Infinite-dimensional idempotent analysis: the role of continuous posets

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The role of continuous posets

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Résumé de la thèse en français

L'analyse idempotente étudie les espaces linéaires de dimension infinie dans lesquels l'opération maximum se substitue à l'addition habituelle. Nous démontrons un ensemble de résultats dans ce cadre, en soulignant l'intérêt des outils d'approximation fournis par la théorie des domaines et des treillis continus. Deux champs d'étude sont considérés : l'intégration et la convexité.

En intégration idempotente, les propriétés des mesures maxitives à valeurs dans un domaine, telles que la régularité au sens topologique, sont revues et complétées ; nous élaborons une réciproque au théorème de Radon–Nikodym idempotent ; avec la généralisation Z de la théorie des domaines nous dépassons différents travaux liés aux représentations de type Riesz des formes linéaires continues sur un module idempotent.

En convexité tropicale, nous obtenons un théorème de type Krein–Milman dans différentes structures algébriques ordonnées, dont les semitreillis et les modules idempotents topologiques localement convexes ; pour cette dernière structure nous prouvons un théorème de représentation intégrale de type Choquet : tout élément d'un compact convexe K peut être représenté par une mesure de possibilité supportée par les points extrêmes de K.

Des réflexions sont finalement abordées sur l'unification de l'analyse classique et de l'analyse idempotente. La principale piste envisagée vient de la notion de semigroupe inverse, qui généralise de façon satisfaisante à la fois les groupes et les semitreillis. Dans cette perspective nous examinons les propriétés « miroir » entre semigroupes inverses et semitreillis, dont la continuité fait partie. Nous élargissons ce point de vue en conclusion.

Abstract

Idempotent analysis involves the study of infinite-dimensional linear spaces in which the usual addition is replaced by the maximum operation. We prove a series of results in this framework and stress the crucial contribution of domain and continuous lattice theory. Two themes are considered: integration and convexity.

In idempotent integration, the properties of domain-valued maxitive measures such as regularity are surveyed and completed in a topological framework; we provide a converse statement to the idempotent Radon–Nikodym theorem; using the Z generalization of domain theory we gather and surpass existing results on the representation of continuous linear forms on an idempotent module.

In tropical convexity, we obtain a Krein–Milman type theorem in several ordered algebraic structures, including locally-convex topological semilattices and idempotent modules; in the latter structure we prove a Choquet integral representation theorem: every point of a compact convex subset K can be represented by a possibility measure supported by the extreme points of K.

The hope for a unification of classical and idempotent analysis is considered in a final step. The notion of inverse semigroup, which fairly generalizes both groups and semilattices, may be the right candidate for this; in this perspective we examine “mirror” properties between inverse semigroups and semilattices, among which continuity. The general conclusion broadens this point of view.

Contents

Maxitivity

CHAPTER I

The idempotent Radon–Nikodym theorem has a converse statement

ABSTRACT. Maxitive integration is an analogue of the Lebesgue integration where σ-additive measures are replaced by σ-maxitive ($\max$-additive) measures. This integral has proved useful in many areas of mathematics such as fuzzy set theory, optimization, idempotent analysis, large deviation theory, or extreme value theory. In all of these applications, the existence of Radon–Nikodym derivatives turns out to be crucial. We gather several existing results of this kind. Then we prove a converse statement to the Radon–Nikodym theorem, i.e. we characterize the σ-maxitive measures that have the Radon–Nikodym property.

I-1. RÉSUMÉ EN FRANÇAIS

Les mesures maxitives ont été à l'origine introduites par Shilkret [**271**]. Elles sont définies de façon analogue aux mesures finiment additives (parfois appelées *charges*), si ce n'est que l'on remplace l'addition habituelle $+$ par l'opération maximum notée $\oplus$. Plus précisément, une *mesure maxitive* sur une tribu $\mathscr{B}$ est une application $\nu : \mathscr{B} \to \overline{\mathbb{R}}_+$ telle que $\nu(\varnothing) = 0$ et

$$\nu(B_1 \cup B_2) = \nu(B_1) \oplus \nu(B_2),$$

pour tous $B_1, B_2 \in \mathscr{B}$. Elle est *σ-maxitive* si elle commute avec les unions de suites croissantes d'éléments de $\mathscr{B}$. Une mesure σ-maxitive ne commute pas forcément avec les *intersections* de suites décroissantes, contrairement à ce qui se passe pour les mesures σ-additives. C'est une différence de taille, qui donne lieu à une notion spécifique : les mesures *optimales*, introduites par Agbeko [**4**].

Relativement aux mesures maxitives, on peut construire une intégrale similaire à celle de Lebesgue. C'est ce qu'a fait Shilkret, et ce qu'ont redécouvert indépendamment Sugeno et Murofushi [**279**] et Maslov [**196**]. Les deux premiers auteurs étaient plutôt attirés par les applications à la théorie des ensembles flous, dans la lignée de l'intégrale de Sugeno [**278**], tandis que Maslov s'intéressait à l'analyse asymptotique.

Depuis lors, cette intégrale a été étudiée et utilisée au sein de plusieurs travaux, avec des motivations variées, liées à la théorie de la dimension et la géométrie fractale, à l'optimisation, aux capacités et aux grandes déviations des processus aléatoires, aux ensembles flous et à la théorie des possibilités, à l'analyse idempotente et à l'algèbre max-plus (tropicale). Notons que le terme « analyse idempotente », qui figure dans le titre de cette thèse, a été inventé par Kolokoltsov et a fait sa première apparition dans les deux articles de Kolokoltsov et Maslov [**154**] et [**155**].

Du fait des multiples champs d'application suscités, le vocabulaire employé autour des mesures maxitives est multiple pour des concepts souvent similaires. À titre d'exemple, indiquons que Maslov et ses successeurs en analyse idempotente parlent plutôt d'*intégrale idempotente* que d'intégrale de Shilkret. Les notations divergent également ; nous suivons le choix de Gerritse [**112**] d'écrire

$$\int_B^\infty f\, d\nu$$

pour désigner l'intégrale de Shilkret de la fonction mesurable f par rapport à la mesure maxitive ν sur l'ensemble B. L'indice supérieur ∞ n'est pas une borne d'intégration, mais rappelle le fait que l'intégrale de Shilkret peut être vue comme une limite d'une suite d'intégrales de Choquet.

Au sein des domaines d'application évoqués ci-dessus, un théorème de type Radon–Nikodym s'avère souvent indispensable. Il serait par exemple difficile de construire une bonne théorie des possibilités (où une mesure de possibilité est l'analogue maxitif d'une mesure de probabilité) sans y insérer la notion de possibilité *conditionnelle* (similaire à l'espérance conditionnelle en théorie des probabilités), dont l'existence est justement garantie par celle de dérivées de Radon–Nikodym ou *densités*. Un tel théorème est par chance disponible : il a été démontré dans [**279**] par Sugeno et Murofushi.

Théorème I-1.1 (Sugeno–Murofushi)**.** *Soient ν et τ des mesures σ-maxitives sur une tribu $\mathscr{B}$. On suppose que τ est σ-finie et σ-principale. Alors ν est absolument continue par rapport à τ si et seulement s'il existe une fonction $\mathscr{B}$-mesurable $c : E \to \overline{\mathbb{R}}_+$ telle que*

$$\nu(B) = \int_B^\infty c\, d\tau,$$

pour tout $B \in \mathscr{B}$.

On voit que pour conserver la forme classique de l'énoncé, i.e. une équivalence entre l'existence d'une densité et la condition d'*absolue continuité*, une condition supplémentaire (en sus de la σ-finitude) est demandée sur la mesure dominante τ : la *σ-principalité*. Celle-ci exprime d'une certaine façon que tout σ-idéal de $\mathscr{B}$ admet un plus grand élément « modulo les ensembles négligeables ». Si les mesures σ-additives σ-finies sont toujours σ-principales, ce n'est pas le cas des mesures σ-maxitives σ-finies. Ainsi toute mesure σ-maxitive ν est absolument continue par rapport à la mesure σ-maxitive $\delta_\#$, définie sur la même tribu $\mathscr{B}$ par $\delta_\#(B) = 1$ si B est non vide et $\delta_\#(\varnothing) = 0$; or ν n'admet pas nécessairement de densité par rapport à $\delta_\#$, qui justement n'est pas σ-principale en général.

D'autres travaux, postérieurs à ceux de Sugeno et Murofushi, ont également abouti à des énoncés de type Radon–Nikodym pour les mesures maxitives. Il s'agit notamment de ceux d'Agbeko [**4**], Akian [**7**], Barron et al.

[**26**] et Drewnowski [**82**]. Dans certains cas, l'existence de l'intégrale de Shilkret ne semblait pas connue des auteurs. En étudiant les liens entre différentes propriétés des mesures maxitives (cf. Tab. 1), il apparaît que ces théorèmes sont en fait des cas particuliers du théorème de Sugeno–Murofushi.

$$
\begin{tikzcd}
 & \text{à variation bornée} \arrow[d, Rightarrow] & \\
 & \text{exhaustive} \arrow[d, Leftrightarrow] & \\
\text{finie} \arrow[d, Rightarrow, shift left] & \text{optimale} \arrow[d, Rightarrow] & \\
\sigma\text{-finie} \arrow[u, Rightarrow, shift left, "optimale"] \arrow[d, Rightarrow, shift left] & \sigma\text{-principale} \arrow[r, Rightarrow] \arrow[d, Rightarrow, shift right] & \text{autocontinue} \\
\text{semi-finie} \arrow[u, Rightarrow, shift left, "optimale"] & \text{CCC} \arrow[u, Rightarrow, shift right, "Zorn"'] \arrow[d, Rightarrow] & \text{essentielle} \arrow[l, Rightarrow] \\
 & \text{localisable} &
\end{tikzcd}
$$

TABLE 1. Nous étudions certaines propriétés des mesures σ-maxitives définies sur une tribu ; les liens entre elles sont ici représentés. En bleu, les conditions de σ-finitude et de σ-principalité prises ensemble sont équivalentes à la propriété de Radon–Nikodym (Théorème I-1.2). Rappelons que pour les mesures σ-additives la σ-finitude implique la σ-principalité.

Au-delà de cette revue et mise en perspective de la littérature, notre contribution mathématique consiste à prouver une réciproque au théorème de Sugeno–Murofushi. Une mesure σ-maxitive τ a la *propriété de Radon–Nikodym* si toute mesure σ-maxitive qu'elle domine admet une densité par rapport à τ. Allié au théorème de Sugeno–Murofushi, notre résultat principal s'énonce alors ainsi :

Théorème I-1.2. *Une mesure σ-maxitive a la propriété de Radon–Nikodym si et seulement si elle est σ-finie et σ-principale.*

Grâce à cela, nous sommes certains que les conditions de σ-finitude et de σ-principalité prises ensemble sont minimales. Ce théorème se démontre en « passant au quotient », c'est-à-dire en s'affranchissant des ensembles de mesure nulle par une relation d'équivalence convenable. Une telle caractérisation permettra en toute confiance de se tourner vers la notion de propriété de Radon–Nikodym d'un *espace* (de type module sur le semicorps idempotent $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \oplus, \times)$), une question dont nous reparlerons au chapitre II (sans chercher à la résoudre).

Pour finir, les *mesures de possibilité* sont redéfinies comme des mesures σ-maxitives normées (de poids total égal à 1) et σ*-principales*. Cela permet de conserver les résultats habituels tout en évitant d'autres hypothèses faites dans la littérature qui « cassent » le parallèle avec la théorie des probabilités. On peut donc d'une part éviter de supposer les mesures de possibilité *complètement maxitives* (ce qui est l'approche d'Akian et al. [**12, 13**], Akian [**6**], Del Moral and Doisy [**77**]) ; d'autre part de les supposer définies sur une τ-algèbre (ce qui est l'approche de de Cooman [**70, 71, 72, 73**] et Puhalskii [**246**]), une τ-algèbre étant une σ-algèbre stable par intersections quelconques.

I-2. Introduction

Maxitive measures, originally introduced by Shilkret [**271**], are defined analogously to classical finitely additive measures or charges with the supremum operation, denoted $\oplus$, in place of the addition $+$. More precisely, a *maxitive measure* on a σ-algebra $\mathscr{B}$ is a map $\nu : \mathscr{B} \rightarrow \overline{\mathbb{R}}_+$ such that $\nu(\varnothing) = 0$ and

$$\nu(B_1 \cup B_2) = \nu(B_1) \oplus \nu(B_2),$$

for all $B_1, B_2 \in \mathscr{B}$. It is σ*-maxitive* if it commutes with unions of nondecreasing sequences of elements of $\mathscr{B}$. One should note that a σ-maxitive measure does not necessarily commute with *intersections* of nonincreasing sequences, unlike σ-additive measures. This feature justifies the specific concept of *optimal* measure introduced by Agbeko [**4**].

A corresponding "maxitive" integral, paralleling Lebesgue's integration theory, was built by Shilkret. It was rediscovered independently by Sugeno and Murofushi [**279**] and by Maslov [**196**]. While Sugeno and Murofushi focused on its virtues for fuzzy set theory (in the line of the Sugeno integral [**278**]), Maslov's concerns aimed at asymptotic analysis.

Since then, this integral has been studied and used by several authors with motivations from dimension theory and fractal geometry, optimization, capacities and large deviations of random processes, fuzzy sets and possibility theory, idempotent analysis and max-plus (tropical) algebra. Note that the term "idempotent analysis", which is used in the title of this thesis, was coined by Kolokoltsov and made its first appearance in the papers by Kolokoltsov and Maslov [**154**] et [**155**].

Because of the numerous fields of application just listed, the wording around maxitive measures is not unique, thus deserves to be reviewed. For instance, Maslov coined the term *idempotent integration*, which is also of wide use. Notations may also diverge; we adopt the choice of Gerritse [**112**] and write

$$\int_B^\infty f \, d\nu$$

for the Shilkret integral of a measurable map f with respect to the maxitive measure ν on B. The index ∞ is not an integration bound, it recalls the fact that the Shilkret integral can be seen as a limit of a sequence of Choquet integrals.

In all of these fields of application, a Radon–Nikodym like theorem is often essential. For instance, a comprehensive theory of possibilities (where a *possibility measure* is the maxitive analogue of a probability measure) cannot do without a notion of *conditional* possibility (just like one needs that of conditional expected value in probability theory). Its existence happens to be ensured by that of Radon–Nikodym derivatives (or *densities*). Such a theorem is actually available: it was proved in [**279**] by Sugeno and Murofushi.

Theorem I-2.1 (Sugeno–Murofushi)**.** *Let ν, τ be σ-maxitive measures on a σ-algebra $\mathscr{B}$. Assume that τ is σ-finite and σ-principal. Then ν is absolutely continuous with respect to τ if and only if there exists some $\mathscr{B}$-measurable map $c : E \to \overline{\mathbb{R}}_+$ such that*

$$\nu(B) = \int_B^\infty c \, d\tau,$$

for all $B \in \mathscr{B}$.

The assertion looks like the classical Radon–Nikodym theorem, except that one needs an unusual condition on the dominating measure τ, namely *σ-principality*. This condition roughly says that every σ-ideal of $\mathscr{B}$ has a greatest element "modulo negligible sets". Although σ-finite σ-additive measures are always σ-principal, this is not true for σ-finite σ-maxitive measures. For instance, every σ-maxitive measure ν is absolutely continuous with respect to the σ-maxitive measure $\delta_\#$, defined on the same σ-algebra $\mathscr{B}$ by $\delta_\#(B) = 1$ if B is nonempty and $\delta_\#(\varnothing) = 0$; however, ν does not always have a density with respect to $\delta_\#$, and this latter measure is not σ-principal in general.

After the article [**279**], many authors have published results of Radon–Nikodym flavour for maxitive measures. This is the case of Agbeko [**4**], Akian [**7**], Barron et al. [**26**], and Drewnowski [**82**]. In some cases, the authors were not aware of the existence of the Shilkret integral. By linking several properties of maxitive measures together (see Table 2), we shall see why these results are already encompassed in the Sugeno–Murofushi theorem.

Beyond this review and clarification of the literature, we prove a converse to the Sugeno–Murofushi theorem. A σ-maxitive measure τ has the *Radon–Nikodym property* if every σ-maxitive measure dominated by τ has a density with respect to τ. Put together with the Sugeno–Murofushi theorem, our main result is the following:

Theorem I-2.2. *A σ-maxitive measure satisfies the Radon–Nikodym property if and only if it is σ-finite and σ-principal.*

This result ensures that the conditions of σ-finiteness and σ-principality together are minimal. We shall prove it with the help of the "quotient space"

$$
\begin{tikzcd}
 & \text{of bounded variation} \arrow[d, Rightarrow] & \\
 & \text{exhaustive} \arrow[d, Leftrightarrow] & \\
\text{finite} \arrow[d, Rightarrow, shift left] & \text{optimal} \arrow[d, Rightarrow] & \\
\sigma\text{-finite} \arrow[u, Rightarrow, shift left, "optimal"] \arrow[d, Rightarrow, shift left] & \sigma\text{-principal} \arrow[r, Rightarrow] \arrow[d, Rightarrow, shift right] & \text{autocontinuous} \\
\text{semi-finite} \arrow[u, Rightarrow, shift left, "optimal"] & \text{CCC} \arrow[u, Rightarrow, shift right, "Zorn"'] \arrow[d, Rightarrow] & \text{essential} \arrow[l, Rightarrow] \\
 & \text{localizable} &
\end{tikzcd}
$$

TABLE 2. Many properties of σ-maxitive measures defined on a σ-algebra will be addressed in this chapter; the links between them are represented here. In blue, the conditions of σ-finiteness and σ-principality together are equivalent to the Radon–Nikodym property, as Theorem I-7.2 will show. Note that for σ-additive measures, σ-finiteness implies σ-principality.

associated with the σ-maxitive measure, i.e. we shall get rid of negligible sets by an appropriate equivalence relation. Such a characterization will be useful in a future work to try to investigate *spaces* (like modules over the idempotent semifield $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \oplus, \times)$) with the Radon–Nikodym property; we shall discuss this problem further in Chapter II (but not solve it).

Finally, we redefine a *possibility measure* as a normed *σ-principal* σ-maxitive measure. The advantage of this definition is that it avoids some other hypothesis that are not satisfactory if one wants to parallel probability theory. Especially, some authors demanded their possibility measures to be *completely maxitive*, e.g. Akian et al. [**12, 13**], Akian [**6**], Del Moral and Doisy [**77**]. Other authors such as de Cooman [**70, 71, 72, 73**] and Puhalskii [**246**] defined these measures on a τ-algebra, i.e. a σ-algebra closed under arbitrary intersections.

The chapter is organized as follows. Section I-3 introduces the notion of σ-maxitive measure and recalls some key theorems and examples. Maxitive measures that can be represented as essential suprema are studied in Section I-4; we also discuss Barron et al.'s theorem whose proof draws a link between maxitive measures and classical additive measures. Section I-5 develops the Shilkret integral and its properties. Section I-6 lists existing Radon–Nikodym theorems for the Shilkret integral, and makes the connection with Section I-4. In Section I-7 we define the quotient space associated with a σ-maxitive measure and characterize maxitive measures satisfying the Radon–Nikodym property. Section I-8 focuses on the important particular case of optimal measures, i.e. maxitive fuzzy measures. Section I-9 proposes new foundations for possibility theory, relying on the concept of σ-principal maxitive measures developed in Section I-6.

I-3. Preliminaries on maxitives measures

I-3.1. Notations. Let E be a nonempty set. A *prepaving* on E is a collection of subsets of E containing the empty set and closed under finite unions. An *ideal* of a prepaving $\mathscr{E}$ is a nonempty subset $\mathscr{I}$ of $\mathscr{E}$ that is closed under finite unions and such that $A \subset G \in \mathscr{I}$ and $A \in \mathscr{E}$ imply $A \in \mathscr{I}$. A collection of subsets of E containing E, the empty set, and closed under finite intersections and countable unions is a *semi-σ-algebra*; in this case, $(E, \mathscr{E})$ is a *semi-measurable space*. In a semi-σ-algebra, a *σ-ideal* is an ideal that is closed under countable unions. A semi-σ-algebra (resp. a topology) closed under the formation of complements is a *σ-algebra* (resp. a *τ-algebra*). When explicitly refering to a σ-algebra, we shall preferentially call it $\mathscr{B}$ instead of $\mathscr{E}$.

Assume in all the sequel that $\mathscr{E}$ is a prepaving on E. A *set function* on $\mathscr{E}$ is a map $\mu : \mathscr{E} \to \overline{\mathbb{R}}_+$ equal to zero at the empty set. A set function μ is

- *monotone* if $\mu(G) \leqslant \mu(G')$ for all $G, G' \in \mathscr{E}$ such that $G \subset G'$,
- *normed* if $\bigoplus_{G \in \mathscr{E}} \mu(G) = 1$,
- *null-additive* if $\mu(G \cup N) = \mu(G)$ for all $G, N \in \mathscr{E}$ with $\mu(N) = 0$,
- *finite* if $\mu(G) < \infty$ for every $G \in \mathscr{E}$,
- *σ-finite* if $\mu(G_n) < \infty$ for all n, where (G_n) is a countable family of elements of $\mathscr{E}$ covering E,
- *continuous from below* if $\mu(G) = \lim_n \mu(G_n)$, for all $G_1 \subset G_2 \subset \ldots \in \mathscr{E}$ such that $G = \bigcup_n G_n \in \mathscr{E}$.

We shall need the following notion of negligibility. If μ is a null-additive monotone set function on $\mathscr{E}$, a subset N of E is *μ-negligible* if it is contained in some $G \in \mathscr{E}$ such that $\mu(G) = 0$. A property $P(x)$ ($x \in E$) is satisfied *μ-almost everywhere* (or *μ-a.e.* for short) if there exists some negligible subset N of E such that $P(x)$ is true, for all $x \in E \backslash N$.

I-3.2. Definition of maxitive measures. In this section, $\mathscr{E}$ will denote a prepaving on some nonempty set E.

A *maxitive* (resp. *completely maxitive*) *measure* on $\mathscr{E}$ is a set function ν on $\mathscr{E}$ such that, for every finite (resp. arbitrary) family $\{G_j\}_{j \in J}$ of elements of $\mathscr{E}$ with $\bigcup_{j \in J} G_j \in \mathscr{E}$,

$$\nu(\bigcup_{j \in J} G_j) = \bigoplus_{j \in J} \nu(G_j). \tag{1}$$

A *σ-maxitive* measure is a continuous from below maxitive measure.

Remark I-3.1. The term "maxitive" qualifying a set function that satisfies Equation (1) was coined by Shilkret [**271**], and has been widely used, especially in the fields of probability theory and fuzzy theory. However, one can find many other terms in the literature for maxitive or σ-maxitive measures, say: *f-additive* or *fuzzy additive measures* [**278, 216, 298**], *contactability measures* [**296**], *measures of type* $\bigvee$ [**54**], *idempotent measures* [**196, 7**], *max-measures* [**279**], *stable measures* [**97**], *optimal measures* [**4, 99**], *cost measures* [**6, 36**], *semi-additive measures* [**111**], *possibility measures* [**201**], *generalized possibility measures* [**87**], *performance measures* [**77**], *sup-decomposable measures* [**202**], *set-additive measures* [**18, 187, 188**]. As for completely maxitive measures, one finds: *sup-measures* [**225, 231**], *idempotent measures* when $\mathscr{E} = 2^E$ or *τ-maxitive measures* for general $\mathscr{E}$ [**246**], (*generalized*) *possibility measures* [**268, 305, 269, 71, 298**], *supremum-preserving measures* [**162**].

Some differences may appear in the definitions, essentially depending on the choice of the range of the measure and on the structure of the space $(E, \mathscr{E})$. See also the historical notes in [**246**, Appendix B].

The definition of the term "possibility measure" remains unclear, and mainly oscillates between "normed σ-maxitive measure" and "normed completely maxitive measure". We shall propose in Section I-9 a different definition, aiming at founding an operational possibility theory.

Note that every maxitive measure is null-additive and monotone. Actually a much stronger property than monotonicity holds, namely the alternating property. For a map $f : \mathscr{E} \to \overline{\mathbb{R}}$ we classically define $\Delta_{G_1} \dots \Delta_{G_n} f(G)$ after Choquet [**60**] by iterating the formula $\Delta_{G_1} f(G) = f(G \cup G_1) - f(G)$ (with the convention that $-\infty + \infty = \infty - \infty = 0$). Then f is *alternating of infinite order* (or *alternating* for short) if

$$(-1)^{n+1} \Delta_{G_1} \dots \Delta_{G_n} f(G) \geqslant 0,$$

for all $n \in \mathbb{N}\backslash\{0\}$, $G, G_1, \dots, G_n \in \mathscr{E}$, where $\mathbb{N}$ denotes the set of non-negative integers. Nguyen et al. [**223**] gave a combinatorial proof of the fact that every finite maxitive measure is alternating (see also Harding et al. [**120**, Theorem 6.2]). This is actually true for every (finite or not) maxitive measure, as the following proposition states.

Proposition I-3.2. *Every maxitive measure on $\mathscr{E}$ is alternating.*

Proof. Recall the convention $\infty - \infty = 0$. We write $s \wedge t$ for the infimum of $\{s, t\}$. Let $G_1, \dots, G_n \in \mathscr{E}$, and define $\nu_0(G) = -\nu(G)$, $\nu_n(G) = (-1)^{n+1} \Delta_{G_n} \dots \Delta_{G_1} \nu(G)$. A proof by induction shows that the property "$\nu_n(G \cup G') = \nu_n(G) \wedge \nu_n(G')$ and $\nu_n(G) = 0 \oplus (\nu_{n-1}(G) - \nu_{n-1}(G_n)) \geqslant 0$, for all $G, G' \in \mathscr{E}$" holds for all $n \in \mathbb{N}\backslash\{0\}$. $\square$

I-3.3. Elementary and advanced examples. Here we collect some examples given in the literature, especially on metric spaces where maxitive measures appear naturally. Some examples are also linked with extreme value theory, which is the branch of probability theory that aims at the modelling of rare events.

Example I-3.3 (Essential supremum)**.** Let μ be a null-additive monotone set function, and let $f : E \to \overline{\mathbb{R}}_+$ be a map. If one sets $\nu(G) = \bigwedge\{t > 0 : G \in \mathscr{I}_t\}$ with $\mathscr{I}_t := \{G \in \mathscr{E} : G \cap \{f > t\}$ is μ-negligible$\}$, where $\bigwedge A$ denotes the infimum of a subset A of $\mathbb{R}_+$, then ν is a maxitive measure, called the *μ-essential supremum* of f, and we write

$$\nu(G) = \bigoplus_{x \in G}^{\mu} f(x). \tag{2}$$

In this case, f is a *relative density* of ν (with respect to μ). Sufficient conditions for the existence of a relative density, when ν and μ are given, are discussed in Section I-4.

Example I-3.4 (Cardinal density of a maxitive measure)**.** In the previous example, one can take for μ the maxitive measure $\delta_\#$ defined by $\delta_\#(G) = 1$ if G is nonempty, $\delta_\#(G) = 0$ otherwise. Then the essential supremum in Equation (2) reduces to an "exact" supremum, i.e.

$$\nu(G) = \bigoplus_{x \in G}^{\delta_\#} f(x) = \bigoplus_{x \in G} f(x). \tag{3}$$

In this special case we say that f is a *cardinal density* of ν. Note also that a maxitive measure with a cardinal density is necessarily completely maxitive. One may ask, conversely, whether complete maxitivity is a sufficient condition for guaranteeing the existence of a cardinal density. This question will be treated in detail in Chapter II.

Examples I-3.5 (Measures of non-compactness)**.** Let E be a Banach space. Following Appell [**18**], a *measure of non-compactness* (or *monc* for short) on E is a maxitive measure ν on the collection of bounded subsets of E, satisfying the following axioms, for all bounded subsets B of E:

- $\nu(B + K) = \nu(B)$, for all compact subsets K in E,
- $\nu(t.B) = t.\nu(B)$, for all $t > 0$,
- $\nu(\overline{\text{co}}(B)) = \nu(B)$, where $\overline{\text{co}}$ denotes the closed convex hull.

The definition may differ from one author to the other, see e.g. Mallet-Paret and Nussbaum [**187, 188**] for a quite different list of axioms. Note that if $E = \mathbb{R}^d$, then $\nu(B) = 0$ for all bounded subsets B. As Appell recalled, three important examples of moncs appear in the literature, namely the *ball monc* (or *Hausdorff monc*)

$$\alpha(B) = \bigwedge\{t > 0 : \text{there are finitely many balls of radius } t \text{ covering } B\};$$

the *set monc* (or *Kuratowski monc*)

$$\beta(B) = \bigwedge\{t > 0 : \text{there are finitely many subsets of diameter at most } t \text{ covering } B\};$$

and the *lattice monc* (or *Istrăţescu monc*)

$$\gamma(B) = \bigoplus\{t > 0 : \text{there is a sequence } (x_n)_n \text{ in } B \\ \text{with } \|x_m - x_n\| \geqslant t \text{ for } m \neq n\},$$

and we have the classical relations $\alpha \leqslant \gamma \leqslant \beta \leqslant 2.\alpha$. Since moncs vanish on compact subsets, hence on singletons, they are a source of examples of maxitive measures with no cardinal density.

Examples I-3.6 (Dimensions)**.**

- If E is a topological space, the topological dimension is a maxitive measure on the collection of its closed subsets (see e.g. Nagata [**218**, Theorem VII-1]). If E is normal, the topological dimension is even σ-maxitive [**218**, Theorem VII-2].
- If E is a metric space, the Hausdorff dimension and the packing-dimension are σ-maxitive measures on 2^E, and the upper box dimension is a maxitive measure on 2^E (see e.g. Falconer [**97**]).
- If E is the Cantor set $\{0,1\}^{\mathbb{N}}$, the constructive Hausdorff dimension and the constructive packing-dimension are completely maxitive measures on 2^E, see Lutz [**185, 186**].
- If E is the set of positive integers, the zeta dimension is a maxitive measure on 2^E, see Doty et al. [**81**].

Example I-3.7 (Random closed sets)**.** Let $(\Omega, \mathscr{A}, P)$ be a probability space and E be a locally-compact, separable, Hausdorff topological space. We denote by $\mathscr{F}$ the collection of closed subsets of E, and by $\mathscr{K}$ the collection of compact subsets. A random closed set is a measurable map $C : \Omega \to \mathscr{F}$. For measurability a σ-algebra on $\mathscr{F}$ is needed. The usual σ-algebra considered is the Borel σ-algebra generated by the Vietoris (or *hit-and-miss*) topology on $\mathscr{F}$. Choquet's fundamental theorem is that the distribution of a random closed set C is characterized by its Choquet capacity $T : \mathscr{K} \to [0,1]$ defined by $T(K) = P[C \cap K \neq \varnothing]$. Moreover, T is an alternating set function that is *continuous from above* on $\mathscr{K}$ (see the definition in Section I-8), and every $[0,1]$-valued alternating, continuous from above set function on $\mathscr{K}$ is the Choquet capacity of some random closed set.

Recall that every maxitive measure is alternating (see Proposition I-3.2). For a given usc map $c : E \to [0,1]$, the following construction explicitly gives a random closed set whose Choquet capacity has cardinal density c [**223**]. Let U be a uniformly distributed random variable on $[0,1]$. Then $C = \{x \in E : c(x) \geqslant U\}$ is a random closed set on E, and its Choquet capacity T is maxitive and satisfies $T(K) = \bigoplus_{x \in K} c(x)$, for all $K \in \mathscr{K}$.

One may observe that this random closed set is such that

$$C(\omega) \subset C(\omega') \text{ or } C(\omega') \subset C(\omega), \tag{4}$$

for all $\omega, \omega' \in \Omega$. More generally, Miranda et al. called *consonant* (of type C2) a random closed set C satisfying Relation (4) for all $\omega, \omega' \in \Omega_0$, for some event Ω_0 of probability 1. These authors showed that a random closed set is consonant if and only if its Choquet capacity is maxitive [**204**, Corollary 5.4].

Elements of random set theory may be found in the reference book by Matheron [**197**], or in the recent monograph by Molchanov [**208**].

Example I-3.8 (Random sup-measures)**.** Let $(\Omega, \mathscr{A})$ and $(E, \mathscr{B})$ be measurable spaces, P be a probability measure on $\mathscr{A}$, and m be a finite σ-additive measure on $\mathscr{B}$. Consider a Poisson point process $(X_k, T_k)_{k\geqslant 1}$ on $\mathbb{R}_+ \times E$ with intensity $\beta x^{-\beta-1}dx \times m(dt)$, where $\beta > 0$. Then the random process defined on $\mathscr{B}$ by

$$M(B) = \bigoplus_{k\geqslant 1} X_k.1_B(T_k)$$

is, ω by ω, a completely maxitive measure. Moreover, this is a *β-Fréchet random sup-measure* with control measure m in the sense of Stoev and Taqqu [**276**, Definition 2.1], for it is a map $M : \Omega \times \mathscr{B} \to \overline{\mathbb{R}}_+$ satisfying the following axioms:

- for all pairwise disjoint collections $(B_j)_{j\in\mathbb{N}}$ of elements of $\mathscr{B}$, the random variables $M(B_j)$, $j \in \mathbb{N}$, are independent, and, almost surely,
$$M(\bigcup_{j\in\mathbb{N}} B_j) = \bigoplus_{j\in\mathbb{N}} M(B_j);$$
- for all $B \in \mathscr{B}$ the random variable $M(B)$ has a Fréchet distribution with shape parameter $1/\beta$, in such a way that, for all $x > 0$,
$$P[M(B) \leqslant x] = \exp(-m(B)x^{-\beta}).$$

The Poisson process $(X_k, T_k)_{k\geqslant 1}$ was introduced by de Haan [**75**] as a tool for representating continuous-time max-stable processes. These processes play an important role in extreme value theory. See also Norberg [**225**] and Resnick and Roy [**252**] for elements on random sup-measures.

Example I-3.9 (The home range)**.** Let $(X_n)_{n\geqslant 1}$ be a sequence of independent, identically distributed $\mathbb{R}^2$-valued random variables, and assume that the common distribution has compact support. We write this sequence in polar coordinates $(R_n, \Theta_n)_{n\geqslant 1}$. Define the map h on Borel subsets B of $[0, 2\pi]$ by:

$$h(B) = \bigoplus\{r \in \mathbb{R}_+ : P[R_1 > r, \Theta_1 \in B] > 0\}.$$

Then, as asserted by de Haan and Resnick [**76**, Proposition 2.1], h is a completely maxitive measure. According to these authors, h may be thought of as the boundary of the natural habitat of some animal, called the *home range* in ecology. The sequence $(X_n)_{n\geqslant 1}$ is then seen as the successive sightings of the animal. De Haan and Resnick aimed at finding consistent estimates of the boundary h.

The following paragraph contradicts an assertion by van de Vel [**284**, Exercise II-3.19.1].

Example I-3.10 (Carathéodory number of a convexity space)**.** A collection $\mathscr{C}$ of subsets of a set X that contains $\varnothing$ and X is a *convexity* on X if it is closed under arbitrary intersections and closed under directed unions. The pair $(X, \mathscr{C})$ is called a *convexity space*, and elements of $\mathscr{C}$ are called *convex* subsets of X. If $A \subset X$, the *convex hull* $\mathrm{co}(A)$ of A is the intersection of all convex subsets containing A. Advanced abstract convexity theory is developed in the monograph by van de Vel [**284**]. The Carathéodory number $c(A)$ of some $A \subset X$ is the least integer n such that, for each subset B of A and $x \in \mathrm{co}(B) \cap A$, there exists some finite subset F of B with cardinality $\leqslant n$ such that $x \in \mathrm{co}(F)$. In [**284**, Exercise II-3.19.1], van de Vel asserted that the map $A \mapsto c(A)$ is a maxitive (integer-valued) measure on $\mathscr{E}$, where $\mathscr{E}$ is the prepaving made up of finite unions of convex subsets of X. However, a simple counterexample is built as follows. Let X be the three-element semilattice $\{x_1, x_2, x_3\}$ with $x_2 = x_1 \wedge x_3$, endowed with the convexity made up of all subsets of X but $\{x_1, x_3\}$. Let $A_i = \{x_i\}$ for $i = 1, 2, 3$. Then $c(A_i) = 1$ for $i = 1, 2, 3$, hence $\max_{i=1,2,3} c(A_i) = 1$. However, $c(\bigcup_{i=1,2,3} A_i) = c(X) = 2$, for if $B := \{x_1, x_3\}$, one has $x_2 \in \mathrm{co}(B) \cap X = X$, while every nonempty subset F of B with cardinality $\leqslant 1$ is either $\{x_1\}$ or $\{x_3\}$, hence does not contain x_2.

Example I-3.11 (Interpretation of maxitive measures)**.** Finkelstein et al. [**100**] suggested to use maxitive measures as a model for a physicist's reasoning and beliefs about probable, possible, and impossible events. Kreinovich et al. [**164**] advocated the use of maxitive measures for modelling rarity of events, for maxitive measures are limits of probability measures in a large deviation sense (for a justification of this affirmation, see e.g. the work by O'Brien and Vervaat [**232**], Gerritse [**112**], O'Brien [**230**], Akian [**7**], Puhalskii [**245, 246**]). This interpretation is in accordance with Bouleau's criticism of extreme value theory [**43**]. This author noted that some events, although possible, are so rare (Bouleau gave the example of the extinction of Neanderthal Man) that they cannot be appropriately modelled by classical probability theory (and in particular by extreme value theory). Since probability theory relies on the frequentist paradigm, the question of the *probability* of such events would make no sense. For further discussion on the intuitive and the formalized distinction between *probable* and *possible* events, see also El Rayes and Morsi [**87**, § 2] and Nguyen et al. [**223**].

I-4. Maxitive measures as essential suprema

I-4.1. Introduction. In this section, we shall be interested in representing a maxitive measure ν as an essential supremum with respect to some null-additive monotone set function μ, i.e. as

$$\nu(G) = \bigoplus_{x \in G}^{\mu} f(x), \tag{5}$$

for all $G \in \mathscr{E}$, as introduced in Example I-3.3. Note that, for such a μ, the set function $\tau := \delta_\mu$, defined by $\tau(G) = 1$ if $\mu(G) > 0$, $\tau(G) = 0$

otherwise, is a maxitive measure, and Equation (5) is satisfied if and only if $\nu(G) = \bigoplus^{\tau}_{x \in G} f(x)$, for all $G \in \mathscr{E}$. Thus, we can restrict our attention to essential suprema with respect to some maxitive measure τ, without loss of generality.

Definition I-4.1. Let ν and τ be null-additive monotone set functions on $\mathscr{E}$. Then ν is *absolutely continuous with respect to* τ (or τ *dominates* ν), in symbols $\nu \dashv \tau$, if for all $G \in \mathscr{E}$, $\tau(G) = 0$ implies $\nu(G) = 0$.

Remark I-4.2. We avoid the more usual notation $\nu \ll \tau$ for absolute continuity; the reason for this will be made clear in Chapter II.

Absolute continuity, although necessary in Equation (5), seems a priori too poor a condition for ensuring the existence of a (relative) density. For instance, every maxitive measure ν satisfies $\nu \dashv \delta_{\#}$, while ν does not necessarily have a cardinal density (see for instance Example I-3.5 on measures of non-compactness). We shall understand in Section I-6 that absolute continuity is actually a necessary and sufficient condition for the existence of a density whenever the dominating measure is *σ-principal* (the measure $\delta_{\#}$ is not σ-principal in general, although this holds if $\mathscr{E}$ is the collection of open subsets of a second-countable topological space).

The next proposition ensures that, under the absolute continuity condition, a relative density exists whenever a cardinal density already exists. Given a σ-algebra $\mathscr{B}$ on E, we say that a maxitive measure ν on $\mathscr{B}$ is *autocontinuous* if there exists some $\mathscr{B}$-measurable map $c : E \to \overline{\mathbb{R}}_+$ such that

$$\nu(B) = \bigoplus^{\nu}_{x \in B} c(x),$$

for all $B \in \mathscr{B}$.

Proposition I-4.3. *Let ν be a maxitive measure on $\mathscr{B}$ with a $\mathscr{B}$-measurable cardinal density c. Then for every maxitive measure τ on $\mathscr{B}$ such that $\nu \dashv \tau$,*

$$\nu(B) = \bigoplus^{\tau}_{x \in B} c(x),$$

for all $B \in \mathscr{B}$. In particular, ν is autocontinuous.

Proof. Let $B \in \mathscr{B}$, and let $x \in B$, $t \in \mathbb{R}_+$ such that $\tau(N) = 0$ with $N \supset B \cap \{c > t\}$. If $c(x) > t$, then $x \in N$. Since $\tau(N) = 0$, we have $\nu(N) = 0$, so that $c(x) = 0$, a contradiction. Thus $c(x) \leqslant t$, and we get $\nu(B) = \bigoplus_{x \in B} c(x) \leqslant \bigoplus^{\tau}_{x \in B} c(x)$.

Now we show the converse inequality. If $\nu(B)$ is infinite, this is evident. If not, let $a > \nu(B) = \bigoplus_{x \in B} c(x)$. Then $B \cap \{c > a\} = \varnothing$ is negligible with respect to τ, hence $a \geqslant \bigoplus^{\tau}_{x \in B} c(x)$, and the result is proved. □

I-4.2. Existence of a relative density. For the remaining part of this section, every measure considered is defined on a σ-algebra $\mathscr{B}$. The following theorem on existence and "uniqueness" of relative densities is due to Barron et al. [**26**, Theorem 3.5]. We add the following component: we define a maxitive measure τ to be *essential* if there exists a σ-finite, σ-additive measure m such that $\tau(B) > 0$ if and only if $m(B) > 0$, for all $B \in \mathscr{B}$.

Theorem I-4.4 (Barron–Cardaliaguet–Jensen). *Let ν, τ be σ-maxitive measures on $\mathscr{B}$. Assume that τ is essential. Then $\nu \dashv \tau$ if and only if there exists some $\mathscr{B}$-measurable map $c : E \to \overline{\mathbb{R}}_+$ such that, for all $B \in \mathscr{B}$,*

$$\nu(B) = \bigoplus_{x \in B}^{\tau} c(x). \tag{6}$$

If these conditions are satisfied, then c is unique τ-almost everywhere.

Sketch of the proof. Since τ is essential we can replace, without loss of generality, τ by some σ-finite, σ-additive measure m in the statement of Theorem I-4.4. We first assume that both m and ν are finite. The ingenious proof given by Barron et al. relies on the following idea: to ν they associate the map m_ν defined on $\mathscr{B}$ by

$$m_\nu(B) = \bigwedge \left\{ \sum_{j \geqslant 1} \nu(B_j) m(B_j) : \bigcup_{j \geqslant 1} B_j = B, B_k \in \mathscr{B}, \forall k \geqslant 1 \right\}.$$

This formula is certainly inspired by the Carathéodory extension procedure in classical measure theory, see e.g. [**15**, Definition 10.21]. As intuition suggests, m_ν turns out to be a σ-additive measure, absolutely continuous with respect to m. Thanks to the classical Radon–Nikodym theorem there is some $\mathbb{R}_+$-valued map $c \in L^1(m)$ such that

$$m_\nu(B) = \int_B c \, dm,$$

for all $B \in \mathscr{B}$. The definition of m_ν actually gives $c \in L^\infty(m)$, and one can prove Equation (6) using the following "reconstruction" formula for ν:

$$\nu(B) = \bigoplus \left\{ \frac{m_\nu(B')}{m(B')} : B' \subset B, B' \in \mathscr{B}, m(B') > 0 \right\},$$

for all $B \in \mathscr{B}$.

Now take some (not necessarily finite) ν, and let $\nu_1 : B \mapsto \arctan \nu(B)$. Then ν_1 is a finite σ-maxitive measure, absolutely continuous with respect to τ, hence one can write $\nu_1(B) = \bigoplus_{x \in B}^{\tau} c_1(x)$. Since $\nu_1(E) \leqslant \pi/2$, we can choose c_1 to be ($\mathscr{B}$-measurable and) such that $0 \leqslant c_1 \leqslant \pi/2$. It is now an easy task to show that, for all $B \in \mathscr{B}$, $\nu(B) = \bigoplus_{x \in B}^{\tau} c(x)$, where $c(x) = \tan(c_1(x))$.

The case where m is σ-finite is easily deduced. □

Corollary I-4.5. *Let ν be an essential σ-maxitive measure on $\mathscr{B}$. Then ν is autocontinuous. Moreover, if the empty set is the only ν-negligible subset, then ν has a cardinal density.*

Barron et al.'s theorem is interesting because of its proof, which points out a correspondence between σ-maxitive and σ-additive measures. However, a part of the mystery persists, for it relies on the classical Radon–Nikodym theorem: the construction of the density remains hidden.

Note that Acerbi et al. [**2**, Theorem 3.2] used Theorem I-4.4 for resolving some non-linear minimization problems. They considered a σ-finite, σ-additive measure m on $(E, \mathscr{B})$, and derived sufficient conditions for a functional $F : L^\infty(m; E, \mathbb{R}^n) \times \mathscr{B} \to \overline{\mathbb{R}}$ to be of the form

$$F(u, B) = \bigoplus_{x \in B}^{m} f(x, u(x)),$$

for some measurable map $f : E \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ such that $f(x, \cdot)$ is lower-semicontinuous on $\mathbb{R}^n$, m-almost everywhere. This study was carried on by Cardaliaguet and Prinari [**55**], with the search for representations of the form

$$F(u, B) = \bigoplus_{x \in B}^{m} f(x, u(x), Du(x)),$$

where u runs over the set of Lipschitz continuous maps on E.

Theorem I-4.4 was rediscovered by Drewnowski [**82**, Theorem 1], with a notably different proof. He applied this result to the representation of Köthe function M-spaces as L^∞-spaces. Actually, we shall see in Section I-6 that Theorem I-4.4 is a direct consequence of a more general result, proved years earlier by Sugeno and Murofushi [**279**], which expresses it as a Radon–Nikodym like theorem with respect to the Shilkret integral (see Theorem I-6.4).

I-4.3. Essential maxitive measures. Considering Theorem I-4.4, a natural interest is to derive sufficient conditions for a maxitive measure to be essential. A null-additive set function on $\mathscr{B}$ satisfies the *countable chain condition* (or is *CCC*) if each family of non-negligible pairwise disjoint elements of $\mathscr{B}$ is countable. (A CCC set function is sometimes called σ-*decomposable*, but this terminology should be avoided, because of possible confusion with the notion of decomposability used e.g. by Weber [**300**].) It is not difficult to show that every essential maxitive measure is CCC. The converse statement was the object of Mesiar's hypothesis, proposed in [**201**]. Murofushi [**214**] showed that this hypothesis as such is wrong, by providing a counterexample; see also Poncet [**242**]. We now give the following sufficient condition for a maxitive measure to be essential. A null-additive set function μ on $\mathscr{B}$ is *of bounded variation* if $\bigoplus_\pi \sum_{B \in \pi} \mu(B) < \infty$, where the supremum is taken over the set of finite $\mathscr{B}$-partitions π of E.

Proposition I-4.6. *Every σ-maxitive measure of bounded variation on $\mathscr{B}$ is essential.*

Proof. Let ν be a σ-maxitive measure of bounded variation on $\mathscr{B}$ and m be the map defined on $\mathscr{B}$ by

$$m(B) = \bigoplus_{\pi} \sum_{B' \in \pi} \nu(B \cap B'),$$

where the supremum is taken over the set of finite $\mathscr{B}$-partitions π of E. Then m, called the *disjoint variation* of ν, is the least σ-additive measure greater than ν (see e.g. Pap [**234**, Theorem 3.2]). Since ν is of bounded variation, m is finite, and $\nu(B) > 0$ if and only if $m(B) > 0$, so that ν is essential. □

I-5. THE SHILKRET INTEGRAL

I-5.1. Introduction. Until today, the Lebesgue integral has given rise to many extensions. The first of them dates back to Vitali [**292**], who proposed to replace σ-additive measures by some more general set functions (see the historical note of Marinacci [**189**]). Decades later, the Choquet integral (see Choquet [**60**] and § I-5.4 for the definition) was born, with the same idea of using "capacities" instead of measures. It has found numerous applications, as in fuzzy set theory, game theory, statistics, or mathematical economics. Inspired by Choquet, many authors have intended to replace operations $(+, \times)$, which constitute the basic algebraic framework of both the Lebesgue and the Choquet integrals, by some more general pair $(\dot{+}, \dot{\times})$ of associative binary relations on $\mathbb{R}_+$ or $\overline{\mathbb{R}}_+$. In the case where $(\dot{+}, \dot{\times})$ is the pair $(\max, \min)$, one gets the *Sugeno integral* or *fuzzy integral* discovered by Sugeno [**278**]. In the general case, one talks about the *pan-integral* or *seminormed fuzzy integral*, see e.g. Weber [**300**], Sugeno and Murofushi [**279**], Wang and Klir [**298**], Pap [**234, 236**]. Interestingly, under reasonable continuity assumptions, one can explicitly describe these general additions $\dot{+}$ (sometimes called *triangular conorms* or *pseudo-additions*) as "mixtures" between classical addition and the maximum operation (see e.g. Sugeno and Murofushi [**279**], Benvenuti and Mesiar [**32**]). Note however that the structure of general multiplications $\dot{\times}$ remains unknown [**32**].

Beyond the simple replacement of arithmetical operations, another direction of generalization is to integrate L-valued functions (giving rise to L-valued integrals) rather than real-valued functions, where L has an appropriate semiring or module structure. In this process, measures can either remain real-valued if L is a (semi)module (as in the Bochner integral which is a well-known extension of the Lebesgue integral, where L is a Banach space), or can also be L-valued if L is a semiring. Maslov [**196**] developed an integration theory for measures with values in an ordered semiring. Other authors considered the case where L is a complete lattice, see e.g. Greco [**117**], Liu and Zhang [**183**], de Cooman et al. [**74**], Kramosil [**159**]. In the line of Maslov, Akian [**7**] focused on defining an integral for dioid-valued functions, and showed how crucial the assumption of *continuity* of the underlying partially ordered set can be (see the monograph by Gierz et al. [**114**] for background on continuous lattices and domain theory; see also Chapter II). Jonasson [**142**] had a similar approach, but managed to mix the powerful tool of continuous poset theory with a general ordered-semiring structure for L. See also Heckmann and Huth [**123**] for the role of continuous posets in integration theory. For extensions of the Riemann integral driven by the idea of approximation and still using arguments from continuous poset theory, see Edalat [**83**], Howroyd [**131**], Lawson and Lu [**173**], and references therein.

A review of integration theory in mathematics should include a number of other prolific developments (e.g. the Birkhoff integral, the Pettis integral, or the stochastic Itô integral among many others). Needless to say this is far beyond the scope of this work; the reader may refer to the book [**235**] for a broad overview of measure and integration theory. Our study here will be limited to the case where $(\dot{+}, \dot{\times})$ is the pair $(\max, \times)$ on $\overline{\mathbb{R}}_+$, which suffices for applications such as large deviations or possibility theory. This section is devoted to the construction of the *Shilkret integral* (or *idempotent integral*). This corresponds to the integral introduced by Shilkret [**271**], who made the earliest attempt of this nature, as far as we know.

I-5.2. Definition and elementary properties. Throughout this section, $\mathscr{E}$ denotes a semi-σ-algebra on E, i.e. a collection of subsets of E containing E and the empty set, and closed under countable unions and finite intersections. A map $f : E \to \overline{\mathbb{R}}_+$ is *$\mathscr{E}$-lower-semimeasurable* (or *$\mathscr{E}$-lsm* for short, or even *lsm* if the context is clear) if $\{f > t\} := \{x \in E : f(x) > t\} \in \mathscr{E}$, for all $t \in \mathbb{R}_+$. This definition generalizes the notion of lower-semicontinuous (lsc) function on a topological space, and that of measurable function on a σ-algebra. It is obvious that the map 1_G, defined by $1_G(x) = 1$ if $x \in G$, $1_G(x) = 0$ otherwise, is lsm, for all $G \in \mathscr{E}$, and the supremum (resp. the sum, the product, the minimum) of a finite family of lsm maps is lsm.

Definition I-5.1. Let ν be a maxitive measure on $\mathscr{E}$, and let $f : E \to \overline{\mathbb{R}}_+$ be an lsm map. The *Shilkret integral* (or *idempotent integral*) of f with respect to ν is defined by

$$\nu(f) = \int_E^\infty f.d\nu = \bigoplus_{t \in \mathbb{R}_+} t.\nu(f > t). \tag{7}$$

The occurrence of ∞ in the notation $\int^\infty$ is *not* an integration bound, see Theorem I-5.7 for a justification.

According to Gerritse [**112**, Proposition 3], the following identity holds:

$$\int_E^\infty f.d\nu = \bigoplus_{G \in \mathscr{E}} \left(f^\wedge(G).\nu(G)\right), \tag{8}$$

where $f^\wedge(A)$ stands for $\bigwedge_{x \in A} f(x)$. Also, notice that the supremum in Equation (7) may be reduced to a countable supremum, for

$$\begin{aligned}\int_E^\infty f \, d\nu &= \bigoplus_{t \in \mathbb{R}_+} t.\nu\left(\bigcup_{r \in \mathbb{Q}_+, r \geqslant t} \{f > r\}\right) = \bigoplus_{t \in \mathbb{R}_+} t. \bigoplus_{t \in \mathbb{Q}_+, r \geqslant t} \nu(f > r) \\ &= \bigoplus_{r \in \mathbb{Q}_+} \bigoplus_{t \in \mathbb{R}_+, t \leqslant r} t.\nu(f > r) = \bigoplus_{r \in \mathbb{Q}_+} r.\nu(f > r),\end{aligned}$$

so that Equation (7) is now given in a countable form.

Maps $f, g : E \to \overline{\mathbb{R}}_+$ are *comonotonic* if, for all $x, y \in E$, $f(x) < f(y) \Rightarrow g(x) \leqslant g(y)$, and *strongly comonotonic* if, for all $x, y \in E$ and $t > 0$, $f(x) < t.f(y) \Rightarrow g(x) \leqslant t.g(y)$. Murofushi and Sugeno [**215**, Lemma 4.3] showed that, in the case of comonotonic f, g,

(9) $$(f + g)^\wedge(G) = f^\wedge(G) + g^\wedge(G),$$

and

(10) $$(f + g)^\oplus(G) = f^\oplus(G) + g^\oplus(G),$$

for all $G \in \mathscr{E}$, where $f^\oplus(G)$ denotes the supremum of f on G. These identities are useful for the next proposition, which summarizes elementary properties of the Shilkret integral.

Proposition I-5.2. *Let ν be a σ-maxitive measure on $(E, \mathscr{E})$. Then, for all lsm maps $f, g : E \to \overline{\mathbb{R}}_+$, $r \in \mathbb{R}_+$, $G \in \mathscr{E}$, the following properties hold:*

- *$\nu(1_G) = \nu(G)$,*
- *homogeneity: $\nu(r.f) = r.\nu(f)$,*
- *subadditivity: $\nu(f + g) \leqslant \nu(f) + \nu(g)$, with equality whenever f and g are strongly comonotonic,*
- *σ-maxitivity: $\nu(\bigoplus_n f_n) = \bigoplus_n \nu(f_n)$, for every sequence $(f_n)_{n \in \mathbb{N}}$ of lsm maps $f_n : E \to \overline{\mathbb{R}}_+$,*
- *$G \mapsto \int_G^\infty f \, d\nu$ is a σ-maxitive measure on $\mathscr{E}$,*
- *if $\nu(f) < \infty$ then $f < \infty$ ν-almost everywhere, and $\nu(f > t) \downarrow 0$ when $t \uparrow \infty$.*

Proof. We prove subadditivity and strongly comonotonic additivity; the other assertions are left to the reader. It is not difficult to see that, for all $r \in \mathbb{Q}_+$,

$$\{f + g > r\} = \{f \oplus g > r\} \cup \bigcup_{r_1 + r_2 = r} \{f > r_1\} \cap \{g > r_2\},$$

where the latter union runs over $\{(r_1, r_2) \in \mathbb{Q}_+^2 : r_1 + r_2 = r\}$. Thus,

$$\begin{aligned}\int^\infty (f + g) \, d\nu &= \int^\infty (f \oplus g) \, d\nu \oplus \bigoplus_{r \in \mathbb{Q}_+} r \bigoplus_{r_1 + r_2 = r} \nu\left(f > r_1, g > r_2\right) \\ &= \int^\infty (f \oplus g) \, d\nu \oplus \bigoplus_{r_1, r_2 \in \mathbb{Q}_+} (r_1 + r_2).\nu\left(f > r_1, g > r_2\right).\end{aligned}$$

But $(r_1 + r_2).\nu\,(f > r_1, g > r_2)$ equals

$$r_1.\nu\,(f > r_1, g > r_2) + r_2.\nu\,(f > r_1, g > r_2)\,,$$

and this last term is lower than $\int^\infty f\,d\nu + \int^\infty g\,d\nu$, so subadditivity is shown. Now suppose that f and g are strongly comonotonic. With the help of Equation (9), we have $(f+g)^\wedge(G)\nu(G) = \alpha(G) + \beta(G)$, where $\alpha(G) := f^\wedge(G)\nu(G)$ and $\beta(G) := g^\wedge(G)\nu(G)$. Notice that strong comonotonicity of f and g implies comonotonicity of α and β, so that we can apply Equation (10) to these maps. Using Equation (8), the proof is complete. □

In order to study the Shilkret integral more deeply, it would be natural to fix a measurable space $(E, \mathscr{B})$ endowed with a σ-maxitive measure ν, and, by analogy with the additive case, to look at the spaces $L^p(\nu)$, $p > 0$. These are Banach spaces, as noticed by Shilkret [**271**], and it is easily seen that the monotone and dominated convergence theorems, the Chebyshev and Hölder inequalities, etc. are satisfied (see [**246**, Lemmata 1.4.5 and 1.4.7] and [**246**, Theorem 1.4.19]). However, these spaces are less interesting to study than their classical counterpart, since $L^p(\nu) = L^1(\nu^{1/p})$, so that all of them can be viewed as L^1 spaces. In particular, $L^2(\nu)$ is not a Hilbert space. Nonetheless, these spaces can be considered as generalizations of the spaces $L^\infty(m)$ (with m a σ-additive measure), since $L^\infty(m) = L^1(\delta_m)$.

Further properties of the Shilkret integral with respect to an optimal measure (see Definition I-8.1) were studied by Agbeko [**5**] and applied to characterizations of boundedness and uniform boundedness of measurable functions. We also refer the reader to Puhalskii [**246**] and to de Cooman [**71**], who both gave a pretty exhaustive treatment of the Shilkret integral. We note however that their approach is essentially limited to completely maxitive measures defined on *τ-algebras* (also called *ample fields*, i.e. σ-algebras closed under arbitrary intersections, see Janssen et al. [**141**]), but this framework has the disadvantage of breaking the parallel with classical measure theory. We shall come back to this debate in Section I-9.

I-5.3. Examples. We pursue the study of two examples introduced above, namely the essential supremum and the Fréchet random sup-measures. We also generalize the latter with the concept of regularly-varying random sup-measure.

Example I-5.3 (Example I-3.3 continued)**.** Let μ be a null-additive monotone set function and let $f : E \to \overline{\mathbb{R}}_+$ be some lsm map. Then the *μ-essential supremum* of f is the maxitive measure $G \mapsto \bigoplus^\mu_{x\in G} f(x)$; it can be seen as a Shilkret integral, i.e.

$$\bigoplus^{\mu}_{x\in G} f(x) = \int^\infty_G f\,d\delta_\mu,$$

where δ_μ is the maxitive measure defined by $\delta_\mu(G) = 1$ if $\mu(G) > 0$, $\delta_\mu(G) = 0$ otherwise. Moreover, integration with respect to the μ-essential supremum (call it τ) gives

$$\int_E^\infty g\,d\tau = \bigoplus_{x\in E}^{\mu} g(x)f(x) = \int_E^\infty gf\,d\delta_\mu.$$

Example I-5.4 (Quantization)**.** The following formula was introduced by Connes and Consani [**67**] as the inverse process of the Maslov "dequantization" of idempotent analysis (on this subject see Kolokoltsov and Maslov [**156**]). If $S : p \to -p\log(p) - (1-p)\log(1-p)$ is the entropy defined on $[0,1]$ and ν_S denotes the maxitive measure with cardinal density $p \mapsto e^{S(p)}$, then

$$x + y = \int_{[0,1]}^\infty x^p y^{1-p}\,d\nu_S(p)$$

for all $x, y \in \mathbb{R}_+$.

Example I-5.5 (Example I-3.8 continued)**.** Let $(\Omega, \mathscr{A})$ and $(E, \mathscr{B})$ be measurable spaces, P be a probability measure on $\mathscr{A}$, and m be a finite σ-additive measure on $\mathscr{B}$. Let M be a β-Fréchet random sup-measure with control measure m. For all measurable maps $f : E \to \overline{\mathbb{R}}_+$, we can consider the Shilkret integral $M(f)$ defined as usual by

$$\int_E^\infty f\,dM = \bigoplus_{t\in\mathbb{R}_+} t.M(f > t).$$

This coincides with the *extremal integral* of Stoev and Taqqu [**276**] (note that these authors did not seem to know about Shilkret's or Maslov's works). It can be seen as a kind of stochastic integral with a deterministic integrand, very similar to the well-known α-stable (or sum-stable) integral (see Samorodnitsky and Taqqu [**263**]). Note that $M(f)$ is indeed a random variable, for the supremum over $\mathbb{R}_+$ can be replaced by a countable supremum (see § I-5.2). Moreover, if $f \in L^\beta(m)$, then $M(f)$ follows a Fréchet distribution with

$$P[M(f) \leqslant x] = \exp(-\|f\|_\beta^\beta\, x^{-\beta}),$$

where $\|f\|_\beta$ denotes the Lebesgue β-norm of f with respect to m, i.e. $\|f\|_\beta = (\int f^\beta\,dm)^{1/\beta}$. This implies that, whenever $\|f\|_\beta < \infty$, $B \mapsto \int_B^\infty f\,dM$ is itself a β-Fréchet random sup-measure with control measure $B \mapsto \int_B f^\beta\,dm$. See [**276**] for additional properties. In the particular case where

$$M(B) = \bigoplus_{k\geqslant 1} X_k.1_B(T_k),$$

for some Poisson point process $(X_k, T_k)_{k\geqslant 1}$ on $\mathbb{R}_+ \times E$ with intensity measure $\beta x^{-\beta-1}dx \times m(dt)$, we have

$$\int_E^\infty f\,dM = \bigoplus_{k\geqslant 1} X_k.f(T_k).$$

De Haan [**75**] introduced this latter integral process and showed that, if $(X_t)_{t\in\mathbb{R}}$ is a continuous-time simple max-stable process, then there exists a

Poisson process with the above properties, and a collection $(f_t)_{t\in\mathbb{R}}$ of nonnegative L^1 maps such that

$$(X_t)_{t\in\mathbb{R}} \stackrel{d}{=} (\int_E^{\infty} f_t \, dM),$$

where $\stackrel{d}{=}$ means equality in finite-dimensional distributions [**75**, Theorem 3].

Example I-5.6 (Regularly-varying sup-measures)**.** A variant on the previous example can be done as follows. Let $(\Omega, \mathscr{A}, P)$ be a probability space, $(E, \mathscr{B})$ be a measurable space, and m be a finite σ-additive measure on $\mathscr{B}$. We define a *β-regularly-varying random sup-measure* with control measure m to be a map $M : \Omega \times \mathscr{B} \to \overline{\mathbb{R}}_+$ satisfying the following conditions:

- for all pairwise disjoint collections $(B_j)_{j\in\mathbb{N}}$ of elements of $\mathscr{B}$, the random variables $M(B_j)$, $j \in \mathbb{N}$, are independent, and, almost surely,

$$M(\bigcup_{j\in\mathbb{N}} B_j) = \bigoplus_{j\in\mathbb{N}} M(B_j);$$

- for all $B \in \mathscr{B}$ the random variable $M(B)$ is regularly-varying of index β, i.e. there exists a function L, slowly-varying at ∞, such that, when $x \to \infty$,

$$P[M(B) > x] \sim m(B) x^{-\beta} L(x).$$

Recall that $L : \mathbb{R}_+ \backslash \{0\} \to \mathbb{R}_+ \backslash \{0\}$ is *slowly-varying at* ∞ if, for all $a > 0$, $\lim_{x\to\infty} L(ax)/L(x) = 1$. See e.g. Resnick [**251**] for more on regularly-varying (and slowly-varying) functions. For all measurable maps $f : E \to \overline{\mathbb{R}}_+$, the random variable $M(f)$ defined as the Shilkret integral of f with respect to M satisfies

$$P[M(f) > x] \sim \|f\|_\beta^\beta x^{-\beta} L(x),$$

when $x \to \infty$, for all $f \in L^\beta(m)$. Let us prove this assertion. First, consider the case where f is a nonnegative (measurable) simple map, i.e. a map of the form $f = \sum_{j=1}^k t_j 1_{B_j}$, where $B_1, \ldots, B_k \in \mathscr{B}$ are disjoint and $t_j > 0$ for $j = 1, \ldots, k$. One can write $f = \bigoplus_{j=1}^k t_j 1_{B_j}$. Thus, $M(f) = \bigoplus_{j=1}^k t_j . M(B_j)$, almost surely, so that

$$P[M(f) > x] \sim -\log P[M(f) \leqslant x] = \sum_{j=1}^{k} -\log P[M(B_j) \leqslant x/t_j],$$

since the random variables $M(B_1), \dots, M(B_k)$ are independent. We get

$$\begin{aligned} P[M(f) > x] &\sim \sum_{j=1}^{k} P[M(B_j) > x/t_j](1 + o(1)) \\ &= \sum_{j=1}^{k} m(B_j) t_j^{\beta} x^{-\beta} L(x/t_j)(1 + o(1)) \\ &= \sum_{j=1}^{k} m(B_j) t_j^{\beta} x^{-\beta} L(x)(1 + o(1)), \end{aligned}$$

since L is slowly-varying. This shows that $P[M(f) > x] \sim \|f\|_{\beta}^{\beta} x^{-\beta} L(x)$. In the general case where f is a nonnegative map in $L^{\beta}(m)$, let (φ_n) be a nondecreasing sequence of nonnegative simple maps that converges pointwise to f. Then $\|\varphi_n\|_{\beta} \to \|f\|_{\beta}$ when $n \to \infty$. As a consequence,

$$P[M(\varphi_n) > x] \sim_{x \to \infty} \|\varphi_n\|_{\beta}^{\beta} x^{-\beta} L(x) \to_n \|f\|_{\beta}^{\beta} x^{-\beta} L(x).$$

But we also have $P[M(\varphi_n) > x] \to_n P[M(f) > x]$, so the desired result follows.

I-5.4. Links with the Choquet integral. One should not mix up idempotent integration introduced above with extensions of the Lebesgue integral such as the Choquet integral. If μ is a monotone set function on $\mathscr{E}$, the *Choquet integral* (see Choquet [**60**]) of an lsm map $f : E \to \overline{\mathbb{R}}_+$ is defined by

$$\int^1 f \, d\mu = \int_{\mathbb{R}_+} \mu(f > t) \, dt.$$

The instance of the index 1 gives the "type" of the integral: this is not an integration bound. This notation was introduced by Gerritse [**112**]. One speaks of a Choquet integral of *level* 1. Note that the following inequality holds:

$$\int^{\infty} f \, d\mu \leqslant \int^1 f \, d\mu,$$

where we extend the Shilkret integral to monotone set functions by

$$\int^{\infty} f \, d\mu := \bigoplus_{t \in \mathbb{R}_+} t.\mu(f > t).$$

The next theorem was given by Gerritse in a topological framework [**112**]. First define the Choquet integral of level $p > 0$ by

$$\int^p f \, d\mu := \left(\int^1 f^p \, d\mu^p \right)^{1/p},$$

where μ^p denotes the set function $G \mapsto \mu(G)^p$.

Theorem I-5.7 (Dequantization of the Choquet integral)**.** *Let μ be a monotone set function on $\mathscr{E}$, and let $f : E \to \overline{\mathbb{R}}_+$ be lsm. Assume that there is some $q > 0$ such that $\int^q f \, d\mu < \infty$. Then*

$$\int^\infty f \, d\mu = \lim_{p \to \infty} \int^p f \, d\mu.$$

Proof. Let $a := \int^\infty f \, d\mu$. The inequality $\leqslant$ follows from $\left(\int^\infty f \, d\mu\right)^p = \int^\infty f^p \, d\mu^p \leqslant \int^1 f^p \, d\mu^p$, for all $p > 0$. In particular, $a \leqslant \int^q f \, d\mu$, hence $a < \infty$. For the converse inequality,

$$\int^p f \, d\mu = \left(\int_0^\infty \mu^p(f^p > t) \, dt \right)^{1/p} = \left(\int_0^\infty p s^{p-1} \mu^p(f > s) \, ds \right)^{1/p},$$

using the change of variable $s = t^{1/p}$. Hence,

$$\begin{aligned} \int^p f \, d\mu &= \left(\int_0^\infty s^{p-q} \mu^{p-q}(f > s) \frac{p}{q} q s^{q-1} \mu^q(f > s) \, ds \right)^{1/p} \\ &\leqslant (a^{p-q})^{1/p} \left(\frac{p}{q} \int^1 f^q \, d\mu^q \right)^{1/p}, \end{aligned}$$

with the right hand side converging to a when $p \to \infty$, which gives the desired result. □

Theorem I-5.7 could be brought together with a result by Mesiar and Pap [**202**, Theorem 3]. These authors showed that, under certain assumptions, the Shilkret integral can be seen as a limit of g-integrals, i.e. integrals of the form

$$\int^g f \, d\mu := g^{-1} \left(\int^1 g \circ f \, d\mu \right),$$

where the map $g : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a *generator* and μ is a σ-additive measure.

The following result provides another link between the Shilkret and the Choquet integrals. Indeed, both reduce to an essential supremum when they are taken with respect to a set function of the form δ_μ.

Proposition I-5.8. *Let μ be a monotone set function on $\mathscr{E}$, and let $f : E \to \overline{\mathbb{R}}_+$ be lsm. Then the map $p \mapsto \int^p f \, d\delta_\mu$ is constant. In particular,*

$$\int^1 f \, d\delta_\mu = \int^\infty f \, \delta_\mu = \bigoplus_{x \in E}^{\mu} f(x).$$

The obvious proof is left to the reader.

