Distributions quasi-stationnaires et méthodes particulaires pour l'approximation de processus conditionnés

Denis Villemonais

École Polytechnique
Centre de Mathématiques Appliquées de l'École Polytechnique

Thèse de doctorat en mathématiques appliquées
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Denis Villemonais

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M	Patrick Cattiaux	Examinateur
M	Pierre Collet	Examinateur
M	Pierre Del Moral	Examinateur
M	Steve N. Evans	Rapporteur
M	Jean-François Le Gall	Examinateur
Mme	Sylvie Méléard	Directrice de thèse
M	Denis Talay	Examinateur & rapporteur

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Introduction

Introduction

Cette thèse porte sur l'étude de la distribution de processus stochastiques avec absorption et de leur approximation. Ces processus trouvent des applications dans de nombreux domaines, tels que l'écologie, la finance ou les études de fiabilité.

Nous étudions en particulier l'évolution en temps long de la distribution de processus de Markov avec absorption. Tandis que l'évolution en temps long d'un processus de Markov récurrent est bien décrite par sa distribution stationnaire, la distribution stationnaire d'un processus de Markov dont l'absorption est presque sûre est concentrée sur l'ensemble des points absorbants, ce qui ne présente qu'un faible intérêt. En revanche, la distribution limite d'un processus conditionné à ne pas être éteint au moment où on l'observe permet de décrire et d'expliquer des comportesments non-triviaux, comme les plateaux de mortalité. Lorsqu'une telle distribution existe, elle est appelée distribution limite conditionnelle, ou limite de Yaglom si elle ne dépend pas de la position initiale du processus.

Dans le premier chapitre, nous montrons en toute généralité que ces distributions conditionnelles limites sont des distributions quasi-stationnaires, c'est-à-dire des distributions qui sont stationnaires pour le processus conditionné à ne pas être absorbé au moment de l'observation. Nous démontrons également que l'existence d'une distribution conditionnelle limite implique la convergence en temps long du taux d'absorption par unité de temps du processus. Enfin nous présentons des méthodes pour prouver l'existence et l'unicité de telles distributions. Le principal outil de ces preuves est un résultat qui lie la propriété de quasi-stationnarité à la théorie spectrale du générateur infinitésimal du processus étudié.

Dans la grande majorité des cas, ces outils de théorie spectrale ne permettent pas d'obtenir explicitement la valeur de la limite de Yaglom du processus étudié. C'est afin de pallier cette difficulté que nous démontrons dans une grande généralité une méthode d'approximation des distributions de processus de Markov conditionnés à ne pas être absorbés et de leur limite de Yaglom. Cette méthode, introduite par Burdzy, Holyst, Ingermann et March en 1996 pour l'étude du cas brownien, est basée sur l'étude d'un système de particules en interaction de type champ moyen. Les particules de ce système évoluent comme des copies indépendantes du processus original avec absorption, jusqu'à ce que l'une d'entre elles soit absorbée. À ce moment, la particule absorbée est remise en jeu à la position d'une des autres particules. Puis le système évolue comme des copies indépendantes du processus avec absorption jusqu'à ce que l'une d'entre elles soit absorbée, et ainsi de suite. Nous montrons que, lorsque le nombre de particules de ce système tend vers l'infini, sa distribution empirique converge vers la distribution conditionnelle du processus et, en temps long, vers sa limite de Yaglom.

La méthode d'approximation démontrée dans cette thèse associée à des méthodes de couplage nous permet d'obtenir de nouveaux résultats d'existence de limite de Yaglom et des propriétés de mélange nouvelles pour des diffusions à coefficients inhomogènes absorbées au bord d'un ouvert borné. Cette propriété de mélange nous permet également de démontrer que la méthode d'approximation converge uniformément en temps dans certains cas.

Les propriétés démontrées et les questions soulevées sont illustrées à l'aide de simulations écrite en C++ durant la thèse.

Distributions quasi-stationnaires et limites de Yaglom

Soit $(X_t)_{t\geq 0}$ un processus de Markov évoluant dans un espace $E\cup\{\partial\}$, où $\partial\notin E$. Nous notons τ_∂ le temps d'atteinte de ∂ par X, qui est le temps d'arrêt défini par

$$\tau_\partial = \inf\{t\geq 0,\, X^\partial_t=\partial\}.$$

Ainsi, $(X_t)_{t\geq 0}$ est un processus avec absorption si et seulement s'il vérifie

$$X_t = X_{\tau_\partial},\ \forall t\geq\tau_\partial \text{ presque sûrement.}$$

Les trajectoires absorbées au temps d'absorption n'apportent pas d'information pertinente sur le processus et nous nous intéresserons en particulier à la distribution du processus X conditionné à ne pas avoir été aborbé, c'est-à-dire à la distribution conditionnelle

$$\mathbb{P}_\mu\left(X^\partial_t\in\cdot|t<\tau_\partial\right),$$

où $\mathbb{P}_\mu$ est la mesure de probabilité associée au processus X de distribution initiale μ sur E.

L'étude de tels processus trouve des applications dans de nombreux domaines, dont, sans être exhaustifs, l'écologie, la biologie, la finance, la chimie et les études de fiabilité. Les objets mathématiques que nous allons introduire nous sont suggérés par des problématiques concrètes. Observons par exemple l'évolution d'une population, dont chaque élément est considéré comme un processus stochastique qui cesse d'évoluer à sa mort. Une quantité qui présente un grand intérêt pour les démographes et les biologistes est le taux de mortalité par tranche d'âge dans la population, c'est-à-dire la proportion d'individus d'une tranche d'âge donnée qui meurent avant d'atteindre la tranche d'âge suivante. Cette quantité est facilement observable et a été relevée dans différentes situation, y compris chez l'être humain - l'INSEE, par exemple, fournit ces données pour la France. Le taux de mortalité par tranche d'âge dépend évidemment de la tranche d'âge considérée: il y a statistiquement plus de risque pour un être humain de 100 ans de mourir dans l'année qui suit que pour un individu de 10 ans. Le graphique de la figure 0.1 représente l'évolution du taux de mortalité en fonction de l'âge chez l'individu masculin en France, dans une échelle logarithmique. L'aspect linéaire de la courbe après trente ans indique que le taux de mortalité évolue exponentiellement avec l'âge. Ce fait a également été énoncé comme une loi générale par Gompertz en 1825 [36] d'après les données dont il disposait. En 1939, Greenwood et Irwin ont remis en cause cette approche, leur étude faisant apparaître que la loi de Gompertz surestime le taux de mortalité de l'être humain aux âges les plus avancés. De plus, leurs observations suggèrent que le taux de mortalité admet une limite avec l'âge. Ce phénomène semble confirmé par des statistiques plus récentes (*cf.* Li et Vaupel [60]) et apparaît de manière distincte chez certains animaux. En effet, à travers des expériences effectuées en laboratoire, Carey, Liedo, Orozco et Vaupel [15] ont mesuré le taux de mortalité de groupes de mouches en fonction de leur âge. Leurs résultats, reproduits ici sur les graphiques de la figure 0.2, montrent sans ambiguïté que les taux de mortalité mesurés décélèrent après un certain âge et même se stabilisent aux âges les plus avancés. Ce phénomène, appelé plateau de mortalité en référence à sa représentation graphique, discrédite la loi de Gompertz pour l'estimation du taux de mortalité chez les individus les plus âgés. D'un manière plus générale, la stabilisation du taux d'absorption est un phénomène courant. Ainsi, Aalen et Gjessing [1] citent le cas surprenant du taux de divorce par nombre d'années de mariage, qui lui aussi présente un "plateau d'absorption", comme on peut l'observer sur le graphique 0.3.

L'existence de plateaux de mortalité nous suggère une forme de stabilisation de la distribution des individus encore vivants à un âge avancé. Dans notre cadre mathématique, ceci nous amène naturellement à l'étude en temps long de la distribution conditionnelle $\mathbb{P}_\mu(X_t \in \cdot | t < \tau_\partial)$ et justifie en particulier l'introduction des propriétés suivantes, qui concernent la convergence en temps long de $\mathbb{P}_\mu(X_t \in \cdot | t < \tau_\partial)$ vers une distribution donnée. Soulignons que les définitions suivantes sont valables pour des processus à temps discret et à temps continu.

Définition 1. *Une* ***distribution conditionnelle limite*** *pour X est une mesure de probabilité α sur E telle qu'il existe une mesure de probabilité μ sur E vérifiant*

$$\alpha(\cdot) = \lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot | t < \tau_\partial).$$

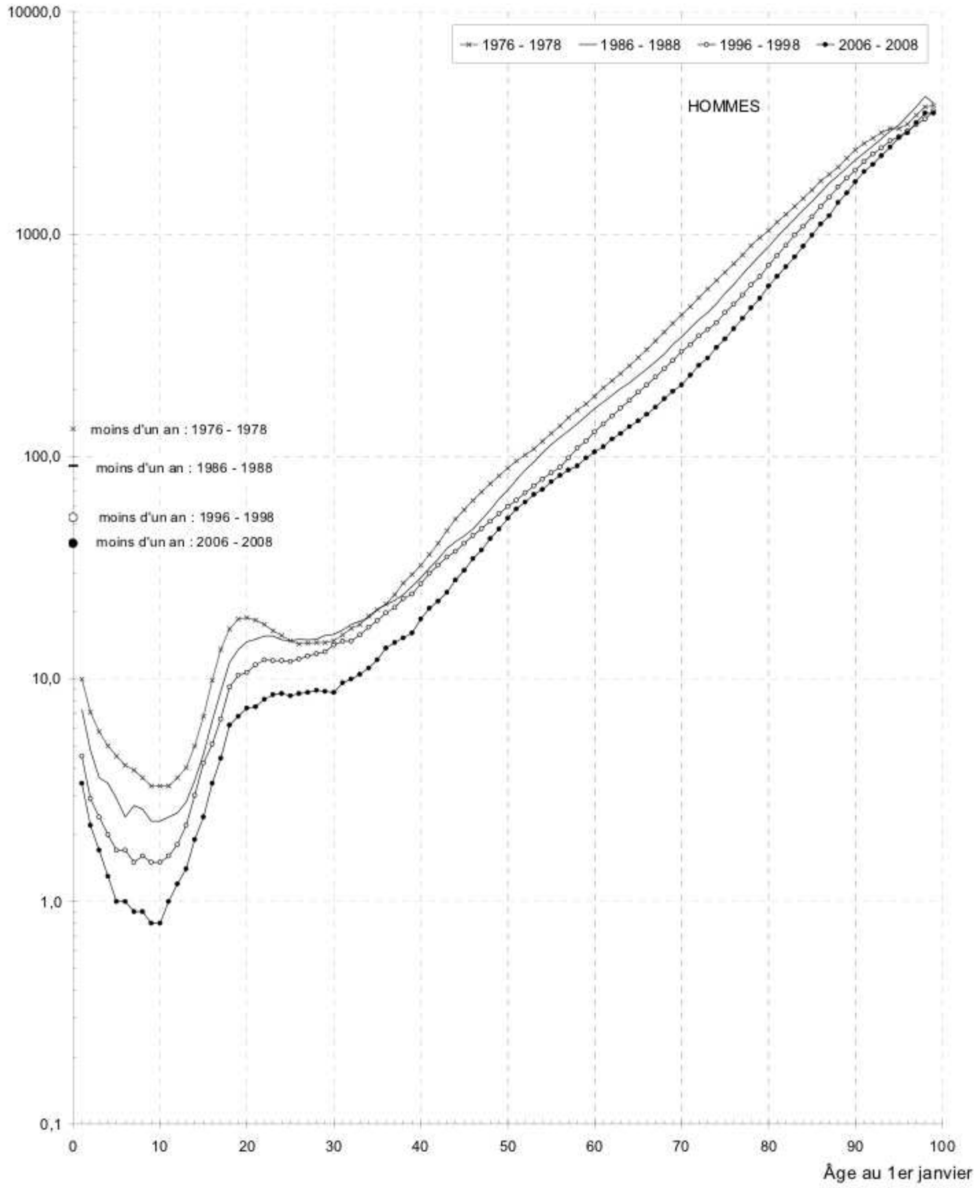


Figure 0.1: Quotients de mortalité par âge chez l'individu de sexe masculin en France - Nombre de décédés pour 10000 personnes de l'âge considéré au 1er janvier [8, p. 125]

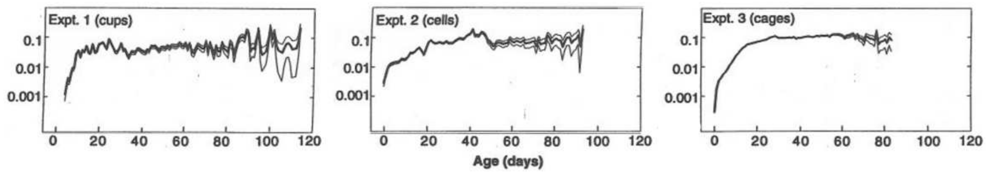


Figure 0.2: Taux de mortalité en fonction de l'âge chez la mouche, avec trois conditions d'expériences distinctes.

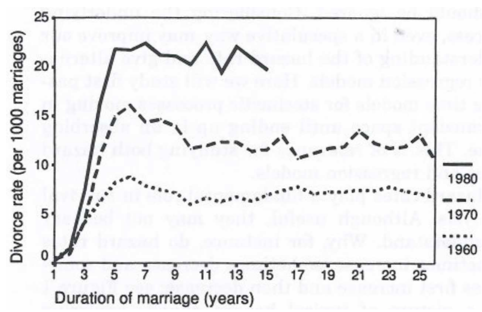


Figure 0.3: Taux de divorce en fonction du nombre d'années de mariage

Dans ce cas, nous dirons que la distribution conditionnelle α attire la distribution initiale μ.

Comme nous le verrons, la distribution conditionnelle limite d'un processus n'est pas nécessairement unique et peut donc dépendre de la distribution initiale du processus. La définition suivante permet de distinguer les cas pour lesquels il existe une distribution conditionelle limite qui ne dépend pas de la position initiale du processus.

Définition 2. *Une **limite de Yaglom** est une mesure de probabilité α sur E telle que, pour tout $x \in E$,*

$$\alpha(\cdot) = \lim_{t\to\infty} \mathbb{P}_x \left(X_t \in \cdot | t < \tau_\partial \right).$$

Comme lors de l'étude des processus de Markov non-absorbés, nous nous intéressons à la stationnarité de ces distributions conditionnelles limites pour le processus conditionné, on parlera alors de *quasi-stationnarité.*

Définition 3. *Une **distribution quasi-stationnaire** pour X est une mesure de probabilité α sur $E \setminus \partial$ telle que, pour tout temps $t \geq 0$,*

$$\alpha(\cdot) = \mathbb{P}_\alpha \left(X_t \in \cdot | t < \tau_\partial \right).$$

L'étude de ces distributions limites a été initiée par le mathématicien russe éponyme Yaglom [86] en 1947 pour des processus de Galton-Watson sous-critiques. Par ses travaux, le chercheur a initié un champ de recherche fertile, sujet d'une intense investigation depuis un demi-siècle, comme l'illustre notamment la bibliographie sur les distributions quasi-stationnaires maintenue par Pollett [69].

Le chapitre 1 de cette thèse est consacré à l'étude de ces objets limites, nous y démontrons des propriétés générales et présentons des preuves d'existence et d'unicité de ces limites. Il s'agit d'un travail en collaboration avec Sylvie Méléard, destiné à être soumis en tant que *survey* sur les distributions quasi-stationnaires.

Nous étudions dans un premier temps la relation entre les trois objets définis ci-dessus. Deux implications sont immédiates au regard des définitions:

1. la limite de Yaglom pour X, si elle existe, est une distribution conditionelle limite

2. toute distribution quasi-stationnaire α pour X est une distribution conditionnelle limite. Elle vérifie en effet

$$\alpha(\cdot) = \lim_{t\to\infty} \mathbb{P}_\alpha\left(X_t \in \cdot | t < \tau_\partial\right).$$

Nous démontrons dans la partie 1 du chapitre 1 que toute distribution conditionnelle limite est également une distribution quasi-stationnaire, ces définitions étant par conséquent équivalentes. Nous donnerons également des exemples de modèles dont les distributions quasi-stationnaires sont multiples, tandis que la limite de Yaglom d'un processus est définie de manière univoque. Nous en déduisons en particulier qu'une distribution limite α conditionnelle n'est pas nécessairement une limite de Yaglom.

Nous démontrons également en toute généralité des propriétés souvent utilisées de manière *ad hoc* dans la littérature sur les distributions quasi-stationnaires. La première propriété concerne la distribution du temps d'absorption.

Proposition 1. *Soit α une distribution quasi-stationnaire associée à un processus X avec absorption presque sûre (c'est-à-dire tel que $\mathbb{P}_x(\tau_\partial < +\infty) = 1$). Alors*

1. *Il existe une constante $\theta(\alpha) > 0$ telle que*

$$\mathbb{P}_\alpha(t < \tau_\partial) = e^{-\theta(\alpha)t},\ \forall t > 0.$$

2. *De plus, pour tout $\gamma < \theta(\alpha)$ et α-presque tout x,*

$$\mathbb{E}_x\left(e^{\gamma\tau_\partial}\right) < +\infty.$$

La deuxième partie de la proposition fournit en particulier une condition nécessaire pour l'existence d'une distribution quasi-stationnaire. Ainsi un processus de Galton-Watson critique est absorbé en temps fini presque sûrement, mais son temps d'absorption vérifie $\mathbb{E}_x(\tau_\gamma) = +\infty$ pour tout $x \geq 1$. Nous en déduisons qu'un processus de Galton-Watson ne possède pas de distribution quasi-stationnaire.

Pour des modèles en temps continu, nous exposons sous une forme nouvelle la relation entre la propriété de quasi-stationnarité et la théorie spectrale du générateur infinitésimal L, résumée dans la proposition suivante.

Proposition 2. *Soit α une mesure de probabilité sur E. Supposons qu'il existe un ensemble $\mathcal{D}$ de fonctions à valeurs réelles bornées, tel que $\mathcal{D}$ est dense dans $L^1(E,\alpha)$ et tel que Lf existe et est borné pour tout $f \in \mathcal{D}$. Alors α est une distribution quasi-stationnaire si et seulement si il existe $\theta(\alpha) > 0$ tel que*

$$\alpha L = -\theta(\alpha)\alpha,$$

où αL est la mesure finie $L^*\alpha$ sur E definie par $(\alpha L)f = \alpha(Lf)$, pour toute fonction $f \in \mathcal{D}$. De plus, $\theta(\alpha)$ est la même constante qu'à la proposition 1.

L'existence de $\mathcal{D}$ est toujours vraie si l'espace E est discret, ou si E est un ouvert de $\mathbb{R}^d$ et Z une diffusion d'Itô à coefficients localement bornés.

Nous démontrons enfin que l'existence d'un distribution conditionnelle limite implique l'existence d'un plateau d'absorption dans une grande généralité. Ce type de résultats ont également été étudiés par Steinsaltz et Evans [78] dans un cadre différent. Nous renvoyons le lecteur à cet article et aux références qui s'y trouvent pour une étude avancée de la relation entre l'existence de plateaux de mortalité en biologie et l'existence théorique d'un plateau d'absorption, sous l'hypothèse d'existence d'une distribution conditionnelle limite.

Une grande part des travaux existants sur les distributions conditionnelles limites concerne les problèmes d'existence et d'unicité de ces limites. Dans le chapitre 1, nous rappelons un panel de preuves employées pour résoudre ces questions à partir de quelques exemples détaillés.

Dans le cas d'un processus de Galton-Watson sous-critique de loi de reproduction individuelle μ, nous reprenons la preuve d'existence et d'unicité proposée par Athreya et Ney [6], qui consiste à démontrer dans un premier temps que toute fonction génératrice $\hat{g}$ d'une fonction conditionnelle limite vérifie, en notant g la fonction génératrice de la loi de reproduction μ et $m < 1$ sa moyenne,

$$\hat{g}(g(s)) = m\hat{g}(s) + 1 - m,\ \forall s \in [0,1]. \tag{1}$$

La preuve est conclue par la démonstration de l'existence et de l'unicité de la solution à une telle équation. Les distributions quasi-stationnaires d'autres processus de branchements ont également été étudiées. Signalons notamment les travaux de Bansaye [7] pour des processus de branchement en environnement aléatoire, où une équation similaire à (1) est obtenue. Nous renvoyons également le lecteur à l'article de Klebaner, Sagitov, Vatutin Patsy et Jagers [52] et aux références qui y sont données pour l'étude de ces objets dans un cadre d'écologie évolutive, ainsi qu'aux travaux de Lambert [58].

Nous développons ensuite le cas des processus de Markov à espace d'états $E \cup \{\partial\}$ fini et en temps continu, étudiés initialement par Darroch et Seneta [22]. Dans ce cas, si le processus peut passer d'un point à un autre de E en temps fini avec une probabilité strictement positive, alors il existe une limite de Yaglom α pour le processus, de plus cette limite attire toute distribution initiale:

$$\alpha(\cdot) = \lim_{t\to\infty} \mathbb{P}_\mu\left(X_t \in \cdot | t < \tau_\partial\right),\ \forall \mu.$$

La preuve repose directement sur la caractérisation spectrale d'une distribution quasi-stationnaire et sur le théorème de Perron-Frobenius. Darroch et Seneta [21] ont également traité le cas de l'étude des chaînes de Markov en temps discret et à espace d'états fini, avec des conditions d'irréductibilité similaires. Les résultats de Van Doorn et Polett [82] généralisent ces résultats à des processus irréductibles.

L'existence d'une limite de Yaglom n'implique toutefois pas l'unicité de la distribution quasi-stationnaire. Le cas des processus de naissance et mort sur $\mathbb{N}$, développé par van Doorn [81] et présenté en détail dans le chapitre 1, est à ce titre très instructif. Certains de ces processus, comme la marche aléatoire simple tuée en 0, ne possèdent pas de distribution quasi-stationnaire. *A contrario* les processus de naissance et mort linéaires, dont les taux de naissance et mort sont donnés par $i\lambda$ et $i\mu$ respectivement, possèdent une limite de Yaglom et une infinité continue de distributions quasi-stationnaires si et seulement si $\lambda < \mu$. Les processus de naissance et mort logistiques, dont les taux sont ceux du processus linéaire auxquels sont ajoutés un terme de compétition quadratique, possèdent une limite de Yaglom qui est l'unique distribution quasi-stationnaire associée au processus.

Pour l'étude des distributions conditionnelles limites de processus de naissance et mort en temps discret, nous renvoyons au travaux de Seneta et Vere-Jones [76] et, pour une approche différente, à un article de Ferrari, Martinez et Picco [31]. Signalons également les travaux de Pakes et Pollett [67] pour les processus de naissance et mort avec catastrophes. Pour l'étude de processus en temps discret plus généraux, voir Coolen-Schrijner et van Doorn [20].

Il existe d'autres situations, qui ne sont pas développées ici, dans lesquelle le nombre de distributions quasi-stationnaires est infini. C'est notamment le cas de certaines diffusions étudiées par Martinez, Picco et San Martin [64] ou du processus d'Orstein-Uhlenbeck étudié par Lladser et San Martín [61], dont la distribution conditionnelle limite dépend de façon non-triviale de la distribution initiale.

Nous terminons le chapitre 1 par l'étude complète du cas des diffusions de Feller logistiques, où nous exposons notamment les arguments de Cattiaux, Collet, Lambert, Martínez, Méléard et San Martín [16]. Une diffusion de Feller logistique est une diffusion d'Itô sur $[0,+\infty[$, qui vérifie l'équation différentielle stochastique

$$dX_t = \sqrt{\gamma X_t} dB_t + (rX_t - cX_t^2),\ X_0 = x > 0. \tag{2}$$

où B est un mouvement brownien uni-dimensionnel et $\gamma > 0$, $r > 0$ et $c > 0$ sont trois constantes fixées (une telle diffusion est un modèle de taille de population, pour une population dont les individus sont en compétition, de très petite taille et dont les événements de naissance et mort se succèdent à grande vitesse, r reflétant une propriété de branchement et c un terme de compétition quadratique). En particulier, 0 est un point absorbant pour toute diffusion de Feller logistique. Nous démontrons que le processus X admet dans ce cas une limite de Yaglom, qui attire toute les distributions initiales et qui est par conséquent l'unique distribution quasi-stationnaire associée au processus. La preuve utilise des arguments fins de théorie spectrale du générateur infinitésimal associé à X et la caractérisation spectrale de l'ensemble des distributions quasi-stationnaires de la proposition 2. L'attraction de toutes les distributions initiales est obtenue grâce à une propriété de retour depuis l'infini vers les compacts, conséquence du terme quadratique cX_t^2. Cattiaux et Méléard [17] ont généralisé ces résultats et ces preuves à des diffusions multi-dimensionnelles tuées au bord d'un ouvert et dont les drifts sont également dégénérés.

Pour l'étude de distributions quasi-stationnaires de diffusions uni-dimensionnelles avec des conditions d'entrées aux bords différentes, nous renvoyons aux travaux de Steinsaltz et Evans [79] et de Kolb et Steinsaltz [54]. Des formules explicites ont également été démontrées dans des cas de diffusions à coefficients dégénérés par Huillet [46].

Les résultats exposés dans cette thèse sont loins d'être exhaustifs tant est fertile le champ d'application et d'étude de la notion de distribution quasi-stationnaire. Toutefois, nous ne pourrions conclure cette partie introductive sans faire référence aux travaux fondateurs de Mandl [63] pour les diffusions uni-dimensionnelles et Pinsky [68] pour des diffusions tuées au bord d'un ouvert borné et à leur généralisation par Gong, Qian et Zhao [37]. Signalons également l'approche originale et générale de Ferrari, Kesten, Martinez et Picco [29].

Avant de passer, dans la partie suivante, à l'étude de l'approximation des limites de Yaglom et des distributions conditionnelles limites telles que définies ci-dessus, signalons qu'il existe un autre type de conditionnement pour des processus de Markov avec absorption, appelé le Q-processus. Le Q-processus Y de X est défini par ses marginales de la façon suivante:

$$\mathbb{P}_x(Y_{t_1} \in \cdot, \cdots, Y_{t_n} \in \cdot) = \lim_{T\to\infty} \mathbb{P}_x(X_{t_1} \in \cdot, \cdots, X_{t_n} \in \cdot \,|\, T < \tau_\partial).$$

De manière intuitive, le Q-processus est le processus X conditionné à ne jamais être absorbé, défini de façon trajectorielle, tandis que nous nous intéressons dans cette thèse à la distribution de X conditionné à ne pas être absorbé quand on l'observe, sans préjuger de son comportement futur. Dans le chapitre 1, nous donnons pour illustration deux exemples de construction de ce Q-processus, pour les processus à espace d'états fini et pour les diffusions de Feller logistiques. Des études approfondies du Q-processus peuvent être trouvées dans [63], [68], [37], [21], [22] cités ci-dessus, nous renvoyons également aux travaux de Collet, Martínez et San Martin [19] et à l'article de Lambert [58], ainsi qu'aux références qui s'y trouvent.

Approximation des distributions conditionnelles et des distributions quasi-stationnaires

Comme l'a souligné Nåsell [66], les méthodes de caractérisation des distributions quasi-stationnaires, telles que leur définition ou la propriété de caractérisation spectrale ci-dessus, ne permettent pas, dans l'immense majorité des cas, d'obtenir des données quantitatives sur les distributions étudiées. Pour cette raison, être en mesure de proposer des schémas d'approximation de la distribution quasi-stationnaire est essentiel. Dans certaines situations, des méthodes *ad hoc* peuvent être développées, comme cela est le cas dans les travaux de Pollett et Stewart [70], de Hart et Pollett [43]. Dans cette thèse, nous nous intéressons à une méthode d'approximation particulaire introduite par Burdzy, Holyst, Ingerman et March [13] en 1996. Nous démontrons que cette méthode, introduite pour l'étude des limites de Yaglom du mouvement Brownien, peut être généralisée à une grande classe de processus.

De plus nous démontrons que cette méthode permet l'approximation, en toute généralité, de la distribution conditionnelle d'un processus en temps fini. Ceci est d'une grande importance. En effet, nous savons que l'existence d'une distribution conditionnelle limite implique l'existence d'un plateau d'absorption et, d'une manière générale, la stabilisation après un certain temps t_0 de la distribution des trajectoires non-absorbées. Mais en pratique, si la probabilité de non-absorption est trop faible au temps t_0, le nombre de trajectoires non-absorbées à observer à ce moment est trop faible pour rendre compte statistiquement de la convergence de la distribution conditionnelle. Dans l'étude des plateaux de mortalité chez l'être humain, Li et Vaupel [59] ont montré, en s'appuyant sur la base de données *Human Mortality Database* [45], que le taux de mortalité humain tend à se stabiliser aux alentours de 105 ans. Toutefois, les statistiques de l'INSEE du graphique 0.1 ne font pas apparaître de plateau de mortalité car il y a trop peu d'individus français mâles d'âge supérieur à 105 ans. Comme nous l'expliquent Li et Vaupel, si l'existence d'un plateau de mortalité chez l'être humain n'a pu être mis en évidence de façon claire que très tardivement, c'est parce que nous manquons de données sur les centenaires. Ainsi nous apercevons nous avec Carey, Liedo, Orozco et Vaupel que, même s'il existe un plateau de mortalité à des âges avancés, il n'a qu'un impact limité sur les secteurs d'activité que ces données intéressent, tels l'actuariat ou la santé publique. Afin de déterminer l'intérêt pratique de l'existence d'une distribution conditionnelle limite, il nous faut donc comparer la vitesse de convergence vers la distribution conditionnelle limite et la vitesse d'absorption du processus. Cette comparaison nécessite d'une part l'approximation de la distribution quasi-stationnaire, d'autre part de la distribution conditionnelle du processus en temps fini, ce que permet la méthode étudiée dans cette thèse.

Avant de développer plus en avant notre méthode, remarquons que, dans des cas suffisamment simples, il existe des outils efficaces pour résoudre numériquement l'équation spectrale présentée ci-dessus. Si, par exemple, l'espace d'état E est fini, alors la résolution numérique du spectre est possible et les problèmes soulevés précédemment peuvent donc être résolus numériquement. En effet, dans le cas où E est fini, l'utilisation du logiciel `scilab` et de sa fonction `spec(`L`)`, nous permet d'obtenir tous les vecteurs propres et toutes les valeurs propres de l'opérateur L (qui est dans ce cas une matrice finie). De plus, le même logiciel permet de calculer numériquement la distribution conditionnelle $\mathbb{P}_x(X_t \in .|t < \tau_\partial)$ en calculant $\exp(tL)/\exp(tL)\mathbf{1}_E$. Il est alors simple d'étudier le système dans sa totalité (vitesse d'extinction, vitesse de convergence vers la limite de Yaglom et valeur du plateau de mortalité par exemple). Des exemples de calculs numériques utilisant cette méthode sont présentés dans le chapitre 1 pour des processus de Markov à espace d'état finis.

Pour approximer directement la distribution conditionnelle $\mathbb{P}_x(X_t \in .|t < \tau_\partial)$, sans passer par les méthodes spectrales décrites ci-dessus, une méthode bien connue est la méthode dite de Monte-Carlo, qui utilise comme principe la loi des grands nombres. Selon cette loi, la moyenne des résultats d'un grand nombre de tirages d'une variable aléatoire approche l'espérance de cette variable aléatoire. Ainsi, en lançant un grand nombre $N \gg 1$ de simulations indépendantes X^i, $i \in \{1,...,N\}$, d'un processus de Markov absorbé X de position initiale $x \in E$, nous obtenons, pour tout ensemble mesurable $A \subset E$ et tout temps $t \geq 0$,

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{X_t^i\in A} \xrightarrow[N\to\infty]{} \mathbb{E}\left(\mathbf{1}_{X_t\in A}\right) = \mathbb{P}_x\left(X_t\in A\right) = \mathbb{P}_x\left(X_t\in A|t<\tau_\partial\right)\mathbb{P}_x\left(t<\tau_\partial\right),$$

De plus, d'après le théorème de la limite centrale, l'erreur commise entre les deux quantités est d'ordre $1/\sqrt{N}$:

$$\sqrt{N}\left|\frac{1}{N\mathbb{P}_x\left(t<\tau_\partial\right)}\sum_{i=1}^{N}\mathbf{1}_{X_t^i\in A} - \mathbb{P}_x\left(X_t\in A|t<\tau_\partial\right)\right| \xrightarrow[N\to\infty]{} \frac{1}{\mathbb{P}_x\left(t<\tau_\partial\right)}.$$

Si la probabilité de non-absorption $\mathbb{P}_x\left(t<\tau_\partial\right)$ est proche de 1, alors cette méthode est bonne. Cependant, pour tous les modèles que nous étudierons, la probabilité

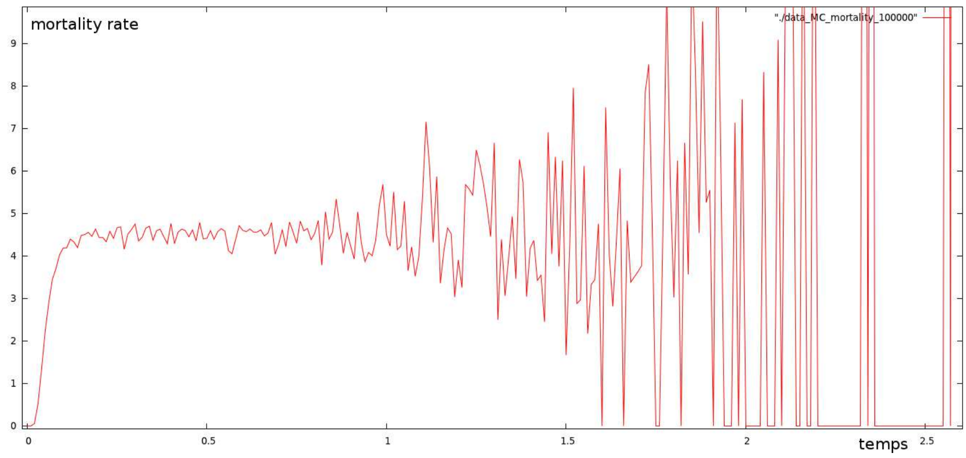


Figure 0.4: Évolution du taux de mortalité d'un mouvement brownien absorbé en 0 et en 1, calculé à l'aide d'une méthode de Monte-Carlo de 100000 simulations.

de non-absorption au temps $t \geq 0$ tend vers 0 quand le temps t tend vers l'infini. Cette propriété reflète le fait naturel qu'un individu finira par mourir, qu'une population finira inexorablement par disparaître et qu'un joueur de casino, ou le casino lui-même, finira par être ruiné en temps fini. Ainsi pour des temps trop importants, l'erreur de la méthode de Monte-Carlo est telle qu'elle nécessite un nombre de simulations d'autant plus grand que l'on s'intéresse à des temps lointains.

Sur la figure 0.4, nous représentons l'évolution du taux de mortalité d'un mouvement brownien uni-dimensionnel absorbé en 0 et en 1, et de position initiale 0.5. Nous utilisons pour cela la méthode de Monte-Carlo avec 100000 simulations, et observons de manière évidente la dégradation de la précision de cette méthode au cours du temps.

Notre méthode d'approximation consiste à simuler des trajectoires dans un premier temps indépendantes, mais, au lieu d'abandonner les trajectoires absorbées, nous les récupérons au moment de leur absorption et les remettons en jeu. La nouvelle position d'une simulation qui vient d'être absorbée est naturellement choisie aléatoirement et uniformément parmis les trajectoires qui n'ont pas été absorbées. Ainsi, l'algorithme est basé sur la simulation d'un système de particules en interaction dont le nombre de particules utiles reste constant au cours du temps. Ferrari et Marić ont appliqué cette méthode avec succès pour l'approximation de chaînes de Markov à espace d'états dénombrable vérifiant certaines conditions de mélange. Elle a également été démontrée par Grigorescu et Kang [40], dont les travaux ont inspiré certaines preuves de cette thèse.

Soit $N \geq 2$ le nombre de particules dont sera constitué notre système. Nous

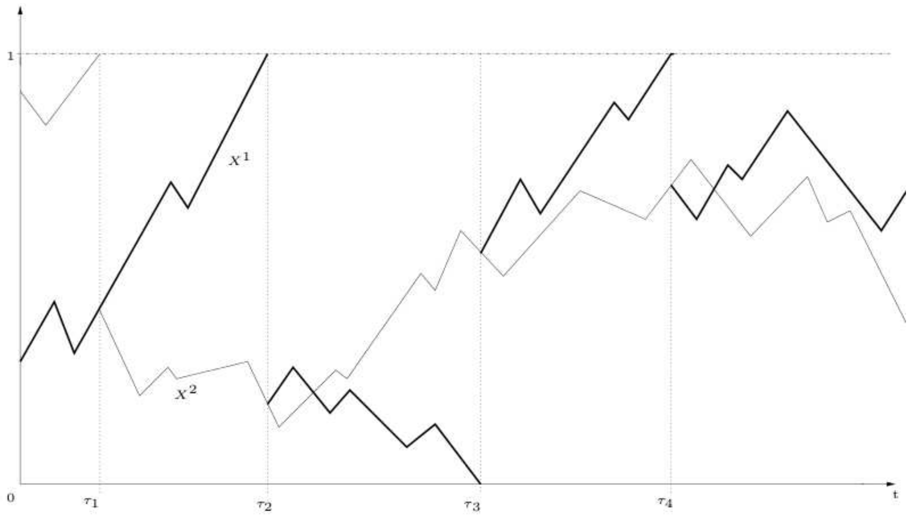


Figure 0.5: Processus de type Fleming-Viot à 2 particules qui évoluent entre leurs sauts comme des mouvements browniens absorbés en 0 et 1.

notons X^i la i-ème particule de ce système. Chaque particule est à valeurs dans l'ensemble E et le système entier $(X^1,...,X^N)$ évolue donc dans E^N. Au temps $t = 0$, chaque particule se trouve à un point $x \in E$ fixé, puis:

- Les particules évoluent indépendamment les unes des autres jusqu'à ce que l'une d'entre elles soit absorbée. Ce temps de première absorption est noté τ_1.
- La particule absorbée (nous supposons qu'elle est unique), est alors envoyée instantanément à la position d'une des $N - 1$ particules non-absorbées. A la fin de cette opération, chacune des particules se trouve dans E.
- Ensuite, les particules évoluent indépendamment les unes des autres jusqu'à ce que l'une d'entre elles soit absorbée. Ce temps de seconde absorption est noté τ_2.
- Comme précédemment, la particule absorbée (supposée unique) est envoyée instantanément à la position d'une des $N - 1$ particules non-absorbées.
- Ensuite, les particules évoluent indépendamment jusqu'au troisième temps d'absorption et ainsi de suite.

