b(x)$ is a vector in $\mathbb{R}^d$, $x = (x_1, x')$, $\varphi(x') \in H^{1/2}(Q)$; $(\cdot, \cdot)$ stands for the standard scalar product in $\mathbb{R}^d$; $\partial u / \partial n_a = \sum_{i,j=1}^{d} a_{ij}(x) n_i\, \partial_j u$ is the conormal derivative, n is the external unit normal. The matrix-valued function $a(x)$ and the vector field $b(x)$ are supposed to be measurable and bounded, that is $a_{ij}(x) \in L^\infty(G)$, $b_i(x) \in L^\infty(G)$, and periodic in x_1 functions. Without loss of generality we assume that the period is equal to 1. For the sake of simplicity the matrix $a(x)$ is supposed to be symmetric. Moreover, we assume that $a(x)$ satisfies the uniform ellipticity condition, that is there exists a positive constant Λ such that, for almost all $x \in \mathbb{R}^d$,

$$\Lambda\, |\xi|^2 \le \sum_{i,j} a_{ij}(x) \xi_i \xi_j, \quad \forall \xi \in \mathbb{R}^d, \tag{2.2}$$

The first goal of this work is to study the behavior of bounded (in a proper sense) solutions of problem (2.1).

3. Auxiliary function $p(x)$

Consider the following periodic problem:

$$\begin{cases} -\mathrm{div}(a(x) \nabla\, u) - (b(x), \nabla\, u) = f(x), & x \in G_0^1 = (0,1) \times Q, \\ \dfrac{\partial u}{\partial n_a} = 0, & x \in \Sigma_0^1 = (0,1) \times \partial Q, \\ u - \ x_1 - \text{periodic}. \end{cases} \tag{3.1}$$

This problem has a unique up to an additive constant solution $u(x)$. We denote by A an unbounded operator in $L^2(G_0^1)$ which maps $u(x)$ into $f(x) \in L^2(G_0^1)$. In view of x_1-periodicity we can identify functions defined on G_0^1 with the corresponding functions defined on the set $Y = \mathfrak{S}_1 \times Q$, where $\mathfrak{S}_1$ is a 1-dimensional circle. Then problem (3.1) reads

$$
\begin{cases} -\mathrm{div}(a(x)\nabla u) - (b(x), \nabla u) = f(x), & x \in Y, \\ \dfrac{\partial u}{\partial n_a} = 0, & x \in \partial Y. \end{cases} \tag{3.2}
$$

The operator A is an unbounded operator from $L^2(Y)$ into itself with dense domain $D(A)$, that consists of the functions $u(x) \in L^2(Y)$ such that there exists $f(x) \in L^2(Y)$: $A\,u = f$ and $\partial u/\partial n_a = 0$ for $x \in \partial Y$. For large $\lambda > 0$ the inverse operator $(A + \lambda I)^{-1}$ exists and it is compact. Moreover, using the De Giorgi–Nash estimates (see, for example, [**5**]) it is easy show that $(A + \lambda I)^{-1}$ is a compact operator in $C(Y)$.

The formally adjoint problem takes the following form:

$$
\begin{cases} -\mathrm{div}(a\nabla v) + \mathrm{div}\,(b\,v) = f, & x \in Y, \\ \dfrac{\partial v}{\partial n_a} - (b, n)\,v = 0, & x \in \partial Y. \end{cases}
$$

In the sequel we will need an auxiliary function $p(x)$ which belongs to the null space of the adjoint operator:

$$
\begin{cases} -\mathrm{div}(a\nabla p) + \mathrm{div}\,(b\,p) = 0, & x \in Y, \\ \dfrac{\partial p}{\partial n_a} - (b, n)\,p = 0, & x \in \partial Y. \end{cases} \tag{3.3}
$$

The goal of this section is to show that such function exists and is positive.

DEFINITION 3.1. We say that the operator B from $L^2(Y)$ ($C(Y)$) into itself is positive if from the inequality $u \geq 0$ it follows that $Bu \geq 0$.

The linear positive operator B is called v-bounded, for some $v \in C(Y)$, $v > 0$, if for every positive function $u \in C(Y)$ there exists two constants $\alpha = \alpha(u)$ and $\beta = \beta(u)$ such that

$$
0 < \alpha(u)\,v \leq B\,u \leq \beta(u)\,v.
$$

First let us show that the operator $(A + \lambda I)^{-1}$ is positive. By the maximum principle, if $f > 0$, then u cannot have a negative minimum in the interior of the domain Y. The assumption that a negative minimum is attained on the boundary ∂Y, will also contradict the maximum principle in view of the positiveness of f. Indeed, since ∂Q is Lipschitz then for every point $\tilde{x} \in \partial Q$ there exists a neighborhood $U(\tilde{x}) \subset \mathbb{R}^d$ such that the surface $\partial Q \cap U(\tilde{x})$ is represented by the equality

$$
x_1 = \mathcal{F}(x_2, ..., x_d),
$$

where $\mathcal{F}$ is a Lipschitz function. Let us make a change of variables straightening the boundary ∂Q, so that the piece of the boundary $\partial Q \cap U(\tilde{x})$ is mapped into the piece of the plane $\xi_1 = 0$ and $Y \cap U(\tilde{x})$ into some domain where $\xi_1 > 0$:

$$
\begin{cases} \xi_1 = x_1 - \mathcal{F}(x_2, ..., x_d), \\ \xi_k = x_k, \quad k = 2, ..., d. \end{cases}
$$

One more change of the variables transfers the co–normal derivative to the normal derivative:

$$\begin{cases} \eta_1 = \xi_1, \\ \eta_k = \xi_k - \dfrac{a_{1k}}{a_{11}}\,\xi_1, \quad k = 2, ..., d. \end{cases}$$

By construction, in the vicinity of the point $\tilde{\eta} = \mathcal{F}(\tilde{x})$ the solution u is only defined for $\eta_1 \geq 0$. We define an extension of u (keeping the notation u for the extended function) by setting $u(\eta_1, \eta') = u(-\eta_1, \eta')$ for negative η_1. One can check that after inverse changing the variables, due to the homogeneous Neumann boundary condition on ∂Q, the extended function $u(x)$ remains a solution of some convection-diffusion equation with a positive right hand side in a neighbourhood of $\tilde{x}$. Thus, in view of the maximum principle, $u(x)$ cannot attend a negative minimum at $\tilde{x}$.

Let us show that the operator $(A + \lambda I)^{-1}$ is 1-bounded in $C(Y)$. First, we note that, in view of the boundedness of the coefficients, the following estimate takes place (see, for example, [**5**]):

$$\|u\|_{C(Y)} \leq C\|f\|_{C(Y)}$$

for some constant C independent of f. Thus,

$$(A + \lambda I)^{-1} f = u \leq C.$$

It remains to show that $(A + \lambda I)^{-1} f > 0$ if $f(x) > 0$, $x \in Y$. Let us suppose that $\min\limits_{x\in Y} u(x) = 0$. In the interior of Y the function $u(x)$ cannot attain a nonpositive minimum unless u is equal to zero. If we assume that u achieves zero minimum on the boundary $\mathfrak{S}^1 \times \partial Q$, then, in the same way as above, we can extend u to a larger domain so that the extended function remains a solution of some elliptic convection-diffusion type equation with a positive right-hand side and, consequently, u cannot achieve its zero minimum on the boundary. Hence, we conclude that $u(x) \geq c(f) > 0$ if $f(x) > 0$.

Now we apply the Krein-Rutman theorem (see, for example, [**12**]) to compact, positive, 1-bounded operator $(A + \lambda I)^{-1}$ in $C(Y)$. According to this theorem, there exists a simple positive eigenvalue λ_0 of the operator $(A + \lambda I)^{-1}$ with a positive eigenvector, and there is no others eigenvalues with positive eigenvectors. Moreover, if we consider $(A + \lambda I)^{-1}$ as an operator in $L^2(Y)$, then there exists a nonnegative periodic in x_1 eigenvector $p(x) \in L^2(Y)$ of the adjoint operator $(A^* + \lambda I)^{-1}$, which corresponds to the same eigenvalue λ_0. Let us note that the operator A has a positive eigenvector (which is equal to 1) corresponding to zero eigenvalue. In view of the uniqueness of eigenvalue with positive eigenfunction $\left(\frac{1}{\lambda_0} - \lambda\right) = 0$, and, therefore, $p(x)$ belongs to the kernel of the adjoint operator A^*.

In order to prove the positiveness and boundedness of the function $p(x)$ up to the boundary $\partial Y = \partial Q$ we extend it to some bigger domain containing $\tilde{G}$.

Since ∂Q is Lipschitz, for any point $\bar{x} \in \partial Q$ there exists a neighborhood $U(\bar{x})$ such that $\Gamma = \partial Q \cap U(\bar{x}) = \{x : x_1 = f(x')\}$, with Lipschitz function $f(x')$. Let us make a change of variables which straightens Γ:

$$\left\{\begin{array}{l} \xi_1 = x_1 - f(x'), \\ \xi_k = x_k, \quad k = 2, ..., d, \end{array}\right.$$

such that the domain $\Omega^+ = Y \cap U(\bar{x})$ is mapped into $\tilde{\Omega}^+$, where $\xi_1 > 0$, and Γ is mapped into $\tilde{\Gamma} = \{\xi : \ \xi_1 = 0\}$.

We define the "extended" coefficients $\tilde{a}_{ij}(\xi)$ and $\tilde{b}_j(\xi)$ in the domain

$$\tilde{\Omega}^- = \left\{\xi = (\xi_1, \xi') : \ \xi_1 < 0, \ (-\xi_1, \xi') \in \tilde{\Omega}^+\right\}$$

as follows:

$$\tilde{a}(\xi_1, \xi') = S\, a(-\xi_1, \xi')\, S^*,$$
$$\tilde{b}(\xi_1, \xi') = S\, b(-\xi_1, \xi'),$$

where the matrix S is given by the expression

$$S = \begin{pmatrix} -1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \end{pmatrix}.$$

If we define the extended function $\tilde{p}$ as

$$\tilde{p}(\xi) = \left\{\begin{array}{ll} p(\xi_1, \xi'), & \xi_1 > 0, \\ p(-\xi_1, \xi'), & \xi_1 < 0, \end{array}\right.$$

then it can be checked that $\tilde{p}$ is a solution of the equation

(3.4) $$-\mathrm{div}\,(\tilde{a}(\xi)\,\nabla\,\tilde{p}(\xi)) + \mathrm{div}\,(\tilde{b}(\xi)\,\tilde{p}(\xi)) = 0, \quad \tilde{\Omega} = \tilde{\Omega}^+ \cup \tilde{\Omega}^-.$$

Indeed, by definition

$$\frac{\partial \tilde{p}}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\,\tilde{p} = 0, \quad \xi \in \tilde{\Gamma},$$

where n is an external normal to $\tilde{\Omega}^+$. Since by construction

$$\frac{\partial \tilde{p}^+}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\,\tilde{p}^+ = -\left(\frac{\partial \tilde{p}^-}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\,\tilde{p}^-\right), \quad \xi \in \tilde{\Gamma},$$

where $\tilde{p}^\pm$ are limit values of the function $\tilde{p}$ on the surface $\tilde{\Gamma}$ from different sides of it, $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$ respectively. Then

$$\int\limits_{\tilde{\Gamma}} \left(\frac{\partial \tilde{p}^+}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\,\tilde{p}^+\right)\,\varphi(\xi)\,d\xi' = \int\limits_{\tilde{\Gamma}} \left(\frac{\partial \tilde{p}^-}{\partial n_{\tilde{a}}} - (\tilde{b}, n)\,\tilde{p}^-\right)\,\varphi(\xi)\,d\xi' = 0,$$

for any function $\varphi(\xi) \in C_0^\infty(\tilde{\Omega})$. Keeping in mind the last equality, one can easily show that $\tilde{p}$ is a solution of (3.4).

By construction the obtained function $\tilde{p}(\xi)$ is nonnegative. In view of the Harnack inequality in any compact subset $\tilde{\Omega}'$ of $\tilde{\Omega}$, $\tilde{p}$ is bounded from below by some positive constant δ which depends on the point $\bar{x}$ and the choice of a compact subset (otherwise it is equal to zero which contradicts the definition of p). Moreover, $\tilde{p}$ is a Holder continuous function in $\tilde{\Omega}'$ (see, e.g. [**21**], [**5**]), where again the upper bound for $\tilde{p}$ depends on the point $\bar{x} \in \tilde{\Gamma}$ and the choice of a compact subset.

Making inverse change of variables we conclude that $p(x)$ is positive and bounded in some neighborhood $U'(\bar{x})$ of any point $\bar{x} \in \partial Q$ up to the boundary ∂Q. Let us take a covering of ∂Q which consists of these neighborhoods $U'(\bar{x})$. Since ∂Q is a compact set, there exist a finite subcovering of it. By means of the standard compactness arguments, one can prove that $p(x)$ is a positive continuous function in the closed set $\overline{Y}$.

4. Existence of bounded solutions

In what follows G_α^β is a finite cylinder $(\alpha, \beta) \times Q$, $\Sigma_\alpha^\beta = (\alpha, \beta) \times \partial Q$ is its lateral boundary and $S_\alpha = \{x = (x_1, x') : x_1 = \alpha,\ x' \in Q\}$.

DEFINITION 4.1. A weak solution of problem (2.1) is called bounded if for any $N > 0$ the following inequality holds:

$$\|u\|_{L^2(G_N^{N+1})} \leq C,$$

where C does not depend on N.

LEMMA 4.2. *A bounded solution* $u(x)$ *of problem* (2.1) *in terms of Definition 4.1 exists. Moreover,*

$$\|\nabla u\|_{L^2(G)} < \infty, \quad \|u\|_{L^\infty(G_1^\infty)} < \infty. \tag{4.1}$$

PROOF. First we consider the following boundary value problem in a finite cylinder

$$\begin{cases} -\mathrm{div}(a(x)\nabla u^k) - (b(x), \nabla u^k) = 0, & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ u^k(0, x') = \varphi(x'), \quad u^k(k, x') = 0, & x' \in Q. \end{cases} \tag{4.2}$$

It is known that the solution to problem (4.2) exists and for any $k > 0$ has finite H^1 and L^∞ norms. Obviously, in view of the maximum principle since $u^k(k, x') = 0$

$$\|u^k\|_{L^\infty(S_1)} \leq \|u^k\|_{L^\infty(S_{1/2})}.$$

Let us consider in the cylinder G_0^1 the following auxiliary problem

$$\begin{cases} -\mathrm{div}(a(x)\nabla z^k) - (b(x), \nabla z^k) = 0, & x \in G_0^1, \\ \dfrac{\partial z^k}{\partial n_a} = 0, & x \in \Sigma_0^1, \\ z^k(0, x') = \varphi(x'), \quad z^k(1, x') = u^k(1, x'), & x' \in Q. \end{cases}$$

Since the last problem is linear, we can represent z^k as a sum $z_1 + z_2^k$, where z_1 and z_2^k satisfy the homogeneous equation and lateral boundary conditions, $z_1(0, x') = \varphi(x')$,

$z_1(1,x') = z_2^k(0,x') = 0$, $z_2^k(1,x') = u^k(1,x')$. It is known that for the function $z_1(x)$ as a solution of elliptic problem in a fixed domain the following estimate holds:

$$\|z_1^k\|_{L^\infty(S_{1/2})} \le C\,\|\varphi\|_{H^{1/2}(Q)}.$$

The $L^\infty(S_{1/2})$ norm of the function z_2^k can be estimated in terms of $L^\infty(S_1)$ norm of u^k as follows

$$\|z_2^k\|_{L^\infty(S_{1/2})} \le \alpha\,\|u^k\|_{L^\infty(S_1)}, \tag{4.3}$$

where $0 < \alpha < 1$, α does not depend on k. Indeed, $|z_2^k| \le v^k$ in G_0^1, where v^k satisfies the same equation and boundary conditions as z_2^k, except for the boundary conditions on S_1, which reads $v^k(1,x') = \|u^k\|_{L^\infty(S_1)}$. Due to the strong maximum principle, $v^k \le \alpha\|u^k\|_{L^\infty(S_1)}$ with $0<\alpha<1$, that yields (4.3). In this way we obtain

$$\|u^k\|_{L^\infty(S_1)} \le \|u^k\|_{L^\infty(S_{1/2})} \le \|z_1\|_{L^\infty(S_{1/2})} + \|z_2^k\|_{L^\infty(S_{1/2})} \le$$

$$\le C\|\varphi\|_{H^{1/2}(Q)} + \alpha\,\|u^k\|_{L^\infty(S_1)}, \quad \alpha<1,$$

and, finally

$$\|u^k\|_{L^\infty(S_1)} \le \frac{C}{1-\alpha}\,\|\varphi\|_{H^{1/2}(Q)}, \quad 0<\alpha<1.$$

Moreover, the $L^2(G_0^1)$ norm of z_1 is bounded

$$\|z_1\|_{L^2(G_0^1)} \le C_1\,\|\varphi\|_{H^{1/2}(Q)},$$

and, since $u^k(1,x') \in L^\infty(S_1)$ then

$$\|z_2^k\|_{L^2(G_0^1)} \le C_2\,\|\varphi\|_{H^{1/2}(Q)},$$

where C_1 and C_2 do not depend on k. Also, in view of the maximum principle,

$$\|u^k\|_{L^\infty(G_1^k)} \le C\,\|\varphi\|_{H^{1/2}(Q)}, \tag{4.4}$$

with C independent on k. Obviously, it follows from the last estimates that

$$\|u^k\|_{L^2(G_N^{N+1})} \le C, \quad N \ge 0. \tag{4.5}$$

Let us note that the estimate (4.4) is valid in $L^\infty(G_\delta^k)$, for any $\delta>0$:

$$\|u^k\|_{L^\infty(G_\delta^k)} \le C(\delta)\|\varphi\|_{H^{1/2}(Q)}, \quad \forall\delta>0, \tag{4.6}$$

with $C(\delta)$ independent on k.

In order to estimate the L^2-norm of the gradient of u^k in G_0^k, notice first that by the standard elliptic estimates in the cylinder G_0^2 we get

$$\|\nabla u^k\|_{L^2(G_0^2)} \le C\|\varphi\|_{H^{1/2}(Q)}. \tag{4.7}$$

Notice also, that $p\,u^k$ is $H^1(G_1^2)$ function because both $p(x)$ and $u^k(x)$ are elements of $H^1(G_1^k) \cap L^\infty(G_1^k)$. Moreover, the estimate holds true

$$\|pu^k\|_{H^1(G_1^2)} \le C\|\varphi\|_{H^{1/2}(Q)}. \tag{4.8}$$

Since $\operatorname{div}\big(a\nabla u^k\big) \in L^2(G_0^k)$ and $\operatorname{div}\big(a\nabla p - bp\big) = 0$, then the normal components of $\big(a\nabla u^k\big)$ and $\big(a\nabla p - bp\big)$ on S_1 are well-defined elements of $H^{-1/2}(Q)$ (see [**2**]), and the inequality holds

$$\|a_{1j}\partial_{x_j} u^k\|_{H^{-1/2}(Q)} \le C\|\varphi\|_{H^{1/2}(Q)}, \quad \|a_{1j}\partial_{x_j} p - b_1 p\|_{H^{-1/2}(Q)} \le C. \tag{4.9}$$

If we multiply the equation in (4.2) by pu^k and integrate the resulting relation over the cylinder G_1^k, then considering (3.3) and integrating several times by parts, we obtain

$$\int\limits_{G_1^k} (a\nabla\, u^k, \nabla\, u^k)\, p\, dx = \int\limits_{S_1} u^k p\, a_{1j}\frac{\partial u^k}{\partial_{x_j}}\, dx' - \frac{1}{2}\int\limits_{S_1} (u^k)^2\, (a_{1j}\frac{\partial p}{\partial_{x_j}} - bp)\, dx.$$

Taking into account (4.4) and (4.7)-(4.9), we estimate the integral on the left-hand side as follows

$$\int\limits_{G_1^k} (a\nabla\, u^k, \nabla\, u^k)\, p\, dx \le C\|\varphi\|^2_{H^{1/2}(Q)}.$$

This estimate and (4.7) imply the desired bound

$$\|\nabla\, u^k\|_{L^2(G_0^k)} \le C\|\varphi\|_{H^{1/2}(Q)}, \tag{4.10}$$

where C does not depend on k.

Finally, using (4.5), (4.10) and compactness arguments, we conclude that $u^k(x)$ converges weakly in $H^1_{\rm loc}(G)$ to a function $u(x)$ which is a solution of problem (2.1) such that (4.1) holds true. Let us note that in view of the Nash–De Giorgi estimates (see [**21**]), for any $\delta > 0$ a solution of problem (2.1) is a Hölder-continuous function in $\overline{G_\delta^\infty}$ up to the lateral boundary of the cylinder. □

REMARK 4.3. Let us note that we did not use the x_1-periodicity of the coefficients $a_{ij}(x)$ and $b_j(x)$ to prove the estimates (4.4) and (4.5). The proof is valid for the case of arbitrary measurable bounded coefficients $a_{ij}(x)$ and $b_j(x)$ and uniformly elliptic matrix $a(x)$.

5. Stabilization of solutions

In this section we are going to show that every bounded solution of problem (2.1) stabilizes to a constant at the exponential rate. To this end let us consider two functions of the variable x_1:

$$M(x_1) = \max_{x'\in Q} u(x_1, x') \quad \text{and} \quad m(x_1) = \min_{x'\in Q} u(x_1, x').$$

By the maximum principle the function $M(x_1)$ does not assume a local maximum point in the open interval $x_1 \in (0, +\infty)$. This implies that $M(x_1)$ has at most one minimum point on $[0, +\infty)$, and that, starting form this minimum point, $M(x_1)$ is monotonous. If $M(x_1)$ does not have minimum point, then it is monotonous on the whole interval $[0, +\infty)$. Similarly, $m(x_1)$ is monotonous, possibly starting from some point.

Therefore, we have only three possibilities for the behavior of the functions $M(x_1)$ and $m(x_1)$:

- $M(x_1)$ monotonously decreases and $m(x_1)$ monotonously increases;
- $M(x_1)$ and $m(x_1)$ monotonously increase (maybe starting from some point);
- $M(x_1)$ and $m(x_1)$ monotonously decrease (maybe starting from some point).

5.1. $M(x_1)$ **monotonously decreases and** $m(x_1)$ **monotonously increases.** Denote $G_N^{N+2} = (N, N+2) \times Q$, $N \geq 0$. Our aim is to estimate the oscillation of $u(x)$ over the cross-section S_{N+1} in terms of the oscillation of $u(x)$ over S_N. Since problem (2.1) is linear, then we can assume without loss of generality that $m(N) = 0$. Then in G_N^{N+2} the function $u(x)$ is nonnegative. As was shown above, the solution $u(x)$ can be extended to a larger domain $(N, N+2) \times \tilde{Q}$, $\overline{Q} \subset \tilde{Q}$, in such a way that the extended function satisfies a convection-diffusion equation in $(N, N+2) \times \tilde{Q}$, and the maximum and the minimum of the extended function over cross-section $\{x_1 = k,\, x' \in \tilde{Q}\}$ coincide with $M(k)$ and $m(k)$, respectively.

Thus, the Harnack inequality holds:

$$m(k) \geq \alpha M(k), \quad \forall k \geq 1,$$

where a constant α depends only on Λ, d and Q. Then

$$M(N+1) - m(N+1) \leq (1-\alpha)\, M(N+1) \leq (1-\alpha)\, M(N). \tag{5.1}$$

Taking into account (5.1) and the assumption $m(N) = 0$, we obtain

$$\operatorname*{osc}_{x_1 = N+1} u(x) = M(N+1) - m(N+1) \leq$$

$$\leq (1-\alpha) \operatorname*{osc}_{x_1 = N} u(x), \quad 0 < (1-\alpha) < 1, \quad N > 0.$$

The last inequality implies that $u(x)$ stabilizes to a constant exponentially. Indeed, since this inequality holds for all $N > 0$, then

$$\operatorname*{osc}_{x_1 = N} u(x) \leq (1-\alpha)^{N-1} \operatorname*{osc}_{x_1 = 1} u(x).$$

Finally, taking into account the boundedness of the function $u(x)$ (see (4.1)) and denoting by C_∞ the limit of $m(x_1)$ as $x_1 \to \infty$, we obtain

$$|u(x) - C_\infty| \leq C_0\, e^{-\gamma_0 x_1}, \quad \gamma_0 = -\log(1-\alpha) > 0. \tag{5.2}$$

REMARK 5.1. One can see that the constant C_0 in (5.2) in this case takes the form

$$C_0 \leq C_1 \, \|\varphi\|_{H^{1/2}(Q)} + C_2\, C_\infty \leq C_1 \|\varphi\|_{H^{1/2}(Q)} + C_2 C \|\varphi\|_{H^{1/2}(Q)}.$$

Indeed, taking into account the linearity of the problem and estimate (4.4), we have

$$|\operatorname*{osc}_{x_1 = 1} u| \leq 2\, \|u\|_{L^\infty(S_1)} \leq C\, \|\varphi\|_{H^{1/2}(Q)}.$$

Let us emphasize also that the constant γ_0 depends only on the ellipticity constant Λ, the space dimension d and the domain Q.

LEMMA 5.2. *There always exists a unique solution $u_0(x)$ of problem* (2.1) *for which $M(x_1)$ decreases and $m(x_1)$ increases.*

Moreover, the function $u_0(x)$ stabilizes to a constant C_φ^∞ at the exponential rate, as $x_1 \to \infty$:

$$|u_0 - C_\varphi^\infty| \le C_0 \, \|\varphi\|_{H^{1/2}(Q)} \, e^{-\gamma_0 x_1}, \quad x_1 > 1.$$

If $\varphi(x') \in L^\infty(Q)$ then for $u_0(x)$ the maximum principle is valid, that is

$$\min_{x' \in Q} \varphi(x') \le u_0(x) \le \max_{x' \in Q} \varphi(x').$$

PROOF. Indeed, such a solution can be constructed with the help of the following auxiliary problems:

$$\begin{cases} -\operatorname{div}(a(x)\nabla u^k) - (b(x), \nabla u^k) = 0, & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ u^k(0, x') = \varphi(x'), \quad \dfrac{\partial u^k}{\partial n_a}(k, x') = 0, & x' \in Q. \end{cases}$$

By the maximum principle, $M^k(x_1) = \max\limits_{x' \in Q} u^k(x_1, x')$ is decreasing and $m^k(x_1) = \min\limits_{x' \in Q} u^k(x')$ is increasing function, for any k. If $\varphi \in L^\infty(Q)$ then

$$\min_{x' \in Q} \varphi(x') \le u^k(x) \le \max_{x' \in Q} \varphi(x'), \quad \forall k > 0.$$

Passing to the limit as $k \to \infty$ completes the proof. Due to the maximum principle the obtained solution u_0 to problem (2.1) is unique. In view of Remark 5.1 the rate of exponential stabilization of u_0 to C_φ^∞ depends only on Λ, d and Q. □

5.2. $M(x_1)$ **and** $m(x_1)$ **are monotonously decreasing (increasing) functions.** If $M(x_1)$ and $m(x_1)$ decrease for sufficiently large x_1, then $M(x_1)$ monotonously decreases on the whole half-line $[0, +\infty)$, while $m(x)$ might have at most one maximum point. One can take N_0 large enough so that on the interval $[N_0, \infty)$ both functions are monotonous. Obviously, it is sufficient to prove the stabilization in the case of monotonously decreasing at infinity functions $M(x_1)$ and $m(x_1)$: the case when $M(x_1)$ and $m(x_1)$ are monotonously increasing functions can be considered in a similar way. As before we assume that $m(N+2) = 0$.

First of all, due to monotonicity and boundedness of $M(x_1)$ and $m(x_1)$ (we consider only bounded solutions) the following limits exist:

$$\lim_{x_1 \to \infty} M(x_1) = M, \qquad \lim_{x_1 \to \infty} m(x_1) = m.$$

For arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, let $N > 0$ be such that

$$M(N) - M(N+2) < \varepsilon_1, \quad m(N) - m(N+2) = m(N) < \varepsilon_2.$$

Then, by the Harnack inequality, in the domain $S_{N+1} = \{N+1\} \times Q$ the estimate holds

$$\varepsilon_2 > m(N+1) \ge \alpha \, M(N+1) \ge \alpha M(N+2) > \alpha M(N) - \alpha \, \varepsilon_1.$$

Thus, we have that

$$\operatorname*{osc}_{x_1=N} \tilde{u}(x) \to 0, \quad N \to \infty.$$

The last equality shows that the functions $M(x_1)$ and $m(x_1)$ converge to the same constant, that is $u(x)$ stabilizes to the constant.

Now we are going to prove that $u(x)$ stabilizes to the constant exponentially. Without loss of generality we can assume that $u(x)$ stabilizes to zero. Instead of the original function $u(x)$ we consider shifted function $\tilde{u}(x) = u(x_1 + N_0, x')$ for $N_0 \geq 0$. Due to the periodicity of the coefficients, $\tilde{u}$ remains a solution of the same problem but with different boundary function at S_0, which we denote by $\psi(x') = \tilde{u}(0, x') = u(N, x')$. Clearly, $\psi(x')$ is a positive continuous function. Let us define a function $v(x)$ as a solution of problem (2.1) with $v(0, x') = 1$ and $v \to 0$ as $x_1 \to \infty$. The existence of such a solution can be justified as follows. As in (4.2), one can construct approximations $\tilde{u}^k$ and v^k for the functions $\tilde{u}(x)$ and $v(x)$. By the maximum principle $\tilde{u}^k(x) \geq \big(\min_{x' \in Q} \psi\big) v^k(x)$. Passing to the limit, as $k \to \infty$, in this inequality, we obtain:

$$v \leq \frac{\tilde{u}(x)}{\min\limits_{x' \in Q} \psi} \to 0, \quad x_1 \to \infty.$$

Thus, the required solution v exists. By the maximum principle

$$\frac{\tilde{u}(x)}{\max\limits_{x' \in Q} \psi} = \frac{\tilde{u}(x)}{M(N)} \leq v(x_1, x'),$$

so, setting $x_1 = 1$ and taking maximum over $x \in Q$ of both sides of the last inequality, we obtain

$$\frac{M(N+1)}{M(N)} \leq \max_{x' \in Q} v(1, x') \leq \beta < 1.$$

Consequently we have:

$$M(N+1) \leq \beta\, M(N), \quad \forall N \geq 0,$$

or

$$M(N) \leq e^{(N-1)\,\ln\beta} \max_{x' \in Q} u(1, x'), \quad N \geq 1.$$

Denoting by γ the positive constant $-\log\beta$ and using estimate (4.4), we obtain

$$M(N) \leq C_1 \, \|\varphi\|_{H^{1/2}(Q)} \, e^{-\gamma N},$$

or, in other words,

$$|u| \leq C_0 \, e^{-\gamma x_1}, \quad x_1 > 1.$$

In the general case, when a bounded solution $u(x)$ to problem (2.1) stabilizes to a nonzero constant C_∞, one can see that C_0 in the last inequality takes the form

$$C_0 = C_1 \, \|\varphi\|_{H^{1/2}(Q)} + C_2 \, C_\infty,$$

with constants C_1 and C_2 which depend only on Λ, d and Q. In this way we have proved the following

LEMMA 5.3. *Under our standing assumptions on* $a_{ij}(x)$ *and* $b_j(x)$, $i, j = 1, ..., d$, *every bounded solution of problem (2.1) stabilizes to a constant at the exponential rate, as* $x_1 \to \infty$.

REMARK 5.4. It should be noted that if we replace in (2.1) the homogeneous Neumann boundary condition on Σ with zero-flux condition

$$\frac{\partial u}{\partial n_a} - (b, n)u = 0,$$

then the corresponding periodic cell problem need not have a nontrivial kernel; in particular, a constant need not be an eigenfunction. In this case the problem

$$\left\{ \begin{array}{ll} -\mathrm{div}\ (a(x)\,\nabla\, u(x)) - (b(x), \nabla\, u(x)) = 0, & x \in G, \\ \dfrac{\partial u}{\partial n_a} - (b, n)u = 0, & x \in \Sigma, \\ u(0, x') = \varphi(x'), & x' \in Q \end{array} \right. \tag{5.3}$$

might have a bounded solution which does not stabilize to a constant at infinity. For example, a function $u(x_1, x_2) = \sin(\sqrt{2}x_1)e^{x_2}$ satisfies the problem

$$\left\{ \begin{array}{ll} -\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial^2 u}{\partial x_2^2} - \dfrac{\partial u}{\partial x_2} = 0, & x \in (0, +\infty) \times (0, 1), \\ \big(\dfrac{\partial u}{\partial x_2} - u\big)(x_1, 0) = \big(\dfrac{\partial u}{\partial x_2} - u\big)(x_1, 1) = 0, & x_1 \in (0, +\infty), \\ u(0, x_2) = 0, & x_2 \in (0, 1), \end{array} \right.$$

clearly, this solution is bounded, but does not converge to a constant, as $x_1 \to \infty$.

The detail analysis of problem (5.3) requires quite delicate arguments of spectral theory and is out of the scope of the present paper.

6. Main result

In order to formulate the main result we introduce the notation

$$\bar{b}_1 = \int\limits_{G_0^1} \left(a_{1j}(x) \frac{\partial p(x)}{\partial x_j} - b_1(x) p(x) \right) dx, \tag{6.1}$$

where the auxiliary function $p(x)$ was introduced in Section 3. Let us notice that in view of the periodicity of the coefficients, the integral on the right-hand side of (6.1) can be taken over G_k^{k+1} for any $k > 0$. Moreover, this integral can be taken over arbitrary cross section $S_\xi = \{\xi\} \times Q$. Indeed, integrating (3.3) over G_ξ^η we obtain the following equality

$$\int\limits_{S_\xi} \left(-a_{1j} \frac{\partial p}{\partial x_j} + b_1\, p \right) dx' = \int\limits_{S_\eta} \left(-a_{1j} \frac{\partial p}{\partial x_j} + b_1\, p \right) dx',$$

for any positive ξ and η. Thus, for any $\xi > 0$

$$\int\limits_{S_\xi} \left(a_{1j}(x) \frac{\partial p(x)}{\partial x_j} - b_1(x) p(x) \right) dx' = \mathrm{Const}.$$

THEOREM 6.1. *Let $a_{ij}(x) \in L^\infty(G)$, $b_j(x) \in L^\infty(G)$ be x_1-periodic functions, and suppose that the condition* (2.2) *is fulfilled. Then the following statements hold:*

(1) *Every bounded (in terms of Definition 4.1) solution $u(x)$ of problem (2.1) stabilizes to a constant at the exponential rate as $x_1 \to \infty$, that is*

$$|u(x) - C_\infty| \le C\, e^{-\gamma x_1}, \quad C,\ \gamma > 0,\ x_1 > 1,$$

where the convergence rate γ does not depend on φ and C_∞.

(2) *$\bar b_1 < 0$ if and only if for any $\varphi(x') \in H^{1/2}(Q)$ and for any $l \in \mathbb{R}$, there exists a bounded solution $u(x)$ of problem (2.1) that converges to the constant l, as $x_1 \to \infty$;*

(3) *$\bar b_1 \ge 0$ if and only if there exists a unique bounded solution $u(x)$ of problem* (2.1) *and it converges to a constant $m = m(\varphi)$, as $x_1 \to \infty$.*

REMARK 6.2. In the case $\bar b_1 \ge 0$ for a solution $u(x)$ of problem (2.1), the function $M(x_1)$ is decreasing and the function $m(x_1)$ is increasing.

Indeed, by virtue of Lemma 5.2, there exists a solution to problem (2.1) such that the corresponding $M(x_1)$ monotonously decreases and $m(x_1)$ monotonously increases. Since $\bar b_1 \ge 0$, the mentioned solution is unique, and the required statement follows.

Although in the case $\bar b_1 < 0$ a bounded solution is not unique, the solution for which $M(x_1)$ decreases and $m(x_1)$ increases remains unique. Such a solution depends continuously on the boundary data $\varphi(x')$ and defines uniquely the constant C^∞_φ, to which it converges. This constant will play an important role in the sequel.

REMARK 6.3. Let us note that in the case when $M(x_1)$ and $m(x_1)$ are both decreasing or increasing functions, the stabilization rate may depend on $\bar b_1$ (cf. Remark 5.1). In general, γ may tend to zero, as $\bar b_1$ goes to zero. Indeed, let us consider the following problem with constant coefficients:

$$\begin{cases} \Delta u + b_1\, \partial_{x_1} u = 0, & x \in G, \\ u(0, x') = 1. \end{cases} \tag{6.2}$$

It is easy to see that in this case $p(x) = \mathrm{Const}$, $\bar b_1 = -b_1$ and all the solutions of problem (6.2) depend only on x_1; furthermore,

$$u(x) = C_1 + C_2\, e^{-b_1 x_1}, \tag{6.3}$$

with some constants C_1 and C_2. Obviously, if $\bar b_1 \ge 0$ then a solution to problem (6.2) is unique and equal to 1; if $\bar b_1 < 0$ then every bounded solution stabilizes to a constant at the exponential rate. As follows from (6.3), the stabilization rate goes to zero as $\bar b_1 = -b_1 \to 0$.

PROOF OF THEOREM 6.1.
1. Stabilization of every bounded solution had been proved above in Section 5.
2. Assume that for any $\varphi(x')$ and for every constant k there exists a solution that converges to this constant. We are going to prove that in this case $\bar{b}_1 < 0$. To this end we denote by $\tilde{u}(x)$ the solution of problem (2.1) with $\varphi(x') \equiv 1$ such that $\tilde{u}(x) \to 0$, as $x_1 \to \infty$. Letting $u(x) = 1 - \tilde{u}$, we obtain a solution of problem (2.1) with $u(0, x') = 0$. If we multiply the equation in (2.1) by $p(x)u(x)$ and integrate the resulting relation over $G_0^\xi = (0, \xi) \times Q$, then we obtain

$$\int\limits_{G_0^\xi} (a\,\nabla\, u, \nabla\, u)\, p\, dx + \frac{1}{2} \int\limits_{S_\xi} \left(\frac{\partial p}{\partial n_a} - (b, n)p \right)\, u^2\, dx' + \int\limits_{S_\xi} a_{1j} \frac{\partial u}{\partial x_j} u\, p\, dx' = 0.$$

Integrating on ξ from N to $N + 1$, for some $N > 0$, gives:

$$\int\limits_N^{N+1} \int\limits_{G_0^\xi} (a\,\nabla\, u, \nabla\, u)\, p(x)\, dx\, d\xi + \frac{1}{2} \int\limits_{G_N^{N+1}} \left(\frac{\partial p}{\partial n_a} - (b, n)p \right)\, u^2\, dx + \\ + \int\limits_{G_N^{N+1}} a_{1j} \frac{\partial u}{\partial x_j} u(x)\, p(x)\, dx = 0.$$

Now, we use the facts that the integral $\int_{G_0^\xi} (a\,\nabla\, u, \nabla\, u)\, p(x)\, dx$ is an increasing function of ξ, $p(x) > 0$ is bounded and, due to our assumption, $u(x)$ stabilizes to 1 at the exponential rate, as $x_1 \to \infty$. Then for sufficiently large N the following inequality holds:

$$\int\limits_{G_0^N} (a\nabla\, u, \nabla u)\, p\, dx + \frac{1}{2}\,\bar{b}_1 \,\le\, C\, \|\nabla u\|_{L^2(G_N^{N+1})}\, \|u\|_{L^2(G_N^{N+1})}$$

Combining standard elliptic estimates for $\tilde{u}(x) = 1 - u(x)$ (extended as in Section 3 to a bigger domain) with the assumption on $\tilde{u}(x)$, one can see that

$$\|\nabla u\|_{L^2(G_N^{N+1})} = \|\nabla \tilde{u}\|_{L^2(G_N^{N+1})} \le C \|\tilde{u}\|_{L^2(G_{N-1}^{N+2})} \le C\, e^{-\gamma N}, \quad \gamma > 0,$$

and, therefore,

$$\int\limits_{G_0^N} (a\nabla\, u, \nabla u)\, p\, dx + \frac{1}{2}\,\bar{b}_1 \,\le C\, e^{-\gamma N}.$$

Passing to the limit as $N \to \infty$ implies that $\bar{b}_1 < 0$. The inverse implication will follow from Lemma 6.4 below.
3.We consider the following sequence of auxiliary boundary value problems:

$$\text{(6.4)} \qquad \begin{cases} -\mathrm{div}\,(a(x)\nabla\, u^k) - (b, \nabla\, u^k) = 0, & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ u^k(0, x') = 1, \quad u^k(k, x') = 0. & \end{cases}$$

First we show that if the sequence $u^k(x)$ of solutions of the auxiliary problems (6.4) converges uniformly to 1 on every compact set as $k \to \infty$, then $\bar{b}_1 \geq 0$.

Let us multiply the first equation in (6.4) by $p(x)u^k(x)$ and integrate the resulting relation over G_0^k. Integrating by parts and taking into account the boundary conditions $u^k(k, x') = 0$, we obtain

$$\int\limits_{G_0^k} (a \nabla u^k, \nabla u^k)\, p\, dx - \frac{1}{2} \int\limits_{S_0} \left(a_{1j} \frac{\partial p}{\partial x_j} - b_1\, p \right) dx' - \int\limits_{S_0} a_{11} \frac{\partial u^k}{\partial x_1}\, p\, dx' = 0.$$

In view of the maximum principle u^k cannot attend its maximum in the interior of the domain G_0^k, so

$$\bar{b}_1 = 2 \int\limits_{G_0^k} (a \nabla u^k, \nabla u^k)\, p\, dx + \int\limits_{S_0} a_{11} \frac{\partial u^k}{\partial x_1}\, p\, dx' \geq 0.$$

Next we prove the following

LEMMA 6.4. *The following two conditions are equivalent:*

(i) *For every boundary condition $\varphi(x')$ there exists a unique bounded solution of problem (2.1) and this solution converges to a constant $m = m(\varphi)$, as $x_1 \to \infty$;*

(ii) *Solutions $u^k(x)$ of problem (6.1) with the boundary condition $\varphi(x') = 1$ converge uniformly on every compact set $K \Subset G_0^k$ to 1, as $k \to \infty$.*

PROOF OF LEMMA 6.4. Let condition (i) be fulfilled. Then, obviously, $u(x) \equiv 1$ if $\varphi = 1$. Since $u^k \to u$, as $k \to \infty$ in the space $H^1_{\rm loc}(G)$, then in view of De Giorgi estimates

$$u^k \rightrightarrows u = 1, \quad k \to \infty$$

on every compact set in G.

Let (ii) hold true. Suppose that there exist $\varphi(x')$, two constants C^1_∞ and C^2_∞ and two bounded solutions u_1 and u_2 of problem (2.1) such that

$$u_1 \to C^1_\infty, \quad u_2 \to C^2_\infty, \quad x_1 \to \infty.$$

Then the function $v = 1 - (u_1 - u_2)/(C^1_\infty - C^2_\infty)$, which stabilizes to zero as $x_1 \to \infty$, solves the following problem:

$$\begin{cases} -\mathrm{div}\ (a(x) \nabla v) - (b(x), \nabla v) = 0, & x \in G, \\ \dfrac{\partial v}{\partial n_a} = 0, & x \in \Sigma, \\ v(0, x') = 1, & x' \in Q. \end{cases}$$

On the other hand, by the maximum principle $v(x) \geq u^k(x)$ where u^k is a solution of problem (6.4). According to (ii), u^k converges to 1 uniformly on every compact set in G_0^k, as $k \to \infty$. Thus $v(x) \geq 1$, $x \in G$. We arrive at contradiction. Lemma 6.4 is proved. □

4. It remains to prove that there are only two possible options for the behaviour of u^k: either $u^k(x)$ decays at the exponential rate, or $u^k(x)$ converges to 1 uniformly on every compact set, as $k \to \infty$.

Obviously, in view of the maximum principle, $\{u^k(x)\}$, for any $x \in G$, is a monotonously increasing sequence and $0 \le u^k(x) \le 1$, for all $x \in G_0^k$. Thus $u^k(x)$ converges uniformly on every compact subset of G to a function $u(x)$, $0 \le u \le 1$, which is a solution of problem (2.1) with $\varphi(x') = 1$. In view of the maximum principle if $u(x) = 1$ in some interior point of G, then $u(x) \equiv 1$, $x \in G$. Hence, either u^k converges uniformly to 1 on every compact subset of G or

$$\lim_{k\to\infty} \max_{x'\in Q} u^k(x_1, x') < 1, \qquad \forall x_1 > 0.$$

Suppose that the latter case takes place, and denote $\lim\limits_{k\to\infty} \max\limits_{x'\in Q} u^{k+1}(1, x') = \beta < 1$. If we introduce

$$v_1^k(x_1, x') = \frac{u^{k+1}(x_1 + 1, x')}{\max\limits_{x'\in Q} u^{k+1}(1, x')},$$

then $v_1^k(0, x') \le 1$ and, due to the maximum principle, $v_1^k(x) \le u^k(x)$. This yields

$$u^{k+1}(x_1 + 1, x') \le u^k(x_1, x') \max_{x'\in Q} u^{k+1}(1, x') \le u^{k+1}(x_1, x') \max_{x'\in Q} u^{k+1}(1, x');$$

thus,

$$u^{k+1}(2, x') \le \beta\, u^{k+1}(1, x') \le \beta^2.$$

Similarly, we can construct

$$v_2^k(x_1, x') = \frac{u^{k+2}(x_1 + 2, x')}{\max\limits_{x'\in Q} u^{k+2}(2, x')}$$

and show that $\lim\limits_{k\to\infty} u^k(3, x') \le \beta^3$. Repeating this procedure, we obtain for any $N > 0$ the inequality $u^k(N, x') \le \beta^N$ which implies the exponential decay for $u^k(x)$, as $x_1 \to \infty$.

Theorem 6.1 is proved. $\square$

Although in the statement of Theorem 6.1 one does not see any difference between the cases $\bar b_1 = 0$ and $\bar b_1 > 0$, the behaviour of the approximations u^k is rather distinct in these two cases. The lemmata below specify the difference.

LEMMA 6.5. *Let $\bar b_1 > 0$. Then the solution u^k to problem* (6.4) *satisfies the estimate*

$$|u^k - 1| \le C\, e^{-\gamma(k - x_1)}, \quad x \in G_0^k, \tag{6.5}$$

where the constant C depends on Λ, d and Q; γ is a positive parameter which may depend on $\bar b_1$.

PROOF. Making change of variables $z_1 = k - x_1$, $z' = x'$ in (6.4) and denoting

$$\tilde{a}(z) = S\,a(k - z_1, z')\,S^* = S\,a(-z_1, z')\,S^*,$$
$$\tilde{b}(z) = s\,b(k - z_1, z') = S\,b(-z_1, z'),$$

with

$$S = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix},$$

we transform problem (6.4) to the form

$$\begin{cases} -\mathrm{div}\,(\tilde{a}(z)\nabla\,\tilde{u}^k) - (\tilde{b}, \nabla\,\tilde{u}^k) = 0, & z \in G_0^k, \\ \dfrac{\partial \tilde{u}^k}{\partial n_{\tilde{a}}} = 0, & z \in \Sigma_0^k, \\ \tilde{u}^k(0, z') = 0, \quad \tilde{u}^k(k, z') = 1. \end{cases}$$

It is easy to see that for the obtained problem the effective drift is negative. As was shown in the proof of Theorem 6.1, the function $(1 - \tilde{u}^k)$ tends to zero exponentially, that is

$$|1 - \tilde{u}^k(z)| \le C\,e^{-\gamma z_1}, \quad z \in G_0^k.$$

Making the inverse change of variables and taking into account that $\tilde{u}^k(z) = u^k(k - z_1, z')$, we obtain (6.5). $\square$

LEMMA 6.6. $\bar{b}_1 = 0$ *if and only if a solution* u^k *of the auxiliary problem (6.4) is close to the linear function on every compact set* $K \Subset G$*, that is if we denote*

$$l^k(x) = \begin{cases} 1 - \dfrac{x_1}{k}, & x_1 \le k, \\ 0, \quad x_1 > k \end{cases}$$

then

$$\|u^k(x) - l^k(x)\|_{L^\infty(G_0^k)} \to 0, \quad k \to \infty.$$

PROOF. The method we use is borrowed from the homogenization theory (see, for example, [**18**]). Let us denote $\varepsilon = 1/k$ in 6.4) and make the change of variables

$$x_1 \mapsto \varepsilon\,x_1, \qquad x_j \mapsto x_j,\ j = 2, ..., d.$$

If we introduce the notation

$$a^\varepsilon(x_1, x') = a\left(\frac{x_1}{\varepsilon}, x'\right), \quad v^\varepsilon(x_1, x') = u\left(\frac{x_1}{\varepsilon}, x'\right),$$

then in the new variables equation (6.4) reads

$$\begin{aligned} &\varepsilon^2\,\partial_{x_1}\,(a_{11}^\varepsilon\,\partial_{x_1}\,v^\varepsilon) + \varepsilon\sum_{i\ne 1}\partial_{x_i}\,(a_{i1}^\varepsilon\,\partial_{x_i}\,v^\varepsilon) + \varepsilon\sum_{k\ne 1}\partial_{x_k}\,(a_{1k}^\varepsilon\,\partial_{x_1}\,v^\varepsilon) + \\ &+ \sum_{i,k\ne 1}\partial_{x_k}\,(a_{ik}^\varepsilon\,\partial_{x_i}\,v^\varepsilon) + \varepsilon\,b_1^\varepsilon\,\partial_{x_1}\,v^\varepsilon + \sum_{k\ne 1} b_k^\varepsilon\,\partial_{x_k}\,v^\varepsilon = 0, \quad x \in G_0^1. \end{aligned} \tag{6.6}$$

The periodicity of the coefficients suggests the following ansatz

$$\tilde v^\varepsilon(x,y_1)=v^0(x,y_1)+\varepsilon\, v^1(x,y_1),\quad y_1=\frac{x_1}{\varepsilon},$$

where the functions v^0 and v^1 are 1-periodic in y_1. Substituting this expression into the equality (6.6), collecting power–like terms related to ε^0 and taking into account the periodicity of all the functions in y_1, we obtain an equation for the function v^0:

$$\begin{gathered}\partial_{y_1}\,(a_{11}(y_1,x')\,\partial_{y_1}\,v^0(x,y_1))+\sum_{i\neq 1}\partial_{y_1}\,(a_{i1}\,\partial_{x_i}\,v^0)+\\ +\sum_{k\neq 1}\partial_{x_k}\,(a^\varepsilon_{1k}\,\partial_{y_1}\,v^0)+\sum_{i,k\neq 1}\partial_{x_k}\,(a^\varepsilon_{ik}\,\partial_{x_i}\,v^0)+\\ +b_1(y_1,x')\,\partial_{y_1}\,v^0+\sum_{k\neq 1}b_k\,\partial_{x_k}\,v^0=0,\quad x\in G^1_0,\ y_1\in(0,1).\end{gathered}\tag{6.7}$$

Since $\partial u/\partial n_a=0$ for $x\in\Sigma^1_0$, then making simple rearrangements we obtain the boundary condition for the function v^0:

$$\begin{gathered}\sum_j a_{1j}(y_1,x')\,n_j\,\partial_{y_1}\,v^0+\sum_{j;i\neq 1}a_{ij}(y_1,x')\,n_j\,\partial_{x_i}\,v^0=0,\\ x'\in\partial Q,\ y_1\in(0,1).\end{gathered}\tag{6.8}$$

The solution of the boundary value problem (6.7)–(6.8) does not depend on the variables y_1 and x', that is:

$$v^0(x,y_1)=v^0(x_1).$$

Following the ideas of the homogenization theory, we represent $\tilde v^\varepsilon(x,y_1)$ as follows:

$$\tilde v^\varepsilon(x,y_1)=v^0(x_1)+\varepsilon\,\chi_1(y_1,x')\,\partial_{x_1}\,v^0(x_1)+\varepsilon^2\,\psi(x_1,y_1,x'),\tag{6.9}$$

where the periodic in y_1 scalar functions χ_1 and ψ are to be found. For convenience let us denote the "fast" variables (y_1,x') by $z=(z_1,z')$. Then, collecting the terms of order ε, one can obtain the following equation for the function $\chi_1(z)$:

$$-\mathrm{div}_z\,(a(z)\,\nabla\,\chi_1)-(b(z),\nabla_z\,\chi_1)=\sum_{k=1}^{d}\partial_{z_k}\,a_{1k}+b_1,\ \ z\in Y=(0,1)\times Q.\tag{6.10}$$

The boundary conditions for the function χ_1 on the lateral boundary of the cylinder take the form

$$(a(z)\nabla_z\,\chi_1,n)=-\sum_i a_{i1}n_i,\quad z\in\partial Y=(0,1)\times\partial Q.\tag{6.11}$$

Due to the Fredholm Alternative, problem (6.10) – (6.11) is solvable if and only if the following equality holds:

$$\int_Y\left(\sum_{k=1}^{d}\partial_{z_k}\,a_{1k}+b_1\right)\,p(z)\,dz-\int_{\partial Y}\sum_{i=1}^{d}a_{i1}n_ip\,d\sigma=0,\tag{6.12}$$

where the function $p(z)$ is a solution of the following problem:

$$
\begin{cases} -\mathrm{div}(a(z)\nabla\, p) + \mathrm{div}\,(b(z)\, p) = 0, \quad z \in Y, \\ (a(z)\nabla p, n) - (b(z), n)\, p = 0, \qquad z \in \partial Y, \end{cases} \tag{6.13}
$$

Since we assume that $\bar{b}_1$ is equal to zero, then the condition (6.12) holds. Indeed, integrating by parts and making simple rearrangements, it is easy to see that the left hand side of (6.12) coincides with $\bar{b}_1$.

Finally, collecting the terms of order ε^2, we obtain the following problem for the function $\psi(x_1, z)$:

$$
\begin{gathered} -\mathrm{div}(a(z), \nabla_z \psi) - (b, \nabla_z \psi) = \\ \left[a_{11} + \sum_{k=1}^{d} \partial_{z_k}(a_{1k}\chi_1) + \sum_{i=1}^{d} a_{i1}\, \partial_{z_i}\chi_1 + a_{11}\partial_{z_1}\chi_1 + b_1\chi_1 \right] \partial_{x_1x_1} v^0, \;\; z \in Y, \end{gathered} \tag{6.14}
$$

$$
(a(z)\nabla_z \psi, n) = -\sum_i a_{i1} n_i \chi_1 \partial_{x_1x_1} v^0, \quad z \in \partial Y. \tag{6.15}
$$

Using one more time the Fredholm Alternative for problem (6.14) – (6.15) we get:

$$
\bar{a}_{11}\, \partial_{x_1x_1} v^0(x_1) = 0, \quad x_1 \in (0,1),
$$

where the constant $\bar{a}_{11}$ is given by

$$
\begin{aligned} \bar{a}_{11} = \int\limits_Y \big(a_{11} + \partial_{z_k}(a_{1k}\chi_1) + a_{i1}\, \partial_{z_i}\chi_1 + a_{11}\partial_{z_1}\chi_1 + b_1\chi_1 \big)\, p(z) dz - \\ - \int\limits_{\partial Y} \sum_i a_{i1} n_i \chi_1 p \, d\sigma. \end{aligned}
$$

Integrating by parts we obtain the following expression for the constant $\bar{a}_{11}$:

$$
\bar{a}_{11} = \int\limits_Y \left(a_{11} - \sum_{k=1}^{d} a_{1k}\chi_1 \partial_{z_k} p + \sum_{i=1}^{d} a_{i1}\, \partial_{z_i}\chi_1\, p + b_1\, \chi_1\, p \right) dz. \tag{6.16}
$$

Let us show that $\bar{a}_{11} > 0$. Then $\partial_{x_1x_1} v^0(x_1) = 0$ and, as a consequence, $v^0(x_1)$ is a linear function on x_1. The scheme of the proof is as follows:

(1) We construct the matrix $\bar{A}$ such that $\bar{A}_{11} = \bar{a}_{11}$. Namely, we set

$$
\bar{A}_{ij} = \int\limits_Y a_{ik}(\delta_{kj} + \partial_{z_k}\chi_j)\, p\, dz - \int\limits_Y \chi_i a_{mj} \partial_{z_m} p\, dz + \int\limits_Y \chi_i \tilde{b}_j p\, dz,
$$

where χ_1 is defined in (6.10) and the functions χ_k for $k \neq 1$ are defined by the equations:

$$\begin{cases} -\mathrm{div}\,(a(z)\nabla\,\chi_k) - (b(z), \nabla\,\chi_k) = -\partial_{z_k}\, a_{k1} - b_k - \bar{b}_k, \quad z \in Y, \\ \dfrac{\partial \chi_k}{\partial n_a} = -a_{ki}\, n_i, \quad z \in \partial Y, \end{cases}$$

and $\bar{b}_k$ are given by the formula:

$$\bar{b}_k = \int\limits_Y (a_{ki}\,\partial_{z_i}\, p - b_k\, p)\, dz.$$

(2) On the second step we prove that $\bar{A}$ is nonnegative definite matrix. For this purpose we show that this matrix can be represented in the form

$$\bar{A} = \int\limits_Y (I + \nabla\,\chi)^T a(z)(I + \nabla\,\chi)\, p(z)\, dz, \tag{6.17}$$

where B^T denotes the adjoint of B.

(3) Then we show that $(I + \nabla\,\chi)\, e_1 \neq 0$.

(4) For an arbitrary nonnegative definite matrix $C = \{c_{ij}\}$ we state that if $c_{11} = 0$, then $c_{1k} = 0$, $k = \overline{2, d}$. Thus $C\, e_1 = 0$. We then show that $\bar{A}e_1 \neq 0$ in our case. Therefore $\bar{a}_{11}$ cannot be equal to zero.

Now we proceed with the detail proof. The fact that $\bar{A}_{11} = \bar{a}_{11}$ readily follows from from the definition of the matrix $\bar{A}$. In order to prove (6.17) let us re-arrange the expression on the right hand side:

$$\int\limits_Y (\delta_{im} + \partial_{z_m}\,\chi_i)\, a_{mk}\, (\delta_{kj} + \partial_{z_k}\,\chi_j)\, p\, dz = \int\limits_Y \delta_{im}\, a_{mk}\, (\delta_{kj} + \partial_{z_k}\,\chi_j)\, p\, dz +$$
$$+ \int\limits_Y \partial_{z_m}\,\chi_i\, a_{mk}\, (\delta_{kj} + \partial_{z_k}\,\chi_j)\, p\, dz.$$

Integrating the second term by parts gives

$$\bar{A}_{ij} = \int\limits_Y \Big(a_{ik}(\delta_{kj} + \partial_{z_k}\,\chi_j)\, p - \frac{1}{2}\chi_i\chi_j\partial_{z_k}(b_k\, p)\Big) dz + \frac{1}{2}\int\limits_{\partial Y} \chi_i\chi_j\, b_k\, n_k\, p\, dz +$$
$$+ \int\limits_Y \chi_i\tilde{b}_j\, p\, dz - \int\limits_Y \chi_i\, a_{mj}\,\partial_{z_m}\, p\, dz + \frac{1}{2}\int\limits_Y \chi_i\,\chi_j\,\partial_{z_k}\,(a_{mk}\,\partial_{z_m}\, p)\, dz -$$
$$-\frac{1}{2}\int\limits_{\partial Y} \chi_i\,\chi_j\, a_{mk}\, n_k\,\partial_{z_m}\, p\, d\sigma + \int\limits_{\partial Y} \chi_i\, a_{mk}\,(\delta_{kj} + \partial_{z_k}\,\chi_j)\, n_m\, p\, d\sigma.$$

Finally, the last equality and (6.12) lead to (6.17).

Let us show that $(I+\nabla\,\chi)\,e_1\neq 0$. Suppose that $(I+\nabla\,\chi)\,e_1=0$. Then $\partial_{z_1}\chi_1=-1$, or

$$\int\limits_Y \partial_{z_1}\chi_1\,dz_1=-1,$$

which contradicts the periodicity of χ.

Consider a nonnegative definite symmetric matrix $C=\{c_{ij}\}_{i,j=1}^d$, and suppose that $c_{11}=0$. Evaluating the quadratic form $c_{ij}\xi_i\xi_j$ at the vector

$$\xi=\{N,1,0,...,0\},\quad N>0,$$

we get

$$c_{ij}\xi_i\xi_j=c_{12}N+c_{21}N+c_{22}=2c_{12}N+c_{22}.$$

If $c_{12}\neq 0$, then for large N (positive if $c_{12}<0$, and negative, if $c_{12}>0$) we obtain $c_{ij}\xi_i\xi_j<0$, which contradicts the non-negativeness of the matrix C. Thus, $c_{12}=c_{21}$ is equal to zero. Similarly we can show that $c_{1k}=0$, $k=2,...,d$. Therefore, $C\,e_1=0$.

In our case $\bar{A}e_1=(I+\nabla\,\chi)\,e_1\neq 0$, and we conclude that $\bar{a}_{11}=\bar{A}_{11}>0$. Consequently, v^0 is a linear function on x_1. Thus, $\psi(x_1,z)$ satisfies the homogeneous problem with respect to the variable z and $\psi=\psi(x_1)$.

Let us return to problem (6.4). We have shown that the expansion (6.9) takes the form:

$$\tilde{v}^\varepsilon=v^0(x_1)+\varepsilon\,\chi_1\left(\frac{x_1}{\varepsilon},x'\right)\partial_{x_1}v^0+\varepsilon^2\psi(x_1)$$

with $v^0(x_1)=1-x_1$ and $\chi(z)$ solving problem (6.10)-(6.11). Denote

$$v_1^\varepsilon=v^0(x_1)+\varepsilon\,\chi_1\left(\frac{x_1}{\varepsilon},x'\right)\partial_{x_1}v^0.$$

One can easily check that the difference $(v^\varepsilon-v_1^\varepsilon)$ satisfies the following problem:

$$\begin{cases} -\mathrm{div}(a^\varepsilon(x)\nabla\,(v^\varepsilon-v_1^\varepsilon))-(b^\varepsilon,\nabla\,(v^\varepsilon-v_1^\varepsilon))=0,\quad x\in G_0^1,\\ (v^\varepsilon-v_1^\varepsilon)(0,x')=\varepsilon\chi_1(0,x'),\quad x'\in Q,\\ (v^\varepsilon-v_1^\varepsilon)(1,x')=-\varepsilon\chi_1(1/\varepsilon,x'),\quad x'\in Q,\\ \dfrac{\partial(v^\varepsilon-v_1^\varepsilon)}{\partial n_{a^\varepsilon}}=0,\quad x\in\Sigma_0^1. \end{cases}$$

We can rewrite problem for the function χ_1 in the form

$$(6.18)\qquad \begin{cases} \sum\limits_{i,k}\partial_{z_k}\left(a_{ik}\partial_{z_i}\left(\chi_1+z_1\right)\right)+\sum\limits_{k}b_k\,\partial_{z_k}\left(\chi_1+z_1\right)=0,\quad z\in Y,\\ \sum\limits_{i,k}a_{ik}\partial_{z_k}\left(\chi_1+z_1\right)n_i=0,\quad z\in\partial Y. \end{cases}$$

By means of the extension techniques in the same way as above, one can show that (χ_1+z_1) is a Hölder-continuous function in $[0,1]\times\overline{Q}$. Consequently,

$$\|v^\varepsilon-v_1^\varepsilon\|_{L^\infty(S_0)}\leq C\varepsilon$$

and

$$\|v^\varepsilon - v_1^\varepsilon\|_{L^\infty(S_1)} \le C\varepsilon,$$

and by the maximum principle

$$\|v^\varepsilon - v_1^\varepsilon\|_{L^\infty(G_0^1)} \le C\varepsilon.$$

$\square$

The following statement characterizes in more general situation the asymptotic behaviour of solutions of auxiliary problems in finite cylinders in the three cases $\bar{b}_1 \lessgtr 0$ and $\bar{b}_1 = 0$. Namely, we consider the following boundary value problem in the finite cylinder G_0^k:

$$\begin{cases} -\mathrm{div}\,\big(a(x)\,\nabla\, v^k\big) - (b(x), \nabla\, v^k) = 0, & x \in G_0^k, \\ \dfrac{\partial v^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ v^k(0, x') = \varphi(x'), \quad v^k(k, x') = M & x' \in Q, \end{cases} \tag{6.19}$$

where $\varphi(x') \in L^\infty(Q)$, M is a constant.

THEOREM 6.7. *Let the assumptions of Theorem 6.1 be fulfilled. Then for the solution* v^k *of problem* (6.19) *the following statements hold:*

(1) *If* $\bar{b}_1 > 0$ *then*

$$\begin{aligned} |v^k - C_\varphi^\infty| \le C_0\,\|\varphi\|_{L^\infty(Q)}\,\big(e^{-\gamma_0 x_1} + e^{-\gamma(k-x_1)}\big) + \\ + C\,M\,e^{-\gamma(k-x_1)}; \end{aligned} \tag{6.20}$$

(2) *If* $\bar{b}_1 < 0$ *then*

$$|v^k - M| \le C_0\,\|\varphi\|_{L^\infty(Q)}\,e^{-\gamma x_1} + C\,M\,e^{-\gamma x_1}; \tag{6.21}$$

(3) *If* $\bar{b}_1 = 0$ *then in* G_0^k *the function* v^k *is close to a linear function:*

$$\begin{aligned} \left| v^k - \frac{C_\varphi^\infty(k - x_1) + M x_1}{k} \right| \le C_0\,\|\varphi\|_{L^\infty(Q)}\,e^{-\gamma_0 x_1} + \\ + \frac{C}{k}\,\big(\|\varphi\|_{L^\infty(Q)} + M\big)\,. \end{aligned} \tag{6.22}$$

The constant C_φ^∞ *is uniquely determined by Lemma 5.2.*

PROOF. Due to Lemma 5.2, if $\varphi \in L^\infty(Q)$ then there always exists a solution $u_0(x)$ of problem (2.1) satisfying the maximum principle. Moreover, such a solution is unique and stabilizes to a constant C_φ^∞ exponentially:

$$|u_0 - C_\varphi^\infty| \le C_0\,\|\varphi\|_{L^\infty(Q)}\,e^{-\gamma_0 x_1}, \quad \gamma_0 > 0,\ x_1 > 1.$$

Recall that γ_0 depends only on Λ, d, Q and does not depend on $\bar{b}_1$.

We represent the solution v^k of problem (6.19) as a sum $v_1^k + v_2^k$, where v_1^k and v_2^k solve the following problems:

$$
\begin{cases}
-\mathrm{div}\,\big(a(x)\,\nabla\, v_1^k\big) - (b(x), \nabla\, v_1^k) = 0, & x \in G_0^k, \\
\dfrac{\partial v_1^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\
v_1^k(0, x') = \varphi(x'), \quad v_1^k(k, x') = C_\varphi^\infty & x' \in Q;
\end{cases}
\tag{6.23}
$$

$$
\begin{cases}
-\mathrm{div}\,\big(a(x)\,\nabla\, v_2^k\big) - (b(x), \nabla\, v_2^k) = 0, & x \in G_0^k, \\
\dfrac{\partial v_2^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\
v_2^k(0, x') = 0, \quad v_2^k(k, x') = -C_\varphi^\infty + M & x' \in Q.
\end{cases}
\tag{6.24}
$$

One can see that, due to the maximum principle, the difference $(u_0 - v_1^k)$ is of order $e^{-\gamma_0 k}$ everywhere in G_0^k, and, consequently,

$$
|v_1^k - C_\varphi^\infty| \le |v_1^k - u_0| + |u_0 - C_\varphi^\infty| \le C_0 \|\varphi\|_{L^\infty(Q)} \left(e^{-\gamma_0 k} + e^{-\gamma_0 x_1}\right), \ x \in G_0^k. \tag{6.25}
$$

- Assume that $\bar{b}_1 > 0$. By Lemma 6.5 v_2^k satisfies the estimate
$$
|v_2^k| \le C_0 (C_\varphi^\infty + |M|)\, e^{-\gamma(k - x_1)}, \quad x \in G_0^k.
$$
Combining the last estimate and (6.25) and taking into account the bound $C_\varphi^\infty \le \|\varphi\|_{L^\infty(Q)}$, we obtain (6.20).
- If $\bar{b}_1 < 0$ then the solution u^k of problem (6.4) decays exponentially, which leads to the estimate for $v_2^k(x)$
$$
|v_2^k - (M - C_\varphi^\infty)| \le C_0 \left(\|\varphi\|_{L^\infty(Q)} + |M|\right)\, e^{-\gamma_0 x_1}, \quad x \in G_0^k,
$$
which proves (6.21).
- In the case $\bar{b}_1 = 0$ to estimate $v_2^k(x)$ we make use of Lemma 6.6. Namely, v_2^k is close to a linear function in this case:
$$
\left| v_2^k - \frac{M - C_\varphi^\infty}{k} x_1 \right| \le \frac{C}{k} \left(\|\varphi\|_{L^\infty(Q)} + |M|\right).
$$

The last estimate and (6.25) implies (6.22). Theorem 6.7 is proved.

$\square$

7. Equivalent definitions of a bounded solution

LEMMA 7.1. *For a solution of problem* (2.1) *the following conditions are equivalent:*

(i)
$$
\|u\|_{L^2(G_N^{N+1})}\, dx \le C, \quad \forall N \ge 0,
$$

where C does not depend on $N \in [0, \infty)$;

(ii) $$\|u\|_{L^\infty(G_1^\infty)} < \infty;$$

(iii) $$\|\nabla\, u\|_{L^2(G)} < \infty.$$

PROOF. $(i) \to (ii)$
Under assumptions of uniform ellipticity of matrix $a(x)$ (2.2) and boundedness of the coefficients, for any compact set G' in $(N, N+1) \times \overline{Q}$ the generalized solution of problem (2.1) satisfies the following estimate:

$$\|u\|_{C^\alpha(G')} \le C\|u\|_{L^2(G_N^{N+1})},$$

for some constants C and $\alpha > 0$ independent of N and, consequently

$$\|u\|_{C^\alpha(G')} \le C,$$

for any compact set G' in G_N^{N+1} with C independent of N. Thus,

$$\|u\|_{L^\infty(G_\delta^\infty)} \le C(\delta) < \infty, \quad \delta > 0.$$

$(i) \to (iii)$
In Section 5 we proved that any bounded solution $u(x)$ stabilizes to a constant C_∞ at the exponential rate with large axial distance. Then the function $(u(x) - C_\infty)$ solves problem (2.1) with boundary condition $(u - C_\infty)(0, x') = (\varphi(x') - C_\infty)$ and vanishes at infinity at the exponential rate. Extending $u(x)$ to a larger domain (as in Section 3) and applying standard elliptic estimates to $(u(x) - C_\infty)$ one deduces that

$$\|\nabla(u - C_\infty)\|_{L^2(G_N^{N+1})} \le C\|u - C_\infty\|_{L^2(G_{N-1}^{N+2})} \le Ce^{-\gamma N},$$

where the constant C does not depend on N. Thus $\nabla\, u(x)$ stabilizes to zero at the exponential rate, as $x_1 \to \infty$, and

$$\int\limits_G |\nabla\, u|^2\, dx = \sum_{N=0}^{\infty} \int\limits_{G_N^{N+1}} |\nabla\, u|^2\, dx \le C_0 \sum_{N=0}^{\infty} e^{-\gamma N} \le C.$$

$(iii) \to (i)$
Let $u(x)$ be a solution of problem (2.1) such that

(7.1) $$\|\nabla\, u\|_{L^2(G)} \le C.$$

The Friedrichs inequality gives an estimate for the L^2-norm of $u(x)$ in the finite cylinder G_N^{N+1}:

$$\|u\|^2_{L^2(G_N^{N+1})} \le C_1 + C_2 N$$

with constants $C_1 = C_1(\varphi)$ and C_2 independent on N. Note that if $0 \le N \le 1$ then

$$\|u\|_{L^2(G_N^{N+1})} \le C;$$

below we suppose that $N \ge 1$.

Let $v(x)$ be a bounded solution of problem (2.1) (it exists by Lemma 4.2). Notice that the difference $(u-v)$ satisfies the estimates

$$\|u-v\|^2_{L^2(G_N^{N+1})} \le C_1 + C_2\, N, \quad \|\nabla\,(u-v)\|^2_{L^2(G)} \le C.$$

If we denote

$$w_N = \frac{1}{\sqrt{N}}(u-v), \tag{7.2}$$

then

$$\|w_N\|^2_{L^2(G_N^{N+1})} \le \frac{C_1}{N} + C_2, \quad \|\nabla\, w_N\|^2_{L^2(G)} \le \frac{C}{N}.$$

Since w_N is a solution of problem (2.1) with zero boundary condition on the base of G, then the last estimates imply

$$\|w_N\|_{L^\infty(G_0^{N+1})} \le \bar{w},$$

with $\bar{w}$ independent of N. By the maximum principle $|w_N|$ does not exceed the solution v_N of the following problem:

$$\begin{cases} -\mathrm{div}\,(a\nabla\, v_N) - (b, \nabla\, v_N) = 0, \quad x \in G_0^{N+1}, \\ \dfrac{\partial v_N}{\partial n_a} = 0, \quad x \in \Sigma_0^{N+1}, \\ v_N(0,x') = 0, \quad v_N(N+1,x') = \bar{w}, \quad x' \in Q. \end{cases} \tag{7.3}$$

We will consider separately the cases $\bar{b}_1 > 0$, $\bar{b}_1 = 0$ and $\bar{b}_1 < 0$.

Let first $\bar{b}_1 > 0$. From Theorem 6.7 for N large enough we conclude that the function w_N is close to zero for $x_1 < N/2$:

$$|w_N| \le v_N \le C_0\, e^{-\gamma N}, \quad x_1 < N/2.$$

Therefore, considering the definition of w_N (see (7.2)) one obtain the following estimate for the difference $(u-v)$:

$$|u-v| \le C_0\sqrt{N}e^{-\gamma N} \to 0, \quad N \to \infty,$$

which implies that $u = v$ and thus

$$\|u\|_{L^2(G_N^{N+1})} \le C.$$

Consider the case $\bar{b}_1 = 0$. As was proved in Lemma 6.5, in this case a solution of problem (7.3) is close in the cylinder G_0^{N+1} to the linear function, namely

$$\|v_N - x_1\, \frac{\bar{w}}{(N+1)}\|_{L^\infty(G_0^{N+1})} \le \frac{C}{N+1}, \quad x \in G_0^{N+1}.$$

Consequently

$$|u-v| \le \frac{C\sqrt{N}}{N+1} + x_1\, \frac{\bar{w}}{(N+1)}\, \sqrt{N}, \quad \forall N > 0.$$

For $x_1 < N^\alpha$, $\alpha < 1/2$, we obtain that

$$|u-v| \to 0, \quad N \to \infty,$$

thus $u(x) = v(x)$ is a bounded solution.

Finally, let us consider the case $\bar{b}_1 < 0$. As was discussed in Section 5, due to the maximum principle, either $m(x_1)$ increases or $M(x_1)$ decreases in the neighbourhood of infinity. Suppose that

$$\min_{x' \in Q} u \to \infty, \quad x_1 \to \infty,$$

the case of decreasing $M(x_1)$ can be studied in a similar way. Subtracting from $u(x)$ a bounded solution $v(x)$ of problem (2.1) with $v(0, x') = u(0, x')$, one can assume without loss of generality that $u(0, x') = 0$. Then $u(N, x')/m(N)$ will be greater than or equal to 1. Let us introduce a function v_N as a solution to the problem

$$\tag{7.4} \begin{cases} -\mathrm{div}\,(a\nabla\, v_N) - (b, \nabla\, v_N) = 0, \quad x \in G_0^N, \\ \dfrac{\partial v_N}{\partial n_a} = 0, \quad x \in \Sigma_0^N, \\ v_N(0, x') = 0, \quad v_N(N, x') = 1, \quad x' \in Q. \end{cases}$$

By the maximum principle $u(x)/m(N) \geq v_N$. As was shown in Theorem 6.1, v_N satisfies the estimate

$$|v_N - 1| \leq C_0\, e^{-\gamma x_1}, \quad x_1 > 1.$$

Thus

$$u(x) - m(N) \geq -C_0 m(N) e^{-\gamma x_1}.$$

Let $\bar{x}_1 = \frac{1}{\gamma}\ln(2C_0)$, then for any $x_1 > \bar{x}_1$ the following estimate holds:

$$C_0 e^{-\gamma x_1} < \frac{1}{2}.$$

Then $v_N > 1/2$ and, consequently,

$$u(\bar{x}_1, x') \geq \frac{1}{2} m(N), \quad x' \in Q.$$

From the last estimate, using Friedrichs inequality, we obtain

$$\|\nabla\, u\|_{L^2(G_0^{\bar{x}_1})} \geq \frac{1}{4\bar{x}_1}\, m^2(N) \to \infty, \quad N \to \infty,$$

that contradicts (iii). Lemma 7.1 is proved. □

8. Inhomogeneous problem with periodic coefficients

We proceed with studying the existence and the stabilization to a constant of a solution to the following boundary value problem:

$$\tag{8.1} \begin{cases} -\mathrm{div}\,(a(x)\nabla\, u) - (b(x),\, \nabla\, u) = f(x) + \mathrm{div}\, F, & x \in G, \\ \dfrac{\partial u}{\partial n_a} = g(x) - (F, n), & x \in \Sigma, \\ u(0, x') = 0, & x' \in Q. \end{cases}$$

Here the assumptions on the coefficients $a_{ij}(x)$, $b_j(x)$ and the cylinder G are the same as in the previous sections. Concerning the functions f, F and g we suppose that $f(x) \in L^2(G)$, $F \in (L^2(G))^d$, $g(x) \in L^2(\Sigma)$, and that these functions decay exponentially as x_1 goes to infinity, i.e.

$$\|f\|_{L^2(G_N^{N+1})} \le C_1\, e^{-\gamma_1 N}, \quad \|F\|_{L^2(G_N^{N+1})} \le C_1\, e^{-\gamma_1 N}, \qquad (8.2)$$
$$\|g\|_{L^2(\Sigma_N^{N+1})} \le C_2\, e^{-\gamma_1 N}$$

for some positive γ_1.

DEFINITION 8.1. We say that $u(x) \in H^1_{loc}(G)$ is a weak solution to problem (8.1) if, for any $\psi(x) \in C_0^\infty((0,\infty); C^\infty(\bar{Q}))$, the following integral equality holds:

$$\int\limits_G (a(x)\nabla u, \nabla\psi)\, dx - \int\limits_\Sigma g(x)\psi(x)\, d\sigma - \int\limits_G (b, \nabla u)\, \psi(x)\, dx = -\int\limits_G (F, \nabla\psi)\, dx.$$

We begin with the case $F = 0$. In this case we can use the integration by parts technic in the weighted space with the weight $p(x)$, as we did above. It turns out that this technic might fail to work if F is not equal to zero. That is why we consider the case of nonzero F separately and reduce it to the case $F = 0$.

LEMMA 8.2. *Let $F = 0$. Then there exists a solution $u(x)$ of problem (8.1), which stabilizes to a constant at the exponential rate, as $x_1 \to \infty$, and satisfies the estimates*

$$\|\nabla u\|_{L^2(G)} \le C(\|(1 + (x_1)^{1+\nu})\, f\|_{L^2(G)} + \|(1 + (x_1)^{1+\nu})\, g\|_{L^2(\Sigma)}); \qquad (8.3)$$

$$\|u\|_{L^2(G_N^{N+1})} \le C(\|(1 + (x_1)^{\frac{3}{2}+\nu})\, f\|_{L^2(G)} + \|(1 + (x_1)^{\frac{3}{2}+\nu})\, g\|_{L^2(\Sigma)}), \ \forall N \ge 0, \qquad (8.4)$$

where $\nu > 0$.

PROOF. Let us consider the sequence of auxiliary problems

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, u_m^k) - (b(x),\, \nabla\, u_m^k) = f_m(x), & x \in G_0^k, \\ \dfrac{\partial u_m^k}{\partial n_a} = g_m(x), & x \in \Sigma_0^k, \\ u_m^k(0, x') = 0, \quad u_m^k(k, x') = 0, & x' \in Q. \end{cases} \qquad (8.5)$$

Here $f_m(x) = f(x)\chi(G_m^{m+1})$ and $g_m(x) = g(x)\chi(G_m^{m+1})$, $\chi(G_\alpha^\beta)$ is a characteristic function of G_α^β. Multiplying the first equation of (8.5) by the product $p(x)u_m^k(x)$, integrating by parts over G_0^k and using boundary conditions for u_m^k, we obtain

$$\int\limits_{G_0^k} (a(x)\nabla\, u_m^k, \nabla\, u_m^k)\, p\, dx - \int\limits_{\Sigma_m^{m+1}} g_m(x)\, u_m^k\, p\, d\sigma = \int\limits_{G_m^{m+1}} f_m(x) u_m^k\, p\, dx. \qquad (8.6)$$

Let us estimate the integral on the right-hand side. Using the boundedness of $p(x)$ and Schwartz inequality one has

$$\left|\int\limits_{G_0^k} f_m(x)\, u_m^k\, p\, dx\right| \le C\|f_m\|_{L^2(G_m^{m+1})}\, \|u_m^k\|_{L^2(G_m^{m+1})}.$$

The Friedrichs inequality yields

$$\int\limits_{G_m^{m+1}} (u_m^k)^2\, dx \le (m+1) \int\limits_{G_0^k} |\nabla\, u_m^k|^2\, dx, \tag{8.7}$$

and, finally,

$$\left|\int\limits_{G_0^k} f_m(x)\, u_m^k\, p\, dx\right| \;\le\; C\, \|\nabla u_m^k\|_{L^2(G_0^k)}(1+\sqrt{m})\, \|f_m\|_{L^2(G_m^{m+1})}.$$

Using analogous arguments one can estimate the integral over the lateral boundary of the cylinder:

$$\left|\int\limits_{\Sigma_m^{m+1}} g_m(x)\, u_m^k\, p(x)\, d\sigma\right| \le C\,(1+\sqrt{m})\, \|g_m\|_{L^2(\Sigma_m^{m+1})}\, \|\nabla\, u_m^k\|_{L^2(G_0^k)}.$$

Combining the above bounds with the integral identity (8.6), we conclude that

$$\|\nabla u_m^k\|_{L^2(G_0^k)} \le C\,(1+\sqrt{m})\left(\|f_m\|_{L^2(G_m^{m+1})} + \|g_m\|_{L^2(\Sigma_m^{m+1})}\right), \tag{8.8}$$

where the constant C does not depend on m, k. Estimate (8.7) implies that the L^2-norm of the function u_m^k is uniformly in k bounded on each G_N^{N+1} for all $N \le m$:

$$\|u_m^k\|_{L^2(G_N^{N+1})} \le C\,(1+m)\left(\|f_m\|_{L^2(G_m^{m+1})} + \|g_m\|_{L^2(\Sigma_m^{m+1})}\right).$$

In the cylinder G_{m+1}^k the function u_m^k satisfies homogeneous equation and

$$\begin{aligned}\|u_m^k\|_{H^{1/2}(S_{m+1})} &\le C\|u_m^k\|_{H^1(G_m^{m+1})} \le \\ &\le C(1+m)\left(\|f_m\|_{L^2(G_m^{m+1})} + \|g_m\|_{L^2(\Sigma_m^{m+1})}\right).\end{aligned}$$

For $u_m^k(x)$ estimate (4.4) obtained while proving Theorem 6.1 takes the form

$$\|u_m^k\|_{L^\infty(G_{m+1}^k)} \le \|u_m^k\|_{H^{1/2}(S_{m+1})}.$$

Consequently, the following inequality holds

$$\|u_m^k\|_{L^2(G_N^{N+1})} \le C\,(1+m)\left(\|f_m\|_{L^2(G_m^{m+1})} + \|g_m\|_{L^2(\Sigma_m^{m+1})}\right) \tag{8.9}$$

for $N \ge 0$ with the constant C independent of k and m.

Since $f(x) = \sum_0^{k-1} f_m$ and $g(x) = \sum_0^{k-1} g_m$, then $u^k = \sum_0^{k-1} u_m^k$ is a solution of problem

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, u^k) - (b(x),\, \nabla\, u^k) = f(x), & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = g(x), & x \in \Sigma_0^k, \\ u^k(0,x') = 0, \quad u^k(k,x') = 0, & x' \in Q, \end{cases}$$

and, in view of (8.8) – (8.9), satisfies the following estimates:

$$\|u^k\|_{L^2(G_N^{N+1})} \le C\, \Big(\|(1+(x_1)^{\frac{3}{2}+\nu})f\|_{L^2(G)} + \|(1+(x_1)^{\frac{3}{2}+\nu})g\|_{L^2(\Sigma)}\Big)\,, \tag{8.10}$$

$$\|\nabla u^k\|_{L^2(G_0^k)} \le C\, \left(\|(1+(x_1)^{1+\nu})f\|_{L^2(G)} + \|(1+(x_1)^{1+\nu})g\|_{L^2(\Sigma)}\right) \tag{8.11}$$

with C independent of k, $\nu > 0$. Hence, up to a subsequence, u^k converges weakly in the space $H^1_{loc}(G)$, as $k \to \infty$, to a function $u(x)$ which satisfies (8.3) and (8.4). We will prove the exponential stabilization to a constant only in the case $f, g = 0$, $F \ne 0$. This proof can be extended to the case of nontrivial f and g with minor modifications. We leave it to the reader. $\square$

REMARK 8.3. Estimate (8.11) can be improved. To this end we recall the classical Hardy inequality for a nonnegative function $v(t)$, $v(0) = 0$.

$$\int_0^\infty \Big(\frac{1}{t}\int_0^t v(\tau)\,d\tau\Big)^2\,dt \le 4\int_0^\infty v(t)^2\,dt.$$

Then, multiplying the equation for u^k by $p\,u^k$ and integrating by parts we get

$$\int_{G_0^k} (a\nabla\, u^k, \nabla\, u^k)\,p\,dx = \int_{G_0^k} f u^k\,p\,dx + \int_{\Sigma_0^k} g\,u^k\,p\,d\sigma. \tag{8.12}$$

Using the ellipticity of the matrix $a(x)$, positiveness of $p(x)$ and the Hardy inequality one gets

$$\begin{aligned} \int_{G_0^k} f u^k\,p\,dx &\le C \int_Q dx' \int_0^k x_1\,|f|\,\frac{1}{x_1}\,|u^k|\,dx \\ &\le C \int_Q dx' \Big(\int_0^k x_1^2\,|f|^2\,dx_1\Big)^{1/2} \Big(4\int_0^\infty |\nabla u^k|^2\,dx_1\Big)^{1/2} \\ &\le C\,\|x_1\,f\|_{L^2(G_0^k)}\,\|\nabla u^k\|_{L^2(G_0^k)}. \end{aligned} \tag{8.13}$$

Then by the interpolation inequality, for any $\beta > 0$,

$$\int\limits_{\Sigma_0^k} g\, u^k\, p\, d\sigma \le \sum_{n=0}^{k-1} \|g\|_{L^2(\Sigma_n^{n+1})} \Big[\beta \|u^k\|_{L^2(G_n^{n+1})} + \beta^{-1}\|\nabla u^k\|_{L^2(G_n^{n+1})}\Big]$$
$$\le \|g\|^2_{L^2(\Sigma_0^k)} + \beta^{-2}\|\nabla u^k\|^2_{L^2(G_0^k)}$$
$$+ C\,\beta \sum_{n=0}^{k-1} \|x_1 g\|_{L^2(\Sigma_n^{n+1})}\, \|x_1^{-1} u^k\|_{L^2(G_n^{n+1})}$$
$$\le \|g\|^2_{L^2(\Sigma_0^k)} + \beta^{-2}\|\nabla u^k\|^2_{L^2(G_0^k)} + C\,\beta \|x_1 g\|_{L^2(\Sigma_0^k)}\, \|x_1^{-1} u^k\|_{L^2(G_0^k)}.$$

Finally, by the Hardy inequality,

$$\int\limits_{\Sigma_0^k} g\, u^k\, p\, d\sigma \le \|g\|^2_{L^2(\Sigma_0^k)} + \beta^{-2}\|\nabla u^k\|^2_{L^2(G_0^k)} + C\,\beta \|x_1 g\|_{L^2(\Sigma_0^k)}\, \|\nabla u^k\|_{L^2(G_0^k)}. \tag{8.14}$$

Taking into account (8.12), (8.13), (8.14) and setting $\beta = \sqrt{2/\Lambda}$, one gets the estimate for ∇u^k

$$\|\nabla u^k\|_{L^2(G_0^k)} \le C\, \|x_1\, f\|_{L^2(G)} + \|(1 + x_1) g\|_{L^2(\Sigma)},$$

and, consequently,

$$\|\nabla u\|_{L^2(G)} \le C\, \|x_1\, f\|_{L^2(G)} + \|(1 + x_1) g\|_{L^2(\Sigma)}.$$

It remains to consider problem (8.1) with a non-trivial F and with $f = g = 0$.

LEMMA 8.4. *There exists a solution of problem* (8.1) *with* $f = g = 0$*, which satisfies the estimates*

$$\begin{aligned} \|\nabla\, u\|_{L^2(G)} &\le C\, \|(1 + x_1^{\frac{3}{2}+\nu}) F\|_{L^2(G)}, \\ \|u\|_{L^2(G_N^{N+1})} &\le C\, \|(1 + x_1^{\frac{3}{2}+\nu}) F\|_{L^2(G)}, \quad \forall N \ge 0, \end{aligned} \tag{8.15}$$

with some positive ν*. This solution stabilizes to a constant at the exponential rate, as* $x_1 \to \infty$.

PROOF. Consider the sequence of auxiliary problems:

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, u^k) - (b(x),\, \nabla\, u^k) = \mathrm{div} F, & x \in G_0^k, \\ \dfrac{\partial u^k}{\partial n_a} = -(F, n), & x \in \Sigma_0^k, \\ u^k(0, x') = u^k(k, x') = 0, & x' \in Q, \end{cases} \tag{8.16}$$

Let us represent the function F in G_0^k as follows:

$$F(x) = \sum_{m=0}^{M} \chi(G_{m\tau}^{(m+1)\tau})\, F(x) \equiv \sum_{m=0}^{M} F^m(x),$$

where $\chi(G_{m\tau}^{(m+1)\tau})(x)$ is a characteristic function of the domain $G_{m\tau}^{(m+1)\tau}$,

$$\tilde{\tau} = \Lambda^4,\ M = \left[\frac{k}{\tilde{\tau}}\right],\ \tau = \frac{k}{M+1}.$$

Clearly, $\mathrm{supp}F^m \subset G_{m\tau}^{(m+1)\tau}$. Due to the linearity of the studied problem, we can represent a solution $u^k(x)$ of (8.16) as the sum $\sum_{m=0}^{M} u_m^k(x)$:

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, u_m^k) - (b(x),\,\nabla\, u_m^k) = \mathrm{div}F^m, & x \in G_0^k, \\ \dfrac{\partial u_m^k}{\partial n_a} = -(F^m, n), & x \in \Sigma_0^k, \\ u_m^k(0,x') = u_m^k(k,x') = 0, & x' \in Q. \end{cases} \tag{8.17}$$

We will first assume that the coefficients a_{ij}, b_j are smooth functions.

Our analysis is based on the properties of the Green function $G^k(x,y)$ of problem (8.16):

$$\begin{cases} -\mathrm{div}_x\,(a(x)\nabla_x\, G^k(x,y)) - (b(x),\,\nabla_x\, G^k(x,y)) = \delta(x-y), & x \in G_0^k, \\ \dfrac{\partial G^k(x,y)}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ G^k(0,x',y) = G^k(k,x',y) = 0, & x' \in Q. \end{cases}$$

Due to our assumptions on $a(x)$ and $b(x)$, the Green function $G^k(x,y)$ is well-defined. If we denote by v^k a solution of (8.16) with the function $\chi(G_{m\tau}^{(m+1)\tau})$ on the right-hand side

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla\, v^k) - (b(x),\,\nabla\, u^k) = \chi(G_{m\tau}^{(m+1)\tau}), & x \in G_0^k, \\ \dfrac{\partial v^k}{\partial n_a} = 0, & x \in \Sigma_0^k, \\ v^k(0,x') = v^k(k,x') = 0, & x' \in Q, \end{cases}$$

then

$$v^k(x) = \int\limits_{G_{m\tau}^{(m+1)\tau}} G^k(x,y)\,dy.$$

As was shown in the proof of Lemma 8.2, the functions $v^k(x)$ satisfy the estimate:

$$\|v^k\|_{L^\infty(G_{(m-1)\tau}^{(m+2)\tau})} \le C(m\tau+1),$$

with the constant C which depends only on Λ, d and Q, but does not depend of k. In particular for all $x \in S_{(m-1)\tau} \cup S_{(m+2)\tau}$, since $y \in G_{m\tau}^{(m+1)\tau}$,

$$\int\limits_{G_{m\tau}^{(m+1)\tau}} G^k(x,y)\,dy \le C(1+m\tau).$$

Recalling the fact that $G^k(y,x)$ is the Green function of the adjoint problem, using the mean value theorem and the Harnack inequality for $G^k(x,\cdot)$, one can easily get the following inequality:

$$|Q|\,G^k(x,y) \le \alpha\,|Q|\,G^k(x,y_0) = \alpha \int\limits_{G_{m\tau}^{(m+1)\tau}} G^k(x,y)\,dy \le C(1+m\tau),$$

for all $x \in S_{(m-1)\tau}$, for all $y \in G_{m\tau}^{(m+1)\tau}$ and some $y_0 \in G_{m\tau}^{(m+1)\tau}$. Here $\alpha > 0$ depends only on the ellipticity constant Λ, the dimension d and the domain Q. Similar inequality is also valid for all $x \in S_{(m+2)\tau}$. Then the standard elliptic estimates read

$$\begin{aligned} \|\nabla_y G^k(x,\cdot)\|_{L^2(G_{m\tau}^{(m+1)\tau})} &\le C\|G^k(x,\cdot)\|_{L^2(G_{(m-1/2)\tau}^{(m+3/2)\tau})} \le \\ &\le C(1+m\tau), \quad x \in S_{(m-1)\tau} \cup S_{(m+2)\tau}. \end{aligned} \tag{8.18}$$

Let us emphasize that the constant C in (8.18) depends on Λ, d, Q and does not depend on k.

Now we turn back to problem (8.17). Considering the representation of u_m^k in terms of the Green function, one can see that

$$u_m^k(x) = -\int\limits_{G_{m\tau}^{(m+1)\tau}} (\nabla_y G^k(x,y), F^m(y))\,dy,$$

and, consequently, in view of (8.18),

$$\begin{aligned} \|u_m^k\|_{L^\infty(S_{(m-1)\tau})} &\le \|\nabla_y G^k(x,\cdot)\|_{L^2(G_{m\tau}^{(m+1)\tau})}\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})} \le \\ &\le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}. \end{aligned}$$

Similar estimate is valid for $x \in S_{(m+2)\tau}$:

$$\|u_m^k\|_{L^\infty(S_{(m+2)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

By virtue of the maximum principle, since $u_m^k(0,x') = u_m^k(k,x') = 0$, we have

$$\|u_m^k\|_{L^\infty(G_0^{(m-1)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}; \tag{8.19}$$

$$\|u_m^k\|_{L^\infty(G_{(m+2)\tau}^k)} \le C(1+m\tau)\,\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}. \tag{8.20}$$

In order to estimate the L^2-norms of u_m^k and ∇u_m^k in $G_{(m-1)\tau}^{(m+2)\tau}$, we represent u_m^k as a sum $v_m^k + w_m^k$, where

v_m^k is a solution of homogeneous equation, $v_m^k((m-1)\tau,x') = u_m^k((m-1)\tau,x')$, $v_m^k((m+2)\tau,x') = u_m^k((m+2)\tau,x')$,

w_m^k satisfies the nonhomogeneous equation and zero Dirichlet boundary conditions on $S_{(m-1)\tau}$ and $S_{(m+2)\tau}$.