I-6. The Radon–Nikodym theorem

I-6.1. Introduction. A widespread proof of the Radon–Nikodym theorem for σ-additive measures, due to von Neumann, uses the representation of bounded linear forms on a Hilbert space (see e.g. Rudin [**260**]). But for σ-maxitive measures the space L^2, as already noticed, actually reduces to an L^1 space, for $L^2(\nu) = L^1(\nu^{1/2})$ for every σ-maxitive measure ν. That is why such an approach is not possible[1], and we have to find another way for proving a Radon–Nikodym theorem for σ-maxitive measures. Sugeno, in relation to the Sugeno integral, was confronted with the same problem in his thesis, and gave sufficient conditions for the existence of a Radon–Nikodym derivative [**278**] at the cost of a topological structure on E. This first result was refined by Candeloro and Pucci [**54**, Theorem 3.7] and Sugeno and Murofushi [**279**, Corollary 8.3].

In this section, we give a general definition of the density of a maxitive measure with respect to the Shilkret integral. Then we recall the main theorem stating the existence of such a density [**279**, Corollary 8.4]. Here, $\mathscr{E}$ still denotes a semi-σ-algebra.

The literature is not unanimous in the meaning of the term "density" applied to maxitive measures. For Akian [**7**], a density is any map c such that $\nu(\cdot) = \bigoplus_{x\in\cdot} c(x)$, i.e. what we called cardinal density. For Barron et al. [**26**] and Drewnowski [**82**], a density corresponds to our concept of relative density (see Section I-4). The following definition encompasses both points of view. Let ν and τ be maxitive measures on $\mathscr{E}$. Then ν *has a density with respect to* τ if there exists some lsm map (called *density*) $c : E \to \overline{\mathbb{R}}_+$ such that

$$\nu(G) = \int_G^\infty c\, d\tau, \tag{11}$$

for all $G \in \mathscr{E}$.

If ν has a density with respect to τ, then ν is absolutely continuous with respect to τ, according to Definition I-4.1. Taking $\tau = \delta_\#$ in Equation (11), one gets $\nu(G) = \bigoplus_{x\in G} c(x)$, i.e. one recovers the notion of cardinal density. If μ is a null-additive monotone set function, then Equation (11) with $\tau = \delta_\mu$ rewrites to $\nu(B) = \bigoplus_{x\in B}^{\mu} c(x)$, which fits with the case of essential suprema and relative densities.

I-6.2. Uniqueness and finiteness of the density. Let $(E, \mathscr{B})$ be a measurable space. A set function $\nu : \mathscr{B} \to \overline{\mathbb{R}}_+$ is *semifinite* if, for all $B \in \mathscr{B}$, $\nu(B) = \bigoplus_{A\subset B} \nu(A)$, where the supremum is taken over $\{A \in \mathscr{B} : A \subset B, \nu(A) < \infty\}$.

Proposition I-6.1. *Let ν, τ be σ-maxitive measures on $\mathscr{B}$. Assume that ν is semifinite and admits a $\mathscr{B}$-measurable density c with respect to τ. Then ν admits a finite-valued $\mathscr{B}$-measurable density with respect to τ.*

[1]Actually, the really significant point in usual L^2 spaces is the ability to *project*. Projections may still be available in ordered algebraic structures, see Remark I-9.3; see also Cohen et al. [**63**].

Proof. Let $c_1 = c.1_{c<\infty}$, which is $\mathscr{B}$-measurable and finite-valued, and let us show that c_1 is still a density of ν with respect to τ. Let $B \in \mathscr{B}$. Then

$$\begin{aligned}\nu(B) &= \int_B^\infty c\,d\tau = \int_B^\infty c_1\,d\tau \oplus \int_B^\infty c.1_{c=\infty}\,d\tau \\ &= \int_B^\infty c_1\,d\tau \oplus (\infty.\tau(B \cap \{c = \infty\}))\,.\end{aligned}$$

In particular, $\nu(B) \geqslant \int_B^\infty c_1\,d\tau$. If $\nu(B) < \infty$, then $\tau(B \cap \{c = \infty\}) = 0$ by the previous equality, hence $\nu(B) = \int_B^\infty c_1\,d\tau$. If $\nu(B) = \infty$, then

$$\nu(B) = \bigoplus_{A \subset B} \nu(A) = \bigoplus_{A \subset B} \int_A^\infty c_1\,d\tau \leqslant \int_B^\infty c_1\,d\tau,$$

where the supremum is taken over $\{A \in \mathscr{B} : A \subset B, \nu(A) < \infty\}$, so that $\nu(B) = \int_B^\infty c_1\,d\tau$, for all $B \in \mathscr{B}$. □

Paralleling the classical case, we have the following result on "uniqueness" of the density.

Proposition I-6.2. *Let ν, τ be σ-maxitive measures on $\mathscr{B}$. If ν admits a $\mathscr{B}$-measurable density with respect to τ, then this density is unique, τ-almost everywhere.*

Proof. The assertion can be proved along the same lines as the case of the Lebesgue integral, see e.g. Rudin [**260**, Theorem 1.39(b)]. □

We can generalize this result to the case where maxitive measures are defined on a semi-σ-algebra $\mathscr{E}$ (rather than a σ-algebra $\mathscr{B}$). A null-additive monotone set function τ on $\mathscr{E}$ is *σ-principal* if, for every σ-ideal $\mathscr{I}$ of $\mathscr{E}$, there exists some $L \in \mathscr{I}$ such that $S\backslash L$ is τ-negligible, for all $S \in \mathscr{I}$. Proposition I-7.1 will justify this terminology.

Proposition I-6.3. *Let ν, τ be σ-maxitive measures on $\mathscr{E}$. Assume that τ is σ-principal, and that ν admits an lsm density with respect to τ. Then ν admits a τ-maximal lsm density with respect to τ.*

Proof. For a nonnegative rational number q, let $\mathscr{I}_q$ be the σ-ideal made up of all subsets $G \in \mathscr{E}$ that are contained in some $\{f > q\}$, where f is an lsm density of ν with respect to τ. Since τ is σ-principal, there is some $L_q \in \mathscr{I}_q$ such that $S\backslash L_q$ is τ-negligible, for all $S \in \mathscr{I}_q$. Note that we can choose L_q of the form $L_q = \{f_q > q\}$, with f_q an lsm density. Now let $c = \bigoplus_{q \in \mathbb{Q}_+} f_q$. Then c is an lsm density, and $\{f > c\} \subset \bigcup_{q \in \mathbb{Q}_+} \{f > q \geqslant f_q\} = \bigcup_{q \in \mathbb{Q}_+} \{f > q\}\backslash L_q$, so that $\{f > c\}$ is negligible with respect to τ, for all lsm densities f. This is what we meant by stating that c is a *τ-maximal* lsm density with respect to τ. □

I-6.3. Principality and existence of a density. Let $(E, \mathscr{B})$ be a measurable space. Sugeno and Murofushi [**279**, Corollary 8.4] proved a Radon–Nikodym theorem for the Shilkret integral when the dominating measure is σ-finite and σ-principal.

Theorem I-6.4 (Sugeno–Murofushi). *Let ν, τ be σ-maxitive measures on $\mathscr{B}$. Assume that τ is σ-finite and σ-principal. Then $\nu \dashv \tau$ if and only if there exists some $\mathscr{B}$-measurable map $c : E \to \overline{\mathbb{R}}_+$ such that*

$$\nu(B) = \int_B^\infty c \, d\tau,$$

for all $B \in \mathscr{B}$. If these conditions are satisfied, then c is unique τ-almost everywhere.

Proof. In the following lines we closely follow the original proof of Sugeno and Murofushi, with some clarifications added.

We can assume, without loss of generality, that τ is finite. The collection

$$\mathscr{B}_t := \{B \in \mathscr{B} : \forall B' \subset B, \nu(B') \geqslant t.\tau(B')\}$$

is a σ-ideal. Since τ is σ-principal, there is some $B_t \in \mathscr{B}_t$ such that $B \backslash B_t$ is τ-negligible, for all $B \in \mathscr{B}_t$.

• Fact 1. For all $t \in \overline{\mathbb{R}}_+$ and $B \in \mathscr{B}$ disjoint from B_t, we have $\nu(B) \leqslant t.\tau(B)$. To see this, let

$$\mathscr{C}_t := \{C \in \mathscr{B} : \exists C' \in \mathscr{B}, C \subset C' \subset B, \nu(C') \leqslant t.\tau(C')\}.$$

Then $\mathscr{C}_t$ is a σ-ideal, and there is some $C_t \in \mathscr{C}_t$ such that $\tau(C \backslash C_t) = 0$ for all $C \in \mathscr{C}_t$. Let us show that $B \backslash C_t \in \mathscr{B}_t$. So let $B' \subset B \backslash C_t$. If $\nu(B') < t.\tau(B')$ then $B' \in \mathscr{C}_t$, hence $\tau(B' \backslash C_t) = 0$. Since $B' = B' \backslash C_t$, this implies $\tau(B') = 0$, a contradiction. Thus, $\nu(B') \geqslant t.\tau(B')$, which proves that $B \backslash C_t \in \mathscr{B}_t$. The definition of B_t implies that $(B \backslash C_t) \backslash B_t$ is τ-negligible. Using the hypothesis $B \cap B_t = \varnothing$, we get $\tau(B \backslash C_t) = 0$, hence $\nu(B \backslash C_t) = 0$. Therefore, $\nu(B) = \nu(C_t) \leqslant t.\tau(C_t) = t.\tau(B)$.

• Fact 2 : $E \backslash (H_\infty \cup \bigcup_k H_j^k)$ is τ-negligible, for all j, if one defines $H_j^k = B_{k2^{-j}} \backslash B_{(k+1)2^{-j}}$ and $H_\infty = B_\infty$. If one notices that $A_j := E \backslash \bigcup_k H_j^k = \bigcap_k B_{k2^{-j}}$, then $A_j \in \mathscr{B}_\infty$, so that $\tau(A_j \backslash H_\infty) = 0$.

• Let c be the map defined by $c = (\infty 1_{H_\infty}) \oplus (\bigoplus_{j,k} k 2^{-j} 1_{H_j^k})$, and let $B \in \mathscr{B}$. Then

$$\int_B^\infty c \, d\tau = \infty.\tau(B \cap H_\infty) \oplus \bigoplus_{j,k} \frac{k}{2^j}.\tau(B \cap H_j^k). \tag{12}$$

It is clear that $\int_B^\infty c \, d\tau \leqslant \nu(B)$. Let us show the converse inequality. If $\nu(B \cap H_\infty) = \infty$, then $\tau(B \cap H_\infty) > 0$ because $\nu \dashv \tau$, so that $\int_B^\infty c \, d\tau = \infty = \nu(B)$. Now suppose that $\nu(B \cap H_\infty) < \infty$. Since $B \cap H_\infty \in \mathscr{B}_\infty$, we deduce that $\tau(B \cap H_\infty) = 0$, which simplifies Equation (12). Since τ is finite, one can write

$$\int_B^\infty c \, d\tau = \bigoplus_{j,k} \left(\frac{k+1}{2^j}.\tau(B \cap H_j^k) - \frac{1}{2^j}.\tau(B \cap H_j^k) \right)$$

so that, by Fact 1,

$$\int_B^\infty c \, d\tau \geqslant \bigoplus_{j,k} \left(\nu(B \cap H_j^k) - \frac{1}{2^j}.\tau(B \cap H_j^k) \right).$$

This implies

$$\begin{aligned}\int_B^\infty c\, d\tau &\geqslant \bigoplus_j \left(-\frac{\tau(B)}{2^j} + \nu(B \cap \bigcup_k H_j^k) \right) \\ &\geqslant \bigoplus_j \left(-\frac{\tau(B)}{2^j} + \nu(B \backslash H_\infty) \right), \text{ by Fact 2 and } \nu \dashv \tau, \\ &= \nu(B \backslash H_\infty),\end{aligned}$$

so that $\int_B^\infty c\, d\tau = \nu(B)$ because $\tau(B \cap H_\infty) = 0$ and $\nu \dashv \tau$. □

The hypothesis of σ-finiteness of τ cannot be removed: consider for instance a finite set E, and let $\nu = \delta_\#$ and $\tau = \infty.\delta_\#$ be σ-maxitive measures defined on the power set of E. Then τ is σ-principal and $\nu \dashv \tau$, but ν never has a density with respect to τ.

Theorem I-6.4 encompasses Theorem I-4.4, for if τ is an essential σ-maxitive measure, then δ_τ is (σ-finite and) σ-principal (use Theorem A.1). We shall give another proof of this theorem in Chapter III with the help of order-theoretical arguments. As for now, it has the following simple consequence, which generalizes Corollary I-4.5.

Corollary I-6.5. *Let ν be a σ-principal σ-maxitive measure on $\mathscr{B}$. Then ν is autocontinuous. Moreover, if the empty set is the only ν-negligible subset, then ν is completely maxitive (and has a cardinal density).*

Proof. Simply take $\tau = \delta_\nu$ in the previous theorem. □

I-7. The quotient space and the Radon–Nikodym property

In this section, we characterize those σ-maxitive measures τ with the *Radon–Nikodym property*, i.e. such that all σ-maxitive measures dominated by τ have a measurable density with respect to τ. At first, we shall introduce the quotient space associated with τ.

Let $(E, \mathscr{B})$ be a measurable space, and let τ be a σ-maxitive measure. On $\mathscr{B}$ we define an equivalence relation $\sim$ by $A \sim B$ if $A \cup N = B \cup N$, for some τ-negligible subset N. We write B^τ for the equivalence class of $B \in \mathscr{B}$. The quotient set derived from $\sim$ is called the *quotient space* associated with τ, and denoted by $\mathscr{B}/\tau$. The quotient space can be equipped with the structure of a σ-complete lattice induced by the partial order $\leqslant$ defined by $A^\tau \leqslant B^\tau$ if $A \subset B \cup N$, for some τ-negligible subset N.

The next proposition, partly due to Sugeno and Murofushi, characterizes σ-principal σ-maxitive measures defined on a σ-algebra.

Proposition I-7.1. *Let τ be a σ-maxitive measure on $\mathscr{B}$. The following conditions are equivalent:*

(1) *τ is σ-principal,*
(2) *τ satisfies the countable chain condition,*
(3) *the quotient space $\mathscr{B}/\tau$ is σ-principal, in the sense that every σ-ideal of $\mathscr{B}/\tau$ is a principal ideal,*

(4) *there is some σ-principal σ-additive measure m on $\mathscr{B}$ such that $m \dashv \tau$ and $\tau \dashv m$.*

Proof. (4) $\Rightarrow$ (1) This implication is clear.

(1) $\Rightarrow$ (2) Assume that τ is σ-principal, and let $\mathscr{A}$ be a family of non-negligible pairwise disjoint elements of $\mathscr{B}$. Let $\mathscr{I}$ be the σ-ideal generated by $\mathscr{A}$, and let $L \in \mathscr{I}$ such that $\tau(I \backslash L) = 0$ for all $I \in \mathscr{I}$. We can choose L of the form $L = \bigcup_{n \in \mathbb{N}} A_n$, with $A_n \in \mathscr{A}$ for all n. Now let us show that $\mathscr{A} = \{A_n : n \in \mathbb{N}\}$, which will prove that $\mathscr{A}$ is countable. So let $A \in \mathscr{A}$, and assume that $A \neq A_n$ for all n. Then $A \cap A_n = \varnothing$ for all n, i.e. $A \subset E \backslash L$. Moreover, the definition of L implies $\tau(A \backslash L) = 0$, so that $\tau(A) = 0$, a contradiction.

(2) $\Rightarrow$ (1) This was proved by Sugeno and Murofushi [**279**, Lemma 4.2] with the help of Zorn's lemma.

(1) $\Rightarrow$ (4) Define m as in the proof of Proposition I-4.6. Let us show that m is σ-principal. If $\mathscr{I}$ is a σ-ideal of $\mathscr{B}$, there exists some $L \in \mathscr{I}$ such that $\tau(B \backslash L) = 0$ for all $B \in \mathscr{I}$. If $B \in \mathscr{I}$, then $\tau(B \cap B' \backslash L) = 0$ for all $B' \in \mathscr{B}$, since $B \cap B' \in \mathscr{I}$. Hence we have $m(B \backslash L) = 0$.

(3) $\Rightarrow$ (1) Let $\mathscr{I}$ be a σ-ideal of $\mathscr{B}$, and let $I = \{B^\tau : B \in \mathscr{I}\}$. Then I is closed under countable suprema, and if $A^\tau \leqslant B^\tau$ with $B \in \mathscr{I}$, then $A \subset B \cup N$ for some negligible subset $N \in \mathscr{B}$. Hence $A \cap (E \backslash N) \subset B$, so that $A \cap (E \backslash N) \in \mathscr{I}$. Since $A \cap (E \backslash N) \sim A$, this implies that $A^\tau \in I$. Thus, I is a σ-ideal of $\mathscr{B}/\tau$. Since $\mathscr{B}/\tau$ is σ-principal, there is some $L \in \mathscr{I}$ such that $B^\tau \in I$ if and only if $B^\tau \leqslant L^\tau$. We deduce that $\tau(B \backslash L) = 0$ for all $B \in \mathscr{I}$, which proves that τ is σ-principal.

(1) $\Rightarrow$ (3) Let I be a σ-ideal of $\mathscr{B}/\tau$. Then $\mathscr{I} = \{B \in \mathscr{B} : B^\tau \in I\}$ is a σ-ideal of $\mathscr{B}$. Since τ is σ-principal, there is some $L \in \mathscr{I}$ such that $\tau(B \backslash L) = 0$ for all $B \in \mathscr{I}$. Then $B^\tau \in I$ if and only if $B^\tau \leqslant L^\tau$, i.e. I is principal. □

Following Segal [**265**], a null-additive monotone set function τ on $\mathscr{B}$ is *localizable* if, for all σ-ideals $\mathscr{I}$ of $\mathscr{B}$, there exists some $L \in \mathscr{B}$ such that

(1) $S \backslash L$ is τ-negligible, for all $S \in \mathscr{I}$,
(2) if there is some $B \in \mathscr{B}$ such that $S \backslash B$ is τ-negligible for all $S \in \mathscr{I}$, then $L \backslash B$ is τ-negligible.

In this case, $\mathscr{I}$ is said to be *localized* in L. It is clear that a null-additive monotone set function is localizable if and only if the associated quotient space is a complete lattice (resp. a *frame*, for it is not difficult to prove that the quotient space is infinitely distributive). Note also that localizability is a weaker condition than σ-principality.

Here comes the characterization of the Radon–Nikodym property.

Theorem I-7.2. *Let τ be a σ-maxitive measure on $\mathscr{B}$. Then τ satisfies the Radon–Nikodym property if and only if τ is σ-finite and σ-principal.*

Proof. Necessity is a reformulation of Theorem I-6.4. Sufficiency is proved in five steps. Let τ be a σ-maxitive measure satisfying the Radon-Nikodym property.

Claim 1: τ is localizable.

Let $\mathscr{I}$ be a σ-ideal of $\mathscr{B}$, and let ν be the map defined on $\mathscr{B}$ by $\nu(B) = \bigoplus_{I \in \mathscr{I}} \tau(B \cap I)$. Then ν is a σ-maxitive measure on $\mathscr{B}$, absolutely continuous with respect to τ, hence we can write

$$\nu(B) = \int_B^\infty c\, d\tau,$$

for some $\mathscr{B}$-measurable map $c : E \to \overline{\mathbb{R}}_+$. Defining $L = \{c \neq 0\} \in \mathscr{B}$, one can see that $\mathscr{I}$ is localized in L.

Claim 2: τ is σ-principal.

Let $\mathscr{I}$ be a σ-ideal in $\mathscr{B}$, and let $L \in \mathscr{B}$ localizing $\mathscr{I}$ with respect to τ. The σ-ideal $\mathscr{J}$ generated by $\mathscr{I}$ and $\{N \in \mathscr{B} : \tau(N) = 0\}$ can be written as $\mathscr{J} = \{I \cup N : I \in \mathscr{I}, \tau(N) = 0\}$. Let us consider the map ν defined on $\mathscr{B}$ by $\nu(B) = 0$ if $B \in \mathscr{J}$, and $\nu(B) = 1$ otherwise. Then ν is a σ-maxitive measure, absolutely continuous with respect to τ, hence we can write

$$\nu(B) = \int_B^\infty c\, d\tau, \tag{13}$$

for some $\mathscr{B}$-measurable map $c : E \to \overline{\mathbb{R}}_+$. If $I \in \mathscr{I}$, then $\nu(I) = 0$, hence, by Equation (13), $\tau(I \cap \{c > t\}) = 0$, for all $t > 0$. This implies that $\tau(I \backslash \{c = 0\}) = 0$, for all $I \in \mathscr{I}$. By definition of L, we deduce that $\tau(L \backslash \{c = 0\}) = 0$. Therefore, $\nu(L) = \nu(L \backslash \{c = 0\}) \oplus \nu(L \cap \{c = 0\}) = 0$. The definition of ν gives $L \in \mathscr{J}$, hence $L = I_0 \cup N_0$, with $I_0 \in \mathscr{I}$ and $\tau(N_0) = 0$. We have found $I_0 \in \mathscr{I}$ such that $\nu(I \backslash I_0) = 0$, for all $I \in \mathscr{I}$, so we have proved that τ is σ-principal.

Claim 3: τ has no spot.

Assume that τ has a *spot*, i.e. an element B_0 of $\mathscr{B}$ such that $\tau(B_0) = \infty$ and, for all $A \subset B_0$, $\tau(A) \in \{0, \infty\}$. Since $\delta_\tau \dashv \tau$, there exists some $\mathscr{B}$-measurable map $f : E \to \overline{\mathbb{R}}_+$ such that $\delta_\tau(B) = \int_B^\infty f\, d\tau$, for all $B \in \mathscr{B}$. Then

$$1 = \bigoplus_{t > 0} t.\tau(B_0 \cap \{f > t\}). \tag{14}$$

In particular, for all $t > 0$, $\tau(B_0 \cap \{f > t\}) < \infty$, so $\tau(B_0 \cap \{f > t\}) = 0$, by definition of B_0. This contradicts Equation (14).

Claim 4: τ is semifinite.

Let ν be the map defined on $\mathscr{B}$ by $\nu(B) = \bigoplus_{A \subset B} \tau(A)$, where the supremum is taken over $\{A \in \mathscr{B} : A \subset B, \tau(A) < \infty\}$. Then ν is a σ-maxitive measure, absolutely continuous with respect to τ, and such that $\nu(B) = \tau(B)$, whenever $\tau(B) < \infty$. Assume that $\nu(B_1) \neq \tau(B_1)$, for some $B_1 \in \mathscr{B}$. Then $\nu(B_1) < \infty = \tau(B_1)$. Let $c : E \to \overline{\mathbb{R}}_+$ be a $\mathscr{B}$-measurable map such that Equation (13) is satisfied. Let $A_t = B_1 \cap \{c > t\}$. We have $\nu(B_1) \geqslant t.\tau(A_t)$, hence $\tau(A_t) < \infty$, for all $t > 0$. Moreover, since $\nu(\{c = 0\}) = 0$ and τ has no spot, we have $\tau(\{c = 0\}) < \infty$. Thus, $\infty = \tau(B_1) = \tau(B_1 \cap \{c > 0\}) = \bigoplus_{q \in \mathbb{Q}_+^*} \tau(A_q)$, and the definition of ν implies $\tau(B_1) \leqslant \nu(B_1)$, a contradiction.

Claim 5: τ is σ-finite.

Let $\mathscr{I}$ be the σ-ideal generated by all $A \in \mathscr{B}$ such that $\tau(A) < \infty$. Let $L \in \mathscr{I}$ such that $\tau(A \backslash L) = 0$, for all $A \in \mathscr{I}$. We can choose L of the form $L = \bigcup_{n \in \mathbb{N}} A_n$, with $\tau(A_n) < \infty$ for all n. Since τ is semifinite, $\tau(B) = \tau(B \cap L)$ for all B, so that $\tau(B) = \bigcup_n \tau(B \cap A_n)$ for all B. This proves that τ is σ-finite. □

I-8. Optimality of maxitive measures

I-8.1. Definition of optimal measures. In this section, we focus on the important particular case of *optimal measures*. We let $(E, \mathscr{B})$ denote a measurable space. A set function ν on $\mathscr{B}$ is *continuous from above* if $\nu(B) = \lim_n \nu(B_n)$, for all $B_1 \supset B_2 \supset \ldots \in \mathscr{B}$ such that $B = \bigcap_n B_n$ (we do not impose the condition $\nu(B_{n_0}) < \infty$ for some n_0). A monotone null-additive set function that is both continuous from above and from below is a *fuzzy measure*. Continuity from above is automatically satisfied for finite σ-additive measures, but this is untrue for (finite) σ-maxitive measures (see Puri and Ralescu [**247**] for a counterexample, see also Wang and Klir [**298**, Example 3.13]), so special care is needed. The following definition is given by Agbeko [**4**].

Definition I-8.1. An *optimal measure* is a maxitive fuzzy measure.

Surprisingly, it suffices for a maxitive measure to be continuous from above in order to satisfy continuity from below:

Proposition I-8.2 (Murofushi–Sugeno–Agbeko)**.** *A set function ν on $\mathscr{B}$ is an optimal measure if and only if it is a continuous from above maxitive measure. In this case, for all sequences (B_n) of elements of $\mathscr{B}$,*

$$\nu\Big(\bigcup_{n \in \mathbb{N}} B_n\Big) = \max_{n \in \mathbb{N}} \nu(B_n), \tag{15}$$

where the max *operator signifies that the supremum is reached.*

Proof. Murofushi and Sugeno [**216**] and after them Agbeko [**4**, Lemma 1.4] and Kramosil [**161**] showed that every continuous from above maxitive measure satisfies Equation (15); the first part of the proposition is then an easy consequence. □

The property of continuity from above in Definition I-8.1 is thus a strong condition. It becomes even more obvious with the following result. It was proved by Agbeko [**4**, Theorem 1.2] using Zorn's lemma, and Fazekas [**99**, Theorem 9] supplied an elementary proof. To formulate it, recall first that a *ν-atom* (called *indecomposable ν-atom* by Agbeko) is an element H of $\mathscr{B}$ such that $\nu(H) > 0$, and for each $B \in \mathscr{B}$ either $\nu(H \backslash B) = 0$, or $\nu(H \cap B) = 0$.

Theorem I-8.3 (Agbeko–Fazekas). *Let ν be an optimal measure on $\mathscr{B}$. Then there exists an at most countable collection $(H_n)_{n \in N}$ of pairwise disjoint ν-atoms $H_n \in \mathscr{B}$ such that*

$$\nu(B) = \max_{n \in N} \nu(B \cap H_n), \tag{16}$$

for all $B \in \mathscr{B}$, where the max *operator signifies that the supremum is reached. In particular, ν takes an at most countable number of values.*

An optimal measure ν satisfies the *exhaustivity* property, according to the terminology used by Pap [**234**], i.e. $\nu(B_n) \to 0$ when $n \to \infty$ for all pairwise disjoint $B_1, B_2, \ldots \in \mathscr{B}$. In fact, exhaustivity is exactly what a σ-maxitive measure needs to be optimal:

Proposition I-8.4. *A σ-maxitive measure is optimal if and only if it is exhaustive.*

Proof. The easy proof is left to the reader. □

Optimal measures were also studied (under various names) by Riečanová [**255**], Murofushi and Sugeno [**216**], Arslanov and Ismail [**19**]. In particular, the last-mentioned authors proved that the cardinality of some nonempty set E is non-measurable[2] if and only if all optimal measures on 2^E have a cardinal density [**19**, Theorem 19].

In the previous section we introduced *semifiniteness* for maxitive measures. For optimal measures, this merely reduces to finiteness.

Proposition I-8.5. *An optimal measure is semifinite if and only if it is finite.*

Proof. Let ν be a semifinite optimal measure on $\mathscr{B}$. If $\nu(E) = 0$, the result is clear. Otherwise, let $0 < s < \nu(E)$. In view of Fazekas [**99**, Remark 5], the set $\{\nu(B) : B \in \mathscr{B}, \nu(B) > s\}$ is finite, thus $\bigoplus_{B \subset E, \nu(B) < \infty} \nu(B) = \nu(B_0)$ for some $B_0 \in \mathscr{B}$ such that $\nu(B_0) < \infty$. By semifiniteness, $\nu(E) = \nu(B_0) < \infty$, so ν is finite. □

I-8.2. Densities of optimal measures. In this paragraph, we use previous results on the existence of densities for σ-maxitive measures, and apply them to optimal measures.

Independently of Sugeno and Murofushi [**279**], Agbeko proved Theorem I-6.4 in the particular case where τ is a normed optimal measure and ν is a finite optimal measure on $\mathscr{B}$ [**4**, Theorem 2.4]. This is indeed a particular case thanks to [**216**, Lemma 2.1], which states that every optimal measure is CCC, hence σ-principal under Zorn's lemma. Without using Zorn's lemma, we show that every optimal measure is σ-principal.

Proposition I-8.6. *Every optimal measure is σ-principal (hence autocontinuous).*

[2] A cardinal $|E|$ is *measurable* if there exists a two-valued probability measure on 2^E making all singletons negligible. The existence of measurable cardinals remains an open question.

Proof. Let ν be an optimal measure on $\mathscr{B}$. In the following lines we use the notations of the Agbeko-Fazekas Theorem (Theorem I-8.3). Let $\mathscr{I}$ be a σ-ideal and $\mathscr{I}_n = \mathscr{I} \cap H_n$, which is also a σ-ideal. There exists some $B_n \in \mathscr{I}_n$ such that $\nu(S \backslash B_n) = 0$ for every $S \in \mathscr{I}_n$. Indeed, this holds if every element of $\mathscr{I}_n$ has a ν-measure equal to zero. Otherwise choose some $B_n \in \mathscr{I}_n$ such that $\nu(B_n) > 0$. Then $\nu(H_n \backslash B_n) = 0$ since H_n is a ν-atom, thus $\nu(S \backslash B_n) \leqslant \nu(H_n \backslash B_n) = 0$ for all $S \in \mathscr{I}_n$. Now let $B = \bigcup_n B_n \in \mathscr{I}$, and let us show that $\nu(S \backslash B) = 0$ for all $S \in \mathscr{I}$. Assume that $\nu(S \backslash B) > 0$, so that there is some n such that $\nu(S \backslash B) = \nu((S \backslash B) \cap H_n) > 0$. On the one hand, $\nu(S \cap H_n) = 0$ since $S \cap H_n \in \mathscr{I}_n$. But on the other hand, $\nu(S \cap H_n) \geqslant \nu((S \backslash B) \cap H_n) > 0$. This proves that ν is σ-principal. □

As a consequence, we derive the Radon–Nikodym like theorem for optimal measures due to Agbeko.

Corollary I-8.7 (Agbeko)**.** *Let ν, τ be σ-maxitive measures on $\mathscr{B}$. Assume that τ is finite and optimal. Then $\nu \dashv \tau$ if and only if there exists some $\mathscr{B}$-measurable map $c : E \to \overline{\mathbb{R}}_+$ such that*

$$\nu(B) = \int_B^\infty c \, d\tau,$$

for all $B \in \mathscr{B}$. If these conditions are satisfied, then c is unique τ-almost everywhere.

Proof. See Theorem I-6.4, or Agbeko [**4**, Theorem 2.4] for the original statement. □

Problem I-8.8. Characterize those σ-maxitive measures τ that satisfy the *optimal* Radon–Nikodym property, i.e. such that all optimal measures that are absolutely continuous with respect to τ, have a measurable density with respect to τ.

We end this section with an analogue of the Lebesgue decomposition theorem for optimal measures, essentially based on a general theorem due to Pap. Two monotone null-additive set functions ν, τ defined on $\mathscr{B}$ are *mutually singular*, denoted by $\nu \perp \tau$, if there is some $A \in \mathscr{B}$ such that $\nu(B \cap A) = \tau(B \backslash A) = 0$, for all $B \in \mathscr{B}$.

Theorem I-8.9. *Let ν, τ be σ-maxitive measures on $\mathscr{B}$. Assume that τ is optimal. Then there exists a unique pair (ν_a, ν_s) of σ-maxitive measures on $\mathscr{B}$ such that $\nu = \nu_a \oplus \nu_s$, $\nu_a \dashv \tau$ and $\nu_s \perp \tau$. Moreover, $\nu_s \perp \nu_a$.*

Proof. Apply the general Lebesgue decomposition theorem due to Pap [**234**, Corollary 6.3] in combination with Proposition I-8.4. □

Problem I-8.10. Does the previous result still hold if τ is a σ-maxitive measure that is not optimal?

I-9. FOUNDATIONS OF POSSIBILITY THEORY

I-9.1. Towards an appropriate definition of possibility measures. Possibility theory is an analogue of probability theory, where probability measures are replaced by their maxitive counterpart. It has been developed over the last few years by several authors including Bellalouna [**31**], Akian et al. [**12, 13**], Akian [**6**], Del Moral and Doisy [**77**], de Cooman [**70, 71, 72, 73**], Puhalskii [**246**], Barron et al. [**27**], Fleming [**101**] among others. See also Baccelli et al. [**20**]. Analogies with probability theory, especially stressed by de Cooman [**70**] and Akian et al. [**13**], arise in the definitional aspects (such as the notion of independent events, or the concept of *maxingale* which replaces that of martingale [**246, 27**]) as well as in important results such as the law of large numbers or the central limit theorem. Nonetheless, possibility theory has its own specificities, for instance the surprising fact that convergence in "possibility" implies almost sure convergence[3] (see [**6**, Proposition 28] and [**246**, Theorem 1.3.5]).

In a stochastic context, the Radon–Nikodym property is highly desirable if one wants to dispose of conditional laws. In the σ-additive case this property is achieved by the classical Radon–Nikodym theorem[4], but in the σ-maxitive case this property may fail in absence of the σ-principality condition. To overcome this drawback, most of the publications require the possibility measure under study $\mathit{\Pi}$ to have a cardinal density, i.e. to be of the form

$$\mathit{\Pi}[A] = \bigoplus_{\omega \in A} c(\omega). \tag{17}$$

This condition was imposed by Akian et al. [**12, 13**], Akian [**6**], Del Moral and Doisy [**77**], de Cooman [**70, 71, 72, 73**], Puhalskii [**246**], Fleming [**101**]. Hypothesis (17) then facilitates the definition of conditioning, for $\mathit{\Pi}[X|Y]$ can be defined by the data of its cardinal density $c_{X|Y}$ given by:

$$c_{X|Y}(x|y) = \frac{c_{(X,Y)}(x,y)}{c_Y(y)},$$

if $c_Y(y) > 0$, and $c_{X|Y}(x|y) = 0$ otherwise, where c_X and c_Y are the respective (maximal) cardinal densities of $\mathit{\Pi}_X := \mathit{\Pi} \circ X^{-1}$ and $\mathit{\Pi}_Y$, and $c_{(X,Y)}$ that of the random variable $(X,Y) : \mathit{\Omega} \times \mathit{\Omega} \to \mathbb{R}_+$. In [**74**] and [**246**], another restrictive hypothesis was adopted, for their authors only considered completely maxitive measures defined on τ-algebras. A τ-algebra $\mathscr{A}$ on $\mathit{\Omega}$ being atomic, every $\omega \in \mathit{\Omega}$ is contained in a smallest event, denoted by $[\omega]_{\mathscr{A}}$. This particularity enables one to give an explicit formula of conditional laws, ω by ω.

The assumption of complete maxitivity and the use of τ-algebras instead of σ-algebras, if easier to handle, are certainly not satisfactory, especially

[3] Recall that probabilists are familiar with the converse implication.

[4] Notice that every probability measure is finite, hence σ-principal, see Theorem A.1 in the Appendix.

if one wants to parallel probability theory. A more general framework is possible, and we suggest to adopt the following definition of a possibility measure.

Definition I-9.1. Let $(\Omega, \mathscr{A})$ be a measurable space. A *possibility measure* (or a *possibility* for short) on $(\Omega, \mathscr{A})$ is a σ-principal σ-maxitive measure Π on $\mathscr{A}$ such that $\Pi[\Omega] = 1$. Then $(\Omega, \mathscr{A}, \Pi)$ is called a *possibility space*.

I-9.2. Conditional law with respect to a possibility measure. A conjunction of factors tends to confirm that this is the right definition. Firstly, properties of Π are transfered to the "laws" of random variables. If $(E, \mathscr{B})$ is a measurable space and $X : \Omega \to E$ is a random variable, its (possibility) law Π_X on $\mathscr{B}$ is the set function defined by $\Pi_X(B) = \Pi[X \in B] := \Pi[X^{-1}(B)]$, and this is a possibility measure. Moreover, if Π is optimal (resp. completely maxitive), then Π_X is optimal (resp. completely maxitive).

Secondly, the σ-principality property ensures that the Radon–Nikodym property is satisfied for the Shilkret integral $\Sigma[X] := \int^\infty X \, d\Pi$ of some random variable $X : \Omega \to \mathbb{R}_+$. Thus, following the classical approach of Halmos and Savage [**119**], conditioning can be defined as follows. Let $X : \Omega \to \mathbb{R}_+$ be a random variable and $\mathscr{D}$ be a sub-σ-algebra of $\mathscr{A}$. The σ-maxitive measure defined on $\mathscr{D}$ by $A \mapsto \Sigma[X.1_A] = \int_A^\infty X \, d\Pi$ is absolutely continuous with respect to the possibility $\Pi|_{\mathscr{D}}$. Thus, there exists some $\mathscr{D}$-measurable random variable from Ω into $\mathbb{R}_+$, written $\Sigma[X|\mathscr{D}]$, such that $\Sigma[X.1_A] = \Sigma\left[\Sigma[X|\mathscr{D}].1_A\right]$ for all $A \in \mathscr{D}$.

Barron et al. [**27**] considered the special case $\Pi := \delta_P$, where P is a probability measure. Then Π is essential, hence σ-principal, so it is a possibility measure, and the Shilkret integral $\Sigma[X]$ of a random variable X coincides with the P-essential supremum of X, i.e. $\Sigma[X] = \bigoplus_{\omega \in \Omega}^{P} X(\omega)$. Also, whenever $\Sigma[X] < \infty$, one has $\Sigma[X|\mathscr{D}] = \lim_{p \to \infty} E[X^p|\mathscr{D}]^{1/p}$, P-almost surely (where $E[X]$ denotes the usual expected value of X with respect to the probability measure P), see [**27**, Proposition 2.12]. Barron et al. derived a number of properties that still work in our more general context, as asserted by the next result (whose proof is left to the reader).

Proposition I-9.2. *Let $X : \Omega \to \mathbb{R}_+$ be a random variable and $\mathscr{D}$ be a sub-σ-algebra of $\mathscr{B}$. Then the following assertions hold:*

- *Y is Π-almost surely equal to $\Sigma[X|\mathscr{D}]$ if and only if $\Sigma[XZ] = \Sigma[YZ]$ for all $\mathscr{D}$-measurable random variables Z,*
- *$X \leqslant \Sigma[X|\mathscr{D}]$, Π-almost surely,*
- *if $Y : \Omega \to \mathbb{R}_+$ is a $\mathscr{D}$-measurable random variable such that $X \leqslant Y$, Π-almost surely, then $\Sigma[X|\mathscr{D}] \leqslant Y$, Π-almost surely,*
- *$X \mapsto \Sigma[X|\mathscr{D}]$ is a $\oplus$-linear form,*
- *$\Sigma\left[\Sigma[X|\mathscr{D}]\right] = \Sigma[X]$,*
- *if X is $\mathscr{D}$-measurable then $\Sigma[X|\mathscr{D}] = X$, Π-almost surely,*

where "Π-almost surely" stands for "Π-almost everywhere".

Remark I-9.3. Considering the second and third properties, $\Sigma[X|\mathscr{D}]$ can be interpreted as a projection (in an order-theoretical sense) of X on the set of $\mathscr{D}$-measurable random variables.

From these properties, Barron et al. deduced an ergodic theorem for maxima and, with the concept of maxingales, developed a theory of optimal stopping in L^∞.

Our new perpective on possibility measures should encourage us to recast possibility theory. The next step would be to confirm that convergence theorems given in [**6**] and [**246**] remain unchanged.

I-10. Conclusion and perspectives

In this work, we emphasized the link between essential suprema representations and Radon–Nikodym like theorems for the Shilkret (idempotent) integral. We showed that the Radon–Nikodym type theorem proved by Sugeno and Murofushi encompasses similar results including those of Agbeko, Barron et al., Drewnowski. Primarily, we were able to derive a converse statement to the Sugeno–Murofushi theorem, i.e. we characterized those σ-maxitive measures satisfying the Radon–Nikodym property as being σ-finite σ-principal. This result does not exist in classical measure theory, at least not in such a concise and exact form.

In Chapters II-III, we shall add a topological structure to the set E, in order to study Choquet-type capacitability theorems and Riesz representations theorems for maxitive measures. Given some partially ordered set L, we shall also focus on L-valued (rather than $\overline{\mathbb{R}}_+$-valued) maxitive measures, that were already considered by authors such as de Cooman [**71**], Akian [**7**], Heckmann and Huth [**123, 122**], Kramosil [**159, 158, 161, 160**]. Following Akian [**7**] and Heckmann and Huth [**123, 122**], we shall develop the links between maxitive measures and domain theory (or continuous lattice theory).

Acknowledgements. I am very grateful to Colas Bardavid who carefully read the whole manuscript and made very accurate suggestions. I also thank Marianne Akian who made some kind remarks and provided a counterexample to [**284**, Exercise II-3.19.1] inserted as Example I-3.10, and Pr. Jimmie D. Lawson for his advice and comments.

Appendix A. Some properties of σ-additive measures

The notions of σ-principal or CCC measures were originally introduced for the study of σ-additive measures. Recall that a σ-additive measure m is *CCC* (resp. *σ-principal*, *localizable*) if the σ-maxitive measure δ_m is.

The next theorem establishes a link between these notions for σ-additive measures. It enlightens the fact that being finite is a very strong condition for a σ-additive measure (while it is of little consequence for a σ-maxitive measure).

Theorem A.1. *Let $(E, \mathscr{B})$ is a measurable space and m be a σ-additive measure on $\mathscr{B}$. Consider the following assertions:*

(1) *m is finite,*
(2) *m is σ-finite,*
(3) *m is σ-principal,*
(4) *m is CCC,*
(5) *m is localizable.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). Moreover, (4) $\Rightarrow$ (3) under Zorn's lemma.

Sketch of the Proof. Assume that m is finite, and let us show that m is σ-principal. Let $\mathscr{I}$ be a σ-ideal of $\mathscr{B}$. Let $a = \bigoplus\{m(S) : S \in \mathscr{I}\}$. We can find some sequence $S_n \in \mathscr{I}$ such that $m(S_n) \uparrow a$. Defining $L := \cup_n S_n \in \mathscr{I}$, we have $m(L) = a$. If there exists some $S \in \mathscr{I}$ such that $m(S \backslash L) > 0$, then $m(S \cup L) > a$ (since m is finite), which contradicts $S \cup L \in \mathscr{I}$. Thus, $m(S \backslash L) = 0$, for all $S \in \mathscr{I}$, which gives σ-principality of m. The other implications in Theorem A.1 can be proved along the same lines as for σ-maxitive measures. $\square$

CHAPTER II

How regular can maxitive measures be?

ABSTRACT. We examine domain-valued maxitive measures defined on the Borel subsets of a topological space. Several characterizations of regularity of maxitive measures are proved, depending on the structure of the topological space. Since every regular maxitive measure is completely maxitive, this yields sufficient conditions for the existence of a cardinal density. We also show that every outer-continuous maxitive measure can be decomposed as the supremum of a regular maxitive measure and a maxitive measure that vanishes on compact subsets.

II-1. RÉSUMÉ EN FRANÇAIS

Les mesures maxitives, aussi connues sous le nom de *mesures idempotentes*, se définissent de la même façon que les mesures finiment additives, à ceci près que l'addition $+$ est remplacée par l'opération suprémum $\oplus$. Dans le chapitre I, nous avons étudié ces mesures et la théorie de l'intégration qui en découle, basée sur l'*intégrale de Shilkret*. Nous nous sommes notamment intéressés à l'analogue idempotent du théorème de Radon–Nikodym. De ce fait, nous avons limité notre étude aux mesures à valeurs réelles ; cependant, cela apparaît insuffisant pour certaines applications.

Pour le comprendre, faisons un détour par l'analyse classique. On sait dans ce cadre que le théorème de Radon–Nikodym est vrai sur certains types d'espaces de Banach (par exemple les espaces réflexifs ou les espaces séparables duaux). Pour formuler un tel théorème, un prérequis est d'étendre l'intégrale de Lebesgue aux fonctions mesurables à valeurs dans ces espaces : c'est l'intégrale de Bochner. Plus généralement, un espace de Banach B a la *propriété de Radon–Nikodym* si, pour tout espace mesuré fini $(\Omega, \mathscr{A}, \mu)$ et pour toute mesure B-valuée m sur $\mathscr{A}$, absolument continue par rapport à μ et à variation bornée, il existe une fonction Bochner-intégrable $f : \Omega \to B$ telle que $m(A) = \int_A f \, d\mu$ pour tout $A \in \mathscr{A}$. Cette propriété a fait l'objet d'une multitude de travaux et a été à la base de nombreuses découvertes sur la structure des espaces de Banach.

Pour espérer obtenir des résultats analogues en analyse idempotente, il est donc nécessaire de disposer d'un outil tel que l'intégrale de Bochner, permettant d'intégrer des fonctions M-valuées, avec M un certain « espace idempotent » (de type module sur le semicorps idempotent $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \max, \times)$, cf. chapitre III) à définir. Ce sont en partie les travaux de Jonasson [**142**] d'une part, d'Akian [**7**] d'autre part, qui s'approchent le plus d'un tel outil. Cependant Akian se limite aux fonctions et mesures à valeurs dans un dioïde (plutôt qu'à valeurs dans un module), et Jonasson se contente du cas additif.

Afin de préparer l'avenir en vue de tels travaux – qui ne seront néanmoins pas abordés dans cette thèse –, nous nous intéressons à la suite d'Akian aux mesures maxitives à valeurs dans un *domaine*. Un domaine est un ensemble ordonné satisfaisant de bonnes propriétés d'approximation. Les ensembles numériques $\mathbb{R}_+$, $\overline{\mathbb{R}}_+$ et $[0, 1]$ sont des exemples de domaines bien connus ; ils sont couramment utilisés comme ensembles d'arrivée pour les mesures maxitives. De nombreuses tentatives ont été faites pour les remplacer par des ensembles ordonnés plus généraux (cf. Maslov [**196**], Greco [**117**], Liu et Zhang [**183**], de Cooman et al. [**74**], Kramosil [**159**]). Néanmoins, l'importance de supposer ces ensembles *continus* (au sens de la théorie des domaines) pour les applications à l'analyse idempotente ou à la théorie des ensembles flous n'a été identifiée que tardivement. Les premiers à avoir intégré cette hypothèse de continuité sont Akian [**6, 7**] et Heckmann et Huth [**122, 123**]. Cf. Lawson [**172**] pour une revue de l'utilisation des domaines en analyse idempotente.

Notons aussi, dans le cas des espaces de Banach, que la propriété de Radon–Nikodym a des liens très étroits avec la *propriété de Krein–Milman*, qui exprime que tout convexe non-vide et fermé-borné est l'enveloppe convexe fermée de ses points extrêmes. En fait, cette dernière propriété implique la propriété de Radon–Nikodym (cf. par exemple Benyamini et Lindenstrauss [**33**, Théorème 5.13]), mais la réciproque reste un problème ouvert. Des questions analogues pourraient être soulevées dans le cas idempotent, et nous aborderons à ce sujet les questions de convexité aux chapitres IV et V.

Une autre application d'importance est le théorème de représentation intégrale de Choquet, dans sa version idempotente. En analyse classique les mesures régulières y jouent un rôle clef. Nous verrons au chapitre V qu'il en est de même dans le cas idempotent ; c'est pourquoi nous nous intéressons ici aux questions de régularité des mesures maxitives, définies sur la tribu $\mathscr{B}$ des Boréliens d'un espace topologique. Sur un espace de Hausdorff, une mesure maxitive ν est *régulière* si elle satisfait les deux conditions suivantes pour tout $B \in \mathscr{B}$:

- *sous-continuité* :
$$\nu(B) = \bigoplus_{K \in \mathscr{K}, K \subset B} \nu(K),$$
- *sur-continuité* :
$$\nu(B) = \bigwedge_{G \in \mathscr{G}, G \supset B} \nu(G),$$

où $\mathscr{K}$ désigne l'ensemble des parties compactes et $\mathscr{G}$ celui des parties ouvertes. Nous prouvons que différents jeux de conditions assurent la sous- ou sur-continuité des mesures maxitives. Les principaux résultats sont résumés

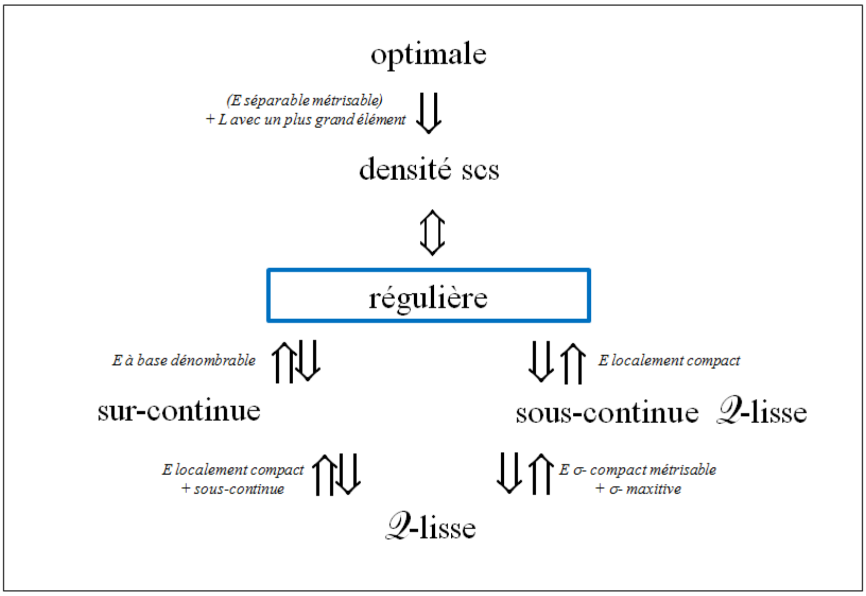

FIGURE 1. Implications et équivalences liées à la régularité des mesures maxitives et démontrées dans ce chapitre, dans le cas où l'espace topologique sous-jacent E est Hausdorff. Certaines assertions dépendent de la structure de E et du domaine L dans lequel la mesure prend ses valeurs. (Densité scs = densité semicontinue supérieurement.)

sur les figures 1 et 2 ; ils généralisent notamment des résultats de Norberg [**225**], Murofushi et Sugeno [**216**], Vervaat [**289**], O'Brien et Watson [**233**], Akian [**7**], Puhalskii [**246**], Miranda et al. [**204**].

L'importance de la condition de régularité est aussi liée au fait que toute mesure maxitive régulière ν admet une densité cardinale, au sens où, pour une certaine fonction c, on a

$$\nu(B) = \bigoplus_{x \in B} c(x),$$

pour toute partie borélienne B. En effet, nombreux sont ceux à s'être attardés sur la question de l'existence d'une telle densité, nous souhaitons donc revisiter celle-ci de façon plus complète.

Nos preuves suivent pour partie la méthode de Riečanová [**255**], qui a étudié la régularité de certaines fonctions d'ensemble S-valuées, avec S un semigroupe ordonné, conditionnellement complet au sens de l'ordre, et satisfaisant un ensemble de conditions telles que la séparation des points par des fonctionnelles continues. Nous n'utilisons pas directement ses résultats, car notre approche est mieux adaptée au cas des mesures maxitives ou optimales à valeurs dans un domaine. En effet un domaine n'est pas nécessairement un semigroupe, ni un ensemble conditionnellement complet.

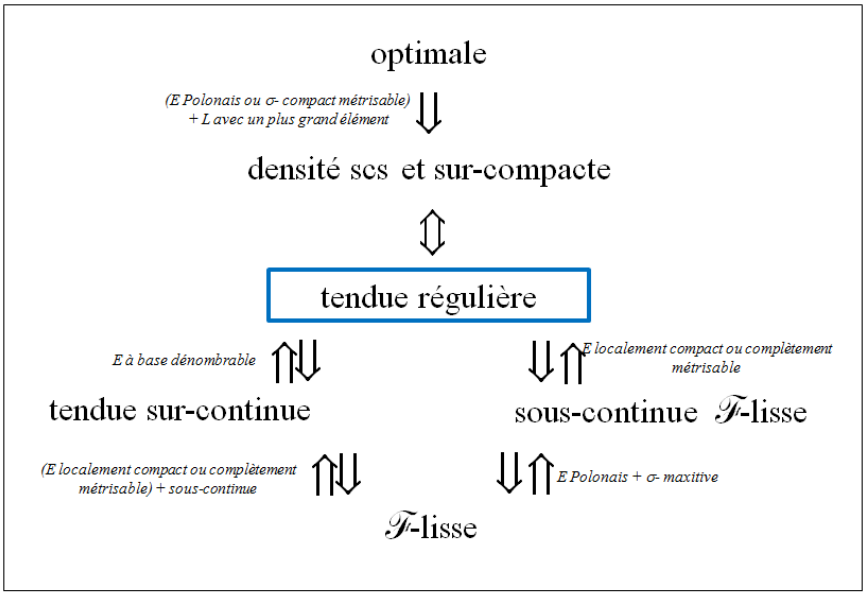

FIGURE 2. Des propriétés similaires sont prouvées autour de la notion de mesure maxitive régulière tendue. (Densité scs = densité semicontinue supérieurement.)

Enfin, nous prouvons un théorème de décomposition des mesures maxitives sur-continues, de la forme suivante :

$$\nu = \lfloor \nu \rfloor \oplus \perp\nu$$

où $\lfloor \nu \rfloor$ est une mesure maxitive régulière (nommée la *partie régulière* de ν) et $\perp\nu$ est une mesure maxitive nulle sur les parties compactes, une propriété que nous nommons *singularité*. Ceci nous donne une autre caractérisation de la régularité, puisque ν est régulière si et seulement si sa partie singulière est nulle. Un énoncé dual est vérifié pour la singularité.

II-2. INTRODUCTION

Maxitive measures, also known as *idempotent measures*, are defined similarly to finitely additive measures with the supremum operation $\oplus$ in place of the addition $+$. In Chapter I, we studied these measures and the related integration theory based on the *Shilkret integral*. We were especially interested in the idempotent analogue of the Radon–Nikodym theorem. In this process, we limited our considerations to maxitive measures taking values in the set of nonnegative real numbers. However, this may be quite restrictive for further applications.

Let us have a look at classical analysis to understand why. In this framework, it is well known that the Radon–Nikodym theorem holds on certain classes of Banach spaces (e.g. reflexive spaces or separable dual spaces). To formulate such a theorem one needs to extend first the Lebesgue integral to measurable functions taking values in these spaces. This is what the

Bochner integral does. More generally, a Banach space B has the *Radon–Nikodym property* if, for all finite measured spaces $(\Omega, \mathscr{A}, \mu)$ and for all B-valued measures m on $\mathscr{A}$, absolutely continuous with respect to μ and of bounded variation, there is a Bochner integrable map $f : \Omega \to B$ such that $m(A) = \int_A f \, d\mu$, for all $A \in \mathscr{A}$. This property has been at the core of a great amount of research and the source of many discoveries on the structure of Banach spaces.

If one hopes for analogous results in idempotent analysis, one must have such a powerful tool as the Bochner integral available, that would integrate M-valued functions, for some "idempotent space" M. One could think of M e.g. as a complete module over the idempotent semifield $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \max, \times)$, see Chapter III, but the appropriate structure still needs to be clarified. Jonasson [**142**] on the one hand, Akian [**7**] on the other hand, both worked in this direction. However, Akian chose to integrate dioid-valued (rather than module-valued) functions, and Jonasson remained in the additive paradigm.

In order to prepare these kinds of future applications -which are not directly in the scope of this thesis-, we study *domain*-valued maxitive measures after Akian. A domain is a partially ordered space with nice approximation properties. Well known examples of domains are $\mathbb{R}_+$, $\overline{\mathbb{R}}_+$, and $[0, 1]$; they are commonly used as target sets for maxitive measures. Many attempts were made for replacing them by more general ordered structures (see Maslov [**196**], Greco [**117**], Liu and Zhang [**183**], de Cooman et al. [**74**], Kramosil [**159**]). Nevertheless, the importance of supposing these ordered structures *continuous* in the sense of domain theory for applications to idempotent analysis or fuzzy set theory has been identified lately. Pioneers were Akian [**6, 7**] and Heckmann and Huth [**122, 123**]. See Lawson [**172**] for a survey on the use of domain theory in idempotent mathematics.

In the case of Banach spaces, it must also be remarked that the Radon–Nikodym property is deeply linked with the *Krein–Milman property*, which says that every nonempty bounded closed convex subset is the closed convex hull of its extreme points. It was proved that the latter property implies the Radon–Nikodym property (see e.g. Benyamini and Lindenstrauss [**33**, Theorem 5.13]), and the converse statement remains an open problem. Similar problems could be raised in the idempotent case, and Chapters IV and V will tackle convexity questionings.

Another application we have in mind is the idempotent analogue of the Choquet integral representation theorem. In classical analysis, regular measures play a key role; we shall see in Chapter V that this is also the case in the idempotent framework. This explains why we deal here with regularity properties of maxitive measures, defined on the Borel σ-algebra $\mathscr{B}$ of some topological space. On a Hausdorff space, a maxitive measure is *regular* if it satisfies both following conditions for all $B \in \mathscr{B}$:

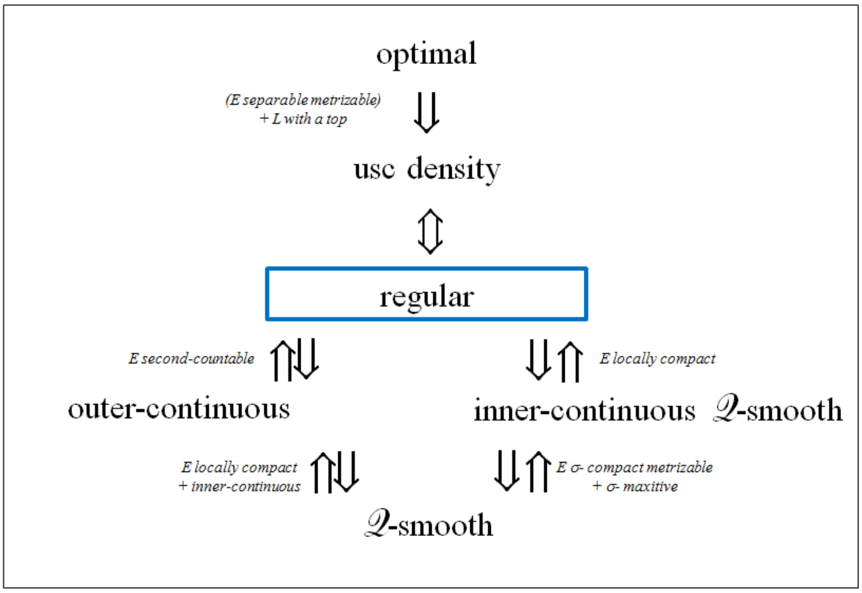

FIGURE 3. Implications and equivalences proved in this chapter in relation to regularity of L-valued maxitive measures, in the case where the underlying topological space E is Hausdorff. Some assertions depend on the structure of E and on the domain L.

- *inner-continuity*:

$$\nu(B) = \bigoplus_{K \in \mathscr{K}, K \subset B} \nu(K),$$

- *outer-continuity*:

$$\nu(B) = \bigwedge_{G \in \mathscr{G}, G \supset B} \nu(G),$$

where $\mathscr{K}$ denotes the collection of compact subsets and $\mathscr{G}$ that of open subsets. We prove a series of conditions that guarantee inner- and/or outer-continuity of maxitive measures. The main results are summarized on Diagrams 3 and 4. They generalize results due to Norberg [**225**], Murofushi and Sugeno [**216**], Vervaat [**289**], O'Brien and Watson [**233**], Akian [**7**], Puhalskii [**246**], Miranda et al. [**204**].

Regularity is an important feature of maxitive measures for a different reason: a regular maxitive measure ν admits a cardinal density in the sense that, for some map c, we have

$$\nu(B) = \bigoplus_{x \in B} c(x),$$

for all Borel sets B. Numerous authors have been interested in conditions that imply the existence of such a density, hence we make the choice to revisit this problem as exhaustively as possible.

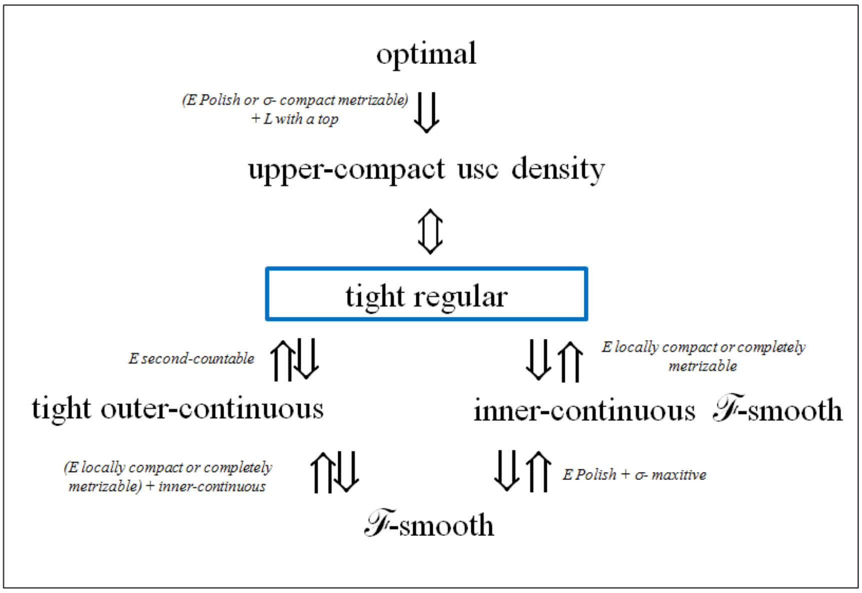

FIGURE 4. Similar properties are derived around tight regular maxitive measures.

For some of our proofs we follow the steps of Riečanová [**255**], who focused on the regularity of certain S-valued set functions, for some conditionally complete ordered semigroup S satisfying a series of conditions, among which the separation of points by continuous functionals. We do not use directly her results, for our approach better suits the special case of domain-valued optimal measures. Indeed, a domain is not necessarily a semigroup, nor is it conditionally complete in general.

As a last step, we prove a decomposition theorem for outer-continuous maxitive measures, that takes the following form:

$$\nu = \lfloor \nu \rfloor \oplus \perp\!\nu$$

where $\lfloor \nu \rfloor$ is a regular maxitive measure called the *regular part* of ν, and $\perp\nu$ is a maxitive measure vanishing on compact subsets, a property that we call *singularity*. This has the consequence that ν is regular (resp. singular) if and only if its singular (resp. regular) part is zero.

The chapter is organized as follows. In Section II-3 we recall basic domain theoretical concepts. Section II-4 introduces the notion of *L-valued* maxitive measure, for some domain L. In Section II-5 we specifically consider maxitive measures defined on the collection of Borel subsets of some quasisober topological space. We prove that regularity and tightness of maxitive measures are linked with different conditions such as existence of a cardinal density, complete maxitivity, smoothness with respect to compact saturated or closed subsets, inner-continuity. We focus on the case where the topological space is metrizable and the maxitive measure is optimal in Section II-6. In Section II-7 we prove the announced decomposition theorem.

II-3. REMINDERS OF DOMAIN THEORY

A nonempty subset F of a partially ordered set or *poset* $(L, \leqslant)$ is *filtered* if, for all $r, s \in F$, one can find $t \in F$ such that $t \leqslant r$ and $t \leqslant s$. A *filter* of L is a filtered subset F such that $F = \{s \in L : \exists r \in F, r \leqslant s\}$. We say that $s \in L$ is *way-above* $r \in L$, written $s \gg r$, if, for every filter F with an infimum $\bigwedge F$, $r \geqslant \bigwedge F$ implies $s \in F$. The *way-above relation*, useful for studying lattice-valued upper-semicontinuous functions (see Gerritse [**113**] and Jonasson [**142**]), is dual to the usual *way-below relation*, but is more appropriate in our context. Coherently, our notions of continuous posets and domains are dual to the traditional ones. We thus say that the poset L is *continuous* if $\twoheaduparrow r := \{s \in L : s \gg r\}$ is a filter and $r = \bigwedge \twoheaduparrow r$, for all $r \in L$. Also, L is *filtered-complete* if every filter has an infimum. A *domain* is then a filtered-complete continuous poset. A poset L has the *interpolation property* if, for all $r, s \in L$ with $s \gg r$, there exists some $t \in L$ such that $s \gg t \gg r$. In continuous posets it is well known that the interpolation property holds, see e.g. [**114**, Theorem I-1.9]. This is a crucial feature that is behind many important results of the theory. For more background on domain theory, see the monograph by Gierz et al. [**114**].

Remark II-3.1. To show that an inequality $r' \geqslant r$ holds in a continuous poset L, it suffices to prove that, whenever $s \gg r'$, we have $s \geqslant r$. This argument will be used many times in this work.

II-4. DOMAIN-VALUED MAXITIVE MEASURES

Let E be a nonempty set. A *prepaving* on E is a collection of subsets of E containing the empty set and closed under finite unions. Assume in all the sequel that $\mathscr{E}$ is a prepaving on E and that L is a poset with a bottom element, that we denote by 0. An L-valued *maxitive measure* (resp. σ-*maxitive measure*, *completely maxitive measure*) on $\mathscr{E}$ is a map $\nu : \mathscr{E} \to L$ such that $\nu(\varnothing) = 0$ and, for every finite (resp. countable, arbitrary) family $\{G_j\}_{j \in J}$ of elements of $\mathscr{E}$ such that $\bigcup_{j \in J} G_j \in \mathscr{E}$, the supremum of $\{\nu(G_j) : j \in J\}$ exists and satisfies

$$\nu(\bigcup_{j \in J} G_j) = \bigoplus_{j \in J} \nu(G_j).$$

The next proposition, inspired by Nguyen et al. [**224**], provides a generic way of constructing a maxitive measure from a nondecreasing family of ideals. An *ideal* of the prepaving $\mathscr{E}$ is a nonempty subset $\mathscr{I}$ of $\mathscr{E}$ that is closed under finite unions and such that $A \subset G \in \mathscr{I}$ and $A \in \mathscr{E}$ imply $A \in \mathscr{I}$.

Proposition II-4.1. *Let $(\mathscr{I}_t)_{t \in L}$ be some family of ideals of $\mathscr{E}$ such that, for all $G \in \mathscr{E}$, $\{t \in L : G \in \mathscr{I}_t\}$ is a filter with infimum. Define $\nu : \mathscr{E} \to L$ by*

$$\nu(G) = \bigwedge \{t \in L : G \in \mathscr{I}_t\}. \tag{18}$$

If $(\mathscr{I}_t)_{t \in L}$ is right-continuous, in the sense that $\mathscr{I}_t = \bigcap_{s \gg t} \mathscr{I}_s$ for all $t \in L$, then ν is maxitive.

Remark II-4.2. Assuming that $\{t \in L : G \in \mathscr{I}_t\}$ is a filter for all $G \in \mathscr{E}$ makes the family $(\mathscr{I}_t)_{t \in L}$ necessarily nondecreasing.

Proof. Let ν be given by Equation (18). Obviously, ν is order-preserving. Let $\{G_j\}_{j \in J}$ be a finite family of elements of $\mathscr{E}$, and let $u \in L$ be an upper-bound of $\{\nu(G_j)\}_{j \in J}$. To prove that ν is maxitive, we show that $u \geqslant \nu(\bigcup_{j \in J} G_j)$. Let $s \gg u$. By definition of the way-above relation $\gg$, one has $G_j \in \mathscr{I}_s$ for all $j \in J$, thus $\bigcup_{j \in J} G_j \in \mathscr{I}_s$. This implies $\bigcup_{j \in J} G_j \in \bigcap_{s \gg u} \mathscr{I}_s = \mathscr{I}_u$. We get $u \geqslant \bigwedge\{r \in L : \bigcup_{j \in J} G_j \in \mathscr{I}_r\} = \nu(\bigcup_{j \in J} G_j)$, so ν is maxitive. $\square$

Supposing that the range L of the maxitive measure is continuous, we can remove the assumption of right-continuity of the family of ideals as follows.

Proposition II-4.3. *Assume that L is a continuous poset. A map $\nu : \mathscr{E} \to L$ is a maxitive measure if and only if there is some family $(\mathscr{I}_t)_{t \in L}$ of ideals of $\mathscr{E}$ such that, for all $G \in \mathscr{E}$, $\{t \in L : G \in \mathscr{I}_t\}$ is a filter with infimum and*

$$\nu(G) = \bigwedge \{t \in L : G \in \mathscr{I}_t\}.$$

In this case, $(\mathscr{I}_t)$ is right-continuous if and only if $\mathscr{I}_t = \{G \in \mathscr{E} : t \geqslant \nu(G)\}$ for all $t \in L$.

Proof. If ν is maxitive, simply take $\mathscr{I}_t = \{G \in \mathscr{E} : t \geqslant \nu(G)\}$, $t \in L$, which is right-continuous since L is continuous. Conversely, assume that Equation (18) is satisfied. Let $\mathscr{J}_t = \bigcap_{s \gg t} \mathscr{I}_s$. Then $(\mathscr{J}_t)_{t \in L}$ is a nondecreasing family of ideals of $\mathscr{E}$ such that $\mathscr{J}_t \supset \mathscr{I}_t$ for all $t \in L$. Moreover, $(\mathscr{J}_t)_{t \in L}$ is right-continuous thanks to the interpolation property, and by continuity of L one has $\nu(G) = \bigwedge\{t \in L : G \in \mathscr{J}_t\}$. Using Proposition II-4.1, ν is maxitive.