Pour tout temps $t \geq 0$, nous noterons A_t^N le nombre (aléatoire) d'absorptions du système de particules avant le temps t. La figure 0.5 est une illustration d'un tel système avec deux particules évoluant entre leurs sauts comme un mouvement brownien tué en 0 et en 1.

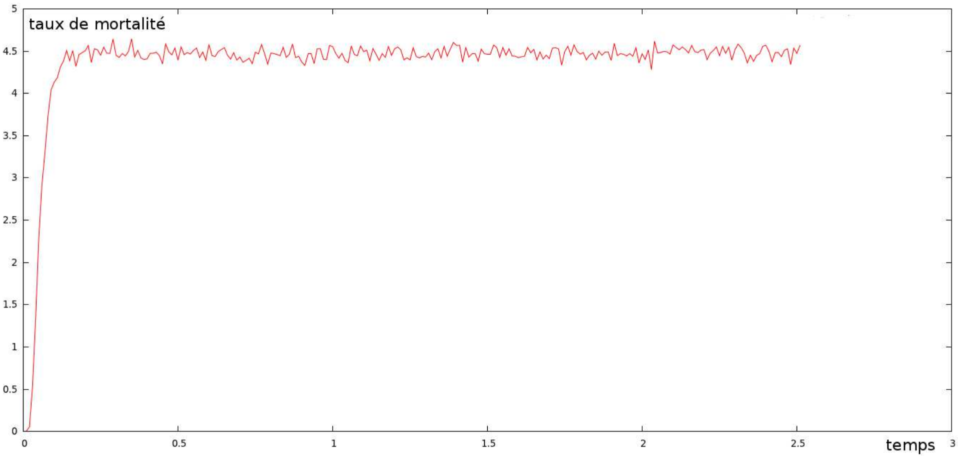


Figure 0.6: Évolution du taux de mortalité d'un mouvement brownien absorbé en 0 et en 1, calculé à l'aide de la méthode particulaire étudiée dans cette thèse, avec un système de 100000 particules.

Un des résultats principaux de cette thèse est le suivant. Il généralise notamment les résultats obtenus par Grigorescu et Kang [40].

Théorème 1 (Théorème 2.1 Chapitre 3)**.**
Soit $t \geq 0$. Si les deux conditions suivantes sont vérifiées

1. *à chaque temps d'absorption, la particule absorbée est unique,*

2. *la variable aléatoire A_t^N est finie presque sûrement pour tout $N \geq 2$,*

alors, pour tout ensemble mesurable $A \subset D$,

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{X_t^i \in A} \xrightarrow[N\to\infty]{loi} \mathbb{P}_x\left(X_t \in A | t < \tau_\partial\right).$$

Contrairement au cas de la méthode de Monte-Carlo, le nombre de particules non-absorbées ne diminue pas au cours du temps. Ainsi, en utilisant cet algorithme pour calculer l'évolution du taux de mortalité d'un mouvement brownien tué en 0 et en 1 au cours du temps, nous obtenons le graphique de la figure 0.6. On observe en particulier que la qualité de la méthode ne se dégrade pas au cours du temps.

Toutefois, les conditions (1) et (2) du théorème de convergence ci-dessus font chacune apparaître une nouvelle difficulté.

La première difficulté concerne l'hypothèse selon laquelle deux particules ne peuvent être absorbées en même temps. Si le processus X évolue en temps continu suivant une équation différentielle stochastique ou suivant un processus de saut pur et à taux de saut borné, alors cette hypothèse est en général vérifiée et ne représente pas de réelle difficulté. Toutefois, si X est donné par un modèle en temps discret, alors la probabilité que deux particules soient absorbées en même temps est *a priori* strictement positive à chaque temps $t \in \mathbb{N}$. Cette difficulté réelle peut être résolue par la construction d'un processus Y en temps continu tel que, en chaque temps entier $t \in \mathbb{N}$, la loi de Y est la même que celle de X. Ce nouveau processus autorise alors la simulation d'un système de particules en interaction $(Y^1,\ldots,Y^N)$ et la généralité du théorème de convergence énoncé ci-dessus permet d'établir que, pour tout temps entier $t \in \mathbb{N}$ et tout ensemble mesurable $A \subset D$,

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{Y_t^i \in A} \xrightarrow[N\to\infty]{} \mathbb{P}_x\left(X_t \in A | t < \tau_\partial\right).$$

La seconde difficulté concerne l'hypothèse selon laquelle le nombre d'absorptions est fini en temps fini presque sûrement, c'est-à-dire $P\left(A_t^N < +\infty\right) = 1$, pour tout $N \geq 2$. Cette propriété est toujours vérifiée pour des processus à temps continu qui ne peuvent être absorbés qu'en des temps aléatoires de loi exponentielle à taux borné, tels que les processus de Galton-Watson en temps continu ou les processus de Markov à espace d'états fini.

En revanche, cette condition n'est pas toujours vérifiée pour des modèles en temps continu et à espace d'états continu tels que les diffusions d'Itô. Voici un contre-exemple simple, pour lequel l'hypothèse de non-explosion du nombre d'absorption n'est pas vérifiée. Supposons que le processus X est un processus de position initiale $(0,0)$ qui se déplace dans $E = \mathbb{R}^2$ avec une vitesse déterministe dans la direction des ordonnées et selon un mouvement Brownien dans la direction des abscisses. Supposons de plus que le processus est absorbé lorsqu'il atteint l'ensemble $\{(x,y) \in \mathbb{R}^2, \text{ tel que } |x| + |y| \geq 2\}$. Dans ce cas, le système à N particules $(X^1,\ldots,X^N)$ effectuera un nombre infini de sauts avant que l'ordonnée (déterministe) du processus n'atteigne 1, ce qui arrivera inexorablement en temps fini. La trajectoire typique d'une particule pour ce système est représentée figure 0.7. Pour de tels processus, la méthode d'approximation présentée dans cette thèse n'est pas valable. Il importe donc de trouver des hypothèses suffisantes pour la condition $P\left(A_t^N < +\infty\right) = 1$.

Dans le cas du mouvement brownien tué au bord d'un ouvert, le problème a été étudié par Burdzy, Hołyst et March [14], par Bieniek, Burdzy et Finch [11], par Löbus [62]. Le second article est le plus général et concerne le mouvement brownien tué au bord d'un ouvert dont la frontière est de régularité Lipschitz, tandis que le premier et le troisième sont restreints à des bords plus réguliers (de plus les auteurs du premier article ont signalé que la preuve proposée est fondamentalement incomplète). Grigorescu et Kang [42] démontrent que la condition de non-explosion $\mathbb{P}_x(A_t^N < +\infty) = 1$ est vérifiée pour toute diffusion X à

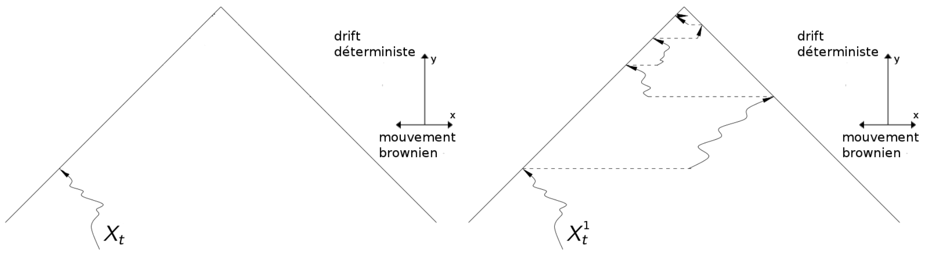


Figure 0.7: Si X est le mouvement avec absorption décrit à gauche, l'explosion du nombre d'absorption en temps fini est presque sûre pour le système de particules, dont une particule et sa trajectoire sont représentées à gauche.

coefficients homogènes en temps, uniformément elliptiques et de classe C^∞, et absorbée au bord d'un ouvert au bord suffisamment régulier. Dans cette thèse, nous donnons un critère de non-explosion pour des mouvements browniens avec drift éventuellement non-bornés et irréguliers (chapitre 2, qui est un article accepté pour publication à *Electronic Journal of Probability*) et pour des diffusions avec sauts évoluant dans un environnement aléatoire avec des coefficients irréguliers et dépendant du temps (chapitre 3, qui est un article récemment soumis à *ESAIM Probability and Statistics*).

Plus précisément, la situation du chapitre 2 est la suivante. Nous étudions un système de particules en interaction dont les particules sautent quand elles atteignent le bord d'un ouvert (comme précédemment), à la différence que la destination de leur saut peut être choisie de façon beaucoup plus souple (ce qui généralise notamment les résultats obtenus par Grigorescu et Kang [42]). Les particules évoluent suivant un mouvement brownien drifté, c'est-à-dire qu'elle sont solution jusqu'à l'atteinte du bord de l'équation différentielle stochastique

$$dX_t = dB_t + q(X_t)dt,\ X_0 = x \in D,$$

où B est un mouvement brownien d-dimensionnel et $q : D \mapsto \mathbb{R}^d$ est une fonction mesurable bornée dans un voisinage de ∂D. Nous supposons que la frontière ∂D est de classe C^2, de telle sorte que la distance d'une particule à la frontière est un mouvement brownien drifté, au moins quand la particule s'approche de la frontière (ceci est une conséquence immédiate du théorème de régularité présenté par Delfour [25, Chapter 5, Section 4] et de la formule d'Itô).

La preuve s'effectue en deux temps. Nous montrons d'abord que l'explosion du nombre de sauts du processus en temps fini implique qu'au moins deux particules convergent vers la frontière, c'est-à-dire que leurs distances à la frontière tendent vers 0 simultanément. Puis nous construisons un couplage entre les distances à la frontière des particules et un système de mouvements browniens à drifts bornés, indépendants et réfléchis en 0, noté $(Y^1,\ldots,Y^N)$ et tel que, pour tout temps t

inférieur au temps d'explosion du nombre de sauts,

$$d(X_t^i,\partial D) \geq Y_t^i \text{ (voir figure 0.8).}$$

Ainsi, l'explosion du nombre de saut implique qu'au moins deux particules ten-

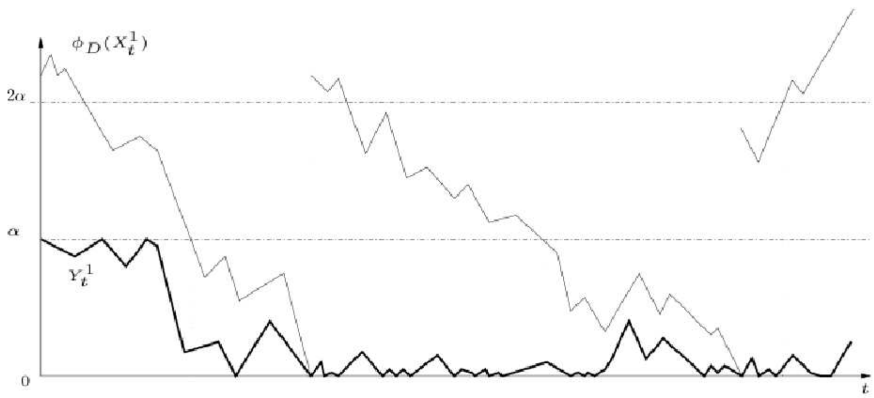


Figure 0.8: La particule X^1 et son mouvement brownien réfléchi Y^1 couplé.

dent vers 0 simultanément, donc que deux mouvement browniens réfléchis indépendant convergent vers 0 simultanément, ce qui arrive avec probabilité nulle et nous permet de conclure la preuve.

Dans le Chapitre 3, nous nous intéressons à des diffusions plus générales, solutions d'équations différentiells stochastiques inhomogènes en temps et dépendant d'un environnement aléatoire. La preuve repose sur le même principe, à savoir que l'explosion du nombre de sauts implique la convergence simultanée vers la frontière de deux particules. Mais dans cette partie, nous montrons directement que leur distance à la frontière ne peut pas converger vers 0 simultanément en utilisant un résultat original de non-atteinte de (0,0) pour des couples de semi-martingales positives. Ce résultat de non-atteinte est prouvé par la définition d'une bonne fonction *puis*, inspirée par les travaux de Delarue [24].

Revenons à présent à l'étude de la limite de Yaglom. Nous avons vu que le système de particules de type Fleming-Viot ci-dessus permet l'approximation de la distribution conditionnelle $\mathbb{P}_x(X_t \in A | t < \tau_\partial)$, pour des temps t finis. Si la limite de Yaglom existe, alors elle est donnée par la limite quand t tend vers l'infini de cette distribution conditionnelle. Or, une fois $N \gg 1$ fixé, nous pouvons simuler pendant un temps long le système de type Fleming-Viot à N particules $(X^1,\ldots,X^N)$. Par conséquent, une approximation de la limite de Yaglom peut être obtenue si le diagramme suivant peut être inversé:

$$\begin{array}{ccc} \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i} & \xrightarrow[N\to+\infty]{} & \mathbb{P}_x\left(X_t\in\cdot|t<\tau_\partial\right) \\ \downarrow_{t\to+\infty} & & \downarrow_{t\to+\infty} \\ ? & \xrightarrow[N\to+\infty]{} & \text{limite de Yaglom} \end{array} \tag{3}$$

Dans le chapitre 2, nous considérons une diffusion d'Itô X évoluant dans un ouvert borné $D\subset\mathbb{R}^d$, $d\geq 1$, jusqu'à son absorption en ∂D. Le processus X est défini jusqu'à son absorption par l'équation différentielle stochastique

$$dX_t = dB_t + q(X_t)dt,\ X_0 = x\in D,$$

où B est un mouvement brownien d-dimensionnel et $q : D\mapsto\mathbb{R}^d$ est une fonction C^1 bornée. Le processus X possède une limite de Yaglom d'après les résultats de Gong, Qian et Zhao [37]. D'après les résultats de non-explosion ci-dessus, le système de particule de la méthode d'approximation effectue un nombre de sauts fini en temps fini presque sûrement. De plus nous montrons que, pour tout $N\geq 2$, le processus $(X_t^1,\ldots,X_t^N)_{t\geq 0}$ à valeurs dans D^N est exponentiellement ergodique, c'est-à-dire qu'il existe une mesure de probabilité M^N sur D^N et deux constantes $C_N,\gamma_N>0$ telles que, pour tout ensemble mesurable $A\subset D^N$,

$$\left|P\left((X_t^1,\ldots,X_t^N)\in A\right) - M^N(A)\right| \leq C_N e^{\gamma_N t}.$$

La preuve de cette propriété, inspirée par [14] utilise à nouveau le couplage construit ci-dessus et les résultats d'ergodicité de Down, Meyn et Tweedie [26].

En particulier, cela implique qu'il existe une mesure de probabilité aléatoire $\mathcal{X}^N$ sur D telle que, pour tout ensemble mesurable $A\subset D$,

$$\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{X_t^i\in A} \xrightarrow[t\to\infty]{loi} \mathcal{X}^N(A).$$

Nous concluons en démontrant que

$$\mathcal{X}^N(A) \xrightarrow[N\to\infty]{loi} \alpha,$$

où α est la limite de Yaglom associée à X. En définitive, le diagramme (3) peut être inversé, ce qui répond au problème d'approximation de la limite de Yaglom pour le cas présenté ci-dessus.

Beaucoup de modèles, en dynamique des populations notamment, sont basés sur des processus à taux de saut non-bornés ou sur des diffusions à coefficients non-bornés. Dans ce cas, les critères assurant $P\left(A_t^N<+\infty\right)=1$ font défaut. Toutefois, nous présentons dans le chapitre 2 une méthode permettant de pallier cette difficulté. La méthode employée consiste à approximer le processus original X par des processus à coefficients bornés $(X^{(m)})_{m\geq 1}$, où m est un paramètre tel que, pour tout $t \geq 0$,

$$X_t^{(m)} \xrightarrow[m\to+\infty]{loi} X_t.$$

Supposons par exemple que X est une diffusion de Feller logistique sur $[0,+\infty[$, c'est-à-dire qu'il existe $\gamma > 0$, $r > 0$ et $c > 0$ tels que X est absorbé en 0 et vérifie l'équation différentielle stochastique (2). D'après les résultats de Cattiaux, Collet, Lambert, Martínez, Méléard et San Martín [16], X possède une limite de Yaglom α. Cependant, les coefficients de l'équation différentielle stochastique (2) ne sont pas elliptiques et savoir si $P\left(A_t^N < +\infty\right) = 1$ est à ce jour un problème ouvert. Afin de pallier cette difficulté, nous définissons le processus $X^{(m)}$ sur $[1/m,m]$ comme solution de l'équation différentielle stochastique (2) et absorbé en $\{1/m,m\}$. Ce processus possède une limite de Yaglom $\alpha^{(m)}$ d'après Pinsky [68]. À un changement de variable près, $X^{(m)}$ est un mouvement brownien absorbé au bord d'un ouvert borné, $\alpha^{(m)}$ peut donc être approchée par la méthode particulaire décrite ci-dessus. Nous concluons en prouvant que, pour tout ensemble mesurable $A \subset]0,+\infty[$,

$$\lim_{m\to+\infty} \alpha^{(m)}(A) = \alpha(A).$$

Ainsi, l'approximation de $\alpha^{(m)}$ pour $m \gg 1$ fournit une approximation de la limite de Yaglom α recherchée. Dans le cas d'une diffusion de Feller logistique, nous obtenons les limites de Yaglom représentées sur la figure 0.9 pour différentes valeurs des paramètres r et c.

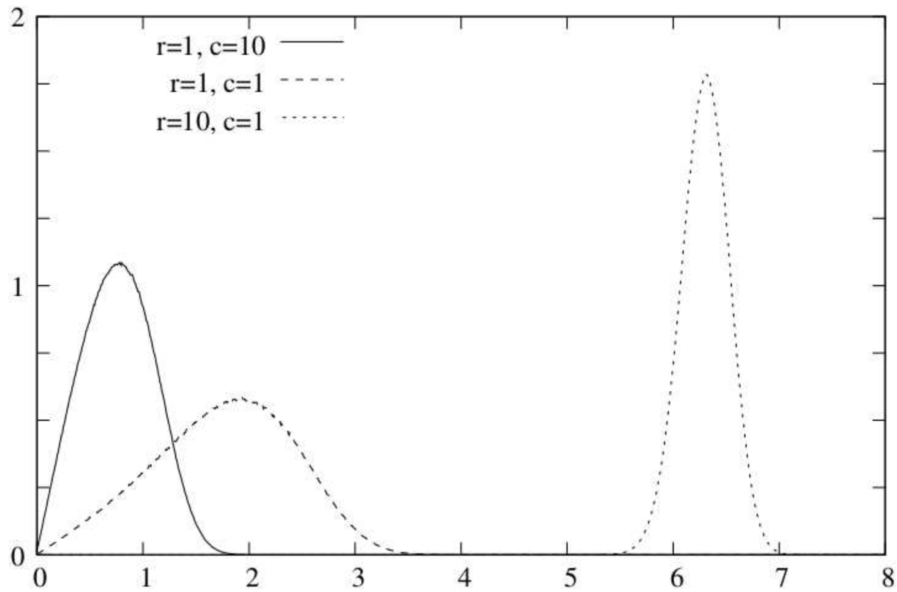


Figure 0.9: Limite de Yaglom pour une diffusion de Feller logistique, avec différentes valeurs de r et c.

Comme nous le voyons, la méthode d'approximation proposée par Burdzy, Holyst, Ingerman et March [13] peut être appliquée dans un cadre qui dépasse largement celui du mouvement Brownien. Ainsi les problèmes posés par l'étude de processus avec absorption peuvent-il être étudiés dans une grande généralité à l'aide de simulations numériques du système de particules de type Fleming-Viot décrit ci-dessus.

Propriétés de mélange pour des modèles inhomogènes en temps

Soit X un processus de Markov inhomogène en temps évoluant dans un espace $E \cup \{\partial\}$ et absorbé en ∂, *i.e.* tel que $(t,X_t)_{t\geq 0}$ est un processus de Markov sur $[0,+\infty[\times E$ absorbé quand il atteint $[0,+\infty[\times\{\partial\}$.

Dans la situation où le processus X est un processus absorbé par des obstacles mous, c'est-à-dire s'il existe une fonction mesurable positive κ uniformément bornée sur $[0,+\infty[\times E$ telle que le temps d'absorption de X a une loi exponentielle de paramètre $\kappa(t,X_t)$, Del Moral et Miclô [23] et Rousset [75] ont montré que la méthode d'approximation particulaire présentée ci-dessus converge vers la distribution conditionelle du processus X.

Le théorème de convergence 1 et le résultat de non-explosion du chapitre 3 nous permettent de généraliser leurs résultats à des diffusions d'Itô inhomogènes en temps et dépendant d'un environnement aléatoire, absorbées au bord d'un ouvert borné de classe C^2.

Un processus dont l'évolution dépend effectivement du temps ne peut pas posséder de distribution quasi-stationnaire. Toutefois, des propriétés de mélange peuvent être établies, similaires aux propriétés de mélange établies pour des processus non-conditionnés, voir par exemple Arnaudon, Coulibaly et Thalmaier [5]. Dans le chapitre 5, fruit d'une collaboration avec Pierre Del Moral, nous étudions la distribution conditionnelle de processus X évoluant dans un ouvert borné $D \subset \mathbb{R}^d$, $d \geq 2$, définis par une équation différentielle stochastique inhomogène à coefficients périodiques en temps. C'est-à-dire qu'il existe deux fonctions mesurables périodiques en temps

$$\sigma : [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \text{ et } b : [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}^d$$

telles que X vérifie l'équation différentielle stochastique

$$dX_t = \sigma(t,X_t)dB_t + b(t,X_t)dt,$$

où B est un mouvement brownien d-dimensionnel. Le processus est absorbé lorsqu'il atteint le bord de D et par des obstacles mous, définis par une fonction mesurable κ uniformément bornée. Nous démontrons un critère suffisant pour la propriété de mélange forte suivante: il existe deux constantes $C > 0$ et $\gamma > 0$ telles que, pour tout $x,y \in D$,

$$\|\mathbb{P}_x(X_t \in \cdot | t < \tau_\partial) - \mathbb{P}_y(X_t \in \cdot | t < \tau_\partial)\|_{TV} \leq Ce^{-t}. \tag{4}$$

La preuve de cette propriété repose sur trois outils importants: un résultat de tension pour la mesure empirique des systèmes de particules avec sauts depuis la frontière de D et démontré au chapitre 4, la méthode d'approximation décrite ci-dessus et un couplage de diffusions multi-dimensionnelles démontré par Priola et Wang [71].

Notre critère autorise en particulier les coefficients σ et b à ne pas être de classe C^1. Dans le cas homogène, ce résultat fournit donc une généralisation des résultats d'existence et d'unicité de la limite de Yaglom démontrés par Pinsky [68], par Gong, Qian et Zhao [37] et par Knobloch et Partzsch [53]. Dans le cadre inhomogène en temps, nous généralisons, en autorisant les obstacles durs, les résultats obtenus par Del Moral et Miclô [23].

De manière intéressante, la démonstration de la propriété de mélange forte utilise la méthode d'approximation présentée ci-dessus. En retour, la propriété de mélange (4) nous permet de démontrer que la méthode d'approximation converge uniformément en temps quand le nombre de particules tend vers $+\infty$. Ceci généralise les travaux de Rousset [75] à des diffusions absorbées au bord d'un ouvert borné.

Programmation

Une part importante de cette thèse a été consacrée à la réalisation de programmes en `c++`, dont les simulations présentées dans ce manuscrit sont issues. La rapidité du langage compilé ainsi que la fine gestion de la mémoire qu'il autorise ont été un outil idéal pour écrire des programmes permettant de simuler aisément des systèmes avec plusieurs centaines de milliers (voire plusieurs millions) de particules dans des temps très raisonnables. L'algorithme utilisé pour simuler le système de particule est décrit dans le chapitre 6 avec des explications sur le code et certaines fonctions utiles. Enfin, le code source est reproduit dans l'annexe de ce dernier chapitre.

Chapter 1

Quasi-stationary distributions and populations size models *.

This chapter is intended to be a survey on quasi-stationary distributions and most of the results in this chapter have been already proved, but not necessarily at the same level of generality. While the survey is by no mean exhaustive, it allows us to present many of the technics usually used to prove the existence and the uniqueness of quasi-stationary distributions.

1.1 Introduction

We are interested in the long time behavior of isolated biological populations with a regulated (density-dependent) reproduction. Competition for limited resources impedes these natural populations without immigration to grow indefinitely and leads them to become extinct. When the population's size attains zero, nothing happens anymore and this population's size process stays at zero. This point 0 is thus an absorbing point for the process. Nevertheless, the time of extinction can be large compared to the individual time scale and it is common that population sizes fluctuate for large amount of time before extinction actually occurs. For example, it has been observed that in populations of endangered species, as for the Arizona ridge-nose rattlesnakes studied in Renault-Ferrière-Porter [73], the statistics of some biological traits seem to stabilize. An other stabilization phenomenon is given by the *mortality plateau*. While demographers thought for a long time that rate of mortality of individuals grows as an exponential function of the age, it has been observed that the rate of mortality slows at advanced ages, or even stabilizes. To capture these phenomenons, we will study the long

*In collaboration with Sylvie Méléard

time behavior of the process conditioned to non extinction and the related notion of **quasi-stationarity**. In particular, we will see that a Markov process with extinction which possess a quasi-stationary distribution has a mortality plateau.

In all the following we will assume that the population's size process $(Z_t, t \geq 0)$ is a Markov process which almost surely goes to extinction. We are interested in looking for characteristics of the process that give more detailed information than the fact that absorption is certain. One way to approach this problem is to study the "quasi-limiting distribution" (QLD) of the process (if it exists), that is the limit, as $t \to +\infty$, of the distribution of Z_t conditioned on non-absorption up to time t. This distribution, which is also called the Yaglom limit, provides particularly useful information if the time scale of absorption is substantially larger than the one of the quasi-limiting distribution. In that case, the process relaxes to the quasi-limiting regime after a relatively short time, and then, after a much longer period, absorption will eventually occur. Thus the quasi-limiting distribution bridges the gap between the known behavior (extinction) and the unknown time-dependent behavior of the process.

There is another point of view concerning quasi-stationarity. A quasi-stationary distribution for the process $(Z_t, t \geq 0)$ denotes any proper initial distribution on the non-absorbing states which is such that the distribution of Z_t conditioned on non-extinction up to time t is independent of $t, t \geq 0$. If the distribution of Z_0 is chosen to be any QSD, then the corresponding QLD exists and equals this QSD. Hence, any quasi-stationary distribution is a quasi-stationary limit, but the converse is not always true.

In Section 1.2 of this course, we will introduce the different notions of QSD and state some general properties. In Section 1.3, we will study the simple case of QSD for processes in continuous time with finite state space. Thus we will concentrate on QSD for several stochastic population models corresponding to different scalings. We will underline the importance of spectral theory as mathematical tool for the research of QSD, in these different contexts. In Section 1.4, we will consider birth and death processes. We will state the results established by Van Doorn [81], giving explicit conditions on the coefficients ensuring the almost sure extinction of the process, and the existence and uniqueness (or not) of a QSD. We will especially focus on the density-dependence case, when the death rate of each individual is proportional to the population's size (called logistic birth and death process). We will show that in that case, the process goes almost surely to extinction, and that there is a unique QSD, coinciding with the unique QLD.

If one assumes that the total amount of resources is fixed and that the initial population's size is large, then the biomass of each individual is small, and we are led to renormalize the birth and death process. In Section 1.5, we show that as the initial population's size tends to infinity, the rescaled birth and death process converges to the unique solution of the famous deterministic logistic equation. In that case the solution converges as time tends to infinity to a nontrivial limit called carrying capacity. This model describes stable large populations whose size stays essentially constant. Another asymptotic consists in assuming that the birth and death rates are proportional to the population's size in a way such that the growth rate does not explode. Hence, as the initial population size increases, the rescaled logistic birth and death process is close to the solution of a stochastic differential equation, called logistic Feller equation. The randomness due to the accumulation of the many birth and death events behaves as a Brownian term with 1/2-Hölder diffusion coefficient (as in the Feller equation) and the density dependence appears through a quadratic drift term. We prove that the unique solution of the logistic Feller equation goes almost surely to zero, and that there is also a unique QSD, equal to the Yaglom limit, obtained as eigenmeasure of the adjoint operator of the killed semi-group. The proof, which is based on spectral theory for this semi-group, is not constructive and cannot be quantitatively exploited. It is thus useful to construct an algorithmic method to simulate the QSD. At this end, we will give the main ideas of a work of Villemonais ([83]), describing a stochastic particle method based on Fleming-Viot systems.

The size $(Z_t, t \geq 0)$ of the population will be modeled by a Markov process taking values in a subset E of $\mathbb{N}$ or $\mathbb{R}_+$, in a discrete or continuous time setting. If the population is isolated, namely without immigration, then the state 0, which describes the extinction of the population, is a trap. Indeed, if there are no more individuals, no reproduction can occur and the population disappears. Thus if the system reaches 0, it stays there forever, that is, if $Z_t = 0$ for some t, then $Z_s = 0$ for any $s \geq t$.

We denote by T_0 the time of extinction, i.e. the stopping time

$$T_0 = \inf\{t > 0, Z_t = 0\}. \tag{1.1}$$

We will consider cases for which the process goes almost surely to zero, whatever the initial state is, namely, for all $z \in E$,

$$\mathbb{P}_z(T_0 < \infty) = 1. \tag{1.2}$$

Before extinction, the process takes its values in the space

$$E^* = E \backslash \{0\}.$$

Any long time distribution of the process conditioned on non-extinction will be supported by E^*.

Notations For any probability measure μ on E^*, we denote by $\mathbb{P}_\mu$ (resp. $\mathbb{E}_\mu$) the probability (resp. the expectation) associated with the process Z initially distributed with respect to μ. For any $x \in E^*$, we set $\mathbb{P}_x = \mathbb{P}_{\delta_x}$ and $\mathbb{E}_x = \mathbb{E}_{\delta_x}$.

We denote by $(P_t)_{t\geq 0}$ the semi-group of the process Z killed at 0. More precisely, for any $z > 0$ and f measurable and bounded on E^*, one defines

$$P_t f(z) = \mathbb{E}_z(f(Z_t)\mathbf{1}_{t<T_0}). \tag{1.3}$$

For any finite measure μ and any bounded measurable function f, we set

$$\mu(f) = \int_{E^*} f(x)\mu(dx).$$

We also define the finite measure μP_t by setting, for all bounded measurable function f,

$$\mu P_t(f) = \mu(P_t f) = \mathbb{E}_\mu(f(Z_t)\mathbf{1}_{t<T_0}).$$

1.2 Definitions, general properties and first examples

There are several natural questions associated with this situation.

Question 1 What is the distribution of the size of a non-extincted population at a large time t ? The mathematical quantity of interest is thus the conditional distribution of Z_t defined, for any Borel subset $A \subset E^*$, by

$$\mathbb{P}_\nu(Z_t \in A | T_0 > t) = \frac{\mathbb{P}_\nu(Z_t \in A; T_0 > t)}{\mathbb{P}_\nu(T_0 > t)} = \frac{\nu P_t(\mathbf{1}_A)}{\nu P_t(\mathbf{1}_{E^*})}, \tag{1.4}$$

where ν is the initial distribution of the population's size Z_0. We want to study the asymptotic behavior of this conditional probability when t tends to infinity. The first definition that we introduce concerns the existence of a limiting conditional distribution.

Definition 1.1. *Let α be a probability measure α on E^*. We say that it is a **quasi-limiting distribution** (QLD) for Z, if there exists a probability measure ν on E^* such that, for any measurable set $A \subset E^*$,*

$$\lim_{t\to\infty} \mathbb{P}_\nu\left(Z_t \in A | T_0 > t\right) = \alpha(A).$$

It is well known that in the ergodic situation, the long time distribution converges and doesn't depend on the initial state. This leads us to the following definition.

Definition 1.2. *We say that Z has a **Yaglom limit** is there exists a probability measure α on E^* such that, for any $x \in E^*$ and any measurable set $A \subset E^*$,*

$$\lim_{t\to\infty} \mathbb{P}_x\left(Z_t \in A | T_0 > t\right) = \alpha(A).$$

When it exists, the Yaglom limit is a QLD. The reverse isn't true in general.

Question 2 As in the ergodic case, we can ask if this Yaglom limit has the conditional stationarity property given by the following definition.

Definition 1.3. *Let α be a probability measure on E^*. We say that α is a* ***quasi-stationary distribution*** *(QSD) if, for all $t \geq 0$ and any measurable set $A \subset E^*$,*

$$\alpha(A) = \mathbb{P}_\alpha(Z_t \in A | T_0 > t).$$

The main questions are the existence and uniqueness of these QSD. We will study examples where a QSD does not exist, or where there is an infinity of QSD, or where there is a unique QSD. The relation between the existence of QSD, QLD and Yaglom limit is clarified in Proposition 1.1 below. Namely, we will prove that

$$\text{Yaglom limit } \Rightarrow \text{QSD } \Leftrightarrow \text{ QLD.}$$

Question 3 Since the processes that we are interested in become extinct in finite time almost surely, the event $t < T_0$ becomes a rare event when t becomes large. An important question is then to know whether the convergence to the Yaglom limit happens before the typical time of extinction, or if it happens only after very large time periods, in which case the populations whose size are distributed with respect to the Yaglom limit are very rare. Both situations can appear, as illustrated by the simple example of Section 1.2.3.

Question 4 While most of theoretical results on QLD, QSD and Yaglom limits are concerned with existence and uniqueness problems, it would be useful in practice to have qualitative information on the Yaglom limit. We present here particle approximation results and numerical computations of the Yaglom limit for some population's size models, providing some enlightenment on Question 3 above.

Question 5 Another mathematical quantity related to this conditioning is based on a pathwise point of view. In the finite state space case of Section 1.3 and the logistic Feller diffusion case of Section 1.5, we will describe the distribution of the trajectories who never attain the trap. This will allow us to define a process, commonly referred to as the Q process for Z. We will prove that the new process defined by this distribution is ergodic, and that its stationary distribution is absolutely continuous with respect to the QSD (but not equal).

The present section is organized as follows. In Subsection 1.2.1, we state general properties of QLD's, QSD's and Yaglom limits. In Subsection 1.2.2, we develop the case of the Galton-Watson process. This discrete time process is of historical importance, since the notion of Yaglom limit has originally been developed for this process by Yaglom itself (see [86]). In Subsection 1.2.3, we develop a very simple example of a process evolving in a finite subset of $\mathbb{N}$. For this process, one can easily prove the existence of the Yaglom limit, the uniqueness of the QSD, and compare the speed of extinction to the speed of convergence to the Yaglom limit. We also provide a numerical computation of the relevant quantities.

1.2.1 General properties

Most of the following results are already known by the community. In this section, we emphasize their generality.

QSD, QLD and Yaglom limit

It is clear that any Yaglom limit and any QSD is also a QLD. We prove here that any QLD is also a QSD. The reverse implication has been proved by Vere-Jones [1969] for continuous time Markov chains evolving in a countable state space. The following proposition extends this result to the general setting.

Proposition 1.1. *Let α be a probability measure on E^*. The distribution α is a QLD for Z if and only if it is a QSD for Z.*

Remark 1.1. When it exists, the Yaglom limit is uniquely defined, while there are processes with an infinity of QSD's (see the birth and death process case of Section 1.4). We immediately deduce that there exists QSD's which aren't a Yaglom limit.

Proof. (1) If α is a QSD then it is a QLD for Z starting with distribution α.

(2) Assume now that α is a QLD for Z and for an initial probability measure μ on E^*. Thus, for any measurable and bounded function f on E^*,

$$\begin{aligned}\alpha(f) &= \lim_{t\to\infty} \mathbb{E}_\mu(f(Z_t)|T_0 > t) \\ &= \lim_{t\to\infty} \frac{\mathbb{E}_\mu(f(Z_t); T_0 > t)}{\mathbb{P}_\mu(T_0 > t)}\end{aligned}$$

Applying the latter with $f(z) = \mathbb{P}_z(T_0 > s)$, we get by the Markov property

$$\mathbb{P}_\alpha(T_0 > s) = \lim_{t\to\infty} \frac{\mathbb{P}_\mu(T_0 > t+s)}{\mathbb{P}_\mu(T_0 > t)}.$$

Let us now consider $f(z) = \mathbb{P}_z(Z_s \in A, T_0 > s)$, with $A \subset E^*$. By the Markov property, we can show that

$$\begin{aligned}\mathbb{P}_\alpha(Z_s \in A; T_0 > s) &= \lim_{t\to\infty} \frac{\mathbb{P}_\mu(Z_{t+s} \in A; T_0 > t+s)}{\mathbb{P}_\mu(T_0 > t)} \\ &= \lim_{t\to\infty} \frac{\mathbb{P}_\mu(Z_{t+s} \in A; T_0 > t+s)}{\mathbb{P}_\mu(T_0 > t+s)} \frac{\mathbb{P}_\mu(T_0 > t+s)}{\mathbb{P}_\mu(T_0 > t)}.\end{aligned}$$

The term $\frac{\mathbb{P}_\mu(Z_{t+s} \in A; T_0 > t+s)}{\mathbb{P}_\mu(T_0 > t+s)}$ converges to $\alpha(A)$ by definition of the QLD α. The term $\frac{\mathbb{P}_\mu(T_0 > t+s)}{\mathbb{P}_\mu(T_0 > t)}$ converges to $\mathbb{P}_\alpha(T_0 > s)$ when t tends to infinity. We deduce that, for any Borel set A of E^* and any $s > 0$,

$$\alpha(A) = \mathbb{P}_\alpha(Z_s \in A | T_0 > s).$$

The probability measure α is then a QSD. □

Exponential extinction rate

Proposition 1.2. *Let us consider a Markov process Z with absorbing point 0 satisfying (1.2). Assume that α is a QSD for the process. Then there exists a positive real number $\theta(\alpha)$ depending on the QSD such that*

$$\mathbb{P}_\alpha(T_0 > t) = e^{-\theta(\alpha)t}. \tag{1.5}$$

This theorem shows us that starting from a QSD, the extinction time has an exponential distribution with parameter $\theta(\alpha)$ given by

$$\theta(\alpha) = -\frac{\ln \mathbb{P}_\alpha(T_0 > t)}{t},$$

which is independent of t.