In view of (8.19), (8.20) and the maximum principle we have

$$\|v_m^k\|_{L^\infty(G_{(m-1)\tau}^{(m+2)\tau})} \le C(1+m\tau)\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

Combining the last estimate with the standard elliptic H^1-estimates in the domain $G^{(m+2)\tau}_{(m-1)\tau}$, and taking into account the fact that the shape of the domain does not depend on m, we conclude that

$$\|v^k_m\|_{H^1(G^{(m+2)\tau}_{(m-1)\tau})} \le C(1+m\tau)\|F^m\|_{L^2(G^{(m+1)\tau}_{m\tau})}.$$

To estimate $w^k_m(x)$ let us multiply the equation by w^k_m and integrate over $G^{(m+2)\tau}_{(m-1)\tau}$. Then exploiting the Friedrichs inequality and taking into account the specific choice of τ, one can see that

$$\|\nabla v^k_m\|_{L^\infty(G^{(m+2)\tau}_{(m-1)\tau})} \le \frac{\Lambda}{2}\|F^m\|_{L^2(G^{(m+1)\tau}_{m\tau})};$$
$$\|v^k_m\|_{L^\infty(G^{(m+2)\tau}_{(m-1)\tau})} \le \frac{\Lambda\,\tau}{2}\|F^m\|_{L^2(G^{(m+1)\tau}_{m\tau})}.$$

Consequently, one has

$$\|u^k_m\|_{H^1(G^{(m+2)\tau}_{(m-1)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G^{(m+1)\tau}_{m\tau})},$$

where C depends only on Λ, d and Q.

Elliptic estimates for u^k_m in $G^{(m-1)\tau}_0$ yield the bound

$$\|\nabla u^k_m\|_{L^2(G^{(m-1)\tau}_0)} \le C(1+m\tau)\,\|F^m\|_{L^2(G^{(m+1)\tau}_{m\tau})}.$$

Since in $G^k_{(m+2)\tau}$ the function u^k_m satisfies the homogeneous equation and homogeneous Neumann boundary conditions on the lateral boundary $\Sigma^k_{(m+2)\tau}$, then inequality (4.10) takes the form

$$\|\nabla u^k_m\|_{L^2(G^k_{(m+2)\tau})} \le C\|u^k_m\|_{H^{1/2}(S_{(m+2)\tau})} \le C(1+m\tau)\,\|F^m\|_{L^2(G^{(m+1)\tau}_{m\tau})}.$$

Thus,

$$\begin{aligned}(8.21)\qquad &\|u^k_m\|_{L^2(G^{N+1}_N)} \le C(1+m\tau)\,\|F^m\|_{L^2(G^{(m+1)\tau}_{m\tau})},\ \forall N\ge 0;\\ &\|\nabla u^k_m\|_{L^2(G^k_0)} \le C(1+m\tau)\,\|F^m\|_{L^2(G^{(m+1)\tau}_{m\tau})}.\end{aligned}$$

And, consequently,

$$\begin{aligned}(8.22)\qquad &\|u^k\|_{L^2(G^{N+1}_N)} \le C\,\|(1+(x_1)^{3/2+\nu})F\|_{L^2(G)},\quad \forall N\ge 0,\\ &\|\nabla u^k\|_{L^2(G^k_0)} \le C\,\|(1+(x_1)^{3/2+\nu})F\|_{L^2(G)},\end{aligned}$$

where the constant C depends on Λ, d and Q.

Using the compactness arguments, we conclude that, along a subsequence, $u^k(x)$ converges weakly in $H^1_{\rm loc}(G)$, as $k\to\infty$, to a function $u(x)$ which solves problem (8.1) and (8.15) holds. This completes the proof of the existence of a bounded solution in the case of smooth coefficients.

In the general case of measurable bounded coefficients a_{ij} and b_j define

$$a_{ij}^\delta(x) = \int\limits_{\mathbb{R}^d} a_{ij}(y)\psi^\delta(x-y)\,dy,$$

$$b_j^\delta(x) = \int\limits_{\mathbb{R}^d} b_j(y)\psi^\delta(x-y)\,dy,$$

where $\psi^\delta(\xi) \in C_0^\infty(\mathbb{R}^d)$ is such that $\psi^\delta(\xi) \geq 0$, $\psi^\delta(-\xi) = \psi^\delta(\xi)$ and $\int_{\mathbb{R}^d} \psi^\delta(\xi)d\xi = 1$. In order to define $a^\delta(x)$ and $b^\delta(x)$ we should extend $a(x)$ and $b(x)$ outside $\mathbb{R}\times Q$ (on $\mathbb{R}_-\times Q$ the coefficients are extended by periodicity). For example, we can set $a(x) = \Lambda I$, I is a unit matrix, $b(x) = \{0, ...0\}$ for $x' \notin Q$. Clearly, the obtained $a(x)$ and $b(x)$ satisfy the same uniform ellipticity and boundedness conditions as before. By construction, a_{ij}^δ converges to a_{ij}, and b_j^δ converges to b_j, as $\delta \to 0$, in $L^p(G_0^k)$, for any $k > 0$ and $p \geq 1$. For the solution $u_\delta^k(x)$ of problem (8.16) with smoothed coefficients a_{ij}^δ, b_j^δ the following bounds are valid:

$$\|u_\delta^k\|_{L^2(G_N^{N+1})} \leq C\,\|(1+(x_1)^{3/2+\nu})F\|_{L^2(G)}, \quad \forall N \geq 0,$$

$$\|\nabla u_\delta^k\|_{L^2(G_0^k)} \leq C\,\|(1+(x_1)^{3/2+\nu})F\|_{L^2(G)},$$

with C independent of δ. Thus, up to a subsequence, $u_\delta^k \to u^k$ in $L^2(G_N^{N+1})$, $\nabla u_\delta^k \rightharpoonup \nabla u^k$ in $L^2(G_0^k)$, as $\delta \to 0$, where u^k solves problem (8.16) with measurable bounded coefficients. Clearly, $u^k(x)$ satisfies the estimates

$$\|u^k\|_{L^2(G_N^{N+1})} = \lim_{\delta\to 0} \|u_\delta^k\|_{L^2(G_N^{N+1})} \leq C\|(1+(x_1)^{3/2+\nu})F\|_{L^2(G)},$$

$$\|\nabla u^k\|_{L^2(G)} \leq \liminf_{\delta\to 0} \|\nabla u_\delta^k\|_{L^2(G)} \leq C\|(1+(x_1)^{3/2+\nu})F\|_{L^2(G)}.$$

Using the compactness arguments, we conclude that, along a subsequence, u^k converges weakly in $H^1_{loc}(G)$, as $k\to\infty$, to a function $u(x)$ which solves (8.1) with $f = g = 0$, and estimates (8.15) are valid.

It is left to prove the stabilization of $u(x)$ at the exponential rate to a constant. It can be easily seen that along a subsequence the functions $\{u_m^k\}$ constructed above converge weakly in $H^1_{\rm loc}(G)$, as $k\to\infty$, to a function $u_m(x)$ which is a solution to the problem

$$\left\{\begin{array}{ll} -\mathrm{div}\,(a(x)\nabla\, u_m) - (b(x),\, \nabla\, u_m) = \mathrm{div} F^m, & x \in G, \\ \dfrac{\partial u_m}{\partial n_a} = -(F^m, n), & x \in \Sigma, \\ u_m(0, x') = 0, & x' \in Q. \end{array}\right. \tag{8.23}$$

It is clear that $u(x) = \sum_{m=0}^\infty u_m(x)$. With regard to Theorem 6.1, one can see that there exists a constant C_m^∞ such that for a solution $u_m(x)$ of problem (8.23) the following estimate holds:

$$|u_m - C_m^\infty| \leq C_0\|u_m\|_{H^{1/2}(S_{(m+2)\tau})}\, e^{-\gamma(x_1-(m+2)\tau)}, \quad x_1 > (m+2)\tau.$$

Notice that by construction, since $u_m^k(k,x')=0$, $|C_m^\infty|\le\|u_m\|_{H^{1/2}(S_{(m+2)\tau})}$. As was shown above,

$$\|u_m\|_{H^{1/2}(S_{(m+2)\tau})}\le C(1+m\tau)\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}.$$

Thus, for $x_1>(m+2)\tau$

$$|u_m-C_m^\infty|\le C(1+m\tau)\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}\,e^{-\gamma(x_1-(m+2)\tau)}. \tag{8.24}$$

Let us check that $u(x)$ converges to $C^\infty=\sum_{m=0}^\infty C_m^\infty$. To this end we estimate the $L^2(G_{N\tau}^{(N+1)\tau})$-norm of the difference $(u-C^\infty)$:

$$\|u-C^\infty\|_{L^2(G_{N\tau}^{(N+1)\tau})}\le\sum_{m=0}^\infty\|u_m-C_m^\infty\|_{L^2(G_{N\tau}^{(N+1)\tau})}.$$

Splitting the sum into two parts and taking into account (8.2), estimates (8.15) and (8.24), we have

$$\|u-C^\infty\|_{L^2(G_{N\tau}^{(N+1)\tau})}\le\left(\sum_{m=0}^{N-3}+\sum_{m=N-2}^\infty\right)\|u_m-C_m^\infty\|_{L^2(G_{N\tau}^{(N+1)\tau})}\le$$

$$\le C\tau\sum_{m=0}^{N-3}(1+m\tau)e^{-\gamma m\tau}\,e^{-\gamma(N\tau-(m+2)\tau)}+$$

$$+C\sum_{m=N-2}^\infty\left(\|u_m\|_{L^2(G_{N\tau}^{(N+1)\tau})}+|C_m^\infty|\right)\le C\,N^2\,e^{-\gamma N\tau}+$$

$$+C\,e^{-\gamma(N-2)\tau/2}\sum_{m=N-2}^\infty(1+m\tau)\,e^{-\gamma m\tau/2}\le\tilde C\,e^{-\tilde\gamma N\tau},\ N\ge 0.$$

The case of nontrivial f and g in (8.1) can be considered analogously. It suffices to use estimates (8.3) – (8.4) instead of (8.15) and notice that bound (8.24) remains valid if we replace $\|F^m\|_{L^2(G_{m\tau}^{(m+1)\tau})}$ with
$\|f_m\|_{L^2(G_{m\tau}^{(m+1)\tau})}+\|g_m\|_{L^2(\Sigma_{m\tau}^{(m+1)\tau})}$. The rest of the proof is exactly the same as above. Lemma 8.4 is proved. □

As in Section 7 we can define a bounded solution of problem (8.1).

DEFINITION 8.5. We say that a weak solution $u(x)$ of problem (8.1) is bounded if one of the following conditions is fulfilled:

(i) $\|u\|_{L^2(G_N^{N+1})}\le C,\quad\forall N\ge 0,$

(ii) $\|\nabla u\|_{L^2(G)}\le C.$

LEMMA 8.6. *The conditions (i) and (ii) are equivalent.*

PROOF. In view of Lemma 8.2 there exists a solution $v(x)$ of problem (8.1) such that the conditions (i) and (ii) hold. Let us consider the difference $(u(x)-v(x))$. It satisfies the homogeneous problem (2.1) with $\varphi = 0$. But for a solution to problem (2.1) conditions (i)–(ii) are equivalent. Lemma 8.6 is proved. $\square$

The rest of this section is devoted to studying the uniqueness of solution to problem (8.1). The result similar to that of Theorem 6.1 takes place. As before, we denote

$$\bar{b}_1 = \int\limits_{G_0^1} \left(a_{1j}(x) \frac{\partial p(x)}{\partial x_j} - b_1(x) p(x) \right) dx,$$

where the function $p(x)$ was introduced in Section 3.

THEOREM 8.7. (1) *Any bounded solution $u(x)$ of problem (8.1) stabilizes to a constant at the exponential rate as $x_1 \to \infty$, that is*

$$\|u(x) - C_\infty\|_{L^2(G_n^\infty)} \le C_0 \, e^{-\gamma n}, \quad \forall n \ge 0,$$

for some $C_0 > 0$ and $\gamma > 0$;

(2) *$\bar{b}_1 < 0$ if and only if for any $\varphi(x') \in H^{1/2}(Q)$ and for any constant $l \in \mathbb{R}$, there exists a bounded solution $u(x)$ of problem (8.1) that converges to the constant l, as $x_1 \to \infty$;*

(3) *$\bar{b}_1 \ge 0$ if and only if for every boundary condition $\varphi(x')$ there exists a unique constant $m(\varphi)$ such that a bounded solution of problem (8.1) converges to this constant as $x_1 \to \infty$.*

PROOF. The existence of a bounded solution that stabilizes to a constant at the exponential rate was proved in Lemma 8.2. Denote this solution by $u_0(x)$. If $u(x)$ is an arbitrary boundary solution of problem (8.1), then Theorem 6.1 applies to the difference $(u(x) - u_0(x))$ and implies the first statement of Theorem 8.7. In order to obtain the second and the third statements, it suffices to observe that the uniqueness of a bounded solution to problem (8.1) is equivalent to that of problem (2.1). Indeed, if there are two distinct bounded solutions, say u_1 and u_2, of problem (8.1), then the difference $(u_1 - u_2) \ne 0$ is a bounded solution of the homogeneous problem, and thus a bounded solution of homogeneous problem is not unique.

Conversely, if we assume that problem (2.1) with $\varphi = 0$ has two distinct bounded solutions, say v_1 and v_2, then $(u_0 + v_1)$ and $(u_0 + v_2)$ are bounded solutions of (8.1). Theorem is proved. $\square$

9. Non-periodic coefficients

The goal of this section is to generalize the results of Section 6 to the case of the coefficients which stabilize exponentially to a periodic regime. We will consider the following boundary value problem:

$$(9.1)\qquad \begin{cases} -\mathrm{div}\ (\hat a(x)\,\nabla\, u(x)) - (\hat b(x), \nabla\, u(x)) = 0, & x \in G,\\ \dfrac{\partial u}{\partial n_{\hat a}} = 0, & x \in \Sigma,\\ u(0,x') = \varphi(x'), & x' \in Q, \end{cases}$$

where Q is a bounded domain in $\mathbb{R}^{d-1}$ with a sufficiently smooth boundary ∂Q. We suppose that the matrix $\hat a(x)$ and vector $\hat b(x)$ admit the representations

$$\hat a(x) = a(x) + a^\circ(x), \quad \hat b(x) = b(x) + b^\circ(x),$$

where $a(x)$ and $b(x)$ are x_1-periodic, while a°_{ij} and b°_j decay exponentially, that is for almost all $x \in G$

$$(9.2)\qquad |a^\circ_{ij}| \le C_1\, e^{-\gamma_1\, x_1}, \quad |b^\circ_j| \le C_2\, e^{-\gamma_1\, x_1}, \quad \gamma_1 > 0.$$

Moreover, as in the previous sections, we assume that $\hat a(x)$ is a symmetric uniformly elliptic matrix, i.e. there exists a positive constant Λ_1 such that for almost all $x \in \mathbb{R}^d$ the following estimate holds:

$$\Lambda_1\, |\xi|^2 \le \hat a_{ij}(x)\, \xi_i\, \xi_j, \quad \xi \in \mathbb{R}^d,$$

and $\hat a_{ij}(x), \hat b_j \in L^\infty(G)$.

LEMMA 9.1. *Let the above conditions be fulfilled. Then a bounded solution to problem* (9.1) *exists and stabilizes to a constant at the exponential rate. Moreover, the following estimates hold:*

$$(9.3)\qquad \|\nabla\, u\|_{L^2(G)} < \infty, \quad \|u\|_{L^\infty(G_1^\infty)} < \infty.$$

PROOF. To prove the existence of a bounded solution we use the sequence of auxiliary problems in growing finite cylinders:

$$(9.4)\qquad \begin{cases} -\mathrm{div}(\hat a(x)\nabla\, u^k) - (\hat b(x), \nabla\, u^k) = 0, & x \in G_0^k,\\ \dfrac{\partial u^k}{\partial n_{\hat a}} = 0, & x \in \Sigma_0^k,\\ u^k(0,x') = \varphi(x'), \quad u^k(k,x') = 0, & x' \in Q. \end{cases}$$

Let us recall that according to Remark 4.3 for measurable bounded coefficients $\hat a_{ij}$ and $\hat b_j$, not necessary periodic, estimates (4.4) and (4.5) hold true. Using the standard elliptic estimates for u^k, we conclude that

$$\|u^k\|_{H^1(G_N^{N+1})} \le C, \quad \forall N > 0,$$

and thus, along a subsequence, u^k converges weakly in $L^2_{\mathrm{loc}}(G)$ to some function $u \in H^1_{\mathrm{loc}}(G)$, as $k \to \infty$, and ∇u^k converges weakly to ∇u in $L^2_{\mathrm{loc}}(G)$. This allows us to pass to the limit in the integral identity and establish the existence of a bounded solution to problem (9.1) such that

$$\|u\|_{H^1(G_N^{N+1})} \le C, \quad \forall N > 0. \tag{9.5}$$

However, these estimates do not imply the finiteness of $L^2(G)$ norm of ∇u.

We will proceed as follows. First, making use of Theorem 8.7, we will show that a bounded solution to problem (9.1) stabilizes to a constant, and then, with the help of this result, we will obtain an estimate for ∇u.

Obviously, problem (9.1) can be rewritten in the form

$$\begin{cases} -\mathrm{div}(a\nabla\, u) - (b, \nabla\, u) = \mathrm{div}(a^\circ \nabla\, u) + (b^\circ, \nabla\, u), & x \in G, \\ \dfrac{\partial u}{\partial n_a} = -\dfrac{\partial u}{\partial n_{a^\circ}}, & x \in \Sigma, \\ u(0, x') = \varphi(x'), & x' \in Q. \end{cases} \tag{9.6}$$

Consider the following problem in G

$$\begin{cases} -\mathrm{div}(a\nabla\, w) - (b, \nabla\, w) = \mathrm{div}(a^\circ \nabla\, u) + (b^\circ, \nabla\, u), & x \in G, \\ \dfrac{\partial w}{\partial n_a} = -\dfrac{\partial u}{\partial n_{a^\circ}}, & x \in \Sigma, \\ w(0, x') = \varphi(x'), & x' \in Q; \end{cases} \tag{9.7}$$

here w is an unknown function and u is the solution of (9.6). Taking into account (9.5), it is easy to see that under our assumptions on a° and b° all the conditions of Theorem 8.7 are fulfilled, and, therefore, any bounded solution $w(x)$ to problem (9.7) stabilizes to a constant at the exponential rate. Since u is a solution of (9.7), it stabilizes to a constant exponentially. Moreover, the inequality holds

$$\int\limits_G |\nabla u|^2\, dx = \sum_{n=0}^{\infty} \int\limits_{G_n^{n+1}} |\nabla u|^2\, dx \le C_0 \sum_{n=0}^{\infty} e^{-\gamma n} \le C.$$

Lemma 9.1 is proved. □

One of the principal results of this section is given by the following lemma, which states that the uniqueness property is invariant under exponentially decaying perturbations of the coefficients.

LEMMA 9.2.

- $\bar{b}_1 < 0$ *iff for any* $\varphi(x') \in H^{1/2}(Q)$ *and any* $l \in \mathbb{R}^1$ *there exists a bounded solution to problem* (9.1) *stabilizes to* l*, as* $x_1 \to \infty$*;*
- $\bar{b}_1 \ge 0$ *iff for any* $\varphi(x') \in H^{1/2}(Q)$ *there exists a unique bounded solution to problem* (9.1) *and it stabilizes to a constant* $m = m(\varphi)$*, as* $x_1 \to \infty$*.*

PROOF. First, assume that for any $\varphi(x')$ there exists a unique solution to problem (9.1) which stabilizes to a constant $m = m(\varphi)$, as $x_1 \to \infty$. In particular, for $\varphi = 1$ a solution u to problem (9.1) is unique and $u \equiv 1$. This solution can be obtained as the limit of solutions u^k of (9.4). Since $u = 1$, then u^k converges to 1 uniformly on each compact subset of G, as $k \to \infty$. Let us show that in this case $\bar{b}_1 \geq 0$. Multiplying the equation in (9.4) by $u^k p$ and integrating by parts over G_ξ^k we obtain

$$\int\limits_{G_\xi^k} (\hat{a}\nabla u^k, \nabla u^k)\, p\, dx + \int\limits_{G_\xi^k} (a^\circ \nabla u^k, \nabla p)\, u^k\, dx - \int\limits_{G_\xi^k} (b^\circ, \nabla u^k)\, p\, u^k\, dx -$$

$$-\frac{1}{2} \int\limits_{S_\xi} \left(\frac{\partial p}{\partial n_a} - (b,n)p \right) (u^k)^2\, dx' + \int\limits_{S_\xi} \frac{\partial u^k}{\partial n_{\hat{a}}}\, u^k\, p\, dx' = 0. \tag{9.8}$$

The integral containing $a^\circ(x)$, admits the following upper bound

$$\Bigg| \int\limits_{G_\xi^k} (a^\circ \nabla u^k, \nabla p)\, u^k\, dx \Bigg| \leq \sum_{n=\xi}^{k-1} \Bigg| \int\limits_{G_n^{n+1}} (a^\circ \nabla u^k, \nabla p)\, u^k\, dx \Bigg| \leq$$

$$\leq C \sum_{n=\xi}^{k-1} e^{-\gamma_1 n} \|\nabla p\|_{L^2(G_n^{n+1})}.$$

For any $\delta > 0$ we can choose sufficiently large ξ_0 so that for all $\xi > \xi_0$

$$C \sum_{n=\xi}^{k-1} e^{-\gamma_1 n} \|\nabla p\|_{L^2(G_n^{n+1})} < \delta.$$

Similarly, for large enough ξ,

$$\Bigg| \int\limits_{G_\xi^k} (b^\circ, \nabla u^k)\, p\, u^k\, dx \Bigg| < \delta.$$

Taking into account the convergence of u^k to 1, coerciveness of the matrix $\hat{a}$ and the definition of $\bar{b}_1$, we obtain the following inequality:

$$\bar{b}_1 \geq -C\delta, \quad \forall \delta > 0,$$

which implies that $\bar{b}_1 \geq 0$.

Now let us suppose that for any constant there exists a solution of (9.1) converging to this constant. Then for any $\xi \geq 0$ there is a solution $v(x)$ to problem

$$\begin{cases} -\mathrm{div}\ (\hat{a}(x)\, \nabla\, v(x)) - (\hat{b}(x), \nabla\, v(x)) = 0, & x \in G_\xi^\infty, \\ \dfrac{\partial v}{\partial n_{\hat{a}}} = 0, & x \in \Sigma_\xi^\infty, \\ v(\xi, x') = 0, & x' \in Q, \end{cases} \tag{9.9}$$

such that

$$|v-1| \le C(\xi)\, e^{-\gamma(x_1-\xi)}, \quad x_1 > \xi.$$

It is clear that uniformly in ξ for any $n>0$

$$\|v\|_{H^1(G_n^{n+1})} \le C.$$

We rewrite problem (9.9) in the form

$$\left\{\begin{array}{ll} -\mathrm{div}(a\,\nabla\, w) - (b, \nabla\, w) = \mathrm{div}(a^\circ \nabla\, v) + (b^\circ, \nabla\, v), & x \in G_\xi^\infty, \\ \dfrac{\partial w}{\partial n_a} = -\dfrac{\partial v}{\partial n_{a^\circ}}, & x \in \Sigma_\xi^\infty, \\ w(\xi, x') = 0, & x' \in Q; \end{array}\right. \tag{9.10}$$

If we assume that $\bar b_1 \ge 0$, then (9.10) has a unique bounded solution which coincides with $v(x)$ and converges to a constant. The uniqueness of solution allows us to estimate $\|v\|_{L^2(G_n^{n+1})}$ in terms of the norm of the right-hand side:

$$\|v\|_{L^2(G_n^{n+1})} \le C\|(1+(x_1)^{3/2+\nu}) a^\circ \nabla v\|_{L^2(G_\xi^\infty)} \le$$

$$\le C \sum_{N=\xi}^{\infty} \|(1+(x_1)^{3/2+\nu}) a^\circ \nabla v\|_{L^2(G_n^{n+1})} \le C \sum_{N=\xi}^{\infty} (1+N^{3/2+\nu})\, e^{-\gamma_1 N}.$$

For any positive δ, choosing sufficiently large ξ, we obtain

$$\|v\|_{L^2(G_n^{n+1})} < \delta, \quad n > \xi.$$

This contradicts our assumption that v converges to 1. Therefore, $\bar b_1 < 0$. Lemma 9.2 is proved. □

REMARK 9.3. It turns out that in the case when $a^\circ_{ij}(x)$ and $b^\circ_j(x)$ do not decay exponentially, the statements of Lemma 9.1 and 9.2 may fail to hold. To illustrate this, let us consider the following problem

$$\left\{\begin{array}{ll} -\Delta u - b_1^\circ(x_1)\partial_1 u = 0, & x \in G, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \Sigma, \\ u(0, x') = 1, & x' \in Q, \end{array}\right. \tag{9.11}$$

with $b_1^\circ = 2/(1+x_1)$. Observe that, in contrast with the non-perturbed problem, which has a unique solution, problem (9.11) possesses two bounded solutions: $u_1 = 1$ and $u_2 = 1/(1+x_1)$. The last one stabilizes to zero, as $x_1 \to \infty$, but not at the exponential rate.

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Paper B

PAPER B

Homogenization of convection-diffusion equation in thin rod structure

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ABSTRACT. The paper deals with the asymptotic behaviour of solutions to a stationary convection-diffusion equation stated in the so-called rod structure, i.e. in a connected open set consisting of the union of a finite number of thin cylinders. It is assumed that each such a cylinder has a periodic microstructure and that the microstructure period is of the same order as the cylinder diameter. Under these assumptions we construct and justify the asymptotic expansion of a solution which consists of the interior asymptotic expansions and the boundary layers. The boundary layer terms arise both in the vicinity of the rod ends and the vicinity of junctions.

The results on the asymptotic behaviour rely on the qualitative study of a convection-diffusion operator in an infinite cylinder. This study is of independent interest.

1. Introduction

The paper is devoted to homogenization of a stationary convection-diffusion model problem in a thin rod structure. More precisely, we study the asymptotic behaviour of solutions to a boundary-value problem for elliptic convection-diffusion equation defined in a thin cylinder being the union of two nonintersecting cylinders with a junction at the origin. We suppose that in each of these cylinders the coefficients are rapidly oscillating functions being periodic in the axial direction, and that the microstructure period is of the same order as the cylinder diameter.

On the lateral boundary of the cylinder we assume Neumann boundary condition, while at the cylinder bases the Dirichlet boundary conditions are posed.

Similar problems for the elasticity system have been intensively studied in the existing literature. We quote here the works [**4**], [**5**], [**7**], [**8**], [**9**], [**16**], [**15**], [**17**]. The contact problem of two heterogeneous bars was considered in [**10**], [**11**], [**14**]. Elliptic equations in divergence form have been addressed, for example, in [**1**] and [**12**].

In contrast to the divergence form operators, in the case of convection-diffusion equation the asymptotic behaviour of solutions depend crucially on the direction of the so-called effective convection which is introduced in Section 2. In the present paper we only consider the case when in each of the two cylinders (being the constituents of the rod) the effective convection is directed from the end of the cylinder towards the junction.

The asymptotic expansion of a solution includes the interior expansion, the boundary layers in the neighbourhood of the cylinder ends, and the interior boundary layer in the vicinity of junction. It should be noted that the leading term of the asymptotics is described in terms of a pair of first order ordinary differential equations. The construction of the interior expansions follows the classical scheme.

The analysis of boundary layers in the neighbourhood of the cylinder ends relies on the results obtained in [**13**].

In order to build the interior boundary layer we study a qualitative problem for convection-diffusion equation in an infinite cylinder. This is done in Section 6. As far as the present authors are aware, none has studied a convection-diffusion equation with first order terms in an infinite cylinder. In the case under consideration, when in each of the two cylinders the effective convection is directed from the end of the cylinder towards the junction, we prove the existence of a solution for such a problem and discuss its qualitative properties. In other cases the situation is much more difficult (especially in the case when effective convections are oppositely directed) and out of the scope of the present work.

The paper is organized as follows. Section 1 contains the problem statement. In Section 2 we are looking for a formal asymptotic solution to problem (2.1) satisfying the Neumann boundary condition on the lateral boundary of the rod. Then, in Sections 3-4 we construct the boundary layer correctors. Finally, in Section 5 we carry out the justification of the presented formalism and derive the estimates for the difference between the asymptotic solution and the exact one. Section 6 is devoted to an auxiliary problem in an infinite cylinder.

2. Problem statement

Let Q be a bounded $C^{2,\alpha}$ domain in $(d-1)$-dimensional Euclidean space $\mathbb{R}^{d-1}$ with points $x' = (x_2, ..., x_d)$. Denote $G_\varepsilon = [-1,1] \times (\varepsilon Q) \subset \mathbb{R}^d$ a thin rod with the lateral boundary $\Gamma_\varepsilon = [-1,1] \times \partial(\varepsilon Q)$; $x = (x_1, x')$. We study the homogenization of a scalar elliptic equation with periodically oscillating coefficients

$$\tag{2.1} \begin{cases} A^\varepsilon u^\varepsilon \equiv -\mathrm{div}\,\big(a^\varepsilon(x)\nabla\, u^\varepsilon\big) - \dfrac{1}{\varepsilon}\big(b^\varepsilon(x), \nabla u^\varepsilon\big) = \dfrac{1}{\varepsilon}\, f(x_1), & x \in G_\varepsilon, \\ B^\varepsilon u^\varepsilon \equiv \dfrac{\partial u^\varepsilon}{\partial n_{a^\varepsilon}} = g(x_1), & x \in \Gamma_\varepsilon, \\ u^\varepsilon(-1, x') = \varphi^-\big(\dfrac{x'}{\varepsilon}\big), \quad u^\varepsilon(1, x') = \varphi^+\big(\dfrac{x'}{\varepsilon}\big), & x' \in \varepsilon Q, \end{cases}$$

where the matrix valued function $a^\varepsilon(x)$ and the vector field $b^\varepsilon(x)$ are given by

$$a^\varepsilon(x) = a\Big(\frac{x}{\varepsilon}\Big), \quad b^\varepsilon(x) = b\Big(\frac{x}{\varepsilon}\Big),$$

and $\varepsilon > 0$ is a small parameter. In (2.1) $(\cdot,\cdot)$ stands for the standard scalar product in $\mathbb{R}^d$; $\partial u^\varepsilon/\partial n_{a^\varepsilon} = (a^\varepsilon \nabla u^\varepsilon, n)$ is the conormal derivative of u^ε, n is an external unit normal. Throughout the paper we denote

$$\mathbb{G} = (-\infty, +\infty) \times Q, \quad \Gamma = (-\infty, +\infty) \times \partial Q;$$

$$G_\alpha^\beta = (\alpha, \beta) \times Q, \quad -\infty \le \alpha \le \beta \le +\infty.$$

We suppose the following conditions to hold:

(H1) The coefficients $a_{ij}(y) \in C^{1,\alpha}(\mathbb{G})$ and $b_j(y) \in C^\alpha(\mathbb{G})$ are periodic outside some compact set $K \Subset G_{-1}^1$. More precisely,

$$a_{ij}(y) = \begin{cases} a_{ij}^+(y), & y_1 > 1, \\ \tilde{a}_{ij}(y), & |y_1| \le 1, \\ a_{ij}^-(y), & y_1 < -1; \end{cases} \qquad b(y) = \begin{cases} b_j^+(y), & y_1 > 1, \\ \tilde{b}_j(y), & |y_1| \le 1, \\ b_j^-(y), & y_1 < -1; \end{cases}$$

where $a^\pm(y)$ and $b^\pm(y)$ are periodic in y_1. Without loss of generality we assume that the period is equal to 1.

(H2) The matrices $a^\pm(y)$ are symmetric.

(H3) We assume that $a^\pm(y)$ satisfy the uniform ellipticity condition, that is there exists a positive constant Λ such that, for almost all $x \in \mathbb{R}^d$,

$$\Lambda\, |\xi|^2 \le \sum_{i,j=1}^d a_{ij}^\pm(y)\, \xi_i\, \xi_j, \quad \forall \xi \in \mathbb{R}^d. \tag{2.2}$$

(H4) $\varphi^\pm(y') \in H^{1/2}(Q)$;

(H5) Functions $f(x_1)$ and $g(x_1)$ are supposed to be smooth, namely, $f(x_1) \in C^2(G_\varepsilon)$ and $g(x_1) \in C^2(\Gamma_\varepsilon)$.

Later we will see that the obtained result can be generalized to the case when f is just $L^2(G_\varepsilon)$ function and $g \in L^2(\Gamma_\varepsilon)$.

The goal of this work is to study the asymptotic behaviour of $u^\varepsilon(x)$, as $\varepsilon \to 0$.

As was noted in the introduction, in contrast to the case of an operator in divergence form, the situation turns out to depend crucially on the signs of the so-called effective fluxes $\bar{b}_1^\pm$, the constants which are defined in terms of the kernel of the adjoint periodic operators and coefficients of the equation. While constructing boundary layer functions, we consider only one case: $\bar{b}_1^+ < 0$, $\bar{b}_1^- > 0$.

3. Formal asymptotic expansion

In the sequel we use the following notations:

$$G_\varepsilon^+ = \{x = (x_1, x') \in G_\varepsilon : x_1 > \varepsilon\}, \quad G_\varepsilon^- = \{x = (x_1, x') \in G_\varepsilon : x_1 < -\varepsilon\};$$
$$A_y^\pm v \equiv -\mathrm{div_y}\,(a^\pm(y)\nabla_y v) - (b^\pm(y), \nabla_y v), \quad y \in Y;$$
$$B_y^\pm v \equiv \frac{\partial v}{\partial n_{a^\pm}} = \sum_{i,j=1}^{d} a_{ij}^\pm(y)\,\partial_{y_j} v\, n_i, \quad y \in Y,$$

where $Y = \mathfrak{S}_1 \times Q$, $\mathfrak{S}_1$ is a unit circle, denotes the cell of periodicity. In what follows we identify y_1-periodic functions with functions defined on Y. Notice that $\partial Y = \mathfrak{S}_1 \times \partial Q$.

In each half-cylinder G_ε^+ and G_ε^- we will seek a formal asymptotic expansion of a solution to equation (2.1) as an asymptotic series (see, for example, [**1**], [**2**])

$$u_\infty^\pm \sim v_0^\pm(x_1) + \varepsilon\, u_1^\pm(x_1, y) + \varepsilon^2\, u_2^\pm(x_1, y) + \varepsilon^3\, u_3^\pm(x_1, y), \quad y = \frac{x}{\varepsilon},\ x \in G_\varepsilon^\pm, \tag{3.1}$$

where $u_k^\pm(x_1, y)$, $k = 1, 2, 3$, are periodic in y_1 functions with period equal to 1. Substituting (3.1) into (2.1), taking into account the relation

$$\frac{\partial u(x, x/\varepsilon)}{\partial x_i} = \frac{\partial u(x, y)}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial u(x, y)}{\partial y_i}, \quad y = \frac{x}{\varepsilon},$$

and the periodicity of $u_k(x_1, y)$, collecting power–like terms related to ε^{-1} we obtain equations for the functions $u_1^\pm$:

$$A_y^\pm\, u_1^\pm(x_1, y) = \big(\partial_{y_i} a_{i1}^\pm(y) + b_1^\pm(y)\big)\,(v_0^\pm)'(x_1) + f(x_1), \quad y \in Y,\ x \in G_\varepsilon^\pm. \tag{3.2}$$

Similarly, substituting (3.1) into the boundary condition on the lateral boundary and collecting power-like terms related to ε^0, one obtains

$$B_y^\pm\, u_1^\pm(x_1, y) = -a_{i1}^\pm(y) n_i\,(v_0^\pm)'(x_1) + g(x_1), \quad y \in \partial Y,\ x \in G_\varepsilon^\pm. \tag{3.3}$$

The compatibility condition for problem (3.2)–(3.3) takes the form:

$$\begin{aligned}&\int\limits_Y f(x_1) p^\pm(y)\,dy + \int\limits_Y \big(\partial_{y_i} a_{i1}^\pm(y) + b_1^\pm(y)\big)\,(v_0^\pm)'(x_1)\,p^\pm(y)\,dy - \\ &- \int\limits_{\partial Y} a_{i1}^\pm(y)\,n_i\,(v_0^\pm)'(x_1) p^\pm(y)\,d\sigma_y + \int\limits_{\partial Y} g(x_1)\,p^\pm(y)\,d\sigma_y = 0,\end{aligned} \tag{3.4}$$

where $p^\pm(y)$ belong to the kernels of adjoint periodic operators defined on Y:

$$\begin{cases} -\mathrm{div}\,(a^\pm(y)\nabla\, p^\pm) + \mathrm{div}\,(b^\pm\, p^\pm) = 0, & y \in Y, \\ \dfrac{\partial p^\pm}{\partial n_{a^\pm}} - (b^\pm, n)\,p^\pm = 0, & y \in \partial Y. \end{cases}$$

In [**13**] (see Section 3) it was shown that under the assumptions made above such functions exist, they are positive and continuous everywhere in $\overline{Y}$, and can be normalized by

$$\int\limits_Y p^{\pm}(y)\,dy = 1.$$

Integrating by parts in (3.4) gives rise an equation for $v_0^{\pm}(x_1)$

$$\bar{b}_1^{\pm}(v_0^{\pm})'(x_1) = f(x_1) + g(x_1)\int\limits_{\partial Y} p^{\pm}(y)d\sigma_y, \tag{3.5}$$

where

$$\bar{b}_1^{\pm} = \int\limits_Y \left(a_{i1}^{\pm}(y)\partial_{y_i}p^{\pm}(y) - b_1^{\pm}(y)p^{\pm}(y)\right)dy$$

is so-called effective axial drift. Throughout the paper we will assume that

(H6) $\quad \bar{b}_1^{-} > 0 \quad and \quad \bar{b}_1^{+} < 0.$

Notice that since $f(x_1), g(x_1) \in C^2([-1,1])$, then $v_0^{+}(x_1) \in C^3(\varepsilon,1)$, $v_0^{-}(x_1) \in C^3(-1,-\varepsilon)$.

Substituting the expression for $f(x_1)$ from (3.5) into (3.2)–(3.3), we obtain

$$\begin{cases} A_y^{\pm}\, u_1^{\pm}(x_1,y) = \left(\partial_{y_i}a_{i1}^{\pm} + b_1^{\pm} + \bar{b}_1^{\pm}\right)(v_0^{\pm})'(x_1) - \\ \qquad\qquad -g(x_1)\displaystyle\int\limits_{\partial Y} p(y)d\sigma_y, \qquad y \in Y, \\ B_y^{\pm}\, u_1^{\pm} = -a_{i1}^{\pm}(y)\, n_i\, (v_0^{\pm})'(x_1) + g(x_1), \;\; y \in \partial Y. \end{cases}$$

The specific form of the right-hand side suggests the following representation of the function $u_1^{\pm}(x_1,y)$:

$$u_1^{\pm}(x_1,y) = N_1^{\pm}(y)\,(v_0^{\pm})'(x_1) + v_1^{\pm}(x_1) + q_1^{\pm}(y)g(x_1), \tag{3.6}$$

where functions $N_1^{\pm}$, $v_1^{\pm}$ and $q_1^{\pm}$ are to be determined. One can see that necessarily the functions $N_1^{\pm}$ and $q_1^{\pm}$ satisfy the problems

$$\begin{cases} A_y^{\pm}\, N_1^{\pm} = \partial_{y_i}a_{i1}^{\pm} + b_1^{\pm} + \bar{b}_1^{\pm}, & y \in Y, \\ B_y^{\pm}\, N_1^{\pm} = -a_{i1}^{\pm}\, n_i, & y \in \partial Y; \end{cases} \tag{3.7}$$

$$\begin{cases} A_y^{\pm}\, q_1^{\pm} = -\displaystyle\int\limits_{\partial Y} p^{\pm}(y)d\sigma_y, & y \in Y, \\ B_y^{\pm}\, q_1^{\pm} = 1, & y \in \partial Y. \end{cases} \tag{3.8}$$

Obviously, by definition of $\bar{b}_1^{\pm}$, the compatibility conditions for (3.7) and (3.8) are satisfied, thus, these problems are uniquely (up to an additive constant) solvable. Since we assume that $a_{ij}(y) \in C^{1,\alpha}(\overline{\mathbb{G}})$ and $b_j(y) \in C^{\alpha}(\overline{\mathbb{G}})$, then $N_1^{\pm}(y)$ and $q_1^{\pm}(y)$ belong to $C^{2,\alpha}(\overline{Y})$ (see, for example [**3**], [**6**]).

Now we return to ansatz (3.1). As before, substituting (3.1) into (2.1), collecting power-like terms related to ε^0 and taking into account the representation (3.6), one can obtain an equation for $u_2^\pm(x_1,y)$

$$
\begin{aligned}
A_y^\pm\, u_2^\pm &= a_{11}^\pm(y)(v_0^\pm)''(x_1) + \partial_{y_i}(a_{i1}^\pm N_1^\pm)\,(v_0^\pm)''(x_1) + \partial_{y_i}(a_{i1}^\pm q_1^\pm)\,g'(x_1)+ \\
&+\partial_{y_i} a_{i1}^\pm\,(v_1^\pm)'(x_1) + b_1^\pm\,N_1^\pm\,(v_0^\pm)''(x_1) + b_1^\pm\,q_1^\pm\,g'(x_1)+ \\
&+b_1^\pm\,(v_1^\pm)'(x_1) + a_{1j}^\pm\,\partial_{y_j} N_1^\pm\,(v_0^\pm)''(x_1) + a_{1j}^\pm\,\partial_{y_j} q_1^\pm\,g'(x_1).
\end{aligned} \tag{3.9}
$$

Similarly, we get the following boundary condition for $u_2(x_1,y)$ on ∂Y

$$
B_y^\pm\, u_2^\pm = -a_{i1}^\pm\,N_1^\pm\,n_i\,(v_0^\pm)''(x_1) - a_{i1}^\pm\,q_1^\pm\,n_i\,g'(x_1) - a_{i1}^\pm\,n_i\,(v_1^\pm)'(x_1). \tag{3.10}
$$

The compatibility condition for (3.9)–(3.10) reads

$$
\bar{b}_1^\pm\,(v_1^\pm)'(x_1) = h_2^\pm\,(v_0^\pm)''(x_1) + \overline{q_1^\pm}\,g'(x_1), \tag{3.11}
$$

where $h_2^\pm$ and $\overline{q_1^\pm}$ are constants given by the following expressions:

$$
h_2^\pm = \int\limits_Y \big(a_{11}^\pm\,p^\pm - a_{i1}^\pm\,N_1^\pm(y)\partial_{y_i}p^\pm + b_1^\pm\,N_1^\pm\,p^\pm + a_{1j}^\pm\,\partial_{y_j}N_1^\pm\,p^\pm\big)\,dy;
$$

$$
\overline{q_1^\pm} = \int\limits_Y \big(- a_{i1}^\pm\,q_1^\pm\partial_{y_i}p^\pm + b_1^\pm\,q_1^\pm p^\pm + a_{1j}^\pm\,\partial_{y_j}q_1^\pm\,p^\pm\big)\,dy.
$$

Let us note that $v_1^\pm(x_1)$, as a solution of (3.11), has continuous derivatives in $\overline{Y}$ up to the second order.

As before, analyzing the right-hand side of (3.9)-(3.10), we seek a solution in the following form:

$$
u_2^\pm(x_1,y) = N_2^\pm(y)\,(v_0^\pm)''(x_1) + \widetilde{N_1^\pm}(y)\,(v_1^\pm)'(x_1) + q_2^\pm(y)\,g'(x_1) + v_2^\pm(x_1).
$$

Substituting this expression into problem (3.9), (3.10) and taking into account (3.11), we obtain that $\widetilde{N_1^\pm}(y) \equiv N_1^\pm(y)$, $N_2^\pm$ and $q_2^\pm$ satisfy the problems

$$
\begin{cases}
A_y^\pm\,N_2^\pm = a_{11}^\pm + \partial_{y_i}(a_{i1}^\pm\,N_1^\pm) + b_1^\pm\,N_1^\pm + a_{1j}^\pm\partial_{y_j}N_1^\pm - h_2^\pm, & y \in Y, \\
B_y^\pm\,N_2^\pm = -a_{i1}^\pm\,N_1^\pm\,n_i, & y \in \partial Y;
\end{cases} \tag{3.12}
$$

$$
\begin{cases}
A_y^\pm\,q_2^\pm = \partial_{y_i}(a_{i1}^\pm\,q_1^\pm) + b_1^\pm\,q_1^\pm + a_{1j}^\pm\,\partial_{y_j}q_1^\pm - \overline{q_1^\pm}, & y \in Y, \\
B_y^\pm\,q_1^\pm = -a_{i1}^\pm\,q_1^\pm\,n_i, & y \in \partial Y.
\end{cases} \tag{3.13}
$$

The compatibility conditions are satisfied and problems (3.12) – (3.13) are uniquely solvable. Smoothness of the coefficients and properties of the functions $N_1^\pm$, $q_1^\pm$ imply that $N_2^\pm(y), q_2^\pm(y) \in C^{2,\alpha}(\overline{Y})$.

Finally, we obtain

$$
u_2^\pm(x_1,y) = N_2^\pm(y)\,(v_0^\pm)''(x_1) + N_1^\pm(y)\,(v_1^\pm)'(x_1) + q_2^\pm(y)\,g'(x_1) + v_2^\pm(x_1). \tag{3.14}
$$

Similarly, one can see that the problem for $u_3(x_1,y)^{\pm}$ takes the form

(3.15)
$$\begin{gathered}A_y^{\pm}\,u_3^{\pm}=\big[a_{11}^{\pm}\,N_1^{\pm}+\partial_{y_i}(a_{i1}^{\pm}N_2^{\pm})+b_1^{\pm}\,N_2^{\pm}+a_{1j}^{\pm}\,\partial_{y_j}N_2^{\pm}\big]\,(v_0^{\pm})^{(3)}(x_1)+\\+\big[a_{11}^{\pm}+\partial_{y_i}(a_{i1}^{\pm}\,N_1^{\pm})+b_1^{\pm}\,N_1^{\pm}+a_{1j}^{\pm}\,\partial_{y_j}N_1^{\pm}\big]\,(v_1^{\pm})''(x_1)+\\+big[a_{11}^{\pm}\,q_1^{\pm}+\partial_{y_i}(a_{i1}^{\pm}\,q_2^{\pm})+b_1^{\pm}\,q_2^{\pm}+a_{y_j}^{\pm}\,\partial_{y_j}q_2^{\pm}\big]\,g''(x_1)\\+\big[\partial_{y_i}a_{i1}^{\pm}+b_1^{\pm}\big]\,(v_2^{\pm})'(x_1);\end{gathered}$$

(3.16)
$$\begin{gathered}B_y^{\pm}\,u_3^{\pm}=-a_{i1}^{\pm}\,N_2^{\pm}n_i(v_0^{\pm})^{(3)}(x_1)-a_{i1}^{\pm}\,N_1^{\pm}\,n_i\,(v_1^{\pm})''(x_1)-\\-a_{i1}^{\pm}\,q_2^{\pm}\,n_i\,g''(x_1)-a_{i1}^{\pm}\,n_i\,(v_2^{\pm})'(x_1).\end{gathered}$$

The compatibility condition for (3.15)-(3.16) gives an equation for $v_2^{\pm}(x_1)$

(3.17)
$$\bar{b}_1^{\pm}\,(v_2^{\pm})'(x_1)=h_3^{\pm}\,(v_0^{\pm})^{(3)}(x_1)+h_2^{\pm}\,(v_1^{\pm})''(x_1)+\overline{q_2^{\pm}}\,g''(x_1),$$

where

$$\begin{gathered}h_3^{\pm}=\int\limits_Y\big(a_{11}^{\pm}N_1^{\pm}p^{\pm}-a_{i1}^{\pm}N_2^{\pm}\partial_{y_i}p^{\pm}+b_1^{\pm}N_2^{\pm}p^{\pm}+a_{1j}^{\pm}\partial_{y_j}N_2^{\pm}p^{\pm}\big)\,dy;\\ \overline{q_2^{\pm}}=\int\limits_Y\big(a_{11}^{\pm}\,q_1^{\pm}p^{\pm}-a_{i1}^{\pm}q_2^{\pm}\partial_{y_i}p^{\pm}+b_1^{\pm}q_2^{\pm}p^{\pm}+a_{1j}^{\pm}\partial_{y_j}q_2^{\pm}p^{\pm}\big)\,dy.\end{gathered}$$

The function $v_2^{\pm}$ as a solution of (3.17) is a continuously differentiable function in $\overline{Y}$.

The final formula for $u_3^{\pm}$ takes the form

(3.18)
$$\begin{gathered}u_3^{\pm}(x_1,y)=N_3^{\pm}(y)(v_0^{\pm})^{(3)}(x_1)+N_2^{\pm}(y)(v_1^{\pm})''(x_1)+\\+N_1^{\pm}(y)(v_2^{\pm})'(x_1)+q_3^{\pm}(y)\,g''(x_1)+v_3^{\pm}(x_1).\end{gathered}$$

Here $N_3^{\pm}$ and $q_3^{\pm}$ satisfy the following problems:

(3.19)
$$\left\{\begin{aligned}&A_y^{\pm}\,N_3^{\pm}=a_{11}^{\pm}\,N_1^{\pm}+\partial_{y_i}(a_{i1}^{\pm}\,N_2^{\pm})+b_1^{\pm}\,N_2^{\pm}+\\&\qquad\qquad+a_{1j}^{\pm}\partial_{y_j}N_2^{\pm}-h_3^{\pm},\quad y\in Y,\\&B_y^{\pm}\,N_3^{\pm}=-a_{i1}^{\pm}N_2^{\pm}\,n_i,\qquad\qquad y\in\partial Y;\end{aligned}\right.$$

(3.20)
$$\left\{\begin{aligned}&A_y^{\pm}\,q_3^{\pm}=a_{11}^{\pm}\,q_1^{\pm}+\partial_{y_i}(a_{i1}^{\pm}\,q_2^{\pm})+b_1^{\pm}\,q_2^{\pm}+\\&\qquad\qquad+a_{1j}^{\pm}\partial_{y_j}q_2^{\pm}-\overline{q_2^{\pm}},\quad y\in Y,\\&B_y^{\pm}\,q_3^{\pm}=-a_{i1}^{\pm}\,q_2^{\pm}\,n_i,\qquad\qquad y\in\partial Y.\end{aligned}\right.$$

The functions $N_3^{\pm}$ and $q_3^{\pm}$ have continuous derivatives in $\overline{Y}$ up to the second order, more precisely $N_3^{\pm},q_3^{\pm}\in C^{2,\alpha}(\overline{Y})$.

In such a way we have constructed the following asymptotic series in G_ε^+ and G_ε^-:

$$
\begin{aligned}
(3.21)\qquad u_\infty^\pm = v_0^\pm(x_1) + \varepsilon\big[N_1^\pm(y)\,(v_0^\pm)'(x_1) + v_1^\pm(x_1) + q_1^\pm(y)g(x_1)\big] + \\
+\varepsilon^2\big[N_2^\pm(y)\,(v_0^\pm)''(x_1) + N_1^\pm(y)\,(v_1^\pm)'(x_1) + v_2^\pm(x_1) + q_2^\pm(y)g'(x_1)\big] + \\
+\varepsilon^3\left[N_3^\pm(y)\,(v_0^\pm)^{(3)}(x_1) + N_2^\pm(y)\,(v_1^\pm)''(x_1)\right] + \\
+\varepsilon^3\left[N_1^\pm(y)\,v_2^\pm(x_1) + v_3^\pm(x_1) + q_3^\pm(y)g''(x_1)\right], \quad y = \frac{x}{\varepsilon}.
\end{aligned}
$$

REMARK 3.1. We have built four terms of the asymptotic expansion, but actually we will use only three. The fourth term is required for deriving the equation for $v_2^\pm(x_1)$ and does not show up in the approximation.

It should be noted that the infinite number of terms in series (3.21) can be constructed. Interested reader can find in [**12**] the description of the general method for such a construction together with some applications and examples.

4. Boundary layers near the rod ends

4.1. Leading term of the asymptotics. Asymptotic series (3.21) do not satisfy the boundary conditions on the bases of the rod. First, let us correct the leading term of the asymptotics with the help of boundary layer functions near the right base; with this correction the leading term takes the form

$$
(4.1)\qquad v_0^+(x_1) + \left(w_0^+\big(\frac{x_1-1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_0^+\right),
$$

where w_0^+ is a solution of a homogeneous problem in a semi-infinite cylinder

$$
(4.2)\qquad \begin{cases} A_y^+ w_0^+(y) = 0, \quad y \in G_{-\infty}^0, \\ B_y^+ w_0^+ = 0, \quad y \in \Gamma_{-\infty}^0, \\ w_0^+(0, y') = \varphi^+(y'). \end{cases}
$$

As was proved in [**13**] (see Theorem 6.1), under assumptions $(\mathbf{H1}) - (\mathbf{H4}), (\mathbf{H6})$, there exists a unique solution w_0^+ to problem (4.2) stabilizing to a constant $\hat{w}_0^+$ at the exponential rate, as $y_1 \to -\infty$. As a boundary condition for v_0^+ let us choose this uniquely defined constant: $v_0^+(1) = \hat{w}_0^+$. In such a way the function defined by (4.1) satisfies the boundary condition as $x_1 = 1$.

Similarly we correct the leading term v_0^- near the left base of the rod:

$$
(4.3)\qquad v_0^-(x_1) + \left(w_0^-\big(\frac{x_1+1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_0^-\right),
$$

where w_0^- is a solution to the following problem:

$$
(4.4)\qquad \begin{cases} A_y^- w_0^-(y) = 0, \quad y \in G_0^{+\infty}, \\ B_y^- w_0^- = 0, \quad y \in \Gamma_0^{+\infty}, \\ w_0^-(0, y') = \varphi^-(y'). \end{cases}
$$

Since $\bar{b}_1^- > 0$ then there exists a unique solution w_0^- to problem (4.4) and this solution stabilizes to a constant, as $y_1 \to \infty$. We denote this constant by $\hat{w}_0^-$ and set $v_0^-(-1) = \hat{w}_0^-$. One can easily see that the function defined by (4.3) satisfies the boundary condition on the left base of the cylinder G_ε.

4.2. Term of order ε. The corrector in the asymptotic expansion (3.21) has the form

$$\varepsilon N_1^{\pm}\big(\frac{x}{\varepsilon}\big)\,(v_0^{\pm})'(x_1) + \varepsilon v_1^{\pm}(x_1) + \varepsilon q_1^{\pm}\big(\frac{x}{\varepsilon}\big) g(x_1).$$

Since the leading term satisfies the boundary conditions as $x_1 = \pm 1$, then our aim is to correct the first term in such a way that the resulting coefficient in front of ε is equal to zero as $x_1 = \pm 1$. First, we show how to construct a boundary layer function near the right base of the rod.

Take a function w_1^+ satisfying the following problem:

$$\begin{cases} A_y^+ w_1^+(y) = 0, \quad y \in G_{-\infty}^0, \\ B_y^+ w_1^+ = 0, \quad y \in \Gamma_{-\infty}^0, \\ w_1^+(0, y') = -N_1^+(\delta, y')\,(v_0^+)'(1) - q_1^+(\delta, y')\, g(1), \end{cases} \tag{4.5}$$

for some fixed $\delta \in [0,1)$. Taking into account that $\bar{b}_1^+ > 0$ one can see that w_1^+ stabilizes to a uniquely defined constant which we denote by $\hat{w}_1^+$ (see [**13**]). Then we take the constant $\hat{w}_1^+$ as a boundary condition for $v_1^+(x_1)$ as $x_1 = 1$: $v_1^+(1) = \hat{w}_1^+$. Obviously, $\varepsilon = (\delta + N)^{-1}$, where δ is the fractional and N is the integer parts of $1/\varepsilon$. By periodicity $N_1^+(1/\varepsilon, x'/\varepsilon) = N_1^+(\delta, x'/\varepsilon)$, $q_1^+(1/\varepsilon, x'/\varepsilon) = q_1^+(\delta, x'/\varepsilon)$.

We correct the first term as follows:

$$\varepsilon N_1^+\big(\frac{x}{\varepsilon}\big)\,(v_0^+)'(x_1) + \varepsilon q_1^+\big(\frac{x}{\varepsilon}\big)\, g(x_1) + \varepsilon v_1^+(x_1) + \varepsilon\big(w_1^+\big(\frac{x_1 - 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_1^+\big). \tag{4.6}$$

Taking into account that $v_1^+(1) = \hat{w}_1^+$, one can see that

$$\varepsilon N_1^+\big(\frac{1}{\varepsilon}, \frac{x'}{\varepsilon}\big)\,(v_0^+)'(1) + \varepsilon q_1^+\big(\frac{1}{\varepsilon}, \frac{x'}{\varepsilon}\big)\, g(1) + \varepsilon v_1^+(1) + \varepsilon\big(w_1^+\big(0, \frac{x'}{\varepsilon}\big) - \hat{w}_1^+\big) = 0.$$

Using the same arguments, we construct a boundary layer function in the neighborhood of the left base of the rod:

$$\varepsilon N_1^-\big(\frac{x}{\varepsilon}\big)\,(v_0^-)'(x_1) + \varepsilon v_1^-(x_1) + \varepsilon\big(w_1^-\big(\frac{x_1 + 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_1^-\big), \tag{4.7}$$

where $w_1^-(y)$ is a unique (due to the assumption that $\bar{b}_1^- > 0$) solution to the following problem:

$$\begin{cases} A_y^- w_1^-(y) = 0, \quad y \in G_0^{+\infty}, \\ B_y^- w_1^- = 0, \quad y \in \Gamma_0^{+\infty}, \\ w_1^-(0, y') = -N_1^-(-\delta, y')\,(v_0^-)'(-1) - q_1^-(-\delta, y')\, g(-1), \end{cases} \tag{4.8}$$

which stabilizes to the constant $\hat{w}_1^-$. And we assign a boundary condition for $v_1(x_1)$ at $x_1 = -1$: $v_1(-1) = \hat{w}_1^-$.

4.3. Term of order ε^2. The term of order ε^2 of the asymptotic expansion takes the form:

$$\varepsilon^2 N_2^{\pm}\big(\frac{x}{\varepsilon}\big)\,(v_0^{\pm})''(x_1) + \varepsilon^2 N_1^{\pm}\big(\frac{x}{\varepsilon}\big)\,(v_1^{\pm})'(x_1) + \varepsilon^2 q_2^{\pm}\big(\frac{x}{\varepsilon}\big)\,g'(x_1) + \varepsilon^2 v_2^{\pm}(x_1).$$

Near the right end of the rod we correct it in the following way:

$$\begin{aligned}\varepsilon^2 N_2^{+}\big(\frac{x}{\varepsilon}\big)\,(v_0^{+})''(x_1) + \varepsilon^2 N_1^{+}\big(\frac{x}{\varepsilon}\big)\,(v_1^{+})'(x_1) + \varepsilon^2 v_2^{+}(x_1) + \\ + \varepsilon^2 q_2^{+}\big(\frac{x}{\varepsilon}\big)\,g'(x_1) + \varepsilon^2\,\big(w_2^{+}\big(\frac{x_1-1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_2^{+}\big),\end{aligned} \tag{4.9}$$

where w_2^+ is a unique solution of the following problem in a semi-infinite cylinder:

$$\begin{cases} A_y^{+} w_2^{+} = 0, \quad y \in G_{-\infty}^{0}, \\ B_y^{+} w_2^{+} = 0, \quad y \in \Gamma_{-\infty}^{0}, \\ w_2^{+}(0, y') = -N_2^{+}(\delta, y')\,(v_0^{+})''(1) \\ -N_1^{+}(\delta, y')\,(v_1^{+})'(1) - q_2^{+}(\delta, y')\,g'(1); \end{cases} \tag{4.10}$$

w_2^+ tends to a constant $\hat{w}_2^+$, as $y_1 \to -\infty$. As before, the existence, uniqueness of a solution and the property of the exponential stabilization to a constant are insured by Theorem 5.1 in [**13**]. Now we can choose a boundary condition for the function v_2^+ as $x_1 = 1$: $v_2^+(1) = \hat{w}_2^+$.

Near the left end of the rod we follow the same scheme:

$$\begin{aligned}\varepsilon^2\, N_2^{-}\big(\frac{x}{\varepsilon}\big)\,(v_0^{-})''(x_1) + \varepsilon^2\, N_1^{-}\big(\frac{x}{\varepsilon}\big)\,(v_1^{-})'(x_1) + \varepsilon^2\, v_2^{-}(x_1) + \\ + \varepsilon^2 q_2^{-}\big(\frac{x}{\varepsilon}\big)\,g'(x_1) + \varepsilon^2\,\big(w_2^{-}\big(\frac{x_1+1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_2^{-}\big),\end{aligned} \tag{4.11}$$

where w_2^- is a unique solution of the following problem:

$$\begin{cases} A_y^{-} w_2^{-}(y) = 0, \quad y \in G_0^{+\infty}, \\ B_y^{-} w_2^{-} = 0, \quad y \in \Gamma_0^{+\infty}, \\ w_2^{-}(0, y') = -N_2^{-}(-\delta, y')\,(v_0^{-})''(-1) - \\ \qquad -N_1^{-}(-\delta, y')\,(v_1^{-})'(-1) - q_2^{-}(-\delta, y')\,g'(-1), \end{cases} \tag{4.12}$$

tending to a constant $\hat{w}_2^-$ as $y_1 \to +\infty$. Now we set the constant $\hat{w}_2^-$ as a boundary condition for the function v_2^- as $x_1 = -1$: $v_2^-(-1) = \hat{w}_2^-$.

Finally, we obtain the following expressions for the boundary layer functions in the neighbourhoods of $S_{\pm 1} = \{x \in G_\varepsilon : x_1 = \pm 1,\ x' \in \varepsilon Q\}$:

$$\begin{aligned} v_{bl}^{\pm} \equiv \big[w_0^{\pm}\big(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_0^{\pm}\big] + \varepsilon\big[w_1^{\pm}\big(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_1^{\pm}\big] + \\ + \varepsilon^2\,\big[w_2^{\pm}\big(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}_2^{\pm}\big], \end{aligned} \tag{4.13}$$

where $w_0^\pm$ are defined by (4.2) and (4.4), $w_1^\pm$ - by (4.5) and (4.8), $w_2^\pm$ - by (4.10) and (4.12), respectively. Let us recall what we assigned as the boundary conditions for $v_0^\pm$, $v_1^\pm$ and $v_2^\pm$

at $x_1 = \pm 1$:

$$v_0^\pm(\pm 1) = \hat{w}_0^\pm; \quad v_1^\pm(\pm 1) = \hat{w}_1^\pm; \quad v_2^\pm(\pm 1) = \hat{w}_2^\pm. \tag{4.14}$$

5. Boundary layer in the middle of the rod

Before constructing the boundary layer functions in the middle of the rod, let us extend $v_0^+(x_1)$ (keeping the same notation) to $(-\infty, \varepsilon)$ as a solution of equation (3.5) satisfying the boundary condition $v_0^+(1) = \hat{w}_0^+$. In the same way we can extend v_1^+, v_2^+ to $(-\infty, \varepsilon)$, and v_0^-, v_1^-, v_2^- to $(-\varepsilon, +\infty)$ as solutions to corresponding ordinary differential equations. Periodic in y_1 functions $N_k^\pm$ and $q_k^\pm$, $k = 1, 2, 3$, we regard as defined everywhere in $\mathbb{G} = \mathbb{R} \times Q$.

Obviously, it suffices to match the formal asymptotic series u_∞^+, defined by (3.21) in G_∞^+, with zero in the vicinity of $S_0^\varepsilon = \{x \in G_\varepsilon : x_1 = 0\}$. Then, in the same way we can match u_∞^- with zero, and, summing up the obtained expressions, arrive at the final boundary layer corrector in the neighbourhood of S_0^ε.

In order to match u_∞^+ with zero in the neibourghood of S_0^ε, we are looking for a "corrected" solution in the form

$$\begin{aligned} v_\varepsilon^+(x) = {} & \chi_0^+(y)\, v_0^+(x_1) + \varepsilon\, N_1^+(y)\, \phi^+(y)\, (v_0^+)'(x_1) + \varepsilon\, \chi_{1,1}^+(y)\, (v_0^+)'(x_1) + \\ & + \varepsilon\, q_1^+(y)\, \phi^+(y) g(x_1) + \varepsilon\, \chi_{1,2}^+\, g(x_1) + \varepsilon\, \chi_1^+(y)\, v_1^+(x_1) + \\ & + \varepsilon^2\, N_2^+(y)\, \phi^+(y)\, (v_0^+)''(x_1) + \varepsilon^2\, \chi_{2,1}^+(y)\, (v_0^+)''(x_1) + \\ & + \varepsilon^2\, N_1^+(y)\, \phi^+(y)\, (v_1^+)'(x_1) + \varepsilon^2\, \chi_{2,2}^+(y)\, (v_1^+)'(x_1) + \\ & + \varepsilon^2\, q_2^+(y)\, \phi^+(y) g'(x_1) + \varepsilon^2\, \chi_{2,3}^+(y)\, g'(x_1) + \varepsilon^2\, \chi_2^+(y)\, v_2^+(x_1), \quad y = x/\varepsilon. \end{aligned} \tag{5.1}$$

where the functions $\chi_1^+(y)$, $\chi_{1,1}^+(y)$, $\chi_{1,2}^+(y)$, $\chi_{2,1}^+(y)$, $\chi_{2,2}^+(y)$, $\chi_{2,3}^+(y)$, $\chi_2^+(y)$ and $\chi_3^+(y)$ are to be determined; $\phi^+(y) = \phi^+(y_1)$ is a smooth cut-off function such that $\phi^+(y) = 0$ if $y_1 < -1$ and $\phi^+(y) = 1$ if $y_1 > 1$. Straightforward calculations show that

$$\begin{aligned} A^\varepsilon v_\varepsilon^+ = {} & -a_{11}(y)\, \partial_{x_1}^2 v_\varepsilon^+ - \frac{1}{\varepsilon}\, b_1(y)\, \partial_{x_1} v_\varepsilon^+ - \frac{1}{\varepsilon}\, a_{1j}(y)\, \partial_{x_1}\, \partial_{y_j} v_\varepsilon^+ - \\ & - \frac{1}{\varepsilon} \partial_{y_i} \left(a_{i1}\, \partial_{x_1} v_\varepsilon^+ \right) - \frac{1}{\varepsilon^2}\, \mathrm{div} \left(a\, \nabla_y v_\varepsilon^+ \right) - \frac{1}{\varepsilon} \left(b, \nabla_y v_\varepsilon^+ \right), \quad y = \frac{x}{\varepsilon}; \end{aligned} \tag{5.2}$$

$$B^\varepsilon v_\varepsilon^+ = a_{i1}(y)\, n_i\, \partial_{x_1} v_\varepsilon^+ + \frac{1}{\varepsilon}\, a_{ij}(y)\, \partial_{y_j} v_\varepsilon^+\, n_i, \quad y = \frac{x}{\varepsilon}. \tag{5.3}$$

Substituting (5.1) into (5.2) and (5.3) and collecting power-like terms of order ε^{-2} and ε^{-1}, respectively, we get an equation and a boundary condition for the function $\chi_0^+(y)$:

$$\begin{cases} A_y\, \chi_0^+ \equiv -\mathrm{div}(a(y) \nabla \chi_0^+) - (b(y), \nabla \chi_0^+) = 0, \quad y \in \mathbb{G}, \\ B_y\, \chi_0^+ \equiv (a(y) \nabla \chi_0^+, n) = 0, \quad y \in \Gamma. \end{cases} \tag{5.4}$$

In the same way, collecting power-like terms of order ε^{-1} and ε^0 in (5.2) and (5.3), correspondingly, gives the following equality:

$$\begin{gathered} A_y\chi_{1,1}\ \cdot (v_0^+)'(x_1) + A_y\chi_{1,2}\ \cdot g(x_1) + A_y\chi_1\ \cdot v_1^+(x_1) = \\ = \left[-A_y(N_1^+\phi^+) + a_{1j}\partial_{y_j}\chi_0^+ + \partial_{y_i}(a_{i1}\chi_0^+) + b_1\chi_0^+\right]\ (v_0^+)'(x_1) + \\ -A_y\big(q_1^+(y)\phi^+(y)\big)\cdot g(x_1) + \phi^+(y)\, f(x_1); \end{gathered}$$

and the following equality on the lateral boundary Γ:

$$\begin{gathered} \left[a_{i1}(y)\,\chi_0^+(y)\, n_i + a_{ij}(y)\,\partial_{y_j}(N_1^+\,\phi^+)\, n_i + a_{ij}\,\partial_{y_j}\chi_{1,1}^+ n_i\right]\ (v_0^+)'(x_1) + \\ + \left[a_{ij}\,\partial_{y_j}(q_1^+\phi^+)\, n_i + a_{ij}\,\partial_{y_j}\chi_{1,2}^+\, n_i\right]\ g(x_1) + \\ +a_{ij}\,\partial_{y_j}\chi_1^+\, n_i\cdot v_1^+(x_1) = \phi^+(y)\, g(x_1). \end{gathered}$$

Recalling the expression for $f(x_1)$ in terms of $v_0^+(x_1)$ (see (3.5)), one can see that $\chi_{1,1}^+(y)$, $\chi_{1,2}^+(y)$ and $\chi_1^+(y)$ satisfy the following problems:

$$\left\{\begin{array}{ll} A_y\chi_{1,1}^+ = -A_y(N_1^+(y)\phi^+(y)) + a_{1j}(y)\partial_{y_j}\chi_0^+(y) + & \\ \qquad +\partial_{y_i}\big(a_{i1}\chi_0^+(y)\big) + b_1(y)\chi_0^+(y) - \bar{b}_1^+\,\phi^+(y), & y\in\mathbb{G}; \\ B_y\chi_{1,1}^+ = -a_{i1}\,\chi_0^+\, n_i - a_{ij}\,\partial_{y_j}(N_1^+\,\phi^+)\, n_i, & y\in\Gamma; \end{array}\right. \tag{5.5}$$

$$\left\{\begin{array}{ll} A_y\chi_{1,2}^+ = -A_y\big(q_1^+(y)\phi^+(y)\big) - \phi^+(y)\displaystyle\int\limits_{\partial Y} p^+(y)\, d\sigma_y, & y\in\mathbb{G}, \\ B_y\chi_{1,2}^+ = -a_{ij}\,\partial_{y_j}\big(q_1^+(y)\,\phi^+(y)\big)\, n_i + \phi^+(y), & y\in\Gamma; \end{array}\right. \tag{5.6}$$

$$\left\{\begin{array}{ll} A_y\chi_1^+(y) = 0, & y\in\mathbb{G}, \\ B_y\chi_1^+ = 0, & y\in\Gamma. \end{array}\right. \tag{5.7}$$

Problems (5.4) –(5.7) were derived by formal calculations which, of course, do not imply the solvability of these problems. Theorem 7.3, proved in Section 7, guarantees the existence of solutions to problems (5.5)–(5.7) in proper classes and, moreover, gives an additional qualitative information about the solutions. Now we claim that problems (5.4)–(5.7) are solvable. Indeed, since $\bar{b}_1^- < 0$ and $\bar{b}_1^+ > 0$, by Theorem 7.3 for any two constants there exists a weak solution to problem (5.4) which stabilizes to these constants at the exponential rate, as $|y_1| \to \infty$. We choose χ_0^+ which stabilizes to 1, as $y_1 \to +\infty$, and to zero, as $y_1 \to -\infty$:

$$\chi_0^+ \underset{y_1\to+\infty}{\longrightarrow} 1, \quad \chi_0^+ \underset{y_1\to-\infty}{\longrightarrow} 0. \tag{5.8}$$

Such a choice of χ_0^+, definitions of $N_1^+(y)$ and $\phi^+(y)$ insure the existence of solution $\chi_{1,1}^+$ of problem (5.5), which stabilizes to the constants at infinity. For the function $\chi_{1,1}^+$ we assign zeros at infinity:

$$\chi_{1,1}^+ \to 0, \quad y_1 \to \pm\infty. \tag{5.9}$$

The motivation for such a choice will be given later. Similarly, taking into account (3.8) and the definition of ϕ^+ one can see that problem (5.6) is solvable. We also choose zeros as constants at infinity for $\chi^+_{1,2}$:

$$\chi^+_{1,2} \to 0, \quad y_1 \to \pm\infty. \tag{5.10}$$

And, finally, Theorem 7.3 implies that there exists $\chi^+_1(y)$ such that

$$\chi^+_1 \underset{y_1\to+\infty}{\longrightarrow} 1, \quad \chi^+_1 \underset{y_1\to-\infty}{\longrightarrow} 0. \tag{5.11}$$

In much the same way, substituting (5.1) into (5.2) and (5.3) and collecting the terms of order ε^0 and ε^1, respectively, we see that the following equalities hold:

$$\begin{gathered} A_y\, \chi^+_{2,1}(v^+_0)''(x_1) + A_y\, \chi^+_{2,2}(v^+_1)'(x_1) + A_y\, \chi^+_{2,3} g'(x_1) + A_y\, \chi^+_2 v^+_2(x_1) = \\ = F^+_1(y)\,(v^+_0)''(x_1) + F^+_2(y)\,(v^+_1)'(x_1) + F^+_3(y)\, g'(x_1), \;\; y \in \mathbb{G}; \end{gathered} \tag{5.12}$$

$$\begin{gathered} B_y\chi_{2,1}\,(v^+_0)''(x_1) + B_y\chi^+_{2,2}\,(v^+_1)'(x_1) + B_y\chi_{2,3}\, g'(x_1) + B_y\chi^+_2\, v^+_2(x_1) = \\ -\left[a_{i1}n_i\,\chi^+_{1,1} + B_y(N^+_2\phi^+) + a_{i1}n_i\, N^+_1\phi^+\right]\,(v^+_0)''(x_1) - \\ -\left[B_y(N^+_1\phi^+) + a_{i1}n_i\chi^+_1\right]\,(v^+_1)'(x_1) - \\ -\left[B_y(q^+_2\phi^+) + a_{i1}n_i\, q^+_1\phi^+ + a_{i1}n_i\,\chi^+_{1,2}\right]\, g'(x_1), \quad y \in \Gamma, \end{gathered} \tag{5.13}$$

where

$$\begin{gathered} F^+_1(y) \equiv -A_y(N^+_2\,\phi^+) + a_{11}\,\chi^+_0 + a_{1j}\partial_{y_j}(N^+_1\phi^+) + \\ +\partial_{y_i}(a_{i1}N^+_1\phi^+) + b_1\,N^+_1\,\phi^+ + a_{1j}\partial_{y_j}\chi^+_{1,1} + \partial_{y_i}(a_{i1}\chi^+_{1,1}) + b_1\,\chi^+_{1,1}; \end{gathered}$$

$$F^+_2(y) \equiv -A_y(N^+_1\,\phi^+) + a_{1j}\partial_{y_j}\chi^+_1 + \partial_{y_i}(a_{i1}\chi^+_1) + b_1\,\chi^+_1;$$

$$\begin{gathered} F^+_3(y) \equiv -A_y(q^+_2\,\phi^+) + a_{1j}\partial_{y_j}(q^+_1\phi^+) + \partial_{y_i}(a_{i1}q^+_1\phi^+) + \\ +b_1\,q^+_1\,\phi^+ + a_{1j}\partial_{y_j}\chi^+_{1,2} + \partial_{y_i}(a_{i1}\chi^+_{1,2}) + b_1\,\chi^+_{1,2}. \end{gathered}$$

Since

$$\bar{b}^+_1\,(v^+_1)'(x_1) + h^+_2\,(v^+_0)''(x_1) + \overline{q^+_1}\, g'(x_1) = 0,$$

we can subtract the last expression multiplied by $\phi^+(y)$ from the right-hand side of (5.12) and obtain

$$\begin{gathered} A_y\,\chi^+_{2,1}\cdot(v^+_0)''(x_1) + A_y\,\chi^+_{2,2}\cdot(v^+_1)'(x_1) + A_y\,\chi^+_{2,3}\cdot g'(x_1) + \\ +A_y\,\chi^+_2\cdot v^+_2(x_1) = \left(F^+_1(y) - h_2\,\phi^+\right)(v^+_0)''(x_1) + \\ \left(F^+_2(y) - \bar{b}^+_1\,\phi^+\right)(v^+_1)'(x_1) + \left(F^+_3(y) - \overline{q^+_1}\,\phi^+\right) g'(x_1). \end{gathered} \tag{5.14}$$

One can see that in view of the specific choice of the functions χ^+_0, $\chi^+_{1,1}$, χ^+_1, $\chi^+_{1,2}$ and definitions of N^+_1, ϕ^+, N^+_2, q^+_1, q^+_2, the functions $(F^+_1(y) - h_2\,\phi^+)$, $(F^+_2(y) - \bar{b}^+_1\,\phi^+)$ and $(F^+_3(y) - \overline{q^+_1}\,\phi^+)$ decay exponentially, as $|y_1| \to \infty$. Taking into account (3.7), (3.12), (3.8), (3.13), (5.5) and (5.6), it can be shown that the bracketed expressions on the right-hand side of (5.14) decay exponentially, as $y_1 \to \pm\infty$. Theorem 7.3 states that there exist $\chi^+_{2,1}$, $\chi^+_{2,2}$, $\chi^+_{2,3}$ stabilizing to zero, as $y_1 \to \pm\infty$, which solve the following problems:

$$\left\{\begin{array}{l} A_y\,\chi^+_{2,1} = -A_y(N_2^+\,\phi^+) + a_{11}\,\chi^+_0 + a_{1j}\,\partial_{y_j}(N_1^+\,\phi^+) + \partial_{y_i}(a_{i1}\,N_1^+\,\phi^+)+ \\ +b_1\,N_1^+\,\phi^+ + a_{1j}\,\partial_{y_j}\chi^+_{1,1} + \partial_{y_i}(a_{i1}\,\chi^+_{1,1}) + b_1\,\chi^+_{1,1} - h_2^+\,\phi^+, \quad y\in\mathbb{G}, \\ B_y\,\chi^+_{2,1} = -B_y\,(N_2^+\,\phi^+) - a_{i1}\,n_i\,\chi^+_{1,1} - a_{i1}\,n_i\,N_1^+\,\phi^+, \quad y\in\Gamma; \end{array}\right. \tag{5.15}$$

$$\left\{\begin{array}{ll} A_y\,\chi^+_{2,2} = -A_y(N_1^+\,\phi^+) + a_{1j}\,\partial_{y_j}\chi^+_1 + & \\ \qquad +\partial_{y_i}(a_{i1}\chi^+_1) + b_1\,\chi^+_1 - \bar{b}^+_1\,\phi^+, & y\in\mathbb{G}, \\ B_y\,\chi^+_{22} = -B_y(N_1^+\,\phi^+) - a_{i1}\,n_i\,\chi^+_1, & y\in\Gamma; \end{array}\right. \tag{5.16}$$

$$\left\{\begin{array}{l} A_y\,\chi^+_{2,3} = -A_y(q_2^+\,\phi^+) + a_{1j}\,\partial_{y_j}(q_1^+\,\phi^+) + \partial_{y_i}(a_{i1}\,q_1^+\,\phi^+) + b_1\,q_1^+\,\phi^+ + \\ +a_{1j}\,\partial_{y_j}\chi^+_{1,2} + \partial_{y_i}(a_{i1}\,\chi^+_{1,2}) + b_1\,\chi^+_{1,2} - \overline{q_1^+}\,\phi^+, \quad y\in\mathbb{G}, \\ B_y\,\chi^+_{2,3} = -B_y\,(q_2^+\,\phi^+) - a_{i1}\,n_i\,\chi^+_{1,2} - a_{i1}\,n_i\,q_1^+\,\phi^+, \quad y\in\Gamma. \end{array}\right. \tag{5.17}$$

Also, there is a solution χ^+_2 of problem

$$\left\{\begin{array}{ll} A_y\,\chi^+_2 = 0, & y\in\mathbb{G}, \\ B_y\,\chi^+_2 = 0, & y\in\Gamma, \end{array}\right. \tag{5.18}$$

which satisfies the following conditions at infinity:

$$\chi^+_2 \underset{y_1\to+\infty}{\longrightarrow} 1, \quad \chi^+_2 \underset{y_1\to-\infty}{\longrightarrow} 0. \tag{5.19}$$

In the same way one can match u^-_∞ with zero in the neighbourhood of S_0:

$$\begin{array}{c} v^-_\varepsilon(x) = \chi^-_0(y)\,v^-_0(x_1) + \varepsilon\,N_1^-(y)\,\phi^-(y)\,(v^-_0)'(x_1) + \varepsilon\,\chi^-_{1,1}(y)\,(v^-_0)'(x_1) + \\ +\varepsilon\,q_1^-(y)\,\phi^-(y)g(x_1) + \varepsilon\,\chi^-_{1,2}\,g(x_1) + \varepsilon\,\chi^-_1(y)\,v^-_1(x_1) + \\ +\varepsilon^2\,N_2^-(y)\,\phi^-(y)\,(v^-_0)''(x_1) + \varepsilon^2\,\chi^-_{2,1}(y)\,(v^-_0)''(x_1) + \\ +\varepsilon^2\,N_1^-(y)\,\phi^-(y)\,(v^-_1)'(x_1) + \varepsilon^2\,\chi^-_{2,2}(y)\,(v^-_1)'(x_1) + \\ +\varepsilon^2\,q_2^-(y)\,\phi^-(y)g'(x_1) + \varepsilon^2\,\chi^-_{2,3}(y)\,g'(x_1) + \varepsilon^2\,\chi_3(y)\,v^-_2(x_1), \;\; y = x/\varepsilon. \end{array} \tag{5.20}$$

Here $\phi^-(y) \equiv 1 - \phi^+(y)$; $\chi^-_0(y)$ is a solution of problem

$$\left\{\begin{array}{ll} A_y\,\chi^-_0 = 0, & y\in\mathbb{G}, \\ B_y\,\chi^-_0 = 0, & y\in\Gamma, \end{array}\right.$$

which converges to 0, as $y_1 \to +\infty$, and to 1, as $y_1 \to -\infty$. Functions $\chi^-_{1,1}(y)$, $\chi^-_{1,2}(y)$ and $\chi^-_1(y)$ satisfy the following problems:

$$\left\{\begin{array}{ll} A_y\chi^-_{1,1} = -A_y(N_1^-(y)\phi^-(y)) + a_{1j}(y)\partial_{y_j}\chi^-_0(y) + \partial_{y_i}\big(a_{i1}\chi^-_0(y)\big) + & \\ \qquad +b_1(y)\chi^-_0(y) - \bar{b}^-_1\,\phi^-(y), & y\in\mathbb{G}; \\ B_y\chi^+_{1,1} = -a_{i1}\,\chi^-_0\,n_i - a_{ij}\,\partial_{y_j}(N_1^-\,\phi^-)\,n_i, & y\in\Gamma; \end{array}\right.$$

$$\begin{cases} A_y\chi_{1,2}^- = -A_y\big(q_1^-(y)\phi^-(y)\big) - \phi^-(y) \int\limits_{\partial Y} p^-(y)\, d\sigma_y, & y \in \mathbb{G}, \\ B_y\chi_{1,2}^- = -a_{ij}\,\partial_{y_j}\big(q_1^-(y)\,\phi^-(y)\big)\, n_i + \phi^-(y), & y \in \Gamma; \end{cases}$$

$$\begin{cases} A_y\chi_1^-(y) = 0, & y \in \mathbb{G}, \\ B_y\chi_1^- = 0, & y \in \Gamma. \end{cases}$$

We choose these functions in such a way that $\chi_{1,1}^-(y)$ and $\chi_{1,2}^-(y)$ tend to zero, as $y_1 \to \pm\infty$; $\chi_1^- \to 0$, as $y_1 \to +\infty$, and $\chi_1^- \to 1$, as $y_1 \to -\infty$. By the same arguments as above, there exist $\chi_{2,1}^-$, $\chi_{2,2}^-$, $\chi_{2,3}^-$ stabilizing to zero, as $y_1 \to \pm\infty$, and solving problems

$$\begin{cases} A_y\,\chi_{2,1}^- = -A_y(N_2^-\,\phi^-) + a_{11}\,\chi_0^- + a_{1j}\,\partial_{y_j}(N_1^-\,\phi^-) + \partial_{y_i}(a_{i1}\,N_1^-\,\phi^-) + \\ +b_1\,N_1^-\,\phi^- + a_{1j}\,\partial_{y_j}\chi_{1,1}^- + \partial_{y_i}(a_{i1}\,\chi_{1,1}^-) + b_1\,\chi_{1,1}^- - h_2^-\,\phi^-, \quad y \in \mathbb{G}, \\ B_y\,\chi_{2,1}^- = -B_y\,(N_2^-\,\phi^-) - a_{i1}\,n_i\,\chi_{1,1}^- - a_{i1}\,n_i\,N_1^-\,\phi^-, \quad y \in \Gamma; \end{cases}$$

$$\begin{cases} A_y\,\chi_{2,2}^- = -A_y(N_1^-\,\phi^-) + a_{1j}\,\partial_{y_j}\chi_1^- + \\ \qquad +\partial_{y_i}(a_{i1}\chi_1^-) + b_1\,\chi_1^- - \overline{b}_1^-\,\phi^-, \quad y \in \mathbb{G}, \\ B_y\,\chi_{22}^- = -B_y(N_1^-\,\phi^-) - a_{i1}\,n_i\,\chi_1^-, \quad y \in \Gamma; \end{cases}$$

$$\begin{cases} A_y\,\chi_{2,3}^- = -A_y(q_2^-\,\phi^-) + a_{1j}\,\partial_{y_j}(q_1^-\,\phi^-) + \partial_{y_j}(a_{i1}\,q_1^-\,\phi^-) + b_1\,q_1^-\,\phi^- + \\ +a_{1j}\,\partial_{y_j}\chi_{1,2}^- + \partial_{y_i}(a_{i1}\,\chi_{1,2}^-) + b_1\,\chi_{1,2}^- - \overline{q_1^-}\,\phi^-, \quad y \in \mathbb{G}, \\ B_y\,\chi_{2,3}^- = -B_y\,(q_2^-\,\phi^-) - a_{i1}\,n_i\,\chi_{1,2}^- - a_{i1}\,n_i\,q_1^-\,\phi^-, \quad y \in \Gamma. \end{cases}$$

Similarly, there exists a function χ_2^- being a solution of problem

$$\begin{cases} A_y\,\chi_2^- = 0, & y \in \mathbb{G}, \\ B_y\,\chi_2^- = 0, & y \in \Gamma, \end{cases}$$

such that

$$\chi_2^- \underset{y_1 \to +\infty}{\longrightarrow} 0, \quad \chi_2^- \underset{y_1 \to -\infty}{\longrightarrow} 1.$$

Finally, taking into account the constructed inner formal asymptotic expansion and boundary layer correctors in the neibourhoods of $S_{\pm 1}$ and S_0, we arrive at the asymptotic solution of problem (2.1):

$$u_\infty^\varepsilon(x) \equiv v_\varepsilon^+(x) + v_{bl}^+(x) + v_\varepsilon^-(x) + v_{bl}^-(x), \tag{5.21}$$

where v_ε^+, v_ε^- and $v_{bl}^\pm$ are defined by (5.1), (5.20) and (4.13), respectively.

REMARK 5.1. Adding the boundary layer functions $v_{bl}^\pm$ to the inner expansions $u_\infty^\pm$ makes it possible to satisfy the boundary conditions on the bases of the rod G_ε with an accuracy up to the third order in ε. Representing (5.21) as the sum of the inner expansions and the boundary layer functions

$$\begin{aligned} u_\infty^\varepsilon &= u_\infty^+(x) + (v_\varepsilon^+(x) - u_\infty^+(x)) + v_{bl}^+(x) \\ &+u_\infty^-(x) + (v_\varepsilon^-(x) - u_\infty^-(x)) + v_{bl}^-(x), \end{aligned}$$

where

$$u_\infty^\pm(x) = v_0^\pm(x_1) + \varepsilon\big[N_1^\pm\big(\frac{x}{\varepsilon}\big)\,(v_0^\pm)'(x_1) + v_1^\pm(x_1) + q_1^\pm\big(\frac{x}{\varepsilon}\big)g(x_1)\big] +$$
$$+\varepsilon^2\big[N_2^\pm\big(\frac{x}{\varepsilon}\big)\,(v_0^\pm)''(x_1) + N_1^\pm\big(\frac{x}{\varepsilon}\big)\,(v_1^\pm)'(x_1) + v_2^\pm(x_1) + q_2^\pm\big(\frac{x}{\varepsilon}\big)g'(x_1)\big],$$

we make $(v_\varepsilon^\pm - u_\infty^\pm)$ exponentially small (but not vanishing) on $S_\pm^\varepsilon$, as well as v_{bl}^+ on S_{-1}^ε and v_{bl}^- on S_{+1}^ε. In order to satisfy exactly the boundary conditions, one can replace (5.21) with

$$\tilde u_\infty^\varepsilon = u_\infty^+(x) + (v_\varepsilon^+(x) - u_\infty^+(x))\,\phi_1(x) + v_{bl}^+(x)\,\phi_1^+(x)$$
$$+u_\infty^-(x) + (v_\varepsilon^-(x) - u_\infty^-(x))\;\phi_1(x) + v_{bl}^-(x)\,\phi_1^-(x),$$

where $\phi_1(x) = 1$ if $|x_1| < 1/3$ and $\phi_1(x) = 0$ otherwise;

$$\phi_1^+(x) = \begin{cases} 1, & x_1 > 2/3, \\ 0, & x_1 < 1/3. \end{cases} \qquad \phi_1^-(x) = \begin{cases} 1, & x_1 < -2/3, \\ 0, & x_1 > -1/3. \end{cases}$$

Substituting $\tilde u_\infty^\varepsilon$ into (2.1), it is straightforward to check that the presence of the cut-off functions results in the appearance of additional exponentially small (with respect to ε^{-1}) terms on the right-hand side. Later on we will prove a priori estimates (6.3) and (6.4) which ensure that the exponentially small perturbation of the right-hand side leads to the exponentially small perturbation of the solution, and, thus, is negligible in any polynomial in ε expansion. To simplify the notations we deal with (5.21) neglecting the discrepancies on $S_{\pm1}^\varepsilon$ which are exponentially small with respect to ε^{-1}.

In order to estimate the difference between the exact solution $u^\varepsilon(x)$ and the approximate one u_∞^ε, we will need the following relations:

$$A^\varepsilon\big(v_\varepsilon^\pm + v_{bl}^\pm\big) - \frac{1}{\varepsilon}\,\phi^\pm\big(\frac{x_1}{\varepsilon}\big)\,f(x_1) =$$
$$= \varepsilon\,\big[-a_{11}(y)\,N_1^\pm(y)\,\phi^\pm(y_1) - a_{1j}(y)\,\partial_{y_j}(N_2^\pm(y)\,\phi^\pm(y))\big]\;(v_0^\pm)^{(3)}(x_1) +$$
$$+\varepsilon\,\big[-\partial_{y_i}(a_{i1}\,N_2^\pm\phi^\pm) - b_1\,N_2^\pm\,\phi^\pm - a_{11}\,\chi_{1,1}^\pm\big]\;(v_0^\pm)^{(3)}(x_1) +$$
$$+\varepsilon\,\big[-a_{1j}\,\partial_{y_j}\chi_{2,1}^\pm - \partial_{y_i}(a_{i1}\,\chi_{2,1}^\pm) - b_1\,\chi_{2,1}^\pm\big]\;(v_0^\pm)^{(3)}(x_1) +$$
$$+\varepsilon\,\big[-\partial_{y_i}(a_{i1}\,N_1^\pm\phi^\pm) - a_{1j}\,\partial_{y_j}(N_1^\pm\,\phi^\pm) - b_1(y)\,N_1^\pm\,\phi^\pm\big]\;(v_1^\pm)''(x_1) +$$
$$+\varepsilon\,\big[-a_{11}\,\chi_1^\pm(y) - a_{1j}\,\partial_{y_j}\chi_{2,2}^\pm - \partial_{y_i}(a_{i1}\,\chi_{2,2}^\pm) - b_1\,\chi_{2,2}^\pm\big]\;(v_1^\pm)''(x_1) +$$
$$+\varepsilon\,\big[-a_{1j}(y)\,\partial_{y_j}\chi_2^\pm(y) - \partial_{y_i}(a_{i1}(y)\,\chi_2^\pm(y)) - b_1(y)\,\chi_2^\pm(y)\big]\;(v_2^\pm)'(x_1) +$$

$$+\varepsilon\,\big[-a_{11}q_1^\pm\phi^\pm - a_{1j}\partial_{y_j}(q_2^\pm\phi^\pm) - \partial_{y_i}(a_{i1}q_2^\pm\phi^\pm) - b_1\,q_2^\pm\phi^\pm\big]\;g''(x_1) +$$
$$+\varepsilon\,\big[-a_{11}\,\chi_{2,2}^\pm - a_{1j}\partial_{y_j}\chi_{2,3}^\pm - \partial_{y_i}(a_{i1}\chi_{2,3}^\pm) - b_1\,\chi_{2,3}^\pm\big]\;g''(x_1) + A^\varepsilon v_{bl}^\pm. \tag{5.22}$$

$$
\begin{aligned}
&B^\varepsilon\big(v_\varepsilon^\pm + v_{bl}^\pm\big) - \phi^\pm\big(\frac{x_1}{\varepsilon}\big)\, g(x_1) = \\
&= \varepsilon^2\, \big[a_{i1}(y)\, n_i\, N_2^\pm(y)\, \phi^\pm(y) + a_{i1}\, n_i\, \chi_{2,1}^\pm(y)\big]\, (v_0^\pm)^{(3)}(x_1)+ \\
&+\varepsilon^2\, \big[a_{i1}\, n_i\, N_1^\pm(y)\, \phi^\pm + a_{i1}\, n_i\, \chi_{2,2}^\pm(y)\big]\, (v_1^\pm)''(x_1)+ \\
&+\varepsilon^2\, \big[a_{i1}\, n_i\, q_2^\pm(y)\, \phi^\pm + a_{i1}\, n_i\, \chi_{2,3}^\pm(y)\big]\, g''(x_1)+ \\
&+\varepsilon^2\, a_{i1}\, n_i\, (v_2^\pm)'(x_1) + B^\varepsilon v_{bl}^\pm, \quad y = x/\varepsilon.
\end{aligned} \tag{5.23}
$$

6. Justification procedure

First we obtain an a priori estimate for a solution to the problem

$$
\begin{cases}
A^\varepsilon u^\varepsilon = f^\varepsilon(x), \quad x \in G_\varepsilon,\\
B^\varepsilon u^\varepsilon = g^\varepsilon(x), \quad x \in \Gamma_\varepsilon,\\
u^\varepsilon(\pm 1, x') = 0, \quad x' \in \varepsilon Q
\end{cases} \tag{6.1}
$$

in terms of $f^\varepsilon(x)$ and $g^\varepsilon(x)$ (for the moment we do not specify the particular structure of these functions). To this end let us make a change of variables in the last problem

$$
x \mapsto y = \frac{x}{\varepsilon}; \quad u^\varepsilon(x) \mapsto v_\varepsilon(y) = u^\varepsilon(\varepsilon y).
$$

Then (6.1) takes the form:

$$
\begin{cases}
A_y v_\varepsilon = \varepsilon^2\, f^\varepsilon(\varepsilon y), & y \in G_{-1/\varepsilon}^{1/\varepsilon},\\
B_y v_\varepsilon = \varepsilon g^\varepsilon(\varepsilon y), & y \in \Gamma_{-1/\varepsilon}^{1/\varepsilon},\\
v_\varepsilon(-1/\varepsilon, y') = v_\varepsilon(1/\varepsilon, y') = 0, & y' \in Q.
\end{cases} \tag{6.2}
$$

It will be shown in Section 7 that the following estimate holds:

$$
\begin{aligned}
\|\nabla v_\varepsilon\|_{L^2(G_{-1/\varepsilon}^{1/\varepsilon})} \le C\, \varepsilon^2 \|(1+|y_1|)\, f^\varepsilon(\varepsilon\cdot)\|_{L^2(G_{-1/\varepsilon}^{1/\varepsilon})} &+ \\
+ C\, \varepsilon \|(1+|y_1|)\, g(\varepsilon\cdot)\|_{L^2(\Gamma_{-1/\varepsilon}^{1/\varepsilon})}&;
\end{aligned}
$$

Making inverse change of variables, one can see that

$$
\|\nabla v_\varepsilon\|^2_{L^2(G_{-1/\varepsilon}^{1/\varepsilon})} = \frac{1}{\varepsilon^{d-2}} \int\limits_{G_\varepsilon} |\nabla u^\varepsilon|^2\, dx,
$$

$$
\int\limits_{G_{-1/\varepsilon}^{1/\varepsilon}} (f^\varepsilon(\varepsilon y))^2\, dy = \frac{1}{\varepsilon^d} \int\limits_{G_\varepsilon} (f^\varepsilon(x))^2\, dx,
$$

$$
\int\limits_{\Gamma_{-1/\varepsilon}^{1/\varepsilon}} (g^\varepsilon(\varepsilon y))^2\, dy = \frac{1}{\varepsilon^{d-1}} \int\limits_{\Gamma_\varepsilon} (g^\varepsilon(x))^2\, dx,
$$

and, consequently,

$$
\|\nabla u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\, \|f^\varepsilon\|_{L^2(G_\varepsilon)} + C\varepsilon^{-\frac{1}{2}}\, \|g^\varepsilon\|_{L^2(\Gamma_\varepsilon)}. \tag{6.3}
$$

Making use of the Friedrichs inequality for the function u^ε in G_ε we obtain

$$\|u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\,\|f^\varepsilon\|_{L^2(G_\varepsilon)} + C\varepsilon^{-\frac{1}{2}}\,\|g^\varepsilon\|_{L^2(\Gamma_\varepsilon)}. \tag{6.4}$$

Estimation of the $L^2(G_\varepsilon)$-norm of $A^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)$ and $L^2(\Gamma_\varepsilon)$-norm of $B^\varepsilon\big((v_\varepsilon^+ + v_{bl}^+) + (v_\varepsilon^- + v_{bl}^-) - u^\varepsilon\big)$ will complete the justification procedure.