Assume that $(\mathscr{I}_t)$ is right-continuous. The inclusion $\mathscr{I}_t \subset \{G \in \mathscr{E} : t \geqslant \nu(G)\}$ is clear. If $t \geqslant \nu(G)$, we want to show that $G \in \mathscr{I}_t$, i.e. $G \in \mathscr{I}_s$ for all $s \gg t$. So let $s \gg t \geqslant \nu(G)$. Equation (18) and the definition of the way-above relation imply that $G \in \mathscr{I}_s$, and the inclusion $\mathscr{I}_t \supset \{G \in \mathscr{E} : t \geqslant \nu(G)\}$ is proved. $\square$

The following corollary is an extension result for maxitive measures. It improves previous results due to Maslov [**196**, Theorem VIII-4.1], Heckmann and Huth [**123**, Proposition 12], Akian [**7**, Proposition 3.1]. We denote by $\mathscr{E}^*$ the collection of all $A \subset E$ such that $\{G \in \mathscr{E} : G \supset A\}$ is a filter. Notice that $\mathscr{E}^*$ is a prepaving containing $\mathscr{E}$. Moreover, if $\mathscr{E}$ is closed under finite intersections and E is in $\mathscr{E}$, then $\mathscr{E}^*$ merely coincides with the power set of E.

Corollary II-4.4. *Assume that* L *is a domain. Let* ν *be an* L*-valued maxitive measure on* $\mathscr{E}$*. The map* $\nu^* : \mathscr{E}^* \to L$ *defined by*

$$\nu^*(A) = \bigwedge_{G \in \mathscr{E}, G \supset A} \nu(G) \tag{19}$$

is a maxitive measure, and this is the maximal extension of ν *to* $\mathscr{E}^*$*.*

Proof. If ν is defined by Equation (18), let $\mathscr{I}_t^*$ denote the collection of all $A \in \mathscr{E}^*$ such that $A \subset B$ for some $B \in \mathscr{I}_t$. Then $(\mathscr{I}_t^*)_{t \in L}$ is a non-decreasing family of ideals of $\mathscr{E}^*$ and, for all $A \in \mathscr{E}^*$, $\{t \in L : A \in \mathscr{I}_t^*\} = \bigcup_{G \in \mathscr{E}, G \supset A} \{t \in L : G \in \mathscr{I}_t\}$ is a filter in L. Now the fact that $\nu^*(A) = \bigwedge \{t \in L : A \in \mathscr{I}_t^*\}$ and Proposition II-4.3 show that ν^* is maxitive. The assertion that ν^* is the maximal maxitive measure extending ν to $\mathscr{E}^*$ is not difficult and left to the reader. $\square$

This corollary also generalizes [**159**, Theorem 15.2], where Kramosil assumed that L is a complete chain (which is necessarily a domain).

II-5. Maxitive measures on topological spaces

II-5.1. Preliminaries on topological spaces. Let E be a topological space. We denote by $\mathscr{G}$ (resp. $\mathscr{F}$) the collection of open (resp. closed) subsets of E. The interior (resp. the closure) of a subset A of E is written A^o (resp. $\overline{A}$). The *specialization order* on E is the quasiorder $\leqslant$ defined on E by $x \leqslant y$ if $x \in G$ implies $y \in G$, for all open subsets G. A subset C of E is *irreducible* if it is nonempty and, for all closed subsets F, F' of E, $C \subset F \cup F'$ implies $C \subset F$ or $C \subset F'$. The closure of a singleton yields an irreducible closed set. We say that E is *quasisober* if every irreducible closed subset is the closure of a singleton. A subset A of E is *saturated* if it is an intersection of open subsets. The *saturation* of A, written $\uparrow A$, is the intersection of all open subsets containing A, and we have

$$\uparrow A = \bigcap_{G \in \mathscr{G}, G \supset A} G = \{x \in E : \exists a \in A, a \leqslant x\}.$$

If A is a singleton $\{x\}$, we write $\uparrow x$ instead of $\uparrow \{x\}$. Note that all open subsets are saturated.

We denote by $\mathscr{Q}$ the collection of (not necessarily Hausdorff) compact saturated subsets of E. For instance, $\uparrow x \in \mathscr{Q}$, for all $x \in E$. We shall need the following theorem, which emphasizes the role of compact saturated subsets for non-Hausdorff spaces.

Theorem II-5.1 (Hofmann–Mislove)**.** *In a quasisober topological space, the collection of compact saturated subsets is closed under finite unions and filtered intersections. Moreover, if* $(Q_j)_{j \in J}$ *is a filtered family of compact saturated subsets such that* $\bigcap_{j \in J} Q_j \subset G$ *for some open* G*, then* $Q_j \subset G$ *for some* $j \in J$*.*

This theorem was proved by Hofmann and Mislove [**128**]. A different proof is due to Keimel and Paseka [**150**]. See Kovár [**157**] for an extension to generalized topological spaces, and Jung and Sünderhauf [**144**] for an enlightenment of this result in the context of proximity lattices. Also, Norberg and Vervaat [**228**] successfully applied this result, in a non-Hausdorff setting, to the theory of capacities which dates back to Choquet [**60**].

II-5.2. The Borel σ-algebra. The *Borel σ-algebra* is the σ-algebra $\mathscr{B}$ generated by $\mathscr{G}$ and $\mathscr{Q}$; its elements are called the *Borel* subsets of E. We also write $\mathscr{K}$ for the collection of compact Borel subsets of E. If E is T_1 (in particular if E is Hausdorff), then $\mathscr{K} = \mathscr{Q}$. In the case where E is T_0, $\mathscr{K}$ contains all singletons $\{x\}$, for $\{x\}$ is the intersection of the compact saturated subset $\uparrow x$ with the closure $\overline{x}$ of $\{x\}$. In the general case (E not necessarily T_0), we let $[x]$ denote the compact Borel subset $\uparrow x \cap \overline{x}$. This is the equivalence class of x with respect to the equivalence relation $x \sim y \Leftrightarrow \overline{x} = \overline{y}$. Notice that $\uparrow[x] = \uparrow x$ for all x.

Lemma II-5.2. *Let E be a topological space. For all Borel subsets B of E, $x \in B$ implies $[x] \subset B$.*

Proof. Let $[\mathscr{B}]$ be the collection of all Borel subsets B such that $[x] \subset B$ for all $x \in B$. We prove that, if $B \in [\mathscr{B}]$, then $E\backslash B \in [\mathscr{B}]$. So suppose that $x \in E\backslash B$, and let us show that $[x] \subset E\backslash B$, i.e. $[x] \cap B = \varnothing$. If $y \in [x] \cap B$, then $[x] = [y]$ on the one hand, and $[y] \subset B$ since $B \in [\mathscr{B}]$ on the other hand. Then $x \in [x] = [y] \subset B$, a contradiction. This proves that $[x] \subset E\backslash B$.

Now it is not difficult to deduce that $[\mathscr{B}]$ is a σ-algebra containing $\mathscr{G}$ and $\mathscr{Q}$, hence $[\mathscr{B}]$ coincides with $\mathscr{B}$. $\square$

II-5.3. Regular maxitive measures. Let E be a topological space with Borel σ-algebra $\mathscr{B}$, and let L be a filtered-complete poset. An L-valued maxitive measure ν on $\mathscr{B}$ is *regular* if it satisfies both following relations for all $B \in \mathscr{B}$:

- *inner-continuity*:

$$\nu(B) = \bigoplus_{K \in \mathscr{K}, K \subset B} \nu(\uparrow K),$$

- *outer-continuity*:

$$\nu(B) = \bigwedge_{G \in \mathscr{G}, G \supset B} \nu(G).$$

Example II-5.3. Assume that L is a domain. For an L-valued (σ-)maxitive measure ν on $\mathscr{G}$, the set $\{\nu(G) : G \in \mathscr{G}, G \supset B\}$ is filtered for all $B \in \mathscr{B}$, so one can define a map ν^+ on $\mathscr{B}$ by

$$\nu^+(B) = \bigwedge_{G \in \mathscr{G}, G \supset B} \nu(G).$$

Then, by Corollary II-4.4, ν^+ is an outer-continuous (σ-)maxitive measure. Moreover, ν^+ is inner-continuous (hence regular) if ν is inner-continuous (combine Lemma II-5.4 and Lemma II-5.7 below).

We shall also use weakened notions of inner- and outer-continuity for an L-valued maxitive measure ν on $\mathscr{B}$:

- *weak inner-continuity*:

$$\nu(G) = \bigoplus_{K \in \mathscr{K}, K \subset G} \nu^+(K), \qquad \text{for all } G \in \mathscr{G},$$

- *weak outer-continuity*:

$$\nu(K) = \bigwedge_{G \in \mathscr{G}, G \supset K} \nu(G), \qquad \text{for all } K \in \mathscr{K}.$$

The following result ensures that the terminology we use is consistent.

Lemma II-5.4. *An inner- (resp. outer-)continuous maxitive measure on $\mathscr{B}$ is weakly inner- (resp. weakly outer-)continuous.*

Proof. The easy proof is left to the reader. □

The notion of weak inner-continuity can be characterized as follows.

Lemma II-5.5. *Assume that L is a domain. Let ν be an L-valued maxitive measure on $\mathscr{B}$. Then ν is weakly inner-continuous if and only if*

$$\nu(\bigcup \mathscr{O}) = \bigoplus \nu(\mathscr{O}), \tag{20}$$

for all families $\mathscr{O}$ of open subsets of E.

Proof. First we suppose that ν is weakly inner-continuous. Let $\mathscr{O}$ be a family of open subsets of E, and let $G = \bigcup \mathscr{O}$. The identity we need to show will be satisfied if we prove that $\nu^+(K) \leqslant \bigoplus \nu(\mathscr{O})$ for all compact Borel subsets $K \subset G$. But for such a K, there are open subsets $O_1, \ldots, O_n$ in $\mathscr{O}$ such that $K \subset O_1 \cup \ldots \cup O_n$, so that $\nu^+(K) \leqslant \nu(O_1) \oplus \ldots \oplus \nu(O_n) \leqslant \bigoplus \nu(\mathscr{O})$.

Conversely, suppose that Equation (20) holds for all families $\mathscr{O}$ of open subsets of E. To prove that ν is weakly inner-continuous, fix some $G \in \mathscr{G}$, let u be an upper-bound of $\{\nu^+(K) : K \in \mathscr{K}, K \subset G\}$, and let $s \gg u$. Then for all $x \in G$, $s \gg \nu^+([x])$, so there is some $G_x \in \mathscr{G}$ such that $G_x \ni x$ and $s \geqslant \nu(G_x)$. By Equation (20) we have $s \geqslant \nu(\bigcup_{x \in G} G_x) \geqslant \nu(G)$. Since L is continuous, wet get $u \geqslant \nu(G)$, and the result follows. □

The following lemma characterizes weak outer-continuity.

Lemma II-5.6. *Assume that L is a domain. Let ν be an L-valued maxitive measure on $\mathscr{B}$. Then*

$$\nu^+(K) = \bigoplus_{x \in K} \nu^+([x]),$$

for all $K \in \mathscr{K}$. As a consequence, ν is weakly outer-continuous if and only if $\nu([x]) = \nu^+([x])$ for all $x \in E$.

Proof. We let $c^+ : x \mapsto \nu^+([x])$. Let $u \in L$ be an upper-bound of $\{c^+(x) : x \in K\}$ and let $s \gg u$. Then, for each $x \in K$, $s \gg c^+(x)$, so there is some open subset $G_x \ni x$ such that $s \geqslant \nu(G_x)$. Since K is compact and $\bigcup_{x \in K} G_x \supset K$, we can extract a finite subcover and write $\bigcup_{j=1}^k G_{x_j} \supset K$. Thus, $s \geqslant \nu^+(K)$. Since L is continuous, this implies that $u \geqslant \nu^+(K)$, so that $\nu^+(K)$ is the least upper-bound of $\{c^+(x) : x \in K\}$. □

It happens that we recover regularity if we combine weak inner- and weak outer-continuity.

Lemma II-5.7. *Assume that L is a domain. Then every L-valued maxitive measure on $\mathscr{B}$ that is both weakly outer-continuous and weakly inner-continuous is regular.*

Proof. Let ν be an L-valued weakly outer-continuous and weakly inner-continuous maxitive measure. Assume that, for some $B \in \mathscr{B}$, $\nu^+(B)$ is not the least upper-bound of $\{\nu(K) : K \in \mathscr{K}, K \subset B\}$. Then there exists some upper-bound $u \in L$ of $\{\nu(K) : K \in \mathscr{K}, K \subset B\}$ such that $u \not\geqslant \nu^+(B)$. Since L is continuous, there exists some $s \gg u$ with $s \not\geqslant \nu^+(B)$. If $x \in B$, then $K_x = [x]$ is a compact Borel subset, and $K_x \subset B$ by Lemma II-5.2. So $s \gg \nu(K_x) = \nu^+(K_x)$ since ν is weakly outer-continuous, hence there exists some $G_x \ni x$ such that $s \geqslant \nu(G_x)$. Since ν is weakly inner-continuous, we deduce $s \geqslant \nu(G)$, where $G = \bigcup_{x \in B} G_x \supset B$, so that $s \geqslant \nu^+(B)$, a contradiction.

So we have proved that $\nu^+(B) = \bigoplus\{\nu(K) : K \in \mathscr{K}, K \subset B\}$, for all $B \in \mathscr{B}$. From this we deduce that $\nu^+(B) = \nu(B)$, i.e. ν is outer-continuous. This implies that $\nu(\uparrow K) = \nu^+(\uparrow K) = \nu^+(K) = \nu(K)$ for all $K \in \mathscr{K}$, and now inner-continuity of ν is clear. □

The following result improves [**7**, Corollary 3.12].

Corollary II-5.8. *Assume that L is a domain. Then, on a second-countable topological space, every L-valued weakly outer-continuous σ-maxitive measure is regular.*

Proof. Let E be second-countable and ν be an L-valued weakly outer-continuous σ-maxitive measure on $\mathscr{B}$. Since E is second-countable, there is some countable base $\mathscr{U}$ for the topology $\mathscr{G}$. To prove that ν is regular, we want to use Lemma II-5.7, thus we show that ν is weakly inner-continuous. So let $\mathscr{O}$ be a family of open subsets of E, and let $G = \bigcup \mathscr{O}$. We let $\mathscr{V} = \{V \in \mathscr{U} : \exists O \in \mathscr{O}, V \subset O\}$. Since $\mathscr{V}$ is countable with union G and ν is σ-maxitive, we deduce that $\nu(G) = \bigoplus \nu(\mathscr{V}) \leqslant \bigoplus \nu(\mathscr{O})$, and the proof is complete. □

Corollary II-5.9. *Assume that L is a domain. Let $\nu : \mathscr{G} \to L$ be a map such that $\nu(\bigcup \mathscr{O}) = \bigoplus \nu(\mathscr{O})$, for all directed families $\mathscr{O}$ of open subsets. Then ν is maxitive if and only if ν extends (uniquely) to a regular maxitive measure on $\mathscr{B}$.*

Proof. Suppose that the map ν is maxitive. The map ν^* defined by Equation (19) on 2^E restricts to an outer-continuous maxitive measure τ on $\mathscr{B}$, and to the completely maxitive map ν on $\mathscr{G}$. Applying Lemma II-5.7, we deduce that τ is regular. Uniqueness of this extension is straightforward. □

II-5.4. Smoothness. From now on, all (L-valued) maxitive measures are assumed to be defined on the Borel σ-algebra $\mathscr{B}$ of a topological space E. If $\mathscr{A}$ is a collection of elements of $\mathscr{B}$ closed under filtered intersections, the maxitive measure ν is $\mathscr{A}$*-smooth* if

$$\bigwedge_{j\in J}\nu(A_j) = \nu(\bigcap_{j\in J} A_j), \tag{21}$$

for every filtered family $(A_j)_{j\in J}$ of elements of $\mathscr{A}$.

An L-valued maxitive measure ν on $\mathscr{B}$ is called *saturated* if for all $K \in \mathscr{K}$ we have $\nu(K) = \nu(\uparrow K)$. Inner-continuous maxitive measures and weakly outer-continuous maxitive measures are always saturated, while weak inner-continuity does not imply saturation in general. Note however that saturation is always satisfied if the space E is T_1.

Variants of Propositions II-5.10 and II-5.13 below were formulated and proved in [**6**] in the case where E is a Hausdorff topological space and L is a continuous lattice, see also [**123**, Proposition 13]. Another variant of the following result is [**228**, Proposition 2.2(a)], which treats the case of real-valued capacities on non-Hausdorff spaces.

Proposition II-5.10. *Assume that L is a domain. Then, on a quasisober space, every L-valued weakly outer-continuous maxitive measure ν is $\mathscr{Q}$-smooth saturated. The converse statement holds in locally-compact quasisober spaces.*

Proof. Assume that E is quasisober, let ν be an L-valued weakly outer-continuous maxitive measure on $\mathscr{B}$, and let $(Q_j)_{j\in J}$ be a filtered family of compact saturated subsets of E. Recall that $Q = \bigcap_{j\in J} Q_j$ is compact saturated, since E is assumed quasisober. The set $\{\nu(Q_j) : j \in J\}$ admits $\nu(Q)$ as a lower-bound. Take another lower-bound ℓ, and let $G \in \mathscr{G}$ such that $G \supset Q$. By the Hofmann–Mislove theorem (Theorem II-5.1), there is some $j_0 \in J$ such that $G \supset Q_{j_0}$. Thus, $\nu(G) \geqslant \nu(Q_{j_0})$, so that $\nu(G) \geqslant \ell$, for all $G \supset Q$. Since ν is weakly outer-continuous, we deduce that $\nu(Q) \geqslant \ell$. We have shown that $\nu(Q)$ is the infimum of $\{\nu(Q_j) : j \in J\}$.

Now assume that E is locally-compact quasisober, and let ν be an L-valued $\mathscr{Q}$-smooth saturated maxitive measure on $\mathscr{B}$. If Q is a compact saturated subset, then by local compactness of E there exists a filtered family $(Q_j)_{j\in J}$ of compact saturated subsets with $\bigcap_{j\in J} Q_j = Q$ and $Q \subset Q_j^o$. Since ν is $\mathscr{Q}$-smooth, this implies that

$$\nu(Q) = \bigwedge_{G\in\mathscr{G}, G\supset Q} \nu(G),$$

i.e. $\nu(Q) = \nu^+(Q)$, for all $Q \in \mathscr{Q}$. Let us show that ν and ν^+ coincide on $\mathscr{K}$. If $K \in \mathscr{K}$, then $\nu(K) = \nu(\uparrow K)$ since ν is saturated. Also, because $G \supset\uparrow K$ if and only if $G \supset K$ for all open subsets G, we have $\nu^+(\uparrow K) = \nu^+(K)$. So this gives $\nu(K) = \nu(\uparrow K) = \nu^+(\uparrow K) = \nu^+(K)$, and we have shown that ν is weakly outer-continuous. □

Remark II-5.11. The first part of Proposition II-5.10 remains true for L-valued weakly outer-continuous monotone set functions.

II-5.5. Tightness. Tightness of maxitive measures can be defined by analogy with tightness of additive measures, so we say that an L-valued maxitive measure ν on $\mathscr{B}$ is *tight* if

$$\bigwedge_{K \in \mathscr{K}} \nu(E \backslash K) = 0.$$

The following lemma slightly extends [**114**, Theorem III-2.11], which states that every continuous semilattice is join-continuous.

Lemma II-5.12. *Assume that L is a domain. Let F be a filter of L and $t \in L$ such that, for all $f \in F$, $t \oplus f$ exists. Then $t \oplus \bigwedge F$ exists and satisfies $t \oplus \bigwedge F = \bigwedge(t \oplus F)$.*

Proof. The subset $t \oplus F$ is filtered, hence has an infimum. Suppose that $\bigwedge(t \oplus F)$ is not the least upper-bound of $\{t, \bigwedge F\}$. Then there exists some upper-bound u of $\{t, \bigwedge F\}$ such that $u \not\geqslant \bigwedge(t \oplus F)$. Since L is continuous, there is some $s \gg u$ such that $s \not\geqslant \bigwedge(t \oplus F)$. Remembering that $u \geqslant \bigwedge F$, there is some $f \in F$ such that $s \geqslant f$. Also, $s \geqslant u \geqslant t$, so that $s \geqslant t \oplus f \geqslant \bigwedge(t \oplus F)$, a contradiction. □

A maxitive measure is *$\mathscr{Q}\mathscr{F}$-smooth* if it is $\mathscr{Q}$-smooth and $\mathscr{F}$-smooth. The second part of the following result was proved by Puhalskii [**246**, Theorem 1.7.8] in the case where $L = \overline{\mathbb{R}}_+$.

Proposition II-5.13. *Assume that L is a domain. Then, on a quasisober space, every L-valued tight weakly outer-continuous maxitive measure is $\mathscr{Q}\mathscr{F}$-smooth saturated. The converse statement holds in locally-compact quasisober spaces and in completely metrizable spaces.*

Proof. Let E be quasisober, let ν be an L-valued tight weakly outer-continuous maxitive measure on $\mathscr{B}$, and let $(F_j)_{j \in J}$ be a filtered family of closed subsets of E. Fix some compact Borel subset K, and let $F = \bigcap_{j \in J} F_j$. Then $F_j \cap K$ and $F \cap K$ are compact, hence $\uparrow(F_j \cap K)$ and $\uparrow(F \cap K)$ are compact saturated. Let us show that

$$\bigcap_{j \in J} \uparrow(F_j \cap K) =\uparrow(F \cap K). \tag{22}$$

The inclusion $\supset$ is clear. For the reverse inclusion, let $x \in E$ such that $x \notin\uparrow(F \cap K)$. Then there is some open subset G containing $F \cap K$ such that $x \notin G$. As a consequence, the compact subset K is included in the union of the directed family $(G \cup (E \backslash F_j))_{j \in J}$, so there exists some $j_0 \in J$

such that $K \subset G \cup (E \backslash F_{j_0})$. This rewrites as $F_{j_0} \cap K \subset G$, so that $\uparrow(F_{j_0} \cap K) \subset G$. Hence, $x \notin \uparrow(F_{j_0} \cap K)$, and Equation (22) is proved.

By Proposition II-5.10, ν is $\mathscr{Q}$-smooth, so

$$\bigwedge_{j \in J} \nu(\uparrow(F_j \cap K)) = \nu(\uparrow(F \cap K)).$$

Since ν is weakly outer-continuous, ν is saturated, hence $\bigwedge_j \nu(F_j \cap K) = \nu(F \cap K)$. Now pick some lower-bound ℓ of the set $\{\nu(F_j) : j \in J\}$. Thanks to the join-continuity of L, $\ell \leqslant \bigwedge_j (\nu(F_j \cap K) \oplus \nu(E \backslash K)) = \nu(F \cap K) \oplus \nu(E \backslash K)$. The tightness of ν and the join-continuity of L imply $\ell \leqslant \nu(F)$, and the result is proved.

For the converse statement, first assume that E is locally-compact quasisober, and let ν be an L-valued $\mathscr{Q}\mathscr{F}$-smooth saturated maxitive measure on $\mathscr{B}$. Then ν is weakly outer-continuous by Proposition II-5.10. Moreover, the collection $\{E \backslash K^o : K \in \mathscr{K}\}$ has empty intersection since E is locally-compact, is filtered, and is made of closed subsets. Since ν is $\mathscr{F}$-smooth, this implies $\bigwedge_{K \in \mathscr{K}} \nu(E \backslash K^o) = 0$. If $t \gg 0$, this gives some $K \in \mathscr{K}$ with $t \geqslant \nu(E \backslash K^o)$, so that $t \geqslant \nu(E \backslash K)$. Since L is continuous, we conclude that ν is tight.

Now if E is a completely metrizable space, the second part of the proof of Proposition II-5.10 still applies to show that an L-valued $\mathscr{F}$-smooth maxitive measure ν is weakly outer-continuous, for every compact subset K is the filtered intersection of some family $(F_j)_j$ of closed subsets with $K \subset F_j^o$. To see why this holds, define $F_j = \bigcap_{k=1}^{j} \overline{G_k}$ where, for all $k \geqslant 1$, G_k is a finite union of open balls of radius $1/k$ covering K. For tightness, one can follow Puhalskii's proof [**246**, Theorem 1.7.8] (although this author considered only $\overline{\mathbb{R}}_+$-valued maxitive measures). ☐

Problem II-5.14. Both completely metrizable spaces and locally-compact quasisober spaces are Baire spaces (see [**15**, Theorem 3.47] and [**114**, Corollary I-3.40.9]). Does the previous result hold for Baire spaces?

Proposition II-5.15. *Assume that L is a domain. Then, on a Polish space, every L-valued $\mathscr{F}$-smooth σ-maxitive measure is tight regular.*

Proof. Assume that E is a Polish space, and let ν be an L-valued $\mathscr{F}$-smooth σ-maxitive measure on $\mathscr{B}$. Since E is separable metrizable, every open subset is Lindelöf, hence the restriction of ν to $\mathscr{G}$ is completely maxitive, i.e. ν is weakly inner-continuous. Now the result follows from Proposition II-5.13. ☐

Remark II-5.16. For the case $L = \overline{\mathbb{R}}_+$, one could prove Proposition II-5.15 with the help of the Choquet capacitability theorem (see e.g. Molchanov [**208**, Theorem E.9] or Aliprantis and Border [**15**, Theorem 12.40]).

Corollary II-5.17. *Assume that L is a domain. Then, on a σ-compact and metrizable space, every L-valued $\mathscr{K}$-smooth σ-maxitive measure is regular.*

Proof. Let E be σ-compact and metrizable, and let ν be an L-valued $\mathscr{K}$-smooth σ-maxitive measure. Since E is σ-compact, there is a sequence $(K_n)_n$ of compact subsets such that $E = \bigcup_n K_n$. Each of these K_n is then a Polish space because E is metrizable. By Proposition II-5.15, this implies that the restriction ν_n of ν to the Borel σ-algebra of K_n is (tight) regular, hence completely maxitive. As a consequence, if $B \in \mathscr{B}$, then $\nu(B) = \bigoplus_n \nu(B \cap K_n) = \bigoplus_n \nu_n(B \cap K_n) = \bigoplus_n \bigoplus_{x \in B \cap K_n} \nu_n(\{x\}) = \bigoplus_n \bigoplus_{x \in B \cap K_n} \nu(\{x\}) = \bigoplus_{x \in B} \nu(\{x\})$, so ν is completely maxitive.

Since complete maxitivity implies weak inner-continuity by Lemma II-5.5, it suffices to prove that ν is weakly outer-continuous in order to conclude that ν is regular. By Lemma II-5.6, we only need to show that $\nu(\{x\}) = \nu^+(\{x\})$ for all x. So let $s \gg \nu(\{x\})$. Then $G := \{y \in E : s \gg \nu(\{y\})\}$ contains x. We prove that G is open, i.e. that $F = E \backslash G$ is closed. Let (y_n) be a sequence in F with $y_n \to y$. If K_n is the topological closure of $\{y_{n'} : n' \geqslant n\}$, then K_n is compact, and $\bigcap_n K_n = \{y\}$ since E is Hausdorff. Since ν is $\mathscr{Q}$-smooth (i.e. $\mathscr{K}$-smooth), this gives $\bigwedge_n \nu(K_n) = \nu(\{y\})$. If $y \notin F$, then $s \gg \bigwedge_n \nu(K_n)$, hence there is some n_0 such that $s \gg \nu(K_{n_0})$. Therefore, $s \gg \nu(\{y_{n_0}\})$, i.e. $y_{n_0} \notin F$, a contradiction. Thus, $y \in F$. Since E is metrizable, it is first-countable, so this proves that F is closed. So G is open, contains x, and $s \geqslant \nu(G)$ because ν is completely maxitive. We deduce that $s \geqslant \nu^+(\{x\})$ and, with the continuity of L, that $\nu(\{x\}) \geqslant \nu^+(\{x\})$. □

Remark II-5.18. Part of the preceding corollary was proved by Miranda et al. [**204**, Proposition 2.6] in the case where $L = \overline{\mathbb{R}}_+$. It also uses ideas from [**246**, Lemma 1.7.4].

II-5.6. Cardinal densities of maxitive measures. In this section we prove new results giving equivalent conditions for a maxitive measure ν on $\mathscr{B}$ to have a *cardinal density*, that is a map $c : E \to L$ such that

$$\nu(B) = \bigoplus_{x \in B} c(x),$$

for all $B \in \mathscr{B}$. As a special case, consider e.g. a finite set E with the discrete topology. Then ν admits a cardinal density defined by $c(x) = \nu(\{x\})$, since $B = \bigcup_{x \in B} \{x\}$, where the union runs over a finite set. In the general case, this reasoning may fail, for we may have $\nu(\{x\}) = 0$ for all $x \in E$, even with a nonzero ν, but it is tempting to consider $c^+(x) := \nu^+([x])$ instead, where ν^+ is defined in Example II-5.3 (see also Corollary II-4.4). This idea, which appeared in [**122, 123**] and [**7**], is effective and leads to Theorem II-5.20.

A map $c : E \to L$ is *upper-semicontinuous* (or *usc* for short) if, for all $t \in L$, the subset $\{t \gg c\}$ is open. We refer the reader to Penot and Théra [**238**], Beer [**29**], van Gool [**285**], Gerritse [**113**], Akian and Singer [**14**] for a wide treatment of upper-semicontinuity of poset-valued and domain-valued maps. Note that, if L is a filtered-complete poset and ν is an L-valued maxitive map on $\mathscr{B}$, then the map c^+ defined by $c^+(x) = \nu^+([x])$ is usc.

A map $c : E \to L$ is *upper-compact* if, for every $t \gg 0$, $\{t \not\gg c\}$ is a compact subset of E.

Proposition II-5.19. *Assume that L is a domain, and let ν be an L-valued maxitive measure on $\mathscr{B}$. If ν is tight and outer-continuous, then $c^+ : x \mapsto \nu^+([x])$ is upper-compact. Conversely, if ν is weakly inner-continuous and c^+ is upper-compact, then ν is tight.*

Proof. Assume that ν is tight and outer-continuous, and let $t \gg 0$. Since $\{\nu(E\backslash K) : K \in \mathscr{K}\}$ is filtered with an infimum equal to 0, the interpolation property implies that there is some $K \in \mathscr{K}$ such that $t \gg \nu(E\backslash K)$. Since ν is outer-continuous, we obtain $t \gg \bigoplus_{x \notin K} c^+(x)$. This shows that $\{t \not\gg c^+\}$ is a subset of K. Since c^+ is usc, $\{t \not\gg c^+\}$ is also closed, hence compact.

Conversely, assume that ν is weakly inner-continuous and that c^+ is upper-compact. Let K_t denote the compact closed subset $\{t \not\gg c^+\}$. Then

$$\bigwedge_{K \in \mathscr{K}} \nu(E\backslash K) \leqslant \bigwedge_{t \gg 0} \nu(E\backslash K_t).$$

Since ν is weakly inner-continuous and $G_t = E\backslash K_t$ is open for all $t \gg 0$, we have by Lemma II-5.6

$$\nu(G_t) = \bigoplus_{K \in \mathscr{K}, K \subset G_t} \nu^+(K) = \bigoplus_{K \in \mathscr{K}, K \subset G_t} \bigoplus_{x \in K} c^+(x),$$

thus $\nu(G_t) = \bigoplus_{x \in G_t} c^+(x)$, so that

$$\bigwedge_{K \in \mathscr{K}} \nu(E\backslash K) \leqslant \bigwedge_{t \gg 0} \bigoplus_{x \in E, t \gg c^+(x)} c^+(x) \leqslant \bigwedge_{t \gg 0} t = 0,$$

so ν is tight. $\square$

The following theorem summarizes many of the above results and highlights the relation between the existence of a density, regularity, and complete maxitivity. Part of it is due to [**7**, Proposition 3.15] and [**243**, Theorem 3.1]. See also Norberg [**225**], Vervaat [**289**]. We also refer the reader to O'Brien and Watson [**233**, Claim 2] and Miranda et al. [**204**, Proposition 2.3, Theorem 2.4] for the case $L = \overline{\mathbb{R}}_+$ and the link with the Choquet capacitability theorem.

Theorem II-5.20. *Assume that L is a domain and E is a quasisober space. Let ν be an L-valued maxitive measure on $\mathscr{B}$. Then ν has a cardinal density if and only if ν is completely maxitive. Also, consider the following assertions:*

(1) *ν is regular,*
(2) *ν has a usc cardinal density,*
(3) *ν is outer-continuous and completely maxitive,*
(4) *ν is weakly outer-continuous and weakly inner-continuous,*
(5) *ν is weakly outer-continuous and σ-maxitive,*
(6) *ν is weakly outer-continuous,*
(7) *ν is $\mathscr{Q}$-smooth and saturated,*

(8) *ν is $\mathscr{Q}$-smooth, weakly inner-continuous, and saturated,*
(9) *ν is $\mathscr{Q}$-smooth, σ-maxitive, and saturated.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Leftarrow$ (8). Moreover,

- *if E is second-countable, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5);*
- *if E is locally-compact, then (8) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) and (6) $\Leftrightarrow$ (7);*
- *if E is σ-compact metrizable, then (9) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5);*
- *if E is locally-compact Polish, then (8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) and (6) $\Leftrightarrow$ (7).*

Proof. If ν is completely maxitive, then $\nu(B) = \nu(\bigcup_{x\in B}[x]) = \bigoplus_{x\in B} c(x)$ by Lemma II-5.2, where $c(x) = \nu([x])$, hence ν has a cardinal density. The reverse assertion is straightforward.

(1) $\Rightarrow$ (2) Assume that ν is regular. Then $\nu(\uparrow K) = \bigoplus_{x\in K} c^+(x)$ for all $K \in \mathscr{K}$ by Lemma II-5.6, where $c^+(x) = \nu^+([x])$. By inner-continuity of ν, $\nu(B) = \bigoplus_{K\in\mathscr{K},K\subset B} \nu(\uparrow K) = \bigoplus_{K\in\mathscr{K},K\subset B} \bigoplus_{x\in K} c^+(x) = \bigoplus_{x\in B} c^+(x)$, for all Borel subsets B, i.e. ν has a usc cardinal density.

(2) $\Rightarrow$ (1) Assume that ν has a usc cardinal density c. Then ν is weakly inner-continuous. Let us show that, if $K \in \mathscr{K}$, then $\nu(K) = \nu^+(K)$. So let $u \gg \nu(K)$. Since $\nu(K) = \bigoplus_{x\in K} c(x)$, we have $K \subset G$ where $G = \{u \gg c\}$ is open. Moreover, $\nu(G) = \bigoplus_{x\in G} c(x) \leqslant u$. Therefore, $\nu(K) = \nu^+(K)$ by continuity of L. This implies that ν is regular by Lemma II-5.7.

So now the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) are clear (use Lemma II-5.5 and Lemma II-5.7). Using Proposition II-5.10, it is also straightforward that (3) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Leftarrow$ (8).

If E is second-countable, use Corollary II-5.8. If E is locally-compact, use Proposition II-5.10. If E is σ-compact metrizable, use Corollary II-5.17. If E is locally-compact Polish, use Proposition II-5.10 and Corollary II-5.17. □

Corollary II-5.21. *Assume that L is a domain and E is a quasisober space. If ν is a regular maxitive measure on E, then $c^+(x) = \nu([x])$ for all $x \in E$, and c^+ is the maximal (usc) cardinal density of ν.*

Theorem II-5.22. *Assume that L is a domain and E is a quasisober space. Let ν be an L-valued maxitive measure on $\mathscr{B}$. Also, consider the following assertions:*

(1) *ν is tight regular,*
(2) *ν has an upper-compact usc cardinal density,*
(3) *ν is tight weakly outer-continuous,*
(4) *ν is $\mathscr{Q}\mathscr{F}$-smooth and saturated,*
(5) *ν is $\mathscr{Q}\mathscr{F}$-smooth, weakly inner-continuous, and saturated,*
(6) *ν is $\mathscr{Q}\mathscr{F}$-smooth, σ-maxitive, and saturated,*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftarrow$ (5). Moreover,

- *if E is locally-compact, then (5) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4);*

- *if E is completely metrizable, then (3) $\Leftrightarrow$ (4);*
- *if E is Polish, then (6) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2);*
- *if E is locally-compact Polish, then (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4).*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is a consequence of Proposition II-5.19. For the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftarrow$ (5), use Proposition II-5.13.

If E is completely metrizable or locally-compact, use Proposition II-5.13. If E is Polish, use Proposition II-5.15. $\square$

II-6. Regularity of optimal measures on metrizable spaces

Let E be a topological space with Borel σ-algebra $\mathscr{B}$. An L-valued maxitive measure ν on $\mathscr{B}$ is *continuous from above* if $\nu(B) = \bigwedge_n \nu(B_n)$, for all $B_1 \supset B_2 \supset \ldots \in \mathscr{B}$ such that $B = \bigcap_n B_n$, and *continuous from below* if $\nu(B) = \bigoplus_n \nu(B_n)$, for all $B_1 \subset B_2 \subset \ldots \in \mathscr{B}$ such that $B = \bigcup_n B_n$. An *optimal measure* is a maxitive measure that is both continuous from above and continuous from below. The following result generalizes the Murofushi–Sugeno–Agebko theorem (see [Chapter I, Proposition 8.2]).

Proposition II-6.1. *Assume that L is a domain with a top. An L-valued maxitive measure ν on $\mathscr{B}$ is an optimal measure if and only if it is a continuous from above.*

Proof. Let ν be an L-valued continuous from above maxitive measure on $\mathscr{B}$, and let us show that ν is continuous from below. So let $B_1 \subset B_2 \subset \ldots \in \mathscr{B}$ and $B = \bigcup_n B_n$, let u be an upper bound of $\{\nu(B_n) : n \geqslant 1\}$, and suppose that $\nu(B) \nleqslant u$. Since L is a domain with a top, there exists a map $\varphi : L \to [0, 1]$ that preserves filtered infima and arbitrary existing suprema such that $\varphi(\nu(B)) = 1$ and $\varphi(u) = 0$ (see e.g. Gierz et al. [**114**, Proposition IV-3.1]). But the map $B' \mapsto \varphi(\nu(B'))$ is clearly a $[0, 1]$-valued optimal measure, so by the Murofushi–Sugeno–Agebko theorem (see [Chapter I, Proposition 8.2]), we have $1 = \varphi(\nu(B)) = \bigoplus_n \varphi(\nu(B_n)) \leqslant \varphi(u) = 0$, a contradiction. $\square$

Riečanová [**255**] studied the regularity of certain S-valued set functions, for some conditionally-complete ordered semigroup S satisfying a series of conditions, among which the separation of points by continuous functionals. In the following lines we closely follow her approach, although we do not use directly her results, for our approach better matches the special case of L-valued optimal measures. In particular, L is not assumed to be a semigroup, nor to be conditionally-complete. Contrarily to Riečanová, we do not examine the case of optimal measures defined on the collection of Baire (rather than Borel) subsets of a metrizable space, but we believe that this could be done with little additional effort.

Proposition II-6.2. *Assume that L is a domain with a top. Then, on a metrizable space, every L-valued optimal measure ν satisfies*

$$\nu(B) = \bigwedge_{G \in \mathscr{G}, G \supset B} \nu(G) = \bigoplus_{F \in \mathscr{F}, F \subset B} \nu(F),$$

for all $B \in \mathscr{B}$.

Proof. Let E be a metrizable space and d be a metric generating the topology. Let $\varphi : L \to [0, 1]$ be a map preserving filtered infima and arbitrary existing suprema, and let ν_φ be the map defined on $\mathscr{B}$ by $\nu_\varphi(B) = \varphi(\nu(B))$. The properties of φ imply that ν_φ is an optimal measure. Let $\mathscr{A}$ be the collection of all $B \in \mathscr{B}$ such that $\nu_\varphi(G\backslash F) \leqslant 1/2$, for some open subset G and closed subset F such that $G \supset B \supset F$. Let us show first that $\mathscr{A}$ contains all open subsets, so let B be open. Let $F_n = \{x \in E : d(x, E\backslash B) \geqslant n^{-1}\}$. Then $(F_n)_{n\geqslant 1}$ is a nondecreasing family of closed subsets whose union is B. Since ν_φ is an optimal measure, $\nu_\varphi(B\backslash F_n)$ tends to 0 when $n \uparrow \infty$. Thus, we can find some closed subset $F \subset B$ with $\nu_\varphi(B\backslash F) \leqslant 1/2$, and this proves that $B \in \mathscr{A}$.

We now show that $\mathscr{A}$ is a σ-algebra. Clearly, $B \in \mathscr{A}$ implies $E\backslash B \in \mathscr{A}$. Let $(B_n)_{n\geqslant 1}$ be a family of elements of $\mathscr{A}$. We prove that $B = \bigcup_n B_n \in \mathscr{A}$. For all n, there are some $G_n \supset B_n \supset F_n$ satisfying $\nu_\varphi(G_n\backslash F_n) \leqslant 1/2$. If $G = \bigcup_n G_n$ and $F = \bigcup_n F_n$, then $G \supset B \supset F$ and $\nu_\varphi(G\backslash F) \leqslant 1/2$. However, F is not closed in general. So let H_n denote the closed subset $\bigcup_{k=1}^n F_k$. As above, $(H_n)_{n\geqslant 1}$ is a nondecreasing family of closed subsets whose union is F, so we can find some closed subset $H \subset F$ with $\nu_\varphi(F\backslash H) \leqslant 1/2$, hence $\nu_\varphi(G\backslash H) \leqslant 1/2$. Consequently, $\mathscr{A}$ coincides with the Borel σ-algebra $\mathscr{B}$.

Assume that, for some $B \in \mathscr{B}$, $\nu^+(B)$ is not the least upper-bound of $\{\nu(F) : F \in \mathscr{F}, F \subset B\}$. Hence, there exists some upper-bound $u \in L$ of $\{\nu(F) : F \in \mathscr{F}, F \subset B\}$ such that $\nu^+(B) \nleqslant u$. Since L is a domain with a top, there exists some $\varphi : L \to [0, 1]$ that preserves filtered infima and arbitrary existing suprema such that $\varphi(\nu^+(B)) = 1$ and $\varphi(u) = 0$ (see e.g. Gierz et al. [**114**, Proposition IV-3.1]). The previous point gives the existence of some $G \supset B \supset F$ such that $\nu_\varphi(G\backslash F) \leqslant 1/2$. Moreover, $\varphi(\nu^+(B)) = 1$ implies $\nu_\varphi(G) = 1$, and $\varphi(u) = 0$ implies $\nu_\varphi(F) = 0$. But $1 = \nu_\varphi(G) = \nu_\varphi(G\backslash F) \oplus \nu_\varphi(F) \leqslant 1/2$, a contradiction. □

Corollary II-6.3. *Assume that L is a domain with a top. Then, on a separable metrizable space, every L-valued optimal measure is regular.*

Proof. Let E be a separable metrizable space, and let ν be an L-valued optimal measure on $\mathscr{B}$. Then ν is outer-continuous by Proposition II-6.2. As a separable metrizable space, E is second-countable, so ν is also inner-continuous by Corollary II-5.8. □

Remark II-6.4. The previous result was proved by Murofushi and Sugeno [**216**, Theorem 4.1] for $\overline{\mathbb{R}}_+$-valued optimal measures.

Remark II-6.5. Recall that a topological space E is separable metrizable in any of the following cases:

(1) if E is second-countable regular Hausdorff (in particular if E is second-countable locally-compact Hausdorff);
(2) if E is σ-compact and metrizable;
(3) if E is Polish (this results from the definition of a Polish space!).

Remark II-6.6. Recall for comparison that a σ-additive measure defined on a second-countable locally-compact Hausdorff space is always regular, whenever it takes finite values on compact subsets (a hypothesis that is not needed in Corollary II-6.3).

Part of the following result is included in Proposition II-5.15.

Proposition II-6.7. *Assume that L is a domain with a top. Then, on a Polish space or on a σ-compact and metrizable space, every L-valued optimal measure is tight regular.*

The proof is inspired by that of [**246**, Theorem 1.7.8].

Proof. We only have to prove that ν is tight. First assume that E is a Polish space and let ν be an L-valued optimal measure on $\mathscr{B}$. Since E is separable, there is some sequence (x_n) dense in E. Let $\epsilon \gg 0$. Let $F_{n,p} = B_{1,p} \cup \ldots \cup B_{n,p}$, where $B_{n,p}$ is the closed ball of radius $1/p$ and center x_n. Then, for all p, $E = \bigcup_n F_{n,p}$. Since ν is optimal, there is some n_p such that $\epsilon \geqslant \nu(E \backslash F_{n_p,p})$. Let K_ϵ denote the subset $\bigcap_p F_{n_p,p}$. For all $\alpha > 0$, K_ϵ can be covered by a finite number of balls of radius at most α, i.e. K_ϵ is totally bounded. Since E is completely metrizable, K_ϵ is compact. Moreover, $\epsilon \geqslant \nu(E \backslash K_\epsilon)$, for all $\epsilon \gg 0$. Thus, ν is tight.

For the case where E is σ-compact and metrizable, a similar proof can be given, for one can write $E = \bigcup_n F_{n,p}$, with $F_{n,p} = F_{n,1}$ compact. $\square$

II-7. Decomposition of maxitive measures

In [**243**], we developed part of the following material in a non-topological framework. Here E is again a quasisober topological space, and $\mathscr{B}$ denotes its collection of Borel subsets. A poset is a *lattice* if every nonempty finite subset has a supremum and an infimum. A lattice is *distributive* if finite infima distribute over finite suprema, and *conditionally-complete* if every nonempty upper-bounded subset has a supremum.

Definition II-7.1. Assume that L is a continuous conditionally-complete lattice. Let ν be an L-valued maxitive measure on $\mathscr{B}$. Then the *regular part* of ν is the map defined on $\mathscr{B}$ by

$$\lfloor \nu \rfloor(B) = \bigoplus_{K \in \mathscr{K}, K \subset B} \nu^+(K).$$

The following proposition confirms that the terminology is appropriate.

Proposition II-7.2. *Assume that L is a continuous conditionally-complete lattice. Let ν be an L-valued maxitive measure on $\mathscr{B}$. Then the regular part of ν is a regular maxitive measure on $\mathscr{B}$, with density $c^+ : x \mapsto \nu^+([x])$. Moreover, $\lfloor\lfloor\nu\rfloor\rfloor = \lfloor\nu\rfloor$.*

Proof. By Lemma II-5.6, $\nu^+(K) = \bigoplus_{x \in K} c^+(x)$ for all compact subsets K of E, so we have $\lfloor\nu\rfloor(B) = \bigoplus_{x \in B} c^+(x)$, for all $B \in \mathscr{B}$. This shows that $\lfloor\nu\rfloor$ has a usc cardinal density, hence is regular by Theorem II-5.20. Outer-continuity of $\lfloor\nu\rfloor$ implies that $\lfloor\nu\rfloor^+(K) = \lfloor\nu\rfloor(K) = \nu^+(K)$, for all $K \in \mathscr{K}$, so $\lfloor\lfloor\nu\rfloor\rfloor = \lfloor\nu\rfloor$. □

The following theorem states the existence of a *singular part* $\perp\nu$ of a maxitive measure ν.

Theorem II-7.3. *Assume that L is a continuous conditionally-complete distributive lattice. Let ν be an L-valued maxitive measure on $\mathscr{B}$. Then there exists a smallest maxitive measure $\perp\nu$ on $\mathscr{B}$, called the singular part of ν, such that the decomposition*

$$\nu^+ = \lfloor\nu\rfloor \oplus \perp\nu \tag{23}$$

holds. Moreover, the singular part of the regular part of ν equals 0, i.e. $\perp\lfloor\nu\rfloor = 0$.

Proof. We give a constructive proof for the existence of $\perp\nu$. Let $\perp\nu(B) = \bigwedge\{t \in L : B \in \mathscr{I}_t\}$, where

$$\mathscr{I}_t := \{B \in \mathscr{B} : \forall A \in \mathscr{B}, A \subset B \Rightarrow \nu^+(A) \leqslant \lfloor\nu\rfloor(A) \oplus t\}.$$

Then $(\mathscr{I}_t)_{t \in L}$ is a nondecreasing family of ideals of $\mathscr{B}$, and distributivity of L implies that $\{t \in L : B \in \mathscr{I}_t\}$ is a filter, for every $B \in \mathscr{B}$. From Proposition II-4.3, we deduce that $\perp\nu$ is a maxitive measure. The fact that $\perp\nu$ is the smallest maxitive measure satisfying Equation (23) is straightforward.

Since $B \in \mathscr{I}_t$ for $t = \nu^+(B)$, we have $\nu^+(B) \geqslant \perp\nu(B)$, thus $\nu^+ \geqslant \lfloor\nu\rfloor \oplus \perp\nu$. For the reverse inequality, one may use the fact that continuity implies join-continuity (see Lemma II-5.12).

The fact that $\perp\lfloor\nu\rfloor = 0$ follows from the definition of the singular part and the fact that $\lfloor\lfloor\nu\rfloor\rfloor = \lfloor\nu\rfloor$. □

As a consequence of the previous result we have the following corollaries. The proof of the first of them is clear.

Corollary II-7.4 (Regularity of the regular part)**.** *Under the conditions of Theorem II-7.3, the following are equivalent if ν is outer-continuous:*

(1) *ν is the regular part of some L-valued maxitive measure,*
(2) *the singular part of ν is identically 0,*
(3) *ν is regular.*

Corollary II-7.5 (Singularity of the singular part)**.** *Under the conditions of Theorem II-7.3, the following are equivalent if ν is outer-continuous:*

(1) *ν is the singular part of some L-valued maxitive measure,*

(2) *the regular part of* ν *is identically* 0,
(3) ν *is singular, in the sense that* $\nu(K) = 0$ *for all* $K \in \mathscr{K}$.

Proof. It is straightforward that (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1). Let us show that (1) $\Rightarrow$ (2), so assume that $\nu = \perp\tau$, for some L-valued maxitive measure τ. Note that $\perp(\perp\tau) \geqslant \perp\tau$, for $\lfloor\perp\tau\rfloor$ is regular and less than τ^+, hence is less than $\lfloor\tau\rfloor$. Thus, $\tau^+ = \lfloor\tau\rfloor \oplus \perp\tau \leqslant \lfloor\tau\rfloor \oplus (\perp\tau)^+ = \lfloor\tau\rfloor \oplus \lfloor\perp\tau\rfloor \oplus \perp(\perp\tau) = \lfloor\tau\rfloor \oplus \perp(\perp\tau) \leqslant \tau^+$. This gives $\tau^+ = \lfloor\tau\rfloor \oplus \perp(\perp\tau)$, hence $\perp(\perp\tau) \geqslant \perp\tau$. Now $\nu = \nu^+ \geqslant \perp\nu = \perp(\perp\tau) \geqslant \perp\tau = \nu$, so that $\nu = \perp\nu$. If $\mathscr{I}_t$ denotes the ideal of $\mathscr{B}$ defined in the proof of Theorem II-7.3, then $[x] \in \mathscr{I}_t$ for all $t \in L$, so that $\nu([x]) = \perp\nu([x]) = 0$, for all $x \in E$. Since ν is outer-continuous, $\nu^+([x]) = \nu([x]) = 0$ for all $x \in E$, so $\lfloor\nu\rfloor(B) = \bigoplus_{x \in B} \nu^+([x]) = 0$, for all $B \in \mathscr{B}$. $\square$

It is worth summarizing calculus rules for operators $\lfloor\cdot\rfloor$, $\perp\cdot$, and $(\cdot)^+$:

Proposition II-7.6. *Assume that* L *is a continuous conditionally-complete distributive lattice, and let* ν, τ *be* L*-valued maxitive measures on* $\mathscr{B}$. *Then the following properties hold:*

(1) $(\nu^+)^+ = \nu^+$,
(2) $\nu^+ = \lfloor\nu\rfloor \oplus \perp\nu$,
(3) $\nu^+ = \lfloor\nu\rfloor \Leftrightarrow \perp\nu = 0$,
(4) $\nu^+ = \perp\nu \Leftrightarrow \lfloor\nu\rfloor = 0$,
(5) $\lfloor\lfloor\nu\rfloor\rfloor = \lfloor\nu\rfloor$,
(6) $\lfloor\nu \oplus \tau\rfloor = \lfloor\nu\rfloor \oplus \lfloor\tau\rfloor$,
(7) $(\nu \oplus \tau)^+ = \nu^+ \oplus \tau^+$,
(8) $\perp(\nu \oplus \tau) \leqslant \perp\nu \oplus \perp\tau$,
(9) $\perp\lfloor\nu\rfloor = 0$,
(10) $\lfloor\nu^+\rfloor = \lfloor\nu\rfloor^+ = \lfloor\nu\rfloor$,
(11) $\perp(\nu^+) \leqslant (\perp\nu)^+$.

Sketch of the proof. Assertions (2), (5) and (9) have already been shown. Assertions (1) and (10) are straightforward. Equivalence (3), resp. (4), is a consequence of Corollary II-7.4, resp. Corollary II-7.5. Identity (7) follows from the continuity of L and the fact that $\mathscr{G}$ is closed under finite intersections. Identity (6) is then an easy consequence. For Inequalities (8) and (11), use the general fact that $\perp\nu$ is the smallest maxitive measure such that $\nu^+ = \lfloor\nu\rfloor \oplus \perp\nu$. $\square$

II-8. Conclusion and perspectives

It would be interesting to reformulate the results of this work in terms of Baire subsets rather than Borel subsets.

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CHAPTER III

What is the role of continuity in continuous linear forms representation?

ABSTRACT. The recent extensions of domain theory have proved particularly efficient to study lattice-valued maxitive measures, when the target lattice is continuous. Maxitive measures are defined analogously to classical measures with the supremum operation in place of the addition. Building further on the links between domain theory and idempotent analysis highlighted by Lawson (2004), we investigate the concept of domain-valued *linear forms* on an idempotent (semi)module. In addition to proving representation theorems for continuous linear forms, we address two applications: the idempotent Radon–Nikodym theorem and the idempotent Riesz representation theorem. To unify similar results from different mathematical areas, our analysis is carried out in the general Z framework of domain theory.

III-1. Résumé en français

Les mesures maxitives sont formellement définies comme les mesures additives classiques, l'opération maximum $\oplus$ venant en remplacement de l'addition $+$. Ces mesures ont été introduites par Shilkret [**271**], puis redécouvertes à plusieurs reprises. Ceci explique que des notions et des résultats similaires apparaissent dans la littérature ; nous avons tenté de les comprendre et de les unifier dans le chapitre I.

Le livre de Maslov [**196**], dans lequel sont considérées des mesures maxitives à valeurs dans un semi-anneau ordonné, atteste de liens forts entre analyse idempotente et théorie des ensembles ordonnées. Des développements similaires ont été entrepris en théorie des ensembles flous, où les mesures de possibilité à valeurs dans $[0, 1]$ ont été peu à peu remplacées par celles à valeurs dans un treillis (cf. Greco [**117**], Liu et Zhang [**183**], de Cooman et al. [**74**], Kramosil [**159**]). Plus récemment, les *treillis continus* et les *domaines* se sont révélés très puissants pour l'étude des mesures maxitives à valeurs dans un treillis ; les travaux sur ce sujet sont ceux de Heckmann et Huth [**122, 123**], intéressés par la théorie des ensembles flous, la théorie des catégories et les treillis continus, et d'Akian [**7**] sur l'analyse idempotente et les grandes déviations de processus aléatoires. Des connexions entre analyse idempotente et treillis continus apparaissent également dans les travaux d'Akian et Singer [**14**], et ont été revus par Lawson [**172**]. Cf. aussi les travaux précurseurs de Norberg [**226, 227**] sur les variables aléatoires à valeurs dans un domaine et l'utilisation des (semi)treillis continus en théorie des ensembles aléatoires.

Le chapitre II avait justement pour objet de considérer des mesures maxitives à valeurs dans un domaine plutôt que dans $\overline{\mathbb{R}}_+$, et par là de renforcer les liens entre analyse idempotente et théorie des domaines. Nous souhaitons dans le présent chapitre consolider encore ces liens et élargir notre approche en étudiant les formes linéaires définies sur un module dont le semi-anneau de base est un semicorps idempotent.

Les travaux qui suivent visent également à résoudre le paradoxe suivant. Soit ν une mesure complètement maxitive définie sur les ouverts $\mathscr{G}(E)$ d'un espace topologique E, et à valeurs dans un treillis complet $\mathbb{k}$. Nous savons depuis Heckmann et Huth [**122, 123**] et Akian [**7**] que si $\mathbb{k}$ est un treillis continu (donc un domaine), alors ν admet une densité cardinale, i.e. s'écrit sous la forme

$$\nu(\cdot) = \bigoplus_{x \in \cdot} c^+(x), \tag{24}$$

pour une certaine application $c^+ : E \to \mathbb{k}$. Cf. aussi [Chapitre II, Corollaire 5.9]. En fait, Heckmann et Huth ont prouvé un résultat plus fort que cela puisqu'ils ont *caractérisé* la continuité de $\mathbb{k}$ en ces termes : si $\mathbb{k}$ est un treillis complet fixé, alors il est continu *si et seulement si*, pour tout espace topologique E, toute mesure complètement maxitive $\nu : \mathscr{G}(E) \to \mathbb{k}$ admet une densité cardinale [**123**, Théorème 5].

Ce résultat scelle donc semble-t-il l'importance de l'hypothèse de continuité sur $\mathbb{k}$. Bizarrement, les travaux de Litvinov et al. [**182**] et Cohen et al. [**63**] semblent se jouer de celui-ci. En effet, ces auteurs ont prouvé un théorème de représentation des formes linéaires *continues* (i.e. complètement maxitives pourrait-on dire) v définies sur un $\mathbb{k}$-module M complet, avec $\mathbb{k}$ un semicorps idempotent complet : on peut écrire

$$v(\cdot) = \langle c, \cdot \rangle, \tag{25}$$

pour un certain élément c de M, où $\langle c, x \rangle$ est une opération définie sur un sous-ensemble de $M \times M$ à valeurs dans $\mathbb{k}$. Cette représentation est formellement et théoriquement très liée à celle de l'Équation (24). Pourtant, dans les hypothèses requises on ne trouve nulle trace d'une éventuelle continuité de $\mathbb{k}$; d'ailleurs les articles [**182**] et [**63**] ne font jamais référence à la littérature sur la théorie des domaines. Comment comprendre ce paradoxe ?

Afin de le résoudre, nous devons aller au-delà de la théorie des domaines habituelle, et utiliser à la place le cadre Z de cette théorie (cf. Bandelt et Erné [**23**]). Il consiste à sélectionner, à la place des parties filtrées, d'autres parties telles que les singletons ou les parties non vides. Rigoureusement, on fixe pour cela un foncteur $\mathsf{Z} : \underline{\mathrm{Po}} \to \underline{\mathrm{Set}}$ de la catégorie des posets dans la catégorie des ensembles. On redéfinit alors notamment les notions de relation « bien au-dessus de » et de poset continu. Dans le cas où Z sélectionne les parties non vides, un poset continu n'est autre qu'un poset *complètement distributif* ou *supercontinu* au sens d'Erné et al. [**93**]. Surtout, dans le cas où Z sélectionne les singletons, on s'aperçoit que la relation « bien au-dessus de » coïncide avec l'ordre $\geqslant$ et que tout poset est continu ! C'est implicitement ce foncteur qui est utilisé dans les articles [**182**] et [**63**], ce qui explique que l'hypothèse de continuité de $\mathbb{k}$ soit masquée ; les résultats sont alors obtenus grâce à d'autres hypothèses compensatrices, qui deviennent inutiles si Z sélectionne par exemple les parties filtrées comme en théorie des domaines habituelle.

Signalons dès à présent que les définitions suivantes dépendront d'un foncteur Z fixé à l'avance :

- relation « bien au-dessus de »,
- semicorps idempotent continu,
- application (ou forme) linéaire lisse,
- application (ou forme) linéaire continue,
- module complétable et module complet,
- coupures et complétion normale d'un module complétable,
- élément fortement archimédien d'un module.

Nous préférons en effet considérer Z comme un *langage* omniprésent qu'on ne rappelle pas systématiquement, contrairement à ce qui est fait dans les articles fondateurs où l'on parle de poset Z-complet, de Z-relation bien au-dessus de, de poset Z-continu, etc. ce qui selon nous alourdit considérablement la graphie et le discours.

Une forme linéaire $v : M \to \mathbb{k}$ est *lisse* si elle commute avec les infima de Z-parties, et *continue* si elle est lisse et commute avec tous les suprema existants. Les formes linéaires lisses peuvent être représentées sous de bonnes conditions par un idéal du module M ; pour les formes linéaires continues cet idéal devient principal, i.e. ne dépend que d'un élément c, et l'on obtient alors la représentation de l'Équation (25) comme l'énonce le théorème suivant.

Théorème III-1.1. *Soit M un module complet sur un semicorps idempotent complet et continu $\mathbb{k} \neq \{0, 1\}$, et soit $v : M \to \mathbb{k}$. Alors v est une forme linéaire continue non-dégénérée sur M si et seulement s'il existe un élément fortement archimédien $c \in M$ tel que $v(\cdot) = \langle c, \cdot\rangle$. Dans ce cas, c est unique et égale le suprémum de $\{1 \geqslant v\}$.*

Ce résultat généralise [**182**, Théorèmes 5.1 et 5.2] et [**63**, Corollaire 39]. Il est obtenu relativement à un foncteur Z *union-complet*, ce qui assure, dès qu'un poset est continu, que la relation « bien au-dessus de » associée, notée $\gg$, est interpolante, i.e. telle que si $t \gg r$, alors il existe s tel que $t \gg s \gg r$.

Grâce au théorème III-1.1 nous retrouvons le théorème de Radon–Nikodym idempotent (ou théorème de Sugeno–Murofushi) dont nous avons parlé au chapitre I. Il faut pour cela utiliser le foncteur Z qui sélectionne les singletons, et travailler dans un module approprié, qui n'est pas tout à fait le module $L^1_+(\tau)$ associé à la mesure maxitive dominante τ, mais un module

$\mathbb{M}$ qui dépend à la fois de τ et de la mesure dominée ν. On montre que τ est localisable (resp. σ-principale) si et seulement si $\mathbb{M}$ est un module complet (resp. σ-principal). Si τ est σ-principale, alors toute forme linéaire σ-continue sur $\mathbb{M}$ est en fait continue. De là, le théorème de Sugeno–Murofushi idempotent s'en déduit facilement.

Le théorème III-1.1 ne suffit pourtant pas pour démontrer la version idempotente du théorème de représentation de Riesz. Celui-ci s'applique typiquement à une forme linéaire $V : M \to \mathbb{R}_+$ définie sur le module M des fonctions positives continues bornées d'un espace de Tychonoff. Il exprime V comme une intégrale de Shilkret par rapport à une certaine mesure maxitive régulière et finie sur les compacts. Comme une telle mesure maxitive admet toujours une densité cardinale finie c^+, cela revient à écrire V sous la forme

$$V(f) = \bigoplus_{x \in E} \frac{f(x)}{c(x)}$$

pour une certaine fonction $c : E \to \mathbb{R}_+^*$ (en fait $c = 1/c^+$), où E est l'espace topologique sous-jacent. Cependant, la fonction c n'est en général pas continue (elle est simplement semicontinue inférieurement), donc *sort* du module de départ M. C'est cela qui rend le théorème III-1.1 insuffisant.

Pour capter ce cas de figure, il faut aller chercher les *extensions* de modules, c'est-à-dire les couples $\overline{M}/M$ avec M un sous-module d'un module complet $\overline{M}$. Ainsi justement le module des fonctions positives semicontinues inférieurement est une extension de celui des fonctions positives continues bornées. On obtient alors le résultat suivant.

Théorème III-1.2. *Soit $\overline{M}/M$ une extension de modules sur un semicorps idempotent complet $\mathbb{k} \neq \{0, 1\}$, et soit $v : M \to \mathbb{k}$ une forme linéaire sur M. On suppose que l'extension est meet-continue. Alors v est continue et non-dégénérée sur $\overline{M}/M$ si et seulement s'il existe un élément archimédien c dans $\overline{M}/M$ tel que $v(\cdot) = \langle c, \cdot \rangle$. Dans ce cas, le suprémum de $\{1 \geqslant v\}$ dans $\overline{M}$ est le plus petit c tel que $v(\cdot) = \langle c, \cdot \rangle$.*

Par souci de simplification, ce théorème est donné uniquement pour le cas où Z sélectionne les singletons. Une hypothèse nouvelle apparaît : on demande que l'extension $\overline{M}/M$ soit *meet-continue*, ce qui exprime une forme de distributivité des infima finis par rapport aux suprema dirigés. Ce résultat permet d'attaquer le théorème de Riesz : on retrouve, avec quelques améliorations, la version énoncée par Choquet [**60**] et prouvée par Kolokoltsov et Maslov [**153**] dans le cas localement compact, ainsi que celle de Breyer et Gulinsky [**48**] reprouvée par Puhalskii [**246**]. On prouve aussi une version du théorème de Riesz dans le cas où l'espace topologique E est séparable métrisable.

III-2. Introduction

Maxitive measures are defined analogously to classical (additive) measures with the supremum operation $\oplus$ in place of the addition $+$. These measures were first introduced by Shilkret [**271**], and rediscovered many times. This explains why similar notions and results coexist in the literature, that we tried to survey, unify, and surpass in Chapter I.

Maslov's monograph [**196**], in which maxitive measures with values in ordered semirings were considered, testifies to deep connections between idempotent analysis and *order theory* or *lattice theory*. Similar initiatives have been undertaken in the framework of fuzzy set theory, where $[0,1]$-valued possibility measures have been replaced by lattice-valued possibility measures (see Greco [**117**], Liu and Zhang [**183**], de Cooman et al. [**74**], Kramosil [**159**]). More recently, the branch of order theory dealing with *continuous lattices* and *domains* turned out to play a crucial role in the study of lattice-valued maxitive measures; see the work of Heckmann and Huth [**122, 123**], treating fuzzy set theory, category theory and continuous lattices, and of Akian [**7**], who favoured applications to idempotent analysis and large deviations of random processes. Connections between idempotent mathematics and continuous lattices (or domain theory) also arose in the work of Akian and Singer [**14**], and were surveyed by Lawson [**172**]. See also the early developments of Norberg [**226, 227**] on domain-valued random variables and the use of continuous (semi)lattices in random set theory.

Chapter II was another contribution to the strengthening of these links; we considered maxitive measures with values in a domain rather than in $\overline{\mathbb{R}}_+$. In the present chapter we shall build further on the role of domain theory in idempotent analysis. We shall be especially interested in linear forms on a module over an idempotent semifield $\mathbb{k}$.

Our motivation partly comes from the following apparent paradox. Let ν be a completely maxitive measure defined on the open subsets $\mathscr{G}(E)$ of a topological space E, and taking its values in a complete lattice $\mathbb{k}$. It is known since Heckmann and Huth [**122, 123**] and Akian [**7**] that, if $\mathbb{k}$ is a continuous lattice (hence a domain), then ν admits a cardinal density, i.e. is of the form

$$\nu(\cdot) = \bigoplus_{x \in \cdot} c^+(x), \tag{26}$$

for some map $c^+ : E \to \mathbb{k}$. See also [Chapter II, Corollary 5.9]. But Heckmann and Huth proved a stronger result, for they *characterized* continuity of $\mathbb{k}$ as follows: if $\mathbb{k}$ is a given complete lattice, then it is continuous *if and only if*, for every topological space E, each completely maxitive measure $\nu : \mathscr{G}(E) \to \mathbb{k}$ admits a cardinal density [**123**, Theorem 5].

Surprisingly, the work of Litvinov et al. [**182**] and Cohen et al. [**63**] seems to contradict this result. Indeed, these authors proved a representation theorem for *continuous* linear forms v defined on a complete $\mathbb{k}$-module M, with $\mathbb{k}$ a complete idempotent semifield: one can write

$$v(\cdot) = \langle c, \cdot \rangle, \tag{27}$$

for some $c \in M$, where $\langle c, x \rangle$ denotes a $\Bbbk$-valued operation defined on a subset of $M \times M$. This representation has formal and theoretical affinities with that of Equation (26). However, its terms require no kind of continuity assumption on $\Bbbk$! Coherently no reference to domain theory appears in the last-mentioned papers. How can one understand this paradox?

To unravel it, we need to go beyond the tools of classical domain theory, and use instead the general Z framework of domain theory (see Bandelt and Erné [**23**]). This is about selecting other subsets than the usual filtered subsets, e.g. singletons or nonempty subsets. This is done by a functor $\mathsf{Z} : \underline{\mathsf{Po}} \to \underline{\mathsf{Set}}$ from the category of posets to the category of sets. Then one can redefine the notions of way-above relation and continuous poset. In the case where Z selects nonempty subsets, a continuous poset is nothing but a *completely distributive* poset or *supercontinuous* poset in the sense of Erné et al. [**93**]. And if Z selects singletons, it happens that the way-above relation coincides with the partial order $\geqslant$ and that every poset is continuous! This functor is implicitly used in [**182**] and [**63**], and this explains why these articles apparently do not ask for continuity of $\Bbbk$.

We warn the reader that the following notions will depend on a given functor Z:

- way-above relation,
- continuous idempotent semifield,
- smooth linear map (or form),
- continuous linear map (or form),
- completable and complete modules,
- cuts and normal completion of a completable module,
- strongly archimedean element of a module.

In works related to Z-theory, it is common practice to constantly recall the dependency on Z (Z-complete poset, Z-way-above relation, Z-continuous poset, etc.); we believe however that it makes the text heavy and is not really useful if the context is clear.

A linear form $v : M \to \Bbbk$ is *smooth* if v commutes with infima of Z-sets, and *continuous* if v is smooth and commutes with arbitrary existing suprema. Under appropriate hypotheses, smooth linear forms can be represented by an ideal of the module M; for continuous linear forms, this ideal becomes principal, i.e. is generated by an element c, and one obtains Equation (27) as stated by the following theorem.

Theorem III-2.1. *Suppose that M is a complete module over a continuous complete idempotent semifield $\Bbbk \neq \{0, 1\}$, and let $v : M \to \Bbbk$. Then v is a non-degenerate continuous linear form on M if and only if there is an strongly archimedean element $c \in M$ such that $v(\cdot) = \langle c, \cdot \rangle$. In this case, c is unique and equals the supremum of the set $\{1 \geqslant v\}$.*

This result generalizes [**182**, Theorems 5.1 and 5.2] and [**63**, Corollary 39]. The implicit functor Z is supposed to be *union-complete*, so that in every continuous poset the way-above relation is interpolating, i.e. such that $t \gg r$ implies $t \gg s \gg r$ for some s.