Proof. By the Markov property,

$$\begin{aligned} \mathbb{P}_\alpha\left(T_0 > t+s\right) &= \mathbb{E}_\alpha\left(\mathbb{P}_{Z_t}(T_0 > s)\mathbf{1}_{T_0>t}\right) \\ &= P_\alpha(T_0 > t)E_\alpha\left(\mathbb{P}_{Z_t}(T_0 > s)|T_0 > t\right), \end{aligned}$$

since $T_0 \leq t$ implies $Z_t = 0$, and $\mathbb{P}_0(T_0 > s) = 0$. By definition of a QSD, we get

$$E_\alpha\left(\mathbb{P}_{Z_t}(T_0 > s)|T_0 > t\right) = P_\alpha(T_0 > s)$$

Hence we obtain that for all s,$t > 0$,

$$\mathbb{P}_\alpha(T_0 > t+s) = \mathbb{P}_\alpha(T_0 > s)\mathbb{P}_\alpha(T_0 > t).$$

Let us denote $g(t) = \mathbb{P}_\alpha(T_0 > t)$. We have $g(0) = 1$ and, because of (1.2), $g(t)$ tends to 0 as t tends to infinity. An elementary proof allows us to conclude that there exists a real number $\theta(\alpha) > 0$ such that

$$\mathbb{P}_\alpha(T_0 > t) = e^{-\theta(\alpha)t}.$$

□

QSD and exponential moments

The following statement gives a necessary condition for the existence of QSD's in terms of existence of exponential moments of the hitting time T_0.

Proposition 1.3. *Assume that α is a QSD. Then any $0 < \gamma < \theta(\alpha)$ and for α-almost all z,*

$$\mathbb{E}_z(e^{\gamma T_0}) < +\infty. \tag{1.6}$$

In particular, there exists a positive number z satisfying (1.6).

Proposition 1.3 suggests that if the population can escape extinction for too long times with positive probability, then the process has no QSD. This is the case for the critical Galton-Watson process: its extinction time is finite almost surely, but its expectation isn't finite.

Proof. We compute the exponential moment in continuous and discrete time settings. In both cases, it is finite if and only if $\theta(\alpha) > \gamma$.

In the continuous time setting, (1.5) says that, under $\mathbb{P}_\alpha$, T_0 has an exponential distribution with parameter $\theta(\alpha)$. We deduce that, for any $\theta(\alpha) > \gamma$,

$$\mathbb{E}_\alpha\left(e^{\gamma T_0}\right) = \frac{\theta(\alpha)}{\theta(\alpha) - \gamma}.$$

In the discrete time setting, (1.5) says that, under $\mathbb{P}_\alpha$, T_0 has a geometric distribution with parameter $e^{-\theta(\alpha)}$. We deduce that

$$\mathbb{E}_\alpha\left(e^{\gamma T_0}\right) = \frac{1 - e^{-\theta(\alpha)}}{e^{-\gamma} - e^{-\theta(\alpha)}}.$$

Since $\mathbb{E}_\alpha(e^{\gamma T_0})$ is equal to $\int_{E^*} E_z(e^{\gamma T_0})\alpha(dz)$, the finiteness of the integral implies the assertion.

□

A spectral point of view

This section is only relevant to the continuous time setting. We define the operator L as the infinitesimal generator of the sub-Markovian semi-group (P_t) associated with the killed process Z. The following result links the existence of QSD's for Z and the spectral properties of the dual L^* of the operator L.

Proposition 1.4. *Let α be a probability measure on E^*. We assume that there exists a set $\mathcal{D}$ of bounded real valued functions such that Lf exists and is bounded for any $f \in \mathcal{D}$ and such that, for any measurable subset $A \subset E^*$, there exists a uniformly bounded sequence (f_n) in $\mathcal{D}$ which converges point-wisely to $\mathbf{1}_A$.*

Then α is a quasi-stationary distribution if and only if there exists $\theta(\alpha) > 0$ such that

$$\alpha(Lf) = -\theta(\alpha)\alpha(f),\ \forall f \in \mathcal{D}.$$

We emphasize that the existence of $\mathcal{D}$ is always true if the state space E^* is discrete. It is also fulfilled if E^* is an open subset of $\mathbb{R}^d$ and if Z is a diffusion with locally bounded coefficients.

Proof. (1) Let α be a QSD for Z. By definition of a QSD, we have, for every Borel set $A \subseteq E^*$,

$$\alpha(A) = \frac{\alpha P_t(\mathbf{1}_A)}{\alpha P_t(\mathbf{1}_{E^*})}.$$

By Theorem 1.5, there exists $\theta(\alpha) > 0$ such that for each $t > 0$,

$$\alpha P_t(\mathbf{1}_{E^*}) = \mathbb{P}_\alpha(T_0 > t) = e^{-\theta(\alpha)t}.$$

We deduce that, for any measurable set $A \subseteq E^*$, $\alpha P_t(\mathbf{1}_A) = e^{-\theta(\alpha)t}\alpha(A)$, which is equivalent to $\alpha P_t = e^{-t\theta(\alpha)}\alpha$. By Kolmogorov's forward equation and by assumption on $\mathcal{D}$, we have

$$\left|\frac{\partial P_t f}{\partial t}(x)\right| = |P_t L f(x)| \leq \|Lf\|_\infty < +\infty, \forall f \in \mathcal{D}.$$

In particular, one can differentiate $\alpha P_t f = \int_{E^*} P_t f(x)\alpha(dx)$ under the integral sign, which implies that

$$\alpha(Lf) = -\theta(\alpha)\alpha(f), \forall f \in \mathcal{D}.$$

(2) Assume know that $\alpha(Lf) = -\theta(\alpha)\alpha(f)$ for all $f \in \mathcal{D}$. By Kolmogorov's backward equation and the same "derivation under the integral sign" argument, we have

$$\frac{\partial \alpha(P_t f)}{\partial t} = \alpha(LP_t f) = -\theta(\alpha)\alpha P_t(f), \forall f \in \mathcal{D}.$$

We deduce that

$$\alpha P_t(f) = e^{-t\theta(\alpha)}\alpha(f), \forall f \in \mathcal{D}.$$

By assumption, there exists, for any measurable subset $A \subset E^*$, a uniformly bounded sequence (f_n) in $\mathcal{D}$ which converges point-wisely to $\mathbf{1}_A$. Finally, we deduce by dominated convergence that

$$\alpha P_t(\mathbf{1}_A) = e^{-t\theta(\alpha)}\alpha(A).$$

This implies immediately that α is a quasi-stationary distribution for Z. □

Long time limit of the extinction rate

Another quantity of interest in the demography and population's dynamics is given by the long time behavior of the killing or extinction rate. In demography setting, the process Z models the vitality of some individual and t its physical age. Thus T_0 is the death time of this individual. This question has been studied in detail by Steinsaltz-Evans [78] for specific cases.

The definition of the extinction rate depends on the time setting:

- In the discrete time setting, the extinction rate of Z starting from μ at time $t \geq 0$ is defined by

$$r_\mu(t) = \mathbb{P}_\mu(T_0 = t + 1 | T_0 > t).$$

- In the continuous time setting, the extinction rate of Z starting from μ at time $t \geq 0$ is defined by

$$r_\mu(t) = -\frac{\frac{\partial}{\partial t}\mathbb{P}_\mu(T_0 > t)}{\mathbb{P}_\mu(T_0 > t)},$$

when the derivative exists and is integrable with respect to μ.

Historically (*cf.* [36]), demographers applied the Gompertz law meaning that this extinction rate was exponentially increasing with time. However in 1932, Greenwood and Irwin [39] observed that in some cases, this behavior was not true. In particular there exist cases where the extinction rate converges to a constant when time increases, leading to the notion of mortality plateau. This behavior of the extinction rate has been observed in experimental situations (see for instance [15]).

The QSD's play a main role in this framework. By Proposition 1.2, if α is a QSD, then the extinction rate $r_\alpha(t)$ is constant and given by

$$r_\alpha(t) = \begin{cases} 1 - e^{-\theta(\alpha)} \text{ in the discrete time setting} \\ \theta(\alpha) \text{ in the continuous time setting} \end{cases}, \forall t \geq 0.$$

We also refer to the introduction of Steinsaltz-Evans [78] for a nice discussion of the notion of QSD in relationship with mortality plateaus.

In the next proposition, we prove that the existence of a QLD for Z started from μ implies the existence of a long term mortality plateau.

Proposition 1.5. *Let α be a QLD for Z, initially distributed with respect to a probability measure μ on E^*. In the continuous time setting, we assume moreover that there exists $h > 0$ such that $L(P_h \mathbf{1}_{E^*})$ is well defined and bounded. In both time settings, the rate of extinction converges in the long term:*

$$\lim_{t\to\infty} r_\mu(t) = r_\alpha(0). \tag{1.7}$$

Proof. The first part of the proposition is a straight forward consequence of Proposition 1.2, in both time settings.

Let us prove the second part of the proposition in the discrete time setting. We have, by the semi-group property and the definition of a QLD,

$$\begin{aligned} r_\mu(t) &= 1 - \frac{\mu P_t(P_1 \mathbf{1}_{E^*})}{\mu P_t(\mathbf{1}_{E^*})} \\ &\xrightarrow[t\to+\infty]{} 1 - \alpha(P_1 \mathbf{1}_{E^*}) = r_\alpha(0). \end{aligned}$$

The limit is by definition the extinction rate at time 0 of Z starting from α, which is $r_\alpha(0)$.

Let us now prove the second part of the proposition in the continuous time setting. Thanks to the Kolmogorov's backward equation, we have

$$\frac{\partial}{\partial t}P_{t+h}\mathbf{1}_{E^*}(x) = P_t L(P_h\mathbf{1}_{E^*})(x),\ \forall x \in E^*.$$

Since $L(P_h\mathbf{1}_{E^*})$ is assumed to be bounded, we deduce that

$$\frac{\partial}{\partial t}\mu P_{t+h}(\mathbf{1}_{E^*}) = \mu P_t L(P_h\mathbf{1}_{E^*}).$$

Then

$$\begin{aligned}\frac{\frac{\partial}{\partial t}\mu P_{t+h}(\mathbf{1}_{E^*})}{\mu P_t(\mathbf{1}_{E^*})} &= \frac{\mu P_t L(P_h\mathbf{1}_{E^*})}{\mu P_t(\mathbf{1}_{E^*})}\\ &\xrightarrow[t\to\infty]{} \alpha(LP_h\mathbf{1}_{E^*}) = -\theta(\alpha)\alpha(P_h\mathbf{1}_{E^*}),\end{aligned}$$

by the definition of a QLD and by Proposition 1.4. We also have

$$\frac{\mu(P_{t+h}\mathbf{1}_{E^*})}{\mu(P_t\mathbf{1}_{E^*})} \xrightarrow[t\to\infty]{} \alpha(P_h\mathbf{1}_{E^*}).$$

Finally, we get

$$r_\mu(t+h) = -\frac{\frac{\partial}{\partial t}\mu(P_{t+h}\mathbf{1}_{E^*})}{\mu(P_{t+h}\mathbf{1}_{E^*})} \xrightarrow[t\to\infty]{} \theta(\alpha),$$

which allows us to conclude the proof of Proposition 1.5. □

1.2.2 An historical example in discrete time: the Galton Watson process

The Galton-Watson process is a population's dynamics model in discrete time, whose size $(Z_n)_{n\geq 0}$ evolves according to the recurrence formula $Z_0 = 1$ and

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)},$$

where $(\xi_i^{(n)})_{i,n}$ is a family of independent random variables, identically distributed with respect to a probability measure μ on $\mathbb{N}$ with generating function g. As defined, Z_n is the size of the n^{th} generation of a population where each individual has a random number of children, chosen with respect to μ and independently of the rest of the population. This process has been introduced by Galton and Watson (see [34]) in order to study the extinction of aristocratic surnames.

The case $\mu(\{1\}) = 1$ is trivial (each individual gives birth to exactly one individual, which leads to a population of constant size 1), and we will assume in the whole section, that

$$0 < \mu(0) + \mu(\{1\}) < 1.$$

We denote by $m = E(\xi_1^{(0)})$ the average number of children by individual in our Galton process. By the branching property, the probability of extinction for the population starting from one individual is given by

$$\begin{aligned}\mathbb{P}_1(\exists n \in \mathbb{N},\, Z_n = 0) &= \lim_{n\to+\infty} \mathbb{E}_1(0^{Z_n})\\ &= \lim_{n\to\infty} g\circ\cdots\circ g(0)\ (n \text{ times}).\end{aligned}$$

There are three different situations (see for instance Athreya-Ney [6]):

- **The sub-critical case** $m < 1$: the process becomes extinct in finite time almost surely and the average extinction time $\mathbb{E}(T_0)$ is finite.
- **The critical case** $m = 1$: the process becomes extinct in finite time almost surely, but $\mathbb{E}(T_0) = +\infty$.
- **The super-critical case** $m > 1$: the process is never extinct with a positive probability, and it yields immediately that $\mathbb{E}(T_0) = +\infty$.

Theorem 1.6 (Yaglom [86], 1947)**.** *Let $(Z_n)_{n\geq 0}$ be a Galton-Watson process with the reproduction generating function g. There is no quasi-stationary distribution in the critical and the super-critical case. In the sub-critical case, the Yaglom limit exists and is the unique QSD of Z. Moreover, its generating function $\hat{g}$ fulfills*

$$\hat{g}(g(s)) = m\hat{g}(s) + 1 - m,\ \forall s \in [0,1]. \tag{1.8}$$

Proof. The proof is adapted from Athreya-Ney [6] p. 13-14. In the critical or the super-critical case, we have $\mathbb{E}_1(T_0) = +\infty$, which implies that $\mathbb{E}_\alpha(T_0) = +\infty$ for all probability measure α on $\mathbb{N}^*$. We deduce from Proposition 1.3 that there is no QSD for the critical or super-critical case.

Assume now that $m < 1$. In a first step we fix an arbitrary probability measure ν on $\mathbb{N}^*$ and we prove that there exists a QLD α for Z starting with distribution ν. We also prove that the generating function of α fulfills (1.8). In a second step, we prove that there is at most one generating function which fulfills equation (1.8), so that α doesn't depend on ν, concluding the proof of Theorem 1.6.

For each $n \geq 0$, we denote by g_n the generating function of Z_n:

$$g_n(s) = \mathbb{E}_\nu\left(s^{Z_n}\right),\ s \in [0,1].$$

Let us also denote by $\hat{g}_n$ the generating function of Z_n conditioned to $\{Z_n > 0\} = \{T_0 > n\}$:

$$\begin{aligned}\hat{g}_n(s) &= \mathbb{E}_\nu(s^{Z_n} |\, Z_n > 0) = \frac{\mathbb{E}_\nu(s^{Z_n}\mathbf{1}_{Z_n>0})}{\mathbb{P}_\nu(Z_n > 0)}\\ &= \frac{\mathbb{E}_\nu(s^{Z_n}) - \mathbb{P}_\nu(Z_n = 0)}{1 - \mathbb{P}_\nu(Z_n = 0)}\\ &= \frac{g_n(s) - g_n(0)}{1 - g_n(0)} = 1 - \frac{1 - g_n(s)}{1 - g_n(0)} \in [0,1].\end{aligned}$$

We note that $\hat{g}_n(0) = 0$, which is quite natural since the conditional law doesn't charge 0.

We set, for a fixed $s \in [0,1)$, $\Gamma(s) = \frac{1-g_1(s)}{1-s}$. Then we have, for all $n \geq 0$,

$$1 - \hat{g}_{n+1}(s) = \frac{\Gamma(g_n(s))}{\Gamma(g_n(0))}\,(1 - \hat{g}_n(s))\,.$$

Since g_1 is convex, Γ is non-decreasing. Moreover $m < 1$ implies that $g_n(x) \geq x$, so that $g_n(s)$ and $1 - \hat{g}_n(s)$ are non-decreasing in n. In particular, $\lim_{n\to\infty} \hat{g}_n(s)$ exists. Let us denote by $\hat{g}(s)$ its limit and by α the corresponding finite measure (whose mass is smaller than one). In order to prove that α is a probability measure on $\mathbb{N}^*$, it is sufficient to prove that $\hat{g}(s) \to 1$ when s goes to 1. We have

$$\Gamma(g_n(0))\,(1 - \hat{g}_{n+1}(s)) = (1 - \hat{g}_n(g_1(s)))$$

Taking the limit for each size, where $\lim_{n\to\infty} \Gamma(g_n(0)) = \Gamma(1) = m$, we deduce that

$$m(1 - \hat{g}(s)) = 1 - \hat{g}(g_1(s)),$$

which implies Equation (1.8). Since $\lim_{s\to 1} g_1(s) = 1$ and $m < 1$, we deduce that $\hat{g}(1) = 1$. Finally, α is a QLD for Z starting with distribution ν.

One could think *a priori* that the function $\hat{g}$ depends on the starting distribution ν. We prove now that it isn't the case, so that there is a unique QLD, and then a unique QSD, which is also the Yaglom limit of the process (indeed, one could choose $\nu = \delta_x$, $x \in \mathbb{N}^*$).

Assume that there exist two generating functions $\hat{g}$ and $\hat{h}$ which fulfill Equation (1.8). By induction, we have, for all $n \geq 1$ and all $s \in [0,1]$,

$$\begin{aligned}
\hat{g}(g_n(s)) &= m^n \hat{g}(s) + \left(m^{n-1} + \cdots + m + 1\right)(m-1),\\
\hat{h}(g_n(s)) &= m^n \hat{h}(s) + \left(m^{n-1} + \cdots + m + 1\right)(m-1).
\end{aligned}$$

We deduce that for $s \in [0,1[$

$$\hat{g}'(g_n(s))\; g_n'(s) = m^n\, \hat{g}'(s)\ ;\ \hat{h}'(g_n(s))\; g_n'(s) = m^n\, \hat{h}'(s).$$

Since for the sub-critical case $g_n(0) \uparrow 1$ when $n \to \infty$, for any $s \in [0,1[$ there will be a k such that

$$g_k(0) \leq s \leq g_{k+1}(0).$$

Hence,

$$\frac{\hat{g}'(s)}{\hat{h}'(s)} = \frac{\hat{g}'(g_n(s))}{\hat{h}'(g_n(s))} \leq \frac{\hat{g}'(g_{n+k+1}(0))}{\hat{h}'(g_{n+k}(0))} = \frac{\hat{g}'(0)}{\hat{h}'(0)}\,\frac{m\, g_{n+k}'(0)}{g_{n+k+1}'(0)} = \frac{\hat{g}'(0)}{\hat{h}'(0)}\,\frac{m}{g'(g_{n+k}(0))}.$$

When n goes to infinity, we obtain $\frac{\hat{g}'(s)}{\hat{h}'(s)} \leq \frac{\hat{g}'(0)}{\hat{h}'(0)}$. The converse inequality is established similarly. Since $\hat{g}$ and $\hat{h}$ are generating functions of probability measures on $\mathbb{N}^*$, we have $\hat{g}(0) = \hat{h}(0) = 0$ and $\hat{g}(1) = \hat{h}(1) = 1$. Finally, the two functions $\hat{g}$ and $\hat{h}$ are equal, which concludes the proof of Theorem 1.6 . □

1.2.3 The simple example of an ergodic process with uniform killing in a finite state space

We present a very simple Markov process with extinction whose quasi-stationary distribution, Yaglom limit, speed of extinction and speed of convergence to the Yaglom limit are very easy to obtain.

Let $(X_t)_{t\geq 0}$ be an exponentially ergodic Markov process which evolves in the state space $E^* = \{1,\cdots,N\}$, $N \geq 1$. By exponentially ergodic, we mean that there exist a probability measure α on E^* and two positive constants $C,\lambda > 0$ such that, for all $z \in \{1,\cdots,N\}$ and all $t \geq 0$,

$$\sup_{i\in E^*} |\mathbb{P}_z(X_t = i) - \alpha(\{i\})| \leq Ce^{-\lambda t}.$$

There is no possible extinction for (X_t). Let $d > 0$ be a positive constant and let τ_d be an exponential random time of parameter d independent of the process (X_t). We define the process (Z_t) by setting

$$Z_t = \begin{cases} X_t, \text{ if } t < \tau_d \\ 0, \text{ if } t \geq \tau_d. \end{cases}$$

This model can be thought as a model for the size of a population which can not be extinct, but at a catastrophic event which happens with rate d. Thus we have

$$\mathbb{P}_z(t < T_0) = e^{-dt},\ \forall t \geq 0.$$

The conditional distribution of Z_t is simply given by the distribution of X_t:

$$\mathbb{P}_z(Z_t = i | Z_t \neq 0) = \mathbb{P}_z(X_t = i),\ \forall z \in E^*.$$

We deduce that the unique QSD is the Yaglom limit α and that for all $z \in E^*$ and all $t \geq 0$,

$$\sup_{i\in E^*} |\mathbb{P}_z(Z_t = i | T_0 > t) - \alpha(\{i\})| \leq Ce^{-\lambda t}.$$

Thus in this case, the conditional distribution of Z converges exponentially fast to the Yaglom limit α, with rate $\lambda > 0$ and the process becomes extinct exponentially fast, with rate $d > 0$.

Hence the comparison between the speed of convergence to the Yaglom limit and the speed of extinction will impact the observables of the process before extinction:

(a) If $\lambda \gg d$, then the convergence to the Yaglom limit happens before the typical time of extinction of the population and the quasi-stationary regime will be observable.

(b) If $\lambda \ll d$, then the extinction of the population occurs before the quasi-stationary regime is reached. As a consequence, we are very unlikely to observe the Yaglom limit.

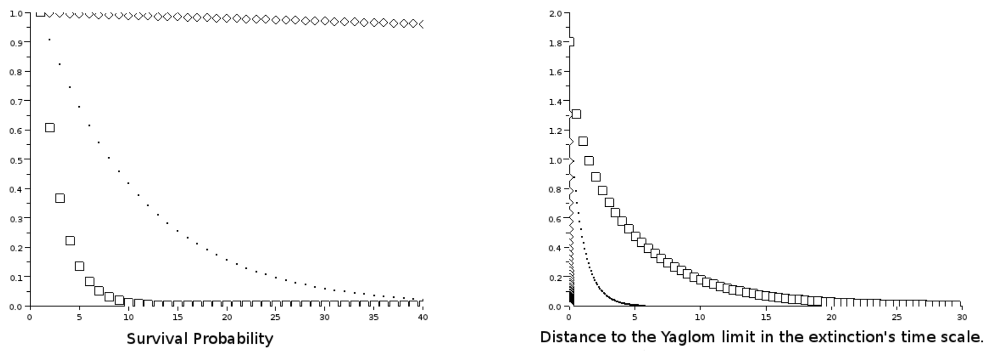


Figure 1.1: Example 1. A numerical computation leads to $\lambda = 0.098$. Three different situations are observed, which leads to three very different patterns for the speed of convergence to the Yaglom limit in the extinction's time scale: ($\diamond$) $\lambda \gg d = 0.001$; ($\square$) $\lambda \ll d = 0.500$; ($\cdot$) $\lambda = d = 0.098$.

(c) If $\lambda \sim d$, the answer is not so immediate and depends on other parameters, as in particular the initial distribution.

Example 1.1. The population size Z is described by a random walk in continuous time evolving in $E = \{0,1,2,\cdots,N\}$ with transition rates given by

$$\begin{aligned}
&i \to i+1 \text{ with rate } 1, \text{ for all } i \in \{1,2,\cdots,N-1\},\\
&i \to i-1 \text{ with rate } 1, \text{ for all } i \in \{2,3,\cdots,N\},\\
&i \to 0 \text{ with rate } d > 0, \text{ for all } i \in \{1,2,\cdots,N\}.
\end{aligned}$$

The boundedness of the population size models a constraint of fixed resources which acts on the growth of the population. We will see more realistic fixed resources models including logistic death rate in the following. One can check that the quasi-stationary probability measure of Z is given by $\alpha_i = 1/N$ for all $i \in E^*$.

Numerical simulations. We fix $N = 100$. Numerical computation using the fact that λ is the spectral gap of the generator of X gives $\lambda = 0.098$. For different values $d = 0.001$, $d = 0.500$ and $d = 0.098$, we compute numerically the mathematical quantities of interest: the extinction probability $\mathbb{P}_z(T_0 > t) = e^{-dt}$ as a function of t (cf. Figure 1.1 left picture) and the distance $\sup_{i\in E^*} |\mathbb{P}_z(Z_t = i | T_0 > t) - \alpha(\{i\})|$ between the conditional distribution of Z_t and α as a function of $-\log \mathbb{P}_z(T_0 > t)$, which gives the extinction's time scale. (cf. Figure 1.1 right picture).

We observe that the convergence to the Yaglom limit happens rapidly in the case ($\diamond$) $\lambda = 0.098 \gg d = 0.001$, indeed the distance to the Yaglom limit is equal 0.05, while the survival probability can't be graphically distinguished from 1. On the contrary, we observe that the convergence happens very slowly in the case ($\square$) $\lambda = 0.098 \ll d = 0.500$, indeed the distance to the Yaglom limit is equal to 0.05 when the survival probability appears to be smaller than $e^{-15} \simeq 3 \times 10^{-7}$. The case ($\cdot$) $\lambda = 0.98 = d$ is an intermediate case, where the distance to the Yaglom limit is equal to 0.05 when the survival probability appears to be equal to $e^{-3} \simeq 0.05$.

1.3 The finite case, with general rate of killing

1.3.1 The quasi-stationary distributions

We generalize Example 1 to a more realistic case where the rate of extinction depends on the size of the population. For instance, the probability of extinction is often higher for a small population than for a big one. The results of this section have been originally proved by Darroch and Seneta ([21] and [22]).

The Markov process $(Z_t)_{t\geq 0}$ evolves in continuous time in $E = \{0,1,...,N\}$, $N \geq 1$ and we still assume that 0 is its unique absorbing state. The semi-group $(P_t)_{t\geq 0}$ is the sub-Markovian semi-group of the killed process and we still denote by L the associated infinitesimal generator. In the finite state space case, the operators L and P_t are matrices, and a probability measure on the finite space E^* is a vector of non-negative entries whose sum is equal to 1.

Theorem 1.7. *Assume that Z is an irreducible and aperiodic process before extinction, which means that there exists $t_0 > 0$ such that the matrix P_{t_0} has only positive entries (in particular, it implies that P_t has positive entries for $t > t_0$). Then the Yaglom limit α exists and is the unique QSD of the process Z_t.*

Moreover, there exists a probability measure π on E^* such that, for any $i,j \in E^*$,

$$\lim_{t\to\infty} e^{\theta(\alpha)t}\mathbb{P}_i(Z_t = j) = \pi_i \alpha_j$$

and

$$\lim_{t\to\infty} \frac{\mathbb{P}_i(T_0 > t+s)}{\mathbb{P}_j(T_0 > t)} = \frac{\pi_i}{\pi_j} e^{-\theta(\alpha)s}.$$

The main tool of the proof of Theorem 1.7 is the Perron-Frobenius Theorem, which gives us a complete description of the spectral properties of P_t and L. The main point is that the matrix P_1 has positive entries. For the proof of the Perron-Frobenius Theorem, we refer to Gantmacher [35] or Serre [77].

Theorem 1.8 (Perron-Frobenius Theorem)**.** *Let (P_t) be a submarkovian semi-group on $\{1,\cdots,N\}$ such that the entries of P_{t_0} are positive for $t_0 > 0$. Thus, there exists a unique positive eigenvalue ρ, which is the maximum of the modulus of the eigenvalues, and there exists a unique left-eigenvector α such that $\alpha_i > 0$*

and $\sum_{i=1}^N \alpha_i = 1$*, and there exists a unique right-eigenvector* π *such that* $\pi_i > 0$ *and* $\sum_{i=1}^N \alpha_i \pi_i = 1$*, satisfying*

$$\alpha P_{t_0} = \rho\alpha \ ; \ P_{t_0}\pi = \rho\pi \tag{1.9}$$

In addition, since (P_t) *is a sub-Markovian semi-group,* $\rho < 1$ *and there exists* $\theta > 0$ *such that* $\rho = e^{-\theta}$*. Therefore*

$$P_t = e^{-\theta t}A + \vartheta(e^{-\chi t}), \tag{1.10}$$

where A *is the matrix defined by* $A_{ij} = \pi_i\alpha_j$*, and* $\chi > \theta$ *and* $\vartheta(e^{-\chi t})$ *denotes a matrix such that none of the entries exceeds* $Ce^{-\chi t}$*, for some constant* $C > 0$*.*

Proof of Theorem 1.7. It is immediate from (1.10) that for any $i,j \in E^*$,

$$e^{\theta t}\mathbb{P}_i(Z_t = j) = e^{\theta t}[P_t]_{ij} = \pi_i\alpha_j + \vartheta(e^{-(\chi-\theta)t}). \tag{1.11}$$

Summing over $j \in E^*$, we deduce that

$$e^{\theta t}\mathbb{P}_i(T_0 > t) = \pi_i + \vartheta(e^{-(\chi-\theta)t}). \tag{1.12}$$

It follows that, for any $i,j \in E^*$,

$$\mathbb{P}_i(Z_t = j | T_0 > t) = \frac{\mathbb{P}_i(Z_t = j)}{\mathbb{P}_i(T_0 > t)} \xrightarrow[t\to\infty]{} \alpha_j.$$

Thus the Yaglom limit exists and is equal to α. Since E is finite, we have for any initial distribution ν on E^*

$$\lim_{t\to\infty} \mathbb{P}_\nu(Z_t = j|T_0 > t) = \sum_{i\in E^*} \nu_i \lim_{t\to\infty} \mathbb{P}_i(Z_t = j|T_0 > t) = \sum_{i\in E^*} \nu_i\alpha_j = \alpha_j.$$

We deduce that the Yaglom limit α is the unique QLD of Z, and thus it is its unique QSD.

By Proposition 1.2, we have $\alpha P_1(\mathbf{1}_{E^*}) = e^{-\theta(\alpha)}$. By (1.9), this quantity is also equal to $e^{-\theta}$, so that $\theta = \theta(\alpha)$. The end of Theorem 1.7 is thus a straightforward consequence of (1.11) and (1.12) $\square$

Remark 1.2. One can deduce from (1.11) and (1.12) that there exists a positive constant C_L such that

$$\sup_{j\in E^*, i\in E^*} |P_i(Z_t = j|Z_t > 0) - \alpha_j| \le C_L e^{-(\chi-\theta(\alpha))},$$

where the quantity $\chi - \theta$ is the second spectral gap of L, *i.e.* the distance between the first and the second eigenvalue of L. Thus if the time-scale $\chi - \theta$ of the convergence to the quasi-limiting distribution is substantially bigger than the time scale of absorption ($\chi - \theta(\alpha) \gg \theta(\alpha)$), the process will relax to the QSD after a relatively short time, and after a much longer period, extinction will occur. On the contrary, if $\chi - \theta(\alpha) \ll \theta(\alpha)$, then the extinction happens before the process had time to relax to the quasi-limiting distribution.

In intermediate cases, where $\lambda - \theta(\alpha) \sim \theta(\alpha)$, the constant C_L, which depends on the whole set of eigenvalues and eigenfunctions of L, plays a main role which need further investigations.

Let us now develop an example in which we can observe these three situations.

Example 1.2. Let Z be a Markov process which models a population whose individuals reproduce and die independently, with individual birth rate $\lambda > 0$ and individual death rate μ, where μ is set as 1 but then kept as arbitrary later. In order to take into account the finiteness of the resources, the process is reflected when it attains a given value N, that we choose here arbitrarily equal to 100. Thus the process Z evolves in the finite state space $\{0,1,\cdots,100\}$, and its transition rates are given by

$$
\begin{aligned}
&i \to i+1 \text{ with rate } \lambda i, \text{ for all } i \in \{1,2,\cdots,99\},\\
&i \to i-1 \text{ with rate } \mu i, \text{ for all } i \in \{1,2,3,\cdots,100\}.
\end{aligned}
$$

The infinitesimal generator of Z is given by

$$
\begin{aligned}
&L_{1,1} = -1-\lambda \text{ and } L_{1,2} = \lambda,\\
&L_{i,i-1} = i, \quad L_{i,i} = -(1+\lambda)i \text{ and } L_{i,i+1} = \lambda\, i, \ \forall i \in \{2,\cdots,99\},\\
&L_{100,99} = 100 \text{ and } L_{100,100} = -100,\\
&L_{i,j} = 0, \ \forall i,j \in \{1,\cdots,100\} \text{ such that } |j-i| > 1.
\end{aligned}
$$

The process Z clearly fulfills the conditions of Theorem 1.7. As a consequence, it has a Yaglom limit α, which is its unique QSD. Moreover, the probability measure α is the unique normalized and positive left eigenvector of L. Since L is a finite matrix of size 100×100, one can numerically compute the whole set of eigenvectors and eigenvalues of the matrix (L_{ij}). This will allow to obtain numerically the Yaglom limit α, its associated extinction rate $\theta(\alpha)$, and the speed of convergence $\chi - \theta(\alpha)$. Moreover, for any $t \geq 0$, one can compute the value of e^{tL}, which is equal to P_t (the semi-group of Z at time t). Hence, we may obtain the numerical value of the conditioned distribution $\mathbb{P}_{Z_0}(Z_t \in .|t < T_0)$, for any initial size Z_0. Finally, we are also able to compute numerically the distance between α and the conditioned distribution $\mathbb{P}_{Z_0}(Z_t \in .|t < T_0)$, for any value of $\lambda > 0$ and $Z_0 \in \{1,\cdots,100\}$.

In Figure 1.2, we represent the Yaglom limit α for different values of λ, namely $\lambda = 0.9$, $\lambda = 1.0$ and $\lambda = 1.1$. Let us comment the numerical results.

(a) In the first case ($\lambda = 0.9$), an individual is more likely to die than to reproduce and we observe that the Yaglom limit is concentrated near the

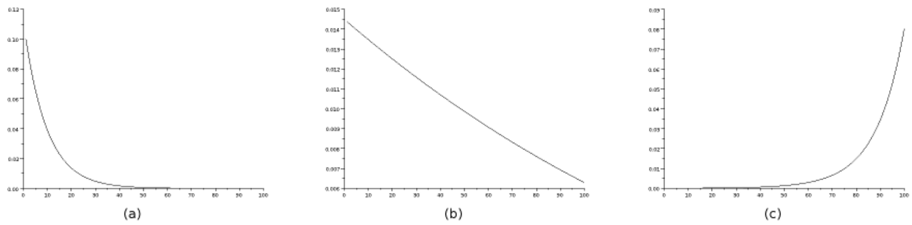


Figure 1.2: Example 2. Yaglom limits for different values of λ. The following values of $\theta(\alpha)$ are obtained by numerical computation. (a) $\lambda = 0.9$, $\theta(\alpha) = 0.100$; (b) $\lambda = 1.0$, $\theta(\alpha) = 0.014$; (c) $\lambda = 1.1$, $\theta(\alpha) = 5.84 \times 10^{-5}$.

absorbing point 0. The rate of extinction $\theta(\alpha)$ is the highest in this case, equal to 0.100. In fact, the process reaches the upper bound 100 very rarely, so that the behavior of the process is very similar to the one of a linear birth and death process with birth and death rates equal to λ and μ respectively. In Section 1.4, we study such linear birth and death processes. We show that the Yaglom limit (which exists if and only if $\lambda < \mu$) is given by a geometric law and $\theta(\alpha) = \mu - \lambda$.

(b) In the second case ($\lambda = \mu = 1$), we observe that α decreases almost linearly from α_1 to α_{100} and the upper bound $N = 100$ plays a crucial role. In fact, letting N tend to $+\infty$, one would observe that for any $i \geq 1$, α_i decreases to 0. The extinction rate $\theta(\alpha)$ which is equal to 0.014 for $N = 100$) would also go to 0. The counterpart of this phenomenon for the linear birth and death process studied in Section 1.4 is that the Yaglom limit doesn't exist when $\mu = \lambda$.

(c) In the third case ($\lambda = 1.1$), the Yaglom limit α is concentrated near the upper bound 100, while the extinction rate is $\theta(\alpha) = 5.84 \times 10^{-5}$. The comparison with the linear birth and death process is no more relevant, since the important factor in this case is the effect of the upper bound $N = 100$, which models the finiteness of the resources in the environment.