REMARK 6.1. The estimates (6.3)–(6.4) actually imply that we can take $f(x_1) \in L^2(G_\varepsilon)$ and $g(x_1) \in L^2(\Gamma_\varepsilon)$.

There are some ”typical” terms in (5.22), (5.23) like

- $$\varepsilon\, a_{11}(y)\, N_1^+(y)\, \phi^+(y_1)(v_0^+)^{(3)}(x_1)_{|y=x/\varepsilon}$$
- $$\varepsilon\, a_{1j}(y)\partial_{y_j}(N_2^+\phi^+)\,(v_0^+)^{(3)}(x_1)_{|y=x/\varepsilon}$$
- $$\varepsilon\, a_{11}\,\chi_1^+(y)\,(v_0^+)^{(3)}(x_1)_{|y=x/\varepsilon}$$
- $$\varepsilon\, a_{1j}\,\partial_{y_j}\chi_{2,2}^+\,(v_0^+)^{(3)}(x_1)_{|y=x/\varepsilon}$$
- $$\varepsilon^2\, a_{i1}(y)\, n_i\, N_2^+(y)\,\phi^+(y)\,(v_0^+)^{(3)}(x_1)_{|y=x/\varepsilon}$$
- $$\varepsilon^2\, a_{i1}(y)\, n_i\,\chi_{2,1}^+(y)\,(v_0^+)^{(3)}(x_1)_{|y=x/\varepsilon}$$
- $$\varepsilon^2\, a_{i1}(y)\, n_i\,(v_2^+)'(x_1)_{|y=x/\varepsilon}$$

Let us estimate them.

1.
$$I_1^2 = \int\limits_{G_\varepsilon} \varepsilon^2\,\Big[a_{11}(\frac{x}{\varepsilon})\,N_1^+(\frac{x}{\varepsilon})\,\phi^+(\frac{x_1}{\varepsilon})(v_0^+)^{(3)}(x_1)\Big]^2\,dx.$$

$v_0^+(x_1)$ is a solution of the following equation:

$$\begin{cases} \bar b_1^+(v_0^+)'(x_1) = f(x_1) + g(x_1)\int\limits_{\partial Y} p^+(y)dy,\ x_1 \in (0,1), \\ v_0^+(1) = \hat w_0^+. \end{cases}$$

If $f(x_1)$ and $g(x_1)$ are two times continuously differentiable, then $(v_0^+)^{(3)}$ is a continuous function on the interval $[0,1]$. Taking into account the fact that $a_{ij}(y)$ are smooth and bounded, $0 \le \phi^+ \le 1$, and $N_1^+(y)$ is a continuous periodic function (see (3.7)), we see that

$$I_1^2 \le C\varepsilon^2\,|G_\varepsilon| \le C\varepsilon^{(d-1)}\,\varepsilon^2.$$

Thus,

$$\Big\|\varepsilon\, a_{11}(\frac{x}{\varepsilon})\,N_1^+(\frac{x}{\varepsilon})\,\phi^+(\frac{x_1}{\varepsilon})(v_0^+)^{(3)}(x_1)\Big\|_{L^2(G_\varepsilon)} \le C\varepsilon^{(d-1)/2}\,\varepsilon.$$

2.
$$I_2=\Big\|\varepsilon^2\, a_{1j}(\frac{x}{\varepsilon})\,\partial_{x_j}\big(N_2^+(\frac{x}{\varepsilon})\phi^+(\frac{x}{\varepsilon})\big)\,(v_0^+)^{(3)}(x_1)\Big\|_{L^2(G_\varepsilon)}.$$
Making change of variables $y=x/\varepsilon$ we obtain:
$$I_2^2=\varepsilon^2\,\varepsilon^d\int\limits_{G_{-1/\varepsilon}^{1/\varepsilon}}\big[a_{1j}(y)\partial_{y_j}\big(N_2^+(y)\phi^+(y)\big)\,(v_0^+)^{(3)}(\varepsilon y_1)\big]^2\,dy.$$
The $L^2(Y)$ norm of the gradient of $N_2^+(y)$ is bounded and the gradient of ϕ^+ is equal to zero in $G_{-1/\varepsilon}^{-1}\cup G_1^{1/\varepsilon}$ due to the definition of the function $\phi^+(y)$.
$$I_2^2\le C\varepsilon^2\,\varepsilon^d\int\limits_{G_{-1/\varepsilon}^{1/\varepsilon}}|\nabla_y\phi^+(y)|^2\,dy+C\varepsilon^2\,\varepsilon^d\int\limits_{G_{-1/\varepsilon}^{1/\varepsilon}}|\nabla_yN_2^+(y)|^2\,dy\le$$
$$\le C\varepsilon^2\,\varepsilon^d\,|G_{-1}^1|+C\varepsilon^2\,\varepsilon^d\,\varepsilon^{-1}\le C\varepsilon^{(d-1)}\big[\varepsilon^2+\varepsilon^3\big].$$
Consequently,
$$\Big\|\varepsilon^2\, a_{1j}(\frac{x}{\varepsilon})\,\partial_{x_j}\big(N_2^+(\frac{x}{\varepsilon})\phi^+(\frac{x}{\varepsilon})\big)\,(v_0^+)^{(3)}(x_1)\Big\|_{L^2(G_\varepsilon)}\le C\varepsilon^{(d-1)/2}\varepsilon.$$
3.
$$I_3^2=\Big\|\varepsilon\, a_{11}(\frac{x}{\varepsilon})\,\chi_1^+(\frac{x}{\varepsilon})\,(v_0^+)^{(3)}(x_1)\Big\|_{L^2(G_\varepsilon)}^2\le C\,\varepsilon^2\,\varepsilon^d\int\limits_{G_{-1/\varepsilon}^{1/\varepsilon}}\big(\chi_1^+(y)\big)^2\,dy.$$
We assign the function $\chi_1^+(y)$ to tend to 1, as $y_1\to+\infty$, and to zero, as $y_1\to-\infty$. For χ_1^+ the following estimate holds (see Theorem 7.3):
$$\|\chi_1^+\|_{L^2(Y)}\le C.$$
Thus,
$$I_3^2\le C\,\varepsilon^2\,\varepsilon^d\,\varepsilon^{-1}=C\,\varepsilon^2\,\varepsilon^{(d-1)},$$
and, finally,
$$\Big\|\varepsilon\, a_{11}(\frac{x}{\varepsilon})\,\chi_1^+(\frac{x}{\varepsilon})\,(v_0^+)^{(3)}(x_1)\Big\|_{L^2(G_\varepsilon)}\le C\,\varepsilon^{(d-1)/2}\,\varepsilon.$$
4.
$$I_4^2=\Big\|\varepsilon^2\, a_{1j}(\frac{x}{\varepsilon})\,\partial_{x_j}(\chi_{2,2}^+(\frac{x}{\varepsilon}))\,(v_0^+)^{(3)}(x_1)\Big\|_{L^2(G_\varepsilon)}^2\le$$
$$\le C\,\varepsilon^2\,\varepsilon^d\int\limits_{G_{-1/\varepsilon}^{1/\varepsilon}}|\nabla_y\chi_{2,2}^+(y)|^2\,dy.$$
Due to Theorem 7.3 the $L^2(\mathbb{G})$ norm of $\chi_{2,2}^+$ is bounded, thus
$$\Big\|\varepsilon^2\, a_{1j}(\frac{x}{\varepsilon})\,\partial_{x_j}(\chi_{2,2}^+(\frac{x}{\varepsilon}))\,(v_0^+)^{(3)}(x_1)\Big\|_{L^2(G_\varepsilon)}\le C\,\varepsilon^{(d-1)/2}\,\varepsilon^{3/2}.$$

5.
$$
\begin{aligned}
I_5^2 = \left\|\varepsilon^2\, a_{i1}(\tfrac{x}{\varepsilon})\, n_i\, N_2^+(\tfrac{x}{\varepsilon})\, \phi^+(\tfrac{x}{\varepsilon})\, (v_0^+)^{(3)}(x_1)\right\|^2_{L^2(\Gamma_\varepsilon)} \le \\
\le C\,\varepsilon^{(d-1)}\,\varepsilon^4 \int\limits_{\Gamma^{1/\varepsilon}_{-1/\varepsilon}} (N_2^+(y))^2\, d\sigma_y \le \\
\le C\,\varepsilon^{(d-1)}\,\varepsilon^4 \int\limits_{G^{1/\varepsilon}_{-1/\varepsilon}} \left[(N_2^+(y))^2 + |\nabla_y N_2^+(y)|^2\right]\, dy.
\end{aligned}
$$
Since $\|N_2^+\|_{L^2(Y)}$ and $\|\nabla N_2^+\|_{L^2(Y)}$ are bounded, then
$$
\left\|\varepsilon^2\, a_{i1}(\frac{x}{\varepsilon})\, n_i\, N_2^+(\frac{x}{\varepsilon})\, \phi^+(\frac{x}{\varepsilon})\, (v_0^+)^{(3)}(x_1)\right\|_{L^2(\Gamma_\varepsilon)} \le C\,\varepsilon^{(d-2)/2}\,\varepsilon^2.
$$
6.
$$
\begin{aligned}
I_6^2 = \left\|\varepsilon^2\, a_{i1}(\frac{x}{\varepsilon})\, n_i\, \chi^+_{2,1}(\frac{x}{\varepsilon})\, (v_0^+)^{(3)}(x_1)\right\|^2_{L^2(\Gamma_\varepsilon)} \le \\
C\,\varepsilon^4\,\varepsilon^{(d-1)} \int\limits_{\Gamma^{1/\varepsilon}_{-1/\varepsilon}} \left(\chi^+_{1,2}(y)\right)^2 d\sigma_y \le \\
\le C\,\varepsilon^4\,\varepsilon^{(d-1)} \int\limits_{G^{1/\varepsilon}_{-1/\varepsilon}} \left[\left(\chi^+_{1,2}(y)\right)^2 + \left|\nabla\chi^+_{1,2}\right|^2\right]\, dy
\end{aligned}
$$
In view of Theorem 7.3, for $\chi^+_{1,2}(y)$ the following estimates are valid:
$$
\|\chi^+_{1,2}\|_{L^2(G^{1/\varepsilon}_{-1/\varepsilon})} \le C\,\varepsilon^{-1}\,\|\chi^+_{1,2}\|_{L^2(Y)} \le C\,\varepsilon^{-1};\quad \|\nabla\chi^+_{1,2}\|_{L^2(\mathbb{G})} \le C.
$$
Consequently,
$$
\left\|\varepsilon^2\, a_{i1}(\frac{x}{\varepsilon})\, n_i\, \chi^+_{2,1}(\frac{x}{\varepsilon})\, (v_0^+)^{(3)}(x_1)\right\|_{L^2(\Gamma_\varepsilon)} \le C\,\varepsilon^{(d-2)/2}\,\varepsilon^2.
$$
7.
$$
I_7 = \left\|\varepsilon^2\, a_{i1}(\frac{x}{\varepsilon})\, n_i\, (v_2^+)'(x_1)\right\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^2 \Big(\int\limits_{\Gamma_\varepsilon} d\sigma_y\Big)^{1/2} \le C\varepsilon^2\,\varepsilon^{(d-2)/2}.
$$
Other terms in (5.22)–(5.23) can be analyzed similarly. Exactly in the same way the $L^2(G_\varepsilon)$ norm of $[A^\varepsilon\big(v^-_\varepsilon + v^-_{bl}\big) - \frac{1}{\varepsilon}\,\phi^-(\frac{x_1}{\varepsilon})\, f(x_1)]$ and $L^2(\Gamma_\varepsilon)$ norm of $[B^\varepsilon\big(v^-_\varepsilon + v^-_{bl}\big) - \phi^-(\frac{x_1}{\varepsilon})\, g(x_1)]$ can be estimated.

Finally, the following estimates have been obtained:
$$
\left\|A^\varepsilon\big((v^+_\varepsilon + v^+_{bl}) + (v^-_\varepsilon + v^-_{bl}) - u^\varepsilon\big)\right\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{(d-1)/2}; \tag{6.5}
$$
$$
\left\|B^\varepsilon\big((v^+_\varepsilon + v^+_{bl}) + (v^-_\varepsilon + v^-_{bl}) - u^\varepsilon\big)\right\|_{L^2(\Gamma_\varepsilon)} \le C\,\varepsilon^2\,\varepsilon^{(d-2)/2}; \tag{6.6}
$$
Taking into account (6.3) – (6.6) we get
$$
\|\nabla(v^+_\varepsilon + v^+_{bl}) + \nabla(v^-_\varepsilon + v^-_{bl}) - \nabla u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{(d-1)/2}; \tag{6.7}
$$
$$
\|(v^+_\varepsilon + v^+_{bl}) + (v^-_\varepsilon + v^-_{bl}) - u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{(d-1)/2}. \tag{6.8}
$$

The estimates (6.7) – (6.8) complete the justification procedure. In such a way we have proved the following theorem.

THEOREM 6.2. *Let the conditions* $(\mathbf{H1})$–$(\mathbf{H6})$ *hold true. Then the approximate solution* u^ε_∞ *given by formula* (5.21) *satisfies the estimates*

$$\|\nabla u^\varepsilon_\infty - \nabla u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{(d-1)/2};$$

$$\|u^\varepsilon_\infty - u^\varepsilon\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{(d-1)/2},$$

where $u^\varepsilon(x)$ *is the exact solution to problem* (2.1).

7. Existence of a solution in an infinite cylinder

Given a bounded domain $Q \subset \mathbb{R}^{d-1}$ with a Lipschitz boundary ∂Q, we denote by $\mathbb{G}$ an infinite cylinder $\mathbb{R}\times Q$ with points $x = (x_1, x')$ and the axis directed along x_1. The lateral boundary of the cylinder is denoted by $\Gamma = \mathbb{R}\times\partial Q$. We study the following boundary value problem for a stationary convection-diffusion equation:

$$\left\{\begin{array}{ll} A\,u \equiv -\mathrm{div}\,(a(x)\,\nabla\,u(x)) - (b(x), \nabla\,u(x)) = f(x), & x\in\mathbb{G}, \\[2ex] B\,u \equiv \dfrac{\partial\,u}{\partial\,n_a} = g(x), & x\in\Gamma. \end{array}\right. \tag{7.1}$$

Here the symbol $(\cdot,\cdot)$ stands for the usual scalar product in $\mathbb{R}^d$; $\partial u/\partial\,n_a = (a\nabla u, n)$ is the conormal derivative of a function u, n is the exterior unit normal. Throughout the section we use the notations

$$G^\beta_\alpha = (\alpha,\beta)\times Q, \quad \Gamma^\beta_\alpha = (\alpha,\beta)\times\partial Q.$$

We assume that the following conditions are satisfied:

(A1) The coefficients $a_{ij}(x)\in L^\infty(\mathbb{G})$, $b_j(x)\in L^\infty(\mathbb{G})$ have the form

$$a_{ij}(x) = \left\{\begin{array}{l} a^+_{ij}(x),\ x\in G^{+\infty}_0, \\ a^-_{ij}(x),\ x\in G^0_{-\infty}; \end{array}\right. \qquad b_j(x) = \left\{\begin{array}{l} b^+_j(x),\ x\in G^{+\infty}_0, \\ b^-_j(x),\ x\in G^0_{-\infty}, \end{array}\right.$$

where a^+_{ij}, b^+_j and a^-_{ij}, b^-_j are x_1-periodic in $G^{+\infty}_0$ and $G^0_{-\infty}$, respectively. Without loss of generality we assume that the period is equal to 1.

(A2) The matrix $a(x)$ is symmetric and satisfies the uniform ellipticity condition, that is there exists a positive constant Λ such that, for almost all $x\in\mathbb{G}$,

$$\Lambda\,|\xi|^2 \le \sum_{i,j=1}^d a^\pm_{ij}(x)\,\xi_i\,\xi_j, \quad \forall\xi\in\mathbb{R}^d. \tag{7.2}$$

(A3) $f(x)\in L^2(\mathbb{G})$ and $g(x)\in L^2(\Gamma)$ decay exponentially at infinity, namely,

$$\|f\|_{L^2(G^{n+1}_n)} \le \tilde{C}\,e^{-\gamma_1\,|n|}, \quad \|g\|_{L^2(\Gamma^{n+1}_n)} \le \tilde{C}\,e^{-\gamma_1\,|n|},\ \gamma_1>0.$$

Denote by $p^{\pm}(y)$ the functions from the kernels of adjoint periodic operators defined on $Y=(0,1]\times Q$:

$$
\left\{
\begin{array}{ll}
(A^{\pm})^{*}p_{\#}^{\pm}\equiv -\mathrm{div}\,(a^{\pm}\nabla\,p_{\#}^{\pm})+\mathrm{div}\,(b^{\pm}\,p_{\#}^{\pm})=0, & y\in Y,\\[2mm]
(B^{\pm})^{*}p_{\#}^{\pm}\equiv \dfrac{\partial p_{\#}^{\pm}}{\partial n_{a^{\pm}}}-(b^{\pm},n)\,p_{\#}^{\pm}=0, & y\in\partial Y.
\end{array}
\right. \tag{7.3}
$$

Each of problems (7.3) related to (a^{+},b^{+}) and (a^{-},b^{-}), respectively, has a unique up to a multiplicative constant solution which is positive and continuous everywhere in $\overline{Y}$ (see, for example, [**13**], Section 3). In both half-cylinders $G_{-\infty}^{0}$ and $G_{0}^{+\infty}$ we introduce the so-called effective drifts:

$$
\bar{b}_{1}^{\pm}=\int\limits_{Y}\Big(a_{1j}^{+}(x)\frac{\partial p_{\#}^{\pm}(x)}{\partial x_{j}}-b_{1}^{\pm}(x)\,p_{\#}^{\pm}(x)\Big)\,dx.
$$

It was shown in [**13**] that the behaviour of solution in half-cylinder crucially depends on the sign of the effective drift. In the case of infinite cylinder the situation is even more difficult: a solution might fail to exist under proper choice of the signs of $\bar{b}_{1}^{\pm}$.

The goal of this section is to show that in the case $\bar{b}_{1}^{+}<0$, $\bar{b}_{1}^{-}>0$ problem (7.1) possess a bounded (in a proper sense) solution, which stabilizes to constants, as $|x_1|\to\infty$.

DEFINITION 7.1. We say that $u(x)$ is a weak solution to problem (7.1) if, for any $\psi(x)\in C_{0}^{\infty}((-\infty,+\infty);C^{\infty}(\overline{Q}))$, the following integral inequality holds:

$$
\int\limits_{\mathbb{G}}(a(x)\nabla u,\nabla\psi)\,dx-\int\limits_{\mathbb{G}}(b,\nabla u)\,\psi\,dx-\int\limits_{\Gamma}g(x)\psi(x)\,d\sigma=\int\limits_{\mathbb{G}}f(x)\,\psi(x)\,dx.
$$

DEFINITION 7.2. A weak solution $u(x)$ of problem (7.1) is said to be bounded if

$$
\|u\|_{L^{2}(G_{n}^{n+1})}\le C,
$$

with a constant C independent of n.

The following theorem contains the main result of the section.

THEOREM 7.3. *Let conditions* $(\mathbf{A1})-(\mathbf{A3})$ *be fulfilled and suppose that* $\bar{b}_{1}^{+}<0$ *and* $\bar{b}_{1}^{-}>0$. *Then, for any constants* K_{∞}^{+} *and* K_{∞}^{-}*, there exists a bounded solution* $u(x)$ *of problem* (7.1) *that converges to these constants, as* $x_1\to\pm\infty$*, respectively,*

$$
\begin{array}{ll}
\|u-K_{\infty}^{-}\|_{L^{2}(G_{-\infty}^{-n})}\le M_{1}\,e^{-\gamma\,n}, & n>0,\\
\|u-K_{\infty}^{+}\|_{L^{2}(G_{n}^{+\infty})}\le M_{1}\,e^{-\gamma\,n}, & n>0;
\end{array} \tag{7.4}
$$

moreover, the following estimates hold

$$
\|u\|_{L^{2}(G_{N}^{N+1})}\le M_{2};\quad \|\nabla\,u\|_{L^{2}(\mathbb{G})}\le M_{3}. \tag{7.5}
$$

The constants M_1, M_2 *and* M_3 *in* (7.4), (7.5) *have the form*

$$
M_{1}=C_{0}\,(|K_{\infty}^{+}|+|K_{\infty}^{-}|)+C_{1}\left(\|(1+|x_{1}|^{\frac{3}{2}+\nu})\,f\|_{L^{2}(\mathbb{G})}+\|(1+|x_{1}|^{\frac{3}{2}+\nu})\,g\|_{L^{2}(\Gamma)}\right);
$$

$$M_2 = C_2 \left(\|(1+|x_1|^{\frac{3}{2}+\nu})\, f\|_{L^2(\mathbb{G})} + \|(1+|x_1|^{\frac{3}{2}+\nu})\, g\|_{L^2(\Gamma)}\right) + C_3 \left(|K_\infty^+| + |K_\infty^-|\right);$$
$$M_3 = C_2 \left(\|(1+|x_1|)\, f\|_{L^2(\mathbb{G})} + \|(1+|x_1|)\, g\|_{L^2(\Gamma)}\right) + C_3\, |K_\infty^+ - K_\infty^-|,$$
where C_0, C_1, C_2 and C_3 only depend on Λ, d and Q, $\nu > 0$.

PROOF. The proof consists of two steps. On the first step we show that there exists u which solves nonhomogeneous problem (7.1) and decays at infinity. On the second step we prove the existence of a solution to the homogeneous equation which stabilizes to some nonzero constants at infinity.

The case $f = g = 0$. Consider the following sequence of the auxiliary boundary value problems:

$$\begin{cases} A\, u^k = 0, & x \in G_{-k}^k, \\ B\, u^k = 0, & x \in \Gamma_{-k}^k, \\ u^k(\pm k, x') = K_\infty^\pm, & x' \in Q \end{cases} \tag{7.6}$$

and denote $v^k = u^k - (K_\infty^+ + K_\infty^-)/2$. Then v^k solves the problem

$$\begin{cases} A\, v^k = 0, & x \in G_{-k}^k, \\ B\, v^k = 0, & x \in \Gamma_{-k}^k, \\ v^k(\pm k, x') = \pm\dfrac{1}{2}\,(K_\infty^+ - K_\infty^-), & x' \in Q. \end{cases}$$

By the maximum principle,

$$|v^k| \le \frac{1}{2}\, |K_\infty^+ - K_\infty^-|, \quad x \in G_{-k}^k, \quad \forall k. \tag{7.7}$$

Indeed, by the maximum principle, a negative minimum can not be attained in the interior of the domain G_{-k}^k. The assumption that a negative minimum is attained on the lateral boundary Γ_{-k}^k also contradicts the maximum principle. One can prove this extending v_k by reflection across the lateral boundary and using the fact that v_k satisfies homogeneous Neumann boundary condition on Γ_{-k}^k. More detailed proof can be found in [**13**], Section 3. This trick is used many times throughout the paper and allows us to apply the maximum principle, the Harnack inequality and Nash estimates up to the lateral boundary of the cylinder.

Consequently, we obtain the following uniform estimate (w.r.t k) for the solution $u^k(x)$ of problem (7.6):

$$\begin{aligned} \|u^k\|_{L^2(G_N^{N+1})} &\le \frac{1}{2}\, |Q|^{1/2} \left(|K_\infty^+ - K_\infty^-| + |K_\infty^+ + K_\infty^-|\right) \\ &\le |Q|^{1/2} \left(|K_\infty^+| + |K_\infty^-|\right), \ \forall N. \end{aligned}$$

It follows from estimate (7.7) that the L^2-norm of the gradient of u^k in every finite cylinder is also bounded (see, e.g. [**3**]):

$$\|\nabla\, u^k\|_{L^2(G_N^{N+1})} = \|\nabla\, v^k\|_{L^2(G_N^{N+1})} \le C_1\, |K_\infty^+ - K_\infty^-|, \ \forall N,$$

where the constant C_1 depends only on Λ, d and Q.

Using the compactness arguments we conclude that, up to a subsequence, u^k converges to a solution u of problem (7.1) (with $f = g = 0$) strongly in $L^2_{loc}(\mathbb{G})$ and $\nabla u^k \rightharpoonup \nabla u$ weakly in $(L^2_{loc}(\mathbb{G}))^d$, as $k \to \infty$.

By the trace theorem

$$\|u^k\|_{H^{1/2}(S_{\pm 1})} \le C\,|K^+_\infty - K^-_\infty|,$$

and moreover (see [**3**], Theorem 8.24)

$$\|u^k\|_{C^\alpha(S_{\pm 1})} \le C\|u^k\|_{L^2(G^2_{-2})} \le C\ |K^+_\infty - K^-_\infty| \tag{7.8}$$

with $\alpha > 0$ and a constant C depends only on d, Q and Λ. Here again we used the same extension arguments as above that allowed us to obtain estimate (7.8) in $S_{\pm 1}$ up to $\{\pm 1\} \times \partial Q$, not only in the interior parts of these domains.

Consider an auxiliary problem

$$\begin{cases} A\,\hat{u}^k = 0, & x \in G^k_1,\\ B\,\hat{u}^k = 0, & x \in \Gamma^k_1,\\ \hat{u}^k(1, x') = \max\{K^-_\infty, K^+_\infty\}, \quad \hat{u}^k(k, x') = K^+_\infty, & x' \in Q. \end{cases} \tag{7.9}$$

In view of (7.8), by the maximum principle, $u^k \le \hat{u}^k$ in G^k_1. As was proved in [**13**] (see Theorem 6.7), in the case $\bar{b}^+_1 < 0$, for any constant K^+_∞, the following estimate is valid:

$$|u^k - K^+_\infty| \le |\hat{u}^k - K^+_\infty| \le C\ \max\{|K^-_\infty|, |K^+_\infty|\}\, e^{-\gamma x_1}, \quad x_1 > 1.$$

Since, up to a subsequence, $\{u^k\}$ converges to u uniformly on every compact set $K \subset \mathbb{G}$, then

$$|u - K^+_\infty| \le C\ \max\{|K^-_\infty|, |K^+_\infty|\}\, e^{-\gamma x_1}, \quad x_1 > 1.$$

The last estimate yields

$$\|u - K^+_\infty\|_{L^2(G^{N+1}_N)} \le C\ \max\{|K^-_\infty|, |K^+_\infty|\}\, e^{-\gamma N}, \quad N = 1, ..., k-1.$$

By the standard elliptic estimates we obtain

$$\|\nabla u\|_{L^2(G^{N+1}_N)} \le C\|u - K^+_\infty\|_{L^2(G^{N+2}_{N-1})} \le C\ \max\{|K^-_\infty|, |K^+_\infty|\}\, e^{-\gamma N},$$

with $N = 1, ..., k-1$. Similarly, in G^1_{-k} we have that $u \to K^-_\infty$ and $\nabla u \to 0$ at the exponential rate, as $x_1 \to -\infty$. This implies that, for any constants $K^\pm_\infty$, there is a solution $u(x)$ of (7.1) with $f = g = 0$ such that $u \to K^\pm_\infty$, as $x_1 \to \pm\infty$ and

$$\begin{gathered} \|u\|_{L^2(G^{N+1}_N)} \le C\left(|K^+_\infty - K^-_\infty| + |K^+_\infty + K^-_\infty|\right) \le C\left(|K^+_\infty| + |K^-_\infty|\right),\\ \|\nabla\, u\|_{L^2(\mathbb{G})} \le C\,|K^+_\infty - K^-_\infty|. \end{gathered} \tag{7.10}$$

The case $f, g \ne 0$. Our next goal is to prove the existence of a solution of problem (7.1) that decays exponentially at infinity. To this end we consider the following sequence of boundary value problems:

$$
\left\{
\begin{array}{ll}
A\,u_k = f(x), & x \in G^k_{-k},\\
B\,u_k = g(x), & x \in \Gamma^k_{-k},\\
u_k(-k,x') = u_k(k,x') = 0, & x' \in Q.
\end{array}
\right.
\tag{7.11}
$$

Without loss of generality we assume that $f(x) > 0$ and $g(x) > 0$, otherwise we represent these functions as the sums of their positive and negative parts. Moreover, we assume that $\operatorname{supp} f, \operatorname{supp} g \subset G_0^{+\infty}$. The case when the supports of f and g are in $G^0_{-\infty}$ can be considered similarly.

Suppose first that the coefficients a_{ij}, b_j and the functions f and g are smooth. Thus, by the strong maximum principle (see, for example, [**3**]), $u^k(x) > 0$, $x \in G_0^k \cup \Gamma_0^k$.

It has been proved in [**13**] (see Theorem 6.7) that in the semi-infinite cylinder $G^{-1}_{-\infty}$ the following estimate for u_k takes place:

$$
u_k(x_1, x') \le C_0\, \|u_k\|_{L^\infty(S_{-1})}\, e^{\gamma\, x_1}, \quad x_1 < -1,\ \gamma > 0,
$$

where C_0 depends only on Λ, d and Q. Since u_k is positive, then the Harnack inequality is valid in the fixed domain G^0_{-1} with a constant α which depends only on d, $|Q|$ and Λ, thus,

$$
u_k(x) \le \alpha\, e^{\gamma\, x_1} \min_{G^0_{-1}} u_k(x), \quad x \in G^{-1}_{-\infty}.
$$

Obviously, there exists $\xi > 1$ such that

$$
u_k(-\xi, x') < \frac{1}{2} \min_{G^0_{-1}} u_k(x).
\tag{7.12}
$$

Due to the linearity of the problem in $G^k_{-\xi}$ we represent u_k as a sum $v_k + w_k$, where v_k is a solution of the homogeneous equation with nonzero Dirichlet boundary conditions

$$
\left\{
\begin{array}{ll}
A\,v_k = 0, & x \in G^k_{-\xi},\\
B\,v_k = 0, & x \in \Gamma^k_{-\xi},\\
v_k(-\xi,x') = u_k(-\xi,x'),\ v_k(k,x') = 0, & x' \in Q;
\end{array}
\right.
\tag{7.13}
$$

and w_k is a solution of the problem

$$
\left\{
\begin{array}{ll}
A\,w_k = f(x), & x \in G^k_{-\xi},\\
B\,w_k = g(x), & x \in \Gamma^k_{-\xi},\\
w_k(-\xi,x') = w_k(k,x') = 0, & x' \in Q.
\end{array}
\right.
\tag{7.14}
$$

By the maximum principle we have

$$
v_k(x) \le \frac{1}{2} \min_{G^0_{-1}} u_k(x), \quad x \in G^k_{-\xi}.
$$

As was proved in [**13**] (see Lemma 8.2), a solution of problem (7.14) satisfies the following estimate:

$$\|\nabla w_k\|_{L^2(G^k_{-\xi})} \le C\,\|(1+|x_1|)\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+|x_1|)\,g\|_{L^2(\Gamma_0^{+\infty})}.$$

Thus, by Friedrichs' inequality

$$\|\nabla w_k\|_{L^2(G^0_{-1})} \le C(\xi)\,\|(1+|x_1|)\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+|x_1|)\,g\|_{L^2(\Gamma_0^{+\infty})}.$$

Obviously,

$$\begin{aligned}&\min_{G^0_{-1}} u_k(x) \le \|u_k\|_{L^2(G^0_{-1})} \le \|v_k\|_{L^2(G^0_{-1})} + \|w_k\|_{L^2(G^0_{-1})}\\ &\le \frac{1}{2}\min_{G^0_{-1}} u_k(x) + \|w_k\|_{L^2(G^0_{-1})}.\end{aligned}$$

It follows from the last inequality that

$$\min_{G^0_{-1}} u_k(x) \le C\,\|(1+|x_1|)\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+|x_1|)\,g\|_{L^2(\Gamma_0^{+\infty})}. \tag{7.15}$$

With the help of the Harnack inequality and (7.15) one has $u_k(-1,x') \le C$. Then, in view of the maximum principle, $u_k(x)$ is bounded in G^{-1}_{-k}. It remains to apply Theorem 8.7 and Lemma 9.2 from [**13**]. According to these results, for $\bar b_1^- > 0$ and $\bar b_1^+ < 0$, we obtain

$$\begin{gathered}|u_k(x)| \le C_0\,\|u_k\|_{L^\infty(S_{-1})}\,e^{\gamma x_1}, \quad x \in G^{-1}_{-k};\\ \|u_k\|_{L^2(G_n^{+\infty})} \le (C_0\,\|u_k\|_{L^\infty(S_{-1})} + C_1)\,e^{-\gamma n},\ \ n>0;\\ \|u_k\|_{L^2(G_N^{N+1})} \le C\,\|(1+|x_1|^{\frac{3}{2}+\nu})\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+|x_1|^{\frac{3}{2}+\nu})\,g\|_{L^2(\Gamma_0^{+\infty})},\ \ \forall N;\\ \|\nabla u_k\|_{L^2(G^k_{-k})} \le C\,\|(1+|x_1|)\,f\|_{L^2(G_0^{+\infty})} + C\,\|(1+|x_1|)\,g\|_{L^2(\Gamma_0^{+\infty})},\end{gathered}$$

and, consequently, for any n,

$$\|u_k\|_{L^2(G_n^{n+1})} \le C_1\left(\|(1+|x_1|^{\frac{3}{2}+\nu})\,f\|_{L^2(\mathbb{G})} + \|(1+|x_1|^{\frac{3}{2}+\nu})\,g\|_{L^2(\Gamma)} + 1\right)e^{-\gamma\,n}.$$

For the nonsmooth data the desired estimates can be justified by means of usual smoothing procedure (see, for instance, [**13**] for details).

Thus, one can see that, up to a subsequence, $\{u_k\}$ (being extended by zero to the whole cylinder $\mathbb{G}$), converges weakly in $H^1_{loc}(\mathbb{G})$ to a solution u of problem (7.1). Moreover, by construction, u decays exponentially at infinity and estimates (7.4), (7.5) hold with $K^\pm_\infty = 0$.

As was shown above, for any constants $K^\pm_\infty$, there exists a solution of homogeneous equation, stabilizing to these constants at infinity and satisfying estimates (7.10). Summing up such a solution with $u(x)$, we obtain the desired solution of nonhomogeneous problem. Theorem 7.3 is proved. □

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Paper C

PAPER C

Homogenization of convection-diffusion equation in infinite cylinder

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ABSTRACT. The paper deals with a periodic homogenization problem for a non-stationary convection-diffusion equation stated in a thin infinite cylindrical domain with homogeneous Neumann boundary condition on the lateral boundary. It is shown that homogenization result holds in moving coordinates, and that the solution admits an asymptotic expansion which consists of the interior expansion being regular in time, and an initial layer.

Keywords: Homogenization, convection-diffusion equation, infinite cylinder, asymptotic behaviour

1. Introduction

The goal of the paper is to study the asymptotic behaviour of a solution to an initial boundary problem for a convection-diffusion equation defined in a thin infinite cylinder with homogeneous Neumann condition on its lateral boundary. We assume that the coefficients of the equation are periodic in the axial direction of the cylinder and that the period is of the same order as the cylinder diameter. The corresponding parabolic operator takes the form

$$\partial_t u - \mathrm{div}\Big(a\Big(\frac{x}{\varepsilon}\Big)\nabla u\Big) + \frac{1}{\varepsilon}\Big(b\Big(\frac{x}{\varepsilon}\Big), \nabla u\Big); \tag{1.1}$$

here ε is a small positive parameter, and we assume the standard uniform ellipticity conditions on $a(y)$ and the boundedness of the entries of $a(y)$ and $b(y)$.

Notice that the scaling factor $1/\varepsilon$ is natural for the convection term. Indeed, if one wants to consider the long-term behaviour of a convection-diffusion process described by the equation

$$\partial_s u - \operatorname{div}\big(a(y)\nabla u\big) + \big(b(y), \nabla u\big) = 0$$

in a fixed infinite cylinder, then making the diffusive change of variables

$$x = \varepsilon y,\ t = \varepsilon^2 s$$

leads to a convection-diffusion problem for operator (1.1) in a thin cylinder.

Closely related problems for a convection-diffusion equation defined in the whole space have been considered in [**6**] and [**1**]. It was proved in particular that the homogenization takes place in moving coordinates $(x,t) \longrightarrow \big(x - (\bar{b}/\varepsilon)t, t\big)$ with a constant vector $\bar{b}$.

Various homogenization problems for divergence form operators and systems in thin bounded domains have been investigated by many authors, we mention here the works [**7**], [**10**] and [**11**]. General homogenization theory results for parabolic equations can be found in [**5**] and [**12**].

In the paper we first prove uniform in ε a priori estimates for the solution. This requires integration in weighted spaces where the solution of the periodic adjoint cell problem is used as a weight. Then we construct the leading terms of the asymptotic expansion in moving coordinates, determine the effective speed, and obtain the estimates for the rate of convergence. Additional difficulty appearing in the problem under consideration is the dimension reduction issue. Indeed, the solutions of the original problem belong to variable Sobolev spaces, which makes the convergence analysis rather involved.

The paper is organized as follows. Section 2 contains the problem setup. In Section 3 we deal with a priori estimates and study auxiliary parabolic cell problems. In Section 4 we construct formal asymptotic expansion which includes the corresponding initial layers, the presence of the initial layer allows us to satisfy the initial condition in higher order approximations. In Section 5 is devoted to the convergence analysis.

2. Problem statement

Let Q be a bounded domain in $\mathbb{R}^{d-1}$ with the Lipschitz boundary ∂Q. For any $\varepsilon > 0$, we denote by $\mathbb{G}_\varepsilon$ a thin infinite cylinder $\mathbb{R} \times \varepsilon\, Q$ with the axis directed along x_1. The lateral boundary of the cylinder $\mathbb{G}_\varepsilon$ is denoted by $\Sigma_\varepsilon = \mathbb{R} \times \partial(\varepsilon Q)$. We study the following non-stationary convection-diffusion equation:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \mathcal{A}_\varepsilon\, u^\varepsilon(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{G}_\varepsilon, \\ \mathcal{B}_\varepsilon\, u^\varepsilon(t,x) = 0, & (t,x) \in (0,T) \times \Sigma_\varepsilon, \\ u^\varepsilon(0,x) = \varphi(x_1), & x \in \mathbb{G}_\varepsilon, \end{cases} \tag{2.1}$$

where

$$
\mathcal{A}_\varepsilon u = -\mathrm{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla_x u\Big) + \frac{1}{\varepsilon}\,(b\big(\frac{x}{\varepsilon}\big), \nabla_x u),
\tag{2.2}
$$

$$
\mathcal{B}_\varepsilon u = (a\big(\frac{x}{\varepsilon}\big)\nabla_x u, n).
$$

We suppose that the following conditions are fulfilled:

(H1) Q is a Lipschitz bounded domain in $\mathbb{R}^{d-1}$;
(H2) $a_{ij}(y), b_j(y) \in L^\infty(\mathbb{G})$, $i, j = 1, ..., d$, are 1-periodic functions with respect to y_1;
(H3) The matrix $a(y)$ satisfies the uniform ellipticity condition, that is there exists a positive constant Λ such that, for almost all $x \in \mathbb{R}^d$,

$$
\Lambda\,|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(y)\,\xi_i\,\xi_j, \quad \forall \xi \in \mathbb{R}^d.
\tag{2.3}
$$

(H4) $\varphi(x_1) \in C_0^\infty(\mathbb{R})$.

DEFINITION 2.1. A function $u^\varepsilon(t,x)$ is said to be a weak solution of problem (2.1) in $(0,T] \times \mathbb{G}_\varepsilon$ if

$$
u^\varepsilon \in L^\infty[\delta, T; L^2_{\mathrm{loc}}(\mathbb{G}_\varepsilon)] \cap L^2[0,T; H^1_{\mathrm{loc}}(\mathbb{G}_\varepsilon)], \quad \delta \in (0,T)
$$

and u^ε satisfies

$$
\int_0^T \int_{\mathbb{G}_\varepsilon} \Big\{ -u^\varepsilon\,\partial_t\psi + (a^\varepsilon \nabla u^\varepsilon, \nabla\psi) + (b^\varepsilon, \nabla u^\varepsilon)\,\psi \Big\}\,dx\,dt = \int_{\mathbb{G}_\varepsilon} \varphi(x_1)\,\psi(0,x)\,dx
$$

for any $\psi \in L^2[0,T; H^1(\mathbb{G}_\varepsilon)]$ such that $\partial_t\psi \in L^2[0,T; L^2(\mathbb{G}_\varepsilon)]$ and $\psi(T,x) = 0$.

We are interested in the asymptotic behaviour of $u^\varepsilon(t,x)$, as $\varepsilon \to 0$. Notice that, for any $\varepsilon > 0$, the existence and the uniqueness of a generalized solution to problem (2.1) is given by classical theory (see, e.g., [**8**]).

3. Some preliminary results

3.1. A priori estimates. In what follows we denote $Y = [0,1) \times Q$,

$$
\mathcal{A}v = -\mathrm{div}_y(a(y)\nabla_y v) + (b(y), \nabla_y v), \quad \mathcal{B}v = (a(y)\nabla_y v, n);
$$

$$
\mathcal{A}^* p^* = -\mathrm{div}(a\nabla p^*) - \mathrm{div}(bp^*), \quad \mathcal{B}^* p^* = (a\nabla p^*, n) + (b,n)p^*,
$$

By the Krein-Rutman theorem and the Harnack inequality, the adjoint periodic problem

$$
\begin{cases}
\mathcal{A}^*\,p^*(y) = 0, & y \in Y, \\
\mathcal{B}^*\,p^*(y) = 0, & y \in \partial Y, \\
p^*(y) & \text{is periodic in } y_1,
\end{cases}
\tag{3.1}
$$

has a positive solution $p^*(y) \in C(Y) \cap H^1(Y)$ such that

$$
0 < p^- \le p^*(y) \le p^+ < \infty.
\tag{3.2}
$$

We fix the choice of p^* by the normalization condition

$$\int\limits_Y p^*(y)\,dy = 1.$$

The goal of this section is to obtain a priori estimates for a non-stationary convection-diffusion equation stated in a thin infinite cylinder. Namely, we consider the following non-homogeneous problem:

$$\left\{\begin{array}{ll} \partial_t u^\varepsilon(t,x) + \mathcal{A}_\varepsilon\, u^\varepsilon(t,x) = f(t,x), & (t,x)\in(0,T)\times\mathbb{G}_\varepsilon,\\ \mathcal{B}_\varepsilon\, u^\varepsilon(t,x) = \varepsilon\, g(t,x), & (t,x)\in(0,T)\times\Sigma_\varepsilon,\\ u^\varepsilon(0,x) = \varphi(x), & x\in\mathbb{G}_\varepsilon, \end{array}\right. \tag{3.3}$$

Multiplying the equation in (3.3) by $p^*(x/\varepsilon)\,u^\varepsilon(x)$ and integrating the resulting relation by parts over $\mathbb{G}_\varepsilon$, we obtain

$$\begin{aligned} &\frac{1}{2}\int\limits_{\mathbb{G}_\varepsilon}\frac{\partial(u^\varepsilon)^2}{\partial t}\,p^*\big(\frac{x}{\varepsilon}\big)\,dx + \frac{1}{2}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon(t,x))^2\,\mathcal{A}^*_\varepsilon p^*\big(\frac{x}{\varepsilon}\big)\,dx+\\ &+\frac{1}{2}\int\limits_{\Sigma_\varepsilon}(u^\varepsilon(t,x))^2\,\mathcal{B}^*_\varepsilon p^*\big(\frac{x}{\varepsilon}\big)\,d\sigma + \int\limits_{\mathbb{G}_\varepsilon}(a^\varepsilon\nabla u^\varepsilon,\nabla u^\varepsilon)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx =\\ &=\varepsilon\int\limits_{\Sigma_\varepsilon} g(t,x)\,u^\varepsilon(t,x)\,p^*\big(\frac{x}{\varepsilon}\big)\,d\sigma + \int\limits_{\mathbb{G}_\varepsilon} f(t,x)\,u^\varepsilon(t,x)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx. \end{aligned}$$

Here we use the notations

$$\begin{aligned} \mathcal{A}^*_\varepsilon q(x) &= -\mathrm{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla q(x)\Big) - \frac{1}{\varepsilon}\,\mathrm{div}\Big(b\big(\frac{x}{\varepsilon}\big)\,q(x)\Big),\\ \mathcal{B}^*_\varepsilon q(x) &= (a\big(\frac{x}{\varepsilon}\big)\nabla q(x), n) + \frac{1}{\varepsilon}\,(b\big(\frac{x}{\varepsilon}\big), n)\,q(x). \end{aligned}$$

Taking into account the definition of $p^*(y)$ we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,p^*\big(\frac{x}{\varepsilon}\big)\,dx + \int\limits_{\mathbb{G}_\varepsilon}(a^\varepsilon\nabla u^\varepsilon,\nabla u^\varepsilon)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx =\\ &=\varepsilon\int\limits_{\Sigma_\varepsilon} g(t,x)\,u^\varepsilon(t,x)\,p^*\big(\frac{x}{\varepsilon}\big)\,d\sigma + \int\limits_{\mathbb{G}_\varepsilon} f(t,x)\,u^\varepsilon(t,x)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx. \end{aligned}$$

Using the positive definiteness of the matrix $a(y)$, bounds (3.2), and the Cauchy-Bunyakovsky inequality, one can obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,p^*\big(\frac{x}{\varepsilon}\big)\,dx+\Lambda\,p^-\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon|^2\,dx\le$$
$$+\frac{1}{2\gamma}\Big\{\int\limits_{\mathbb{G}_\varepsilon}(f(t,x))^2\,dx+\varepsilon\int\limits_{\Sigma_\varepsilon}(g(t,x))^2\,d\sigma\Big\}$$
$$+\frac{\gamma}{2}\,p^+\,\varepsilon\int\limits_{\Sigma_\varepsilon}(u^\varepsilon)^2\,d\sigma+\frac{\gamma}{2}\,p^+\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,dx$$

for any $\gamma>0$. By the trace theorem

$$\int\limits_{\Sigma_\varepsilon}(u^\varepsilon)^2\,d\sigma\le\frac{C_1}{\varepsilon}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,dx+C_2\,\varepsilon\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon|^2\,dx$$

with constants C_1, C_2 independent of ε. Consequently, for a sufficiently small γ,

$$\frac{d}{dt}\left[\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,p^*\big(\frac{x}{\varepsilon}\big)\,dx+\int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon(s,x)|^2\,dxds\right]$$
$$\le C\left\{\|f\|^2_{L^2(\mathbb{G}_\varepsilon)}+\varepsilon\,\|g\|^2_{L^2(\Sigma_\varepsilon)}\right\}+\left[\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,p^*\big(\frac{x}{\varepsilon}\big)\,dx+\int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon(s,x)|^2\,dxds\right].$$

Integrating with respect to t and applying the Grönwall lemma and the positiveness of the function p^*, one can see that

$$\begin{aligned}&\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,dx+\int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon(s,x)|^2\,dxds\le\\&\le C\,e^t\left\{\|f\|^2_{L^2[0,T;L^2(\mathbb{G}_\varepsilon)]}+\varepsilon\,\|g\|^2_{L^2[0,T;L^2(\Sigma_\varepsilon)]}+\|\varphi\|^2_{L^2(\mathbb{G}_\varepsilon)}\right\},\quad t\in(0,T]\end{aligned}\tag{3.4}$$

where the constant C does not depend on ε, and depends only on Λ, d and Q.

For the justification procedure we also need a priori estimates in the case of right-hand side being the divergence of a bounded vector-function. Namely, consider the following problem:

$$\begin{cases}\partial_t u^\varepsilon(t,x)+\mathcal{A}_\varepsilon\,u^\varepsilon(t,x)=\mathrm{div}_x F(t,x), & (t,x)\in(0,T)\times\mathbb{G}_\varepsilon,\\ \mathcal{B}_\varepsilon\,u^\varepsilon(t,x)=-(F,n), & (t,x)\in(0,T)\times\Sigma_\varepsilon,\\ u^\varepsilon(0,x)=0, & x\in\mathbb{G}_\varepsilon,\end{cases}\tag{3.5}$$

with $F(t,x)$ such that

$$|F(t,x)|\le f_1(t,x)\,e^{-\gamma\,|x_1|},\quad f_1\in L^\infty((0,T)\times\mathbb{G}_\varepsilon).$$

Multiplying equation in (3.5) by $p^*(\frac{x}{\varepsilon})u^\varepsilon(x)$ and integrating by parts over $\mathbb{G}_\varepsilon$, we obtain

$$
\begin{aligned}
(3.6)\qquad &\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\, p^*\big(\frac{x}{\varepsilon}\big)\, dx + \int\limits_{\mathbb{G}_\varepsilon}(a^\varepsilon\nabla u^\varepsilon, \nabla u^\varepsilon)\, p^*\big(\frac{x}{\varepsilon}\big)\, dx = \\
&= -\int\limits_{\mathbb{G}_\varepsilon}(F(t,x),\ \nabla u^\varepsilon(t,x))\, p^*\big(\frac{x}{\varepsilon}\big)\, dx \\
&-\int\limits_{\mathbb{G}_\varepsilon}(F(t,x),\ \nabla p^*\big(\frac{x}{\varepsilon}\big))\, u^\varepsilon(t,x)\, dx \equiv I_1^\varepsilon + I_2^\varepsilon.
\end{aligned}
$$

Exploiting the Couchy-Bunyakovsky inequality and taking into account (3.2) one gets

$$
\begin{aligned}
|I_1^\varepsilon| &\le \frac{p^+}{2\delta}\|F\|^2_{L^2(\mathbb{G}_\varepsilon)} + \frac{p^+\,\delta}{2}\,\|\nabla u^\varepsilon\|_{L^2(\mathbb{G}_\varepsilon)} \\
&\le \frac{p^+}{\delta}\|f_1\|^2_{L^\infty((0,T)\times\mathbb{G}_\varepsilon)}\,\varepsilon^{d-1} + p^+\,\delta\,\|\nabla u^\varepsilon\|_{L^2(\mathbb{G}_\varepsilon)}, \quad \delta = \frac{p^-\Lambda}{p^+}.
\end{aligned}
$$

The integral I_2^ε is estimated as follows

$$
\begin{aligned}
|I_2^\varepsilon| &\le \frac{1}{2}\int\limits_{\mathbb{G}_\varepsilon}|\big(F, \nabla p^*\big(\frac{x}{\varepsilon}\big)\big)|^2\, dx + \frac{1}{2}\,\|u^\varepsilon\|^2_{L^2(\mathbb{G}_\varepsilon)} \\
&\le \sum_{n=-\infty}^{+\infty}\|F\|^2_{L^\infty(\varepsilon Y_n)}\int\limits_{\varepsilon Y_n}|\nabla p^*\big(\frac{x}{\varepsilon}\big)|^2\, dx + \frac{1}{2}\,\|u^\varepsilon\|^2_{L^2(\mathbb{G}_\varepsilon)} \\
&\le \sum_{n=-\infty}^{+\infty}\|F\|^2_{L^\infty(\varepsilon Y_n)}\,\varepsilon^{d-2}\int\limits_{Y_n}|\nabla p^*(y)|^2\, dy + \frac{1}{2}\,\|u^\varepsilon\|^2_{L^2(\mathbb{G}_\varepsilon)} \\
&\le C\,\varepsilon^{d-1}\,\frac{1}{\varepsilon}\,\|f_1\|^2_{L^\infty((0,T)\times\mathbb{G}_\varepsilon)} + \frac{1}{2}\,\|u^\varepsilon\|^2_{L^2(\mathbb{G}_\varepsilon)},
\end{aligned}
$$

where $Y_n = (n, n+1] \times Q$.

Finally, combining the obtained estimates for I_1^ε and I_2^ε with (3.6), and using the Grönwall's lemma, for $t \in (0,T]$ one has

$$
(3.7)\qquad \int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\, dx + \int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon(s,x)|^2\, dxds \le \frac{C\,e^t}{\varepsilon}\,\varepsilon^{d-1}\,\|f_1\|^2_{L^\infty((0,T)\times\mathbb{G}_\varepsilon)}.
$$

3.2. Auxiliary results. In the sequel we will need the information about the asymptotic behaviour of solutions to parabolic equations, as $t \to \infty$. Consider the initial boundary value problem

$$
(3.8)\qquad \begin{cases}
\partial_\tau v(\tau,y) + \mathcal{A}\, v(\tau,y) = 0, & (\tau,y) \in (0,\infty)\times Y, \\
\mathcal{B}\, v(\tau,y) = 0, & (\tau,y) \in (0,\infty)\times \partial Y, \\
v(\tau,y) - y_1 - \ \text{periodic}, & \\
v(0,y) = \psi(y), & y \in Y,
\end{cases}
$$

where $Y = [0,1) \times Q$.

LEMMA 3.1. *Suppose conditions* $(\mathbf{H1}) - (\mathbf{H3})$ *are fulfilled and* $\psi(y) \in L^2(Y)$. *Then there exists a unique weak solution* v *to problem* (3.8)*, and it stabilizes to a constant* v^∞ *at the exponential rate, as* $\tau \to \infty$*, that is*

$$|v(\tau, y) - v^\infty| \le C_0 \, \|\psi\|_{L^2(Y)} e^{-\gamma\tau}, \quad y \in Y, \ \tau > 0, \tag{3.9}$$

with positive constants C_0 *and* γ *depending only on* Λ, d *and* Q. *Moreover,* $\nabla_y v$ *stabilizes exponentially to* 0*, as* $\tau \to \infty$*:*

$$\int\limits_{\tau}^{\tau+1} \int\limits_{Y} |\nabla_y v(s,y)|^2 \, dyds \le C \, e^{-2\gamma\tau}. \tag{3.10}$$

The constant v^∞ *is defined by*

$$v^\infty = \int\limits_{Y} \psi(y) \, p^*(y) \, dy,$$

where $p^*(y)$ *solves problem* (3.1).

PROOF. Let us consider two functions

$$m(\tau) = \min_{y \in Y} v(\tau, y), \quad M(\tau) = \max_{y \in Y} v(\tau, y).$$

By the maximum principle, $M(\tau)$ decreases and $m(\tau)$ increases. In view of the linearity of the problem, without loss of generality we assume that $m(\tau_0) = 0$. Since $v \ge 0$, then we can use the Harnack inequality

$$m(\tau_0 + 1) \ge \alpha \, M(\tau_0 + 1), \quad \alpha < 1$$

to obtain the estimate

$$\mathrm{osc}_{\tau=\tau_0+1} v(\tau, y) \equiv M(\tau_0+1) - m(\tau_0+1) \le (1-\alpha) M(\tau_0) \equiv \mathrm{osc}_{\tau=\tau_0} v(\tau, y).$$

Consequently,

$$\mathrm{osc}_{\tau=\tau_0+1} v(\tau, y) \le (1-\alpha) \mathrm{osc}_{\tau=\tau_0} v(\tau, y), \quad \tau_0 \ge 0$$

and, obviously, v converges to some constant v^∞, as $\tau \to \infty$

$$|v(\tau, y) - v^\infty| \le C_0 \, \|\psi\|_{L^2(Y)} e^{-\gamma\tau},$$

where C and γ depend only on Λ, d and Q.

Let us calculate the constant v^∞. To this end we multiply the equation in (3.8) by p^* and integrate by parts over the set $(0, \tau) \times Y$. As a result we obtain the following equality:

$$\int\limits_{Y} v(\tau, y) \, p^*(y) \, dy = \int\limits_{Y} \psi(y) \, p^*(y) \, dy.$$

Since v converges uniformly to the constant v^∞, as $\tau \to \infty$, then it follows from the last equality that

$$v^\infty = \int\limits_Y \psi(y)\, p^*(y)\, dy,$$

if p^* is normalized by $\int_Y p^* dy = 1$.

Now we prove estimate (3.10). Note that the function $w = v - v^\infty$ solves the same equation as v, and satisfies the initial condition $w(0,y) = \psi - v^\infty$. Multiplying the equation by w, integrating by parts and applying the Cauchy-bunyakovski inequality gives

$$\int\limits_Y |w(\tau+1,y)|^2\, dy + \Lambda \int\limits_\tau^{\tau+1}\int\limits_Y |\nabla w|^2\, dyds \le$$

$$\Lambda^{-1}\frac{1}{2\eta}\int\limits_\tau^{\tau+1}\int\limits_Y |w(s,y)|^2\, dyds + \Lambda^{-1}\frac{\eta}{2}\int\limits_\tau^{\tau+1}\int\limits_Y |\nabla_y w(s,y)|^2\, dyds + \int\limits_Y |w(\tau,y)|^2\, dy,$$

and, consequently, choosing $\eta < 2\Lambda^2$ and using (3.9), we obtain

$$\int\limits_\tau^{\tau+1}\int\limits_Y |\nabla w|^2\, dyds \le C\, e^{-2\gamma\tau}.$$

$\square$

The next lemma generalizes the result of Lemma 3.1 to the non-homogeneous case. Consider the boundary value problem

$$\left\{\begin{array}{ll} \partial_\tau v(\tau,y) + \mathcal{A}\, v(\tau,y) = f(\tau,y) + \mathrm{div}_y F(\tau,y), & (\tau,y)\in(0,\infty)\times Y,\\ \mathcal{B}\, v(\tau,y) = g(\tau,y) - (F(\tau,y),n), & (\tau,y)\in(0,\infty)\times\partial Y,\\ v(0,y) = 0, & y\in Y, \end{array}\right. \tag{3.11}$$

where $f \in L^2[0,\infty; L^2(Y)]$, $F \in L^2[0,\infty; L^2(Y)^d]$ and $g \in L^2[0,\infty; L^2(\partial Y)]$ decay exponentially, as $\tau\to\infty$, that is

$$\int\limits_\tau^{\tau+1} \|f(s,\cdot)\|^2_{L^2(Y)}\, ds \le C\, e^{-\gamma_1\tau};\ \int\limits_\tau^{\tau+1} \|F(s,\cdot)\|^2_{L^2(Y)^d}\, ds \le C\, e^{-\gamma_1\tau};$$

$$\int\limits_\tau^{\tau+1} \|g(s,\cdot)\|^2_{L^2(\partial Y)}\, ds \le C\, e^{-\gamma_1\tau}, \quad \gamma_1 > 0.$$

LEMMA 3.2. *Under the assumptions being made, a solution of problem* (3.11) *satisfies the estimates*

$$\int\limits_\tau^{\tau+1} \|v(s,\cdot) - v^\infty\|^2_{L^2(Y)}\, ds \le C\, e^{-\tilde\gamma\tau}, \tag{3.12}$$

$$\int_{\tau}^{\tau+1} \|\nabla v(s,\cdot)\|^2_{L^2(Y)}\,ds \le C\,e^{-\tilde\gamma\tau}, \quad \tilde\gamma > 0. \tag{3.13}$$

Here C depends on Λ, d and Q. The constant v^∞ is determined by

$$v^\infty = \int_0^\infty \int_Y f(\tau,y)\,p^*(y)\,dy\,d\tau - \int_0^\infty \int_Y (F(\tau,y),\nabla p^*(y))\,dy\,d\tau$$
$$+ \int_0^\infty \int_{\partial Y} g(\tau,y)\,p^*(y)\,d\sigma\,d\tau,$$

p^* being a solution of (3.1).

PROOF. First of all we represent the functions on the right-hand side of (3.11) as the sums of functions with finite supports, that is

$$f(\tau,y) = \sum_{m=-\infty}^{+\infty} f_m(\tau,y), \quad F(\tau,y) = \sum_{m=-\infty}^{+\infty} F_m(\tau,y), \quad g(\tau,y) = \sum_{m=-\infty}^{+\infty} g_m(\tau,y),$$

where $f_m(\tau,y) = f(\tau,y)\,\chi_{[m,m+1)}$, $F_m(\tau,y) = F(\tau,y)\,\chi_{[m,m+1)}$, $g_m(\tau,y) = g(\tau,y)\,\chi_{[m,m+1)}$, and $\chi_{[m,m+1)} = \chi_{[m,m+1)}(\tau)$ is the characteristic function of the interval $[m,m+1)$.

Due to the linearity of the problem, the solution v of (3.11) can be represented in the form

$$v(\tau,y) = \sum_{m=-\infty}^{+\infty} v_m(\tau,y),$$

where v_m solves the problem

$$\begin{cases} \partial_\tau v_m + \mathcal{A}\,v_m = f_m(\tau,y) + \mathrm{div}_y F_m(\tau,y), & (\tau,y)\in(0,\infty)\times Y, \\ \mathcal{B}\,v_m = g_m(\tau,y) - (F_m(\tau,y),n), & (\tau,y)\in(0,\infty)\times\partial Y, \\ v_m(0,y) = 0, & y\in Y. \end{cases} \tag{3.14}$$

Notice that, in view of the uniqueness of the solution, $v_m(\tau,y) = 0$ for $\tau\in[0,m)$. Then, multiplying the equation in (3.14) by v_m and integrating over $(m-1,m+2)\times Y$, we obtain

$$\int_Y (v_m(m+2,y))^2\,dy \le C\Big(\int_m^{m+1} \|f(s,\cdot)\|^2_{L^2(Y)}\,ds +$$
$$+ \int_m^{m+1} \|g(s,\cdot)\|^2_{L^2(\partial Y)}\,ds + \int_m^{m+1} \|F(s,\cdot)\|^2_{L^2(Y)}\,ds\Big) \le C\,e^{-\gamma_1 m}.$$

By Lemma 3.1, v_m satisfies estimate (3.9) in $(m+2,\infty)\times Y$, namely,

$$|v_m(\tau,y) - v_m^\infty| \le C\,e^{-\gamma_1 m}\,e^{-\gamma(\tau-m-2)} \le C\,e^{-\tilde\gamma\tau}, \quad \tau > m+2,$$

for some constant v_m^∞, where $\tilde\gamma = \min\{\gamma, \gamma_1\}$. In view of the maximum principle,

$$v_m^\infty \le C e^{-\gamma_1 m}.$$

Let us show that $v = \sum v_m$ stabilizes to $v^\infty = \sum v_m^\infty$, as $\tau \to \infty$. To this end we estimate the L^2-norm of the difference $v - v^\infty$.

$$\begin{aligned} \int\limits_N^{N+1} \|v(s,\cdot) - v^\infty\|^2_{L^2(Y)}\, ds &= \int\limits_N^{N+1} \|\sum_{m=0}^{+\infty} (v_m(s,\cdot) - v_m^\infty)\|^2_{L^2(Y)}\, ds = \\ &= \int\limits_N^{N+1} \|\Big\{\sum_{m\le N-2} + \sum_{m\ge N-1}\Big\}(v_m(s,\cdot) - v_m^\infty)\|^2_{L^2(Y)}\, ds \le \\ &\le C_1\, N^2\, e^{-2\gamma N} + C_2\, e^{-2\gamma_1 N} \le C\, e^{-\tilde\gamma N}, \quad \tilde\gamma > 0. \end{aligned}$$

The exponential decay of ∇v can be proved in much the same way as in the homogeneous case. □

4. Asymptotic expansion

4.1. Formal inner expansion. Following the ideas in [**6**] and [**1**], we are looking for an approximate solution in the form

$$u^\varepsilon \sim u_0\big(t\,,\, x_1 - \varepsilon^{-1}\,\bar b_1 t\big) + \sum_{k=1}^{\infty} \varepsilon^k\, v_k\big(t\,,\, x_1 - \varepsilon^{-1}\,\bar b_1 t\,,\, y\big), \quad y = \frac{x}{\varepsilon}, \tag{4.1}$$

where v_k, $k \ge 1$, are unknown functions which are 1-periodic in y_1; the constant $\bar b_1$ is to be determined.

Substituting (4.1) into (2.1) and collecting power-like terms in front of ε^{-1} in the equation and of ε^0 in the boundary condition, we obtain the following periodic problem for the unknown function v_1:

$$\begin{cases} \mathcal{A}_y\, v_1(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t\,,\, y) \\ = \big(\partial_{y_i} a_{i1}(y) - b_1(y) + \bar b_1\big)\ \partial_{x_1} u_0(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t), & y \in Y, \\ \mathcal{B}_y\, v_1(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t\,,\, y) = -a_{i1}(y)\, n_i\, \partial_{x_1} u_0(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t), & y \in \partial Y; \end{cases} \tag{4.2}$$

Setting

$$\bar b_1 = \int\limits_Y (a_{i1}(y)\, \partial_{y_i} p^*(y) + b_1(y)\, p^*(y))\, dy, \tag{4.3}$$

we guarantee that a solution to problem (4.2) exists. The specific form of the right-hand side of (4.2) suggests the following representation of v_1:

$$v_1(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t\,,\, y) = N_1(y)\, \partial_{x_1} u_0(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t) + u_1(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t),$$

where a Y-periodic function N_1 solves the problem

$$\text{(4.4)} \qquad \begin{cases} \mathcal{A}_y \, N_1(y) = \partial_{y_i} a_{i1}(y) - b_1(y) + \bar{b}_1, & y \in Y, \\ \mathcal{B}_y \, N_1(y) = -a_{i1}(y) \, n_i, & y \in \partial Y; \end{cases}$$

Similarly, we get the problem for v_2

$$\text{(4.5)} \qquad \begin{cases} \mathcal{A}_y \, v_2(t\,,\, x_1 - \varepsilon^{-1}\bar{b}_1 t\,,\, y) = -\partial_t u_0\big(t, x_1 - \dfrac{\bar{b}_1}{\varepsilon}\, r\big)\Big|_{r=t} \\ + \big\{a_{11}(y) + \partial_{y_i}(a_{i1}(y)N_1(y)) + a_{1j}(y)\partial_{y_j} N_1(y) \\ -b_1(y)\, N_1(y) + \bar{b}_1 \, N_1(y)\big\} \; \partial^2_{x_1} u_0\big(t, x_1 - \dfrac{\bar{b}_1}{\varepsilon}\, t\big) \\ + \big\{\partial_{y_i} a_{i1}(y) - b_1(y) + \bar{b}_1\big\} \; \partial_{x_1} u_1\big(t, x_1 - \dfrac{\bar{b}_1}{\varepsilon}\, t\big), & y \in Y, \\ \mathcal{B}_y \, v_2(y) = -a_{i1}(y)\, N_1(y)\, n_i \; \partial^2_{x_1} u_0\big(t, x_1 - \dfrac{\bar{b}_1}{\varepsilon}\, t\big), & y \in \partial Y. \end{cases}$$

The compatibility condition for (4.5) gives rise to the Cauchy problem for u_0

$$\text{(4.6)} \qquad \begin{cases} \partial_t u_0(t, x_1) = a^{\mathrm{hom}}_{11} \, \partial^2_{x_1} u_0(t, x_1), & (t, x_1) \in (0, T) \times \mathbb{R}, \\ u_0(0, x_1) = \varphi(x_1), & x_1 \in \mathbb{R}, \end{cases}$$

where the constant a^{hom}_{11} is defined by

$$\begin{aligned} a^{\mathrm{hom}}_{11} = & \int\limits_Y \big[a_{11}(y) + a_{1j}(y)\partial_{y_j} N_1(y) - b_1(y)\, N_1(y)\big]\, p^*(y)\, dy \\ & + \int\limits_Y \big[\bar{b}_1 \, N_1(y) p^*(y) - a_{i1}(y)\, N_1(y)\, \partial_{y_i} p^*(y)\big]\, dy. \end{aligned}$$

The positiveness of a^{hom}_{11} has been proved in [**9**].

LEMMA 4.1. *The constant a^{hom}_{11} is strictly positive.*

The form of the right-hand side of the equation in (4.5) suggests the following representation for the solution v_2:

$$\begin{aligned} & v_2(t\,,\, x_1 - \varepsilon^{-1}\bar{b}_1 t\,,\, y) = N_2(y)\, \partial^2_{x_1} u_0(t\,,\, x_1 - \varepsilon^{-1}\bar{b}_1 t) \\ & + N_1(y)\, \partial_{x_1} u_1(t\,,\, x_1 - \varepsilon^{-1}\bar{b}_1 t) + u_2(t\,,\, x_1 - \varepsilon^{-1}\bar{b}_1 t) \end{aligned}$$

with y_1-periodic function N_2 being a solution of the problem

$$\text{(4.7)} \qquad \begin{cases} \mathcal{A}\, N_2(y) = a_{11}(y) + \partial_{y_i}(a_{i1}(y)N_1(y)) + a_{1j}(y)\partial_{y_j} N_1(y) \\ \qquad\qquad -b_1(y)\, N_1(y) + \bar{b}_1 \, N_1(y) - a^{\mathrm{hom}}_{11}, & y \in Y, \\ \mathcal{B}\, N_2(y) = -a_{i1}(y)\, n_i \, N_1(y), & y \in \partial Y; \end{cases}$$

Similarly, we obtain a boundary value problem for v_3

$$(4.8)\qquad \begin{cases} \mathcal{A}_y\, v_3(t\,,\, x_1-\varepsilon^{-1}\bar b_1 t\,,\, y) = -N_1(y)\partial_t\partial_{x_1}u_0\big(t, x_1-\dfrac{\bar b_1}{\varepsilon}\, r\big)\Big|_{r=t} \\ -\partial_t u_1\big(t, x_1-\dfrac{\bar b_1}{\varepsilon}\, r\big)\Big|_{r=t} + \Big[a_{11}(y)N_1(y)+\partial_{y_i}(a_{i1}(y)N_2(y)) \\ +a_{1j}(y)\partial_{y_j}N_2(y) - b_1(y)\,N_2(y)+\bar b_1\,N_2(y)\Big]\,\partial^3_{x_1}u_0\big(t, x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ +\Big[a_{11}(y)+\partial_{y_i}(a_{i1}(y)N_1(y))+a_{1j}(y)\partial_{y_j}N_1(y) \\ \ +b_1(y)\,N_1(y)+\bar b_1\,N_1(y)\Big]\,\partial^2_{x_1}u_1\big(t, x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ +\Big[\partial_{y_i}a_{i1}(y)-b_1(y)+\bar b_1\Big]\,\partial_{x_1}u_2\big(t, x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big), \qquad y\in Y, \\ \mathcal{B}_y\, v_3(y) = -a_{i1}(y)\,N_2(y)\,n_i\ \partial^3_{x_1}u_0\big(t, x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ -a_{i1}(y)\,N_1(y)\,n_i\ \partial^2_{x_1}u_1\big(t, x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ -a_{i1}(y)\,n_i\ \partial_{x_1}u_2\big(t, x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big), \qquad y\in\partial Y. \end{cases}$$

From the compatibility condition for (4.8) we derive the equation for u_1:

$$\partial_t u_1(t,x_1) = a_{11}^{\rm hom}\partial^2_{x_1}u_1(t,x_1) + h_3\,\partial^3_{x_1}u_0(t,x_1), \quad (t,x_1)\in(0,T)\times\mathbb{R},$$

where

$$(4.9)\qquad h_3 = \int\limits_Y \big(-a_{11}^{\rm hom}N_1p^* + a_{11}N_1p^* - a_{i1}N_2\partial_{y_i}p^* + b_1N_2p^* \\ +a_{1j}\partial_{y_j}N_2p^* + \bar b_1\,N_2p^*\big)\,dy.$$

Naturally, v_3 can be represented as the sum

$$v_3\big(t, x_1-\frac{\bar b_1}{\varepsilon}\,t\,,\,y\big) = N_3(y)\,\partial^3_{x_1}u_0\big(t, x_1-\frac{\bar b_1}{\varepsilon}\,t\big) + N_2(y)\,\partial^2_{x_1}u_1\big(t, x_1-\frac{\bar b_1}{\varepsilon}\,t\big) \\ +N_1(y)\,\partial_{x_1}u_2\big(t, x_1-\frac{\bar b_1}{\varepsilon}\,t\big) + u_3\big(t, x_1-\frac{\bar b_1}{\varepsilon}\,t\big),$$

with N_3 being a y_1-periodic solution of the cell problem

$$(4.10)\qquad \begin{cases} \mathcal{A}\,N_3(y) = a_{11}(y)N_1(y)+\partial_{y_i}(a_{i1}(y)N_2(y))+a_{1j}(y)\partial_{y_j}N_2(y) \\ \qquad -b_1(y)\,N_2(y)+\bar b_1\,N_2(y)-a_{11}^{\rm hom}N_1(y)-h_3, \quad y\in Y, \\ \mathcal{B}\,N_3(y) = -a_{i1}(y)\,n_i\,N_2(y), \qquad y\in\partial Y. \end{cases}$$

Arguing as above, one can derive the equation for u_2

$$\partial_t u_2(t,x_1) = a_{11}^{\rm hom}\partial^2_{x_1}u_2(t,x_1) \\ +h_4\,\partial^4_{x_1}u_0(t,x_1) + h_3\,\partial^3_{x_1}u_1(t,x_1),$$

where the constant h_4 is defined by

$$h_4 = \int_Y \big(- a_{11}^{\mathrm{hom}} N_2 p^* + a_{11} N_2 p^* - a_{i1} N_3 \partial_{y_i} p^* + b_1 N_3 p^* \\ + a_{1j} \partial_{y_j} N_3 p^* + \bar{b}_1\, N_3 p^* - h_3 N_1 \big)\, dy. \tag{4.11}$$

Notice that determining initial conditions for u_1 and u_2 requires constructing initial layer correctors, which is done in Section 4.2.

Finally, as an inner approximate solution we take first three terms of (4.1)

$$u^\varepsilon_\infty(t,x) = u_0\big(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t\big) + \varepsilon\, N_1\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} u_0(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t)$$
$$+\varepsilon\, u_1\big(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t\big) + \varepsilon^2\, N_2\big(\frac{x}{\varepsilon}\big)\, \partial^2_{x_1} u_0(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t)$$
$$+\varepsilon^2\, N_1\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} u_1(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t) + \varepsilon^2 u_2(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t).$$

4.2. Initial layers. The leading term of the asymptotics $u_0(t,x_1)$ satisfies the initial condition $u_0(0,x_1) = \varphi(x_1)$. We introduce the initial layer functions, which will allow us to satisfy the initial condition up to the second power of ε. Consider the function $\phi_1(\tau,y)$ which is a solution to the problem

$$\begin{cases} \partial_\tau \phi_1 + \mathcal{A}_y \phi_1 = 0, & (\tau,y) \in (0,\infty)\times Y, \\ \mathcal{B}_y \phi_1 = 0, & (\tau,y) \in (0,\infty)\times \partial Y, \\ \phi_1(0,y) = -N_1(y). \end{cases} \tag{4.12}$$

By Lemma 3.1, ϕ_1 stabilizes to a constant $\overline{\phi_1}$, as $\tau \to \infty$, at the exponential rate. The constant $\overline{\phi_1}$ can be calculated as follows

$$\overline{\phi_1} = - \int_Y N_1(y)\, p^*(y)\, dy. \tag{4.13}$$

We use this constant to set the initial value for u_1: $u_1(0,x_1) = \overline{\phi_1}\, \varphi'(x_1)$. In this way

$$\Big[u_0\big(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t\big) + \varepsilon\, N_1\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} u_0(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t) + \varepsilon\, u_1\big(t\,,\, x_1 - \frac{\bar{b}_1}{\varepsilon}\, t\big)$$
$$+ \varepsilon\big(\phi_1\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\, \varphi'(x_1)\Big]\Big|_{t=0} = \varphi(x_1).$$

Similarly, we introduce $\phi_2(\tau,y)$ such that:

$$\begin{cases} \partial_\tau \phi_2 + \mathcal{A}_y \phi_2 = 0, & (\tau,y) \in (0,\infty)\times Y, \\ \mathcal{B}_y \phi_2 = 0, & (\tau,y) \in (0,\infty)\times \partial Y, \\ \phi_2(0,y) = -N_2(y); \end{cases} \tag{4.14}$$

The constant to which ϕ_2 stabilizes, as $\tau \to \infty$, we denote by $\overline{\phi_2}$

$$\overline{\phi_2} = -\int\limits_Y N_2(y)\, p^*(y)\, dy, \tag{4.15}$$

and set

$$u_2(0, x_1) = \overline{\phi_2}\, \varphi''(x_1) + \overline{\phi_1}\, \varphi''(x_1).$$

In this way the boundary value problems for u_1 and u_2 take the form

$$\begin{cases} \partial_t u_1(t, x_1) = a_{11}^{\mathrm{hom}} \partial^2_{x_1} u_1(t, x_1) + h_3\, \partial^3_{x_1} u_0(t, x_1), & (t, x_1) \in (0, T) \times \mathbb{R}, \\ u_1(0, x_1) = \overline{\phi_1}\, \varphi'(x_1), \quad x_1 \in \mathbb{R}; \end{cases} \tag{4.16}$$

$$\begin{cases} \partial_t u_2(t, x_1) = a_{11}^{\mathrm{hom}} \partial^2_{x_1} u_2(t, x_1) \\ +h_4\, \partial^4_{x_1} u_0(t, x_1) + h_3\, \partial^3_{x_1} u_1(t, x_1), , \quad (t, x_1) \in (0, T) \times \mathbb{R}, \\ u_2(0, x_1) = \overline{\phi_2}\, \varphi''(x_1) + \overline{\phi_1}\, \varphi''(x_1) \end{cases} \tag{4.17}$$

with the constants h_3, h_4 defined in (4.9), (4.11). Then

$$\Big[\varepsilon^2\, N_2\big(\frac{x}{\varepsilon}\big)\, \partial^2_{x_1} u_0(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) + \varepsilon^2\, N_1\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} u_1(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) +$$
$$+ +\varepsilon^2\, q_2\big(\frac{x}{\varepsilon}\big)\, g(x_1) + \varepsilon^2 u_2(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) + \varepsilon^2\big(\phi_2\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big) - \overline{\phi_2}\big)\, \varphi''(x_1) +$$
$$+\varepsilon^2\big(\phi_1\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\, \varphi''(x_1)\Big]\Big|_{t=0} = 0.$$

Denote

$$\begin{aligned} u^\varepsilon_{il}(t, x) = {} & \varepsilon\big(\phi_1\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\, \varphi'(x_1) + \varepsilon^2\big(\phi_2\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big) - \overline{\phi_2}\big)\, \varphi''(x_1) + \\ & +\varepsilon^2\big(\phi_1\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\, \varphi''(x_1). \end{aligned} \tag{4.18}$$

We summarize this section by writing down the formal asymptotic expansion for a solution u^ε of problem (2.1) which has been constructed above. It reads

$$\begin{aligned} U^\varepsilon(t, x) = {} & u_0\big(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t\big) + \varepsilon\, N_1\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} u_0(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) \\ & +\varepsilon\, u_1\big(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t\big) + \varepsilon^2\, N_2\big(\frac{x}{\varepsilon}\big)\, \partial^2_{x_1} u_0(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) \\ & +\varepsilon^2\, N_1\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} u_1(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) + \varepsilon^2 u_2(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) + u^\varepsilon_{il}(t, x). \end{aligned} \tag{4.19}$$

Here u_0 is a solution of the homogenized problem (4.6); N_1, N_2 solve auxiliary cell problems (4.4), (4.7); u_1 and u_2 are solutions of nonhomogeneous Cauchy problems (4.16), (4.17); the initial layer u^ε_{il} is given by (4.12)-(4.15) and (4.18). Notice that the approximate solution satisfies the initial condition: $U^\varepsilon(0, x) = \varphi(x_1)$.

5. Justification procedure

In thin domain $\mathbb{G}_\varepsilon$ it is natural to introduce the following notion of convergence (see, for example, [**4**], [**13**]).

DEFINITION 5.1. We say that $f_\varepsilon(t,x)$ converges strongly to zero in $L^2[0,T;H^1(\mathbb{G}_\varepsilon)]$ if

$$\varepsilon^{-\frac{(d-1)}{2}} \, \|f_\varepsilon\|_{L^2[0,T;H^1(\mathbb{G}_\varepsilon)]} \longrightarrow 0, \quad \varepsilon \to 0.$$

The normalization factor $\varepsilon^{-\frac{(d-1)}{2}}$ appears due to the fact that the norm of a fixed non-trivial $C_0^\infty(\mathbb{R})$ function $\varphi(x_1)$ in the space $L^2[0,T;H^1(\mathbb{G}_\varepsilon)]$ is of order $\varepsilon^{\frac{(d-1)}{2}}$.

The following theorem is the main result of the paper.

THEOREM 5.2. *Let conditions* $(\mathbf{H1})-(\mathbf{H4})$ *be fulfilled. Then the difference between the exact solution* u^ε *of problem* (2.1) *and the approximate solution* U^ε *given by* (4.19), *converges in* $L^2[0,T;H^1_{loc}(\mathbb{G}_\varepsilon)]$ *to zero, as* $\varepsilon \to 0$. *Moreover, the following estimate holds:*

$$\int\limits_{\mathbb{G}_\varepsilon} (u^\varepsilon - U^\varepsilon)^2 \, dx + \int\limits_0^t \int\limits_{\mathbb{G}_\varepsilon} |\nabla(u^\varepsilon(s,x) - U^\varepsilon(s,x))|^2 \, dxds \le C\,\varepsilon^2\,\varepsilon^{d-1}. \tag{5.1}$$

PROOF. In order to estimate the norm (in the appropriate space) of the difference $u^\varepsilon - U^\varepsilon$ between the exact and the approximate solutions, we calculate first $\mathcal{A}_\varepsilon(u^\varepsilon - U^\varepsilon)$ and $\mathcal{B}_\varepsilon(u^\varepsilon - U^\varepsilon)$, and then make use of a priori estimates (3.4), (3.7). Straightforward computations yield

$$\mathcal{A}_\varepsilon(u^\varepsilon(t,x) - U^\varepsilon(t,x)) = \varepsilon\,(R_1^\varepsilon(t,x) + R_2^\varepsilon(t,x)) + o(\varepsilon), \quad \varepsilon \to 0,$$

where

$$\begin{aligned} R_1^\varepsilon(t,x) = &- \sum_{k=0}^{1} N_k\big(\frac{x}{\varepsilon}\big)\partial_t \partial_{x_1}^k u_{1-k}(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, r)\Big|_{r=t} \\ &+\bar b_1 \, \sum_{k=0}^{2} N_k\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1}^{k+1} u_{2-k}(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) \\ &\sum_{k=0}^{1} a_{11}\big(\frac{x}{\varepsilon}\big)\, N_k\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1}^{k+2} u_{1-k}(t\,,\, x_1 - \frac{\bar b_1}{\varepsilon}\, t) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{2} a_{1j}\big(\frac{x}{\varepsilon}\big)\,\partial_{y_j}N_k(y)\Big|_{y=x/\varepsilon}\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t)\\
&+\sum_{k=0}^{2} b_1\big(\frac{x}{\varepsilon}\big)\,N_k\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t)\\
&+\bar b_1\sum_{k=1}^{2}\big(\phi_k(\tau,y)-\overline{\phi_k}\big)\,\varphi'''(x_1)+a_{11}(y)\,\big(\phi_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)-\overline{\phi_1}\big)\,\varphi'''(x_1)\\
&+\sum_{k=1}^{2} a_{1j}(y)\,\nabla_y\phi_k(\tau,y)\Big|_{y=x/\varepsilon,\tau=t/\varepsilon^2}\varphi'''(x_1)\\
&+b_1(y)\sum_{k=1}^{2}\big(\phi_k\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)-\overline{\phi_k}\big)\,\varphi'''(x_1).
\end{aligned}
$$

$$
\begin{aligned}
R_2^\varepsilon(t,x)=&\sum_{k=0}^{2}\partial_{y_i}(a_{i1}(y)\,N_k(y))\Big|_{y=x/\varepsilon}\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t)\\
&+\sum_{k=1}^{2}\partial_{y_i}(a_{i1}(y)(\phi_k(\tau,y)-\overline{\phi_k}))\Big|_{y=x/\varepsilon,\tau=t/\varepsilon^2}\varphi'''(x_1).
\end{aligned}
$$

Similarly,

$$
\mathcal{B}_\varepsilon(u^\varepsilon(t,x)-U^\varepsilon(t,x))=\varepsilon^2\,R_3^\varepsilon(t,x)
$$

with

$$
\begin{aligned}
R_3^\varepsilon(t,x)=&-\sum_{k=0}^{2} a_{i1}\big(\frac{x}{\varepsilon}\big)\,n_i\,N_k\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t)\\
&a_{i1}(y)n_i\sum_{k=1}^{2}\big(\phi_k\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)-\overline{\phi_k}\big)\,\varphi'''(x_1).
\end{aligned}
$$

By a priori estimates (3.4) and (3.7),

$$
\begin{aligned}
&\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon-U^\varepsilon)^2\,dx+\int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla(u^\varepsilon(s,x)-U^\varepsilon(s,x))|^2\,dxds\\
&\le C\,e^T\Big\{\|\varepsilon\,R_1^\varepsilon\|^2_{L^2[0,T;L^2(\mathbb{G}_\varepsilon)]}+\varepsilon\,\|\varepsilon^2\,R_3^\varepsilon\|^2_{L^2[0,T;L^2(\Sigma_\varepsilon)]}+\frac{1}{\varepsilon}\,\|\varepsilon R_2^\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{G}_\varepsilon)}\Big\}.
\end{aligned}
$$

In order to estimate R_1^ε, R_2^ε and R_3^ε, we analyze properties of the solutions u_0, u_1 and u_2 of problems (4.6), (4.16) and (4.17). For u_0 the well-known integral Poisson formula takes place:

$$
u_0(t,x_1)=\frac{\theta(t)}{\sqrt{4\pi a_{11}^{\hom}t}}\int\limits_{\mathbb{R}}\varphi(\xi)\,e^{-\frac{|x_1-\xi|^2}{4a_{11}^{\hom}t}}\,d\xi.
$$

Here θ is the unit step function, that is $\theta(t) = 1$ for $t \geq 0$, and $\theta(t) = 0$ when $t < 0$. Moreover, similar formula is valid for any derivative of u_0 with respect to x_1:

$$\partial_{x_1}^{(k)} u_0(t, x_1) = \frac{\theta(t)}{\sqrt{4\pi a_{11}^{\hom} t}} \int\limits_{\mathbb{R}} \partial_\xi^{(k)} \varphi(\xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{\hom} t}}\, d\xi.$$

Bearing in mind that φ has finite support, one can see that

$$|\partial_{x_1}^{(k)} u_0(t, x_1)| \leq C\, e^{-\alpha|x_1|^2}, \quad C, \alpha > 0, \tag{5.2}$$

where α depends on T. Similarly, the following integral representation of $\partial_{x_1}^{(k)} u_1$ is valid:

$$\begin{aligned}\partial_{x_1}^{(k)} u_1(t, x_1) = &\frac{\theta(t)\,\bar{\phi}_1}{\sqrt{4\pi a_{11}^{\hom} t}} \int\limits_{\mathbb{R}} \partial_\xi^{(k+1)} \varphi(\xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{\hom} t}}\, d\xi \\ &+ \int\limits_0^t \frac{h_3}{\sqrt{4\pi a_{11}^{hom}(t-\tau)}}\, d\tau \int\limits_{\mathbb{R}} \partial_{x_1}^{3+k} u_0(\tau, \xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}}\, d\xi = I_1 + I_2.\end{aligned}$$

Arguing as above we obtain

$$|I_1| \leq C\, e^{-\alpha|x_1|^2}, \quad \alpha > 0.$$

Let us estimate I_2.

$$|I_2| \leq \Big\{ \int\limits_0^t d\tau \int\limits_{|x_1-\xi|\leq 2|x_1|} d\xi + \int\limits_0^t d\tau \int\limits_{|x_1-\xi|>2|x_1|} d\xi \Big\} \frac{h_3\, \partial_{x_1}^{3+k} u_0(\tau, \xi)}{\sqrt{4\pi a_{11}^{hom}(t-\tau)}}\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}}.$$

In view of (5.2), for ξ satisfying $|x_1 - \xi| \leq 2|x_1|$,

$$\big|\partial_{x_1}^{3+k} u_0(\tau, \xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}}\big| \leq C\, e^{-\alpha|\xi|^2}\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom} T}} \leq C\, e^{-\alpha_1|x_1|^2}, \ \alpha_1 > 0,$$

thus,

$$\begin{aligned}&\int\limits_0^t d\tau \int\limits_{|x_1-\xi|\leq 2|x_1|} \frac{h_3\, \partial_{x_1}^{3+k} u_0(\tau, \xi)}{\sqrt{4\pi a_{11}^{hom}(t-\tau)}}\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}} d\xi \\ &\leq C\, |x_1|\, e^{-\alpha_1|x_1|^2} \int\limits_0^t \frac{1}{\sqrt{t-\tau}} d\tau \leq C\, e^{-\alpha_1|x_1|^2}.\end{aligned}$$

Noticing that

$$\frac{1}{\sqrt{4\pi a_{11}^{\hom} t}} \int\limits_{\mathbb{R}} e^{-\frac{|x_1-\xi|^2}{4a_{11}^{\hom} t}}\, d\xi = 1,$$

one has

$$\int_0^t d\tau \int_{|x_1-\xi|>2|x_1|} \frac{h_3\, \partial_{x_1}^{3+k} u_0(\tau,\xi)}{\sqrt{4\pi a_{11}^{hom}(t-\tau)}}\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}} d\xi \le C\, e^{-\alpha_1|x_1|^2},\ \alpha_1>0.$$

In this way we see that u_1 satisfies the estimate

$$|\partial_{x_1}^k u_1(t,x_1)| \le C\, e^{-\alpha_1|x_1|^2},\ \alpha_1>0,\ t\ge 0,\ x\in\mathbb{R}. \tag{5.3}$$

Arguing as above, one can see that analogous estimate holds for u_2 solving problem (4.17).

$$|\partial_{x_1}^k u_2(t,x_1)| \le C\, e^{-\alpha_1|x_1|^2},\ \alpha_1>0,\ t\ge 0,\ x\in\mathbb{R}. \tag{5.4}$$

Bearing in mind the boundedness of the coefficients a_{ij}, b_j, properties of N_1 and N_2 as the solutions of (4.4), (4.7), and bounds (5.2)-(5.4), one can check the validity of (5.1). Note that the exponential decay of the initial layer functions is used while estimating the corresponding terms. □

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Paper D

PAPER D

Homogenization and concentration for a diffusion equation with large convection in a bounded domain

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ABSTRACT. We consider the homogenization of a non-stationary convection-diffusion equation posed in a bounded domain with periodically oscillating coefficients and homogeneous Dirichlet boundary conditions. Assuming that the convection term is large, we give the asymptotic profile of the solution and determine its rate of decay. In particular, it allows us to characterize the "hot spot", i.e., the precise asymptotic location of the solution maximum which lies close to the domain boundary and is also the point of concentration. Due to the competition between convection and diffusion the position of the "hot spot" is not always intuitive as exemplified in some numerical tests.

Keywords: Homogenization, convection-diffusion, localization.

1. Introduction

The goal of the paper is to study the homogenization of a convection-diffusion equation with rapidly periodically oscillating coefficients defined in a bounded domain. Namely, we consider the following initial boundary problem:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + A^\varepsilon\, u^\varepsilon(t,x) = 0, & \text{in } (0,T)\times\Omega, \\ u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times\partial\Omega, \\ u^\varepsilon(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, u_0 belongs to $L^2(\Omega)$ and A^ε is an operator defined by

$$A^\varepsilon u^\varepsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_j}\right) + \frac{1}{\varepsilon}\, b_j\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_j},$$

where we employ the convention of summation over repeated Latin indices. As usual ε, which denotes the period of the coefficients, is a small positive parameter intended to tend to zero. Note the large scaling in front of the convective term which corresponds to the convective and diffusive terms having both the same order of magnitude at the small scale ε (this is a classical assumption in homogenization [**5**], [**12**], [**13**], [**21**]). We make the following assumptions on the coefficients of the operator A^ε.

(H1) The coefficients $a_{ij}(y), b_j(y)$ are measurable bounded functions defined on the unit cell $Y = (0,1]^d$, that is $a_{ij}, b_j \in L^\infty(Y)$. Moreover, $a_{ij}(y), b_j(y)$ are Y-periodic.

(H2) The $d \times d$ matrix $a(y)$ is uniformly elliptic, that is there exists $\Lambda > 0$ such that, for all $\xi \in \mathbb{R}^d$ and for almost all $y \in \Omega$,

$$a_{ij}(y)\xi_i\xi_j \geq \Lambda|\xi|^2.$$

For the large convection term we do not suppose that the effective drift (the weighted average of b defined below by (2.4)) is zero, nor that the vector field $b(y)$ is divergence-free. Some additional assumptions on the smoothness and compact support of the initial data u_0 will be made in Section 2 after introducing auxiliary spectral cell problems. In view of **(H1)** and **(H2)**, for any $\varepsilon > 0$, problem (1.1) has a unique weak solution $u^\varepsilon \in L^\infty[0,T; L^2(\Omega)] \cap L^2[0,T; H^1(\Omega)]$ (see [**6**]).

Our main goal is to describe the asymptotic behavior of the solution $u^\varepsilon(t,x)$ of problem (1.1) as ε goes to zero. There are of course many motivations to study such a problem (one of them being the transport of solutes in porous media [**17**]). However, if (1.1) is interpreted as the heat equation in a fluid domain (the fluid velocity being given by $\varepsilon^{-1}b(x/\varepsilon)$), we can paraphrase the famous “hot spot” conjecture of J. Rauch [**23**], [**7**], [**10**], and ask a simple question in plain words. If the initial temperature u_0 has its maximum inside the domain Ω, where shall this maximum or “hot spot” go as time evolves ? More precisely, we want to answer this question asymptotically as ε goes to zero. Theorem 2.3 (and the discussion following it) gives a complete answer to this question. The “hot spot” is a concentration point x_c, located asymptotically close to the boundary $\partial\Omega$ (see Figure 1), which maximizes the linear function $\Theta \cdot x$ on Ω where the vector parameter Θ is determined as an optimal parameter in an auxiliary cell problem (see Lemma 2.1). Surprisingly Θ is not some average of the velocity field but is the result of an intricate interaction between convection and diffusion in the periodicity cell (even in the case of constant coefficients ; see the numerical examples of Section 7). Furthermore, Theorem 2.3 gives the asymptotic profile of the solution, which is localized in the vicinity of the “hot spot” x_c, in terms of a homogenized equation with an initial condition that depends on the geometry of the support of the initial data u_0.

Before we explain our results in greater details, we briefly review previous results in the literature. In the case when the vector-field $b(y)$ is solenoidal and has zero mean-value, problem (1.1) has been studied by the classical homogenization methods (see, e.g, [**8**], [**25**]). In particular, the sequence of solutions is bounded in $L^\infty[0,T;L^2(\Omega)]\cap L^2[0,T;H^1(\Omega)]$ and converges, as $\varepsilon\to 0$, to the solution of an effective or homogenized problem in which there is no convective term. For general vector-fields $b(y)$, and if the domain Ω is the whole space $\mathbb{R}^d$, the convection might dominate the diffusion and we cannot expect a usual convergence of the sequence of solutions $u^\varepsilon(t,x)$ in the fixed spatial reference frame. Rather, introducing a frame of moving coordinates $(t, x-\bar{b}t/\varepsilon)$, where the constant vector $\bar{b}$ is the so-called effective drift (or effective convection) which is defined by (2.4) as a weighted average of b, it is known that the translated sequence $u^\varepsilon(t, x-\bar{b}t/\varepsilon)$ converges to the solution of an homogenized parabolic equation [**5**], [**13**]. Note that the notion of effective drift was first introduced in [**21**]. Of course, the convergence in moving coordinates cannot work in a bounded domain. The purpose of the present work is to study the asymptotic behavior of (1.1) in the case of a bounded domain Ω.