Using Theorem III-2.1 we reprove the idempotent Radon–Nikodym theorem (or Sugeno–Murofushi theorem, see Chapter I). For this purpose we choose for Z the functor that selects singletons. If τ (resp. ν) denotes the dominating (resp. dominated) σ-maxitive measure, the module we work with is not $L^1_+(\tau)$ but a module $\mathbf{M}$ that depends on both τ and ν. We show that τ is localizable (resp. σ-principal) if and only if $\mathbf{M}$ is a complete module (resp. a σ-principal module). Moreover, if τ is σ-principal, then every σ-continuous linear form on $\mathbf{M}$ is continuous. With this result, the idempotent Radon–Nikodym theorem can be deduced easily.

Unfortunately, Theorem III-2.1 is not sufficient for proving an idempotent version of the Riesz representation theorem. The idempotent Riesz theorem usually applies to a linear form $V : M \to \mathbb{R}_+$ defined on the module M of nonnegative bounded continuous maps of a Tychonoff space. It asserts that V can be expressed as a Shilkret integral with respect to some regular maxitive measure that is finite on compact subsets. Since such a measure always admits a finite cardinal density c^+, this amounts to writing V as

$$V(f) = \bigoplus_{x \in E} \frac{f(x)}{c(x)},$$

for some map $c : E \to \mathbb{R}_+^*$ (and in fact $c = 1/c^+$), where E is the underlying topological space. But c does not need to be continuous, it is only lower-semicontinuous in general. This means that c is *outside* M, a case that is not treated by Theorem III-2.1.

To take account of this situation, we introduce *module extensions*, i.e. pairs $\overline{M}/M$ with M a submodule of a complete module $\overline{M}$. For instance the module of nonnegative lower-semicontinuous maps is an extension of the module of nonnegative bounded continuous maps. We obtain the following result.

Theorem III-2.2. *Suppose that $\overline{M}/M$ is a extension over a complete idempotent semifield $\Bbbk$, and let $v : M \to \Bbbk$ be a linear form on M. Assume that the extension is meet-continuous. Then v is non-degenerate continuous on $\overline{M}/M$ if and only if there is an archimedean element c in $\overline{M}/M$ such that $v(\cdot) = \langle c, \cdot \rangle$. In this case, the supremum of $\{1 \geqslant v\}$ in $\overline{M}$ is the least c satisfying $v(\cdot) = \langle c, \cdot \rangle$.*

For simplification purposes this theorem is limited to the case where Z selects singletons. A novel assumption is introduced: we ask for the extension $\overline{M}/M$ to be *meet-continuous*. This specifies that finite infima distribute over directed suprema. This result enables one to tackle the idempotent Riesz theorem. We reprove, with a few improvements, a version of this theorem given by Choquet [**60**] and proved by Kolokoltsov and Maslov [**153**] in the locally-compact case, and a version due to Breyer and Gulinsky [**48**] and also reproved by Puhalskii [**246**]. We also prove a Riesz like theorem in the case where the topological space E is separable metrizable.

The chapter is organized as follows. Section III-3 recalls basics of domains and continuous posets, in the categorical framework of Z-theory. Section III-4 deals with the concepts of idempotent semifields and modules over semirings. In Section III-5 we introduce the notion of linear forms defined on a $\mathbb{k}$-module, where $\mathbb{k}$ is an idempotent semifield. We propose a generic way of constructing such maps using ideals of the underlying module. When continuity assumptions on $\mathbb{k}$ are required, we use the tools of Z-theory introduced in Section III-3. In Section III-6 our main theorem on representation of continuous linear forms on a complete module is proved. In Section III-7 we go through some applications to maxitive measures and the idempotent Radon–Nikodym theorem. Section III-8 provides necessary and sufficient conditions for a module to be embeddable into a complete module. In Section III-9 we prove a representation theorem for residuated forms on a module extension. In Section III-10 the idempotent Riesz representation theorem is proved.

III-3. A PRIMER ON Z-THEORY FOR CONTINUOUS POSETS AND DOMAINS

A *poset* or *partially ordered set* $(P, \leqslant)$ is a set P equipped with a reflexive, antisymmetric and transitive binary relation $\leqslant$. Let us denote by $\underline{\mathsf{Po}}$ the category of all posets with order-preserving maps as morphisms. A *subset selection* is a function that assigns to each poset P a certain collection $\mathsf{Z}[P]$ of subsets of P called the Z-*sets* of P. A *subset system* is a subset selection Z such that

- $i)$ at least one $\mathsf{Z}[P]$ has a nonempty element,
- $ii)$ for each order-preserving map $f : P \to Q$, $f(Z) \in \mathsf{Z}[Q]$ for every $Z \in \mathsf{Z}[P]$,

the point $ii)$ meaning that Z is a covariant functor from $\underline{\mathsf{Po}}$ to $\underline{\mathsf{Set}}$ (the category of sets) with $\mathsf{Z}[f]$ defined by $\mathsf{Z}[f](Z) = f(Z)$ if $Z \in \mathsf{Z}[P]$, for every order-preserving map $f : P \to Q$. To this definition, first given by Wright et al. [**303**], we add a third (unusual but useful in the framework of this chapter) condition:

- $iii)$ the empty set is not in $\mathsf{Z}[P]$, for all posets P.

The suggestion of [**303**] to apply subset systems to the theory of continuous posets was followed by Nelson [**222**], Novak [**229**], Bandelt [**22**], Bandelt and Erné [**23**], [**24**], and this research was carried on by Venugopalan [**287**], [**288**], Xu [**304**], Baranga [**25**], Menon [**200**], Shi and Wang [**270**], Erné [**92**], [**96**] among others. Conditions $i)$ and $ii)$ together ensure that each $\mathsf{Z}[P]$ contains all singletons.

The basic example of subset system is the set of directed subsets of P. This subset system is behind the classical theory of continuous posets and domains, see the monograph by Gierz et al. [**114**]. Here are some further examples:

(1) Taking $\mathsf{Z}[P]$ as the set of all nonempty subsets of P works well for investigating completely distributive lattices, see Erné et al. [**93**]. Completely distributive lattices were initially examined by Raney [**248**], [**249**].
(2) The case where $\mathsf{Z}[P]$ is the set of filtered subsets of P was used for instance by G. Gerritse [**113**], Jonasson [**142**], Akian and Singer [**14**]. See also Chapter II.
(3) If $\mathsf{Z}[P]$ is the set of all singletons of P, then Z is also a subset selection.
(4) A series of papers deals with the case where $\mathsf{Z}[P]$ is the set of chains of P, see Markowsky and Rosen [**194**], and Markowsky [**191**], [**192**], [**193**]. Using the Hausdorff maximality theorem, relations between directed subsets and chains were explored by Iwamura [**134**], Bruns [**52**], and Markowsky [**190**]. See also Erné [**92**, p. 54].
(5) The case where $\mathsf{Z}[P]$ is the set of nonempty finite subsets of P was investigated by Martinez [**195**]. See also Frink [**104**] and Erné [**89**].

Rather than Z, we shall often deal with the subset selection F, defined by $\mathsf{F}[P] = \{\uparrow Z : Z \in \mathsf{Z}[P]\}$, where $\uparrow Z$ is the upper subset generated by Z, i.e. $\uparrow Z := \{y \in P : \exists x \in Z, x \leqslant y\}$. The elements of $\mathsf{F}[P]$ are the F-*sets*, or the (Z-)*filters*, of P. Although F is not a subset system in general, it satisfies the following conditions:

i) at least one $\mathsf{F}[P]$ has a nonempty element,
ii') for each order-preserving map $f : P \to Q$, $\uparrow f(F) \in \mathsf{F}[Q]$ for every $F \in \mathsf{F}[P]$,
iii) an F-set is never empty.

A subset selection F derived from a subset system Z as above will be called a *filter selection*. Note that, like Z, F is functorial, i.e. $\mathsf{F}[g \circ f] = \mathsf{F}[g] \circ \mathsf{F}[f]$ for all order-preserving maps $f : P \to Q$ and $g : Q \to R$, if one naturally defines $\mathsf{F}[f](F) = \uparrow f(F)$ for all $F \in \mathsf{F}[P]$.

Translation III-3.1 (Filter selections)**.** The first three examples of subset systems given above lead to the following filter selections, respectively:

(1) $\mathsf{F}[P]$ is the set $\mathsf{Up}^*[P]$ of *nonempty upper subsets* of P,
(2) $\mathsf{F}[P]$ is the set $\mathsf{Fi}[P]$ of *filters* (in the sense of [**114**]) of P,
(3) $\mathsf{F}[P]$ is the set $\mathsf{PFi}[P]$ of *principal filters* of P.

We now introduce the *way-above relation*, which in our context is more relevant than the usual *way-below relation*. Thus, our notions of continuous posets and domains are dual to the traditional definitions. The way-above relation has already been used to study lattice-valued upper-semicontinuous functions, see for instance [**113**] and [**142**]; see also Chapter II. We say that

$y \in P$ is *way-above* $x \in P$, written $y \gg x$, if, for every F-set F with infimum, $x \geqslant \bigwedge F$ implies $y \in F$. We use the notations $\Downarrow x = \{y \in P : x \gg y\}$, $\Uparrow x = \{y \in P : y \gg x\}$, and for $A \subset P$, $\Downarrow A = \{y \in P : \exists x \in A, x \gg y\}$, $\Uparrow A = \{y \in P : \exists x \in A, y \gg x\}$. The poset P is *continuous* if every element is the F infimum of elements way-above it, i.e. $\Uparrow x \in \mathsf{F}[P]$ and $x = \bigwedge \Uparrow x$ for all $x \in P$. A *domain* is a continuous poset in which every F-set has an infimum.

Translation III-3.2 (Continuous posets)**.** For our three examples of subset systems, the notion of continuous posets translates respectively as follows:

(1) if $\mathsf{F} = \mathsf{Up}^*$, then a poset is continuous if and only if it is completely distributive (complete distributivity is sometimes called *supercontinuity*),
(2) if $\mathsf{F} = \mathsf{Fi}$, then a poset is continuous if and only if it is continuous in the sense of Chapter II,
(3) if $\mathsf{F} = \mathsf{PFi}$, then the way-above relation $y \gg x$ reduces to the partial order $y \geqslant x$, and every poset is continuous.

For a poset P, the way-above relation is *additive* if, for all $x \in P$, the subset $\Downarrow x$ is either empty or directed, i.e. if whenever $x \gg y$ and $x \gg y'$, we have $x \gg z$ for some $z \in P$ such that $z \geqslant y$ and $z \geqslant y'$. A continuous poset with an additive way-above relation is *stably-continuous*. With respect to the filter selection PFi, every poset is stably-continuous.

A poset P has the *interpolation property* if, for all $x, y \in P$ with $y \gg x$, there exists some $z \in P$ such that $y \gg z \gg x$. For continuous posets in the classical sense, it is well known that the interpolation property holds, see e.g. [**114**, Theorem I-1.9]. This is a crucial feature that is behind many important results of the theory. For an arbitrary choice of Z, however, this needs no longer to be true. Deriving sufficient conditions on Z to recover the interpolation property is the goal of the following theorem. The subset selection F is *union-complete* if, for every $V \in \mathsf{F}[\mathsf{F}[P]]$ (where $\mathsf{F}[P]$ is considered as a poset ordered by reverse inclusion $\supset$), $\bigcup V \in \mathsf{F}[P]$. As explained in [**92**], this condition embodies the fact that finite unions of finite sets are finite, $\supset$-filtered unions of filtered sets are filtered, etc. The following theorem restates a result due to [**229**] and [**23**] in its dual form. We give the proof here for the sake of completeness.

Theorem III-3.3. [**229, 23**] *If* F *is a union-complete filter selection, then every continuous poset has the interpolation property.*

Remark III-3.4. In the context of Z-theory, many authors (see [**229**], [**23**], [**287**]) call *strongly continuous* a continuous poset with the interpolation property.

Proof. Let P be a continuous poset, and let $x \in P$. We need to show that $F \subset \Uparrow F$, where F denotes the F-set $F = \Uparrow x$. For this purpose we first prove that $\Uparrow F$ is an F-set. Write $\Uparrow F = \bigcup_{y \in F} \Uparrow y = \bigcup V$, where V is the collection of subsets contained in some $\Uparrow y$, $y \in F$. Considering the order-preserving map $f : P \ni y \mapsto \Uparrow y \in \mathsf{F}[P]$ (recall that $\mathsf{F}[P]$ is ordered by reverse inclusion) and using Property ii') above, we have $V = \uparrow f(F) \in \mathsf{F}[\mathsf{F}[P]]$. Since F is union-complete, one has $\Uparrow F = \bigcup V \in \mathsf{F}[P]$. Since P is continuous,

$$x = \bigwedge \Uparrow x = \bigwedge F = \bigwedge_{y \in F} y = \bigwedge_{y \in F} (\bigwedge \Uparrow y) = \bigwedge (\bigcup_{y \in F} \Uparrow y) = \bigwedge \Uparrow F.$$

The definition of the way-above relation and the fact that $\Uparrow F \in \mathsf{F}[P]$ give $y \in \Uparrow F = \Uparrow(\Uparrow x)$, for all $y \in \Uparrow x$. This proves that P has the interpolation property. $\square$

All subset systems mentioned above are union-complete. It remains an open problem to exhibit a continuous poset with respect to some subset system that does not satisfy the interpolation property.

We should stress the fact that the machinery of category theory is justified as long as relations between posets are examined. If a single poset P is at stake, having just a collection of subsets of P at disposal could be sufficient, as in the works [**22**], [**24**], [**304**] (where the letter $\mathfrak{M}$ is used for the collection of selected subsets). In the present work, we hope that the relevance of using functorial (filter) selections will be made clear.

III-4. Semirings, semifields, modules over a semiring

III-4.1. Semirings, semifields. A *semiring* is an abelian monoid $(\Bbbk, \oplus, 0)$ endowed with an additional binary relation $\times$ (the multiplication) that is associative, has a unit $1 \neq 0$, distributes over $\oplus$, and admits 0 as absorbing element. A semiring is *idempotent* (or is a *dioid*, see Baccelli et al. [**20**] or Gondran and Minoux [**115**]) if $\oplus$ is idempotent, i.e. $t \oplus t = t$ for all t, and *commutative* if the multiplication is commutative. An (*idempotent*) *semifield* is an (idempotent) semiring in which every non-zero element has a multiplicative inverse. We do not assume a semifield to be commutative in general (see however Remark III-4.2). Notice that, if $\Bbbk$ is an idempotent semifield, then $\Bbbk \backslash \{0\}$ is a *lattice-group*. This implies that $\Bbbk$ is a distributive lattice; in particular, every nonempty finite subset of $\Bbbk$ has an infimum, and we have

$$s \wedge t = (s^{-1} \oplus t^{-1})^{-1}, \tag{28}$$

for all $s, t \in \Bbbk \backslash \{0\}$.

A dioid has a natural structure of partially ordered set with $s \leqslant t \Leftrightarrow s \oplus t = t$, whose bottom element is 0. With this point of view $s \oplus t$ is nothing but the supremum of $\{s, t\}$, hence every dioid is a commutative idempotent monoid, i.e. a *semilattice*. Given a filter selection F, a dioid is *complete* if every upper-bounded subset T has a supremum such that

$$s(\bigoplus T) = \bigoplus_{t \in T} st, \qquad (\bigoplus T)s = \bigoplus_{t \in T} ts, \tag{29}$$

for all s, and if every F-set F has an infimum such that

$$(30) \qquad s(\bigwedge F) = \bigwedge_{f \in F} sf, \qquad (\bigwedge F)s = \bigwedge_{f \in F} fs,$$

for all s. With respect to the filter selection PFi that selects principal ideals, Equations (30) are trivial.

In an idempotent semifield, Equations (29) and (30) are satisfied for all subsets T (resp. F) with supremum (resp. with infimum). Thus, an idempotent semifield is complete if and only if every upper-bounded subset has a supremum. This makes the notion of complete idempotent semifield independent of the filter selection F.

The following lemma, which will be used many times in this chapter, mimics a result by Akian and Singer [**14**, Lemma 2.1].

Lemma III-4.1 (Extends [**14**, Lemma 2.1]). *Let F be a filter selection, and let $\Bbbk$ be an idempotent semifield. For all $r, s, t \in \Bbbk$ with $r \neq 0$, $t \gg s$ implies $tr \gg sr$.*

Proof. Let F be an F-set of $\Bbbk$ with infimum such that $sr \geqslant \bigwedge F$. The map $f : \Bbbk \to \Bbbk$ defined by $f(u) = ur^{-1}$ is order-preserving, hence the set $\uparrow f(F) = f(F)$ is an F-set. Since $t \gg s \geqslant (\bigwedge F)r^{-1} = \bigwedge f(F)$, we have $t \in f(F)$, so $tr \in F$. This shows that $tr \gg sr$. $\square$

From now on we use the acronym *cis* for a complete idempotent semifield distinct from $\{0, 1\}$. A cis is never a complete lattice; if it were, there would be a greatest element $\top$, and we would have $\top \geqslant 1 \Rightarrow \top^2 \geqslant \top \Rightarrow \top^2 = \top \Rightarrow \top = 1$, while $\top = 1$ is only possible if the cis coincides with $\{0, 1\}$ (a case that is excluded in the definition of a cis).

Remark III-4.2. It is worth recalling that, by the Iwasawa theorem, every cis is commutative (see e.g. Birkhoff [**39**, Theorem 28]).

Remark III-4.3 (On quasifields)**.** Litvinov et al. [**182**] defined a *quasifield* as a dioid in which every non-zero element is the supremum of invertible elements and such that $t \leqslant 1$ whenever the subset $\{t^n : n = 1, 2, \ldots\}$ is upper-bounded. They showed that every quasifield distinct from $\{0, 1\}$ can be embedded into a cis, and asserted that, conversely, every cis is a quasifield. This latter point indeed holds, for if, for some $t \neq 0$, the subset $\{t^n : n = 1, 2, \ldots\}$ is upper-bounded, and if s is its supremum, then with Equations (29) we have $ts \leqslant s$; since $s \neq 0$ we deduce that $t \leqslant 1$.

III-4.2. Modules over a semiring. We now turn our attention to modules over a semiring.

Definition III-4.4. Let $\Bbbk$ be a semiring. A *right* $\Bbbk$*-module* is a commutative monoid $(M, \oplus, 0)$ equipped with a right action $M \times \Bbbk \ni (x, t) \mapsto x.t \in M$

such that, for all $x \in M$, $x.0 = 0$, $x.1 = x$, and for all $y \in M$, $s, t \in \Bbbk$,

$$x.(st) = (x.s).t,$$
$$(x \oplus y).t = x.t \oplus y.t,$$
$$x.(s \oplus t) = x.s \oplus x.t.$$

A subset of M is a *submodule* if it contains 0 and is closed under addition and external multiplication.

In the sequel we shall say $\Bbbk$-*module* or *module over* $\Bbbk$ for *right* $\Bbbk$-*module*, and we shall only deal with modules over an idempotent semifield $\Bbbk$. Then the previous axioms imply that $x \oplus x = x$ for all $x \in M$, and $0.t = 0$ for all $t \in \Bbbk$, so the addition $x \oplus y$ of two elements x, y of M is the supremum of $\{x, y\}$ with respect to the induced partial order $x \leqslant y \Leftrightarrow x \oplus y = y$. In other words, $(M, \oplus, 0)$ is a semilattice.

For background on -or applications of- modules over dioids or quantales, see Zimmermann [**307**], Samborskiĭ and Shpiz [**262**], Abramsky and Vickers [**1**], Rosenthal [**258**], [**132**], Kruml [**165**], Cohen et al. [**63**], Litvinov et al. [**182**], Shpiz [**272**], Shpiz and Litvinov [**272**], Gondran and Minoux [**115**], Russo [**261**], Castella [**56**].

Remark III-4.5. Some authors, especially in the area of idempotent analysis and max-plus algebra, prefer to call *semimodule* a module over a semiring, and *idempotent semimodule* a module over a dioid or over an idempotent semifield. However, one can see that the axioms given in Definition III-4.4 do not differ from the axioms defining a classical module (over a ring), and distinctions only appear in the choice of the base semiring. The same remark can be made for axioms defining a *morphism* between modules (see Section III-5 for the precise definition). Hence, from a categorical (and also from a historical) point of view, we see no reason not to keep on with the term *module*.

The following example is inspired by extreme value theory.

Example III-4.6. We equip the set $\mathbb{R}_+$ of nonnegative real numbers with its idempotent semifield structure, i.e. with the maximum operation for $\oplus$, and the usual multiplication. We write $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \oplus, \times)$. Let μ, σ, ξ be real numbers with $\sigma > 0$, and consider

$$M_{\mu,\sigma,\xi} = \left\{ x \in \mathbb{R} \cup \{-\infty\} : 1 + \xi \frac{x - \mu}{\sigma} > 0 \right\}.$$

Then $(M_{\mu,\sigma,\xi}, \oplus, \mathbf{0})$ is an $\mathbb{R}_+^{\max}$-module if $\oplus$ denotes the usual maximum operation, if $\mathbf{0}$ denotes $\mu - \sigma/\xi$ if ξ is positive, $-\infty$ otherwise, and if we consider the external multiplication defined by

$$x.t = \mu - \frac{\sigma}{\xi} + \frac{\sigma t^{\xi}}{\xi}(1 + \xi \frac{x - \mu}{\sigma}),$$

if ξ is non-zero, and

$$x.t = x + \sigma \log(t)$$

otherwise, for all $x \in M_{\mu,\sigma,\xi}$, $t \in \mathbb{R}_+$.

III-5. Morphisms and linear forms

In this section, F is a union-complete filter selection, and $\Bbbk$ is an idempotent semifield. When continuity assumptions on $\Bbbk$ are required, we use the tools of Z-theory introduced in Section III-3.

A *morphism* (or *linear map*) between two $\Bbbk$-modules M and N is a map $f : M \to N$ satisfying both following conditions:

- *homogeneity*: $f(x.t) = f(x).t$,
- *maxitivity*: $f(x \oplus y) = f(x) \oplus f(y)$,

for all $x, y \in M$, $t \in \Bbbk$. Or equivalently, $f(0) = 0$ and $f(x \oplus y.t) = f(x) \oplus f(y).t$, for all $x, y \in M$ and $t \in \Bbbk$. A morphism f is *smooth* if, for all F-sets F of M with infimum, $f(F)$ has an infimum in N such that $f(\bigwedge F) = \bigwedge f(F)$. A (*smooth*) *linear form* on a $\Bbbk$-module M is a (smooth) morphism $v : M \to \Bbbk$, where $\Bbbk$ is considered as a $\Bbbk$-module.

Translation III-5.1 (Smoothness)**.**

1. If $\mathsf{F} = \mathsf{Up}^*$, then a morphism is smooth if and only if it preserves all nonempty existing infima.
2. If $\mathsf{F} = \mathsf{Fi}$, then a morphism is smooth if and only if it is Scott-continuous.
3. If $\mathsf{F} = \mathsf{PFi}$, then every morphism is smooth.

Example III-5.2 (Example III-4.6 continued)**.** The map $v : M_{\mu,\sigma,\xi} \to \mathbb{R}_+$ defined by $v(x) = (1 + \xi\frac{x-\mu}{\sigma})^{1/\xi}$ if ξ is non-zero, $v(x) = \exp(\frac{x-\mu}{\sigma})$ otherwise, is a linear form on $M_{\mu,\sigma,\xi}$, smooth with respect to Up^*.

Example III-5.3. The set $\mathbb{R}_+$ is still equipped with its idempotent semifield structure. Let $\mathscr{E}$ be a semi-σ-algebra on a nonempty set E. In Chapter I we saw that the *Shilkret integral* (or *idempotent integral*) of some lower-semimeasurable map $f : E \to \overline{\mathbb{R}}_+$ with respect to a σ-maxitive measure ν on $\mathscr{E}$ is defined by

$$\int_E^{\infty} f.d\nu = \bigoplus_{t \in \mathbb{R}_+} t.\nu(f > t).$$

Such a map f is *ν-integrable* if its Shilkret integral is finite. Then the set M of ν-integrable maps is an $\mathbb{R}_+^{\max}$-module, and the Shilkret integral is a linear form on M.

Notations III-5.4. Let I be a subset of a $\Bbbk$-module M.

- For all $t \in \Bbbk\backslash\{0\}$, we write $I.t = \{x.t : x \in I\}$, and $I.0 = \bigcap_{t \neq 0} I.t$. The reader is warned that $I.0$ does not coincide with $\{0\}$ in general.
- For all $x \in M$, we denote by $\langle I, x\rangle$ the set $\{t \in \Bbbk : x \in I.t\}$.

A subset X is *lower* if $X = \downarrow X$, where $\downarrow X := \{y : \exists x \in X, y \leqslant x\}$. An *ideal* I of M is a lower subset of M such that $x \oplus y \in I$, for all $x, y \in I$. An ideal I is *smooth* if, for all F-sets F of M with infimum, $\bigwedge F \in I$ implies

$F \cap I \neq \emptyset$. An ideal I is *right-continuous* if $I.t = \bigcap_{s \gg t} I.s$ for all $t \in \mathbb{k}$, and *left-continuous* if $I.t = \bigcup_{t \gg s} I.s$ for all $t \in \mathbb{k}$.

The next proposition, which is inspired by the concept of *Minkowski functional* (or *gauge*) in convex analysis and by a remark of Nguyen et al. [**224**] on maxitive measures, provides a generic way of constructing a linear form from an ideal. We first prove a useful lemma.

Lemma III-5.5 (Compare with [**182**, Lemma 5.1])**.** *Let $\mathbb{k}$ be an idempotent semifield. Then $\mathbb{k}\backslash\{0\}$ has an infimum, and*

- $\bigwedge \mathbb{k}\backslash\{0\} = 1$ *if and only if* $\mathbb{k} = \{0, 1\}$,
- $\bigwedge \mathbb{k}\backslash\{0\} = 0$ *if and only if* $\mathbb{k} \neq \{0, 1\}$.

Proof. If $\mathbb{k} = \{0, 1\}$, the result is clear, so suppose that $\mathbb{k} \neq \{0, 1\}$. This implies the existence of some $t \in \mathbb{k}, t > 0$ and $t \neq 1$. To show that 0 is the infimum of $\mathbb{k}\backslash\{0\}$, we pick some lower bound s of $\mathbb{k}\backslash\{0\}$, and we assume that $s > 0$. If $s = 1$, the definition of s gives $t > 1$, hence $0 < t^{-1} < 1 = s$, a contradiction. As a consequence, $s \neq 1$. Since $1 \in \mathbb{k}\backslash\{0\}$, we have $s < 1$, by definition of s. Hence, $0 < s^2 < s$, another contradiction. We conclude that $s = 0$, which proves that 0 is the infimum of $\mathbb{k}\backslash\{0\}$ whenever $\mathbb{k} \neq \{0, 1\}$. ☐

Proposition III-5.6. *Let $\mathbb{k}$ be an idempotent semifield, $\mathbb{k} \neq \{0, 1\}$, and let I be an ideal (resp. a smooth ideal) of M such that, for all $x \in M$, $\langle I, x\rangle$ is an F-set with infimum. Define $v : M \to \mathbb{k}$ by*

$$v(x) = \bigwedge \langle I, x\rangle, \tag{31}$$

for all $x \in M$. If I is right-continuous, then $I = \{1 \geqslant v\}$ and v is a linear form (resp. a smooth linear form) on M.

Proof. If v is given by Equation (31) with a right-continuous ideal I, then v is order-preserving, for if $x \leqslant y$, $y \in I.t$, and $t \neq 0$, then $x.t^{-1} \leqslant y.t^{-1} \in I$, so that $x = (x.t^{-1}).t \in I.t$. Thus, one has $\{t \in \mathbb{k} : x \in I.t\} \supset \{t \in \mathbb{k} : y \in I.t\}$, so that $v(x) \leqslant v(y)$.

Now let us show that $v(x) \oplus v(x') \geqslant v(x \oplus x')$. So let $s \gg v(x) \oplus v(x')$. Then $s \gg v(x)$, so there exists some $t \in \mathbb{k}$ such that $s \geqslant t$ and $x \in I.t$. There is also some t' with the corresponding properties with respect to x'. Note that $x, x' \in I.s$, which implies $x \oplus x' \in I.s$. Since I is right-continuous, we have $x \oplus x' \in I.s_0$, where $s_0 := v(x) \oplus v(x')$. If $s_0 = 0$, then $x \oplus x' \in I.0$, so that $v(x \oplus x') = 0 = s_0 = v(x) \oplus v(x')$ by definition of v. Otherwise, we can write $x \oplus x' = y.s_0$, with $y \in I$. We thus have $v(x \oplus x') = v(y).s_0$. Since $y \in I$, $v(y) \leqslant 1$. This leads to $v(x \oplus x') \leqslant s_0$, i.e. $v(x \oplus x') \leqslant v(x) \oplus v(x')$.

For v to be a linear form, it remains to show that $v(x.t) = v(x).t$, for all $x \in M$, $t \in \mathbb{k}$. This step is not difficult and left to the reader.

It is clear that $I \subset \{1 \geqslant v\}$. For the reverse inclusion, let $x \in \{1 \geqslant v\}$, i.e. $1 \geqslant v(x)$. To prove that $x \in I$, we use the right-continuity of I, i.e. we show that $x \in I.s$ for all $s \gg 1$. We have $s \gg v(x) = \bigwedge \langle I, x\rangle$. The subset

$\langle I, x\rangle$ is assumed to be an F-set, so $s \in \langle I, x\rangle$, i.e. $x \in I.s$, which is the desired result.

Suppose in addition that I is smooth, and let us show that v is smooth. First recall that, if F is an F-set of M with infimum f_0 and $t \in \mathbb{k}\backslash\{0\}$, then $F.t^{-1}$ is an F-set such that $\bigwedge(F.t^{-1}) = f_0.t^{-1}$. We obtain

$$\begin{aligned} v(f_0) &= \bigwedge\{t \in \mathbb{k}\backslash\{0\} : f_0 \in I.t\} \\ &= \bigwedge\{t \in \mathbb{k}\backslash\{0\} : \bigwedge(F.t^{-1}) \in I\} \\ &= \bigwedge \bigcup_{f \in F} \{t \in \mathbb{k}\backslash\{0\} : f.t^{-1} \in I\}, \end{aligned}$$

since I is smooth. We deduce that $v(f_0) = \bigwedge_{f \in F} \bigwedge\{t \in \mathbb{k}\backslash\{0\} : f \in I.t\} = \bigwedge_{f \in F} v(f)$, so v is smooth. □

When the range $\mathbb{k}$ of the map v is continuous, one can remove the assumption of right-continuity of I. This leads to the converse statement as follows.

Proposition III-5.7. *Assume that $\mathbb{k}$ is a (stably-)continuous cis. A map $v : M \to \mathbb{k}$ is a (smooth) linear form on M if and only if there is some (smooth) ideal I of M such that $\langle I, x\rangle$ is an F-set and*

$$v(x) = \bigwedge\langle I, x\rangle,$$

for all $x \in M$. In this case:

(1) *I is right-continuous if and only if $I = \{x \in M : 1 \geqslant v(x)\}$;*
(2) *I is left-continuous if and only if $I = \{x \in M : 1 \gg v(x)\}$.*

Proof. At first we consider the case where $\mathbb{k}$ is a continuous cis. If v is a linear form, define $I := \{1 \geqslant v\}$. This is an ideal such that, for all $t \neq 0$, $I.t = \{t \geqslant v\}$, and, by Lemma III-5.5, $I.0 = \{x \in M : v(x) = 0\}$. Since $\mathbb{k}$ is continuous, I is right-continuous. Moreover, $\langle I, x\rangle$ equals the principal filter generated by $v(x)$, hence is an F-set, and $v(x) = \bigwedge\langle I, x\rangle$ for all x.

If $\mathbb{k}$ is stably-continuous and v is smooth, we can rather define $I := \{1 \gg v\}$. By hypothesis the way-above relation is additive, so that I is an ideal. Also, $I.t = \{t \gg v\}$ by Lemma III-4.1, and $I.0 = \{x \in M : v(x) = 0\}$ since $\mathbb{k}$ is continuous. Left-continuity of I holds by the interpolation property. Moreover, for all x, $\langle I, x\rangle = \Uparrow v(x)$, which is an F-set whose infimum is $v(x)$ since $\mathbb{k}$ is continuous. Smoothness of I is a consequence of the smoothness of v and of the fact that $\langle I, x\rangle$ is nonempty.

Conversely, assume that Equation (31) is satisfied, and let us show that v is a linear form. Let $J = \bigcap_{s \gg 1} I.s$. Then J is an ideal of M containing I. We prove that, for all $t \in \mathbb{k}$,

$$t \geqslant v(x) \Leftrightarrow x \in J.t. \tag{32}$$

Using Lemma III-5.5, it suffices to prove Equivalence (32) for $t \neq 0$. If $t \geqslant v(x)$ and $s \gg 1$, then $st \gg t$ by Lemma III-4.1, so $st \gg v(x)$. This gives $x \in I.st$, hence $x.t^{-1} \in I.s$, for all $s \gg 1$. By definition of J we get

$x.t^{-1} \in J$, i.e. $x \in J.t$. Now we suppose that $x \in J.t$, and we want to show that $t \geqslant v(x)$. So let $s \gg t$. If $u = st^{-1}$, then $u \gg 1$ (see again Lemma III-4.1), so $xt^{-1} \in I.u$. Thus, $x \in I.s$. The definition of v implies $s \geqslant v(x)$, for all $s \gg t$. By continuity of $\mathbb{k}$ we have $t \geqslant v(x)$. So Equivalence (32) is proved. This also shows that J is right-continuous and that $\langle J, x\rangle$ is an F-set whose infimum is $v(x)$, for all $x \in M$. By Proposition III-5.6, v is a linear form, and, as in the proof of Proposition III-5.6, v is smooth if I is smooth.

To finish the proof, suppose again that v is a linear form defined by Equation (31). If I is right-continuous, then the previous point implies $I = J$ and $I.t = \{t \geqslant v\}$, for all $t \in \mathbb{k}$, so Item (1) is proved. If I is left-continuous, the inclusion $I \supset \{1 \gg v\}$ is clear, by definition of v and $\gg$. If $x \in I$, then $x \in I.s$ for some $s \in \mathbb{k}$ such that $1 \gg s$, by left-continuity of I. This implies that $s \geqslant v(x)$, so that $1 \gg v(x)$, and Item (2) is proved. $\square$

Translation III-5.8. Back to the three main instances of filter selections, the assumptions of Proposition III-5.7 translate as follows.

(1) $\langle I, x\rangle$ is an Up^*-set if and only if $\langle I, x\rangle$ is nonempty. This condition is satisfied for all $x \in M$ as soon as $I \neq \{0\}$ and, for all $x, y \in M$ with $y \neq 0$, there exists some $t \in \mathbb{k}$ with $x \leqslant y.t$.
(2) $\langle I, x\rangle$ is a Fi-set if and only if $\langle I, x\rangle$ is nonempty. As above, this condition is satisfied for all $x \in M$ as soon as $I \neq \{0\}$ and, for all $x, y \in M$ with $y \neq 0$, there exists some $t \in \mathbb{k}$ with $x \leqslant y.t$.
(3) $\langle I, x\rangle$ is a PFi-set if and only if $\langle I, x\rangle$ has a least element.

The last case leads to the following corollary.

Corollary III-5.9. *Let $\mathbb{k}$ be an idempotent semifield, $\mathbb{k} \neq \{0, 1\}$. A map $v : M \to \mathbb{k}$ is a linear form on M if and only if there is some ideal I of M such that*

$$x \in I.t \Longleftrightarrow t \geqslant v(x),$$

for all $x \in M$, $t \in \mathbb{k}$. In this case, I equals $\{1 \geqslant v\}$.

Proof. Let F be the filter selection PFi that selects principal filters. With this choice, $\mathbb{k}$ is continuous, and the way-above relation coincides with $\geqslant$, so that I is necessarily right-continuous. To conclude, use Proposition III-5.7. $\square$

At this early stage, the reader may already understand, from the previous proof, where the "paradox" evoked in the Introduction comes from: no continuity assumption seems to be needed in the terms of Corollary III-5.9, but this is simply due to the fact that, with respect to the filter selection PFi, the cis $\mathbb{k}$ is always continuous. This will be made even clearer in Section III-6, where we shall deal with the representation of continuous linear forms.

III-6. Continuous linear forms on a complete module

III-6.1. Continuity, residuation. In this section, we prove a representation theorem for continuous linear forms on a complete module. Let F be a union-complete filter selection, and let M, N be modules over an idempotent semifield $\mathbb{k}$. A morphism $f : M \to N$ is *continuous* if it is smooth and such that, for every subset $X \subset M$ with a supremum in M, $f(X)$ has a supremum in N satisfying $f(\bigoplus X) = \bigoplus f(X)$.

Translation III-6.1 (Continuity)**.**

(1) If $\mathsf{F} = \mathsf{Up}^*$, then a morphism is continuous if and only if it preserves all existing infima and suprema.
(2) If $\mathsf{F} = \mathsf{Fi}$, then a morphism is continuous if and only if it is bi-Scott-continuous (this notion was called *wo-continuity* by Shpiz [**272**]).
(3) If $\mathsf{F} = \mathsf{PFi}$, then a morphism is continuous if and only if it preserves all existing suprema (this notion of continuity is the one adopted by Cohen et al. [**63**]; Litvinov et al. [**182**] and Shpiz [**272**] called such a morphism a *b-morphism*).

We say that M is *completable* if, for all $x \in M$, the map $\mathbb{k} \to M, t \mapsto x.t$ is a continuous morphism, i.e. if, for all $x \in M$ and all $T \subset \mathbb{k}$ with supremum,

$$x.\bigoplus T = \bigoplus_{t \in T} x.t, \tag{33}$$

and if for all $x \in M$ and all F-sets F of $\mathbb{k}$ with infimum,

$$x.\bigwedge F = \bigwedge_{f \in F} x.f. \tag{34}$$

Remark III-6.2. Note that, if Equation (33) is satisfied for every $T \subset \mathbb{k}$ with supremum, then Equation (34) is also satisfied for every $F \subset \mathbb{k}$ with *non-zero infimum* (one does not need F be to an F-set in this case). Hence in the definition of completability the role of Equation (34) is to control the behaviour of $\mathbb{k} \to M, t \mapsto x.t$ around zero. Therefore, the case $\mathsf{F} = \mathsf{Up}^*$ is demanding, while with $\mathsf{F} = \mathsf{PFi}$ this behaviour is unconstrained.

Also, M is *complete* if it is completable and such that every upper-bounded subset (resp. every F-set) has a supremum (resp. an infimum). In Section III-8, Theorem III-8.1 will define the concept of *normal completion* of a completable module and show that completability is equivalent to embeddability into a complete module.

A map $f : M \to N$ between $\mathbb{k}$-modules M, N is *residuated* if there exists a (necessarily unique) map $f^{\#} : N \to M$, called the *adjoint* of f, and satisfying

$$x \leqslant f^{\#}(y) \iff y \geqslant f(x),$$

for all $x \in M$, $y \in N$. Residuated maps are related to Galois connections, see Erné et al. [**94**]. A *residuated form* on M is a homogeneous residuated map from M to $\Bbbk$. A map $v : M \to \Bbbk$ is *non-degenerate* if $\{x \in M : 1 \geqslant v(x)\}$ is upper-bounded. For instance the map $M \ni x \mapsto 0 \in \Bbbk$ is a non-degenerate (continuous) linear form if and only if M has a greatest element.

Lemma III-6.3. *Let M be a complete module over a cis $\Bbbk$. Then a map $v : M \to \Bbbk$ is a smooth residuated form if and only if it is a non-degenerate continuous linear form.*

Proof. Necessity is clear. For sufficiency, let v be a non-degenerate continuous linear form, and let $w : \Bbbk \to M, t \mapsto \bigoplus\{x \in M : t \geqslant v(x)\}$. This map is well-defined since v is non-degenerate and homogeneous. If $t \geqslant v(x)$, then $x \leqslant w(t)$ by definition of w. Conversely, if $x \leqslant w(t)$, then $v(x) \leqslant v(w(t))$, and since v preserves arbitrary existing suprema, $v(x) \leqslant \bigoplus\{v(x') : x' \in M, t \geqslant v(x')\} \leqslant t$. This proves that w is the adjoint of v, hence v is a (smooth) residuated form. □

Remark III-6.4. If $\mathsf{F} = \mathsf{PFi}$, the previous lemma identifies residuated forms with non-degenerate *b-linear functionals* in the sense of Litvinov et al. [**182**].

III-6.2. Archimedean elements and scalar product. As explained in the § after Lemma III-4.1, a cis $\Bbbk$ has no greatest element. However, we can conventionally add a top $\top$ to $\Bbbk$ and define $\overline{\Bbbk} = \Bbbk \cup \{\top\}$. Naturally extending $\oplus$ and $\times$ to $\overline{\Bbbk}$ by $t \oplus \top = \top \oplus t = \top$, $t.\top = \top.t = \top$ if $t \neq 0$, and $0.\top = \top.0 = 0$, we see that $\overline{\Bbbk}$ has the structure of a complete dioid. We also define $\top^{-1} = 0$ and $0^{-1} = \top$. If $x, c \in M$, we let

$$x \backslash c = \bigoplus\{t \in \Bbbk : c \geqslant x.t\}$$

whenever this set is upper-bounded, and $x \backslash c = \top$ otherwise. Also, the infimum of the subset $F_c(x) = \{t \in \Bbbk : c.t \geqslant x\}$ in $\overline{\Bbbk}$ is denoted by $\langle c, x\rangle$, and one can easily check that

$$\langle c, x\rangle = (x \backslash c)^{-1},$$

for all $x, c \in M$. We are interested in conditions on c ensuring that the map $x \mapsto \langle c, x\rangle$ is a continuous or a residuated linear form on M. So we need the

Definition III-6.5. Let M be a module over an idempotent semifield $\Bbbk$. An element $c \in M$ is called *archimedean* if both following conditions are satisfied:

- the subset $F_c(x)$ is nonempty for all $x \in M$,
- if the infimum of $F_c(x)$ is zero, then $F_c(x) \supset \Bbbk \backslash \{0\}$,

where $F_c(x)$ denotes the subset $\{t \in \Bbbk : c.t \geqslant x\}$.

The first condition implies that the bracket $\langle c, x\rangle$ is well-defined in $\Bbbk$ for all $x \in M$. The second condition may seem unnatural and so deserves some explanation. First observe that it will automatically be satisfied in any of the following standard situations:

- if $\mathsf{F} \in \{\mathsf{Up}^*, \mathsf{Fi}\}$ and M is completable,
- if the cis $\Bbbk$ is continuous with respect Fi,
- in particular if $\Bbbk$ is a totally ordered cis (e.g. $\Bbbk = \mathbb{R}_+^{\max}$).

To see how it works, let us suppose in the following lines that M is a completable module over a cis $\Bbbk$, and that c is such that $F_c(x)$ is nonempty for all $x \in M$. How far is then $x \mapsto \langle c, x\rangle$ from being a residuated map?

If $\langle c, x\rangle$ is non-zero, then the infimum of the subset $F_c(x)$ is reached (see Remark III-6.2), so that $c.\langle c, x\rangle \geqslant x$ or, in other words,

$$x \leqslant c.t \Longleftrightarrow t \geqslant \langle c, x\rangle, \tag{35}$$

for all $t \in \Bbbk$. However, this equivalence is no longer guaranteed if $\langle c, x\rangle = 0$. This is where a "smooth" behaviour of $\Bbbk \to M, t \mapsto x.t$ around zero is needed, in accordance with Remark III-6.2; so at this stage we must distinguish between the different filter selections.

The important fact is that the subset $F_c(x)$ is always filtered (if nonempty) by Equation (28). This makes Fi *the most natural filter selection* to use on $\Bbbk$-modules. As a consequence, if $\mathsf{F} \in \{\mathsf{Up}^*, \mathsf{Fi}\}$, Equivalence (35) is satisfied by completability of M, even if $\langle c, x\rangle = 0$. Thus, the second condition in Definition III-6.5 is fullfilled, and the map $x \mapsto \langle c, x\rangle$ is residuated. Moreover, $\langle c, x\rangle = 0$ implies $x = 0$.

The case $\mathsf{F} = \mathsf{PFi}$ is more delicate. We might include in the definition of c that $F_c(x)$ be an F-set, so here a principal filter; but this would imply that $\langle c, x\rangle \neq 0$ whenever $x \neq 0$, a property that is not desirable for applications (see e.g. the case of the Riesz representation theorem in Section III-10). That is why we introduced a second *ad hoc* condition in the definition of an archimedean element. The following proposition gives sufficient conditions on $\Bbbk$ for this condition to hold.

Proposition III-6.6. *Let $\Bbbk$ be an idempotent semifield. Consider the following conditions:*

1. *1 is way-above 0 with respect to Fi;*
2. *there is a $t \in \Bbbk$ way-above 0 with respect to Fi;*
3. *every filter with a zero infimum contains $\Bbbk\backslash\{0\}$;*
4. *every unbounded ideal coincides with $\Bbbk$;*
5. *$\Bbbk$ is continuous with respect to Fi;*
6. *$\Bbbk$ is totally ordered;*

Then (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1). If any of these conditions is satisfied, then

- *$\Bbbk$ is join-continuous (with respect to Fi), i.e. satisfies $s \oplus \bigwedge F = \bigwedge(s \oplus F)$ for all filters F and $s \in \Bbbk$;*
- *the second condition of Definition III-6.5 always holds;*
- *for all $t \in \Bbbk\backslash\{0\}$ and $s < 1$, there is some $n \in \mathbb{N}$ such that $t \geqslant s^n$.*

An archimedean element c of M is *strongly archimedean* if $t \gg s$ implies $c.t \gg c.s$ for all $s, t \in \Bbbk$. For an archimedean element c, the map

$x \mapsto \langle c, x \rangle$ is smooth only if c is strongly archimedean; the converse statement holds as soon as $\Bbbk$ is continuous.

The following result justifies the term *scalar product* for the bracket $\langle \cdot, \cdot \rangle$ (see also Cohen et al. [**63**, Section 3] for more on this topic; note that these authors preferred to call scalar product the bracket $(\cdot \backslash \cdot)$ rather than $\langle \cdot, \cdot \rangle$).

Lemma III-6.7. *Let M, N be modules over a cis $\Bbbk$, and let $f : M \to N$ be a (smooth) residuated linear map. If c is a (strongly) archimedean element of N then $f^{\#}(c)$ is a (strongly) archimedean element of M, and we have*

$$\langle c, f(x) \rangle = \langle f^{\#}(c), x \rangle,$$

for all $x \in M$, where $f^{\#}$ denotes the upper adjoint of f.

Proof. If $t \in \Bbbk \backslash \{0\}$, we have $c.t \geqslant f(x) \Leftrightarrow c \geqslant f(x.t^{-1}) \Leftrightarrow f^{\#}(c) \geqslant x.t^{-1} \Leftrightarrow f^{\#}(c).t \geqslant x$. Since c is archimedean, $F_c(f(x))$ contains some non-zero element, so that $F_{f^{\#}(c)}(x)$ is nonempty, so the first condition for $f^{\#}(c)$ to be archimedean is checked. Now if the subset $F_{f^{\#}(c)}(x)$ has zero infimum, then either it contains 0 (and in this case $x = 0$ so that $F_{f^{\#}(c)}(x) = \Bbbk$) or it does not. In the latter case, the series of equivalence at the beginning of the proof shows that $F_{f^{\#}(c)}(x) = F_c(f(x)) \backslash \{0\}$. This implies that $F_c(f(x))$ has zero infimum, thus contains $\Bbbk \backslash \{0\}$ since c is archimedean. Therefore, $F_{f^{\#}(c)}(x)$ also contains $\Bbbk \backslash \{0\}$, and we have proved that $f^{\#}(c)$ is archimedean. Using again the equivalence $c.t \geqslant f(x) \Leftrightarrow f^{\#}(c).t \geqslant x$ for all $t \in \Bbbk \backslash \{0\}$, we deduce that $t \geqslant \langle c, f(x) \rangle \Leftrightarrow t \geqslant \langle f^{\#}(c), x \rangle$ for all $t \in \Bbbk \backslash \{0\}$, so

$$\langle c, f(x) \rangle = \langle f^{\#}(c), x \rangle.$$

For the rest of the proof, assume that f is smooth and that c is strongly archimedean. Let $t \gg s$, and let us show that $f^{\#}(c).t \gg f^{\#}(c).s$. For this purpose, let F be an F-set of M with infimum such that $f^{\#}(c).s \geqslant \bigwedge F$. Then $f^{\#}(c.s) \geqslant \bigwedge F$, hence $c.s \geqslant f(\bigwedge F)$. Since f is smooth, this implies that $c.s \geqslant \bigwedge f(F)$. Since $\uparrow f(F)$ is an F-set of $\Bbbk$ and $c.t \geqslant c.s$, we obtain $c.t \geqslant f(x)$ for some $x \in F$. This gives $f^{\#}(c).t = f^{\#}(c.t) \geqslant x$, and the result is proved. ∎

We are now in a position to prove the main result of this section.

Theorem III-6.8 (Compare [**182**, Theorems 5.1-5.2], [**63**, Corollary 39])**.** *Suppose that M is a complete module over a continuous cis $\Bbbk$, and let $v : M \to \Bbbk$. The following conditions are equivalent:*

(1) *v is a smooth residuated form on M,*
(2) *v is a non-degenerate continuous linear form on M,*
(3) *$v(\cdot) = \langle c, \cdot \rangle$, for some strongly archimedean element $c \in M$.*

If these conditions are satisfied, then c is unique and equals the supremum of the set $\{x \in M : 1 \geqslant v(x)\}$.

Proof. Equivalence between (1) and (2) is given by Lemma III-6.3, and the implication (3) $\Rightarrow$ (1) was the purpose of § III-6.2. So let us prove that (1) implies (3). Let v be a smooth residuated form on M. Define c as the supremum of the set $\{x \in M : 1 \geqslant v(x)\}$, i.e. $c = v^{\#}(1)$, where $v^{\#}$ is the adjoint of v. If one notices that 1 is a strongly archimedean element in $\Bbbk$, then c is a strongly archimedean element in M by Lemma III-6.7, and one has

$$\langle 1, v(x) \rangle = \langle v^{\#}(1), x \rangle,$$

for all $x \in M$, that is $v(x) = \langle c, x \rangle$, for all $x \in M$. Uniqueness is deduced from the fact that $x \leqslant c \Leftrightarrow 1 \geqslant \langle c, x \rangle$. $\square$

Example III-6.9 (Example III-5.2 continued)**.** We introduced the linear form v on $M_{\mu,\sigma,\xi}$ defined by $v(x) = (1 + \xi\frac{x-\mu}{\sigma})^{1/\xi}$ if ξ is non-zero, $v(x) = \exp(\frac{x-\mu}{\sigma})$ otherwise. An easy computation shows that μ is the supremum of $\{x \in M_{\mu,\sigma,\xi} : 1 \geqslant v(x)\}$, and that $v(x) = \langle \mu, x \rangle$ for all $x \in M_{\mu,\sigma\,\xi}$. Moreover, μ is strongly archimedean (with respect to $\mathsf{F} = \mathsf{Up}^*$), for if $t > s$, then $\mu.t = \mu + \sigma\frac{t^{\xi}-1}{\xi} > \mu.s$ if ξ is non-zero, and $\mu.t = \mu + \sigma \log(t) > \mu.s$ otherwise. Hence, v is a smooth residuated form on $M_{\mu,\sigma,\xi}$ (where smoothness is understood with respect to $\mathsf{F} = \mathsf{Up}^*$).

This theorem upgrades a result by Cohen et al. [**63**, Corollary 39] deduced from a geometric Hahn–Banach type theorem [**63**, Theorem 34]. A different formulation will be proved in Section III-9, in the framework of *module extensions*. As for now, we use Theorem III-6.8 to reprove the Radon–Nikodym theorem for the Shilkret integral.

III-7. The Radon–Nikodym theorem: a different perspective

We come back to the Radon–Nikodym theorem for the Shilkret integral surveyed in Chapter I; here we deduce this result from the order-theoretical developments of the previous section. See Chapters I-II for definitions and notations related to maxitive measures. The filter selection used throughout this section is PFi, i.e. the one that selects principal filters.

III-7.1. Complements on σ-complete modules. The next result prepares applications to the Radon–Nikodym theorem. It gives sufficient conditions on a module M over a cis $\Bbbk$ in order that every linear form on M be continuous. The module M is *σ-complete* if every upper-bounded countable subset has a supremum and if M is completable, i.e. if

$$x.\bigoplus T = \bigoplus_{t \in T} x.t,$$

for all $x \in M$, $T \subset \Bbbk$ with supremum. We say that M is *σ-principal* if every upper-bounded σ-ideal is principal, i.e. of the form $\downarrow x$ for some $x \in M$. A subset G of M is *generating* if, for all $x \in M$, $x = \bigoplus \downarrow x \cap G$. Also, the module is *countably generated* if there exists a generating subset G such that $\downarrow x \cap G$ is countable, for all $x \in M$. A linear form v on M is

σ-continuous if, for every countable subset $X \subset M$ admitting a supremum in M, $v(X)$ has a supremum in $\Bbbk$ satisfying $v(\bigoplus X) = \bigoplus v(X)$.

Proposition III-7.1. *Let M be a σ-complete module over a cis $\Bbbk$.*

(1) *If M is countably generated, then M is σ-principal.*
(2) *If M is σ-principal, then M is complete and every σ-continuous linear form is continuous.*

Proof. (1) Assume that M is countably generated by some subset G. Let I be an upper-bounded σ-ideal of M. If u is an upper-bound of I, the subset $I \cap G$ is included in the countable subset $\downarrow u \cap G$, hence is countable. So let $x := \bigoplus G \cap I \in I$. It is easily seen that $x = \bigoplus I$, hence $I = \downarrow x$, i.e. I is a principal ideal.

(2) Assume that M is σ-principal, and let X be an upper-bounded subset of M. The σ-ideal I generated by X is made up of elements lower than joins of countable subsets of X. Since I is upper-bounded, it is principal, so we have $I = \downarrow x$ for some $x \in I$, and x is of the form $x = \bigoplus G$ for some countable subset G of X. Thus, we have $x = \bigoplus G = \bigoplus I = \bigoplus X$, so that M is complete. Moreover, if $v : M \to \Bbbk$ is a σ-continuous linear form, then $v(\bigoplus X) = v(\bigoplus G) = \bigoplus v(G) \leqslant \bigoplus v(X)$, hence v is continuous. $\square$

III-7.2. Maxitive measures as linear forms. Let $\mathscr{E}$ be a semi-σ-algebra on some nonempty set E and ν, τ be σ-maxitive measures on $\mathscr{E}$. We shall assume that ν is finite and absolutely continuous with respect to τ, in symbols $\nu \dashv \tau$. In order to apply the results of Section III-6, we merely want to get rid of the collection of τ-negligible subsets. We could consider the quotient space $\mathscr{E}/\tau$, but this would not give us the structure of module over the idempotent semifield $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \max, \times)$ that we need. A better idea is the following.

Let $\mathscr{E}_+^{\nu} = \mathscr{L}_+^1(E, \mathscr{E}, \nu)$ be the set of all ν-integrable lsm maps $g : E \to \overline{\mathbb{R}}_+$. A map n in $\mathscr{E}_+^{\nu}$ is *τ-negligible* if the subset $\{n > 0\}$ is τ-negligible. We define on $\mathscr{E}_+^{\nu}$ the equivalence relation $\langle\rangle$ by $f \langle\rangle g$ if and only if, for some τ-negligible map n, we have $f \oplus n = g \oplus n$. We denote by $\langle g \rangle$ the equivalence class of a $g \in \mathscr{E}_+^{\nu}$. Then the quotient set $\mathbf{M} := \mathscr{E}_+^{\nu}/\tau := \mathscr{E}_+^{\nu}/\langle\rangle$ is a σ-complete module over $\mathbb{R}_+^{\max}$ with external multiplication $\mathbf{f}.t := \langle t.f \rangle$ and countable addition $\bigoplus_{j=1}^{\infty} \mathbf{g}_j = \langle \bigoplus_{j=1}^{\infty} g_j \rangle$, for all $t \in \mathbb{R}_+$ and $\mathbf{f} = \langle f \rangle, \mathbf{g}_j = \langle g_j \rangle \in \mathbf{M}$. The induced partial order is $\mathbf{f} \leqslant \mathbf{g}$ if and only if $\{f > g\}$ is τ-negligible. The reader can check that the previous definitions do not depend on the choice of the representatives f, g, etc.

Recall that τ on $\mathscr{E}$ is *localizable* if, for each σ-ideal $\mathscr{I}$ of $\mathscr{E}$, there exists some $L \in \mathscr{E}$ such that

- $S \backslash L$ is τ-negligible, for all $S \in \mathscr{I}$,
- if there is some $G \in \mathscr{E}$ such that $S \backslash G$ is τ-negligible for all $S \in \mathscr{I}$, then $L \backslash G$ is τ-negligible.

In this case, $\mathscr{I}$ is said to be *localized* in L.

Proposition III-7.2. *Let ν, τ be σ-maxitive measures on $\mathscr{E}$. Assume that ν is finite and such that $\nu \dashv \tau$. Then τ is localizable (resp. σ-principal) if and only if $\mathscr{E}_+^\nu/\tau$ is a complete module (resp. a σ-principal module).*

Proof. A preliminary remark is that, since ν is finite, the lsm map 1_G is ν-integrable for all $G \in \mathscr{E}$. Assume that $\mathbf{M} = \mathscr{E}_+^\nu/\tau$ is a complete module, and let $\mathscr{I}$ be a σ-ideal of $\mathscr{E}$. Then the σ-ideal I generated by $\{\langle 1_S \rangle : S \in \mathscr{I}\}$ is upper-bounded (by $\langle 1_E \rangle$) in $\mathbf{M}$. Hence there is some $f \in \mathscr{E}_+^\nu$ such that $\langle f \rangle$ is the supremum of I. In particular, if $S \in \mathscr{I}$, there is some τ-negligible lsm map n such that $1_S \leqslant f \oplus n$, so that $S \subset L \cup \{n > 0\}$, where $L := \{f > 2^{-1}\}$. As a consequence, $S \backslash L$ is τ-negligible for all $S \in \mathscr{I}$. To show that $\mathscr{I}$ is localized in L, let $G \in \mathscr{E}$ such that $S \backslash G$ is τ-negligible for all $S \in \mathscr{I}$. Then $\langle 1_G \rangle$ is an upper-bound of I, so that $\langle f \rangle \leqslant \langle 1_G \rangle$ by definition of f. Since $2^{-1}.1_L \leqslant f$, we deduce that $L \backslash G$ is τ-negligible, hence that τ is localizable.

If $\mathbf{M}$ is σ-principal, we can impose L to belong to $\mathscr{I}$ and to be such that $\langle 1_L \rangle$ generates I. Then L generates $\mathscr{I}$, and this proves that τ is σ-principal.

Conversely, suppose that τ is localizable, and let I be an upper-bounded σ-ideal of $\mathbf{M}$. If $q \in \mathbb{Q}_+$, let $\mathscr{I}_q = \{\{f > q\} : \langle f \rangle \in I\}$. This is a σ-ideal, hence it is localized in some $L_q \in \mathscr{E}$. Let $\langle g \rangle$ be an upper-bound of I. Then $S \backslash \{g > q\}$ is τ-negligible, for all $S \in \mathscr{I}_q$ and all $q \in \mathbb{Q}_+$. Since $\mathscr{I}_q$ is localized in L_q we deduce that $L_q \backslash \{g > q\}$ is τ-negligible. This implies that the map ℓ defined by $\ell = \bigoplus_{q \in \mathbb{Q}_+} q.1_{L_q}$ is ν-integrable and satisfies $\langle \ell \rangle \leqslant \langle g \rangle$. To show that $\langle \ell \rangle$ is the supremum of I, it suffices to prove that $\langle \ell \rangle$ is an upper-bound of I. If $\langle f \rangle \in I$, there exists some τ-negligible subset $N_q \in \mathscr{E}$ such that $\{f > q\} \subset L_q \cup N_q$. If $n = \bigoplus_{q \in \mathbb{Q}} q.1_{N_q}$, then $\{n > 0\} \subset \bigcup_{q \in \mathbb{Q}_+} N_q$, so n is τ-negligible. We have $f \leqslant \ell \oplus n$, so that $\langle f \rangle \leqslant \langle \ell \rangle$. This proves that $\langle \ell \rangle$ is the supremum of I, and that $\mathbf{M}$ is complete.

If τ is σ-principal, then the set L_q can be choosen of the form $\{\ell_q > q\}$, where $\langle \ell_q \rangle \in I$. It can be seen that ℓ and $\bigoplus_{q \in \mathbb{Q}_+} \ell_q$ are equivalent, so that $\langle \ell \rangle = \langle \bigoplus_{q \in \mathbb{Q}_+} \ell_q \rangle \in I$. This shows that I is principal, so that $\mathbf{M}$ is a σ-principal module. □

We denote by $\mathbf{v}$ the map induced by ν on $\mathbf{M}$, i.e.

$$\mathbf{v}(\mathbf{f}) = \int^\infty f \, d\nu,$$

for all $\mathbf{f} = \langle f \rangle \in \mathbf{M}$. Since $\mathscr{E}_+^\nu$ demands ν-integrable maps, we have $\mathbf{v}(\mathbf{f}) < \infty$, so $\mathbf{v}$ is a σ-continuous linear form on $\mathbf{M}$. We shall say that ν is *τ-continuous* if $\mathbf{v}$ is continuous. As a corollary of Theorem III-6.8 we have the following result. Recall that a map $g : E \to \overline{\mathbb{R}}_+$ is *upper-semimeasurable* or *usm* if $\{g < t\} \in \mathscr{E}$ for all $t \in \mathbb{R}_+$.

Theorem III-7.3. *Let ν, τ be σ-maxitive measures on $\mathscr{E}$. Assume that ν is finite and τ is localizable. Then the following assertions are equivalent:*

- *$\nu \dashv \tau$ and ν is τ-continuous,*

- *ν has a usm relative density with respect to τ.*

Proof. Since ν is finite and τ is localizable, $\mathbf{M} = \mathscr{E}_+^\nu/\tau$ is a complete module by Proposition III-7.2. From the identity $\langle \mathbf{c}, \mathbf{f} \rangle = \bigoplus_{x \in E}^\tau \frac{f(x)}{c(x)}$, which holds for all $\mathbf{f} = \langle f \rangle, \mathbf{c} = \langle c \rangle \in \mathbf{M}$, we deduce that ν has a usm relative density with respect to τ if and only if there is some $\mathbf{c} \in \mathbf{M}$ such that $\mathbf{v}(\cdot) = \langle \mathbf{c}, \cdot \rangle$. This situation implies that $\nu \dashv \tau$ and ν is τ-continuous by Theorem III-6.8.

For the converse statement, assume that $\nu \dashv \tau$ and ν is τ-continuous. We only have to prove that $\mathbf{v}$ is non-degenerate, for then Theorem III-6.8 gives the desired result. So let $f \in \mathscr{E}_+^\nu$ such that $\mathbf{v}(\mathbf{f}) \leqslant 1$, where $\mathbf{f} = \langle f \rangle$. Then, for all rational numbers $q > 0$, the subset $\{f > q\}$ is in the σ-ideal $\mathscr{I}_q = \{G \in \mathscr{E} : \nu(G) \leqslant q^{-1}\}$. Since τ is localizable, $\mathscr{I}_q$ is localized in some $L_q \in \mathscr{E}$, and since ν is τ-continuous, we have $\nu(L_q) \leqslant q^{-1}$. As a consequence, the map $g = \bigoplus_{q \in \mathbb{Q}_+} q.1_{L_q}$ is lsm, ν-integrable (with $\int^\infty g \, d\nu \leqslant 1$), and such that $\langle f \rangle \leqslant \langle g \rangle$. This proves that the subset $\{\mathbf{f} \in \mathbf{M} : 1 \geqslant \mathbf{v}(\mathbf{f})\}$ is upper-bounded in $\mathbf{M}$, i.e. that $\mathbf{v}$ is non-degenerate. □

Corollary III-7.4. *Let ν, τ be σ-maxitive measures on $\mathscr{E}$. Assume that τ is σ-principal. Then $\nu \dashv \tau$ if and only if ν has a usm relative density with respect to τ.*

Proof. We first suppose that ν is finite. If τ is σ-principal, then $\mathscr{E}_+^\nu/\tau$ is a σ-principal module by Proposition III-7.2. By Proposition III-7.1, ν is τ-continuous and τ is localizable. So Theorem III-7.3 applies, and ν admits a usm relative density with respect to τ. In the case where ν is non-finite, we replace ν by $\nu_1 : B \mapsto \arctan \nu(B)$, which has a usm relative density c_1 with respect to τ. Therefore, $\tan c_1$ is a usm relative density of ν with respect to τ. □

Example III-7.5. Let E be a topological space, $\mathscr{E}$ be the collection $\mathscr{G}$ of open subsets of E, and $\tau = \delta_\#$. Then $\delta_\#$ is localizable and $\nu \dashv \delta_\#$, for all maxitive measures ν on $\mathscr{G}$. Moreover, ν is $\delta_\#$-continuous if and only if ν is completely maxitive if and only if ν has a usc cardinal density. Also, $\delta_\#$ is σ-principal if and only if every subset of E is Lindelöf (then E is usually said to be *hereditarily Lindelöf*, a property that is implied by second-countability), in which case every ν on $\mathscr{G}$ has a cardinal density.

To conclude this section we propose a new proof of the Sugeno–Murofushi theorem, which is a Radon–Nikodym like theorem for the Shilkret integral (see [Chapter I, Theorem 6.4]).

Theorem III-7.6 (Sugeno–Murofushi)**.** *Let ν, τ be σ-maxitive measures on a σ-algebra $\mathscr{B}$. Assume that τ is σ-finite and σ-principal. Then $\nu \dashv \tau$ if and only if there exists some $\mathscr{B}$-measurable map $c : E \to \overline{\mathbb{R}}_+$ such that*

$$\nu(B) = \int_B^\infty c \, d\tau,$$

for all $B \in \mathscr{B}$. If these conditions are satisfied, then c is unique τ-almost everywhere.

Proof. If $\nu \dashv \tau$, then by Corollary III-7.4 there are $\mathscr{B}$-measurable maps $c_1, c_2 : E \to \overline{\mathbb{R}}_+$ such that $\nu(B) = \bigoplus_{x\in B}^{\tau} c_1(x)$ and $\tau(B) = \bigoplus_{x\in B}^{\tau} c_2(x)$, for all $B \in \mathscr{B}$. Since τ is σ-finite, one can choose a map c_2 that takes only finite values (see [Chapter I, Proposition 6.1]). Using the fact that $\tau(\{c_2 = 0\}) = 0$, it is easy to show that $\nu(\cdot) = \int_{\cdot}^{\infty} c \, d\tau$ on $\mathscr{B}$, where c is the measurable map defined by $c(x) = c_1(x)/c_2(x)$ if $c_2(x) \neq 0$, $c(x) = 0$ otherwise. □

III-8. Completable modules and the normal completion

III-8.1. The normal completion of a completable module. The following theorem defines the concept of *normal completion* of a completable module, alias *Dedekind–MacNeille completion* or *completion by cuts*. See e.g. Erné [**91**] for the normal completion of quasiordered sets.

Theorem III-8.1. *Let $\Bbbk$ be an idempotent semifield. A $\Bbbk$-module is completable if and only if it can be continuously embedded into a complete $\Bbbk$-module.*

Sketch of the proof. Sufficiency is obvious. For necessity, let M be a completable $\Bbbk$-module. We follow the usual Dedekind–MacNeille completion method for partially ordered sets. If $X \subset M$, we write $X^{\downarrow}$ (resp. $X^{\uparrow}$) for the subset of lower (resp. upper) bounds of X in M, and we write $X^{\uparrow\downarrow}$ instead of $(X^{\uparrow})^{\downarrow}$. A subset X of M is *closed* if $X^{\uparrow\downarrow} = X$, and *proper* if either $X \neq M$ or M has a greatest element. Let $\mathscr{N}(M)$ be the collection of all proper closed subsets X of M. If $X \oplus X' := (X \cup X')^{\uparrow\downarrow}$ for all proper closed subsets X, X', then $X \oplus X'$ is closed, proper (to prove this, note that a closed subset is proper if and only if it is upper-bounded) and $(\mathscr{N}(M), \oplus, \{0\})$ is a commutative idempotent monoid. The partial order induced by $\oplus$ on $\mathscr{N}(M)$ is the inclusion, i.e. $X \leqslant X' \Leftrightarrow X \subset X'$. For the external multiplication we let $X.t := \{x.t : x \in X\}$ if $t \neq 0$ and $X.0 = \{0\}$, and one can check that $X.t$ is proper closed for all proper closed subsets X. Also, since M is completable, the following relations hold:

$$(X \oplus X').t = X.t \oplus X'.t,$$
$$X.\bigoplus T = \bigoplus_{t\in T} X.t,$$

for all $X, X' \in \mathscr{N}(M)$, $t \in \Bbbk$ and $T \subset \Bbbk$ with supremum, and

$$X.\bigwedge F = \bigwedge_{f\in F} X.f,$$

for all $X \in \mathscr{N}(M)$ and F-sets F in $\Bbbk$ with infimum. Thus, $\mathscr{N}(M)$ is a completable $\Bbbk$-module, which is actually complete for

$$\bigoplus_{j\in J} X_j = (\bigcup_{j\in J} X_j)^{\uparrow\downarrow},$$

for all upper-bounded families $(X_j)_{j\in J}$ of proper closed subsets. Note that the infimum in $\mathscr{N}(M)$ satisfies

$$\bigwedge_{j\in J} X_j = \bigcap_{j\in J} X_j,$$

for all families $(X_j)_{j\in J}$ of proper closed subsets.

To embed M into $\mathscr{N}(M)$, let $i_M : M \to \mathscr{N}(M), x \mapsto \downarrow x$. This map i_M is well defined, for $\downarrow x$ is proper closed for all $x \in M$. Clearly, we have $i_M(x.t) = i_M(x).t$ and $i_M(x \oplus y) = i_M(x) \oplus i_M(y)$ for all $x, y \in M$ and $t \in \Bbbk$, so that i_M is an injective morphism. Moreover, for all subsets X of M with supremum (resp. with infimum), we have $i_M(\bigoplus X) = \bigoplus i_M(X)$ (resp. $i_M(\bigwedge X) = \bigwedge i_M(X)$), so that i_M is continuous. $\square$

Remark III-8.2. If $\mathsf{F} = \mathsf{PFi}$, then a module is completable if and only if it is *b-regular* in the sense of Litvinov et al. [**182**, Definition 3.9].