In Figure 1.3, we study the effect of the initial position and of the value of the parameter λ on the speed of convergence to the Yaglom limit and on the speed of extinction. We choose the positions $Z_0 = 1$, $Z_0 = 10$ and $Z_0 = 100$, and we look at the two different cases $\lambda = 0.9$ and $\lambda = 1.1$, which correspond to the subcritical case (a) and to the supercritical case (c) respectively. We represent, for each set of values of (λ, Z_0), the distance to the Yaglom limit $\sup_{i\in\{1,\cdots,100\}} |\mathbb{P}_{Z_0}(Z_t = i | t < T_0) - \alpha_i|$ as a function of the time, and the same distance as a function of the logarithm of the survival probability $-\log \mathbb{P}_{Z_0}(t < T_0)$ (i.e. the extinction time scale). By numerical computation, we also obtain that

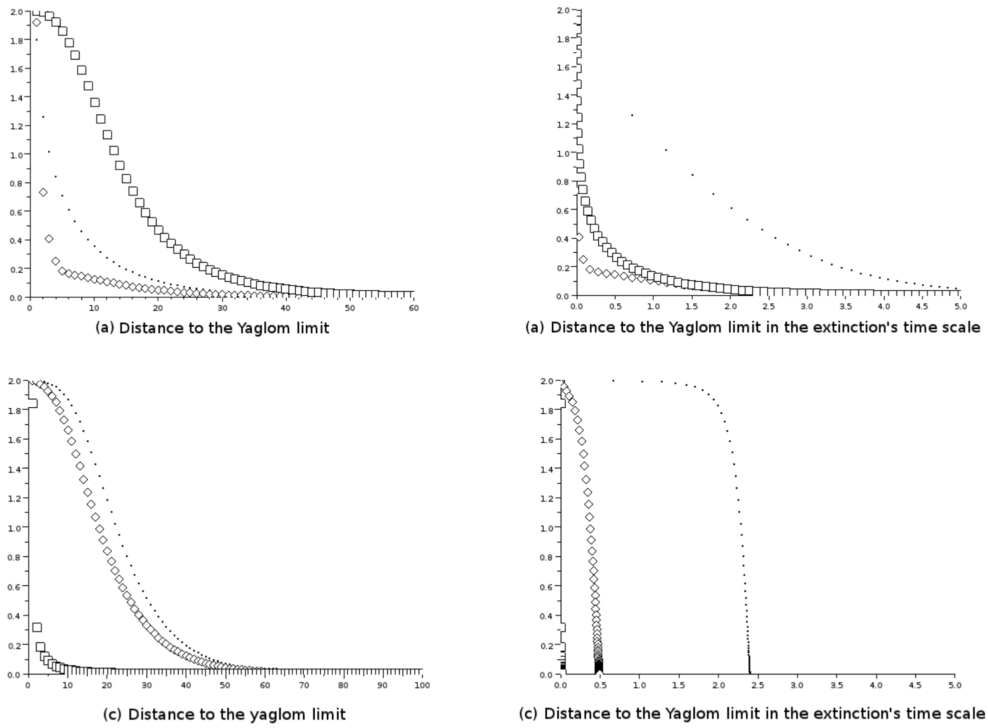


Figure 1.3: Example 2. Pictures (a) and (c) correspond to different values of λ (the following values of $\theta(\alpha) - \chi$ have been obtained by numerical computation): (a) $\lambda = 0.9$, $\theta(\alpha) = 0.100$, $\theta(\alpha) - \chi = 0.102$; (c) $\lambda = 1.1$, $\theta(\alpha) = 5.84 \times 10^{-5}$, $\theta(\alpha) - \chi = 0.103$; each curve corresponds to a given initial size of the population: $(\cdot)$ $Z_0 = 1$; $(\diamond)$ $Z_0 = 10$; $(\Box)$ $Z_0 = 100$.

(a) $\lambda = 0.9$: $\theta(\alpha) = 0.100$ and $\theta(\alpha) - \chi = 0.102$.

(c) $\lambda = 1.1$: $\theta(\alpha) = 5.84 \times 10^{-5}$ and $\theta(\alpha) - \chi = 0.103$.

In the case (a), we have $\theta(\alpha) = 0.100 \simeq \chi - \theta(\alpha) = 0.102$ and we observe that the speed of convergence depends on the initial position in a non-trivial way: while the survival probability is smaller for the process starting from 10 than for the process starting from 100, the convergence to the Yaglom limit in the extinction's time scale happens faster in the case $Z_0 = 10$.

In the case (c), we have $\theta(\alpha) = 5.84 \times 10^{-5} \ll \chi - \theta(\alpha) = 0.103$. The speed of convergence to the Yaglom in the extinction's time scale depends on the initial position: if $(\Box)$ $Z_0 = 100$, then it is almost immediate; if $(\diamond)$ $Z_0 = 10$, the distance between the conditional distribution and the Yaglom limit is equal to 0.05 when the survival probability is around $e^{-0.5} \simeq 0.61$; if $(\cdot)$ $Z_0 = 1$, then this distance is equal to 0.05 when the survival probability is around $e^{-2.4} \simeq 0.091$.

1.3.2 The Q-process

Let us now study the marginals of the process conditioned to never be extinct.

Theorem 1.9. *Assume that we are in the conditions of Theorem 1.7. For any* $i_0, i_1, \cdots, i_k \in E^*$, *any* $0 < s_1 < \cdots, s_k < t$, *the limiting value* $\lim_{t\to\infty} \mathbb{P}_{i_0}(Z_{s_1} = i_1, \cdots, Z_{s_k} = i_k | T_0 > t)$ *exists.*

Let $(Y_t, t \geq 0)$ *be the process starting from* $i_0 \in E^*$ *and defined by its finite dimensional distributions*

$$\mathbb{P}_{i_0}(Y_{s_1} = i_1, \cdots, Y_{s_k} = i_k) = \lim_{t\to\infty} \mathbb{P}_{i_0}(Z_{s_1} = i_1, \cdots, Z_{s_k} = i_k | T_0 > t). \quad (1.13)$$

Then Y *is a Markov process with values in* E^* *and transition probabilities given by*

$$\mathbb{P}_i(Y_t = j) = e^{\theta(\alpha)t} \frac{\pi_i}{\pi_j} P_{ij}(t).$$

It is conservative, and has a unique stationary probability measure $(\alpha_j \pi_j)_j$.

Remark that contrary to intuition, the stationary probability measure is not the QSD but is absolutely continuous with respect to the QSD.

Proof. Let us denote $\theta(\alpha)$ by θ for simplicity. Let $i_0, i_1, \cdots, i_k \in E^*$ and $0 < s_1 < \cdots < s_k < t$. We introduce the filtration $\mathcal{F}_s = \sigma(Z_u, u \leq s)$. Then

$$\begin{aligned}
\mathbb{P}_{i_0}(Z_{s_1} = i_1, \cdots, Z_{s_k} = i_k \;;\; T_0 > t) &= \mathbb{E}_{i_0}\left(\mathbf{1}_{Z_{s_1}=i_1,\cdots,Z_{s_k}=i_k}\, \mathbb{E}_{i_0}\left(\mathbf{1}_{T_0>t} | \mathcal{F}_{s_k}\right)\right) \\
&= \mathbb{E}_{i_0}(\mathbf{1}_{Z_{s_1}=i_1,\cdots,Z_{s_k}=i_k}\, \mathbb{E}_{i_k}(\mathbf{1}_{T_0>t-s_k})) \\
&\quad (\text{ by Markov property}) \\
&= \mathbb{P}_{i_0}\left(Z_{s_1} = i_1, \cdots, Z_{s_k} = i_k\right) \mathbb{P}_{i_k}(T_0 > t - s_k).
\end{aligned}$$

By Theorem 1.7,

$$\lim_{t\to\infty} \frac{\mathbb{P}_{i_k}(T_0 > t - s_k)}{\mathbb{P}_{i_0}(T_0 > t)} = \frac{\pi_{i_k}}{\pi_{i_0}} e^{\theta s_k}.$$

Thus

$$\lim_{t\to\infty} \mathbb{P}_{i_0}(Z_{s_1} = i_1, \cdots, Z_{s_k} = i_k | T_0 > t) = \mathbb{P}_{i_0}(Z_{s_1} = i_1, \cdots, Z_{s_k} = i_k) \frac{\pi_{i_k}}{\pi_{i_0}} e^{\theta s_k}. \quad (1.14)$$

Let us now show that Y is a Markov process. We have

$$\begin{aligned}
\mathbb{P}_{i_0}(Y_{s_1} = i_1, \cdots, Y_{s_k} = i_k, Y_t = j) &= e^{\theta t} \frac{\pi_j}{\pi_{i_0}} \mathbb{P}_{i_0}(Z_{s_1} = i_1, \cdots, Z_{s_k} = i_k, Z_t = j) \\
&= e^{\theta(t-s_k)}\, e^{\theta s_k} \frac{\pi_j}{\pi_{i_k}} \frac{\pi_{i_k}}{\pi_{i_0}} \mathbb{P}_{i_0}(Z_{s_1} = i_1, \cdots, Z_{s_k} = i_k) \\
&\quad \times \mathbb{P}_{i_k}(Z_{t-s_k} = j) \quad (\text{by Markov property of } Z) \\
&= \mathbb{P}_{i_0}(Y_{s_1} = i_1, \cdots, Y_{s_k} = i_k)\, \mathbb{P}_{i_k}(Y_{t-s_k} = j),
\end{aligned}$$

and thus $\mathbb{P}(Y_t = j | Y_{s_1} = i_1, \cdots, Y_{s_k} = i_k) = \mathbb{P}_{i_k}(Y_{t-s_k} = j)$.

By (1.14) and Theorem 1.7, we have

$$\mathbb{P}_i(Y_t = j) = \frac{\pi_j}{\pi_i}\, \mathbb{P}_i(Z_t = j)\, e^{\theta t} \xrightarrow[t\to+\infty]{} \frac{\pi_j}{\pi_i}\, \alpha_j\, \pi_i = \alpha_j \pi_j.$$

Moreover let us compute the infinitesimal generator $\hat{L}$ of Y from the infinitesimal generator L of Z. We have for $j \neq i$,

$$\hat{L}_{ij} = \lim_{s\to 0} \hat{P}_{ij}(s) = \frac{\pi_j}{\pi_i}\, L_{ij}.$$

For $j = i$,

$$\begin{aligned}
\hat{L}_{ii} &= -\lim_{s\to 0} \frac{1-\hat{P}_{ii}(s)}{s} = -\lim_{s\to 0} \frac{1-e^{\theta s}P_{ii}(s)}{s} \\
&= -\lim_{s\to 0} \frac{1-e^{\theta s} + e^{\theta s}(1-p_{ii}(s))}{s} = \theta + L_{ii}.
\end{aligned}$$

Let us finish the proof by showing that the process Y is conservative.

$$\sum_{j\in E^*} \hat{L}_{ij} = \sum_{j\in E^*} \frac{\pi_j}{\pi_i} L_{ij} + \theta.$$

Since $L\pi = -\theta\pi$, then $\sum_{j\in E^*} \pi_j L_{ij} = -\theta \pi_i$ and thus $\sum_{j\in E^*} \hat{L}_{ij} = 0$. □

1.4 QSD for birth and death processes

We are describing here the dynamics of isolated asexual populations, as for example populations of bacteria with cell binary division, in continuous time. Individuals may reproduce or die, and there is only one child per birth. The population size dynamics will be modeled by a birth and death process in continuous time. The individuals may interact, competing (for example) for resources and therefore the individual rate of death will depend on the total size of the population. In a first part, we recall and partially prove some results on the non-explosion of continuous time birth and death processes. We will also recall conditions on the birth and death rates which ensure that the process goes to extinction in finite time almost surely. In a second part, we concentrate on the cases where the process goes almost surely to zero and we study the existence and uniqueness of quasi-stationary distributions.

1.4.1 Birth and death processes

We consider here birth and death processes with rates $(\lambda_i)_i$ and $(\mu_i)_i$, that is $\mathbb{N}$-valued pure jump Markov processes, whose jumps are $+1$ or -1, with transition rates given by:

$$\begin{array}{ccccc} i & \to & i+1 & \text{with rate} & \lambda_i\ , \\ i & \to & i-1 & \text{with rate} & \mu_i, \end{array}$$

where λ_i and μ_i, $i \in \mathbb{N}$, are non-negative real numbers.

Knowing that the process is at state i at a certain time, the process will wait for an exponential time of parameter λ_i before jumping to $i+1$ or independently, will wait for an exponential time of parameter μ_i before jumping to $i-1$. The total jump rate from state i is thus $\lambda_i + \mu_i$. We will assume in what follows that $\lambda_0 = \mu_0 = 0$. This condition ensures that 0 is an absorbing point, modeling the extinction of the population.

The most standard examples are the following ones.

1. **The Yule process.** For each $i \in \mathbb{N}$, $\lambda_i = \lambda i$ for a positive real number λ, and $\mu_i = 0$. There are no deaths. It's a fission model.

2. **The linear birth and death process, or binary branching process.** There exist positive numbers λ and μ such that $\lambda_i = \lambda i$ and $\mu_i = \mu i$. This model holds if individuals reproduce and die independently, with birth rate equal to λ and death rate equal to μ.

3. **The logistic birth and death process.** We assume that every individual in the population has a constant birth rate $\lambda > 0$ and a natural death rate $\mu > 0$. Moreover the individuals compete to share fixed resources, and each individual $j \neq i$ creates a competition pressure on individual i with rate $c > 0$. Thus, given that the population's size is i, the individual death rate is given by $c(i-1)$ and the total death rate is $\mu_i = \mu i + ci(i-1)$.

In the following, we will assume that $\lambda_i > 0$ and $\mu_i > 0$ for any $i \in \mathbb{N}^*$.

We denote by $(\tau_n)_n$ the sequence of the jump times of the process, either births or deaths. Let us first see under which conditions on the birth and death rates the process is well defined for all time $t \geq 0$, i.e. $\tau = \lim_n \tau_n = +\infty$ almost surely. Indeed, if $\tau = \lim_n \tau_n < \infty$ with a positive probability, the process would only be defined for $t < \tau$ on this event. There would be an accumulation of jumps near τ and the process could increase until infinity in finite time.

Let us give a necessary and sufficient condition ensuring that a birth and death process does not explode in finite time. The result is already stated in Anderson [4], but the proof that we give is far much shorter and easier to follow.

Theorem 1.10. *The birth and death process does not explode in finite time, almost surely, if and only if* $\sum_n r_n = +\infty$, *where*

$$r_n = \frac{1}{\lambda_n} + \sum_{k=1}^{n-1} \frac{\mu_{k+1}\cdots\mu_n}{\lambda_k\lambda_{k+1}\cdots\lambda_n} + \frac{\mu_1\cdots\mu_n}{\lambda_1\cdots\lambda_n}.$$

Proof. 1) Let us more generally consider a pure jump Markov process $(X_t, t \geq 0)$ with values in $\mathbb{N}$, and generator $(L_{ij}, i, j \in \mathbb{N})$. We set $q_i = -L_{ii}$. Let $(\tau_n)_n$ be the sequence of jump times of the process and $(S_n)_n$ the sequence of inter-times defined by

$$S_n = \tau_n - \tau_{n-1}, \quad \forall n \geq 1; \quad \tau_0 = 0, \quad S_0 = 0.$$

We also set $\tau_\infty = \lim_{n\to\infty} \tau_n \in [0, +\infty]$. The process does not explode in finite time almost surely (and is well defined for all time $t \in \mathbb{R}_+$), if and only if for each $i \in \mathbb{N}$

$$\mathbb{P}_i(\tau_\infty < \infty) = 0.$$

Let us show that this property is satisfied if and only if the unique non-negative and bounded solution $x = (x_i)_{i\in\mathbb{N}}$ of $L\,x = x$ is the null solution.

For any i, we set $h_i^{(0)} = 1$ and for $n \in \mathbb{N}^*$, $h_i^{(n)} = \mathbb{E}_i(\exp(-\sum_{k=1}^n S_k))$. For any $n \in \mathbb{N}$, we have

$$h_i^{(n+1)} = \sum_{j\neq i} \frac{L_{ij}}{q_i}\, h_j^{(n)}\, \mathbb{E}_i(\exp(-S_1)).$$

Indeed, the property is true for $n = 0$ since $\sum_{i\neq j} \frac{L_{ij}}{q_i} = 1$. Moreover, by conditioning with respect to S_1 and using the strong Markov property, we get

$$\mathbb{E}_i\left(\exp(-\sum_{k=1}^{n+1} S_k)\Big|\, S_1\right) \;=\; \exp(-S_1)\; \mathbb{E}_{X_{S_1}}\left(\exp(-\sum_{k=1}^{n} S_k)\right), \tag{1.15}$$

since the jump times of the S_1-translated process are the $\tau_n - S_1, n \in \mathbb{N}^*$. We have

$$\begin{aligned} \mathbb{E}_i\left(\mathbb{E}_{X_{S_1}}\left(\exp(-\sum_{k=1}^{n} S_k)\right)\right) &= \sum_{j\neq i} \mathbb{P}_i(X_{S_1} = j)\; \mathbb{E}_j\left(\exp(-\sum_{k=1}^{n} S_k)\right) \\ &= \sum_{j\neq i} \frac{L_{ij}}{q_i}\; \mathbb{E}_j(\exp(-\sum_{k=1}^{n} S_k)), \end{aligned}$$

since $\mathbb{P}_i(X_{S_1} = j) = \frac{L_{ij}}{q_i}$. By independence of S_1 and X_{S_1}, we deduce from (1.15) that

$$\mathbb{E}_i\left(\exp(-\sum_{k=1}^{n+1} S_k)\right) = \sum_{j\neq i} \frac{L_{ij}}{q_i}\; \mathbb{E}_j\left(\exp(-\sum_{k=1}^{n} S_k)\right)\; \mathbb{E}_i\left(\exp(-S_1)\right).$$

As

$$\mathbb{E}_i(\exp(-S_1)) = \int_0^\infty q_i e^{-q_i s} e^{-s} ds = \frac{q_i}{1+q_i},$$

it turns out that

$$h_i^{(n+1)} = \sum_{j \neq i} \frac{L_{ij}}{1+q_i}\, h_j^{(n)}. \tag{1.16}$$

Let $(x_i)_i$ be a nonnegative solution of $Lx = x$ bounded by 1, then $x_i = \sum_j L_{ij}\, x_j = L_{ii}x_i + \sum_{j\neq i} L_{ij}x_j = -q_i x_i + \sum_{j \neq i} L_{ij} x_j$, so that

$$x_i = \sum_{j\neq i} \frac{L_{ij}}{1+q_i} x_j. \tag{1.17}$$

Since $h_i^{(0)} = 1 \geq x_i \geq 0$ and $\frac{L_{ij}}{1+q_i} \geq 0$ for all $i,j \in E$, we deduce by iteration from (1.16) and (1.17) that $h_i^{(n)} \geq x_i \geq 0$, for any $n \in \mathbb{N}$.

Let us in another hand define for any j the quantity $z_j = \mathbb{E}_j(e^{-\tau_\infty})$. Using $\tau_\infty = \lim_n \tau_n$, and $\tau_n = \sum_{k=1}^n S_k$, we deduce by monotone convergence that $z_j = \lim_n h_j^{(n)}$.

If the process does not explode a.s., then $\tau_\infty = \infty$ a.s., and $\lim_n h_i^{(n)} = z_i = 0$. Since $h_i^{(n)} \geq x_i \geq 0$, we deduce that $x_i = 0$. Thus, the unique nonnegative bounded solution of $Lx = x$ is zero.

If the process explodes with positive probability, then there exists i such that $\mathbb{P}_i(\tau_\infty < \infty) > 0$. Making n tend to infinity in (1.16), we get

$$z_i = \sum_{j\neq i} \frac{L_{ij}}{1+q_i}\, z_j.$$

Since $z_i > 0$, z is a positive and bounded solution of $Lz = z$.

2) Let us now apply this result to the birth and death process with $\lambda_0 = \mu_0 = 0$. We have, for $i \geq 1$, $L_{i,i+1} = \lambda_i$, $L_{i,i-1} = \mu_i$, $L_{i,i} = -(\lambda_i + \mu_i)$. The equation $Lx = x$ is given by $x_0 = 0$ and for all $n \geq 1$ by

$$\lambda_n x_{n+1} - (\lambda_n + \mu_n) x_n + \mu_n x_{n-1} = x_n.$$

Thus, if we set $\Delta_n = x_n - x_{n-1}$, we have $\Delta_1 = x_1$ and for $n \geq 1$,

$$\Delta_{n+1} = \Delta_n \frac{\mu_n}{\lambda_n} + \frac{1}{\lambda_n} x_n.$$

Let us remark that for any n, $\Delta_n \geq 0$, and thus the sequence $(x_n)_n$ is nondecreasing. If $x_1 = 0$, the solution is zero. If not, we get by induction

$$\Delta_{n+1} = \frac{1}{\lambda_n} x_n + \sum_{k=1}^{n-1} \frac{1}{\lambda_k} \frac{\mu_{k+1}}{\lambda_{k+1}} \cdots \frac{\mu_n}{\lambda_n}\, x_k + \frac{\mu_1}{\lambda_1} \cdots \frac{\mu_n}{\lambda_n}\, x_1.$$

Letting

$$r_n = \frac{1}{\lambda_n} + \sum_{k=1}^{n-1} \frac{\mu_{k+1}\cdots\mu_n}{\lambda_k\lambda_{k+1}\cdots\lambda_n} + \frac{\mu_1\cdots\mu_n}{\lambda_1\cdots\lambda_n},$$

we deduce that

$$r_n\ x_1 \le \Delta_{n+1} \le r_n\ x_n.$$

Then

$$x_1(1 + r_1 + \cdots r_n) \le x_{n+1} \le x_1\ \prod_{k=1}^{n}(1+r_k).$$

The boundedness of the sequence $(x_n)_n$ is thus equivalent to the convergence of the series $\sum_k r_k$. □

Corollary 1.11. *Let us consider a BD-process with birth rates $(\lambda_i)_i$. If there exists a constant $\lambda > 0$ such that*

$$\lambda_i \le \lambda i,$$

then the process is well defined on $\mathbb{R}_+$.

The proof is immediate. It turns out that the linear BD-processes and the logistic processes are well defined on $\mathbb{R}_+$.

Let us now study under which assumption a BD-process goes to extinction almost surely.

Proposition 1.12. *The BD-process goes almost-surely to extinction if and only if*

$$\sum_{k=1}^{\infty} \frac{\mu_1\cdots\mu_k}{\lambda_1\cdots\lambda_k} = +\infty. \tag{1.18}$$

It yields that the condition for extinction is given by

$$\sum_{k\ge 1} \frac{1}{\lambda_k \pi_k} = +\infty.$$

Proof. Let us introduce the quantity

$$u_i := \mathbb{P}(Extinction|Z_0 = i) = \mathbb{P}_i(T_0 < \infty),$$

which is the probability to attain 0 in finite time, starting from i. We have denoted as before by T_0 the extinction time. Then, using the Markov property and the fact that the jumps have amplitude ± 1, we get the induction formula

$$\lambda_i\ u_{i+1} - (\lambda_i + \mu_i)\ u_i + \mu_i\ u_{i-1} = 0,\ \forall i \ge 1.$$

To resolve this equation, we firstly assume that the rates λ_i, μ_i are nonzero until some fixed level I such that $\lambda_I = \mu_I = 0$. Let us define for each i, $u_i^{(I)} := \mathbb{P}_i(T_0 < T_I)$. Thus $u_i = \lim_{I\to\infty} u_i^{(I)}$. If we define

$$U_I := \sum_{k=1}^{I-1} \frac{\mu_1 \cdots \mu_k}{\lambda_1 \cdots \lambda_k},$$

an easy computation shows that for $i \in \{1, \cdots, I-1\}$,

$$u_i^{(I)} = (1+U_I)^{-1} \sum_{k=i}^{I-1} \frac{\mu_1 \cdots \mu_k}{\lambda_1 \cdots \lambda_k}.$$

In particular, $u_1^{(I)} = \frac{U_I}{1+U_I}$. Hence, if $(U_I)_I$ tends to infinity when $I \to \infty$, then any extinction probability u_i is equal to 1. If $(U_I)_I$ converges to a finite limit U_∞, then for $i \geq 1$,

$$u_i = (1+U_\infty)^{-1} \sum_{k=i}^{\infty} \frac{\mu_1 \cdots \mu_k}{\lambda_1 \cdots \lambda_k},$$

which is strictly less than 1. □

Corollary 1.13. *1. The linear BD-process with rates λi and μi goes almost surely to extinction if and only if $\lambda \leq \mu$.*

2. The logistic BD-process goes almost surely to extinction.

Proof. 1) If $\lambda \leq \mu$, *i.e.* when the process is sub-critical or critical, we obtain $U_I \geq I - 1$ for any $I \geq 1$. Then $(U_I)_I$ goes to infinity when $I \to \infty$ and the process goes to extinction with probability 1. Conversely, if $\lambda > \mu$, the sequence $(U_I)_I$ converges to $\frac{\mu}{\lambda-\mu}$, and an easy computation gives $u_i = (\lambda/\mu)^i$.

2) Here we have

$$\lambda_i = \lambda i \ ; \ \mu_i = \mu i + ci(i-1). \tag{1.19}$$

It is easy to check that (1.18) is satisfied. □

1.4.2 Quasi-stationary distributions for birth and death processes

We consider a BD-process (Z_t) with almost sure extinction. A probability measure α on $\mathbb{N}^*$ is given by a sequence $(\alpha_j)_{j\geq 1}$ of non-negative numbers such that $\sum_{j\geq 1} \alpha_j = 1$.

Our first result is a necessary and sufficient condition for such a sequence $(\alpha_j)_{j\geq 1}$ to be a QSD for Z. Thereafter we will study the set of sequences which fulfill this condition (we refer the reader to Van Doorn [81] for details).

Theorem 1.14. *The sequence $(\alpha_j)_{j\geq 1}$ is a QSD if and only if*

1. $\alpha_j \geq 0, \ \forall j \geq 1,$ *and* $\sum_{j\geq 1} \alpha_j = 1$.
2. $\forall \ j \geq 1,$

$$\begin{aligned}
&\lambda_{j-1}\alpha_{j-1} - (\lambda_j + \mu_j)\alpha_j + \mu_{j+1}\alpha_{j+1} = -\mu_1\alpha_1\alpha_j; \\
&-(\lambda_1 + \mu_1)\alpha_1 + \mu_2\alpha_2 = -\mu_1\alpha_1^2.
\end{aligned} \tag{1.20}$$

The next result follows immediately.

Corollary 1.15. *Let us define inductively the sequence of polynomials* $(H_n(x))_n$ *as follows:* $H_1(x) = 1$ *for all* $x \in \mathbb{R}$ *and*

$$\begin{aligned}
\text{For } n \geq 2, \quad \lambda_n \ H_{n+1}(x) &= (\lambda_n + \mu_n - x) \ H_n(x) - \mu_{n-1} \ H_{n-1}(x); \\
\lambda_1 \ H_2(x) &= \lambda_1 + \mu_1 - x.
\end{aligned} \tag{1.21}$$

Then, any quasi-stationary distribution $(\alpha_j)_j$ *satisfies for all* $j \geq 1$,

$$\alpha_j = \alpha_1 \ \pi_j \ H_j(\mu_1\alpha_1),$$

where

$$\pi_1 = 1 \ ; \ \pi_n = \frac{\lambda_1 \cdots \lambda_{n-1}}{\mu_2 \cdots \mu_n}. \tag{1.22}$$

Proof of Theorem 1.14. By Proposition 1.4 and for a QSD α, there exists $\theta > 0$ such that

$$\alpha L = -\theta \, \alpha,$$

where L is the infinitesimal generator of Z restricted to $\mathbb{N}^*$. Taking the j^{th} component of this equation, we get

$$\begin{aligned}
&\lambda_{j-1}\alpha_{j-1} - (\lambda_j + \mu_j)\alpha_j + \mu_{j+1}\alpha_{j+1} = -\theta \, \alpha_j, \ \forall j \geq 2 \\
&-(\lambda_1 + \mu_1)\alpha_1 + \mu_2\alpha_2 = -\theta \, \alpha_1.
\end{aligned}$$

Summing over $j \geq 1$, we get after re-indexing

$$0 = \sum_{j\geq 1} \lambda_j\alpha_j - (\lambda_j + \mu_j)\alpha_j + \mu_j\alpha_j = -\theta \sum_{j\geq 1} \alpha_j + \mu_1\alpha_1.$$

We deduce that $\theta = \mu_1\alpha_1$, which concludes the proof of Theorem 1.14.

□

The study of the polynomials (H_n) has been detailed in Van Doorn [81]. In particular it is shown that there exists a non-negative number ξ_1 such that

$$x \leq \xi_1 \iff H_n(x) > 0, \ \forall n \geq 1.$$

By Corollary 1.15, $\alpha_j = \alpha_1 \, \pi_j \, H_j(\mu_1\alpha_1)$. Since for any j, $\alpha_j > 0$, we have $H_j(\mu_1\alpha_1) > 0$ for all $j \geq 1$ and then

$$0 < \mu_1\alpha_1 \leq \xi_1.$$

We can immediately deduce from this property that if $\xi_1 = 0$, then there is no quasi-stationary distribution.

To go further, one has to study more carefully the spectral properties of the semi-group (P_t) and the polynomials $(H_n)_n$, as it has been done in [51], [38] and [81]. From these papers, the polynomials $(H_n)_n$ are shown to be orthogonal with respect to the spectral measure of (P_t). In addition, it yields a tractable necessary and sufficient condition for the existence of QSD based on the birth and death rates. The series (S) with general term $S_n = \frac{1}{\lambda_n\pi_n}\sum_{i=n+1}^{\infty}\pi_i$ plays a crucial role. Remark that (S) converges if and only if $\sum_{n=1}^{\infty}\pi_n\left(\frac{1}{\mu_1}+\sum_{i=1}^{n-1}\frac{1}{\lambda_i\pi_i}\right) < +\infty$.

Theorem 1.16. *([81], Theorems 3.2 and 4.1). We have the convergence*

$$\lim_{t\to\infty}\mathbb{P}_i(Z_t = j | T_0 > t) = \frac{1}{\mu_1}\, \pi_j \, \xi_1 \, H_j(\xi_1).$$

In particular, we obtain

$$\xi_1 = \lim_{t\to\infty}\mu_1\mathbb{P}_1(Z_t = j | T_0 > t) \tag{1.23}$$

1. *If $\xi_1 = 0$, there is no QSD.*

2. *If (S) converges, then $\xi_1 > 0$ and the Yaglom limit is the unique QSD.*

3. *If (S) diverges and $\xi_1 \neq 0$, then there is a continuum of QSD, given by the one parameter family $(\hat{\alpha}_j(x))_{0<x\leq\xi_1}$:*

$$\hat{\alpha}_j(x) = \frac{1}{\mu_1}\, \pi_j \, x \, H_j(x).$$

Let us now develop some examples.

The linear case. We assume $\lambda_i = \lambda\, i$; $\mu_i = \mu\, i$ and $\lambda \leq \mu$. In that case, the BD-process is a branching process, where each individual reproduces with rate λ and dies with rate μ. A straight forward computation shows that the series (S) diverges. Setting $f_s : k \mapsto s^k$, we get by the Kolmogorov forward equation,

$$\frac{\partial P_t f_s(1)}{\partial t} = \mu P_t f_s(0) - (\lambda+\mu)P_t f_s(1) + \lambda P_t f_s(2).$$

But the branching property of the process implies $P_t f_s(2) = (P_t f_s(1))^2$, while $f_s(0) = 1$ so that

$$\frac{\partial P_t f_s(1)}{\partial t} = \mu - (\lambda+\mu) P_t f_s(1) + \lambda \left(P_t f_s(1)\right)^2.$$

Setting $m = 2\frac{\lambda}{\lambda+\mu}$, we deduce that for $s < 1$,

$$P_t f_s(1) = 1 - \frac{2(1-s)(2m-1)}{(ms+m-2)e^{-(\lambda+\mu)(m-1)t} + (1-s)m}.$$

In particular, we deduce that the generating function $F_t : s \mapsto \mathbb{E}(s^{Z_t} | Z_t > 0)$ of Z_t conditioned to $Z_t > 0$ converges when t goes to infinity:

$$F_t(s) = \frac{P_t f_s(1) - P_t f_0(1)}{1 - P_t f_0(1)} \xrightarrow[t\to\infty]{} \frac{(\lambda-\mu)s}{\lambda s - \mu}.$$

We deduce that the Yaglom limit of Z does not exist if $\lambda = \mu$ and is given by the geometric distribution with parameter $\frac{\lambda}{\mu}$ if $\lambda < \mu$:

$$\alpha_k = \left(\frac{\lambda}{\mu}\right)^{k-1} \left(1 - \frac{\lambda}{\mu}\right).$$

An easy computation yields $\xi_1 = \mu - \lambda$, since by (1.23), $\alpha_1 = \frac{\xi_1}{\mu}$. But the series (S) diverges so that for $\lambda < \mu$, $\xi_1 > 0$ and there is an infinite number of QSD. If $\lambda = \mu$, $\xi_1 = 0$ and there is no QSD.

The logistic case. We assume $\lambda_i = \lambda i$; $\mu_i = \mu i + c i(i-1)$. Because of the quadratic term, the branching property is lost and we can not compute the Yaglom limit as above. Therefore, we have no other choice than to study the convergence of the series (S).

We have

$$\begin{aligned}\sum_{i=n+1}^{\infty} \pi_i &\leq \sum_{i=n+1}^{\infty} \left(\frac{\lambda}{c}\right)^{i-1} \frac{1}{i!} = \sum_{p=0}^{\infty} \left(\frac{\lambda}{c}\right)^{n+p} \frac{1}{(n+p+1)!} \\ &\leq \left(\frac{\lambda}{c}\right)^{n} \frac{1}{(n+1)!} \sum_{p=0}^{\infty} \left(\frac{\lambda}{c}\right)^{p} \frac{1}{p!} = \left(\frac{\lambda}{c}\right)^{n} \frac{1}{(n+1)!}\, e^{\frac{\lambda}{c}},\end{aligned}$$

since $\frac{(n+1)!}{(n+p+1)!} \leq \frac{1}{p!}$. Thus as $\frac{1}{\pi_n} \leq C\left(\frac{c}{\lambda}\right)^{n-1} n!$, we get

$$\frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i \leq \frac{C}{c} \frac{1}{n(n+1)}\, e^{\frac{\lambda}{c}}.$$

Hence the series converges. Thus the Yaglom limit exists and is the unique quasi-stationary distribution.

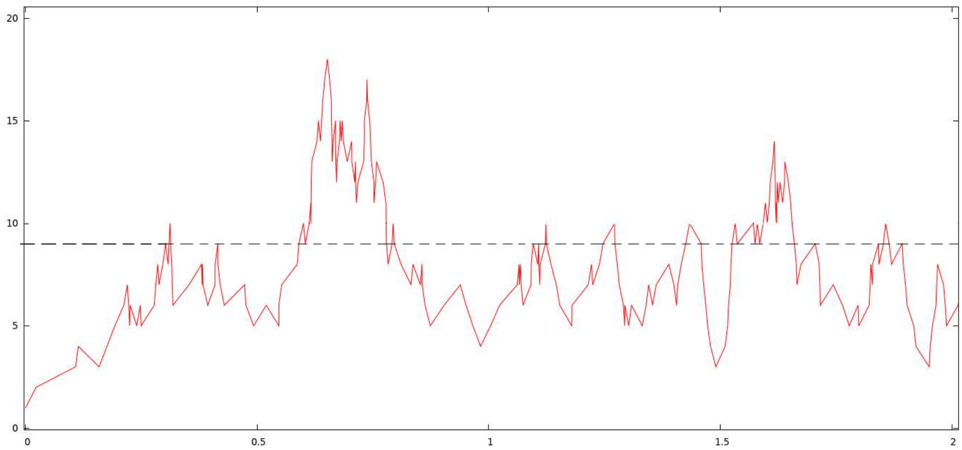


Figure 1.4: Example 3. A random path of a logistic birth and death process with initial size $Z_0 = 1$ and with parameters $\lambda = 10$, $\mu = 1$ and $c = 1$

In the logistic case, we're not able to compute explicitly the Yaglom limit. However, one can obtain substantial qualitative information by looking closer to the jump rates of the process. For instance, $(\lambda - \mu)/c$ is a key value for the process. Indeed, given a population size i, the expectation of the next step is equal to $\frac{i(\lambda-\mu-c(i-1))}{\lambda i+\mu i+ci(i-1)}$. Then the sign of this expectation depends on the position of $i-1$ with respect to $(\lambda - \mu)/c$:

- If $i \leq (\lambda - \mu)/c + 1$, then the expectation of the next step will be positive.
- If $i = (\lambda - \mu)/c + 1$, then it will be 0.
- If $i > (\lambda - \mu)/c + 1$, then it will be negative.

We deduce that the region around $(\lambda - \mu)/c$ is stable: it plays the role of a typical size for the population and we expect that the mass of the Yaglom limit is concentrated around it. The value $(\lambda - \mu)/c$ is called (by the biologists) the charge capacity of the logistic BD-process with parameters λ, μ and c. In the next section, we consider processes with large populations, which means that we study logistic BD-processes with large charge capacity.

Example 1.3. We develop now a numerical illustration of the logistic BD-process case. Across the whole example, the value of the charge capacity $\frac{\lambda-\mu}{c}$ is fixed, arbitrarily chosen equal to 9.

In order to illustrate the concept of charge capacity, we represent in Figure 1.4 a random path of a logistic birth and death process with initial size $Z_0 = 1$ and with parameters $\lambda = 10$, $\mu = 1$ and $c = 1$. We observe that the process remains for long times in a region around the charge capacity. Moreover, we see that the process remains mainly below the charge capacity; this is because the jumps rate

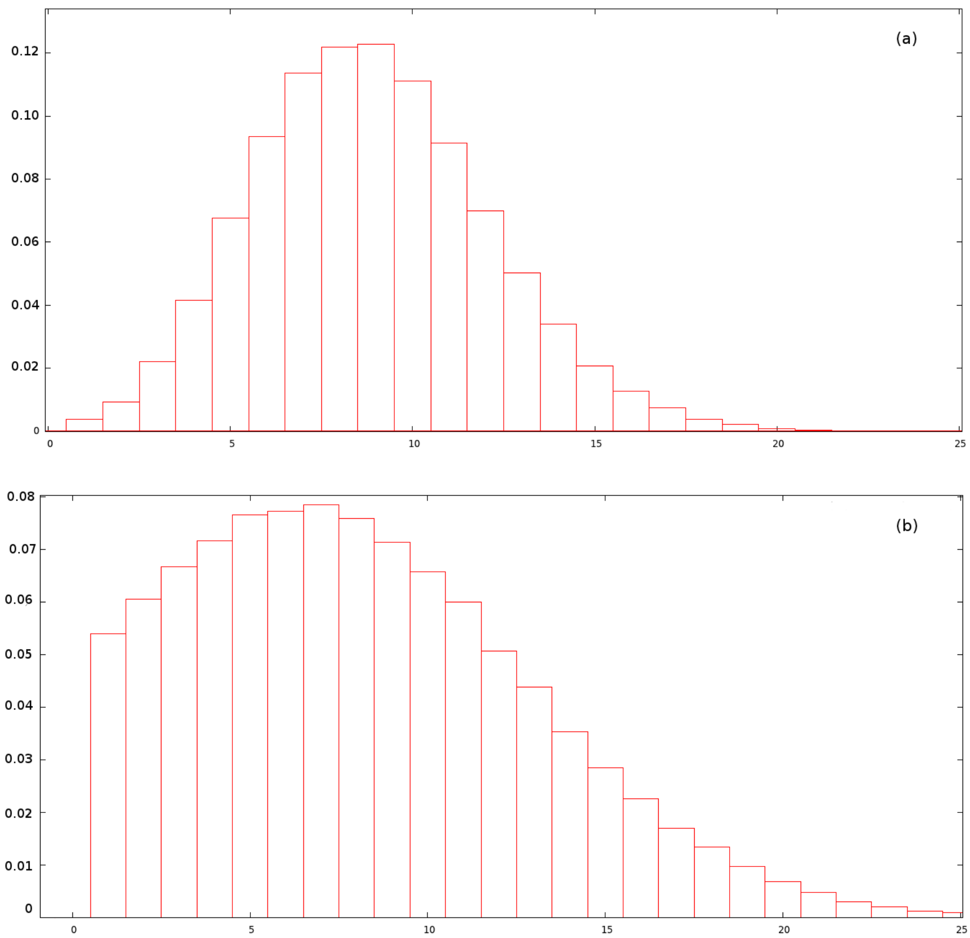


Figure 1.5: Example 3. The Yaglom limits of two logistic birth and death processes with the same charge capacity $\frac{\lambda-\mu}{c} = 9$: (a) $\lambda = 10$, $\mu = 1$ and $c = 1$; (b) $\lambda = 10$, $\mu = 7$ and $c = 1/3$.