Bearing these previous results in mind, intuitively, it is clear that in a bounded domain the initial profile should move rapidly in the direction of the effective drift $\bar{b}$ until it reaches the boundary, and then dissipate due to the homogeneous Dirichlet boundary condition, as t grows. Since the convection term is large, the dissipation increases, as $\varepsilon\to 0$, so that the solution asymptotically converges to zero at finite time. Indeed, introducing a rescaled (short) time $\tau=\varepsilon^{-1}t$, we rewrite problem (1.1) in the form

$$\begin{cases} \partial_\tau u^\varepsilon - \varepsilon\,\mathrm{div}\big(a^\varepsilon\,\nabla u^\varepsilon\big) + b^\varepsilon\cdot\nabla u^\varepsilon = 0, & \text{in } (0,\varepsilon^{-1}T)\times\Omega,\\ u^\varepsilon(\tau,x)=0, & \text{on } (0,\varepsilon^{-1}T)\times\partial\Omega,\\ u^\varepsilon(0,x)=u_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

Applying the classical two-scale asymptotic expansion method [**8**], one can show that, for any $\tau\geq 0$

$$\int_\Omega |u^\varepsilon(\tau,x)-u^0(\tau,x)|^2\,dx\to 0,\quad \varepsilon\to 0,$$

where the leading term of the asymptotics u^0 satisfies the following first-order equation

$$\begin{cases} \partial_\tau u^0(\tau,x) + \bar{b}\cdot\nabla u^0(\tau,x) = 0, & \text{in } (0,+\infty)\times\Omega,\\ u^0(\tau,x)=0, & \text{on } (0,+\infty)\times\partial\Omega_{\bar{b}},\\ u^0(0,x)=u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

with $\bar{b}$ being the vector of effective convection defined by (2.4). Here $\partial\Omega_{\bar{b}}$ is the subset of $\partial\Omega$ such that $\bar{b}\cdot n<0$ where n stands for the exterior unit normal on $\partial\Omega$. One can construct higher order terms in the asymptotic expansion for u^ε. This expansion will contain a boundary layer corrector in the vicinity of $\partial\Omega\setminus\partial\Omega_{\bar{b}}$. A similar problem in a more general setting has been studied in [**9**].

The solution of problem (1.3) can be found explicitly,

$$u^0(\tau, x) = \begin{cases} u_0(x - \bar{b}\tau), & \text{for } (\tau, x) \text{ such that } x,\ (x - \bar{b}\tau) \in \Omega, \\ 0, & \text{otherwise}, \end{cases}$$

which shows that u^0 vanishes after a finite time $\tau_0 = O(1)$. In the original coordinates (t, x) we have

$$\int_\Omega |u^\varepsilon(t, x) - u_0(x - \varepsilon^{-1}\,\bar{b}\,t)|^2\,dx \to 0, \quad \varepsilon \to 0.$$

Thus, for $t = O(\varepsilon)$ the initial profile of u^ε moves with the velocity $\varepsilon^{-1}\,\bar{b}$ until it reaches the boundary of Ω and then dissipates. Furthermore, any finite number of terms in the two-scale asymptotic expansion of $u^\varepsilon(\tau, x)$ vanish for $\tau \geq \tau_0 = O(1)$ and thus for $t \geq t_0$ with an arbitrary small $t_0 > 0$. On the other hand, if u_0 is positive, then by the maximum principle, $u^\varepsilon > 0$ for all t. Thus, the method of two-scale asymptotic expansion in this short-time scaling is unable to capture the limit behaviour of $u^\varepsilon(t, x)$ for positive time. The goal of the present paper is therefore to perform a more delicate analysis and to determine the rate of vanishing of u^ε, as $\varepsilon \to 0$.

The homogenization of the spectral problem corresponding to (1.1) in a bounded domain for a general velocity $b(y)$ was performed in [**11**], [**12**]. Interestingly enough the effective drift does not play any role in such a case but rather the key parameter is another constant vector $\Theta \in \mathbb{R}^d$ which is defined as an optimal exponential parameter in a spectral cell problem (see Lemma 2.1). More precisely, it is proved in [**11**], [**12**] that the first eigenfunction concentrates as a boundary layer on $\partial\Omega$ in the direction of Θ. We shall prove that the same vector parameter Θ is also crucial in the asymptotic analysis of (1.1).

Notice that for large time and after a proper rescaling the solution of (1.1) should behave like the first eigenfunction of the corresponding elliptic operator, and thus concentrates in a small neighbourhood of $\partial\Omega$ in the direction of Θ. We prove that this guess is correct, not only for large time but also for any time $t = O(1)$, namely that $u^\varepsilon(t, x)$ concentrates in the neighbourhood of the “hot spot” or concentration point $x_c \in \partial\Omega$ which depends on Θ. The value of Θ can be determined in terms of some optimality property of the first eigenvalue of an auxiliary periodic spectral problem (see Section 2). It should be stressed that, in general, Θ does not coincide with $\bar{b}$. As a consequence, it may happen that the concentration point x_c does not even belong to the subset of $\partial\Omega$ consisting of points which are attained by translation of the initial data support along $\bar{b}$. This phenomenon is illustrated by numerical examples in Section 7.

The paper is organized as follows. In Section 2 we introduce auxiliary spectral problems in the unit cell Y and impose additional conditions on the geometry of the compact support of u_0. We then state our main result (see Theorem 2.3) and give its geometric interpretation. In Section 3, in order to simplify the original problem (1.1), we use a factorization principle, as in [**24**], [**18**], [**26**], [**11**], based on the first eigenfunctions of the auxiliary spectral problems. As a result, we obtain a reduced problem, where the new convection is divergence-free and has zero mean-value. Studying the asymptotic behaviour of the Green function of the reduced problem, performed in Section 4, is an important part of the proof. It is based on the result obtained in [**1**] for a fundamental solution of a parabolic operator with lower order terms. Asymptotics of u^ε is derived in Section 5. In Section 6 we study the case when the boundary of the support of u_0 has a flat part. To illustrate the main result of the paper, in Section 7 we present direct computations of u^ε using the software FreeFEM++ [**15**]. A number of basic facts from the theory of almost periodic functions is given in Section 8.

2. Auxiliary spectral problems and main result

We define an operator A and its adjoint A^* by

$$Au = -\mathrm{div}(a\nabla u) + b\cdot\nabla u, \quad A^*v = -\mathrm{div}(a^T\nabla v) - \mathrm{div}(b\,v),$$

where a^T is the transposed matrix of a. Following [**8**], for $\theta \in \mathbb{R}^d$, we introduce two parameterized families of spectral problems (direct and adjoint) in the periodicity cell $Y = [0,1)^d$.

$$\begin{cases} e^{-\theta\cdot y}\,A\,e^{\theta\cdot y}\,p_\theta(y) = \lambda(\theta)\,p_\theta(y), & Y, \\ y \to p_\theta(y) \quad \text{Y-periodic}. \end{cases} \tag{2.1}$$

$$\begin{cases} e^{\theta\cdot y}\,A^*\,e^{-\theta\cdot y}\,p^*_\theta(y) = \lambda(\theta)\,p^*_\theta(y), & Y, \\ y \to p^*_\theta(y) \quad \text{Y-periodic}. \end{cases} \tag{2.2}$$

The next result, based on the Krein-Rutman theorem, was proved in [**11**], [**12**].

LEMMA 2.1. *For each $\theta \in \mathbb{R}^d$, the first eigenvalue $\lambda_1(\theta)$ of problem* (2.1) *is real, simple, and the corresponding eigenfunctions p_θ and p^*_θ can be chosen positive. Moreover, $\theta \to \lambda_1(\theta)$ is twice differentiable, strictly concave and admits a maximum which is obtained for a unique $\theta = \Theta$.*

The eigenfunctions p_θ and p^*_θ defined by Lemma 2.1, can be normalized by

$$\int_Y |p_\theta(y)|^2 dy = 1 \quad \text{and} \quad \int_Y p_\theta(y)\,p^*_\theta(y)\,dy = 1.$$

Differentiating equation (2.1) with respect to θ_i, integrating against p^*_θ and writing down the compatibility condition for the obtained equation yield

$$\frac{\partial\lambda_1}{\partial\theta_i} = \int_Y \big(b_i\,p_\theta\,p^*_\theta + a_{ij}(p_\theta\,\partial_{y_j}p^*_\theta - p^*_\theta\,\partial_{y_j}p_\theta) - 2\,\theta_j\,a_{ij}\,p_\theta\,p^*_\theta\big)\,dy. \tag{2.3}$$

Obviously, $p_{\theta=0} = 1$, and, thus,

$$\frac{\partial \lambda_1}{\partial \theta_i}(\theta = 0) = \int\limits_{Y} \big(b_i\, p^*_{\theta=0} + a_{ij}\, \partial_{y_j} p^*_{\theta=0}\big)\, dy := \bar{b}_i, \tag{2.4}$$

which defines the components $\bar{b}_i$ of the so-called effective drift. In the present paper we assume that $\bar{b} \neq 0$ (or, equivalently, $\Theta \neq 0$). The case $\bar{b} = 0$ can be studied by classical methods (see, for example, [**25**]). The equivalence of $\bar{b} = 0$ and $\Theta = 0$ is obvious since $\lambda_1(\theta)$ is strictly concave with a unique maximum.

We need to make some assumptions on the geometry of the support ω (a closed set as usual) of the initial data u_0 with respect to the direction of Θ. One possible set of conditions is the following.

(H3) The initial data $u_0(x)$ is a continuous function in Ω, has a compact support $\omega \Subset \Omega$ and belongs to $C^2(\omega)$. Moreover, ω is a C^2-class domain.

(H4) The "source" point $\bar{x} \in \partial\omega$, at which the minimum in $\min_{x\in\omega} \Theta \cdot x$ is achieved, is unique (see Figure 1(a)). In other words

$$\Theta \cdot (x - \bar{x}) > 0, \quad x \in \omega \setminus \{\bar{x}\}.$$

(H5) The point $\bar{x}$ is elliptic and $\partial\omega$ is locally convex at $\bar{x}$, that is the principal curvatures at $\bar{x}$ have the same sign. More precisely, in local coordinates the boundary of ω in some neighborhood $U_\delta(\bar{x})$ of the point $\bar{x}$ can be defined by

$$z_d = (Sz', z') + o(|z'|^2)$$

for some positive definite $(d-1)\times(d-1)$ matrix S. Here $z' = (z_1, \cdots z_{d-1})$ are the orthonormal coordinates in the tangential hyperplane at $\bar{x}$, and z_d is the coordinate in the normal direction.

(H6) $\nabla u_0(\bar{x}) \cdot \Theta \neq 0$.

REMARK 2.2. In assumption $(\mathbf{H3})$ it is essential that the support ω is a strict subset of Ω, i.e., does not touch the boundary $\partial\Omega$ (see Remark 5.5 for further comments on this issue). However, the continuity assumption on the initial function u_0 is not necessary. It will be relaxed in Theorem 5.6 where $u_0(x)$ still belongs to $C^2(\overline{\omega})$ but is discontinuous through $\partial\omega$. Of course, assuming continuity or not will change the order of convergence and the multiplicative constant in front of the asymptotic solution.

Note that assumption $(\mathbf{H4})$ implies that $\Theta \neq 0$ is a normal vector to $\partial\omega$ at $\bar{x}$.

Eventually, assumption $(\mathbf{H6})$ is required because, u_0 being continuous in Ω, we have $u_0(\bar{x}) = 0$.

To avoid excessive technicalities for the moment, we state our main result in a loose way (see Theorem 5.1 for a precise statement).

THEOREM 2.3. *Suppose conditions* $(\mathbf{H1}) - (\mathbf{H6})$ *are satisfied and* $\Theta \neq 0$. *If* u^ε *is a solution of problem* (1.1)*, then, for any* $t_0 > 0$ *and* $t \geq t_0$

$$u^\varepsilon(t,x) \approx \varepsilon^2\, \varepsilon^{\frac{d-1}{2}}\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\, M_\varepsilon\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, u(t,x), \quad \varepsilon \to 0,$$

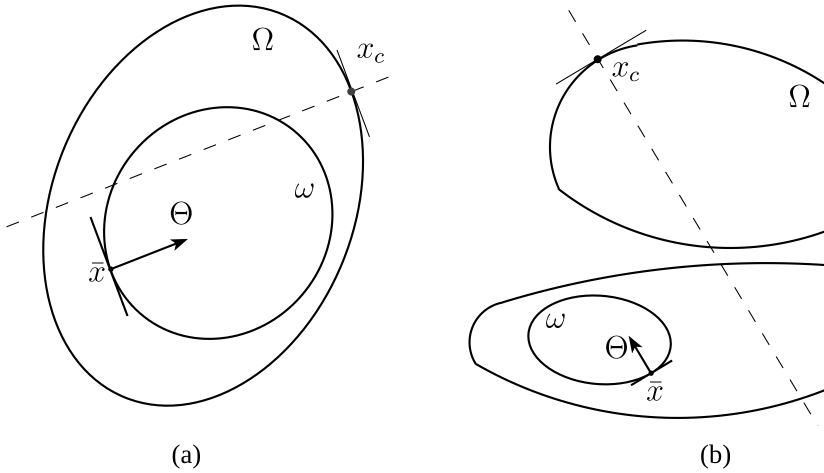


FIGURE 1. Definition of the source point $\bar{x}$ and of the concentration point x_c.

where $(\lambda_1(\Theta), p_\Theta)$ is the first eigenpair defined by Lemma 2.1 and $u(t,x)$ solves the homogenized problem

$$\left\{\begin{array}{ll} \partial_t u = \operatorname{div}(a^{\text{eff}}\,\nabla u), & (t,x) \in (0,T)\times\Omega, \\ u(t,x) = 0, & (t,x) \in (0,T)\times\partial\Omega, \\ u(0,x) = \nabla u_0(\bar{x})\cdot \dfrac{\Theta}{|\Theta|}\,\delta(x-\bar{x}), & x\in\Omega. \end{array}\right. \tag{2.5}$$

Here a^{eff} is a positive definite matrix, defined by (4.7), M_ε is a constant, defined in Theorem 5.1, depending on p_Θ, on the geometry of $\partial\omega$ at $\bar{x}$ and on the relative position of $\bar{x}$ in εY (see Remark 5.2 and Figure 2), and $\delta(x-\bar{x})$ is the Dirac delta-function at the point $\bar{x}$.

The interpretation of Theorem 2.3 in terms of concentration or finding the "hot spot" is the following. Up to a multiplicative constant $\varepsilon^2\,\varepsilon^{\frac{d-1}{2}} M_\varepsilon$, the solution u^ε is asymptotically equal to the product of two exponential terms, a periodically oscillating function $p_\Theta\big(\frac{x}{\varepsilon}\big)$ (which is uniformly positive and bounded) and the homogenized function $u(t,x)$ (which is independent of ε). The first exponential term $e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}$ indicates a fast decay in time, uniform in space. The second exponential term $e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}$ is the root of a localization phenomenon. Indeed, it is maximum at those points on the boundary, $x_c \in \partial\Omega$, which have a maximal coordinate $\Theta\cdot x$, independently of the position of $\bar{x}$ (see Figure 1(b)). These (possibly multiple) points x_c are the "hot spots". Everywhere else in Ω the solution is exponentially smaller, for any positive time. This behaviour can clearly be checked on the numerical examples of Section 7. It is of course similar to the behavior of the corresponding first eigenfunction as studied in [**12**].

The proof of Theorem 2.3 consists of several steps. First, using a factorization principle (see, for example, [**24**], [**18**], [**26**], [**11**]) in Section 3 we make a change of unknown function in such a way that the resulting equation is amenable to homogenization. After that, the new unknown function $v^\varepsilon(t,x)$ is represented in terms of the corresponding Green function $K^\varepsilon(t,x,\xi)$. Studying the asymptotic behaviour of K^ε is performed in Section 4. Finally, we turn back to the original problem and write down the asymptotics for u^ε in Section 5 which finishes the proof of Theorem 2.3.

REMARK 2.4. Theorem 2.3 holds true even if we add a singular zero-order term of the type $\varepsilon^{-2}c(\frac{x}{\varepsilon})u^\varepsilon$ in the equation (1.1). This zero-order term will be removed by the factorization principle and the rest of the proof is identical. With some additional work Theorem 2.3 can be generalized to the case of so-called cooperative systems for which a maximum principle holds. Such systems of diffusion equations arise in nuclear reactor physics and their homogenization (for the spectral problem) was studied in [**12**].

3. Factorization

We represent a solution u^ε of the original problem (1.1) in the form

$$u^\varepsilon(t,x) = e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, v^\varepsilon(t,x), \tag{3.1}$$

where Θ and p_Θ are defined in Lemma 2.1. Notice that the change of unknowns is well-defined since p_Θ is positive and continuous. Substituting (3.1) into (1.1), multiplying the resulting equation by $p^*_\Theta\big(\frac{x}{\varepsilon}\big)$ and using (2.2), one obtains the following problem for v^ε:

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\,\partial_t v^\varepsilon + A^\varepsilon_\Theta\, v^\varepsilon = 0, & (t,x)\in(0,T)\times\Omega,\\ v^\varepsilon(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega,\\ v^\varepsilon(0,x) = \dfrac{u_0(x)}{p_\Theta\big(\frac{x}{\varepsilon}\big)}\, e^{-\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}, & x\in\Omega, \end{cases} \tag{3.2}$$

where $\varrho_\Theta(y) = p_\Theta(y)\, p^*_\Theta(y)$ and

$$A^\varepsilon_\Theta v = -\frac{\partial}{\partial x_i}\Big(a^\Theta_{ij}\big(\frac{x}{\varepsilon}\big)\,\frac{\partial v}{\partial x_j}\Big) + \frac{1}{\varepsilon}\, b^\Theta_i\big(\frac{x}{\varepsilon}\big)\,\frac{\partial v}{\partial x_i},$$

and the coefficients of the operator A^ε_Θ are given by

$$\begin{aligned} &a^\Theta_{ij}(y) = \varrho_\Theta(y)\, a_{ij}(y);\\ &b^\Theta_i(y) = \varrho_\Theta(y)\, b_j(y) - 2\,\varrho_\Theta(y)\, a_{ij}(y)\,\Theta_j\\ &+a_{ij}(y)\left[p_\Theta(y)\,\partial_{y_j}p^*_\Theta(y) - p^*_\Theta(y)\,\partial_{y_j}p_\Theta(y)\right]. \end{aligned} \tag{3.3}$$

Obviously, the matrix a^Θ is positive definite since both p_Θ and p^*_Θ are positive functions. Moreover, it has been shown in [**11**] that, for any $\theta\in\mathbb{R}^d$, the vector-field b^θ is divergence-free and that, for $\theta=\Theta$, it has zero mean-value

$$\int\limits_Y b^\Theta(y)\,dy = 0;\quad \operatorname{div} b^\theta = 0,\ \forall\,\theta. \tag{3.4}$$

REMARK 3.1. This computation leading to the simple problem (3.2) for v^ε does not work if the coefficients are merely locally periodic, namely of the type $a(x, x/\varepsilon)$, $b(x, x/\varepsilon)$. Indeed there would be additional terms in (3.2) due to the partial derivatives with respect to the slow variable x because $\lambda_1(\Theta)$ and p_Θ would depend on x.

Although problem (3.2) is not self-adjoint, the classical approach of homogenization (based on energy estimates in Sobolev spaces) would apply, thanks to (3.4), if the initial condition were not singular (the limit of $e^{-\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}$ is 0 or $+\infty$ almost everywhere). This singular behavior of the initial data (which formally has a limit merely in the sense of distributions) requires a different methodology for homogenizing (3.2). In order to overcome this difficulty, we use the representation of v^ε in terms of the corresponding Green function

$$v^\varepsilon(t,x) = \int\limits_\Omega K_\varepsilon(t,x,\xi)\,\frac{u_0(\xi)}{p_\Theta\big(\frac{\xi}{\varepsilon}\big)}\,e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\,d\xi, \tag{3.5}$$

where, for any given ξ, K_ε, as a function of (t,x), solves the problem

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\,\partial_t K_\varepsilon(t,x,\xi) + A^\varepsilon_\Theta\, K_\varepsilon(t,x,\xi) = 0, & (t,x)\in(0,T)\times\Omega,\\ K_\varepsilon(t,x,\xi) = 0, & (t,x)\in(0,T)\times\partial\Omega,\\ K_\varepsilon(0,x,\xi) = \delta(x-\xi), & x\in\Omega, \end{cases} \tag{3.6}$$

The strategy is now to replace the Green function K_ε by an ansatz in (3.5) and to study the limit, as $\varepsilon\to 0$, of the resulting singular integral. The next section is devoted to the study of the asymptotic behavior of K_ε.

4. Asymptotics of the Green function K_ε

The main goal of this section is to prove the following statement.

LEMMA 4.1. *Assume that conditions* $(\mathbf{H1})-(\mathbf{H2})$ *are satisfied. Let K_ε be the Green function of problem* (3.2)*. Then, for any $t_0>0$ and any compact subset $B\Subset\Omega$, there exists a constant C such that, for all $t\ge t_0>0$, $\xi\in B$,*

$$\int\limits_\Omega |K_\varepsilon(t,x,\xi)-K_0(t,x,\xi)|^2\,dx \le C\,\varepsilon^2,$$
$$|K_\varepsilon(t,x,\xi)-K_0(t,x,\xi)| \le C\,\varepsilon^\gamma, \quad x\in\Omega,$$

where the constant C depends on $t_0, \operatorname{dist}(B,\partial\Omega), \Omega, \Lambda, d$ and is independent of ε, $\gamma=\gamma(\Omega,\Lambda,d)>0$, and K_0 is the Green function of the homogenized problem (2.5), i.e., as a function of (t,x), it solves

$$\begin{cases} \partial_t K_0(t,x,\xi) = \operatorname{div}(a^{\rm eff}\nabla K_0(t,x,\xi)), & (t,x)\in(0,T)\times\Omega,\\ K_0(t,x,\xi)=0, & (t,x)\in(0,T)\times\partial\Omega,\\ K_0(0,x,\xi)=\delta(x-\xi), & x\in\Omega, \end{cases} \tag{4.1}$$

with the constant positive definite matrix $a^{\rm eff}$ defined by (4.7).

PROOF. The main difficulty in studying the asymptotics of the Green function K_ε, defined as a solution of (3.6), is the presence of the delta function in the initial condition. To overcome this difficulty, we consider the difference

$$V_\varepsilon(t,x,\xi) = \Phi_\varepsilon(t,x,\xi) - K_\varepsilon(t,x,\xi),$$

where Φ_ε is the Green function of the same parabolic equation in the whole space, that is, for $\xi \in \mathbb{R}^d$, Φ_ε, as a function of (t,x), is a solution of the problem

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\ \partial_t \Phi_\varepsilon(t,x,\xi) + A_\Theta^\varepsilon \Phi_\varepsilon(t,x,\xi) = 0, & (t,x) \in (0,T)\times\mathbb{R}^d, \\ \Phi_\varepsilon(0,x,\xi) = \delta(x-\xi), & x \in \mathbb{R}^d. \end{cases} \tag{4.2}$$

In this way, for all $\xi \in \Omega$, V_ε, as a function of (t,x), solves the problem

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\, \partial_t V_\varepsilon(t,x,\xi) + A_\Theta^\varepsilon\, V_\varepsilon(t,x,\xi) = 0, & (t,x) \in (0,T)\times\Omega, \\ V_\varepsilon(t,x,\xi) = \Phi_\varepsilon(t,x,\xi), & (t,x) \in (0,T)\times\partial\Omega, \\ V_\varepsilon(0,x,\xi) = 0, & x \in \Omega. \end{cases} \tag{4.3}$$

We emphasize that V_ε, in contrast with K_ε, is Hölder continuous for all $t \geq 0$ provided that $\xi \notin \partial\Omega$.

Notice that, by a proper rescaling in time and space, Φ_ε can be identified with the fundamental solution of an operator which is independent of ε. Indeed,

$$\Phi_\varepsilon(t,x,\xi) = \varepsilon^{-d}\, \Phi\Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\Big), \tag{4.4}$$

where $\Phi(\tau,y,\eta)$ is defined, for $\eta \in \mathbb{R}^d$, as the solution in (τ,y) of

$$\begin{cases} \varrho_\Theta(y)\ \partial_\tau \Phi(\tau,y,\eta) + A_\Theta \Phi(\tau,y,\eta) = 0, & \tau > 0,\ y \in \mathbb{R}^d, \\ \Phi(0,y,\eta) = \delta(y-\eta), & y \in \mathbb{R}^d. \end{cases} \tag{4.5}$$

Here, for brevity, we denote by A_Θ the rescaled version of A_Θ^ε

$$A_\Theta \Phi(\tau,y,\eta) = -\mathrm{div}_y(a^\Theta(y)\nabla_y \Phi(\tau,y,\eta)) + b^\Theta(y)\cdot\nabla_y\Phi(\tau,y,\eta).$$

We also introduce the fundamental solution $\Phi_0(t,x,\xi)$ for the effective operator

$$\begin{cases} \partial_t \Phi_0 = \mathrm{div}_x(a^{\rm eff}\nabla_x \Phi_0), & (t,x) \in (0,T)\times\mathbb{R}^d, \\ \Phi_0(0,x,\xi) = \delta(x-\xi), & x \in \mathbb{R}^d. \end{cases} \tag{4.6}$$

The homogenized matrix $a^{\rm eff}$ is classically **[8]**, **[25]** given by

$$\begin{aligned} a_{ij}^{\rm eff} &= \int_Y \big(a_{ij}^\Theta(y) + a_{ik}^\Theta(y)\partial_{y_k}N_j(y) - b_i^\Theta(y)\,N_j(y)\big)\ dy \\ &= \int_Y \big(a_{ji}^\Theta(\eta) + a_{ki}^\Theta(\eta)\partial_{y_k}N_j^*(\eta) + b_i^\Theta(\eta)\,N_j^*(\eta)\big)\ d\eta, \end{aligned} \tag{4.7}$$

where the vector-valued functions $N=(N_i)_{1\le i\le d}$ and $N^*=(N_i^*)_{1\le i\le d}$ solve the direct and adjoint cell problems, respectively,

$$
\left\{
\begin{array}{l}
-\mathrm{div}(a^\Theta\nabla N_i)+b^\Theta\cdot\nabla N_i=\partial_{y_j}a^\Theta_{ij}(y)-b^\Theta_i(y),\quad Y,\\[2mm]
y\mapsto N_i\quad Y-\text{periodic};
\end{array}
\right.
\tag{4.8}
$$

$$
\left\{
\begin{array}{l}
-\mathrm{div}((a^\Theta)^T\nabla N_i^*)-b^\Theta\cdot\nabla N_i^*=\partial_{y_j}a^\Theta_{ji}(y)+b^\Theta_i(y),\quad Y,\\[2mm]
y\mapsto N_i^*\quad Y-\text{periodic}.
\end{array}
\right.
\tag{4.9}
$$

The matrix $a^{\rm eff}$ is positive definite (see, for example, [**8**], [**20**], [**25**]) and is exactly the same homogenized matrix as in the homogenization of the spectral problem [**11**]. Note that N and N^* are Hölder continuous functions (see [**16**]). The solution of problem (4.6) can be written explicitly:

$$
\Phi_0(t,x,\xi)=\frac{1}{(4\pi t)^{d/2}}\,\frac{1}{\det a^{\rm eff}}\,\exp\Big\{-\frac{(x-\xi)^T(a^{\rm eff})^{-1}(x-\xi)}{4t}\Big\}.
$$

The first-order approximation for the Green function Φ, solution of (4.5), is defined as follows

$$
\Phi_1(\tau,y,\eta)=\Phi_0(\tau,y,\eta)+N(y)\cdot\nabla_x\Phi_0(\tau,y,\eta)+N^*(\eta)\cdot\nabla_\xi\Phi_0(\tau,y,\eta).
\tag{4.10}
$$

By means of Bloch wave analysis it has been shown in [**1**] that, under assumption (3.4), there exists a constant C such that, for any $\tau\ge 1$ and $y,\eta\in\mathbb{R}^d$,

$$
\begin{aligned}
|\Phi(\tau,y,\eta)-\Phi_0(\tau,y,\eta)|&\le\frac{C}{\tau^{(d+1)/2}},\\
|\Phi(\tau,y,\eta)-\Phi_1(\tau,y,\eta)|&\le\frac{C}{\tau^{(d+2)/2}}.
\end{aligned}
\tag{4.11}
$$

Thus, in view of the rescaling (4.4), there exists a constant $C>0$, which does not depend on ε, such that, for any $t\ge\varepsilon^2$, $x,\xi\in\mathbb{R}^d$,

$$
\begin{aligned}
|\Phi_\varepsilon(t,x,\xi)-\Phi_0(t,x,\xi)|&\le\frac{C\,\varepsilon}{t^{(d+1)/2}};\\
|\Phi_\varepsilon(t,x,\xi)-\Phi_1^\varepsilon(t,x,\xi)|&\le\frac{C\,\varepsilon^2}{t^{(d+2)/2}}.
\end{aligned}
\tag{4.12}
$$

Here $\Phi_1^\varepsilon(t,x,\xi)=\varepsilon^{-d}\,\Phi_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big)$, namely

$$
\Phi_1^\varepsilon(t,x,\xi)=\Phi_0(t,x,\xi)+\varepsilon\,N\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla_x\Phi_0(t,x,\xi)+\varepsilon\,N^*\Big(\frac{\xi}{\varepsilon}\Big)\cdot\nabla_\xi\Phi_0(t,x,\xi).
\tag{4.13}
$$

Next, we study the asymptotic behavior of V_ε, solution of (4.3). The (formal) two-scale asymptotic expansion method suggests to approximate V_ε by a first-order ansatz defined by

$$
V_1^\varepsilon(t,x,\xi)=V_0(t,x,\xi)+\varepsilon\,N\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla_xV_0(t,x,\xi)+\varepsilon\,N^*\Big(\frac{\xi}{\varepsilon}\Big)\cdot\nabla_\xi V_0(t,x,\xi),
\tag{4.14}
$$

where N and N^* are the solutions of cell problems (4.8) and (4.9), respectively, and, for fixed ξ, V_0, as a function of (t,x), is the solution of the effective problem

$$\begin{cases} \partial_t V_0(t,x,\xi) = \operatorname{div}_x(a^{\text{eff}}\nabla_x V_0(t,x,\xi)), & (t,x)\in(0,T)\times\Omega,\\ V_0(t,x,\xi) = \Phi_0(t,x,\xi), & (t,x)\in(0,T)\times\partial\Omega,\\ V_0(0,x,\xi)=0, & x\in\Omega. \end{cases} \tag{4.15}$$

Due to the maximum principle and to the explicit formula for Φ_0, there exists a constant C, which depends only on Λ and d, such that, for any compact subset $B \Subset \Omega$, $\xi\in B$, $(t,x)\in [0,T]\times\Omega$,

$$0\le V_0(t,x,\xi)\le \max_{(t,x)\in[0,T)\times\partial\Omega}\Phi_0(t,x,\xi)\le \frac{C}{\operatorname{dist}(B,\partial\Omega)^d}. \tag{4.16}$$

Moreover, combining (4.16) with the local estimates of the derivatives of V_0 gives

$$\left|\partial_t^k\partial_{x_j}^l\partial_{\xi_j}^m V_0(t,x,\xi)\right|\le \frac{C}{\operatorname{dist}(B,\partial\Omega)^{d+2k+l+m}},\ (t,x,\xi)\in[0,T]\times\Omega\times B. \tag{4.17}$$

To finish the proof of Lemma 4.1 we need the following intermediate result.

LEMMA 4.2. *Let V_ε and V_0 be solutions of problems* (4.3) *and* (4.15)*, respectively. Then, for any compact subset $B\Subset\Omega$, there exists a positive constant C, only depending on* $\operatorname{dist}(B,\partial\Omega),\Omega,d,\Lambda$*, such that, for any* $(t,\xi)\in[0,T]\times B$,

$$\int\limits_\Omega |V_\varepsilon(t,x,\xi)-V_0(t,x,\xi)|^2\,dx\le C\,\varepsilon^2.$$

PROOF. Let V_1^ε be the first-order approximation of V_ε defined by (4.14). Evaluating the remainder after substituting the difference $\widetilde{V}^\varepsilon = V_1^\varepsilon - V_\varepsilon$ into problem (4.3), we get

$$\begin{cases} \varrho_\Theta\big(\frac{x}{\varepsilon}\big)\,\partial_t\widetilde{V}^\varepsilon + A_\Theta^\varepsilon\widetilde{V}^\varepsilon = F\big(t,x,\xi;\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big)\\ \qquad +\varepsilon\, f\big(t,x,\xi;\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big), \quad (t,x)\in(0,T)\times\Omega,\\ \widetilde{V}^\varepsilon = G_\varepsilon\big(t,x,\xi;\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big), \quad (t,x)\in(0,T)\times\partial\Omega,\\ \widetilde{V}^\varepsilon(0,x,\xi)=0, \quad x\in\Omega, \end{cases} \tag{4.18}$$

with F, f and G defined by

$$F(t,x,\xi;y,\eta) = \varrho_\Theta(y)\,\partial_t V_0 - \mathrm{div}_y(a^\Theta(y)\nabla_x(N(y)\,\nabla_x V_0(t,x,\xi)))$$
$$-\mathrm{div}_y(a^\Theta(y)\nabla_x(N^*(\eta)\,\nabla_\xi V_0(t,x,\xi))) - \mathrm{div}_x(a^\Theta(y)\nabla_x V_0(t,x,\xi))$$
$$-\mathrm{div}_x(a^\Theta(y)\nabla_y(N(y)\nabla_x V_0(t,x,\xi))) + b^\Theta(y)\cdot\nabla_x(N(y)\,\nabla_x V_0(t,x,\xi)))$$
$$+b^\Theta(y)\cdot\nabla_x(N^*(\eta)\,\nabla_\xi V_0(t,x,\xi));$$
$$f(t,x,\xi;y,\eta) = N(y)\cdot\partial_t\nabla_x V_0(t,x,\xi) + N^*(\eta)\cdot\partial_t\nabla_\xi V_0(t,x,\xi)$$
$$-\mathrm{div}_x(a^\Theta(y)\nabla_x(N(y)\cdot\nabla_x V_0(t,x,\xi)))$$
$$-\mathrm{div}_x(a^\Theta(y)\nabla_x(N^*(y)\cdot\nabla_\xi V_0(t,x,\xi)));$$
$$G_\varepsilon(t,x,\xi;y,\eta) = \Phi_0(t,x,\xi) - \Phi_\varepsilon(t,x,\xi)$$
$$+\varepsilon\,N(y)\cdot\nabla_x V_0(t,x,\xi) + \varepsilon\,N^*(\eta)\cdot\nabla_\xi V_0(t,x,\xi).$$

By linearity, we represent $\widetilde{V}^\varepsilon$ as a sum $\widetilde{V}^\varepsilon = \widetilde{V}_1^\varepsilon + \widetilde{V}_2^\varepsilon$, where $\widetilde{V}_1^\varepsilon$ and $\widetilde{V}_2^\varepsilon$ are solutions of the follwoing problems

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\,\partial_t\widetilde{V}_1^\varepsilon + A_\Theta^\varepsilon\widetilde{V}_1^\varepsilon = F\big(t,x,\xi;\dfrac{x}{\varepsilon},\dfrac{\xi}{\varepsilon}\big) \\ \qquad +\varepsilon\,f\big(t,x,\xi;\dfrac{x}{\varepsilon},\dfrac{\xi}{\varepsilon}\big), \ (t,x)\in(0,T)\times\Omega, \\ \widetilde{V}_1^\varepsilon = 0, \qquad (t,x)\in(0,T)\times\partial\Omega, \\ \widetilde{V}_1^\varepsilon(0,x,\xi) = 0, \qquad x\in\Omega; \end{cases} \tag{4.19}$$

$$\begin{cases} \varrho_\Theta\big(\dfrac{x}{\varepsilon}\big)\,\partial_t\widetilde{V}_2^\varepsilon + A_\Theta^\varepsilon\widetilde{V}_2^\varepsilon = 0, \ (t,x)\in(0,T)\times\Omega, \\ \widetilde{V}_2^\varepsilon = G_\varepsilon\big(t,x,\xi;\dfrac{x}{\varepsilon},\dfrac{\xi}{\varepsilon}\big), \quad (t,x)\in(0,T)\times\partial\Omega, \\ \widetilde{V}_2^\varepsilon(0,x,\xi) = 0, \qquad x\in\Omega. \end{cases} \tag{4.20}$$

The trick is to estimate $\widetilde{V}_1^\varepsilon$ by standard energy estimates and $\widetilde{V}_2^\varepsilon$ by the maximum principle. First, we estimate $\widetilde{V}_1^\varepsilon$. Taking into account (4.17) and the boundedness of N, N^*, after integration by parts one has, for $\xi\in B\Subset\Omega$,

$$\left|\int\limits_Y F(t,x,\xi;y,\eta)w(y)\,dy\right| \le C\|w\|_{H^1_\#(Y)}, \quad \forall w\in H^1_\#(Y),$$

where $H^1_\#(Y)$ stands for the closure of Y-periodic smooth functions with respect to the $H^1(Y)$ norm. Thus, as a function of y, F belongs to the dual space $H^{-1}_\#(Y)$ uniformly in

(t,x,ξ,η). As is usual in the method of two-scale asymptotic expansion, equating the Y-average of F to zero yields the homogenized equation (4.15). Therefore, it is no surprise that, in view of (3.4), (4.15) and the periodicity of a_{ij}^Θ, N, N^*, we compute

$$\int\limits_Y F(t,x,\xi;y,\eta)\,dy = 0.$$

Thus, for any t,x,ξ there exists a Y-periodic with respect to y vector function $\chi = \chi(t,x,\xi;y,\eta)$, which belongs to $L^2_\#(Y;\mathbb{R}^d)$, such that

$$\begin{aligned} &F(t,x,\xi;y,\eta) = \operatorname{div}_y \chi(t,x,\xi;y,\eta) \\ &\int\limits_Y |\chi(t,x,\xi;y,\eta)|^2\,dy \le C,\ \xi \in B \Subset \Omega. \end{aligned} \tag{4.21}$$

By rescaling we obtain

$$F(t,x,\xi;y,\xi/\varepsilon) = \varepsilon \operatorname{div}_x\Big(\chi(t,x,\xi;x/\varepsilon,\eta)\Big) - \varepsilon\,\Big(\operatorname{div}_x\chi\Big)(t,x,\xi;x/\varepsilon,\eta). \tag{4.22}$$

Since b^Θ is divergence-free, the a priori estimates are then obtained in the classical way. Multiplying the equation in (4.19) by $\widetilde{V}_1^\varepsilon$, integrating by parts and using (4.21), (4.22) yield

$$\int\limits_\Omega |\widetilde{V}_1^\varepsilon(t,x,\xi)|^2\,dx \le C\,\varepsilon^2, \quad (t,x)\in[0,T]\times\Omega,\ \xi\in B\Subset\Omega. \tag{4.23}$$

Second, we estimate $\widetilde{V}_2^\varepsilon$, solution of (4.20), by using the maximum principle. Our next goal is to prove that

$$|G_\varepsilon\big(t,x,\xi;\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\big)| \le C\,\varepsilon, \quad (t,x)\in[0,T]\times\partial\Omega,\ \xi\in B\Subset\Omega. \tag{4.24}$$

By (4.12), for any $\beta \le 2$ and $t \ge \varepsilon^\beta$,

$$|\Phi_\varepsilon(t,x,\xi) - \Phi_1^\varepsilon(t,x,\xi)| \le C\,\varepsilon^{2-(d+2)\beta/2}. \tag{4.25}$$

In (4.25) we find $2-(d+2)\beta/2 \ge 1$ if and only if $\beta \le (1+d/2)^{-1}$ which is always smaller than 2. For $x\in\partial\Omega, \xi\in B\Subset\Omega$, uniformly with respect to $t\ge 0$, we have

$$|\nabla_x\Phi_0(t,x,\xi)| \le \frac{C\,|x-\xi|}{t^{1+d/2}}\,e^{-\frac{C_0|x-\xi|^2}{t}} \le C$$

and a similar bound for $\nabla_\xi\Phi_0$. Thus, from (4.13) we deduce

$$|\Phi_1^\varepsilon(t,x,\xi) - \Phi_0(t,x,\xi)| \le C\,\varepsilon, \quad t\ge 0,\ x\in\partial\Omega,\ \xi\in B\Subset\Omega. \tag{4.26}$$

Combining (4.25) and (4.26) yields, for any $0<\beta\le(1+d/2)^{-1}$,

$$|\Phi_\varepsilon(t,x,\xi) - \Phi_0(t,x,\xi)| \le C\,\varepsilon, \quad t\ge\varepsilon^\beta, x\in\partial\Omega, \xi\in B\Subset\Omega. \tag{4.27}$$

To estimate $\Phi_\varepsilon - \Phi_0$ for small $t \in [0, \varepsilon^\beta)$ we make use of the Aronson estimates [**6**]. Taking into account (3.4) and (4.4), we see that Φ_ε admits the following bound

$$0 \le \Phi_\varepsilon(t,x,\xi) = \varepsilon^{-d}\,\Phi\Big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\Big) \le \frac{C}{t^{d/2}} \exp\Big\{ - \frac{C_0|x-\xi|^2}{t}\Big\}$$

with the constants C_0, C independent of ε. Thus, for sufficiently small ε, we obtain

$$\begin{aligned} &|\Phi_\varepsilon(t,x,\xi) - \Phi_0(t,x,\xi)| \le |\Phi_\varepsilon(t,x,\xi)| + |\Phi_0(t,x,\xi)| \\ &\le \frac{C}{t^{d/2}} \exp\Big\{ - \frac{C_0|x-\xi|^2}{t}\Big\} \le \frac{C}{\varepsilon^{d\beta/2}} \exp\Big\{ - \frac{C_0|x-\xi|^2}{\varepsilon^\beta}\Big\}. \end{aligned} \tag{4.28}$$

Thus, for $t \in [0,\varepsilon^\beta)$, $x \in \partial\Omega$ and $\xi \in B \Subset \Omega$, the difference $|\Phi_\varepsilon(t,x,\xi) - \Phi_0(t,x,\xi)|$ is exponentially small if $\beta > 0$. Combining (4.27) and (4.28) yields

$$|\Phi_\varepsilon(t,x,\xi) - \Phi_0(t,x,\xi)| \le C\,\varepsilon, \quad (t,x) \in [0,T] \times \partial\Omega,\ \xi \in B \Subset \Omega, \tag{4.29}$$

with the constant C depending on $\mathrm{dist}(B,\Omega), \Lambda, d$. The boundedness of N, N^*, estimates (4.17) and (4.29) imply (4.24).

Then, we use the maximum principle in (4.20) to deduce from (4.24) that

$$|\widetilde{V}_2^\varepsilon(t,x,\xi)| \le C\,\varepsilon, \quad (t,x,\xi) \in [0,T) \times \Omega \times B. \tag{4.30}$$

In view of (4.23) and (4.30), we conclude

$$\int\limits_\Omega |V_\varepsilon(t,x,\xi) - V_1^\varepsilon(t,x,\xi)|^2\,dx \le C\,\varepsilon^2, \quad t \in [0,T],\ \xi \in B \Subset \Omega.$$

Recalling the definition of V_1^ε and using estimate (4.17) complete the proof of Lemma 4.2. $\square$

Turning back to the proof of Lemma 4.1, the Green function $K_0(t,x,\xi)$, which is defined as the solution of (4.1), satisfies $K_0 = V_0 - \Phi_0$. Similarly, by definition, $K_\varepsilon = V_\varepsilon - \Phi_\varepsilon$. Taking into account (4.12), Lemma 4.2 implies

$$\int\limits_\Omega |K_\varepsilon(t,x,\xi) - K_0(t,x,\xi)|^2\,dx \le C\,\varepsilon^2, \quad t \ge t_0 > 0,\ \xi \in B \Subset \Omega.$$

We would like to emphasize that the constant C in the last estimate only depends on t_0, $\mathrm{dist}(B,\partial\Omega)$, Λ, d, Ω. Due to the Nash-De Giorgi estimates for the parabolic equations (see, for example, [**19**]), K_ε is Hölder continuous (of course K_0 is), and, thus, one can deduce a uniform estimate

$$|K_\varepsilon(t,x,\xi) - K_0(t,x,\xi)| \le C\,\varepsilon^\gamma, \quad t \ge t_0 > 0,\ x \in \Omega,\ \xi \in B \Subset \Omega \tag{4.31}$$

for some $\gamma > 0$ depending on Ω, Λ and d. We emphasize that the constants C, γ do not depend on ε. Indeed, due to condition (3.4), problem (3.2) can be rewritten in divergence form, without any convective term and without any ε-factor in front of the coefficients. The proof of Lemma 4.1 is complete. $\square$

REMARK 4.3. Estimate (4.31) is enough for our purpose, but we emphasize that it can be improved. Namely, constructing sufficiently many terms in the asymptotic expansion for V_ε, one can show that

$$|K_\varepsilon(t,x,\xi) - K_0(t,x,\xi)| \le C\,\varepsilon, \quad t \ge t_0 > 0,\ x \in \Omega,\ \xi \in B \Subset \Omega.$$

5. Asymptotics of u^ε or v^ε

The goal of this section is to prove our main result Theorem 2.3 and actually to give a more precise statement of it in Theorem 5.1. By the factorization principle (3.1) it is equivalent to find a precise asymptotic expansion of v^ε. Recall that v^ε, as a solution of (3.2), can be represented in terms of the corresponding Green function K_ε by using formula (3.5). Bearing in mind Lemma 4.1, we rearrange (3.5) as follows

$$v^\varepsilon(t,x) = I_1^\varepsilon + I_2^\varepsilon \tag{5.1}$$

with

$$I_1^\varepsilon = \int\limits_\Omega K_0(t,x,\xi)\,\frac{u_0(\xi)}{p_\Theta\left(\frac{\xi}{\varepsilon}\right)}\,e^{-\frac{\Theta\cdot(\xi-\bar x)}{\varepsilon}}\,d\xi,$$
$$I_2^\varepsilon = \int\limits_\Omega \big(K_\varepsilon(t,x,\xi) - K_0(t,x,\xi)\big)\,\frac{u_0(\xi)}{p_\Theta\left(\frac{\xi}{\varepsilon}\right)}\,e^{-\frac{\Theta\cdot(\xi-\bar x)}{\varepsilon}}\,d\xi.$$

Of course, because of (4.31), the second integral in (5.1) is going to be, at least, ε^γ times smaller that the first one. Recall that, by assumption **(H3)**, u_0 has a compact support $\omega \Subset \Omega$ so we are able to use the previous estimates of Lemma 4.1. Let us compute approximately the first integral I_1^ε. Since $\Theta\cdot(x-\bar x) > 0$ for $x \in \omega \setminus \{\bar x\}$, it is clear that the main contribution is given by integrating over a neighborhood of the point $\bar x$. We consider the case of general position, when condition **(H5)** is fulfilled, that is, in local coordinates in a neighborhood $U_\delta(\bar x)$ of the point $\bar x$, $\partial\omega$ can be defined by

$$z_d = (Sz', z') + o(|z'|^2)$$

for some positive definite $(d-1)\times(d-1)$ matrix S. Here $(z_1,\cdots,z_d)$ is an orthonormal basis such that the coordinates $z' = (z_1,\cdots,z_{d-1})$ are tangential to $\partial\omega$ and the axis z_d is the interior normal at $\bar x$. Note that, by assumption **(H4)**, Θ is directed along z_d. The neighborhood of $\bar x$ is defined by

$$U_\delta(\bar x) = \{z \in \omega :\ |z'| \le \delta,\ 0 \le z_d \le \delta^2\|S\|\},$$

where $\|S\| = \max_{|x'|=1}|Sx'|$. Choosing $\delta = \varepsilon^{1/4}$ guaranties that the integral over the complement to $U_\delta(\bar x)$ is negligible. Indeed,

$$\Big|\int\limits_{\omega\setminus U_\delta(\bar x)} K_0(t,x,\xi)\,\frac{u_0(\xi)}{p_\Theta\left(\frac{\xi}{\varepsilon}\right)}\,e^{-\frac{\Theta\cdot(\xi-\bar x)}{\varepsilon}}\,d\xi\Big| = O(e^{-\frac{1}{\sqrt{\varepsilon}}}).$$

Let us now compute the integral over $U_\delta(\bar{x})$, $\delta = \varepsilon^{1/4}$. Expanding K_0 and u_0 (which is of class C^2 in ω) into Taylor series about $\bar{x}$ and taking into account assumption (**H6**), for $t \geq t_0 > 0$, we obtain

$$\begin{aligned} I_1^\varepsilon &= K_0(t,x,\bar{x}) \frac{\partial u_0}{\partial \Theta}(\bar{x}) \int\limits_{U_\delta(\bar{x})} \frac{\Theta}{|\Theta|} \cdot (\xi - \bar{x}) \big(p_\Theta\big(\frac{\xi}{\varepsilon}\big)\big)^{-1} e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}} \, d\xi + O(\varepsilon^3 \varepsilon^{\frac{d-1}{2}}) \\ &= K_0(t,x,\bar{x}) \frac{\partial u_0}{\partial \Theta}(\bar{x}) \int\limits_{U_\delta(0)} \frac{\Theta}{|\Theta|} \cdot \xi \big(p_\Theta\big(\frac{\xi}{\varepsilon} + \frac{\bar{x}}{\varepsilon}\big)\big)^{-1} e^{-\frac{\Theta\cdot\xi}{\varepsilon}} \, d\xi + O(\varepsilon^3 \varepsilon^{\frac{d-1}{2}}). \end{aligned}$$

where $\partial u_0/\partial\Theta := \nabla u_0 \cdot \Theta/|\Theta|$ is the directional derivative of u_0 along Θ (the tangential derivative of u_0 vanishes at $\bar{x}$ because u_0 is continuous and equal to 0 outside ω). Note that we have anticipated the precise order of the remainder term which will be clear once we compute the leading integral. Let us introduce the rotation matrix $\mathfrak{R}$ which defines the local coordinate system $(z_1, z_2, \cdots, z_d) = (z', z_d)$ previously defined. By definition it satisfies $\xi = \mathfrak{R}^{-1} z$ and $\Theta \cdot \xi = |\Theta| \, z_d$. Applying this change of variables we get

$$p_\Theta\big(\frac{\xi}{\varepsilon} + \{\frac{\bar{x}}{\varepsilon}\}\big) = p_\Theta\big(\mathfrak{R}^{-1}\big(\frac{z}{\varepsilon} + \mathfrak{R}\{\frac{\bar{x}}{\varepsilon}\}\big)\big) \equiv P_\Theta\big(\frac{z}{\varepsilon} + \bar{z}^\varepsilon\big), \tag{5.2}$$

where $\{\bar{x}/\varepsilon\}$ is the fractional part of $\bar{x}/\varepsilon$ and $\bar{z}^\varepsilon = \mathfrak{R}\,\{\bar{x}/\varepsilon\}$. In the case when $\Theta_1, \Theta_2, \cdots, \Theta_d$ are rationally dependent in pairs, P_Θ remains periodic with another period. Otherwise P_Θ is merely almost periodic. It happens, for example, when all $\Theta_k, k = 1, ..., d$ are rationally independent in pairs.

We turn to the computation of the integral over $U_\delta(0)$. By the above change of variables we get

$$\begin{aligned} I_1^\varepsilon &= K_0(t,x,\bar{x}) \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\ &\times \int\limits_{|z'|\leq\delta} dz' \int\limits_{(Sz',z')}^{\delta^2\|S\|} z_d \, P_\Theta^{-1}\big(\frac{z}{\varepsilon} + \bar{z}^\varepsilon\big) \, e^{-\frac{|\Theta| z_d}{\varepsilon}} \, dz_d + o(\varepsilon^2 \varepsilon^{\frac{d-1}{2}}). \end{aligned} \tag{5.3}$$

To blow-up the integral in (5.3) we make a (parabolic) rescaling of the space variables

$$\zeta' = \frac{z'}{\sqrt{\varepsilon}}, \quad \zeta_d = \frac{z_d}{\varepsilon},$$

and recalling that $\delta = \varepsilon^{1/4}$, we arrive at the following integral

$$\begin{aligned} I_1^\varepsilon &= \varepsilon^2 \varepsilon^{\frac{(d-1)}{2}} K_0(t,x,\bar{x}) \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\ &\times \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d \, P_\Theta^{-1}\big(\frac{\zeta'}{\sqrt{\varepsilon}} + (\bar{z}^\varepsilon)', \zeta_d + \bar{z}_d^\varepsilon\big) \, e^{-|\Theta|\zeta_d} \, d\zeta_d + o(\varepsilon^2 \varepsilon^{\frac{d-1}{2}}), \end{aligned}$$

where the reaminder term takes into account the fact that the domain of integration is now infinite. Changing the order of integration we have

$$
\begin{aligned}
I_1^\varepsilon &= \varepsilon^2 \, \varepsilon^{\frac{(d-1)}{2}} \, K_0(t,x,\bar{x}) \, \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\
&\times \int_0^{+\infty} \zeta_d \, e^{-|\Theta| \zeta_d} \, d\zeta_d \int_{(S\zeta',\zeta')\leq \zeta_d} P_\Theta^{-1}\big(\frac{\zeta'}{\sqrt{\varepsilon}} + (\bar{z}^\varepsilon)', \zeta_d + \bar{z}_d^\varepsilon\big) \, d\zeta' + o(\varepsilon^2 \, \varepsilon^{\frac{d-1}{2}}).
\end{aligned}
$$

The function $P_\Theta^{-1}\big(\eta', \tau_d\big)$ is uniformly continuous; moreover, it is almost periodic with respect to the first variable. Thus, for any bounded Borel set $B \subset \mathbb{R}^{d-1}$, the following limit exists

$$
\mathcal{M}\{P_\Theta^{-1}\big(\cdot, \tau_d\big)\} = \lim_{t\to\infty} \frac{1}{|tB|} \int_{tB} P_\Theta^{-1}\big(\eta' + \tau', \tau_d\big) \, d\eta'. \tag{5.4}
$$

We emphasize that the convergence is uniform with respect to τ' and τ_d, and the limit does not depend on τ'. Therefore, by Lemma 8.2, as $\varepsilon \to 0$, we eventually deduce

$$
\begin{aligned}
I_1^\varepsilon &= \varepsilon^2 \, \varepsilon^{\frac{d-1}{2}} \, K_0(t,x,\bar{x}) \, \frac{\partial u_0}{\partial \Theta}(\bar{x}) \\
&\times \int_{\mathbb{R}^{d-1}} d\zeta' \int_{(S\zeta',\zeta')}^{+\infty} \zeta_d \, e^{-|\Theta| \zeta_d} \, \mathcal{M}\{P_\Theta^{-1}\big(\cdot, \zeta_d + \bar{z}_d^\varepsilon\big)\} \, d\zeta_d + o(\varepsilon^2 \, \varepsilon^{\frac{d-1}{2}}),
\end{aligned} \tag{5.5}
$$

where the remainder term is asymptotically smaller than the leading order term (uniformly in $t \geq 0, x \in \overline{\Omega}$) but we cannot say how much since there is no precise speed of convergence for averages of almost periodic functions in Lemma 8.2.

The case of the second integral I_2^ε is then very similar. Taking into account the positiveness of p_Θ, and Lemma 4.1, for $t \geq t_0 > 0$, we obtain

$$
|I_2^\varepsilon| \leq \, C \, \varepsilon^\gamma \int_\omega |u_0(x)| \, e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}} \, d\xi,
$$

where C does not depend on ε. The same computation as above (but without the necessity of considering almost periodic functions) yields

$$\begin{aligned}
|I_2^\varepsilon| &\le C\,\varepsilon^\gamma \left|\frac{\partial u_0}{\partial\Theta}(\bar x)\right| \int\limits_\omega \left|\frac{\Theta}{|\Theta|}\cdot(\xi-\bar x)\right| e^{-\frac{\Theta\cdot(\xi-\bar x)}{\varepsilon}}\,d\xi \\
&\le C\,\varepsilon^\gamma \left|\frac{\partial u_0}{\partial\Theta}(\bar x)\right| \int\limits_{\mathbb{R}^{d-1}} dz' \int\limits_{S_0|z'|^2}^{+\infty} z_d\, e^{-\frac{|\Theta| z_d}{\varepsilon}}\,dz_d \\
&\le C\,\varepsilon^{2+\gamma} \left|\frac{\partial u_0}{\partial\Theta}(\bar x)\right| \int\limits_{\mathbb{R}^{d-1}} (1+S_0\,|z'|^2\,\varepsilon^{-1})\, e^{-\frac{|\Theta| S_0 |z'|^2}{\varepsilon}} dz' \\
&\le C\,\varepsilon^{2+\gamma}\,\varepsilon^{\frac{d-1}{2}},
\end{aligned}$$

for some constant $S_0>0$ and $C=C(S_0,\Theta)$. Finally, we have derived the following asymptotics of v^ε, as $\varepsilon\to 0$,

$$\begin{aligned}
v^\varepsilon(t,x) &= \varepsilon^2\,\varepsilon^{\frac{d-1}{2}}\Big(1+r_\varepsilon(t,x)\Big)K_0(t,x,\bar x)\,\frac{\partial u_0}{\partial\Theta}(\bar x) \\
&\times \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d\, e^{-|\Theta|\zeta_d}\,\mathcal{M}\{P_\Theta^{-1}\big(\cdot,\zeta_d+\bar z_d^\varepsilon\big)\}\,d\zeta_d,
\end{aligned}$$

where $r_\varepsilon(t,x)$ converges to zero uniformly with respect to $(t,x)\in[t_0,T]\times\overline{\Omega}$ with any $t_0>0$.

We summarize the result, just obtained, by formulating a more precise version of Theorem 2.3, describing the asymptotics of $u^\varepsilon(t,x)$.

THEOREM 5.1. *Suppose conditions* $(\mathbf{H1})-(\mathbf{H6})$ *are satisfied and* $\Theta\neq 0$. *Let* u^ε *be the solution of problem* (1.1). *Then, for* $t\ge t_0>0$,

$$u^\varepsilon(t,x) = \varepsilon^2\,\varepsilon^{\frac{d-1}{2}}\Big(1+r_\varepsilon(t,x)\Big)e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\,e^{\frac{\Theta\cdot(x-\bar x)}{\varepsilon}}\,M_\varepsilon\,p_\Theta\big(\frac{x}{\varepsilon}\big)\,u(t,x),$$

where $(\lambda_1(\Theta),p_\Theta)$ *is the first eigenpair defined by Lemma 2.1 and* $r_\varepsilon(t,x)\to 0$, *as* $\varepsilon\to 0$, *uniformly with respect to* $(t,x)\in[t_0,T]\times\overline{\Omega}$. *The function* $u(t,x)$ *solves the homogenized problem*

$$\begin{cases}
\partial_t u = \operatorname{div}(a^{\text{eff}}\,\nabla u), & (t,x)\in(0,T)\times\Omega,\\
u(t,x)=0, & (t,x)\in(0,T)\times\partial\Omega,\\
u(0,x)=\nabla u_0(\bar x)\cdot\dfrac{\Theta}{|\Theta|}\,\delta(x-\bar x), & x\in\Omega,
\end{cases} \tag{5.6}$$

with a^{eff} being a positive definite matrix given by (4.7)*, $\delta(x-\bar{x})$ is the Dirac delta-function at the point $\bar{x}$. The constant M_ε is defined by*

$$M_\varepsilon = \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d \, e^{-|\Theta|\zeta_d} \, \mathcal{M}\{P_\Theta^{-1}\big(\cdot, \zeta_d + \bar{z}_d^\varepsilon\big)\} \, d\zeta_d, \tag{5.7}$$

where $\mathcal{M}\{P_\Theta^{-1}\big(\cdot, \tau_d\big)\}$ is the mean-value of the almost periodic function $\eta' \to P_\Theta^{-1}(\eta', \tau_d)$ (see (5.4)*), P_Θ is given by* (5.2) *and $\bar{z}_d^\varepsilon = \mathfrak{R}\{\bar{x}/\varepsilon\} \cdot \frac{\Theta}{|\Theta|}$.*

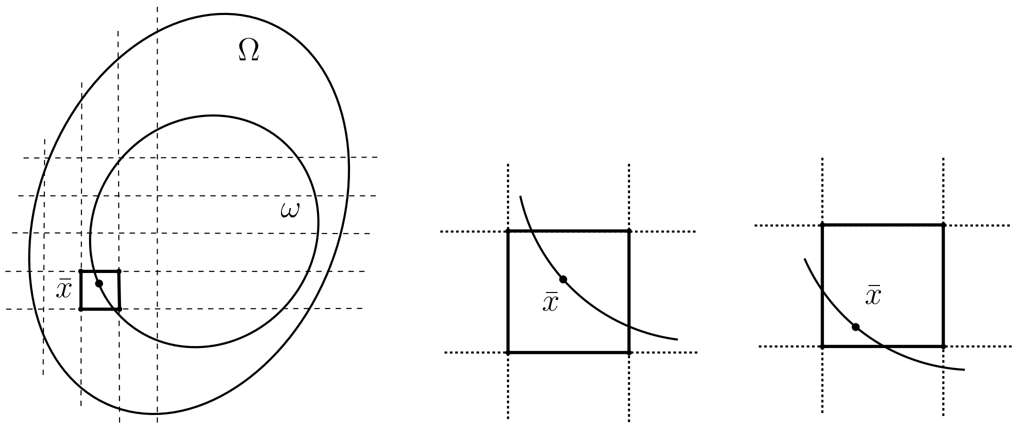


FIGURE 2. Position of $\bar{x}$ in εY for different values of ε

REMARK 5.2. The constant M_ε defined by (5.7) depends on $\bar{z}_d^\varepsilon = \mathfrak{R}\{\bar{x}/\varepsilon\} \cdot \frac{\Theta}{|\Theta|}$, that is on the component, parallel to Θ, of the fractional part of $\bar{x}/\varepsilon$, or, in other words, on the relative position of $\bar{x}$ inside the cell εY (see Figure 2). Notice that M_ε is bounded, thus, up to a subsequence, it converges to some M^*, as $\varepsilon \to 0$. The choice of the converging subsequence is only a matter of the geometric definition of the periodic medium. For example, if $\bar{x}$ is known, we may decide to make it the origin and to define the periodic microstructure relative to this origin. Then $\bar{x} = 0$, $\bar{z}^\varepsilon = 0$ is fixed in the periodicity cell, and $M_\varepsilon = M$ is independent of ε.

It might happen that the vector Θ is such that its components Θ_d and Θ_k are rationally independent for all $k \neq d$. In such a case, it turns out that the constant M_ε does not depend on ε and, moreover, can be explicitly computed. This is the topic of the following result.

COROLLARY 5.3. *Let conditions of Theorem 5.1 be satisfied. And assume that the vector Θ is such that Θ_d and Θ_k, for any $k = 1, \cdots, (d-1)$, are rationally independent.*

Then M_ε is independent of ε and is given by

$$M_\varepsilon = \frac{(d-1)}{|\Theta|^2}\Big(\frac{\pi}{|\Theta|}\Big)^{\frac{d-1}{2}} (\det S)^{1/2} \int\limits_Y p_\Theta^{-1}(y)\,dy.$$

In other words, for $t \geq t_0 > 0$,

$$u^\varepsilon(t,x) = \Big(\frac{\varepsilon}{|\Theta|}\Big)^{2+\frac{d-1}{2}} K\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, u(t,x)\,\Big(1 + r_\varepsilon(t,x)\Big),$$

where $r_\varepsilon(t,x) \to 0$, as $\varepsilon \to 0$, uniformly with respect to $(t,x) \in [t_0,T]\times\overline{\Omega}$; $u(t,x)$ solves the homogenized problem (5.6)*. The constant K is given by*

$$K = (d-1)\,\pi^{\frac{d-1}{2}}\,(\det S)^{1/2} \int\limits_Y p_\Theta^{-1}(y)\,dy.$$

PROOF. It is sufficient to notice that in the case when Θ_d and Θ_k, $k = 1,2,\cdots(d-1)$, are rationally independent, the mean value of the almost periodic function $P_\Theta^{-1}(\zeta',\tau_d)$ with respect to the first variable ζ', for any τ_d, coincides with its volume average

$$\mathcal{M}\{P_\Theta^{-1}(\cdot,\tau_d)\} = \int\limits_Y p_\Theta^{-1}(y)\,dy.$$

Thus, the constant M_ε given by (5.7) does not depend on ε and has the following form

$$M_\varepsilon = \Big(\int\limits_Y p_\Theta^{-1}(y)\,dy\Big) \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d\, e^{-|\Theta|\zeta_d}\,d\zeta_d.$$

Evaluating the last integral we obtain

$$M_\varepsilon = \frac{(d-1)}{|\Theta|^2}\Big(\frac{\pi}{|\Theta|}\Big)^{\frac{d-1}{2}} (\det S)^{1/2} \int\limits_Y p_\Theta^{-1}(y)\,dy$$

that implies the desired result. □

REMARK 5.4. Theorem 5.1 does not provide any rate of convergence due to several reasons. First of all, without specifying the remainder in hypothesis **(H5)**, one cannot expect any estimate in (5.3). One possible option would be to assume that in local coordinates, in the neighbourhood of the point $\bar{x}$, $\partial\omega$ is defined by

$$z_d = (Sz', z') + O(|z|^3).$$

Then in (5.3) one would obtain the error $O(\varepsilon^3\,\varepsilon^{(d-1)/2})$.

The second reason for the lack of estimates is concealed in Lemma 8.2. In contrast with the classical mean value theorem for periodic functions, Lemma 8.2 does not provide any rate of convergence. However, if all the components of the vector Θ are rationally dependent, then P_Θ remains periodic (maybe with another period), and one can apply the mean value theorem for smooth periodic functions that gives an error $O(\varepsilon)$, and, consequently, $O(\varepsilon^3\,\varepsilon^{(d-1)/2})$ in (5.5).

Finally, estimate (4.31) guaranties that the second integral in (5.1) is ε^γ smaller than the first one, where $0<\gamma\le 1$ depends on Λ,Ω,d.

REMARK 5.5. We stress that if condition **(H3)** is violated and the support of u_0 touches the boundary of Ω, then the two integrals in (5.1) are of the same order, and we cannot neglect the second integral any more. In this case it is necessary to construct not only the leading term of the asymptotics for K_ε, but also a corrector term together with a boundary layer corrector. It is possible in some particular cases, for example, when $\bar{x}$ belongs to a flat part of the boundary of Ω, or when the coefficients of the equation are constant. But it is well known that boundary layers in homogenization are very difficult to build in the case of a non flat boundary. Simple cases (flat boundaries, cylindrical domains) will be considered in our forthcoming paper [**3**].

Another typical situation arises when we do not assume anymore that the initial data u_0 is continuous on Ω but merely that it has compact support and is C^2 inside its support. In particular, in this new situation we may have $u_0(\bar{x})\neq 0$. The next theorem, characterizing the asymptotic behaviour of u^ε in this case, can be proved in exactly the same way as Theorem 5.1.

THEOREM 5.6. *Suppose conditions* $(\mathbf{H1}),(\mathbf{H2}),(\mathbf{H4}),(\mathbf{H5})$ *are satisfied and* $\Theta\neq 0$. *Assume that* u_0 *has compact support* $\omega\Subset\Omega$, $u_0\in C^2(\overline{\omega})$ *and* $u_0(\bar{x})\neq 0$. *If* u^ε *is a solution of problem* (1.1), *then, for* $t\ge t_0>0$

$$u^\varepsilon(t,x)=\varepsilon\,\varepsilon^{\frac{d-1}{2}}\Big(1+r_\varepsilon(t,x)\Big)e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\,e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\,M_\varepsilon\,p_\Theta\big(\frac{x}{\varepsilon}\big)\,u(t,x),$$

where $r_\varepsilon(t,x)\to 0$, *as* $\varepsilon\to 0$, *uniformly with respect to* $(t,x)\in[t_0,T]\times\overline{\Omega}$. *Here,* $u(t,x)$ *solves the effective problem*

$$\begin{cases}\partial_t u=\operatorname{div}(a^{\text{eff}}\,\nabla u), & (t,x)\in(0,T)\times\Omega,\\ u(t,x)=0, & (t,x)\in(0,T)\times\partial\Omega,\\ u(0,x)=u_0(\bar{x})\,\delta(x-\bar{x}), & x\in\Omega.\end{cases}$$

The constant M_ε *is now given by*

$$M_\varepsilon=\int\limits_{\mathbb{R}^{d-1}}d\zeta'\int\limits_{(S\zeta',\zeta')}^{+\infty}e^{-|\Theta|\zeta_d}\,\mathcal{M}\{P_\Theta^{-1}\big(\cdot,\zeta_d+\bar{z}_d^\varepsilon\big)\}\,d\zeta_d,$$

with the same definitions of the mean-value $\mathcal{M}$, *of the almost periodic function* P_Θ *and of* $\bar{z}_d^\varepsilon$ *as in Theorem 5.1.*

REMARK 5.7. Yet another possible situation is that $u_0=\partial u_0/\partial\Theta=0$ in the neighborhood of $\bar{x}$. If we assume that $u_0\in C^3(\omega)$ and replace condition $(\mathbf{H6})$ by

$$\frac{\partial^2 u_0}{\partial\Theta^2}(\bar{x})=\frac{\partial}{\partial\Theta}\Big(\frac{\partial u_0}{\partial\Theta}\Big)(\bar{x})\neq 0,$$

where $\partial u_0/\partial\Theta$ is the directional derivative of u_0 in the direction of Θ, then we can prove in this case that, for $t \ge t_0 > 0$,

$$u^\varepsilon(t,x) = \varepsilon^3\,\varepsilon^{\frac{d-1}{2}}\Big(1 + r_\varepsilon(t,x)\Big)e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\,e^{-\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}}\,M_\varepsilon\,p_\Theta\big(\frac{x}{\varepsilon}\big)\,u(t,x),$$

where $r_\varepsilon(t,x) \to 0$, as $\varepsilon \to 0$, uniformly with respect to $(t,x) \in [t_0,T] \times \overline{\Omega}$ and $u(t,x)$ is a solution of

$$\begin{cases} \partial_t u = \mathrm{div}(a^{\mathrm{eff}}\,\nabla u), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = \dfrac{1}{2}\,\dfrac{\partial^2 u_0}{\partial\Theta^2}(\bar{x})\,\delta(x-\bar{x}), & x \in \Omega. \end{cases}$$

The constant M_ε is now given by

$$M_\varepsilon = \int\limits_{\mathbb{R}^{d-1}} d\zeta' \int\limits_{(S\zeta',\zeta')}^{+\infty} \zeta_d^2\,e^{-|\Theta|\zeta_d}\,\mathcal{M}\{P_\Theta^{-1}\big(\cdot,\zeta_d + \bar{z}_d^\varepsilon\big)\}\,d\zeta_d.$$

The case when u_0 vanishes on the boundary of ω together with its derivatives up to order k, can be treated similarly.

It should be noticed that a statement similar to that of Corollary 5.3 remains valid for Theorem 5.6 and Remark 5.7.

6. The case of a flat boundary of ω

In the previous sections we analyzed the case when the quadratic form of the surface $\partial\omega$ is non-degenerate at the point $\bar{x}$. The asymptotics of the solution of problem (1.1) can also be constructed when $\bar{x}$ belongs to a flat part Σ of $\partial\omega$ and the vector Θ is orthogonal to Σ.

More precisely, we replace the previous assumptions **(H4), (H5), (H6)** with the following ones.

- **(H4')** The set of points $\bar{x}$ which provide the minimum in $\min_{x\in\omega}\Theta\cdot x$ is a subset Σ of $\partial\omega$ which is included in an hyperplane of $\mathbb{R}^d$ and Σ has a positive $(d-1)$-measure.
- **(H5')** $u_0(y) = 0$ for all $y \in \Sigma$. There exists $\bar{x} \in \Sigma$ such that $\dfrac{\partial u_0}{\partial\Theta}(\bar{x}) \neq 0$.

REMARK 6.1. Assumption **(H4')** implies that

$$\Theta\cdot(x-\bar{x}) > 0 \text{ for all } x \in \omega\setminus\Sigma,\ \ \bar{x} \in \Sigma,$$

and Θ is orthogonal to Σ and directed inside ω (see Figure 3). Furthermore, $\bar{x}_\Theta = \bar{x}\cdot\frac{\Theta}{|\Theta|}$ is the same for all $\bar{x} \in \Sigma$.

In this case we prove the following result.

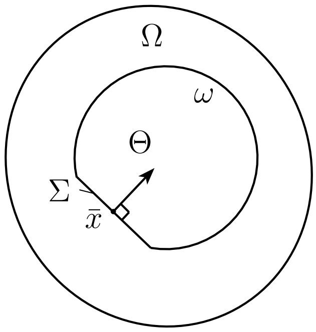


FIGURE 3. The case of a flat part of the boundary $\partial\omega$

THEOREM 6.2. *Assume that conditions **(H1)-(H3)** and **(H4')-(H5')** are fulfilled, and* $\Theta \neq 0$*. Then, for* $t \geq t_0 > 0$*, the asymptotic behaviour of the solution* u^ε *of problem* (1.1) *is described by*

$$u^\varepsilon(t,x) = \varepsilon^2 \, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}} \, e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}} \, (1 + r_\varepsilon(t,x)) \, M_\varepsilon \, p_\Theta\big(\frac{x}{\varepsilon}\big) \, u(t,x),$$

where $r_\varepsilon(t,x) \to 0$*, as* $\varepsilon \to 0$*, uniformly with respect to* $(t,x) \in [t_0,T] \times \overline{\Omega}$*,* $(\lambda_1(\Theta), p_\Theta)$ *is the first eigenpair defined by Lemma 2.1,* $\bar{x}$ *is an arbitrary point on* Σ *and* $u(t,x)$ *solves the homogenized problem*

$$(6.1)\qquad \begin{cases} \partial_t u = \operatorname{div}(a^{\text{eff}} \, \nabla u), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = \dfrac{\partial u_0}{\partial \Theta}(x) \, \delta_\Sigma, & x \in \Omega. \end{cases}$$

Here a^{eff} *is still defined by* (4.7)*,* δ_Σ *is the Dirac delta-function on* Σ *and the constant* M_ε *is given by*

$$M_\varepsilon = \int_0^{+\infty} \zeta_d \, e^{-|\Theta|\,\zeta_d} \, \mathcal{M}\{P_\Theta^{-1}\big(\cdot, \zeta_d + \frac{\bar{x}_\Theta}{\varepsilon}\big)\} d\zeta_d$$

with $\mathcal{M}\{P_\Theta^{-1}(\cdot,\tau_d)\}$ *being the mean value of the almost periodic function* $P_\Theta^{-1}(\cdot,\tau_d)$ *(see* (5.4)*),* $P_\Theta(z)$ *being the rotation of* p_Θ *in the local coordinates of* Σ*:* $P_\Theta(\zeta) = p_\Theta(\mathcal{R}^{-1}\zeta)$*, where* $\mathcal{R}$ *is the rotation matrix.*

PROOF. The proof starts, like that of Theorem 5.1, by using the representation formula (3.5) for the solution v^ε of (3.2) in terms of the Green function K_ε. Writing $K_\varepsilon = K_0 + (K_\varepsilon - K_0)$ we arrive at (5.1), namely

$$v^\varepsilon(t,x) = I_1^\varepsilon + I_2^\varepsilon.$$

By Lemma 4.1, we can estimate I_2^ε, passing to local coordinates, as in the proof of Theorem 5.1,

$$|I_2^\varepsilon| \le C\,\varepsilon^\gamma \int\limits_\omega |u_0(\xi)|\, e^{-\frac{\Theta\cdot(\xi-\bar{x})}{\varepsilon}}\, d\xi$$

$$\le C\,\varepsilon^\gamma \int\limits_\Sigma \Big|\frac{\partial u_0}{\partial\Theta}(z',\bar{x}_\Theta)\Big| dz' \int\limits_0^{+\infty} z_d\, e^{-\frac{|\Theta| z_d}{\varepsilon}}\, dz_d$$

for some $\gamma = \gamma(\Lambda,\Omega,d) > 0$ defined in (4.31). Making the change of variables $\zeta_d = z_d/\varepsilon$, we see that

$$|I_2^\varepsilon| \le C\,\varepsilon^{2+\gamma} \int\limits_\Sigma \Big|\frac{\partial u_0}{\partial\Theta}(z',\bar{x_\Theta})\Big| dz' \int\limits_0^{+\infty} \zeta_d\, e^{-|\Theta|\zeta_d}\, d\zeta_d \le C\,\varepsilon^{2+\gamma}.$$

In order to compute approximately I_1^ε, we again pass to the local coordinates. Namely, we rotate coordinates $z = \Re\,\xi$ in such a way that Θ is directed along z_d. It is obvious that only the neighborhood of Σ contributes in I_1^ε. Expanding K_0 and u_0 into a Taylor series with respect to z_d and making the change of variables $\zeta_d = z_d/\varepsilon$ leads to

$$I_1^\varepsilon = \varepsilon^2 \int\limits_0^{+\infty} \zeta_d\, e^{-|\Theta|\,\zeta_d}\, d\zeta_d \int\limits_\Sigma K_0(t,x,z',\bar{x}_\Theta)\, \frac{\partial u_0}{\partial\Theta}(z',\bar{x}_\Theta)\, P_\Theta^{-1}\big(\frac{z'}{\varepsilon}, \zeta_d + \frac{\bar{x}_\Theta}{\varepsilon}\big)\, dz' + o(\varepsilon^2).$$

where $P_\Theta(\zeta) \equiv p_\Theta(\mathcal{R}^{-1}\zeta)$ with $\mathcal{R}$ being the rotation matrix.

Since $P_\Theta^{-1}(\zeta',\tau_d)$ is uniformly continuous, and, moreover, almost periodic with respect to ζ', by Lemma 8.1, we have

$$I_1^\varepsilon = \varepsilon^2\, M_\varepsilon \int\limits_\Sigma K_0(t,x,z',\bar{x}_\Theta)\, \frac{\partial u_0}{\partial\Theta}(z',\bar{x}_\Theta)\ dz' + o(\varepsilon^2),$$

where

$$M_\varepsilon = \int\limits_0^{+\infty} \zeta_d\, e^{-|\Theta|\,\zeta_d}\, \mathcal{M}\{P_\Theta^{-1}\big(\cdot,\zeta_d + \frac{\bar{x}_\Theta}{\varepsilon}\big)\} d\zeta_d.$$

Here $\mathcal{M}\{P_\Theta^{-1}(\cdot,\tau_d)\}$ is the mean value of the almost periodic function $P_\Theta^{-1}(\cdot,\tau_d)$ (see (5.4)).

Consequently, as $\varepsilon \to 0$,

$$v^\varepsilon(t,x) = \varepsilon^2\, M_\varepsilon \int\limits_\Sigma K_0(t,x,z',\bar{x}_\Theta)\, \frac{\partial u_0}{\partial\Theta}(z',\bar{x}_\Theta)\ dz' + o(\varepsilon^2).$$

Recalling that K_0 is the Green function of the effective problem (4.1) completes the proof. $\square$

COROLLARY 6.3. *Let conditions of Theorem 6.2 be fulfilled. Assume that the vector Θ is such that Θ_d and Θ_k, for any $k = 1, \cdots, (d-1)$, are rationally independent. Then, for $t \geq t_0 > 0$,*

$$u^\varepsilon(t,x) = \Big(\frac{\varepsilon}{|\Theta|}\Big)^2 e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}} e^{\frac{\Theta\cdot(x-\bar{x})}{\varepsilon}} (1 + r_\varepsilon(t,x))\, p_\Theta\big(\frac{x}{\varepsilon}\big) \Big(\int_Y p_\Theta^{-1}\, dy\Big)\, u(t,x),$$

where $r_\varepsilon(t,x) \to 0$, as $\varepsilon \to 0$, uniformly with respect to $(t,x) \in [t_0,T] \times \overline{\Omega}$ and $u(t,x)$ solves the homogenized problem (6.1).

Corollary 6.3 is proved in the same way as Corollary 5.3.

7. Numerical examples

In this section we illustrate the results obtained in the previous sections by direct computations performed with the free software FreeFEM++ ([**15**]).

When studying convection-diffusion equation, the so-called effective convection (effective drift) defined by (2.4) plays an important role. As was already noticed, condition $\bar{b}_i \neq 0$ yields $\Theta_i \neq 0$. The question arises, if $\bar{b}$ coincide with Θ or not. The answer is negative, and the corresponding example is given below.

Example 1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Consider the following boundary value problem with constant coefficients:

$$\text{(7.1)} \qquad \begin{cases} \partial_t u^\varepsilon - \dfrac{\partial^2 u^\varepsilon}{\partial x_1^2} - 2\,\dfrac{\partial^2 u^\varepsilon}{\partial x_1\,\partial x_2} - 2\,\dfrac{\partial^2 u^\varepsilon}{\partial x_2^2} + \dfrac{1}{\varepsilon}\, b\, \dfrac{\partial u^\varepsilon}{\partial x_2} = 0, & \text{in } (0,T)\times\Omega, \\ u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times\partial\Omega, \\ u^\varepsilon(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

Here $b > 0$ is a real parameter and it is obvious that the effective drift is $\bar{b} = \{0, b\}$. To find Θ, one should consider the spectral problem (2.1) on the periodicity cell. Since the coefficients of the equation are constant, $\lambda_1(\theta)$ can be found easily:

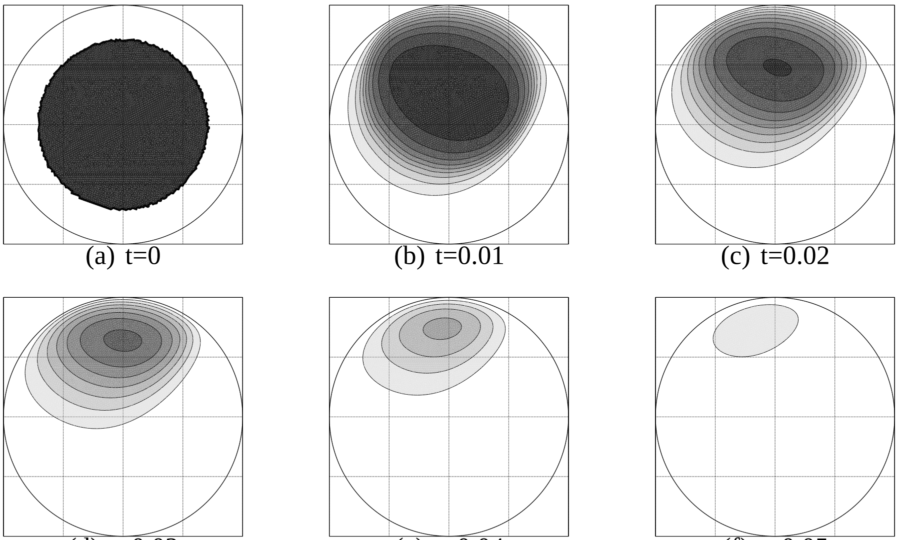
$$\lambda_1(\theta) = -\theta_1^2 - 2\,\theta_1\,\theta_2 - 2\,\theta_2^2 + b\,\theta_2.$$

The maximum of λ_1 is attained at $\Theta = \{-b/2, b/2\} \neq \bar{b}$.

For the numerical computations, we choose Ω to be the unit circle $\Omega = \{x : |x_1 - 1|^2 + |x_2 - 1|^2 \leq 1\}$, u_0 being the characteristic function of the smaller circle $\{x : |x_1 - 1|^2 + |x_2 - 1|^2 \leq 0.5\}$ (see Figure 4(a)), $b = 1$ and $\varepsilon = 0.03$. Theorem 2.3 predicts that the "hot spot" or concentration point of the solution u_ε will be at the point $x_c = (1 - \sqrt{2}/2, 1 + \sqrt{2}/2)$ where Θ is orthogonal to $\partial\Omega$.

The presence of the large parameter in front of the convection in (1.1) suggests to use Characteristics-Galerkin Method (see [**14**], [**22**]). As a finite element space, a space of piecewise linear continuous functions has been chosen. The number of triangles is 21192. The result of the direct computations at different times are presented on Figure 4.

Splitting each triangle of the mesh in 9, we have compared two solutions, u_1 defined on the original mesh and u_2 on the refined one, and computed the relative L^2-error for



(a) t=0 (b) t=0.01 (c) t=0.02

(d) t=0.03 (e) t=0.04 (f) t=0.05

FIGURE 4. Isolines of u^ε for small values of t

small t

$$\sup_t \frac{\|u_1 - u_2\|_{L^2(\Omega)}}{\|u_1\|_{L^2(\Omega)}} \approx 0.002.$$

It is small enough so we can conclude that convergence under mesh refinement is attained. It can be seen from Figure 4 that the solution profile, vanishing with time, moves first in the vertical direction (along the effective drift) and then to the left. Because of the very fast decay, it is not possible to plot the solution itself at large time. Thus, instead of u^ε we consider $\tilde{u}^\varepsilon = u^\varepsilon / \max_\Omega u^\varepsilon$. On Figure 5 the isolines of $\tilde{u}^\varepsilon$ are presented. One can see that indeed the concentration occurs at the point $(1-\sqrt{2}/2, 1+\sqrt{2}/2)$, not the point $(1,2)$ where $\bar{b}$ is normal to $\partial\Omega$.

We perform another numerical test in a nonconvex domain for the same values of the parameters in (7.1). The isolines of the rescaled solution $\tilde{u}^\varepsilon$ are ploted on Figure 6. It is interesting to see how the initial profile first moves in the direction of the effective drift, then vanishes and reappear afterwards to concentrate at the "hot spot" where $\Theta \cdot x$ attains its maximum, as predicted by Theorem 2.3. Such an example is clearly non-intuitive (at least to the authors).

8. Some results from the theory of almost periodic functions.

Denote by $\mathrm{Trig}(\mathbb{R}^d)$ the set of all trigonometric polynomials

$$\mathrm{Trig}(\mathbb{R}^d) = \big\{\mathcal{P}(x) \big|\ \mathcal{P}(x) = \sum_{\xi\in\mathbb{R}^d} c_\xi\, e^{ix\cdot\xi}\big\},$$

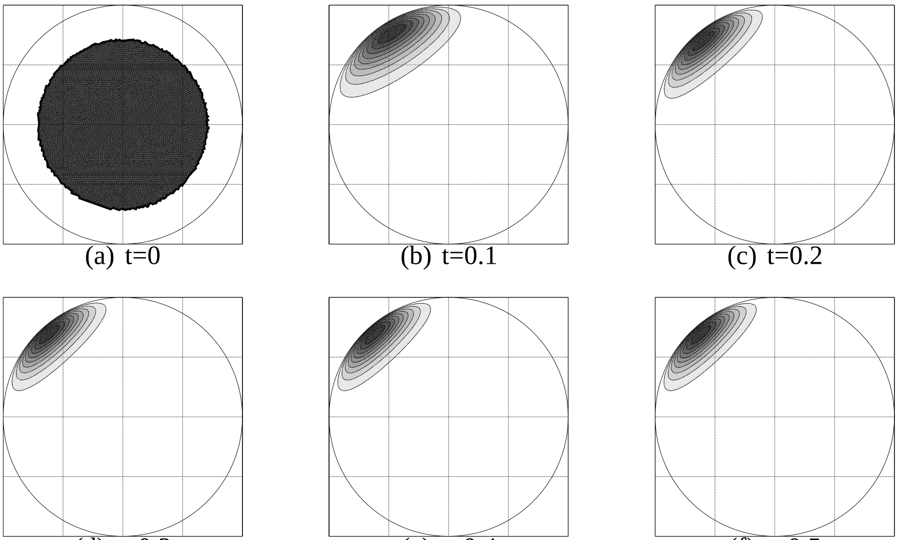


FIGURE 5. Isolines of rescaled u^ε for different values of t

where in the sum only finite number of $c_\xi \neq 0$. We designate by $\mathrm{CAP}(\mathbb{R}^d)$ (set of almost periodic functions) a closure of $\mathrm{Trig}(\mathbb{R}^d)$ with respect to the norm $\sup\limits_{\mathbb{R}^d} |\mathcal{P}(x)|$. For any almost periodic function $g \in \mathrm{CAP}(\mathbb{R}^d)$, there exists a mean value

$$\mathcal{M}\{g\} = \lim_{t\to\infty} \frac{1}{|t\,\mathcal{B}|} \int\limits_{t\,\mathcal{B}} g(x)\,dx, \tag{8.1}$$

where $\mathcal{B} \subset \mathbb{R}^d$ is a Borel set, $|\mathcal{B}|$ - its volume. The mean-value theorem takes place for almost periodic functions ([**18**]).

LEMMA 8.1. *Given* $g \in \mathrm{CAP}(\mathbb{R}^d)$ *and* $v \in L^2(Q)$, $Q \subset \mathbb{R}^d$, *the following equality holds true:*

$$\lim_{\varepsilon\to 0} \int\limits_Q g\big(\frac{x}{\varepsilon}\big)\, v(x)\,dx = \mathcal{M}\{g\} \int\limits_Q v(x)\,dx,$$

where $\mathcal{M}\{g\}$ *is given by formula* (8.1).

Lemma 8.1 can be formulated also in more general form.

LEMMA 8.2. *Given a function* $g(x,y) \in C[\overline{Q}; \mathrm{CAP}(\mathbb{R}^d)]$, $Q \subset \mathbb{R}^d$, *the following equality holds:*

$$\lim_{\varepsilon\to 0} \int\limits_Q g\big(x, \frac{x}{\varepsilon}\big)\,dx = \int\limits_Q \mathcal{M}\{g(x,\cdot)\}\,dx,$$

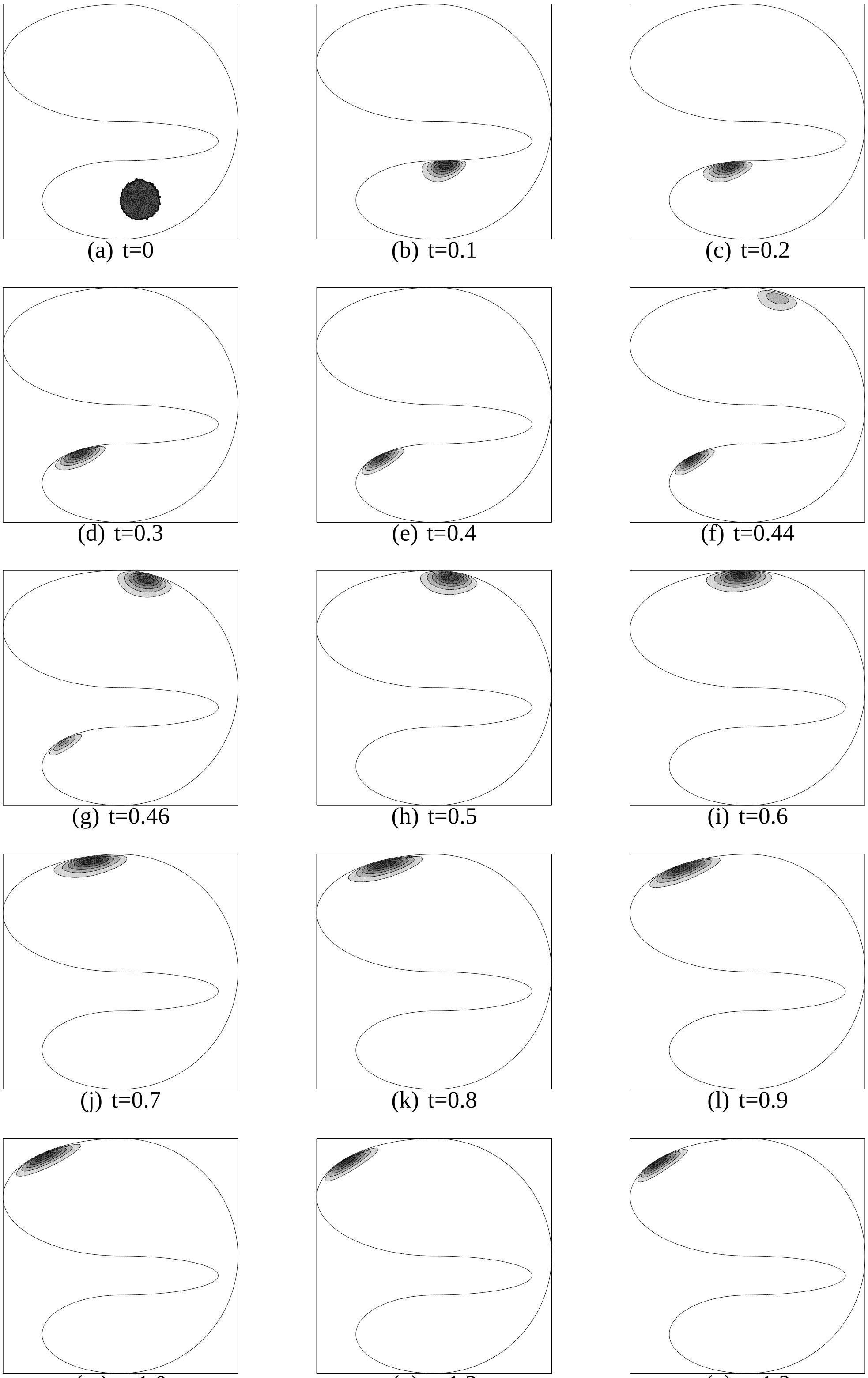

(a) t=0 (b) t=0.1 (c) t=0.2 (d) t=0.3 (e) t=0.4 (f) t=0.44 (g) t=0.46 (h) t=0.5 (i) t=0.6 (j) t=0.7 (k) t=0.8 (l) t=0.9 (m) t=1.0 (n) t=1.2 (o) t=1.3

FIGURE 6. Isolines of rescaled u^{ε} for different values of t in a non-convex domain

where

$$\mathcal{M}\{g(x,\cdot)\} = \lim_{t\to\infty} \frac{1}{|t\,\mathcal{B}|} \int\limits_{t\,\mathcal{B}} g(x,y)\,dy.$$

The last statement can be proved combining the approximation of $g(x,y)$ by finite sums of the type $\sum f_1(x)\, f_2(y)$ and the result of Lemma 8.1.

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Paper E

PAPER E

Homogenization of a nonstationary convection-diffusion equation in a thin rod and in a layer

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ABSTRACT. The paper deals with a homogenization problem for a non-stationary convection-diffusion equation defined in a thin rod or in a layer. Under the assumption that the convection term is large, we describe the evolution of the solution's profile and determine the rate of its decay. Due to the specific geometry of the domains under consideration, it is possible to construct the boundary layer correctors.

Keywords: Homogenization, convection-diffusion, localization.

1. Introduction

The paper deals with the homogenization of a nonstationary convection-diffusion equation with large convection stated either in a thin rod or in a layer. In the previous work [**4**] the authors addressed a similar homogenization problem for an equation defined in a general bounded domain Ω. Namely, the following initial-boundary value problem has been considered:

$$
(1.1)\qquad \begin{cases} \partial_t u^\varepsilon - \operatorname{div}\big(a\big(\dfrac{x}{\varepsilon}\big)\nabla u^\varepsilon\big) + \dfrac{1}{\varepsilon}\, b\big(\dfrac{x}{\varepsilon}\big)\cdot\nabla u^\varepsilon = 0, & \text{in } (0,T)\times\Omega,\\ u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times\partial\Omega,\\ u^\varepsilon(0,x) = u_0(x), & x\in\Omega, \end{cases}
$$

with periodic coefficients a_{ij}, b_j and a small parameter ε. Notice that in the case of solenoidal vector-field $b(y)$ with zero mean-value the problem can be studied by the classical homogenization methods (see, for example, [**5**], [**15**]). In particular, the sequence of solutions is bounded in $L^\infty[0,T;L^2(\Omega)]\cap L^2[0,T;H^1(\Omega)]$ and converges, as $\varepsilon\to 0$, to a solution of an effective or homogenized problem in which there is no convective term. A similar behaviour of u^ε is observed if the so-called effective drift is equal to zero. The behaviour of the solution changes essentially if the effective drift is nontrivial. Problem (1.1) with nonzero effective drift has been considered in [**4**] under the crucial assumption that the initial function u_0 has a compact support in Ω. In this case the initial profile moves towards the boundary during the time of order ε, and then, upon reaching the boundary, starts dissipating. As a result, the solution is asymptotically small for time $t \gg \varepsilon$. Paper [**4**] focuses on the asymptotics of u^ε for such times.

Without the assumption that u_0 has a compact support in Ω, one faces the necessity to construct boundary layer correctors in the neighbourhood of $\partial\Omega$. It is well known that this problem cannot be solved in the case of general bounded domain. However, it is getting feasible if the periodic structure agrees with the geometry of the boundary of Ω. In the present paper we consider two types of domains which possess this property. Namely, we study a convection-diffusion models in a thin rod and in a layer in $\mathbb{R}^d$.