Remark III-8.3. Identifying M and $i_M(M)$, every element of $\mathscr{N}(M)$ can be expressed as a supremum (resp. an infimum) of elements of M. In particular, every element of $\mathscr{N}(M)$ is upper-bounded by some element of M.

Remark III-8.4. Every idempotent semifield considered as a module over itself is completable. However, an idempotent semifield can be embedded into a complete idempotent semifield if and only if it is commutative (see Remark III-4.2).

III-8.2. Cut-stability and extensions. In this paragraph we give two categorical results on the normal completion. Since every $\{0, 1\}$-module is completable, they extend that of Erné [**91**].

Erné introduced the concept of cut-stability, that we modify as follows. A map $f : M \to N$ is *lower cut-stable* if

$$f(X^{\uparrow})^{\downarrow} = f(X)^{\uparrow\downarrow},$$

for all subsets X of M, and *cut-stable* if it is lower cut-stable and such that

$$f(F^{\downarrow})^{\uparrow} = f(F)^{\downarrow\uparrow},$$

for all F-sets F of M. For instance, the map i_M that embeds a completable module into its normal completion (see the proof of Theorem III-8.1) is cut-stable. Note that every cut-stable morphism is continuous.

Proposition III-8.5 (Compare with [**91**, Theorem 3.1])**.** *Let $\Bbbk$ be an idempotent semifield. A morphism f between completable $\Bbbk$-modules M and N is lower cut-stable if and only if there exists a (unique) morphism $\mathscr{N}(f)$ between the normal completions $\mathscr{N}(M)$ and $\mathscr{N}(N)$ that preserves arbitrary* *existing suprema and extending f, i.e. such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"'] & N \arrow[d, "i_N"] \\ \mathscr{N}(M) \arrow[r, "\mathscr{N}(f)"] & \mathscr{N}(N) \end{tikzcd}$$

Moreover, in the cases $\mathsf{F} = \mathsf{Up}^*$ *and* $\mathsf{F} = \mathsf{PFi}$*, the morphism f is cut-stable if and only if $\mathscr{N}(f)$ is continuous, and the normal completion extends to a functor $\mathscr{N}$ on the category of completable $\Bbbk$-modules with continuous morphisms.*

Proof. Analogous to that of [**91**, Theorem 3.1]. □

Remark III-8.6. In the case $\mathsf{F} = \mathsf{Fi}$, it is only possible to say that $\mathscr{N}(f)$ preserves infima of filters of $\mathscr{N}(M)$ of the form $\mathscr{F}_F = \{X \in \mathscr{N}(M) : X \cap F \neq \varnothing\}$, with F a filter of M. This is not enough to make $\mathscr{N}(f)$ smooth (i.e. continuous) in general.

The following universal property of the normal completion is deduced immediately.

Corollary III-8.7 (Compare with [**91**, Corollary 3.2])**.** *Let $\Bbbk$ be an idempotent semifield. A morphism f from a completable $\Bbbk$-module M into a complete $\Bbbk$-module N is lower cut-stable if and only if there exists a (unique) morphism from $\mathscr{N}(M)$ into N arbitrary existing suprema and extending f, i.e. such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"'] & N \\ \mathscr{N}(M) \arrow[ur, dashed, "\mathscr{N}(f)"'] & \end{tikzcd}$$

Moreover, in the cases $\mathsf{F} = \mathsf{Up}^*$ *and* $\mathsf{F} = \mathsf{PFi}$*, the morphism f is cut-stable if and only if $\mathscr{N}(f)$ is continuous.*

We call a pair $\overline{M}/M$ an *extension* over $\Bbbk$ (we shall also speak of the extension $\overline{M}$ of M over $\Bbbk$) if $\overline{M}$ is a complete module over $\Bbbk$ and M is a submodule of $\overline{M}$. In this situation, M is necessarily completable. The extension is *short* if, for all $y \in \overline{M}$, there is some $x \in M$ such that $y \leqslant x$. This condition restricts the "size" of $\overline{M}$ and will reveal its importance in the next section. Also, the extension is *cut-stable* if the map $i : M \ni x \mapsto x \in \overline{M}$ is cut-stable; in this case, if a subset (resp. an F-set) of M has a supremum (resp. an infimum) in M, then it coincides with its supremum (resp. its infimum) in $\overline{M}$. For a completable module M, the normal completion leads to a short and cut-stable extension $\mathscr{N}(M)/i_M(M)$ (see Remark III-8.3).

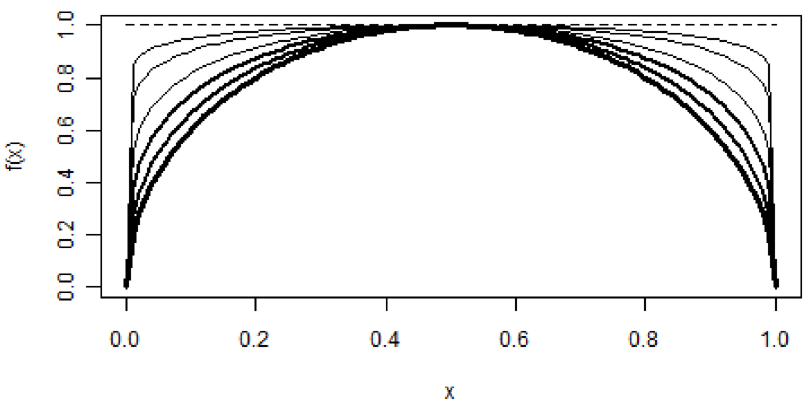

FIGURE 1. A nondecreasing sequence of continuous functions on $[0, 1]$.

Example III-8.8. Let E be a Hausdorff topological space, let C_c^+ be the set of compactly-supported continuous maps from E to $\mathbb{R}_+$, and let L^+ be the set of lower-semicontinuous maps from E to $\overline{\mathbb{R}}_+$. Then L^+/C_c^+ is an extension over $\mathbb{R}_+^{\max}$ that is neither short nor cut-stable in general. Figure 1 gives a sequence of continuous functions on $E = [0, 1]$, whose supremum is $x \mapsto 1$ in C_c^+, and is $x \mapsto 1_{(0,1)}(x)$ in L^+.

III-9. Residuated forms on a module extension

This section is expressed in the language PFi of principal filters, and $\overline{M}/M$ is an extension over an idempotent semifield $\mathbb{k}$. Henceforth, all suprema of subsets of M or $\overline{M}$ are taken in $\overline{M}$. A map $v : M \to \mathbb{k}$ is *residuated on* $\overline{M}/M$ if there exists a map $w : \mathbb{k} \to \overline{M}$ satisfying

$$x \leqslant w(t) \iff t \geqslant v(x), \tag{36}$$

for all $x \in M, t \in \mathbb{k}$. In this case, there exists a least map w such that Equivalence (36) holds, called the *adjoint* of v with respect to $\overline{M}/M$, denoted by $v^\#$, and defined by

$$v^\#(t) = \bigoplus \{x \in M : t \geqslant v(x)\},$$

for all $t \in \mathbb{k}$, where the supremum is taken in $\overline{M}$.

Lemma III-9.1. *A map that is residuated on a short extension of M is residuated on each extension of M.*

Proof. Let $\overline{M}/M$ be a short extension. Consider the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "i_M"'] & \overline{M} \\ \mathscr{N}(M) \arrow[ur, dashed, "\bar{i}"'] & \end{tikzcd}$$

where $\bar{i}$ is defined by $\bar{i}(X) = \bigoplus X$. Let $v : M \to \mathbb{k}$ be a residuated map on $\overline{M}/M$. Then v admits an adjoint $v^{\#} : \mathbb{k} \to \overline{M}$. We first show that v is residuated on $\mathscr{N}(M)/M$. If $t \in \mathbb{k}$, the subset $I_t = \{x \in M : t \geqslant v(x)\}$ is upper-bounded (by $v^{\#}(t)$) in $\overline{M}$, hence also in M since $\overline{M}/M$ is short. Thus, I_t admits a supremum in $\mathscr{N}(M)$, that we denote by $w(t)$. Since $\bar{i}$ preserves arbitrary existing suprema, $\bar{i}(w(t)) = \bigoplus\{x \in M : t \geqslant v(x)\}$, where the supremum is taken in $\overline{M}$. Thus, $\bar{i} \circ w = v^{\#}$. We show that Equivalence (36) holds. Clearly, $t \geqslant v(x)$ implies $x \leqslant w(t)$. Conversely, assume that $x \leqslant w(t)$. Composing by $\bar{i}$, we get $x \leqslant v^{\#}(t)$, so that $t \geqslant v(x)$. This proves that v is residuated on $\mathscr{N}(M)/M$.

Now let $\tilde{M}/M$ be some extension of M, and consider the related commutative diagram:

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[d, "i_M"'] & \tilde{M} \\ \mathscr{N}(M) \arrow[ur, dashed, "\tilde{j}"'] & \end{tikzcd}$$

where $j : M \ni x \mapsto x \in \tilde{M}$ and $\tilde{j} : X \mapsto \bigoplus X$. Then one can show that

$$x \leqslant \tilde{j} \circ w(t) \Leftrightarrow t \geqslant v(x),$$

for all $x \in M, t \in \mathbb{k}$, so v is residuated on $\tilde{M}/M$. □

We define a *residuated form* on $\overline{M}/M$ as a homogeneous residuated map on $\overline{M}/M$. A map $v : M \to \mathbb{k}$ is *non-degenerate* on $\overline{M}/M$ if $\{x \in M : 1 \geqslant v(x)\}$ has an upper-bound in $\overline{M}$.

Lemma III-9.2. *Let $\overline{M}/M$ be a short extension over a cis $\mathbb{k}$. We suppose that every element of $\overline{M}$ can be expressed as the supremum in $\overline{M}$ of elements of M. Then a map $v : M \to \mathbb{k}$ is a residuated form on $\overline{M}/M$ if and only if*

v is non-degenerate on $\overline{M}/M$ and extends to a residuated form on $\overline{M}$.

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[d, "i"'] & \mathbb{k} \\ \overline{M} \arrow[ur, dashed, "\overline{v}"'] & \end{tikzcd}$$

Proof. Let v be a residuated form on $\overline{M}/M$ with adjoint $v^{\#} : \mathbb{k} \to \overline{M}$. Then $\{x \in M : 1 \geqslant v(x)\}$ is upper-bounded (by $v^{\#}(1)$) in $\overline{M}$, so v is non-degenerate on $\overline{M}/M$. Moreover, if $y \in \overline{M}$, then y is upper-bounded by some $x \in M$ since $\overline{M}/M$ is short, so the subset $\{t \in \mathbb{k} : y \leqslant v^{\#}(t)\}$ of $\mathbb{k}$, which contains $t = v(x)$, is nonempty. Thus, we can define the map $\overline{v} : \overline{M} \to \mathbb{k}$ by $\overline{v}(y) = \bigwedge\{t \in \mathbb{k} : y \leqslant v^{\#}(t)\}$. Since every element of $\overline{M}$ can be expressed as the supremum in $\overline{M}$ of elements of M, we have

$$y \leqslant v^{\#}(t) \Leftrightarrow t \geqslant \overline{v}(y),$$

for all $y \in \overline{M}, t \in \mathbb{k}$. So we obtain that $\overline{v}$ is a residuated form on $\overline{M}$.

Conversely, assume that a map $v : M \to \mathbb{k}$ is non-degenerate on $\overline{M}/M$ and extends to a residuated form $\overline{v}$ on $\overline{M}$. If w is the adjoint of $\overline{v}$ (with respect to $\overline{M}/\overline{M}$), then

$$x \leqslant w(t) \Leftrightarrow t \geqslant v(x),$$

for all $x \in M, t \in \mathbb{k}$, so v is a residuated form on $\overline{M}/M$. □

With respect to the filter selection PFi, we say that an element $c \in \overline{M}$ is *archimedean in* $\overline{M}/M$ if

- the subset $F_c(x)$ is nonempty for all $x \in M$,
- if the infimum of $F_c(x)$ is zero, then $F_c(x) \supset \mathbb{k}\backslash\{0\}$,

where $F_c(x)$ denotes the subset $\{t \in \mathbb{k} : c.t \geqslant x\}$. The next lemma can be proved along the same lines as Lemma III-6.7.

Lemma III-9.3. *Let $\overline{M}/M$ be an extension over a cis $\mathbb{k}$, and let $v : M \to \mathbb{k}$ be a residuated form on $\overline{M}/M$. Then the supremum of $\{x \in M : 1 \geqslant v(x)\}$ is archimedean in $\overline{M}/M$.*

The innovation of this section mainly relies on highlighting the role of the following concept in the representation of continuous linear forms. An extension $\overline{M}/M$ is *meet-continuous* if

$$x \wedge \bigoplus I = \bigoplus \downarrow x \cap I,$$

for all $x \in M$ and all ideals I of M with an upper-bound in $\overline{M}$.

Example III-9.4 (Example III-8.8 continued)**.** Let E be a topological space. We still denote by L^+ the set of lower-semicontinuous maps from E to $\overline{\mathbb{R}}_+$. If M is some submodule of the set of continuous maps from E to $\mathbb{R}_+$, then the extension L^+/M is meet-continuous. Before proving this assertion, note that the supremum in L^+ coincides with the pointwise supremum. Now let $f \in M$ and let I be an ideal in M. We want to show that $f \wedge \bigoplus I \leqslant \bigoplus \downarrow f \cap I$, i.e. that $f(x) \wedge \bigoplus_{g \in I} g(x) \leqslant \bigoplus_{h \in I, h \leqslant f} h(x)$. For this purpose, let $s < f(x) \wedge \bigoplus_{g \in I} g(x)$. There is some $g \in I$ such that $s < f(x) \wedge g(x)$. Then the map $h = f \wedge g$ is continuous, is in I and satisfies $h \leqslant f$ and $s < h(x)$, so the claim follows.

A map $f : M \to \Bbbk$ is *continuous* on $\overline{M}/M$ if, for every subset $X \subset M$ such that $\bigoplus X \in M$, $f(X)$ has a supremum in $\Bbbk$ satisfying $f(\bigoplus X) = \bigoplus f(X)$. If $x \in M$ and $c \in \overline{M}$, we write $\langle c, x \rangle$ for the infimum of $\{t \in \Bbbk : c.t \geqslant x\}$ whenever this set is nonempty, and $\langle c, x \rangle = \top$ otherwise. The following theorem shows that, under conditions different than those of Theorem III-6.8, the representation $v(\cdot) = \langle c, \cdot \rangle$ still holds, at the price that c no longer needs to belong to M. But it is actually important to authorize c to be outside M, in order to encompass the (idempotent) Riesz representation theorem (see Theorem III-10.3 below).

Theorem III-9.5. *Suppose that $\overline{M}/M$ is a meet-continuous extension over a cis $\Bbbk$, and let $v : M \to \Bbbk$ be a linear form on M. Then the following conditions are equivalent:*

(1) *v is residuated on $\overline{M}/M$,*
(2) *v is non-degenerate continuous on $\overline{M}/M$,*
(3) *$v(\cdot) = \langle c, \cdot \rangle$, for some archimedean element c in $\overline{M}/M$.*

If any of these conditions is satisfied, then the supremum of $\{1 \geqslant v\}$ in $\overline{M}$ is the least c satisfying $v(\cdot) = \langle c, \cdot \rangle$.

Proof. Assume that v is non-degenerate and continuous. Let $t \in \Bbbk$ and $I_t = \{x \in M : t \geqslant v(x)\}$. Clearly I_t is an ideal, let $w(t)$ denote its supremum in $\overline{M}$. To prove that v is residuated on $\overline{M}/M$, we have to show that, if $x \in M$ and $x \leqslant w(t)$, then $t \geqslant v(x)$, i.e. $x \in I_t$. If $\overline{M}/M$ is meet-continuous, then $x = x \wedge w(t) = x \wedge \bigoplus I_t = \bigoplus \downarrow x \cap I_t \in M$, and using the fact that v is continuous we get $v(x) = \bigoplus_{y \in \downarrow x \cap I_t} v(y) \leqslant t$, i.e. $x \in I_t$. This proves that (2) implies (1), and the converse implication is clear. The equivalence between (1) and (3) can be proved along the same lines as in the proof of Theorem III-6.8. $\square$

Remark III-9.6. If every element of $\overline{M}$ can be expressed as the supremum in $\overline{M}$ of elements of M, there is a *unique* such element c in the previous theorem. Indeed, assume that, for some archimedean elements b, c in $\overline{M}/M$, we have $\langle b, x \rangle = \langle c, x \rangle$ for all $x \in M$. Then $x \leqslant b \Leftrightarrow 1 \geqslant \langle b, x \rangle \Leftrightarrow 1 \geqslant \langle c, x \rangle \Leftrightarrow x \leqslant c$, for all $x \in M$, so that $\downarrow b \cap M = \downarrow c \cap M$. This implies that $b = \bigoplus \downarrow b \cap M = \bigoplus \downarrow c \cap M = c$.

As a direct application, we shall reprove an idempotent version of the Riesz representation theorem in Section III-10.

III-10. The Riesz representation theorem

In this section, we aim at proving Riesz representation theorems for the Shilkret integral with the help of Theorem III-9.5. The filter selection implicitly used here is PFi, i.e. the one that selects principal filters.

Let E be a Hausdorff topological space, $\mathscr{G}$ (resp. $\mathscr{B}$) be the collection of open subsets (resp. Borel subsets) of E, and C_c^+ be the set of nonnegative compactly-supported continuous maps from E to $\mathbb{R}_+$. The next theorem is of historical importance, for (part of) it was originally stated by Choquet [**60**, Paragraph 53.1] without proof; the first proof is due to Kolokoltsov and Maslov [**153**, Theorem 1]. The reader can also refer to Puhalskii [**246**, Theorem 1.7.21].

Lemma III-10.1 (Urysohn)**.** *Let E be a locally-compact Hausdorff space. If $K \subset U \subset E$ with K compact and U open, then there exists a compactly-supported continuous map $f : E \to [0, 1]$ such that $f(x) = 1$ for all $x \in K$ and $\overline{\{x \in E : f(x) > 0\}} \subset U$.*

Proof. This is customarily proved by using the fact that the one-point compactification of E is a normal space, see e.g. Aliprantis and Border [**15**, Corollary 2.74]. □

Lemma III-10.2. *Let E be a locally-compact Hausdorff space. Every lower-semicontinuous map $g : E \to \overline{\mathbb{R}}_+$ is a supremum of elements of C_c^+.*

Proof. Let $s \in \mathbb{R}_+$ and $x \in E$ be such that $s < g(x)$. Since $\{g > s\}$ is an open subset, there is some $f_1 \in C_c^+$ such that $s < f_1(x) < g(x)$ and $f_1 = 0$ on $\{g \leqslant s\}$, by Urysohn's lemma. Now $\{g > f_1\}$ is also open, so there is some $f_2 \in C_c^+$ such that $f_2(x) = f_1(x)$ and $f_2 = 0$ on $\{g \leqslant f_1\}$. This proves that the map $f_{s,x} = f_1 \wedge f_2$ is in C_c^+ and satisfies $f_{s,x} \leqslant g$ and $s < f_{s,x}(x)$. As a consequence, one can see that g is the pointwise supremum of all such maps $f_{s,x}$. □

Theorem III-10.3 (Improves [**246**, Theorem 1.7.21])**.** *Let E be a locally-compact Hausdorff space, and let $V : C_c^+ \to \mathbb{R}_+$ be a linear form on C_c^+. Then there exists a unique regular maxitive measure ν on $\mathscr{B}$ such that*

$$V(f) = \int^\infty f \, d\nu,$$

for all $f \in C_c^+$. Moreover, ν takes finite values on compact subsets of E.

Proof. The functional V is a linear form on the $\mathbb{R}_+^{\max}$-module C_c^+. If $\overline{M} = L^+$ is the module of $\overline{\mathbb{R}}_+$-valued lower-semicontinuous maps and $M = C_c^+$, then $\overline{M}/M$ is a meet-continuous extension by Example III-9.4, so Theorem III-9.5 applies if we show that V is continuous and non-degenerate on $\overline{M}/M$. Non-degeneracy of V on $\overline{M}/M$ is ensured by the existence of arbitrary suprema in $\overline{M}$. For the continuity of V, let $(f_j)_{j \in J}$ be a nondecreasing net of elements of C_c^+ such that $f := \bigoplus_{j \in J} f_j \in M$, where the supremum is taken in $\overline{M}$. In particular, $(f_j)_{j\in J}$ converges pointwise to f, see Example III-9.4. We want to prove that $V(f) = \bigoplus_{j\in J} V(f_j)$. So let $1 > \varepsilon > 0$, let K_ε be the compact set $\{f \geqslant \varepsilon\}$, and define h_j on K_ε by $h_j(x) = f_j(x)/f(x)$. Then $h_j \in C_c^+(K_\varepsilon)$ and $(h_j)_{j\in J}$ is a nondecreasing net converging to 1 pointwise. Applying Dini's Theorem, the convergence is uniform on K_ε, hence there is some $j_0 \in J$ such that $1 \leqslant \varepsilon + h_{j_0}$ on K_ε. Thus, $f \leqslant \varepsilon.f + f_{j_0}$ on K_ε. Let K be a compact set containing $\{0 < f < 1\}$. By Urysohn's lemma, we may find a compactly-supported continuous map h such that $h = 1$ on K. Then $f \leqslant (\frac{1}{1-\varepsilon} f_{j_0}) \oplus (\varepsilon h)$ on E. This implies $V(f) \leqslant (\frac{1}{1-\varepsilon} \bigoplus_{j\in J} V(f_j)) \oplus (\varepsilon.V(h))$, for all $1 > \varepsilon > 0$, so V is continuous. By Theorem III-9.5, there exists some archimedean element c in $\overline{M}/M$ such that $V(f) = \langle c, f\rangle$, for all $f \in M$. Defining the usc map c^+ by $c^+(x) = 1/c(x)$, we have $V(f) = \bigoplus_{x\in E} f(x).c^+(x)$, for all $f \in M$. The maxitive measure ν defined on $\mathscr{B}$ by $\nu(B) = \bigoplus_{x\in B} c^+(x)$ for all $B \in \mathscr{B}$ is regular by [Chapter II, Theorem 5.20], and we have

$$V(f) = \int^\infty f \, d\nu,$$

for all $f \in M$. If K is a compact subset of E, Urysohn's lemma provides a map $f \in M$ such that $f(x) = 1$ for all $x \in K$, so that $c^+(x) \leqslant V(f)$ for all $x \in K$. This ensures that $\nu(K) = \bigoplus_{x\in K} c^+(x)$ is finite.

Uniqueness of ν is a direct consequence of the uniqueness of c, which itself derives from Lemma III-10.2 and Remark III-9.6. □

In the same line, one can formulate Riesz like theorems for a functional V defined on the set C_b^+ of nonnegative bounded continuous maps instead of C_c^+. Breyer and Gulinsky [**48**] proved the next theorem[1], see also Puhalskii [**246**, Theorem 1.7.25] and Gulinsky [**118**, Theorem 3.4]. A functional $V : C_b^+ \to \mathbb{R}_+$ is *tight* if, for all $\varepsilon > 0$, there is some compact subset K of E such that $V(f) \leqslant \varepsilon \|f\|$, for each $f \in C_b^+$ that equals 0 on K.

Theorem III-10.4. [**48**] *Assume that E is a Tychonoff space, and let $V : C_b^+ \to \mathbb{R}_+$ be a tight linear form on C_b^+ that preserves countable pointwise suprema. Then there exists a unique (finite) tight regular maxitive measure ν on $\mathscr{B}$ such that*

$$V(f) = \int^\infty f \, d\nu,$$

for all $f \in C_b^+$.

Proof. See e.g. Puhalskii [**246**, Theorem 1.7.25]. The idea of the proof is to use the Stone–Čech compactification of E and to apply Theorem III-10.3. □

In order to treat the case of a non-tight linear form on C_b^+, we shall assume that the Tychnoff space is also second-countable, i.e. is a separable

[1] We were not in a position to access this article.

metrizable space. A functional $V : C_b^+ \to \mathbb{R}_+$ is *optimal* if, for all nonincreasing sequences $(f_n)_{n\in\mathbb{N}}$ of elements of C_b^+ tending pointwise to 0, the sequence $(V(f_n))_{n\in\mathbb{N}}$ tends to 0.

Theorem III-10.5. *Assume that E is a separable metrizable space, and let $V : C_b^+ \to \mathbb{R}_+$ be a linear form on C_b^+ that preserves countable pointwise suprema. Then there exists a unique (finite) regular maxitive measure ν on $\mathscr{B}$ such that*

$$V(f) = \int^\infty f \, d\nu,$$

for all $f \in C_b^+$. Moreover, if E is Polish, then the following conditions are equivalent:

- *V is optimal,*
- *V is tight,*
- *ν is tight,*
- *c^+ is upper-compact,*

where c^+ is the maximal cardinal density of ν.

Proof. We denote L^+ by $\overline{M}$ and C_b^+ by M. As in the proof of Theorem III-10.3, $\overline{M}/M$ is a meet-continuous extension by Example III-9.4, V is a nondegenerate linear form on $\overline{M}/M$, and we want to prove that

$$V(f) = \bigoplus_{j\in J} V(f_j), \tag{37}$$

for all nondecreasing nets $(f_j)_{j\in J}$ in M such that $f := \bigoplus_{j\in J} f_j \in M$, where the supremum is taken in $\overline{M}$. Let q be a nonnegative rational number. The open subset $\{f > q\}$ is covered by the family of open subsets $\{f_j > q\}$, $j \in J$. Since E is separable metrizable, it is second-countable, so we can extract a countable subcover and write $\{f > q\} = \bigcup_{j\in N_q} \{f_j > q\}$, where N_q is a countable subset of J. Defining N as the union of all N_q, which is countable, we see that $f = \bigoplus_{j\in N} f_j$. Since V preserves countable pointwise suprema in M, Equation (37) holds, so V is continuous on $\overline{M}/M$. The existence of ν now follows from the same argument as in the proof of Theorem III-10.3. Since E is separable metrizable, E is normal; using Urysohn's lemma for normal spaces (see e.g. [**15**, Theorem 2.46]), we can show that every $\overline{\mathbb{R}}_+$-valued lower-semicontinuous map is a supremum of elements of C_b^+, as a perfect analogue of Lemma III-10.2, and this leads to the uniqueness of ν.

Now suppose that E is Polish. Assume that $\{c^+ \geqslant t\}$ is not compact, for some $t > 0$. Then there exists some $\varepsilon > 0$ and some sequence (x_n) of elements of $\{c^+ \geqslant t\}$ such that $d(x_m, x_n) > \varepsilon$, for all $m \neq n$. Since E is Polish, one can find some countable family of open balls with radius $\varepsilon/2$ covering E. Let $B_k \ni x_k$ be one of these balls containing x_k. Let $f_k \in C_b^+$ such that $f_k(x_k) = 1/t$ and $f_k = 0$ on $E\backslash B_k$. Then $g_n := \bigoplus_{k\geqslant n} f_k$ tends pointwise to 0. But $V(g_n) \geqslant g_n(x_n).c^+(x_n) \geqslant 1$, so V is not optimal.

Conversely, assume that c^+ is upper-compact and that V is not optimal. So let (f_n) be a nonincreasing sequence of elements of C_b^+ that tends pointwise to zero, and assume that $\bigwedge_{n\in\mathbb{N}} V(f_n) > 0$. Then there exists some $t > 0$ such that $V(f_n) > t$, for all n. We deduce the existence of some $x_n \in E$ such that $f_n(x_n).c^+(x_n) > t$. Since the sequence (f_n) is nonincreasing, $f_n \leqslant f_0$. Also, f_0 is upper-bounded by some $u > 0$, so that x_n is in the compact subset $\{c^+ \geqslant t/u\}$. This implies that (x_n) clusters to some x, but this contradicts the fact that $f_m(x_n) \geqslant f_m(x_n).c^+(x_n) \geqslant f_n(x_n).c^+(x_n) > t$ for all $n \geqslant m$.

By [Chapter II, Proposition 5.19], we know that ν is tight if and only if c^+ is upper-compact. If V is tight, then ν is tight by uniqueness of ν in Theorem III-10.4; the converse statement is obvious. □

Bell and Bryc also investigated the case where E is Polish, their result [**30**, Theorem 2.1] is encompassed in the previous theorem.

Akian proved a slightly different result for normal spaces that we merely recall for the sake of completeness.

Theorem III-10.6. [**7**, Theorem 4.8] *Assume that E is a normal space, and let $V : C_b^+ \to \mathbb{R}_+$ be a linear form on C_b^+ that preserves countable pointwise suprema. Then there exists a unique σ-maxitive measure ν on $\mathscr{B}$ such that*

$$V(f) = \int^\infty f \, d\nu,$$

for all $f \in C_b^+$.

Remark III-10.7 (On large deviations)**.** The idempotent Riesz representation theorem partly originates from large deviation questionings. Varadhan [**286**] was interested in the functional defined on the set $C_b^+(E)$ of nonnegative bounded continuous maps, for some Polish space E, by

$$V(f) = \lim_{n\to\infty} \left(\int_E f^{1/\alpha_n} \, d\mu_n \right)^{\alpha_n},$$

whenever the limit exists, where (μ_n) is a sequence of probability measures on E satisfying a large deviation principle, and $\alpha_n \downarrow 0$ when $n \uparrow \infty$. He proved the representation

$$V(f) = \bigoplus_{x\in E} (f(x) e^{-I(x)}), \tag{38}$$

where $I : E \to \overline{\mathbb{R}}_+$ is the (lower-semicontinuous) *rate function*, that governs large deviations. For more on the links between large deviation principles and maxitive measures, we refer the reader to Puhalskii [**245, 246**], B. Gerritse [**112**], Akian [**7**], Nedović et al. [**220**].

III-11. Conclusion and perspectives

Following Cohen et al. [**63**], we could certainly have pushed on the generalization to the use of *reflexive* idempotent semirings[2] instead of idempotent semifields, but our main interest here, at least in the first part of the chapter, was to stress the role of Z theory in the gathering of similar but a priori distinct results from different mathematical areas.

Some results on modules and continuous linear forms are of topological flavour; this aspect will be sharpened in Chapter V, where topological $\mathbb{R}_+^{\max}$-modules will be at stake.

Acknowledgements. I would like to thank Marc Leandri for his advice on a preliminary version of the manuscript. I am also indebted in Pr. Jimmie D. Lawson who made numerous comments and pointed out a mistake in an ealier version of Section III-8.

[2]Or even to the use of idempotent semirings in which every element is the supremum of reflexive elements, so as to generalize both Cohen et al.'s and Litvinov et al.'s approaches.

Convexity in a tropical setting

CHAPTER IV

Convexities on ordered structures have their Krein–Milman theorem

ABSTRACT. We show analogues of the classical Krein–Milman theorem for several ordered algebraic structures, especially in a semilattice (non-linear) framework. In that case, subsemilattices are seen as convex subsets, and for our proofs we use arguments from continuous lattice theory and abstract convexity theory.

IV-1. Résumé en français

Un semitreillis est un monoïde commutatif $(S, \oplus)$ dont tous les éléments t sont idempotents, i.e. tels que $t \oplus t = t$. Dans ce cas S possède une relation d'ordre naturelle définie par $s \leqslant t \Leftrightarrow s \oplus t = t$, qui fait de $s \oplus t$ la borne supérieure de $\{s, t\}$. La structure de semitreillis a été largement étudiée au cours des dernières décennies ; l'un des résultats clefs de la théorie est l'identification de la catégorie des semitreillis continus complets avec celle des semitreillis topologiques compacts avec petits semitreillis. Il est dû dans sa forme complète à Hofmann and Stralka [**129**]. Les travaux de Lawson ont fortement contribué à sa découverte (cf. [**168**], [**170**]). Le lecteur pourra aussi se reporter à Lea [**176**] pour une preuve alternative, et à Gierz et al. [**114**, Theorem VI-3.4]. Ce théorème (dit théorème fondamental des semitreillis compacts) jette un pont entre des objets a priori de nature *algébrique* et des objets a priori de nature *topologique*.

Une « troisième voie » est d'appréhender un semitreillis comme un objet *géométrique*, dans lequel les sous-semitreillis sont considérés comme des parties convexes. De façon surprenante, un tel point de vue a été très peu abordé dans la littérature. Les seuls travaux recencés sur la question sont ceux de Jamison ([**135**], [**136**], [**139**, Appendix], [**140**]), repris partiellement par van de Vel ([**282**], [**284**], [**283**]), ainsi qu'un commentaire dans le livre de Gierz et al. [**114**, p. 403].

Une raison à cela est sans doute qu'un semitreillis avec zéro peut être vu comme un module sur le semicorps idempotent $\mathbb{B} = \{0, 1\}$, et s'inscrit donc dans le cadre plus général des modules sur un semicorps idempotent quelconque $(\Bbbk, \oplus, \times)$ (cf. chapitre III). Or, pour ces structures les questions de convexité ont été largement explorées : c'est le point de vue « tropical ». Le chapitre V reviendra en détail sur ces questions et founira les références appropriées à la littérature.

Pourtant, se contenter de voir les semitreillis comme des modules particuliers est réducteur. En effet, l'utilisation de l'ensemble $\mathbb{B}$ en tant que semicorps idempotent fini (et l'unique tel d'ailleurs) engendre, comme en géométrie sur les corps finis, des phénomènes atypiques. Ces manifestations sont par exemple qu'un $\mathbb{B}$-module de type fini est fini, qu'une partie convexe d'un $\mathbb{B}$-module n'est pas connexe en général, ou que l'unique élément archimédien s'il existe est le plus grand élément. Cette dichotomie entre cas fini et infini est apparue au chapitre III, où nous avons souvent dû exclure le cas $\mathbb{k} = \mathbb{B}$ dans les énoncés ; elle est visible dès le Lemme 5.5 du chapitre III : pour un semicorps idempotent $\mathbb{k}$, l'infimum de $\mathbb{k}\backslash\{0\}$ est 1 si $\mathbb{k} = \mathbb{B}$, et 0 sinon. Ainsi les $\mathbb{B}$-modules vont présenter des phénomènes de « discontinuité » qu'on ne retrouve pas dans des modules tels que $\mathbb{R}_+^n$ (sur le semicorps idempotent $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \max, \times)$).

Une autre justification à l'étude des semitreillis préalablement à celle des modules sur un semicorps idempotent est qu'à une partie convexe K d'un $\mathbb{R}_+^{\max}$-module M peut être associé l'ensemble $\{(r.x, r) : x \in K, r \in [0,1]\}$ qui est un sous-semitreillis du semitreillis $M \times [0,1]$. De ce fait, travailler d'abord sur les semitreillis permettra d'accélérer l'étude géométrique des $\mathbb{R}_+^{\max}$-modules.

Nous examinons également dans ce chapitre les autres convexités naturelles qui apparaissent sur différents types de structures ordonnées telles que les ensembles ordonnés, les semitreillis, les treillis. Ainsi une autre convexité naturelle sur les semitreillis est celle constituée des sous-semitreillis qui sont *convexes pour l'ordre*, i.e. qui contiennent un intervalle dès qu'ils en contiennent les bornes. Cette convexité a été plus étudiée que la première, cf. Jamison [**136**] et van de Vel ([**282**], [**283**], [**284**]) ; cf. également Horvath et Llinares Ciscar [**130**] et Nguyen The Vinh [**290**] qui s'intéressent au cas des semitreillis topologiques connexes par arcs.

Pour ces différentes structures, nous démontrons des analogues de résultats classiques tels que le théorème de Krein–Milman (Krein et Milman [**163**], cf. aussi Bourbaki [**45**]) et la réciproque de celui-ci due à Milman [**203**]. Dans le cas des semitreillis munis de la convexité des sous-semitreillis, notre résultat principal s'énonce ainsi :

Théorème IV-1.1. *Dans un semitreillis topologique localement convexe, toute partie convexe, faiblement fermée, localement compacte et sans ligne est l'enveloppe convexe faiblement fermée de ses points extrêmes.*

L'hypothèse de locale convexité revient à dire que le semitreillis a des *petits semitreillis* au sens de Lawson [**168**]. Le concept de *ligne*, qui est clair dans le cas classique, est à définir de façon appropriée dans ce contexte. La topologie faible renvoie à la topologie engendrée par la famille des morphismes continus du semitreillis S dans $[0,1]$. Du fait du théorème fondamental des semitreillis compacts, des méthodes ou des éléments de théorie des domaines transparaissent fortement dans nos preuves.

De nombreux théorèmes de Krein–Milman sont prouvés dans la littérature ; néanmoins ils ne suffisent pas pour démontrer directement le théorème ci-dessus. Ainsi Fan [**98**, Lemma 3] définit de façon abstraite le concept d'*extrémalité*, proche de celui de *face* qui apparaît en analyse classique, et l'utilise pour prouver une version abstraite du théorème de Krein–Milman (cf. aussi [**284**, Theorem IV-2.6]). Cependant le cadre qu'il définit, augmenté des compléments d'auteurs comme Lassak [**166**], s'il généralise le cadre classique, où l'addition est une opération *simplifiable*, ne fonctionne pas dans notre contexte idempotent.

Wieczorek [**301**] fournit un autre résultat de ce type. Deux hypothèses sont requises pour celui-ci : que tous les singletons soient convexes d'une part, et que la famille des fonctions réelles semicontinues supérieurement et *strictement convexes* sépare les points des parties convexes fermées (cf. aussi [**284**, Topic IV-2.30]). Cependant, pour les structures ordonnées, la première condition n'est pas toujours vérifiée (ainsi de la convexité des parties *montantes* d'un ensemble ordonné, ou de la convexité des idéaux d'un semitreillis), et la deuxième condition apparaît trop complexe à vérifier en pratique.

Enfin nous examinons le cas des semitreillis topologiques d'*ampleur* finie b (pour *breadth*), qui ont la propriété d'être toujours localement convexes. Comme déjà remarqué par Jamison [**140**, § 4.D], l'ampleur s'interprète directement comme le nombre de Carathéodory associé à la convexité des sous-semitreillis. Nous prouvons un théorème de type Minkowski, qui exprime le fait que, avec de bonnes hypothèses, tout point est le suprémum d'au plus b points extrêmes. La *profondeur* du semitreillis peut également être vue comme un invariant de convexité – son nombre de Helly –, et nous montrons qu'elle a des liens étroits avec le nombre de points extrêmes d'une partie compacte convexe.

IV-2. Introduction

A semilattice is a commutative semigroup $(S, \oplus)$ in which all elements t are idempotent, i.e. such that $t \oplus t = t$. Then S is endowed with a natural partial order defined by $s \leqslant t \Leftrightarrow s \oplus t = t$, so that $s \oplus t$ is the supremum of the pair $\{s, t\}$. Semilattices have been widely explored in the last decades; a key result of the theory is the "fundamental theorem of compact semilattices", that identifies the category of complete continuous semilattices with that of compact topological semilattices with small semilattices. The statement is due to Hofmann and Stralka [**129**]. Lawson's contribution was decisive for its discovery (see [**168**], [**170**]). See also Lea [**176**] for an alternative proof and Gierz et al. [**114**, Theorem VI-3.4]. This theorem draws a link between the *algebraic* and the *topological* natures of semilattices.

But semilattices can also be regarded as *geometric* objects, where subsemilattices are treated as convex subsets. Surprisingly, this point of view has been hardly considered in the literature. Exceptions are the work of Jamison ([**135**], [**136**], [**139**, Appendix], [**140**]) cited by van de Vel ([**282**], [**284**], [**283**]), and a comment by Gierz et al. [**114**, p. 403].

One reason is certainly that a semilattice with a least element can be seen as a module over the idempotent semifield $\mathbb{B} = \{0, 1\}$. Therefore, it belongs to the more general class of modules over an idempotent semifield $(\mathbb{k}, \oplus, \times)$ (see Chapter III), and it happens that these structures have been deeply studied in the framework of "max-plus" or "tropical" convexity. We shall address these aspects and give appropriate references in Chapter V.

However, semilattices should not be reduced to a special case of modules over an idempotent semifield. Indeed, the use of the set $\mathbb{B}$ as a finite idempotent semifield creates unusual phenomena: for instance, a $\mathbb{B}$-module of finite type is finite; a convex subset of a $\mathbb{B}$-module is not connected in general. These differences between the finite and the infinite case were already perceptible in Chapter III: we often had to exclude the case $\mathbb{k} = \mathbb{B}$. With [Chapter III, Lemma 5.5] we saw that, given an idempotent semifield $\mathbb{k}$, the infimum of $\mathbb{k}\backslash\{0\}$ is 1 if $\mathbb{k} = \mathbb{B}$, and 0 otherwise. Consequently, one should expect $\mathbb{B}$-modules to present discontinuous phenomena, that one does not usually observe in modules such as $\mathbb{R}_+^n$ (over the idempotent semifield $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \max, \times)$).

It is also worth studying semilattices before modules, because if K is a convex subset of an $\mathbb{R}_+^{\max}$-module M, then the set $\{(r.x, r) : x \in K, r \in [0,1]\}$ is a subsemilattice of the semilattice $M \times [0,1]$. This partly explains why results on semilattices shall be useful for applications to the geometry of $\mathbb{R}_+^{\max}$-modules.

Other convexities naturally arise on ordered structures such as partially ordered sets, semilattices and lattices. For instance, a semilattice can also be endowed with the convexity made up of its *order-convex* subsemilattices; this case was notably studied by Jamison [**136**] and van de Vel ([**282**], [**283**], [**284**]). See also Horvath and Llinares Ciscar [**130**] and Nguyen The Vinh [**290**] for investigations on path-connected topological semilattices.

For this series of convexity structures, we prove analogues of classical results of convex analysis such as the Krein–Milman theorem (Krein and Milman [**163**], see also Bourbaki [**45**]) and Milman's converse [**203**]. For semilattices equipped with the convexity of subsemilattices, our main result is the following:

Theorem IV-2.1. *Let S be a locally-convex topological semilattice. Then every locally-compact, weakly-closed, convex subset of S containing no line is the weakly-closed convex hull of its extreme points.*

Local convexity here is another way to say that S has *small semilattices* in the sense of Lawson [**168**]. The concept of *line*, which is intuitive in the classical case, needs to be properly defined in this non-linear context. The *weak topology* refers to the topology generated by the family of continuous semilattice-morphisms from S to $[0,1]$. Because of the fundamental theorem of compact semilattices, our proofs directly or indirecly use methods and elements from domain theory.

Numerous Krein–Milman theorems have been proved in the literature. Yet they do not enable one to deduce directly the above theorem. For instance Fan [**98**, Lemma 3] gave a set-theoretic definition of an *extremality*, a concept close to the notion of *face* in classical analysis; then he used it to prove an abstract Krein–Milman type theorem (see also [**284**, Theorem IV-2.6]). However, his definition and adds-on by others such as Lassak [**166**] remain driven by classical convexity theory, where addition is a *cancellative* binary relation; it does not work in an idempotent setting.

Another result of this kind is due to Wieczorek [**301**]. It requires two conditions: that every singleton be convex, and that the family of upper-semicontinuous *strictly convex* real-valued maps separate convex closed subsets and points (see also [**284**, Topic IV-2.30]). However, for ordered structures, the former condition may not be satisfied (consider e.g. the upper convexity on a poset, or the ideal convexity on a semilattice), and the latter seems too complex for practical verification.

We also examine the case of topological semilattices with finite *breadth* b, that happen to be always locally-convex. Jamison [**140**, § 4.D] remarked that breadth coincides with the Carathéodory number associated with the convexity of subsemilattices. We prove a Minkowski type theorem, which asserts that under appropriate hypothesis every point is the join of at most b extreme points. The *depth* of the semilattice also coincides with an interesting convexity invariant, namely the Helly number, and we establish links between depth and the number of extreme points of a compact convex subset.

The chapter is organized as follows. Section IV-3 gives basics of abstract convexity theory. In Section IV-4 we recall Wallace's lemma on the existence of minimal elements in compact partially ordered sets, which will reveal its importance for the existence of extreme points in compact ordered structures. We also expose a Krein–Milman type theorem in partially ordered sets. Section IV-5 introduces the main convexity examined in this work, which is the convexity made up of the subsemilattices of a semilattice. We prove that a Krein–Milman type theorem also holds, and see that it essentially comes from the result that coirreducible elements are order-generating in continuous semilattices. An analogous form of Bauer's principle is also proved. Section IV-6 goes one step further: after the work of Klee in classical convex analysis, we prove that the Krein–Milman theorem holds for locally-compact weakly-closed convex subsets containing no line, with an adequate definition of line in topological semilattices. Also, Milman's converse is proved. Topological semilattices with finite breadth or with finite depth are considered in Section IV-7. We recall that the breadth and the Carathéodory number of a semilattice coincide, and we prove a Minkowski type theorem. Other convexities on semilattices and lattices are proposed in Section IV-8. We provide necessary and sufficient conditions for these convexities to be convex geometries, which is a minimal requirement for Krein–Milman type theorems.

IV-3. Reminders of abstract convexity

A collection $\mathscr{C}$ of subsets of a set X is a *convexity* (or an *alignment*) on X if it satisfies the following axioms:

- $\emptyset, X \in \mathscr{C}$,
- $\mathscr{C}$ is closed under arbitrary intersections,
- $\mathscr{C}$ is closed under directed unions.

The last condition means the following: if $\mathscr{D} \subset \mathscr{C}$ is such that, for all $C_1, C_2 \in \mathscr{D}$, there is some $C \in \mathscr{D}$ containing both C_1 and C_2, then $\bigcup \mathscr{D} \in \mathscr{C}$. The pair $(X, \mathscr{C})$ is then a *convexity space*. Elements of $\mathscr{C}$ are called *convex* subsets of X. If $A \subset X$, the *convex hull* $\mathrm{co}(A)$ of A is the intersection of all convex subsets containing A. Convex subsets that are the convex hull of some finite subset are called *polytopes*. They are of special importance for they generate the whole convexity, in the sense that $C \subset X$ is convex if and only if, for every finite subset F of C, $\mathrm{co}(F) \subset C$.

The wording of the Krein–Milman theorem includes the notion of *extreme point* of a subset $A \subset X$, which is an element x of A such that $x \notin \mathrm{co}(A \backslash \{x\})$, or equivalently, if A is convex, such that $A \backslash \{x\}$ is convex. The set of extreme points of A is denoted by $\mathrm{ex}\, A$.

In practice, X will be a convexity space endowed with a *compatible* topology, that is a topology making every polytope (topologically) closed. Then X will be called a *topological convexity space*.

For more background on abstract convexity, see the monograph of van de Vel [**284**]. Other attempts and approaches, that we do not consider here, have been made by mathematicians to generalize the concept of convexity; see for instance Singer [**274**] or Park [**237**].

IV-4. Convexities on partially ordered sets

In this section we recall (and discuss) Wallace's lemma (see Wallace [**294**, § 2]), that we shall use several times later on, and we interpret it as a Krein–Milman type theorem for partially ordered sets. We also prove the converse statement, known as Milman's converse in the framework of locally-convex topological vector spaces.

IV-4.1. Wallace's lemma. A *partially ordered set* or *poset* $(P, \leqslant)$ is a set P equipped with a reflexive, transitive, and antisymmetric binary relation $\leqslant$. If $A \subset P$, we denote by $\downarrow A$ the *lower* subset generated by A, i.e. $\downarrow A := \{x \in P : \exists a \in A, x \leqslant a\}$, and we write $\downarrow x$ for the *principal ideal* $\downarrow \{x\}$. *Upper* subsets $\uparrow A$ and *principal filters* $\uparrow x$ are defined dually. A topology on a poset is *lower semiclosed* (resp. *upper semiclosed*) if each principal ideal (resp. principal filter) is a closed subset. It is *semiclosed* if it is both lower semiclosed and upper semiclosed.

Note that our definition of a *compact* subset of a topological space does not assume Hausdorffness.

Proposition IV-4.1 (Wallace's lemma, [**294**, § 2])**.** *Let a poset be equipped with a lower semiclosed topology. Then every nonempty compact subset has a minimal element.*

We take advantage of this reminder to stress that we found no explicit statement in the literature of the following equivalence. Recall first that the Ultrafilter Principle (alias the Prime Ideal Theorem), which says that every filter on a set is contained in an ultrafilter, is strictly weaker than the axiom of choice.

Proposition IV-4.2. *Wallace's lemma for all posets together with the Ultrafilter Principle are equivalent to the axiom of choice.*

Proof. Necessity is made clear by the proof of [**114**, Proposition VI-5.3], which makes use of Hausdorff's maximality principle to prove Wallace's lemma. For sufficiency, let P be a poset, and let L be a linearly ordered subset (or *chain*) of P. Let $\mathscr{L}$ be the (nonempty) collection of chains of P containing L, ordered by reverse inclusion. Then $\mathscr{L}$ is a complete semilattice (i.e. a semilattice in which every nonempty subset has a supremum and every filtered subset has an infimum, see Section IV-5), hence is compact when equipped with the Lawson topology (see [**114**, Theorem III-1.9]; its proof uses Alexander's lemma, which itself is known to be implied by the Ultrafilter Principle). By Wallace's lemma, $\mathscr{L}$ has a minimal element, i.e. there is a maximal chain in P containing L. This proves Hausdorff's maximality principle, which is equivalent to the axiom of choice. □

IV-4.2. Krein–Milman theorems for posets. Actually, the result [**114**, Proposition VI-5.3], used in the previous proof, refines Wallace's lemma: under the same hypothesis, it concludes that, if K is a compact subset and $x \in K$, there is some minimal element of K below x. We interpret this version as a Krein–Milman type theorem for partially ordered sets endowed with the *upper convexity* made up of upper subsets (see Edelman and Jamison [**86**, Theorem 3.2] for a characterization of this convexity). In this setting, extreme points of a convex subset K coincide with its minimal elements $\operatorname{Min} K$, and if K is compact convex, then $K = \operatorname{co}(\operatorname{ex} K) = \uparrow(\operatorname{Min} K)$ (see Figure 1). Note the absence of topological closure in this equality.

Theorem IV-4.3 (Krein–Milman for posets I)**.** *Consider a poset with the upper (resp. lower) convexity, and equipped with a lower semiclosed (resp. an upper semiclosed) topology. Then every compact subset K satisfies*

$$\operatorname{co}(K) = \operatorname{co}(\operatorname{ex} K).$$

Proof. A direct consequence (and actually, an equivalent form) of Wallace's lemma is the following: if K is a compact subset and $x \in K$, there is some minimal element of K below x. To see this, it suffices to apply Wallace's lemma to the nonempty compact subset $K \cap \downarrow x$. Then $K \subset \uparrow(\operatorname{Min} K) = \operatorname{co}(\operatorname{ex} K)$, so that $\operatorname{co}(K) = \uparrow K = \operatorname{co}(\operatorname{ex} K)$. □

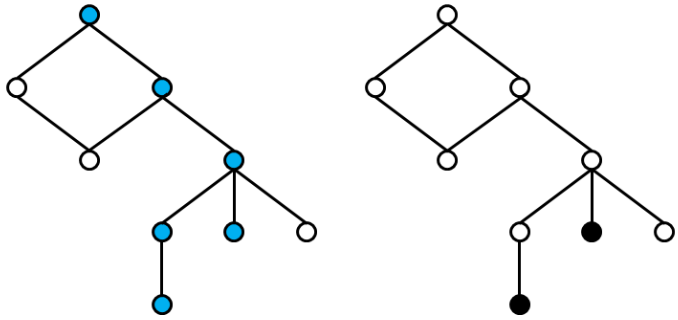

FIGURE 1. Hasse diagram of a finite partially ordered set (with the discrete topology). The blue points (on the left) define a convex subset (with respect to the upper convexity); the black points (on the right) are its minimal elements.

Franklin [**103**], Baker [**21**] or Jamison [**139**] preferentially consider posets endowed with their *order convexity*. This convexity, introduced by Birkhoff [**38**], is generated by intervals $[x, y] = \uparrow x \cap \downarrow y = \{z : x \leqslant z \leqslant y\}$. See Jamison [**138**] for various characterizations of order convexity. See also Birkhoff and Bennett [**40**]. Here, convex subsets are subsets of the form $\uparrow A \cap \downarrow A$, extreme points are the elements e such that $e \in [x, y] \Rightarrow e \in \{x, y\}$, i.e. are either minimal elements or maximal elements, and Franklin [**103**, Theorem III] and Baker [**21**, Theorem 1] proved that a Krein–Milman type theorem also holds.

Theorem IV-4.4 (Krein–Milman for posets II, [**103**, Theorem III] and [**21**, Theorem 1])**.** *Consider a poset with the order convexity, and equipped with a semiclosed topology. Then every compact subset K satisfies*

$$\mathrm{co}(K) = \mathrm{co}(\mathrm{ex}\, K).$$

See also Wirth [**302**, Theorem 1] for a Krein–Milman type theorem in certain posets equipped with the open-interval topology.

It is remarkable that, in Theorems IV-4.3 and IV-4.4, the Krein–Milman property holds without local convexity. Local convexity is certainly automatic in every compact *pospace* (defined as a poset P equipped with a topology making the partial order closed in $P \times P$), as is well known since the work of Nachbin [**217**], but Theorems IV-4.3 and IV-4.4 do not need this assumption.

IV-4.3. Milman's converse. In classical convex analysis, Milman's theorem [**203**] is probably as important as the Krein–Milman theorem itself, for it asserts that the representation of a compact convex subset as the closed convex hull of its extreme points is, in some sense, optimal. That is, for every such representation, the "representing" subset, if closed, contains the subset of extreme points. Fortunately, a similar result holds in pospaces. For the next assertion, we write $\overline{A}$ for the topological closure of a subset A.

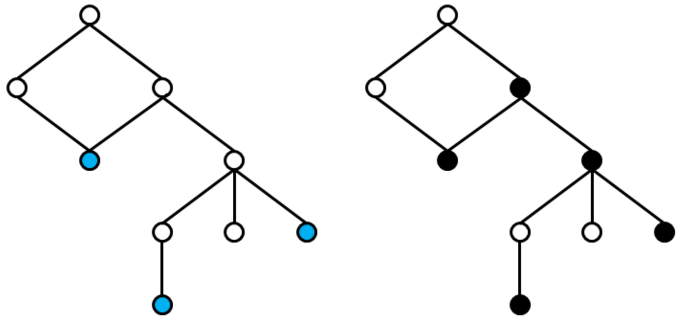

FIGURE 2. Hasse diagram of a finite semilattice. The blue points (on the left) define a subset; the black points (on the right) are its convex hull (with respect to the algebraic convexity), which here is not connected.

Theorem IV-4.5 (Milman for posets)**.** *Let P be a pospace with the upper (resp. lower, order) convexity, and K be a closed convex subset of P. Then, for every compact subset A of K such that $K = \overline{\mathrm{co}}(A)$, we have $A \supset \mathrm{ex}\, K$.*

Proof. We consider the case of upper convexity only. Since P is a pospace and A is compact in P, $\mathrm{co}(A) = \uparrow A$ is closed in P by [**114**, Proposition VI-1.6(ii)], hence $K = \uparrow A$. Thus, $\mathrm{ex}\, K = \mathrm{Min}\, K = \mathrm{Min}(\uparrow A) \subset A$. □

IV-5. THE ALGEBRAIC CONVEXITY OF A SEMILATTICE

IV-5.1. Introduction. A *semilattice* S is a poset in which every nonempty finite subset F has a supremum, denoted by $\bigoplus_S F$ (or by $\bigoplus F$ when the context is clear). If $x, y \in S$, we write $x \oplus y$ for $\bigoplus\{x, y\}$.

We endow the semilattice S with its *algebraic convexity* made up of its subsemilattices, i.e. the subsets T of S such that $x \oplus y \in T$ whenever $x, y \in T$ (in particular the empty set is a subsemilattice). We shall also say that subsemilattices are *convex* subsets of S. If $A \subset S$, the *convex hull* $\mathrm{co}(A)$ of A is the subsemilattice generated by A (see Figure 2).

The algebraic convexity of a semilattice deserves special attention, for it has been hardly considered in the literature. Exceptions are the work of Jamison ([**135**], [**136**], [**139**, Appendix], [**140**]) and a comment by Gierz et al. [**114**, p. 403]. However, recall from the Introduction that a semilattice is equivalently described as a $\mathbb{B}$-module with $\mathbb{B} = \{0, 1\}$, thus is a special case of module over an idempotent semifield. Consequently, the algebraic convexity of a semilattice is the same as the *tropical convexity* of the associated $\mathbb{B}$-module. Tropical convexity has been the subject of a great amount of research, and we refer the reader to Chapter V for background and references.

It should be stressed that other interesting convexities can be defined on semilattices, for instance the *ideal convexity* consisting of lower subsemilattices, or the *order-algebraic convexity* made up of *order-convex* subsemilattices, that is subsemilattices T such that $x \leqslant y \leqslant z$ and $x, z \in T$ imply

$y \in T$. Information on the latter convexity may be gathered from Jamison [**136**] and van de Vel ([**282**], [**283**], [**284**]), and we shall discuss several convexities in more detail in Section IV-8.

If K is a subset of the semilattice S, then $x \in K$ is an extreme point of K if and only if x is *coirreducible* in K, i.e., for every nonempty finite subset F of K, $x = \bigoplus F \Rightarrow x \in F$ (see Figure 3).

The semilattice S is *topological* if it is endowed with a Hausdorff topology such that $S \times S \ni (x, y) \mapsto x \oplus y \in S$ is continuous (where $S \times S$ is equipped with the product topology). Be careful that, in [**114**], a topological semilattice is not supposed Hausdorff, although this hypothesis is made in all other references cited in this work. A topological semilattice S can then be seen as a topological convexity space, in which the topological closure of every convex subset remains convex (this is what van de Vel called *closure stability* [**284**, Definition III-1.7]). This latter property can be easily proved using nets. Also, S is *locally-convex* if every point has a basis of convex neighbourhoods, that is if it has small semilattices in the sense of Lawson [**167**] (see also Gierz et al. [**114**, Definition VI-3.1]).

IV-5.2. Compact local convexity or complete continuity? At this stage it is worth recalling the fundamental theorem of compact semilattices (see Hofmann and Stralka [**129**, Theorem 2.23], Lea [**176**, Theorem], and Gierz et al. [**114**, Theorem VI-3.4]). For this purpose we briefly recall some basic definitions of continuous poset theory. A subset F of a poset $(P, \leqslant)$ is *filtered* if it is nonempty and, for all $x, y \in F$, there is a lower bound of $\{x, y\}$ in F. We say that $y \in P$ is *way-above* $x \in P$, written $y \gg x$, if, for every filtered subset F with an infimum $\bigwedge F$, $x \geqslant \bigwedge F$ implies $y \in \uparrow F$. The poset P is *continuous* if $\Uparrow x := \{y \in P : y \gg x\}$ is filtered and $x = \bigwedge \Uparrow x$, for all $x \in P$. A *domain* is a continuous poset in which every filtered subset has an infimum. A domain that is also a semilattice is a *continuous semilattice*. A semilattice is *complete* if every nonempty subset has a supremum and every filtered subset has an infimum.

Intervals of (extended) real numbers, with the usual order, for instance $[0, 1]$, $[0, 1)$, $(0, 1)$, are all continuous posets, and the way-above relation coincides with the strict order $>$, except at the top element when it exists (e.g. $1 \gg 1$ in $[0, 1]$). All these examples are also semilattices, but only $[0, 1]$ and $[0, 1)$ are domains (thus continuous semilattices), and $[0, 1]$ is the only complete semilattice (or complete lattice).

A subset A of the poset P is *Scott-open* if it is lower and if, whenever $\bigwedge F \in A$ for some filtered subset F of P with infimum, then $F \cap A \neq \emptyset$. The collection of Scott-open subsets of P is a topology, called the *Scott topology*. The *Lawson topology* on P is then the topology generated by the Scott topology and the subsets of the form $P \setminus \downarrow x$, $x \in P$.

Here comes the announced fundamental theorem of compact semilattices (we skip the identification of morphisms between the two categories at stake).

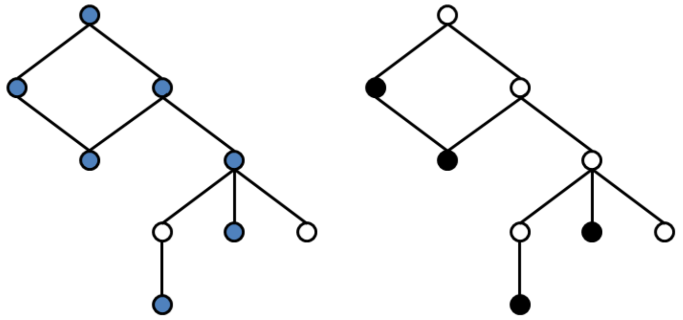

FIGURE 3. Hasse diagram of a finite semilattice. The blue points (on the left) define a subsemilattice; the black points (on the right) are its coirreducible elements.

Theorem IV-5.1. [**114**, Theorem VI-3.4]

(1) *Let K be a complete continuous semilattice. Then, with respect to the Lawson topology K is a compact locally-convex topological semilattice.*
(2) *Conversely, let K be a compact locally-convex topological semilattice. Then, with respect to its semilattice structure K is a complete continuous semilattice. Furthermore, the topology of K is the Lawson topology.* □

We warn the reader that, considering a locally-convex topological semilattice with a complete semilattice structure, the previous theorem cannot be used to assert that S is continuous, nor that the topology is the Lawson topology.

Problem IV-5.2. Gierz et al. asserted that a (not necessarily complete) continuous semilattice is a *strictly locally-convex* topological semilattice (meaning that every point has a basis of convex open neighbourhoods) for the Lawson topology (see [**114**, Exercise III-2.17]). Is there any kind of converse statement?

IV-5.3. The Krein–Milman theorem. With the correspondence given by the fundamental theorem IV-5.1, we now prove an analogue of the Krein–Milman theorem for semilattices:

Theorem IV-5.3 (Krein–Milman for semilattices)**.** *Let S be a locally-convex topological semilattice. Then every nonempty compact subset of S has at least one extreme point, and every compact convex subset of S is the closed convex hull of its extreme points.*

Proof. The former assertion is a direct consequence of Wallace's lemma since every minimal point is extreme. The latter comes from an interpretation of [**114**, Corollary I-3.10]. Let K be a nonempty compact convex subset of S. By [**114**, Proposition VI-3.2(i)], since S is a topological semilattice with small semilattices, K is, in its own right, a compact topological semilattice with small semilattices when equipped with the relative topology. Hence by the fundamental theorem of compact semilattices, K is a complete continuous semilattice.

Now, a consequence of [**114**, Corollary I-3.10] is that, in the continuous semilattice K, the subset of coirreducible elements (i.e., extreme points) of K is order-generating (see also Hofmann and Lawson [**126**, Proposition 2.7]). This means that every x in K equals $\bigoplus_K(\operatorname{ex} K \cap \downarrow x)$, where the supremum is taken in K.

To conclude the proof, let T be the topological closure in S of the subsemilattice $\operatorname{co}(\operatorname{ex} K \cap \downarrow x)$ of K. Since K is closed in S, T is also closed in K. By the closure stability property (see the Introduction of Section IV-5), T is then a closed subsemilattice of the compact semilattice K. By [**114**, Proposition VI-2.9], T is stable by suprema in K of nonempty subsets, hence $x = \bigoplus_K(\operatorname{ex} K \cap \downarrow x)$ is in T. This proves that $x \in \overline{\operatorname{co}}(\operatorname{ex} K)$, so that $K = \overline{\operatorname{co}}(\operatorname{ex} K)$. □

Remark IV-5.4. We can weaken the assumptions of Theorem IV-5.3, and only suppose that S is a locally-convex Hausdorff *semitopological* semilattice, i.e. a semilattice equipped with a locally-convex Hausdorff topology and a separately continuous addition. Indeed, [**114**, Theorem VII-4.8] then ensures that every compact convex subset of S is still a *topological* semilattice (see also the original paper by Lawson [**171**] on semitopological semigroups).

The hypothesis of the preceding theorem can be weakened in a different manner. We say that a subset K of a semilattice is *principally-compact* if $K \cap \downarrow x$ is compact for all $x \in K$.

Corollary IV-5.5. *Let S be a locally-convex topological semilattice. Then every nonempty principally-compact subset of S has at least one extreme point, and every principally-compact closed convex subset of S is the closed convex hull of its extreme points.*

Proof. Let K be a nonempty principally-compact subset of S, and let $x \in K$. If one notices that $\operatorname{ex}(K \cap \downarrow x) = \operatorname{ex}(K) \cap \downarrow x$, then the first assertion of the corollary is obvious. Now suppose also that K is convex, and let $L = K \cap \downarrow x$, which is nonempty compact convex. Then, by the Krein–Milman theorem, $x \in L = \overline{\operatorname{co}}(\operatorname{ex} L) = \overline{\operatorname{co}}(\operatorname{ex}(K) \cap \downarrow x) \subset \overline{\operatorname{co}}(\operatorname{ex} K)$, so that $K = \overline{\operatorname{co}}(\operatorname{ex} K)$. □

IV-5.4. Bauer's principle. Let S be a topological semilattice and K be a convex subset of S, and let L be a chain (considered as a semilattice). A map $f : K \to L$ such that $f(x \oplus y) \leqslant f(x) \oplus f(y)$ (resp. $f(x \oplus y) \geqslant f(x) \oplus f(y)$), for all $x, y \in K$, is called *convex* (resp. *concave*). An *affine* map is a convex and concave map, i.e. a semilattice-morphism. It is easily checked that f is concave if and only if it is order-preserving. Also, f is convex (resp. concave) if and only if its *epigraph* $\{(x, t) \in K \times L : f(x) \leqslant t\}$ (resp. its *hypograph* $\{(x, t) \in K \times L : f(x) \geqslant t\}$) is convex in $K \times L$.

We also say that a map $f : K \to L$ is *lower-semicontinuous* or *lsc* (resp. *upper-semicontinuous* or *usc*) if $\{f > t\}$ (resp. $\{f < t\}$) is open in K for all $t \in L$.

Let K be a nonempty subset of S. A subset E of K is *extreme* in K if, for all $x, y \in K$, $x \oplus y \in E \Rightarrow (x \in E$ or $y \in E)$, and E is a *face* of K if E is a nonempty compact subset of K that is extreme in K. The next result is a semilattice-version of the classical Bauer's maximum principle [**28**].

Proposition IV-5.6 (Bauer's maximum principle)**.** *Let S be a topological semilattice, K be a nonempty compact convex subset of S, and L be a chain. Let $f : K \to L$ be a convex, usc map. Then* $\operatorname{argmax} f$ *is a face of K, and f attains its maximum on* $\operatorname{ex} K$.

Proof. By compactness of K, we classically know that f attains its maximum on K. Now let $a = \max_{x \in K} f(x)$, and let $\operatorname{argmax} f$ be the nonempty set $\{x \in K : f(x) = a\}$. The fact that $\operatorname{argmax} f = \{x \in K : f(x) \geqslant a\}$ and the upper-semicontinuity of f tell us that $\operatorname{argmax} f$ is closed, hence (nonempty) compact. Also, by convexity of f and the fact that L is a chain, $\operatorname{argmax} f$ is extreme in K, thus a face of K. Hence, every minimal element of $\operatorname{argmax} f$ (which exists by Wallace's lemma) belongs to $\operatorname{ex} K$. □

Remark IV-5.7. Lassak [**166**] gave, in an abstract convexity setting, a set-theoretic notion of *extreme subset* as follows. For a convexity space X and a subset K, he called $E \subset K$ an extreme subset of K if

$$E \cap \operatorname{co}(F) \subset \operatorname{co}(E \cap F), \tag{39}$$

for all finite subsets F of K. However, with this definition, we do not recover the intuitive notion of extreme subset introduced above for semilattices. Even if Lassak's approach is appropriate for generalizing convexity of vector spaces, it does not fit with the setting of ordered structures that we want to study. The following modification in the definition actually untangles this problem, i.e. is adequate for both classical and "ordered" applications: one should replace (39) by

$$E \cap \operatorname{co}(F) \neq \varnothing \Rightarrow E \cap F \neq \varnothing,$$

for all finite subsets F of K. The transitivity of the relation "is extreme in" is then lost, but this is indeed what happens in ordered structures. In particular, in the previous proof, an extreme point of $\operatorname{argmax} f$ would not necessarily give an extreme point of K.

For completeness, we also give a dual version of Bauer's principle. Here the hypothesis can be weakened. A map $f : K \to L$ is called *quasiconcave* if $\{x \in K : f(x) > a\}$ is convex for every $a \in L$. Notice that there is no need to introduce the dual notion of *quasiconvex* map, for it simply coincides with that of convex map. However, a quasiconcave map may be non-concave (consider for instance f defined on $\mathbb{B}$ by $f(0) = 1$, $f(1) = 0$).

Proposition IV-5.8. *Let S be a topological semilattice, K be a nonempty closed convex subset of S, and L be a chain with a greatest element $\top$. Let*

$f : K \to L$ be a quasiconcave map, $f \not\equiv \top$. We also suppose that f is lower-compact, in the sense that the subset

$$\{x \in K : f(x) \leqslant a\}$$

is compact for all $a \in L \backslash \{\top\}$. Then $\operatorname{argmin} f$ *is a face of K, and f attains its minimum on* $\operatorname{ex} K$.

Proof. Let $x_0 \in K$ such that $f(x_0) \neq \top$. The subset $F = \{f \leqslant f(x_0)\}$ is nonempty compact, so that f attains its minimum on F, hence on K. Let $a := \min_{x \in K} f(x) < \top$. Then $\operatorname{argmin} f = \{x \in K : f(x) \leqslant a\}$ is nonempty compact. With the quasiconcavity of f, $\operatorname{argmin} f$ is also an extreme subset of K. Thus, every minimal element of $\operatorname{argmin} f$ (which exists by Wallace's lemma) belongs to $\operatorname{ex} K$. □

IV-6. Extension of the Krein–Milman theorem in semilattices

IV-6.1. Introduction. It is natural to ask whether the Krein–Milman theorem also holds in locally-compact closed convex subsets of some locally-convex topological semilattice. As such, the answer is negative. For instance, the set $S = K = (-\infty, 0] \times (-\infty, 0]$ equipped with its usual (componentwise) semilattice structure and its usual topology is a locally-convex, locally-compact topological semilattice, but it has no extreme point.