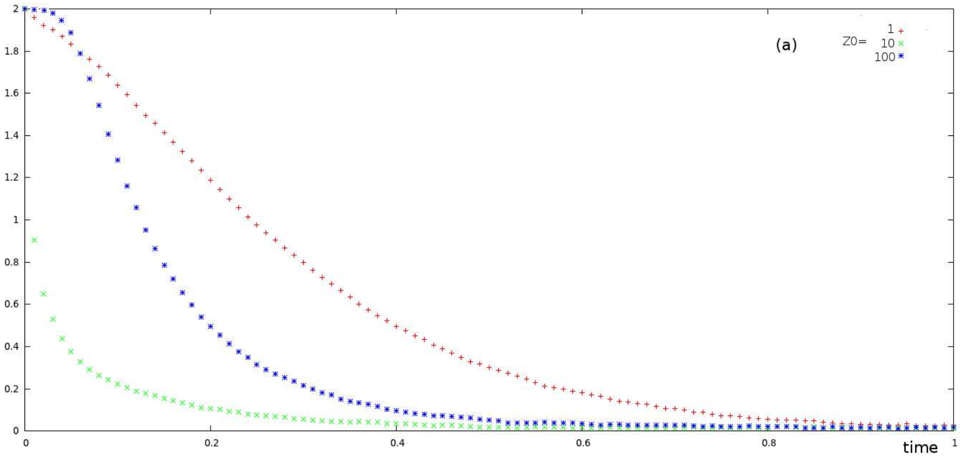
are higher in the upper region, so that it is less stable than the region below the charge capacity.

Let us now compare the Yaglom limits (which are numerically computed using the approximation method studied in the following chapters of this thesis) of two different logistic BD processes whose charge capacity is equal to 9 (see Figure 1.5):

(a) $Z^{(a)}$, whose parameters are $\lambda = 9$, $\mu = 1$ and $c = 1$,

(b) $Z^{(b)}$, whose parameters are $\lambda = 10$, $\mu = 7$ and $c = 1/3$.

We observe that the Yaglom limits of $Z^{(a)}$ and $Z^{(b)}$ are supported by a region which is around the charge capacity, this was expected because of the definition of



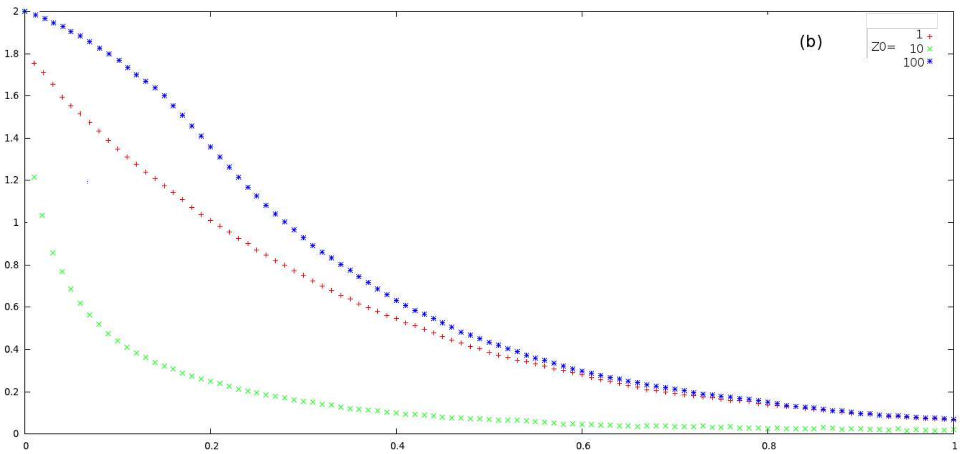


Figure 1.6: Example 3. Evolution of the distance between the conditioned distribution and the Yaglom limits of two logistic birth and death processes with the same charge capacity $\frac{\lambda-\mu}{c} = 9$: (a) $\lambda = 10$, $\mu = 1$ and $c = 1$; (b) $\lambda = 10$, $\mu = 7$ and $c = 1/3$.

the charge capacity. We also remark that the Yaglom limit of the process $Z^{(b)}$ has a more flat shape than the Yaglom limit of $Z^{(a)}$. This is because the competition parameter of $Z^{(b)}$ is small in comparison with the birth and death parameters, so that the drift toward the charge capacity is small to, both above and below the charge capacity.

We compute now the distance between the conditioned distribution and the Yaglom limit for the two processes $Z^{(a)}$ and $Z^{(b)}$ for different values of the initial state, namely $Z_0 = 1$, $Z_0 = 10$ and $Z_0 = 100$. The numerical results are represented Figure 1.6. We observe a strong dependence between the speed of convergence and the initial position of the processes. In the case of $Z^{(a)}$, it takes only a very short time to the process starting from 100 to reach the charge capacity, because the competition parameter is relatively high and so is the drift downward the charge capacity. On the contrary, in the case of $Z^{(b)}$, it takes a longer time for the process to come back from 100 to the charge capacity, so that the speed of convergence to the Yaglom limit is slow. In both cases, the convergence to the Yaglom limit happens very rapidly when starting from the value 10, because it is near the charge capacity.

1.5 The logistic Feller diffusion process

1.5.1 A large population model

We are now considering logistic birth and death processes modeling a large population with small individuals, that is with a large starting size and a large charge capacity assumption. We introduce a parameter $K \in \mathbb{N}^*$ and assume that the individual's weights (or biomasses) are equal to $\frac{1}{K}$. We also assume that the initial population size Z^K is of order K and we study the limiting behavior of the total biomass Z^K_t/K when K tends to infinity. Such a rescaling is classical and we recall here the methods used to prove it. In what follows, λ, μ and c are fixed positive constants.

In Subsection 1.5.1, individual birth and death rates are assumed to be constant and the competition rate depends linearly on the individual biomass $\frac{1}{K}$. In Subsection 1.5.1, we investigate the qualitative differences of evolutionary dynamics across populations with allometric demographics: life-lengths and reproduction times are assumed to be proportional to the individual's weights.

In both cases, the charge capacity of (Z^K) will be $(\lambda-\mu)K/c$.

Convergence to the logistic equation

Given a parameter K which scales the population's size, we consider the logistic BD-process Z^K with birth, death and competition parameters λ, μ and c/K respectively. We assume that the initial value of Z^K is of order K, in the sense that there exists a non-negative real random variable X_0 such that

$$\frac{Z^K_0}{K} \xrightarrow[K\to\infty]{} X_0 \quad \text{with } \mathbb{E}(X_0^3) < +\infty.$$

We consider the total biomass process defined by $X^K = Z^K/K$ for all $K \geq 1$ and are interested in the limit of X^K when $K \to \infty$. The transitions of the process $(X^K_t, t \geq 0)$ are the following ones:

$$\begin{aligned}
\frac{i}{K} &\to \frac{i+1}{K} \quad \text{with rate } \lambda i = \lambda K \frac{i}{K};\\
\frac{i}{K} &\to \frac{i-1}{K} \quad \text{with rate } \mu i + \frac{c}{K}\, i(i-1) = K\frac{i}{K}\left(\mu + c(\frac{i}{K} - \frac{1}{K})\right).
\end{aligned}$$

Theorem 1.17. *Assume that X_0 is a positive number x_0. Then, the process $(X^K_t, t \geq 0)$ converges in law in $\mathbb{D}([0,T], \mathbb{R}_+)$ to the unique continuous (in time) deterministic function solution of*

$$x(t) = x_0 + \int_0^t (\lambda - \mu - cx(s))x(s)ds.$$

Remark 1.3. The function x is thus solution of the ordinary differential equation

$$\dot{x} = (\lambda - \mu)x - cx^2 \; ; \; x(0) = x_0, \tag{1.24}$$

called the logistic equation. This equation has been historically introduced as the first macroscopic model describing populations regulated by competition between individuals (Verhulst 1938). In Theorem 1.17 above, it appears as the limit of properly scaled stochastic jump models.

Remark that the function x solution of (1.24) hits 0 in finite time if $\lambda < \mu$, while it remains positive forever if $\lambda \geq \mu$, converging in the long term to its unique stable equilibrium $\frac{\lambda-\mu}{c}$, also called charge capacity. Thus at this scale extinction does not happen.

Proof of Theorem 1.17. The Markov process $(X^K_t, t \geq 0)$ is well defined and its infinitesimal generator is given, for any measurable and bounded function ϕ, by

$$\begin{aligned} L_K\phi(x) &= \lambda K x \left(\phi(x + \frac{1}{K}) - \phi(x)\right) \qquad (1.25)\\ &\quad + K(\mu x + cx(x - \frac{1}{K}))\left(\phi(x - \frac{1}{K}) - \phi(x)\right). \qquad (1.26)\end{aligned}$$

Hence, we deduce by Dynkin's theorem ([28] Prop. IV-1.7) that

$$\phi(X^K_t) - \phi(X^K_0) - \int_0^t L_K\phi(X^k_s)ds \tag{1.27}$$

is a local martingale, and a martingale, as soon as each term in (1.27) is integrable. In particular, taking $\phi(x) = x$ leads that $(X^K_t, t \geq 0)$ is a semimartingale and there exists a local martingale M^K such that

$$X^K_t = X^K_0 + M^K_t + \int_0^t X^K_s \left(\lambda - \mu - c\left(X^K_s - \frac{1}{K}\right)\right) ds. \tag{1.28}$$

Since x_0 is deterministic and using a localization argument, we deduce that

$$\mathbb{E}(\sup_{t \leq T}(X^K_t)^2) < \infty.$$

Moreover, taking $\phi(x) = x^2$ applied to (1.27), and comparing with Itô's formula applied to $(X^K)^2$, we deduce that (M^K) is a square-integrable martingale with quadratic variation process

$$\langle M^K \rangle_t = \frac{1}{K} \int_0^t \left(\lambda + \mu + c\left(X^K_s - \frac{1}{K}\right)\right) X^K_s ds. \tag{1.29}$$

We now study the convergence in law of the sequence (X^K), when K tends to infinity. For any K, the law of X^K is a probability measure on the trajectory space $\mathbb{D}_T = \mathbb{D}([0,T],\mathbb{R}_+)$, that is the Skorohod space of left-limited and right-continuous functions from $[0,T]$ into $\mathbb{R}_+$, endowed with the Skorohod topology. This topology makes $\mathbb{D}_T$ a Polish state, that is a metrizable complete and separable space, which is not true if $\mathbb{D}_T$ is endowed with the uniform topology. See Billingsley [12] for details.

The proof of Theorem 1.17 is obtained by a compactness-uniqueness argument. The uniqueness of the solution of (1.24) is immediate.

By a natural coupling, one may bound the birth and death process X^K stochastically from above by the Yule process Y^K started from x_0, which jumps from x to $x+\frac{1}{K}$, at the same birth time than X^K. It is easy to show that $\sup_K \mathbb{E}(\sup_{t\leq T}(Y_t^K)^3) < \infty$, thus it turns out that

$$\sup_K \mathbb{E}(\sup_{t\leq T}(X_t^K)^3) < \infty.$$

From this uniform estimate, we deduce the uniform tightness of the laws of X^K (as probability measures on $\mathbb{D}_T$), using the Aldous criterion (cf. Aldous [2], Joffe-Métivier [49]). By Prokhorov's theorem, the compactness of the laws of (X^K) is thus proved. To get an intuition of the limit, we can firstly remark that

$$\sup_{t\leq T}|X_t^K - X_{t-}^K| \leq \frac{1}{K}.$$

Since the function $x \mapsto \sup_{t\leq T}|x_t - x_{t-}|$ is continuous on $\mathbb{D}_T$, we may conclude that each limiting value (in law) of the sequence (X^K) is a pathwise continuous process. In addition using (1.29) and (1.5.1), we easily get that

$$\lim_{K\to\infty} \mathbb{E}(\langle M^K\rangle_t) = 0.$$

The random fluctuations disappear when K tends to infinity and the limiting values are deterministic functions. Now it remains to show that these limiting values are solutions of (1.24), which can be done similarly to the proof of Theorem 1.18 (4) stated below. □

The logistic Feller diffusion process

In this section, we study the logistic BD-processes Z^K with birth and death rates given by $\gamma K + \lambda$ and $\gamma + \mu$ respectively. Here γ, λ and μ are still positive constants. We assume that the competition parameter is given by c/K, so that the charge capacity of Z^K is still $(\lambda - \mu)K/c$.

Remark 1.4. *This BD-process $(Z_t^K)_t$ can also be interpreted as a time-rescaled BD-process Y_{Kt}^K, whose birth, death and competition parameters are given by* $\gamma +$

λ/K, $\gamma+\mu/K$ and c/K^2 respectively, that is a critical BD-process with small pertubations.

We consider the sequence of processes X^K defined for all $t \geq 0$ by

$$X_t^K = \frac{Z_t^K}{K}.$$

The transitions of the process are given by

$$\begin{aligned} \frac{i}{K} &\rightarrow \frac{i+1}{K} \text{ with rate } \gamma K i + \lambda i \\ \frac{i}{K} &\rightarrow \frac{i-1}{K} \text{ with rate } \gamma K i + \mu i + \frac{c}{K} i(i-1) \end{aligned} \tag{1.30}$$

Formula (1.28) giving the semi-martingale decomposition of X^K will stay true with a martingale part N^K such that

$$\langle N^K \rangle_t = \frac{1}{K} \int_0^t (2\gamma K + \lambda + \mu + c\left(X_s^K - \frac{1}{K}\right) X_s^K ds.$$

One immediately observes that the expectation of this quantity will not tend to zero as K tends to infinity. Hence the fluctuations will not disappear at infinity and the limit will be random. Let us now state the theorem.

Theorem 1.18. *i) Consider the sequence of processes (X^K) with transitions (1.30) and initial condition X_0 such that $\mathbb{E}(X_0^3) < \infty$. It converges in law in $\mathcal{P}(\mathbb{D}_T)$ to the continuous process X, defined as the unique solution of the stochastic differential equation*

$$dX_t = \sqrt{2\gamma X_t} dB_t + \left((\lambda - \mu)X_t - cX_t^2\right) dt, X_0 \in]0, +\infty[, \tag{1.31}$$

where $(B_t)_{t\in[0,+\infty[}$ is a standard Brownian motion.

ii) Let introduce for each $y \geq 0$ the stopping time

$$T_y = \inf\{t \in \mathbb{R}_+, X_t = y\}. \tag{1.32}$$

For any $x \geq 0$, we get

$$\mathbb{P}_x(T_0 < \infty) = 1.$$

When $c = 0$, Equation (1.31) is the Feller stochastic differential equation. In the general case where $c \neq 0$, it will be called logistic Feller stochastic differential equation following the terminology introduced by Etheridge [27] and Lambert [57]. Let us remark that the solution of (1.31) is non-negative, and that 0 is an absorbing point.

Remark 1.5. Theorem 1.18 shows that the accumulation of a large amount of birth and death events creates stochasticity, often called by biologists ecological drift or demographic stochasticity. Contrarily to the previous case (Theorem 1.17), the limiting process suffers extinction almost surely.

Proof. As for Theorem 1.17, the proof is based on a uniqueness-compactness argument.

(1) The uniqueness of the solution of (1.31) follows from a general existence and pathwise uniqueness result in Ikeda-Watanabe [47] Section IV-3 or Karatzas-Shreve [50]. For a stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt,$$

with σ and b smooth enough, the existence and pathwise uniqueness are determined thanks to the following scale functions: for $x > 0$,

$$\begin{aligned} Q(x) &= -\int_1^x \frac{2b(y)}{\sigma^2(y)}dy \ ; \ \Lambda(x) = \int_1^x e^{Q(z)}dz \ ; \\ \kappa(x) &= \int_1^x e^{Q(y)} \left(\int_1^y e^{-Q(z)}dz \right) dy. \end{aligned} \tag{1.33}$$

More precisely, it is proved that (i) $\forall x > 0$, $\mathbb{P}_x(T_0 < T_\infty) = 1$ and (ii) $\Lambda(+\infty) = \infty$; $\kappa(0^+) < +\infty$ are equivalent. In that case, one has pathwise uniqueness of the process, and then uniqueness in law.

In our situation, the coefficients are given by

$$\sigma(x) = \sqrt{2\gamma x} \ ; \ b(x) = (\lambda - \mu)x - cx^2,$$

so that the functions Λ and κ satisfy (ii). Thus the SDE (1.31) has a unique pathwise solution which reaches 0 in finite time almost surely.

(2) Let us assume that $\mathbb{E}(X_0^3) < \infty$ and let us prove that

$$\sup_K \mathbb{E}(\sup_{t\leq T} (X_t^K)^2) < \infty.$$

The generator of X^K is given by

$$\begin{aligned} \tilde{L}_K\phi(x) &= (\gamma Kx + \lambda x)K \left(\phi(x + \frac{1}{K}) - \phi(x) \right) \\ &\quad + (\gamma Kx + \mu x + cx(x - \frac{1}{K}))K \left(\phi(x - \frac{1}{K}) - \phi(x) \right). \end{aligned} \tag{1.34}$$

With $\phi(x) = x^3$, we obtain that

$$\begin{aligned}(X_t^K)^3 &= X_0^3 + M_t^K + \int_0^t \gamma K^2 X_s^K \left[\left(X_s^K + \frac{1}{K}\right)^3 - \left(X_s^K - \frac{1}{K}\right)^3 - (X_s^K)^3\right] ds \\ &+ \int_0^t \lambda K X_s^K \left[\left(X_s^K + \frac{1}{K}\right)^3 - (X_s^K)^3\right] ds \\ &+ \int_0^t (\mu + c(X_s^K - 1)) K X_s^K \left[\left(X_s^K - \frac{1}{K}\right)^3 - (X_s^K)^3\right] ds,\end{aligned}$$

where M^K is a local martingale. Using a standard localization argument and that

$$\left(X_s^K + \frac{1}{K}\right)^3 - \left(X_s^K - \frac{1}{K}\right)^3 - (X_s^K)^3 = 6\,\frac{X_s^K}{K^2},$$

we get

$$\mathbb{E}((X_t^K)^3) \leq \mathbb{E}(X_0^3) + C \int_0^t \mathbb{E}((X_s^K)^3) ds,$$

where C is independent of K. By Gronwall's lemma, we deduce that

$$\sup_{t\leq T} \sup_K \mathbb{E}(|X_t^K|^3) < \infty. \tag{1.35}$$

Now, thanks to this result and to Doob's inequality, we may deduce from the semi-martingale decomposition of $(X_t^K)^2$ (obtained using $\phi(x) = x^2$), that

$$\sup_K \mathbb{E}(\sup_{t\leq T} |X_t^K|^2) < \infty. \tag{1.36}$$

(3) The uniform tightness of the laws of (X^K) is obtained, thanks to (1.36), using as before the Aldous criterion [2]. Then the sequence of laws is relatively compact and it remains to characterize its limit values.

(4) As in the proof of Theorem 1.17, we remark that the limiting values only charge the set of continuous trajectories, since $\sup_{t\leq T} |\Delta X_t^K| \leq \frac{1}{K}$. Let $\mathbb{Q} \in \mathcal{P}(C([0,T], \mathbb{R}_+))$ be a limiting value of the sequence of laws of the processes X^K. We will identify $\mathbb{Q}$ as the (unique) law of the solution of the logistic Feller stochastic differential equation and the convergence will be proved. Let us denote $C_T = C([0,T], \mathbb{R}_+)$ and define, for $\phi \in C_b^2$ and $t > 0$, the function

$$\begin{aligned}\psi_t : C_T &\to \mathbb{R} \\ X &\mapsto \phi(X_t) - \phi(X_0) - \int_0^t \left(\gamma X_s \phi''(X_s) + ((\lambda - \mu) X_s - c X_s^2) \phi'(X_s)\right) ds,\end{aligned}$$

which is continuous $\mathbb{Q}$-a.s.. Our aim is firstly to show that the process $(\psi_t(X))_t$ is a $\mathbb{Q}$-martingale.

For $x \in \mathbb{R}_+$, let us define

$$L\phi(x) = \gamma x\phi''(x) + ((\lambda - \mu)x - cx^2)\phi'(x).$$

Using the Taylor expansion, we immediately get (where $\tilde{L}_K$ has been defined in (1.34))

$$\begin{aligned}
|\tilde{L}_K\phi(x) - L\phi(x)| &= \gamma K^2\ x \left|\phi(x+\frac{1}{K}) + \phi(x-\frac{1}{K}) - 2\phi(x) - \frac{1}{K^2}\phi''(x)\right| \\
&\quad +\lambda\ K\ x \left|\phi(x+\frac{1}{K}) - \phi(x) - \frac{1}{K}\phi'(x)\right| \\
&\quad +K\ (\mu x + cx(x-1)) \left|\phi(x-\frac{1}{K}) - \phi(x) + \frac{1}{K}\phi'(x)\right| \\
&\le \frac{C}{K}\ (x^2+1), \qquad (1.37)
\end{aligned}$$

where C doesn't depend on x and K. By (1.36), we deduce that $\mathbb{E}\left(|\tilde{L}_K\phi(X_t^K) - L\phi(X_t^K)|\right)$ tends to 0 as K tends to infinity, uniformly for $t \in [0,T]$.

For $s_1 < \cdots < s_k < s < t$, for $g_1, \cdots, g_k \in C_b$, we introduce the function H defined on the path space by

$$H(X) = g_1(X_{s_1})\cdots g_k(X_{s_k})\left(\psi_t(X) - \psi_s(X)\right).$$

Let us show that

$$\mathbb{E}_{\mathbb{Q}}(H(X)) = 0, \qquad (1.38)$$

which implies that $(\psi_t(X))_t$ is a $\mathbb{Q}$-martingale.

By construction, $\psi_t^K(X^K) = \phi(X_t^K) - \phi(X_0) - \int_0^t \tilde{L}_K\phi(X_s^K)ds$ defines a martingale, then

$$\mathbb{E}\left[g_1(X_{s_1}^K)\cdots g_k(X_{s_k}^K)\left(\psi_t^K(X^K) - \psi_s^K(X^K)\right)\right] = 0.$$

In another way, this quantity is equal to

$$\begin{aligned}
&\mathbb{E}\left[g_1(X_{s_1}^K)\cdots g_k(X_{s_k}^K)\left(\psi_t^K(X^K) - \psi_s^K(X^K) - \psi_t(X^K) + \psi_s(X^K)\right)\right] \\
&+\mathbb{E}\left[g_1(X_{s_1}^K)\cdots g_k(X_{s_k}^K)\left(\psi_t(X^K) - \psi_s(X^K)\right) - g_1(X_{s_1})\cdots g_k(X_{s_k})\left(\psi_t(X) - \psi_s(X)\right)\right] \\
&+\mathbb{E}\left[g_1(X_{s_1})\cdots g_k(X_{s_k})\left(\psi_t(X) - \psi_s(X)\right)\right].
\end{aligned}$$

The first term is equal to $\mathbb{E}\left[g_1(X_{s_1}^K)\cdots g_k(X_{s_k}^K)\int_0^t\left(\tilde{L}_K\phi(X_s^K) - L\phi(X_s^K)\right)ds\right]$ and tends to 0 by (1.36) and (1.37).
The second term is equal to $\mathbb{E}(H(X^K) - H(X))$. Since $X \mapsto H(X)$ is continuous and since $H(X) \le C\left(1 + \int_s^t(1+X_u^2)du\right)$ and is therefore uniformly integrable by (1.35), this term tends to 0 as K tends to infinity.
Then (1.38) is fulfilled and the process $\psi_t(X) = \phi(X_t) - \phi(X_0) - \int_0^t L\phi(X_s)ds$ is a $\mathbb{Q}$-martingale.

(5) The last step consists in proving that under $\mathbb{Q}$, the process X is solution of the logistic Feller stochastic differential equation (1.31). By (1.36) and taking $\phi(x) = x$ leads to $X_t = X_0 + M_t + \int_0^t ((\lambda - \mu)X_s - cX_s^2)ds$, where M is a martingale. Taking $\phi(x) = x^2$ in one hand and applying Itô's formula for X_t^2 in another hand allows us to identify

$$\langle M \rangle_t = \int_0^t 2\,\gamma\, X_s\, ds.$$

By the representation theorem proved in [50] Theorem III-4.2 or in [47], there exists a Brownian motion B such that

$$M_t = \int_0^t \sqrt{2\gamma X_s}\, dB_s.$$

That concludes the proof.

□

1.5.2 QSD for logistic Feller diffusion processes

We are now interested in studying the quasi-stationarity for the logistic Feller diffusion process solution of the equation

$$dZ_t = \sqrt{Z_t} dB_t + (rZ_t - cZ_t^2)dt, \quad Z_0 > 0,$$

where the Brownian motion B and the initial state Z_0 are given, and r and c are assumed to be positive. (We have assumed that $\gamma = 1/2$). The results and proofs that are presented in this section have been obtained by Cattiaux, Collet, Lambert, Martínez, Méléard and San Martín in [16].

The main theorem of this part is the following.

Theorem 1.19. *Assume that Z_0, r and c are positive. Then the Yaglom limit α of the process Z exists and is the unique QSD of Z.*

Moreover, there exists a positive function η_1 on $\mathbb{R}_+^*$ such that

1. $$\alpha(dx) = \frac{\eta_1(x)e^{-Q(x)}}{\int_{\mathbb{R}_+^*} \eta_1(y)e^{-Q(y)}dy} dx, \tag{1.39}$$

2. $\forall x \in \mathbb{R}_+^*$, $\lim_{t\to\infty} e^{\theta(\alpha)t}\mathbb{P}_x(T_0 > t) = \eta_1(x)$,

3. *there exists $\chi > 0$ such that, $\forall x \in \mathbb{R}_+^*$,*

$$\lim_{t\to+\infty} e^{-(\chi-\theta(\alpha))t} \left|\mathbb{P}_x\left(X_t \in A | T_0 > t\right) - \alpha(A)\right| < +\infty.$$

4. *the QSD α attracts all initial distribution, which means that α is a QLD for Z starting from any initial distribution.*

Remark 1.6. *The theory studying the quasi-stationary distributions for one-dimensional diffusion processes started with Mandl [63] and has been developed by many authors. See in particular [19], [65], [79], [53]. Nevertheless in most of the papers, the diffusion and drift coefficients are regular and the "Mandl's condition" $\kappa(+\infty) = \infty$ (see (1.33)) is assumed. This condition is not satisfied in our case because of the degeneracy of the diffusion and the non boundedness of the drift coefficient.*

Theorem 1.19 differs from the results obtained in case of smooth drifts or going slower to infinity. For example, Lambert [58] proves that if $c = 0$ and $r \leq 0$, then either $r = 0$ and there is no QSD, or $r < 0$ and there is an infinite number of QSD. Lladser and San Martin [61] show that in the case of the Ornstein-Uhlenbeck process

$$dY_t = dB_t - Y_t dt,$$

killed at 0, there is a continuum of QSD.

In the logistic Feller diffusion situation as in the logistic BD-process, the uniqueness is given by the competition term $c X_t^2$ induced by the ecological constraints.

The proof of Theorem 1.19 will be declined in the next subsections.

spectral theory for the killed semi-group

We firstly make a change of variable. Let us introduce the process $(X_t; , t \geq 0)$ defined by $X_t = 2\sqrt{Z_t}$. Of course, X is absorbed at 0 (like Z) and the research of QSD for Z will be easily deduced from the one obtained for X.

An elementary computation using Itô's formula shows that

$$dX_t = dB_t - q(X_t)dt, \tag{1.40}$$

where the function $q(x)$ is given by

$$q(x) = \frac{1}{2x} - \frac{rx}{2} + \frac{cx^3}{8}.$$

Such a process X driven by a Brownian motion is called a Kolmogorov diffusion process. Let us remark that the function q is continuous on $\mathbb{R}_+^*$ but explodes at 0 as $\frac{1}{2x}$ and at infinity as $\frac{c}{8}x^3$. The strong (cubic) downward drift at infinity will force the process to live essentially in compact sets. That will provide the uniqueness of the QSD, as seen below.

Let us study the Kolmogorov diffusion process (1.40). As before we are interested in the semi-group of the killed process, that is, for any $x > 0$, for any $t > 0$, for any $f \in C_b(\mathbb{R}_+^*)$,

$$P_t f(x) = \mathbb{E}_x(f(X_t)\mathbf{1}_{t<T_0}), \tag{1.41}$$

with the associated infinitesimal generator given for $\phi \in C_c^2((0,+\infty))$ by

$$L\phi = \frac{1}{2}\phi'' - q\phi'.$$

We are led to develop a spectral theory for this generator. Firstly, we introduce the measure μ, defined by

$$\mu(dy) = e^{-Q(y)}dy,$$

where Q is defined by

$$Q(y) = \int_1^y 2q(z)dz = \ln y + \frac{r}{2}(1-y^2) + \frac{c}{16}(y^4-1). \tag{1.42}$$

In particular $-Q/2$ is a potential of the drift $-q$. Let us remark that in our case, the measure μ is not finite. Nevertheless, through the unity function 1 does not belong to $\mathbb{L}^2(\mu)$, this space is the good functional space in which to work. The key point we firstly show is that, starting from $x > 0$, the law of the killed process at time t is absolutely continuous with respect to μ with a density belonging to $\mathbb{L}^2(\mu)$. The first step of the proof is a Girsanov Theorem.

Proposition 1.20. *For any bounded Borel function F defined on $\Omega = C([0,t],\mathbb{R}_+^*)$ it holds*

$$\mathbb{E}_x\left[F(\omega)\boldsymbol{1}_{t<T_0(\omega)}\right] = \mathbb{E}^{\mathbb{W}_x}\left[F(\omega)\boldsymbol{1}_{t<T_0(\omega)}\exp\left(\frac{1}{2}Q(x) - \frac{1}{2}Q(\omega_t) - \frac{1}{2}\int_0^t (q^2-q')(\omega_s)ds\right)\right]$$

where $\mathbb{E}^{\mathbb{W}_x}$ denotes the expectation with respect to the Wiener measure starting from x and ω the current point in Ω.

Proof. It is enough to show the result for non-negative and bounded functions F. Let $\varepsilon \in (0,1)$ and $\tau_\varepsilon = T_\varepsilon \wedge T_{1/\varepsilon}$. Let us choose some ψ_ε which is a non-negative C^∞ function with compact support included in $]\varepsilon/2, 2/\varepsilon[$ such that $\psi_\varepsilon(u) = 1$ if $\varepsilon \le u \le 1/\varepsilon$. For all x such that $\varepsilon \le x \le 1/\varepsilon$ the law of the diffusion (1.40) coincides up to τ_ε with the law of a similar diffusion process X^ε obtained by replacing q with the cutoff function $q_\varepsilon = q\psi_\varepsilon$. For the latter we may apply Novikov criterion (cf. [74] p.332), ensuring that the law of X^ε is given via Girsanov's formula. Hence

$$\begin{aligned}
\mathbb{E}_x\left[F(\omega)\mathbf{1}_{t<\tau_\varepsilon(\omega)}\right] &= \mathbb{E}^{\mathbb{W}_x}\left[F(\omega)\,\mathbf{1}_{t<\tau_\varepsilon(\omega)}\,\exp\left(\int_0^t -q_\varepsilon(\omega_s)d\omega_s - \frac{1}{2}\int_0^t (q_\varepsilon)^2(\omega_s)ds\right)\right]\\
&= \mathbb{E}^{\mathbb{W}_x}\left[F(\omega)\,\mathbf{1}_{t<\tau_\varepsilon(\omega)}\,\exp\left(\int_0^t -q(\omega_s)d\omega_s - \frac{1}{2}\int_0^t q^2(\omega_s)ds\right)\right]\\
&= \mathbb{E}^{\mathbb{W}_x}\left[F(\omega)\mathbf{1}_{t<\tau_\varepsilon(\omega)}\exp\left(\frac{1}{2}Q(x) - \frac{1}{2}Q(\omega_t) - \frac{1}{2}\int_0^t (q^2-q')(\omega_s)ds\right)\right]
\end{aligned}$$

integrating by parts the stochastic integral. But $\mathbf{1}_{t<\tau_\varepsilon}$ is non-decreasing in ε and converges almost surely to $\mathbf{1}_{t<T_0}$ both for $\mathbb{W}_x$ and for $\mathbb{P}_x$ (since $\mathbb{P}_x(T_0 < \infty) = 1$)).

Indeed, almost surely,

$$\lim_{\varepsilon\to 0} X_{\tau_\varepsilon} = \lim_{\varepsilon\to 0} X_{\tau_\varepsilon} = \lim_{\varepsilon\to 0} \varepsilon = 0$$

so that $\lim_{\varepsilon\to 0} \tau_\varepsilon \geq T_0$. But $\tau_\varepsilon \leq T_0$ yielding the equality. It remains to use Lebesgue monotone convergence theorem to finish the proof. □

Theorem 1.21. *For all $x > 0$ and all $t > 0$ there exists a density function $r(t,x,.)$ that satisfies*

$$\mathbb{E}_x[f(X_t)\,\boldsymbol{1}_{t<T_0}] = \int_0^{+\infty} f(y)\, r(t,x,y)\, \mu(dy)$$

for all bounded Borel function f. In addition, for all $t > 0$ and all $x > 0$,

$$\int_0^{+\infty} r^2(t,x,y)\,\mu(dy) \leq (1/2\pi t)^{\frac{1}{2}}\, e^{Ct}\, e^{Q(x)}\,,$$

where

$$C = -\inf_{y>0}(q^2(y) - q'(y)) < +\infty.$$

Proof. Define

$$G(\omega) = \mathbf{1}_{t<T_0(\omega)} \exp\left(\frac{1}{2}\,Q(\omega_0) - \frac{1}{2}\,Q(\omega_t) - \frac{1}{2}\int_0^t (q^2 - q')(\omega_s)ds\right).$$

Denote by

$$e^{-v(t,x,y)} = (2\pi t)^{-\frac{1}{2}} \exp\left(-\frac{(x-y)^2}{2t}\right)$$

the density at time t of the Brownian motion starting from x. According to Proposition 1.20, we have

$$\begin{aligned}
\mathbb{E}_x\left(f(X_t)\,\mathbf{1}_{t<T_0}\right) &= \mathbb{E}^{\mathbb{W}_x}\left(f(\omega_t)\;\mathbb{E}^{\mathbb{W}_x}(G|\omega_t)\right)\\
&= \int_0^{+\infty} f(y)\,\mathbb{E}^{\mathbb{W}_x}(G|\omega_t = y)\,e^{-v(t,x,y)}\,dy\\
&= \int_0^{+\infty} f(y)\,\mathbb{E}^{\mathbb{W}_x}(G|\omega_t = y)\,e^{-v(t,x,y)+Q(y)}\,\mu(dy)\,,
\end{aligned}$$

because $\mathbb{E}^{\mathbb{W}_x}(G|\omega_t = y) = 0$ if $y \leq 0$. In other words, the law of X_t restricted to non extinction has a density with respect to μ given by

$$r(t,x,y) = \mathbb{E}^{\mathbb{W}_x}(G|\omega_t = y)\,e^{-v(t,x,y)+Q(y)}\,.$$

Hence

$$\begin{aligned}
\int_0^{+\infty} r^2(t,x,y)\,\mu(dy) &= \int_0^{+\infty} \left(\mathbb{E}^{\mathbb{W}_x}(G|\omega_t = y)\,e^{-v(t,x,y)+Q(y)}\right)^2\\
&\qquad \times e^{-Q(y)+v(t,x,y)}\,e^{-v(t,x,y)}\,dy\\
&= \mathbb{E}^{\mathbb{W}_x}\left(e^{-v(t,x,\omega_t)+Q(\omega_t)}\,\left(\mathbb{E}^{\mathbb{W}_x}(G|\omega_t)\right)^2\right)\\
&\leq \mathbb{E}^{\mathbb{W}_x}\left(e^{-v(t,x,\omega_t)+Q(\omega_t)}\,\mathbb{E}^{\mathbb{W}_x}(G^2|\omega_t)\right)\\
&\leq e^{Q(x)}\,\mathbb{E}^{\mathbb{W}_x}\left(\mathbf{1}_{t<T_0(\omega)}\,e^{-v(t,x,\omega_t)}\,e^{-\int_0^t (q^2-q')(\omega_s)ds}\right),
\end{aligned}$$

where we have used Cauchy-Schwarz's inequality. Since $e^{-v(t,x,.)} \leq (1/2\pi t)^{\frac{1}{2}}$, the proof is completed. $\square$

Thanks to Theorem 1.21, we can show, using the theory of Dirichlet forms (cf. Fukushima's theory [33]) that the infinitesimal generator L of X, defined by (1.5.2), can be extended to the generator of a continuous symmetric semi-group of contractions of $\mathbb{L}^2(\mu)$ denoted by $(P_t)_{t\geq 0}$. Then we can develop a spectral theory for L and P_t in $\mathbb{L}^2(\mu)$. In all what follows, and for $f,g \in \mathbb{L}^2(\mu)$, we will denote

$$\langle f, g\rangle_\mu = \int_{\mathbb{R}_+} f(x)g(x)\mu(dx).$$

The symmetry of P_t means that

$$\langle P_t f, g\rangle_\mu = \langle f, P_t g\rangle_\mu.$$

In Cattiaux et al. [16], the following spectral theorem in $\mathbb{L}^2(\mu)$ is proved.