In the case of a thin rod (Section 2) we impose homogeneous Neumann boundary conditions on the lateral boundary of the rod and homogeneous Dirichlet boundary conditions on its bases. As was noticed above, the solution vanishes for time $t \gg \varepsilon$. We determine the rate of vanishing of the solution and describe the evolution of its profile (see Theorem 2.2). If the effective axial drift is not zero (we study only this case), the rescaled solution concentrates in the vicinity of one of the rod ends, and the choice of the end depends on the sign of the effective convection. In order to characterize the rate of decay we introduce a 1-parameter family of auxiliary cell spectral problems (see [**5**], [**6**]). The asymptotic behaviour of the solution is then governed by the first eigenpair of the said family of spectral problems.

Among the technical tools used in the paper, are factorization principle (see [**10**], [**14**], [**15**], [**6**]), dimension reduction arguments and qualitative results required for constructing boundary layer correctors.

In the case of a layer addressed in Section 3, in addition to the factorization principle, we also have to introduce moving coordinates. More precisely, we use a parameterized cell spectral problem and factorization principle to suppress the component of the effective drift which is perpendicular to the layer boundary. While, due to the presence of the longitudinal components of the effective convection we have to introduce the moving coordinates (see [**8**], [**3**]). The main result in this case is given by Theorem 3.5.

2. Convection-diffusion equation in a thin rod

The section concerns the homogenization of a nonstationary convection-diffusion equation stated in a thin rod $G_\varepsilon = (-1,1) \times \varepsilon Q$. Here $Q \subset \mathbb{R}^{d-1}$ is a bounded domain with the Lipschitz boundary ∂Q, $\varepsilon > 0$ is a small parameter. Throughout the paper the points in $\mathbb{R}^d$ are denoted $x = (x_1, x')$. The lateral boundary of the rod G_ε is denoted $\Sigma_\varepsilon = (-1,1) \times \partial Q$.

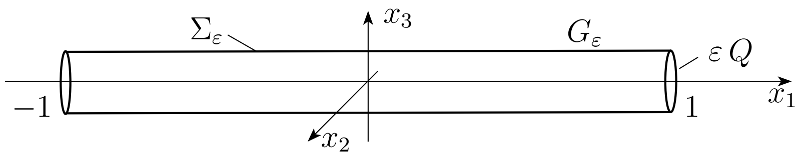

FIGURE 1. The rod G_ε

For $T > 0$, we consider the following model:

$$
\left\{
\begin{array}{ll}
\partial_t u^\varepsilon(t,x) + A_\varepsilon\, u^\varepsilon(t,x) = 0, & \text{in } (0,T)\times G_\varepsilon,\\
B_\varepsilon u^\varepsilon(t,x) = 0, & \text{on } (0,T)\times \Sigma_\varepsilon,\\
u^\varepsilon(t,\pm 1, x') = 0, & \text{on } (0,T)\times Q,\\
u^\varepsilon(0,x) = u_0(x_1), & x \in G_\varepsilon
\end{array}
\right. \tag{2.1}
$$

with

$$
A_\varepsilon u^\varepsilon = -\mathrm{div}\big(a^\varepsilon \nabla u^\varepsilon\big) + \frac{1}{\varepsilon} b^\varepsilon \cdot \nabla u^\varepsilon; \quad B_\varepsilon u^\varepsilon = a^\varepsilon \nabla u^\varepsilon \cdot n.
$$

The coefficients of the equation are given by

$$
a^\varepsilon_{ij} = a_{ij}\big(\frac{x}{\varepsilon}\big), \quad b^\varepsilon_j = b_j\big(\frac{x}{\varepsilon}\big).
$$

We assume that:

(H1) The coefficients of A_ε are measurable bounded functions, that is $a_{ij}, b_j \in L^\infty(\mathbb{R}\times Q)$. Moreover, $a_{ij}(y_1, y'), b_j(y_1, y')$ are periodic with respect to y_1.

(H2) The $d \times d$ matrix $a(y)$ is symmetric and satisfies the uniform ellipticity condition, that is there exists $\Lambda > 0$ such that

$$
a_{ij}(y)\xi_i\xi_j \geq \Lambda |\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d.
$$

(H3) The initial function $u_0(x_1) \in C[-1,1]$.

For simplicity, in what follows we assume that $\varepsilon = 1/N$, $N \in \mathbb{Z}_+$.

Under the stated assumptions we study the asymptotic behaviour of solutions $u^\varepsilon(t,x)$ of problem (2.1), as $\varepsilon \to 0$.

2.1. Auxiliary spectral problems and main result. In what follows we denote

$$
Au = -\mathrm{div}(a\nabla u) + b\cdot\nabla u, \quad Bu = a\nabla u \cdot n;
$$

$$
A^* u = -\mathrm{div}(a\nabla u) - \mathrm{div}(b\, u), \quad B^* u = a\nabla u\cdot n + (b\cdot n)\, u.
$$

Following [**5**], for $\theta \in \mathbb{R}$, we introduce two parameterized families of spectral problems (direct and adjoint).

$$\left\{\begin{array}{l} e^{-\theta y_1} A\, e^{\theta y_1}\, p_\theta(y) = \lambda(\theta)\, p_\theta(y), \quad Y = \mathfrak{S}_1 \times Q, \\ e^{-\theta y_1} B\, e^{\theta y_1}\, p_\theta(y) = 0, \quad \partial Y = \mathfrak{S}_1 \times \partial Q, \\ y_1 \to p_\theta(y) \quad \text{1-periodic.} \end{array}\right. \tag{2.2}$$

$$\left\{\begin{array}{l} e^{\theta y_1} A^*\, e^{-\theta y_1}\, p^*_\theta(y) = \lambda(\theta)\, p^*_\theta(y), \quad Y, \\ e^{\theta y_1} B^*\, e^{-\theta y_1}\, p^*_\theta(y) = 0, \quad \partial Y, \\ y_1 \to p^*_\theta(y) \quad \text{1-periodic.} \end{array}\right.$$

Here $\mathfrak{S}_1$ is the 1-dimensional unit circle. Note that the exponential transform is applied with respect to only one variable x_1. Next result, based on the Krein-Rutman theorem, has been proved in [**6**].

LEMMA 2.1. *For each $\theta \in \mathbb{R}$, the first eigenvalue $\lambda_1(\theta)$ of problem* (2.2) *is real, simple, and the corresponding eigenfunctions p_θ and p^*_θ can be chosen positive. Moreover, $\theta \to \lambda_1(\theta)$ is twice differentiable, strictly concave and admits a maximum which is obtained for a unique $\theta = \Theta$.*

The eigenfunctions p_θ and p^*_θ defined by Lemma 2.1, can be normalized by

$$\int\limits_Y p_\theta(y)\, p^*_\theta(y)\, dy = 1.$$

Differentiating equation (2.2) with respect to θ, integrating against p^*_θ and writing down the compatibility condition for the resulting equation, yield

$$\frac{d\lambda_1}{d\theta} = \int\limits_Y \big(b_1\, p_\theta\, p^*_\theta + a_{ij}(p_\theta\, \partial_{y_j} p^*_\theta - p^*_\theta\, \partial_{y_j} p_\theta) - 2\,\theta\, p_\theta\, p^*_\theta\, a_{11}\big)\, dy. \tag{2.3}$$

Noticing that $\lambda_1(0) = 0$ and $p_\theta(y)|_{\theta=0} = 1$, one obtains

$$\frac{d\lambda_1}{d\theta}\Big|_{\theta=0} = \int\limits_Y \big(a_{ij}\, \partial_{y_j} p^* + b_1\, p^*\big)\, dy \equiv \bar{b}_1, \tag{2.4}$$

where $p^*(y) = p^*_\theta(y)|_{\theta=0}$. The last expression is the so-called effective axial drift.

In what follows we assume that $\bar{b}_1 > 0$ that yields $\Theta > 0$. The case $\bar{b}_1 < 0$ can be considered in the same way.

To avoid the technicalities, we formulate, first, the main result of the section in a loose way.

THEOREM 2.2. *Let conditions* $(\mathbf{H1}) - (\mathbf{H3})$ *be fulfilled and $\bar{b}_1 > 0$ (see* (2.4)*). Suppose $u_0 \in C^1[-1,1]$ is such that $u_0(-1) \neq 0$. Then there exist constants a^{eff} and M such* *that, for $t > 0$ and $x \in G_\varepsilon$, the asymptotics of the solution u^ε of problem* (2.1) *takes the form*

$$u^\varepsilon(t,x) = \varepsilon^{d+1}\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_1+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, \big[u(t,x_1) + r_\varepsilon(t,x)\big],$$

where u is a solution of the one-dimensional effective problem

$$\begin{cases} \partial_t u = a^{\rm eff}\, \partial^2_{x_1} u, & (t,x_1) \in (0,T)\times(-1,1),\\ u(t,\pm 1) = 0, & t \in (0,T),\\ u(0,x_1) = M\, u_0(-1)\, \delta'(x_1+1), & x_1 \in (-1,1). \end{cases}$$

Here $r_\varepsilon(t,x)$ is such that $|r_\varepsilon(t,\cdot)| \le C\sqrt{\varepsilon}$ for $t \ge t_0 > 0$, $x \in G_\varepsilon$, and the constant C depends on $\min\{(x_1+1),(1-x_1)\}$.

The proof of Theorem 2.2 is perform is several steps. First, we make use of the factorization principle in order to simplify the original problem. Then, we represent new unknown function in terms of the corresponding Green's function. And, finally, we study the asymptotic behaviour of the mentioned Green's function, as $\varepsilon \to 0$.

2.2. Proof of Theorem 2.2.

2.2.1. *Factorization.* In order to simplify the original problem we perform the change of unknowns, as was suggested in [**7**], [**3**].

$$u^\varepsilon(t,x) = e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_1+1)}{\varepsilon}}\, p_\Theta\big(\frac{x}{\varepsilon}\big)\, v^\varepsilon(t,x). \tag{2.5}$$

Substituting (2.5) into (2.1) yields the problem for the new unknown function v^ε

$$\begin{cases} \rho^\varepsilon_\Theta(x)\, \partial_t v^\varepsilon - {\rm div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v^\varepsilon\big) + \frac{1}{\varepsilon}\, b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v^\varepsilon = 0, & (0,T)\times G_\varepsilon,\\ a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v^\varepsilon\cdot n = 0, & (0,T)\times\Sigma_\varepsilon,\\ v^\varepsilon(t,\pm1,x') = 0, & (0,T)\times Q,\\ v^\varepsilon(0,x) = u_0(x_1)\, p_\Theta^{-1}\big(\frac{x}{\varepsilon}\big)\, e^{-\frac{\Theta(x_1+1)}{\varepsilon}}, & x \in G_\varepsilon. \end{cases} \tag{2.6}$$

Here

$$\begin{aligned} &\rho^\varepsilon_\Theta(x) = p_\Theta\big(\frac{x}{\varepsilon}\big)\, p^*_\Theta\big(\frac{x}{\varepsilon}\big); \quad a^\Theta(y) = p_\Theta(y)\, p^*_\Theta(y)\, a(y);\\ &b^\Theta(y) = p_\Theta(y)\, p^*_\Theta(y)\, b(y) - 2\,\Theta\, p_\Theta(y)\, p^*_\Theta(y)\, a_{1\cdot}(y)\\ &+a(y)\, \big[p_\Theta(y)\, \nabla_y p^*_\Theta(y) - p^*_\Theta(y)\, \nabla_y p_\Theta(y)\big]. \end{aligned} \tag{2.7}$$

For brevity, in what follows we denote

$$A^\varepsilon_\Theta v = -{\rm div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\big) + \frac{1}{\varepsilon}\, b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v, \quad B^\varepsilon_\Theta v = a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\cdot n;$$

$$A_\Theta v = -{\rm div}(a^\Theta\nabla v) + b^\Theta\cdot\nabla v, \quad B_\Theta v = a^\Theta\nabla v\cdot n.$$

$$A^{*,\varepsilon}_\Theta v = -\mathrm{div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\big) - \frac{1}{\varepsilon}\, b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v;$$

$$A^*_\Theta v = -\mathrm{div}(a^\Theta\nabla v) - b^\Theta\cdot\nabla v.$$

Straightforward calculations yield

$$\forall\theta\in\mathbb{R}:\quad \mathrm{div}_y b^\theta(y) = 0;\quad b^\theta\cdot n = 0\quad\text{on }\Sigma_\varepsilon. \tag{2.8}$$

Taking into account the fact that Θ is the maximum point of λ_1 and equality (2.3), we obtain that the first component of b^Θ has zero mean:

$$\int\limits_Y b_1^\Theta(y)\,dy = 0. \tag{2.9}$$

Due to (2.8), (2.9), problem (2.6) admits homogenization. However, the presence of an asymptotically singular initial condition in (2.6) brings some difficulties into the homogenization procedure. In order to study the asymptotic behaviour of v^ε, we will use its representation in terms of the corresponding Green's function $K_\varepsilon(t,x,\xi)$

$$v^\varepsilon(t,x) = \int\limits_{G_\varepsilon} K_\varepsilon(t,x,\xi)\,u_0(\xi_1)\,p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\,e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\,d\xi. \tag{2.10}$$

Here K_ε, for each $\xi\in G_\varepsilon$, solves the problem

$$\begin{cases} \rho^\varepsilon_\Theta\,\partial_t K_\varepsilon + A^\varepsilon_\Theta K_\varepsilon = 0, & (0,T)\times G_\varepsilon,\\ B^\varepsilon_\Theta K_\varepsilon = 0, & (0,T)\times\Sigma_\varepsilon,\\ K_\varepsilon(t,x,\xi)\Big|_{x_1=\pm1} = 0, & (0,T)\times Q,\\ K_\varepsilon(0,x,\xi) = \delta(x-\xi), & x\in G_\varepsilon. \end{cases} \tag{2.11}$$

Note that K_ε with respect to (t,ξ) is a solution of the formally adjoint problem, which differs from (2.11) in the sign in front of the first-order terms.

Because of the presence of the delta-function in the initial condition, it is difficult to construct the asymptotics for K_ε directly. Let us introduce a function

$$V_\varepsilon(t,x,\xi) = \Phi_\varepsilon(t,x,\xi) - K_\varepsilon(t,x,\xi),$$

where Φ_ε is Green's function in the thin infinite cylinder $\mathbb{G}_\varepsilon = \mathbb{R}\times\varepsilon Q$

$$\begin{cases} \rho^\varepsilon_\Theta(\xi)\,\partial_t\Phi_\varepsilon + A^{*,\varepsilon}_\Theta\Phi_\varepsilon = 0, & (t,\xi)\in(0,T)\times\mathbb{G}_\varepsilon,\\ B^\varepsilon_\Theta\Phi_\varepsilon = 0, & (t,\xi)\in(0,T)\times\Gamma_\varepsilon,\\ \Phi_\varepsilon(0,x,\xi) = \delta(x-\xi), & \xi\in\mathbb{G}_\varepsilon. \end{cases}$$

By Γ_ε we denote the lateral boundary $\mathbb{R} \times \partial(\varepsilon Q)$ of the cylinder $\mathbb{G}_\varepsilon$. Thus, for each $x \in G_\varepsilon$, V_ε solves the problem

$$\begin{cases} \rho_\Theta^\varepsilon(x)\,\partial_t V_\varepsilon + A_\Theta^{*,\varepsilon} V_\varepsilon = 0, & (t,\xi) \in (0,T) \times G_\varepsilon, \\ B_\Theta^\varepsilon V_\varepsilon = 0, & (t,\xi) \in (0,T) \times \Sigma_\varepsilon, \\ V_\varepsilon(t,x,\xi)\Big|_{\xi_1=\pm 1} = \Phi_\varepsilon(t,x,\xi)\Big|_{\xi_1=\pm 1}, & (t,\xi) \in (0,T) \times Q, \\ V_\varepsilon(0,x,\xi) = 0, & \xi \in G_\varepsilon. \end{cases} \tag{2.12}$$

2.2.2. *Asymptotics for* $\Phi_\varepsilon(t,x,\xi)$. Denote by Φ_0 a fundamental solution of the 1-dimensional homogenized problem

$$\begin{cases} \partial_t \Phi_0 = a^{\text{eff}}\,\partial_\xi^2 \Phi_0(t,x_1,\xi_1), \quad \xi \in \mathbb{R}, \\ \Phi_0(0,x_1,\xi_1) = \delta(x_1 - \xi_1). \end{cases}$$

Here the effective coefficient a^{eff} is given by

$$\begin{aligned} a^{\text{eff}} &= \int_Y (a_{11}^\Theta + a_{1j}^\Theta \partial_{y_j} N - b_1^\Theta\, N)\, dy \\ &= \int_Y (a_{11}^\Theta + a_{1j}^\Theta \partial_{y_j} N^* + b_1^\Theta\, N^*)\, dy, \end{aligned} \tag{2.13}$$

1-periodic in y_1 functions N and N^* solve the standard cell problems (direct and adjoint, respectively):

$$\begin{cases} A_\Theta N = \partial_{y_j} a_{j1}^\Theta(y) - b_1^\Theta(y), \quad Y, \\ B_\Theta N = 0, \quad \partial Y; \end{cases} \tag{2.14}$$

$$\begin{cases} A_\Theta^* N^* = \partial_{y_j} a_{j1}^\Theta + b_1^\Theta, \quad Y, \\ B_\Theta N^* = 0, \quad \partial Y. \end{cases} \tag{2.15}$$

It can be shown that $a^{\text{eff}} > 0$ (see, for example, [**12**]). Note that N and N^* are Hölder continuous functions (see [**9**]).

The fundamental solution Φ_0 admits the explicit formula

$$\Phi_0(t,x_1,\xi_1) = \frac{1}{2\sqrt{\pi\, t}}\,\frac{1}{a^{\text{eff}}}\, e^{-\frac{|x_1-\xi_1|^2}{4a^{\text{eff}}\,t}}. \tag{2.16}$$

We also introduce the first- and second-order approximation of Φ_ε by

$$\begin{aligned} &\Phi_1^\varepsilon(t,x,\xi) = \Phi_0(t,x_1,\xi_1) + \varepsilon\, N\Big(\frac{x}{\varepsilon}\Big)\,\partial_{x_1}\Phi_0(t,x_1,\xi_1) \\ &+\varepsilon\, N^*\Big(\frac{\xi}{\varepsilon}\Big)\,\partial_{\xi_1}\Phi_0(t,x_1,\xi_1). \end{aligned} \tag{2.17}$$

$$
\begin{aligned}
\Phi_2^\varepsilon(t,x,\xi) &= \Phi_1^\varepsilon(t,x,\xi) + \varepsilon^2 N_2\big(\frac{x}{\varepsilon}\big)\, \partial^2_{x_1}\Phi_0(t,x_1,\xi_1) \\
&+\varepsilon^2 N_2^*\big(\frac{\xi}{\varepsilon}\big)\, \partial^2_{\xi_1}\Phi_0(t,x_1,\xi_1) + \varepsilon^2 N\big(\frac{x}{\varepsilon}\big)\, N^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{x_1}\partial_{\xi_1}\Phi_0(t,x_1,\xi_1).
\end{aligned} \tag{2.18}
$$

Next result, which concerns the asymptotic behaviour of Φ_ε, is similar to one announced in [**15**] (see Chapter II) and then proved rigorously in [**1**].

LEMMA 2.3. *For any* $x, \xi \in \mathbb{G}_\varepsilon$ *and* $t \geq \varepsilon^2$*,*

$$
|\Phi_\varepsilon(t,x,\xi) - \Phi_k(t,x_1,\xi_1)| \leq C\, \frac{\varepsilon^{k+1}}{t^{(k+1)/2}}, \quad k = 0,1,2. \tag{2.19}
$$

2.2.3. *Asymptotics for* $V_\varepsilon(t,x,\xi)$. The formal asymptotic expansion for V_ε takes the form

$$
\begin{aligned}
W_\varepsilon(t,x,\xi) &= V_0(t,x_1,\xi_1) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} V_0(t,x_1,\xi_1) \\
&+\varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_1} V_0(t,x_1,\xi_1) + \varepsilon\, V_1(t,x_1,\xi_1) + \varepsilon\, V_{bl}^\varepsilon(t,x,\xi) \\
&+\varepsilon^2\, V_2(t,x_1,\xi_1;y,\eta)\Big|_{y=x/\varepsilon,\eta=\xi/\varepsilon} + \varepsilon^3\, W_{bl}^\varepsilon(t,x,\xi),
\end{aligned} \tag{2.20}
$$

where

$$
\begin{aligned}
V_2(t,x_1,\xi_1;y,\eta) &= N_2(y)\, \partial^2_{x_1} V_0(t,x_1,\xi_1) \\
&+N_2^*(\eta)\, \partial^2_{\xi_1} V_0(t,x_1,\xi_1) + N(y)\, N^*(\eta)\, \partial_{x_1}\partial_{\xi_1} V_0(t,x_1,\xi_1) \\
&+N(y)\, \partial_{x_1} V_1(t,x_1,\xi_1) + N^*(\eta)\, \partial_{\xi_1} V_1(t,x_1,\xi_1).
\end{aligned} \tag{2.21}
$$

In (2.20) the terms of order ε will be used as an approximation of V_ε, while V_2 and W_{bl}^ε are constructed to guarantee the required accuracy.

In the expansion (2.20), the function V_0, for each $x_1 \in (-1,1)$, is a solution of the homogenized problem

$$
\begin{cases}
\partial_t V_0 = a^{\text{eff}}\, \partial^2_{\xi_1} V_0, & (t,\xi_1) \in (0,T) \times (-1,1), \\
V_0(t,x_1,\pm 1) = \Phi_0(t,x_1,\pm 1), & t \in (0,T), \\
V_0(0,x_1,\xi_1) = 0, & \xi_1 \in (-1,1).
\end{cases} \tag{2.22}
$$

The effective coefficient a^{eff} is defined by (2.13); N and N^* are solutions of (2.14) and (2.15), respectively. For $x_1 \neq \pm 1$, the function V_0 belongs to $C^\infty([0,T]\times(-1,1)\times[-1,1])$ and for $t \in [0,T],\ x \in I \Subset (-1,1)$ we have

$$
0 \leq \partial_t^k\, \partial_{x_1}^l\, \partial_{\xi_1}^m\, V_0(t,x_1,\xi_1) \leq \frac{C}{\min\{(x_1-1),(x_1+1)\}^{2k+l+m+1}}. \tag{2.23}
$$

In order to define V_1 and the boundary layer corrector V_{bl}^{ε}, we introduce functions $v^{\pm}$ stated in semi-infinite cylinders $G^{+}=(0,+\infty)\times Q$ and $G^{-}=(-\infty,0)\times Q$:

$$\left\{\begin{array}{ll} A_{\Theta}^{*}v^{\pm}=0, & \eta\in G^{\mp},\\ B_{\Theta}v^{\pm}=0, & \eta\in\Sigma^{\mp},\\ v^{\pm}(0,\eta')=-N^{*}(0,\eta'), & \end{array}\right. \tag{2.24}$$

where $\Sigma^{\pm}$ are the lateral boundaries of $G^{\pm}$. It has been proved in [**12**] that bounded solutions $v^{\pm}$ exist, they are uniquely defined and stabilize to some constants $\hat{v}^{\pm}$ at the exponential rate, as $\eta_1\to\pm\infty$:

$$\begin{array}{l} |v^{\pm}(\eta_1,\eta')-\hat{v}^{\pm}|\le C_0\,e^{-\gamma\,|\eta_1|},\quad C_0,\gamma>0;\\ \|\nabla v^{+}\|_{L^2((n,n+1)\times Q)}\le C\,e^{-\gamma\,n},\quad \forall n>0,\\ \|\nabla v^{-}\|_{L^2((-(n+1),-n)\times Q)}\le C\,e^{-\gamma\,n},\quad \forall n>0. \end{array} \tag{2.25}$$

Then, V_1, for $x_1\in(-1,1)$, satisfies the problem

$$\left\{\begin{array}{lr} \partial_t V_1=a^{\text{eff}}\,\partial_{\xi_1}^2 V_1+F(t,x_1,\xi_1), & (t,\xi_1)\in(0,T)\times(-1,1),\\ V_1(t,x_1,\pm1)=\hat{v}^{\pm}\,\partial_{\xi_1}\,(V_0-\Phi_0)\Big|_{\xi_1=\pm1}, & t\in(0,T),\\ V_1(0,x_1,\xi_1)=0, & \xi_1\in(-1,1), \end{array}\right. \tag{2.26}$$

where

$$\begin{array}{l} F(t,x_1,\xi_1)=\partial_{\xi_1}^3 V_0(t,x_1,\xi_1)\displaystyle\int\limits_{Y}\big[a_{1j}^{\Theta}(\eta)\partial_{\eta_j}N_2^{*}(\eta)\\ \quad+a_{11}^{\Theta}(\eta)N^{*}(\eta)+b_1^{\Theta}(\eta)N_2^{*}(\eta)-a^{\text{eff}}\,\rho_{\Theta}(\eta)\,N^{*}(\eta)\big]\,d\eta, \end{array} \tag{2.27}$$

and N_2^{*} solves the problem

$$\left\{\begin{array}{l} A_{\Theta}^{*}N_2^{*}=\partial_{\eta_i}(a_{i1}^{\Theta}N^{*})+a_{1j}^{\Theta}\partial_{\eta_j}N^{*}\\ +a_{11}^{\Theta}+b_1^{\Theta}\,N^{*}-a^{\text{eff}}\,\rho_{\Theta},\quad Y,\\ B_{\Theta}N_2^{*}=-a_{i1}^{\Theta}\,n_i\,N^{*},\quad \partial Y. \end{array}\right.$$

In the definition of V_2, the 1-periodic (w.r.t y_1) function $N_2(y)$ solves the following problem:

$$\left\{\begin{array}{l} A_{\Theta}N_2=\partial_{\eta_i}(a_{i1}^{\Theta}N)+a_{1j}^{\Theta}\partial_{\eta_j}N\\ +a_{11}^{\Theta}-b_1^{\Theta}\,N-a^{\text{eff}}\,\rho_{\Theta},\quad Y,\\ B_{\Theta}N_2=-a_{i1}^{\Theta}\,n_i\,N,\quad \partial Y. \end{array}\right.$$

Notice that V_1 is a smooth function of its variables for $x \in I \Subset (-1,1)$, and N_2^* is Hölder continuous. The first boundary layer corrector is given by

$$
\begin{aligned}
V_{bl}^\varepsilon(t,x,\xi) &= \Big[v^-\big(\frac{\xi_1+1}{\varepsilon},\frac{\xi'}{\varepsilon}\big) - \hat v^-\Big]\,\partial_{\xi_1}(V_0-\Phi_0)\Big|_{\xi_1=-1} \\
&+\Big[v^+\big(\frac{\xi_1+1}{\varepsilon},\frac{\xi'}{\varepsilon}\big) - \hat v^+\Big]\,\partial_{\xi_1}(V_0-\Phi_0)\Big|_{\xi_1=1}.
\end{aligned}
\tag{2.28}
$$

Due to the above constructions,

$$
\begin{aligned}
&\Big\{V_0(t,x_1,\xi_1) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}V_0(t,x_1,\xi_1) + \varepsilon\, N\big(\frac{\xi}{\varepsilon}\big)\,\partial_{\xi_1}V_0(t,x_1,\xi_1) \\
&+\varepsilon\, V_1(t,x_1,\xi_1) + \varepsilon\, V_{bl}^\varepsilon(t,x_1,\xi_1)\Big\}\Big|_{\xi_1=\pm1} = \Phi_1^\varepsilon(t,x,\xi)\Big|_{\xi_1=\pm1},
\end{aligned}
$$

where Φ_1^ε is defined by (2.17).

The second boundary layer corrector W_{bl}^ε is designed to compensate the time derivative of V_{bl}^ε and is defined by

$$
\begin{aligned}
W_{bl}^\varepsilon(t,x,\xi) &= \Big[w^-\big(\frac{\xi_1+1}{\varepsilon},\frac{\xi'}{\varepsilon}\big) - \hat w^-\Big]\,\partial_t\partial_{\xi_1}(V_0-\Phi_0)\Big|_{\xi_1=-1} \\
&+\Big[w^+\big(\frac{\xi_1+1}{\varepsilon},\frac{\xi'}{\varepsilon}\big) - \hat w^+\Big]\,\partial_t\partial_{\xi_1}(V_0-\Phi_0)\Big|_{\xi_1=1}.
\end{aligned}
$$

The functions $w^\pm$ solve nonhomogeneous problems

$$
\begin{cases}
A_\Theta^* w^\pm = (\hat v^\pm - v^\pm(\eta))\,\rho_\Theta(\eta), & \eta \in G^\mp, \\
B_\Theta w^\pm = 0, & \eta \in \Sigma^\mp, \\
w^\pm(0,\eta') = 0.
\end{cases}
$$

Bounded solutions $w^\pm$ exist, they are uniquely defined and stabilize to some constants $\hat w^\pm$ at the exponential rate, as $\eta_1 \to \pm\infty$ (see [**12**]).

LEMMA 2.4. *Denote by V_1^ε the first-order approximation of V_ε*

$$
\begin{aligned}
V_1^\varepsilon(t,x,\xi) &= V_0(t,x_1,\xi_1) + \varepsilon\, N\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}V_0(t,x_1,\xi_1) \\
&+\varepsilon\, N^*\big(\frac{\xi}{\varepsilon}\big)\,\partial_{\xi_1}V_0(t,x_1,\xi_1) + \varepsilon\, V_1(t,x_1,\xi_1) + \varepsilon\, V_{bl}^\varepsilon(t,x,\xi).
\end{aligned}
$$

Then, for $x \in G_\varepsilon$ and $t \geq 0$, the following estimate is valid:

$$
\int\limits_{G_\varepsilon} |V_\varepsilon - V_1^\varepsilon|^2\, d\xi \leq C\,\varepsilon^4\,\varepsilon^{d-1}
\tag{2.29}
$$

with the constant C depending on $\min((x_1-1),(x_1+1))$, Λ, d and independent of ε.

PROOF. The proof of Lemma 2.4 consists of several steps. First, we calculate the discrepancy appearing on the right-hand side of the equation and boundary condition when substituting the difference $W_\varepsilon - V_\varepsilon$ into (2.12). Then, by linearity, by represent $W_\varepsilon - V_\varepsilon$

as a sum $\tilde{V}_1^\varepsilon + \tilde{V}_2^\varepsilon$, where $\tilde{V}_1^\varepsilon$ is estimated using standard a priori estimates, and $\tilde{V}_2^\varepsilon$ - with the help of the maximum principle. Finally, combining the obtained estimates for $\tilde{V}_1^\varepsilon$, $\tilde{V}_2^\varepsilon$ with an estimate for the difference $W_\varepsilon - V_1^\varepsilon$, we end up with the estimate for $V_\varepsilon - V_1^\varepsilon$.
1. **Auxiliary a priori estimates**
Consider the following problem:

$$\begin{cases} \rho_\Theta^\varepsilon \partial_t w^\varepsilon + A_\Theta^{*,\varepsilon}\, w^\varepsilon = f(t,x) + \mathrm{div} F(t,x), & (0,T)\times G_\varepsilon, \\ B_\Theta^\varepsilon\, w^\varepsilon = \varepsilon g(t,x) - (F,n), & (0,T)\times \Sigma_\varepsilon, \\ w^\varepsilon(t,\pm 1, x') = 0, & (t,x') \in (0,T)\times Q, \\ w^\varepsilon(0,x) = 0, & x \in G_\varepsilon. \end{cases} \tag{2.30}$$

Since $\mathrm{div}\, b_\Theta^\varepsilon = 0$, a priori estimates are obtained in the standard way. Multiplying the equation in (2.30) by w^ε and integrating by parts and exploiting the Cauchy-Bunyakovsky inequality and Grönwall's lemma, we obtain

$$\begin{aligned} &\int\limits_{G_\varepsilon} (w^\varepsilon)^2\, dx + \int\limits_0^t \int\limits_{G_\varepsilon} |\nabla w^\varepsilon|^2\, dx\, d\tau \\ &\le C\, e^{C_1 t} \left(\|f\|^2_{L^2((0,T)\times G_\varepsilon)} + \varepsilon \|g\|_{L^2((0,T)\times\Sigma_\varepsilon)} + \|F\|_{L^2((0,T)\times G_\varepsilon)} \right), \end{aligned} \tag{2.31}$$

where the constants C, C_1 are independent of ε.

2. To estimate the $L^2(G_\varepsilon)$ norm of $W_\varepsilon - V_\varepsilon$, we calculate first the discrepancy appearing on the right-hand side in the equation and lateral boundary condition.

$$\begin{aligned} &\rho_\Theta\, \partial_t (W_\varepsilon - V_\varepsilon) + A_\Theta^{*\varepsilon} (W_\varepsilon - V_\varepsilon) \\ &= \varepsilon\, R_1(t,x_1,\xi_1;y,\eta) + \varepsilon^2\, R_2(t,x_1,\xi_1;y,\eta)\Big|_{y=x/\varepsilon,\eta=\xi/\varepsilon}, \\ &B_\Theta^\varepsilon (W_\varepsilon - V_\varepsilon) = \varepsilon^2\, a_{i1}^\Theta(\eta)\, n_i\, \partial_{\xi_1} V_2(t,x_1,\xi_1;y,\eta)\Big|_{y=x/\varepsilon,\eta=\xi/\varepsilon} \end{aligned}$$

where

$$\begin{aligned} &R_1(t,x_1,\xi_1;y,\eta) = \rho_\Theta(\eta) N(y) \partial_t \partial_{x_1} V_0(t,x_1,\xi_1) \\ &+\rho_\Theta(\eta) N^*(\eta) \partial_t \partial_{\xi_1} V_0 + \rho_\Theta(\eta)\partial_t V_1 - a_{11}^\Theta(\eta) N(y) \partial_{\xi_1}^2 \partial_{x_1} V_0(t,x_1,\xi_1) \\ &-a_{11}^\Theta(\eta) N^*(\eta) \partial_{\xi_1}^3 V_0(t,x_1,\xi_1) - a_{11}^\Theta(\eta) \partial_{\xi_1}^2 V_1(t,x_1,\xi_1) \\ &-a_{1j}^\Theta(\eta) \partial_{\xi_1} \partial_{\eta_j} V_2(t,x_1,\xi_1;y,\eta) - \partial_{\eta_i}(a_{i1}^\Theta(\eta) \partial_{\xi_1} V_2(t,x_1,\xi_1;y,\eta)) \\ &-b_1^\Theta(\eta) \partial_{\xi_1} V_2(t,x_1,\xi_1;y,\eta). \end{aligned}$$

$$R_2(t,x_1,\xi_1;y,\eta) = \rho_\Theta\, \partial_t V_2(t,x_1,\xi_1;y,\eta) - a_{11}^\Theta(\eta) \partial_{\xi_1}^2 V_2(t,x_1,\xi_1;y,\eta).$$

By linearity, we represent $W_\varepsilon - V_\varepsilon$ as a sum $W_\varepsilon - V_\varepsilon = \tilde{V}_1^\varepsilon + \tilde{V}_2^\varepsilon$, where $\tilde{V}_1^\varepsilon$ and $\tilde{V}_2^\varepsilon$, for each $x \in G_\varepsilon$, solve the following problems:

$$\begin{cases} \rho_\Theta\, \partial_t \tilde{V}_1^\varepsilon + A_\Theta^{*\varepsilon} \tilde{V}_1^\varepsilon = \varepsilon\, R_1(t, x_1, \xi_1; y, \eta) & \\ +\varepsilon^2\, R_2(t, x_1, \xi_1; y, \eta)\Big|_{y=x/\varepsilon, \eta=\xi/\varepsilon}, & (t,\xi) \in (0,T) \times G_\varepsilon, \\ B_\Theta^\varepsilon \tilde{V}_1^\varepsilon = \varepsilon^2\, a_{i1}^\Theta(\eta)\, n_i\, \partial_{\xi_1} V_2(t, x_1, \xi_1; y, \eta)\Big|_{y=x/\varepsilon, \eta=\xi/\varepsilon}, & (t,\xi) \in (0,T) \times \Sigma_\varepsilon, \\ \tilde{V}_1^\varepsilon(t, x, \xi)\Big|_{\xi_1=\pm 1} = 0, & t \in (0,T) \\ \tilde{V}_1^\varepsilon(0, x, \xi) = 0, & \xi \in G_\varepsilon; \end{cases}$$

$$\begin{cases} \rho_\Theta\, \partial_t \tilde{V}_2^\varepsilon + A_\Theta^{*\varepsilon} \tilde{V}_2^\varepsilon = 0, & (t,\xi) \in (0,T) \times G_\varepsilon, \\ B_\Theta^\varepsilon \tilde{V}_2^\varepsilon = 0, & (t,\xi) \in (0,T) \times \Sigma_\varepsilon, \\ \tilde{V}_2^\varepsilon(t, x, \xi)\Big|_{\xi_1=\pm 1} = (W_\varepsilon - V_\varepsilon)(t, x, \xi)\Big|_{\xi_1=\pm 1}, & t \in (0,T) \\ \tilde{V}_2^\varepsilon(0, x, \xi) = 0, & \xi \in G_\varepsilon. \end{cases}$$

3. We estimate first $\tilde{V}_1^\varepsilon$ using a priori estimates (2.31). To this end, we notice that, in view of (2.22) and (2.26),

$$\int\limits_Y R_1(t, x_1, \xi_1; y, \eta)\, d\eta + \int\limits_{\partial Y} a_{i1}^\Theta(\eta) \partial_{\xi_1} V_2(t, x_1, \xi_1; y, \eta)\, d\sigma = 0,$$

thus, there exists a 1-periodic with respect to η_1 vector-function $\chi = \chi(t, x_1, \xi_1; y, \eta)$ such that

$$\begin{cases} -\mathrm{div}_\eta \chi = R_1(t, x_1, \xi_1; y, \eta), & \eta \in Y, \\ (\chi, n) = a_{i1}^\Theta(\eta)\, n_i\, \partial_{\xi_1} V_2(t, x_1, \xi_1; y, \eta), & \eta \in \partial Y. \end{cases}$$

Obviously,

$$R_1(t, x_1, \xi_1; y, \eta)\Big|_{\eta=\xi/\varepsilon} = -\varepsilon \mathrm{div}_\xi \chi\big(t, x_1, \xi_1; y, \frac{\xi}{\varepsilon}\big)$$
$$+\varepsilon \partial_{\xi_1} \chi_1\big(t, x_1, \xi_1; y, \eta\big)\Big|_{\eta=\xi/\varepsilon};$$

and

$$\int\limits_{G_\varepsilon} \chi^2(t, x_1, \xi_1; y, \frac{\xi}{\varepsilon})\, d\xi \le C \int\limits_{G_\varepsilon} R_1^2(t, x_1, \xi_1; y, \frac{\xi}{\varepsilon})\, d\xi.$$

Then, in view of (2.23), we have

$$\int\limits_{G_\varepsilon} \chi^2(t, x_1, \xi_1; y, \frac{\xi}{\varepsilon})\, d\xi \le C\, \varepsilon^4\, \varepsilon^{d-1}, \;\; x_1 \in (-1,1), \;\; x' \in \varepsilon Q \tag{2.32}$$

with the constant C depending on $\min\{(x_1-1), (x_1+1)\}$, Λ, d.

Taking into account (2.21) and (2.23), we see that

$$\int\limits_{G_\varepsilon} R_2^2(t,x_1,\xi_1;y,\frac{\xi}{\varepsilon})\,d\xi \le C\,\varepsilon^4\,\varepsilon^{d-1},\ x_1\in(-1,1),\ x'\in\varepsilon Q \tag{2.33}$$

where C depends on $\min\{(x_1-1),(x_1+1)\}$, Λ, d. Thus, for $x\in G_\varepsilon$ such that $x_1\in I\Subset(-1,1)$,

$$\int\limits_{G_\varepsilon} |\tilde{V_1^\varepsilon}(t,x,\xi)|^2\,d\xi \le C\,\varepsilon^4\,\varepsilon^{d-1},\quad t\ge 0, \tag{2.34}$$

with the constant C which depends on $\min\{(x_1-1),(x_1+1)\}$, Λ, d.

4. We proceed by estimating $\tilde{V_2^\varepsilon}$. Due to the presence of the boundary layer corrector V_{bl}^ε,

$$\begin{aligned}&W_\varepsilon(t,x,\xi',\pm1)-\Phi_\varepsilon(t,x,\xi',\pm1)\\&=\Big(\varepsilon^2 V_2(t,x_1,\xi_1;y,\frac{\xi}{\varepsilon})+\varepsilon^3 W_{bl}^\varepsilon(t,x,\xi)\Big)+\big(\Phi_1^\varepsilon(t,x,\xi)-\Phi_\varepsilon(t,x,\xi)\big).\end{aligned}$$

Taking into account (2.23) and the fact that N, N^*, N_2, N_2^* are Hölder continuous functions, we see that

$$\Big|\varepsilon^2 V_2(t,x_1,\xi_1;y,\frac{\xi}{\varepsilon})+\varepsilon^3 W_{bl}^\varepsilon(t,x,\xi)\Big|\le C\,\varepsilon^2,\quad t\ge 0,\ \xi\in G_\varepsilon,$$

where C depends on $\min\{(x_1-1),(x_1+1)\}$, Λ, d.

To estimate $|\Phi_1^\varepsilon-\Phi_\varepsilon|$ for $t\le\varepsilon^\beta$, $\beta>0$, we use Aronson's estimates. Namely, thanks to (2.8)-(2.9), for $\xi_1=\pm1$, $x_1\in I\Subset(-1,1)$ and $t\le\varepsilon^\beta$

$$|\Phi_\varepsilon|\le\frac{C}{\sqrt{t}}\,e^{-\frac{C_0|x_1-\xi_1|^2}{t}}=O(e^{-C_1/\varepsilon^\beta})$$

with some constants C_0, C, C_1.

Similarly, Φ_0 is exponentially small if $\xi_1=\pm1$, $x_1\in I\Subset(-1,1)$ and $t\le\varepsilon^\beta$, $\beta>0$.

To estimate $|\Phi_1^\varepsilon-\Phi_\varepsilon|$ for $t\ge\varepsilon^\beta$, we make use of Lemma 2.3. Namely, for $t\ge\varepsilon^\beta$, the following estimate holds true:

$$|\Phi_\varepsilon(t,x,\xi)-\Phi_2^\varepsilon(t,x,\xi)|\le C\,\varepsilon^{3-2\beta},\quad\forall\beta>0,$$

with the constant C independent of ε. Then, by (2.16),

$$|\Phi_2^\varepsilon(t,x,\pm1,\xi')-\Phi_1^\varepsilon(t,x,\pm1,\xi')|\le C\,\varepsilon^2,\quad x_1\in I\Subset(-1,1),$$

with some constant C depending on the distance from x to the rod ends.

Finally, we obtain that

$$|\Phi_\varepsilon(t,x,\pm1,\xi')-\Phi_1^\varepsilon(t,x,\pm1,\xi')|\le C\,\varepsilon^2,$$

where C depends on $\min\{(x_1-1),(x_1+1)\}$, Λ, d.

Combining the last estimate with the estimates for Φ_ε and Φ_0 for $t \le \varepsilon^\beta$, $\beta > 0$, we obtain that the boundary conditions on the bases of the rod are satisfied up to the second order in ε:

$$|W_\varepsilon(t,x,\pm 1,\xi') - \Phi_\varepsilon(t,x,\pm 1,\xi')| \le C\,\varepsilon^2, \quad t \ge 0,\ x_1 \in I \Subset (-1,1) \tag{2.35}$$

where C depends on $\min\{(x_1-1),(x_1+1)\}, \Lambda, d$. Thus, by the maximum principle, for $x \in G_\varepsilon$ such that $x_1 \in I \Subset (-1,1)$,

$$|\tilde{V_2^\varepsilon}(t,x,\xi)| \le C\,\varepsilon^2, \quad t \ge 0,\ \xi \in G_\varepsilon, \tag{2.36}$$

where C depends on $\min\{(x_1-1),(x_1+1)\}, \Lambda, d$.

5. Finally, recalling that $W_\varepsilon - V_\varepsilon = \tilde{V_1^\varepsilon} + \tilde{V_2^\varepsilon}$, by (2.34), (2.36), for any $t \in [0,T]$, we obtain the following estimate:

$$\int\limits_{G_\varepsilon} |V_\varepsilon - W_\varepsilon|^2\,dx \le C\,\varepsilon^4\,\varepsilon^{d-1}, \quad x_1 \in I \Subset (-1,1), x' \in \varepsilon Q.$$

It is easy to see that for $x_1 \in I \Subset (-1,1), x' \in \varepsilon Q$,

$$\int\limits_{G_\varepsilon} |V_2(t,x_1,\xi_1;y,\eta)|^2\,\Big|_{\eta=\xi/\varepsilon} d\xi + \int\limits_{G_\varepsilon} |W_{bl}^\varepsilon|^2 \le C\,\varepsilon^{d-1}.$$

Consequently, last two estimates yield (2.29). Lemma 2.4 is proved. □

2.2.4. *Asymptotics for v^ε.* Recalling the definition of V_ε, we obtain the following approximation for Green's function K_ε:

$$\begin{aligned} K_1^\varepsilon(t,x,\xi) &= K_0(t,x_1,\xi_1) + \varepsilon\,N\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}K_0(t,x_1,\xi_1) \\ &+\varepsilon\,N^*\big(\frac{\xi}{\varepsilon}\big)\,\partial_{\xi_1}K_0(t,x_1,\xi_1) + \varepsilon\,K_1(t,x_1,\xi_1) - \varepsilon\,V_{bl}^\varepsilon(t,x,\xi), \end{aligned} \tag{2.37}$$

where K_0 is the Green function of the one-dimensional effective problem

$$\begin{cases} \partial_t K_0 = a^{\text{eff}}\,\partial_{\xi_1}^2 K_0, & (t,\xi_1) \in (0,T) \times (-1,1), \\ K_0(t,x_1,\pm 1) = 0, & t \in (0,T), \\ K_0(0,x_1,\xi_1) = \delta(x_1 - \xi_1), & \xi_1 \in (-1,1), \end{cases} \tag{2.38}$$

K_1 is a solution of the nonhomogeneous problem

$$\begin{cases} \partial_t K_1 = a^{\text{eff}}\,\partial_{\xi_1}^2 K_1 - F(t,x_1,\xi_1), & (t,\xi_1) \in (0,T) \times (-1,1), \\ K_1(t,x_1,\pm 1) = \hat{v}^\pm\,\partial_{\xi_1} K_0\Big|_{\xi_1=\pm 1}, & t \in (0,T), \\ K_1(0,x_1,\xi_1) = 0, & \xi_1 \in (-1,1) \end{cases}$$

with the right-hand side F given by (2.27). The boundary layer corrector V_{bl}^ε is defined by (2.24) and (2.28), respectively. Taking into account Lemmata 2.4 and 2.3, we obtain the following statement.

LEMMA 2.5. *For each $x \in G_\varepsilon$ and $t > 0$, the following estimates hold true:*

$$\|K_\varepsilon - K_0\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{\frac{d-1}{2}},$$
$$\|K_\varepsilon - K_1^\varepsilon\|_{L^2(G_\varepsilon)} \le C\,\varepsilon^2\,\varepsilon^{\frac{d-1}{2}}, \tag{2.39}$$

where the constant C depends on $\min\{(1-x_1),(1+x_1)\}$, Λ, d and does not depend on ε.

Now we turn to analyzing the asymptotics of v^ε which is given by (2.10) in terms of the corresponding Green function K_ε. Obviously, (2.10) can be rewritten in the following form:

$$v^\varepsilon(t,x) = \int\limits_{G_\varepsilon} K_1^\varepsilon(t,x,\xi)\,u_0(\xi_1)\,p_\Theta^{-1}\Big(\frac{\xi}{\varepsilon}\Big)\,e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\,d\xi$$
$$+\int\limits_{G_\varepsilon}(K_\varepsilon(t,x,\xi) - K_1^\varepsilon(t,x,\xi))\,u_0(\xi_1)\,p_\Theta^{-1}\Big(\frac{\xi}{\varepsilon}\Big)\,e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\,d\xi. \tag{2.40}$$

Thanks to (2.39) we have

$$\Big|\int\limits_{G_\varepsilon}(K_\varepsilon(t,x,\xi) - K_1^\varepsilon(t,x,\xi))\,u_0(\xi_1)\,p_\Theta^{-1}\Big(\frac{\xi}{\varepsilon}\Big)\,e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\,d\xi\Big| \le C\,\varepsilon^{1/2}\,\varepsilon^{d+1},$$

where the constant C depends on $\min\{(1-x_1),(1+x_1)\}$, Λ, d and does not depend on ε.

We proceed with analyzing the second integral in (2.40). We compute separately the contributions of each summand in (2.37).

Expanding K_0 and u_0 into the Teylor series in the neighbourhood of $\xi_1 = -1$, we see that, for $t \ge t_0 > 0$,

$$\int\limits_{G_\varepsilon} K_0(t,x_1,\xi_1)\,u_0(\xi_1)\,p_\Theta^{-1}\Big(\frac{\xi}{\varepsilon}\Big)\,e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\,d\xi$$
$$= u_0(-1)\,\partial_{\xi_1}K_0(t,x_1,-1)\int\limits_{G_\varepsilon}(\xi_1+1)\,p_\Theta^{-1}\Big(\frac{\xi}{\varepsilon}\Big)\,e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\,d\xi + O(\varepsilon^{d+2}),\ \ \varepsilon\to 0.$$

Making change of variables

$$z_1 = \frac{\xi_1+1}{\varepsilon},\ \ z' = \frac{\xi'}{\varepsilon},$$

and using the periodicity of p_Θ, one gets

$$\int\limits_{G_\varepsilon} K_0(t,x_1,\xi_1)\,u_0(\xi_1)\,p_\Theta^{-1}\Big(\frac{\xi}{\varepsilon}\Big)\,e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\,d\xi$$
$$= \varepsilon^{d+1}\,u_0(-1)\,\partial_{\xi_1}K_0(t,x_1,-1)$$
$$\times\int\limits_0^{+\infty}\int\limits_Q z_1\,p_\Theta(z)^{-1}\,e^{-\Theta z_1}\,dz'dz_1 + O(\varepsilon^{d+2}). \tag{2.41}$$

Recall that, for simplicity, we assume $\varepsilon = 1/N$, $N \in \mathbb{Z}_+$.

Similarly we have that for $t \ge t_0 > 0$,

$$\begin{aligned} &\varepsilon \int\limits_{G_\varepsilon} N^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_1} K_0(t,x,\xi)\, \frac{u_0(\xi_1)}{p_\Theta(\xi/\varepsilon)}\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi \\ &= \varepsilon^{d+1}\, u_0(-1)\, \partial_{\xi_1} K_0(t,x_1,-1) \\ &\times \int\limits_0^{+\infty} \int\limits_Q N^*(z)\, p_\Theta(z)^{-1}\, e^{-\Theta z_1}\, dz' dz_1 + O(\varepsilon^{d+2}). \end{aligned} \tag{2.42}$$

Taking into account that $\partial_{x_1} K_0(t,x_1,\pm 1) = 0$, we obtain

$$\varepsilon \int\limits_{G_\varepsilon} N\big(\frac{x}{\varepsilon}\big)\, \partial_{x_1} K_0(t,x_1,\xi_1)\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi = O(\varepsilon^{d+2}),$$

thus, this term can be neglected.

By the same argument, since $K_1(t,x_1,-1) = \hat{v}^- \partial_{\xi_1} K_0(t,x_1,-1)$,

$$\varepsilon \int\limits_{G_\varepsilon} (K_1(t,x_1,\xi_1) - \hat{v}^- \partial_{\xi_1} K_0(t,x_1,-1))\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi = O(\varepsilon^{d+2}).$$

Performing change of variables as above and using the periodicity of p_Θ, we get

$$\begin{aligned} &\varepsilon \int\limits_{G_\varepsilon} v^-\big(\frac{\xi_1+1}{\varepsilon}, \frac{\xi'}{\varepsilon}\big)\, \partial_{\xi_1} K_0(t,x_1,-1)\, u_0(\xi_1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_1+1)}{\varepsilon}}\, d\xi \\ &= \varepsilon^{d+1}\, u_0(-1)\, \partial_{\xi_1} K_0(t,x_1,-1) \\ &\times \int\limits_0^{+\infty} \int\limits_Q v^-(z)\, p_\Theta^{-1}(z)\, e^{-\Theta z_1}\, dz' dz_1 + O(\varepsilon^{d+2}). \end{aligned} \tag{2.43}$$

Thanks to (2.25), the term containing the boundary layer corrector near the right base of the rod is exponentially small. Combining (2.41)–(2.43) yields

$$v^\varepsilon(t,x) = \varepsilon^{d+1}\, M\, u_0(-1)\, \partial_{\xi_1} K_0(t,x_1,-1) + r_\varepsilon(t,x), \ \ t \ge t_0 > 0,$$

where $|r_\varepsilon(t,x)| \le C_0\, \varepsilon^{1/2}\, \varepsilon^{d+1}$ with the constant C_0 depending on $\min\{(x_1+1),(1-x_1)\}, \Lambda, d$; the constant M is defined by

$$M = \int\limits_0^{+\infty} \int\limits_Q (z_1 + N^*(z) + v^-(z))\, p_\Theta^{-1}(z)\, e^{-\Theta z_1}\, dz' dz_1.$$

In this way we have proved the following theorem.

THEOREM 2.6. *Let conditions* $(\mathbf{H1}) - (\mathbf{H3})$ *be fulfilled and* $\bar{b}_1 > 0$.

(1) *Suppose* $u_0 \in C^1[-1,1]$ *is such that* $u_0(-1) \neq 0$. *The asymptotics of the solution* u^ε *of problem* (2.1), *for* $t \geq t_0 > 0$ *and* $x \in G_\varepsilon$, *takes the form*

$$u^\varepsilon(t,x) = \varepsilon^2\, \varepsilon^{d-1}\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_1+1)}{\varepsilon}}\, p_\Theta\Big(\frac{x}{\varepsilon}\Big)\, \big[u(t,x) + r_\varepsilon(t,x)\big],$$

where u *is a solution of the effective problem*

$$\begin{cases} \partial_t u = a^{\rm eff}\, \partial^2_{x_1} u, & (t,x_1) \in (0,T)\times(-1,1),\\ u(t,\pm 1) = 0, & t \in (0,T),\\ u(0,x_1) = M\, u_0(-1)\, \delta'(x_1+1), & x_1 \in (-1,1). \end{cases}$$

Here the effective coefficient $a^{\rm eff}$ *is defined by* (2.13), $r_\varepsilon(t,x)$ *is such that* $|r_\varepsilon(t,x)| \leq C\sqrt{\varepsilon}$, *and the estimate is uniform for* $t \geq t_0 > 0$, $x_1 \in I \Subset (-1,1)$, *with* $C = C(\min\{(1-x_1),(1+x_1)\},\Lambda,d)$. *The constant* M *is given by*

$$M = \int\limits_0^{+\infty} \int\limits_Q (z_1 + N^*(z) + v^-(z))\, p_\Theta^{-1}(z)\, e^{-\Theta z_1}\, dz' dz_1,$$

where p_θ *is defined by Lemma 2.1,* Θ *is the maximum point of* $\lambda_1(\theta)$*; the functions* N^* *and* v^- *solve* (2.15) *and* (2.24), *respectively.*

(2) *If* $u_0 \in C^{k+1}(-1,1)$ *is such that* $u_0^{(l)}(-1) = 0$, $l = 0,\cdots,k-1$, *and* $u_0^{(k)}(-1) \neq 0$, *then*

$$u^\varepsilon(t,x) = \varepsilon^{k+2}\, \varepsilon^{d-1}\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_1+1)}{\varepsilon}}\, p_\Theta\Big(\frac{x}{\varepsilon}\Big)\, \big[\widetilde{u}(t,x) + \widetilde{r}^\varepsilon(t,x)\big],$$

where $\widetilde{u}$ *is a solution of the homogenized problem*

$$\begin{cases} \partial_t \widetilde{u} = a^{\rm eff}\, \partial^2_{x_1} \widetilde{u}, & (t,x_1) \in (0,T)\times(-1,1),\\ \widetilde{u}(t,\pm 1) = 0, & t \in (0,T),\\ \widetilde{u}(0,x_1) = \widetilde{M}\, u_0^{(k)}(-1)\, \delta'(x_1+1), & x_1 \in (-1,1). \end{cases}$$

Here the constant $\widetilde{M}$ *is given by*

$$\widetilde{M} = \frac{1}{k!} \int\limits_0^{+\infty} \int\limits_Q (z_1)^k\, (z_1 + N^*(z) + v^-(z))\, p_\Theta^{-1}(z)\, e^{-\Theta z_1}\, dz' dz_1.$$

$\widetilde{r}^\varepsilon(t,x)$ *is such that* $|\widetilde{r}^\varepsilon(t,x)| \leq C\sqrt{\varepsilon}$, *and the estimate is uniform for for* $t \geq t_0 > 0$, $x_1 \in I \Subset (-1,1)$, $x' \in \varepsilon Q$, *with* $C = C(\min\{(1-x_1),(1+x_1)\},\Lambda,d)$.

The second statement of Theorem 2.6 can be proved in exactly the same way as the first one.

3. Problem in a layer

Denote by Ω the layer $\{x \in \mathbb{R}^d : x' = (x_1, \cdots, x_{d-1}) \in \mathbb{R}^{d-1},\ -1 \le x_d \le 1\}$ (see Figure 2). Then the boundary of Ω consists of two planes $\Gamma^{\pm} = \{x \in \mathbb{R}^d : x_d = \pm 1\}$. We study a homogenization problem for a non-stationary convection-diffusion equation stated in Ω:

$$\begin{cases} \partial_t u^\varepsilon + A_\varepsilon\, u^\varepsilon = 0, & (t,x) \in (0,T) \times \Omega, \\ u^\varepsilon = 0, & (t,x) \in \Gamma^+ \cap \Gamma^-, \\ u^\varepsilon(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where

$$A_\varepsilon u^\varepsilon = -\mathrm{div}\big(a^\varepsilon \nabla u^\varepsilon\big) + \frac{1}{\varepsilon} b^\varepsilon \cdot \nabla u^\varepsilon,$$

and the coefficients of the equation are given by

$$a^\varepsilon_{ij} = a_{ij}\big(\frac{x}{\varepsilon}\big); \quad b^\varepsilon_j = b_j\big(\frac{x}{\varepsilon}\big).$$

In the sequel we assume that the following conditions are satisfied:

- **(A1)** The coefficients of the equation $a_{ij}, b_j \in L^\infty(\Omega)$ are Y-periodic, $Y = (0,1]^d$ being the periodicity cell.
- **(A2)** The $d \times d$ matrix $a(y)$ satisfies the uniform ellipticity condition, that is there exists $\Lambda > 0$ such that
$$a_{ij}(y)\xi_i\xi_j \ge \Lambda|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d.$$
- **(A3)** $u_0(x) \in C_0(\mathbb{R}^{d-1}; C[-1,1])$, that is the initial function u_0 has compact support with respect to $x' = (x_1, \cdots, x_{d-1})$.
- **(A4)** For simplicity we assume that $\varepsilon = 1/N$, so that the periodic structure agrees with the thickness of the layer Ω.

REMARK 3.1. Instead of the function u_0 having compact support in $\overline{\Omega}$, one can consider a more general situation, when $u_0 \in C(\overline{\Omega}) \cap L^1(\Omega) \cap L^2(\Omega)$.

As in the case of a thin rod, we study the asymptotic behaviour of solutions $u^\varepsilon(t,x)$ of problem (3.1), as $\varepsilon \to 0$. Note that if the effective convection in the direction x_d is zero, the homogenization in moving coordinates takes place (see [**8**], [**3**]).

3.1. Auxiliary spectral problems.

For brevity, in what follows we denote

$$Au = -\mathrm{div}(a\nabla u) + b \cdot \nabla u, \quad A^* v = -\mathrm{div}(a\nabla v) - \mathrm{div}(b\, v).$$

In order to simplify the original problem, we make use of the factorization principle, as in Section 2, with respect to x_d, and then construct the asymptotics of the new unknown function in moving coordinates.

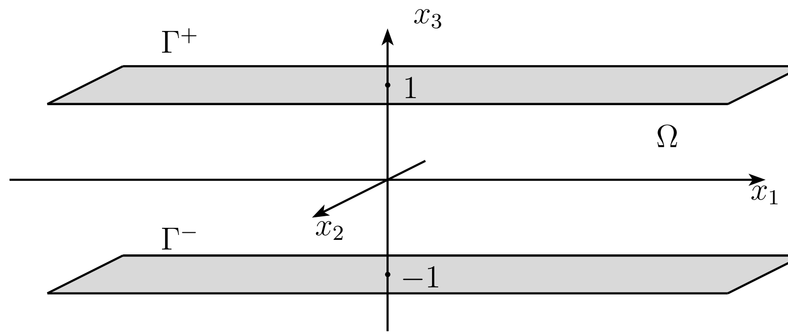

FIGURE 2. The layer Ω

For $\theta \in \mathbb{R}$, we introduce two parameterized families of spectral problems:

$$\begin{cases} e^{-\theta\, y_d}\, A\, e^{\theta\, y_d}\, p_\theta(y) = \lambda(\theta)\, p_\theta(y), \quad Y, \\ y \to p_\theta(y) \quad \text{Y-periodic.} \end{cases} \tag{3.2}$$

$$\begin{cases} e^{\theta\, y_d}\, A^*\, e^{-\theta\, y_d}\, p^*_\theta(y) = \lambda(\theta)\, p^*_\theta(y), \quad Y, \\ y \to p^*_\theta(y) \quad \text{Y-periodic.} \end{cases}$$

By the Krein-Rutman theorem, for each $\theta \in \mathbb{R}$, the first eigenvalue $\lambda_1(\theta)$ of problem (3.2) is real, simple, and the corresponding eigenfunctions p_θ and p^*_θ can be chosen positive. Moreover, as was proved in [**6**], $\theta \to \lambda_1(\theta)$ is twice differentiable, strictly concave and admits a maximum which is obtained for a unique $\theta = \Theta$. The eigenfunctions p_θ and p^*_θ can be normalized by

$$\int\limits_Y p_\theta(y)\, p^*_\theta(y)\, dy = 1.$$

Arguments similar to those in Section 2 yield

$$\frac{d\lambda_1}{d\theta}\Big|_{\theta=0} = \int\limits_Y \big(b_d\, p^*_\theta + a_{dj}\, \partial_{y_j} p^*_\theta\big)\, dy = \bar{b}_d, \tag{3.3}$$

where $\bar{b}_d$ is the last component of the effective drift. Hence, if $\bar{b}_d = 0$, then $\Theta = 0$. As was noticed already, we assume that $\bar{b}_d \neq 0$ (or, equivalently, $\Theta \neq 0$). In the case $\bar{b}_d = 0$ the method of homogenization in moving coordinates can be applied directly.

3.2. Factorization. If $\bar{b}_d > 0$, then we perform the change of unknown function as follows

$$u^\varepsilon(t,x) = e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta\,(x_d+1)}{\varepsilon}}\, p_\Theta\Big(\frac{x}{\varepsilon}\Big)\, v^\varepsilon(t,x). \tag{3.4}$$

If $\bar b_d < 0$, then instead of $\exp\Big(\frac{\Theta\,(x_d+1)}{\varepsilon}\Big)$ we take $\exp\Big(\frac{\Theta\,(x_d-1)}{\varepsilon}\Big)$. Substituting (3.4) into (3.1), one obtains that the new unknown function v^ε satisfies the problem which reads

$$
\begin{cases}
\rho^\varepsilon_\Theta\,\partial_t v^\varepsilon + A^\varepsilon_\Theta\, v^\varepsilon = 0, & (t,x)\in(0,T)\times\Omega,\\
v^\varepsilon = 0, & (t,x)\in\Gamma^+\cup\Gamma^-,\\
v^\varepsilon(0,x) = u_0(x)\,p_\Theta^{-1}\big(\frac{x}{\varepsilon}\big)\,e^{-\frac{\Theta\,(x_d+1)}{\varepsilon}}, & x\in\Omega,
\end{cases}
\tag{3.5}
$$

where

$$
A^\varepsilon_\Theta v = -\mathrm{div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\big) + \frac{1}{\varepsilon}\,b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v,\qquad \varrho_\Theta(y) = p_\Theta(y)\,p^*_\Theta(y)
$$

and the coefficients of the operator are given by

$$
\begin{aligned}
&a^\Theta_{ij}(y) = \varrho_\Theta(y)\,a_{ij}(y);\\
&b^\Theta_i(y) = \varrho_\Theta(y)\,b_i(y) - 2\,\varrho_\Theta(y)\,a_{id}(y)\,\Theta\\
&+a_{ij}(y)\,\big[p_\Theta(y)\,\partial_{y_j}p^*_\Theta(y) - p^*_\Theta(y)\,\partial_{y_j}p_\Theta(y)\big].
\end{aligned}
\tag{3.6}
$$

The matrix a^Θ is positive definite since both p_Θ and p^*_Θ are positive functions. The vector-field b^Θ, for each $\theta\in\mathbb{R}$, is divergence-free and b^Θ_d has zero mean, that is

$$
\int\limits_Y b^\Theta_d(y)\,dy = 0;\qquad \mathrm{div}\,b^\theta = 0,\ \forall\,\theta.
\tag{3.7}
$$

Notice that if the average of b^Θ_k, $k=1,\cdots,d-1$, is not equal to zero, then, in contrast with the case of a thin rod, the classical homogenization methods do not apply to problem (3.5). To overcome this difficulty, we use of the homogenization in moving coordinates. Denoting the Green function of problem (3.5) by $K_\varepsilon(t,x,\xi)$, we represent v^ε in the form

$$
v^\varepsilon(t,x) = \int\limits_\Omega K_\varepsilon(t,x,\xi)\,u_0(\xi)\,p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\,e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\,d\xi.
\tag{3.8}
$$

For any x, K_ε solves the adjoint problem

$$
\begin{cases}
\varrho_\Theta\big(\frac{\xi}{\varepsilon}\big)\,\partial_t K_\varepsilon(t,x,\xi) + A^{*,\varepsilon}_\Theta\,K_\varepsilon(t,x,\xi) = 0, & (t,\xi)\in(0,T)\times\Omega,\\
K_\varepsilon(t,x,\xi) = 0, & (t,\xi)\in(0,T)\times(\Gamma^-\cup\Gamma^+),\\
K_\varepsilon(0,x,\xi) = \delta(x-\xi), & \xi\in\Omega,
\end{cases}
\tag{3.9}
$$

$$
A^{*,\varepsilon}_\Theta v = -\mathrm{div}\big(a^\Theta\big(\frac{x}{\varepsilon}\big)\nabla v\big) - \frac{1}{\varepsilon}\,b^\Theta\big(\frac{x}{\varepsilon}\big)\cdot\nabla v.
$$

Since b_Θ is divergence-free, $A^{*,\varepsilon}_\Theta$ differs from A^ε_Θ by the sign in front of the first-order term. For any $\xi\in\Omega$, K_ε solves the direct problem with respect to (t,x), but since we are interested in the asymptotics of K_ε w.r.t ξ, we prefer to interpret it from the very beginning as a solution of adjoint problem (3.9).

We study the asymptotic behaviour of K_ε, as $\varepsilon \to 0$, and then from (3.8) derive the asymptotics for v^ε.

3.3. Asymptotic behaviour of $K_\varepsilon(t, x, \xi)$. As in the proof of Theorem 2.2, instead of analyzing directly K_ε, we consider the difference

$$V_\varepsilon(t, x, \xi) = \Phi_\varepsilon(t, x, \xi) - K_\varepsilon(t, x, \xi),$$

where Φ_ε is the fundamental solution in $\mathbb{R}^d$, that is, for any $x \in \mathbb{R}^d$, Φ_ε solves the problem

$$\begin{cases} \varrho_\Theta\big(\dfrac{\xi}{\varepsilon}\big)\, \partial_t \Phi_\varepsilon + A_\Theta^{*,\varepsilon} \Phi_\varepsilon = 0, & (t, \xi) \in (0, T) \times \mathbb{R}^d, \\ \Phi_\varepsilon(0, x, \xi) = \delta(x - \xi), & \xi \in \mathbb{R}^d. \end{cases}$$

In this way, for all $x \in \Omega$, V_ε satisfies the problem

$$\begin{cases} \varrho_\Theta\big(\dfrac{\xi}{\varepsilon}\big)\, \partial_t V_\varepsilon(t, x, \xi) + A_\Theta^{*,\varepsilon}\, V_\varepsilon(t, x, \xi) = 0, & (t, \xi) \in (0, T) \times \Omega, \\ V_\varepsilon(t, x, \xi) = \Phi_\varepsilon(t, x, \xi), & (t, \xi) \in (0, T) \times (\Gamma^- \cup \Gamma^+), \\ V_\varepsilon(0, x, \xi) = 0, & \xi \in \Omega. \end{cases} \tag{3.10}$$

We would like to emphasize that V_ε is a smooth function of ξ, for x such that $x_d \neq \pm 1$.

To describe the asymoptotics of Φ_ε, we introduce $\Phi_0(t, x, \xi)$, the fundamental solution of the effective problem

$$\begin{cases} \partial_t \Phi_0 = \mathrm{div}_\xi(a^{\text{eff}} \nabla_\xi \Phi_0), & (t, \xi) \in (0, T) \times \mathbb{R}^d, \\ \Phi_0(0, x, \xi) = \delta(x - \xi), & \xi \in \mathbb{R}^d \end{cases}$$

with a^{eff} given by

$$\begin{aligned} a_{ij}^{\text{eff}} &= \int\limits_Y (a_{ij}^\Theta(y) + a_{ik}^\Theta(y) \partial_{y_k} N_j(y) - b_i^\Theta(y)\, N_j(y) + \beta_j^\Theta\, \rho_\Theta\, N_j(y))\, dy \\ &= \int\limits_Y (a_{ij}^\Theta(\eta) + a_{ik}^\Theta(\eta) \partial_{y_k} N_j^*(\eta) + b_i^\Theta(\eta)\, N_j^*(\eta) - \beta_j^\Theta\, \rho_\Theta\, N_j^*(\eta))\, d\eta. \end{aligned}$$

The constant

$$\beta_j^\Theta = \int_Y b_j^\Theta(y) dy, \quad j = 1, \cdots, d-1,$$

is the jth component of the effective convection ($\beta_d^\Theta = 0$); vector functions N and N^* solve the following cell problems (direct and adjoint, respectively):

$$\begin{cases} -\mathrm{div}(a^\Theta \nabla N_i) + b^\Theta \cdot \nabla N_i = \partial_{y_j} a_{ij}^\Theta(y) - b_i^\Theta(y) + \beta^\Theta, & Y, \\ y \mapsto N_i \quad Y-\text{periodic}; \end{cases} \tag{3.11}$$

$$\begin{cases} -\mathrm{div}(a^\Theta \nabla N_i^*) - b^\Theta \cdot \nabla N_i^* = \partial_{y_j} a_{ij}^\Theta(y) + b_i^\Theta(y) - \beta^\Theta, & Y, \\ y \mapsto N_i^* \quad Y-\text{periodic}. \end{cases} \tag{3.12}$$

Note that, in view of the definition of β^Θ, the compatibility conditions for (3.11) and (3.12) are satisfied. It can be seen that the matrix a^{eff} is positive definite, and the functions N and N^* are Hölder continuous functions (see [**9**]).

The following important result characterizing the asymptotic behaviour of Φ_ε can be proved in the same way as in [**1**].

LEMMA 3.2. *Assume that conditions* **(A1)-(A2)** *are fulfilled. Then, for* $x, \xi \in \mathbb{R}^d$ *and* $t \geq \varepsilon^2$*, the estimate holds*

$$\Big|\Phi_\varepsilon(t,x,\xi) - \Phi_1^\varepsilon\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}\,t\big)\Big| \leq C\,\frac{\varepsilon^2}{t^{(d+2)/2}},$$

where $\beta^\Theta = \int_Y b^\Theta(y)\,dy$*, the constant* C *does not depend on* ε*, and the first-order approximation* Φ_1^ε *is given by*

$$\Phi_1^\varepsilon\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}\,t\big) = \Phi_0\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}\,t\big) + \varepsilon\,N\big(\frac{x}{\varepsilon}\big)\,\nabla_x\Phi_0\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}\,t\big)$$
$$+\varepsilon\,N^*\big(\frac{\xi}{\varepsilon}\big)\,\nabla_\xi\Phi_0\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}\,t\big).$$

Turning back to V_ε, for each $x \in \Omega$, we denote by V_0 a solution of the homogenized problem

$$\begin{cases} \partial_t V_0 = \operatorname{div}_\xi(a^{\text{eff}}\nabla_\xi V_0), & (t,\xi)\in(0,T)\times\Omega, \\ V_0(t,x,\xi) = \Phi_0(t,x,\xi), & (t,\xi)\in(0,T)\times(\Gamma^-\cup\Gamma^+), \\ V_0(0,x,\xi) = 0, & \xi\in\Omega. \end{cases}$$

Note that, for $x \notin (\Gamma^-\cup\Gamma^+)$, $V_0 \in C^\infty([0,T]\times\Omega\times\overline{\Omega})$ and for $(t,\xi)\in[0,T]\times\Omega$ one has

$$0 \leq \partial_t^k\,\partial_x^l\,\partial_\xi^m\,V_0(t,x,\xi) \leq \frac{C}{\operatorname{dist}(K,(\Gamma^-\cup\Gamma^+))^{2k+l+m+d-1}}, \quad x\in K \Subset \Omega.$$

The formal asymptotic expansion for V_ε takes the form

$$\begin{aligned} W_\varepsilon\big(t,x,\widetilde{\xi}\big) &= V_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\,N\big(\frac{x}{\varepsilon}\big)\cdot\nabla_x V_0\big(t,x,\widetilde{\xi}\big) \\ &+\varepsilon\,N^*\big(\frac{\xi}{\varepsilon}\big)\cdot\nabla_\xi V_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\,V_1\big(t,x,\widetilde{\xi}\big) \\ &+\varepsilon V_{\text{bl}}^\varepsilon(t,x,\xi) + \varepsilon^2\,V_2^\varepsilon\big(t,x,\widetilde{\xi}\big) + \varepsilon^2\,\varphi_{\text{bl}}^\varepsilon(t,x,\xi) \\ &+\varepsilon^3\,\psi_{\text{bl}}^\varepsilon(t,x,\xi), \quad \widetilde{\xi} = \xi + \frac{\beta^\Theta}{\varepsilon}t, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} V_2^\varepsilon\big(t,x,\widetilde{\xi}\big) &= \Big\{Q_{ij}(y)\,\partial_{x_i}\partial_{x_j}V_0\big(t,x,\widetilde{\xi}\big) \\ &+Q_{ij}^*(\eta)\,\partial_{\xi_i}\partial_{\xi_j}V_0\big(t,x,\widetilde{\xi}\big) + N_i(y)\,N_j^*(\eta)\,\partial_{x_i}\partial_{\xi_j}V_0\big(t,x_1,\widetilde{\xi}\big) \\ &+N_i(y)\,\partial_{x_i}V_1\big(t,x_1,\widetilde{\xi}\big) + N_i^*(\eta)\,\partial_{\xi_i}V_1\big(t,x_1,\widetilde{\xi}\big)\Big\}\Big|_{y=\frac{x}{\varepsilon},\eta=\frac{\xi}{\varepsilon}}. \end{aligned} \tag{3.14}$$

The functions Q_{ij}, Q^*_{ij} solve the problems

$$\begin{cases} A_\Theta Q_{ij} = \partial_{y_k}(a^\Theta_{ki} N_j) + a^\Theta_{ik}\partial_{y_k} N_j + a^\Theta_{ij} \\ -b^\Theta_i\, N_j + \beta^\Theta_i\, \rho_\Theta\, N_j - a^{\text{eff}}_{ij}\, \rho_\Theta, \quad Y, \\ y \mapsto Q_{ij} \text{ is periodic;} \end{cases}$$

$$\begin{cases} A^*_\Theta Q^*_{ij} = \partial_{y_k}(a^\Theta_{ki} N^*_j) + a^\Theta_{ik}\partial_{y_k} N^*_j + a^\Theta_{ij} \\ +b^\Theta_i\, N^*_j - \beta^\Theta_i\, \rho_\Theta\, N^*_j - a^{\text{eff}}_{ij}\, \rho_\Theta, \quad Y, \\ y \mapsto Q^*_{ij} \text{ is periodic.} \end{cases}$$

Here for brevity we denote

$$A_\Theta v = -\text{div}_y(a(y)\nabla_y v) + b(y)\cdot\nabla_y v,$$
$$A^*_\Theta v = -\text{div}_y(a(y)\nabla_y v) - b(y)\cdot\nabla_y v.$$

In order to define V_1 and the first boundary layer corrector $V^\varepsilon_{\text{bl}}$, we consider auxiliary problems in semi-infinite cylinders $G^\mp = (0,1]^{d-1}\times(0,\mp\infty)$:

$$\begin{cases} A^*_\Theta v^\pm = 0, & \eta \in G^\mp, \\ v^\pm(\eta', 0) = -N^*_d(\eta', 0), \\ \eta' \mapsto v^\pm(\eta', \eta_d) \text{ is } (0,1]^{d-1} - \text{periodic.} \end{cases} \tag{3.15}$$

Since $\beta_d = 0$, then such functions $v^\pm$ exist, they are uniquely defined and stabilize to some constants $\hat v^\pm$ at the exponential rate, as $\eta_d \to \mp\infty$ (see [**13**]):

$$\begin{aligned} &|v^\pm(\eta', \eta_d) - \hat v^\pm| \le C_0\, e^{-\gamma\,|\eta_d|}, \quad C_0, \gamma > 0; \\ &\|\nabla v^+\|_{L^2((n-1,n)\times Q)} \le C\, e^{-\gamma\, n}, \quad \forall n < 0, \\ &\|\nabla v^-\|_{L^2((n,n+1)\times Q)} \le C\, e^{-\gamma\, n}, \quad \forall n > 0. \end{aligned} \tag{3.16}$$

The first boundary layer corrector is given by

$$\begin{aligned} V^\varepsilon_{bl}(t,x,\xi) = &\Big[v^-\big(\frac{\xi'}{\varepsilon}, \frac{\xi_d + 1}{\varepsilon}\big) - \hat v^-\Big]\, \partial_{\xi_d}(V_0 - \Phi_0)\big(t, x, \xi + \frac{\beta^\Theta}{\varepsilon} t\big)\Big|_{\xi_d = -1} \\ + &\Big[v^+\big(\frac{\xi'}{\varepsilon}, \frac{\xi_d - 1}{\varepsilon}\big) - \hat v^+\Big]\, \partial_{\xi_d}(V_0 - \Phi_0)\big(t, x, \xi + \frac{\beta^\Theta}{\varepsilon} t\big)\Big|_{\xi_d = 1}. \end{aligned} \tag{3.17}$$

Then, V_1, for $x \in \Omega$, satisfies the problem

$$\begin{cases} \partial_t V_1 = \text{div}_\xi(a^{\text{eff}}\, \nabla_\xi V_1) + F(t,x,\xi), & (t,\xi) \in (0,T)\times\Omega, \\ V_1(t,x,\xi',\pm 1) = \hat v^\pm\, \partial_{\xi_d}\,(V_0 - \Phi_0)(t,x,\xi',\pm 1), & (t,\xi') \in (0,T)\times\mathbb{R}^{d-1}, \\ V_1(0,x,\xi) = 0, & \xi \in \Omega, \end{cases} \tag{3.18}$$

where

$$F(t,x,\xi) = \partial_{\xi_k}\partial_{\xi_i}\partial_{\xi_j}V_0(t,x,\xi)\int\limits_Y \big[a_{kl}^\Theta\partial_{\eta_l}Q_{ij}^* + a_{kj}^\Theta N^* + b_k^\Theta Q_{ij}^* - \beta_k^\Theta\rho_\Theta Q_{ij}^* - a_{ij}^{\text{eff}}\,\rho_\Theta\,N_k^*\big]\,d\eta.$$

The second boundary layer corrector $\varphi_{\text{bl}}^\varepsilon$ is defined as follows

$$\begin{aligned}
&\varphi_{bl}^\varepsilon(t,x,\xi)\\
&= \Big[\varphi_k^-\big(\frac{\xi'}{\varepsilon},\frac{\xi_d+1}{\varepsilon}\big) - \hat\varphi_k^-\Big]\,\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)\Big|_{\xi_d=-1}\Big)\\
&+\Big[\varphi_k^+\big(\frac{\xi'}{\varepsilon},\frac{\xi_d-1}{\varepsilon}\big) - \hat\varphi_k^+\Big]\,\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)\Big|_{\xi_d=1}\Big).
\end{aligned}$$

The functions $\varphi_k^\pm$ solve nonhomogeneous problems

$$\begin{cases}
A_\Theta^*\varphi_k^\pm = \partial_{\eta_i}(a_{ik}^\Theta(v^\pm-\hat v^\pm)) + a_{ik}^\Theta\,\partial_{\eta_i}v^\pm\\
+(b_k^\Theta-\beta_k^\Theta\rho_\Theta)(v^\pm-\hat v^\pm), \qquad \eta\in G^\mp,\\
\varphi_k^\pm(\eta',0)=0,\\
\eta'\mapsto\varphi_k^\pm(\eta',\eta_d)\ \text{is}\ (0,1]^{d-1}-\text{periodic}.
\end{cases}$$

The right-hand side of the last equation, due to (3.16), is an exponentially decaying function. Since $\beta_d^\Theta=0$, the functions $\varphi_k^\pm$ exist, they are uniquely defined and stabilize to some constants $\hat w_k^\pm$ at the exponential rate, as $\eta_d\to\pm\infty$ (see [**13**]). The corrector $\varphi_{\text{bl}}^\varepsilon$ has been introduced to compensate the terms of order ε^0 which will appear on the right-hand side after substituting $V_{\text{bl}}^\varepsilon$ into the original equation.

The last boundary layer corrector ψ_{bl}^ε is defined by

$$\begin{aligned}
&\psi_{bl}^\varepsilon(t,x,\xi)\\
&= \Big[\psi_{ik}^-\big(\frac{\xi'}{\varepsilon},\frac{\xi_d+1}{\varepsilon}\big) - \hat\psi_{ik}^-\Big]\,\partial_{\xi_i}\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)\Big|_{\xi_d=-1}\Big)\\
&+\Big[\psi_{ik}^+\big(\frac{\xi'}{\varepsilon},\frac{\xi_d-1}{\varepsilon}\big) - \hat\psi_{ik}^+\Big]\,\partial_{\xi_i}\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)\Big|_{\xi_d=1}\Big).
\end{aligned}$$

The functions $\psi_{ik}^\pm$ solve nonhomogeneous problems

$$\begin{cases}
A_\Theta^*\psi_{ik}^\pm = (a_{ik}^\Theta - a_{ik}^{\text{eff}}\rho_\Theta)(v^\pm-\hat v^\pm) + \partial_{\eta_i}(a_{ij}^\Theta(\varphi_k-\hat\varphi_k))\\
+a_{ij}^\Theta\partial_{\eta_j}\varphi_k + (b_i^\Theta-\beta_i^\Theta)(\varphi_k-\hat\varphi_k), \qquad \eta\in G^\mp,\\
\psi_{ik}^\pm(\eta',0)=0,\\
\eta'\mapsto\psi_{ik}^\pm(\eta',\eta_d)\ \text{is}\ (0,1]^{d-1}-\text{periodic}.
\end{cases}$$

The right-hand side of the last equation is an exponentially decaying function. Thus, the functions $\psi_{ik}^\pm$ exist, they are uniquely defined and stabilize to some constants $\hat\psi_j^\pm$ at the exponential rate, as $\eta_d\to\pm\infty$. The boundary layer corrector $\psi_{\text{bl}}^\varepsilon$ has been designed in order to compensate the terms of order ε on the right-hand side of the equation, which come from $V^\varepsilon_{\rm bl}$ and $\varphi^\varepsilon_{\rm bl}$ being substituted into the equation.

Now the formal expansion is defined and we can proceed with justifying the obtained formal asymptotics for V_ε. We would like to emphasize that the functions V_1 and $V^\varepsilon_{\rm bl}$ are introduced to satisfy the boundary conditions on $\Gamma^\pm$ up to the second order in ε, while V^ε_2, $\varphi^\varepsilon_{\rm bl}$ and $\psi^\varepsilon_{\rm bl}$ serve to guarantee the required accuracy and will not show up in the final expansion (see Proposition 3.3 below).

Our next goal is to prove the following statement.

PROPOSITION 3.3. *Denote by V^ε_1 the first-order approximation of V_ε*

$$\begin{aligned}&V^\varepsilon_1(t,x,\xi)=V_0\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)+\varepsilon\, N_j\big(\frac{x}{\varepsilon}\big)\,\partial_{x_j}V_0\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)\\&+\varepsilon\, N^*_j\big(\frac{\xi}{\varepsilon}\big)\,\partial_{\xi_j}V_0\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)+\varepsilon\, V_1\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)+\varepsilon\, V^\varepsilon_{bl}(t,x,\xi).\end{aligned}\tag{3.19}$$

Then, for x such that $x_d\in I\Subset(-1,1)$ and for $t\geq 0$, the following estimate is valid:

$$\int\limits_\Omega |V_\varepsilon-V^\varepsilon_1|^2\leq C\,\varepsilon^4\tag{3.20}$$

with the constant C depends on $\operatorname{dist}(x,\Gamma^-\cup\Gamma^+),\Lambda,d$ and is independent of ε.

PROOF. Let us substitute ansatz (3.13) into (3.10) and compute the discrepancy.

$$\begin{aligned}&\rho^\varepsilon_\Theta\partial_t(W_\varepsilon-V_\varepsilon)+A^{*,\varepsilon}_\Theta(W_\varepsilon-V_\varepsilon)\\&=\varepsilon\, R_1\big(t,x,\widetilde{\xi};y,\eta\big)+\varepsilon\operatorname{div}_\eta(a^\Theta(\eta)\nabla_{\widetilde{\xi}}V_2(t,x,\widetilde{\xi};y,\eta))\Big|_{y=\frac{x}{\varepsilon},\eta=\frac{\xi}{\varepsilon}}\\&+\varepsilon^2\, R_2\big(t,x,\widetilde{\xi};\eta\big)+\varepsilon^3\, R_3\big(t,x,\widetilde{\xi};\eta\big)\Big|_{y=\frac{x}{\varepsilon},\eta=\frac{\xi}{\varepsilon}},\quad \widetilde{\xi}=\xi+\frac{\beta^\Theta}{\varepsilon}t,\end{aligned}\tag{3.21}$$

where

$$\begin{aligned}&R_1\big(t,x,\widetilde{\xi};y,\eta\big)=-\rho_\Theta(\eta)\partial_tV_1(t,x,\widetilde{\xi})-\rho_\Theta(\eta)N^*_j(\eta)\partial_t\partial_{\xi_j}V_0(t,x,\widetilde{\xi})\\&-\rho_\Theta(\eta)N_j(y)\partial_t\partial_{x_j}V_0(t,x,\widetilde{\xi})-\rho_\Theta(\eta)\beta^\Theta_j\partial_{\widetilde{\xi}_j}V_2(t,x,\widetilde{\xi};y,\eta)\\&+\operatorname{div}_\xi(a^\Theta(\eta)\nabla_\eta V_2(t,x,\widetilde{\xi};y,\eta))+\operatorname{div}_\xi(a^\Theta(\eta)\nabla_\xi(N^*(\eta)\cdot\nabla_\xi V_0(t,x,\widetilde{\xi}))\\&+\operatorname{div}_\xi(a^\Theta(\eta)\nabla_\xi(N(y)\cdot\nabla_x V_0(t,x,\widetilde{\xi}))+\operatorname{div}_\xi(a^\Theta(\eta)\nabla_\xi V_1(t,x,\widetilde{\xi}))\\&+b^\Theta_j(\eta)\partial_{\xi_j}V_2(t,x,\widetilde{\xi};y,\eta),\quad \widetilde{\xi}=\xi+\frac{\beta^\Theta}{\varepsilon}t;\end{aligned}$$

$$
\begin{aligned}
&R_2\big(t,x,\widetilde{\xi};\eta\big)=\Big\{(a^{\rm eff}_{ij}-a^\Theta_{ij}(\eta))(\varphi_k(\eta)-\hat{\varphi}_k)\\
&-\partial_{\eta_j}(a^\Theta_{jl}(\eta)(\psi_{ik}(\eta)-\hat{\psi}_{ik}))-a^\Theta_{jl}(\eta)\partial_{\eta_l}\psi_{ik}(\eta)\\
&+(\beta^\Theta_j-b^\Theta_j(\eta))(\psi_{ik}(\eta)-\hat{\psi}_{ik})\Big\}\\
&\times\partial_{\xi_j}\partial_{\xi_i}\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)\Big|_{\xi_d=1}\Big);\\
&R_3\big(t,x,\widetilde{\xi};\eta\big)=(\rho_\Theta(\eta)a^{\rm eff}_{jl}-a^\Theta_{jl})(\psi_{ik}(\eta)-\hat{\psi}_{ik})\\
&\times\partial_{\xi_l}\partial_{\xi_j}\partial_{\xi_i}\partial_{\xi_k}\Big(\partial_{\xi_d}(V_0-\Phi_0)\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t\big)\Big|_{\xi_d=1}\Big).
\end{aligned}
$$

Notice that, in view of (3.14) and (3.18),

$$
\int\limits_Y R_1\big(t,x,\widetilde{\xi};y,\eta\big)\,d\eta=0.
$$

Thus, there exists $\chi\big(t,x,\widetilde{\xi};y,\eta\big)$, periodic in η, such that

$$
-\mathrm{div}_\eta\chi=R_1\big(t,x,\widetilde{\xi};y,\eta\big).
$$

Consequently,

$$
\begin{aligned}
&R_1\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t;y,\frac{\xi}{\varepsilon}\big)=-\varepsilon\mathrm{div}_\xi\chi\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t;y,\frac{\xi}{\varepsilon}\big)\\
&+\varepsilon\mathrm{div}_\xi\chi\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t;y,\eta\big)\Big|_{\eta=\frac{\xi}{\varepsilon}}.
\end{aligned}
$$

It is easy to see that, for sufficiently small ε,

$$
\int\limits_\Omega\big[\chi\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t;y,\frac{\xi}{\varepsilon}\big)\big]^2\,d\xi\le C\int\limits_\Omega\int\limits_Y\big[R_1\big(t,x,\xi;y,\eta\big)\big]^2\,d\eta\,d\xi
$$

with the constant C independent of ε. To estimate the norm on the right-hand side of the last inequality, we notice that each term in R_1 is a product of the form

$$
F(y,\eta)\,\partial_t^r\partial_{\xi_j}^m V_0(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t)
$$

with a bounded periodic function $F(y,\eta)$. Hence, it is sufficient to estimate the derivatives of the function V_0.

Since the effective operator has constant coefficients, Φ_0 is given by the explicit formula

$$
\Phi_0(t,x,\xi)=\frac{1}{(2\pi t)^{d/2}}\,\frac{1}{\det a^{\rm eff}}\exp\Big(-\frac{(x-\xi)^T(a^{\rm eff})^{-1}(x-\xi)}{4t}\Big). \tag{3.22}
$$

Denote by $K_0(t,x,\xi)$ the Green function of the effective problem

$$\left\{\begin{array}{ll}\partial_t w(t,x)=\operatorname{div}(a^{\rm eff}\nabla w(t,x)), & (t,x)\in(0,T)\times\Omega,\\ w(t,x)=0, & (t,x)\in(0,T)\times(\Gamma^-\cup\Gamma^+),\\ w(0,x)=w_0(x), & x\in\Omega.\end{array}\right. \tag{3.23}$$

By the maximum principle,

$$K_0(t,x,\xi)\le\Phi_0(t,x,\xi)\le\frac{C}{t^{d/2}}\exp\big(-\gamma_0\frac{(x-\xi)^2}{t}\big),$$

and, thus,

$$V_0(t,x,\xi)=\Phi_0(t,x,\xi)-K_0(t,x,\xi)\le\frac{C}{t^{d/2}}\exp\big(-\gamma_0\frac{(x-\xi)^2}{t}\big).$$

The last estimate together with the local estimates for the derivatives of V_0 (see, for example, [**11**]) imply

$$|\partial_t^r\partial_\xi^m V_0(t,x,\xi)|\le C\,e^{-\gamma(x-\xi)^2},\quad t\ge t_0>0. \tag{3.24}$$

Consequently,

$$\int\limits_\Omega\big[\chi\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}t;y,\frac{\xi}{\varepsilon}\big)\big]^2\,d\xi\le C.$$

Multiplying equation (3.21) by $W_\varepsilon-V_\varepsilon$, integrating by parts and taking into account (3.7), exponential decay of the boundary layer correctors, (3.22) and (3.24), we obtain

$$\int\limits_\Omega|W_\varepsilon-V_\varepsilon|^2\,d\xi\le C\,\varepsilon^4,\quad t\ge0. \tag{3.25}$$

Note that due to the presence of the boundary layer correctors, the boundary conditions on $\Gamma^+\cap\Gamma^-$ in (3.10) are satisfied up to the second order in ε. It is left to notice that for $t\ge0$ and $x\in\Omega$ such that $x_d\in I\Subset(-1,1)$

$$\int\limits_\Omega|W_\varepsilon(t,x,\xi)-V_1^\varepsilon(t,x,\xi)|^2d\xi\le C\,\varepsilon^4,$$

where V_1^ε is the first-order approximation of V_ε defined by (3.19). Combining the last two estimates yields the desired result. Proposition 3.3 is proved. $\square$

We turn back to the asymptotics of $K_\varepsilon(t,x,\xi)$. Let us recall that

$$K_0(t,x,\xi)=\Phi_0(t,x,\xi)-V_0(t,x,\xi)$$

is the Green function of the effective problem (3.23). The following lemma constitutes the main result of the section.