An additional hypothesis is certainly needed, and classical convex analysis helps to intuit it. Recall that, in 1957, Klee [**151**, Theorem 3.4] notably improved the classical Krein–Milman theorem, for he showed that, in a locally-convex Hausdorff topological vector space, every locally-compact closed convex subset *containing no line* is the closed convex hull of its extreme points and rays. In semilattices, the concept of extreme ray reduces to that of extreme point, but how could we define a suitable notion of line? Before coming to our proposal, we introduce *locally-convact semilattices*, where a *convact* subset is a compact convex subset.

IV-6.2. Separation in locally-convact semilattices. A topological semilattice in which every element has a basis of convact neighbourhoods is called a *locally-convact* topological semilattice. This is equivalent to requiring the semilattice to be both locally-convex and locally-compact, since the topological closure of a convex subset remains convex.

Example IV-6.1. If X is a locally-compact Hausdorff topological space, the *upper space* $(U[X], \subset)$ of X is the semilattice of all nonempty compact subsets of X topologized with the Lawson topology. The term *upper space* was coined by Edalat [**84**]. Recall that $U[X]$ is a continuous semilattice [**84**, Proposition 3.3], hence a strictly locally-convex topological semilattice [**114**, Exercise III-2.17]. It is also known that $U[X]$ is locally-compact (see Liukkonen and Mislove [**184**, § I]).

Problem IV-6.2. By [**171**, Proposition 7.1] a locally-compact Hausdorff semitopological group is topological. Is a locally-convact Hausdorff semitopological semilattice with closed order necessarily a topological semilattice?

The next lemma is implicit in the paper by Liukkonen and Mislove [**184**], but it deserves a specific statement.

Lemma IV-6.3. *In a locally-convact topological semilattice, every nonempty relatively compact subset has a supremum, and every nonempty compact convex subset has a greatest element.*

Proof. For the second assertion see e.g. [**114**, Proposition VI-1.13(v)] (it suffices for the ambient semilattice to be Hausdorff semitopological). Let K be a locally-convact semilattice, and A be a nonempty relatively compact subset. Then $\overline{A}$ is compact, so by [**168**, Lemma 5.2] we can find a compact convex subset C of K containing $\overline{A}$. Then we know by the fundamental theorem that C is a complete semilattice, so A has a supremum $a_0 = \bigoplus_C A$ in C. We show that a_0 is also the supremum of A in K. So let $x \in K$ be an upper bound of A in K. Since $x \in\uparrow C$, the set $C \cap \downarrow x$ is nonempty compact convex, so it has a greatest element c. Then $a_0 \leqslant c \leqslant x$. This proves that A has a supremum in K. □

Remark IV-6.4. The previous proof actually uses the concept of projection. To see this, let S be a locally-convex topological semilattice, and K be a nonempty compact convex subset of S. Then, for every $x \in\uparrow K$, the set $K \cap \downarrow x = \{k \in K : k \leqslant x\}$ is nonempty compact convex so has a greatest element, so we can define the *projection of x on K* by

$$p_K(x) := \bigoplus_K \{k \in K : k \leqslant x\}.$$

The partial map p_K deserves to be called a projection for it satisfies $p_K \circ p_K = p_K$ and $p_K(x) \leqslant x$ for all $x \in\uparrow K$. Moreover, if $x \notin K$, the set

$$H = \{y \in S : y \leqslant x \Rightarrow y \leqslant p_K(x)\}$$

is a halfspace (i.e. a convex subset with a convex complement) separating K and x. Compare with Cohen et al. [**63**, Theorem 8], where a similar statement is given for complete idempotent modules.

Now we can legitimately recall the results of Liukkonen and Mislove [**184**, Proposition 1.1].

Proposition IV-6.5. [**184**, Proposition 1.1] *Let K be a locally-compact topological semilattice. Then K is locally-convex if and only if the map $U[K] \ni A \mapsto \bigoplus_K A \in K$ is a continuous surmorphism. In this case, if $A \subset K$ is compact, then A has a compact convex neighbourhood in K, and there is a minimal subset $B \subset A$ such that $\bigoplus_K A = \bigoplus_K B$.*

An additional ingredient will be needed for our advanced Krein–Milman type theorem, namely a result for separating the points in locally-convact semilattices. This role is played by the following result.

Proposition IV-6.6 (Compare with [**168**, Theorem 4.1]). *In a locally-convact topological semilattice K, let A be a nonempty closed upper subset and $x \notin A$. Then there exists a continuous semilattice-morphism $\varphi : K \to [0,1]$ that commutes with arbitrary existing suprema and such that $\varphi(A) = \{1\}$ and $\varphi(x) = 0$.*

Proof. With a proof similar to that of Urysohn's lemma and utilizing the axiom of choice, Lawson [**168**, Theorem 4.1] built a continuous semilattice-morphism $\varphi : K \to [0,1]$ such that $\varphi(x) = 0$, $\varphi(A) = \{1\}$, and of the form $\varphi(z) = \bigoplus\{t : z \in V_t\}$, where t runs over the set of dyadic numbers in $[0,1]$, and $V_t = K\setminus \downarrow z_t$ for some $z_t \in K$. We show that φ preserves arbitrary existing suprema. If F is a nonempty subset of K with supremum, then $\bigoplus F \notin V_t \Leftrightarrow \bigoplus F \leqslant z_t \Leftrightarrow (\forall f \in F)(f \leqslant z_t) \Leftrightarrow (\forall f \in F)(f \notin V_t)$. We deduce that $\{t : \bigoplus F \in V_t\} = \bigcup_{f\in F}\{t : f \in V_t\}$, so that $\varphi(\bigoplus F) = \bigoplus_{f\in F}\bigoplus\{t : f \in V_t\} = \bigoplus_{f\in F}\varphi(f)$, i.e. φ preserves existing suprema. □

Note that, under the same hypothesis, if $x, y \in K$ such that $x \nleqslant y$, this proposition provides a continuous semilattice-morphism $\varphi : K \to [0,1]$ that commutes with arbitrary nonempty suprema and such that $\varphi(x) = 1$ and $\varphi(y) = 0$. In particular, the φ's separate the points of K.

We state two additional results on separation in semilattices. They will not be used later on, but we believe they are of independent interest.

Proposition IV-6.7. *In a locally-convact topological semilattice K, let A be a compact convex subset and $x \notin A$. Then there exists an open convex neighbourhood V of A such that $x \notin \overline{V}$.*

Proof. If $x \notin\uparrow A$, then, considering that $\uparrow A$ is a closed (use e.g. [**114**, Proposition VI-1.6(ii)]) and upper subset of K, we can apply Proposition IV-6.6 and take $V = \{\varphi > 1/2\}$. Otherwise, $B := A\cap \downarrow x$ is nonempty compact convex, so its has a greatest element $b = \bigoplus_K B \in B$. Since $x \notin A$, $x \neq b$. Thus $x \nleqslant b$, so there is some $\psi : K \to [0,1]$ such that $\psi(x) = 1$ and $\psi(b) = 0$. Hence, the set $U := \{\psi < 1/2\}$ is open in K and contains B. Now consider $C = A\setminus U$. For every $c \in C$, $c \nleqslant x$, i.e. $x \notin\uparrow C$. But C is closed in A, hence compact, so $\uparrow C$ is closed by [**114**, Proposition VI-1.6(ii)], and Proposition IV-6.6 applies again: there is some $\varphi : K \to [0,1]$ such that $\varphi(C) = \{1\}$ and $\varphi(x) = 0$. To conclude the proof, choose $V = \{\psi < 1/2\} \cup \{\varphi > 1/2\}$. This is an open convex subset containing A, and $x \notin \overline{V} \subset \{\psi \leqslant 1/2\} \cup \{\varphi \geqslant 1/2\}$. □

Remark IV-6.8. In the proof the convex set $\{\psi \leqslant 1/2\} \cup \{\varphi \geqslant 1/2\}$ has a convex complement, i.e. is a (closed) halfspace.

Corollary IV-6.9 (Axiom NS_4). *In a compact locally-convex topological semilattice K, let A, B be disjoint closed convex subsets. Then there exists a closed convex neighbourhood of A that is disjoint from B.*

Proof. Let $x \in B$. Since A is disjoint from B, $x \notin A$, hence there exists some open convex neighbourhood V_x of A such that $x \notin \overline{V}_x$ by Proposition IV-6.7. The family of open subsets $(K\setminus\overline{V}_x)_{x\in B}$ covers the compact subset B, hence admits a finite subcover $(K\backslash\overline{V}_x)_{x\in F}$, with $F \subset B$ finite. Therefore, $K\backslash B \supset \bigcap_{x\in F}\overline{V}_x \supset \bigcap_{x\in F} V_x \supset A$. Thus, $\bigcap_{x\in F}\overline{V}_x$ is a closed convex neighbourhood of A that is disjoint from B. □

IV-6.3. Extension to the locally-compact case. To resolve the problem raised in the introduction (§ IV-6.1), we define a *line* of a topological semilattice S as an upper-bounded chain in S that is not relatively compact. Hence a closed subset K of S *contains no line* if every upper-bounded chain in K is contained in a compact subset of K.

Lemma IV-6.10. *Let S be a locally-convex topological semilattice, and let K be a locally-compact closed convex subset of S containing no line. Then every element x of K is the supremum in K of the extreme points of K below x.*

One may find some similarities between the following proof and that of the tropical analogue of Minkowski's theorem in $\mathbb{R}_+^n$ (see Gaubert and Katz [**107**, Theorem 3.2] and Butkovic, Schneider, and Sergeev [**53**, Proposition 24], see also Helbig [**124**, Theorem IV.5] for a first but less precise statement, and Develin and Sturmfels [**78**, Proposition 5] for an analogue of Carathéorory's theorem). We shall come back to this issue in Chapter V.

Proof. If $\varphi : K \to [0,1]$ is a semilattice-morphism, we let $K_\varphi = \{u \in K : u \leqslant x, \varphi(u) = \varphi(x)\}$. Let C be a maximal chain in K_φ (containing x). Then C is upper-bounded, contained in K, but must not be a line, hence is relatively compact. Since a maximal chain in a poset with a semiclosed topology is always closed [**114**, Proposition VI-5.1], and since K_φ is closed, we deduce that C is compact. In particular, C has a least element $u_\varphi \in C$ by Wallace's lemma. We show that u_φ is an extreme point of K. If there are $v, w \in K$ such that $u_\varphi = v \oplus w$, then $\varphi(x) = \varphi(u_\varphi) = \max(\varphi(v), \varphi(w))$. Let us assume, without loss of generality, that $\varphi(x) = \varphi(v)$. It follows that $v \in K_\varphi$. Also, $u_\varphi \geqslant v$, so that $u_\varphi = v$ by definition of u_φ. This proves that $u_\varphi \in \operatorname{ex} K$.

Now let $y \in K$ be some upper bound of the set $\operatorname{ex} K \cap \downarrow x$ in K. Then $y \geqslant u_\varphi$ for all φ, so that $\varphi(y) \geqslant \varphi(u_\varphi) = \varphi(x)$ for all φ. By Proposition IV-6.6, this implies that $y \geqslant x$. This proves that $x = \bigoplus_K \operatorname{ex} K \cap \downarrow x$. □

Remark IV-6.11. We can be more restrictive in the definition of a line. Redefine a line in S as an upper-bounded chain C that is not relatively compact and that satisfies $\downarrow c \not\subset C$, for all $c \in C$. One can check that the previous proof still works. Consequently, the lemma now encompasses the case where the set S is itself a chain (considered as a locally-convex topological semilattice when equipped with its interval topology).

Let S be a locally-convex topological semilattice. If Ψ is the set of continuous semilattice-morphisms $\psi : S \to [0,1]$, there is a natural mapping $S \to [0,1]^\Psi$. This is not an injective map in general, for the ψ's do not necessarily separate the points of S. If we equip the set $[0,1]^\Psi$ with the (compact Hausdorff) product topology, which amounts to the topology of pointwise convergence, we define the *weak topology* $\sigma(S, \Psi)$ as the topology on S generated by the family

$$\{\psi^{-1}(V) : \psi \in \Psi, V \text{ open in } [0, 1]\}.$$

It is coarser than the original topology. Moreover, if K is a subset of S, then the topology induced on K by $\sigma(S, \Psi)$ coincides with $\sigma(K, \Psi|_K)$, where $\Psi|_K$ denotes the family of restrictions of the functions in Ψ to K (see [**15**, Lemma 2.53]). Note that, if K is a locally-compact closed convex subset of S, we cannot conclude that $\sigma(K, \Psi|_K)$ coincides with the original topology (one would like to use e.g. [**15**, Theorem 2.55]) because the ψ's restricted to K do not separate points and closed subsets in general. We now restate the Klee–Krein–Milman type theorem given in the Introduction (Theorem IV-2.1).

Theorem IV-6.12 (Klee–Krein–Milman for semilattices)**.** *In a locally-convex topological semilattice, every locally-compact weakly-closed convex subset containing no line is the weakly-closed convex hull of its extreme points.*

Proof. Let S be a locally-convex topological semilattice, let K be a locally-compact weakly-closed convex subset of S containing no line, and let $x \in K$. Since K is closed in the weak topology, it is closed in the original topology, so the previous lemma applies: we have $x = \bigoplus_K D$, where D is the directed subset $\mathrm{co}(\mathrm{ex}\, K \cap \downarrow\! x)$. We have to show that x is in the weak closure D^* of D. So assume that $x \notin D^*$. By definition of the weak topology, there are open subsets $V_1, \ldots, V_k$ of $[0, 1]$ and continuous semilattice-morphisms $\psi_1, \ldots, \psi_k : S \to [0, 1]$ such that $x \in \bigcap_{j=1}^{k} \psi_j^{-1}(V_j) \subset S \backslash D^*$. Let us denote by $\varphi_1, \ldots, \varphi_k$ the respective restrictions of $\psi_1, \ldots, \psi_k$ to K. Using the notations of the proof of Lemma IV-6.10, we let $u = u_{\varphi_1} \oplus \ldots \oplus u_{\varphi_k}$. Then u is in D as a finite join of extreme points of K below x. Remembering that $\varphi_j(u_{\varphi_j}) = \varphi_j(x)$ for all j, one can see that $\varphi_j(u) = \varphi_j(x)$ for all j. This implies that $\varphi_j(u) \in V_j$ for all j, thus $u \in S\backslash D^*$, a contradiction. □

IV-6.4. Milman's converse. In Section IV-4 we have proved Milman's theorem in pospaces with the lower, upper, or order convexity. For topological semilattices, this result is less evident, since the convex hull of a compact subset does not need to be closed in general. Fortunately, it does work, even for locally-compact convex subsets. The next lemma is interesting in its own right.

Lemma IV-6.13. *Let S be a locally-convex topological semilattice, and K be a locally-compact convex subset of S. Then, for every compact subset A of K and every $x \in \mathrm{ex}\, K$, $x = \bigoplus_K A$ implies $x \in A$.*

First proof. By Proposition IV-6.5, there is a minimal subset B of A such that $\bigoplus_K B = \bigoplus_K A$. If B is empty or a singleton, then $x \in A$ is clear. Otherwise, let $b \in B$. Then $B\backslash\{b\}$, as a nonempty relatively compact subset of A, has a supremum b_0 in K. Moreover, $x = \bigoplus_K B = b_0 \oplus b$. Since $x \in \operatorname{ex} K$, we get $x \in \{b_0, b\}$. By minimality of B, $x \neq b_0$, so $x = b \in A$. $\square$

Second proof. Assume that $x \notin A$. One may wish to apply Proposition IV-6.7, but here we do not assume A to be convex. For every $a \in A$, $a \not\geqslant x$, and by Proposition IV-6.6 there exists a continuous semilattice-morphism $\varphi_a : K \to [0,1]$ such that $\varphi_a(a) = 0$ and $\varphi_a(x) = 1$. Let V_a be the open subset $\{\varphi_a < 1/2\}$ of K. The compact set A is covered by the open family $\{V_a\}_{a\in A}$, so we can extract a finite subfamily $\{V_a\}_{a\in F}$ still covering A. If $H_a := \{\varphi_a \leqslant 1/2\}$, we deduce that $A = \bigcup_{a\in F}(A \cap H_a)$. Every $A \cap H_a$ is compact and can be supposed nonempty, hence has a supremum in K by Lemma IV-6.3, thus $x = \bigoplus_K A = \bigoplus_{a\in F}(\bigoplus_K A \cap H_a)$. But x is an extreme point of K, so that $x = \bigoplus_K (A \cap H_{a_0})$ for some $a_0 \in F$. Proposition IV-6.6 also says that φ_{a_0} can be choosen so as to preserve arbitrary nonempty suprema in K, so $1 = \varphi_{a_0}(x) = \varphi_{a_0}(\bigoplus_K A \cap H_{a_0}) = \bigoplus \varphi_{a_0}(A \cap H_{a_0}) \leqslant 1/2$, a contradiction. $\square$

Remark IV-6.14. Compare Lemma IV-6.13 with [**114**, Corollary V-1.4], which is a similar result that holds in continuous lattices. See also [**114**, p. 403].

Theorem IV-6.15 (Milman for semilattices)**.** *Let S be a locally-convex topological semilattice, and K be a locally-compact closed convex subset of S. Then, for each compact subset A of K such that $K = \overline{\mathrm{co}}(A)$, we have $A \supset \operatorname{ex} K$.*

Proof. Let $x \in \operatorname{ex} K$, and assume that $x \notin A$. Let $B := A \cap \downarrow x$, and suppose at first that B is nonempty. Then B is nonempty compact, so admits a supremum $b = \bigoplus_K B$ in K by Lemma IV-6.3. Moreover, $x \neq b$ by the preceding lemma. Now the same method used in the proof of Proposition IV-6.7 provides a closed convex neighbourhood $\overline{V}$ of A such that $x \notin \overline{V}$. But $\overline{V} \supset \overline{\mathrm{co}}(A) = K$, a contradiction. If B is empty, then $x \notin \uparrow A$, and we can separate x and the upper closed subset $\uparrow A$ by a continuous semilattice-morphism, and the same contradiction appears. $\square$

IV-7. Semilattices with finite breadth

IV-7.1. Breadth and Minkowski's theorem. In locally-convex topological semilattices, an important subclass is that of topological semilattices with finite breadth. The *breadth* is defined as the least integer b such that, for all nonempty finite subsets F, there exists some $G \subset F$ with at most b elements such that $\bigoplus F = \bigoplus G$. It turns out that the breadth has a direct geometric interpretation, for as noticed by Jamison [**140**, § 4.D] it coincides with the Carathéodory number of the semilattice equipped with its algebraic convexity. The next lemma prepares a series of results on topological semilattices with finite breadth.

Lemma IV-7.1. *Let S be a topological semilattice with finite breadth b. If A is a compact subset of S, so is* $\mathrm{co}(A)$.

Proof. First remark that $A_b := \{x_1 \oplus \ldots \oplus x_b : x_1, \ldots, x_b \in A\}$ is a set between A and $\mathrm{co}(A)$. Moreover, this is a semilattice by definition of breadth, hence $\mathrm{co}(A) = A_b$. This also means that $\mathrm{co}(A)$ is the image of $A \times \ldots \times A$ by the continuous map $\phi : S \times \ldots \times S \to S$, $(x_1, \ldots, x_b) \mapsto x_1 \oplus \ldots \oplus x_b$. Hence, if A is compact, $\mathrm{co}(A) = \phi(A \times \ldots \times A)$ is compact . □

The following result, due to Lawson [**169**, Theorem 1.1], is a consequence of Lemma IV-7.1.

Proposition IV-7.2. [**169**, Theorem 1.1] *Every topological semilattice with finite breadth b is locally-convex.*

Proof. Let G be an open subset containing some point x. The continuity of ϕ defined above and the fact that $\phi(x, \ldots, x) \in G$ imply that $x \in V \subset V_b \subset G$ for some open subset V, where $V_b := \{x_1 \oplus \ldots \oplus x_b : x_1, \ldots, x_b \in V\}$. Thus $V_b = \mathrm{co}(V)$ is a convex neighbourhood of x contained in G. □

A topological semilattice has *compactly finite breadth* if every nonempty compact subset A contains a finite subset F with $\bigoplus A = \bigoplus F$. See Liukkonen and Mislove [**184**, Theorem 1.5] for equivalent conditions in locally-convact topological semilattices, and Lawson et al. [**174**, Theorem 1.11] for additional conditions. Another consequence of Lemma IV-7.1 is that "finite breadth" is usually stronger than "compactly finite breadth".

Corollary IV-7.3. *Every locally-compact topological semilattice with finite breadth has compactly finite breadth.*

Proof. If A is a nonempty compact subset, then A has a supremum a by Lemma IV-6.3, and $a \in \overline{\mathrm{co}}(A) = \mathrm{co}(A)$ by Lemma IV-7.1, so that $a = \bigoplus F$ for some finite $F \subset A$. □

A semilattice is *distributive* if, for all $x, y, z \in S$ with $x \leqslant y \oplus z$, there exists some $y' \leqslant y$, $z' \leqslant z$, such that $x = y' \oplus z'$. Also recall that a (distributive) *lattice* is a (distributive) semilattice in which every nonempty finite subset has an infimum.

Theorem IV-7.4. *In a topological distributive lattice S with finite breadth b (still equipped with the algebraic semilattice convexity), let K be a compact convex subset of S. Then every $x \in K$ can be written as the convex combination of at most b extreme points.*

Proof. Let L be the lattice generated by K in S. By Lemma IV-7.1 (applied to S and L with the opposite order), L is compact, so this is a compact locally-convex topological semilattice. Using either [**284**, Proposition 1.13.3] or a combination of [**114**, Theorem III-2.15] and the proof of [**114**, Proposition III-2.13], one can assert that L is also locally-convex with respect to the order convexity. Thus, [**277**, Theorem 3.1], due to Stralka, can be applied: L as a topological lattice can be embedded (algebraically and topologically) in a product of b compact (connected) chains $C = \prod_{j=1}^{b} C_j$. As a consequence, K as a topological semilattice also embeds in C. For all $j = 1, \dots, b$, we denote by $\varphi_j : K \to C_j$ the jth projection, which is a continuous semilattice-morphism.

The remaining part of the proof can now mimic that of Lemma IV-6.10, using the finite collection of maps $\{\varphi_j : j = 1, \dots, b\}$, which separates the points of K, instead of the whole collection of continuous semilattice-morphisms $\varphi : K \to [0, 1]$. This leads to the fact that, for all $x \in K$, one can write $x = u_1 \oplus \dots \oplus u_b$ for some extreme points $u_1, \dots, u_b$ of K. $\square$

Problem IV-7.5. Does the conclusion of this theorem still hold for S a topological distributive *semilattice* with finite breadth?

As a final remark, it should be emphasized that, in a locally-convex topological semilattice S, the set $\operatorname{ex} K$ of extreme points of some compact convex subset K is not necessarily closed. Actually, if S is distributive, it is known that $\operatorname{ex} K$ is closed if and only if the way-above relation on K is additive [**114**, Proposition V-3.7].

IV-7.2. Depth of a semilattice. The *depth* of a semilattice, defined as the supreme cardinality of a chain, is another important convex invariant, as highlighted by the following result[1]. Recall that the Helly number is the least integer h such that each finite family of convex subsets meeting h by h has a nonempty intersection.

Proposition IV-7.6. *The Helly number of a semilattice equals its depth.*

To prove this assertion, we shall need a result due to Jamison [**139**, Theorem 7], which says that in a finite *convex geometry* (see the definition in § IV-8.1), the Helly number equals the clique number, so first we give some definitions. Let X be a convexity space. A subset K of X is *free* if it is both convex and *independent*, i.e. such that $K = \operatorname{ex} K$. A *clique* is a maximal free subset, and the *clique number* of X is the supremum of the cardinalities of all cliques.

Lemma IV-7.7. *The free subsets (resp. the cliques) of a semilattice coincide with its chains (resp. its maximal chains), and the clique number of a semilattice equals its depth.*

Proof. Let C be a free subset of a semilattice, let $x, y \in C$, and let us prove that x and y are comparable. Since C is convex, $z := x \oplus y \in C$. But $C = \operatorname{ex} C$, so z is a extreme point of C, hence $z \in \{x, y\}$, i.e. $x \leqslant y$ or $y \leqslant x$. This proves that C is a chain. The converse statement is straightforward, and the rest of the proof follows. $\square$

Proof of Proposition IV-7.6. Write d for the depth of S. Let n be an integer $\leqslant d$, and let C be a chain with cardinality n. Then the finite family $(K_c)_{c \in C}$ of convex subsets $K_c = C \backslash \{c\}$ meets $n - 1$ by $n - 1$ but is of empty

[1]This result is left as an exercise in [**284**, Exercise II-1.23]. As far as we know, no proof of it exists in the literature.

intersection, so that $h > n-1$. This implies that $h \geqslant d$ (even if $d = \infty$). If $d = \infty$, we get $h = d$.

Now assume that d is finite. Let $(K_j)_{j \in J}$ be a finite family of convex subsets meeting d by d. For every $I \subset J$ with cardinality d, let $x_I \in \bigcap_{j \in I} K_j$. Denote by X the subsemilattice of S generated by $\{x_I\}_{I \subset J, |I|=d}$. Note that the depth d_X of X is less than d. Moreover, X is a finite set, and the algebraic convexity on X is a *convex geometry* (see the definition in § IV-8.1). Thus, [**139**, Theorem 7] applies, i.e. the clique number of X equals its Helly number h_X. By the previous lemma, this rewrites to $h_X = d_X$, hence $h_X \leqslant d$. Now, let X_j be the subsemilattice of X generated by $\{x_I\}_{I \subset J, |I|=d, j \in I}$. Then $x_I \in \bigcap_{j \in I} X_j$ for all I, so that $(X_j)_{j \in J}$ is a finite family of convex subsets of X meeting d by d. Since $h_X \leqslant d$, we have $\bigcap_{j \in J} X_j \neq \varnothing$, by definition of the Helly number. Morever, $X_j \subset K_j$, so we get $\bigcap_{j \in J} K_j \neq \varnothing$. This shows that $h \leqslant d$. □

The next result connects the depth with the extreme points of convex subsets and can be seen as a corollary of Lemma IV-6.10.

Proposition IV-7.8. *Let S be a locally-convex topological distributive semilattice, and K be a locally-compact closed convex subset of S. Assume that K has finite depth d. Then K is finite and has exactly d extreme points.*

Proof. We follow the proof given by Blyth [**41**, Theorem 5.3] for finite distributive lattices. Let C be a chain of maximal length d in K. For convenience, we write $c_1 < \ldots < c_d$ for elements of C. Let $\theta : \operatorname{ex} K \to C$ such that $\theta(p) = \min\{c \in C : c \geqslant p\}$. Note that c_1 is necessarily the least element of K, hence is in $\operatorname{ex} K$, and $\theta(c_1) = c_1$. If $c_k \in C \backslash \{c_1\}$, there exists some $p \in \operatorname{ex} K$ such that $p \leqslant c_k$ and $p \nleqslant c_{k-1}$, since $\operatorname{ex} K$ order-generates K by Lemma IV-6.10. This implies $\theta(p) = c_k$. We have shown that θ is surjective.

Let us prove that θ is injective. Assume that $\theta(p) = \theta(q) = c_k \in C$ for some $p, q \in \operatorname{ex} K$. If $c_k = c_1$, then $p = q = c_k$, so suppose that $c_k \neq c_1$. Then $c_{k-1} \oplus p \leqslant c_k$ is clear, and one also has $c_{k-1} \oplus p \geqslant c_k$, otherwise $c_{k-1} < c_{k-1} \oplus p < c_k$ which is impossible because of the maximality of C. We get $c_{k-1} \oplus p = c_k$, and symmetrically $c_k = c_{k-1} \oplus q$. Thus, $p \leqslant c_{k-1} \oplus p = c_{k-1} \oplus q$. The distributivity of S and the fact that p is an extreme point of K imply $p \leqslant c_{k-1}$ (which would contradict $\theta(p) = c_k$) or $p \leqslant q$. Similarly, $p \geqslant q$, so $p = q$, and θ is injective, hence bijective. This proves that the cardinality of $\operatorname{ex} K$ equals d.

Since K has finite depth, every (upper-bounded) chain in K is finite hence compact, so K contains no line. By Lemma IV-6.10, the finite subset $\operatorname{ex} K$ order-generates K, so that K is finite. □

IV-8. Convex geometries on semilattices and lattices

IV-8.1. Introduction. Some convexities may not satisfy a Krein–Milman type theorem and, for some of them, even polytopes may not coincide with the convex hull of their extreme points. This last property actually characterizes convexities that are convex geometries, whose usual definition follows. A convexity space X is a *convex geometry* (or an *antimatroid*) if, given a convex subset K, and two unequal points x and y, neither in K, then $y \in \mathrm{co}(K \cup \{x\})$ implies $x \notin \mathrm{co}(K \cup \{y\})$. This amounts to say that the relation $\leqslant_K$ defined on $X \backslash K$ by $x \leqslant_K y \Leftrightarrow y \in \mathrm{co}(K \cup \{x\})$ is a partial order. The convexities previously introduced, namely the order (resp. lower, upper) convexity for posets, and the algebraic convexity for semilattices, are indeed convex geometries (see [**284**, Exercise I-2.24]). In this section, we investigate some other convexities on semilattices and lattices that are not convex geometries in general.

Let X be a convexity space and $x \in X$. A *copoint* at x is a convex set $C \subset X$ maximal with the property $x \notin C$, in which case x is an *attaching point* of C.

Lemma IV-8.1. *Let X be a convexity space. If C is a convex subset and $x \notin C$, there is some copoint at x containing C.*

Proof. This is an easy consequence of Zorn's lemma. □

The next important theorem, due to Jamison [**137**], and to Edelman and Jamison [**86**] for the case where the set X is finite, lists several equivalent conditions for a convexity to be a convex geometry. For the sake of completeness, we shall give a proof of this result.

Theorem IV-8.2 (Jamison–Edelman)**.** *Let X be a convexity space. Then the following are equivalent:*

(1) *X is a convex geometry,*
(2) *each polytope is the convex hull of its extreme points,*
(3) *for each copoint C at x, the set $C \cup \{x\}$ is convex,*
(4) *each copoint C has a unique attaching point.*

Proof. (1) ⇒ (3). Assume that X is a convex geometry, and let C be a copoint at x. Assume that $C \cup \{x\}$ is not convex, i.e. there is some $y \in \mathrm{co}(C \cup \{x\})$, $y \notin C \cup \{x\}$. Then $\mathrm{co}(C \cup \{y\})$ is a convex set avoiding x and strictly greater than C, a contradiction.

(3) ⇒ (4). Let C be a copoint at x, and assume that it has another attaching point $y \neq x$. Then, by (3), $C \cup \{y\}$ is a convex set avoiding x and strictly greater than C, a contradiction.

(4) ⇒ (2). Let K be a polytope, and let F be a minimal finite subset such that $K = \mathrm{co}(F)$. Consider some $x \in F$ that is not an extreme point of K. By minimality of F, $x \notin \mathrm{co}(F \backslash \{x\})$, so there is some copoint C at x containing $\mathrm{co}(F \backslash \{x\})$. Since x is not an extreme point, C is strictly contained in $K \backslash \{x\}$, so there is some $y \neq x$, $y \notin C$. Let D be a copoint at y containing C. If $x \notin D$, then $C = D$ by maximality of C, but then, by (4), $x = y$, a contradiction. Hence, $x \in D$, so that $D = K$, which contradicts $y \notin D$. So we have shown that $F \subset \mathrm{ex}\, K$, i.e. $K = \mathrm{co}(\mathrm{ex}\, K)$.

(2) $\Rightarrow$ (1). Assume that, for some $x \neq y$ and some convex subset K, $x \in \operatorname{co}(K \cup \{y\}) \backslash K$ and $y \in \operatorname{co}(K \cup \{x\}) \backslash K$. It is easy to see that there exists some finite subset $F \subset K$ such that $x \in \operatorname{co}(F \cup \{y\})$ and $y \in \operatorname{co}(F \cup \{x\})$. Then the polytope $L = \operatorname{co}(F \cup \{x\}) = \operatorname{co}(F \cup \{y\})$ is the convex hull of its extreme points $\operatorname{ex} L$, and we deduce $\operatorname{ex} L \subset F \cup \{x\}$ and $\operatorname{ex} L \subset F \cup \{y\}$, hence $\operatorname{ex} L \subset F$, so that $L = \operatorname{co}(\operatorname{ex} L) \subset \operatorname{co}(F) \subset K$. This contradicts $x \notin K$. □

For one more equivalent condition using the concept of *meet-distributive lattice*, see Edelman [**85**, Theorem 3.3], Birkhoff and Bennett [**40**], and Monjardet [**211**].

In the following paragraphs, we say that a topological convexity space *satisfies the Krein–Milman property* if every compact convex subset is the convex hull of its extreme points.

IV-8.2. The ideal convexity of a semilattice. Recall from Section IV-5 that the *ideal convexity* of a semilattice consists of its lower subsemilattices. An element of a convex subset K is then an extreme point of K if and only if it is at the same time maximal and coprime in K (x is *coprime* if, for every nonempty finite subset F with $x \leqslant \bigoplus F$, $x \leqslant f$ for some $f \in F$). We call *max-coprime* an element that is both maximal and coprime.

Proposition IV-8.3. *A semilattice with the ideal convexity is a convex geometry if and only if it is a chain. In this case, when endowed with a compatible topology, it satisfies the Krein–Milman property.*

Proof. For a chain, the ideal convexity coincides with the lower convexity, thus is a convex geometry. The Krein–Milman property is then the terms of Theorem IV-4.4.

Now assume that the ideal convexity of some semilattice S is a convex geometry, and let us show that S is a chain. So let $x, y \in S$ with $x \not\leqslant y$. Then $x \notin \downarrow y$, which is a convex subset. Thus, by Lemma IV-8.1, there is some copoint C at x containing $\downarrow y$. By Theorem IV-8.2, $C \cup \{x\}$ is convex, and $y \in C$, so we have $y \oplus x \in C \cup \{x\}$, i.e. $y \oplus x \in C$ or $y < x$. The former case has to be rejected, otherwise $x \in \downarrow(y \oplus x) \subset C$. Hence, $y < x$, which concludes the proof. □

We seize the opportunity to mention here that the ideal convexity was considered by Martinez [**195**], whose main result [**195**, Theorem 1.2] can be rephrased in the langage of abstract convexity as follows:

Theorem IV-8.4 (Martinez). *Consider a semilattice with the ideal convexity. Then the following are equivalent:*

- *the ideal convexity is completely distributive,*
- *each copoint admits an attaching point with a unique copoint,*
- *each element can be uniquely decomposed as the join of a finite number of pairwise incomparable coprime elements.*

Decomposing elements as joins of coirreducible or coprime elements has been the subject of a great amount of research in order theory (see e.g. Erné [**90, 93**] and references therein, see also Bińczak et al. [**37**, Theorem 5.4] on presentable semilattices), and this theorem invites us to look at these past results from an abstract convexity point of view.

Remark IV-8.5. Martinez' theorem actually characterizes semilattices that are free $\mathbb{B}$-modules. Indeed, consider in the following lines a semilattice S with a least element 0, and assume for convenience that $S \neq \{0\}$. The last condition in Martinez' theorem says that a subset of the family of coprime elements is a basis (i.e. a subset B such that, for every x there is a unique finite -possibly empty- subset of B whose join is x). Conversely, assume that the semilattice admits a basis B, and let us show that every $b \in B$ is a non-zero coprime element. So let F be a finite subset such that $b \leqslant \bigoplus F$. For all $x \in F$, there is a finite subset F_x of B such that $x = \bigoplus F_x$. Hence, $F' := \bigcup_{x \in F} F_x$ is a finite subset of B whose join is $\bigoplus F$. Since $b \leqslant \bigoplus F$, $F' \cup \{b\}$ is another such subset, so $F' = F' \cup \{b\}$ by definition of B. This gives $b \in F'$, i.e. $b \in F_x$ for some $x \in F$. This shows that $b \leqslant x$ for some $x \in F$, i.e. that b is a coprime element. Also, b is non-zero, otherwise $0 \in B$ would be the join of both the empty set and $\{0\}$.

Another consequence is that every semilattice that is a free $\mathbb{B}$-module is distributive. For suppose that $x \leqslant y \oplus z$, and let F be a finite subset of a basis B such that $x = \bigoplus F$. Since every element of F is coprime, we have $f \leqslant y$ or $f \leqslant z$ for all $f \in F$. Then, if $y' = \bigoplus \{f \in F : f \leqslant y\}$ and $z' = \bigoplus \{f \in F : f \leqslant z\}$, we get $y' \leqslant y$, $z' \leqslant z$, and $x = y' \oplus z'$, which shows distributivity.

Therefore, if a semilattice S is a free $\mathbb{B}$-module, then it has a unique basis, equal to the subset of its non-zero coprime elements. To see this, let B be a basis of S. Since S is distributive, the subset of its coprime elements is $\operatorname{ex} S$, and we have seen that $B \subset (\operatorname{ex} S) \backslash \{0\}$. If $x \in (\operatorname{ex} S) \backslash \{0\}$, there exists a nonempty finite subset F of B such that $x = \bigoplus F$. Since x is an extreme point of S, we deduce $x \in F$, so that $x \in B$.

If now we define the *rank* r of a distributive semilattice S as the cardinality of $(\operatorname{ex} S) \backslash \{0\}$, then, applying Proposition IV-7.8 to S equipped with the discrete topology, one can see that the following conditions are equivalent:

- S is a $\mathbb{B}$-module of finite type,
- S has finite depth,
- S has finite rank,
- S is finite.

In this case, the depth d of S equals $r + 1$. Moreover, if S is free, then S is in bijection with the collection of subsets of $(\operatorname{ex} S) \backslash \{0\}$, hence has exactly 2^r elements.

IV-8.3. The order-algebraic convexity of a semilattice. Quite different from the previous case is the one of the *order-algebraic convexity* of a semilattice, made up of its order-convex subsemilattices, for it involves trees instead of chains. A *tree* is a semilattice in which every principal filter $\uparrow x$ is a chain. It is an easy task to see that the set of extreme points of a convex subset is the union of its minimal elements and max-coprime elements.

Proposition IV-8.6. *A semilattice with the order-algebraic convexity is a convex geometry if and only if it is a tree. In this case, when endowed with a Hausdorff semitopological topology, it satisfies the Krein–Milman property.*

Proof. Assume that the order-algebraic convexity of some semilattice S is a convex geometry, and let us show that S is a tree. So let $a, b, x \in S$ such that $a \geqslant x$ and $b \geqslant x$. We want to prove that a and b are comparable, so suppose that $b \nleqslant a$, i.e. $b \notin \downarrow a$. The subset $\downarrow a$ is convex, so by Lemma IV-8.1 there exists some copoint C at b containing $\downarrow a$. In particular, $a, x \in C$. Now use the fact that the convexity is a convex geometry: this implies that $C \cup \{b\}$ is convex (Theorem IV-8.2), hence $a \oplus b \in C \cup \{b\}$. If $a \oplus b \in C$, then $b \in [x, a \oplus b] \subset C$, whereas $b \notin C$. Thus, $a \oplus b \in \{b\}$, i.e. $b \geqslant a$.

Conversely, consider a Hausdorff semitopological tree, and let K be a compact convex subset. We (implicitly) follow the suggestion of proof from [**284**, Exercise I-5.26]. Denote by $\leqslant_b$ the relation $\leqslant_{\{b\}}$ defined on $K \backslash \{b\}$ (see § IV-8.1), obviously extended to K. Then, for all $x, y \in K$, $y \leqslant_b x$ if and only if $x \leqslant b \oplus y$ and ($x \geqslant b$ or $x \geqslant y$). Since the tree is semitopological, the subsets $\uparrow x$ and $\downarrow x$ are closed by [**114**, Proposition VI-1.13(ii)]. Also, the map $y \mapsto b \oplus y$ is continuous, so $\leqslant_b$-principal ideals $\downarrow_b x = \{y \in K : y \leqslant_b x\}$ are closed in K.

Now let $x \in K$. If x is minimal in K, then $x \in \operatorname{ex} K$. Otherwise, applying Wallace's lemma, there exists some minimal element b of K such that $b < x$ (in particular, $b \in \operatorname{ex} K$). Using Wallace's lemma once more, we find an element $y \in \downarrow_b x \cap \uparrow b$, minimal with respect to the partial order $\leqslant_b$. If we show that $y \in \operatorname{ex} K$, we shall have proved that $x \in \operatorname{co}(\{b, y\}) \subset \operatorname{co}(\operatorname{ex} K)$. So write $y \leqslant \bigoplus F$ for some nonempty finite subset F of K, and let us see why $y \in F$. In the ambiant tree, $\uparrow b$ is a chain, hence the supremum of $\{b \oplus f : f \in F\}$ is actually a maximum, i.e. there is some $f_0 \in F$ such that $b \oplus f_0 = \bigoplus(b \oplus F) = b \oplus \bigoplus F$. This implies that $b \oplus f_0 \geqslant y \geqslant b$, so that $f_0 \leqslant_b y$. We obtain $y = f_0 \in F$ by minimality of y. We conclude that y is max-coprime in K, i.e. $y \in \operatorname{ex} K$. ☐

IV-8.4. The order-algebraic convexity of a lattice. Similarly to the above example, the *order-algebraic convexity* of a lattice comprises its order-convex sublattices. The corresponding set of extreme points of a convex subset is the union of its max-coprime and its min-prime (defined dually) elements. Here the condition to get a convex geometry is the same as for ideal convexity.

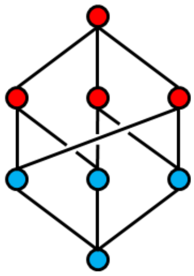

FIGURE 4. Hasse diagram of the power set of $\{1, 2, 3\}$. The blue (resp. red) points are the coirreducible elements with respect to inclusion (resp. reverse inclusion). This poset has no doubly-irreducible elements.

Proposition IV-8.7. *A lattice with the order-algebraic convexity is a convex geometry if and only if it is a chain. In this case, when endowed with a compatible topology, it satisfies the Krein–Milman property.*

Proof. Following the lines of the proof of Proposition IV-8.3, if $y \nleqslant x$, there is some copoint C at x containing the convex subset $\{y\}$. The subset $C \cup \{x\}$ must be convex if the convexity is a convex geometry, so $y \wedge x \in C \cup \{x\}$ and $y \oplus x \in C \cup \{x\}$. If both $y \wedge x$ and $y \oplus x$ are in C, then $x \in [y \wedge x, y \oplus x] \subset C$ by order-convexity, which contradicts $x \notin C$. Thus, either $y \oplus x \in \{x\}$ (which is not possible for we assumed $y \nleqslant x$) or $y \wedge x \in \{x\}$, i.e. $y > x$. $\square$

IV-8.5. The algebraic convexity of a lattice. A final, still challenging example should be evoked. On a lattice, one can consider the *algebraic convexity* made up of its sublattices. An abundant literature of topological flavour exists on lattices, and the toolkit of results on locally-convex lattices and compact lattices could let one think that the approach adopted for semilattices in Section IV-5 could be reedited without pain. For instance, [**114**, Proposition VII-2.8] gives a lattice counterpart to the fundamental theorem IV-5.1. Also, Choe [**58, 59**] and Stralka [**277**] among others studied topological lattices with small lattices, which are nothing but locally-convex topological lattices.

Unfortunately, a deeper examination of this convexity leads to special difficulties. Simply consider the fact that extreme points are the *doubly-irreducible* elements (elements that are simultaneously coirreducible for $\leqslant$ and for $\geqslant$), the existence of which is not guaranteed in general, even in finite distributive lattices (look at the power set, ordered by inclusion, of a set with cardinality > 2 for instance, see Figure 4). On that subject, see "Annexe 1: Pruning a poset with veins", at the end of this thesis.

The work of Erné [**90**], after that of Monjardet and Wille [**212**], although difficult to interpret, gives some hope in this direction (see also the paper by Berman and Bordalo [**35**]). Rephrasing [**90**, Theorem 4.14] for the finite case, one has:

Proposition IV-8.8 (Monjardet–Wille–Erné)**.** *In a finite distributive lattice, the following conditions are equivalent:*

- *P is principally separated,*
- *the normal completion of P is a distributive lattice,*
- *the lattice is generated by its doubly-irreducible elements,*
- *each coprime is a meet of doubly-irreducible elements,*
- *for all $p \in P, q \in Q$ with $p \leqslant q$, there exists $r \in P \cap Q : p \leqslant r \leqslant q$,*

where P (resp. Q) denotes the subset of coprime (resp. prime) elements.

The normal completion refers to the smallest complete lattice in which a poset embeds (also called Dedekind–MacNeille completion, or completion by cuts, see Chapter III). Principal separation in a poset is the assertion that, for all $x \nleqslant y$, there are some $p \leqslant x$, $q \geqslant y$ such that $p \nleqslant q$ and $\uparrow p \cup \downarrow q$ is the whole poset; for complete lattices, this is equivalent to complete distributivity.

A distributive lattice with the algebraic convexity is then a convex geometry if and only if *every* finite sublattice satisfies the conditions of Proposition IV-8.8 (because a polytope is here necessarily finite).

Problem IV-8.9. Does every compact locally-convex distributive lattice (i.e. every completely distributive lattice) satisfy the Krein–Milman property as soon as it is a convex geometry ?

IV-9. Conclusion and perspectives

In Chapter V, we shall consider the natural (algebraic) convexity on *idempotent modules*. In a future work, we shall also aim at relaxing the Hausdorff hypothesis after the work of Goubault-Larrecq [**116**], who proved a Krein–Milman type theorem for non-Hausdorff *cones* (in the sense of Keimel [**149**]).

Acknowledgements. I am very grateful to Marianne Akian for her crucial help in the proof of Milman's converse for semilattices, and to Stéphane Gaubert who pointed out to me the direction of proof of the Krein–Milman theorem. I also gratefully thank Pr. Jimmie D. Lawson for his very motivating suggestions on the non-Hausdorff setting.

Appendix A. Some properties of convexities on ordered structures

A.1. Arity. If the convex sets of a convexity are exactly the subsets C such that $\mathrm{co}(F) \subset C$ for all $F \subset C$ with cardinality $\leqslant n$, then the convexity is of *arity* $\leqslant n$. All the convexities considered in this chapter are of arity $\leqslant 2$. The following table summarizes special cases.

Structure	Convexity	Arity	Arity = 1?
poset	upper	1	+
poset	order	$\leqslant 2$	iff depth $= 2$
semilattice	algebraic	$\leqslant 2$	iff chain
semilattice	ideal	$\leqslant 2$	iff chain
semilattice	order-alg.	$\leqslant 2$	iff chain
lattice	order-alg.	$\leqslant 2$	iff chain
lattice	algebraic	$\leqslant 2$	iff chain

TABLE 1. Arity.

A.2. Separation axioms. Convexities are classically classified according to five basic separation axioms, mimicking the usual conditions $T_0, \ldots, T_4$ in topology:

- S_0. for each pair of distinct points, there exists a convex set containing one point but not the other,
- S_1. all singletons are convex,
- S_2. two distinct points extend to complementary halfspaces,
- S_3. each convex subset is an intersection of halfspaces,
- S_4. two disjoint convex subsets extend to complementary halfspaces,

where a *halfspace* is a convex subset with a convex complement.

The S_4 separation axiom is also called the Kakutani separation property, since Kakutani [**146**] proved its validity in real vector spaces with their usual (Euclidian) convexity. Ellis [**88**] gave an abstract version of Kakutani's result, that we recall below. Briec et al. [**51**, Theorem 2.1] give a self-contained proof in the framework of finite-dimensional tropical geometry, restating arguments due to van de Vel.

Proposition A.1. *On a poset, the upper convexity (resp. the lower convexity) is S_0 (but not S_1, unless the partial order is trivial), the order convexity is S_3, and the order convexity on a chain is S_4.*

Proof. Let C be an order-convex subset and $x \notin C$. If $C \cap \downarrow x = \varnothing$, then $\downarrow x$ is a halfspace separating C and x. The case $C \cap \uparrow x = \varnothing$ is similar. Otherwise, there exists some $y \in C \cap \downarrow x$ and $z \in C \cap \uparrow x$, hence $y \leqslant x \leqslant z$. Since C is order-convex, we have $x \in C$, a contradiction. □

Proposition A.2. *On a semilattice, the algebraic and the order-algebraic convexities are S_4.*

Proof. The case of the order-algebraic convexity is treated in van de Vel [**284**, Proposition I-3.12.2]. The algebraic convexity is of arity 2 and clearly satisfies the Pasch property (see the definition in [**284**, § I-4.9]), hence is S_4 by [**284**, Theorem 4.12]. □

Proposition A.3. *On a lattice that is a distributive continuous semilattice (or dually), in particular on a completely distributive lattice, the algebraic convexity is S_2.*

Proof. By [**114**, Corollary I-3.13], if L is a distributive continuous semilattice, then its subset of coprime elements is order-generating. Hence, if $x \nleqslant y$, one can find some coprime element p with $p \leqslant x$ and $p \nleqslant y$. This implies that $\uparrow p$, which is a halfspace, separates x and y. $\square$

Structure	Convexity	S_1	S_2
poset	upper	iff antichain	iff antichain
poset	order	+	+
semilattice	algebraic	+	+
semilattice	ideal	iff antichain	iff antichain
semilattice	order-alg.	+	+
lattice	order-alg.	+	iff distributive
lattice	algebraic	+	if distrib. continuous

TABLE 2. S_1 and S_2 axioms. All structures satisfy the S_0 axiom. For lattices with the order-algebraic convexity, see [**284**, Proposition I-3.12.3].

Structure	Convexity	S_3	S_4
poset	upper	iff antichain	iff antichain
poset	order	+	if chain
semilattice	algebraic	+	+
semilattice	ideal	iff antichain	iff antichain
semilattice	order-alg.	+	+
lattice	order-alg.	iff distributive	iff distributive
lattice	algebraic	?	?

TABLE 3. S_3 and S_4 axioms. For lattices with the order-algebraic convexity, see [**284**, Proposition I-3.12.3].

A.3. The convex geometry property. Table 4 recalls the results of Section IV-8.

Structure	Convexity	Convex geometry	Extreme points
poset	upper	+	minimal
poset	order	+	minimal or maximal
semilattice	algebraic	+	coirreducible
semilattice	ideal	iff chain	max-coprime
semilattice	order-alg.	iff tree	minimal or max-coprime
lattice	order-alg.	iff chain	min-prime or max-coprime
lattice	algebraic	?	doubly-irreducible

TABLE 4. Convex geometry property and extreme points.

CHAPTER V

Krein–Milman's and Choquet's theorems in the max-plus world

ABSTRACT. We prove an idempotent version of the Choquet representation theorem. More precisely, we show that, in a locally-convex topological $\mathbb{R}_+^{\max}$-module, every point of a compact convex subset K can be represented by a possibility measure supported by the extreme points of K. We also obtain a Krein–Milman type theorem in locally-convex topological $\mathbb{R}_+^{\max}$-modules.

V-1. RÉSUMÉ EN FRANÇAIS

La prolongation naturelle du chapitre IV, au cours duquel nous avons étudié la convexité dans les semitreillis, est l'étude de la convexité dans les modules « idempotents », c'est-à-dire les modules sur le semicorps idempotent $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \max, \times)$.

Nous choisissons comme référence le *semi-anneau max-times* $\mathbb{R}_+^{\max}$ plutôt que le *semi-anneau max-plus* $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. Ce dernier ainsi que le semi-anneau $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$ sont plus couramment employés dans la littérature du fait de l'interprétation qu'en offrent la plupart des applications (théorie du contrôle, planification), mais l'on peut passer d'un cadre à l'autre par une simple transformation logarithmique. Le choix de $\mathbb{R}_+^{\max}$ a l'avantage de rendre plus clair le parallèle entre modules idempotents et espaces vectoriels sur $\mathbb{R}$.

Nous avons étudié au chapitre III les modules sur un semicorps idempotent d'un point de vue algébrique, à la suite des travaux de Cohen et al. [**63**] et Litvinov, Maslov et Shpiz [**182**] notamment. Pour approfondir notre examen, notamment en dimension infinie, c'est à présent un point de vue topologique que nous souhaitons apporter.

On trouve peu de travaux qui se placent dans le cadre abstrait des modules idempotents topologiques de dimension infinie ; dans les modules de dimension infinie considérés dans la littérature c'est souvent la topologie de la convergence simple qui est employée. Une exception est l'article [**273**] de Shpiz et Litvinov, où un théorème de point fixe de type Schauder est démontré. Relevons également l'article de Shpiz [**272**] qui démontre l'existence de vecteurs propres pour certains endomorphismes des modules idempotents dits *archimédiens* ; s'il n'est pas explicitement topologique, ce travail s'appuie en fait indirectement sur la topologie de Lawson du module d'intérêt (considéré comme un semitreillis).

De nombreux travaux ont porté sur la convexité max-plus en dimension finie. Ils ont été initiés dans les années 1970 par Zimmermann qui parlait alors d' « algèbre extrêmale » [**307**]. Au-delà des travaux de Cohen et al. déjà cités, une nouvelle approche a émergé avec la publication de Develin et Sturmfels [**78**], qui a fait le lien avec la géométrie tropicale et a pointé une connexion surprenante avec l'analyse phylogénétique. Ce travail a vu l'émergence de l'étude combinatoire des polyhèdres max-plus convexes, qui s'est ensuite poursuivie avec Joswig [**143**], Develin et Yu [**79**], Allamigeon [**16**], Allamigeon, Gaubert et Katz [**17**], Akian, Gaubert et Guterman [**9**] parmi d'autres.

Dans ce cadre les théorèmes classiques de convexité discrète ont trouvé leurs analogues. Il en est ainsi du théorème de Carathéodory, cf. Helbig [**124**], Develin et Sturmfels [**78**], Briec et Horvath [**49**]. Un théorème de Minkowski max-plus a été prouvé indépendamment par Gaubert et Katz [**106, 107**] et Butkovič, Schneider et Sergeev [**53**]. Le théorème de Helly max-plus est dû à Briec et Horvath [**49**], et a été reprouvé par Gaubert et Sergeev [**110**]. Gaubert et Meunier ont fait la synthèse de ces résultats, en les complétant notamment par un théorème de Tverberg et un théorème de Carathéodory « coloré » [**109**]. Nous renvoyons à ce même article pour des commentaires sur les preuves possibles permettant d'aboutir à un théorème de Radon max-plus.

Citons enfin les travaux de Briec et Horvath [**49**], Briec, Horvath et Rubinov [**51**], Adilov et Rubinov [**3**] ; ces auteurs se sont intéressés à la convexité max-plus en dimension finie, en la redécouvrant indépendamment sous le nom de B-convexité. Certaines de leurs démonstrations ont l'originalité d'appliquer explicitement la déquantification de Maslov : les convexes max-plus sont directement vus comme des déformés à l'infini de convexes usuels (cf. Figures 1 et 2).

Pour une introduction à l'algèbre max-plus, cf. Akian, Bapat et Gaubert [**8**]. Pour une introduction à la géométrie tropicale, cf. par exemple Develin et Sturmfels [**78**], Richter-Gebert et al. [**254**], Itenberg [**133**]. Pour une revue des champs d'intérêt et d'application de l'analyse idempotente, cf. Kolokoltsov [**152**] et Litvinov [**180, 181**].

Les théorèmes de Carathéodory et de Minkowski précités nous interpellent sur la possibilité d'obtenir un théorème de type Krein–Milman dans les $\mathbb{R}_+^{\max}$-modules topologiques. C'est l'un des principaux objets de ce chapitre de prouver un tel résultat. Nous supposerons pour cela que le module considéré est localement convexe, comme nous l'avions fait au chapitre IV pour le cas des semitreillis. Nous allons d'ailleurs mettre directement à profit les résultats du chapitre IV pour arriver à nos fins. Notre autre résultat marquant est l'analogue du théorème de représentation intégrale de Choquet :

Théorème V-1.1 (Théorème de Choquet, version idempotente)**.** *Soit M un $\mathbb{R}_+^{\max}$-module topologique localement convexe et K une partie compacte convexe non vide. Tout élément de K est représenté par une mesure de possibilité sur K supportée par les points extrêmes de K.*

On peut donc écrire $x \in K$ sous la forme

$$x = \bigoplus_{y \in K} p(y).y, \tag{40}$$

où p est une possibilité sur K telle que $p(B) = 0$ pour tout borélien $B \subset K \backslash \operatorname{ex} K$. Dans ce chapitre, une possibilité désigne une mesure complètement maxitive de poids total égal à 1 ; cette définition diffère de celle utilisée au chapitre I.

Ce résultat de représentation des éléments d'un module idempotent fait écho aux travaux de Akian, Gaubert et Walsh [**11**] sur la frontière de Martin max-plus. Ceux-ci prouvent justement la représentation (40) pour les vecteurs x max-plus harmoniques ou max-plus surharmoniques du module $(\mathbb{R}_+^{\max})^S$. Ils montrent que la possibilité p peut être prise *maximale*. Cf. aussi l'article de Walsh [**295**], où l'auteur s'attache à démontrer, dans le cas max-plus harmonique, l'existence de mesures p *minimales* satisfaisant l'équation (40).

Pour prouver ces résultats, de nouveaux théorèmes de séparation dans les $\mathbb{R}_+^{\max}$-modules topologiques sont nécessaires. Il faut noter que des résultats de séparation et de type Hahn–Banach sont déjà présents dans [**307**]. On trouve aussi des théorèmes de Hahn–Banach (formes géométrique et analytique) ainsi que des résultats de séparation pour les parties convexes fermées dans Cohen et al. [**65**], que les auteurs appliquent à la représentation des fonctionnelles semicontinues inférieurement et max-plus convexes. Cf. aussi sur ce sujet Samborskii et Shpiz [**262**], Develin et Sturmfels [**78**], Briec et al. [**51**], Gaubert et Katz [**106, 108**], Gaubert et Sergeev [**110**], et les nouveaux résultats de séparation obtenus récemment par Akian, Gaubert et Nitica [**10**] et Briec et Horvath [**50**].

V-2. Introduction

In Chapter IV we explored convexities on various algebraic structures such as partially ordered sets, semilattices, and lattices. Especially, we focused on semilattices equipped with the convexity made up of all subsemilattices, and derived a Krein–Milman type theorem. A natural continuation, that we shall tackle here, is the study of the convexity in modules over the idempotent semifield $\mathbb{R}_+^{\max} = (\mathbb{R}_+, \max, \times)$.

In the sequel we shall use the *max-times semiring* $\mathbb{R}_+^{\max}$ rather than *max-plus semiring* $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. The latter and the isomorphic semiring $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$ are more common in the literature and more meaningful for applications such as control theory or scheduling. Yet both settings are isomorphic up to a logarithmic transformation. The choice of $\mathbb{R}_+^{\max}$ has the advantage of clarifying the parallel between idempotent modules and real vector spaces.

Chapter III gave us an algebraic point of view on modules over an idempotent semifield, especially after the works of Cohen et al. [**63**] and Litvinov, Maslov, and Shpiz [**182**]. We now wish to stress the topological point of view, so as to better deal with infinite-dimensionality.

The abstract concept of an infinite-dimensional topological $\mathbb{R}_+^{\max}$-module has been hardly examined in the literature; and when general idempotent modules were at stake, the topology of simple convergence was most often considered. One notable exception is the article [**273**] by Shpiz and Litvinov, where a Schauder type fixed-point theorem was proved. We also notice Shpiz' work [**272**], where it was shown that certain endomorphisms of *archimedean* idempotent modules have eigenvectors; although it does not explicitly use topological arguments, the Lawson topology of the underlying module is at work anyway.

On the other hand, finite-dimensional modules have been widely investigated. Zimmermann [**307**] initiated this research in the 1970's. Develin and Sturmfels [**78**] made the link with tropical geometry and pointed out an unexpected connection with phylogenetic analysis. This work originated the combinatorial study of max-plus polyhedra, which was pursued by Joswig [**143**], Develin and Yu [**79**], Allamigeon [**16**], Allamigeon, Gaubert, and Katz [**17**], Akian, Gaubert, and Guterman [**9**] among other.

In this framework, many classical theorems of discrete geometry have found their idempotent counterpart. This is the case of the Carathéodory theorem, see Helbig [**124**], Develin and Sturmfels [**78**], Briec and Horvath [**49**]. A max-plus Minkowski theorem was proved independently by Gaubert and Katz [**106, 107**] and Butkovič, Schneider, and Sergeev [**53**]. The idempotent version of Helly's theorem is due to Briec and Horvath [**49**], and was reproved by Gaubert and Sergeev [**110**]. Gaubert and Meunier gathered these results and added a Tverberg type theorem and a "colorful" Carathéodory theorem [**109**]; the reader may also consult this article for information on possible proofs leading to a max-plus Radon theorem.

The work of Briec and Horvath [**49**], Briec, Horvath, and Rubinov [**51**], Adilov and Rubinov [**3**] deserves also to be cited; it deals with finite-dimensional max-plus convexity, rediscovered under the name of B-convexity. The originality lies in the method of proof, where Maslov's dequantization is used: max-plus convex subsets are directly seen as asymptotic deformations of usual convex subsets (cf. Figures 1 and 2).

For an introduction to max-plus algebra, see Akian, Bapat, and Gaubert [**8**]. For an introduction to tropical geometry, see e.g. Develin and Sturmfels [**78**], Richter-Gebert et al. [**254**], Itenberg [**133**]. For a survey on idempotent analysis and its applications, see Kolokoltsov [**152**] and Litvinov [**180, 181**].

Considering the Carathéodory and Minkowski type theorems mentioned above, could we hope for a Krein–Milman type theorem in topological $\mathbb{R}_+^{\max}$-modules? This chapter aims at proving such a result. For this purpose we shall need a hypothesis of local convexity and use results from

Chapter IV. Another striking result we shall prove is the following analogue of the Choquet integral representation theorem.

Theorem V-2.1 (Idempotent Choquet theorem)**.** *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module and K be a nonempty compact convex subset of M. Then every $x \in K$ is represented by a possibility supported by the extreme points of K.*

Thus, we can write $x \in K$ as

$$x = \bigoplus_{y \in K} p(y).y, \tag{41}$$

where p is a possibility on K such that $p(B) = 0$ for all Borel subsets B of $K \setminus \operatorname{ex} K$. Unlike Chapter I, a possibility is here a completely maxitive measure with total mass equal to 1.

This representation result echoes the work of Akian, Gaubert et Walsh [**11**] on the max-plus Martin boundary. These authors proved that the representation of Equation (41) holds for max-plus harmonic or superharmonic vectors of the module $(\mathbb{R}_+^{\max})^S$. They also showed the existence of a *maximal* representing measure p. The existence of a *minimal* representing measure was proved by Walsh [**295**].

To show our theorems, we shall rely on new separation theorems in topological $\mathbb{R}_+^{\max}$-modules. Note that separation results and Hahn–Banach type theorems were proved in [**307**]. One may also find Hahn–Banach theorems (geometric and analytic forms) and separation results for closed convex subsets in Cohen et al. [**65**], with applications to the representation of lsc, max-plus convex functionals. On that subject see also Samborskii and Shpiz [**262**], Develin and Sturmfels [**78**], Briec et al. [**51**], Gaubert and Katz [**106, 108**], Gaubert and Sergeev [**110**], and the recent separation results obtained by Akian, Gaubert, and Nitica [**10**] and Briec and Horvath [**50**].

The chapter is organized as follows. Section V-3 introduces the convexity considered on $\mathbb{R}_+^{\max}$-modules. In Section V-4 we transpose the classical tool of Minkowski functional (or gauge) to our max-times context. We establish correspondences between properties of subsets of the (topological) $\mathbb{R}_+^{\max}$-module at stake and properties of their gauges. To characterize continuous linear forms as gauges, we define at this occasion the concept of *straightness* distinguishing certain lower subsets. Section V-5 deals with separation theorems. We prove a sort of geometric Hahn–Banach type theorem. We also present a series of results with sufficient conditions to separate points in topological $\mathbb{R}_+^{\max}$-modules. In particular, linear lsc functional separate points in topological $\mathbb{R}_+^{\max}$-modules, while usc linear forms separate points in strictly locally-convex $\mathbb{R}_+^{\max}$-modules. The separation of points enables us to define the concept of representation of points by possibilities in Section V-6, then to prove our Choquet type representation theorem after a series of lemmata. We also expose a Krein–Milman type theorem and, in the case where continuous linear forms separate points, we show that a converse to the Krein–Milman theorem also holds.

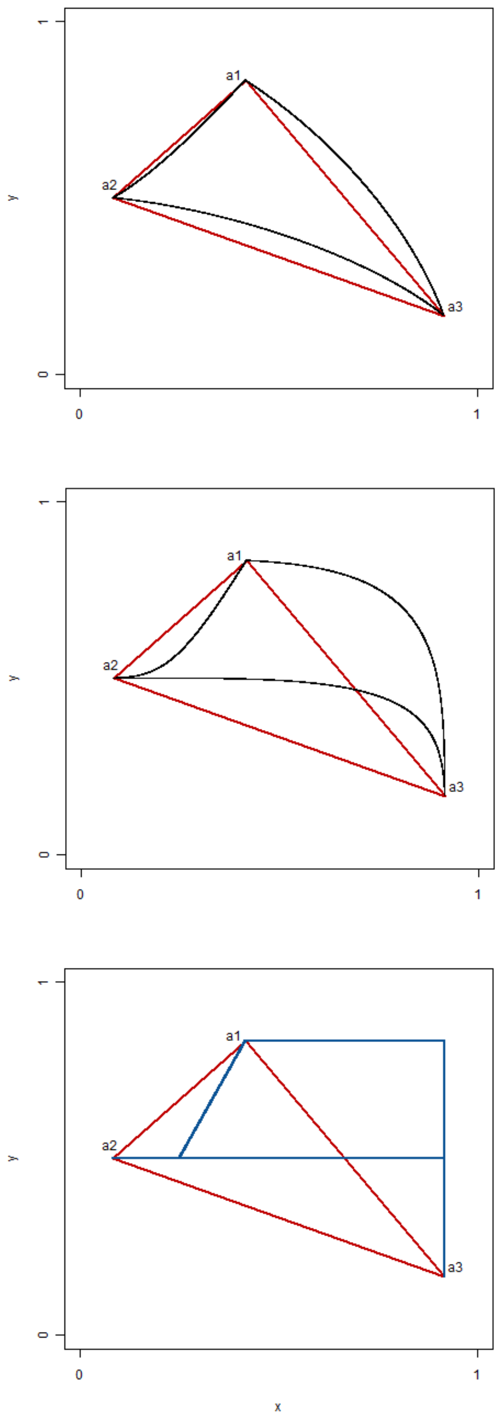

FIGURE 1. The red classical triangle is dequantized into a blue tropical triangle.

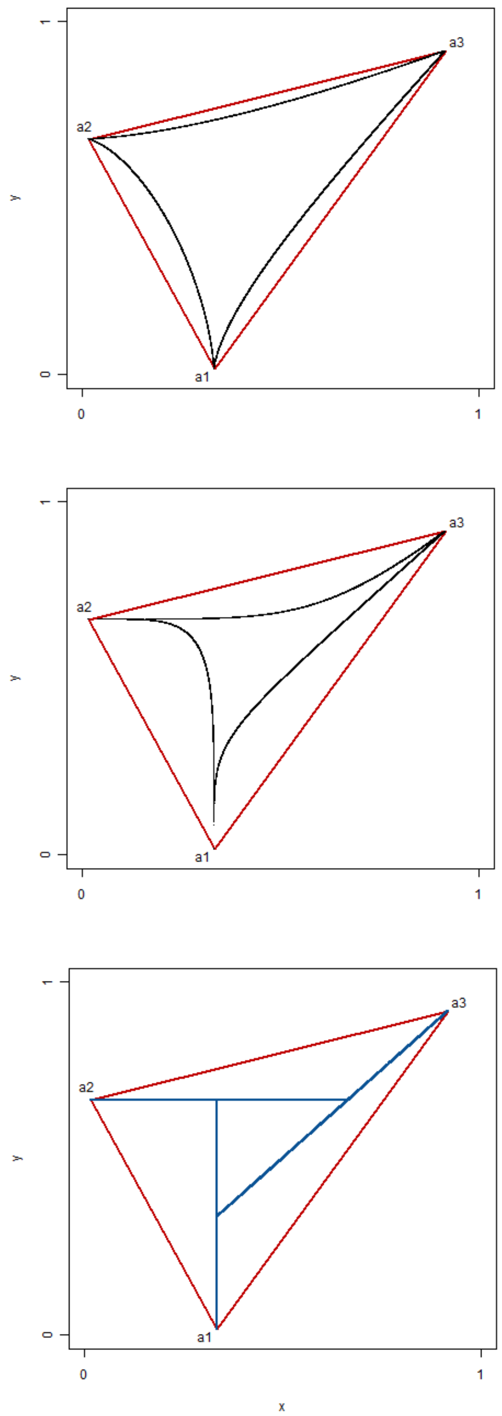

FIGURE 2. Another classical triangle (in red) is dequantized into a tropical triangle (in blue).

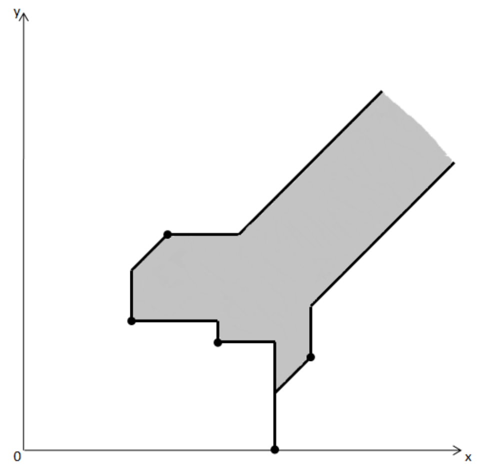

FIGURE 3. A convex subset of the module $(\mathbb{R}_+^{\max})^2$, reproduced from Gaubert and Katz [**106**].