Theorem 1.22. *The operator $-L$ has a purely discrete spectrum $0 < \lambda_1 < \lambda_2 < \cdots$. Furthermore each λ_i ($i \in \mathbb{N}^*$) is associated with a unique (up to a multiplicative constant) eigenfunction η_i of class $C^2((0,\infty))$, which satisfies the ODE*

$$\frac{1}{2}\eta_i'' - q\eta_i' = -\lambda_i \eta_i. \tag{1.43}$$

The sequence $(\eta_i)_{i\geq 1}$ is an orthonormal basis of $\mathbb{L}^2(\mu)$ and $\eta_1(x) > 0$ for all $x > 0$. In addition, for each i, $\eta_i \in \mathbb{L}^1(\mu)$.

The proof of this theorem is based on a relation between the Fokker-Planck operator L and a Schrödinger operator. Indeed, let us set for $g \in \mathbb{L}^2(dx)$,

$$\tilde{P}_t g = e^{-Q/2}\, P_t(g\, e^{Q/2}).$$

$\tilde{P}_t$ is a strongly semi-group on $\mathbb{L}^2(dx)$ with generator defined for $g \in C_c^\infty((0,+\infty))$ by

$$\tilde{L}g = \frac{1}{2}\triangle g - \frac{1}{2}(q^2 - q')\, g.$$

The spectral theory for such Schrödinger operator with potential $\frac{(q^2-q')}{2}$ on the line (or the half-line) is well known (see for example the book of Berezin-Shubin [10]), but the potential $\frac{(q^2-q')}{2}$ does not belong to $\mathbb{L}^\infty_{loc}$ as generally assumed. Nevertheless, in our case $\inf(q^2 - q') > -\infty$, which ensures the compactness of these operators.

The following corollary of Theorem 1.22 is a generalization of the Perron-Frobenius Theorem in the infinite-dimensional case.

Corollary 1.23. *For all bounded and measurable function f, we have*

$$P_t f =_{\mathbb{L}^2(\mu)} \sum_{i\in\mathbb{N}^*} e^{-\lambda_i t} \ \langle \eta_i, f\rangle_\mu \ \eta_i. \tag{1.44}$$

Proof. Fix $t > 0$ and let f be a bounded measurable function on $\mathbb{R}_+^*$. Let us first prove that $P_t f$ belongs to $\mathbb{L}^2(\mu)$. On the one hand, we have

$$\int_1^{+\infty} (P_t f(x))^2 d\mu(x) \le \|f\|_\infty^2 \int_1^\infty e^{-Q(x)} dx < \infty.$$

On the other hand, by Proposition 1.20, we have, for all $x \in \mathbb{R}_+^*$,

$$\begin{aligned} P_t f(x) &\le \|f\|_\infty e^{\frac{1}{2}Q(x)+\frac{1}{2}Ct} \mathbb{E}^{\mathbb{W}_x}\left[\mathbf{1}_{t<T_0(\omega)} e^{-\frac{1}{2}Q(\omega_t)}\right] \\ &\le \|f\|_\infty e^{\frac{1}{2}Q(x)+\frac{1}{2}Ct} \int_0^\infty e^{-\frac{1}{2}Q(y)} \frac{e^{-\frac{1}{2t}(y-x)^2}}{\sqrt{2\pi t}} dy. \end{aligned}$$

But the function

$$y \mapsto e^{-\frac{1}{2}Q(y)} = \frac{1}{\sqrt{y}} e^{-\frac{r}{4}(1-y^2)-\frac{c}{32}(y^4-1)},$$

is integrable on $]0, +\infty[$. Since $e^{-\frac{1}{2t}(y-x)^2} \le 1$, we deduce that there exists a constant $K_t > 0$ independent of x and f such that

$$P_t f(x) \le K_t \|f\|_\infty e^{\frac{1}{2}Q(x)},$$

and thus

$$\int_0^1 (P_t f(x))^2 d\mu(x) \le K_t^2 \|f\|_\infty^2.$$

Finally $(P_t f)^2$ is integrable with respect to μ, so that $P_t f \in \mathbb{L}^2(\mu)$.

Now we deduce from Theorem 1.22 that

$$P_t f =_{\mathbb{L}^2(\mu)} \sum_{i\in\mathbb{N}^*} \langle P_t f, \eta_i\rangle_\mu \ \eta_i \tag{1.45}$$

If f belongs to $\mathbb{L}^2(\mu)$, then the symmetry of P_t implies that

$$\begin{aligned} \langle P_t f, \eta_i\rangle_\mu &= \langle f, P_t \eta_i\rangle_\mu \\ &= e^{-\lambda_i t} \ \langle f, \eta_i\rangle_\mu. \end{aligned}$$

Since $\eta_i \in \mathbb{L}^1(\mu)$, we deduce from the Dominated Convergence Theorem that the equality $\langle P_t f, \eta_i\rangle_\mu = e^{-\lambda_i t} \ \langle f, \eta_i\rangle_\mu$ extends to all measurable bounded functions. This and the equality (1.45) allow us to conclude the proof of Corollary 1.23. □

Existence of the Yaglom limit

By Corollary 1.23, we have for any bounded and measurable function f,

$$\begin{aligned}\|e^{\lambda_1 t}P_t f - \langle \eta_1, f\rangle \eta_1\|^2_{\mathbb{L}^2(\mu)} &= \sum_{i\geq 2} e^{-2t(\lambda_i-\lambda_1)}|\langle \eta_i,f\rangle|^2\\ &\leq e^{-2(t-1)(\lambda_2-\lambda_1)}\sum_{i\in\mathbb{N}^*} e^{-2(\lambda_i-\lambda_1)}|\langle \eta_i,f\rangle|^2\\ &\leq e^{-2(t-1)(\lambda_2-\lambda_1)}\, e^{2\lambda_1}\, \|P_1 f\|^2_{\mathbb{L}^2(\mu)}\end{aligned}$$

Using Cauchy-Schwartz inequality, we deduce that, for any function $h \in \mathbb{L}^2(\mu)$,

$$\left|e^{\lambda_1 t}\langle P_t f,h\rangle - \langle \eta_1, f\rangle\langle \eta_1, h\rangle_\mu\right| \leq e^{-2(t-1)(\lambda_2-\lambda_1)}\|P_1 f\|_{\mathbb{L}^2(\mu)}\|h\|_{\mathbb{L}^2(\mu)}. \tag{1.46}$$

By Theorem 1.21, $\delta_x P_1$ has the density $r(1,x,.) \in \mathbb{L}^2(\mu)$ with respect to μ, so that

$$\left\|e^{\lambda_1 t}P_{t+1}f(x) - \langle \eta_1, f\rangle\langle \eta_1, r(1,x,\cdot)\rangle_\mu\right\| \leq e^{-2(t-1)(\lambda_2-\lambda_1)}\|P_1 f\|_{\mathbb{L}^2(\mu)}\|r(1,c,\cdot)\|_{\mathbb{L}^2(\mu)}.$$

By definition of η_1, we have $\langle \eta_1, r(1,x,\cdot)\rangle_\mu = e^{-\lambda_1}\eta_1(x)$. Thus we have

$$e^{\lambda_1 t}P_{t+1}f(x) \xrightarrow[t\to+\infty]{} \langle \eta_1, f\rangle_\mu\, e^{-\lambda_1}\eta_1(x)$$

and

$$e^{\lambda_1 t}P_{t+1}\mathbf{1}_{\mathbb{R}_+^*}(x) \xrightarrow[t\to\infty]{} \langle \eta_1, \mathbf{1}_{\mathbb{R}_+^*}\rangle_\mu\, e^{-\lambda_1}\eta_1(x)$$

Finally, $\eta_1(x)$ being positive,

$$\frac{P_t f(x)}{P_t\mathbf{1}_{\mathbb{R}_+^*}(x)} \xrightarrow[t\to+\infty]{} \frac{\langle \eta_1, f\rangle_\mu}{\langle \eta_1, \mathbf{1}_{\mathbb{R}_+^*}\rangle_\mu} = \alpha(f),$$

where α is defined in (1.39). We conclude that α is a Yaglom limit for Z. We also deduce parts (2) and (3) of Theorem 1.19.

Attractiveness of any initial distributions

We begin by showing the attractiveness of compactly supported probability measures. Let ν be a compactly supported probability measure on $(0,+\infty)$. By Theorem 1.21, $y \mapsto \int_{E^*} r(1,x,y)d\nu(x)$ is the density of νP_1 with respect to μ. By [16, Lemma 5.3], there exists a locally bounded function Θ such that

$$r(1,x,y) \leq \Theta(x)\eta_1(y),\ \forall x,y \in (0,+\infty).$$

In particular, $h : y \mapsto \int_{E^*} r(1,x,y)d\nu(x)$ belongs to $\mathbb{L}^2$. Then we deduce from (1.46) that

$$\mathbb{E}_\nu\left(f(X_{t+1})\middle|\, T_0 > t+1\right) = \frac{\nu P_{t+1}(f)}{\nu P_{t+1}(\mathbf{1}_{E^*})} \xrightarrow[t\to\infty]{} \alpha(f).$$

We conclude that α attracts any compactly supported probability measure.

Let us now prove that α attracts all initial distributions ν supported in $(0,\infty)$, which means that, for any probability measure ν on $\mathbb{R}_+^*$, for any Borel set A, we get

$$\lim_{t\to\infty} \mathbb{P}_\nu(X_t \in A | T_0 > t) = \alpha(A). \tag{1.47}$$

This is part (4) of Theorem 1.19 and it clearly implies the uniqueness of the QSD for Z.

Proposition 1.24. *For any $a > 0$, there exists $y_a > 0$ such that $\sup_{x>y_a} \mathbb{E}_x(e^{aT_{y_a}}) < \infty$.*

Proof. Let us remark that

$$\int_1^\infty e^{Q(y)} \int_y^\infty e^{-Q(z)}\, dz\, dy < \infty.$$

Let $a > 0$, and pick x_a large enough so that

$$\int_{x_a}^\infty e^{Q(x)} \int_x^\infty e^{-Q(z)}\, dz\, dx \le \frac{1}{2a}\,.$$

Let J be the nonnegative increasing function defined on $[x_a,\infty)$ by

$$J(x) = \int_{x_a}^x e^{Q(y)} \int_y^\infty e^{-Q(z)}\, dz\, dy.$$

Then check that $J'' = 2qJ' - 1$, so that $LJ = -1/2$. Set now $y_a = 1 + x_a$, and consider a large $M > x$. Itô's formula gives

$$\mathbb{E}_x(e^{a(t\wedge T_M \wedge T_{y_a})}\, J(X_{t\wedge T_M\wedge T_{y_a}})) = J(x) + \mathbb{E}_x\left(\int_0^{t\wedge T_M\wedge T_{y_a}} e^{as}\,(aJ(X_s) + LJ(X_s))\, ds\right).$$

But $LJ = -1/2$, and $J(X_s) < J(\infty) \le 1/(2a)$ for any $s \le T_{y_a}$, so that

$$\mathbb{E}_x(e^{a(t\wedge T_M\wedge T_{y_a})}\, J(X_{t\wedge T_M\wedge T_{y_a}})) \le J(x)\,.$$

For $x \ge y_a$, one gets $1/(2a) > J(x) \ge J(y_a) > 0$. It follows that $\mathbb{E}_x(e^{a(t\wedge T_M\wedge T_{y_a})}) \le 1/(2aJ(y_a))$. Letting $M \to \infty$ then $t \to \infty$, we deduce $\mathbb{E}_x(e^{aT_{y_a}}) \le 1/(2aJ(y_a))$, by the monotone convergence theorem. So Proposition 1.24 is proved. ☐

Proving that α attracts all initial distribution requires the following estimates near 0 and ∞.

Lemma 1.25. *For $h \in \mathbb{L}^1(\mu)$ strictly positive on $(0,\infty)$ we have*

$$\lim_{\varepsilon \downarrow 0} \limsup_{t\to\infty} \frac{\int_0^\varepsilon h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}{\int_0^{+\infty} h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} = 0 \tag{1.48}$$

$$\lim_{M\uparrow\infty} \limsup_{t\to\infty} \frac{\int_M^\infty h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}{\int_0^{+\infty} h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} = 0 \tag{1.49}$$

Proof. We start with (1.48). Using Harnack's inequality (see [80, Theorem 1.1]), we have for $\varepsilon < 1$ and large t

$$\frac{\int_0^\varepsilon h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}{\int_0^{+\infty} h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} \leq \frac{\mathbb{P}_1(T_0 > t)\int_0^\varepsilon h(z)\mu(dz)}{C\int_1^{3/2} h(x)\mu(dx)\mathbb{P}_1(T_0 > t-1)},$$

then

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\int_0^\varepsilon h(x)\mathbb{P}_x(T_0 > t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0 > t)\mu(dx)} &\leq \limsup_{t\to\infty} \frac{\mathbb{P}_1(T_0 > t)\int_0^\varepsilon h(z)\mu(dz)}{C\int_1^{3/2} h(x)\mu(dx)\mathbb{P}_1(T_0 > t-1)} \\
&= \frac{e^{-\lambda_1}\int_0^\varepsilon h(z)\mu(dz)}{C\int_1^{3/2} h(x)\mu(dx)},
\end{aligned}$$

and the first assertion of the lemma is proved.

For the second limit, we set $A_0 := \sup_{x \geq y_{\lambda_1}} \mathbb{E}_x(e^{\lambda_1 T_{y_{\lambda_1}}}) < \infty$, where y_{λ_1} is taken from Proposition 1.24. Then for large $M > y_{\lambda_1}$, we have

$$\mathbb{P}_x(T_0 > t) = \int_0^t \mathbb{P}_{x_0}(T_0 > u)\mathbb{P}_x(T_{x_0} \in d(t-u)) + \mathbb{P}_x(T_{x_0} > t).$$

Using $\lim_{u\to\infty} e^{\lambda_1 u}\mathbb{P}_{x_0}(T_0 > u) = \eta_1(x_0)\langle \eta_1, 1\rangle_\mu$, we obtain

$$B_0 := \sup_{u \geq 0} e^{\lambda_1 u}\mathbb{P}_{x_0}(T_0 > u) < \infty.$$

Then

$$\begin{aligned}
\mathbb{P}_x(T_0 > t) &\leq B_0 \int_0^t e^{-\lambda_1 u}\mathbb{P}_x(T_{x_0} \in d(t-u)) + \mathbb{P}_x(T_{x_0} > t) \\
&\leq B_0\, e^{-\lambda_1 t}\, \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) + e^{-\lambda_1 t}\, \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) \leq e^{-\lambda_1 t} A_0(B_0 + 1),
\end{aligned}$$

and (1.49) follows immediately (since $x \geq x_0 \geq y_{\lambda_1} \Rightarrow T_{x_0} \leq T_{y_{\lambda_1}}$). □

Let ν be any fixed probability distribution whose support is contained in $(0,\infty)$. We must show that the conditional evolution of ν converges to α. We begin by claiming that ν can be assumed to have a strictly positive density h, with respect to μ. Indeed, let

$$\ell(y) = \int_0^{+\infty} r(1,x,y)\nu(dx).$$

Using Tonelli's theorem we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} r(1,x,y)\nu(dx)\,\mu(dy) &= \int_0^{+\infty}\int_0^{+\infty} r(1,x,y)\,\mu(dy)\,\nu(dx)\\ &= \int_0^{+\infty} \mathbb{P}_x(T_0>1)\nu(dx)\le 1,\end{aligned}$$

which implies that $\int r(1,x,y)\nu(dx)$ is finite $dy-$a.s.. Finally, define $h=\ell/\int \ell d\mu$. Notice that for $d\rho = hd\mu$

$$\mathbb{P}_\nu(X_{t+1}\in\cdot \mid T_0>t+1) = \mathbb{P}_\rho(X_t\in\cdot\mid T_0>t),$$

showing the claim.

Consider $M>\varepsilon>0$ and any Borel set A included in $(0,\infty)$. Then

$$\left|\frac{\int \mathbb{P}_x(X_t\in A, T_0>t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)} - \frac{\int_\varepsilon^M \mathbb{P}_x(X_t\in A, T_0>t)h(x)\,\mu(dx)}{\int_\varepsilon^M \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)}\right|$$

is bounded by the sum of the following two terms

$$\begin{aligned}I1 &= \left|\frac{\int \mathbb{P}_x(X_t\in A, T_0>t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)} - \frac{\int_\varepsilon^M \mathbb{P}_x(X_t\in A, T_0>t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)}\right|\\ I2 &= \left|\frac{\int_\varepsilon^M \mathbb{P}_x(X_t\in A, T_0>t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)} - \frac{\int_\varepsilon^M \mathbb{P}_x(X_t\in A, T_0>t)h(x)\,\mu(dx)}{\int_\varepsilon^M \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)}\right|.\end{aligned}$$

We have the bound

$$I1\vee I2 \le \frac{\int_0^\varepsilon \mathbb{P}_x(T_0>t)h(x)\,\mu(dx) + \int_M^\infty \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)}.$$

Thus, from Lemma 1.25 we get

$$\lim_{\varepsilon\downarrow 0,\,M\uparrow\infty}\limsup_{t\to\infty}\left|\frac{\int \mathbb{P}_x(X_t\in A, T_0>t)h(x)\mu(dx)}{\int \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)}\right. \left. - \frac{\int_\varepsilon^M \mathbb{P}_x(X_t\in A, T_0>t)h(x)\mu(dx)}{\int_\varepsilon^M \mathbb{P}_x(T_0>t)h(x)\mu(dx)}\right| = 0.$$

On the other hand we have

$$\lim_{t\to\infty}\frac{\int_\varepsilon^M \mathbb{P}_x(X_t\in A, T_0>t)h(x)\,\mu(dx)}{\int_\varepsilon^M \mathbb{P}_x(T_0>t)h(x)\,\mu(dx)} = \frac{\int_A \eta_1(z)\mu(dz)}{\int_{\mathbb{R}^+}\eta_1(z)\mu(dz)} = \alpha(A),$$

since α attracts any compactly supported probability measures , and the result follows.

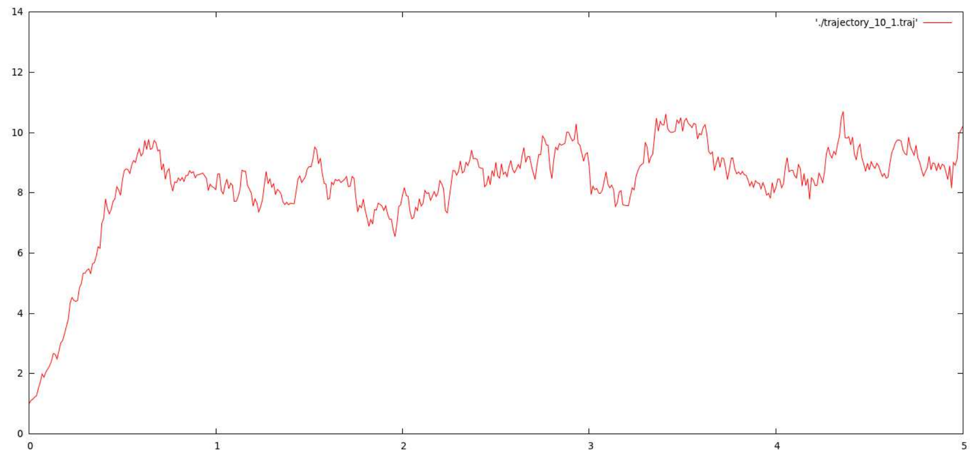


Figure 1.7: Example 4. A random path of a logistic Feller diffusion process with initial size $Z_0 = 1$ and with parameters $r = 9$ and $c = 1$

Example 1.4. We develop now a numerical illustration of the logistic Feller diffusion process case. As in the logistic birth and death process case (see Example 3, Section 1.4), the value of the charge capacity $\frac{r}{c}$ will remain across the whole example equal to the fixed value 9.

We begin by showing in Figure 1.7 a random path of a logistic Feller diffusion process with initial size $Z_0 = 1$ and with parameters $r = 9$ and $c = 1$. We observe that the process quickly attains the value of the charge capacity and remains around it for a long time.

We compare now the Yaglom limits of two different logistic Feller diffusion processes whose charge capacity is equal to 9 (see Figure 1.8):

(a) $Z^{(a)}$, whose parameters are $r = 9$ and $c = 1$,

(c) $Z^{(b)}$, whose parameters are $r = 3$ and $c = 1/3$.

As for the logistic BD-processes, we observe that the three Yaglom limits are *centered* around the charge capacity. But as a consequence of the relatively *weak* noise around the charge capacity, the Yaglom limit have clearly a smaller variation around this value in the logistic Feller diffusion case than in the logistic BD process case. We also observe that the smaller are the parameters, the flatter is the Yaglom limit and the explanation is the same as in the logistic BD-process case: the drift toward the charge capacity is smaller in the case of $Z^{(b)}$, because the competition parameter is also smaller in that case than for $Z^{(a)}$ (while the charge capacities are the same).

We observe now the distance between the conditional distribution of $Z^{(a)}$ and $Z^{(b)}$ and their respective Yaglom limits, for different initial states, namely $Z_0 = 1$,

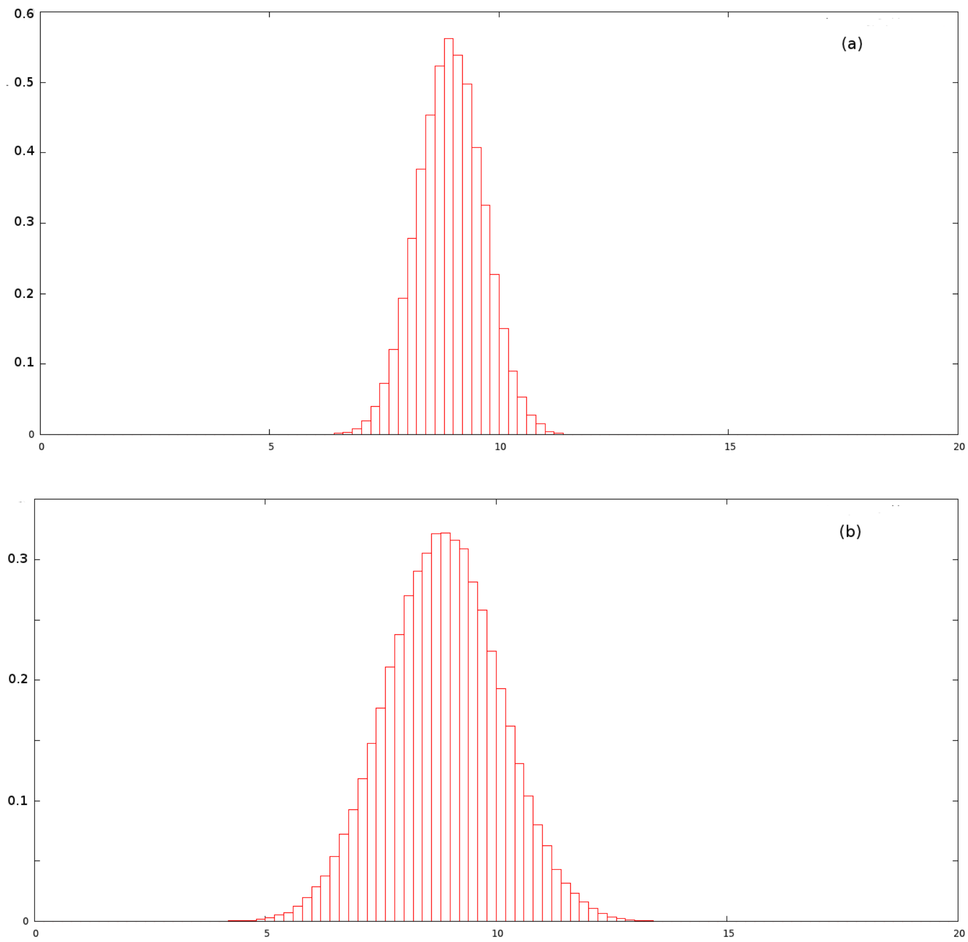


Figure 1.8: Example 4. The Yaglom limits of two logistic Feller diffusion processes with the same charge capacity: (a) $r = 9$ and $c = 1$; (b) $r = 3$ and $c = 1/3$.

$Z_0 = 10$ and $Z_0 = 100$. The results, computed with the help of the approximation method studied later in the manuscript, are represented on figure 1.9. For both $Z^{(a)}$ and $Z^{(b)}$, the speed of convergence is the highest for $Z_0 = 10$, which is quite intuitive since the value of the charge capacity is 9, and it is higher for the processes starting from 100 than from 1. In particular, this behavior is different than in the logistic birth and death process case.

1.5.3 The Q-process

Let us now describe the law of the trajectories conditioned to never attain 0.

Theorem 1.26. *Let us fix a time s and consider B a measurable subset of $C([0,s],\mathbb{R}_+)$. Then for any $x \in \mathbb{R}_+^*$,*

$$\lim_{t\to\infty} \mathbb{P}_x(X \in B | t < T_0) = Q_x(X \in B),$$

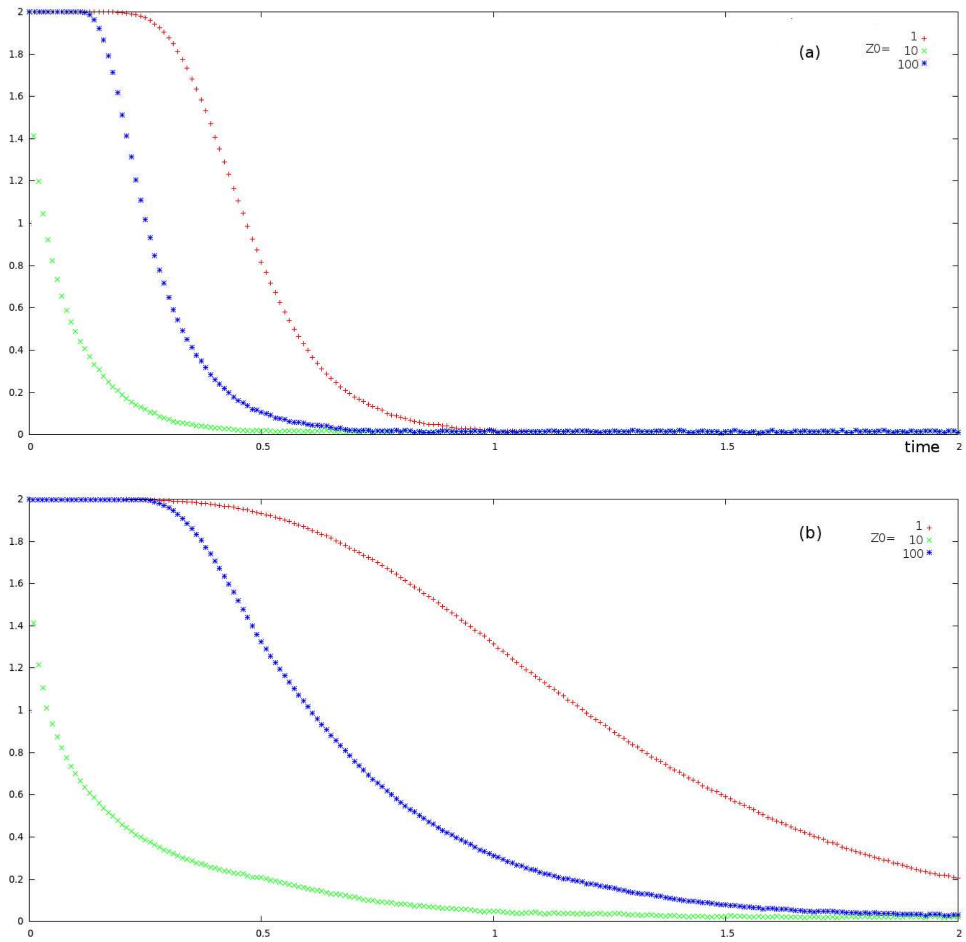


Figure 1.9: Example 4. Evolution of the distance between the conditioned distribution and the Yaglom limits of two logistic Feller diffusion processes with the same charge capacity $\frac{r}{c} = 9$: (a) $r = 9$ and $c = 1$; (b) $r = 3$ and $c = 1/3$.

where Q_x is the law of a continuous process with transition probabilities given by $q(s,x,y)dy$, where

$$q(s,x,y) = e^{\lambda_1 s}\ \frac{\eta_1(y)}{\eta_1(x)}\ r(s,x,y)\ e^{-Q(y)}.$$

Proof. Since $r(s,x,y)\,e^{-Q(y)}dy$ is the law of X_s started from x before extinction, we have to prove that

$$Q_x(X \in B) = \mathbb{E}_x\left(\mathbf{1}_B(X)\ \frac{\eta_1(X_s)}{\eta_1(x)}\ \mathbf{1}_{T_0>s}\right).$$

We have, for $t > s$,

$$\frac{\mathbb{P}_x(X \in B; T_0 > t)}{\mathbb{P}_x(T_0 > t)} = \frac{\mathbb{P}_x(X \in B; T_0 > s; \mathbb{E}_{X_s}(T_0 > t-s))}{\mathbb{P}_x(T_0 > t)},$$

and we have proved that

$$\lim_{t\to\infty} \frac{\mathbb{P}_y(T_0 > t-s)}{\mathbb{P}_x(T_0 > t)} = e^{\lambda_1 s}\ \frac{\eta_1(y)}{\eta_1(x)}.$$

Then,

$$\lim_{t\to\infty} \frac{\mathbb{P}_x(X \in B; T_0 > t)}{\mathbb{P}_x(T_0 > t)} = \frac{e^{\lambda_1 s}}{\eta_1(x)}\ \mathbb{P}_x\left(\mathbf{1}_B(X)\ \frac{\eta_1(X_s)}{\eta_1(x)}\ \mathbf{1}_{T_0>s}\right).$$

□

Corollary 1.27. *For any Borel set $A \subset (0,\infty)$ and any x,*

$$\lim_{s\to\infty} Q_x(X_s \in A) = \int_A \eta_1^2(y)\mu(dy) =< \eta_1, 1 >_\mu \int_A \eta_1(y)\alpha(dy).$$

Proof. Since $\mathbf{1}_A\,\eta_1 \in \mathbb{L}^2(\mu)$, thus

$$\eta_1(x)\ Q_x(X_s \in A) = \int \mathbf{1}_A(y)\ \eta_1(y)\,e^{\lambda_1 s}\ r(s,x,y)\ \mu(dy)$$

converges to $\eta_1(x)\ \int_B \eta_1^2(y)\mu(dy)$ as $s \to +\infty$, since $e^{\lambda_1 s} r(s,x,.)$ converges to $\eta_1(x)\,\eta_1(.)$ in $\mathbb{L}^2(d\mu)$. □

Remark 1.7. The stationary measure of the Q-process is absolutely continuous with respect to α, with Radon-Nikodym derivative $< \eta_1, 1 >_\mu\ \eta_1$.

Chapter 2

Interacting particle systems and Yaglom limit approximation of diffusions with unbounded drift *

Abstract

We study the existence and the exponential ergodicity of a general interacting particle system, whose components are driven by independent diffusion processes with values in an open subset of $\mathbb{R}^d$, $d \geq 1$. The interaction occurs when a particle hits the boundary: it jumps to a position chosen with respect to a probability measure depending on the position of the whole system.

Then we study the behavior of such a system when the number of particles goes to infinity. This leads us to an approximation method for the Yaglom limit of multi-dimensional diffusion processes with unbounded drift defined on an unbounded open set. While most of known results on such limits are obtained by spectral theory arguments and are concerned with existence and uniqueness problems, our approximation method allows us to get numerical values of quasi-stationary distributions, which find applications to many disciplines. We end the paper with numerical illustrations of our approximation method for stochastic processes related to biological population models.

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2.1 Introduction

Let $D \subset \mathbb{R}^d$ be an open set with a regular boundary (see Hypothesis 2.1). The first part of this paper is devoted to the study of interacting particle systems $(X^1,\dots,X^N)$, whose components X^i evolve in D as diffusion processes and jump when they hit the boundary ∂D. More precisely, let $N \geq 2$ be the number of particles in our system. Let us consider N independent d-dimensional Brownian motions $B^1,\dots,B^N$ and a jump measure $\mathcal{J}^{(N)} : \partial(D^N) \mapsto \mathcal{M}_1(D^N)$, where $\mathcal{M}_1(D^N)$ denotes the set of probability measures on D^N. We build the interacting particle system $(X^1,\dots,X^N)$ with values in D^N as follows. At the beginning, the particles X^i evolve as independent diffusion processes with values in D defined by

$$dX_t^{(i)} = dB_t^i + q_i^{(N)}(X_t^{(i)})dt,\ X_0^{(i)} \in D, \tag{2.1}$$

where $q_i^{(N)}$ is locally Lipschitz on D, such that the diffusion process doesn't explode in finite time. When a particle hits the boundary, say at time τ_1, it jumps to a position chosen with respect to $\mathcal{J}^{(N)}(X^1_{\tau_1\text{-}},\dots,X^N_{\tau_1\text{-}})$. Then the particles evolve independently with respect to (2.1) until one of them hits the boundary and so on. In the whole study, we require the jumping particle to be attracted away from the boundary by the other ones during the jump (in the sense of Hypothesis 2.2 on $\mathcal{J}^{(N)}$ in Section 2.2.2). We emphasize the fact that the diffusion processes which drive the particles between the jumps can depend on the particles and their coefficients aren't necessarily bounded (see Hypothesis 2.1). This construction is a generalization of the Fleming-Viot type model introduced in [14] for Brownian particles and in [42] for diffusion particles. Diffusions with jumps from the boundary have also been studied in [9], with a continuity condition on $\mathcal{J}^{(N)}$ that isn't required in our case, and in [41], where fine properties of a Brownian motion with rebirth have been established (see also the recent works of Kolb and Würkber [56], [55]).

In a first step, we show that the interacting particle system is well defined, which means that accumulation of jumps doesn't occur before the interacting particles system goes to infinity. Under additional conditions on $q_i^{(N)}$ and D, we prove that the interacting particle system doesn't reach infinity in finite time almost surely. In a second step, we give suitable conditions ensuring the system to be exponentially ergodic. The whole study is made possible thanks to a coupling between $(X^1,\dots,X^N)$ and a system of N independent 1-dimensional reflected diffusion processes. The coupling is built in Section 2.2.3.

Assume that D is bounded. For all $N \geq 2$, let $\mathcal{J}^{(N)}$ be a jump measure and $(q_i^{(N)})_{1\leq i\leq N}$ a family of drifts. Assume that the conditions for existence and ergodicity of the interacting process are fulfilled for all $N \geq 2$. Let M^N be its stationary distribution. We denote by $\mathcal{X}^N$ the associated empirical stationary distribution, which is defined by $\mathcal{X}^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, where $(x_1,\dots,x_N) \in D^N$ is distributed following M^N. Under some bound assumptions on $(q_i^{(N)})_{1\leq i\leq N, 2\leq N}$ (see Hypothesis 2.4), we prove in Section 2.2.4 that the family of laws of the random measures $\mathcal{X}^N$ is uniformly tight.

In Section 2.3, we study a particular case: $q_i^{(N)} = q$ doesn't depend on i,N and

$$\mathcal{J}^{(N)}(x_1,\ldots,x_N) = \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j},\ x_i \in \partial D. \tag{2.2}$$

It means that at each jump time, the jumping particle is sent to the position of a particle chosen uniformly between the $N-1$ remaining ones. In this situation, we identify the limit of the family of empirical stationary distributions $(\mathcal{X}^N)_{N\geq 2}$. This leads us to an approximation method of limiting conditional distributions of diffusion processes absorbed at the boundary of an open set of $\mathbb{R}^d$, studied by Cattiaux and Méléard in [17] and defined as follows. Let $U_\infty \subset \mathbb{R}^d$ be an open set and $\mathbb{P}^\infty$ be the law of the diffusion process defined by the SDE

$$dX_t^\infty = dB_t + \nabla V(X_t^\infty)dt,\ X^\infty \in U_\infty \tag{2.3}$$

and absorbed at the boundary ∂U_∞. Here B is a d-dimensional Brownian motion and $V \in C^2(U_\infty,\mathbb{R})$. We denote by τ_∂ the absorption time of the diffusion process (2.3). As proved in [17], the limiting conditional distribution

$$\nu_\infty = \lim_{t\to\infty}\mathbb{P}_x^\infty\left(X_t^\infty \in .|t<\tau_\partial\right) \tag{2.4}$$

exists and doesn't depend on $x \in U_\infty$, under suitable conditions which allow the drift ∇V and the set U_∞ to not fulfill the conditions of Section 2.2 (see Hypothesis 2.5 in Section 2.3). This probability is called the Yaglom limit associated with $\mathbb{P}^\infty$. It is a quasi-stationary distribution for the diffusion process (2.3), which means that $\mathbb{P}_{\nu_\infty}^\infty(X_t^\infty \in dx|t<\tau_\partial) = \nu_\infty$ for all $t\geq 0$. We refer to [16, 54, 61] and references therein for existence or uniqueness results on quasi-stationary distributions in other settings.

Yaglom limits are an important tool in the theory of Markov processes with absorbing states, which are commonly used in stochastic models of biological populations, epidemics, chemical reactions and market dynamics (see the bibliography [69, Applications]). Indeed, while the long time behavior of a recurrent Markov process is well described by its stationary distribution, the stationary distribution of an absorbed Markov process is concentrated on the absorbing states, which is of poor interest. In contrast, the limiting distribution of the process conditioned to not being absorbed when it is observed can explain some complex behavior, as the mortality plateau at advanced ages (see [1] and [78]), which leads to new applications of Markov processes with absorbing states in biology (see [60]). As stressed in [66], such distributions are in most cases not explicitly computable. In [17], the existence of the Yaglom limit is proved by spectral theory arguments, which doesn't allow us to get its explicit value. The main motivation of Section

2.3 is to prove an approximation method of ν_∞, even when the drift ∇V and the domain U_∞ don't fulfill the conditions of Section 2.2.