LEMMA 3.4. *Assume that conditions* $(\mathbf{A1})-(\mathbf{A2})$ *are satisfied. If* K_ε *is a Green function solving* (3.9), *then, for* $t \geq t_0 > 0$ *and* $x \in \Omega$ *such that* $x_d \in I \Subset (-1,1)$, *the following estimate holds:*

$$\int\limits_\Omega |K_\varepsilon(t,x,\xi) - K_1^\varepsilon\big(t,x,\xi+\frac{\beta^\Theta}{\varepsilon}\,t\big)|^2 d\xi \leq C\,\varepsilon^4,$$

where the constant C *depends on* $\operatorname{dist}(x,\Gamma^-\cup\Gamma^+)$ *and is independent of* ε*;* K_1^ε *is a first-order approximation of* K_ε *given by*

$$\begin{aligned} K_1^\varepsilon\big(t,x,\widetilde{\xi}\big) &= K_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\,N\big(\frac{x}{\varepsilon}\big)\,\nabla_x K_0\big(t,x,\widetilde{\xi}\big) \\ &+ \varepsilon\,N\big(\frac{\xi}{\varepsilon}\big)\,\nabla_\xi K_0\big(t,x,\widetilde{\xi}\big) + \varepsilon\,K_1(t,x,\widetilde{\xi}) - \varepsilon\,V_{bl}^\varepsilon(t,x,\widetilde{\xi}), \quad \widetilde{\xi} = \xi + \frac{\beta^\Theta}{\varepsilon}\,t, \end{aligned} \tag{3.26}$$

K_0 *is the Green function of the effective problem* (3.23), N, N^* *are the solutions of* (3.11), (3.12)*, respectively. The effective convection* $\beta_j^\Theta = \int_Y b_j^\Theta(y)dy$ *with* $\beta_d^\Theta = 0$*. The function* $K_1(t,x,\xi)$ *satisfies the problem*

$$\begin{cases} \partial_t K_1(t,x,\xi) = \operatorname{div}_\xi(a^{\text{eff}}\,\nabla_\xi K_1(t,x,\xi)) - F(t,x,\xi), & (t,\xi) \in (0,T)\times\Omega, \\ K_1(t,x,\xi',\pm1) = \hat{v}^\pm\,\partial_{\xi_d} K_0(t,x,\xi',\pm1), & (t,\xi') \in (0,T)\times\mathbb{R}^{d-1}, \\ K_1(0,x,\xi) = 0, & \xi \in \Omega, \end{cases}$$

where

$$\begin{aligned} F(t,x,\xi) = \partial_{\xi_k}\partial_{\xi_i}\partial_{\xi_j} V_0(t,x,\xi) \int\limits_Y \big[a_{kl}^\Theta \partial_{\eta_l} Q_{ij}^* \\ + a_{kj}^\Theta N^* + b_k^\Theta Q_{ij}^* - \beta_k^\Theta \rho_\Theta Q_{ij}^* - a_{ij}^{\text{eff}}\,\rho_\Theta\,N_k^*\big]\,d\eta. \end{aligned}$$

The boundary layer corrector $V_{\text{bl}}^\varepsilon$ *is defined by* (3.17)*.*

3.4. Asymptotics of $u^\varepsilon(t,x)$**.** Recall that v^ε as a solution of (3.4), is represented in terms of the Green function K_ε as follows

$$v^\varepsilon(t,x) = \int\limits_\Omega K_\varepsilon(t,x,\xi)\,u_0(\xi)\,p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\,e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\,d\xi.$$

Obviously,

$$\begin{aligned} v^\varepsilon(t,x) &= \int\limits_\Omega K_1^\varepsilon(t,x,\xi)\,u_0(\xi)\,p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\,e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\,d\xi \\ &+ \int\limits_\Omega (K_\varepsilon - K_1^\varepsilon)\,u_0(\xi)\,p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\,e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\,d\xi, \end{aligned} \tag{3.27}$$

where K_1^ε is the first order approximation of K_ε given by (3.26). Suppose that the initial function is such that $u_0(x',\pm1) \neq 0$. The case $u_0(x',\pm1) = \cdots = \partial_{\xi_d}^{k-1} u_0(x',\pm1) = 0$, $\partial_{\xi_d}^k u_0(x',\pm1) \neq 0$ can be considered similarly. With the help of Lemma 3.4 we estimate the second integral in (3.27). Taking into account the boundedness of p_Θ, and expanding u_0 into Teylor series in the neighbourhood of $\xi_d = -1$, we have

$$
\begin{aligned}
&\Big| \int\limits_{\Omega} (K_\varepsilon - K_1^\varepsilon)\, u_0(\xi)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}\, d\xi \Big| \\
&\le C\,\varepsilon^3 \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, d\xi' \int\limits_0^{+\infty} e^{-\Theta z_d}\, dz_d \le C\,\varepsilon^3.
\end{aligned}
$$

Our next goal is to derive the asymptotics for the first integral in (3.27). Denote

$$
v_0^\varepsilon(\xi) = u_0(\xi)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta\,(\xi_d+1)}{\varepsilon}}.
$$

Then, by (3.26),

$$
\begin{aligned}
I = &\int\limits_\Omega K_1^\varepsilon(t,x,\xi)\, v_0^\varepsilon(\xi)\, d\xi = \int\limits_\Omega K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_\Omega N_j^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_j} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_\Omega v^-\big(\frac{\xi'}{\varepsilon}, \frac{\xi_d+1}{\varepsilon}\big)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_\Omega N_j\big(\frac{x}{\varepsilon}\big)\, \partial_{x_j} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_\Omega \Big(K_1\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big) - \hat v^-\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} \Big)\, v_0^\varepsilon(\xi)\, d\xi \\
&+\varepsilon \int\limits_\Omega \Big(v^+\big(\frac{\xi'}{\varepsilon}, \frac{\xi_d-1}{\varepsilon}\big) - \hat v^+\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d=1} \Big)\, v_0^\varepsilon(\xi)\, d\xi.
\end{aligned}
\tag{3.28}
$$

Notice that $K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big) = K_0\big(t, x, \xi + \frac{\beta^\Theta}{\varepsilon} t\big)$ since $\beta_d^\Theta = 0$ and x' live in $\mathbb{R}^{d-1}$.

Expanding K_0 and u_0 into the Teylor series with respect to ξ_d, for $t \geq t_0 > 0$, we obtain

$$\int\limits_{\Omega} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi$$

$$= \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d = -1} d\xi'$$

$$\times \int\limits_{-1}^{1} (\xi_d + 1)\, p_\Theta^{-1}\big(\frac{\xi}{\varepsilon}\big)\, e^{-\frac{\Theta(\xi_d + 1)}{\varepsilon}} d\xi_d + O(\varepsilon^3)$$

$$= \varepsilon^2 \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d = -1} d\xi'$$

$$\times \int\limits_{0}^{+\infty} z_d\, p_\Theta^{-1}\big(\frac{\xi'}{\varepsilon}, z_d\big)\, e^{-\Theta z_d} dz_d + O(\varepsilon^3).$$

The integral

$$\int\limits_{0}^{+\infty} z_d\, p_\Theta^{-1}\big(\frac{\xi'}{\varepsilon}, z_d\big)\, e^{-\Theta z_d} dz_d$$

is $\varepsilon(0,1]^{d-1}$-periodic with respect to ξ'. By the classical mean-value theorem (see, for example, [**15**]),

$$\int\limits_{\Omega} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi$$

$$= \varepsilon^2 \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d = -1} d\xi'$$

$$\times \int\limits_{(0,1]^{d-1}} \int\limits_{0}^{+\infty} z_d\, p_\Theta^{-1}(z', z_d)\, e^{-\Theta z_d} dz_d\, dz' + O(\varepsilon^3).$$

By similar arguments,

$$\int\limits_{\Omega} N_j^*\big(\frac{\xi}{\varepsilon}\big)\, \partial_{\xi_j} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\, v_0^\varepsilon(\xi)\, d\xi$$

$$= \varepsilon^2 \int\limits_{\mathbb{R}^{d-1}} u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^\Theta}{\varepsilon} t, \xi\big)\Big|_{\xi_d = -1} d\xi'$$

$$\times \int\limits_{(0,1]^{d-1}} \int\limits_{0}^{+\infty} N_d^*(z)\, p_\Theta^{-1}(z)\, e^{-\Theta z_d} dz_d\, dz' + O(\varepsilon^3);$$

and

$$\int\limits_{\Omega} v^-\big(\frac{\xi'}{\varepsilon},\frac{\xi_d+1}{\varepsilon}\big)\,\partial_{\xi_d}K_0\big(t,x-\frac{\beta^\Theta}{\varepsilon}t,\xi\big)\,v_0^\varepsilon(\xi)\,d\xi$$
$$=\varepsilon^2\int\limits_{\mathbb{R}^{d-1}} u_0(\xi',-1)\,\partial_{\xi_d}K_0\big(t,x-\frac{\beta^\Theta}{\varepsilon}t,\xi\big)\Big|_{\xi_d=-1}d\xi'$$
$$\times\int\limits_{(0,1]^{d-1}}\int\limits_0^{+\infty} v^-(z)\,p_\Theta^{-1}(z)\,e^{-\Theta z_d}dz_d\,dz'+O(\varepsilon^3).$$

Noticing that $K_1\big|_{\xi_d=-1}=\hat{v}^-\partial_{\xi_d}K_0\big|_{\xi_d=-1}$, and $\partial_{x_j}K_0\big|_{\xi_d=-1}=0$, one can see that the last two integrals in (3.28) are of order ε^3. We would like to emphasize that, in view of (3.16), the terms containing boundary layer correctors near Γ^+ are negligible.

Finally,

$$v^\varepsilon(t,x)=\varepsilon^2\,M\int\limits_{\mathbb{R}^{d-1}} u_0(\xi',-1)\,\partial_{\xi_d}K_0\big(t,x-\frac{\beta^\Theta}{\varepsilon}t,\xi\big)\Big|_{\xi_d=-1}d\xi'+O(\varepsilon^3),$$

where the constant M is given by

$$M=\int\limits_{(0,1]^{d-1}}\int\limits_0^{+\infty}\big[z_d+N_d^*(z)+v^-(z)\big]\,p_\Theta^{-1}(z)\,e^{-\Theta z_d}dz_d\,dz'.$$

To summarize, we formulate the following theorem.

THEOREM 3.5. *Suppose that conditions* **(A1)-(A4)** *are fulfilled,* $\bar{b}_d\neq 0$ *(see* (3.3)*) and* $u_0(x',-1)\neq 0$*. Then, for* $t\geq t_0>0$*, the asymptotics of the solution* u^ε *of problem* (3.1)*, as* $\varepsilon\to 0$*, takes the form*

$$u^\varepsilon(t,x)=\varepsilon^2\,e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\,e^{\frac{\Theta(x_d+1)}{\varepsilon}}\,p_\Theta\big(\frac{x}{\varepsilon}\big)\,\big[u(t,x-\frac{\beta^\Theta}{\varepsilon}t)+r_\varepsilon(t,x)\big],$$

where $r_\varepsilon(t,x)\leq C\,\varepsilon$*, for* $t\geq t_0>0$ *and* $x\in\Omega$ *such that* $x_d\in I\Subset(-1,1)$*, and the constant* C *depends on* $\mathrm{dist}(x,\Gamma^-\cup\Gamma^+)$.

$$u(t,x-\frac{\beta^\Theta}{\varepsilon}t)=M\int\limits_{\mathbb{R}^{d-1}} u_0(\xi',-1)\,\partial_{\xi_d}K_0\big(t,x-\frac{\beta^\Theta}{\varepsilon}t,\xi\big)\Big|_{\xi_d=-1}d\xi'$$

and the constant M *is given by*

$$M=\int\limits_{(0,1]^{d-1}}\int\limits_0^{+\infty}\big[z_d+N_d^*(z)+v^-(z)\big]\,p_\Theta^{-1}(z)\,e^{-\Theta z_d}dz_d\,dz',$$

$\lambda_1(\Theta),p_\Theta$ *have been defined in Section 3.1;* K_0 *is the Green function of the effective problem* (3.23)*;* N_d^* *is a solution of the cell problem* (3.12)*;* v^- *is defined by* (3.15)*. The* *effective convection* β^{Θ} *is defined by*

$$\beta_j^{\Theta} = \int_Y b_j^{\Theta}(y)dy, \quad \beta_d^{\Theta} = 0,$$

with b_j^{Θ} *given in* (3.6).

REMARK 3.6. In the case $u_0(x', -1) = \cdots = \partial_{\xi_d}^{k-1} u_0(x', -1) = 0$ and $\partial_{x_d}^k u_0(x', -1) \neq 0$ for some k, the asymptotics of u^{ε} takes the form

$$u^{\varepsilon}(t,x) = \varepsilon^{2+k}\, e^{-\frac{\lambda_1(\Theta)t}{\varepsilon^2}}\, e^{\frac{\Theta(x_d+1)}{\varepsilon}}\, p_{\Theta}\big(\frac{x}{\varepsilon}\big)\, \big[u(t, x - \frac{\beta^{\Theta}}{\varepsilon} t) + r_{\varepsilon}(t,x)\big],$$

where $r_{\varepsilon}(t,x) \leq C\,\varepsilon$, for $t \geq t_0 > 0$ and $x \in \Omega$ such that $x_d \in I \Subset (-1,1)$; $C = C(\mathrm{dist}(x, \Gamma^- \cup \Gamma^+))$.

$$u(t, x - \frac{\beta^{\Theta}}{\varepsilon} t) = \frac{M}{k!} \int\limits_{\mathbb{R}^{d-1}} \partial_{x_d}^k u_0(\xi', -1)\, \partial_{\xi_d} K_0\big(t, x - \frac{\beta^{\Theta}}{\varepsilon} t, \xi\big)\Big|_{\xi_d=-1} d\xi'$$

and the constant M is given by

$$M = \int\limits_{(0,1]^{d-1}} \int\limits_0^{+\infty} (z_d)^k \left[z_d + N_d^*(z) + v^-(z)\right] p_{\Theta}^{-1}(z)\, e^{-\Theta z_d} dz_d\, dz'.$$

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Paper F

PAPER F

Homogenization of spectral problem for periodic elliptic operators with sign-changing density function

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ABSTRACT. The work deals with the asymptotic behaviour of spectra of second order self-adjoint elliptic operators with periodic rapidly oscillating coefficients in the case when the density function (the factor on the spectral parameter) changes sign. We study the Dirichlet problem in a regular bounded domain and show that the spectrum of this problem is discrete and consists of two series, one of them tends towards $+\infty$ and another towards $-\infty$. The asymptotic behaviour of positive and negative eigenvalues and the corresponding eigenfunctions depends crucially on whether the average of the weight function is equal to zero or not. We construct the asymptotics of eigenpairs in both cases.

Keywords: Spectral problem, sign-changing density, homogenization.

1. Introduction

The paper focuses on the homogenization of the Dirichlet spectral problem

$$(1.1)\qquad \begin{aligned} -\mathrm{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla u\Big) &= \rho\big(\frac{x}{\varepsilon}\big)\,\lambda\, u \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

stated in a regular bounded domain $\Omega \subset \mathbb{R}^n$ for a second order symmetric uniformly elliptic operator

$$\mathcal{L}^\varepsilon = \mathrm{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla\Big)$$

with periodic rapidly oscillating coefficients, ε being a small positive parameter.

Regarding the density function ρ we assume that it is periodic and changes sign, that is both the sets $\{x \in \Omega \,:\, \rho(x/\varepsilon) < 0\}$ and $\{x \in \Omega \,:\, \rho(x/\varepsilon) > 0\}$ are of positive measure. The last assumption makes the problem under consideration nonstandard.

It is shown that for each $\varepsilon > 0$ the spectrum of (1.1) is discrete and consists of the following two infinite sequences

$$0 < \lambda_1^{\varepsilon,+} \le \lambda_2^{\varepsilon,+} \le \cdots \le \lambda_j^{\varepsilon,+} \le ..., \qquad \lim_{j\to\infty} \lambda_j^{\varepsilon,+} = +\infty,$$

and

$$0 > \lambda_1^{\varepsilon,-} \ge \lambda_2^{\varepsilon,-} \ge \cdots \ge \lambda_j^{\varepsilon,-} \ge ..., \qquad \lim_{j\to\infty} \lambda_j^{\varepsilon,-} = -\infty.$$

The asymptotic behaviour of the eigenpairs depends crucially on whether the average of ρ is positive, or negative, or equal to zero.

Previously, a spectral problem with sign-changing density for the Laplace operator has been considered in [**10**]; in this work the limit behaviour of spectrum has been studied under the assumption that the density consists of a fixed positive part and asymptotically vanishing negative part.

There is a vast literature on homogenization of spectral problems in the case of point-wise positive weight ρ. These problems have been studied in [**7**], [**8**] and then in many other papers. The homogenization of spectral problems in perforated domains has been studied in [**13**] followed by many other works on the subject. The limit behaviour of spectrum of elasticity system in perforated domain has been considered in [**12**]. In [**4**] the authors have generalized the results obtained in [**12**] by making weaker the assumptions on the regularity of the inclusions and external forces.

The spectral problems for locally periodic symmetric second order elliptic operators with large potential have been studied in [**1**]. The work [**2**] dealt with the asymptotic behaviour of spectrum for a periodic symmetric elliptic system with large potential.

As was noticed above, the asymptotic behaviour of the spectrum depends crucially on the sign of the average of ρ. In the present paper we construct the asymptotics in the following cases.

I. If the average of ρ is strictly positive, then the positive eigenvalues and the corresponding eigenfunctions show the same regular limit behaviour as in the case of point-wise positive spectral density. Namely, for any $j \ge 1$ the eigenvalue $\lambda_j^{\varepsilon,+}$ converges, as $\varepsilon \to 0$, to the j-th eigenvalue of the limit spectral problem. The corresponding eigenfunctions converge along subsequences.

II. If the mean value of ρ is equal to zero, then the limit spectral problem generates a quadratic operator pencil, and the eigenvalues $\lambda_j^{\varepsilon,+}$, $j = 1, 2, \ldots$, are of order $1/\varepsilon$ so that the normalized quantities $\varepsilon\lambda_j^{\varepsilon,+}$ converge to eigenvalues of the limit spectral problem.

2. Problem setup

Let Ω be a $C^{2,\delta}$ bounded domain in $\mathbb{R}^d$ with a boundary $\partial\Omega$. We consider the following spectral problem:

$$\begin{cases} \mathcal{L}^\varepsilon u^\varepsilon(x) \equiv -\mathrm{div}\Big(a^\varepsilon(x)\nabla u^\varepsilon(x)\Big) = \rho^\varepsilon(x)\,\lambda^\varepsilon\, u^\varepsilon(x), & x \in \Omega, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where

$$a^\varepsilon(x) = a\Big(\frac{x}{\varepsilon}\Big), \quad \rho^\varepsilon(x) = \rho\Big(\frac{x}{\varepsilon}\Big), \quad \varepsilon > 0.$$

Here $a(y)$ is a symmetric $d \times d$ matrix satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^d a_{ij}(y)\xi_i\xi_j \geq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

for some $\Lambda > 0$. We assume that $a_{ij}(y) \in L^\infty(Y)$ are periodic functions (from this time onward Y denotes the periodicity cell); $\rho(y) \in L^\infty(Y)$ is Y-periodic and changes sign, that is the sets $\{y \in Y : \rho(y) < 0\}$ and $\{y \in Y : \rho(y) > 0\}$ have positive Lebesgue measures. The weak formulation of spectral problem (2.1) reads: to find $\lambda^\varepsilon \in \mathbb{C}$ (eigenvalues) and $u^\varepsilon \in H_0^1(\Omega)$, $u^\varepsilon \neq 0$, (eigenfunctions) such that

$$a_\varepsilon(u^\varepsilon, v) = \lambda^\varepsilon\,(\rho^\varepsilon\, u^\varepsilon, v)_\Omega, \quad v \in H_0^1(\Omega), \tag{2.2}$$

where $a_\varepsilon(u,v) = (a^\varepsilon\nabla u, \nabla v)_\Omega$ is a bilinear quadratic form; $(\cdot,\cdot)_\Omega$ is a scalar product in $L^2(\Omega)$. Since function ρ^ε changes sign, problem (2.2) is not a standard spectral problem, and the existing results on the spectrum of semi-bounded self-adjoint operators with compact resolvent do not apply. To overcome this difficulty, we reduce the studied problem (2.2) to an equivalent spectral problem for a compact self-adjoint operator.

Denote by $\mathcal{H}$ a space $H_0^1(\Omega)$ equipped with the norm

$$\|u\|_{\mathcal{H}} = \langle u, u\rangle = a_\varepsilon(u,u). \tag{2.3}$$

The bilinear form $(\rho^\varepsilon u, v)_\Omega$ on $\mathcal{H}$ defines a bounded linear operator $\mathcal{K}^\varepsilon : \mathcal{H} \to \mathcal{H}$ such that

$$(\rho^\varepsilon u, v)_\Omega = \langle \mathcal{K}^\varepsilon u, v\rangle.$$

By definition, the operator $\mathcal{K}^\varepsilon$ is symmetric and its domain $D(\mathcal{K}^\varepsilon)$ coincides with the whole space $\mathcal{H}$, thus it is self-adjoint. Using the representation of $\mathcal{K}^\varepsilon u$ as a solution of the boundary value problem

$$\begin{cases} -\mathrm{div}\Big(a^\varepsilon(x)\nabla(\mathcal{K}^\varepsilon u(x))\Big) = \rho^\varepsilon(x)\, u(x), & x \in \Omega, \\ \mathcal{K}^\varepsilon u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

and the compactness of the imbedding $H_0^1(\Omega)$ in $L^2(\Omega)$, one can see that $\mathcal{K}^\varepsilon$ is a compact operator.

In terms of the operator $\mathcal{K}^\varepsilon$ problem (2.2) takes the form

$$\mathcal{K}^\varepsilon u^\varepsilon = \mu^\varepsilon\, u^\varepsilon, \quad \mu^\varepsilon = 1/\lambda^\varepsilon. \tag{2.5}$$

REMARK 2.1. Note that in the case $\rho(y) \geq 0$ the operator $\mathcal{K}^\varepsilon$ is positive and its spectrum $\sigma(\mathcal{K}^\varepsilon)$ belongs to the segment $[0, k^\varepsilon] \subset \mathbb{R}$, $k^\varepsilon = \|\mathcal{K}^\varepsilon\|$. Moreover, $\mu^\varepsilon = 0$ belongs to the essential spectrum $\sigma_e(\mathcal{K}^\varepsilon)$.

Recall that the essential spectrum of a self-adjoint operator A is by definition

$$\sigma_e(A) = \sigma_p^\infty(A) \cup \sigma_c(A),$$

where $\sigma_p^\infty(A)$ is a set of eigenvalues of infinite multiplicity and $\sigma_c(A)$ is the continuous spectrum (see, e.g., [**3**]).

The spectrum of the operator $\mathcal{K}^\varepsilon$ is described by the following statement.

LEMMA 2.2. *Let $\rho(y)$ be such that the Lebesgue's measure of the sets where ρ is positive and negative is greater than zero, in other words*

$$|\{y : \ \rho(y) \lesseqgtr 0\}| > 0. \tag{2.6}$$

*Then $\sigma(\mathcal{K}^\varepsilon) \subset [-k^\varepsilon, k^\varepsilon]$, $k^\varepsilon = \|\mathcal{K}^\varepsilon\|$; the point $\mu = 0$ is the only element of the essential spectrum $\sigma_e(\mathcal{K}^\varepsilon)$ (see, for example, [**3**]). Moreover, the discrete spectrum of the operator $\mathcal{K}^\varepsilon$ consists of two infinite monotone sequences*

$$\mu_1^{\varepsilon,+} \geq \mu_2^{\varepsilon,+} \geq \cdots \geq \mu_j^{\varepsilon,+} \geq \cdots \to +0,$$

$$\mu_1^{\varepsilon,-} \leq \mu_2^{\varepsilon,-} \leq \cdots \leq \mu_j^{\varepsilon,-} \leq \cdots \to -0.$$

Proof. Since the operator $\mathcal{K}^\varepsilon$ is compact and self-adjoint, its spectrum $\sigma(\mathcal{K}^\varepsilon)$ is a countable set of points in $\mathbb{R}$ which does not have any accumulation points except maybe for $\mu = 0$. Every nonzero eigenvalue has finite multiplicity.

Let us show that the families $\{\mu_j^{\varepsilon,\pm}\}$ are infinite, and, thus, converge to 0. We make use of the minimum principle (see [**3**]) which implies that the eigenvalues $\mu_j^{\varepsilon,\pm}$ can be found from the formula

$$\mu_j^{\varepsilon,\pm} = \max_{\substack{u\in\mathcal{H},\\ \langle u,u\rangle=1}} \pm\langle \mathcal{K}^\varepsilon\, u, u\rangle,$$

where the minimum is taken over vectors $u \in \mathcal{H}$ which are orthogonal to the linear span of $\{u_k^{\varepsilon,\pm}\}_{k=1}^{j-1}$, and $u_j^{\varepsilon,\pm}$ is a point on the unit sphere in $\mathcal{H}$ at which the minimum is attained. In particular, for the first negative eigenvalue $\mu_1^{\varepsilon,-}$ we have

$$\mu_1^{\varepsilon,-} = \min_{\substack{u\in\mathcal{H},\\ \langle u,u\rangle=1}} \langle \mathcal{K}^\varepsilon\, u, u\rangle = \min_{\langle u,u\rangle=1} (\rho^\varepsilon u, u)_\Omega.$$

Due to our assumption, the measure of the set $M_1^\varepsilon \equiv \{x \in \Omega : \ \rho^\varepsilon(x) < 0\}$ is positive. Denote by $\chi(M_1^\varepsilon)$ the characteristic function of M_1^ε, and let φ be a $C_0^\infty(\mathbb{R}^n)$ function such that $\varphi \geq 0$, $\varphi(x) = 0$ if $|x| \geq 1$; $\int_\Omega \varphi(x)dx = 1$. We set $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$. Then

$\chi_\delta(M_1^\varepsilon) \equiv \chi(M_1^\varepsilon) * \varphi_\delta$ converges to $\chi(M_1^\varepsilon)$ in $L^2(\Omega)$ as $\delta \to 0$. As usually, the sign " $*$ " indicates the convolution of two functions, that is

$$\chi_\delta(M_1^\varepsilon) = \int\limits_{\mathbb{R}^d} \chi(M_1^\varepsilon)(x-z)\,\varphi_\delta(z)\,dz.$$

If we choose a constant c_0 such that

$$c_0^2 \langle \chi_\delta(M_1^\varepsilon), \chi_\delta(M_1^\varepsilon)\rangle = 1,$$

then the function $c_0\chi_\delta(M_1^\varepsilon)$ can be used as a test function in the expression for $\mu_1^{\varepsilon,-}$, and thus

$$\mu_1^{\varepsilon,-} \le c_0^2(\rho^\varepsilon \chi_\delta(M_1^\varepsilon), \chi_\delta(M_1^\varepsilon))_\Omega \underset{\delta\to 0}{\longrightarrow} c_0^2(\rho^\varepsilon \chi(M_1^\varepsilon), \chi(M_1^\varepsilon))_\Omega.$$

Hence, in view of definition of M_1^ε, the last expression implies $\mu_1^{\varepsilon,-} < 0$.

The second eigenvalue can be found from the formula

$$\mu_2^{\varepsilon,-} = \min_{\substack{\langle u,u\rangle=1\\ \langle u,u_1^{\varepsilon,-}\rangle=0}} \langle \mathcal{K}^\varepsilon u, u\rangle = \min_{\substack{\langle u,u\rangle=1\\ \langle u,u_1^{\varepsilon,-}\rangle=0}} (\rho^\varepsilon u, u)_\Omega.$$

It is not difficult to see that the set

$$\Theta = \{w \in H_0^1(\Omega):\ \langle w, u_1^{\varepsilon,-}\rangle = 0,\ (\rho^\varepsilon w, w)_\Omega < 0\}$$

is not empty. Indeed, it suffices to consider two functions in $H_0^1(\Omega)$, say $\psi_1^\varepsilon(x)$ and $\psi_2^\varepsilon(x)$, with disjoint supports such that $(\rho^\varepsilon\psi_1^\varepsilon, \psi_1^\varepsilon)_\Omega < 0$ and $(\rho^\varepsilon\psi_2^\varepsilon, \psi_2^\varepsilon)_\Omega < 0$. Choosing a suitable linear combination $\gamma_1\psi_1^\varepsilon + \gamma_2\psi_2^\varepsilon$, one readily gets an element of Θ. This yields $\mu_2^{\varepsilon,-} < 0$.

In the same way one proves that for any $m \ge 1$ the set

$$\{w \in H_0^1(\Omega):\ (\rho^\varepsilon w, w)_\Omega < 0,\ \langle w, u_k^{\varepsilon,-}\rangle = 0,\ k = 1, ..., m\}$$

is not empty. Thus, we have shown that $\mu_j^{\varepsilon,-} < 0$ for any j.

Due to the compactness of $\mathcal{K}^\varepsilon$, for any $\varepsilon > 0$ we have

$$\lim_{j\to\infty} \mu_j^{\varepsilon,-} = 0.$$

Similar arguments for positive eigenvalues $\mu_j^{\varepsilon,+}$ give the desired statement. Lemma 2.2 is proved. □

Taking into account the relation $\mu_j^{\varepsilon,\pm} = 1/\lambda_j^{\varepsilon,\pm}$, we obtain the following theorem.

THEOREM 2.3. *Under the assumption* (2.6) *the operator* $\mathcal{L}^\varepsilon$ *has a discreet spectrum which consists of two sequences*

$$\begin{aligned} &0 < \lambda_1^{\varepsilon,+} \le \lambda_2^{\varepsilon,+} \le \cdots \le \lambda_j^{\varepsilon,+} \le \cdots \to +\infty,\\ &0 > \lambda_1^{\varepsilon,-} \ge \lambda_2^{\varepsilon,-} \ge \cdots \ge \lambda_j^{\varepsilon,-} \ge \cdots \to -\infty. \end{aligned} \tag{2.7}$$

The corresponding eigenfunctions $u_j^{\varepsilon,\pm}$ *satisfy the orthogonality and normalization condition*

$$\langle u_i^{\varepsilon,\pm}, u_j^{\varepsilon,\pm}\rangle = \delta_{i,j}. \tag{2.8}$$

3. The case $\langle\rho\rangle > 0$

We represent a solution $(\lambda^\varepsilon, u^\varepsilon)$ of problem (2.2) in the form

$$
\begin{aligned}
u^\varepsilon(x) &= u^0(x) + \varepsilon\, N(y)^T\, \nabla u^0(x) + \varepsilon^2 w(x,y) + \cdots\,, \quad y = \frac{x}{\varepsilon}, \\
\lambda^\varepsilon &= \lambda^0 + \cdots\,;
\end{aligned}
\tag{3.1}
$$

here $N(y)$ and $w(x,y)$ are Y-periodic in y. Let us substitute ansätze (3.1) into (2.1) and collect power-like with respect to ε terms. This yields the following problems on the periodicity cell Y:

$$
\begin{cases}
-\mathrm{div}_y(a(y)\nabla_y N_k(y)) = \mathrm{div}_y a_{\cdot k}(y), \quad k = 1, ..., d, \quad y \in Y, \\
N_k \in H^1_\#(Y),
\end{cases}
\tag{3.2}
$$

where $a_{\cdot k}$ is a kth column of the matrix $a(y)$,

$$
\begin{cases}
-\mathrm{div}_y(a(y)\nabla_y w(x,y)) & = \mathrm{div}_x(a(y)\nabla_x u^0(x)) + \lambda^0 \rho(y) u^0(x) \\
 & \quad +\mathrm{div}_x\big[a(y)\nabla_y(N(y)^T\, \nabla_x u^0(x))\big] \\
 & \quad +\mathrm{div}_y\big[a(y)\nabla_x(N(y)^T\, \nabla_x u^0(x))\big], \quad y \in Y, \\
w(x,y) & \quad \text{is periodic in } y.
\end{cases}
\tag{3.3}
$$

If the mean-value of the right-hand side in (3.3) is equal to zero, then a solution of periodic problem (3.3) exists and is unique up to an additive constant (see, for example, Section 1.1 in [**15**]). The compatibility condition in problem (3.3) reads

$$
\begin{aligned}
&\int\limits_Y \big\{\mathrm{div}_x(a(y)\nabla_x u^0(x)) + \lambda^0 \rho(y) u^0(x) \\
&+\mathrm{div}_x\big[a(y)\nabla_y(N(y)^T\, \nabla_x u^0(x))\big] + \\
&+\mathrm{div}_y\big[a(y)\nabla_x(N(y)^T\, \nabla_x u^0(x))\big]\big\}\; dy = 0.
\end{aligned}
$$

From the last equality one derives the following equation for $u^0(x)$:

$$
\begin{cases}
\mathcal{L}^{\mathrm{hom}} u^0(x) \equiv -\mathrm{div}(a^{\mathrm{hom}} \nabla u^0(x)) = \lambda^0\, \langle\rho\rangle\, u^0(x), \quad x \in \Omega, \\
u^0(x) = 0, \quad x \in \partial\Omega,
\end{cases}
\tag{3.4}
$$

where the constant matrix a^{hom} has the form

$$
a^{\mathrm{hom}} = \int\limits_Y \big[a(y) + a(y)\nabla_y N(y)^T\big]\, dy,
\tag{3.5}
$$

in other words,

$$
a^{\mathrm{hom}}_{ij} = \int\limits_Y \big[a_{ij}(y) + a_{ik}(y)\partial_{y_k} N_j(y)\big]\, dy, \quad i,j = 1, ..., d.
$$

This matrix a^{hom} is symmetric and positive definite (see, for instance [**15**]).

In view of Remark 2.1, Dirichlet problem (3.4) has the discreet spectrum

$$0<\lambda_1^{0,+}<\lambda_2^{0,+}\le\cdots\le\lambda_j^{0,+}\le\cdots\to+\infty. \tag{3.6}$$

Note that the first eigenvalue $\lambda_1^{0,+}$ is simple (see, for example, [**6**]). The corresponding eigenfunctions can be chosen to satisfy the orthogonality and normalization condition

$$(a^{\mathrm{hom}}\nabla u_i^{0,+},\nabla u_j^{0,+})_\Omega=\langle\rho\rangle\,|\lambda_i^{0,+}|\,(u_i^{0,+},u_j^{0,+})_\Omega=\delta_{i,j},\quad i,j=1,...,d. \tag{3.7}$$

REMARK 3.1. Since Ω is a $C^{2,\delta}$ domain, then $u_j^{0,+}$ are $C^{2,\delta}(\overline{\Omega})$ functions (see [**6**]). Moreover, in the interior of the domain Ω the eigenfunctions $u_j^{0,+}$ are C^∞ functions (see [**6**]).

The next statement characterizes the asymptotic behaviour of the positive part of the spectrum $\sigma(\mathcal{L}^\varepsilon)$ as $\varepsilon\to0$.

THEOREM 3.2. *Assume that $a_{ij},\rho\in L^\infty(Y)$ are periodic functions, and $\langle\rho\rangle>0$. Then the following statements hold true:*

(1) *Let $\lambda_j^{0,+}$ be an eigenvalue of the limit spectral problem (3.4), and assume that the multiplicity of λ_j^0 is equal to $\varkappa_j^+$, $\varkappa_j^+\ge1$, so that $\lambda_{j-1}^{0,+}<\lambda_j^{0,+}=\lambda_{j+1}^{0,+}=\cdots=\lambda_{j+\varkappa_j^+-1}^{0,+}<\lambda_{j+\varkappa_j^+}^{0,+}$. Then there exist $\varepsilon_j>0$ and a constant c_j such that for $\varkappa$ eigenvalues $\lambda_j^{\varepsilon,+},\cdots,\lambda_{j+\varkappa_j^+-1}^{\varepsilon,+}$ of problem (2.1) and only for them the inequality holds*

$$|\lambda_q^{\varepsilon,+}-\lambda_j^{0,+}|\le c_j\,\varepsilon^{1/2},\quad \varepsilon\in(0,\varepsilon_j).$$

Moreover, for $q\notin\{j,j+1,\dots,j+\varkappa_j^+-1\}$ the inequality holds

$$|\lambda_q^{\varepsilon,+}-\lambda_j^{0,+}|\ge\tilde c_j,\quad \varepsilon\in(0,\varepsilon_j),$$

with some $\tilde c_j>0$.

(2) *There exists an unitary $\varkappa_j^+\times\varkappa_j^+$ matrix β^ε such that*

$$\left\|u_p^{\varepsilon,+}-\sum_{k=j}^{j+\varkappa_j^+-1}\beta_{kp}^\varepsilon\,\widetilde U_k^{\varepsilon,+}\right\|_{H^1(\Omega)}\le C_j\,\varepsilon^{1/2},\quad p=j,\cdots,j+\varkappa_j^+-1, \tag{3.8}$$

where

$$\widetilde U_k^{\varepsilon,\pm}(x)=u_k^{0,+}(x)+\varepsilon\,N^\varepsilon(x)^T\,\nabla u_k^{0,+}(x). \tag{3.9}$$

Here $N^\varepsilon(x)=N(x/\varepsilon)$, the vector-function $N(y)$ is a solution of problem (3.2)*; the eigenfunctions $u_k^{0,+}$ of limit spectral problem* (3.4) *satisfy normalization condition* (3.7)*.*

"Almost eigenfunctions" $\{\widetilde U_k^{\varepsilon,+}\}$ are "almost" orthogonal and normalized in the sense of the following inequality:

$$\left|\langle\widetilde U_k^{\varepsilon,+},\widetilde U_l^{\varepsilon,+}\rangle-\delta_{k,l}\right|\le C\,\varepsilon^{1/2}. \tag{3.10}$$

REMARK 3.3. Since $\lambda_1^{0,+}$ is simple, then by Theorem 3.2, for sufficiently small ε the eigenvalue $\lambda_1^{\varepsilon,+}$ is simple as well.

Theorem 3.2 can be also reformulated as a convergence result.

COROLLARY 3.4. *For the positive eigenvalues* (2.7) *and* (3.6) *the following convergence result holds:*

$$\lambda_j^{\varepsilon,+} \to \lambda_j^{0,+}, \quad \varepsilon \to 0.$$

If $\lambda_j^{0,+}$ is simple, then $\lambda_j^{\varepsilon,+}$ is also simple and the corresponding eigenfunctions satisfy the relations:

- $u_j^{\varepsilon,+} \underset{\varepsilon\to 0}{\longrightarrow} u_j^{0,+}$ *strongly in* $L^2(\Omega)$;
- $u_j^{\varepsilon,+} - \varepsilon (N^\varepsilon)^T \nabla u_j^{0,+} \underset{\varepsilon\to 0}{\longrightarrow} u_j^{0,+}$ *strongly in* $H^1(\Omega)$;
- $a^\varepsilon \nabla u_j^{\varepsilon,+} \underset{\varepsilon\to 0}{\longrightarrow} a^{\rm hom} \nabla u_j^{0,+}$ *weakly in* $L^2(\Omega)$, *where $\langle\cdot\rangle$ denotes the mean value over Y.*

The proofs of Theorem 3.2, as well as Corollary 3.4, are similar to the proofs of the corresponding statements in the case $\langle\rho\rangle = 0$ but a little bit less technical, that is why we omit them here.

REMARK 3.5. Theorem 3.2 also applies to the negative part of the spectrum in the case $\langle\rho\rangle < 0$. Indeed, it suffices to replace ρ with $-\rho$ in (2.1).

4. The asymptotics of spectrum in the case $\langle\rho\rangle = 0$

4.1. Preliminary notes. We begin by estimating $|\lambda_1^{\varepsilon,\pm}|$ from below. Multiplying the equality $\mathcal{L}^\varepsilon u_1^{\varepsilon,+} = \lambda_1^{\varepsilon,+}\rho^\varepsilon u_1^{\varepsilon,+}$ by $u_1^{\varepsilon,+}$ and integrating the result over Ω, we obtain

$$\int\limits_\Omega (\nabla u_1^{\varepsilon,+})^T a^\varepsilon (\nabla u_1^{\varepsilon,+}) dx = \lambda_1^{\varepsilon,+} \int\limits_\Omega \rho^\varepsilon (u_1^{\varepsilon,+})^2\, dx. \tag{4.1}$$

Since $\langle\rho\rangle = 0$, there is a periodic vector-function $J \in (L^\infty_\#(Y))^n \cap (H^1_\#(Y))^n$ such that $\rho(y) = \mathrm{div} J(y)$ and $\langle J\rangle = 0$. This yields

$$\lambda_1^{\varepsilon,+} \int\limits_\Omega \rho^\varepsilon(x)(u_1^{\varepsilon,+}(x))^2\, dx = 2\,\varepsilon\,\lambda_1^{\varepsilon,+} \int\limits_\Omega u_1^{\varepsilon,+}(x)\, J\Big(\frac{x}{\varepsilon}\Big)^T \nabla\, u_1^{\varepsilon,+}(x)\, dx \le$$

$$\le C\,\varepsilon\,\lambda_1^{\varepsilon,+} \int\limits_\Omega |u_1^{\varepsilon,+}(x)|\,|\nabla\, u_1^{\varepsilon,+}(x)|\, dx \le C\,\varepsilon\,\lambda_1^{\varepsilon,+}\, \|\nabla u_1^{\varepsilon,+}\|^2_{L^2(\Omega)}.$$

Combining this inequality with (4.1), we conclude that $\lambda_1^{\varepsilon,+} \ge c\varepsilon^{-1}, \qquad c > 0$. Similarly, $|\lambda_1^{\varepsilon,-}| \ge c\varepsilon^{-1}$.

Let us show that if $\langle\rho\rangle = 0$, then $|\lambda_1^{\varepsilon,\pm}| \le C\varepsilon^{-1}$. To this end we use the variational principle (see, e.g., [**3**]):

$$\lambda_1^{\varepsilon,\pm} = \pm \min_{\substack{v\in\mathcal{H}\\ (\rho^\varepsilon v,v)_\Omega=\pm1}} \int\limits_\Omega \nabla^T v(x)\, a^\varepsilon(x)\, \nabla v(x)\, dx. \tag{4.2}$$

Denote

$$v^\varepsilon(x) = c^\varepsilon \varphi(x)\big[1+\varepsilon\,\rho^\varepsilon(x)\big],$$

with $\varphi\in C_0^\infty(\Omega),\ \varphi\not\equiv 0$. We choose the constant c^ε such that

$$1 = (c^\varepsilon)^2 \int\limits_\Omega \rho^\varepsilon(x)\,(\varphi(x))^2\left[1+2\varepsilon\,\rho^\varepsilon(x)+\varepsilon^2\,(\rho^\varepsilon(x))^2\right]dx.$$

Using again the representation $\rho(y) = \mathrm{div} J(y)$, neglecting the terms of order ε^2, and integrating by parts, we obtain

$$\begin{aligned}1 &= (c^\varepsilon)^2\varepsilon \int\limits_\Omega \left\{\mathrm{div} J\big(\frac{x}{\varepsilon}\big)\,(\varphi(x))^2 + 2(\rho^\varepsilon(x))^2\,(\varphi(x))^2 + O(\varepsilon)\right\}dx = \\ &= -(c^\varepsilon)^2\left\{\varepsilon\int\limits_\Omega \Big(J\big(\frac{x}{\varepsilon}\big)\Big)^T \nabla(\varphi(x))^2\,dx + 2\varepsilon\int\limits_\Omega (\rho^\varepsilon(x))^2\,(\varphi(x))^2\,dx + O(\varepsilon^2)\right\}.\end{aligned} \tag{4.3}$$

Since $\langle J\rangle = 0$, each component of this vector-function admits the representation $J_k = \mathrm{div}\widetilde{J}_{\cdot k}$, where $\widetilde{J}_{\cdot k}$ are periodic vectror-functions. This allows us to integrate by parts in (4.3) once more and to derive

$$1 = (c^\varepsilon)^2\big(2\,\varepsilon\,\langle\rho^2\rangle\,\|\varphi\|^2_{L^2(\Omega)} + O(\varepsilon^2)\big).$$

Therefore,

$$(c^\varepsilon)^2 = \big(2\,\varepsilon\,\langle\rho^2\rangle\,\|\varphi\|^2_{L^2(\Omega)}\big)^{-1} + O(\varepsilon).$$

Taking v^ε as a test function in (4.2), one sees that

$$\lambda_1^{\varepsilon,+} \le \int\limits_\Omega \nabla^T v^\varepsilon(x)\, a^\varepsilon(x)\nabla v^\varepsilon(x)\,dx \le C\,\varepsilon^{-1}.$$

The negative eigenvalue $\lambda_1^{\varepsilon,-}$ can be estimated in the same way.

4.2. Formal asymptotic expansion. Bearing in mind the estimates from the previous subsection, we are looking for a solution of problem (2.1) in the form

$$\begin{gathered} u^\varepsilon(x) = u^0(x) + \varepsilon\, u_1(x,y) + \varepsilon^2\, u_2(x,y) + \cdots, \quad y = \frac{x}{\varepsilon}, \\ \lambda^\varepsilon = \varepsilon^{-1}\nu + \cdots, \end{gathered} \tag{4.4}$$

where ν, $u^0(x)$, $u_1(x,y)$ and $u_2(x,y)$ are to be determined. We suppose that $u_1(x,y)$ and $u_2(x,y)$ are Y-periodic in y. Substituting asymptotic ansätze (4.4) into (2.1) and collecting terms of order ε^{-1}, we obtain the following equation for the unknown function $u_1(x,y)$:

$$\begin{cases} -\mathrm{div}_y\big(a(y)\nabla_y u_1(x,y)\big) = \mathrm{div}_y\big(a(y)\nabla_x u^0(x)\big) + \nu\,\rho(y)\,u^0(x), \quad y\in Y,\\ u_1(x,\cdot)\in H^1_\#(Y). \end{cases}$$

Note that, since $a(y)$ is periodic and $\langle\rho\rangle = 0$, the compatibility condition is satisfied and the last problem has a unique solution of zero average. The specific form of the right-hand side of the equation suggests the following representation for the solution:

$$u_1(x,y) = N(y)^T\,\nabla_x u^0(x) + \nu\,N^0(y)\,u^0(x).$$

Thus, the unknown vector-function N and the function N^0 solve the problems

$$\begin{cases} -\mathrm{div}_y(a(y)\nabla_y N_k(y)) = \mathrm{div}_y a_{\cdot k}(y), \quad k=1,...,d, \quad y\in Y,\\ N_k\in H^1_\#(Y), \end{cases} \tag{4.5}$$

where $a_{\cdot k}$ is a kth column in the matrix $a(y)$,

$$\begin{cases} -\mathrm{div}_y(a(y)\nabla_y N^0(y)) = \rho(y), \quad y\in Y,\\ N^0\in H^1_\#(Y). \end{cases} \tag{4.6}$$

Notice that the compatibility conditions in problems (4.5) and (4.6) are satisfied.

Then, collecting the terms of order ε^0, we get an equation for the function $u_2(x,y)$ on the periodicity cell Y, namely

$$\begin{cases} -\mathrm{div}_y\big(a(y)\nabla_y u_2(x,y)\big) = \mathrm{div}_y\big(a(y)\nabla_x(N(y)^T\nabla_x u^0(x))\big)+\\ +\nu\,\mathrm{div}_y\big(a(y)\nabla_x u^0(x)\,N^0(y)\big) + \mathrm{div}_x\big(a(y)\nabla_y N(y)\,\nabla_x u^0(x)\big)+\\ +\nu\,\mathrm{div}_x\big(a(y)\nabla_y N^0(y)\,u^0(x)\big) + \mathrm{div}_x\big(a(y)\,\nabla_x u^0(x)\big)+\\ +\nu\,\rho(y)\,N(y)^T\,\nabla_x u^0(x) + \nu^2\,\rho(y)\,N^0(y)\,u^0(x), \quad y\in Y,\\ u_2(x,\cdot)\in H^1_\#(Y). \end{cases} \tag{4.7}$$

Owing to the periodicity of the matrix $a(y)$, the vector function $N(y)$ and the function $N^0(y)$, we have

$$\int\limits_Y \mathrm{div}_y\big(a(y)\nabla_x(N(y)^T\nabla_x u^0(x))\big) + \nu\,\mathrm{div}_y\big(a(y)\nabla_x u^0(x)\,N^0(y)\big)\,dy = 0.$$

Thus, the compatibility condition in problem (4.7) reads

$$\begin{aligned} &\mathrm{div}_x\Big\{\int\limits_Y \big(a(y) + a(y)\,\nabla_y N(y)\big)\,dy\,\nabla_x u^0(x)\Big\}\\ &+\nu\,\mathrm{div}_x\Big\{\int\limits_Y a(y)\,\nabla_y N^0(y)\,dy\,u^0(x)\Big\}+\\ &+\nu\int\limits_Y \rho(y)\,N(y)^T\,dy\,\nabla_x u^0(x) + \nu^2\int\limits_Y \rho(y)\,N^0(y)\,dy\,u^0(x) = 0. \end{aligned} \tag{4.8}$$

Let us rearrange (4.8) using equations for N and N^0:

$$\begin{aligned}
&\int_Y \rho(y)\, N(y)^T\, dy = -\int_Y \operatorname{div}_y\big(a(y)\nabla_y N^0(y)\big)\, N(y)^T\, dy \\
&= \int_Y \nabla_y^T N^0(y)\, a(y)\, \nabla_y N(y)^T\, dy,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\operatorname{div}_x\Big\{ \int_Y a(y)\, \nabla_y N^0(y)\, dy\, u^0(x) \Big\} \\
&= -\operatorname{div}_x\Big\{ \textstyle\int_Y \operatorname{div}_y a(y)\, N^0(y)\, dy\, u^0(x) \Big\} = \\
&= \operatorname{div}_x\Big\{ \int_Y \operatorname{div}_y\big(a(y)\nabla_y N(y)^T\big)\, N^0(y)\, dy\, u^0(x) \Big\} = \\
&= -\int_Y \nabla_y^T N^0(y)\, a(y)\, \nabla_y N(y)^T\, dy,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&\nu^2 \int_Y \rho(y)\, N^0(y)\, dy\, u^0(x) = -\nu^2 \int_Y \operatorname{div}_y\big(a(y)\nabla_y N^0(y)\big)\, N^0(y)\, dy\, u^0(x) = \\
&= \nu^2 \int_Y \nabla_y^T N^0(y)\, a(y)\, \nabla_y N^0(y)\, dy\, u^0(x) \equiv \nu^2\, \kappa^2\, u^0(x),
\end{aligned} \tag{4.11}$$

where we have set

$$\kappa^2 = \int_Y \nabla_y^T N^0(y)\, a(y)\, \nabla_y N^0(y)\, dy. \tag{4.12}$$

Notice that the sum of the right-hand sides in (4.9) and (4.10) is equal to zero. Consequently, (4.8) (supplemented with an appropriate boundary condition) takes the form

$$\begin{cases} \mathcal{L}^{\mathrm{hom}} u^0(x) \equiv -\operatorname{div}_x\big(a^{\mathrm{hom}} \nabla_x u^0(x)\big) = \nu^2\, \kappa^2\, u^0(x), & x \in \Omega, \\ u^0(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.13}$$

with a positive definite symmetric homogenized matrix a^{hom} defined by the formula

$$a^{\mathrm{hom}} \equiv \int_Y \big(a(y) + a(y)\, \nabla_y N(y)\big)\, dy. \tag{4.14}$$

Although (4.13) is a spectral problem for a quadratic operator pencil with respect to ν, it is not difficult to characterize its spectrum introducing the new spectral parameter $\tau = \nu^2$. Indeed, the spectrum of (4.13) consists of two sequences

$$\begin{aligned}
&0 < \nu_1^{0,+} < \nu_2^{0,+} \le \cdots \le \nu_j^{0,+} \le \cdots \to +\infty, \\
&0 > \nu_1^{0,-} > \nu_2^{0,-} \ge \cdots \ge \nu_j^{0,-} \ge \cdots \to -\infty.
\end{aligned} \tag{4.15}$$

with $\nu_j^{0,-} = -\nu_j^{0,+}$, $j = 1, 2, \dots$, and with the corresponding eigenfunctions $u_j^{0,+} = u_j^{0,-}$. In what follows, omitting the indices $\pm$, we will denote them u_j^0. The notation $\varkappa_j$ will be used for the multiplicity of $\nu_j^{0,\pm}$.

For the eigenfunctions u_j^0 we choose the following orthogonality and normalization condition:

$$(a^{\mathrm{hom}}\nabla u_i^0, \nabla u_j^0)_\Omega + \nu_i^0\,\nu_i^0\,\kappa^2\,(u_i^0, u_j^0)_\Omega = \delta_{i,j}. \tag{4.16}$$

Although, at first sight, such a choice seems to be odd, it ensures the convergence of energies. It should be noted that the first positive and negative eigenvalues $\nu_1^{0,\pm}$ are simple.

REMARK 4.1. Since Ω is a $C^{2,\delta}$-domain, then, as in the case $\langle\rho\rangle > 0$, u_j^0 are $C^{2,\delta}(\overline{\Omega})$-functions (see [**6**]). Moreover, in the interior of the domain Ω the eigenfunctions u_j^0 are C^∞-functions (see [**6**]).

4.3. Justification procedure in the case $\langle\rho\rangle = 0$. Let $\nu_j^{0,\pm}$ be eigenvalues of the operator pencil (4.13) of multiplicity $\varkappa_j$ that is $\pm\nu_{j-1}^{0,\pm} < \pm\nu_j^{0,\pm} = \pm\nu_{j+1}^{0,\pm} = \cdots = \pm\nu_{j+\varkappa_j-1}^{0,\pm} < \pm\nu_{j+\varkappa_j}^{0,\pm}$, and $\{u_p^0\}$, $p = j, \cdots, j+\varkappa_j-1$, be the eigenfunctions of the limit spectral problem (4.13) corresponding to $\nu_j^{0,\pm}$. We denote $\Omega_\gamma = \{x \in \Omega :\ \mathrm{dist}(x, \partial\Omega) \geq \gamma\}$. Let χ_ε be a cut-off function which is equal to 0 in $\Omega \setminus \Omega_{h\varepsilon}$, $h > \sqrt{d}$, equal to 1 in $\Omega_{2h\varepsilon}$, and is such that

$$0 \leq \chi_\varepsilon(x) \leq 1, \quad |\nabla\chi_\varepsilon(x)| \leq C\varepsilon^{-1}. \tag{4.17}$$

The justification procedure will rely on the lemma about "almost eigenvalues and eigenfunctions" (see, for example, [**14**], [**12**] and [**9**]).

LEMMA 4.2. *Given a self-adjoint operator $\mathcal{K}^\varepsilon : \mathcal{H} \to \mathcal{H}$, let $\nu \in \mathbb{R}$ and $v \in \mathcal{H}$ be such that*

$$\|v\|_{\mathcal{H}} = 1, \quad \delta \equiv \|\mathcal{K}^\varepsilon\, v - \nu\, v\|_{\mathcal{H}} < |\nu|.$$

Then there exists an eigenvalue μ_l^ε of the operator $\mathcal{K}^\varepsilon$ such that

$$|\mu_l^\varepsilon - \nu| \leq \delta.$$

Moreover, for any $\delta_1 \in (\delta, |\nu|)$ there exist coefficients $\{b_j^\varepsilon\} \in \mathbb{R}$ satisfying

$$\|v - \sum b_j^\varepsilon u_j^\varepsilon\|_{\mathcal{H}} \leq 2\frac{\delta}{\delta_1},$$

where the sum is taken over all the eigenvalues of the operator $\mathcal{K}^\varepsilon$ in the segment $[\nu - \delta_1, \nu + \delta_1]$, and $\{u_j^\varepsilon\}$ are the corresponding orthonormalized in $\mathcal{H}$ eigenfunctions. The coefficients b_j^ε are normalized by $\sum |b_j^\varepsilon|^2 = 1$.

For an arbitrary $p \in \{j, \dots, j+\varkappa_j-1\}$, denote

$$U_p^{\varepsilon,\pm}(x) \equiv u_p^0(x) + \varepsilon\,\chi_\varepsilon(x)\,N\Big(\frac{x}{\varepsilon}\Big)^T\,\nabla u_p^0(x) + \varepsilon\,\nu_j^{0,\pm}\,N^0\Big(\frac{x}{\varepsilon}\Big)\,u_p^0(x), \tag{4.18}$$

where $N(y)$ and $N^0(y)$ solve problems (4.5) and (4.6), respectively. The normalized functions $\mathcal{U}_p^{\varepsilon,+} \equiv \|U_p^{\varepsilon,+}\|_{\mathcal{H}}^{-1}\, U_p^{\varepsilon,+}$ and the numbers $\varepsilon(\nu_j^{0,+})^{-1}$ will play the role of $v \in \mathcal{H}$ and $\nu \in \mathbb{R}$ in Lemma 4.2. Notice that ∇u_p^0 need not be equal to zero on the boundary $\partial\Omega$; the cut-off function has been introduced in order to make the approximate solution (4.18) an element of the space $\mathcal{H}$.

We are going to estimate

$$\big\|\mathcal{K}^\varepsilon \mathcal{U}_p^{\varepsilon,\pm} - \varepsilon(\nu_j^{0,\pm})^{-1}\mathcal{U}_p^{\varepsilon,\pm}\big\|_{\mathcal{H}} = \sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}} \big|\langle \mathcal{K}^\varepsilon \mathcal{U}_p^{\varepsilon,\pm} - \varepsilon(\nu_j^{0,\pm})^{-1}\mathcal{U}_p^{\varepsilon,\pm}\,,\, v\rangle\big| =$$

$$= \varepsilon(\nu_j^{0,\pm})^{-1}\,\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1} \sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}} \big|\langle \varepsilon^{-1}\,\nu_j^{0,\pm}\ \mathcal{K}^\varepsilon\, U_p^{\varepsilon,\pm} - U_p^{\varepsilon,\pm}\,, v\,\rangle\big| =$$

$$= \varepsilon(\nu_j^{0,\pm})^{-1}\,\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1} \sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}} \big|a_\varepsilon(U_p^{\varepsilon,\pm}, v) - \varepsilon^{-1}\,\nu_j^{0,\pm}\,\big(\rho^\varepsilon\, U_p^{\varepsilon,\pm}, v\,\big)_\Omega\big| =$$

$$= \varepsilon(\nu_j^{0,\pm})^{-1}\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1} \sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}} \big|\big(\mathcal{L}^\varepsilon\, U_p^{\varepsilon,\pm} - \varepsilon^{-1}\,\nu_j^{0,\pm}\,\rho^\varepsilon\, U_p^{\varepsilon,\pm}, v\,\big)_\Omega\big| =$$

$$= \varepsilon(\nu_j^{0,\pm})^{-1}\,\|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1} \sup_{\substack{v\in\mathcal{H}\\ \langle v,v\rangle=1}} \big|\varepsilon^{-1}(I_1^\varepsilon, v)_\Omega + \varepsilon^0(I_2^\varepsilon, v)_\Omega + \varepsilon^1(I_3^\varepsilon, v)_\Omega\big|,$$

where

$$I_1^\varepsilon(x) = \Big\{ -\operatorname{div}_y\big[a(y)\nabla_y N(y)\,\nabla_x u_p^0(x))\big]\,\chi_\varepsilon(x) - \operatorname{div}_y\big[a(y)\nabla_x u_p^0(x)\big] -$$
$$-\,\nu_j^{0,\pm}\operatorname{div}_y\big[a(y)\nabla_y N^0(y)\,u_p^0(x)\big] - \nu_j^{0,\pm}\,\rho(y)\,u_p^0(x)\Big\}\Big|_{y=x/\varepsilon} =$$
$$= -\operatorname{div}_y a_{\cdot i}(y)\,\partial_{x_i} u_p^0(x)\,\big(1-\chi_\varepsilon(x)\big)\Big|_{y=x/\varepsilon};$$

$$I_2^\varepsilon(x) = \Big\{ -\operatorname{div}_x\big[a(y)\nabla_y N(y)\,\nabla u_p^0(x)\big]\,\chi_\varepsilon(x)$$
$$-\nabla_x^T\chi_\varepsilon(x)\,a(y)\nabla_y N(y)\,\nabla u_p^0(x) -$$
$$-\,\nu_j^{0,\pm}\operatorname{div}_x\big[a(y)\nabla_y N^0(y)\,u_p^0(x)\big] - \operatorname{div}_x\big[a(y)\nabla u_p^0(x)\big] -$$
$$-\,\nu_j^{0,\pm}\rho(y)\,\chi_\varepsilon(x)\,N(y)^T\,\nabla_x u_p^0(x) - (\nu_j^{0,\pm})^2\,\rho(y)\,N^0(y)\,u_p^0(x)\Big\}\Big|_{y=x/\varepsilon};$$

$$I_3^\varepsilon(x) = \Big\{ -\big[\operatorname{div}_x + \varepsilon^{-1}\operatorname{div}_y\big]\,\big(a(y)\nabla_x\big(\chi_\varepsilon(x)N(y)^T\,\nabla u_p^0(x)\big)\big) -$$
$$-\,\nu_j^{0,\pm}\,\big[\operatorname{div}_x + \varepsilon^{-1}\operatorname{div}_y\big]\big(a(y)\,N^0(y)\,\nabla_x u_p^0(x)\big)\Big\}\Big|_{y=x/\varepsilon}.$$

Integrating by parts, and considering the regularity of u_p^0, we obtain

$$\varepsilon^{-1}\left|(I_1^\varepsilon, v)_\Omega\right| = \varepsilon^{-1}\Big|\int\limits_\Omega \varepsilon\,\nabla_y^T\big((1-\chi_\varepsilon(x))\,v(x)\big)\,a_{\cdot i}\big(\frac{x}{\varepsilon}\big)\,\partial_{x_i}u_p^0(x)\,dx\Big| \le$$
$$\le C_1 \int\limits_{\Omega\setminus\Omega_{h\varepsilon}} \big\{|v(x)| + |\nabla_x v(x)|\big\}\,(1-\chi_\varepsilon(x))\,dx$$
$$+C_2 \int\limits_{\Omega\setminus\Omega_{h\varepsilon}} |v(x)|\,|\nabla_x\chi_\varepsilon(x)|\,dx.$$

By (4.17), Lemma 4.3 formulated below, and the Cauchy-Bunyakovsky inequality we get

$$\varepsilon^{-1}\left|(I_1^\varepsilon, v)_\Omega\right| \le C\sqrt{\varepsilon}; \tag{4.19}$$

here we have also used the fact that the measure of $\Omega\setminus\Omega_{h\varepsilon}$ is of order ε. The proofs of the following auxiliary inequalities of Hardy's type can be found, for example, in [**9**].

LEMMA 4.3. *Let $v\in H_0^1(\Omega)$. Then*

$$\|v\|_{L^2(\partial\Omega_\gamma)} \le C\,\sqrt{\gamma}\,\|\nabla v\|_{L^2(\Omega)};$$

$$\|v\|_{L^2(\Omega\setminus\Omega_\gamma)} \le C\,\gamma\,\|\nabla v\|_{L^2(\Omega)}.$$

Denote by $W_\#^{1,\infty}(Y)$ a space of periodic functions v with the norm

$$\|v\|_{W_\#^{1,\infty}(Y)} = \|v\|_{L^\infty(Y)} + \|\nabla v\|_{L^\infty(Y)}.$$

LEMMA 4.4. *Let $\langle g\rangle$ be the mean value of g over the periodicity cell Y, $f\in H^1(\Omega)$ and $g\in L^2(Y)$ (or, alternatively, $f\in W_\#^{1,\infty}(Y)$ and $g\in L^1(Y)$). Then the following inequality is valid:*

$$\left|\int\limits_\Omega \chi_\varepsilon(x)f(x)g\big(\frac{x}{\varepsilon}\big)\,dx - \langle g\rangle\int\limits_\Omega f(x)\,dx\right| \le C\,\varepsilon\|f\|\,\|g\|$$

with the corresponding norms of the functions f and g, the constant C does not depend on ε, f and g.

The proof of Lemma 4.4 can be found, for example, in [**5**] and [**11**].

In order to estimate $(I_2^\varepsilon, v)_\Omega$ we rearrange I_2^ε as follows:

$$
\begin{aligned}
& I_2^\varepsilon(x) = -\Big\{ \mathrm{div}_x\big[a(y)\nabla_y N(y)\,\nabla_x u_p^0(x)\big] + \mathrm{div}_x\big[a(y)\nabla_x u_p^0(x)\big]+ \\
& + \nu_j^{0,\pm}\,\mathrm{div}_x\big[a(y)\nabla_y N^0(y)\,u_p^0(x)\big] + \nu_j^{0,\pm}\rho(y)\,N(y)^T\,\nabla_x u_p^0(x) + \\
& +(\nu_j^{0,\pm})^2\,\rho(y)\,N^0(y)\,u_p^0(x)\Big\}\Big|_{y=x/\varepsilon}\chi_\varepsilon(x)- \\
& -\Big[\nu_j^{0,\pm}\,\mathrm{div}_x\big[a(y)\nabla_y N^0(y)\,u_p^0(x)\big] + \mathrm{div}_x\big[a(y)\nabla_x u_p^0(x)\big] + \\
& + (\nu_j^{0,\pm})^2\,\rho(y)\,N^0(y)\,u_p^0(x)\Big]\Big|_{y=x/\varepsilon}\big(1-\chi_\varepsilon(x)\big) \\
& -\nabla_x^T\chi_\varepsilon(x)\,a(y)\nabla_y N(y)\,\nabla_x u_p^0(x)\Big|_{y=x/\varepsilon} \equiv \\
& \equiv f_1^\varepsilon(x)\,\chi_\varepsilon(x) + f_2^\varepsilon(x)\,(1-\chi_\varepsilon(x)) + f_3^\varepsilon(x);
\end{aligned}
$$

Since the expression in the braces has zero mean (see (4.8)), we, by Lemma 4.4, have

$$
\big|\big(f_1^\varepsilon\,\chi_\varepsilon, v\big)_\Omega\big| \le C\,\varepsilon. \tag{4.20}
$$

Taking into account the boundedness of the coefficients, Remark 4.1, formula (4.17) and the properties of N^0 as a solution of (4.6), one can check up that

$$
\begin{aligned}
& \Big|\big(f_2^\varepsilon\,(1-\chi_\varepsilon), v\big)_\Omega\Big| \le C\,\sqrt{\varepsilon}\,\|\nabla_y N^0\|_{L^2(\Omega\setminus\Omega_{h\varepsilon})}\,\|v\|_{L^2(\Omega\setminus\Omega_{h\varepsilon})}+ \\
& +C|\Omega\setminus\Omega_{h\varepsilon}|^{1/2}\,\|v\|_{L^2(\Omega\setminus\Omega_{h\varepsilon})} \le C\,\varepsilon^{3/2}\|v\|_{H^1(\Omega)}.
\end{aligned}
$$

Here $|\Omega\setminus\Omega_{h\varepsilon}|$ is the Lebesgue measure of the set $\Omega\setminus\Omega_{h\varepsilon}$. By similar arguments, we derive the estimate

$$
\begin{aligned}
& \Big|\big(f_3^\varepsilon\,\nabla\chi_\varepsilon, v\big)_\Omega\Big| \le C\,\varepsilon^{-1}\Big(\int\limits_{\Omega\setminus\Omega_{h\varepsilon}} |\nabla_y N(y)^T|^2\big|_{y=x/\varepsilon}\,dx\Big)^{1/2}\,\|v\|_{L^2(\Omega\setminus\Omega_{h\varepsilon})} \\
& \le C\,\sqrt{\varepsilon}\,\|v\|_{H^1(\Omega)}.
\end{aligned}
$$

Consequently,

$$
|(I_2^\varepsilon, v)_\Omega| \le C\,\sqrt{\varepsilon}. \tag{4.21}
$$

In view of (4.17) integrating by parts yields

$$
\begin{aligned}
&|\varepsilon\,(I_3^\varepsilon, v)_\Omega| \le \varepsilon\left|\int\limits_\Omega \nabla^T v(x)\,a^\varepsilon(x)\,\nabla\chi_\varepsilon(x)\,N^T\big(\frac{x}{\varepsilon}\big)\,\nabla u_p^0(x)\,dx\right| +\\
&+\varepsilon\left|\int\limits_\Omega \nabla^T v(x)\,a^\varepsilon(x)\,\nabla_x(N(y)^T\,\nabla u_p^0(x))\big|_{y=x/\varepsilon}\,\chi_\varepsilon(x)\,dx\right| +\\
&+\varepsilon\left|\int\limits_\Omega \nabla^T v(x)\,a^\varepsilon(x)\,\nabla u_p^0(x)\,N^0\big(\frac{x}{\varepsilon}\big)dx\right| \le\\
&\le C\,|\Omega\setminus\Omega_{h\varepsilon}|^{1/2}\|\nabla v\|_{L^2(\Omega)} + C\varepsilon\|\nabla v\|_{L^2(\Omega)}.
\end{aligned}
$$

Finally,

$$
|\varepsilon\,(I_3^\varepsilon, v)_\Omega| \le C\,\sqrt{\varepsilon}. \tag{4.22}
$$

LEMMA 4.5. *"Almost eigenfunctions"* $\mathcal{U}_p^{\varepsilon,\pm} = \|U_p^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\,U_p^{\varepsilon,\pm}$, $p = j,\cdots,j+\varkappa_j-1$, *where* $U_p^{\varepsilon,\pm}$ *is defined by* (4.18), *are "almost orthonormal". Namely, the following inequality holds true:*

$$
\big|\langle\mathcal{U}_p^{\varepsilon,\pm},\mathcal{U}_q^{\varepsilon,\pm}\rangle - \delta_{p,q}\big| \le C\,\varepsilon, \quad p,q = j,\cdots,j+\varkappa_j-1. \tag{4.23}
$$

Proof. 1. First, we calculate the gradient of the function $U_p^{\varepsilon,\pm}$:

$$
\begin{aligned}
\nabla U_p^{\varepsilon,\pm}(x) = \Big\{&\nabla_x u_p^0(x) + \nabla_y N(y)\,\nabla_x u_p^0(x)\\
&+\nu_j^{0,\pm}\nabla_y N^0(y)\,u_p^0(x)\Big\}\big|_{y=x/\varepsilon}\,\chi_\varepsilon(x)+\\
&+\varepsilon\Big\{\nabla_x(N(y)^T\nabla_x u_p^0(x)) + \nu_j^{0,\pm}\nabla_y N^0(y)\,u_p^0(x)\Big\}\big|_{y=x/\varepsilon}\,\chi_\varepsilon(x)+\\
&+\Big[\Big\{\nabla_x u_p^0(x) + \nu_j^{0,\pm}\nabla_y N^0(y)\,u_p^0(x)\Big\}(1-\chi_\varepsilon(x))\\
&+\varepsilon\,N(y)^T\nabla_x u_p^0(x)\,\nabla_x\chi_\varepsilon(x)\Big]\big|_{y=x/\varepsilon} \equiv\\
&\equiv \chi_\varepsilon(x)\,J_{1p}^\varepsilon(x) + \varepsilon\chi_\varepsilon(x)J_{2p}^\varepsilon(x) + J_{3p}^\varepsilon(x).
\end{aligned}
$$

Then $\langle U_p^{\varepsilon,\pm}, U_q^{\varepsilon,\pm}\rangle$ takes the form

$$
\begin{aligned}
&(a^\varepsilon\,\nabla U_p^{\varepsilon,\pm}, \nabla U_q^{\varepsilon,\pm})_\Omega = (a^\varepsilon\chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon)_\Omega + \varepsilon(a^\varepsilon\chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{2q}^\varepsilon)_\Omega\\
&+(a^\varepsilon\chi_\varepsilon J_{1p}^\varepsilon, J_{3q}^\varepsilon)_\Omega + \varepsilon(a^\varepsilon\chi_\varepsilon J_{2p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon)_\Omega + \varepsilon^2(a^\varepsilon\chi_\varepsilon J_{2p}^\varepsilon, \chi_\varepsilon J_{2q}^\varepsilon)_\Omega\\
&+\varepsilon(a^\varepsilon\chi_\varepsilon J_{2p}^\varepsilon, J_{3q}^\varepsilon)_\Omega + (a^\varepsilon J_{3p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon)_\Omega + \varepsilon(a^\varepsilon J_{3p}^\varepsilon, \chi_\varepsilon J_{2q}^\varepsilon)_\Omega\\
&+(a^\varepsilon J_{3p}^\varepsilon, J_{3q}^\varepsilon)_\Omega.
\end{aligned}
$$

2. Let us proceed with proving that

$$
\big|(a^\varepsilon\chi_\varepsilon J_{1p}^\varepsilon, \chi_\varepsilon J_{1q}^\varepsilon)_\Omega - \delta_{p,q}\big| \le C\,\varepsilon. \tag{4.24}
$$

We have

$$
\begin{aligned}
(a^\varepsilon \chi_\varepsilon J^\varepsilon_{1p}, \chi_\varepsilon J^\varepsilon_{1q})_\Omega &= \int\limits_\Omega \chi^2_\varepsilon(x)\, \nabla^T_x u^0_p \left[a + (\nabla_y N^T)^T\, a\right]\Big|_{y=x/\varepsilon} \nabla_x u^0_p\, dx+ \\
&+ \int\limits_\Omega \chi^2_\varepsilon(x)\, \nu_j^{0,\pm} \left\{\nabla^T_y N^0(y)\, a(y) + \nabla^T_y N^0(y)\, a(y)\, \nabla_y N(y)^T\right\}\Big|_{y=x/\varepsilon} \\
&\quad \times u^0_q(x)\, \nabla_x u^0_p(x)\, dx+ \\
&+ \int\limits_\Omega \chi^2_\varepsilon(x)\, \nabla^T_x u^0_q(x) \left\{a(y)\, \nabla_y N(y)^T\right. \\
&\left.+(\nabla_y N(y)^T)^T\, a(y)\, \nabla_y N(y)^T\right\}\Big|_{y=x/\varepsilon} \nabla_x u^0_p(x)\, dx+ \\
&+ \int\limits_\Omega \chi^2_\varepsilon(x)\, \nu_j^{0,\pm}\, \nabla^T_x u^0_q(x) \left\{a(y)\, \nabla^T_y N^0(y)\right. \\
&\quad \left.+\nabla^T_y N^0(y)\, a(y)\, \nabla_y N(y)^T\right\}\Big|_{y=x/\varepsilon} u^0_p(x)\, dx+ \\
&\quad + \int\limits_\Omega \chi^2_\varepsilon(x)\, (\nu_j^{0,\pm})^2\, \nabla^T_y N^0(y)\, a(y)\, \nabla_y N^0(y)\Big|_{y=x/\varepsilon} u^0_p(x)\, u^0_q(x)\, dx.
\end{aligned}
$$

Notice that the mean value of the expression in square brackets coincides with the homogenized matrix (see (4.14)). Integrating by parts one can show that all expressions in braces have zero mean, and, by definition,

$$\int\limits_Y \nabla^T_y N^0(y)\, a(y)\, \nabla_y N^0(y)\, dy \equiv \kappa^2.$$

Thus, by Lemma 4.4,

$$\left|(a^\varepsilon\, \chi_\varepsilon J^\varepsilon_{1p}, \chi_\varepsilon J^\varepsilon_{1q})_\Omega - (a^{\text{hom}}\, \nabla u^0_p, \nabla u^0_q)_\Omega - (\nu_j^{0,\pm})^2\, \kappa^2\, (u^0_p, u^0_q)_\Omega\right| \le C\, \varepsilon.$$

Taking into account the orthogonality and normalization condition (4.16), we obtain

$$\left|(a^\varepsilon\, \chi_\varepsilon J^\varepsilon_{1p}, \chi_\varepsilon J^\varepsilon_{1q})_\Omega - \delta_{p,q}\right| \le C\, \varepsilon.$$

In particular, the $L^2(\Omega)$-norm of $\chi_\varepsilon J^\varepsilon_{1p}$ is bounded for a small $\varepsilon > 0$:

$$\|\chi_\varepsilon J^\varepsilon_{1p}\|^2_{L^2(\Omega)} \le 1 + C\,\varepsilon \le \tilde{C}. \tag{4.25}$$

3. At this step we show that

$$\left|(a^\varepsilon\, \nabla U^{\varepsilon,\pm}_p, \nabla U^{\varepsilon,\pm}_q)_\Omega - (a^\varepsilon \chi_\varepsilon J^\varepsilon_{1p}, \chi_\varepsilon J^\varepsilon_{1q})_\Omega\right| \le C\,\varepsilon. \tag{4.26}$$

Combining (4.25) with the evident estimate

$$\left|\varepsilon^2 \int\limits_\Omega \chi^2_\varepsilon(x)\, (J^\varepsilon_{2p}(x))^2\, dx\right| \le C\varepsilon^2, \tag{4.27}$$

we obtain

$$2\varepsilon\left|(a^\varepsilon\,\chi_\varepsilon\,J^\varepsilon_{1p},\chi_\varepsilon\,J^\varepsilon_{2q})_\Omega\right|\leq C\,\varepsilon. \tag{4.28}$$

From (4.17), Remark 4.1 and the bound $|\Omega\setminus\Omega_{h\varepsilon}|\leq c\varepsilon$, it follows that

$$\begin{aligned}\|J^\varepsilon_{3p}\|^2_{L^2(\Omega)}\leq&\int_\Omega(1-\chi_\varepsilon(x))^2\,(|\nabla_x u^0_p(x)|^2\\&+\nu_j^{0,\pm}|\nabla_y N^0(y)|^2\Big|_{y=x/\varepsilon}(u^0_p(x))^2)\,dx\\&+\varepsilon^2\int_\Omega|\nabla_x\chi_\varepsilon(x)|^2\,|N(x/\varepsilon)^T\,\nabla_x u^0_p(x)|^2\,dx\\&\leq C(|\Omega\setminus\Omega_{h\varepsilon}|+\varepsilon+\varepsilon^2\,\varepsilon^{-2}|\Omega\setminus\Omega_{h\varepsilon}|)\leq C\,\varepsilon.\end{aligned}$$

The last estimate together with (4.27) gives

$$\varepsilon\left|(a^\varepsilon\,\chi_\varepsilon\,J^\varepsilon_{2p},J^\varepsilon_{3q})_\Omega\right|\leq C\,\varepsilon^{3/2}. \tag{4.29}$$

As regards to the term $(a^\varepsilon\chi_\varepsilon J^\varepsilon_{1p},J^\varepsilon_{3q})_\Omega$, it is not difficult to show that

$$\left|(a^\varepsilon\chi_\varepsilon J^\varepsilon_{1p},J^\varepsilon_{3q})_\Omega\right|\leq C\varepsilon. \tag{4.30}$$

Now (4.27)–(4.30) imply (4.26) which, in turn, together with (4.24) leads to the inequality

$$\left|(a^\varepsilon\nabla U^{\varepsilon,\pm}_p,\nabla U^{\varepsilon,\pm}_q)_\Omega-\delta_{p,q}\right|\leq C\,\varepsilon. \tag{4.31}$$

In particular, the last estimate yields

$$\|U^{\varepsilon,\pm}_p\|^2_{\mathcal{H}}\geq\frac{1}{2},\quad\varepsilon\in(0,\varepsilon_0). \tag{4.32}$$

Since $\mathcal{U}^{\varepsilon,\pm}_p=\|U^{\varepsilon,\pm}_p\|^{-1}_{\mathcal{H}}\;U^{\varepsilon,\pm}_p$, we have

$$\begin{aligned}\left|\langle\mathcal{U}^{\varepsilon,\pm}_p,\mathcal{U}^{\varepsilon,\pm}_q\rangle-\delta_{p,q}\right|&=\|U^{\varepsilon,\pm}_p\|^{-2}_{\mathcal{H}}\left|\langle U^{\varepsilon,\pm}_p,U^{\varepsilon,\pm}_q\rangle-\|U^{\varepsilon,\pm}_p\|^2_{\mathcal{H}}\,\delta_{p,q}\right|\leq\\&\leq\|U^{\varepsilon,\pm}_p\|^{-2}_{\mathcal{H}}\left|\langle U^{\varepsilon,\pm}_p,U^{\varepsilon,\pm}_q\rangle-\delta_{p,q}\right|+\|U^{\varepsilon,\pm}_p\|^{-2}_{\mathcal{H}}\,\delta_{p,q}\left|1-\|U^{\varepsilon,\pm}_p\|^{-2}_{\mathcal{H}}\right|.\end{aligned}$$

The last inequality, (4.31) and (4.32) result in (4.23). Lemma is proved. □

Taking into account (4.19), (4.21), (4.22) and (4.32), we obtain the estimate

$$\left\|\mathcal{K}^\varepsilon\,\mathcal{U}^{\varepsilon,\pm}_p-\varepsilon(\nu_j^{0,\pm})^{-1}\,\mathcal{U}^{\varepsilon,\pm}_p\right\|_{\mathcal{H}}\leq C\,\varepsilon^{3/2}. \tag{4.33}$$

By Lemma 4.2, there exists an eigenvalue $\mu^{\varepsilon,\pm}_{q_j}$ of the operator $\mathcal{K}^\varepsilon$, where q_j might depend on ε, such that

$$|\mu^{\varepsilon,\pm}_{q_j}-\varepsilon(\nu_j^{0,\pm})^{-1}|\leq C\,\varepsilon^{3/2}. \tag{4.34}$$

Since $\lambda^{\varepsilon,\pm}_{q_j}=(\mu^{\varepsilon,\pm}_{q_j})^{-1}$, there exist $\varepsilon_j>0$ and a constant c_j such that

$$|\lambda^{\varepsilon,\pm}_{q_j}-\varepsilon^{-1}\nu_j^{0,\pm}|\leq c_j\,\varepsilon^{-1/2},\quad\varepsilon\in(0,\varepsilon_j).$$

Moreover, letting δ_1 in Lemma 4.2 be equal to $\Theta_j\,\varepsilon^{3/2}$ (the constant Θ_j will be chosen below), we conclude that there exists a $K_j(\varepsilon)\times\varkappa_j$ constant matrix α^ε such that

$$\Big\|\mathcal{U}_p^{\varepsilon,\pm}-\sum_{k=J_j}^{J_j+K_J(\varepsilon)-1}\alpha_{kp}^\varepsilon\,u_k^{\varepsilon,\pm}\Big\|_{\mathcal{H}}\le 2\frac{C\varepsilon^{3/2}}{\delta_1}\le C_j\Theta_j^{-1},\quad p=j,\cdots,j+\varkappa_j-1,$$

here $\mu_{J_j(\varepsilon)}^{\varepsilon,\pm},\ldots,\mu_{J_j(\varepsilon)+K_j(\varepsilon)-1}^{\varepsilon,\pm}$ are all the eigenvalues of the operator $\mathcal{K}^\varepsilon$ which satisfy the estimate

$$|\mu_k^{\varepsilon,\pm}-\varepsilon(\nu_j^{0,\pm})^{-1}|\le\Theta_j\,\varepsilon^{3/2}. \tag{4.35}$$

Since the eigenvalues $\nu_j^{0,\pm}$ do not depend on ε, one can choose the constants $\varepsilon_j>0$ such that the intervals $(\varepsilon(\nu_j^{0,\pm})^{-1}-\Theta_j\varepsilon^{3/2}\,,\,\varepsilon(\nu_j^{0,\pm})^{-1}+\Theta_j\varepsilon^{3/2})$ and $(\varepsilon(\nu_k^{0,\pm})^{-1}-\Theta_k\varepsilon^{3/2}\,,\,\varepsilon(\nu_k^{0,\pm})^{-1}+\Theta_k\varepsilon^{3/2})$ do not intersect under the condition $\nu_j^{0,\pm}\neq\nu_k^{0,\pm}$ and $\varepsilon<\min\{\varepsilon_j,\varepsilon_k\}$. Then the eigenvalue sets $\{\mu_k^{\varepsilon,\pm}\}$ related to different $\nu_j^{0,\pm}$ in (4.35) do not intersect for a small ε.

Thus, we conclude that, for any $\nu_j^{0,\pm}$ of multiplicity $\varkappa_j$, there exist $K_J(\varepsilon)$ eigenvalues $\lambda_k^{\varepsilon,\pm}$ of problem (2.1) such that

$$|\varepsilon\,\lambda_k^{\varepsilon,\pm}-\nu_j^{0,\pm}|\le\Theta_j\,\varepsilon^{1/2},\quad \varepsilon\in(0,\varepsilon_j), \tag{4.36}$$

and the functions $\mathcal{U}_p^{\varepsilon\pm}$ admit the approximation

$$\Big\|\mathcal{U}_p^{\varepsilon,\pm}-\sum_{k=J_j(\varepsilon)}^{J_j(\varepsilon)+K_J(\varepsilon)-1}\alpha_{kp}^\varepsilon\,u_k^{\varepsilon,\pm}\Big\|_{\mathcal{H}}\le C_j\,\Theta_j^{-1},\quad p=j,\cdots,j+\varkappa_j-1, \tag{4.37}$$

Denote $J(j)=\min\{i\in\mathbb{Z}^+\,:\,\nu_i^{0,\pm}=\nu_j^{0,\pm}\}$. The main result of this section is given in the following theorem.

THEOREM 4.6. *Assume that $a_{ij},\rho\in L^\infty(Y)$ are periodic functions, and the function ρ has zero mean. Let $\nu_j^{0,\pm}$ be an eigenvalue of the Dirichlet problem* (4.13) *of multiplicity $\varkappa_j$. Then the following statements hold true:*

(1) *For each $j=1,2,\ldots,$ there exist $\varepsilon_j>0$ and a constant c_j such that only the eigenvalues $\lambda_{J(j)}^{\varepsilon,\pm},\cdots,\lambda_{J(j)+\varkappa_j-1}^{\varepsilon,\pm}$ of problem* (2.1) *satisfy the inequality*

$$|\varepsilon\,\lambda_q^{\varepsilon,\pm}-\nu_j^{0,\pm}|\le c_j\,\varepsilon^{1/2},\quad \varepsilon\in(0,\varepsilon_j).$$

(2) *There exists a unitary $\varkappa_j\times\varkappa_j$ matrix β^ε such that*

$$\Big\|u_p^{\varepsilon,\pm}-\sum_{k=J(j)}^{J(j)+\varkappa_j-1}\beta_{kp}^\varepsilon\,\widetilde{U}_k^{\varepsilon,\pm}\Big\|_{H^1(\Omega)}\le C_j\,\varepsilon^{1/2}, \tag{4.38}$$

where $p=J(j),\cdots,J(j)+\varkappa_j-1$ and

$$\widetilde{U}_k^{\varepsilon,\pm}(x)=u_k^0(x)+\varepsilon\,N\big(\frac{x}{\varepsilon}\big)^T\,\nabla u_k^0(x)+\varepsilon\,\nu_j^{0,\pm}\,N^0\big(\frac{x}{\varepsilon}\big)\,u_k^0(x). \tag{4.39}$$

Here the functions N, N^0 solve problems (4.5) *and* (4.6)*, respectively; eigenfunctions u_k^0 of the limit problem* (4.13) *satisfy the orthogonality and normalization condition* (4.16).

"Almost eigenfunctions" $\{\widetilde{U}_k^{\varepsilon,\pm}\}$ are "almost" orthogonal and normalized in the following sense:

$$\Big|\langle \widetilde{U}_k^{\varepsilon,\pm}, \widetilde{U}_l^{\varepsilon,\pm}\rangle - \delta_{k,l}\Big| \le C\,\varepsilon^{1/2}. \tag{4.40}$$

REMARK 4.7. Since both $\nu_1^{0,+}$ and $\nu_1^{0,-}$ are simple, for $\varepsilon \in (0,\varepsilon_1)$, eigenvalues $\lambda_1^{\varepsilon,\pm}$ are simple owing to Theorem 4.6.

Proof of Theorem 4.6. The proof consists of the several steps. First, we show that columns of the matrix α^ε are "almost" orthonormal, and from this deduce that $K_J(\varepsilon) \ge \varkappa_j$. Then we prove that $J_j(\varepsilon) = J(j)$ and $K_J(\varepsilon) = \varkappa_j$. Finally, using properties of the matrix α^ε we derive (4.38). 1. A simple transformation gives

$$\langle \mathcal{U}_p^{\varepsilon,\pm}, \mathcal{U}_q^{\varepsilon,\pm}\rangle = \Big\langle \mathcal{U}_p^{\varepsilon,\pm} - \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon u_k^{\varepsilon,\pm}\,, \mathcal{U}_q^{\varepsilon,\pm}\Big\rangle$$
$$+\Big\langle \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon u_k^{\varepsilon,\pm}\,, \mathcal{U}_q^{\varepsilon,\pm} - \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kq}^\varepsilon u_k^{\varepsilon,\pm}\Big\rangle + \sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon\,\alpha_{kq}^\varepsilon.$$

Taking estimates (4.23) and (4.37) into account, we obtain

$$\Big|\sum_{k=J_j}^{J_j+K_J(\varepsilon)-1} \alpha_{kp}^\varepsilon\,\alpha_{kq}^\varepsilon - \delta_{p,q}\Big| \le C\,\Theta_j^{-1},$$

and, in other words,

$$\big|(\alpha_{\cdot p}^\varepsilon)^T\,\alpha_{\cdot q}^\varepsilon - \delta_{p,q}\big| \le C\,\Theta_j^{-1}, \tag{4.41}$$

where $\alpha_{\cdot p}^\varepsilon$ denotes a pth column in the matrix α^ε, and $p, q = J(j), \cdots, J(j)+\varkappa_j-1$. The last inequality means that the vectors $\{\alpha_{\cdot p}^\varepsilon\}_{p=J(j)}^{J(j)+\varkappa_j-1}$ are asymptotically orthonormal. This property implies the linear independence of the vectors $\{\alpha_{\cdot p}^\varepsilon\}$. Indeed, assume that $\{\alpha_{\cdot p}^\varepsilon\}_{p=J(j)}^{J(j)+\varkappa_j-1}$ are not linearly independent. Then there exist constants $c_{J(j)}, \cdots, c_{J(j)+\varkappa_j-1}$ such that

$$\sum_{k=J(j)}^{J(j)+\varkappa_j-1} c_k\,\alpha_{\cdot k}^\varepsilon = 0.$$

Without loss of generality we assume that $c_{J(j)} = 1 \ge \max_k |c_k|$. Then

$$\alpha_{\cdot,J(j)}^\varepsilon + \sum_{k>J(j)} c_k\,\alpha_{\cdot k}^\varepsilon = 0.$$

Multiplying the last equality by $\alpha_{\cdot,J(j)}^\varepsilon$ and using (4.41) we obtain the inequality

$$\big|(\alpha_{\cdot,J(j)}^\varepsilon)^T\,\alpha_{\cdot,J(j)}^\varepsilon\big| \le C_j\,\Theta_j^{-1},$$

that contradicts (4.41) if Θ_j is small. Thus, the vectors $\{\alpha^\varepsilon_{\cdot p}\}_{p=J(j)}^{J(j)+\varkappa_j-1}$ of length $K_J(\varepsilon)$ are linearly independent. Obviously, it is possible only in the case $K_J(\varepsilon)\geq\varkappa_j$.

2. Our next goal is to prove that any accumulating point of the sequence $\varepsilon\lambda_j^{\varepsilon,\pm}$, as $\varepsilon\to 0$, is an eigenvalue of problem (4.13).

LEMMA 4.8. *Assume that, for an infinitesimal positive sequence $\{\varepsilon_k\}$, there exists a sequence $\{j(k)\}$ such that*

$$\varepsilon_k\lambda_{j(k)}^{\varepsilon_k,+}\underset{k\to\infty}{\longrightarrow}\beta\quad\textit{or}\quad\varepsilon_k\lambda_{j(k)}^{\varepsilon_k,-}\underset{k\to\infty}{\longrightarrow}\beta.$$

Then β is an eigenvalue of the limit problem (4.13).

Furthermore, for any j, perhaps along a subsequence, $\varepsilon\lambda_j^{\varepsilon,\pm}$ does converge to an eigenvalue of the Dirichlet problem (4.13).

Proof of Lemma 4.8. Note first that due to the inequality $K_j(\varepsilon)\geq\varkappa_j$ and (4.36), the sequence $\{\varepsilon\lambda_j^{\varepsilon,\pm}\}$ is bounded for any j. Therefore, the second statement of Lemma follows from the first one.

Since the eigenpair $\{\lambda_{j(k)}^{\varepsilon_k,\pm},u_{j(k)}^{\varepsilon_k,\pm}\}$ solves problem (2.2), integrating by parts yields

$$\left(u_{j(k)}^{\varepsilon_k,\pm},\mathcal{L}^{\varepsilon_k}V-\lambda_{j(k)}^{\varepsilon_k,\pm}\rho^{\varepsilon_k}V\right)_\Omega=0,\quad V\in H_0^1(\Omega).\tag{4.42}$$

In view of the normalization condition (2.6), up to a subsequence, $u_{j(k)}^{\varepsilon_k,\pm}$ converges weakly in $H_0^1(\Omega)$ to some function $\bar u^\pm$:

$$u_{j(k)}^{\varepsilon_k,\pm}\to\bar u^\pm\quad\text{weakly in } H_0^1(\Omega),\ \varepsilon_k\to 0.\tag{4.43}$$

In order to show that β is an eigenvalue of problem (4.13), for any $v\in C_0^\infty(\Omega)$ we substitute into (4.42) a test function in the form

$$V^\varepsilon(x)\equiv v(x)+\varepsilon\,N\Big(\frac{x}{\varepsilon}\Big)^T\nabla_x v(x)+\varepsilon^2\,\lambda_j^{\varepsilon,\pm}\,N^0\Big(\frac{x}{\varepsilon}\Big)\,v(x)$$

where N and N^0 solve problems (4.5) and (4.6), respectively. Let us calculate the expression $\mathcal{L}^\varepsilon V^\varepsilon-\lambda_j^{\varepsilon,\pm}\rho^\varepsilon V^\varepsilon$:

$$\begin{aligned}
&\mathcal{L}^\varepsilon V^\varepsilon(x)-\lambda_j^{\varepsilon,\pm}\rho^\varepsilon(x)V^\varepsilon(x)\\
&=\Big\{-\operatorname{div}_x\big(a(y)\nabla_x v(x)\big)-\operatorname{div}_x\big(a(y)\,\nabla_y N(y)\,\nabla_x v(x)\big)\\
&-\varepsilon\,\lambda_j^{\varepsilon,\pm}\operatorname{div}_x\big(a(y)\nabla_y N^0(y)\,v(x)\big)-\varepsilon\,\lambda_j^{\varepsilon,\pm}\,\rho(y)\,N(y)^T\,\nabla_x v(x)\\
&-(\varepsilon\,\lambda_j^{\varepsilon,\pm})^2\rho(y)\,N^0(y)\,v(x)\Big\}\Big|_{y=x/\varepsilon}\\
&-\varepsilon\,[\operatorname{div}_x+\varepsilon^{-1}\operatorname{div}_y]\big(a(y)\nabla_x(N(y)^T\,\nabla_x v(x))\big)\Big|_{y=x/\varepsilon}\\
&-\varepsilon^2\lambda_j^{\varepsilon,\pm}\,[\operatorname{div}_x+\varepsilon^{-1}\operatorname{div}_y]\big(a(y)\,N^0(y)\,\nabla_x v(x))\big)\Big|_{y=x/\varepsilon}\\
&\equiv I_1^\varepsilon(x,y)\Big|_{y=x/\varepsilon}+I_2^\varepsilon(x)+I_3^\varepsilon(x).
\end{aligned}$$

Recalling the definition of $a^{\rm hom}$ and κ, one sees that the mean value of the expression in braces takes the form

$$\int\limits_Y I_1^\varepsilon(x,y)\,dy = -\mathrm{div}(a^{\rm hom}\nabla v) - (\varepsilon\,\lambda_j^{\varepsilon,\pm})^2\kappa^2\,v(x).$$

In view of (4.43), we have

$$\big(u_{j(k)}^{\varepsilon_k,\pm}, \int_Y I_1^\varepsilon(\cdot,y)\,dy\big)_\Omega \underset{\varepsilon_k\to 0}{\longrightarrow} \big(\bar u^\pm, -\mathrm{div}(a^{\rm hom}\nabla v) - \beta^2\kappa^2\,v(x)\big)_\Omega.$$

Considering (4.43) and the smoothness of v, Lemma 4.4 provides that

$$\Big(u_{j(k)}^{\varepsilon_k,\pm}\,, \mathcal{I}_1^\varepsilon - \int_Y I_1^\varepsilon(\cdot,y)\,dy\Big)_\Omega \to 0, \quad \varepsilon\to 0, \quad \mathcal{I}_1^\varepsilon(x) \equiv I_1^\varepsilon\big(x,\frac{x}{\varepsilon}\big).$$

Then, integrating by parts and using the boundedness of $a(y)$ and regularity properties of N and N^0, we estimate $(u_{j(k)}^{\varepsilon_k,\pm}, I_2^\varepsilon)_\Omega$ and $(u_{j(k)}^{\varepsilon_k,\pm}, I_3^\varepsilon)_\Omega$ as follows

$$\big|(u_{j(k)}^{\varepsilon_k,\pm}, I_2^\varepsilon)_\Omega\big| = \Big|\varepsilon_k \int\limits_\Omega \nabla^T u_{j(k)}^{\varepsilon_k,\pm}\, a^{\varepsilon_k}(x)\,\nabla_x(N(y)^T\nabla_x v(x))\big|_{y=x/\varepsilon_k}\,dx\Big| \le$$
$$\le C\,\varepsilon_k\|\nabla u_{j(k)}^{\varepsilon_k,\pm}\|_{L^2(\Omega)} \le C\varepsilon_k;$$
$$\big|(u_{j(k)}^{\varepsilon_k,\pm}, I_3^\varepsilon)_\Omega\big| = \Big|\varepsilon_k^2\,\lambda_{j(k)}^{\varepsilon_k,\pm} \int\limits_\Omega \nabla^T u_{j(k)}^{\varepsilon_k,\pm}\, a^{\varepsilon_k}(x)\,N^0\big(\frac{x}{\varepsilon_k}\big)\,\nabla_x v(x)\,dx\Big| \le$$
$$\le C\,\varepsilon_k\|\nabla u_{j(k)}^{\varepsilon_k,\pm}\|_{L^2(\Omega)} \le C\varepsilon_k.$$

In such a way passing to the limit in the integral identity (4.42) leads to the equality

$$\big(\bar u^\pm, \mathcal{L}^{\rm hom} v - (\beta)^2\,\kappa^2\,v\big)_\Omega = 0, \quad v\in C_0^\infty(\Omega).$$

Integrating by parts gives

$$\big(\mathcal{L}^{\rm hom}\bar u^\pm - (\beta)^2\,\kappa^2\,\bar u^\pm, v\big)_\Omega = 0, \quad v\in C_0^\infty(\Omega).$$

Since the space $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, the last equality holds for any $v\in H_0^1(\Omega)$, that means $\{\beta, \bar u^\pm\}$ to be an eigenpair of problem (4.13) if $\bar u^\pm \ne 0$. Let us assume that $u_{j(k)}^{\varepsilon_k,\pm}$ converges weakly in $H^1(\Omega)$ to $\bar u^\pm \equiv 0$, as $\varepsilon_k\to 0$. By the definition of the eigenpair $\{\lambda_{j(k)}^{\varepsilon_k,\pm}, u_{j(k)}^{\varepsilon_k,\pm}\}$ and normalization condition (2.6) we have

$$1 = a^{\varepsilon_k}(u_{j(k)}^{\varepsilon_k,\pm}, u_{j(k)}^{\varepsilon_k,\pm}) = \lambda_{j(k)}^{\varepsilon_k,\pm}(\rho^{\varepsilon_k}u_{j(k)}^{\varepsilon_k,\pm}, u_{j(k)}^{\varepsilon_k,\pm})_\Omega.$$

In the same way as in the proof of Lemma 4.4 one shows that

$$(\rho^{\varepsilon_k}u_{j(k)}^{\varepsilon_k,\pm}, u_{j(k)}^{\varepsilon_k,\pm})_\Omega \le C\varepsilon_k\|u_{j(k)}^{\varepsilon_k,\pm}\|_{L^2(\Omega)}\|\nabla u_{j(k)}^{\varepsilon_k,\pm}\|_{L^2(\Omega)}$$

Combining the last two relations and taking into account the estimate $|\varepsilon_k\lambda_{j(k)}^\pm| \le C$, we conclude that

$$\|u_{j(k)}^{\varepsilon_k,\pm}\|_{L^2(\Omega)} \ge C > 0.$$

Thus, $\|\bar u^\pm\| \ge C > 0$. We arrive at contradiction. Lemma 4.8 is proved. $\square$

Assume that $K_J(\varepsilon_k) > \varkappa_j$ for some sequence $\varepsilon_k \to 0$. It means that there exist $c_j > 0$ and at least $\varkappa_j + 1$ eigenvalues $\lambda_l^{\varepsilon_k,\pm}$ of problem (2.1) such that

$$|\varepsilon_k \lambda_l^{\varepsilon_k,\pm} - \nu_j^{0,\pm}| \le c_j \varepsilon_k^{1/2}, \quad l = J_j(\varepsilon_k), \cdots, J_j(\varepsilon_k) + \varkappa_j.$$

Then by Lemma 4.8 the corresponding eigenfunctions $u_l^{\varepsilon_k,\pm}$ converge to eigenfunctions $\{\bar u_r^\pm\}_{r=1}^{\varkappa_j+1}$ of the Dirichlet problem (4.13):

$$\mathcal{L}^{\rm hom} \bar u_r^\pm = (\nu_j^{0,\pm})^2 \,\kappa^2\, \bar u_r^\pm, \quad r = 1,\, 2, \dots, \varkappa_j + 1.$$

It is straightforward to check that the functions $\{\bar u_r^\pm\}_{r=1}^{\varkappa_j+1}$ are linearly independent. In order to prove the linear independence of the obtained eigenfunctions $\bar u_l^\pm$ we consider a linear combination $\tilde u^{\varepsilon,\pm} = \sum_{l=J_j}^{J_j+\varkappa_j} c_l u_l^{\varepsilon,\pm}$, where the constants c_l are chosen in such a way that $\sum_{l=J_j}^{J_j+\varkappa_j} |c_l|^2 = 1$. Then $\tilde u^{\varepsilon,\pm}$ is an eigenfunction of $\mathcal{L}^\varepsilon$. By Lemma 4.8 it converges to the function $\tilde u^\pm = \sum_{l=J_j}^{J_j+\varkappa_j} c_l \bar u_l^\pm \ne 0$, which implies the linear independence of the eigenfunctions functions $\{\bar u_l^\pm\}_{l=J_j}^{J_j+\varkappa_j}$ corresponding to the eigenvalue $\nu_j^\pm$. Therefore, the multiplicity of $\nu_j^{0,\pm}$ is greater than or equal to $\varkappa_j + 1$ which contradicts our assumption. Thus, $K_J(\varepsilon) = \varkappa_j$. Combining this relation with the fact that for each $j \in \mathbb{Z}^+$, any accumulating point of the sequence $\varepsilon\lambda_j^{\varepsilon,\pm}$, as $\varepsilon \to 0$, is an eigenvalue of the homogenized problem, we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon \lambda_j^{\varepsilon,\pm} = \nu_j^{0,\pm}.$$

This completes the proof of the first statement of Theorem 4.6.