V-3. CONVEXITY ON MODULES OVER THE TROPICAL ALGEBRA $\mathbb{R}_+^{\max}$

In this section we consider modules over $\mathbb{R}_+^{\max}$, i.e. over $\mathbb{R}_+$ equipped with its usual idempotent semifield structure and its usual topology. We carry on with our investigations on convexity started in Chapter IV. Recall that a *module* over $\mathbb{R}_+^{\max}$ is a semilattice with bottom $(M, \oplus, 0)$ together with an external multiplication $(r, x) \mapsto r.x : \mathbb{R}_+ \times M \to M$ such that $0.x = 0$, $1.x = x$ for all $x \in M$, and

$$\begin{aligned} r.(x \oplus y) &= r.x \oplus r.y, \\ (r \oplus s).x &= r.x \oplus s.x, \\ (rs).x &= r.(s.x), \end{aligned}$$

for all $r, s \in \mathbb{R}_+$, $x, y \in M$. Unlike Chapter III, we make the choice of a *left* action. A nonempty subset K of M is *convex* if, for all $x, y \in K$ and $r, s \in \mathbb{R}_+$, $r \oplus s = 1$ implies $r.x \oplus s.y \in K$. Note that every convex subset of M is a subsemilattice. See Figure 3 for an instance of a convex set in $(\mathbb{R}_+^{\max})^2$; see also Figure 4 to better understand how convexity works visually. If $A \subset M$, the convex hull $\mathrm{co}(A)$ of A is the intersection of all convex subsets containing A. An extreme point of a convex subset $K \subset M$ is a point $x \in K$ such that, if $x = s.y \oplus t.z$ for some $y, z \in K$ and $s, t \in \mathbb{R}_+$ with $s \oplus t = 1$, then $x = y$ or $x = z$.

A module M is *topological* if it is endowed with a Hausdorff topology such that the maps $M \times M \ni (x, y) \mapsto x \oplus y \in M$, and $\mathbb{R}_+ \times M \ni (r, x) \mapsto r.x \in M$, are continuous. A topological module is *locally-convex* (resp. *strictly locally-convex*) if every point has a basis of convex (resp. open convex) neighbourhoods.

Remark V-3.1. We warn the reader that topological $\mathbb{R}_+^{\max}$-modules as defined above do *not* generalize topological semilattices. To encompass this

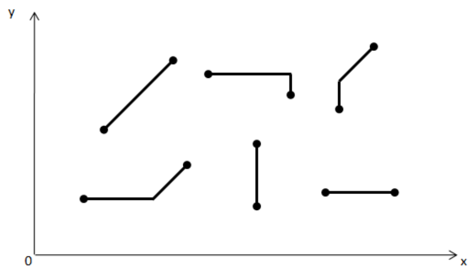

FIGURE 4. The six sorts of tropical segments in the $\mathbb{R}_+^{\max}$-module $(\mathbb{R}_+^{\max})^2$.

notion we should require the external multiplication to be continuous on $\mathbb{R}_+\backslash\{0\} \times M$ rather than on $\mathbb{R}_+ \times M$. Shpiz and Litvinov preferred this more permissive definition in [**273**], where they proved a Schauder fixed point theorem in $\mathbb{R}_+^{\max}$-modules. Davydov et al. [**69**] also introduced this more general definition.

Notice that the continuity of the left action ensures that every topological $\mathbb{R}_+^{\max}$-module M is *completable*[1], in the sense that

$$(\bigoplus T).x = \bigoplus_{t \in T} t.x, \qquad (\bigwedge T).x = \bigwedge_{t \in T} t.x,$$

for all $x \in M$ and all nonempty bounded subsets T of $\mathbb{R}_+$.

The following lemma gives a useful property of convex subsets.

Lemma V-3.2. *In a topological $\mathbb{R}_+^{\max}$-module, the topological closure of a convex subset is convex.*

Proof. Let C be a convex subset, and let $x, y \in \overline{C}$ and $r \in [0, 1]$. There are nets $(x_\alpha)_\alpha$ and $(y_\beta)_\beta$ in C such that $x_\alpha \to x$ and $y_\beta \to y$. Let $z_{\alpha,\beta} = r.x_\alpha \oplus y_\beta$, which belongs to C since C is convex. The net $(z_{\alpha,\beta})_{(\alpha,\beta)}$ tends to $r.x \oplus y$, so that $r.x \oplus y \in \overline{C}$, and we have proved that $\overline{C}$ is convex. □

V-4. MINKOWSKI FUNCTIONALS

V-4.1. Definition and first properties. Minkowski functionals are a powerful tool used in classical convex analysis to describe or characterize the properties of subsets of the vector space at stake. Following the developments of Keimel [**149**] made for "abstract" cones, we shall see that they still have an interesting role to play in idempotent analysis.

Let M be an $\mathbb{R}_+^{\max}$-module. A functional $p : M \to \overline{\mathbb{R}}_+$ is *homogeneous* if $p(r.x) = r.p(x)$, for all $r \in \mathbb{R}_+$, $x \in M$. It is *sublinear* if homogeneous and such that $p(x \oplus y) \leqslant p(x) \oplus p(y)$, for all $x, y \in M$. *Superlinearity* and *linearity* are defined accordingly; note that for homogeneous functionals,

[1] The definition of completability here is given with respect to Up*, or equivalently with respect to Fi since $\mathbb{R}_+$ is a chain. See Chapter III.

superlinearity is equivalent to order preservation. A *linear form* on M is a finite linear functional (from M to $\mathbb{R}_+$).

If A is a subset of M, we define its *Minkowski functional* (or *gauge*) $p_A : M \to \overline{\mathbb{R}}_+$ by

$$p_A(x) = \bigwedge\{t > 0 : x \in t.A\},$$

for all $x \in M$, i.e. $p_A(x) = \bigwedge\langle A, x\rangle$ in the notations of Chapter III. The functional p_A is homogeneous if A is nonempty. Conversely, if we have a functional $p : M \to \overline{\mathbb{R}}_+$, we define the *support* of p by

$$A(p) = \{x \in M : p(x) \leqslant 1\},$$

which is nonempty if p is homogeneous.

Example V-4.1. Let S be a semilattice, considered as an $\mathbb{R}_+^{\max}$-module (with $t.x = x$ if $t > 0$ and $0.x = 0$, for all $x \in S$). If $A \subset S$, then $p_A(x) = 0$ if $x \in A$, and $p_A(x) = \infty$ otherwise.

If p takes only finite values, then $A = A(p)$ is *absorbing*, in the sense that, for all $x \in M$, there is some $t > 0$ such that $x \in t.A$. Conversely, if A is an absorbing subset, then p_A takes only finite values. Note that every open subset containing 0 is absorbing.

Lemma V-4.2. *Let M be an $\mathbb{R}_+^{\max}$-module. There is a Galois connection between homogeneous functionals $p : M \to \overline{\mathbb{R}}_+$ and nonempty subsets A of M, in the sense that*

$$A \subset A(p) \Longleftrightarrow p \leqslant p_A.$$

Moreover, $A \subset A(p_A)$ and $p = p_{A(p)}$.

Proof. First assume that $A \subset A(p)$, and let $x \in M$. If $t > 0$ is such that $x \in t.A$, then $t^{-1}.x \in A(p)$, hence $p(t^{-1}.x) \leqslant 1$, which implies $p(x) \leqslant t$ since p is homogeneous. Then $p(x) \leqslant p_A(x)$ for all $x \in M$, i.e. $p \leqslant p_A$. Conversely, assume that $p \leqslant p_A$, and let $x \in A$. Then $p(x) \leqslant p_A(x) \leqslant 1$, hence $x \in A(p)$, which proves that $A \subset A(p)$.

The inequalities $A \subset A(p_A)$ and $p \leqslant p_{A(p)}$ are a straightforward consequence of properties of Galois connections. Equality $p = p_{A(p)}$ actually holds for $p_{A(p)}(x) = \bigwedge\{t > 0 : t^{-1}.x \in A(p)\} = \bigwedge\{t > 0 : p(x) \leqslant t\} = p(x)$. □

The previous result raises the problem of knowing when the equality $A = A(p_A)$ holds. We shall give a sufficient condition below, see Remark V-4.7.

V-4.2. Characterizing special subsets by their gauges. Here we establish a correspondence between nonempty lower (resp. convex, closed lower, closed straight) subsets of M and superlinear (resp. sublinear, superlinear lsc, superlinear continuous) functionals $p : M \to \overline{\mathbb{R}}_+$.

Proposition V-4.3 (Gauges for lower subsets)**.** *Let M be an $\mathbb{R}_+^{\max}$-module.*

- *The gauge p_A of a nonempty lower subset A of M is superlinear.*
- *The support $A(p)$ of a superlinear functional p is lower.*

Proof. Let A be a nonempty lower subset of M. Since $\overline{\mathbb{R}}_+$ is totally ordered, p_A is superlinear if and only if it is (homogeneous and) order-preserving. If $x \leqslant y$ and $y \in t.A$ for some $t > 0$, then $t^{-1}.x \leqslant t^{-1}.y \in A$, hence $t^{-1}.x \in A$, i.e. $x \in t.A$, so that $p_A(x) \leqslant t$. This shows that $p_A(x) \leqslant p_A(y)$. Conversely, if p is superlinear, $A(p)$ is clearly a lower subset. □

Proposition V-4.4 (Gauges for convex subsets)**.** *Let M be an $\mathbb{R}_+^{\max}$-module.*

- *The gauge p_A of a nonempty convex subset A of M is sublinear.*
- *The support $A(p)$ of a sublinear functional p is convex.*

Proof. Let A be a nonempty convex subset of M. Let $x, y \in M$ and $t > p_A(x) \oplus p_A(y)$. Then $x \in r.A$ and $y \in s.A$ for some $r < t$, $s < t$. This implies that $x \oplus y \in (r.A) \oplus (s.A) \subset (r \oplus s).A$ since A is convex. As a consequence, $p_A(x \oplus y) \leqslant t$, which proves that $p_A(x \oplus y) \leqslant p_A(x) \oplus p_A(y)$, i.e. that p_A is sublinear. Conversely, assume that p is sublinear. To show that $A(p)$ is convex, we prove that $(r.A(p)) \oplus (s.A(p)) \subset (r \oplus s).A(p)$ for all $r, s > 0$. If $z \in (r.A(p)) \oplus (s.A(p))$, then $z = r.x \oplus s.y$ for some $x, y \in A(p)$. Thus, $p(z) \leqslant r.p(x) \oplus s.p(y) \leqslant r \oplus s$. This gives $z = (r \oplus s)a \in (r \oplus s).A(p)$, where $a := (r \oplus s)^{-1} z \in A(p)$. □

Since a convex lower subset of M is nothing but an ideal, we have the following corollary.

Corollary V-4.5 (Gauges for ideals)**.** *Let M be an $\mathbb{R}_+^{\max}$-module.*

- *The gauge p_A of an ideal A of M is linear.*
- *The support $A(p)$ of a linear functional p is an ideal of M.*

Proposition V-4.6 (Gauges for closed lower subsets)**.** *Let M be a topological $\mathbb{R}_+^{\max}$-module.*

- *The gauge p_A of a nonempty closed lower subset A of M is superlinear lsc.*
- *The support $A(p)$ of a superlinear lsc functional p is closed lower in M.*

Proof. Let A be a nonempty closed lower subset of M, and let $x \in M$. By continuity of the map $\mathbb{R}_+ \backslash \{0\} \ni r \mapsto r.x \in M$, the subset $B = \{r > 0 : r.x \in A\}$ of $\mathbb{R}_+ \backslash \{0\}$ is closed in $\mathbb{R}_+ \backslash \{0\}$. Also, B is lower, so it is either empty, or equal to $\mathbb{R}_+ \backslash \{0\}$, or equal to $(0, r_0]$ for some $r_0 > 0$. This implies that either $p_A(x) = \infty$, or $p_A(x) = 0$, or $p_A(x) = 1/r_0$, respectively. Thus, whenever $p_A(x) > 0$, we have $p_A(x) = \min\{t > 0 : x \in t.A\}$, with the convention $\infty.A = M$. To prove that p_A is lsc, one needs to show that $F_t := \{p_A \leqslant t\}$ is closed for all $t > 0$ (then $F_0 = \bigcap_{t>0} F_t$ will be closed too). But since p_A is homogeneous and $\mathbb{R}_+ \backslash \{0\} \ni r \mapsto r.x \in M$ is continuous for all $x \in M$, it suffices to show that $\{p_A \leqslant 1\} = A(p_A)$ is closed. For this purpose, we shall prove that $A(p_A) \subset A$. So let $x \in M$ such that $p_A(x) \leqslant 1$.

If $p_A(x) < 1$, then $x \in t.A$ for some $t < 1$, hence $x \in A$ since A is lower. If $p_A(x) = 1$, we use the fact that $p_A(x) = \min\{t > 0 : x \in t.A\}$, which gives directly $x \in A$. Hence, $A(p_A) = A$, and $A(p_A)$ is closed. Conversely, if p is a homogeneous lsc functional, $A(p) = \{p \leqslant 1\}$ is closed. ☐

Remark V-4.7. We have proved that, if A is a nonempty closed lower subset of M, then $A = A(p_A)$.

Let A be a nonempty lower subset of M. Then, for all $t > 1$, $A \subset t.A$. Such a subset is called *straight* if, for all $t > 1$, $A \subset t.A^o$, where A^o denotes the topological interior of A. Note that if A is closed and straight, then it is regular-closed, i.e. such that $A = \overline{A^o}$.

Proposition V-4.8 (Gauges for closed straight subsets)**.** *Let M be a topological $\mathbb{R}_+^{\max}$-module.*

- *The gauge p_A of a closed straight subset A of M is superlinear continuous.*
- *The support $A(p)$ of a superlinear continuous functional p is closed straight in M.*

Proof. Let A be a nonempty closed straight subset of M. Since A is a nonempty closed lower subset, we already know that p_A is superlinear lsc. It remains to prove that $V_t = \{x \in M : p_A(x) < t\}$ is open for all $t > 0$, so let $x \in V_t$, $t > 0$. Let $s > 0$ such that $p_A(x) < s < t$, and let $y = s^{-1}.x$. We have $p_A(y) < 1$, hence there is some $u > 0$ such that $p_A(y) < u < 1$. By Remark V-4.7, $A = A(p_A)$; since $p_A(u^{-1}y) \leqslant 1$, we deduce that $y \in u.A$. Since A is straight, we have $y \in A^o$. We obtain $x \in s.A^o \subset V_t$, the latter inclusion coming from $A^o \subset A \subset \{p_A \leqslant 1\}$. Since $s.A^o$ is open, this concludes the first part of the proof.

Conversely, let p be a superlinear continuous functional. The set $A(p)$ is a nonempty closed lower subset of M, and we want to show that it is also straight. Let $t > 1$, let $x \in A(p)$, and let us show that $x \in t.A(p)^o$. From $p(x) \leqslant 1$, we deduce that $t^{-1}.x \in \{p < 1\}$. By continuity of p, $\{p < 1\}$ is open. Since $\{p < 1\} \subset A(p)$, we have $t^{-1}.x \in A(p)^o$, and the result is proved. ☐

Proposition V-4.9 (Gauges for open convex subsets)**.** *Let M be a topological $\mathbb{R}_+^{\max}$-module.*

- *The gauge p_A of an open convex subset A of M containing 0 is a finite sublinear usc functional, and satisfies $A = \{p_A < 1\}$.*
- *The set $\{p < 1\}$ associated with a finite sublinear usc functional p is open convex in M and contains 0.*

Proof. The proof is left to the reader. ☐

V-5. Separation theorems in idempotent modules

V-5.1. A geometric Hahn–Banach theorem. The non-topological assertion of the following theorem was proved by Roth [**259**, Theorem 2.1] for the case of modules over $(\mathbb{R}_+, +, \times)$ (called *cones*). It was used by Tix [**281**] and Keimel [**148, 149**] as a crucial intermediate result for deriving Hahn–Banach type theorems in such cones.

Theorem V-5.1 (Sandwich theorem for idempotent modules)**.** *Let M be a (topological) $\mathbb{R}_+^{\max}$-module. Let $p : M \to \overline{\mathbb{R}}_+$ be a sublinear functional and $q : M \to \overline{\mathbb{R}}_+$ be a superlinear (lsc) functional such that $q \leqslant p$. Then there are linear (lsc) functionals $\varphi : M \to \overline{\mathbb{R}}_+$ that are minimal among those that satisfy $q \leqslant \varphi \leqslant p$.*

Proof. We work on supports of functionals to prove this theorem. Let $\mathscr{I}$ be the collection of ideals I of M such that $A(q) \supset I \supset A(p)$. The support $A(p)$ is convex (Proposition V-4.4), while $A(q)$ is lower (Proposition V-4.3). Hence the subset $\downarrow A(p)$ is an ideal between $A(p)$ and $A(q)$, which proves that $\mathscr{I}$ is nonempty. Since the union of a chain of ideals remains an ideal, $\mathscr{I}$ admits a maximal element I_0 by Zorn's lemma. Then by Corollary V-4.5 the functional $\varphi = p_{I_0}$ is linear and satisfies $p_{A(q)} = q \leqslant \varphi \leqslant p = p_{A(p)}$. Using Corollary V-4.5 once more, the support $A(\varphi)$ of φ is an ideal containing I_0, hence $I_0 = A(\varphi)$ by maximality of I_0. With this last fact we deduce the minimality of φ.

Now assume that M is topological and that q is lsc. Then $A(q)$ is not only lower but also closed. Consequently, the lower subset I_1 generated by the topological closure of I_0 is between $A(q)$ and $A(p)$ and contains I_0. In addition, I_1 is an ideal, for the closure $\overline{I_0}$ of I_0 is convex by Lemma V-3.2. By maximality of I_0, we deduce that $I_0 = I_1$. Moreover, I_1 contains $\overline{I_0}$, so that $I_0 = \overline{I_0}$. This shows that I_0 is a closed ideal, so that φ is lsc by Proposition V-4.6. □

Our next result is a sort of geometric Hahn–Banach theorem; it separates functionally convex subsets and open upper subsets.

Theorem V-5.2 (Geometric Hahn–Banach theorem for idempotent modules)**.** *Let M be a topological $\mathbb{R}_+^{\max}$-module. Let A be a nonempty convex subset of M, and let U be an open upper subset. If A and U are disjoint, then there exists some linear lsc functional $\varphi : M \to \overline{\mathbb{R}}_+$ such that $\varphi(a) \leqslant 1 < \varphi(u)$ for all $a \in A$, $u \in U$.*

Proof. By Proposition V-4.6, the gauge q of $M \backslash U$ is superlinear lsc, and by Proposition V-4.4 the gauge p of A is sublinear. Since $A \cap U = \varnothing$, we also have $q \leqslant p$. Applying the Sandwich theorem (Theorem V-5.1), we have a linear lsc functional $\varphi : M \to \overline{\mathbb{R}}_+$ such that $q \leqslant \varphi \leqslant p$, and we deduce that $\varphi(a) \leqslant 1$, for all $a \in A$. We still must show that $1 < \varphi(u)$, for all $u \in U$. So assume that $\varphi(u_0) \leqslant 1$ for some $u_0 \in U$. Then $q(u_0) \leqslant 1$, i.e. $u_0 \in A(q)$. Since $M \backslash U$ is a lower closed subset, Remark V-4.7 implies that $M \backslash U = A(q) \ni u_0$, a contradiction. □

Remark V-5.3. We say that a topological $\mathbb{R}_+^{\max}$-module M is *all-straight* if every closed lower subset with nonempty interior is straight. This is equivalent to saying that $s.x \in (\downarrow x)^o$, whenever $s < 1$ and $(\downarrow x)^o$ is nonempty. If we assume in Theorem V-5.2 that M is all-straight and that A has nonempty interior, then φ is continuous. Indeed, the support $A(\varphi)$ of φ is closed lower by Proposition V-4.6. Moreover, $A(\varphi)$ contains A thus has nonempty interior. Hence $A(\varphi)$ is straight, so that $\varphi = p_{A(\varphi)}$ is a continuous linear form by Proposition V-4.8.

Corollary V-5.4. *Let M be a topological $\mathbb{R}_+^{\max}$-module. Let K be a compact convex subset of M, and let A be a nonempty upper subset. If K and A are disjoint, then there exists some linear lsc functional $\varphi : M \to \overline{\mathbb{R}}_+$ such that $\varphi(x) \leqslant 1 < \varphi(a)$ for all $x \in K$, $a \in A$.*

Proof. Since K is compact, the subset $\downarrow K$ is closed by [**114**, Proposition VI-1.6(ii)], so that $U = M \backslash \downarrow K$ is an open upper subset disjoint from the convex subset K. Applying Theorem V-5.2, there exists some linear lsc functional $\varphi : M \to \overline{\mathbb{R}}_+$ such that $\varphi(x) \leqslant 1 < \varphi(u)$, for all $x \in K$, $u \in U$. Since A is upper, U contains A and the result follows. □

Remark V-5.5. If we assume in Theorem V-5.2 that M is all-straight and that K has nonempty interior, then φ is continuous.

Corollary V-5.6. *Let M be a topological $\mathbb{R}_+^{\max}$-module. Let A be a nonempty upper subset of M, and let $x \in M \backslash A$. Then there exists some linear lsc functional $\varphi : M \to \overline{\mathbb{R}}_+$ such that $\varphi(x) \leqslant 1 < \varphi(a)$ for all $a \in A$.*

Proof. Take $K = \{x\}$ in the previous corollary. □

V-5.2. Separating points in locally-convex $\mathbb{R}_+^{\max}$-modules. We shall now prove some sufficient conditions for separating points by lsc, usc, or continuous linear functionals or linear forms. As a direct consequence of Corollary V-5.6 we have the following result.

Corollary V-5.7. *In a topological $\mathbb{R}_+^{\max}$-module, linear lsc functionals separate points.*

Proof. If $y \nleqslant x$, take $A = \uparrow y$ in Corollary V-5.6. □

Proposition V-5.8. *In a topological $\mathbb{R}_+^{\max}$-module, continuous linear forms separate points if and only if every principal ideal is an intersection of closed straight ideals.*

Proof. Let M be a topological $\mathbb{R}_+^{\max}$-module. Suppose that every principal ideal is an intersection of closed straight ideals, and let $x, y \in M$ such that $y \nleqslant x$. This implies the existence of a closed straight ideal I such that $x \in I$ and $y \notin I$. Let φ be the functional defined by $\varphi(z) = \bigwedge \langle I, z \rangle$ in the notations of Chapter III, meaning that φ is the gauge of the subset I. Then by Corollary V-4.5 and Proposition V-4.8, φ is a continuous linear form, and we have $\varphi(x) \leqslant 1 < \varphi(y)$.

Conversely, suppose that continuous linear forms separate points. Let $x \in M$ and $y \in \bigcap_{I \ni x} I$, where I runs over the closed straight ideals containing x. Assume that $y \notin \downarrow x$. Then there is some continuous linear form

φ such that $\varphi(y) > \varphi(x)$. Let $t \in \mathbb{R}_+$ such that $\varphi(y) > t > \varphi(x)$. Then $t^{-1}.x$ is in the support $A(\varphi)$ of φ, while $t^{-1}.y$ is not. By Corollary V-4.5 and Proposition V-4.8, $I = t.A(\varphi)$ is a closed straight ideal containing x, while $y \notin I$: a contradiction. $\square$

Remark V-5.9. It is also true that continuous linear forms that preserve arbitrary existing suprema separate points if and only if every point is an infimum of straight elements (where c is called *straight* if the subset $\downarrow c$ is straight), for in this case I may be chosen of the form $\downarrow c$.

Proposition V-5.10. *In a (strictly) locally-convex topological $\mathbb{R}_+^{\max}$-module, (usc) linear forms separate points.*

Proof. Let x, y be elements of a locally-convex topological $\mathbb{R}_+^{\max}$-module M with $x \neq y$. Without loss of generality, we suppose that $x \nleqslant y$. Since the map $t \mapsto t.y$ is continuous and $\uparrow x$ is closed, there exists some $t > 1$ such that $t.y \in M \setminus \uparrow x$. By local convexity of M, we have $t.y \in C^o \subset C \subset M \setminus \uparrow x$ for some convex subset C of M. Let $G = \downarrow C^o$. By [**114**, Proposition VI-1.13(iii)], G is open. Moreover, $\downarrow C$ is convex, and we have $t.y \in G \subset \downarrow C \subset M \setminus \uparrow x$.

Now let p be the gauge of $\downarrow C$. Since G is open and contains 0, it is an absorbing subset, so $\downarrow C$ is an absorbing ideal. We deduce that p is a finite linear functional by Corollary V-4.5, i.e. a linear form. Also, $t.y \in \downarrow C$, so $p(t.y) \leqslant 1$. This implies that $p(y) < 1$. To conclude we want to show that $1 \leqslant p(x)$. But if $1 > p(x)$, there is some $s < 1$ such that $x \in s. \downarrow C$. This gives $s^{-1}.x \in M \setminus \uparrow x$, a contradiction.

If moreover M is *strictly* locally-convex, we can suppose that C is open convex, so that $\downarrow C$ is open convex too. In this case, p is usc by Proposition V-4.9. $\square$

Corollary V-5.11. *In a locally-convex topological $\mathbb{R}_+^{\max}$-module, the identity $x = r.x \oplus y$ implies ($x = y$ or $r = 1$).*

Proof. Let x, y be elements of a locally-convex topological $\mathbb{R}_+^{\max}$-module M such that $x = r.x \oplus y$ and $x \neq y$. Then $x \nleqslant y$, so by Proposition V-5.10 there exists a linear form $\varphi : M \to \mathbb{R}_+$ such that $\varphi(x) > \varphi(y)$. Since $x = r.x \oplus y$, we get $\varphi(x) = r\varphi(x) \oplus \varphi(y)$. This implies that $\varphi(x) = r\varphi(x)$. Since φ takes only finite values and $\varphi(x) > 0$, we get $r = 1$. $\square$

We have stronger separation results in *locally-convact* topological $\mathbb{R}_+^{\max}$-modules (i.e. topological $\mathbb{R}_+^{\max}$-modules that are at the same time locally-compact and locally-convex). We first remark that a locally-convact topological $\mathbb{R}_+^{\max}$-module is strictly locally-convex.

Lemma V-5.12. *Every locally-convact topological $\mathbb{R}_+^{\max}$-module is strictly locally-convex.*

Proof. Let G be an open subset of a locally-convact topological $\mathbb{R}_+^{\max}$-module M containing some point x. Then there is a compact convex subset

K such that $G \supset K \supset K^o \ni x$. By the fundamental theorem of compact semilattices (see the discussion of Chapter IV on that subject), K is a complete continuous semilattice whose topology is the Lawson topology. In particular, every point of K has a basis of open order-convex subsemilattices as neighbourhoods (to see why this holds, the reader may refer to [**114**, Theorem III-2.15] and to the proof of [**114**, Proposition III-2.13]). So in K there exists an open order-convex subsemilattice V such that $K^o \supset V \ni x$. Since V is open in K and contained in K^o, it is open in K^o, hence open. But V is also a convex subset of M: if $x, y \in V$ and $r \in [0, 1]$, then $V \ni y \leqslant r.x \oplus y \leqslant x \oplus y \in V$; since V is order-convex in K, we get $r.x \oplus y \in V$. So V is the desired open convex neighbourhood of x contained in G. □

Corollary V-5.13. *In a locally-convact topological $\mathbb{R}_+^{\max}$-module, both lsc linear forms and usc linear forms separate points.*

Proof. In the proof of Proposition V-5.10, one can now choose a compact convex C. Hence, $\downarrow C$ is a closed ideal by [**114**, Proposition VI-1.6(ii)], so p is lsc by Proposition V-4.6. □

Theorem V-5.14. *In an all-straight locally-convact topological $\mathbb{R}_+^{\max}$-module, continuous linear forms that preserve arbitrary existing suprema separate points.*

Proof. Let x be an element of an all-straight locally-convact topological $\mathbb{R}_+^{\max}$-module M. Using the local-convacity of M, one can (classically) show that x is the filtered infimum of $\{y \in M : x \in (\downarrow y)^o\}$, see e.g. Gierz et al. [**114**, p. 452]. Moreover, if $x \in (\downarrow y)^o$, then $(\downarrow y)^o$ is nonempty, so y is straight since M is assumed to be all-straight. This shows that every x is an infimum of straight elements, and the result follows from Proposition V-5.8 and Remark V-5.9. □

V-6. Choquet's representation in a tropical setting

V-6.1. A result on the Shilkret integral. Let E be a Hausdorff topological space. We denote by $\mathscr{G}$ (resp. $\mathscr{K}$, $\mathscr{B}$) the collection of open (resp. compact, Borel) subsets of E. We call $p : \mathscr{B} \to \overline{\mathbb{R}}_+$ a *possibility measure* (or a *possibility*) on E if p is a completely maxitive measure on E such that $p(E) = 1$ (see Chapter II). Be aware that this definition of a possibility differs from [Chapter I, Definition 9.1].

In Chapter I we recalled the definition and properties of the Shilkret integral. In the case where the measure is completely maxitive, the identity that usually defines the Shilkret integral greatly simplifies.

Proposition V-6.1. *Let p be a completely maxitive measure on $\mathscr{B}$. Then the Shilkret integral is given by*

$$\int_E^\infty f.dp = \bigoplus_{x \in E} p(x) f(x),$$

for all measurable maps $f : E \to \overline{\mathbb{R}}_+$, where $p(x)$ stands for $p(\{x\})$.

Proof. Since p is completely maxitive we have $p(f > r) = \bigoplus_{x:f(x)>r} p(x)$ for all $r \in \mathbb{R}_+$. Thus,

$$\begin{aligned}\int_E^\infty f.dp &= \bigoplus_{r\in\mathbb{R}_+} p(f > r)r = \bigoplus_{r\in\mathbb{R}_+} \bigoplus_{x:f(x)>r} p(x)r \\ &= \bigoplus_{x\in E} p(x) \bigoplus_{r<f(x)} r = \bigoplus_{x\in E} p(x)f(x),\end{aligned}$$

for all measurable maps $f : E \to \overline{\mathbb{R}}_+$. □

V-6.2. The collection of regular maxitive measures. Here are some facts on regular maxitive measures that will be useful for proving a converse statement to the idempotent Krein–Milman theorem. Let E be a Hausdorff topological space. We denote by $\mathscr{M}$ the collection of regular maxitive measures on E.

Lemma V-6.2. *The set $\mathscr{M}$ is a complete lattice, and there is an isomorphism of complete lattices between $\mathscr{M}$ and the set of usc maps from E to $\overline{\mathbb{R}}_+$.*

Sketch of the proof. The proof is not difficult, but we take advantage of it to stress the following Galois connection (Ψ, Φ) between maps $f : E \to \overline{\mathbb{R}}_+$ and set functions $\mu : \mathscr{B} \to \overline{\mathbb{R}}_+$:

$$\begin{aligned}\Psi &: f \mapsto (B \mapsto \bigoplus_{x\in B} f(x)) \\ \Phi &: \mu \mapsto (x \mapsto \bigwedge_{G\ni x} \mu(G)),\end{aligned}$$

where G runs over the open subsets containing x. The pair (Ψ, Φ) is indeed a Galois connection, for we have $\Psi(f) \leqslant \mu \Leftrightarrow f \leqslant \Phi(\mu)$, for all f, μ. Moreover, the fixed points of the Galois connection are described as follows:

$$\begin{aligned}\Psi(\Phi(\mu)) = \mu &\Longleftrightarrow \mu \text{ is regular maxitive;} \\ \Phi(\Psi(f)) = f &\Longleftrightarrow f \text{ is usc.}\end{aligned}$$

For more on the categorical side of this Galois connection and the link with continuous posets, see Heckmann and Huth [**123**]. □

The *vague* (or *sup-vague*) topology on $\mathscr{M}$ is classically defined as the coarsest topology that makes the maps $\nu \mapsto \nu(B)$ lsc for open B and usc for compact B. Equivalently, this is the topology generated by the subsets of the form $\{\nu : \nu(G) > t\}$ and $\{\nu : \nu(K) < t\}$ for open G, compact K, and $t > 0$. This implies that a net $(\nu_n)_{n\in N}$ converges vaguely to ν in $\mathscr{M}$ if and only if $\liminf_n \nu_n(G) \geqslant \nu(G)$ and $\limsup_n \nu_n(K) \leqslant \nu(K)$ for all open subsets G and compact subsets K of E. See e.g. B. Gerritse [**112**].

Lemma V-6.3 (G. Gerritse)**.** *Assume that E is a locally-compact Hausdorff space. Then $\mathscr{M}$ endowed with the Lawson topology is compact Hausdorff.*

Proof. Combine G. Gerritse's theorem [**113**, Theorem 8.4], which asserts that $\mathscr{M}$ is a continuous lattice, and Gierz et al. [**114**, Corollary III-1.11]. □

Lemma V-6.4. *Assume that E is a locally-compact Hausdorff space. The Lawson topology and the vague topology on $\mathscr{M}$ agree.*

Proof. Let $t > 0$, let K be a nonempty compact subset, and let us show that $\{\nu : \nu(K) < t\}$ is Scott-open in $\mathscr{M}$. So let $(\nu_j)_{j\in J}$ be a filtered family of elements of $\mathscr{M}$ such that $w(K) < t$, where $w = \bigwedge_{j\in J} \nu_j$. Since w is outer-continuous, there exists some open subset V containing K such that $w(V) < t$. By local compactness of E, we can find a relatively compact, open subset G such that $K \subset G \subset \overline{G} \subset V$. Let τ be the element of $\mathscr{M}$ defined by $\tau(x) = t$ if $x \in (\overline{G})^o$ and $\tau(x) = \infty$ otherwise. By [**113**, Lemma 8.3], we have $\tau \gg w$. This implies that $\tau \gg \nu_{j_0}$ for some $j_0 \in J$, by definition of the way-above relation $\gg$. Hence by [**113**, Lemma 8.2], $\tau(x) = t > \nu_{j_0}(x)$, for all $x \in K$. Since K is nonempty compact and $x \mapsto \nu_{j_0}(x)$ is usc, the supremum $\nu_{j_0}(K)$ of $\{\nu_{j_0}(x) : x \in K\}$ is reached, so that $t > \nu_{j_0}(K)$. So we have proved that $\nu_{j_0} \in \{\nu : \nu(K) < t\}$. The subset $\{\nu : \nu(K) < t\}$ is also lower, so it is Scott-open (hence Lawson-open) in $\mathscr{M}$.

If G' is an open subset of E, then $\{\nu : \nu(G') \leqslant t\}$ coincides with $\{\nu : \nu \leqslant \tau'\}$, where τ' is the element of $\mathscr{M}$ defined by $\tau'(x) = t$ if $x \in G'$ and $\tau'(x) = \infty$ otherwise. This shows that $\{\nu : \nu(G') \leqslant t\}$ is Lawson-closed.

Consequently, the vague topology is coarser than the Lawson topology. Since the former is Hausdorff and the latter is compact, both topologies agree (see Bourbaki [**44**, Corollaire 3, p. 63]). □

Theorem V-6.5 (Norberg)**.** *Assume that E is a locally-compact Hausdorff space. The vague topology on $\mathscr{M}$ is generated by the maps*

$$\nu \mapsto \int_E^\infty f.d\nu,$$

where $f : E \to \mathbb{R}_+$ runs over the nonnegative compactly-supported continuous maps on E.

Proof. See [**225**, Theorem 2.6]; Norberg in [**225**] also assumes that E is second-countable, a hypothesis that is actually not needed here. □

V-6.3. Representing points by possibilities. In a topological $\mathbb{R}_+^{\max}$-module M, let K be a nonempty subset, and p a possibility on K. A point x in M is said to be *represented by* p if $\varphi(x) = \int_K^\infty \varphi.dp$, for all linear lsc functionals $\varphi : M \to \overline{\mathbb{R}}_+$. Note that, if $x \in K$, the possibility ε_x defined by $\varepsilon_x(B) = 1$ if $x \in B$, $\varepsilon_x(B) = 0$ otherwise, represents x. Proposition V-6.20 below will ensure the existence of a representing (regular) possibility in many cases, even when $x \notin K$.

Lemma V-6.6 (Shpiz–Litvinov)**.** *Let M be a topological $\mathbb{R}_+^{\max}$-module and K a nonempty compact convex subset of M. Then the supremum of $\{p(y).y : y \in K\}$ exists and is in K, for all possibilities p on K.*

Proof. First suppose that there exists some $y_0 \in K$ such that $p(y_0) = 1$. We consider the set D of convex combinations defined by

$$D = \{y_0 \oplus p(y_1).y_1 \oplus \ldots \oplus p(y_n).y_n : y_1, \ldots, y_n \in K, n \geqslant 1\}.$$

Then D is a directed subset of K. By [**114**, Proposition VI-1.3], D has a supremum x in K, and x is obviously the supremum of $\{p(y).y : y \in K\}$.

Now suppose that $p(y) < 1$ for all $y \in K$. If $0 < t < 1$, let $p_t(y) = t^{-1}(p(y) \wedge t)$. Then p_t is a possibility on K and there exists some $y_t \in K$ such that $p_t(y_t) = 1$. The beginning of the proof shows that the supremum x_t of $\{p_t(y).y : y \in K\}$ exists and is in K. Since K is compact, we can assume without loss of generality that x_t tends to some $x \in K$ when $t \to 1$. Then one can see that x is the supremum of $\{p(y).y : y \in K\}$. □

Remark V-6.7. The previous result is due to Shpiz and Litvinov [**273**, Proposition 6]. However, in their proof the authors implicitly made the assumption that the supremum of $\{p(y).y : y \in K\}$ always exists; our own proof explains why this holds.

Lemma V-6.8. *Let M be a topological $\mathbb{R}_+^{\max}$-module, K a nonempty compact convex subset of M, and p a possibility on K. For all $x \in K$,*

$$x = \bigoplus_{y \in K} p(y).y$$

if and only if x is represented by p.

Proof. Assume that $x = \bigoplus_{y \in K} p(y).y$, and let φ be a linear lsc functional on M. As in the proof of Lemma V-6.6 we first suppose that there exists some $y_0 \in K$ such that $p(y_0) = 1$, and we define

$$D = \{y_0 \oplus p(y_1).y_1 \oplus \ldots \oplus p(y_n).y_n : y_1, \ldots, y_n \in K, n \geqslant 1\},$$

which is a directed subset of K with supremum x in K. By [**114**, Proposition VI-1.3], D as a directed net converges to x in K. Let φ be a linear lsc functional on M. The linearity of φ gives $\varphi(d) \leqslant \bigoplus_{y \in K} p(y)\varphi(y)$, for all $d \in D$. By lower-semicontinuity of the restriction of φ to K, $\varphi(x) \leqslant \bigoplus_{y \in K} p(y)\varphi(y)$. The reverse inequality being clear, we get $\varphi(x) = \bigoplus_{y \in K} p(y)\varphi(y)$.

Now suppose that $p(y) < 1$ for all $y \in K$. Again we consider the possibility p_t on K defined by $p_t(y) = t^{-1}(p(y) \wedge t)$, if $0 < t < 1$. The first part of the proof applies, so $\varphi(x_t) = \bigoplus_{y \in K} p_t(y)\varphi(y)$ for all linear lsc functionals φ on M, where $x_t = \bigoplus_{y \in K} p_t(y).y$. Since $x \leqslant x_t \leqslant t^{-1}.x$, we have $\varphi(x) \leqslant \varphi(x_t) \leqslant t^{-1}\varphi(x)$, so $\varphi(x_t)$ tends to $\varphi(x)$ when $t \to 1$. From this we deduce that $\varphi(x) = \bigoplus_{y \in K} p(y)\varphi(y)$, i.e., using Proposition V-6.1, $\varphi(x) = \int_K^\infty \varphi.dp$. This means that x is represented by p.

Conversely, assume that $\varphi(x) = \bigoplus_{y\in K} p(y)\varphi(y)$ for all linear lsc functionals φ on M. We let $z := \bigoplus_{y\in K} p(y).y$. By Lemma V-6.6, this supremum exists and $z \in K$. The previous argument implies that $\varphi(z) = \bigoplus_{y\in K} p(y)\varphi(y)$, for all linear lsc functionals φ on M. This shows that $x = z$ for, by Corollary V-5.7, linear lsc functionals on M separate points. □

Lemma V-6.9. *Let M be a topological $\mathbb{R}_+^{\max}$-module and K be a nonempty compact convex subset of M. If $x \in K$ is represented by a possibility p on K, then x is represented by the regular possibility defined on K as the usc regularization of p.*

Proof. The usc regularization ν of p is defined on K by

$$\nu(u) = \bigwedge_{G\ni u} \bigoplus_{y\in G} p(y),$$

where G runs over the open subsets of K containing u. Note that ν is indeed usc and satisfies $\bigoplus_{u\in K} \nu(u) = 1$ since $p \leqslant \nu \leqslant 1$, so it defines a regular possibility on K.

Let $z = \bigoplus_{u\in K} \nu(u).u$ (this supremum exists and is in K by Lemma V-6.6). Since $p \leqslant \nu$, we have $x \leqslant z$. We have to show that $x \geqslant z$. Equivalently, we want $\varphi(x) \geqslant \varphi(z)$ for all linear lsc functionals φ on M. For this purpose, we can suppose that $\varphi(x) < \infty$. For all $y \in K$, $x \geqslant p(y).y$, so that $\varphi(x) \geqslant p(y)\varphi(y)$. If $\varphi(x) > 0$, then $\varphi(x)/\varphi(y) \geqslant p(y)$ for all y. Since the map $y \mapsto \varphi(x)/\varphi(y)$ is already usc, this implies that $\varphi(x)/\varphi(y) \geqslant \nu(y)$ for all y, so that $\varphi(x) \geqslant \varphi(z)$. Now suppose that $\varphi(x) = 0$, and let G_0 denote the subset $K \cap \{\varphi > 0\}$, which is open in K. If $\varphi(z) > 0$, there is some $y_0 \in K$ such that $\nu(y_0)\varphi(y_0) > 0$. Thus, $y_0 \in G_0$ and $\nu(y_0) > 0$, so by definition of ν we get $\bigoplus_{y\in G_0} p(y) > 0$. So there exists $y_1 \in K$ with $\varphi(y_1) > 0$ and $p(y_1) > 0$, which contradicts $0 = \varphi(x) \geqslant p(y_1)\varphi(y_1)$. So we have shown that $\varphi(x) = \varphi(z) = 0$. This proves that $\varphi(x) = \varphi(z)$ for all linear lsc functionals φ, hence $x = z = \bigoplus_{u\in K} \nu(u).u$, i.e. x is represented by ν. □

Remark V-6.10. See the proof of Akian et al. [**11**, Lemma 6.5] to compare with the finite-dimensional case.

To formulate the Choquet representation problem, we say that the possibility p on K is *supported* by the Borel subset A of K if $p(B) = 0$ for all Borel subsets B of $K\backslash A$. Then the question we want to solve in a tropical context is the following: if K is a nonempty compact convex subset of some locally-convex topological $\mathbb{R}_+^{\max}$-module, and $x \in K$, can we find some possibility on K representing x and supported by the extreme points of K?

Let M be a topological $\mathbb{R}_+^{\max}$-module and K be a nonempty convex subset of M. Consider the subset $\widehat{K}$ of $M \times [0,1]$ defined by

$$\widehat{K} = \{(r.u, r) : u \in K, r \in [0,1]\}. \tag{42}$$

This technique of studying a convex set by adding a dimension to it was called *tropical homogenization* by Allamigeon [**16**]. The identity $(t.w, t) = (r.u, r) \oplus (s.v, s)$ with $w = (rt^{-1}).u \oplus (st^{-1}).v$ and $t = r \oplus s$, whenever $r, s > 0$, and the convexity of K show that $\widehat{K}$ is a subsemilattice of the topological semilattice $M \times [0, 1]$.

Lemma V-6.11. *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module and K be a nonempty convex subset of M. Then, for all $r \in (0, 1]$, x is an extreme point of K if and only if $(r.x, r)$ is coirreducible in $\widehat{K}$.*

Proof. Assume that $(r.x, r)$ is coirreducible in $\widehat{K}$, and write $x = s.y \oplus t.z$, for some $y, z \in K$ and $s, t \in \mathbb{R}_+$ with $s \oplus t = 1$. Then, in $\widehat{K}$,

$$(r.x, r) = (rs.y, rs) \oplus (rt.z, rt).$$

Since $(r.x, r)$ is coirreducible in $\widehat{K}$, we can assume, without loss of generality, that $(r.x, r) = (rs.y, rs)$, i.e. $s = 1$ and $x = y$. This shows that $x \in \operatorname{ex} K$.

Conversely, assume that x is an extreme point of K. Let $r \in (0, 1]$ and write $(r.x, r) = (s.y, s) \oplus (t.z, t)$, with $y, z \in K$ and $s, t \in [0, 1]$. Then $r = s \oplus t$ and $x = s'.y \oplus t'.z$, where $s' = r^{-1}s$ and $t' = r^{-1}t$. Since $s' \oplus t' = 1$ and $x \in \operatorname{ex} K$, we can assume, without loss of generality, that $x = y$ and, by Corollary V-5.11, that $s' = 1$. This gives $(r.x, r) = (s.y, s)$ and shows that $(r.x, r)$ is coirreducible. $\square$

Here comes the key result of this chapter, namely the idempotent analogue of the Choquet theorem.

Theorem V-6.12 (Idempotent Choquet theorem)**.** *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module and K be a nonempty compact convex subset of M. Then every $x \in K$ is represented by a possibility supported by the extreme points of K, i.e.,*

$$x = \bigoplus_{y \in K} p(y).y,$$

for some possibility p on K such that $p(B) = 0$ for all Borel subsets B of $K \setminus \operatorname{ex} K$.

Proof. Let $x \in K$. We have seen that $\widehat{K}$ defined by Equation (42) is a subsemilattice of the locally-convex (Hausdorff) topological semilattice $M \times [0, 1]$. But $\widehat{K}$ is also compact as the image of the compact set $K \times [0, 1]$ by the continuous map $(u, r) \mapsto (r.u, r)$. Since $(x, 1) \in \widehat{K}$, the proof of the Krein–Milman theorem for semilattices [Chapter IV, Theorem 5.3] shows that we can write

$$(x, 1) = \bigoplus_{j \in J} (p_j.u_j, p_j),$$

for some nonempty family of coirreducible elements $(p_j.u_j, p_j)$ of $\widehat{K}$ with $u_j \in K$ and $p_j \neq 0$. We deduce that

$$x = \bigoplus_{j \in J} p_j.u_j \qquad \text{and} \qquad \bigoplus_{j \in J} p_j = 1. \tag{43}$$

Lemma V-6.11 also shows that u_j is an extreme point of K for all j, and the proof is complete. □

Corollary V-6.13. *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module and K be a nonempty compact convex subset of M. Then every $x \in K$ is represented by a regular possibility ν on K satisfying*

$$x = \bigoplus_{y \in \operatorname{ex} K} \nu(y).y,$$

supported by the topological closure $\overline{\operatorname{ex} K}$ of the extreme points of K, and such that $\nu(B) = 1$ for all Borel subsets B of K containing $\operatorname{ex} K$.

Proof. From Theorem V-6.12 and Lemma V-6.9, we can deduce that

$$x = \bigoplus_{y \in \operatorname{ex} K} \nu(y).y = \bigoplus_{y \in K} \nu(y).y,$$

where ν is defined as the usc regularization of p. □

Problem V-6.14. If we assume moreover that continuous linear forms separate points in M, can we represent every $x \in K$ by a *regular* possibility supported by *the extreme points* of K?

To conclude this section, we show that the subset of extreme points is measurable as soon as the underlying topological space is metrizable. This remarkably agrees with the classical case [**240**, Proposition 1.3].

Proposition V-6.15. *Let M be a topological $\mathbb{R}_+^{\max}$-module and K be a nonempty compact convex subset of M. If K is metrizable, then the extreme points of K form a $\mathscr{G}_\delta$ (in particular, a Borel) set.*

Proof. Let d be a metric generating the topology of K. For each integer $n \geqslant 1$, let

$$F_n = \{x = r.y \oplus z : y, z \in K, r \in [0,1], d(y,x) \wedge d(x,z) \geqslant n^{-1}\}.$$

Then F_n is closed, and x is an extreme point of K if and only if it is in all $K \backslash F_n$. The subset of extreme points of K, as a countable intersection of open subsets, is $\mathscr{G}_\delta$. □

V-6.4. The Krein–Milman theorem in $\mathbb{R}_+^{\max}$-modules. As an application of the Choquet representation theorem, we have the following Krein–Milman type theorem.

Theorem V-6.16 (Idempotent Krein–Milman theorem)**.** *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module. Then every nonempty compact subset of M has at least one extreme point, and every compact convex subset of M is the closed convex hull of its extreme points.*

Proof. Let K be a nonempty compact subset of M. By Wallace's lemma [**294**, § 2], K admits a minimal element, which is easily seen to be an extreme point of K. Hence, $\operatorname{ex} K \neq \varnothing$. Now we also suppose that K is convex. We have to show that $K = \overline{\operatorname{co}}(\operatorname{ex} K)$, so let $x \in K$. By Theorem V-6.12, we can write $x = \bigoplus_{u \in K} p(u).u$, for some possibility p on K satisfying $p(u) = 0$ whenever $u \in K \setminus \operatorname{ex} K$. Let y_0 be minimal in $K \cap \downarrow x$. For all finite subsets F of $\{u \in K : p(u) > 0\}$, we define

$$x_F = \bigoplus_{u \in F} y_0 \oplus p(u).u.$$

Note that $x_F \in \operatorname{co}(\operatorname{ex} K)$, so it suffices to prove that the directed net $(x_F)_F$ converges to x in order to conclude that $x \in \overline{\operatorname{co}}(\operatorname{ex} K)$. This indeed holds by [**114**, Proposition VI-1.3], which applies since x is the supremum of $(x_F)_F$ and K is compact. □

For an idempotent analogue of Bauer's principle [**28**], we need to recall the concepts of convexity and concavity of maps. Let M be an $\mathbb{R}_+^{\max}$-module and K be a convex subset of M. A map $f : K \to \mathbb{R}_+$ such that $f(r.x \oplus s.y) \leqslant rf(x) \oplus sf(y)$ (resp. $f(r.x \oplus s.y) \geqslant rf(x) \oplus sf(y)$), for all $x, y \in K$ and $r, s \in [0, 1]$ with $r \oplus s = 1$, is called *convex* (resp. *concave*). One can see that f is convex (resp. concave) if and only if its *epigraph* $\{(x, r) \in K \times \mathbb{R}_+ : f(x) \leqslant r\}$ (resp. its *hypograph* $\{(x, r) \in K \times \mathbb{R}_+ : f(x) \geqslant r\}$) is convex in $K \times \mathbb{R}_+^{\max}$.

Theorem V-6.17 (Bauer's maximum principle)**.** *Let K be a nonempty compact convex subset of an $\mathbb{R}_+^{\max}$-module, and let $f : K \to \mathbb{R}_+$ be a convex usc map. Then f attains its maximum on* $\operatorname{ex} K$.

Proof. By compactness of K, we classically know that f attains its maximum on K. Now let $a = \max_{x \in K} f(x)$, and let K_f be the nonempty set $\{x \in K : f(x) = a\}$. The fact that $K_f = \{x \in K : f(x) \geqslant a\}$ and the upper-semicontinuity of f tell us that K_f is closed, hence (nonempty) compact. Using Wallace's lemma, we let u_0 be a minimal point of K_f. We show that u_0 is an extreme point of K. So let $x, y \in K$ and $r, s \in \mathbb{R}_+$ with $r \oplus s = 1$ such that $u_0 = r.x \oplus s.y$. Then

$$\begin{aligned} a &= f(r.x \oplus s.y) \\ &\leqslant rf(x) \oplus sf(y) \\ &\leqslant ra \oplus sa \\ &= a. \end{aligned}$$

This shows that $a = rf(x) \oplus sf(y)$. Assume, without loss of generality, that $a = rf(x)$. Then $a = rf(x) \leqslant f(x) \leqslant a$, hence $a = ra$. This implies that $a = 0$ (in which case the theorem is clear since $\operatorname{ex} K \neq \varnothing$) or that $r = 1$. In the latter case, we have $a = f(x)$, i.e. $x \in K_f$. Also, $r = 1$ gives $u_0 \geqslant x$. But u_0 is minimal in K_f, so that $u_0 = x$. This proves that $u_0 \in \operatorname{ex} K$ and completes the proof. □

Theorem V-6.18 (Bauer's minimum principle)**.** *Let K be a nonempty compact convex subset of an $\mathbb{R}_+^{\max}$-module, and let $f : K \to \mathbb{R}_+$ be a concave lsc map. Then f attains its minimum on $\operatorname{ex} K$.*

Proof. The proof is quite similar to the previous one. Let $a = \min_{x \in K} f(x)$, and let $K_f = \{x \in K : f(x) = a\}$. Since K_f equals $\{x \in K : f(x) \leqslant a\}$ and f is lsc, K_f is closed, hence (nonempty) compact. By Wallace's lemma, there is a minimal element in K_f, call it u_0. As in the proof of Theorem V-6.17, one can show that u_0 is an extreme point of K. □

V-6.5. Milman's converse in $\mathbb{R}_+^{\max}$-modules. In this paragraph we prove a converse to the above Krein–Milman type theorem. The following lemma is prefigured by [**11**, Lemma 6.5].

Lemma V-6.19. *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module, and K be a compact convex subset of M. Then, for all nonempty subsets A of K, all $x \in \operatorname{ex} K$, and all regular possibilities ν on A, $x = \bigoplus_{a \in A} \nu(a).a$ implies $x \in \overline{A}$. Moreover, there is a net $(a_\alpha)_\alpha$ of elements of A such that $a_\alpha \to x$ and $\nu(a_\alpha) \to 1$.*

Proof. We apply the same technique as in the proof of Theorem V-6.12: we consider the set

$$\widehat{K} = \{(r.u, r) : u \in K, r \in [0,1]\},$$

which is a nonempty compact subsemilattice of the locally-convex topological semilattice $M \times [0,1]$. Also define

$$A_\nu = \{(\nu(a).a, \nu(a)) : a \in A\},$$

and let B_ν be the topological closure of A_ν in $M \times [0,1]$ (hence in $\widehat{K}$). Then B_ν is a nonempty compact subset of $\widehat{K}$. Since $x = \bigoplus_{a \in A} \nu(a).a$, we have $(x,1) = \bigoplus_{\widehat{K}} A_\nu$. But the principal ideal generated by $(x,1)$ is closed in $M \times [0,1]$, so that $(x,1) = \bigoplus_{\widehat{K}} B_\nu$. Moreover, $(x,1) \in \operatorname{ex} \widehat{K}$ by Lemma V-6.11, so by Milman's converse for semilattices [Chapter IV, Lemma 6.13] we have $(x,1) \in B_\nu$. Let $(a_\alpha)_\alpha$ be a net of elements of A such that $(\nu(a_\alpha).a_\alpha, \nu(a_\alpha))$ tends to $(x,1)$. Then $\nu(a_\alpha)$ tends to 1. In addition, K is compact, so we can suppose without loss of generality that a_α tends to some $y \in K$. This shows that $\nu(a_\alpha).a_\alpha$ tends to $x = y$. In particular, $x \in \overline{A}$. □

Proposition V-6.20. *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module in which continuous linear forms separate points, and A be a nonempty compact subset of M. For all $x \in \overline{\operatorname{co}}(A)$, there exists a regular possibility on A that represents x.*

First proof. Let $x \in \overline{\operatorname{co}}(A)$ and let $(x_\alpha)_\alpha$ be a net in $\operatorname{co}(A)$ that converges to x. We can write $x_\alpha = \bigoplus_{a \in A} \nu_\alpha(a).a$, where ν_α is a regular possibility on A supported by a finite set $F_\alpha \subset A$. Then, for all linear functionals φ on M, $\varphi(x_\alpha) = \bigoplus_{a \in A} \nu_\alpha(a)\varphi(a)$.

We denote by $\mathscr{M}(A)$ the set of regular maxitive measures on A. By G. Gerritse's theorem (see Lemma V-6.3), $\mathscr{M}(A)$ is compact with respect to the Lawson topology (or equivalently with respect to the vague topology). Thus, we can assume without loss of generality that the net $(\nu_\alpha)_\alpha$ converges in $\mathscr{M}(A)$ to some regular maxitive measure ν on A. Using Norberg's theorem (see Theorem V-6.5), this implies that

$$\bigoplus_{a\in A} \nu_\alpha(a) f(a) \to_\alpha \bigoplus_{a\in A} \nu(a) f(a),$$

for all continuous maps $f : A \to \mathbb{R}_+$. Taking in particular $f : a \mapsto 1$, we have $\bigoplus_{a\in A} \nu(a) = 1$, so ν is actually a regular possibility. We conclude that

$$\varphi(x) = \bigoplus_{a\in A} \nu(a)\varphi(a),$$

for all continuous linear forms φ on M, i.e. x is represented by ν. □

Second proof. As in the first proof we write $x_\alpha = \bigoplus_{a\in A} \nu_\alpha(a).a$. Now we consider the functional V_α defined on the set $C^+(A)$ of continuous maps $f : A \to \mathbb{R}_+$ by

$$V_\alpha(f) = \bigoplus_{a\in A} (\bigoplus_{\beta\geqslant\alpha} \nu_\beta(a)) f(a).$$

This functional is a linear form on $C^+(A)$, so by the Riesz representation theorem (see [Chapter III, Theorem 10.3]) there exists a regular possibility w_α on A such that

$$V_\alpha(f) = \bigoplus_{a\in A} w_\alpha(a) f(a),$$

for all $f \in C^+(A)$.

Let $\varphi : M \to \mathbb{R}_+$ be a continuous linear form. We have $\varphi(x) = \limsup_\alpha \varphi(x_\alpha)$, so that

$$\begin{aligned}\varphi(x) &= \bigwedge_\alpha \bigoplus_{\beta\geqslant\alpha} \varphi(x_\beta)\\ &= \bigwedge_\alpha \bigoplus_{\beta\geqslant\alpha} \bigoplus_{a\in A} \nu_\beta(a)\varphi(a)\\ &= \bigwedge_\alpha V_\alpha(\varphi_{|A})\\ &= \bigwedge_\alpha \bigoplus_{a\in A} w_\alpha(a)\varphi(a).\end{aligned}$$

Let us show that x is represented by the regular possibility ν on A defined by $\nu(a) = \bigwedge_\alpha w_\alpha(a)$. (The map ν is usc as an infimum of usc maps, and the arguments that follow can be reused to show that $\bigoplus_{a\in A} \nu(a) = 1$, so ν is indeed a regular possibility.) To reach this goal we must prove that $\varphi(x) = \bigoplus_{a\in A} \nu(a)\varphi(a)$. The inequality $\geqslant$ is clear. For the converse inequality, let $t > \bigoplus_{a\in A} \nu(a)\varphi(a)$. Then $A \subset \bigcup_\alpha \{t > w_\alpha\varphi\}$, and since A is compact we have $A \subset \{t > w_{\alpha_0}\varphi\}$ for some α_0. This implies

$$t \geqslant \bigoplus_{a\in A} w_{\alpha_0}(a)\varphi(a) \geqslant \bigwedge_\alpha \bigoplus_{a\in A} w_\alpha(a)\varphi(a),$$

so that $t \geqslant \varphi(x)$, and the desired result is proved. $\square$

Proposition V-6.21. *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module and K be a nonempty compact convex subset of M. For all $x \in K$, the following conditions are equivalent:*

(1) *x is an extreme point of K,*
(2) *for every regular possibility ν on K that represents x, $\nu(x) = 1$,*
(3) *ε_x is the least regular possibility on K that represents x.*

Proof. Equivalence (2) $\Leftrightarrow$ (3) is clear. To prove that (1) implies (2), assume that x is an extreme point of K, and write $x = \bigoplus_{y \in K} \nu(y).y$ for some regular possibility ν on K. Let $A = \{y \in K : \nu(y) > 0\}$. Then A is a nonempty subset of K such that $x = \bigoplus_{a \in A} \nu(a).a$, so by Lemma V-6.19 there is a net $(a_\alpha)_\alpha$ of elements of A such that $a_\alpha \to x$ and $\nu(a_\alpha) \to 1$. Assume that $\nu(x) < s$, for some $s < 1$. Then x is in the open subset $\{\nu < s\}$, so there is some α_0 such that, for all $\alpha \geqslant \alpha_0$, $a_\alpha \in \{\nu < s\}$. Thus, $\nu(a_\alpha) < s$ for all $\alpha \geqslant \alpha_0$, which contradicts $\nu(a_\alpha) \to 1$. As a consequence, $\nu(x) = 1$.

Conversely, let us show that (2) implies (1). So write x as $x = r.y \oplus z$, for some $y, z \in K$ and $r \in [0, 1]$. Consider the regular possibility ν defined on K by the usc map $\nu(y) = r$, $\nu(z) = 1$, and $\nu(u) = 0$ if $u \notin \{y, z\}$. Then ν represents x, hence $\nu(x) = 1$, i.e. $x \in \{y, z\}$ (the case $x = y$ remains possible if $r = 1$), so x is an extreme point of K. $\square$

Corollary V-6.22 (Milman's converse)**.** *Let M be a locally-convex topological $\mathbb{R}_+^{\max}$-module in which continuous linear forms separate points, and K be a compact convex subset of M. Then, for every closed subset A of K such that $K = \overline{\mathrm{co}}(A)$, we have $A \supset \mathrm{ex}\, K$.*

Proof. Let $x \in \mathrm{ex}\, K$. By Proposition V-6.20, there exists a regular possibility ν on A that represents x. Let τ be the regular possibility on K defined by $\tau(B) = \nu(B \cap A)$, for all Borel subsets B of K. Then τ represents x, so that, by Proposition V-6.21, $\tau(x) = \nu(\{x\} \cap A) = 1$. Thus, $\{x\} \cap A \neq \varnothing$, i.e. $x \in A$. $\square$

When the extreme point of K at stake is actually minimal in K, Proposition V-6.21 can be revised as follows (note that M needs not to be locally-convex).

Proposition V-6.23. *Let M be a topological $\mathbb{R}_+^{\max}$-module and K be a nonempty compact convex subset of M. For all $x \in K$, x is minimal in K if and only if*

$$\nu(y) = 1 \Leftrightarrow y = x,$$

for every regular possibility ν on K that represents x.

Proof. First suppose that x is minimal in K. Assume that $x = \bigoplus_{y \in K} \nu(y).y$ for some regular possibility ν on K. Since $y \mapsto \nu(y)$ is usc on K, it reaches its maximum 1 at some point $y_0 \in K$. This implies that $x \geqslant \nu(y_0).y_0 = y_0$, so $x = y_0$ by minimality of x. This shows that $\nu(x) = 1$. Also, if y is any element of K such that $\nu(y) = 1$, the same argument implies that $x = y$.

Conversely, suppose that $\nu(y) = 1 \Leftrightarrow y = x$, for every regular possibility ν on K that represents x, and let us show that x is minimal in K. So let $y \leqslant x$, $y \in K$. Then the regular possibility ν defined on K by $\nu(z) = 1$ if $z \in \{x, y\}$, $\nu(z) = 0$ otherwise, represents x. Moreover, $\nu(y) = 1$, so that $x = y$. □

V-7. Conclusion and perspectives

We mention here various possible continuations of this work. In our opinion, the main observation that follows from the present work is that we are today at the very beginning of the study of topological $\mathbb{R}_+^{\max}$-modules. We still have to understand the links between different properties such as continuity in the sense of domain theory (that partly arise in this chapter, e.g. through the notion of *straightness*); completeness; local compactness and local convexity; the existence of sufficiently many straight elements; finite-dimensionality; the existence of some *compact* principal ideal different from $\{0\}$; the properties of the topology made up of open lower subsets.

In Chapter IV we have proved a Klee–Krein–Milman type theorem in semilattices, i.e. we have extended the Krein–Milman theorem to *locally-compact* closed convex subsets containing no line. The question remains open whether such a generalization is possible in topological $\mathbb{R}_+^{\max}$-modules. Also, an idempotent analogue of the Banach–Alaoglu theorem remains to be done; note that Plotkin [**241**] solved this problem in a non-Hausdorff, domain-theoreric framework.

As a very first application of the Choquet type representation theorem proved in this chapter, one could revisit some classical tools such as the Fenchel transform, the representation of topical functions, and the (max-plus) Martin boundary.

Another challenge is to develop the theory of modules over the dioid $(\overline{\mathbb{R}}_+, \max, \min)$. One could hope to have also analogues of many classical theorems of infinite-dimensional functional analysis.

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Can classical analysis and idempotent analysis be unified?

CHAPTER VI

Mirror properties in inverse semigroups

ABSTRACT. Inverse semigroups are a class of semigroups whose structure induces a compatible partial order. This partial order is examined so as to establish *mirror properties* between an inverse semigroup and the semilattice of its idempotent elements, such as continuity in the sense of domain theory.

VI-1. RÉSUMÉ EN FRANÇAIS

La *théorie des domaines* est une branche de la théorie des ensembles ordonnés née dans les années 1970 avec les travaux précurseurs de Scott [**264**]. Au sein de la classe des domaines, les *semitreillis continus* occupent une place à part. Cela tient notamment au « théorème fondamental des semitreillis compacts », qui identifie deux catégories, l'une à caractère algébrique (celle des semitreillis continus complets) et l'autre à caractère géométrique (celle des semitreillis topologiques compacts avec petits semitreillis). Cf. chapitre IV.

Une généralisation des semitreillis est la notion de *semigroupe inverse*, qui remonte aux années 1950 avec les travaux de Wagner [**293**], Liber [**178**] et Preston [**244**]. Un semigroupe inverse dispose d'une relation d'ordre « intrinsèque » compatible avec la loi de semigroupe, et plusieurs auteurs ont cherché à analyser cette structure du point de vue de la théorie des ensembles ordonnés, tels que Wagner [**293**], Mitsch [**205**], Mark V. Lawson [**175**], Resende [**250**]. Cf. aussi Mitsch [**206, 207**] pour une extension de cette relation d'ordre à *tout* semigroupe, à la suite des travaux de Hartwig [**121**] et Nambooripad [**219**] sur les semigroupes réguliers.

Les semigroupes inverses généralisent également les groupes. Mais si de nombreux outils de la théorie des groupes ont été exportés vers celle des semigroupes inverses, la situation est différente concernant les outils de la théorie des semitreillis. Ainsi, aucune tentative n'a été faite pour appliquer aux semigroupes inverses les concepts (d'approximation et de continuité notamment) issus de la théorie des domaines. C'est le but du présent chapitre que de combler ce manque. Plus précisément, nous prouvons ce que nous appelons des *propriétés miroir*, c'est-à-dire des propriétés qui sont vraies pour un semigroupe inverse S si et seulement si elles sont vraies pour son semitreillis $\Sigma(S)$ des éléments idempotents. Notre théorème principal est le suivant :

Théorème VI-1.1. *Soit S un semigroupe miroir. Alors S est continu (resp. algébrique) si et seulement si $\Sigma(S)$ est continu (resp. algébrique).*

Le fait pour un semigroupe inverse d'être *miroir* consiste en une simple condition technique sur les infima de sous-ensembles filtrés. Nous appliquons le résultat précédent à différents exemples de semigroupes inverses, tels que le pseudogroupe symétrique d'un espace topologique, le monoïde bicyclique ou le semigroupe des caractères d'un semigroupe inverse. Nous redéfinissons au passage la notion de caractère pour qu'elle englobe à la fois celle utilisée classiquement en théorie des groupes et la notion réellement opérationnelle en théorie des semitreillis (où les « caractères » sont à valeurs dans $[0, 1]$ plutôt qu'à valeurs dans le disque complexe).

Cet examen des semigroupes inverses du point de vue de la théorie des domaines est motivé par les récents travaux de Castella [**57**]. Celui-ci s'est aperçu que les semigroupes inverses (qu'il redécouvre et nomme *quasi-groupes*) constituent une *bonne* généralisation des groupes et des semitreillis, au sens où elle n'est pas trop forte (contrairement à la théorie générale des semigroupes). Il implémente ce point de vue à travers l'étude des *semi-anneaux inverses*, des *semicorps inverses* et des polynômes à coefficients dans ces semicorps, et esquisse donc une théorie unique rassemblant algèbre classique et algèbre max-plus. D'autres travaux très récents sont sur une piste similaire : ainsi Lescot [**177**], Connes et Consani [**67**] et Connes [**66**], dans leur très sérieuse quête du « corps à un élément » $\mathbb{F}_1$, un concept non défini introduit en géométrie par Tits [**280**], sont emmenés sur le terrain de l'algèbre max-plus et de la géométrie tropicale ; les semitreillis qu'ils considèrent, une fois munis d'une multiplication distributive par rapport à la loi de semigroupe, sont alors vus comme des semicorps de caractéristique 1, donc comme un cas particulier de l'étude plus générale des semicorps de caractéristique non nulle.

Une porte s'ouvre sur un vaste champ d'étude, qui dépasse le simple cadre algébrique, et qui viserait à unifier les mathématiques classiques (où l'addition règne en maître) et les mathématiques idempotentes (basées sur l'opération maximum). Mais un tel programme ne se fera pas sans la théorie des domaines. En effet, si elle ne joue pas ou peu de rôle en mathématique classique (la théorie des semigroupes inverses explique cela par le fait que la relation d'ordre intrinsèque est réduite à l'égalité), elle devient incontournable en mathématique idempotente. C'est ce qu'observe Jimmie D. Lawson dans son article [**172**] et que nous avons constaté dans les chapitres II, IV et V : démontrer des analogues idempotents de théorèmes tels que celui de Krein–Milman ou de Choquet sur les représentations intégrales rend ce bagage utile et nécessaire.

Une suite naturelle au présent travail sera d'une part de développer l'étude des semigroupes inverses topologiques, et d'autre part d'initier une théorie des modules sur un semicorps inverse (d'un point de vue algébrique puis topologique et géométrique).

VI-2. INTRODUCTION

The branch of order theory called *domain theory* was initiated in the early 1970's with the pioneering work of Dana S. Scott on a model of untyped lambda-calculus [**264**]. Of special interest among the class of domains are *continuous semilattices*. This is partly due to the "fundamental theorem of compact semilattices", which identifies continuous complete semilattices and compact topological semilattices with small semilattices, see Chapter IV.

A generalization of semilattices is the concept of *inverse semigroup*, which dates back to the 1950's with the works of Wagner [**293**], Liber [**178**], and Preston [**244**]. An inverse semigroup is naturally endowed with a compatible partial order (called *intrinsic*), and many authors have investigated its structure from the point of view of order theory, including Wagner [**293**], Mitsch [**205**], Mark V. Lawson [**175**], Resende [**250**]. See also Mitsch [**206, 207**] for the extension of this partial order to *every* semigroup (continuing the works of Hartwig [**121**] and Nambooripad [**219**] on regular semigroups).