The approximation method is based on a sequence of interacting particle systems defined with the jump measures (2.2), for all $N \geq 2$. In the case of a Brownian motion absorbed at the boundary of a bounded open set (*i.e.* $q = 0$), Burdzy *et al.* conjectured in [13] that the unique limiting measure of the sequence $(\mathcal{X}^N)_{N\in\mathbb{N}}$ is the Yaglom limit ν_∞. This has been confirmed in the Brownian motion case (see [14], [40] and [62]) and proved in [30] for some Markov processes defined on discrete spaces. New difficulties arise from our case. For instance, the interacting particle process introduced above isn't necessarily well defined, since it doesn't fulfill the conditions of Section 2.2. To avoid this difficulty, we introduce a cut-off of U_∞ near its boundary. More precisely, let $(U_m)_{m\geq 0}$ be an increasing family of regular bounded subsets of U_∞, such that ∇V is bounded on each U_m and such that $U_\infty = \bigcup_{m\geq 0} U_m$. We define an interacting particle process $(X^{m,1},\ldots,X^{m,N})$ on each subset U_m^N, by setting $q_i^{(N)} = \nabla V$ and $D = U_m$ in (2.1). For all $m \geq 0$ and $N \geq 2$, $(X^{m,1},\ldots,X^{m,N})$ is well defined and exponentially ergodic. Denoting by $\mathcal{X}^{m,N}$ its empirical stationary distribution, we prove that

$$\lim_{m\to\infty} \lim_{N\to\infty} \mathcal{X}^{m,N} = \nu_\infty.$$

We conclude in Section 2.3.3 with some numerical illustrations of our method applied to the 1-dimensional Wright-Fisher diffusion conditioned to be absorbed at 0, to the Logistic Feller diffusion and to the 2-dimensional stochastic Lotka-Volterra diffusion.

2.2 A general interacting particle process with jumps from the boundary

2.2.1 Construction of the interacting process

Let D be an open subset of $\mathbb{R}^d$, $d \geq 1$. Let $N \geq 2$ be fixed. For all $i \in \{1,\ldots,N\}$, we denote by $\mathbb{P}^i$ the law of the diffusion process $X^{(i)}$, which is defined on D by

$$dX_t^{(i)} = dB_t^i - q_i^{(N)}(X_t^{(i)})dt,\ X_0^{(i)} = x^i \in D \tag{2.5}$$

and is absorbed at the boundary ∂D. Here $B^1,\ldots,B^N$ are N independent d-dimensional Brownian motions and $q_i^{(N)} = (q_{i,1}^{(N)},\ldots,q_{i,d}^{(N)})$ is locally Lipschitz. We assume that the process is absorbed in finite time almost surely and that it doesn't explode to infinity in finite time almost surely.

The infinitesimal generator associated with the diffusion process (2.5) will be denoted by $\mathcal{L}_i^{(N)}$, with

$$\mathcal{L}_i^{(N)} = \frac{1}{2}\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} - q_{i,j}^{(N)}\frac{\partial}{\partial x_j}$$

on its domain $\mathcal{D}_{\mathcal{L}_i^{(N)}}$.

For each $i \in \{1,...,N\}$, we set

$$\mathcal{D}_i = \{(x_1,...,x_N) \in \partial(D^N), \text{ such that } x_i \in \partial D, \text{ and, } \forall j \neq i,\, x_j \in D\}.$$

We define a system of particles $(X^1,...,X^N)$ with values in D^N, which is càdlàg and whose components jump from $\bigcup_i \mathcal{D}_i$. Between the jumps, each particle evolves independently of the other ones with respect to $\mathbb{P}^i$.

Let $\mathcal{J}^{(N)} : \bigcup_{i=0}^N \mathcal{D}_i \to \mathcal{M}_1(D)$ be the jump measure, which associates a probability measure $\mathcal{J}^{(N)}(x_1,...,x_N)$ on D to each point $(x_1,...,x_N) \in \bigcup_{i=1}^N \mathcal{D}_i$. Let $(X_0^1,...,X_0^N) \in D^N$ be the starting point of the interacting particle process $(X^1,...,X^N)$, which is built as follows:

- Each particle evolves following the SDE (2.5) independently of the other ones, until one particle, say X^{i_1}, hits the boundary at a time which is denoted by τ_1. On the one hand, we have $\tau_1 > 0$ almost surely, because each particle starts in D. On the other hand, the particle which hits the boundary at time τ_1 is unique, because the particles evolves as independent Itô's diffusion processes in D. It follows that $(X_{\tau_1\text{-}}^1,...,X_{\tau_1\text{-}}^N)$ belongs to $\mathcal{D}_{i_1}$.

- The position of X^{i_1} at time τ_1 is then chosen with respect to the probability measure $\mathcal{J}^{(N)}(X_{\tau_1\text{-}}^1,...,X_{\tau_1\text{-}}^N)$.

- At time τ_1 and after proceeding to the jump, all the particles are in D. Then the particles evolve with respect to (2.5) and independently of each other, until one of them, say X^{i_2}, hits the boundary, at a time which is denoted by τ_2. As above, we have $\tau_1 < \tau_2$ and $(X_{\tau_2\text{-}}^1,...,X_{\tau_2\text{-}}^N) \in \mathcal{D}_{i_2}$.

- The position of X^{i_2} at time τ_2 is then chosen with respect to the probability measure $\mathcal{J}^{(N)}(X_{\tau_2\text{-}}^1,...,X_{\tau_2\text{-}}^N)$.

- Then the particles evolve with law $\mathbb{P}^i$ and independently of each other, and so on.

The law of the interacting particle process with initial distribution $m \in \mathcal{M}_1(D^N)$ will be denoted by P_m^N, or by P_x^N if $m = \delta_x$, with $x \in D^N$. The associated expectation will be denoted by E_m^N, or by E_x if $m = \delta_x$. For all $\beta > 0$, we denote by $S_\beta = \inf\{t \geq 0, |(X_t^1,...,X_t^N)| \geq \beta\}$ the first exit time from $\{x \in D^N, |x|_2 < \beta\}$, where $|\cdot|$ denotes the Euclidean norm. We set $S_\infty = \lim_{\beta\to\infty} S_\beta$.

The sequence of successive jumping particles is denoted by $(i_n)_{n\geq 1}$, and

$$0 < \tau_1 < \tau_2 < ...$$

denotes the strictly increasing sequence of jumping times (which is well defined for all $n \geq 0$ since the process is supposed to be absorbed in finite time almost surely).

Thanks to the non-explosion assumption on each $\mathbb{P}^i$, we have $\tau_n < S_\infty$ for all $n \geq 1$ almost surely. We set $\tau_\infty = \lim_{n\to\infty} \tau_n \leq S_\infty$. The process described above isn't necessarily well defined for all $t \in [0,S_\infty[$, and we need more assumptions on D and on the jump measure $\mathcal{J}^{(N)}$ to conclude that $\tau_\infty = S_\infty$ almost surely.

In the sequel, we denote by ϕ_D the Euclidean distance to the boundary ∂D:

$$\phi_D(x) = \inf_{y\in\partial D} |y - x|, \text{ for all } x \in D.$$

For all $r > 0$, we define the collection of open subsets $D_r = \{x \in D,\, \phi_D(x) > r\}$. For all $\beta > 0$, we set $B_\beta = \{x \in D, |x| < \beta\}$.

Hypothesis 2.1. *There exists a neighborhood U of ∂D such that*

1. *the distance ϕ_D is of class C^2 on U,*
2. *for all $\beta > 0$,*
$$\inf_{x\in U\cap B_\beta,\ i\in\{1,\dots,N\}} \mathcal{L}_i^{(N)}\phi_D(x) > -\infty.$$

In particular, Hypothesis 2.1 implies

$$|\nabla\phi_D(x)| = 1,\ \forall x \in U. \tag{2.6}$$

Remark 2.1. The first part of Hypothesis 2.1 is fulfilled if and only if D is an open set whose boundary is of class C^2 (see [25, Chapter 5, Section 4]).

The following assumption ensures that the jumping particle is attracted away from the boundary by the other ones.

Hypothesis 2.2. *There exists a non-decreasing continuous function $f^{(N)} : \mathbb{R}_+ \to \mathbb{R}_+$ vanishing at 0 and strictly increasing in a neighborhood of 0 such that, $\forall i \in \{1,\dots,N\}$,*

$$\inf_{(x_1,\dots,x_N)\in\mathcal{D}_i} \mathcal{J}^{(N)}(x_1,\dots,x_N)(\{y \in D,\ \phi_D(y) \geq \min_{j\neq i} f^{(N)}(\phi_D(x_j))\}) \geq p_0^{(N)},$$

$p_0^{(N)} > 0$ is a positive constant.

Informally, $f^{(N)}(\phi_D)$ is a kind of distance from the boundary and we assume that at each jump time τ_n, the probability of the event "the jump position $X^{i_n}_{\tau_n}$ is chosen farther from the boundary than at least one another particle" is bounded below by a positive constant $p_0^{(N)}$.

Remark 2.2. Hypothesis 2.2 is very general and allows a lot of choices for $\mathcal{J}^{(N)}(x_1,\dots,x_N)$. For instance, for all $\mu \in \mathcal{M}_1(D)$, one can find a compact set $K \subset D$ such that $\mu(K) > 0$. Then $\mathcal{J}^{(N)}(x_1,\dots,x_N) = \mu$ fulfills the assumption with $p_0^{(N)} = \mu(K)$ and $f^{(N)}(\phi_D) = \phi_D \wedge d(K,\partial D)$.

Hypothesis 2.2 also includes the case studied by Grigorescu and Kang in [42], where

$$\mathcal{J}^{(N)}(x_1,\dots,x_N) = \sum_{j\neq i} p_{ij}(x_i)\delta_{x_j},\ \forall (x_1,\dots,x_N)\in\mathcal{D}_i.$$

with $\sum_{j\neq i} p_{ij}(x_i) = 1$ and $\inf_{i\in\{1,\dots,N\}, j\neq i, x_i\in\partial D} p_{ij}(x_i) > 0$. In that case, the particle on the boundary jumps to one of the other ones, with positive weights. It yields that Hypothesis 2.2 is fulfilled with $p_0^{(N)} = 1$ and $f^{(N)}(\phi_D) = \phi_D$. In Section 2.3, we will focus on the particular case

$$\mathcal{J}^{(N)}(x_1,\dots,x_N) = \frac{1}{N-1}\sum_{j=1,\dots,N,\ j\neq i} \delta_{x_j},\ \forall (x_1,\dots,x_N)\in\mathcal{D}_i.$$

That will lead us to an approximation method of the Yaglom limit (2.4).

Finally, given a jump measure $\mathcal{J}^{(N)}$ satisfying Hypothesis 2 (with $p_0^{(N)}$ and $f^{(N)}$), any $\sigma^{(N)} : \bigcup_{i=0}^N \mathcal{D}_i \to \mathcal{M}_1(D)$ and a constant $\alpha^{(N)} > 0$, the jump measure

$$\mathcal{J}_\sigma^{(N)}(x_1,\dots,x_N) = \alpha^{(N)}\mathcal{J}^{(N)}(x_1,\dots,x_N) + (1-\alpha^{(N)})\sigma^{(N)}(x_1,\dots,x_N),\ \forall (x_1,\dots,x_N)\in\mathcal{D}_i,$$

fulfills the Hypothesis 2.2 with $p_{0,\sigma}^{(N)} = \alpha^{(N)}p_0^{(N)}$ and $f_\sigma^{(N)}(\phi_D) = f^{(N)}(\phi_D)$.

Finally, we give a condition which ensures the exponential ergodicity of the process. In particular, this condition is satisfied if D is bounded and fulfills Hypothesis 2.1.

Hypothesis 2.3. *There exists $\alpha > 0$, $t_0^{(N)} > 0$ and a compact set $K_0^{(N)} \subset D$ such that*

1. *the distance ϕ_D is of class C^2 on $D\setminus D_{2\alpha}$ and*
$$\inf_{x\in D\setminus D_{2\alpha},\ i\in\{1,\dots,N\}} \mathcal{L}_i^{(N)}\phi_D(x) > -\infty.$$
2. *for all $i\in\{1,\dots,N\}$, we have*
$$p_1^{(N)} = \prod_{i=1}^N \inf_{x\in D_{\alpha/2}} \mathbb{P}_x^i(X^{(i)}_{t_0^{(N)}} \in K_0^{(N)}) > 0.$$

Theorem 2.1. *Assume that Hypotheses 2.1 and 2.2 are fulfilled. Then the process $(X^1,\dots,X^N)$ is well defined, which means that $\tau_\infty = S_\infty$ almost surely.*

If Hypothesis 2.2 and the first point of Hypothesis 2.3 are fulfilled, then $\tau_\infty = S_\infty = +\infty$ almost surely.

If Hypotheses 2.2 and 2.3 are fulfilled, then the process $(X^1,\dots,X^N)$ is exponentially ergodic, which means that there exists a probability measure M^N on D^N such that,

$$||P_x^N((X_t^1,\dots,X_t^N)\in .) - M^N||_{TV} \leq C^{(N)}(x)\left(\rho^{(N)}\right)^t,\ \forall x\in D^N,\ \forall t\in\mathbb{R}_+,$$

where $C^{(N)}(x)$ is finite, $\rho^{(N)} < 1$ and $||.||_{TV}$ is the total variation norm. In particular, M^N is a stationary measure for the process $(X^1,\dots,X^N)$.

The main tool of the proof is a coupling between $(X^1_t,\dots,X^N_t)_{t\in[0,S_\beta]}$ and a system of N independent one-dimensional diffusion processes $(Y^{\beta,1}_t,\dots,Y^{\beta,N}_t)_{t\in[0,S_\beta]}$, for each $\beta>0$. The system is built in order to satisfy

$$0\leq Y^{\beta,i}_t\leq \phi_D(X^i_t)\ \text{a.s.}$$

for all $t\in[0,\tau_\infty\wedge S_\beta]$ and each $i\in\{1,\dots,N\}$. We build this coupling in Subsection 2.2.2 and we conclude the proof of Theorem 2.1 in Subsection 2.2.3 .

In Subsection 2.2.4, we assume that D is bounded and that, for all $N\geq 2$, we're given $\mathcal{J}^{(N)}$ and a family of drifts $(q^{(N)}_i)_{1\leq i\leq N}$, such that Hypotheses 2.1, 2.2 and 2.3 are fulfilled. Moreover, we assume that α in Hypothesis 2.3 doesn't depend on N. Under some suitable bounds on the family $(q^{(N)}_i)_{1\leq i\leq N,\ N\geq 2}$, we prove that the family of laws of the empirical distributions $(\mathcal{X}^N)_{N\geq 2}$ is uniformly tight. In our case, this is equivalent to the property: $\forall\epsilon\geq 0$, there exists a compact set $K\subset D$ such that $E(\mathcal{X}^N(D\setminus K))\leq\epsilon$ for all $N\geq 2$ (see [48]). In particular, this implies that $(\mathcal{X}^N)_{N\geq 2}$ is weakly sequentially compact. Let us recall that a sequence of random measures $(\gamma_N)_N$ on D converges weakly to a random measure γ on D, if $\gamma_N(f)$ converges to $\gamma(f)$ for all continuous bounded functions $f:D\to\mathbb{R}$. This property will be crucial in Section 2.3.

2.2.2 Coupling's construction

Proposition 2.2. *Assume that Hypothesis 2.1 is fulfilled and fix $\beta>0$. Then there exists $a>0$, a N-dimensional Brownian motion $(W^1,\dots,W^N)$ and positive constants $Q_1,\dots,Q_N$ such that, for each $i\in\{1,\dots,N\}$, the reflected diffusion process with values in $[0,a]$ defined by the reflection equation (cf. [18])*

$$Y^{\beta,i}_t=Y^{\beta,i}_0+W^i_t-Q_it+L^{i,0}_t-L^{i,a}_t,\ Y^{\beta,i}_0=\min(a,\phi_D(X^i_0))\tag{2.7}$$

satisfies

$$0\leq Y^{\beta,i}_t\leq\phi_D(X^i_t)\wedge a\ \ a.s.\tag{2.8}$$

for all $t\in[0,\tau_\infty\wedge S_\beta[$ (see Figure 2.1). In (2.7), $L^{i,0}$ (resp. $L^{i,a}$) denotes the local time of $Y^{\beta,i}$ at $\{0\}$ (resp. $\{a\}$).

Remark 2.3. If the first part of Hypothesis 2.3 is fulfilled, then the proof remains valid with $\beta=\infty$ and $a=\alpha$ (where $\alpha>0$ is defined in Hypothesis 2.3). This leads us to a coupling between X^i and $Y^{\infty,i}$, valid for all $t\in[0,\tau_\infty\wedge S_\infty[=[0,\tau_\infty[$.

Proof of Proposition 2.2 : The set $\overline{B_\beta\setminus U}$ is a compact subset of D, then there exists $a>0$ such that $B_\beta\setminus U\subset D_{2a}$. In particular, we have $B_\beta\setminus D_{2a}\subset U$, so that ϕ_D is of class C^2 in $B_\beta\setminus D_{2a}$.

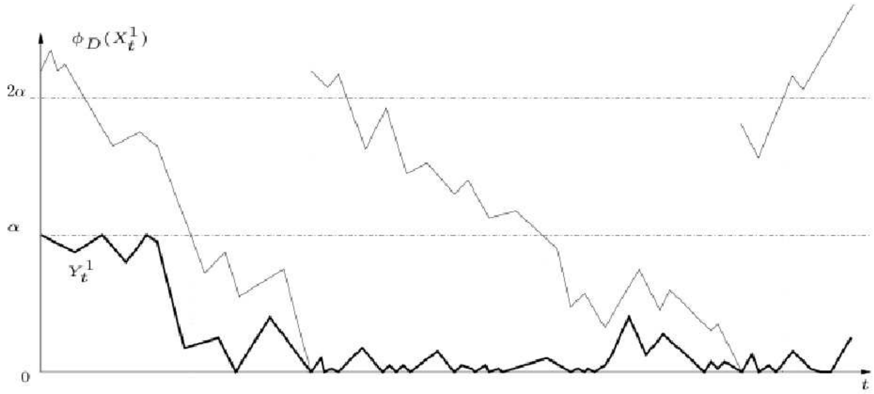

Figure 2.1: The particle X^1 and its coupled reflected diffusion process Y^1

Fix $i \in \{1,...,N\}$. We define a sequence of stopping times $(\theta_n^i)_n$ such that $X_t^i \in B_\beta \setminus D_{2a}$ for all $t \in [\theta_{2n}^i,\theta_{2n+1}^i[$ and $X_t^i \in \overline{D_a}$ for all $t \in [\theta_{2n+1}^i,\theta_{2n+2}^i[$. More precisely, we set (see Figure 2.2)

$$\theta_0^i = \inf\{t \in [0,+\infty[,\ X_t^i \in B_\beta \setminus D_a\} \wedge \tau_\infty \wedge S_\beta,$$
$$\theta_1^i = \inf\{t \in [t_0,+\infty[,\ X_t^i \in \overline{D_{2a}}\} \wedge \tau_\infty \wedge S_\beta,$$

and, for $n \geq 1$,

$$\theta_{2n}^i = \inf\{t \in [t_{2n-1}^i,+\infty[,\ X_t^i \in B_\beta \setminus D_a\} \wedge \tau_\infty \wedge S_\beta,$$
$$\theta_{2n+1}^i = \inf\{t \in [t_{2n}^i,+\infty[,\ X_t^i \in \overline{D_{2a}}\} \wedge \tau_\infty \wedge S_\beta.$$

The sequence (θ_n^i) is non-decreasing and goes to $\tau_\infty \wedge S_\beta$ almost surely.

Let γ^i be a 1-dimensional Brownian motion independent of the process $(X^1,...,X^N)$ and of the Brownian motion $(B^1,...,B^N)$. We set

$$W_t^i = \gamma_t^i, \text{ for } t \in [0,\theta_0^i[,$$

and, for all $n \geq 0$,

$$W_t^i = W_{\theta_{2n}^i}^i + \int_{\theta_{2n}^i}^t \nabla\phi_D(X_{s\text{-}}^i)\cdot dB_s^i \text{ for } t \in [\theta_{2n}^i,\theta_{2n+1}^i[,$$
$$W_t^i = W_{\theta_{2n+1}^i}^i + (\gamma_t^i - \gamma_{\theta_{2n+1}^i}^i) \text{ for } t \in [\theta_{2n+1}^i,\theta_{2n+2}^i[,$$

where $\int_{\theta_{2n}^i}^t \nabla\phi_D(X_{s\text{-}}^i)\cdot dB_s^i$ has the law of a Brownian motion between times θ_{2n}^i and θ_{2n+1}^i, thanks to (2.6). The process $(W^1,...,W^N)$ is yet defined for all $t \in [0,\tau_\infty \wedge S_\beta[$. We set

$$W_t^i = W_{\tau_\infty \wedge S_\beta -}^i + (\gamma_t^i - \gamma_{\tau_\infty \wedge S_\beta}^i) \text{ for } t \in [\tau_\infty \wedge S_\beta, +\infty[$$

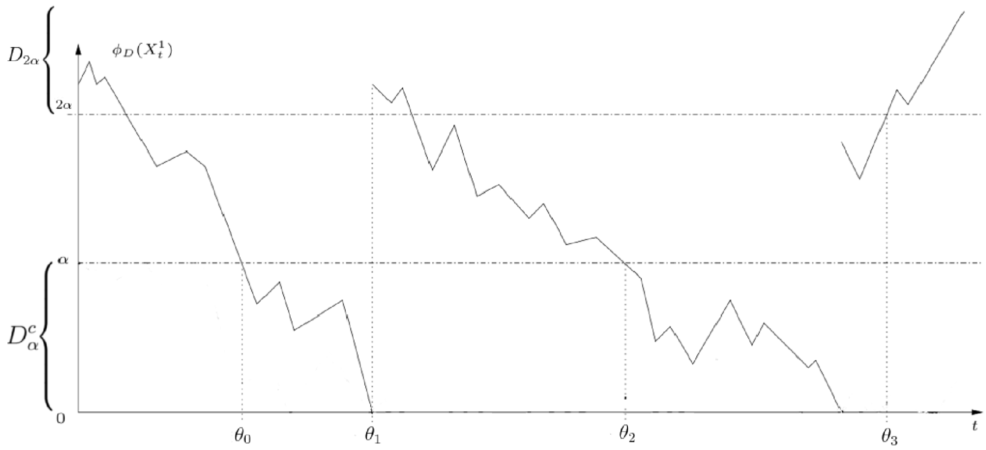

Figure 2.2: Definition of the sequence of stopping times $(\theta_n^i)_{n\geq 0}$

It is immediate that $(W^1,...,W^N)$ is a N-dimensional Brownian motion.

Fix $i \in \{1,...,N\}$. Thanks to Hypothesis 2.2, there exists $Q_i^{(N)} \geq 0$ such that

$$\inf_{x\in B_\beta\setminus D_{2a}} \mathcal{L}_i^{(N)}\phi_D(x) \geq -Q_i^{(N)}.$$

Let us prove that the reflected diffusion process $Y^{\beta,i}$ defined by (2.7) fulfills inequality (2.8) for all $t \in [0,\tau_\infty \wedge S_\beta[$.

We set $\zeta = \inf\left\{0 \leq t < \tau_\infty \wedge S_\beta, Y_t^{\beta,i} > \phi_D(X_t^i)\right\}$ and we work conditionally to $\zeta < \tau_\infty \wedge S_\beta$. By right continuity of the two processes,

$$0 < \phi_D(X_\zeta^i) \leq Y_\zeta^{\beta,i} \leq a \text{ a.s.}$$

One can find a stopping time $\zeta' \in]\zeta,\tau_\infty \wedge S_\beta[$, such that X^i doesn't jump between ζ and ζ' and such that $Y_t^{\beta,i} > 0$ and $X_t^i \in B_\beta \setminus D_{2a}$ for all $t \in [\zeta,\zeta']$ almost surely.

Thanks to the regularity of ϕ_D on $B_\beta \setminus D_{2a}$, we can apply Itô's formula to $(\phi_D(X_t^i))_{t\in[\zeta,\zeta']}$, and we get, for all stopping time $t \in [\zeta,\zeta']$,

$$\phi_D(X_t^i) = \phi_D(X_\zeta^i) + \int_\zeta^t \nabla\phi_D(X_s^i)\cdot dB_s^i + \int_\zeta^t \mathcal{L}_i^{(N)}\phi_D(X_s^i)ds.$$

But ζ and ζ' lie between an entry time of X^i to $B_\beta \setminus D_a$ and the following entry time to $\overline{D_{2a}}$. It yields that there exists $n \geq 0$ such that $[\zeta,\zeta'] \subset [\theta_{2n}^i,\theta_{2n+1}^i[$. We deduce that

$$\phi_D(X_t^i) - Y_t^{\beta,i} = \phi_D(X_\zeta^i) - Y_\zeta^{\beta,i} + \int_\zeta^t (\mathcal{L}_i^{(N)}\phi_D(X_s^i) + Q_i^{(N)})ds - L_t^{i,0} + L_\zeta^{i,0} + L_t^{i,a} - L_\zeta^{i,a},$$

where $\mathcal{L}_i^{(N)}\phi_D(X_s^i) + Q_i^{(N)} \geq 0$, $(L_s^{i,a})_{s\geq 0}$ is increasing and $L_t^{i,0} = L_\zeta^{i,0}$, since $Y^{\beta,i}$ doesn't hit 0 between times ζ and t. It follows that, for all $t \in [\zeta,\zeta']$,

$$\begin{aligned}\phi_D(X_t^i) - Y_t^{\beta,i} &\geq \phi_D(X_\zeta^i) - Y_\zeta^{\beta,i}\\ &\geq \phi_D(X_{\zeta-}^i) - Y_{\zeta-}^{\beta,i} \geq 0.\end{aligned}$$

where the second inequality comes from the positivity of the jumps of $\phi_D(X^i)$ and from the left continuity of $Y^{\beta,i}$, while the third inequality is due to the definition of ζ. Then $\phi_D(X^i) - Y^{\beta,i}$ stays non-negative between times ζ and ζ', what contradicts the definition of ζ. Finally, $\zeta = \tau_\infty \wedge S_\beta$ almost surely, which means that the coupling inequality (2.8) remains true for all $t \in [0,\tau_\infty \wedge S_\beta[$. □

2.2.3 Proof of Theorem 2.1

Proof that $(X^1,\dots,X^N)$ is well defined under Hypotheses 2.1 and 2.2. Let $N \geq 2$ be the size of the interacting particle system and fix arbitrarily its starting point $x \in D^N$. Thanks to the non explosiveness of each diffusion process $\mathbb{P}^i$, the interacting particle process can't escape to infinity in finite time after a finite number of jumps. It yields that $\tau_\infty \leq S_\infty$ almost surely.

Fix $\beta > 0$ such that $x \in B_\beta$ and define the event $C_\beta = \{\tau_\infty < S_\beta\}$. Assume that C_β occurs with positive probability. Conditionally to C_β, the total number of jumps is equal to $+\infty$ before the finite time τ_∞. There is a finite number of particles, then at least one particle makes an infinite number of jumps before τ_∞. We denote it by i_0 (which is a random index).

For each jumping time τ_n, we denote by $\sigma_n^{i_0}$ the next jumping time of i_0, with $\tau_n < \sigma_n^{i_0} < \tau_\infty$. Conditionally to C_β, we get $\sigma_n^{i_0} - \tau_n \to 0$ when $n \to \infty$. For all C^2 function f with compact support in $]0,2a[$, the process $f(\phi_D(X^{i_0}))$ is a continuous diffusion process with bounded coefficients between τ_n and $\sigma_n^{i_0}$-, then

$$\sup_{t\in[\tau_n,\sigma_n^{i_0}[} |f(\phi_D(X_t^{i_0}))| = \sup_{t\in[\tau_n,\sigma_n^{i_0}[} |f(\phi_D(X_t^{i_0})) - f(\phi_D(X_{\sigma_n^{i_0}\text{-}}^{i_0}))| \xrightarrow[n\to\infty]{} 0,\ a.s.$$

Since the process $\phi_D(X^{i_0})$ is continuous between τ_n and $\sigma_n^{i_0}-$, we conclude that $\phi_D(X_{\tau_n}^{i_0})$ doesn't lie above the support of f, for n big enough almost surely. But the support of f can be chosen arbitrarily close to 0, it yields that $\phi_D(X_{\tau_n}^{i_0})$ goes to 0 almost surely conditionally to C_β.

Let us denote by $(\tau_n^{i_0})_n$ the sequence of jumping times of the particle i_0. We denote by A_n the event

$$A_n = \left\{\exists i \neq i_0 \,|\, \phi_D(X_{\tau_n^{i_0}}^i) \leq f^{(N)}(\phi_D(X_{\tau_n^{i_0}}^{i_0}))\right\},$$

where $f^{(N)}$ is the function of Hypothesis 2.2 . We have, for all $1 \leq k \leq l$,

$$P\left(\bigcap_{n=k}^{l+1} A_n^c\right) = E\left(E\left(\prod_{n=k}^{l+1} \mathbb{1}_{A_n^c} \,|\, (X_t^1,\dots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right)\right)$$
$$= E\left(\prod_{n=k}^{l} \mathbb{1}_{A_n^c} E\left(\mathbb{1}_{A_{l+1}^c} \,|\, (X_t^1,\dots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right)\right),$$

where, by definition of the jump mechanism of the interacting particle system,

$$E\left(\mathbb{1}_{A_{l+1}^c} \,|\, (X_t^1,\dots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right) = \mathcal{J}^{(N)}(X^1_{\tau_{l+1}^{i_0}},\dots,X^N_{\tau_{l+1}^{i_0}})\left(A_{l+1}^c\right)$$
$$\leq 1 - p_0^{(N)},$$

by Hypothesis 2.2. By induction on l, we get

$$P\left(\bigcap_{n=k}^{l} A_n^c\right) \leq (1-p_0^{(N)})^{l-k},\ \forall 1 \leq k \leq l.$$

Since $p_0^{(N)} > 0$, it yields that

$$P\left(\bigcup_{k\geq 1}\bigcap_{n=k}^{\infty} A_n^c\right) = 0.$$

It means that, for infinitely many jumps τ_n almost surely, one can find a particle j such that $f^{(N)}(\phi_D(X^j_{\tau_n})) \leq \phi_D(X^{i_0}_{\tau_n})$. Because there is only a finite number of other particles, one can find a particle, say j_0 (which is a random variable), such that

$$f^{(N)}(\phi_D(X^{j_0}_{\tau_n})) \leq \phi_D(X^{i_0}_{\tau_n}),\ \text{for infinitely many } n \geq 1.$$

In particular, $\lim_{n\to\infty}\left(\phi_D(X^{i_0}_{\tau_n}), f^{(N)}(\phi_D(X^{j_0}_{\tau_n}))\right) = (0,0)$ almost surely. But $(f^{(N)})^{-1}$ is well defined and continuous near 0, then

$$\lim_{n\to\infty}\left(\phi_D(X^{i_0}_{\tau_n}),\phi_D(X^{j_0}_{\tau_n})\right) = (0,0)\ a.s.$$

Using the coupling inequality of Proposition 2.2, we deduce that

$$C_\beta \subset \left\{\lim_{t\to\tau_\infty}(Y_t^{\beta,i_0},Y_t^{\beta,j_0}) = (0,0)\right\}.$$

Then, conditionally to C_β, Y^{β,i_0} and Y^{β,j_0} are independent reflected diffusion processes with bounded drift, which hit 0 at the same time. This occurs for two independent reflected Brownian motions with probability 0, and then for Y^{β,i_0} and Y^{β,j_0} too, by the Girsanov's Theorem. That implies $P_x(C_\beta) = 0$.

We have proved that $\tau_\infty \geq S_\beta$ almost surely for all $\beta > 0$, which leads to $\tau_\infty \geq S_\infty$ almost surely. Finally, we get $\tau_\infty = S_\infty$ almost surely.

If the first part of Hypothesis 2.3 is fulfilled, one can defined the coupled reflected diffusion $Y^{\infty,i}$, which fulfills inequality (2.8) with $a = \alpha$ and for all $t \in [0,\tau_\infty \wedge S_\infty[= [0,\tau_\infty[$. Then the same proof leads to

$$\{\tau_\infty < +\infty\} \subset \left\{ \lim_{t\to\tau_\infty} (Y_t^{\infty,i_0},Y_t^{\infty,j_0}) = (0,0) \right\}.$$

Finally, we deduce that $\tau_\infty = \infty$ almost surely. □

Remark 2.4. One could wonder if the previous coupling argument can be generalized, replacing (2.5) by uniformly elliptic diffusion processes. In fact, such arguments lead to the definition of Y^i as the reflected diffusion $Y_t^i = \int_0^t \phi(X_s^i)dW_s^i - Q_i t + L_t^0 - L_t^\alpha$, where ϕ is a regular function. In our case of a drifted Brownian motion, ϕ is equal to 1 and Y^i is a reflected drifted Brownian motion independent of the others particles. But in the general case, the Y^i are general orthogonal semi-martingales. It yields that the generalization of the previous proof reduces to the following hard problem (see [74, Question 2, page 217] and references therein): "Which are the two-dimensional continuous semi-martingales for which the one point sets are polar ?". Since this question has no general answer, it seems that the previous proof doesn't generalize immediately to general uniformly elliptic diffusion processes.

We emphasize the fact that the proof of the exponential ergodicity can be generalized (as soon as $\tau_\infty = S_\infty = +\infty$ is proved), using the fact that $(Y_t^1,\ldots,Y_t^N)_{t\geq 0}$ is a time changed Brownian motion with drift and reflection (see [74, Theorem 1.9 (Knight)]). This time change argument has been developed in [42], with a different coupling construction. This change of time can also be used in order to generalize Theorem 2.3 below, as soon as the exponential ergodicity is proved.

Proof of the exponential ergodicity. It is sufficient to prove that there exists $n \geq 1$, $\epsilon > 0$ and a non-trivial probability ϑ on D^N such that

$$P_x((X^1_{nt_0^{(N)}},\ldots,X^N_{nt_0^{(N)}}) \in A) \geq \epsilon\vartheta(A),\ \forall x \in K_0,\ A \in \mathcal{B}(D^N), \tag{2.9}$$

with $K_0 = \left(K_0^{(N)}\right)^N$, where $t_0^{(N)}$ and $K_0^{(N)}$ are defined in Hypothesis 2.3, and such that

$$\sup_{x\in K_0} E_x(\kappa^{\tau'}) < \infty, \tag{2.10}$$

where κ is a positive constant and $\tau' = \min\{n \geq 1, (X^1_{nt_0^{(N)}},\ldots,X^N_{nt_0^{(N)}})_{n\in\mathbb{N}} \in K_0\}$ is the return time to K_0 of the Markov chain $(X^1_{nt_0^{(N)}},\ldots,X^N_{nt_0^{(N)}})_{n\in\mathbb{N}}$. Indeed, Down, Meyn and Tweedie proved in [26, Theorem 2.1 p.1673] that if the Markov chain $(X^1_{nt_0^{(N)}},\ldots,X^N_{nt_0^{(N)}})_{n\in\mathbb{N}}$ is aperiodic (which is obvious in our case) and fulfills (2.9) and (2.10), then it is geometrically ergodic. But, thanks to [26, Theorem

5.3 p.1681], the geometric ergodicity of this Markov chain is a sufficient condition for $(X^1,\ldots,X^N)$ to be exponentially ergodic.

We assume without loss of generality that $K_0^{(N)} \subset D_{\alpha/2}$ (where α is defined in Hypothesis 2.3). Let us set

$$\vartheta(A) = \frac{\prod_{i=1}^N \inf_{x\in D_{\alpha/2}} \mathbb{P}^i(X^{(i)}_{t_0^{(N)}} \in A\cap K_0^{(N)})}{\prod_{i=1}^N \inf_{x\in D_{\alpha/2}} \mathbb{P}^i(X^{(i)}_{t_0^{(N)}} \in K_0^{(N)})}.$$

Thanks to Hypothesis 2.3, ϑ is a non-trivial probability measure. Moreover, (2.9) is clearly fulfilled with $n=1$ and $\epsilon = \prod_{i=1}^N \inf_{x\in D_\alpha} \mathbb{P}^i(X^{(i)}_{t_0^{(N)}} \in K_0^{(N)})$.

Let us prove that $\exists\kappa>0$ such that (2.10) holds. One can define the N-dimensional diffusion $(Y^{\infty,1},\ldots,Y^{\infty,N})$ reflected on $\{0,\alpha\}$ and coupled with $(X^1,\ldots,X^N)$, so that inequality (2.8) is fulfilled for all $t\in[0,+\infty[$ and $a=\alpha$. For all $x_0\in D^N$, we have by the Markov property

$$\begin{aligned}
\mathbb{P}_{x_0}((X^1_{2t_0^{(N)}},\ldots,X^N_{2t_0^{(N)}}) \in K_0^N) &\geq \mathbb{P}_{x_0}(X^i_{t_0^{(N)}} \in D_{\alpha/2},\forall i) \inf_{x\in D^N_{\alpha/2}} \mathbb{P}_x(X^i_{t_0^{(N)}} \in K_0^{(N)},\forall i)\\
&\geq \mathbb{P}_{x_0}(X^i_{t_0^{(N)}} \in D_{\alpha/2},\forall i) \prod_{i=1}^N \inf_{x\in D_{\alpha/2}} \mathbb{P}_x(X^i_{t_0^{(N)}} \in K_0^{(N)})\\
&\geq \mathbb{P}_{x_0}(X^i_{t_0^{(N)}} \in D_{\alpha/2},\forall i) p_1^{(N)},
\end{aligned}$$

where $p_1^{(N)}>0$ is defined in Hypothesis 2.3. It yields that

$$\begin{aligned}
\mathbb{P}_{x_0}((X^1_{2t_0^{(N)}},\ldots,X^N_{2t_0^{(N)}}) \in K_0^N) &\geq p_1^{(N)} \mathbb{P}_{x_0}(\phi_D(X^i_{t_0^{(N)}}) > \alpha/2,\forall i)\\
&\geq p_1^{(N)} \prod_{i=1}^N \mathbb{P}_{Y_0^{\infty,i}}(Y^{\infty,i}_{t_0^{(N)}} > \alpha/2),
\end{aligned}$$

thanks to Proposition 2.2. A comparison argument shows that $\mathbb{P}_{Y_0^{\infty,i}}(Y^{\infty,i}_{t_0^{(N)}} > \alpha/2) \geq \mathbb{P}_0(Y^{\infty,i} > \alpha/2)$. Then

$$\inf_{x_0\in D} \mathbb{P}_{x_0}((X^1_{2t_0^{(N)}},\ldots,X^N_{2t_0^{(N)}}) \in K_0^N) \geq p_1^{(N)} \prod_{i=1}^N \mathbb{P}_0(Y^{\infty,i}_{t_0^{(N)}} > \alpha/2) > 0,$$

thanks to the strict positivity of the density of the law of $Y^{\infty,i}_{t_0^{(N)}}$, for all $i\in\{1,\ldots,N\}$. Using the Markov property, we get, $\forall n\geq 1$,

$$\begin{aligned}
P(\tau' \geq 2nt_0^{(N)}) &\geq (1 - \inf_{x_0\in D} \mathbb{P}_{x_0}((X^1_{2t_0^{(N)}},\ldots,X^N_{2t_0^{(N)}}) \in K_0^N)) P(\tau' \geq 2(n-1)t_0^{(N)})\\
&\geq (1 - \inf_{x_0\in D} \mathbb{P}_{x_0}((X^1_{2t_0^{(N)}},\ldots,X^N_{2t_0^{(N)}}) \in K_0^N))^n,
\end{aligned}$$

where $0 < \inf_{x_0\in D} \mathbb{P}_{x_0}((X^1_{2t_0^{(N)}},\ldots,X^N_{2t_0^{(N)}}) \in K_0^N) \leq 1$. It yields that there exists $\kappa>0$ such that (2.10) is fulfilled. $\square$

2.2.4 Uniform tightness of the empirical stationary distributions

In this part, the open set D is supposed to be bounded. Assume that a jump measure $\mathcal{J}^{(N)}$ and a family of drifts $(q_i^{(N)})_{i=1,\dots,N}$ are given for each $N \geq 2$.