3. In order to prove the second statement in the theorem, we come back to bound (4.33) and apply the estimate in Lemma 4.2 with $\delta_1 = c_j\varepsilon$, c_j being a sufficiently small constant. In our case this estimate reads

$$\Big\| \mathcal{U}_p^{\varepsilon,\pm} - \sum_{k\in S(j,\varepsilon)} \alpha_{kp}^\varepsilon\, u_k^{\varepsilon,\pm} \Big\|_{\mathcal{H}} \le 2\frac{C\varepsilon^{3/2}}{\delta_1} \le C_j \varepsilon^{1/2}\,,$$

here $p = j, \cdots, j + \varkappa_j - 1$; $S(j,\varepsilon)$ is the set of eigenvalues $\mu_k^{\varepsilon,\pm}$ of the operator $\mathcal{K}^\varepsilon$ which satisfy the estimate

$$|\mu_k^{\varepsilon,\pm} - \varepsilon(\nu_j^{0,\pm})^{-1}| \le c_j\, \varepsilon; \tag{4.44}$$

the constant matrix α^ε is such that

$$\big|(\alpha^\varepsilon_{\cdot p})^T\, \alpha^\varepsilon_{\cdot q} - \delta_{p,q}\big| \le C_j\, \varepsilon^{1/2}, \quad p, q = J(j), \cdots, J(j) + \varkappa_j - 1, \tag{4.45}$$

From the first statement of the theorem we deduce that for sufficiently small $\varepsilon > 0$ the set $S(j,\varepsilon)$ coincides with the set $\{J(j), \dots, J(j)+\varkappa_j - 1\}$. Hence, for $p = j, \cdots, j+\varkappa_j-1$, we have

$$\Big\| \mathcal{U}_p^{\varepsilon,\pm} - \sum_{k=J(j)}^{J(j)+\varkappa_j-1} \alpha_{kp}^\varepsilon\, u_k^{\varepsilon,\pm} \Big\|_{\mathcal{H}} \le 2\frac{C\varepsilon^{3/2}}{\delta_1} \le C_j \varepsilon^{1/2}, \tag{4.46}$$

with $\varkappa_j \times \varkappa_j$ matrix α^ε which meets (4.45). It remains to use the following simple statement.

LEMMA 4.9. *For any $n \times n$ matrix A satisfying an equality*

$$\|A^T A - \mathbb{I}; \mathbb{R}^n \to \mathbb{R}^n\| = \gamma \in (0,1),$$

there exists a unitary matrix B such that

$$\|AB - \mathbb{I}; \mathbb{R}^n \to \mathbb{R}^n\| \leq \gamma;$$

here $\mathbb{I}$ is a unit matrix and

$$\|D; \mathbb{R}^n \to \mathbb{R}^n\| = \sup_{\substack{\xi \in \mathbb{R}^n \\ \|\xi; \mathbb{R}^n\| = 1}} \|D\xi; \mathbb{R}^n\|.$$

We omit the proof of this lemma which can be found in [**9**].

According to (4.45) and Lemma 4.9, there exists a unitary $\varkappa_j \times \varkappa_j$ matrix β^ε such that

$$\|\alpha^\varepsilon \, \beta^\varepsilon - \mathbb{I}; \mathbb{R}^{\varkappa_j} \to \mathbb{R}^{\varkappa_j}\| \leq C\varepsilon^{1/2}. \tag{4.47}$$

If we denote by $\mathcal{U}_{\bar{J}}^{\varepsilon,\pm}$, $U_{\bar{J}}^{\varepsilon,\pm}$ and $u_{\bar{J}}^{\varepsilon,\pm}$ the vectors $(\mathcal{U}_{J(j)}^{\varepsilon,\pm}, \ldots, \mathcal{U}_{J(j)+\varkappa_j-1}^{\varepsilon,\pm})^T$, $(U_{J(j)}^{\varepsilon,\pm}, \ldots, U_{J(j)+\varkappa_j-1}^{\varepsilon,\pm})^T$ and $(u_{J(j)}^{\varepsilon,\pm}, \ldots, u_{J(j)+\varkappa_j-1}^{\varepsilon,\pm})^T$, respectively, then

$$\begin{aligned}
&\Big\|u_{\bar{J}}^{\varepsilon,\pm} - \beta^\varepsilon U_{\bar{J}}^{\varepsilon,\pm}\Big\|_{\mathcal{H}^{\varkappa_j}} \leq 2\Big\|\alpha^\varepsilon u_{\bar{J}}^{\varepsilon,\pm} - \alpha^\varepsilon \beta^\varepsilon U_{\bar{J}}^{\varepsilon,\pm}\Big\|_{\mathcal{H}^{\varkappa_j}} \\
&\leq \Big\|\alpha^\varepsilon u_{\bar{J}}^{\varepsilon,\pm} - \mathcal{U}_{\bar{J}}^{\varepsilon,\pm}\Big\|_{\mathcal{H}^{\varkappa_j}} + \Big\|U_{\bar{J}}^{\varepsilon,\pm} - \mathcal{U}_{\bar{J}}^{\varepsilon,\pm}\Big\|_{\mathcal{H}^{\varkappa_j}} \\
&+ \Big\|U_{\bar{J}}^{\varepsilon,\pm} - \alpha^\varepsilon \beta^\varepsilon U_{\bar{J}}^{\varepsilon,\pm}\Big\|_{\mathcal{H}^{\varkappa_j}} \leq C_j \varepsilon^{1/2};
\end{aligned}$$

here we have also used (4.31), (4.47) and (4.46). The last inequality implies that

$$\Big\|u_p^{\varepsilon,\pm} - \sum_{m=j}^{j+\varkappa_j-1} \beta_{mp}^\varepsilon \, U_m^{\varepsilon,\pm}\Big\|_{\mathcal{H}} \leq C_j \varepsilon^{1/2}.$$

In order to replace here $U_m^{\varepsilon,\pm}$ given by formula (4.18) with $\widetilde{U}_m^{\varepsilon,\pm}$ defined by (4.39), we estimate the $H^1(\Omega)$ norm of the difference

$$\|U_m^{\varepsilon,\pm} - \widetilde{U}_m^{\varepsilon,\pm}\|_{H^1(\Omega)} = \varepsilon^2 \, \|(1-\chi_\varepsilon)\,(N^\varepsilon)^T \, \nabla u_m^0\|_{H^1(\Omega)}$$

with $N^\varepsilon(x) = N^\varepsilon(x/\varepsilon)$. Considering the properties of χ_ε and $N(y)$, it is straightforward to check that

$$\|U_m^{\varepsilon,\pm} - \widetilde{U}_m^{\varepsilon,\pm}\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}, \tag{4.48}$$

which, in turn, results in (4.38).

Lemma 4.5 states that the functions $\{\mathcal{U}_p^{\varepsilon,\pm}\}_{p=j}^{j+\varkappa_j-1}$ corresponding to the same eigenvalue $\nu_j^{0,\pm}$ are almost orthonormal. Let u_q^0 be an eigenfunction of the limit problem (4.13) which corresponds to $\nu_m^{0,\pm}$. Using formula (4.4) we construct $\mathcal{U}_q^{\varepsilon,\pm} = \|U_q^{\varepsilon,\pm}\|_{\mathcal{H}}^{-1}\, U_q^{\varepsilon,\pm}$. By (4.33), we have

$$\mathcal{K}^\varepsilon \mathcal{U}_p^{\varepsilon,\pm} = \varepsilon(\nu_j^{\varepsilon,\pm})^{-1}\mathcal{U}_p^{\varepsilon,\pm} + \theta_p^\varepsilon(x), \quad \|\theta_p^\varepsilon\|_{\mathcal{H}} \le C\varepsilon^{3/2};$$

$$\mathcal{K}^\varepsilon \mathcal{U}_q^{\varepsilon,\pm} = \varepsilon(\nu_m^{\varepsilon,\pm})^{-1}\mathcal{U}_q^{\varepsilon,\pm} + \theta_q^\varepsilon(x), \quad \|\theta_q^\varepsilon\|_{\mathcal{H}} \le C\varepsilon^{3/2}.$$

Multiplying the last two relations by $\mathcal{U}_q^{\varepsilon,\pm}$ and $\mathcal{U}_p^{\varepsilon,\pm}$ in $\mathcal{H}$, respectively, and subtracting them from each other, we obtain

$$\langle \mathcal{U}_p^{\varepsilon,\pm}, \mathcal{U}_q^{\varepsilon,\pm}\rangle = \varepsilon^{-1}\, \frac{\nu_j^{0,\pm}\,\nu_m^{0,\pm}}{\nu_j^{0,\pm} - \nu_m^{0,\pm}}\, \big[\langle \theta_q^\varepsilon, \mathcal{U}_p^{\varepsilon,\pm}\rangle - \langle \theta_p^\varepsilon, \mathcal{U}_q^{\varepsilon,\pm}\rangle\big] \le C\varepsilon^{1/2}. \tag{4.49}$$

This completes the proof of Theorem 4.6. □

From Theorem 4.6 we obtain the following convergence result.

COROLLARY 4.10. *For the sequences of eigenvalues* (2.7) *and* (4.15) *the following convergence result holds:*

$$\varepsilon\lambda_j^{\varepsilon,\pm} \to \nu_j^{0,\pm}, \quad \varepsilon \to 0.$$

Moreover, if $\nu_j^{0,\pm}$ is a simple eigenvalue, then $\lambda_j^{\varepsilon,\pm}$ is also simple, for a small ε, and the corresponding eigenfunctions satisfy the relations:

$$\begin{aligned}
& u_j^{\varepsilon,\pm} \underset{\varepsilon\to 0}{\longrightarrow} u_j^0 \quad \textit{strongly in } L^2(\Omega);\\
& u_j^{\varepsilon,\pm} - \varepsilon N\Big(\frac{x}{\varepsilon}\Big)^T \nabla u_j^0 - \varepsilon\, \nu_j^{0,\pm}\, N^0\Big(\frac{x}{\varepsilon}\Big)\, u_j^0 \underset{\varepsilon\to 0}{\longrightarrow} u_j^0 \quad strongly\ \ in\ \ H^1(\Omega);\\
& a^\varepsilon\ \nabla u_j^{\varepsilon,\pm} \underset{\varepsilon\to 0}{\rightharpoonup} a^{\text{hom}}\,\nabla u_j^0 + \nu_j^{0,\pm}\,\langle a\,\nabla N^0\rangle\, u_j^0 \quad weakly\ \ in\ \ L^2(\Omega),
\end{aligned}$$

where $\langle\cdot\rangle$ denotes the mean value over Y.

Proof. All the statements, except for the last one, are immediate consequences of Theorem 4.6. In order to prove the convergence of fluxes, we estimate, for an arbitrary function $v \in C_0^\infty(\Omega)$, the following expression

$$\begin{aligned}
& \Big|\big(a^\varepsilon \nabla u_j^{\varepsilon,\pm} - a^{\text{hom}}\,\nabla u_j^0 - \nu_j^{0,\pm}\,\langle a\,\nabla N^0\rangle\, u_j^0\ ,\ v\big)_\Omega\Big| \le\\
& \le C\,\big\|\nabla(u_j^{\varepsilon,\pm} - \widetilde{U}_j^{\varepsilon,\pm})\big\|_{L^2(\Omega)}\,\big\|v\big\|_{L^2(\Omega)}\\
& + \big|\big(a^\varepsilon\,\nabla\widetilde{U}_j^{\varepsilon,\pm} - a^{\text{hom}}\,\nabla u_j^0 - \nu_j^{0,\pm}\,\langle a\,\nabla N^0\rangle\, u_j^0\ ,\ v\big)_\Omega\big|.
\end{aligned}$$

By Theorem 4.6,

$$\Big\|\nabla(u_j^{\varepsilon,\pm} - \widetilde{U}_j^{\varepsilon,\pm})\Big\|_{L^2(\Omega)} \le C_j\varepsilon^{1/2}.$$

A straightforward calculation gives

$$a(y)\,\nabla\widetilde{U}_j^{\varepsilon,\pm}(x) - a^{\rm hom}\,\nabla u_j^0(x) - \nu_j^{0,\pm}\,\langle a\,\nabla N^0\rangle\, u_j^0(x) =$$

$$= \big\{(a(y) + a(y)\nabla_y N(y)) - a^{\rm hom}\big\}\,\Big|_{y=x/\varepsilon} \nabla_x u_j^0(x) +$$

$$+\nu_j^{0,\pm}\,\big\{a(y)\nabla_y N^0(y) - \langle a\,\nabla N^0\rangle\big\}\,\Big|_{y=x/\varepsilon} u_j^0(x) +$$

$$+\varepsilon\, a(y)\,\nabla_x(N(y)^T\,\nabla_x u_j^0(x))\,\Big|_{y=x/\varepsilon} + \varepsilon\,\nu_j^{0,\pm}\,a(y)\,N^0(y)\,\nabla_x u_j^0(x)\,\Big|_{y=x/\varepsilon}.$$

The first two items on the right-hand side have zero mean, thus, by Lemma 4.4

$$\Big|\int\limits_\Omega \big\{(a(y) + a(y)\nabla_y N(y)) - a^{\rm hom}\big\}\Big|_{y=x/\varepsilon} v(x)\,\nabla u_j^0(x)\,dx\Big| \le C\varepsilon;$$

$$\Big|\int\limits_\Omega \big\{a(y)\nabla_y N^0(y) - \langle a\,\nabla N^0\rangle\big\}\Big|_{y=x/\varepsilon} u_j^0(x)\;v(x)\,dx\Big| \le C\varepsilon;$$

Finally, using the boundedness of $a_{ij}(y)$, properties of $N^0(y)$ and the smoothness of $u_j^0(x)$ we have

$$\varepsilon\Big|\int\limits_\Omega a(y)\,\nabla_x(N(y)^T\,\nabla_x u_j^0(x))\Big|_{y=x/\varepsilon} v(x)\,dx\Big| \le C\varepsilon;$$

$$\varepsilon\,\nu_j^{0,\pm}\Big|\int\limits_\Omega a(y)\,N^0(y)\Big|_{y=x/\varepsilon}\nabla_x u_j^0(x)\;v(x)\,dx\Big| \le C\varepsilon.$$

Summing up the obtained estimates, we arrive at the last statement in the corollary. □

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Paper G

PAPER G

Homogenization of spectral problem for locally periodic elliptic operators with sign-changing density function

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ABSTRACT. The paper deals with homogenization of a spectral problem for a second order self-adjoint elliptic operator stated in a thin cylinder with homogeneous Neumann boundary condition on the lateral boundary and Dirichlet condition on the bases of the cylinder. We assume that the operator coefficients and the spectral density function are locally periodic in the axial direction of the cylinder, and that the spectral density function changes sign. We show that the behaviour of the spectrum depends essentially on whether the average of the density function is zero or not. In both cases we construct the effective 1-dimensional spectral problem and prove the convergence of spectra.

Keywords: Spectral problem, sign-changing density, homogenization, thin cylinder.

1. Introduction

The paper is aimed at homogenization of a spectral problem for a second order divergence form elliptic operator defined in a thin cylinder of finite length with homogeneous Neumann boundary condition on the lateral boundary of the cylinder and Dirichlet conditions on the cylinder bases. We make a crucial assumption that the spectral weight function changes sign and assume that both operator coefficients and the weight function are locally periodic in the axial direction of the cylinder.

Under the said conditions we show that the asymptotic behaviour of the spectrum depends essentially on whether the average of the weight function over the period is equal to zero or not. In both cases we construct an effective model and prove the convergence result; the estimates for the rate of convergence are also obtained.

The studied spectral problem might have interesting and important applications in the modern theory of metamaterials, that is artificial composite materials engineered to produce a desired electromagnetic behavior with significantly enhanced performance over ”natural” structures. For example, when the world is observed through conventional lenses, the sharpness of the image is determined by and limited to the wavelength of light. Metamaterials with negative refractive index aimed at creation of ”perfect” lenses, that is lenses with capabilities beyond conventional (positive index) ones.

First initiated by L.S. Pontrgyagin in [**15**], the qualitative theory of spectral problems in spaces with indefinite metric was further developed by M.G. Krein ([**7**]), I.S. Iokhvidov ([**4**]) and other mathematicians. The detailed presentation of this theory can be found, for example, in books [**1**], [**16**].

The homogenization of spectral problems in the case of positive weight functions was considered in [**5**], [**6**], [**17**], then in [**13**] for elasticity system and then in many other works. However, the presence of sign-changing weight function makes the problem nonstandard and leads to new interesting phenomena. For operators with pure periodic coefficients defined in a fixed (not asymptotically thin) domains similar problems have been studied in the recent works [**11**], [**12**]. In contrast with problems investigated in these works, for the model considered in the present paper the limit spectral problem is one-dimensional, so that dimension reduction arguments are to be used. We combine the asymptotic expansion technique with the singular measure approach developed in [**20**] and [**2**].

For the density function having positive average the effective spectral problem happens to be a Sturm-Liouville problem. In this case the convergence of the positive part of the spectrum is justified by means of convergence in variable spaces with singular measures.

In the case of zero average weight function the limit spectral problem is that for a quadratic operator pencil. To study this operator pencil we apply the results from [**8**] combined with usual arguments used when studying Sturm-Liouville problems. It should be noted that in contrast with [**12**], the presence of slow variable in the coefficients makes the limit operator pencil nontrivial, so that it can not be reduced to the standard Sturm-Liouville problem.

The fact that the considered operator is defined in a thin cylinder allows us to build boundary layer correctors in the neighbourhood of the cylinder bases and, as a result, improve essentially the asymptotics. As a matter of fact, if the coefficients are sufficiently regular, then arbitrary many terms in the asymptotic expansion can be constructed. The existence of exponentially decaying boundary layer correctors is assured by the results obtained in [**14**].

In the last section we address the case when the local average of the weight function changes sign. In this case the convergence of both, positive and negative parts of the spectrum is justified.

The asymptotics of negative part of the spectrum in the case of positive average of the density function will be treated in a separate publication.

The paper is organized as follows. Section 2 contains the statement of the problem together with some preliminary results concerning the structure of the spectrum of the original operator. In Section 3.1 we construct the formal asymptotic expansion in the case when the average of the weight function over the period is positive. The justification of the homogenization procedure is given in Section 3.2. Section 4 is devoted to the case when the average of the weight function is equal to zero. In Section 5 the case when the average of the weight function changes sign is considered.

2. Problem setup and main results

Let Q be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^{d-1}$ with a boundary ∂Q. The points in $\mathbb{R}^d$ are denoted $x = (x_1, x')$, where $x' = x_2, ..., x_d$. Denote by G_ε a thin rod $[-1,1] \times \varepsilon Q$ with the lateral boundary $\Sigma_\varepsilon = (-1,1) \times \partial(\varepsilon Q)$ and the bases $S_{\pm 1} = \{\pm 1\} \times \varepsilon Q$. In the cylinder G_ε we consider the following spectral problem:

$$(2.1) \qquad \begin{cases} \mathcal{A}^\varepsilon u^\varepsilon(x) \equiv -\mathrm{div}\Big(a^\varepsilon(x)\nabla u^\varepsilon(x)\Big) = \lambda^\varepsilon\, \rho^\varepsilon(x)\, u^\varepsilon(x), & x \in G_\varepsilon, \\ \mathcal{B}^\varepsilon u^\varepsilon(x) \equiv (a^\varepsilon\, \nabla u^\varepsilon, n) = 0, & x \in \Sigma_\varepsilon, \\ u^\varepsilon(-1, x') = u^\varepsilon(1, x') = 0, & x \in \partial(\varepsilon Q). \end{cases}$$

with

$$a^\varepsilon(x) = a\big(x_1, \frac{x}{\varepsilon}\big), \quad \rho^\varepsilon(x) = \rho\big(x_1, \frac{x}{\varepsilon}\big),$$

where $a(x_1, y)$ is a symmetric $d \times d$ matrix and $\rho(x_1, y)$ is a scalar function; $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^d$. We assume the following conditions to hold:

- $(\mathbf{H0})$ $a_{ij}(x_1, y), \rho(x_1, y) \in C^{1,\alpha}([-1,1]; C^\alpha(\overline{Y}))$ for some $\alpha > 0$. Here $Y = \mathcal{S}_1 \times Q$ denotes the periodicity cell, $\mathcal{S}_1$ is a unit circle;
- $(\mathbf{H1})$ Functions $a_{ij}(x_1, y)$ and $\rho(x_1, y)$, are 1-periodic in y_1;
- $(\mathbf{H2})$ The matrix $a(x_1, y)$ satisfies the uniform ellipticity condition, that is for any $x_1 \in [-1,1]$ and $y \in Y$

$$\sum_{i,j=1}^{d} a_{ij}(x_1, y)\xi_i\xi_j \geq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad \Lambda > 0;$$

- $(\mathbf{H3})$ The weight function $\rho(x_1, y)$ changes sign, that is for any $x_1 \in [-1,1]$ the sets $\{y \in Y \,:\, \rho(x_1, y) < 0\}$ and $\{y \in Y \,:\, \rho(x_1, y) > 0\}$ have positive Lebesgue measures, i.e.

$$\big|\{y \in Y \,:\, \rho(x_1, y) \lessgtr 0\}\big| > 0.$$

Also, for presentation simplicity we assume that

$$(2.2) \qquad \varepsilon = 1/L, \qquad L = 1, 2, \ldots.$$

The general case can be treated in the same way, see Remark 3.4 in Section 3 for further discussion.

REMARK 2.1. It follows from condition (**H3**) that, for sufficiently small ε, the sets $\{x \in G_\varepsilon \ : \ \rho\big(x_1, \frac{x}{\varepsilon}\big) < 0\}$ and $\{x \in G_\varepsilon \ : \ \rho\big(x_1, \frac{x}{\varepsilon}\big) > 0\}$ have positive Lebesgue measures.

The weak formulation of problem (2.1) is as follows: find $\lambda^\varepsilon \in \mathbb{C}$ (eigenvalues) and $u^\varepsilon \in H^1(G_\varepsilon) \setminus \{0\}$ (eiegenfunctions) such that $u^\varepsilon(\pm 1, x') = 0$ and

$$(a^\varepsilon \nabla u^\varepsilon, \nabla v)_{L^2(G_\varepsilon)} = \lambda^\varepsilon \, (\rho^\varepsilon \, u^\varepsilon, v)_{L^2(G_\varepsilon)}, \tag{2.3}$$

where $v \in C^\infty(G_\varepsilon)$ such that $v(\pm 1, x') = 0$, $(\cdot, \cdot)_{L^2(G_\varepsilon)}$ denotes the usual scalar product in $L^2(G_\varepsilon)$.

First we study the qualitative properties of problem (2.1) for a fixed value of ε. For this aim, following the ideas in [**12**], we are going to reduce the problem under consideration to an equivalent spectral problem for a compact self-adjoint operator. To this end let us introduce the space

$$\mathcal{H}^\varepsilon = \{u \in H^1(G_\varepsilon) : \ u\big|_{S_{\pm 1}} = 0\}$$

equipped with the norm

$$\|u\|^2_{\mathcal{H}^\varepsilon} = (u, u)_{\mathcal{H}^\varepsilon} = (a^\varepsilon \nabla u, \nabla u)_{L^2(G_\varepsilon)}.$$

Thanks to the Friedrischs inequality

$$\|v\|_{L^2(G^\varepsilon)} \le 2\|\nabla v\|_{L^2(G_\varepsilon)}, \quad v \in \mathcal{H}^\varepsilon,$$

the quadratic form $(a^\varepsilon \nabla u, \nabla u)_{L^2(G_\varepsilon)}$ defines a norm in $\mathcal{H}^\varepsilon$, which is equivalent to the standard $H^1(G_\varepsilon)$ norm.

In view of condition (**H0**), the bilinear form $(\rho^\varepsilon u, v)_{L^2(G_\varepsilon)}$ defines on $\mathcal{H}^\varepsilon$ a bounded linear operator $\mathcal{K}^\varepsilon : \mathcal{H}^\varepsilon \to \mathcal{H}^\varepsilon$ by the following rule:

$$(\mathcal{K}^\varepsilon u, v)_{\mathcal{H}^\varepsilon} = (\rho^\varepsilon u, v)_{L^2(G_\varepsilon)}.$$

By definition, the operator $\mathcal{K}^\varepsilon$ is symmetric and, since it is bounded, it is self-adjoint. Notice that $\mathcal{K}^\varepsilon u$ can be also introduced as a solution of the boundary value problem

$$\begin{cases} \mathcal{A}^\varepsilon(\mathcal{K}^\varepsilon u(x)) = \rho^\varepsilon(x)\, u(x), & x \in G_\varepsilon, \\ B^\varepsilon(\mathcal{K}^\varepsilon u(x)) = 0, & x \in \Sigma_\varepsilon, \\ \mathcal{K}^\varepsilon u(x) = 0, & x \in S_{\pm 1}. \end{cases} \tag{2.4}$$

Considering this representation and the compactness of the imbedding $H^1(G_\varepsilon)$ in $L^2(G_\varepsilon)$, one can see that $\mathcal{K}^\varepsilon$ is a compact operator, both in $\mathcal{H}^\varepsilon$ and in $L^2(G_\varepsilon)$.

REMARK 2.2. Since for any $u \in L^2(G_\varepsilon)$ the function $\mathcal{K}^\varepsilon u$ belongs to $\mathcal{H}^\varepsilon$, then the spectrum of $\mathcal{K}^\varepsilon$ in $L^2(G_\varepsilon)$ coincides with that in $\mathcal{H}^\varepsilon$. We prefer to study the spectrum of $\mathcal{K}^\varepsilon$ in the space $\mathcal{H}^\varepsilon$ because in this space $\mathcal{K}^\varepsilon$ is self-adjoint.

In terms of the operator $\mathcal{K}^\varepsilon$ problem (2.1) takes the form

$$\mathcal{K}^\varepsilon u^\varepsilon = \mu^\varepsilon \, u^\varepsilon, \quad \mu^\varepsilon = 1/\lambda^\varepsilon. \tag{2.5}$$

Exactly in the same way as in **[12]** (see Lemma 2.1) one can show that the discrete spectrum of the operator $\mathcal{K}^\varepsilon$ consists of two infinite sequences. The following statement holds.

LEMMA 2.3. *Suppose that conditions* $(\mathbf{H0}) - (\mathbf{H3})$ *are fulfilled. Then the spectrum* $\sigma(\mathcal{K}^\varepsilon)$ *of the operator* $\mathcal{K}^\varepsilon$ *belongs to the interval* $[-k^\varepsilon, k^\varepsilon]$, $k^\varepsilon = \|\mathcal{K}^\varepsilon\|$; *the point* $\mu = 0$ *is the only element of the essential spectrum* $\sigma_e(\mathcal{K}^\varepsilon)$. *Moreover, the discrete spectrum of the operator* $\mathcal{K}^\varepsilon$ *consists of two infinite sequences*

$$\mu_1^{\varepsilon,+} \geq \mu_2^{\varepsilon,+} \geq \cdots \geq \mu_j^{\varepsilon,+} \geq \cdots \to +0,$$
$$\mu_1^{\varepsilon,-} \leq \mu_2^{\varepsilon,-} \leq \cdots \leq \mu_j^{\varepsilon,-} \leq \cdots \to -0.$$

Taking into account (2.5), we conclude that problem (2.1) has a discrete spectrum which consists also of two infinite sequences. More precisely, we have proved the following result.

THEOREM 2.4. *Under the assumptions* $(\mathbf{H0}) - (\mathbf{H3})$ *spectral problem* (2.1) *has a discrete spectrum which consists of two sequences*

$$0 < \lambda_1^{\varepsilon,+} \leq \lambda_2^{\varepsilon,+} \leq \cdots \leq \lambda_j^{\varepsilon,+} \leq \cdots \to +\infty,$$
$$0 > \lambda_1^{\varepsilon,-} \geq \lambda_2^{\varepsilon,-} \geq \cdots \geq \lambda_j^{\varepsilon,-} \geq \cdots \to -\infty.$$

Under proper normalization, the corresponding eigenfunctions $u_j^{\varepsilon,\pm}$ *satisfy the orthogonality condition*

$$(u_i^{\varepsilon,\pm}, u_j^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon} = \varepsilon^{d-1} \, |Q| \, \delta_{ij}, \tag{2.6}$$

where $|Q|$ *is the Lebesgue measure of* Q *and* δ_{ij} *is the Kronecker delta.*

The goal of the present work is to study the asymptotic behaviour of the spectrum of problem (2.1), as $\varepsilon \to 0$. As was already pointed out, the asymptotic behaviour of the spectrum depends crucially on whether the local average of $\rho(x_1, \cdot)$ is zero on $[-1, 1]$ or not. To avoid the technicalities for the moment, we formulate the main result of the paper in a loose way.

THEOREM 2.5. *Let conditions* $(\mathbf{H0}) - (\mathbf{H3})$ *be fulfilled. If* $\lambda_j^{\varepsilon,+}$ *(*$\lambda_j^{\varepsilon,-}$*) stands for the* j*th positive (negative) eigenvalue of problem* (2.1), *and* $u_j^{\varepsilon,+}$ *(*$u_j^{\varepsilon,-}$*) for the corresponding eigenfunction, then the following convergence results hold:*

(1) *If* $\langle \rho(x_1, \cdot) \rangle > 0$ *for all* $x_1 \in [-1, 1]$, *then, for any* j,

$$\lambda_j^{\varepsilon,+} \to \lambda_j^{0,+}, \quad \varepsilon \to 0;$$
$$\varepsilon^{\frac{d-1}{2}} \, \|u_j^{\varepsilon,+} - u_j^{0,+}\|_{L^2(G_\varepsilon)} \to 0, \quad \varepsilon \to 0,$$

where $(\lambda_j^{0,+}, u_j^{0,+})$ *is the* j*th eigenpair of the effective Sturm-Liouville problem*

$$\begin{cases} -\dfrac{d}{dx_1}\Big(a^{\text{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\Big) = \lambda^0 \, \langle \rho(x_1, \cdot) \rangle \, u^0(x_1), \quad x_1 \in (-1, 1), \\ u^0(\pm 1) = 0, \end{cases} \tag{2.7}$$

with some strictly positive continuous function $a^{\text{eff}}(x_1)$ *(see* (3.3)*).*

(2) *If* $\langle\rho(x_1,\cdot)\rangle = 0$ *for all* $x_1 \in [-1,1]$*, then, for any* j*,*

$$\varepsilon\,\lambda_j^{\varepsilon,\pm} - \nu_j^{0,\pm} \to 0, \quad \varepsilon \to 0;$$
$$\varepsilon^{\frac{d-1}{2}}\,\|u_j^{\varepsilon,\pm} - v_j^{0,\pm}\|_{L^2(G_\varepsilon)} \to 0, \quad \varepsilon \to 0,$$

where $(\nu_j^{0,\pm}, v_j^{0,\pm})$ *are the* j*th eigenpairs of the following quadratic operator pencil:*

$$\begin{cases} -\dfrac{d}{dx_1}\Big(a^{\text{eff}}(x_1)\dfrac{dv^0(x_1)}{dx_1}\Big) + \nu^0\,\mathbf{B}(x_1)\,v^0(x_1) \\ -(\nu^0)^2\,\mathbf{C}(x_1)\,v^0(x_1) = 0, \quad x_1 \in (-1,1), \\ v^0(-1) = v^0(1) = 0, \end{cases} \tag{2.8}$$

with the functions $\mathbf{B}(x_1), \mathbf{C}(x_1) > 0$ *defined by* (4.8) *and* (4.7)*, respectively.*

(3) *If* $\langle\rho(x_1,\cdot)\rangle$ *changes sign , then, for any* j*,*

$$\lambda_j^{\varepsilon,\pm} \to \lambda_j^{0,\pm}, \quad \varepsilon \to 0;$$
$$\varepsilon^{\frac{d-1}{2}}\,\|u_j^{\varepsilon,\pm} - u_j^{0,\pm}\|_{L^2(G_\varepsilon)} \to 0, \quad \varepsilon \to 0,$$

where $(\lambda_j^{0,\pm}, u_j^{0,\pm})$ *are the* j*th eigenpairs of the effective spectral problem*

$$\begin{cases} -\dfrac{d}{dx_1}\big(a^{\text{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\big) = \lambda^0\,\langle\rho(x_1,\cdot)\rangle\,u^0(x_1), \quad x_1 \in (-1,1), \\ u^0(\pm 1) = 0, \end{cases} \tag{2.9}$$

with the function $a^{\text{eff}}(x_1) > 0$ *defined by* (3.3)*.*

Notice that in the case $\langle\rho(x_1,\cdot)\rangle > 0$ the eigenvalues of the effective problem form a monotone sequence $\lambda_j^{0,+} \to +\infty$, as $j \to +\infty$, while in the cases $\langle\rho(x_1,\cdot)\rangle = 0$ and when $\langle\rho(x_1,\cdot)\rangle$ changes sign the spectra of the effective spectral problems (2.8) and (2.9) consist of two infinite sequences, tending to $+\infty$ and $-\infty$ (see Theorems 3.2, 4.1 and Section 5). Thus, one cannot characterize the asymptotic behaviour of the negative part of the spectrum in the case $\langle\rho(x_1,\cdot)\rangle > 0$ in terms of the effective problem (2.7). The negative part of the spectrum will be considered elsewhere.

Theorem 2.5 follows from stronger results given in Sections 3-5 (see Theorem 3.6, 4.3, 5.1). In all cases we construct interior correctors, boundary layer correctors in the vicinity of the cylinder bases, and obtain estimates for the rate of convergence.

3. The case $\langle\rho(x_1,\cdot)\rangle > 0$

3.1. Formal asymptotic expansion. In what follows we denote $\nabla_y = \{\partial_{y_1}, \cdots, \partial_{y_d}\}^T$,

$$\langle\rho(x_1,\cdot)\rangle = \int\limits_Y \rho(x_1,y)\,dy;$$
$$\mathcal{A}_y\,u \equiv -\text{div}_y(a(x_1,y)\nabla_y u); \;\; \mathcal{B}_y\,u \equiv (a(x_1,y)\nabla_y u, n).$$

We are looking for a solution $(\lambda^\varepsilon, u^\varepsilon)$ of problem (2.1) in the form

$$u^\varepsilon(x) = u^0(x_1) + \varepsilon\, u^1(x_1, y) + \varepsilon^2\, u^2(x_1, y) + \varepsilon^3\, u^3(x_1, y) + \cdots\,, \qquad (3.1)$$
$$\lambda^\varepsilon = \lambda^0 + \varepsilon\, \lambda^1 + \cdots\,, \quad y = \frac{x}{\varepsilon},$$

where unknown functions $u^k(x_1, y)$ are 1-periodic in y_1. Let us substitute ansätze (3.1) into (2.1) and collect power-like with respect to ε terms. Equating the coefficient in front of ε^{-1} to 0, we obtain an equation for $u^1(x_1, \cdot)$, $x_1 \in (-1, 1)$:

$$\begin{cases} \mathcal{A}_y u^1(x_1, y) = \mathrm{div}_y a_{\cdot 1}(x_1, y)\, \dfrac{du^0}{dx_1}, & y \in Y, \\ \mathcal{B}_y u^1(x_1, y) = -(a_{\cdot 1}(x_1, y), n)\, \dfrac{du^0}{dx_1}, & y \in \partial Y, \\ u^1(x_1, \cdot) - \ y_1 - \text{periodic}, \end{cases}$$

where $a_{\cdot k}$ is a kth column of the matrix $a(y)$. Note that $\partial Y = \mathcal{S}_1 \times \partial Q$. Particular form of the right-hand side in the last equation suggests the representation for u^1

$$u^1(x_1, y) = N^{1,1}(x_1, y)\, \frac{du^0(x_1)}{dx_1} + v^1(x_1),$$

with $N^{1,1}$ being, for any $x_1 \in (-1, 1)$, a solution of the problem

$$\begin{cases} \mathcal{A}_y N^{1,1}(x_1, y) = \mathrm{div}_y a_{\cdot 1}(x_1, y), & y \in Y, \\ \mathcal{B}_y N^{1,1}(x_1, y) = -(a_{\cdot 1}(x_1, y), n), & y \in \partial Y, \\ N^{1,1}(x_1, \cdot) - \ y_1 - \text{periodic}. \end{cases} \qquad (3.2)$$

Under assumption (**H0**), $N^{1,1}(x_1, y) \in C^{1,\alpha}([-1, 1]; C^{1,\alpha}(\overline{Y}))$.

Similarly, collecting the terms of order ε^0 we obtain the problem for u^2:

$$\begin{cases} \mathcal{A}_y u^2(x_1, y) = \dfrac{\partial}{\partial x_1}\big(a_{1\cdot}(x_1, y) \nabla_y N^{1,1}(x_1, y) \dfrac{du^0(x_1)}{dx_1}\big) \\ + \dfrac{\partial}{\partial x_1}(a_{11}(x_1, y) \dfrac{du^0(x_1)}{dx_1}) + \mathrm{div}_y \big(a_{\cdot 1}(x_1, y)\, \dfrac{\partial}{\partial x_1}(N^{1,1}(x_1, y) u^0(x_1))\big) \\ + \mathrm{div}_y a_{\cdot 1}(x_1, y)\, \dfrac{dv^1(x_1)}{dx_1} + \lambda^0\, \rho(x_1, y)\, u^0(x_1), \quad x_1 \in (-1, 1),\ y \in Y, \\ \mathcal{B}_y u^2(x_1, y) = -(a_{\cdot 1}(x_1, y), n)\, \dfrac{\partial}{\partial x_1}(N^{1,1}(x_1, y) u^0(x_1)) \\ -(a_{\cdot 1}(x_1, y), n)\, \dfrac{dv^1}{dx_1}, \qquad x_1 \in (-1, 1),\ y \in \partial Y, \\ u^2(x_1, \cdot) - \ y_1 - \text{periodic}. \end{cases}$$

The compatibility condition for the last problem reads

$$\frac{d}{dx_1}\int\limits_Y \big(a_{11}(x_1,y) + a_{1\cdot}(x_1,y)\nabla_y N^{1,1}(x_1,y)\big)\,dy\frac{du^0(x_1)}{dx_1}$$
$$+\lambda^0\int\limits_Y \rho(x_1,y)\,dy\,u^0(x_1) = 0, \quad x_1\in(-1,1).$$

Denoting

$$a^{\text{eff}}(x_1) = \int\limits_Y a_{1j}(x_1,y)\big(\delta_{1j} + \partial_{y_j}N^{1,1}(x_1,y)\big)\,dy, \tag{3.3}$$

we derive the following problem for u^0:

$$\begin{cases} \mathcal{A}^0\,u^0(x_1) \equiv -\dfrac{d}{dx_1}\big(a^{\text{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\big) \\ = \lambda^0\,\langle\rho(x_1,\cdot)\rangle\,u^0(x_1), \quad x_1\in(-1,1), \\ u^0(\pm 1) = 0. \end{cases} \tag{3.4}$$

LEMMA 3.1. *The effective coefficient* $a^{\text{eff}}(x_1)\in C^{1,\alpha}[-1,1]$ *is positive for all* $x_1\in[-1,1]$.

PROOF. Obviously, $a^{\text{eff}}(x_1)$ is an element $\{A^{\text{eff}}(x_1)\}_{11}$ of the matrix $A^{\text{eff}}(x_1)$ given by

$$A^{\text{eff}}_{ij}(x_1) = \int\limits_Y \big(a_{ij}(x_1,y) + a_{ik}\,\partial_{y_k}N_k^{1,1}(x_1,y)\big)\,dy,$$

where functions $N_k^{1,1}$ solve the problems

$$\begin{cases} \mathcal{A}_y N_k^{1,1}(x_1,y) = \text{div}_y a_{\cdot k}(x_1,y), \quad k=2,...,d, \quad y\in Y, \\ \mathcal{B}_y N_k^{1,1}(x_1,y) = -(a_{\cdot k}(x_1,y),n), \qquad\qquad\qquad y\in\partial Y, \\ N_k^{1,1} - \ y_1 - \text{periodic}. \end{cases}$$

Let us show that the matrix $A^{\text{eff}}(x_1)$ is positive definite. Notice that

$$0 = \int\limits_Y \partial_{y_m}(a_{jm}N_i^{1,1})\,dy - \int\limits_{\partial Y} a_{jm}N_i^{1,1}n_m\,d\sigma.$$

Reorganizing the last expression yields

$$
\begin{aligned}
0 &= \int\limits_Y \partial_{y_m}(a_{jm}N_i^{1,1})\,dy - \int\limits_{\partial Y} a_{jm}N_i^{1,1}n_m\,d\sigma \\
&= \int\limits_Y \big(a_{jm}\partial_{y_m}N_i^{1,1} + \partial_{y_m}a_{mj}\,N_i^{1,1}\big)\,dy - \int\limits_{\partial Y} a_{jm}N_i^{1,1}n_m\,d\sigma \\
&= \int\limits_Y \big(a_{jm}\partial_{y_m}N_i^{1,1} - \partial_{y_m}(a_{mk}\,\partial_{y_k}N_j^{1,1})N_i^{1,1}\big)\,dy - \int\limits_{\partial Y} a_{jm}N_i^{1,1}n_m\,d\sigma \\
&= \int\limits_Y \big(a_{jm}\partial_{y_m}N_i^{1,1} + a_{mk}\,\partial_{y_k}N_j^{1,1}\,\partial_{y_m}N_i^{1,1}\big)\,dy.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
A^{\text{eff}}(x_1) &= \int\limits_Y \big(a_{ij}(x_1,y) + a_{ik}\,\partial_{y_k}N_k^{1,1}(x_1,y)\big)\,dy \\
&+ \int\limits_Y \big(a_{jm}\partial_{y_m}N_i^{1,1} + a_{mk}\,\partial_{y_k}N_j^{1,1}\,\partial_{y_m}N_i^{1,1}\big)\,dy \\
&= \int\limits_Y \big(\delta_{im} + \partial_{y_m}N_i^{1,1}\big)\,a_{mk}\,\big(\delta_{kj} + \partial_{y_k}N_j^{1,1}\big)\,dy,
\end{aligned}
$$

thus, the matrix A^{eff} is nonnegative. Let us show that $a^{\text{eff}} > 0$. For an arbitrary nonnegative matrix C we state that if $C_{11} = 0$, then $C_{1k} = 0, k = 2, \cdots, d$, and, consequently, $Ce_1 = 0$. Assuming that $(\delta_{1j}+\partial_{y_1}N_j^{1,1}) = 0$ we arrive at contradiction with the periodicity of $N_1^{1,1}$ in y_1. Thus, $a^{\text{eff}} > 0$. $\square$

For the reader's convenience we formulate here the classical result on Sturm-Liouville spectral problem (see, for instance, [**10**]).

THEOREM 3.2. *The eigenvalues of the Sturm-Liouville problem* (3.4) *are real and form a monotone sequence*

$$
0 < \lambda_1^{0,+} < \lambda_2^{0,+} < \cdots < \lambda_j^{0,+} \cdots \to +\infty.
$$

Moreover, all the eigenvalues are simple.

REMARK 3.3. The corresponding eigenfunctions $u_i^{0,+} \in C^{2,\alpha}[-1,1]$ of problem (3.4) can be normalized by

$$
\int\limits_{-1}^{1} a^{\text{eff}}(x_1)\,\frac{du_i^{0,+}}{dx_1}\,\frac{du_j^{0,+}}{dx_1}\,dx_1 = \delta_{ij}. \tag{3.5}
$$

Our next goal is to derive the equation for the unknown function $v^1(x_1)$. To this end we analyze the right-hand side of the equation for $u^2(x_1, y)$. The structure of the right-hand side suggests the following representation:

$$
\begin{aligned}
u^2(x_1,y) &= N^{2,2}(x_1,y)\,\frac{d^2u^0(x_1)}{dx_1^2} + N^{2,1}(x_1,y)\,\frac{du^0(x_1)}{dx_1}\\
&+N^{2,0}(x_1,y)\,u^0(x_1) + N^{1,1}(x_1,y)\,\frac{dv^1(x_1)}{dx_1} + v^2(x_1),
\end{aligned}
\tag{3.6}
$$

where $N^{2,2}$, $N^{2,1}$ and $N^{2,0}$ are y_1-periodic solutions of the problems

$$
\begin{cases}
\mathcal{A}_y N^{2,2}(x_1,y) = \mathrm{div}_y(a(x_1,y)N^{1,1}(x_1,y))\\
+a_{1j}(x_1,y)(\delta_{1j} + \partial_{y_j}N^{1,1}(x_1,y)) - a^{\text{eff}}(x_1), \qquad y \in Y,\\
\mathcal{B}_y N^{2,2}(x_1,y) = -(a_{\cdot 1}(x_1,y),n)N^{1,1}(x_1,y), \quad y \in \partial Y;
\end{cases}
\tag{3.7}
$$

$$
\begin{cases}
\mathcal{A}_y N^{2,1}(x_1,y) = \mathrm{div}_y(a_{\cdot 1}(x_1,y)\dfrac{\partial}{\partial x_1}N^{1,1}(x_1,y))\\
+\dfrac{\partial}{\partial x_1}\big[a_{1j}(x_1,y)(\delta_{1j} + \partial_{y_j}N^{1,1}(x_1,y))\big] - \dfrac{da^{\text{eff}}(x_1)}{dx_1}, \quad y \in Y,\\
\mathcal{B}_y N^{2,1}(x_1,y) = -(a_{\cdot 1}(x_1,y),n)\,\dfrac{\partial}{\partial x_1}N^{1,1}(x_1,y), \qquad y \in \partial Y;
\end{cases}
\tag{3.8}
$$

$$
\begin{cases}
\mathcal{A}_y N^{2,0}(x_1,y) = \lambda^0\big(\rho(x_1,y) - \langle\rho(x_1,\cdot)\rangle\big), \quad y \in Y,\\
\mathcal{B}_y N^{2,0}(x_1,y) = 0, \qquad y \in \partial Y.
\end{cases}
\tag{3.9}
$$

Equating the coefficients in front of ε^1, we get the equation for u^3:

$$
\begin{cases}
\mathcal{A}_y u^3(x_1,y) = \mathrm{div}_y\big(a_{\cdot 1}(x_1,y)\dfrac{\partial u^2}{\partial x_1}\big) + \dfrac{\partial}{\partial x_1}\big(a_{11}(x_1,y)\dfrac{\partial u^1}{\partial x_1}\big)\\
+\lambda^0\,\rho(x_1,y)\,u^1(x_1,y) + \lambda^1\,\rho(x_1,y)\,u^0(x_1), \qquad y \in Y,\\
\mathcal{B}_y u^3(x_1,y) = -(a_{\cdot 1}(x_1,y),n)\,\dfrac{\partial u^2}{\partial x_1}.
\end{cases}
$$

The compatibility condition for the last equation reads

$$
-\frac{d}{dx_1}\big(a^{\text{eff}}(x_1)\frac{dv^1}{dx_1}\big) - \lambda^0\,\langle\rho(x_1,\cdot)\rangle\,v^1(x_1) = F(x_1) + \lambda^1\,\langle\rho(x_1,\cdot)\rangle\,u^0,
\tag{3.10}
$$

where

$$
\begin{aligned}
F(x_1) &= \sum_{k=0}^{2}\frac{d}{dx_1}\int\limits_Y a_{1\cdot}(x_1,y)\nabla_y N^{2,k}(x_1,y)\,\frac{d^ku^0(x_1)}{dx_1^k}\,dy\\
&+\lambda^0\int\limits_Y \rho(x_1,y)\,N^{1,1}(x_1,y)\,\frac{du^0(x_1)}{dx_1}\,dy.
\end{aligned}
\tag{3.11}
$$

Determining the boundary conditions for $v^1(x_1)$ at the points $x_1 = \pm 1$ requires constructing boundary layer correctors in the vicinity of these points.

Let $G^- = (0,+\infty)\times Q$ and $G^+ = (-\infty,0)\times Q$ be semi-infinite cylinders with the axis directed along y_1 and lateral boundaries $\Sigma^- = (0,+\infty)\times\partial Q$ and $\Sigma^+ = (-\infty,0)\times\partial Q$. We denote by $w^\pm(y)$ solutions to the following boundary value problems:

$$\left\{\begin{array}{ll} -\mathrm{div}_y(a(\pm 1, y_1 \pm \delta, y')\nabla_y w^\pm) = 0, & y\in G^\pm, \\ (a(\pm 1, y_1\pm\delta, y')\nabla_y w^\pm, n) = 0, & y\in\Sigma^\pm, \\ w^\pm(0,y') = -N^{1,1}(\pm 1, \pm\delta, y')\,\dfrac{du^0}{dx_1}(\pm 1),\ y'\in Q. \end{array}\right. \tag{3.12}$$

where $\delta = \delta(\varepsilon)$ is the fractional part of ε^{-1}. Due to our assumption (2.2) we have $\delta = 0$ so that problem (3.12) reads

$$\left\{\begin{array}{ll} -\mathrm{div}_y(a(\pm 1, y)\nabla_y w^\pm) = 0, & y\in G^\pm, \\ (a(\pm 1, y)\nabla_y w^\pm, n) = 0, & y\in\Sigma^\pm, \\ w^\pm(0,y') = -N^{1,1}(\pm 1, 0, y')\,\dfrac{du^0}{dx_1}(\pm 1),\ y'\in Q, \end{array}\right. \tag{3.13}$$

According to [**14**] there exists a unique bounded solution $w^\pm \in H^1_{loc}(G^\pm)\cap C^{1,\alpha}(\overline{G^\pm})$ of problem (3.13). It stabilizes to some constant $\hat w^\pm$, as $|y_1|\to+\infty$:

$$\begin{gathered} |w^\pm(y_1,y') - \hat w^\pm| \le C_0\, e^{-\gamma|y_1|}, \quad C_0,\gamma>0; \\ \|\nabla w^+\|_{L^2((n,n+1)\times Q)} \le C\, e^{-\gamma n}, \quad \forall n>0, \\ \|\nabla w^-\|_{L^2((-(n+1),-n)\times Q)} \le C\, e^{-\gamma n}, \quad \forall n>0, \end{gathered} \tag{3.14}$$

for some $\gamma > 0$. As a boundary condition for $v^1(x_1)$ we choose the uniquely defined constants $\hat w^\pm$: $v^1(\pm 1) = \hat w^\pm$. Thus, the problem for v^1 takes the form

$$\left\{\begin{array}{l} -\dfrac{d}{dx_1}\big(a^{\rm eff}(x_1)\dfrac{dv^1}{dx_1}\big) - \lambda^0\,\langle\rho(x_1,\cdot)\rangle\, v^1(x_1) \\ = F(x_1) + \lambda^1\,\langle\rho(x_1,\cdot)\rangle\, u^0, \quad x_1\in(-1,1), \\ v^1(\pm 1) = \hat w^\pm, \end{array}\right. \tag{3.15}$$

where $F(x_1)$ is defined by (3.11).

Due to the Fredholm alternative, problem (3.15) is solvable in $H^1(-1,1)$ if and only if the right-hand side is orthogonal to the kernel of the adjoint operator, that is to the function $u^0(x_1)$ (see (3.4)). Thus, taking into account the normalization condition (3.5), we have

$$\begin{aligned} \lambda^1 = &-\lambda^0 \int_{-1}^{1} F(x_1)\, u^0(x_1)\, dx_1 \\ &+\lambda^0\big(a^{\rm eff}(1)\,\frac{du^0}{dx_1}(1)\,\hat w^+ - a^{\rm eff}(-1)\,\frac{du^0}{dx_1}(-1)\,\hat w^-\big). \end{aligned} \tag{3.16}$$

Under our standing assumptions $v^1 \in C^{2,\alpha}[-1,1]$. Notice that $v^1(x_1)$ is defined up to a function of the form $C\,u^0(x_1)$, where C is a constant. We fix the choice of v^1 setting

$$\int_{-1}^{1} v^1(x_1)\,u^0(x_1)\,dx_1 = 0.$$

In this way the function

$$u^0(x_1) + \varepsilon\left[N^{1,1}\big(x_1, \frac{x'}{\varepsilon}\big)\frac{du^0(x_1)}{dx_1} + v^1(x_1)\right]$$
$$+\varepsilon\left[w^+\big(\frac{x_1-1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}^+\right] + \varepsilon\left[w^-\big(\frac{x_1+1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}^-\right]$$

satisfies the homogeneous Dirichlet boundary conditions on $S_{\pm 1}$. We denote

$$u_{\rm bl}^{\varepsilon,\pm}(x) \equiv \tilde{u}_{\rm bl}^{\varepsilon,\pm}(y)\Big|_{y=\frac{x}{\varepsilon}} = w^{\pm}\big(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}^{\pm}, \tag{3.17}$$

where

$$\tilde{u}_{\rm bl}^{\varepsilon,\pm}(y) = w^{\pm}\big(y_1 \mp \frac{1}{\varepsilon}, y'\big) - \hat{w}^{\pm}.$$

REMARK 3.4. If assumption (2.2) does not hold, then problem (3.12) depends on a parameter $\delta = \delta(\varepsilon) \in [0,1)$ being the fractional part of $1/\varepsilon$. In this case the boundary layer functions $w^{\pm}(y)$ also depend on δ, so do $\hat{w}^{\pm}$, v^1 and λ^1. Nevertheless, all the results of Theorem 2.5 remain valid. We assume (2.2) just for presentation simplicity. The dependence on $\delta(\varepsilon)$ does not create any additional technical difficulties.

REMARK 3.5. We succeeded in constructing exponential boundary layer correctors $u_{\rm bl}^{\varepsilon,\pm}$ owing to the special structure of the domain G_ε. This allowed us to define v^1, λ^1 and other higher order terms of the asymptotic expansion (3.1). In the case of a generic smooth bounded domain one is unable to construct such a boundary layer due to the disagreement between the periodic structure and the domain boundary. By this reason in [**11**] and [**12**] only two leading terms of the expansion have been constructed.

3.2. Justification procedure in the case $\langle \rho(x_1,\cdot)\rangle > 0$.

Let $\lambda_j^{0,+}$ be the jth eigenvalue and $u_j^{0,+}$ the corresponding eigenfunction of problem (3.4). For any $j \in \mathbb{N}$ we denote

$$U_j^{\varepsilon,+}(x) = u_j^{0,+}(x_1) + \varepsilon\, N^{1,1}\big(x_1, \frac{x}{\varepsilon}\big)\,\frac{du_j^{0,+}(x_1)}{dx_1}$$
$$+\varepsilon\, v_j^{1,+}(x_1) + \varepsilon\,\big(u_{\rm bl}^{\varepsilon,+}(x) + u_{\rm bl}^{\varepsilon,-}(x)\big), \tag{3.18}$$

where $u_j^{0,+}$, $N^{1,1}$ and $v_j^{1,+}$ solve problems (3.4), (3.2) and (3.15), respectively (with $u^0 = u_j^{0,+}$ and $\lambda^0 = \lambda_j^{0,+}$). The boundary layer functions $u_{\rm bl}^{\varepsilon,\pm}$ are defined by (3.17) and (3.13). Let us emphasize that, due to the presence of the boundary layer terms, the function $U_j^{\varepsilon,+}$ satisfies the homogeneous Dirichlet boundary conditions on $S_{\pm 1}$, and, as a consequence, belong to the space $\mathcal{H}^\varepsilon$.

The goal of this section is to prove the following result.

THEOREM 3.6. *Let conditions* $(\mathbf{H0})-(\mathbf{H3})$ *be fulfilled, and suppose that* $\langle\rho(x_1,\cdot)\rangle > 0$ *for any* $x_1 \in [-1,1]$. *If* $\lambda_j^{\varepsilon,+}$ *is the* j*th positive eigenvalue of problem* (2.1) *and* $u_j^{\varepsilon,+}$ *is the corresponding eigenfunction, then the following statements hold:*

(i) *For any* $j \in \mathbb{N}$, *there exist* ε_j *and* $C_j > 0$ *such that*

$$|\lambda_j^{\varepsilon,+} - \lambda_j^{0,+}| \le C_j\,\varepsilon, \quad \forall\varepsilon \in (0,\varepsilon_j].$$

(ii) *For any* $j \in \mathbb{N}$

$$\|u_j^{\varepsilon,+} - U_j^{\varepsilon,+}\|_{H^1(G_\varepsilon)} \le C_j\,\varepsilon\,\varepsilon^{\frac{d-1}{2}}$$

where $U_j^{\varepsilon,+}$ *is defined by* (3.18), *and* $(\lambda_j^{0,+}, u_j^{0,+})$ *is the jth eigenpair of the limit problem* (3.4). *Moreover, the "almost eigenfunctions" are almost orthonormal, that is*

$$\left|\frac{\varepsilon^{-(d-1)}}{|Q|}\,(a^\varepsilon\,\nabla U_i^{\varepsilon,+}, \nabla U_j^{\varepsilon,+})_{L^2(G_\varepsilon)} - \delta_{ij}\right| \le C_j\,\varepsilon.$$

(iii) *For any* $j \in \mathbb{N}$, $\lambda_j^{\varepsilon,+}$ *is simple, for sufficiently small* $\varepsilon > 0$.

REMARK 3.7. The estimates of Theorem 3.6 rely on the presence of the boundary layer correctors in the asymptotics of $u_j^{\varepsilon,+}$. The estimates obtained in [**11**] and [**12**] for a generic smooth domain are of order $\sqrt{\varepsilon}$.

PROOF OF THEOREM 3.6. We make use of the following statement about "almost eigenvalues and eigenfunctions" (see [**18**], [**19**]).

LEMMA 3.8. *Given a compact self-adjoint operator* $\mathcal{K}^\varepsilon : \mathcal{H}^\varepsilon \to \mathcal{H}^\varepsilon$, *let* $\nu \in \mathbb{R}$ *and* $v \in \mathcal{H}^\varepsilon$ *be such that*

$$\|v\|_{\mathcal{H}^\varepsilon} = 1, \quad \delta \equiv \|\mathcal{K}^\varepsilon\,v - \nu\,v\|_{\mathcal{H}^\varepsilon} < |\nu|.$$

Then there exists an eigenvalue μ_l^ε *of the operator* $\mathcal{K}^\varepsilon$ *such that*

$$|\mu_l^\varepsilon - \nu| \le \delta.$$

Moreover, for any $\delta_1 \in (\delta, |\nu|)$ *there exist coefficients* $\{b_j^\varepsilon\} \in \mathbb{R}$ *satisfying*

$$\|v - \sum b_j^\varepsilon u_j^\varepsilon\|_{\mathcal{H}^\varepsilon} \le 2\frac{\delta}{\delta_1},$$

where the sum is taken over all the eigenvalues of the operator $\mathcal{K}^\varepsilon$ *in the segment* $[\nu - \delta_1, \nu + \delta_1]$, *and* $\{u_j^\varepsilon\}$ *are the corresponding orthonormalized in* $\mathcal{H}^\varepsilon$ *eigenfunctions. The coefficients* b_j^ε *are normalized by* $\sum |b_j^\varepsilon|^2 = 1$.

As $v \in \mathcal{H}^\varepsilon$ and $\nu \in \mathbb{R}$ in Lemma 3.8 we use the normalized ansatz (3.18)

$$\mathcal{U}_j^{\varepsilon,+} = \frac{U_j^{\varepsilon,+}}{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}}$$

and the numbers $(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})^{-1}$, respectively. Here $\lambda_j^{1,+}$ is defined by formula (3.16) with $u^0=u_j^{0,+}$.

LEMMA 3.9. *For any $j\in\mathbb{N}$ there is $\varepsilon_j>0$ such that*

$$\|\mathcal{K}^\varepsilon\,\mathcal{U}_j^{\varepsilon,+}-(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})^{-1}\,\mathcal{U}_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}\le C_j\,\varepsilon,\qquad \varepsilon<\varepsilon_j, \tag{3.19}$$

for some constant C_j that does not depend on ε.

PROOF. Letting

$$I^\varepsilon\equiv\|\mathcal{K}^\varepsilon\,\mathcal{U}_j^{\varepsilon,+}-(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})^{-1}\,\mathcal{U}_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon},$$

after straightforward rearrangements we have

$$\begin{aligned}
I^\varepsilon&=\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}}\Big|\Big(\mathcal{K}^\varepsilon\,\mathcal{U}_j^{\varepsilon,+}-(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})^{-1}\,\mathcal{U}_j^{\varepsilon,+}\,,\,w\Big)_{\mathcal{H}^\varepsilon}\Big|\\
&=\frac{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}^{-1}}{(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})}\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}}\Big|\Big((\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})\,\mathcal{K}^\varepsilon\,U_j^{\varepsilon,+}-U_j^{\varepsilon,+}\,,\,w\Big)_{\mathcal{H}^\varepsilon}\Big|\\
&=\frac{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}^{-1}}{(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})}\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}}\Big|(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})\,\big(\rho^\varepsilon\,U_j^{\varepsilon,+},w\big)_{L^2(G_\varepsilon)}\\
&\qquad\qquad-\big(a^\varepsilon\,\nabla U_j^{\varepsilon,+}\,,\,\nabla w\big)_{L^2(G_\varepsilon)}\Big|.
\end{aligned}$$

Integrating by parts and using the boundary conditions for $N^{1,1}$ yield

$$\begin{aligned}
I^\varepsilon&=\frac{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}^{-1}}{(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})}\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}}\Big|\Big(\mathcal{A}^\varepsilon\,U_j^{\varepsilon,+}-(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})\,\rho^\varepsilon\,U_j^{\varepsilon,+}\,,\,w\Big)_{L^2(G_\varepsilon)}\\
&+\varepsilon\int\limits_{\Sigma_\varepsilon}(a_{\cdot 1}^\varepsilon(x),n)\,\frac{\partial}{\partial x_1}\big(N^{1,1}(x_1,y)\,\frac{du_j^{0,+}(x_1)}{dx_1}+v_j^{1,+}(x_1)\big)\big|_{y=x/\varepsilon}\,d\sigma\\
&+\varepsilon\int\limits_{\Sigma_\varepsilon}\big(a^\varepsilon\nabla_y(\tilde u_{\rm bl}^{\varepsilon,-}+\tilde u_{\rm bl}^{\varepsilon,+}),n\big)\,w\,d\sigma\Big|\\
&=\frac{\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}^{-1}}{(\lambda_j^{0,+}+\varepsilon\lambda_j^{1,+})}\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}}\Big|(\varepsilon^0I_0^\varepsilon+\varepsilon^1I_1^\varepsilon+\varepsilon^2I_2^\varepsilon,w)_{L^2(G_\varepsilon)}\\
&+\varepsilon\int\limits_{\Sigma_\varepsilon}(a_{\cdot 1}^\varepsilon(x),n)\,\frac{\partial}{\partial x_1}\big(N^{1,1}(x_1,y)\,\frac{du_j^{0,+}(x_1)}{dx_1}+v_j^{1,+}(x_1)\big)\big|_{y=x/\varepsilon}\,d\sigma\\
&+\varepsilon\int\limits_{\Sigma_\varepsilon}\big(a^\varepsilon\nabla_y(\tilde u_{\rm bl}^{\varepsilon,-}+\tilde u_{\rm bl}^{\varepsilon,+}),n\big)\,w\,d\sigma\Big|.
\end{aligned}$$

Here

$$I_0^\varepsilon(x) = I_0(x_1,y)\big|_{y=x/\varepsilon} = -\frac{\partial}{\partial x_1}\Big(a_{1\cdot}\nabla_y N^{1,1}\,\frac{du_j^{0,+}}{dx_1}\Big)$$
$$-\frac{\partial}{\partial x_1}\Big(a_{11}\frac{du_j^{0,+}}{dx_1}\Big) - \lambda_j^{0,+}\,\rho\,u_j^{0,+}\Big|_{y=x/\varepsilon}.$$

$$I_1^\varepsilon(x) = I_{\rm bl}^\varepsilon(x) - \{\mathrm{div}_x + \frac{1}{\varepsilon}\mathrm{div}_y\}\,\big(a_{\cdot 1}\frac{\partial}{\partial x_1}\big(N^{1,1}\,\frac{du_j^{0,+}}{dx_1} + v_j^{1,+}\big)\big|_{y=x/\varepsilon}\big)$$
$$-\lambda_j^{1,+}\,\rho\,\big(N^{1,1}\,\frac{du_j^{0,+}}{dx_1} + v_j^{1,+}\big)\big|_{y=x/\varepsilon} - \lambda_j^{1,+}\,\rho^\varepsilon\,u_j^{0,+},$$

$$I_{\rm bl}^\varepsilon(x) = \mathcal{A}^\varepsilon\,(u_{\rm bl}^{\varepsilon,-} + u_{\rm bl}^{\varepsilon,+}) - \lambda_j^{0,+}\,\rho^\varepsilon\,(u_{\rm bl}^{\varepsilon,-} + u_{\rm bl}^{\varepsilon,+}),$$
$$I_2^\varepsilon(x) = \lambda_j^{1,+}\,\rho(x_1,y)\,N^{1,1}(x_1,y)\,\frac{du_j^{0,+}}{dx_1}\big|_{y=x/\varepsilon}$$
$$+\lambda_j^{1,+}\,\rho(x_1,y)\,v_j^{1,+}(x_1)\big|_{y=x/\varepsilon} + \lambda_j^{1,+}\,(u_{\rm bl}^{\varepsilon,-}(x) + u_{\rm bl}^{\varepsilon,+}(x)).$$

PROPOSITION 3.10. *The boundary layer functions $u_{\rm bl}^{\varepsilon,\pm}$ satisfy the estimate*

$$\Big|\varepsilon(\mathcal{A}^\varepsilon\,u_{\rm bl}^{\varepsilon,\pm}, v)_{L^2(G_\varepsilon)} + \varepsilon(a^\varepsilon\nabla u_{\rm bl}^{\varepsilon,\pm}\,v, n)_{L^2(\Sigma_\varepsilon)} - \varepsilon\,\lambda_j^{0,+}\,(\rho^\varepsilon\,u_{\rm bl}^{\varepsilon,\pm}, v)_{L^2(G_\varepsilon)}\Big|$$
$$\le C\,\varepsilon\,\varepsilon^{(d-1)/2}\,\|v\|_{H^1(G_\varepsilon)}, \quad v\in\mathcal{H}^\varepsilon.$$

PROOF. We prove the proposition for $u_{\rm bl}^{\varepsilon,-}$, a similar proof can be performed for $u_{\rm bl}^{\varepsilon,+}$. Due to the definition of $u_{\rm bl}^{\varepsilon,-}$, up to the terms of higher order,

$$\varepsilon\,\mathcal{A}^\varepsilon\,u_{\rm bl}^{\varepsilon,-}(x) = -(\mathrm{div}_x + \frac{1}{\varepsilon}\mathrm{div}_y)\big((x_1+1)\,\frac{\partial a}{\partial x_1}(x_1,y)\,\nabla_y\tilde u_{\rm bl}^{\varepsilon,-}(y)\big)\Big|_{y=x/\varepsilon}.$$

Integrating by parts yields

$$\varepsilon(\mathcal{A}^\varepsilon\,u_{\rm bl}^{\varepsilon,-}, v)_{L^2(G_\varepsilon)} + \varepsilon(a^\varepsilon\nabla_y u_{\rm bl}^{\varepsilon,-}\,v, n)_{L^2(\Sigma_\varepsilon)}$$
$$= \varepsilon\int\limits_{G_\varepsilon}(y_1 + 1/\varepsilon)\,\frac{\partial a}{\partial x_1}(-1,y)\,(\nabla_y\tilde u_{\rm bl}^{\varepsilon,-}(y), \nabla v(x))\big|_{y=x/\varepsilon}\,dx.$$

Schwartz inequality and the exponential decay of $u_{\rm bl}^-$ give

$$\Big|\varepsilon(\mathcal{A}^\varepsilon\,u_{\rm bl}^{\varepsilon,-}, v)_{L^2(G_\varepsilon)} + \varepsilon(a^\varepsilon\nabla_y u_{\rm bl}^{\varepsilon,-}\,v, n)_{L^2(\Sigma_\varepsilon)}\Big| \le C\,\varepsilon\,\varepsilon^{(d-1)/2}\,\|v\|_{H^1(G_\varepsilon)}$$

with the constant C depending only on Λ and Q. Then, due to the boundedness of ρ and the Schwartz inequality,

$$\Big|\varepsilon\,\lambda_j^{0,+}\,(\rho^\varepsilon\,u_{\rm bl}^{\varepsilon,\pm}, v)_{L^2(G_\varepsilon)}\Big| \le C\,\varepsilon\int\limits_{G_\varepsilon}|u_{\rm bl}^{\varepsilon,-}|\,|v|\,dx.$$

By the exponential decay property of $u_{\rm bl}^-$,

$$\|u_{\rm bl}^{\varepsilon,-}\|_{L^2(G_\varepsilon)} \le C\,\sqrt{\varepsilon}\,\varepsilon^{\frac{d-1}{2}}.$$

The last estimate completes the proof. □

Further analysis essentially relies on the following statement.

LEMMA 3.11. *Let $g(x_1,y) \in C^{1,\alpha}([-1,1];C^\alpha(\overline{Y}))$ be such that*

$$\langle g(x_1,\cdot)\rangle = \int\limits_Y g(x_1,y)\,dy = 0.$$

Then, for any $w \in H^1(G_\varepsilon)$, the following estimate is valid:

$$\Big|\int\limits_{G_\varepsilon} g\big(x_1,\frac{x}{\varepsilon}\big)\,w(x)\,dx\Big| \le C\,\varepsilon\,\varepsilon^{\frac{d-1}{2}}\,\|w\|_{H^1(G_\varepsilon)}$$

with a constant C independent of ε.

PROOF. Since $\langle g(x_1,\cdot)\rangle = 0$, then there exists a y_1-periodic function $\psi(x_1,y) \in C^{1,\alpha}([-1,1];C^{2,\alpha}(\overline{Y}))$ being a solution of the problem

$$\begin{cases} -\Delta_y\,\psi(x_1,y) = g(x_1,y), & y \in Y,\\ (\nabla_y\Psi(x_1,y),n) = 0, & y \in \partial Y.\end{cases}$$

Then we have

$$\begin{aligned}
&\int\limits_{G_\varepsilon} g(x_1,y)\,w(x)\,dx = \varepsilon\int\limits_{G_\varepsilon}(\nabla_y\psi(x_1,y)\,,\,\nabla w(x))\big|_{y=x/\varepsilon}\,dx\\
&+\varepsilon\int\limits_{G_\varepsilon} w(x)\,{\rm div}_x(\nabla_y\psi(x_1,y))\big|_{y=x/\varepsilon}\,dx\\
&\le C\,\varepsilon\,\varepsilon^{\frac{d-1}{2}}\,\|w\|_{H^1(G_\varepsilon)}.
\end{aligned}$$

□

Let us turn back to the proof of Lemma 3.9. Since $u_j^{0,+}$ is a solution of problem (3.4), then $I_0(x_1,y) \in C^{1,\alpha}([-1,1];C^\alpha(\overline{Y}))$ and

$$\int\limits_Y I_0(x_1,y)\,dy = 0.$$

Thus, by Lemma 3.11,

$$\Big|\int\limits_{G_\varepsilon} I_0^\varepsilon(x)\,w(x)\,dx\Big| \le C\,\varepsilon\,\varepsilon^{\frac{d-1}{2}}\,\|w\|_{H^1(G_\varepsilon)}. \tag{3.20}$$

The terms containing $u^\varepsilon_{\rm bl}$ have been estimated in Proposition 3.10. Integrating by parts the remaining terms of $(I^\varepsilon_1, w)_{L^2(G_\varepsilon)}$, using $(\mathbf{H0})$ and the regularity properties of $u_j^{0,+}$,$N^{1,1}$ and $v_j^{1,+}$, one can show that

$$\tag{3.21} \begin{aligned} &\Big|\varepsilon\,(I^\varepsilon_1, w)_{L^2(G_\varepsilon)} + \varepsilon \int\limits_{\Sigma_\varepsilon} (a^\varepsilon_{\cdot 1}, n)\,\frac{\partial}{\partial x_1}\big(N^{1,1}\,\frac{du_j^{0,+}}{dx_1} + v_j^{1,+}\big)\big|_{y=x/\varepsilon}\, w d\sigma\Big| \\ &\le C\,\varepsilon\,\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}. \end{aligned}$$

The quantity $(I^\varepsilon_2, w)_{L^2(G_\varepsilon)}$ is estimated in a similar way:

$$\tag{3.22} \varepsilon^2\,|(I^\varepsilon_2, w)_{L^2(G_\varepsilon)}| \le C\,\varepsilon^2\,\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}.$$

It remains to estimate the norm $\|U_j^{\varepsilon,+}\|_{\mathcal{H}^\varepsilon}$. To this end we compute first the gradient of $U_j^{\varepsilon,+}$:

$$\begin{aligned} &\frac{\partial}{\partial x_1}U_j^{\varepsilon,+} = \frac{du_j^{0,+}(x_1)}{dx_1} + \varepsilon\,\frac{\partial u_j^{1,+}}{\partial x_1}(x_1,y) \\ &+\frac{\partial}{\partial y_1}N^{1,1}(x_1,y)\,\frac{du_j^{0,+}(x_1)}{dx_1} + \frac{\partial}{\partial y_1}\,(\tilde u^{\varepsilon,+}_{\rm bl}(y) + \tilde u^{\varepsilon,-}_{\rm bl}(y))\Big|_{y=\frac{x}{\varepsilon}}; \\ &\frac{\partial}{\partial x_k}U_j^{\varepsilon,+} = \frac{\partial}{\partial y_k}N^{1,1}(x_1,y)\,\frac{du_j^{0,+}(x_1)}{dx_1} + \frac{\partial}{\partial y_k}(\tilde u^{\varepsilon,+}_{\rm bl}(y) + \tilde u^{\varepsilon,-}_{\rm bl}(y))\Big|_{y=\frac{x}{\varepsilon}},\ \ k\neq 1. \end{aligned}$$

where

$$u_j^{1,+}(x_1,y) = N^{1,1}(x_1,y)\,\frac{du_j^{0,+}(x_1)}{dx_1} + v_j^{1,+}(x_1).$$

It is easy to see that

$$\begin{aligned} &(a^\varepsilon\nabla U_i^{\varepsilon,+}, \nabla U_j^{\varepsilon,+}) = \Big[a_{11}(x_1,y) + a_{\cdot 1}(x_1,y)\nabla_y N^{1,1}(x_1,y)\Big]\,\frac{du_i^{0,+}}{dx_1}\,\frac{du_j^{0,+}}{dx_1} \\ &+\Big[a_{1\cdot}(x_1,y) + a(x_1,y)\nabla_y N^{1,1}(x_1,y)\Big]\,\nabla_y N^{1,1}(x_1,y)\,\frac{du_i^{0,+}}{dx_1}\,\frac{du_j^{0,+}}{dx_1} \\ &+J^\varepsilon_{xx}(x_1,y) + J^\varepsilon_{xy}(x_1,y) + J^\varepsilon_{yy}(x_1,y), \quad y = \frac{x}{\varepsilon} \end{aligned}$$

where

$$J^\varepsilon_{xx}(x_1,y) = \varepsilon\,a_{11}\frac{du_i^{0,+}}{dx_1}\frac{du_j^{1,+}}{dx_1} + \varepsilon\,a_{11}\frac{\partial u_i^{1,+}}{\partial x_1}\,\frac{du_j^{0,+}}{dx_1} + \varepsilon^2\,a_{11}\frac{\partial u_i^{1,+}}{\partial x_1}\,\frac{\partial u_j^{1,+}}{\partial x_1};$$

$$J^\varepsilon_{xy}(x_1,y) = \varepsilon a_{\cdot 1}\,\nabla_y N^{1,1}\,\frac{\partial u_i^{1,+}}{\partial x_1}\,\frac{du_j^{0,+}}{dx_1} + \varepsilon\,a_{1\cdot}\,\nabla_y N^{1,1}\,\frac{\partial u_j^{1,+}}{\partial x_1}\,\frac{du_i^{0,+}}{dx_1};$$

$$
\begin{aligned}
J^{\varepsilon}_{yy}(x_1,y) &= a_{\cdot 1}\,\nabla_y(\tilde u^{\varepsilon,+}_{\rm bl}+\tilde u^{\varepsilon,-}_{\rm bl})\,\frac{du_i^{0,+}}{dx_1}+\varepsilon\, a_{\cdot 1}\,\nabla_y(\tilde u^{\varepsilon,+}_{\rm bl}+\tilde u^{\varepsilon,-}_{\rm bl})\,\frac{\partial u_i^{1,+}}{\partial x_1}\\
&+\varepsilon\, a_{1\cdot}\,\nabla_y(\tilde u^{\varepsilon,+}_{\rm bl}+\tilde u^{\varepsilon,-}_{\rm bl})\,\frac{\partial u_j^{1,+}}{\partial x_1}+(a\,\nabla_y N^{1,1},\nabla_y(\tilde u^{\varepsilon,+}_{\rm bl}+\tilde u^{\varepsilon,-}_{\rm bl}))\,\frac{du_i^{0,+}}{dx_1}\\
&+(a\,\nabla_y N^{1,1},\nabla_y(\tilde u^{\varepsilon,+}_{\rm bl}+\tilde u^{\varepsilon,-}_{\rm bl}))\,\frac{du_j^{0,+}}{dx_1}+(a\,\nabla_y(\tilde u^{\varepsilon,+}_{\rm bl}+\tilde u^{\varepsilon,-}_{\rm bl}),\nabla_y(\tilde u^{\varepsilon,+}_{\rm bl}+\tilde u^{\varepsilon,-}_{\rm bl})).
\end{aligned}
$$

Using the regularity properties of $u_i^{0,+}$ and $N^{1,1}$ one can easily see that

$$
\Big|\int\limits_{G_\varepsilon} J^{\varepsilon}_{xx}(x_1,\frac{x}{\varepsilon})\,dx\Big|\le C\,\varepsilon\,|G_\varepsilon|\le C\,\varepsilon\,\varepsilon^{d-1}.
$$

Then, by the periodicity of $N^{1,1}$ in y_1, we have

$$
\begin{aligned}
|\int\limits_{G_\varepsilon} J^{\varepsilon}_{xy}(x_1,\frac{x}{\varepsilon})\,dx| &\le C\,\varepsilon\int\limits_{G_\varepsilon}|\nabla_y N^{1,1}|_{y=x/\varepsilon}\,dx\\
&= C\,\varepsilon\,\varepsilon^{-1}\,\varepsilon^{d}\int\limits_{Y}|\nabla_y N^{1,1}|\,dy\le C\,\varepsilon\,\varepsilon^{d-1}.
\end{aligned}
$$

Taking into account the exponential decay of $u^{\varepsilon,\pm}_{\rm bl}$ (see Proposition 3.10) we obtain the estimate

$$
|\int\limits_{G_\varepsilon} J^{\varepsilon}_{yy}(x_1,\frac{x}{\varepsilon})\,dx|\le C\,\varepsilon\,\varepsilon^{d-1}.
$$

Thus,

$$
\begin{aligned}
&\Bigg|(a^{\varepsilon}\nabla U_i^{\varepsilon,+},\nabla U_j^{\varepsilon,+})_{L^2(G_\varepsilon)}-\int\limits_{G_\varepsilon}\big\{a_{11}+a_{\cdot 1}\nabla_y N^{1,1}\big\}_{y=x/\varepsilon}\,\frac{du_i^{0,+}}{dx_1}\,\frac{du_j^{0,+}}{dx_1}\,dx\\
&-\int\limits_{G_\varepsilon}\big\{a_{1\cdot}+a\nabla_y N^{1,1}\big\}\,\nabla_y N^{1,1}\big|_{y=x/\varepsilon}\,\frac{du_i^{0,+}}{dx_1}\,\frac{du_j^{0,+}}{dx_1}\,dx\Bigg|\le C\,\varepsilon\,\varepsilon^{d-1}.
\end{aligned}
$$

Considering (3.3) and Lemma 3.11, we get

$$
\Big|(a^{\varepsilon}\nabla U_i^{\varepsilon,+},\nabla U_j^{\varepsilon,+})_{L^2(G_\varepsilon)}-\int\limits_{G_\varepsilon} a^{\rm eff}(x_1)\frac{du_i^{0,+}}{dx_1}\,\frac{du_j^{0,+}}{dx_1}\,dx\Big|\le C\,\varepsilon\,\varepsilon^{d-1}.
$$

Consequently, in view of the normalization condition (3.5), one has

$$
\Big|(a^{\varepsilon}\nabla U_i^{\varepsilon,+},\nabla U_j^{\varepsilon,+})_{L^2(G_\varepsilon)}-|Q|\,\varepsilon^{d-1}\,\delta_{ij}\Big|\le C\,\varepsilon\,\varepsilon^{d-1}, \tag{3.23}
$$

and, for sufficiently small ε

$$
\varepsilon^{-\frac{(d-1)}{2}}\,\|U_i^{\varepsilon,+}\|_{\mathcal H^{\varepsilon}}\ge\frac{|Q|^{1/2}}{2},\quad \varepsilon<\varepsilon_i. \tag{3.24}
$$

Combining estimates (3.20), (3.21), (3.22), (4.35) and Proposition 3.10 yields the desired bound (3.19). Lemma 3.9 is proved. $\square$

Combining Lemma 3.9 and Lemma 3.8, we conclude that for any eigenvalue $\lambda_j^{0,+}$ of problem (3.4) there exists an eigenvalue $\mu_q^{\varepsilon,+}$ of the operator $\mathcal{K}^\varepsilon$ such that

$$|\mu_q^{\varepsilon,+} - (\lambda_j^{0,+} + \varepsilon\lambda_j^{1,+})^{-1}| \le \tilde{c}_j\varepsilon.$$

Considering the fact that $\lambda_q^{\varepsilon,+} = (\mu_q^{\varepsilon,+})^{-1}$, we have

$$|\lambda_q^{\varepsilon,+} - \lambda_j^{0,+}| \le c_j\,\varepsilon, \quad \varepsilon < \varepsilon_j. \tag{3.25}$$

Generally speaking, there might be more than one eigenvalue of the operator $\mathcal{A}^\varepsilon$ (problem (2.1)) satisfying inequality (5.14), but we will show that in the case under consideration such an eigenvalue $\lambda_j^{\varepsilon,+}$ is unique if $\varepsilon < \varepsilon_j$.

LEMMA 3.12. *For any* q, *the estimate holds*

$$0 < m \le \lambda_q^{\varepsilon,+} \le M_q.$$

PROOF. Let us first estimate the norm of the operator $\mathcal{K}^\varepsilon$.

$$\|\mathcal{K}^\varepsilon\| = \sup_{\|u\|_{\mathcal{H}^\varepsilon}=1} (\mathcal{K}^\varepsilon u, u)_{\mathcal{H}^\varepsilon} = \sup_{\|u\|_{\mathcal{H}^\varepsilon}=1} (\rho^\varepsilon u, u)_{L^2(G_\varepsilon)} \le C\,\|u\|_{L^2(G_\varepsilon)}$$

where C does not depend on ε. Thus, $\mu_q^{\varepsilon,+} \le C$, for any q, and, consequently, $\lambda_q^{\varepsilon,+} \ge m$ with m independent of ε.

In order to show that the inverse inequality is valid, we recall that for any $\lambda_j^{0,+}$ there is an eigenvalue of $\mathcal{K}^\varepsilon$ such that

$$\mu(\varepsilon, j) \to (\lambda_j^{0,+})^{-1}, \quad \varepsilon \to 0.$$

It implies that $\mu(\varepsilon, j) \ge c_j$ and, moreover, $\mu_k^{\varepsilon,+} \ge c_j$ for all $k \ge j$. Lemma 3.12 is proved. $\square$

It follows from Lemma 3.12 that, up to a subsequence, $\lambda_j^{\varepsilon,+}$ converges to some λ_*, as $\varepsilon \to 0$.

LEMMA 3.13. *Suppose that (perhaps for a subsequence)*

$$\lambda_j^{\varepsilon,+} \to \lambda_*, \quad \varepsilon \to 0.$$

Then λ_* *is an eigenvalue of problem* (3.4).

There are several different ways of proving Lemma 3.13. Here we expose the proof based on the technique of convergence in variable spaces with singular measures.

Introduce the "universal domain" $K_d = [-1,1]^d$. For ε small enough, $G_\varepsilon \subset K_d$. In what follows, for arbitrary Borel set $B \subset K_d$, we denote

$$\mu_\varepsilon(B) = \frac{\varepsilon^{-(d-1)}}{|Q|} \int_B \chi(G_\varepsilon)\,dx, \tag{3.26}$$

where $\chi(G_\varepsilon)$ is the characteristic function of G_ε; dx is a usual d-dimensional Lebesgue measure. Then μ_ε converges weakly to a measure $\mu_* = dx_1 \times \delta(x')$, as $\varepsilon \to 0$. For any ε, the space of Borel measurable functions $g(x)$ such that

$$\int\limits_{K_d} (g(x))^2 \, d\mu_\varepsilon(x) < \infty$$

is denoted $L^2(K_d, \mu_\varepsilon)$.

Let us also recall the definition of the Sobolev space with measure.

DEFINITION 3.14. We say that a function $g \in L^2(K_d, \mu_\varepsilon)$ belongs to the space $H^1(K_d, \mu_\varepsilon)$ if there exists a vector function $z \in L^2(K_d, \mu_\varepsilon)^d$ and a sequence $\varphi_k \in C^\infty(K_d)$ such that

$$\begin{aligned} \varphi_k \to g \quad &\text{in } L^2(K_d, \mu_\varepsilon), \quad k \to \infty, \\ \nabla\varphi_k \to z \quad &\text{in } L^2(K_d, \mu_\varepsilon)^d, \quad k \to \infty. \end{aligned}$$

In this case z is called the gradient of g and is denoted by $\nabla^{\mu_\varepsilon} g$.

Since in our case the measure μ_ε is a weighted Lebesgue measure, then $\nabla^{\mu_\varepsilon} g = \nabla g$ and the space $H^1(K_d, \mu_\varepsilon)$ coincides with the usual Sobolev space $H^1(G_\varepsilon)$.

DEFINITION 3.15.
We say that a sequence of functions $\{g^\varepsilon(x)\} \subset L^2(K_d, \mu_\varepsilon)$ weakly converges in $L^2(K_d, \mu_\varepsilon)$ to a function $g(x_1) \in L^2(K_d, \mu_*)$, as $\varepsilon \to 0$, if

(i) $\|g^\varepsilon\|_{L^2(K_d,\mu_\varepsilon)} \le C$;

(ii) for any $\varphi \in C^\infty(\mathbb{R})$ the following limit relation holds:

$$\lim_{\varepsilon \to 0} \int\limits_{K_d} g^\varepsilon(x) \, \varphi(x) \, d\mu_\varepsilon(x) = \int\limits_{K_d} g(x_1) \, \varphi(x) \, d\mu_*(x).$$

A sequence $\{g^\varepsilon\}$ is said to converge strongly to $g(x_1)$ in $L^2(K_d, \mu_\varepsilon)$, as $\varepsilon \to 0$, if it converges weakly and

$$\lim_{\varepsilon \to 0} \int\limits_{K_d} g^\varepsilon(x) \, \psi^\varepsilon(x) \, d\mu_\varepsilon(x) = \int\limits_{K_d} g(x_1) \, \psi(x_1) \, d\mu_*(x)$$

for any sequence $\{\psi^\varepsilon(x)\}$ weakly converging to $\psi(x_1)$ in $L^2(K_d, \mu_\varepsilon)$.

Notice that the property of weak compactness of a bounded sequence in a separable Hilbert space remains valid with respect to the convergence in variable spaces.

In order to prove Lemma 3.13 we use the technique of two-scale convergence in variable spaces with measure, so for the reader's convenience we recall the relevant definition.

DEFINITION 3.16. We say that $g^\varepsilon \in L^2(K_d, \mu_\varepsilon)$ two-scale converges in $L^2(K_d, \mu_\varepsilon)$ to a function $\tilde{g}(x_1, y) \in L^2(K_d \times Y, \mu_* \times dy)$, as $\varepsilon \to 0$, if

(i)

$$\|g^\varepsilon\|_{L^2(K_d,\mu_\varepsilon)} \le C, \quad \varepsilon > 0;$$

(ii) The following limit relation holds:

$$\lim_{\varepsilon\to 0}\int\limits_{K_d} g^\varepsilon(x)\,\varphi(x)\,\psi(\frac{x}{\varepsilon})d\mu_\varepsilon(x) = \int\limits_{K_d}\int\limits_{Y} \tilde g(x_1,y)\,\varphi(x)\,\psi(y)\,dy\,d\mu_*(x)$$

for any $\varphi\in C^\infty(K_d)$, $\varphi\big|_{x_1=\pm 1}=0$, and $\psi(y)\in C^\infty(Y)$ periodic in y_1.

PROOF OF LEMMA 3.13 By the normalization condition (2.6)

$$\|u_j^{\varepsilon,+}\|_{L^2(K_d,\mu_\varepsilon)} + \|\nabla u_j^{\varepsilon,+}\|_{L^2(K_d,\mu_\varepsilon)^d} \le C, \tag{3.27}$$

thus, up to a subsequence, $u_j^{\varepsilon,+}(x)$ converges weakly in $L^2(K_d,\mu_\varepsilon)$ to a function $u_*(x_1)\in L^2(K_d,\mu_*)$, as $\varepsilon\to 0$. Let us show that in fact the convergence is strong. Denote

$$\bar u_j^\varepsilon(x_1) = \int\limits_{\varepsilon Q} u_j^{\varepsilon,+}(x_1,x')\,dx'.$$

Then, due to the Poincaré inequality,

$$\int\limits_{\varepsilon Q} (u_j^{\varepsilon,+}(x)-\bar u_j^\varepsilon(x_1))^2\,dx' \le C\,\varepsilon^2 \int\limits_{\varepsilon Q} |\nabla(u_j^{\varepsilon,+}(x)-\bar u_j^\varepsilon(x_1))|^2\,dx'.$$

Integrating with respect to x_1 and taking into account (3.27), we get

$$\int\limits_{K_d} (u_j^{\varepsilon,+}(x)-\bar u_j^\varepsilon(x_1))^2\,d\mu_\varepsilon \le C\,\varepsilon.$$

On the other hand, $\bar u_j^\varepsilon(x_1)$ is uniformly bounded in $H^1(-1,1)$, thus there exists $\bar u(x_1)$ such that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{-(d-1)}}{|Q|}\int\limits_{G_\varepsilon} (\bar u_j^\varepsilon(x_1))^2\,dx = \int\limits_{-1}^{1} (\bar u(x_1))^2\,dx_1.$$

The strong convergence of $u_j^{\varepsilon,+}(x)$ to $\bar u(x_1) = u_*(x_1)$ in $L^2(K_d,\mu_\varepsilon)$ is the immediate consequence of the last two formulae.

By Lemma 3.11, $\rho^\varepsilon(x)$ converges weakly to $\langle\rho(x_1,\cdot)\rangle$ in $L^2(K_d,\mu_\varepsilon)$. Thus,

$$\lambda_j^{\varepsilon,+}\,\rho^\varepsilon(x)\,u_j^{\varepsilon,+}(x) \longrightarrow \lambda_*\,\langle\rho(x_1,\cdot)\rangle\,u_*(x_1) \quad \textit{weakly in } L^2(K_d,\mu_\varepsilon),\ \varepsilon\to 0.$$

Denoting

$$f^\varepsilon(x) = \lambda_j^{\varepsilon,+}\,\rho^\varepsilon(x)\,u_j^{\varepsilon,+}(x), \quad f^0(x_1) = \lambda_*\,\langle\rho(x_1,\cdot)\rangle\,u_*(x_1),$$

we arrive at the following boundary value problem:

$$\begin{cases} \mathcal{A}^\varepsilon u_j^{\varepsilon,+}(x) = f^\varepsilon(x), & x\in G_\varepsilon,\\ \mathcal{B}^\varepsilon u_j^{\varepsilon,+}(x) = 0, & x\in \Sigma_\varepsilon,\\ u_j^{\varepsilon,+}(\pm 1,x') = 0, & x'\in\varepsilon Q. \end{cases} \tag{3.28}$$

The homogenization theorem for locally periodic elliptic equations in variable spaces (see [**2**], [**20**]) implies that

$$u_j^{\varepsilon,+}(x) \longrightarrow u_*(x_1) \quad wealky\ in\ L^2(K_d,\mu_\varepsilon), \quad \varepsilon\to 0,$$

$$a^\varepsilon(x)\,\nabla u_j^{\varepsilon,+}(x) \longrightarrow \{a^{\rm eff}(x_1)\frac{du_*}{dx_1}(x_1),0,\cdots 0\}^T \quad wealky\ in\ L^2(K_d,\mu_\varepsilon)^d, \quad \varepsilon\to 0,$$

where $u_*(x_1)\in H_0^1(-1,1)$ is a solution of problem (3.4).