Inverse semigroups also form a nice generalization of groups. While many tools of group theory have been successfully exported to inverse semigroup theory, the contribution of semilattice theory is barely visible. Especially, no attempt has been made to apply the concepts (of approximation and continuity in particular) of domain theory to the framework of inverse semigroups. The purpose of this work is to fill this gap. More precisely, we aim at proving what we call *mirror properties*, i.e. properties that hold for an inverse semigroup S if and only if they hold for its semilattice $\Sigma(S)$ of idempotent elements. Our main theorem asserts that continuity and algebraicity in the sense of domain theory are mirror properties:

Theorem VI-2.1. *Assume that S is a mirror semigroup. Then S is continuous (resp. algebraic) if and only if $\Sigma(S)$ is continuous (resp. algebraic).*

The hypothesis of being a *mirror* semigroup is a simple technical condition on infima of filtered subsets. We apply this result to a series of examples such as the symmetric pseudogroup of a topological space, the bicyclic monoid, or the semigroup of characters of an inverse semigroup. In passing we redefine the notion of character to encompass both the classical one used in group theory and the one that plays this role in semilattice theory (where "characters" take their values in $[0, 1]$ rather than in the complex disc).

This study of inverse semigroups from the point of view of domain theory is motivated by the recent work of Castella [**57**]. He realized that inverse semigroups (that he rediscovered under the name of *quasigroups*) are a *good* (i.e. not a too strong, unlike general semigroup theory) generalization of groups and semilattices. He therefore applied this concept to the study of *inverse semirings*, *inverse semifields* and polynomials with coefficients in these semifields. He thus sketched a global theory gathering both classical algebra and max-plus algebra. This echos with other very recent work by Lescot [**177**], Connes and Consani [**67**], and Connes [**66**]. These authors looked for the "one-element field" $\mathbb{F}_1$ which is an undefined concept introduced in geometry by Tits [**280**]. This quest led them to max-plus algebra and tropical geometry, and the semilattices they considered, once endowed with an additional binary relation, are seen as semifields of characteristic one, hence as a particular case of the whole class of semifields of non-zero characteristic.

Little by little, a new field of investigation takes shape, not only connected with algebraic questionings, but also with analytic ones. It would aim at unifying classical mathematics (where addition predominates) and idempotent mathematics (based on the maximum operation). But such a program will not be effective without domain theory. For even if domains are of limited importance in classical mathematics (and inverse semigroup theory tells us that the intrinsic order reduces to equality when groups are at stake), it becomes a crucial tool in idempotent mathematics, as Jimmie D. Lawson explained in [**172**]. We also observed this in Chapters II, IV and V: proving idempotent analogues of the Krein–Milman theorem or the Choquet integral representation theorem necessitates such a theory.

The chapter is organized as follows. Section VI-3 gives some basics of inverse semigroup theory and recalls or builds some key examples. In Section VI-4 we focus on completeness properties of inverse semigroups, recalling some known results due to Rinow [**256**] and Domanov [**80**] and bringing up some new ones. In Section VI-5 we prove mirror properties related to sum-continuity / join-continuity. Our main theorem on continuous (resp. algebraic) inverse semigroups is the purpose of Section VI-6.

VI-3. Preliminaries on inverse semigroups

A *semigroup* $(S, \oplus)$ is a set S equipped with an associative binary relation $\oplus$. An element ϵ of S is *idempotent* if $\epsilon \oplus \epsilon = \epsilon$. The set of idempotent elements of S is denoted by $\Sigma(S)$. The semigroup S is *inverse* if the idempotents of S commute and if, for all $s \in S$, there is some $t \in S$, called an *inverse* of s, such that $s \oplus t \oplus s = s$ and $t \oplus s \oplus t = t$. An *inverse monoid* is an inverse semigroup with an identity element. It is well-known that a semigroup is inverse if and only if every element s has a unique inverse, denoted by s^*. Our main reference for inverse semigroup theory is the monograph by Mark V. Lawson [**175**]. The reader may also consult Petrich's book [**239**].

It is worth recalling some basic rules for inverse semigroups.

Proposition VI-3.1. *Let S be an inverse semigroup, and let $s, t \in S$. Then*

(1) *$s \oplus s^*$ and $s^* \oplus s$ are idempotent.*
(2) *$(s^*)^* = s$ and $(s \oplus t)^* = t^* \oplus s^*$.*
(3) *$s^* = s$ if s is idempotent.*

A partial order $\leqslant$ can be defined on the idempotents of a semigroup as follows: $\epsilon \leqslant \phi$ if $\epsilon \oplus \phi = \phi \oplus \epsilon = \phi$. For an inverse semigroup, this partial order can be naturally extended to the whole underlying set, if we define $s \leqslant t$ by $t = s \oplus \epsilon$ for some idempotent ϵ. We shall refer to $\leqslant$ as the *intrinsic (partial) order* of the inverse semigroup S. In this case, $(\Sigma(S), \leqslant)$ is a *semilattice* [**175**, Proposition 1-8], i.e. a partially ordered set in which every nonempty finite subset has a supremum (or, equivalently, a commutative idempotent semigroup). Also, the intrinsic order is compatible with the structure of semigroup, in the sense that $s \leqslant t$ and $s' \leqslant t'$ imply $s \oplus s' \leqslant t \oplus t'$ [**175**, Proposition 1-7]. We recall some equivalent characterizations of $\leqslant$.

Lemma VI-3.2. *Let S be an inverse semigroup, and let $s, t \in S$. Then $(s \leqslant t) \Leftrightarrow (s^* \leqslant t^*) \Leftrightarrow (t = s \oplus t^* \oplus t) \Leftrightarrow (t = t \oplus t^* \oplus s) \Leftrightarrow (t = \epsilon \oplus s$ for some idempotent $\epsilon)$.*

Proof. See e.g. [**175**, Lemma 1-6]. □

The following examples are extracted from the literature, with the exception of the last two of them.

Example VI-3.3 (Groups)**.** An inverse semigroup is a group if and only if its intrinsic order coincides with equality. The only idempotent element is the identity of the group.

Example VI-3.4 (Symmetric pseudogroup)**.** Let X be a nonempty set. The *symmetric pseudogroup* $\mathscr{I}(X)$ on X is the set made up of all the partial bijections on X, i.e. the bijections $f : U \to V$ where U and V are subsets of X (in this situation we write $\mathrm{dom}(f)$ for U). This is an inverse semigroup when endowed with the composition defined by $f_1 f : x \mapsto f_1(f(x)) : f^{-1}(V \cap U_1) \to f_1(V \cap U_1)$, where $f : U \to V$ and $f_1 : U_1 \to V_1$. The involution is the inversion, given by $f^* = f^{-1} : V \to U$. Idempotent elements of $\mathscr{I}(X)$ are of the form id_U for some subset U, and the partial order on $\mathscr{I}(X)$ is given by $f \leqslant g$ if and only if $g = f|_U$ for some subset $U \subset \mathrm{dom}(f)$.

Example VI-3.5 (Cosets of a group, see e.g. McAlister [**199**])**.** Let G be a group. A *coset* of G is a subset of the form Hg, where H is a subgroup of G and $g \in G$. Any nonempty intersection of cosets is a coset, so, for all cosets C, C', we can consider the smallest coset $C \otimes C'$ containing CC'. The product $\otimes$ makes the collection $\mathscr{C}(G)$ of cosets of G into an inverse monoid called the *coset monoid* of G. The intrinsic order of $\mathscr{C}(G)$ coincides with inclusion, and the idempotents of $\mathscr{C}(G)$ are exactly the subgroups of G.

Example VI-3.6 (Bicyclic monoid, see e.g. Mark V. Lawson [**175**, Sec. 3.4])**.** Let P be the positive cone of a lattice-ordered group. On $P \times P$, one can define the binary relation $\oplus$ by $(a, b) \oplus (c, d) = (a - b + b \vee c, d - c + b \vee c)$. This makes $P \times P$ into an inverse monoid with identity $(0, 0)$, called the *bicyclic monoid*, such that $(a, b)^* = (b, a)$. An element (a, b) is idempotent if and only if $a = b$, and the intrinsic order satisfies $(a, b) \leqslant (c, d)$ if and only if $a = b + c - d$ and $b \leqslant d$. In particular, for idempotents, $(a, a) \leqslant (b, b)$ if and only if $a \leqslant b$.

Example VI-3.7 (Rotation semigroup)**.** On the unit disc B^2 of the complex numbers $\mathbb{C}$, let us consider the binary relation defined by

$$z \otimes z' = (r \wedge r') \exp(\mathbf{i}(\theta + \theta')),$$

if one write $z = re^{\mathbf{i}\theta}$ and $z' = r'e^{\mathbf{i}\theta'}$ with $r, r' \in [0, 1]$. Then $(B^2, \otimes)$ is an inverse monoid, z^* coincides with the conjugate $\bar{z}$ of z, and z is idempotent if and only if $z \in [0, 1]$. Moreover, the intrinsic order satisfies $z \leqslant z'$ if and only if $r \geqslant r'$ and $\theta \equiv \theta'[2\pi]$.

Example VI-3.8 (Characters)**.** Let S be a commutative inverse monoid. We define a *character* on S as a morphism $\chi : S \to B^2$ of inverse monoids, where B^2 is equipped with the inverse monoid structure of the previous example. If 1 denotes the identity of S, this means that $\chi(1) = 1$ and $\chi(st) = \chi(s) \otimes \chi(t)$ for all $s, t \in S$ (the fact that $\chi(s^*) = \chi(s)^*$ then automatically holds). This definition differs from the one used e.g. by Warne and Williams [**299**] and Fulp [**105**], for these authors equipped $\mathbb{C}$ or B^2 with the usual multiplication. The set $S^\wedge$ of characters on S has itself a natural structure of inverse monoid. A character χ is idempotent in $S^\wedge$ if and only if it is $[0, 1]$-valued. Moreover, if S is actually a group (resp. a semilattice), then every non-zero χ takes its values into the unit circle (resp. into $[0, 1]$).

Henceforth, S denotes an inverse semigroup and we let $\Sigma = \Sigma(S)$. The purpose of the next section is to investigate completeness properties of inverse semigroups with respect to their intrinsic order.

VI-4. Completeness properties of inverse semigroups

Of particular usefulness in the framework of domain theory is the property of being filtered-complete. The term *poset* is an abbreviation for "partially ordered set". A subset F of a poset is *filtered* if F is nonempty and, for each pair $s, t \in F$, there exists some $r \in F$ such that $s \geqslant r$ and $t \geqslant r$. The poset is *filtered-complete* if every filtered subset has an infimum. It is *conditionally filtered-complete* if every principal filter $\{t : t \geqslant s\}$ is a filtered-complete poset, i.e. if every lower-bounded filtered subset has an infimum.

In the framework of inverse semigroups, a subset of the semilattice of idempotents Σ may have an infimum in Σ but no infimum in the whole inverse semigroup. However, this property will appear desirable for establishing mirror properties, hence a definition is needed.

Definition VI-4.1. The inverse semigroup S is a *mirror semigroup* if every filtered subset of Σ with an infimum in Σ also has an infimum in S.

In this case, both infima coincide and belong to Σ. Indeed, if Φ is a filtered subset of Σ with infimum ϕ in Σ and infimum f in S, then $f \geqslant \phi$, i.e. $f = \phi \oplus f^* \oplus f$. As a sum of idempotent elements, f is idempotent itself, so that $f = \phi$.

Example VI-4.2. As an example of inverse semigroup that is not mirror, we can consider the set $S = \{\omega\} \cup [0, 1]$ equipped with the binary relation $\oplus$ defined by $s \oplus t = \max(s, t)$ if $s, t \in [0, 1]$, $\omega \oplus s = s \oplus \omega = s$ if $s \in (0, 1]$, $\omega \oplus 0 = 0 \oplus \omega = \omega$, $\omega \oplus \omega = 0$. Then $(S, \oplus)$ is an inverse semigroup whose subset of idempotents is $\Sigma = [0, 1]$. Moreover, the filtered subset $(0, 1]$ admits 0 as infimum in Σ, but has two incomparable lower bounds in S, namely 0 and ω, hence has no infimum in S.

Fortunately, mirror semigroups are rather numerous, as the following proposition shows.

Proposition VI-4.3. *The inverse semigroup S is a mirror semigroup in any of the following cases:*

(1) *S projects onto Σ, i.e. there is an order-preserving map $j : S \to \Sigma$ such that $j \circ j = j$ and $j \geqslant \mathrm{id}_S$.*
(2) *S is reduced, i.e. $s \leqslant \epsilon$ and $\epsilon \in \Sigma$ imply $s \in \Sigma$.*
(3) *S is (conditionally) filtered-complete,*
(4) *$(S, \oplus)$ is a semilattice, i.e. S coincides with Σ,*
(5) *$(S, \oplus)$ is a group.*
(6) *S is finite.*

Proof. Cases (3), (4) and (5) are straightforward. For (1), (2), and (6), let Φ be a filtered subset of Σ. Assume that Φ has an infimum ϕ in Σ, and let $\ell \in S$ be a lower bound of Φ. We need to show that $\phi \geqslant \ell$.

(1) Assume the existence of $j : S \to \Sigma$. For all $\alpha \in \Phi$, $\alpha \geqslant \ell$, hence $j(\alpha) = \alpha \geqslant j(\ell)$. Since $j(\ell)$ is idempotent, this is a lower bound of Φ in Σ, and we deduce that $\phi \geqslant j(\ell) \geqslant \ell$.

(2) If S is reduced, then $\ell \in \Sigma$ since Φ is supposed nonempty, so that $\phi \geqslant \ell$.

(6) If S is finite, then the filtered subset Φ is finite, so $\phi \in \Phi$. This implies $\phi \geqslant \ell$. □

Examples VI-4.4. The symmetric pseudogroup is filtered-complete; the coset monoid of a group is conditionally filtered-complete; the bicyclic monoid, the rotation semigroup, and the character monoid are reduced. Thus, all the examples of inverse semigroups that we introduced in the previous section are mirror semigroups.

The reader may ask why the definition of a mirror semigroup does not require that every filtered subset of Σ with an infimum in S also has an infimum in Σ. This property turns out to hold in every inverse semigroup. The following lemma gives a stronger statement. If $A \subset S$, we write $\bigwedge A$ for the infimum of A in S, whenever it exists. Also, we denote by σ the *source map* $S \to \Sigma$ defined by $s \mapsto \sigma(s) = s^* \oplus s$.

Lemma VI-4.5. [**256**] *Let A be a nonempty subset of S. If $\bigwedge A$ exists, then $\bigwedge \sigma(A)$ exists and $\bigwedge \sigma(A) = \sigma(\bigwedge A)$.*

Proof. The reader may refer to [**175**, Lemma 1-17]. □

As a consequence, the map σ is Scott-continuous (see e.g. [**114**, Proposition II-2.1] and adapt the proof to the case of posets that are not filtered-complete), and Σ is a Scott-closed subset of S and is a retract of S when S is endowed with its Scott topology.

Another important feature of infima that we shall need later on is a kind of conditional distributivity property.

Lemma VI-4.6. [**80**] *Let A be a nonempty subset of S and $s \in S$. If $\bigwedge A$ exists and $a \oplus a^* \geqslant s^* \oplus s$ for all $a \in A$, then $\bigwedge(s \oplus A)$ exists and $\bigwedge(s \oplus A) = s \oplus (\bigwedge A)$.*

Proof. See e.g. [**175**, Proposition 1-18]. □

As a corollary, we get our first mirror property.

Proposition VI-4.7. *Assume that S is a mirror semigroup. Then S is conditionally filtered-complete if and only if Σ is conditionally filtered-complete.*

Proof. Assume that Σ is conditionally filtered-complete, and let F be a lower-bounded filtered subset of S. Then $\Phi = \sigma(F)$ is lower-bounded and filtered in Σ. Let $\phi = \bigwedge \Phi$. We show that, if ℓ is a lower bound of F in S, then $\bigwedge F$ (exists and) equals $\ell \oplus \phi$. For every $\alpha \in \Phi$, $\alpha = \alpha \oplus \alpha^* \geqslant \ell^* \oplus \ell$, so by Lemma VI-4.6, $\bigwedge(\ell \oplus \Phi)$ exists and equals $\ell \oplus \phi$. But, for every $f \in F$, $f \geqslant \ell$, i.e. $\ell \oplus f^* \oplus f = f$ by Lemma VI-3.2, so that $\ell \oplus \Phi = \{\ell \oplus f^* \oplus f : f \in F\} = \{f : f \in F\} = F$. Hence $\bigwedge F = \ell \oplus \phi$. □

VI-5. Sum-continuity and additive way-above relation

The second mirror property concerns sum-continuity. We call a mirror semigroup S *sum-continuous* if, for all filtered subsets $F \subset S$ with infimum, and for all $s \in S$, $\bigwedge(F \oplus s)$ exists and equals $(\bigwedge F) \oplus s$. This is tantamount to saying that the map $t \mapsto t \oplus s$ is Scott-continuous for all $s \in S$. Restricted to the case of semilattices, sum-continuity can be called *join-continuity* (although [**114**, Definition III-2.1] proposes a slightly different notion of join-continuity).

Proposition VI-5.1. *Assume that S is a mirror semigroup. Then S is sum-continuous if and only if Σ is join-continuous.*

Proof. Recalling that Σ is a Scott-closed subset of S, sum-continuity of S clearly implies join-continuity of Σ. For the converse statement, we mainly follow the lines of the proof of [**175**, Proposition 1-20]. Assume that Σ is join-continuous. Let F be a filtered subset of S with infimum f, and let $s \in S$. The set $F \oplus s$ is lower-bounded by $f \oplus s$. Now let ℓ be some lower bound of $F \oplus s$. By Lemma VI-4.5 we have

$$f^* \oplus f \oplus s \oplus s^* = \sigma(\bigwedge F) \oplus s \oplus s^* = (\bigwedge \sigma(F)) \oplus s \oplus s^*.$$

Since $\sigma(F)$ is filtered in Σ and Σ is join-continuous, this gives

$$f^* \oplus f \oplus s \oplus s^* = \bigwedge(\sigma(F) \oplus s \oplus s^*).$$

Now $F \oplus s$ is lower-bounded by ℓ, thus $\sigma(F) \oplus s \oplus s^*$ is lower-bounded by $f^* \oplus \ell \oplus s^*$, so that $f^* \oplus f \oplus s \oplus s^* \geqslant f^* \oplus \ell \oplus s^*$. We get

$$f \oplus s = f \oplus (f^* \oplus f \oplus s \oplus s^*) \oplus s \geqslant f \oplus f^* \oplus \ell \oplus s^* \oplus s \geqslant \ell.$$

Hence, $f \oplus s = \bigwedge(F \oplus s)$. □

Problem VI-5.2 (Example VI-3.5 continued)**.** Is the coset monoid $\mathscr{C}(G)$ of a group G sum-continuous?

Since Σ is commutative, we deduce that a mirror semigroup is sum-continuous if and only if, for all filtered subsets F with infimum, and for all s, $\bigwedge(s \oplus F)$ exists and equals $s \oplus (\bigwedge F)$. This will help in demonstrating Lemma VI-5.4.

The following result, although easily proved, will be of crucial importance for establishing Proposition VI-5.6 and Theorem VI-6.3. It highlights the role played by the *subset system* made up of filtered subsets for deriving mirror properties. For instance, mirror properties on complete distributivity (also called *supercontinuity*), where arbitrary subsets replace filtered subsets, would probably fail to be true (see however the mirror property [**175**, Proposition 1-20] and the one on infinite distributivity proved by Resende [**250**]).

Lemma VI-5.3. *Let F be a filtered subset of S. Then f is the least element of $F \oplus f^* \oplus f$, for all $f \in F$.*

Proof. Let $f, f_1 \in F$. Since F is filtered, there is some $f_2 \in F$ such that $f \geqslant f_2$ and $f_1 \geqslant f_2$. Thus, $f_1 \oplus f^* \oplus f \geqslant f_2 \oplus f_2^* \oplus f \geqslant f$, so f is a lower bound of $F \oplus f^* \oplus f$. Since f is in $F \oplus f^* \oplus f$, the assertion is proved. □

Now we recall the way-above relation. If s, t are elements of a poset, we say that t is *way-above* s, if, for every filtered subset F with infimum, $s \geqslant \bigwedge F$ implies $t \geqslant f$ for some $f \in F$. Denoting by $\gg$ the way-above relation on S and by $\ggcurly$ the way-above relation on Σ, we have the following lemma.

Lemma VI-5.4. *Assume that S is a sum-continuous mirror semigroup. Then for all $s, t \in S$, $t \gg s$ if and only if $t \geqslant s$ and $\sigma(t) \ggcurly \sigma(s)$.*

Proof. Assume that $t \gg s$, and let Φ be some filtered subset of Σ with infimum such that $\sigma(s) \geqslant \bigwedge \Phi$. Then $s = s \oplus \sigma(s) \geqslant \bigwedge(s \oplus \Phi)$. Since $s \oplus \Phi$ is filtered and $t \gg s$, there is some $\phi \in \Phi$ such that $t \geqslant s \oplus \phi$. This also gives $t^* \geqslant \phi \oplus s^*$, thus $\sigma(t) = t^* \oplus t \geqslant \phi \oplus \sigma(s) \oplus \phi = \sigma(s) \oplus \phi \geqslant \phi$. We therefore get $\sigma(t) \ggcurly \sigma(s)$.

Conversely, assume that $t \geqslant s$ and $\sigma(t) \ggcurly \sigma(s)$, and let F be a filtered subset of S with infimum such that $s \geqslant \bigwedge F$. We have $\sigma(s) \geqslant \sigma(\bigwedge F) = \bigwedge \sigma(F)$, and $\sigma(F)$ is filtered in Σ, thus there is some $f \in F$ such that $\sigma(t) \geqslant \sigma(f)$. Now $t = t \oplus \sigma(t) \geqslant t \oplus \sigma(f) \geqslant s \oplus \sigma(f) \geqslant (\bigwedge F) \oplus \sigma(f) = \bigwedge(F \oplus f^* \oplus f)$. Using Lemma VI-5.3, we have $t \geqslant f$, and this shows that $t \gg s$. □

Corollary VI-5.5. *Assume that S is a sum-continuous mirror semigroup. Then for all $\epsilon, \phi \in \Sigma$, $\epsilon \ggg \phi$ if and only if $\epsilon \gg \phi$.*

The way-above relation on S is *additive* if $t \gg s$ and $t' \gg s'$ imply $t \oplus t' \gg s \oplus s'$. If S reduces to a semilattice, this amounts to the usual definition (see [**114**, Definition III-5.8]).

Proposition VI-5.6. *Assume that S is a sum-continuous mirror semigroup. Then the way-above relation on S is additive if and only if the way-above relation on Σ is additive.*

Proof. If S has an additive way-above relation, the previous corollary ensures that Σ also has an additive way-above relation. Conversely, assume that $\ggg$ is additive, and let $r, s, t, u \in S$ such that $r \gg s$ and $t \gg u$. Let F be a filtered subset of S with infimum f_0 such that $s \oplus u \geqslant f_0$. Then $s^* \oplus s \oplus u \oplus u^* \geqslant s^* \oplus f_0 \oplus u^*$, and since $s^* \oplus s \oplus u \oplus u^*$ is idempotent, $s^* \oplus s \oplus u \oplus u^* \geqslant \sigma(s^* \oplus f_0 \oplus u^*)$. Since S is sum-continuous and $s^* \oplus F \oplus u^*$ is filtered, we have $s^* \oplus f_0 \oplus u^* = \bigwedge(s^* \oplus F \oplus u^*)$, and by Scott-continuity of σ we deduce

$$s^* \oplus s \oplus u \oplus u^* \geqslant \bigwedge \sigma(s^* \oplus F \oplus u^*). \tag{44}$$

By Lemma VI-5.4, $r^* \oplus r \ggg s^* \oplus s$, and similarly $t \oplus t^* \ggg u \oplus u^*$. Since $\ggg$ is additive, this gives $r^* \oplus r \oplus t \oplus t^* \ggg s^* \oplus s \oplus u \oplus u^*$. Combining this with Equation (44), we see that there is some $f \in F$ such that $r^* \oplus r \oplus t \oplus t^* \geqslant \sigma(s^* \oplus f \oplus u^*)$. Hence,

$$\begin{aligned} r \oplus t &= r \oplus (r^* \oplus r \oplus t \oplus t^*) \oplus t \\ &\geqslant s \oplus \sigma(s^* \oplus f \oplus u^*) \oplus u \\ &\geqslant ((s \oplus u) \oplus f^*) \oplus (s \oplus s^*) \oplus f \oplus (u^* \oplus u) \\ &\geqslant (f_0 \oplus f_0^*) \oplus (s \oplus s^*) \oplus f \oplus (u^* \oplus u) \\ &\geqslant f. \end{aligned}$$

This proves that $r \oplus t \gg s \oplus u$, i.e. that $\gg$ is additive. □

VI-6. Continuity, algebraicity

A poset is *continuous* if $\{t : t \text{ way-above } s\}$ is filtered with infimum equal to s, for all s. An *fcpo* is a filtered-complete poset, and a *domain* is a continuous fcpo. A poset is *algebraic* if every element s is the filtered infimum of the compact elements above it (where an element is *compact* if it is way-above itself). Or equivalently, by [**114**, Proposition I-4.3], if the poset is continuous and if $t \gg s$ implies $t \geqslant k \geqslant s$ for some compact element k. A mirror semigroup is *continuous* (resp. *algebraic*) if it is continuous (resp. *algebraic*) with respect to its intrinsic partial order.

Lemma VI-6.1. *A continuous mirror semigroup is sum-continuous.*

Proof. Assume that S is a continuous mirror semigroup. We show that S is sum-continuous, or equivalently by Proposition VI-5.1 that Σ is join-continuous. Let Φ be a filtered subset of Σ with infimum $\phi = \bigwedge \Phi$, and let $\epsilon \in \Sigma$. Then $\epsilon \oplus \Phi$ is lower-bounded by $\epsilon \oplus \phi$. Now let ℓ be some lower bound of $\epsilon \oplus \Phi$ in S. We prove that $\epsilon \oplus \phi \geqslant \ell$. For this purpose, let $s \gg \epsilon \oplus \phi$. Since s is greater than the idempotent element $\epsilon \oplus \phi$, s is idempotent itself. Moreover, we have $s \gg \phi$, so there exists some $\phi_1 \in \Phi$ such that $s \geqslant \phi_1$. Thus, $s = s \oplus s \geqslant \epsilon \oplus \phi_1 \geqslant \ell$. Since S is continuous, we deduce that $\epsilon \oplus \phi \geqslant \ell$, hence $\epsilon \oplus \phi = \bigwedge(\epsilon \oplus \Phi)$, so Σ is continuous, and the result follows. □

Lemma VI-6.2. *Assume that S is a mirror semigroup. If S is a continuous poset (resp. an fcpo, a domain, an algebraic poset), then Σ is a continuous poset (resp. an fcpo, a domain, an algebraic poset).*

Proof. Assume that S is filtered-complete. Since Σ is Scott-closed in S, this is a sub-fcpo of S by [**114**, Exercise II-1.26(ii)].

Assume that S is a continuous poset, and let $\epsilon \in \Sigma$. Every element way-above ϵ in S belongs to Σ, and is way-above ϵ in Σ (this merely results from the fact that S is mirror). Thus, ϵ is the infimum (in Σ) of a filtered subset of elements way-above it, which gives continuity of Σ.

Assume that S is algebraic. To prove that Σ is algebraic, we need to show that, whenever $\epsilon \ggg \phi$, there is some $\kappa \in \Sigma$ with $\kappa \ggg \kappa$ and $\epsilon \geqslant \kappa \geqslant \phi$. But S is sum-continuous by Lemma VI-6.1, so by Corollary VI-5.5 $\epsilon \ggg \phi$ implies $\epsilon \gg \phi$. Since S is algebraic, there is some compact element $k \in S$ such that $\epsilon \geqslant k \geqslant \phi$. With Lemma VI-5.4, we see that $\kappa = k^* \oplus k$ is a compact element in Σ, and $\epsilon \geqslant \kappa \geqslant \phi$. □

Here comes the most important of our mirror properties.

Theorem VI-6.3. *Assume that S is a mirror semigroup. Then S is continuous (resp. algebraic) if and only if Σ is continuous (resp. algebraic).*

Proof. Assume that Σ is a continuous poset. Then Σ is join-continuous by Lemma VI-6.1, so S is sum-continuous by Proposition VI-5.1. Let us show that S is continuous. Let $s \in S$. There exists some filtered subset Φ of Σ such that

$$\bigwedge \Phi = s^* \oplus s \tag{45}$$

and $\phi \ggg s^* \oplus s$ for all $\phi \in \Phi$. Let $\phi \in \Phi$. Then $s \oplus \phi \geqslant s$ on the one hand, and $\phi \oplus s^* \oplus s \geqslant \phi \ggg s^* \oplus s$, so $\sigma(s \oplus \phi) \ggg \sigma(s)$, on the other hand. By Lemma VI-5.4, we have $s \oplus \phi \gg s$, for all $\phi \in \Phi$. Also, S is sum-continuous, so from Equation (45) we deduce that s is the infimum of $s \oplus \Phi$, and this set is filtered and consists of elements way-above s. This establishes the continuity of S.

Assume that Σ is algebraic. Let $t \gg s$ and let us show that $t \geqslant k \geqslant s$ for some compact element $k \in S$. By Lemma VI-5.4 we have $\sigma(t) \ggg \sigma(s)$, so there is some compact element κ in Σ such that $\sigma(t) \geqslant \kappa \geqslant \sigma(s)$. We get

$s \oplus \sigma(t) \geqslant s \oplus \kappa \geqslant s$, and since $t \geqslant s$ we have $t \geqslant s \oplus \kappa \geqslant s$. The element $k = s \oplus \kappa$ satifies $k^* \oplus k = \kappa$, so that k is compact in S by Lemma VI-5.4. This proves that S is algebraic. $\square$

A continuous inverse semigroup with an additive way-above relation is called a *stably continuous inverse semigroup*.

Corollary VI-6.4. *Assume that S is a mirror semigroup. Then S is stably continuous if and only if Σ is stably continuous.*

If $\epsilon \in \Sigma$, we write H_ϵ for the subset $\{s \in S : s^* \oplus s = \epsilon\}$. (If S is a *Clifford inverse semigroup*, i.e. an inverse semigroup such that $s^* \oplus s = s \oplus s^*$ for all $s \in S$, then H_ϵ is the maximal subgroup of S with identity element ϵ.)

Theorem VI-6.5. *Assume that S is an inverse semigroup such that Σ is a continuous poset. Then S is a mirror semigroup if and only if, for all $\epsilon \in \Sigma$, and each pair of distinct points $s, t \in H_\epsilon$, there exists some $\varphi \in \Sigma$, $\varphi \gg \epsilon$, such that $s \oplus \varphi \neq t \oplus \varphi$. In this case, S is a continuous poset.*

Proof. Assume that S is a mirror semigroup, and let $\epsilon \in \Sigma$ and $s, t \in H_\epsilon$. Suppose that, for all $\varphi \gg \epsilon$, we have $s \oplus \varphi = t \oplus \varphi$. Since Σ is continuous, $A := \{\varphi \in \Sigma : \varphi \gg \epsilon\}$ is a filtered subset of Σ that admits ϵ as infimum in Σ. Hence, ϵ is also the infimum of A in S, for S is mirror. Moreover, we have $\varphi \oplus \varphi^* = \varphi \geqslant \epsilon = s^* \oplus s$, for all $\varphi \in A$, so we can apply Lemma VI-4.6, which gives

$$\begin{aligned} s \oplus \epsilon = s \oplus \left(\bigwedge A\right) = \bigwedge (s \oplus A) = \bigwedge_{\varphi \in A} (s \oplus \varphi) \\ = \bigwedge_{\varphi \in A} (t \oplus \varphi) = t \oplus \left(\bigwedge A\right) = t \oplus \epsilon. \end{aligned}$$

Since $s, t \in H_\epsilon$, we get $s = s \oplus s^* \oplus s = s \oplus \epsilon = t \oplus \epsilon = t \oplus t^* \oplus t = t$.

Conversely, assume that the property given in the theorem is satisfied. We want to show that S is a mirror semigroup, so let Φ be a filtered subset of Σ with an infimum ϵ in Σ. We want to prove that ϵ is also the infimum of Φ in S, so let $\ell \in S$ be a lower bound of Φ. Then $\ell^* \oplus \ell$ is a lower bound of Φ in Σ, so that $\epsilon \geqslant \ell^* \oplus \ell$ by definition of ϵ. This implies that $(\ell \oplus \epsilon)^* \oplus (\ell \oplus \epsilon) = \epsilon$, so we have both ϵ and $\ell \oplus \epsilon$ in H_ϵ. If we suppose that $\epsilon \neq \ell \oplus \epsilon$, there exists some $\varphi \in \Sigma$, $\varphi \gg \epsilon$ such that $\epsilon \oplus \varphi \neq \ell \oplus \epsilon \oplus \varphi$, i.e. $\varphi \neq \ell \oplus \varphi$. But the fact that $\varphi \gg \epsilon$ and the definition of ϵ imply that there is some $\epsilon_1 \in \Phi$ such that $\varphi \geqslant \epsilon_1$, so that $\varphi \geqslant \ell$, i.e. $\varphi = \ell \oplus \varphi$, a contradiction. We have thus proved that $\epsilon = \ell \oplus \epsilon$, which rewrites to $\epsilon \geqslant \ell$. This shows that ϵ is the greatest lower bound of Φ in S. $\square$

Problem VI-6.6. Find a non-continuous inverse semigroup whose semilattice of idempotents is continuous. (Note that in the inverse semigroup $S = \{\omega\} \cup [0, 1]$ given in Example VI-4.2, the element ω is compact, thus S is continuous.)

Example VI-6.7 (Example VI-3.6 continued)**.** In the bicyclic monoid, the semilattice of idempotents is isomorphic to $(P, \leqslant)$. In the particular cases where P equals $\mathbb{N}$ or $\mathbb{R}_+$, which both are stably continuous semilattices, the associated bicyclic monoid is stably continuous.

Example VI-6.8 (Example VI-3.7 continued)**.** In the rotation semigroup, the semilattice of idempotents $[0, 1]$ is stably continuous, so $(B^2, \otimes)$ is stably continuous.

Example VI-6.9 (Example VI-3.8 continued)**.** Let S be a finite commutative inverse monoid. The cube $[0, 1]^{\Sigma(S)}$, as a finite cartesian product of continuous lattices, is a continuous lattice [**114**, Proposition I-2.1]. Considering the semilattice $\Sigma(S)^\wedge$ of characters on $\Sigma(S)$ as a subset of $[0, 1]^{\Sigma(S)}$, it is closed under arbitrary infima and suprema, so it is a continuous lattice [**114**, Theorem I-2.6]. Since $\Sigma(S)^\wedge$ and $\Sigma(S^\wedge)$ are isomorphic, the semilattice $\Sigma(S^\wedge)$ is also continuous. Now the character monoid $S^\wedge$ of S is mirror, so $S^\wedge$ is continuous.

The case of the symmetric pseudogroup introduced in Example VI-3.4 is extended to topological spaces as follows. If X is a topological space, the *symmetric pseudogroup* $\mathscr{I}(X)$ on X is the set made up of all the partial homeomorphisms on X, i.e. the homeomorphisms $f : U \to V$ where U and V are open sets of X. The law of inverse semigroup is defined as in the discrete case. The symmetric pseudogroup is filtered-complete, hence is a mirror semigroup.

A topological space X is *core-compact* if its collection of closed subsets $(\mathscr{F}(X), \subset)$ is a continuous poset. We then have the following characterization.

Corollary VI-6.10. *Let X be a topological space. Then X is core-compact if and only if its symmetric pseudogroup $\mathscr{I}(X)$ is continuous.*

Proof. Let Σ be the semilattice of idempotents of $\mathscr{I}(X)$. Defining the maps $i : \mathscr{F}(X) \to \Sigma$ and $j : \Sigma \to \mathscr{F}(X)$ respectively by $i(F) = \mathrm{id}_{X\setminus F}$ and $j(f) = X \setminus \mathrm{dom}(f)$, it is easy to show that both i and j are upper adjoints of each other, i.e. that $i(F) \leqslant f$ if and only if $F \subset j(f)$, and $f \leqslant i(F)$ if and only if $j(f) \subset F$, for all $F \in \mathscr{F}(X)$ and $f \in \Sigma$. Thus, i and j are both isomorphisms of complete lattices. By [**114**, Theorem I-2.11] we deduce that X is core-compact if and only if Σ is continuous, and, by Theorem VI-6.3, if and only if $\mathscr{I}(X)$ is continuous. □

The topologist may grant more appeal to the following corollary.

Corollary VI-6.11. *Let X be a Hausdorff topological space. Then X is locally compact if and only if its symmetric pseudogroup $\mathscr{I}(X)$ is continuous.*

Proof. Given that the space X is Hausdorff, it is known since Hofmann and Mislove [**128**] that local compactness and core-compactness are equivalent properties. □

Corollary VI-6.12. *Let X be a Hausdorff topological space. Then X is totally disconnected and locally compact if and only if its symmetric pseudogroup $\mathscr{I}(X)$ is algebraic.*

Proof. See [**114**, Exercise I-4.28(iv)], where it is asserted that the lattice of closed subsets of a Hausdorff space X is algebraic if and only if X is totally disconnected and locally compact, then apply Theorem VI-6.3. □

VI-7. Conclusion and perspectives

The work presented in this chapter is a first step in the study of inverse semigroups from a domain theoretical perspective. In future work we shall aim at topological considerations, using Scott's and Lawson's topologies. We shall also examine in more detail (compact) topological inverse semigroups.

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CHAPTER VII

Conclusion heuristique en français

Dans les introductions ou conclusions propres à chaque chapitre nous avons indiqué des éléments de perspectives. Nous ouvrons ici en quelques lignes des pistes de recherche dédiées à l'unification des mathématiques classique et idempotente.

VII-1. Un premier point de vue : la déquantification de Maslov

Les mathématiques idempotentes voient en général l'opération maximum comme une situation limite de l'addition, du fait de l'égalité suivante :

$$\lim_{p\to\infty}(x^p + y^p)^{1/p} = \max(x, y), \tag{46}$$

pour tous $x, y \in \mathbb{R}_+$. Ainsi un « espace idempotent », quel que soit le sens précis qu'on lui donne, est considéré comme une déformation asymptotique de l'espace euclidien traditionnel. C'est particulièrement visible dans les travaux de Briec et Horvath [**49**] sur la convexité max-plus (que les auteurs nomment « B-convexité »), où les parties max-plus convexes sont étudiées directement en tant que déformées à l'infini des parties convexes au sens classique, cf. [Chapitre V, Figure 1].

Ce point de vue est nommé *déquantification* de Maslov [**196**] ; ce dernier l'a utilisé pour transformer certaines équations aux dérivées partielles de la physique par un changement de variable $x \mapsto x^h$; l'indice h, que l'on fait tendre vers 0, joue le rôle de la constante de Planck (cf. la revue [**180**] de Litvinov). L'égalité (46) peut en effet être récrite sous la forme

$$\lim_{h\to 0}(x \oplus_h y) = x \oplus_0 y,$$

où $\oplus_h$ est l'opération de semigroupe sur $K_h^+ = \mathbb{R}_+$ définie par $x \oplus_h y = (x^{1/h} + y^{1/h})^h$ si $h > 0$, et $x \oplus_0 y = \max(x, y)$.

Rappelons que la *quantification* comme procédé inverse de la déquantification de Maslov a été proposée par Connes et Consani [**67**], cf. [Chapitre I, Exemple 5.4]. À partir d'un semicorps idempotent commutatif K_0^+ vérifiant certaines propriétés adéquates, les auteurs reconstruisent une famille (K_h) de corps dont K_0^+ apparaît comme la déquantification. Leur point de départ est pour cela la formule suivante, valable dans $\mathbb{R}_+$:

$$x + y = \sup_{p\in[0,1]} x^p y^{1-p} e^{S(p)}, \tag{47}$$

où l'entropie est définie sur $[0,1]$ par $S(p) = -p\log(p) - (1-p)\log(1-p)$. La loi d'addition $\oplus_h$ de K_h est alors définie pour $x, y \in K_h^+ = \{z \in K_h : z \geqslant 0\}$ en remplaçant $e^{S(p)}$ par $h^{-S(p)}$ dans la formule (47), où $h \in K_0^+$, $h < 1$.

VII-2. Un second point de vue : la caractéristique 1

Reprenant l'idée de Castella [**57**], nous avons présenté au chapitre VI les semigroupes inverses comme la structure algébrique clef pour une unification des mathématiques classique et idempotente. Essayons de voir plus précisément comment cela pourrait se passer. Soit $(A, \oplus, 0, \times, 1)$ un semi-anneau *inverse*, c'est-à-dire un semi-anneau tel que $(A, \oplus, 0)$ soit un monoïde inverse commutatif. Les semi-anneaux inverses ont été étudiés à l'origine par Karvellas [**147**].

À la suite de Castella [**57**] et de Connes et Consani [**67**], on définit la *caractéristique* de A, notée $c(A)$, comme le plus petit entier k non nul tel que $k.1_A \oplus 1_A = 1_A$, s'il existe (dans le cas contraire, on pose $k = 0$). Soit à présent M un *module* sur le semi-anneau A, c'est-à-dire un monoïde commutatif $(M, \oplus, 0)$ muni d'une loi externe $A \times M \to M, (a, x) \mapsto a.x$ telle que, pour tout $x \in M$, $0.x = 0$, $1.x = x$, et pour tous $y \in M$, $a, b \in A$,

$$(ab).x = a.(b.x),$$
$$a.(x \oplus y) = a.x \oplus a.y,$$
$$(a \oplus b).x = a.x \oplus b.x.$$

Si on suppose de plus que A est *intègre* (ou *simplifiable*), c'est-à-dire que l'application $A \to A, b \mapsto a.b$ est injective pour tout a non nul, alors, suivant Zeleznikow [**306**, Lemme 13] ou Castella [**57**, Proposition 2(c)], deux cas se présentent :

- si $c(A) = 1$, alors A est un semi-anneau idempotent, i.e. tel que $a \oplus a = a$ pour tout $a \in A$, et M est un module idempotent ;
- si $c(A) \neq 1$, alors A est un anneau (intègre), et M est un module sur A au sens classique.

Cette dichotomie simplifie le problème et invite à approfondir le cas $c(A) = 1$, i.e. la théorie algébrique des semi-anneaux idempotents, qui reste encore mal connue aujourd'hui (malgré des travaux significatifs, cf. les livres de Baccelli et al. [**20**] et Gondran et Minoux [**115**]). Cependant elle n'efface pas l'intérêt d'une théorie unifiée. En effet, comment par exemple définir correctement la notion de *racine* d'un polynôme à coefficients dans un semi-anneau inverse ?

VII-3. Un dernier point de vue : les pseudo-opérations

Une autre façon de généraliser le maximum et l'addition, à tout le moins sur les nombres réels, est d'y introduire une pseudo-addition. Une loi de composition interne sur $[0, \infty]$ ou $[0, \infty)$ qui est associative, de neutre 0, est appelée une *conorme triangulaire* (ou une *t-conorme*). Suivant Sugeno et

Murofushi [**279**], une *pseudo-addition* est une t-cnorme continue à gauche et à droite (faisant ainsi de $[0, \infty]$ ou $[0, \infty)$ un monoïde ordonné semi-topologique).

Il faut à cet égard relever le [**279**, Théorème 2.1], qui fournit une représentation des pseudo-additions sur $[0, \infty]$ comme une sorte de « mélange » de la loi d'addition $+$ et de la loi du maximum $\vee$. Sa preuve repose sur une combinaison de résultats antérieurs dus à Mostert et Shields [**213**] et à Ling [**179**]. Des éléments complémentaires sont fournis par Benvenuti et Mesiar [**32**] dans le cas où une *pseudo-multiplication* (distributive par rapport à la pseudo-addition) est fournie.

L'étude des t-conormes qui satisfont certaines propriétés de continuité fait partie de l'étude plus générale des *I-semigroupes* (avec I pour « intervalle »), qui a intéressé des auteurs tels que Clifford [**61, 62**] ou Berglund [**34**]. Une partie des résultats généraux est elle-même issue des théories des semigroupes totalement ordonnés et des semigroupes compacts, cf. les revues de Hofmann et Lawson [**127**] et de Hofmann [**125**].

Avec l'influence de la théorie des capacités due à Choquet [**60**], les pseudo-additions, et plus généralement les *pseudo-opérations* ou *opérations pseudo-arithmétiques*, ont trouvé des terrains d'application privilégiés en théorie des ensembles flous, avec le développement des mesures non-additives (cf. Pap [**234**]) et de la pan-intégration (cf. Weber [**300**], Sugeno et Murofushi [**279**], Wang et Klir [**298**]). On peut rapprocher cela de l'avènement des *copules* en théorie des probabilités, favorisé notamment par le théorème de Sklar [**275**] (sur les copules, cf. aussi le livre de Nelsen [**221**]).

Finalement, si $\mathbb{R}_+$ est muni d'un couple d'opérations pseudo-arithmétiques $(\oplus, \times)$, où $\times$ désigne la multiplication habituelle, supposée distributive par rapport à $\oplus$, et si M est un module sur $(\mathbb{R}_+, \oplus, \times)$ tel que $(M, \oplus, 0)$ soit un monoïde inverse commutatif, alors, comme à la Section VII-2, deux cas se présentent :

- si $(\mathbb{R}_+, \oplus)$ possède au moins un élément idempotent non nul, alors $(M, \oplus, 0)$ est un monoïde idempotent, et $(r \oplus s).x = \max(r, s).x$ pour tous $r, s \in \mathbb{R}_+$ et $x \in M$ (bien qu'on n'ait *pas* nécessairement $r \oplus s = \max(r, s)$).
- si $(\mathbb{R}_+, \oplus)$ ne possède aucun élément idempotent autre que zéro, alors la loi $\oplus$ sur $\mathbb{R}_+$ est de la forme $s \oplus t = (s^{1/h} + t^{1/h})^h$, pour un certain $h > 0$ [**32**, Corollaire 2]. De plus à chaque élément idempotent ϵ de M, on peut associer l'ensemble $\mathbb{R}_+^\epsilon = \{t \in \mathbb{R}_+ : t.\epsilon = \epsilon\}$, qui par symétrisation s'étend en un corps $\mathbb{R}^\epsilon$ sur lequel $M^\epsilon = \{x \in M : x \oplus x^* = \epsilon\}$ possède une structure d'espace vectoriel.

Dans le premier cas, M est donc isomorphe à un $\mathbb{R}_+^{\max}$-module ; dans le second cas, M est une union disjointe d'espaces vectoriels.

VII-4. Que reste-t-il à unifier ?

Dans le dépassement du paradigme linéaire classique, les théories de la mesure et de l'intégration font donc figure de bons élèves. Cependant, les points de vue « additif » et « maxitif » restent encore à rassembler quand un tel effort n'a pas été entrepris ; il s'agit par exemple :

(1) des processus stochastiques somme-stables[1] et max-stables ;
(2) de l'analyse classique et de l'analyse idempotente ;
(3) de la géométrie algébrique et de la géométrie tropicale ;
(4) de l'algèbre linéaire et de l'algèbre max-plus.

Sur le thème (1), qui fait partie du champ d'intérêt de la théorie des valeurs extrêmes, les travaux récents de Stoev et Taqqu [**276**], Fougères et al. [**102**], Kabluchko [**145**], et Wang et Stoev [**297**] ouvrent la voie. L'étude de la convexité et des cônes abstraits appliquée aux processus somme-stables et max-stables, due à Davydov et al. [**68, 69**] et Molchanov [**209, 210**], vient renforcer la démarche dans une autre direction.

Comme nous l'avons vu sur le thème (2), l'analyse idempotente s'est définie au départ comme une limite de l'analyse classique ; ce point de vue n'a pas encouragé la convergence entre les deux théories, qui reste à ce jour embryonnaire. L'émergence de cette question passe sans doute à nouveau par l'étude des cônes abstraits, comme dans les travaux de Tix [**281**] et de Keimel [**149**].

De même sur le thème (3), le point de vue de la déquantification prédomine : la géométrie tropicale est décrite comme le « squelette » de la géométrie algébrique. Certains tels Viro [**291**] espèrent que la compréhension de ce squelette permettra, par une inversion de la déquantification de Maslov, d'importer des résultats tropicaux vers la géométrie algébrique. Il s'agirait pour cela de rendre rigoureuse l'idée de quantification de Connes et Consani. Dans ce cadre, toute unification devra de plus s'appuyer sur celle de l'algèbre linéaire et de l'algèbre max-plus.

Concernant cette dernière, c'est-à-dire le thème (4), la situation est différente. En effet l'algèbre max-plus s'est construite de façon « directe », et non comme une asymptotique de l'algèbre linéaire. Elle est née en réponse à des besoins spécifiques émis par certains domaines d'application comme la productique, l'informatique ou les réseaux de transport (cf. Cohen et al. [**64**]). Les seuls travaux dont nous avons connaissance qui se penchent consciemment sur la question d'une unification sont ceux de Castella [**57, 56**], cf. chapitre VI.

[1] Appelés plus communément processus α-*stables*.

Appendix

CHAPTER VIII

Pruning a poset with veins

ABSTRACT. We recall some abstract connectivity concepts, and apply them to special chains in partially ordered sets, called veins, that are defined as order-convex chains that are contained in every maximal chain they meet. Veins enable us to define a new partial order on the same underlying set, called the pruning order. The associated pruned poset is simpler than the initial poset, but irreducible, coirreducible, and doubly-irreducible elements are preserved by the operation of pruning.

VIII-1. RÉSUMÉ EN FRANÇAIS

Alors que tout semitreillis fini est engendré par ses éléments irréductibles, il est faux de dire que tout treillis fini est engendré par ses éléments doublement irréductibles. Ainsi, l'ensemble P_3 des parties de $\{1, 2, 3\}$ ordonné par inclusion est un treillis (distributif) fini dont aucun élément n'est doublement irréductible. Il est vrai que P_3 n'est pas un treillis planaire, et on sait par ailleurs que les treillis planaires finis ont au moins un élément doublement irréductible. Dans le même ordre d'idée, plusieurs résultats ont été démontrés permettant d'affirmer l'existence d'un ou plusieurs tels éléments pour certains types d'ensembles ordonnés.

Mais sur le fait d'être engendré par ceux-ci, il faut aller chercher un théorème dû à Monjardet et Wille [**212**] et complété par Erné [**90**] : on y trouve des conditions nécessaires et suffisantes pour qu'un treillis distributif fini soit engendré par ses éléments doublement irréductibles (cf. à ce sujet le théorème de Monjardet–Wille–Erné [Chapitre IV, Proposition 8.8]).

Cependant, les conditions fournies par ce théorème portent notamment sur la complétion normale du treillis d'intérêt, et semblent donc peu opérationnelles. Nous fournissons ici un moyen d'« élaguer » un ensemble ordonné fini (en définissant une nouvelle relation d'ordre qui supprime certaines relations entre éléments) de façon à le simplifier, tout en conservant par élagage les éléments irréductibles et co-irréductibles. Cette opération d'élagage est basée sur la notion de *veine*, que nous définissons comme une chaîne convexe qui est contenue dans toute chaîne maximale qu'elle intersecte.

VIII-2. INTRODUCTION

While every finite semilattice is generated by its irreducible elements, a finite lattice is not always generated by its doubly-irreducible elements.

For instance, the power set P_3 of $\{1, 2, 3\}$ ordered by inclusion is a finite (distributive) lattice with no doubly-irreducible element. It turns out that P_3 is not a planar lattice, and we know that every finite planar lattice has at least one doubly-irreducible element. In the same line, different results were proved that assert the existence of one or several such elements for special types of posets.

But we need a theorem due to Monjardet and Wille [**212**] augmented by Erné [**90**] to get necessary and sufficient conditions on a finite distributive lattice to be generated by its doubly-irreducible elements (see the Monjardet–Wille–Erné theorem [Chapter IV, Proposition 8.8]).

However, the conditions provided by this theorem are related to the normal completion of the lattice at stake, hence seem hardly operational. Here we propose a way to "prune" a finite poset; this means that we define a new partial order on the same underlying set, called the pruning order. The associated pruned poset is simpler than the initial poset, but irreducible, coirreducible, and doubly-irreducible elements are preserved by the operation of pruning. This pruning operation is based on the notion of *vein*, which is an order-convex chain contained in every maximal chain it meets.

VIII-3. Short preliminaries on connectivities

Here we recall the axiomatic concept of connectivity, which nicely generalizes the corresponding notions used in topological spaces or graphs. This will offer an appropriate framework for the study of *veins* in the next section. A *connectivity* on a set E is a nonempty collection $\mathscr{C}$ of subsets of E covering E and satisfying

$$\bigcap \mathscr{A} \neq \varnothing \Rightarrow \bigcup \mathscr{A} \in \mathscr{C},$$

for all subsets $\mathscr{A} \subset \mathscr{C}$. The elements of $\mathscr{C}$ are the *connected* subsets of E, and $(E, \mathscr{C})$ is called a *connectivity space*. The space is *point-connected* if all singletons are connected. All connectivities considered here will be point-connected. The *connected components* of a connectivity space E are the maximal connected subsets.

We owe this axiomatisation to Börger [**42**]. Matheron and Serra [**198**] and Serra [**266, 267**], interested in applications to mathematical morphology and image analysis, rediscovered this concept, and their work was pursued by Ronse [**257**] and Braga-Neto and Goutsias [**46, 47**] among others, for similar purposes. At the same time, analogous work arising from order-theoretic interests was developed by Richmond and Vainio [**253**] and Erné and Vainio [**95**].

VIII-4. Irreducible chains in partially ordered sets

VIII-4.1. Irreducible chains. A *partially ordered set* or *poset* $(P, \leqslant)$ is a set P equipped with a reflexive, transitive, and antisymmetric binary relation $\leqslant$. A nonempty subset C of P is a *chain* (or a *totally ordered subset*) if, for all $x, y \in C$, $x \leqslant y$ or $y \leqslant x$. A chain M is *maximal* if $C \supset M$ implies $C = M$, for all chains C in P.

We call *irreducible* a chain C such that, for all maximal chains M,

$$C \cap M \neq \varnothing \Longrightarrow C \subset M.$$

Note that every nonempty subset of an irreducible chain is an irreducible chain. The next proposition gives a characterization.

Proposition VIII-4.1. *A chain C is irreducible if and only if, for all nonempty finite (resp. arbibrary) families of maximal chains covering C, one of them contains C.*

Proof. Assume that C is irreducible, and let $(M_j)_{j \in J}$ be some family of maximal chains covering C. Let $x \in C$. Then $x \in M_{j_0}$ for some $j_0 \in J$, hence $C \cap M_{j_0} \neq \varnothing$. This implies $C \subset M_{j_0}$.

Conversely, assume that the property given by the proposition is satisfied for some chain C, and let M be a maximal chain meeting C at x. If C is not contained in M, then $C \cap M^c \neq \varnothing$. By Zorn's lemma, there exists some maximal chain N containing $C \cap M^c$ and avoiding x. Then $C \subset M \cup N$, hence $C \subset N$, a contradiction. $\square$

Here comes the link with connectivities.

Proposition VIII-4.2. *On a poset, the family of irreducible chains is a connectivity, and maximal irreducible chains correspond to connected components.*

Proof. First notice that every singleton is an irreducible chain. Let $(C_j)_{j \in J}$ be a family of irreducible chains with nonempty intersection, and let M be a maximal chain meeting $\bigcup_{j \in J} C_j$ (note that such an M always exists). There is some $j_0 \in J$ such that $C_{j_0} \cap M \neq \varnothing$, so that $C_{j_0} \subset M$. Now for all $j \in J$, $\varnothing \neq C_j \cap C_{j_0} \subset C_j \cap M$, which implies $C_j \subset M$. Therefore, $K = \bigcup_{j \in J} C_j \subset M$, which proves that K is a (nonempty) chain and that this chain is irreducible. $\square$

VIII-4.2. Veins as irreducible convex chains. A subset C of a poset is *convex* if, for all $x, y \in C$ with $x \leqslant y$, $[x, y] \subset C$, where the interval $[x, y]$ is the set $\{z : x \leqslant z \leqslant y\}$. Note that an irreducible chain is not necessarily convex. We define a *vein* as an irreducible convex chain. One can see a vein as a "path" with no diversion.

Proposition VIII-4.3. *On a poset, the family of veins is a connectivity, and maximal veins correspond to connected components.*

Proof. Each singleton is a vein. Let $(C_j)_{j \in J}$ be a family of veins with nonempty intersection. We already know that $K = \bigcup_{j \in J} C_j$ is an irreducible chain, let us show that K is convex. So let $x, y \in K$ and z such that $x \leqslant z \leqslant y$. There is some j_0 such that $x \in C_{j_0}$ and some k_0 with $y \in C_{k_0}$. Take a point t in the intersection of all C_j, and let M be a maximal chain containing $\{x, z, y\}$. Since M meets the irreducible chain K, it contains

K. Both z and t are in M, so these points are comparable. If $z \leqslant t$, then $z \in [x, t] \subset C_{j_0}$ since C_{j_0} is convex. If $t \leqslant z$, then $z \in [t, y] \subset C_{k_0}$. In either case, $z \in K$, and the convexity of K is proved. □

Proposition VIII-4.4. *Let P be a poset and Q be a subset of P. If V is a vein of P meeting Q, then $V \cap Q$ is a vein of Q.*

Proof. The set $V \cap Q$ is clearly a nonempty convex chain in Q. Assume that M is a maximal chain in Q such that $V \cap Q \cap M \neq \varnothing$. Let N be a maximal chain in P containing M. Then $V \cap N \neq \varnothing$, so that $V \subset N$ since V is a vein in P. This implies that $V \cap Q \subset N \cap Q$. But $N \cap Q$ is a chain in Q containing M, so that $M = N \cap Q$ by maximality of M. This proves that $V \cap Q \subset M$, i.e. that $V \cap Q$ is a vein in Q. □

VIII-4.3. Pruning of a poset. A vein is *strict* if it is not a singleton. On a poset P we can define a new binary relation $\leqslant_*$ by $x \leqslant_* y$ if $x = y$ or ($x < y$ and there is some maximal chain in $[x, y]$ that contains no strict vein). We call this relation the *pruning order* of P. The *pruning* P^* of P is the set P equipped with the pruning order. The following results will justify this wording.

Theorem VIII-4.5. *On every poset, the pruning order is a partial order.*

Proof. The matter is to show the transitivity of $\leqslant_*$. Assume that $x \leqslant_* y$ and $y \leqslant_* z$. If two points among x, y, z are equal, then $x \leqslant_* z$, so consider that $x < y < z$. Let M be a maximal chain in $[x, y]$ containing no strict vein, and define $N \subset [y, z]$ similarly. Then $M \cup N$ is a chain, and we show that it is maximal in $[x, z]$. So let C be a chain such that $M \cup N \subset C \subset [x, z]$. Then $C \cap [x, y]$ is a chain in $[x, y]$ containing M, hence $M = C \cap [x, y]$ by maximality of M. Analogously, $N = C \cap [y, z]$. This gives $M \cup N = C \cap ([x, y] \cup [y, z])$. But since $y \in C$, every $c \in C$ is comparable with y, so that $C \subset [x, y] \cup [y, z]$. We get $M \cup N = C$, which proves the maximality of $M \cup N$ in $[x, z]$.

To finish the proof, we show that $M \cup N$ contains no strict vein. Let V be a vein in $M \cup N$, and suppose that we can find some $v, w \in V$ with $v \neq w$ (for instance $v < w$). If both v, w are in M, then $[v, w] \subset V$ by order-convexity of V, and $[v, w]$ is a strict vein, necessarily contained in M, a contradiction. Thus, we must have $v \in M$ and $w \in N$. This gives $v \leqslant y \leqslant w$. Since $v < w$, we can say, for instance, that $v < y$, so that $[v, y] \subset V$ is a strict vein contained in M, a contradiction. □

Lemma VIII-4.6. *Let P be a poset, and let $x, y \in P$ such that $x <_* y$. If M is a maximal chain in $[x, y]$ containing no strict vein, then M is also a chain with respect to the pruning order.*

Proof. Let $x', y' \in M$. Assume for instance that $x' < y'$, and let us prove that $x' <_* y'$. This will be the case if we prove that $M' := M \cap [x', y']$ is a maximal chain in $[x', y']$ (containing no strict vein). Let C be a chain such that $M' \subset C \subset [x', y']$, and let $c \in C$. Then $M \cup \{c\}$ satisfies

$M \subset M \cup \{c\} \subset [x, y]$. Also, $M \cup \{c\}$ is a chain: if $z \in M$, then z and c are comparable, for either $z < x'$ (in which case $z < c$), or $z > y'$ (in which case $z > c$), or $z \in [x', y']$ (in which case $z \in M'$, hence $z \in C$, and z and c are again comparable as elements of the chain C). By maximality of M, $M = M \cup \{c\}$, i.e. $c \in M$, so that $c \in M \cap [x', y'] = M'$. This means that $M' = C$, i.e. M' is a maximal chain in $[x', y']$, hence $x' <_* y'$. This shows that M is a chain with respect to $\leqslant_*$. □

In a poset P, we classically write $x \prec y$ whenever y *covers* x, which means that $x < y$ and $[x, y] = \{x, y\}$.

Lemma VIII-4.7. *Let P be a poset, and let $x, y \in P$ such that $x <_* y$. If an element $c \in [x, y]$ satisfies $x \prec c$ (resp. $c \prec y$), then $x <_* c$ (resp. $c <_* y$).*

Proof. Assume for instance that $c \in [x, y]$ is such that $x \prec c$ (the case $c \prec y$ is similar). Note that $C = \{x, c\}$ is a convex chain. If C is not a vein, then C is a maximal chain in $[x, c]$ containing no strict vein, so that $x <_* c$. Suppose on the contrary that C is a vein. Since $x <_* y$, there is some maximal chain M in $[x, y]$ containing no strict vein. Let N be a maximal chain in P containing M. Since $x \in C \cap N \neq \varnothing$, we deduce that $C \subset N$, i.e. $c \in N$. Thus, $M \cup \{c\}$ is a subchain of $[x, y]$ containing M, so that $c \in M$ by maximality of M. Then Lemma VIII-4.6 implies $x <_* c$, i.e. C is not a vein, a contradiction. □

Theorem VIII-4.8. *Let P be a poset in which every bounded chain is finite. Then $(P^*)^* = P^*$.*

Proof. We use the terms and notations $*$-chain, $*$-vein, $[x, y]_*$, etc. with obvious definitions. Assume that $x <_* y$, for some $x, y \in P$. We want to show that $x <_{**} y$. By definition of $<_*$, there exists some maximal chain M in $[x, y]$ containing no strict vein. By Lemma VIII-4.6, M is a (maximal) $*$-chain (in $[x, y]_*$). To conclude that $x <_{**} y$, it remains to show that M contains no strict $*$-vein. Suppose on the contrary that there is some strict $*$-vein V contained in M. With the assumption that every bounded chain in P is finite, we may suppose that V is a two-element $*$-vein, i.e. $V = \{a, b\}$ with $a <_* b$.

Let us show that V is convex. So let $c \in P$ such that $a \leqslant c \leqslant b$. Since $a <_* b$, there is some maximal chain N in $[a, b]$ containing c. We assumed that every bounded chain in P is finite, so we can write N as $a = n_0 < n_1 < \ldots < n_m = b$, and $n_k = c$ for some k. By maximality of N, we have $n_0 \prec n_1$, so that $n_0 <_* n_1$ by Lemma VIII-4.7. If N^* is a maximal $*$-chain containing $\{n_0, n_1\}$, then $a \in V \cap N^* \neq \varnothing$, so $V \subset N^*$ since V is a $*$-vein. This implies that $b \in N^*$, so either $b \leqslant_* n_1$ or $n_1 \leqslant_* b$. But we also know that $n_1 \leqslant b$, so $n_1 \leqslant_* b$. We see that $n_1 \in [a, b]_*$; since V is $*$-convex, this proves that $n_1 \in V$. We deduce by induction that $n_j \in V$ for all j, so in particular $c \in V$, and we have shown that V is convex.

Now let us show that V is irreducible. So let M' be a maximal chain in P such that $V \cap M' \neq \varnothing$. We want to show that $V \subset M'$. We may suppose, without loss of generality, that $a \in V \cap M'$. The hypothesis made on P implies the existence of some $\beta \in M'$ such that $a \prec \beta$. Then $M'' = \{a, \beta\}$ is a maximal chain in $[a, \beta]$.

First case: M'' contains a strict vein. Then M'' is itself a vein. If N'' is a maximal chain containing $\{a, b\}$, then $a \in M'' \cap N'' \neq \emptyset$, so that $M'' \subset N''$. Thus, $\beta \in N''$, so β and b are comparable.

Second case: M'' contains no strict vein. Then $a <_* \beta$. Now if N^* is a maximal $*$-chain containing $\{a, \beta\}$, then $V \cap N^* \neq \emptyset$. Since V is a $*$-vein, this implies $V \subset N^*$, so $b \leqslant_* \beta$ or $\beta \leqslant_* b$. Again, β and b are comparable.

Since V is convex and $a \prec \beta$, both cases imply that $b = \beta$. So we have $b \in M'$, i.e. $V \subset M'$, which shows that V is irreducible.

We have proved that V is a strict vein contained in M, a contradiction. So M contains no strict $*$-vein, and $x <_{**} y$.

Conversely, if $x <_{**} y$, then $x <_* y$ is obvious, so we have proved that $x \leqslant_{**} y \Leftrightarrow x \leqslant_* y$ for all $x, y \in P$, i.e. $(P^*)^* = P^*$. □

Remark VIII-4.9. If P contains an infinite chain, we may have $(P^*)^* \neq P^*$. Consider for instance $P = [0, 1] \cup \{\omega\}$, where ω is an additional element such that $0 < \omega < 1$. In P^*, no relation holds but $0 <_* \omega <_* 1$ and, in $(P^*)^*$, no elements are comparable.

Problem VIII-4.10. Is it true that $((P^*)^*)^* = (P^*)^*$ for every poset P?

In a poset, an element x is *irreducible* if x is a maximal element or $\uparrow x \backslash \{x\}$ is a filter, *coirreducible* if it is irreducible in the poset $(P, \geqslant)$ dual to P, and doubly-irreducible if it is both irreducible and coirreducible. Remark that if P is conditionally complete, then x is irreducible if and only if $x = a \wedge b$ implies $x \in \{a, b\}$, for all a, b.

Proposition VIII-4.11. *Let P be a finite conditionally complete poset and $x \in P$. Then*

- *x is irreducible in P if and only if x is irreducible in P^*,*
- *x is coirreducible in P if and only if x is coirreducible in P^*.*

Proof. Let $x \in P$ and assume that x is not irreducible. Then there are a, b such that $x = a \wedge b$ and $x \notin \{a, b\}$, and we can assume that $x \prec a$ and $x \prec b$ since P is finite. Then $\{x, a\}$ is a maximal chain in $[x, a]$. Moreover, it contains no strict vein: if V is a strict vein included in $\{x, a\}$, then $V = \{x, a\}$; but if M is a maximal chain containing $\{x, b\}$, then $V \cap M = \{x\} \neq \emptyset$, while $V \not\subset M$. Hence $x \leqslant_* a$, and symmetrically $x \leqslant_* b$. Moreover, if u satisfies $u \leqslant_* a$ and $u \leqslant_* b$, then $u \leqslant a$ and $u \leqslant b$, so that $u \leqslant a \wedge b = x$. This shows that x is the infimum in P^* of $\{a, b\}$, so x is not irreducible in P^*.

Conversely, let x be irreducible in P, and let us show that x is irreducible in P^*. So let a, b such that $x <_* a$ and $x <_* b$. This implies that $x \leqslant a \wedge b$, and even $x < a \wedge b$ since x is irreducible in P. Let c such that $x \prec c \leqslant a \wedge b$. We show that $c \leqslant_* a$. Since $x <_* a$, there is a maximal chain M in $[x, a]$ containing no strict vein. Considering that $x \prec c$, we see that $M \backslash \{x\}$ is

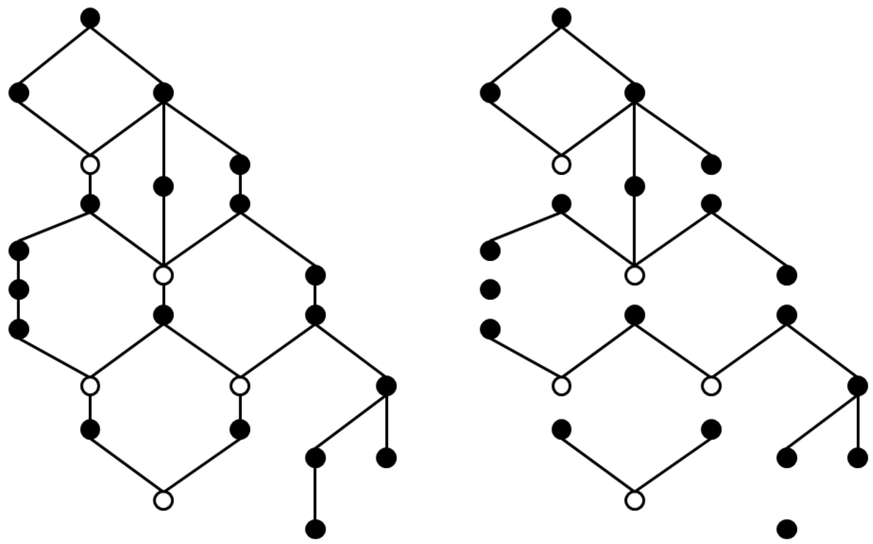

FIGURE 1. On the left, a poset P with nineteen irreducible elements (in black); on the right, the pruned poset P^* has the same irreducible elements as P.

a maximal chain in $[c, a]$ containing no strict vein. Consequently, $c \leqslant_* a$. Similarly, $c \leqslant_* b$. This proves that the subset $\{a \in P : x <_* a\}$ is either empty or filtered, i.e. that x is irreducible in P^*. □

VIII-5. CONCLUSION AND PERSPECTIVES

A future work may consist in finding an algorithm to efficiently prune a given poset.

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