It follows from the normalization condition (2.6), boundedness of $\rho(x_1,y)$ and $\lambda_j^{\varepsilon,+}$ that

$$1=\frac{\varepsilon^{-(d-1)}}{|Q|}\,(a^\varepsilon\,\nabla u_j^{\varepsilon,+},\nabla u_j^{\varepsilon,+})_{L^2(G_\varepsilon)}=\lambda_j^{\varepsilon,+}\int\limits_{K_d}\rho^\varepsilon\,(u_j^{\varepsilon,+})^2\,d\mu_\varepsilon\le C_j\,\|u_j^{\varepsilon,+}\|^2_{L^2(K_d,\mu_\varepsilon)}.$$

Considering the strong convergence of $u_j^{\varepsilon,+}$ to u_* in $L^2(K_d,\mu_\varepsilon)$, we conclude that $u_*\neq 0$. Thus, (λ_*,u_*) is an eigenpair of the effective problem (3.4). Lemma 3.13 is proved.

Turning back to the proof of Theorem 3.6, suppose that there exist two different eigenvalues $\lambda_i^{\varepsilon,+}\neq\lambda_j^{\varepsilon,+}$ satisfying inequality (5.14) with $\lambda^{0,+}$ being an eigenvalue of the operator $\mathcal{A}^0$. As was proved in Lemma 3.13, in this case the corresponding eigenfunctions $u_i^{\varepsilon,+}$ and $u_j^{\varepsilon,+}$ converge strongly in $L^2(K_d,\mu_\varepsilon)$ to the eigenfunctions $u_i^{0,+}$ and $u_j^{0,+}$ of $\mathcal{A}^0$, which correspond to $\lambda^{0,+}$. Let us show that $u_i^{0,+}$ and $u_j^{0,+}$ are linearly independent. By the normalization condition

$$\lambda_i^{\varepsilon,+}\,(\rho^\varepsilon u_i^{\varepsilon,+},u_j^{\varepsilon,+})_{L^2(K_d,\mu_\varepsilon)}=\delta_{ij}.$$

Notice that, by Lemma 3.11, ρ^ε converges weakly in $L^2(K_d,\mu_\varepsilon)$ to its average $\langle\rho(x_1,\cdot)\rangle$. Thus, passing to the limit in the last identity, we obtain

$$\lambda^{0,+}\int\limits_{K_d}\langle\rho(x_1,\cdot)\rangle\,u_i^{0,+}(x_1)\,u_j^{0,+}(x_1)\,d\mu_*=\delta_{ij}$$

that implies the linear independence of $u_i^{0,+}$ and $u_j^{0,+}$. But $\lambda^{0,+}$ as an eigenvalue of $\mathcal{A}^0$ is simple by Theorem 3.2. We arrive at contradiction, thus, for any j there exists a unique $\lambda_j^{\varepsilon,+}$ satisfying (5.14). In particular, it means that for sufficiently small ε the eigenvalues $\lambda_j^{\varepsilon,+}$ are simple.

Combining Lemma 3.8, Lemma 3.12 and Lemma 3.13 one obtains the first statement of Theorem 3.6.

The second statement (ii) of Theorem 3.6 follows immediately from Lemma 3.8 and (i). This completes the proof. $\square$

Theorem 3.6 might be formulated in terms of convergence in variable spaces with measure.

COROLLARY 3.17. *Suppose that conditions* $(\mathbf{H0})-(\mathbf{H3})$ *hold true and* $\langle\rho(x_1,\cdot)\rangle>0$. *Let* $(\lambda_j^{\varepsilon,+},u_j^{\varepsilon,+})$ *and* $(\nu_j^{0,+},u_j^{0,+})$ *be eigenpairs of problems* (2.1) *and* (3.4), *respectively. Then*

(a) *For any* $j \in \mathbb{N}$, $\lambda_j^{\varepsilon,+} \to \lambda_j^{0,+}$, *as* $\varepsilon \to 0$, *and*

$$u_j^{\varepsilon,+}(x) \longrightarrow u_j^{0,+}(x_1) \quad strongly\ in\ L^2(K_d, \mu_\varepsilon), \quad \varepsilon \to 0$$

in terms of Definition 3.15.

(b) *The convergence of fluxes takes place, that is*

$$a^\varepsilon(x)\, \nabla u_j^{\varepsilon,+}(x) \longrightarrow \{a^{\text{eff}}(x_1)\, \frac{du_j^{0,+}}{dx_1}(x_1), 0, \cdots 0\}^T$$

weakly in $L^2(K_d, \mu_\varepsilon)^d$, *as* $\varepsilon \to 0$.

PROOF. The first statement follows from the normalization condition (2.6) (see proof of Lemma 3.13). The convergence of fluxes is a consequence of the homogenization result used while proving Lemma 3.13. □

4. The case $\langle \rho(x_1, \cdot) \rangle = 0$

4.1. Formal asymptotic expansion. Using the arguments similar to those in Section 3.4.1, [**12**], yields

$$c\,\varepsilon^{-1} \le \lambda_1^{\varepsilon,\pm} \le C\,\varepsilon^{-1},$$

for some constants c and C.

Considering the last estimate, we look for a solution of problem (2.1) in the form

$$\begin{aligned} u^\varepsilon(x) &= u^0(x_1) + \varepsilon\, u^1(x_1, y) + \varepsilon^2\, u^2(x_1, y) + \cdots, \quad y = \frac{x}{\varepsilon}, \\ \lambda^\varepsilon &= \varepsilon^{-1}\nu^0 + \nu^1 + \cdots, \end{aligned} \tag{4.1}$$

where ν^0, ν^1, $u^0(x_1)$, $u^1(x_1, y)$ and $u^2(x_1, y)$ are to be determined. We suppose that $u^1(x_1, y)$ and $u^2(x_1, y)$ are 1-periodic in y_1. Substituting asymptotic ansätze (4.1) into (2.1) and collecting terms of order ε^{-1}, we obtain the following equation for the unknown function $u^1(x_1, y)$:

$$\begin{cases} \mathcal{A}_y u^1(x_1, y) = \mathrm{div}_y a_{\cdot 1}(x_1, y)\, \dfrac{du^0(x_1)}{dx_1} + \nu^0\, \rho(x_1, y)\, u^0(x_1), \quad y \in Y, \\ \mathcal{B}_y u^1(x_1, y) = -a_{i1}(x_1, y)\, n_i\, \dfrac{du^0(x_1)}{dx_1}, \quad y \in \partial Y, \\ u^1(x_1, y) \quad \text{is } 1-\text{periodic in } y_1. \end{cases}$$

Note that, since $\langle \rho(x_1, \cdot) \rangle = 0$, the compatibility condition is satisfied. The structure of the right-hand side of the last equation suggests the following representation for $u^1(x_1, y)$:

$$u^1(x_1, y) = N^{1,1}(x_1, y)\, \frac{du^0(x_1)}{dx_1} + \nu^0\, N^{1,0}(x_1, y)\, u^0(x_1) + v^1(x_1). \tag{4.2}$$

Then the functions $N^{1,1}$ and $N^{1,0}$ are 1-periodic in y_1 solutions of the problems

$$\begin{cases} \mathcal{A}_y N^{1,1}(x_1, y) = \mathrm{div}_y a_{\cdot 1}(x_1, y), \quad y \in Y, \\ \mathcal{B}_y N^{1,1}(x_1, y) = -a_{i1}(x_1, y)\, n_i, \quad y \in \partial Y, \\ N^{1,1}(x_1, y) \text{ is } 1-\text{periodic in } y_1; \end{cases} \tag{4.3}$$

$$
(4.4) \qquad \begin{cases} \mathcal{A}_y N^{1,0}(x,y) = \rho(x_1,y), \quad y \in Y, \\ \mathcal{B}_y N^{1,0}(x_1,y) = 0, \quad y \in \partial Y, \\ N^{1,0}(x_1,y) \text{ is } 1-\text{periodic in } y_1. \end{cases}
$$

Under assumption (**H0**) the functions $N^{1,1}(x_1,y)$, $N^{1,0}(x_1,y)$ belong to the space $C^{1,\alpha}([-1,1]\times\overline{Y})$.

Similarly, substituting (4.1) into (2.1) and collecting the terms in front of ε^0, we have

$$
(4.5) \qquad \begin{cases} \mathcal{A}_y u^2(x_1,y) & = \operatorname{div}_y(a_{\cdot 1}(x_1,y)\,\dfrac{\partial u^1}{\partial x_1}(x_1,y) \\ & +\dfrac{\partial}{\partial x_1}(a_{1\cdot}(x_1,y)\nabla_y u^1(x_1,y)) \\ & +\dfrac{\partial}{\partial x_1}(a_{11}(x_1,y)\dfrac{du^0(x_1)}{dx_1}) + \nu^1\,\rho(x_1,y)u^0(x_1) \\ & +\nu^0\,\rho(x_1,y)\,u^1(x_1), \quad y \in Y, \\ \mathcal{B}_y u^2(x_1,y) & = -a_{i1}(x_1,y)\,n_i\,\dfrac{\partial}{\partial x_1}u^1(x_1,y), \quad y \in \partial Y, \\ u^2(x_1,y) & \text{is } \ 1-\text{periodic in } y_1. \end{cases}
$$

The compatibility condition for the last problem reads

$$
(4.6) \qquad \begin{aligned} & \frac{d}{dx_1}\int\limits_Y (a_{11} + a_{1\cdot}(x_1,y)\,\nabla_y N^{1,1}(x_1,y))\,\frac{du^0(x_1)}{dx_1}\,dy \\ & +\nu^0\,\frac{d}{dx_1}\int\limits_Y a_{1\cdot}(x_1,y)\,\nabla_y N^{1,0}(x_1,y))\,u^0(x_1)\,dy \\ & +\nu^0\int\limits_Y \rho(x_1,y)\,N^{1,1}(x_1,y)\,dy\,\frac{du^0(x_1)}{dx_1} \\ & +(\nu^0)^2\int\limits_Y \rho(x_1,y)\,N^{1,0}(x_1,y)\,u^0(x_1)dy = 0 \end{aligned}
$$

Rearranging the last three terms in (4.6) gives

$$
\begin{aligned} & \nu^0\,\frac{d}{dx_1}\int\limits_Y a_{1\cdot}(x_1,y)\,\nabla_y N^{1,0}(x_1,y))\,u^0(x_1)\,dy \\ & +\nu^0\int\limits_Y \rho(x_1,y)\,N^{1,1}(x_1,y)\,dy\,\frac{du^0(x_1)}{dx_1} \\ & +(\nu^0)^2\int\limits_Y \rho(x_1,y)\,N^{1,0}(x_1,y)\,u^0(x_1)dy \\ & = (\nu^0)^2\,u^0(x_1)\int\limits_Y (a(x_1,y)\,\nabla_y N^{1,0}, \nabla_y N^{1,0})\,dy \end{aligned}
$$

$$+\nu^0 u^0(x_1)\frac{d}{dx_1}\int\limits_Y (a(x_1,y)\nabla_y N^{1,1},\nabla_y N^{1,0})\,dy.$$

Denote

$$\mathbf{C}(x_1)=\int\limits_Y (a(x_1,y)\nabla_y N^{1,0},\nabla_y N^{1,0})\,dy; \tag{4.7}$$

$$\mathbf{B}(x_1)=\frac{\partial}{\partial x_1}\int\limits_Y (a(x_1,y)\nabla_y N^{1,1},\nabla_y N^{1,0})\,dy. \tag{4.8}$$

In view of the regularity properties of $N^{1,1}$ and $N^{1,0}$, $\mathbf{C}\in C^{1,\alpha}[-1,1]$ and $\mathbf{B}\in C^{\alpha}[-1,1]$. Thus, (4.6) supplemented with an appropriate boundary condition takes the form of a quadratic operator pencil

$$\begin{cases}\mathbf{\Pi}(\nu^0)u^0(x_1)\equiv -\dfrac{d}{dx_1}\Big(a^{\text{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\Big)+ & \nu^0\,\mathbf{B}(x_1)\,u^0(x_1)\\ -(\nu^0)^2\,\mathbf{C}(x_1)\,u^0(x_1)=0, & x_1\in(-1,1),\\ u^0(-1)=u^0(1)=0. \end{cases} \tag{4.9}$$

The variational formulation of problem (4.9) reads: find $u^0\in H_0^1(-1,1)$, $u^0\neq 0$, such that

$$\int\limits_{-1}^{1} a^{\text{eff}}\frac{du^0}{dx_1}\frac{dv}{dx_1}\,dx_1+\nu^0\int\limits_{-1}^{1}\mathbf{B}\,u^0\,v\,dx_1-(\nu^0)^2\int\limits_{-1}^{1}\mathbf{C}\,u^0\,v\,dx_1=0, \tag{4.10}$$

for any $v\in H_0^1(-1,1)$.

The next theorem characterizes the spectrum of the quadratic operator pencil (4.9).

THEOREM 4.1. *The spectrum of problem* (4.9) *is discrete. The eigenvalues are real, algebraically and geometrically simple, and form two infinite sequences*

$$0<\nu_1^{0,+}<\nu_2^{0,+}<\cdots\nu_j^{0,+}\cdots\to+\infty,$$

$$0>\nu_1^{0,-}>\nu_2^{0,-}>\cdots\nu_j^{0,-}\cdots\to-\infty.$$

The corresponding eigenfunctions can be normalized by

$$\int\limits_{-1}^{1} a^{\text{eff}}\frac{du_i^{0,\pm}}{dx_1}\frac{du_j^{0,\pm}}{dx_1}\,dx_1+\nu_i^{0,\pm}\,\nu_j^{0,\pm}\int\limits_{-1}^{1}\mathbf{C}\,u_i^{0,\pm}\,u_j^{0,\pm}\,dx_1=\delta_{ij}, \tag{4.11}$$

where a^{eff} and $\mathbf{C}$ are defined by (3.3) *and* (4.7)*, respectively.*

PROOF. The existence of infinite number of eigenvalues is given by the following classical theorem (see [**3**], [**8**]).

THEOREM 4.2. Keldysh Theorem

Given compact operators T and H, such that H is a normal operator with $KerH = \{0\}$ ($HH^* = H^*H$) and H^2 is self-adjoint. Consider the Keldysh operator pencil

$$B(\lambda) = Id - \lambda T\, H - \lambda^2\, H^2,$$

where Id is the identity operator. The following statements hold:

(1) *For any $\delta > 0$, there is only finite number of eigenvalues outside the angle*

$$\{\lambda : \big|\arg\lambda - \frac{k\pi}{2}\big| < \delta\}, \quad k = 0, 2;$$

(2) *Denote $N_+(r)$ the number of eigenvalues counted according to their multiplicity of the operator H^2 in the interval $(1/r^2, +\infty)$. Let $N_k(r, B(\lambda))$ be a number of eigenvalues of the operator pencil $B(\lambda)$ contained in the sector*

$$\{\lambda : \big|\arg\lambda - \frac{k\pi}{2}\big| < \frac{\pi}{4}, \quad |\lambda| < r\}, \quad k = 0, 1, 2, 3.$$

If

$$\liminf_{r\to\infty} \frac{\log N_+(r)}{\log r} < \infty, \tag{4.12}$$

then

$$\liminf_{r\to\infty} \Big|\frac{N_{2k}(r, B(\lambda))}{N_+(r)} - 1\Big| = 0, \quad k = 0, 1.$$

In our case the operator pencil has the form

$$\mathbf{\Pi}(\nu^0) = \mathcal{A}^0 + \nu^0\,\mathbf{B}(x_1)\,Id - (\nu^0)^2\,\mathbf{C}(x_1)\,Id.$$

Since $(\mathcal{A}^0)^{-1}$ is a self-adjoint compact positive operator from $L^2(-1,1)$ into itself, then there exists a self-adjoint positive operator $S = (\mathcal{A}^0)^{-1/2}$. It is compact as an operator from $L^2(-1,1)$ into itself, bounded if we consider it as an operator from $L^2(-1,1)$ into $H_0^1(-1,1)$, and compact if it acts on $H_0^1(-1,1)$ with values in $H_0^1(-1,1)$. We apply the operator S to both sides of the operator pencil $\mathbf{\Pi}(\nu^0)$. As a result we obtain

$$\widetilde{\mathbf{\Pi}}(\nu^0) = Id + \nu^0\, S\,\mathbf{B}(x_1)\,S - (\nu^0)^2\, S\,\mathbf{C}(x_1)\,S. \tag{4.13}$$

One can check that $H^2 = S\,\mathbf{C}(x_1)\,S : L^2(-1,1) \to L^2(-1,1)$ is a self-adjoint compact positive operator. Then $H = (S\,\mathbf{C}(x_1)\,S)^{1/2}$ is also compact positive and self-adjoint with $KerH = \{0\}$. Introducing

$$T = S\,\mathbf{B}(x_1)\,S\,(S\,\mathbf{C}(x_1)\,S)^{-1/2},$$

we see that T is a compact operator from $L^2(-1,1)$ into itself. Indeed, $S\,\mathbf{B}(x_1)\,S$ is a compact operator from $L^2(-1,1)$ into $H_0^1(-1,1)$, and $H^{-1} = (S\,\mathbf{C}(x_1)\,S)^{-1/2} : H_0^1(-1,1) \to L^2(-1,1)$ is bounded.

The spectrum of the quadratic operator pencil (4.13) is discrete and consists of eigenvalues of finite multiplicity possibly accumulating at ∞.

Let us estimate the number of eigenvalues of H^2 in the interval $(1/r^2, +\infty)$. Let L be a subspace of $L^2(-1,1)$. Then due to the minimax principle, the kth eigenvalue of H^2 can be found from the formula

$$\begin{aligned}\nu_k^+ &= \min_L \max_{x\in L\setminus\{0\}} \frac{(H^2x,x)_{L^2(-1,1)}}{(x,x)_{L^2(-1,1)}}\\ &\le \underline{C} \min_L \max_{x\in L\setminus\{0\}} \frac{(Sx,Sx)_{L^2(-1,1)}}{(x,x)_{L^2(-1,1)}} = \underline{C}\, \mu_k^+,\end{aligned}$$

where μ_k^+ is the kth eigenvalue of the operator $(\mathcal{A}^0)^{-1}$. Similarly, since $\mathbf{C}(x_1)$ is bounded from below, we get the lower bound for ν_k^+, and, consequently,

$$\overline{C}\mu_k^+ \le \nu_k^+ \le \underline{C}\, \mu_k^+.$$

Thus, we conclude that the number of eigenvalues of the operators H^2 and $(\mathcal{A}^0)^{-1}$ in $(1/r^2, +\infty)$ is asymptotically equivalent. The following inequality characterizes the growth of the eigenvalues of the Sturm-Liouville problem for the operator $\mathcal{A}^0$ (see, for example, [**9**], [**10**]):

$$\frac{C_1\,\pi^2\,k^2}{4} \le \frac{1}{\mu_k^+} \le \frac{C_2\,\pi^2\,k^2}{4},$$

where the constants C_1 and C_2 are lower and upper bounds for $a^{\text{eff}}(x_1)$, respectively.

Thus, we conclude that the number of eigenvalues of H^2 in the interval $(1/r^2, +\infty)$ is proportional to r, and, consequently, condition (4.12) is satisfied. By the Keldysh theorem, $N_0(r, \widetilde{\mathbf{\Pi}}(\nu^0))$, as well as $N_2(r, \widetilde{\mathbf{\Pi}}(\nu^0))$, goes to infinity, as $r \to \infty$, thus, it is true also for $\mathbf{\Pi}(\nu^0)$.

Let us show that the eigenvalues of problem (4.9) are real. Suppose

$$\nu^0 = \Re(\nu^0) + i\Im(\nu^0),$$

where $\Re(\nu^0)$ and $\Im(\nu^0)$ represent the real and imaginary parts of ν^0, respectively. Substituting the last expression in (4.10) and setting $v = \overline{u^0}$ we obtain

$$\begin{cases} \displaystyle\int_{-1}^{1} a^{\text{eff}} \left|\frac{du^0}{dx_1}\right|^2 dx_1 + \Re(\nu^0) \int_{-1}^{1} \mathbf{B}\,|u^0|^2\,dx_1 \\ \displaystyle -\left[(\Re(\nu^0))^2 - (\Im(\nu^0))^2\right] \int_{-1}^{1} \mathbf{C}\,|u^0|^2\,dx_1 = 0, \\ \displaystyle \Im(\nu^0) \int_{-1}^{1} \mathbf{B}\,|u^0|^2\,dx_1 - 2\Im(\nu^0)\,\Re(\nu^0) \int_{-1}^{1} \mathbf{C}\,|u^0|^2\,dx_1 = 0. \end{cases}$$

By our assumption $\Im(\nu^0) \neq 0$. Thus, it follows from the last equation that

$$\int_{-1}^{1} \mathbf{B}\,|u^0|^2\,dx_1 = 2\Re(\nu^0)\int_{-1}^{1} \mathbf{C}\,|u^0|^2\,dx_1,$$

and, therefore,

$$\int_{-1}^{1} a^{\text{eff}}\left|\frac{du^0}{dx_1}\right|^2 dx_1 + \left[(\Re(\nu^0))^2 + (\Im(\nu^0))^2\right]\int_{-1}^{1} \mathbf{C}\,|u^0|^2\,dx_1 = 0$$

that contradicts the positiveness of a^{eff} and $\mathbf{C}$, and, consequently, ν^0 is real. In this way the existence of two infinite sequences of eigenvalues tending to $\pm\infty$ is proved.

Let us show that the algebraic multiplicity of ν^0 is equal to 1. Suppose there exists $\varphi^1 \in H_0^1(-1,1)$ such that

$$\mathbf{\Pi}(\nu^0)\varphi^1(x_1) = -\mathbf{B}(x_1)\,u^0(x_1) + 2\nu^0\mathbf{C}(x_1)\,u^0(x_1),$$

where $\mathbf{\Pi}$ is defined by (4.9). Using φ^1 as a test function in (4.10) and substituting the resulting equality into the last formula yields

$$2\nu^0\int_{-1}^{1} \mathbf{C}\,(u^0)^2\,dx_1 - \int_{-1}^{1} \mathbf{B}\,(u^0)^2\,dx_1 = 0.$$

In view of (4.9),

$$\begin{aligned} 0 &= 2(\nu^0)^2\int_{-1}^{1} \mathbf{C}\,(u^0)^2\,dx_1 - \nu^0\int_{-1}^{1} \mathbf{B}\,(u^0)^2\,dx_1 \\ &= (\nu^0)^2\int_{-1}^{1} \mathbf{C}\,(u^0)^2\,dx_1 + \int_{-1}^{1} a^{\text{eff}}\left|\frac{du^0}{dx_1}\right|^2 dx_1 > 0. \end{aligned}$$

We arrive at contradiction. Thus, the eigenvalues of problem (4.9) are algebraically simple.

Suppose the geometric multiplicity of ν^0 is greater than 1, in other words, there exist two linearly independent eigenfunctions u_1^0 and u_2^0 corresponding to the same ν^0. Choosing C_1 and C_2 in such a way that the function $\tilde{u}^0 = C_1u_1^0 + C_2u_2^0$ satisfies the boundary conditions

$$\tilde{u}^0(-1) = \frac{d\tilde{u}^0}{dx_1}(-1) = 0,$$

we see that, by the uniqueness result for ordinary differential equations, $\tilde{u}^0 = 0$, that contradicts the linear independence of u_1^0 and u_2^0. $\square$

We turn back to constructing the asymptotic expansion. The specific form of the right-hand side of (3.6) suggests the following representation for $u^2(x_1, y)$:

$$
\begin{aligned}
u^2(x_1,y) \;&= N^{2,2}(x_1,y)\,\frac{d^2u^0(x_1)}{dx_1^2} + N^{2,1}(x_1,y)\frac{du^0(x_1)}{dx_1}\\
&+\nu^0\, q_2(x_1,y)\frac{du^0(x_1)}{dx_1} + \nu^0 N^{2,0}(x_1,y)u^0(x_1) + \nu^1 N^{1,0}\, u^0\\
&+(\nu^0)^2 r_2(x_1,y)u^0(x_1) + N^{1,1}(x_1,y)\frac{dv^1(x_1)}{dx_1}\\
&+\nu^0 N^{1,0}(x_1,y)v^1(x_1) + v^2(x_1),
\end{aligned}
\tag{4.14}
$$

where $N^{2,2}$, $N^{2,1}$ and $N^{2,0}$ are y_1-periodic solutions of the problems

$$
\left\{
\begin{array}{l}
\mathcal{A}_y N^{2,2}(x_1,y) = \mathrm{div}_y(a(x_1,y)N^{1,1}(x_1,y))\\
+a_{1j}(x_1,y)(\delta_{1j} + \partial_{y_j}N^{1,1}(x_1,y)) - a^{\mathrm{eff}}(x_1), \qquad y \in Y,\\
\mathcal{B}_y N^{2,2}(x_1,y) = -(a_{\cdot 1}(x_1,y),n)N^{1,1}(x_1,y), \quad y \in \partial Y;
\end{array}
\right.
\tag{4.15}
$$

$$
\left\{
\begin{array}{l}
\mathcal{A}_y N^{2,1}(x_1,y) = \mathrm{div}_y(a_{\cdot 1}(x_1,y)\dfrac{\partial}{\partial x_1}N^{1,1}(x_1,y))\\
+\dfrac{\partial}{\partial x_1}\big[a_{1j}(x_1,y)(\delta_{1j} + \partial_{y_j}N^{1,1}(x_1,y))\big] - \dfrac{da^{\mathrm{eff}}(x_1)}{dx_1}, \quad y \in Y,\\
\mathcal{B}_y N^{2,1}(x_1,y) = -(a_{\cdot 1}(x_1,y),n)\,\dfrac{\partial}{\partial x_1}N^{1,1}(x_1,y), \qquad y \in \partial Y;
\end{array}
\right.
\tag{4.16}
$$

$$
\left\{
\begin{array}{ll}
\mathcal{A}_y N^{2,0}(x_1,y) = & \mathrm{div}_y(a_{\cdot 1}(x_1,y)\dfrac{\partial}{\partial x_1}N^{1,0}(x_1,y)\\
& +\dfrac{\partial}{\partial x_1}\big(a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y)\big)\\
& -\dfrac{d}{dx_1}\displaystyle\int_{\partial Y} a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y)\,dy, \quad y \in Y,\\
\mathcal{B}_y N^{2,0}(x_1,y) = & (a_{\cdot 1}(x_1,y),n)\,\dfrac{\partial}{\partial x_1}N^{1,0}(x_1,y), \qquad\qquad y \in \partial Y.
\end{array}
\right.
\tag{4.17}
$$

The y_1-periodic functions $q_2(x_1, y)$ and $r_2(x_1, y)$ solve the problems

$$
\left\{
\begin{array}{l}
\mathcal{A}_y q_2(x_1,y) = \mathrm{div}_y(a_{\cdot 1}(x_1,y)N^{1,0}(x_1,y))\\
+a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y) + \rho(x_1,y)N^{1,1}(x_1,y), \quad y \in Y,\\
\mathcal{B}_y q_2(x_1,y) = -(a_{\cdot 1}(x_1,y),n)\,N^{1,0}(x_1,y), \qquad y \in \partial Y;
\end{array}
\right.
\tag{4.18}
$$

$$
\left\{
\begin{array}{ll}
\mathcal{A}_y r_2(x_1,y) = \rho(x_1,y)N^{1,0}(x_1,y) - \mathbf{C}(x_1), & y \in Y,\\
\mathcal{B}_y r_2(x_1,y) = 0, & y \in \partial Y.
\end{array}
\right.
\tag{4.19}
$$

Bearing in mind (4.3) and (4.4), we see that the compatibility condition for (4.18) is satisfied. Similarly, by (4.7), problem (4.19) is solvable.

Our next goal is to obtain an equation for $v^1(x_1)$. To this end we substitute (4.1) into (2.1) and collect terms of order ε^1 in the equation and of order ε^2 in the boundary condition. In this way we get the problem for $u^3(x_1,y)$.

$$
(4.20)\qquad \begin{cases} \mathcal{A}_y u^3(x_1,y) & = \mathrm{div}_y(a_{\cdot 1}(x_1,y)\,\dfrac{\partial u^2}{\partial x_1}(x_1,y) \\ & \quad +\dfrac{\partial}{\partial x_1}(a_{1\cdot}(x_1,y)\nabla_y u^2(x_1,y)) \\ & \quad +\dfrac{\partial}{\partial x_1}(a_{11}(x_1,y)\dfrac{du^1(x_1)}{dx_1}) \\ & \quad +\nu^1\,\rho(x_1,y)u^1(x_1)+\nu^0\,\rho(x_1,y)u^2(x_1), \quad y\in Y, \\ \mathcal{B}_y u^3(x_1,y) & = -a_{i1}(x_1,y)\,n_i\,\dfrac{\partial}{\partial x_1}u^2(x_1,y), \quad y\in\partial Y, \\ u^3(x_1,y) & \text{ is } 1-\text{periodic in } y_1. \end{cases}
$$

The compatibility condition for the last problem reads

$$
(4.21)\qquad -\frac{d}{dx_1}\Big(a^{\mathrm{eff}}\frac{dv^1}{dx_1}\Big)+\nu^0\mathbf{B}v^1-(\nu^0)^2\mathbf{C}v^1=F_1-\nu^1\mathbf{B}u^0+2\nu^1\nu^0\mathbf{C}u^0,
$$

where $\mathbf{B}(x_1)$ and $\mathbf{C}(x_1)$ are defined by (4.8) and (4.7), respectively, and

$$
(4.22)\qquad \begin{aligned} F_1(x_1) &= \frac{d}{dx_1}\int\limits_Y a_{1\cdot}(x_1,y)\nabla_y\tilde u_2(x_1,y)\,dy \\ &+\frac{d}{dx_1}\int\limits_Y a_{11}(x_1,y)\frac{\partial\tilde u_1}{\partial x_1}(x_1,y)\,dy+\nu^0\int\limits_Y \rho(x_1,y)\tilde u_2(x_1,y)\,dy. \end{aligned}
$$

Here for brevity we denote

$$
\tilde u_1(x_1,y)=N^{1,1}(x_1,y)\,\frac{du^0(x_1)}{dx_1}+\nu^0\,N^{1,0}(x_1,y)\,u^0(x_1);
$$

$$
\begin{aligned} \tilde u_2(x_1,y) \; &= N^{2,2}(x_1,y)\,\frac{d^2u^0(x_1)}{dx_1^2}+N^{2,1}(x_1,y)\frac{du^0(x_1)}{dx_1} \\ &+\nu^0\,q_2(x_1,y)\frac{du^0(x_1)}{dx_1}+\nu^0N^{2,0}(x_1,y)u^0(x_1) \\ &+(\nu^0)^2r_2(x_1,y)u^0(x_1) \end{aligned}
$$

with the functions $N^{2,2},N^{2,1},N^{2,0},q_2,r_2$ defined in (3.7), (3.8), (3.9), (4.18), (4.19).

As in Section 3, determining the boundary conditions for $v^1(x_1)$ requires constructing the boundary layer correctors in the neighbourhood of the points $x=\pm 1$.

Denote, as before, $G^-=(0,+\infty)\times Q$ and $G^+=(-\infty,0)\times Q$ the semi-infinite cylinders with the axis directed along y_1 and lateral boundaries $\Sigma^-=(0,+\infty)\times\partial Q$ and

$\Sigma^+ = (-\infty, 0) \times \partial Q$. Consider the following boundary value problem:

$$
(4.23) \qquad \begin{cases} -\mathrm{div}_y(a(\pm 1, y_1+\delta, y')\nabla_y w^\pm) = 0, & y \in G^\pm, \\ (a(\pm 1, y_1+\delta, y')\nabla_y w^\pm, n) = 0, & y \in \Sigma^\pm, \\ w^\pm(0, y') = -N^{1,1}(\pm 1, \delta, y')\,\dfrac{du^0}{dx_1}(\pm 1) \\ -\nu^0 N^{1,0}(\pm 1, \delta, y')\, u^0(\pm 1), \end{cases}
$$

with δ being the fractional part of ε^{-1}, which is equal to zero in view of condition (2.2). There exists a unique bounded solution $w^\pm \in C^{1,\alpha}(\overline{G^\pm})$ of problem (4.23) stabilizing to some constant $\hat{w}^\pm$, as $|y_1| \to +\infty$ (see [**14**]):

$$
(4.24) \qquad \begin{aligned} |w^\pm(y_1, y') - \hat{w}^\pm| &\le C_0\, e^{-\gamma |y_1|}, \quad C_0, \gamma > 0; \\ \|\nabla w^+\|_{L^2((n,n+1)\times Q)} &\le C\, e^{-\gamma n}, \quad \forall n \ge 0, \\ \|\nabla w^-\|_{L^2((-(n+1),-n)\times Q)} &\le C\, e^{-\gamma n}, \quad \forall n \ge 0, \end{aligned}
$$

for some $\gamma > 0$. As a boundary condition for $v^1(x_1)$ we choose the uniquely defined constants $\hat{w}^\pm$: $v^1(\pm 1) = \hat{w}^\pm$. Thus, the problem for v^1 takes the form

$$
(4.25) \qquad \begin{cases} \mathbf{\Pi}(\nu^0) v^1(x_1) = F_1 - \nu^1 \mathbf{B} u^0 + 2\nu^1\nu^0 \mathbf{C} u^0, & x_1 \in (-1, 1), \\ v^1(\pm 1) = \hat{w}^\pm. \end{cases}
$$

Since $\mathbf{\Pi}(\nu^0) u^0 = 0$, problem (4.25) is solvable if the right-hand side is orthogonal to u^0, that is

$$
\int_{-1}^{1} F_1\, u^0 dx_1 = \nu^1 \int_{-1}^{1} \mathbf{B}(u^0)^2\, dx_1 - 2\nu^0\, \nu^1 \int_{-1}^{1} \mathbf{C}(u^0)^2\, dx_1 + \bar{F},
$$

where the constant $\bar{F}$ is given by

$$
(4.26) \qquad \bar{F} = \Big(a^{\mathrm{eff}}(1)\frac{du^0}{dx_1}(1)\hat{w}^+ - a^{\mathrm{eff}}(-1)\frac{du^0}{dx_1}(-1)\hat{w}^-\Big).
$$

It follows easily from (4.9) that

$$
\int_{-1}^{1} \mathbf{B}(x_1)(u^0(x_1))^2\, dx_1 - 2\nu^0 \int_{-1}^{1} \mathbf{C}(x_1)(u^0(x_1))^2\, dx_1 \ne 0.
$$

Thus, ν^1 can be defined in such a way that (4.25) possesses a solution. Namely,

$$
(4.27) \qquad \nu^1 = \Big\{ \int_{-1}^{1} F_1\, u^0\, dx_1 - \bar{F} \Big\} \Big\{ \int_{-1}^{1} \big[\mathbf{B}(x_1) - 2\nu^0 \mathbf{C}(x_1)\big]\, (u^0(x_1))^2\, dx_1 \Big\}^{-1}.
$$

We fix the choice of the function v^1 by setting

$$\int_{-1}^{1} v^1(x_1)\, u^0(x_1)\, dx_1 = 0.$$

Note that, in view of the regularity assumptions (**H0**), $v^1 \in C^{2,\alpha}[-1,1]$, $\alpha > 0$. In this way the function

$$\begin{aligned} &u^0(x_1) + \varepsilon\, N^{1,1}\big(x_1, \frac{x}{\varepsilon}\big)\, \frac{du^0(x_1)}{dx_1} \\ &+\varepsilon\, \nu^0\, N^{1,0}\big(x_1, \frac{x}{\varepsilon}\big)\, u^0(x_1) + \varepsilon\, v^1(x_1) + \varepsilon\, u^\varepsilon_{\rm bl}(x), \end{aligned}$$

with

$$\begin{aligned} u^\varepsilon_{\rm bl}(x) &= \tilde u^\varepsilon_{\rm bl}(y)\Big|_{y=x/\varepsilon} \\ &= \Big(w^+\big(y_1 - \frac{1}{\varepsilon}, y'\big) - \hat w^+\Big) + \Big(w^-\big(y_1 + \frac{1}{\varepsilon}, y'\big) - \hat w^-\Big)\Big|_{y=x/\varepsilon}, \end{aligned} \tag{4.28}$$

satisfies the homogeneous Dirichlet boundary conditions at $x_1 = \pm 1$.

4.2. Justification procedure in the case $\langle \rho(x_1, \cdot)\rangle = 0$.
Let $\nu_j^{0,\pm}$ be the eigenvalues and $u_j^{0,\pm}$ the corresponding eigenfunctions of problem (4.9). For any $j \in \mathbb{N}$ we denote

$$\begin{aligned} U_j^{\varepsilon,\pm}(x) &= u_j^{0,\pm}(x_1) + \varepsilon\, N^{1,1}\big(x_1, \frac{x}{\varepsilon}\big)\, \frac{du_j^{0,\pm}(x_1)}{dx_1} \\ &+\varepsilon\, \nu_j^{0,\pm}\, N^{1,0}\big(x_1, \frac{x}{\varepsilon}\big)\, u_j^{0,\pm}(x_1) + \varepsilon\, v_j^{1,\pm}(x_1) + \varepsilon\, u^\varepsilon_{\rm bl}(x), \end{aligned} \tag{4.29}$$

where $u_j^{0,\pm}$, $N^{1,1}$, $N^{1,0}$ and $v_j^{1,\pm}$ solve problems (4.9), (4.3), (4.4) and (4.25), respectively (with $u^0 = u_j^{0,\pm}$ and $\nu^0 = \nu_j^{0,\pm}$). The boundary layer corrector $u^\varepsilon_{\rm bl}$ is defined by (4.28) and (4.23).

Let us emphasize that, due to the presence of the boundary layer terms, the function $U_j^{\varepsilon,\pm}$ satisfies the homogeneous Dirichlet boundary conditions on $S_{\pm 1}$, and, as a consequence, belong to the space $\mathcal{H}^\varepsilon$.

We denote by $\nu_j^{1,\pm}$ a constant defined by (4.27) with $u^0 = u_j^{0,\pm}$ and $\nu^0 = \nu_j^{0,\pm}$. For the readers convenience we recall its definition.

$$\nu_j^{1,\pm} = \Big\{ \int_{-1}^{1} F_1\, u_j^{0,\pm}\, dx_1 - \bar F \Big\}\, \Big\{ \int_{-1}^{1} \big[\mathbf{B} - 2\nu_j^{0,\pm}\mathbf{C}\big]\, (u_j^{0,\pm})^2\, dx_1 \Big\}^{-1}, \tag{4.30}$$

where $(\nu_j^{0,\pm}, u_j^{0,\pm})$ are eigenpairs of problem (4.9), the functions $\mathbf{B}(x_1), \mathbf{C}(x_1)$ are defined by (4.8) and (4.7), respectively; the function $F_1(x_1)$ and the constant $\bar F$ are given by (4.22) and (4.26) with $u^0 = u_j^{0,\pm}$ and $\nu^0 = \nu_j^{0,\pm}$.

The goal of this section is to prove the following result.

THEOREM 4.3. *Let conditions* $(\mathbf{H0})-(\mathbf{H3})$ *be fulfilled, and suppose that* $\langle\rho(x_1,y)\rangle = 0$ *for any* $x_1\in[-1,1]$. *If* $(\lambda_j^{\varepsilon,\pm}, u_j^{\varepsilon,\pm})$ *are eigenpairs of problem* (2.1)*, and* $(\nu_j^{0,\pm}, u_j^{0,\pm})$ *are eigenpairs of the operator pencil* (4.9)*, then*

(i) *For any* j*, there exist* ε_j *and* $C_j>0$ *such that*

$$|\lambda_j^{\varepsilon,\pm}-(\varepsilon^{-1}\nu_j^{0,\pm}+\nu_j^{1,\pm})|\le C_j\,\varepsilon,\quad \forall\varepsilon\in(0,\varepsilon_j].$$

Here $\nu_j^{1,\pm}$ *is defined in* (4.30).

(ii) *For any* j

$$\|u_j^{\varepsilon,\pm}-U_j^{\varepsilon,\pm}\|_{H^1(G_\varepsilon)}\le C_j\,\varepsilon\,\varepsilon^{\frac{d-1}{2}},$$

where $U_j^{\varepsilon,\pm}$ *is defined by* (4.29)*. Moreover, the "almost eigenfunctions" satisfy the almost orthogonality and normalization condition*

$$\Big|\frac{\varepsilon^{-(d-1)}}{|Q|}\,(a^\varepsilon\,\nabla U_i^{\varepsilon,\pm},\nabla U_j^{\varepsilon,\pm})_{L^2(G_\varepsilon)}-\delta_{ij}\Big|\le C_j\,\varepsilon.$$

(iii) *For any* $j\in\mathbb{N}$, $\lambda_j^{\varepsilon,\pm}$ *are simple, for sufficiently small* $\varepsilon>0$.

PROOF OF THEOREM 3.6. As in Section 3, we make use of Lemma 3.8. Denote

$$\mathcal{U}_j^{\varepsilon,\pm}=\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}\,U_j^{\varepsilon,\pm}.$$

LEMMA 4.4. *For any* $j\in\mathbb{N}$ *there is* $\varepsilon_j>0$ *such that*

$$\big\|\mathcal{K}^\varepsilon\mathcal{U}_j^{\varepsilon,\pm}-(\varepsilon^{-1}\nu_j^{0,\pm}+\nu_j^{1,\pm})^{-1}\mathcal{U}_j^{\varepsilon,\pm}\big\|_{\mathcal{H}^\varepsilon}\le C_j\,\varepsilon^2,\qquad \varepsilon<\varepsilon_j, \tag{4.31}$$

where the constant C_j *depends only on* j.

PROOF. After straightforward rearrangements and integration by parts we have

$$I^\varepsilon\equiv\big\|\mathcal{K}^\varepsilon\,\mathcal{U}_j^{\varepsilon,\pm}-(\varepsilon^{-1}\nu_j^{0,\pm}+\nu_j^{1,\pm})^{-1}\,\mathcal{U}_j^{\varepsilon,\pm}\big\|_{\mathcal{H}^\varepsilon}$$

$$=\frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{\big|\varepsilon^{-1}\nu_j^{0,\pm}+\nu_j^{1,\pm}\big|}\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}}\Bigg|\Big(\mathcal{A}^\varepsilon\,U_j^{\varepsilon,\pm},w\Big)_{L^2(G_\varepsilon)}$$

$$-(\varepsilon^{-1}\nu_j^{0,\pm}+\nu_j^{1,\pm})\,\Big(\rho^\varepsilon\,U_j^{\varepsilon,\pm}\,,\,w\Big)_{L^2(G_\varepsilon)}+\int\limits_{\Sigma_\varepsilon}\big(a^\varepsilon\nabla U_j^{\varepsilon,\pm},n\big)\,w\,d\sigma\Bigg|.$$

It is convenient to use the notation

$$U_j^{\varepsilon,\pm}(x)=u_j^{0,\pm}(x_1)+\varepsilon\,u_j^{1,\pm}(x_1,y)\big|_{y=x/\varepsilon}+\varepsilon\,u_{\rm bl}^\varepsilon(x).$$

Recall that $u_j^{0,\pm}(x_1) \in C^{2,\alpha}[-1,1]$ and $u_j^{1,\pm}(x_1,y) \in C^{1,\alpha}([-1,1]\times\overline{Y})$. In this way we obtain

$$I^\varepsilon = \frac{\|U_j^{\varepsilon,\pm}\|^{-1}_{\mathcal{H}^\varepsilon}}{\left|\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}\right|} \sup_{\substack{w\in\mathcal{H}^\varepsilon \\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \Bigg| \int\limits_{G_\varepsilon} \mathcal{A}^\varepsilon \left(u_j^{0,\pm}(x_1) + \varepsilon\, u_j^{1,\pm}(x_1, \frac{x}{\varepsilon})\right) w(x)\, dx$$

$$-(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}) \int\limits_{G_\varepsilon} \rho^\varepsilon(x) \left(u_j^{0,\pm}(x_1) + \varepsilon\, u_j^{1,\pm}(x_1,y)\big|_{y=x/\varepsilon}\right) w(x)\, dx$$

$$+\varepsilon \int\limits_{\Sigma_\varepsilon} (a^\varepsilon_{\cdot 1}, n)\, \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\big|_{y=x/\varepsilon}\, w\, d\sigma + \varepsilon \left(\mathcal{A}^\varepsilon\, u^\varepsilon_{\mathrm{bl}}, w\right)_{L^2(G_\varepsilon)}$$

$$-(\nu_j^{0,\pm} + \varepsilon\, \nu_j^{1,\pm}) \left(\rho^\varepsilon\, u^\varepsilon_{\mathrm{bl}}, w\right)_{L^2(G_\varepsilon)} + \int\limits_{\Sigma_\varepsilon} (a^\varepsilon_{\cdot 1}, n)\, \nabla_y \tilde{u}^\varepsilon_{\mathrm{bl}}(y)\big|_{y=x/\varepsilon}\, w\, d\sigma \Bigg|.$$

The last three terms containing $u^\varepsilon_{\mathrm{bl}}$ can be estimated exactly like in Lemma 3.10.

$$\begin{aligned} &\Big|\varepsilon(\mathcal{A}^\varepsilon\, u^\varepsilon_{\mathrm{bl}}, w)_{L^2(G_\varepsilon)} + \varepsilon(a^\varepsilon \nabla u^\varepsilon_{\mathrm{bl}}\, w, n)_{L^2(\Sigma_\varepsilon)} \\ &\qquad -(\nu_j^{0,\pm} + \varepsilon\, \nu_j^{1,\pm})\, (\rho^\varepsilon\, u^\varepsilon_{\mathrm{bl}}, w)_{L^2(G_\varepsilon)}\Big| \\ &\le C\, \varepsilon\, \varepsilon^{(d-1)/2}\, \|w\|_{H^1(G_\varepsilon)}, \quad w \in \mathcal{H}^\varepsilon. \end{aligned} \tag{4.32}$$

Then

$$\begin{aligned} &\int\limits_{G_\varepsilon} \mathcal{A}^\varepsilon \left(u_j^{0,\pm}(x_1) + \varepsilon\, u_j^{1,\pm}(x_1, \frac{x}{\varepsilon})\right) w(x)\, dx \\ &-(\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm}) \int\limits_{G_\varepsilon} \rho^\varepsilon(x) \left(u_j^{0,\pm}(x_1) + \varepsilon\, u_j^{1,\pm}(x_1,y)\big|_{y=x/\varepsilon}\right) w(x)\, dx \\ &+\varepsilon \int\limits_{\Sigma_\varepsilon} (a^\varepsilon_{\cdot 1}, n)\, \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\big|_{y=x/\varepsilon}\, w\, d\sigma = \\ &= \varepsilon^0\, (I_0^\varepsilon, w)_{L^2(G_\varepsilon)} + \varepsilon^1\, (I_1^\varepsilon, w)_{L^2(G_\varepsilon)} \\ &+\varepsilon \int\limits_{\Sigma_\varepsilon} (a^\varepsilon_{\cdot 1}, n)\, \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\big|_{y=x/\varepsilon}\, w\, d\sigma; \end{aligned}$$

here

$$I_0^\varepsilon(x)=I_0(x_1,y)\big|_{y=x/\varepsilon}=-\frac{\partial}{\partial x_1}(a_{1\cdot}(x_1,y)\nabla_y u_j^{1,\pm}(x_1,y)$$
$$-\frac{\partial}{\partial x_1}\big(a_{11}(x_1,y)\frac{du_j^{0,\pm}}{dx_1}(x_1)\big)-\nu_j^{1,\pm}\rho(x_1,y)u_j^{0,\pm}(x_1)$$
$$-\nu_j^{0,\pm}u_j^{1,\pm}(x_1,y)\,\rho(x_1,y)\Big|_{y=x/\varepsilon};$$

$$I_1^\varepsilon(x)=-\left\{\mathrm{div}_x+\frac{1}{\varepsilon}\mathrm{div}_y\right\}\big(a_{\cdot 1}(x_1,y)\,\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\big)$$
$$-\nu_j^{1,\pm}\,\rho(x_1,y)\,u_j^{1,\pm}(x_1,y)\Big|_{y=x/\varepsilon}.$$

By (4.9), the average of $I_0(x_1,y)\in C^{1,\alpha}([-1,1];C^\alpha(\overline{Y}))$ over Y is equal to zero, thus, by Lemma 3.11

$$\big|\big(I_0^\varepsilon,w\big)_{L^2(G_\varepsilon)}\big|\le C\,\varepsilon\,\varepsilon^{\frac{d-1}{2}}\|w\|_{H^1(G_\varepsilon)}.$$

Integrating by parts and bearing in mind the regularity properties of $u_j^{1,\pm}$ and assumption $(\mathbf{H0})$, one can see that

$$\Big|\big(I_1^\varepsilon,w\big)_{L^2(G_\varepsilon)}+\int\limits_{\Sigma_\varepsilon}(a_{\cdot 1}^\varepsilon,n)\,\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\big|_{y=x/\varepsilon}\,w\,d\sigma\Big|$$
$$=\Big|\int\limits_{G_\varepsilon}(a_{\cdot 1}^\varepsilon(x),\nabla w)\,\frac{\partial u_j^{1,\pm}}{\partial x_1}\Big|_{y=x/\varepsilon}dx-\nu_j^{1,\pm}\int\limits_{G_\varepsilon}\rho^\varepsilon(x)\,u_j^{1,\pm}(x_1,\frac{x}{\varepsilon})\,w(x)\,dx\Big|$$
$$\le C\,\varepsilon^{\frac{d-1}{2}}\,\|w\|_{H^1(G_\varepsilon)}.$$

Thus,

$$I^\varepsilon\le C\,\frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{\big|\varepsilon^{-1}\nu_j^{0,\pm}+\nu_j^{1,\pm}\big|}\,\varepsilon\,\varepsilon^{\frac{d-1}{2}}.\tag{4.33}$$

Let us estimate $\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}$. Rearranging the terms in the expression for $\big(U_i^{\varepsilon,\pm},U_j^{\varepsilon,\pm}\big)_{\mathcal{H}^\varepsilon}$ yields

$$\big(U_i^{\varepsilon,\pm},U_j^{\varepsilon,\pm}\big)_{\mathcal{H}^\varepsilon}=J_{xx}^\varepsilon+J_{xy}^\varepsilon+J_{yx}^\varepsilon+J_{yy}^\varepsilon,$$

where

$$J_{xx}^\varepsilon=\int\limits_{G_\varepsilon}a_{11}^\varepsilon\frac{du_i^{0,\pm}}{dx_1}\,\frac{du_j^{0,\pm}}{dx_1}\,dx+\varepsilon\int\limits_{G_\varepsilon}a_{11}^\varepsilon\frac{du_i^{0,\pm}}{dx_1}\,\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\,dx$$
$$+\varepsilon\int\limits_{G_\varepsilon}a_{11}^\varepsilon\frac{du_j^{0,\pm}}{dx_1}\,\frac{\partial u_i^{1,\pm}}{\partial x_1}(x_1,y)\,dx+\varepsilon^2\int\limits_{G_\varepsilon}a_{11}^\varepsilon\frac{\partial u_i^{1,\pm}}{\partial x_1}\,\frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1,y)\,dx.$$

$$
\begin{aligned}
J^{\varepsilon}_{xy} &= \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y N^{1,1})\big|_{y=x/\varepsilon} \frac{du_i^{0,\pm}}{dx_1}\,\frac{du_j^{0,\pm}}{dx_1}\,dx\\
&+ \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y \tilde{u}^{\varepsilon}_{\mathrm{bl}})\big|_{y=x/\varepsilon} \frac{du_i^{0,\pm}}{dx_1}\,dx\\
&+\nu_j^{0,\pm} \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y N^{1,0})\big|_{y=x/\varepsilon} \frac{du_i^{0,\pm}}{dx_1}\,u_j^{0,\pm}\,dx\\
&+\varepsilon \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y u_j^{1,\pm})\big|_{y=x/\varepsilon} \frac{\partial u_i^{1,\pm}}{\partial x_1}(x_1, y)\,dx\\
&+\varepsilon \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y \tilde{u}^{\varepsilon}_{\mathrm{bl}})\big|_{y=x/\varepsilon} \frac{\partial u_i^{1,\pm}}{\partial x_1}(x_1, y)\,dx.
\end{aligned}
$$

$$
\begin{aligned}
J^{\varepsilon}_{yx} &= \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y N^{1,1})\big|_{y=x/\varepsilon} \frac{du_i^{0,\pm}}{dx_1}\,\frac{du_j^{0,\pm}}{dx_1}\,dx\\
&+ \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \tilde{u}^{\varepsilon}_{\mathrm{bl}})\big|_{y=x/\varepsilon} \frac{du_j^{0,\pm}}{dx_1}\,dx\\
&+\nu_i^{0,\pm} \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y N^{1,0})\big|_{y=x/\varepsilon} \frac{du_j^{0,\pm}}{dx_1}\,u_i^{0,\pm}\,dx+\\
&+\varepsilon \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y u_i^{1,\pm})\big|_{y=x/\varepsilon} \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1, y)\,dx\\
&+\varepsilon \int\limits_{G_\varepsilon} (a^{\varepsilon}_{\cdot 1}, \nabla_y \tilde{u}^{\varepsilon}_{\mathrm{bl}})\big|_{y=x/\varepsilon} \frac{\partial u_j^{1,\pm}}{\partial x_1}(x_1, y)\,dx.
\end{aligned}
$$

$$
\begin{aligned}
J^{\varepsilon}_{yy} &= \int\limits_{G_\varepsilon} (a^{\varepsilon}\nabla_y N^{1,1}, \nabla_y N^{1,1})\big|_{y=x/\varepsilon} \frac{du_i^{0,\pm}}{dx_1}\,\frac{du_j^{0,\pm}}{dx_1}\,dx\\
&+\nu_i^{0,\pm} \int\limits_{G_\varepsilon} (a^{\varepsilon}\nabla_y N^{1,0}, \nabla_y N^{1,1})\big|_{y=x/\varepsilon} u_i^{0,\pm}\,\frac{du_j^{0,\pm}}{dx_1}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\nu_j^{0,\pm} \int\limits_{G_\varepsilon} (a^{\varepsilon}\nabla_y N^{1,0}, \nabla_y N^{1,1})\big|_{y=x/\varepsilon} u_j^{0,\pm}\,\frac{du_i^{0,\pm}}{dx_1}\,dx\\
&+\nu_i^{0,\pm}\nu_j^{0,\pm} \int\limits_{G_\varepsilon} (a^{\varepsilon}\nabla_y N^{1,0}, \nabla_y N^{1,0})\big|_{y=x/\varepsilon} u_i^{0,\pm}\,u_j^{0,\pm}\,dx.
\end{aligned}
$$

There are several "typical" terms in the expressions for J^ε_{xx}, J^ε_{xy}, J^ε_{yx} and J^ε_{yy} to be estimated. For example, using the regularity properties of $a(x_1,y)$, $u^{0,\pm}_j$ and $u^{1,\pm}_j$ we get

$$\Big|\varepsilon\int\limits_{G_\varepsilon} a^\varepsilon_{11}\,\frac{du^{0,\pm}_i}{dx_1}\,\frac{\partial u^{1,\pm}_j}{\partial x_1}(x_1,y)\,dx\Big| \le C\,\varepsilon\,|G_\varepsilon| = C\,\varepsilon\,\varepsilon^{d-1}.$$

Then, taking into account the exponential decay of $\tilde u^\varepsilon_{\rm bl}$ one can see that

$$\begin{aligned}&\Big|\varepsilon\int\limits_{G_\varepsilon}(a_{\cdot 1},\nabla_y\tilde u^\varepsilon_{\rm bl})\big|_{y=x/\varepsilon}\,\frac{du^{0,\pm}_i}{dx_1}\,dx\Big|\\ &\le C\,\varepsilon^d\int\limits_{-1/\varepsilon}^{1/\varepsilon}dx_1\int\limits_Q|\nabla_y\tilde u^\varepsilon_{\rm bl}|\,dy' \le C\,\varepsilon\,\varepsilon^{d-1}.\end{aligned}$$

In view of boundedness of $\partial u^{1,\pm}_j/\partial x_1$ and periodicity of $N^{1,1}$, $N^{1,0}$

$$\begin{aligned}&\Big|\varepsilon\int\limits_{G_\varepsilon}(a_{\cdot 1},\nabla_y u^{1,\pm}_j)\big|_{y=x/\varepsilon}\,\frac{\partial u^{1,\pm}_i}{\partial x_1}(x_1,y)\,dx\Big|\\ &\le C\,\varepsilon\int\limits_{G_\varepsilon}\big[|\nabla_y N^{1,1}(x_1,y)|+|\nabla_y N^{1,0}(x_1,y)|\big]\big|_{y=x/\varepsilon}\,dx\\ &\le C\,\varepsilon^d\max_{x_1\in[-1,1]}\Big[\int\limits_Y|\nabla_y N^{1,1}(x_1,y)|\,dy+\int\limits_Y|\nabla_y N^{1,0}(x_1,y)|\,dy\Big] \le C\,\varepsilon\,\varepsilon^{d-1}.\end{aligned}$$

Notice that

$$\begin{aligned}&\int\limits_Y\Big\{(a_{1\cdot}(x_1,y),\nabla_y N^{1,0}(x_1,y))\\ &+(a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y),\nabla_y N^{1,1}(x_1,y))\Big\}\,dy = 0,\end{aligned}$$

and, thus, by Lemma 3.11

$$\begin{aligned}&\Big|\nu^{0,\pm}_i\int\limits_{G_\varepsilon}\Big\{\big(a_{1\cdot}(x_1,y),\nabla_y N^{1,0}(x_1,y)\big)\\ &+\big(a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y),\nabla_y N^{1,1}(x_1,y)\big)\Big\}\Big|_{y=x/\varepsilon}u^{0,\pm}_i(x_1)\,\frac{du^{0,\pm}_j}{dx_1}(x_1)\,dx\Big|\\ &\le C\,\varepsilon\,\varepsilon^{d-1}.\end{aligned}$$

Similarly,

$$\Big|\nu_j^{0,\pm}\int\limits_{G_\varepsilon}\Big\{\big(a_{1\cdot}(x_1,y),\nabla_y N^{1,0}(x_1,y)\big)$$
$$+\big(a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y),\nabla_y N^{1,1}(x_1,y)\big)\Big\}\Big|_{y=x/\varepsilon}u_j^{0,\pm}(x_1)\,\frac{du_i^{0,\pm}}{dx_1}(x_1)\,dx\Big|$$
$$\le C\,\varepsilon\,\varepsilon^{d-1}.$$

Consequently,

$$(U_i^{\varepsilon,\pm},U_j^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon}=\int\limits_{G_\varepsilon}\big(a_{11}^\varepsilon+a_{\cdot 1}^\varepsilon\nabla_y N^{1,1}(x_1,y)\big)\big|_{y=x/\varepsilon}\frac{du_i^{0,\pm}}{dx_1}\,\frac{du_j^{0,\pm}}{dx_1}\,dx$$
$$+\int\limits_{G_\varepsilon}\Big\{\big(a_{1\cdot}^\varepsilon,\nabla_y N^{1,1}\big)+\big(a^\varepsilon\nabla_y N^{1,1},\nabla_y N^{1,1}\big)\Big\}\big|_{y=x/\varepsilon}\frac{du_i^{0,\pm}}{dx_1}\,\frac{du_j^{0,\pm}}{dx_1}\,dx$$
$$+\nu_i^{0,\pm}\,\nu_j^{0,\pm}\int\limits_{G_\varepsilon}\big(a^\varepsilon\nabla_y N^{1,0},\nabla_y N^{1,0}\big)\big|_{y=x/\varepsilon}u_i^{0,\pm}\,u_j^{0,\pm}\,dx+\mathcal{R}_\varepsilon,$$

where $|\mathcal{R}_\varepsilon|\le C\,\varepsilon\,\varepsilon^{d-1}$.

Recalling the definition of the effective coefficient a^{eff} and of the function $\mathbf{C}(x_1)$ (see (3.3) and (4.7), respectively), by Lemma 3.11, we have

$$\Big|(U_i^{\varepsilon,\pm},U_j^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon}-\varepsilon^{d-1}\,|Q|\int\limits_{-1}^{1}a^{\text{eff}}(x_1)\frac{du_i^{0,\pm}}{dx_1}\,\frac{du_j^{0,\pm}}{dx_1}\,dx_1$$
$$-\nu_i^{0,\pm}\,\nu_j^{0,\pm}\,\varepsilon^{d-1}\,|Q|\int\limits_{-1}^{1}\mathbf{C}(x_1)u_i^{0,\pm}\,u_j^{0,\pm}\,dx_1\Big|\le C\,\varepsilon\,\varepsilon^{d-1}.$$

In view of the normalization condition (4.11),

$$\Big|\frac{\varepsilon^{-(d-1)}}{|Q|}\,(U_i^{\varepsilon,\pm},U_j^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon}-\delta_{ij}\Big|\le C\,\varepsilon. \tag{4.34}$$

Estimate (4.34) implies the lower bound for the norm $\|U_i^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}$:

$$\|U_i^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}\ge\frac{|Q|^{1/2}}{2}\,\varepsilon^{\frac{d-1}{2}},\quad \varepsilon<\varepsilon_i. \tag{4.35}$$

Combining (4.33) and (4.35) yields the desired estimate (4.31). Lemma 4.4 is proved. $\square$

We turn back to the proof of Theorem 4.3. By Lemma 3.8, in view of estimate (4.31), for any j there exists an eigenvalue $\mu_q^{\varepsilon,\pm}$ of the operator $\mathcal{K}^\varepsilon$ such that

$$\big|\mu_q^{\varepsilon,\pm}-(\varepsilon^{-1}\nu_j^{0,\pm}+\nu_j^{1,\pm})^{-1}\big|<C_j\varepsilon^2,\quad \varepsilon<\varepsilon_j.$$

Considering the relation $\lambda_j^{\varepsilon,\pm} = (\mu_j^{\varepsilon,\pm})^{-1}$, we get

$$\big|\lambda_q^{\varepsilon,\pm} - (\varepsilon^{-1}\nu_j^{0,\pm} + \nu_j^{1,\pm})\big| < C_j\varepsilon, \quad \varepsilon < \varepsilon_j. \tag{4.36}$$

Our next goal is to prove that, for any j, there is a unique $\lambda_j^{\varepsilon,\pm}$ satisfying inequality (4.36). The proof consists of three steps presented below. Lemma 4.5 gives the lower and upper bounds for $\lambda_j^{\varepsilon,\pm}$. Lemma 4.6 claims that, up to a subsequence, $\varepsilon\lambda_q^{\varepsilon,\pm}$ converges to an eigenvalue of the operator pencil (4.9). Then we show that there exists a unique eigenvalue $\lambda_j^{\varepsilon,\pm}$ satisfying (4.36).

LEMMA 4.5. *For any j, the estimate holds true*

$$0 < m \le \varepsilon\,|\lambda_j^{\varepsilon,\pm}| \le M_j \tag{4.37}$$

with some constants m and M_j.

PROOF. By the definition of the operator $\mathcal{K}^\varepsilon$,

$$\|\mathcal{K}^\varepsilon\| = \sup_{(v,v)_{\mathcal{H}^\varepsilon}=1} (\mathcal{K}^\varepsilon v, v)_{\mathcal{H}^\varepsilon} = \sup_{(v,v)_{\mathcal{H}^\varepsilon}=1} (\rho^\varepsilon v, v)_{L^2(G_\varepsilon)}.$$

Arguments similar to those in Lemma 3.11 yield

$$\Big|\int\limits_{G_\varepsilon} \rho^\varepsilon\,(v)^2\,dx\Big| \le C\,\varepsilon\,\|v\|^2_{H^1(G_\varepsilon)}.$$

Thus,

$$\|\mathcal{K}^\varepsilon\| \le C\,\varepsilon, \quad |\mu_j^{\varepsilon,\pm}| \le C\,\varepsilon,\ \forall j.$$

Considering the equality $\lambda_j^{\varepsilon,\pm} = (\mu_j^{\varepsilon,\pm})^{-1}$, we obtain the lower bound in (4.37). The upper bound in (4.37) follows easily from estimate (4.36). Lemma 4.5 is proved. □

LEMMA 4.6. *For any j, up to a subsequence, $\varepsilon\lambda_j^{\varepsilon,\pm}$ converges to an eigenvalue ν_* of problem* (4.9).

PROOF. In view of Lemma 4.5, $\varepsilon\lambda_j^{\varepsilon,\pm}$ converges to some $\nu_* \in \mathbb{R}\setminus\{0\}$. Let us show that ν_* is an eigenvalue of the operator pencil (4.9). The weak formulation of problem (2.1) has the form

$$\Big(\mathcal{A}^\varepsilon u_i^{\varepsilon,\pm} - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon u_i^{\varepsilon,\pm}\,,\,w\Big)_{L^2(G_\varepsilon)} = 0, \quad w \in \mathcal{H}^\varepsilon.$$

Integrating by parts leads to the equality

$$\Big(u_i^{\varepsilon,\pm}\,,\,\mathcal{A}^\varepsilon w - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon w\Big)_{L^2(G_\varepsilon)} + \int\limits_{\Sigma^\varepsilon} (a^\varepsilon\nabla w, n)\,u_i^{\varepsilon,\pm}\,d\sigma = 0, \quad w \in \mathcal{H}^\varepsilon. \tag{4.38}$$

By the normalization condition (2.6), $u_i^{\varepsilon,\pm}(x) \in L^2(K_d,\mu_\varepsilon)$ converges strongly in the variable space $L^2(K_d,\mu_\varepsilon)$ to a function $u_*(x_1) \in L^2(K_d,\mu_*)$, $K_d = [-1,1]^d$ (see Lemma 3.13

for the details). Thus, showing that $\mathcal{A}^\varepsilon w - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon w$ converges weakly in $L^2(K_d,\mu_\varepsilon)$ will allow us to pass to the limit in (4.38). For this purpose we construct a test function

$$\begin{aligned} V^\varepsilon(x) = \;& v(x_1) + \varepsilon\, N^{1,1}(x_1,\frac{x}{\varepsilon})\frac{dv(x_1)}{dx_1} \\ & +\varepsilon^2\,\lambda_i^{\varepsilon,\pm}\, N^{1,0}(x_1,\frac{x}{\varepsilon})\, v(x_1), \quad v\in C_0^\infty[-1,1]. \end{aligned}$$

We would like to emphasize that, in contrast with anzats (4.29), we do not add the boundary layer corrector here. The reason is that $v(x_1)$ is equal to zero at points ± 1 together with all its derivatives, that yields $V^\varepsilon(\pm 1, x') = 0$.

Simple transformations yield

$$\mathcal{A}^\varepsilon V^\varepsilon - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon\, V^\varepsilon = J_1^\varepsilon(x_1,y) + J_2^\varepsilon(x_1,y)\Big|_{y=x/\varepsilon},$$

where

$$\begin{aligned} J_1^\varepsilon(x_1,y) = \;& -\frac{\partial}{\partial x_1}\big(a_{1\cdot}(x_1,y)\nabla_y N^{1,1}(x_1,y)\,\frac{dv(x_1)}{dx_1}\big) \\ & -\frac{\partial}{\partial x_1}\big(a_{11}(x_1,y)\,\frac{dv(x_1)}{dx_1}\big) \\ & -\varepsilon\,\lambda_i^{\varepsilon,\pm}\,\frac{\partial}{\partial x_1}\big(a_{1\cdot}(x_1,y)\nabla_y N^{1,0}(x_1,y)\,v(x_1)\big) \\ & -\varepsilon\,\lambda_i^{\varepsilon,\pm}\,\rho(x_1,y)N^{1,1}(x_1,y)\,\frac{dv(x_1)}{dx_1} \\ & -(\varepsilon\,\lambda_i^{\varepsilon,\pm})^2\,\rho(x_1,y)N^{1,0}(x_1,y)\,v(x_1); \end{aligned}$$

$$\begin{aligned} & J_2^\varepsilon(x_1,y) = -\varepsilon\Big\{\mathrm{div}_x + \frac{1}{\varepsilon}\mathrm{div}_y\Big\}\Big[a_{\cdot 1}(x_1,y)\frac{\partial}{\partial x_1}(N^{1,1}(x_1,y)\,\frac{dv(x_1)}{dx_1})\Big] \\ & -\varepsilon^2\lambda_i^{\varepsilon,\pm}\Big\{\mathrm{div}_x + \frac{1}{\varepsilon}\mathrm{div}_y\Big\}\Big[a_{\cdot 1}(x_1,y)\frac{\partial}{\partial x_1}(N^{1,0}(x_1,y)\,v(x_1))\Big]. \end{aligned}$$

In view of (3.3), (4.8) and (4.7),

$$\begin{aligned} & \int\limits_Y J_1^\varepsilon(x_1,y)\,dy = -\frac{\partial}{\partial x_1}\big(a^{\text{eff}}(x_1)\,\frac{dv(x_1)}{dx_1}\big) \\ & +\varepsilon\,\lambda_i^{\varepsilon,\pm}\,\mathbf{B}(x_1)\,v(x_1) - (\varepsilon\,\lambda_i^{\varepsilon,\pm})^2\,\mathbf{C}(x_1)\,v(x_1). \end{aligned}$$

Using Lemma 3.11 and normalization condition (2.6), we obtain

$$\begin{aligned} & \Big|\int\limits_{G_\varepsilon} J_1^\varepsilon(x_1,y)\big|_{y=x/\varepsilon}\,u_i^{\varepsilon,\pm}(x)dx - \int\limits_{G_\varepsilon}\int\limits_Y J_1^\varepsilon(x_1,y)u_i^{\varepsilon,\pm}(x)\,dy\,dx\Big| \\ & \le C\,\varepsilon\,\varepsilon^{\frac{(d-1)}{2}}\|u_i^{\varepsilon,\pm}\|_{H^1(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{d-1}. \end{aligned}$$

Then, integrating by parts one gets

$$\int\limits_{G_\varepsilon} J_2^\varepsilon(x_1,y)\big|_{y=x/\varepsilon}\, u_i^{\varepsilon,\pm}(x)dx + \int\limits_{\Sigma_\varepsilon} (a^\varepsilon \nabla V^\varepsilon, n)\, u_i^{\varepsilon,\pm}\, d\sigma$$
$$= \varepsilon \int\limits_{G_\varepsilon} a_{\cdot 1}(x_1,y)\frac{\partial}{\partial x_1}\big(N^{1,1}(x_1,y)\,\frac{dv(x_1)}{dx_1}\big)\Big|_{y=x/\varepsilon} \nabla u_i^{\varepsilon,\pm}(x)\,dx$$
$$+\varepsilon^2\lambda_i^{\varepsilon,\pm}\int\limits_{G_\varepsilon} a_{\cdot 1}(x_1,y)\frac{\partial}{\partial x_1}\big(N^{1,0}(x_1,y)\,v(x_1)\big)\Big|_{y=x/\varepsilon} \nabla u_i^{\varepsilon,\pm}(x)\,dx.$$

Estimating the terms on the right-hand side of the last equality yields

$$\Big|\int\limits_{G_\varepsilon} J_2^\varepsilon(x_1,y)\big|_{y=x/\varepsilon}\, u_i^{\varepsilon,\pm}(x)dx + \int\limits_{\Sigma_\varepsilon} (a^\varepsilon \nabla V^\varepsilon, n)\, u_i^{\varepsilon,\pm}\, d\sigma\Big|$$
$$\le C\,\varepsilon\,|G_\varepsilon|^{1/2}\,\|\nabla u_i^{\varepsilon,\pm}\|_{L^2(G_\varepsilon)} \le C\,\varepsilon\,\varepsilon^{d-1}.$$

Consequently,

$$0 = \Big(u_i^{\varepsilon,\pm}\,,\,\mathcal{A}^\varepsilon w - \lambda_i^{\varepsilon,\pm}\rho^\varepsilon w\Big)_{L^2(G_\varepsilon)} + \int\limits_{\Sigma^\varepsilon}(a^\varepsilon\nabla w, n)\,u_i^{\varepsilon,\pm}\,d\sigma$$
$$= \Big(u_i^{\varepsilon,\pm}\,,\,\mathbf{\Pi}(\varepsilon\lambda_i^{\varepsilon,\pm})\,v\Big)_{L^2(G_\varepsilon)} + r^\varepsilon, \quad |r^\varepsilon| \le C\,\varepsilon\,\varepsilon^{d-1}.$$

By definition of the measure μ_ε (see Section 3)

$$\int\limits_{K_d} u_i^{\varepsilon,\pm}(x)\,\mathbf{\Pi}(\varepsilon\lambda_i^{\varepsilon,\pm})\,v(x_1)\,d\mu_\varepsilon + \frac{r^\varepsilon}{\varepsilon^{d-1}|Q|} = 0.$$

Passing to the limit in the last equality, taking into account the strong convergence of $u_i^{\varepsilon,\pm}$ in $L^2(K_d,\mu_\varepsilon)$, yields

$$\int\limits_{K_d} u_*(x_1)\,\mathbf{\Pi}(\nu_*)\,v(x_1)\,d\mu_*(x) = 0.$$

Integration by parts gives

$$\int\limits_{K_d} v(x_1)\,\mathbf{\Pi}(\nu_*)\,u_*(x_1)\,d\mu_*(x) = 0, \quad v \in C_0^\infty[-1,1].$$

Thus, u_* satisfies the equation

$$\mathbf{\Pi}(\nu_*)\,u_*(x_1) = -\frac{d}{dx_1}\big(a^{\rm eff}\,\frac{du_*}{dx_1}\big) + \nu_*\,\mathbf{B}u_* - (\nu_*)^2\,\mathbf{C}\,u_* = 0. \tag{4.39}$$

By the definition of $u_i^{\varepsilon,\pm}$ and $\lambda_i^{\varepsilon,\pm}$ we have

$$\|u_i^{\varepsilon,\pm}\|^2_{\mathcal{H}^\varepsilon} = \lambda_i^{\varepsilon,\pm}\,(\rho^\varepsilon\,u_i^{\varepsilon,\pm}, u_i^{\varepsilon,\pm})_{L^2(G_\varepsilon)}.$$

Since $\langle \rho(x_1, \cdot)\rangle = 0$, then

$$\Big| \int\limits_{G_\varepsilon} \rho^\varepsilon (u_i^{\varepsilon,\pm})^2\, dx \Big| \le C\,\varepsilon\, \|u_i^{\varepsilon,\pm}\|_{L^2(G_\varepsilon)}\, \|u_i^{\varepsilon,\pm}\|_{H^1(G_\varepsilon)},$$

and, consequently,

$$\|u_i^{\varepsilon,\pm}\|^2_{\mathcal{H}^\varepsilon} \le C\,\varepsilon\,\lambda_i^{\varepsilon,\pm}\, \|u_i^{\varepsilon,\pm}\|_{L^2(G_\varepsilon)}\, \|u_i^{\varepsilon,\pm}\|_{H^1(G_\varepsilon)}.$$

Taking into account estimate (4.37) and the defintion of the measure μ_ε, we have

$$\|u_i^{\varepsilon,\pm}\|_{L^2(K_d,\mu_\varepsilon)} \ge c > 0.$$

Considering the strong convergence of $u_i^{\varepsilon,\pm}$ in $L^2(K_d,\mu_\varepsilon)$ leads to the inequality

$$\|u_*\|_{L^2(-1,1)} \ge c > 0,$$

which means, together with (4.39), that (ν_*, u_*) is an eigenpair of the operator pencil (4.9). Lemma 4.6 is proved. $\square$

Assume that

$$\varepsilon\lambda_i^{\varepsilon,\pm} \to \nu_j^{0,\pm}, \quad \varepsilon \to 0,$$
$$\varepsilon\lambda_k^{\varepsilon,\pm} \to \nu_j^{0,\pm}, \quad \varepsilon \to 0.$$

Then necessarily $i = k$. Indeed, by Lemma 4.6 the eigenfunctions $u_i^{\varepsilon,\pm}$ and $u_k^{\varepsilon,\pm}$ converge to the eigenfunctions $u_1^{*,\pm}$ and $u_2^{*,\pm}$ of (4.9) corresponding to $\nu_j^{0,\pm}$, and, as was proved above, $u_1^{*,\pm} \ne 0$ and $u_2^{*,\pm} \ne 0$. Since the eigenvalue $\nu_j^{0,\pm}$ is simple, we have

$$u_1^{*,\pm} + \bar c_1 u_2^{*,\pm} = 0.$$

for some $\bar c_1 \ne 0$. Assume that $i \ne k$, and consider the expression

$$T^\varepsilon = \frac{1}{\varepsilon}\big(\rho^\varepsilon(u_i^{\varepsilon,\pm} + \bar c_1 u_k^{\varepsilon,\pm}), (u_i^{\varepsilon,\pm} + \bar c_1 u_k^{\varepsilon,\pm})\big)_{L^2(K_d,\mu^\varepsilon)}$$

Considering (2.6), (4.37) and (3.26), we obtain

$$T^\varepsilon = \frac{1}{\varepsilon\lambda_i^{\varepsilon,\pm}}\,\frac{(u_i^{\varepsilon,\pm}, u_i^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon}}{\varepsilon^{d-1}|Q|} + \frac{c_1^2}{\varepsilon\lambda_k^{\varepsilon,\pm}}\,\frac{(u_k^{\varepsilon,\pm}, u_k^{\varepsilon,\pm})_{\mathcal{H}^\varepsilon}}{\varepsilon^{d-1}|Q|}$$

$$= \frac{1}{\varepsilon\lambda_i^{\varepsilon,\pm}} + \frac{c_1^2}{\varepsilon\lambda_k^{\varepsilon,\pm}} \quad\longrightarrow\quad \frac{1}{\nu_j^{0,\pm}} + \frac{c_1^2}{\nu_j^{0,\pm}} = \frac{1+c_1^2}{\nu_j^{0,\pm}} \ne 0. \tag{4.40}$$

It was shown in the proof of Lemma 3.13 that $u_i^{\varepsilon,\pm}$ and $u_k^{\varepsilon,\pm}$ converges strongly in $L^2(K_d,\mu^\varepsilon)$, therefore,

$$\|u_i^{\varepsilon,\pm} + c_1 u_k^{\varepsilon,\pm}\|_{L^2(K_d,\mu^\varepsilon)} \to 0,$$

as $\varepsilon \to 0$. Denote by $S(x,y)$ a solution to the following problem

$$\begin{cases} -\Delta_y S(x_1,y) = \rho(x_1,y), & y \in G^\pm, \\ \nabla_y S(x_1,y)\cdot n(y) = 0, & y \in \Sigma^\pm, \\ S(x_1,y) \text{ is 1-periodic in } y_1. \end{cases}$$

Since $\langle\rho(x_1,\cdot)\rangle = 0$, this problem is solvable. Setting $R(x_1,y) = \nabla_y S(x_1,y)$ we have

$$\frac{1}{\varepsilon}\rho\Big(x_1,\frac{x}{\varepsilon}\Big) = \mathrm{div} R\Big(x_1,\frac{x}{\varepsilon}\Big) - \frac{\partial}{\partial x_1}R(x_1,y)\big|_{y=x/\varepsilon}.$$

Denoting $R_1^\varepsilon(x) = \frac{\partial}{\partial x_1}R(x_1,y)\big|_{y=x/\varepsilon}$ and $R^\varepsilon(x) = R\Big(x_1,\frac{x}{\varepsilon}\Big)$, we rewrite T^ε as follows

$$T^\varepsilon = \big(\mathrm{div}R^\varepsilon(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}),(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm})\big)_{L^2(K_d,\mu^\varepsilon)}$$
$$-\big(R_1^\varepsilon(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}),(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm})\big)_{L^2(K_d,\mu^\varepsilon)}.$$

Clearly, R_1^ε is uniformly in ε bounded. Therefore, the second term on the right-hand side tends to zero, as $\varepsilon\to 0$. Integration by parts in the first term yields

$$\big(\mathrm{div}R^\varepsilon(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}),(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm})\big)_{L^2(K_d,\mu^\varepsilon)} =$$
$$-2\big(R^\varepsilon(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm}),\nabla(u_i^{\varepsilon,\pm}+c_1u_k^{\varepsilon,\pm})\big)_{L^2(K_d,\mu^\varepsilon)}.$$

Since $\|\nabla u_i^{\varepsilon,\pm}\|_{L^2(K_d,\mu^\varepsilon)}$ and $\|\nabla u_i^{\varepsilon,\pm}\|_{L^2(K_d,\mu^\varepsilon)}$ are uniformly in ε bounded, the first term also tends to zero, as $\varepsilon\to 0$, which implies that $\lim\limits_{\varepsilon\to 0}T^\varepsilon = 0$. This contradicts (4.40). We conclude that $i=k$.

Finally, we conclude that for any j there is only one $\lambda_j^{\varepsilon,\pm}$ satisfying inequality (4.36), and thus, it is simple for sufficiently small ε. In view of the geometric simplicity of $\nu_j^{0,\pm}$ and Lemma 3.8, the corresponding eigenfunction $u_j^{\varepsilon,\pm}$ can be approximated by the "almost eigenfunction" $U_j^{\varepsilon,\pm}$:

$$\|u_j^{\varepsilon,\pm} - U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon} \le c_j\,\varepsilon, \quad \varepsilon<\varepsilon_j.$$

The proof of Theorem 4.3 is complete. □

5. The case of sign-changing $\langle\rho(x_1,\cdot)\rangle$

In the case of sign-changing $\langle\rho(x_1,\cdot)\rangle$ the limit spectral problem takes the form

$$\begin{cases} \mathcal{A}^0\,u^0(x_1) \equiv -\dfrac{d}{dx_1}\Big(a^{\text{eff}}(x_1)\dfrac{du^0(x_1)}{dx_1}\Big) \\ = \lambda^0\,\langle\rho(x_1,\cdot)\rangle\,u^0(x_1), \quad x_1\in(-1,1), \\ u^0(\pm1) = 0. \end{cases} \tag{5.1}$$

Here the effective coefficient a^{eff} is defined by (3.3). By Lemma 3.1 the coefficient $a^{\text{eff}}(\cdot)$ is a $C^{1,\alpha}[-1,1]$ function such that $a^{\text{eff}}(x_1)>0$ for all $x_1\in[-1,1]$.

Since $\langle\rho(x_1,\cdot)\rangle$ changes sign, one can see in the same way as in Theorem 2.4 that the spectrum of problem (5.1) is discrete and consists of two infinite sequences

$$0<\lambda_1^{0,+}<\lambda_2^{0,+}<\cdots<\lambda_j^{0,+}\cdots\to+\infty,$$
$$0>\lambda_1^{0,-}>\lambda_2^{0,-}>\cdots>\lambda_j^{0,-}\cdots\to-\infty.$$

Moreover, since problem (5.1) is one-dimensional, all the eigenvalues $\lambda_j^{0,\pm}$ are simple. The corresponding eigenfunctions $u_i^{0,\pm} \in C^{2,\alpha}[-1,1]$ of problem (5.1) can be normalized by

$$\int_{-1}^{1} a^{\text{eff}}(x_1)\, \frac{du_i^{0,\pm}}{dx_1}\, \frac{du_j^{0,\pm}}{dx_1}\, dx_1 = \delta_{ij}. \tag{5.2}$$

For any $j \in \mathbb{N}$ we denote

$$\begin{aligned} U_j^{\varepsilon,\pm}(x) = u_j^{0,\pm}(x_1) + \varepsilon\, N^{1,1}(x_1,y)\, \frac{du_j^{0,\pm}(x_1)}{dx_1}\,\Big|_{y=x/\varepsilon} \\ +\varepsilon\, v_j^{1,\pm}(x_1) + \varepsilon\big(u_{\text{bl}}^{\varepsilon,+}(x) + u_{\text{bl}}^{\varepsilon,-}(x)\big), \end{aligned} \tag{5.3}$$

where $u_j^{0,\pm}$, $N^{1,1}$ and $v_j^{1,\pm}$ solve problems (5.1), (3.2) and (3.15), respectively, with $u^0 = u_j^{0,\pm}$ and $\lambda^0 = \lambda_j^{0,\pm}$. The boundary layer functions $u_{\text{bl}}^{\varepsilon,\pm}$ are defined by (3.17) and (3.13) with $u^0 = u_j^{0,\pm}$.

THEOREM 5.1. *Let conditions* $(\mathbf{H0}) - (\mathbf{H3})$ *be fulfilled, and suppose that* $\langle \rho(x_1,\cdot)\rangle$ *changes its sign on* $[-1,1]$*. If* $(\lambda_j^{\varepsilon,\pm}, u_j^{\varepsilon,\pm})$ *are eigenpairs of problem* (2.1)*, and* $(\lambda_j^{0,\pm}, u_j^{0,\pm})$ *are those of problem* (5.1)*, then the following statements hold:*

(i) *For any* $j \in \mathbb{N}$*, there exist* ε_j *and* $C_j > 0$ *such that*

$$|\lambda_j^{\varepsilon,\pm} - \lambda_j^{0,\pm}| \le C_j\, \varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_j].$$

(ii) *For any* $j \in \mathbb{N}$

$$\|u_j^{\varepsilon,\pm} - U_j^{\varepsilon,\pm}\|_{H^1(G_\varepsilon)} \le C_j\, \varepsilon\, \varepsilon^{\frac{d-1}{2}}$$

where $U_j^{\varepsilon,\pm}$ *is defined by* (5.3)*. Moreover, the "almost eigenfunctions" satisfy the almost orthogonality and normalization condition*

$$\Big|\frac{\varepsilon^{-(d-1)}}{|Q|}\,(a^\varepsilon\, \nabla U_i^{\varepsilon,\pm}, \nabla U_j^{\varepsilon,\pm})_{L^2(G_\varepsilon)} - \delta_{ij}\Big| \le C_j\, \varepsilon.$$

(iii) *For* $j \in \mathbb{N}$*,* $\lambda_j^{\varepsilon,\pm}$ *are simple, for sufficiently small* $\varepsilon > 0$*.*

PROOF. Since the proof of Theorem 5.1 is similar to that of Theorem 3.6, we give here just a sketch of this proof.

First, we construct a formal asymptotic expansion for a solution $(\lambda^\varepsilon, u^\varepsilon)$ of problem (2.1). In the case under consideration it takes the same form as in the case $\langle \rho(x_1,\cdot)\rangle > 0$ (see (3.1)). Namely,

$$\begin{aligned} u^\varepsilon(x) = u^0(x_1) + \varepsilon\, u^1(x_1,y) + \varepsilon^2\, u^2(x_1,y) + \varepsilon^3\, u^3(x_1,y) + \cdots, \\ \lambda^\varepsilon = \lambda^0 + \varepsilon\, \lambda^1 + \cdots, \quad y = \frac{x}{\varepsilon}, \end{aligned} \tag{5.4}$$

where unknown functions $u^k(x_1, y)$ are 1-periodic in y_1. We substitute these ansätze for u^ε and λ^ε in (2.1), collect power-like terms, and repeat the computations of Section 3.1. At the first step we obtain that

$$u^1(x_1, y) = N^{1,1}(x_1, y)\,\frac{du^0(x_1)}{dx_1} + v^1(x_1)$$

with $N^{1,1}$ defined in (3.2). At the second step this yields problem (3.4), that is the pair $(\lambda^0,\, u^0)$ solves problem (5.1).

Notice that, since 0 does not belong to the spectrum of (5.1), for each $u^0 \neq 0$ we have

$$\lambda^0 \neq 0, \qquad \int_{-1}^{1} \langle \rho(x_1, \cdot)\rangle (u^0(x_1))^2\, dx_1 \neq 0. \tag{5.5}$$

In order to determine the function $v^1(x_1)$ we set, like in (3.6),

$$\begin{aligned} u^2(x_1, y) &= N^{2,2}(x_1, y)\,\frac{d^2u^0(x_1)}{dx_1^2} + N^{2,1}(x_1, y)\,\frac{du^0(x_1)}{dx_1} \\ &+ N^{2,0}(x_1, y)\, u^0(x_1) + N^{1,1}(x_1, y)\,\frac{dv^1(x_1)}{dx_1} + v^2(x_1), \end{aligned}$$

where $N^{2,2}$, $N^{2,1}$ and $N^{2,0}$ are y_1-periodic functions defined in (3.7)–(3.9). Recalling the definition of the boundary layer functions $w^\pm(y_1, y')$ (see (3.13)) and the corresponding constants $\hat{w}^\pm$, and repeating once again the computations of Section 3.1, we arrive at problem (3.15) that reads

$$\left\{ \begin{array}{l} -\dfrac{d}{dx_1}\Big(a^{\rm eff}(x_1)\dfrac{dv^1}{dx_1}\Big) - \lambda^0\,\langle \rho(x_1, \cdot)\rangle\, v^1(x_1) \\ = F(x_1) + \lambda^1\,\langle \rho(x_1, \cdot)\rangle\, u^0, \quad\; x_1 \in (-1, 1), \\ v^1(\pm 1) = \hat{w}^\pm \end{array} \right. \tag{5.6}$$

with $F(x_1)$ defined by (3.11).

In view of (5.5), normalization condition (5.2), and by the Fredholm theorem, the solvability condition of the last problem reads

$$\begin{aligned} \lambda^1 &= -\lambda^0 \int_{-1}^{1} F(x_1)\, u^0(x_1)\, dx_1 \\ &+ \lambda^0\Big(a^{\rm eff}(1)\,\frac{du^0}{dx_1}(1)\,\hat{w}^+ - a^{\rm eff}(-1)\,\frac{du^0}{dx_1}(-1)\,\hat{w}^-\Big). \end{aligned} \tag{5.7}$$

Imposing the normalization condition

$$\int_{-1}^{1} v^1(x_1)\, u^0(x_1)\, dx_1 = 0$$

and letting

$$u_{\mathrm{bl}}^{\varepsilon,\pm}(x) = w^{\pm}\big(\frac{x_1 \mp 1}{\varepsilon}, \frac{x'}{\varepsilon}\big) - \hat{w}^{\pm}, \tag{5.8}$$

we finally obtain a formal asymptotic expansion of u^ε:

$$\begin{aligned} U^\varepsilon(x) &= u^0(x_1) + \varepsilon\, N^{1,1}\big(x_1, \frac{x'}{\varepsilon}\big)\frac{du^0(x_1)}{dx_1} \\ &+\varepsilon\, v^1(x_1) + \varepsilon\big(u_{\mathrm{bl}}^{\varepsilon,+}(x) + u_{\mathrm{bl}}^{\varepsilon,-}(x)\big). \end{aligned} \tag{5.9}$$

Let $\lambda_j^{0,+}$ ($\lambda_j^{0,-}$) be the jth positive (negative) eigenvalue of problem (3.4). We substitute the corresponding eigenfunction $u_j^{0,+}$ ($u_j^{0,-}$) for u^0 in (5.9) and denote

$$\begin{aligned} U_j^{\varepsilon,\pm}(x) &= u_j^{0,\pm}(x_1) + \varepsilon\, N^{1,1}(x_1,y)\,\frac{du_j^{0,\pm}(x_1)}{dx_1}\,\Big|_{y=x/\varepsilon} \\ &+\varepsilon\, v_j^{1,\pm}(x_1) + \varepsilon\,\Big(\big(u_{\mathrm{bl}}^{\varepsilon,+}(x) + u_{\mathrm{bl}}^{\varepsilon,-}(x)\big)\Big), \end{aligned} \tag{5.10}$$

where $u_j^{0,\pm}$, $N^{1,1}$ and $v_j^{1,\pm}$ solve problems (5.1), (3.2) and (5.6), respectively, with $u^0 = u_j^{0,\pm}$ and $\lambda^0 = \lambda_j^{0,\pm}$. The boundary layer functions $u_{\mathrm{bl}}^{\varepsilon,\pm}$ are defined by (3.17) and (3.13) again with $u^0 = u_j^{0,\pm}$.

Notice that by construction the function $U_j^{\varepsilon,\pm}$ are elements of the space $\mathcal{H}^\varepsilon$.

Consider the normalized ansatz (5.10)

$$\mathcal{U}_j^{\varepsilon,\pm} = \big(\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}\big)^{-1}\, U_j^{\varepsilon,\pm}$$

and the numbers $(\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm})^{-1}$ with $\lambda_j^{1,\pm}$ defined by formula (5.7) with $u^0 = u_j^{0,\pm}$ and $\lambda^0 = \lambda_j^{0,\pm}$.

The statement of Lemma 3.9 remains valid in the case under consideration both for positive and negative parts of the spectrum.

LEMMA 5.2. *For any $j \in \mathbb{N}$ there are $\varepsilon_j > 0$ and $C_j > 0$ that only depend on j, such that*

$$\|\mathcal{K}^\varepsilon\, \mathcal{U}_j^{\varepsilon,\pm} - (\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm})^{-1}\mathcal{U}_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon} \le C_j\,\varepsilon \qquad \textit{for all } \varepsilon < \varepsilon_j. \tag{5.11}$$

PROOF. As in the proof of Lemma 3.9 we set

$$I_j^{\varepsilon,\pm} \equiv \|\mathcal{K}^\varepsilon\, \mathcal{U}_j^{\varepsilon,\pm} - (\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm})^{-1}\mathcal{U}_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon},$$

and after straightforward rearrangements get

$$
\begin{aligned}
I_j^{\varepsilon,\pm} &= \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left| \Big(\mathcal{K}^\varepsilon \,\mathcal{U}_j^{\varepsilon,\pm} - (\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm})^{-1}\,\mathcal{U}_j^{\varepsilon,\pm}\,,\, w \Big)_{\mathcal{H}^\varepsilon} \right| \\
&= \frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{\left|\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm}\right|} \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left| \Big((\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm})\,\mathcal{K}^\varepsilon\, U_j^{\varepsilon,\pm} - U_j^{\varepsilon,\pm}\,,\, w \Big)_{\mathcal{H}^\varepsilon} \right| \\
&= \frac{\|U_j^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon}^{-1}}{\left|\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm}\right|} \sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \Big| (\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm})\,\big(\rho^\varepsilon\, U_j^{\varepsilon,\pm}, w\big)_{L^2(G_\varepsilon)} \\
&\qquad\qquad - \big(a^\varepsilon\,\nabla U_j^{\varepsilon,\pm}\,,\,\nabla w\big)_{L^2(G_\varepsilon)} \Big|.
\end{aligned}
$$

Estimate (4.35)) justified in the proof of Lemma 3.9 did not rely on the positiveness of $\langle\rho(x_1,\cdot)\rangle$. Thus it also holds in the case of sign-changing $\langle\rho(x_1,\cdot)\rangle$. Namely, for all sufficiently small $\varepsilon > 0$ we have

$$
\|U_i^{\varepsilon,\pm}\|_{\mathcal{H}^\varepsilon} \geq \frac{|Q|^{1/2}}{2}\,\varepsilon^{\frac{(d-1)}{2}}\,. \tag{5.12}
$$

Analogously, in the same way as in the proof of Lemma 3.9, we obtain

$$
\sup_{\substack{w\in\mathcal{H}^\varepsilon\\ \|w\|_{\mathcal{H}^\varepsilon}=1}} \left| (\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm})\,\big(\rho^\varepsilon\, U_j^{\varepsilon,\pm}, w\big)_{L^2(G_\varepsilon)} - \big(a^\varepsilon\,\nabla U_j^{\varepsilon,\pm}\,,\,\nabla w\big)_{L^2(G_\varepsilon)} \right| \leq C\varepsilon\varepsilon^{\frac{(d-1)}{2}}. \tag{5.13}
$$

Since $\lambda_j^{0,\pm} \neq 0$, then for sufficiently small $\varepsilon > 0$ we have $\left|\lambda_j^{0,\pm} + \varepsilon\lambda_j^{1,\pm}\right| \geq C$ with some $C > 0$. Combining this estimate with (5.12) and (5.13) yields (5.11). □

From Lemma 5.2 and Lemma 3.8 it follows that for any $j \in \mathbb{N}$ there are $\varepsilon_j > 0$ and $q^\pm$ such that

$$
|\lambda_{q^\pm}^{\varepsilon,\pm} - \lambda_j^{0,\pm}| \leq c_j\,\varepsilon, \qquad \varepsilon < \varepsilon_j. \tag{5.14}
$$

By the same arguments as in Lemmata 3.12 and 3.13 it is easy to deduce that for any $q \in \mathbb{N}$

$$
0 < m \leq |\lambda_q^{\varepsilon,\pm}| \leq M_q,
$$

and that any limit point λ_* of a sequence $\{\lambda_j^{\varepsilon,+}\}$ or $\{\lambda_j^{\varepsilon,-}\}$ is an eigenvalue of problem (5.1).

In the same way as in the proof of Theorem 3.6 this readily implies all the statements of Theorem 5.1. □

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