Hypothesis 2.4. *Hypotheses 2.1 and 2.2 are fulfilled for each $N \geq 2$ and Hypothesis 2.3 is fulfilled with the same α for each $N \geq 2$. Moreover, there exists $r > 1$ such that*

$$\sup_{N\geq 2} \frac{1}{N} \sum_{i=1}^{N} r^{(Q_i^{(N)})^2} < +\infty,$$

where $Q_i^{(N)} = -\inf_{x\in D\setminus D_\alpha} \mathcal{L}_i^{(N)} \phi_D(x)$.

For all $N \geq 2$, we denote by $m^N \in \mathcal{M}_1(D^N)$ the initial distribution and by $\mu^N(t,dx)$ the empirical distribution of the N-particles process defined by the jump measure $\mathcal{J}^{(N)}$ and the family $(q_i^{(N)})_{i\in\{1,\dots,N\}}$. Its stationary distribution is denoted by M^N and its empirical stationary distribution is denoted by $\mathcal{X}^N$:

$$\mathcal{X}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$$

where $(x^1,\dots,x^N)$ is a random vector in D^N distributed following M^N.

Theorem 2.3. *Assume that Hypothesis 2.4 is fulfilled. For all sequence of measures $m^N \in \mathcal{M}_1(D^N)$ and all $t > 0$, the family of laws of the random measures $\left(\mu^N(t,dx)\right)_{N\geq 2}$ is uniformly tight. In particular, the family of laws of the empirical stationary distributions $\left(\mathcal{X}^N\right)_{N\geq 2}$ is uniformly tight.*

Proof. Let us consider the process $(X^1,\dots,X^N)$ starting with a distribution m^N and its coupled process $(Y^{\infty,1},\dots,Y^{\infty,N})$. For all $t \in [0,\tau_\infty[$, we denote by $\mu'^N(t,dx)$ the empirical measure of $(Y_t^{\infty,1},\dots,Y_t^{\infty,N})$. By the coupling inequality (2.8), we get

$$\mu^N(t,D_r^c) \leq \mu'^N(t,[0,r]),\ \forall r \in [0,\alpha].$$

Using the Markov property, we deduce that, for all $s < t$,

$$E_{X_s^1,\dots,X_s^N}\left(\mu^N(t-s,D_r^c)\right) \leq E_{Y_s^{\infty,1},\dots,Y_s^{\infty,N}}\left(\mu'^N(t-s,[0,r])\right) \text{ a.s.}$$

Then, by a comparison argument,

$$\begin{aligned} E_{X_s^1,\dots,X_s^N}\left(\mu^N(t-s,D_r^c)\right) &\leq E_{0,\dots,0}\left(\mu'^N(t-s,[0,r])\right) \text{ a.s.} \\ &\leq \frac{1}{N}\sum_{i=1}^{N} P_0(Y_{t-s}^{\infty,i} \leq r) \text{ a.s.} \end{aligned} \tag{2.11}$$

Thanks to the Girsanov's Theorem, we have

$$P_0(Y_{t-s}^{\infty,i} \leq r) = E_0\left(\delta_{w_{t-s}^i + L_{t-s}^{i,\alpha} - L_{t-s}^{i,0}}([0,r]) e^{Q_i^{(N)} w_{t-s}^i - (Q_i^{(N)})^2 (t-s)}\right) e^{\frac{3}{2}(Q_i^{(N)})^2 (t-s)},$$

where $(w^1,\dots,w^N)$ is a N-dimensional Brownian motion. By the Cauchy Schwartz inequality, we get

$$P_0(Y^{\infty,i}_{t-s} \leq r) \leq \sqrt{E_0\left(\left(\delta_{w^i_{t-s}+L^{i,\alpha}_{t-s}-L^{i,0}_{t-s}}([0,r])\right)^2\right) E_0\left(\left(e^{Q^{(N)}_i w^i_{t-s}-(Q^{(N)}_i)^2(t-s)}\right)^2\right)},$$
$$\leq \sqrt{E_0\left(\delta_{w^i_{t-s}+L^{i,\alpha}_{t-s}-L^{i,0}_{t-s}}([0,r])\right)}$$

where the second inequality occurs, since $0 \leq \delta_{w^i_{t-s}+L^{i,\alpha}_{t-s}-L^{i,0}_{t-s}}([0,r]) \leq 1$ almost surely and the process $e^{2Q^N_i w^i_t-2(Q^{(N)}_i)^2 t}$ is the Doléans exponential of $2Q^{(N)}_i w^i_t$, whose expectation is 1. Taking the expectation in (2.11), it yields that

$$E_{m^N}\left(\mu^N(t,D^c_r)\right) \leq \sqrt{P_0\left(\delta_{w^i_{t-s}+L^{i,\alpha}_{t-s}-L^{i,0}_{t-s}}([0,r])\right)}\frac{1}{N}\sum_{i=1}^N e^{\frac{3}{2}(Q^{(N)}_i)^2(t-s)},\ \forall 0<s<t.$$

Thanks to Hypothesis 2.4, there exists $s_0 \in]0,t[$ such that $\frac{1}{N}\sum_{i=1}^N e^{\frac{3}{2}(Q^{(N)}_i)^2(t-s_0)}$ is uniformly bounded in $N \geq 2$. But $P_0\left(\delta_{w^i_{t-s_0}+L^{i,\alpha}_{t-s_0}-L^{i,0}_{t-s_0}}([0,r])\right)$ goes to 0 when $r \to 0$, so that the family of the laws of random measures $(\mu^N(t,dx))_{N\geq 2}$ is uniformly tight.

If we set m^N equal to the stationary distribution M^N, then we get by stationarity that $\mathcal{X}^N$ is distributed as $\mu^N(t,.)$, for all $N \geq 2$ and $t > 0$. Finally, the family of laws of the empirical stationary distributions $(\mathcal{X}^N)_{N\geq 2}$ is uniformly tight. □

2.3 Yaglom limit's approximation

We consider now the particular case $\mathcal{J}^{(N)}(x_1,\dots,x_N) = \frac{1}{N-1}\sum_{k=1,k\neq i}^N \delta_{x_k}$: at each jump time, the particle which hits the boundary jumps to the position of a particle chosen uniformly between the $N-1$ remaining ones. We assume moreover that $q^{(N)}_i = q$ doesn't depend on i,N. In this framework, we are able to identify the limiting distribution of the empirical stationary distribution sequence, when the number of particles tends to infinity. This leads us to an approximation method of the Yaglom limits (2.4), including cases where the drift of the diffusion process isn't bounded and where the boundary is neither regular nor bounded.

Let U_∞ be an open domain of $\mathbb{R}^d$, with $d \geq 1$. We denote by $\mathbb{P}^\infty$ the law of the diffusion process defined on U_∞ by

$$dX^\infty_t = dB_t - \nabla V(X^\infty_t)dt,\ X^\infty_0 = x \in U_\infty \tag{2.12}$$

and absorbed at the boundary ∂U_∞. Here B is a d-dimensional Brownian motion and $V \in C^2(U_\infty,\mathbb{R})$. We assume that Hypothesis 2.5 below is fulfilled, so that the Yaglom limit

$$\nu_\infty = \lim_{t\to+\infty} \mathbb{P}_x^\infty \left(X_t^\infty \in .|t \leq \tau_\partial \right), \forall x \in U_\infty \tag{2.13}$$

exists and doesn't depend on x, as proved by Cattiaux and Méléard in [17, Theorem B.2]. We emphasize the fact that this hypothesis allows the drift ∇V of the diffusion process (2.12) to be unbounded and the boundary ∂U_∞ to be neither of class C^2 nor bounded. In particular, the results of the previous section aren't available in all generality for diffusion processes with law $\mathbb{P}^\infty$.

Hypothesis 2.5. *We assume that*

1. $\mathbb{P}_x^\infty(\tau_\partial < +\infty) = 1$,
2. $\exists C > 0$ *such that* $G(x) = |\nabla V|^2(x) - \Delta V(x) \geq -C > -\infty$, $\forall x \in U_\infty$,
3. $\overline{G}(R) \to +\infty$ *as* $R \to \infty$, *where*
$$\overline{G}(R) = \inf \left\{ G(x); |x| \geq R \text{ and } x \in U_\infty \right\},$$
4. *There exists an increasing sequence* $(U_m)_{m\geq 0}$ *of bounded open subsets of* U_∞, *such that the boundary of* U_m *is of class* C^2 *for all* $m \geq 0$, *and such that* $\bigcup_{m\geq 0} \overline{U_m} = U_\infty$.
5. *There exists* $R_0 > 0$ *such that*
$$\int_{U_\infty \cap \{d(x,\partial U_\infty) > R_0\}} e^{-2V(x)} dx < \infty \text{ and}$$
$$\int_{U_\infty \cap \{d(x,\partial U_\infty) \leq R_0\}} \left(\int_{U_\infty} p_1^{U_\infty}(x,y) dy \right) e^{-V(x)} dx < \infty.$$

   Here $p_1^{U_\infty}$ *is the transition density of the diffusion process* (2.12) *with respect to the Lebesgue measure.*

According to [17], the second point implies that the semi-group induced by $\mathbb{P}^\infty$ is ultra-contractive. The assumptions 1-4 imply that the generator associated with $\mathbb{P}^\infty$ has a purely discrete spectrum and that its minimal eigenvalue $-\lambda_\infty$ is simple and negative. The last assumption ensures that the eigenfunction associated with $-\lambda_\infty$ is integrable with respect to $e^{-2V(x)}dx$. Finally, Hypothesis 2.5 is sufficient for the existence of the Yaglom limit (2.13).

Remark 2.5. For example, it is proved in [17] that Hypothesis 2.5 is fulfilled by the Lotka-Volterra system studied numerically in Subsection 2.3.3. Up to a change of variable, this system is defined by the diffusion process with values in $U_\infty = \mathbb{R}_+^2$, which satisfies

$$\begin{aligned} dY_t^1 &= dB_t^1 + \left(\frac{r_1 Y_t^1}{2} - \frac{c_{11}\gamma_1 \left(Y_t^1\right)^3}{8} - \frac{c_{12}\gamma_2 Y_t^1 \left(Y_t^2\right)^2}{8} - \frac{1}{2Y_t^1} \right) dt \\ dY_t^2 &= dB_t^2 + \left(\frac{r_2 Y_t^2}{2} - \frac{c_{22}\gamma_2 \left(Y_t^2\right)^3}{8} - \frac{c_{21}\gamma_1 Y_t^2 \left(Y_t^1\right)^2}{8} - \frac{1}{2Y_t^2} \right) dt \end{aligned} \tag{2.14}$$

and is absorbed at ∂U_∞. Here B^1,B^2 are two independent one-dimensional Brownian motions and the parameters of the diffusion process fulfill condition (2.30).

In order to define the interacting particle process of the previous section, we work with diffusion processes defined on U_m, $m \geq 0$. More precisely, for all $m \geq 0$, we denote by $\mathbb{P}^m$ the law of the diffusion process defined on U_m by

$$dX_t^{U_m} = dB_t - q_m(X_t^{U_m})dt,\ X_0^{U_m} = x \in U_m \tag{2.15}$$

and absorbed at the boundary ∂U_m. Here B is a d-dimensional Brownian motion and $q_m : \overline{U_m} \mapsto \mathbb{R}$ is a continuous function. We denote by $\mathcal{L}^m$ the infinitesimal generator of the diffusion process with law $\mathbb{P}^m$. For all $m \geq 0$, the diffusion process with law $\mathbb{P}^m$ clearly fulfills the conditions of Section 2.2. For all $N \geq 2$, let $(X^{m,1},\ldots,X^{m,N})$ be the interacting particle process defined by the law $\mathbb{P}^m$ between the jumps and by the jump measure $\mathcal{J}^{(m,N)}(x_1,\ldots,x_N) = \frac{1}{N-1}\sum_{k=1,k\neq i}^N \delta_{x_k}$. By Theorem 2.1, this process is well defined and exponentially ergodic.

For all $m \geq 0$ and all $N \geq 2$, we denote by $\mu^{m,N}(t,dx)$ the empirical distribution of $(X_t^{m,1},\ldots,X_t^{m,N})$, by $M^{m,N}$ the stationary distribution of $(X^{m,1},\ldots,X^{m,N})$ and by $\mathcal{X}^{m,N}$ the associated empirical stationary distribution.

We are now able to state the main result of this section.

Theorem 2.4. *Assume that Hypothesis 2.5 is satisfied and that $q_m = \nabla V \mathbb{1}_{\overline{U_m}}$ for all $m \geq 0$. Then*

$$\lim_{m\to\infty} \lim_{N\to\infty} \mathcal{X}^{m,N} = \nu_\infty,$$

in the weak topology of random measures, which means that, for all bounded continuous function $f : U_\infty \mapsto \mathbb{R}_+$,

$$\lim_{m\to\infty} \lim_{N\to\infty} E(\mathcal{X}^{m,N}(f)) = \nu_\infty(f).$$

In Section 2.3.1, we fix $m \geq 0$ and we prove that the sequence $(\mathcal{X}^{m,N})_{N\geq 2}$ converges to a deterministic probability ν_m when N goes to infinity. In particular, we prove that ν_m is the Yaglom limit associated with $\mathbb{P}^m$, which exists by [17]. In Section 2.3.2, we conclude the proof, proceeding by a compactness/uniqueness argument: we prove that $(\nu_m)_{m\geq 0}$ is a uniformly tight family and we show that each limiting probability of the family $(\nu_m)_{m\geq 0}$ is equal to the Yaglom limit ν_∞. The last Section 2.3.3 is devoted to numerical illustrations of Theorem 2.4.

2.3.1 Convergence of $(\mathcal{X}^{m,N})_{N\geq 2}$, when $m \geq 0$ is fixed

Proposition 2.5. *Let $m \geq 0$ be fixed and let $q_m : \overline{U_m} \mapsto \mathbb{R}$ be a continuous function. Assume that $\mu^{m,N}(0,dx)$ converges in the weak topology of random measure to a random probability measure μ_m with values in $\mathcal{M}_1(U_m)$, when $N \to \infty$. Then, for all $T \geq 0$, $\mu^{m,N}(T,dx)$ converges in the weak topology of random measure to $\mathbb{P}^m_{\mu_m}(X_T \in .|X_T \in U_m)$ when N goes to infinity.*

Moreover, if there exists $\nu_m \in \mathcal{M}_1(U_m)$ such that

$$\nu_m = \lim_{t\to\infty} \mathbb{P}^m_\mu \left(X^{U_m}_t \in .|X^{U_m}_t \in U_m \right), \forall \mu \in \mathcal{M}_1(U_m), \tag{2.16}$$

then the sequence of empirical stationary distributions $(\mathcal{X}^{m,N})_{N\geq 2}$ converges to ν_m in the weak topology of random measures when N goes to infinity.

Remark 2.6. In Proposition 2.5, ν_m is the Yaglom limit and the unique quasi-stationary distribution associated with $\mathbb{P}^m$. For instance, each of the two following conditions is sufficient for the existence of such a measure:

1. If $q_m = \mathbb{1}_{\overline{U_m}} \nabla V$, by [17]. This is the case of Theorem 2.4.

2. If q_m belongs to $C^{1,\alpha}(\overline{U_m})$ with $\alpha > 0$, by [37].

Proof of Proposition 2.5. We set

$$\nu^{m,N}(t,dx) = \left(\frac{N-1}{N}\right)^{A^N_t} \mu^{m,N}(t,dx),$$

where $A^N_t = \sum_{n=1}^{\infty} \mathbb{1}_{\tau_n \leq t}$ denotes the number of jumps before time t. Intuitively, we introduce a loss of $1/N$ of the total mass at each jump, in order to approximate the distribution of the diffusion process (2.15) without conditioning. We will come back to the study of $\mu^{m,N}$ and the conditioned diffusion process by normalizing $\nu^{m,N}$ (a similar method is used in [40], with a different value of $\nu^{m,N}$).

Similarly to [40, Proposition 1], we apply the Itô's formula to the semi-martingale $\mu^{m,N}(t,\psi) = \frac{1}{N}\sum_{i=1}^{N} \psi(X^{m,i}_t)$, where $\psi \in C^2(U_m,\mathbb{R})$ vanishes at ∂D. We get

$$\mu^{m,N}(t,\psi) = \mu^{m,N}(0,\psi) + \int_0^t \mu^{m,N}(s,\mathcal{L}^m\psi)ds + \mathcal{M}^{c,N}(t,\psi) + \mathcal{M}^{j,N}(t,\psi) + \frac{1}{N-1}\sum_{0\leq\tau_n\leq t} \mu^{m,N}(\tau_n\text{-},\psi), \tag{2.17}$$

where $\mathcal{M}^{c,N}(t,\psi)$ is the continuous martingale

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{d}\int_0^t \frac{\partial\psi}{\partial x_j}(X^{m,i}_s)dB^{i,j}_s$$

and $\mathcal{M}^{j,N}(t,\psi)$ is the pure jump martingale

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{0\leq\tau_n^i\leq t}\left(\psi(X^{m,i}_{\tau_n^i})-\frac{N}{N-1}\mu^{m,N}(\tau_n^i\text{-},\psi)\right).$$

Applying the Itô's formula to the semi-martingale $\nu^{m,N}(t,\psi)$, we deduce from (2.17) that

$$\begin{aligned}\nu^{m,N}(t,\psi)=\nu^{m,N}(0,\psi)+\int_0^t\nu^{m,N}(s,\mathcal{L}^m\psi)ds+\int_0^t\left(\frac{N-1}{N}\right)^{A_s^N}d\mathcal{M}^{c,N}(s,\psi)\\+\sum_{0\leq\tau_n\leq t}(\nu^{m,N}(\tau_n,\psi)-\nu^{m,N}(\tau_n\text{-},\psi)).\end{aligned}$$

Where we have

$$\begin{aligned}\nu^{m,N}(\tau_n,\psi)-\nu^{m,N}(\tau_n\text{-},\psi)=\left(\frac{N-1}{N}\right)^{A^N_{\tau_n}}\left(\mu^{m,N}(\tau_n,\psi)-\mu^{m,N}(\tau_n\text{-},\psi)\right)\\+\mu^{m,N}(\tau_n\text{-},\psi)\left(\left(\frac{N-1}{N}\right)^{A^N_{\tau_n}}-\left(\frac{N-1}{N}\right)^{A^N_{\tau_n\text{-}}}\right).\end{aligned}$$

But

$$\mu^{m,N}(\tau_n,\psi)-\mu^{m,N}(\tau_n\text{-},\psi)=\frac{1}{N-1}\mu^{m,N}(\tau_n\text{-},\psi)+\mathcal{M}^{j,N}(\tau_n,\psi)-\mathcal{M}^{j,N}(\tau_n\text{-},\psi)$$

and

$$\left(\frac{N-1}{N}\right)^{A^N_{\tau_n}}-\left(\frac{N-1}{N}\right)^{A^N_{\tau_n\text{-}}}=-\frac{1}{N-1}\left(\frac{N-1}{N}\right)^{A^N_{\tau_n}}.$$

Then

$$\begin{aligned}\nu^{m,N}(\tau_n,\psi)-\nu^{m,N}(\tau_n\text{-},\psi)&=\left(\frac{N-1}{N}\right)^{A^N_{\tau_n}}\left(\mathcal{M}^{j,N}(\tau_n,\psi)-\mathcal{M}^{j,N}(\tau_n\text{-},\psi)\right).\\&=\frac{N-1}{N}\left(\frac{N-1}{N}\right)^{A^N_{\tau_n\text{-}}}\left(\mathcal{M}^{j,N}(\tau_n,\psi)-\mathcal{M}^{j,N}(\tau_n\text{-},\psi)\right).\end{aligned}$$

That implies

$$\begin{aligned}\nu^{m,N}(t,\psi)-\nu^{m,N}(0,\psi)=\int_0^t\nu^{m,N}(s,\mathcal{L}^m\psi)ds+\int_0^t\left(\frac{N-1}{N}\right)^{A_s^N}d\mathcal{M}^{c,N}(s,\psi)\\+\frac{N-1}{N}\sum_{0\leq\tau_n\leq t}\left(\frac{N-1}{N}\right)^{A^N_{\tau_n\text{-}}}\left(\mathcal{M}^{j,N}(\tau_n,\psi)-\mathcal{M}^{j,N}(\tau_n\text{-},\psi)\right).\end{aligned}$$

It yields that, for all smooth functions $\Psi(t,x)$ vanishing at the boundary of U_m,

$$\begin{aligned}\nu^{m,N}(t,\Psi(t,.))-\nu^{m,N}(0,\Psi(0,.))=\int_0^t\nu^{m,N}(s,\frac{\partial\Psi(s,.)}{\partial s}+\mathcal{L}^m\Psi(s,.))ds\\+\mathcal{N}^{c,N}(t,\Psi)+\mathcal{N}^{j,N}(t,\Psi),\end{aligned}$$

where $\mathcal{N}^{c,N}(t,\Psi)$ is the continuous martingale

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{d}\int_0^t\left(\frac{N-1}{N}\right)^{A_s^N}\frac{\partial\Psi}{\partial x_j}(s,X_s^{m,i})dB_s^{i,j}$$

and $\mathcal{N}^{j,N}(t,\Psi)$ is the pure jump martingale

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{0\leq\tau_n^i\leq t}\left(\frac{N-1}{N}\right)^{A^N_{\tau_n^i\text{-}}}\left(\Psi(\tau_n^i,X^i_{\tau_n^i})-\frac{N}{N-1}\mu^{m,N}(\tau_n^i\text{-},\Psi(\tau_n^i\text{-},.))\right).$$

Let $T>0$ be fixed. For all $\delta>0$, define $\Psi^\delta(t,x)=P^m_{T-t}P^m_\delta f(x)$, where $f\in C^2(\overline{U_m})$ and $(P^m_s)_{s\geq 0}$ is the semigroup associated with $\mathbb{P}^m$: $P^m_s f(x) = E_x(f(X_s^{U_m}))$. Then Ψ^δ vanishes on the boundary, is smooth, and fulfills

$$\frac{\partial}{\partial s}\Psi^\delta(s,x)+\frac{1}{2}\Delta\Psi^\delta(s,x)+q_m(x)\nabla\Psi^\delta(s,x)=0,$$

thanks to Kolmogorov's equation (see [28, Proposition 1.5 p.9]). It yields that

$$\nu^{m,N}(t,\Psi^\delta(t,.))-\nu^{m,N}(0,\Psi^\delta(0,.))=\mathcal{N}^{c,N}(t,\Psi^\delta)+\mathcal{N}^{j,N}(t,\Psi^\delta). \tag{2.18}$$

Since $\left(\frac{N-1}{N}\right)^{A_s^N}\leq 1$ a.s., we get

$$\begin{aligned}E\left(\mathcal{N}^{c,N}(T,\Psi^\delta)^2\right)&\leq\frac{T}{N}\|\nabla\Psi^\delta\|_\infty^2\\&\leq\frac{T}{N}\frac{c_m}{\sqrt{(T-t+\delta)\wedge 1}}\|f\|^2_\infty\end{aligned}\tag{2.19}$$

where $c_m>0$ is a positive constant. The last inequality comes from [71, Theorem 4.5] on gradient estimates in regular domains of $\mathbb{R}^d$. The jumps of the martingale $\mathcal{M}^{j,N}(t,\Psi^\delta)$ are smaller than $\frac{2}{N}\|\Psi^\delta\|_\infty$, then

$$\begin{aligned}&E\left[\sum_{0\leq\tau_n\leq T}\left(\frac{N-1}{N}\right)^{2A_{\tau_n\text{-}}}\left(\mathcal{M}^{j,N}(\tau_n,\Psi^\delta(\tau_n,.))-\mathcal{M}^{j,N}(\tau_n\text{-},\Psi^\delta(\tau_n\text{-},.))\right)^2\right]\\&\qquad\leq\frac{4}{N^2}\|\Psi^\delta\|^2_\infty E\left[\sum_{0\leq\tau_n\leq T}\left(\frac{N-1}{N}\right)^{2A_{\tau_n\text{-}}}\right]\\&\qquad\leq\frac{4}{N}\|\Psi^\delta\|^2_\infty.\end{aligned}$$

Then

$$E\left(\mathcal{N}^{j,N}(\Psi,T)^2\right)\leq\frac{4}{N}\|\Psi\|^2_\infty\leq\frac{4}{N}\|f\|^2_\infty. \tag{2.20}$$

Taking $t=T$ and $\delta=\frac{1}{N}$, we get from (2.18), (2.19) and (2.20) that

$$\sqrt{E\left(\left|\nu^{m,N}(t,P^m_{\frac{1}{N}}f)-\nu^{m,N}(0,P^m_{T+\frac{1}{N}}f)\right|^2\right)}\leq\sqrt{\frac{c_mT+4}{\sqrt{N}}}\|f\|_\infty.$$

Assume that f vanishes at ∂U_m, so that f belongs to the domain of $\mathcal{L}^m$. Then $\|P^m_{\frac{1}{N}} f - f\|_\infty \leq \frac{1}{N}\|\mathcal{L}^m f\|_\infty$ and we have

$$\sqrt{E\left(\left|\nu^{m,N}(T,f) - \nu^{m,N}(0,P^m_T f)\right|^2\right)} \leq \sqrt{\frac{c_m T + 4}{\sqrt{N}}}\|f\|_\infty + \frac{2}{N}\|\mathcal{L}^m f\|_\infty \xrightarrow{N\to\infty} 0. \tag{2.21}$$

By assumption, the family of random probabilities $(\nu^{m,N}(0,.))_{N\geq 2} = (\mu^{m,N}(0,.))_{N\geq 2}$ converges to μ_m. We deduce from (2.21) that

$$\nu^{m,N}(T,f) \xrightarrow[N\to\infty]{} \mu_m(P^m_T f), \tag{2.22}$$

for all $f \in C^2(U_m)$ vanishing at boundary. But the family of laws of $\left(\nu^{m,N}(T,.)\right)_{N\geq 2}$ is uniformly tight by Theorem 2.3 . It yields from (2.22) that its unique limiting distribution is $\mu_m(P^m_T .)$. In particular,

$$\left(\nu^{m,N}(T,U_m),\nu^{m,N}(T,.)\right) \xrightarrow[N\to\infty]{law} \left(\mu_m(P^m_T \mathbb{1}_{U_m}),\mu_m(P^m_T .)\right).$$

But $\mu_m(P^m_T \mathbb{1}_{U_m})$ never vanishes almost surely, so that

$$\mu^{m,N}(T,.) = \frac{\nu^{m,N}(T,.)}{\nu^{m,N}(T,U_m)} \xrightarrow[N\to\infty]{law} \frac{\mu_m(P^m_T .)}{\mu_m(P^m_T \mathbb{1}_{U_m})} = \mathbb{P}^m_{\mu_m}(X^{U_m}_T \in .|X^{U_m}_T \in U_m). \tag{2.23}$$

The family of laws of the random probabilities $(\mathcal{X}^{m,N})_{N\geq 0}$ is uniformly tight, by Theorem 2.3. Let $\mathcal{X}^m$ be one of its limiting probabilities. By definition, there exists a strictly increasing map $\varphi : \mathbb{N} \mapsto \mathbb{N}$, such that $\mathcal{X}^{m,\varphi(N)}$ converges in distribution to $\mathcal{X}^m$ when $N \to \infty$. By stationarity, $\mathcal{X}^{m,\varphi(N)}$ has the same law as $\mu^{m,\varphi(N)}(T,.)$, which converges in distribution to $\mathbb{P}^m_{\mathcal{X}}(X^{U_m}_T \in .|X^{U_m}_T \in U_m)$, thanks to (2.23). But $\mathbb{P}^m_{\mathcal{X}^m}(X^{U_m}_T \in .|X^{U_m}_T \in U_m)$ converges almost surely to ν_m when $T \to \infty$, by (2.16). We deduce from this that $\mathcal{X}^m$ has the same law as ν_m. As a consequence, the unique limiting probability of the uniformly tight family $(\mathcal{X}^m)_N$ is ν_m, which allows us to conclude the proof of Proposition 2.5. □

2.3.2 Convergence of the family $(\nu_m)_{m\geq 0}$

Proposition 2.6. *Assume that Hypothesis 2.5 is fulfilled and that $q_m = \nabla V \mathbb{1}_{\overline{U_m}}$. Then the sequence $(\nu_m)_{m\geq 0}$ converges weakly to the Yaglom limit ν_∞ when $m \to \infty$.*

Remark 2.7. Since $q_m = \nabla V \mathbb{1}_{\overline{U_m}}$, the operator $\mathcal{L}^m$ is symmetric with respect to the measure $e^{-2V(x)}dx$, but this isn't directly used in the proof of Proposition 2.6. We mainly use inequalities from [17] that are implied by the ultra-contractivity of $\mathbb{P}^\infty$ and the third point of Hypothesis 2.5. However, it seems hard to generalize this last hypothesis and its implications to diffusions with non-gradient drifts.

Proof of Proposition 2.6. For all $m \geq 0$ and $m = \infty$, it has been proved in [17] that $-\mathcal{L}^{m*}$ has a simple eigenvalue $\lambda_m > 0$ with minimal real part, where $\mathcal{L}^{m*}$ is the adjoint operator of $\mathcal{L}^m$. The corresponding normalized eigenfunction η_m is strictly positive on U_m, belongs to $C^2(U_m,\mathbb{R})$ and fulfills

$$\mathcal{L}^{m*}\eta_m = -\lambda_m \eta_m \text{ and } \int_{U_m} \eta_m(x)^2 d\sigma(x) = 1, \tag{2.24}$$

where

$$d\sigma(x) = e^{-2V(x)}dx.$$

The Yaglom limit ν_m is given by

$$d\nu_m = \frac{\eta_m \mathbb{1}_{U_m} d\sigma}{\int_{U_m} \eta_m(x) d\sigma(x)},\ \forall m \geq 0 \text{ or } m = \infty.$$

In order to prove that $(\nu_m)_{m\geq 0}$ converges to ν_∞, we show that $(\lambda_m)_{m\geq 0}$ converges to λ_∞. Then we prove that $(\eta_m \mathbb{1}_{U_m} d\sigma)_{m\geq 0}$ is uniformly tight. We conclude by proving that every limiting point $\eta d\sigma$ is a nonzero measure proportional to $\eta_\infty d\sigma$.

For all $m \geq 0$ or $m = \infty$, the eigenvalue λ_m of $-\mathcal{L}^{m*}$ is given by (see for instance [87, chapter XI, part 8])

$$\lambda_m = \inf_{\phi \in C_0^\infty(U_m),\ \langle \phi,\phi\rangle_{\sigma,m}=1} \langle \mathcal{L}^{m*}\phi,\phi\rangle_{\sigma,m}.$$

where $C_0^\infty(U_m)$ is the vector space of infinitely differentiable functions with compact support in U_m and $\langle f,g\rangle_{\sigma,m} = \int_{U_m} f(u)g(u)d\sigma(u)$. For all $\phi \in C_0^\infty(U_\infty)$, the support of ϕ belongs to U_m for m big enough, then $C_0^\infty(U_\infty) = \bigcup_{m\geq 0} C_0^\infty(U_m)$ since the reverse inclusion is clear. Moreover, if $\phi \in C_0^\infty(U_m)$, then $\mathcal{L}^{\infty *}\phi(x) = \mathcal{L}^{m*}\phi(x)$ for all $x \in U_m$. Finally,

$$\begin{aligned}
\lambda_\infty &= \inf_{m\geq 0} \inf_{\phi \in C_0^\infty(U_m),\ \langle \phi,\phi\rangle_{\sigma,m}=1} \langle \mathcal{L}^{m*}\phi,\phi\rangle_{\sigma,m} \\
&= \lim_{m\geq 0} \searrow \lambda_m.
\end{aligned}$$

Let us show that the family $(\eta_m \mathbb{1}_{U_m} d\sigma)_{m\geq 0}$ is uniformly tight. Fix an arbitrary positive constant $\epsilon > 0$ and let us prove that one can find a compact set $K_\epsilon \subset U_\infty$ which fulfills

$$\int_{U_\infty \setminus K_\epsilon} \epsilon_m \mathbb{1}_{U_m} d\sigma \leq \epsilon,\ \forall m \geq 0. \tag{2.25}$$

Let R_0 be the positive constant of the fifth part of Hypothesis 2.5. For all compact set K, we have

$$\int_{U_\infty \setminus K} \eta_m \mathbb{1}_{U_m} d\sigma = \int_{\{d(x,\partial U_m) > R_0\}\cap U_m \setminus K} \eta_m d\sigma + \int_{\{d(x,\partial U_m)\leq R_0\}\cap U_m\setminus K} \eta_m d\sigma.$$

From the proof of [17, Proposition B.6], we have on the one hand

$$\int_{\{d(x,\partial U_m)>R_0\}\cap U_m\setminus K}\eta_m d\sigma\leq\sqrt{\int_{\{d(x,\partial U_\infty)>R_0\}\cap U_\infty\setminus K}e^{-2V(x)}dx},$$

which is smaller than $\epsilon/2$ for a good choice of K, say K'_ϵ, since the integral at the right-hand side is finite by Hypothesis 2.5. On the other hand

$$\int_{\{d(x,\partial U_m)\leq R_0\}\cap U_m\setminus K}\eta_m d\sigma\leq e^{C/2}e^{\lambda_m}\kappa\int_{\{d(x,\partial U_\infty)\leq R_0\}\cap U_\infty\setminus K}\left(\int_{U_\infty}p_1^{U_\infty}(x,y)dy\right)dx,\tag{2.26}$$

where $\kappa=\sup_{m\geq 0}\|\eta_m e^{-V}\|_\infty<\infty$ thanks to [17], and $\lambda_m\leq\lambda_\infty$ for all $m\geq 0$. But the integral on the right-hand side is well defined by Hypothesis 2.5, then one can find a compact set K''_ϵ such that (2.26) is bounded by $\epsilon/2$. We set $K_\epsilon=K'_\epsilon\cup K''_\epsilon$ so that (2.25) is fulfilled. Since inequality (2.25) occurs for all $\epsilon>0$, the family $(\eta_m d\sigma)_{m\geq 0}$ is uniformly tight. Moreover, $\eta_m d\sigma$ has a density with respect to the Lebesgue measure, which is bounded by κe^{-V}, uniformly in $m\geq 0$. Then it is uniformly bounded on every compact set, so that every limiting distribution is absolutely continuous with respect to the Lebesgue measure.

Let $\eta d\sigma$ be a limiting measure of $(\eta_m d\sigma)_{m\geq 0}$. For all $\phi\in C_0^\infty(U_\infty,\mathbb{R})$, the support of ϕ belongs to U_m for m big enough, then

$$\begin{aligned}\langle\eta,\mathcal{L}^\infty\phi\rangle_{\sigma,\infty}&=\lim_{m\to\infty}\langle\eta_m,\mathcal{L}^m\phi\rangle_{\sigma,m}\\&=\lim_{m\to\infty}\langle\mathcal{L}^{m*}\eta_m,\phi\rangle_{\sigma,m}\\&=\lim_{m\to\infty}-\lambda_m\langle\eta_m,\phi\rangle_{\sigma,m}\\&=-\lambda_\infty\langle\eta,\phi\rangle_{\sigma,\infty}.\end{aligned}$$

Thanks to the elliptic regularity Theorem, η is of class C^2 and fulfills $\mathcal{L}^{\infty*}\eta=-\lambda_\infty\eta$. But the eigenvalue λ_∞ is simple, then η is proportional to η_∞. Let $\beta\geq 0$ be the non-negative constant such that $\eta=\beta\eta_\infty$. In particular, there exists an increasing function $\phi:\mathbb{N}\mapsto\mathbb{N}$ such that $\eta_{\phi(m)}d\sigma$ converges weakly to $\beta\eta_\infty d\sigma$.

Let us prove that β is positive. For all compact subset $K\subset U_\infty$, we have

$$\begin{aligned}\beta\left\langle\eta_\infty,e^V\mathbb{1}_K\right\rangle_{\sigma,\infty}&=\lim_{m\to\infty}\left\langle\eta_{\phi(m)},\mathbb{1}_K e^V\right\rangle_{\sigma,\phi(m)}\\&\geq\lim_{m\to\infty}\frac{1}{\kappa}\left\langle\eta_{\phi(m)},\mathbb{1}_K\eta_{\phi(m)}\right\rangle_{\sigma,\phi(m)}\\&\geq\frac{1}{\kappa}\left(1-\sup_{m\geq 0}\left\langle\eta_m,\mathbb{1}_{U_m\setminus K}\eta_m\right\rangle_{\sigma,m}\right),\end{aligned}\tag{2.27}$$

where $\kappa=\sup_{m\geq 0}\|\eta_m e^{-V}\|_\infty<\infty$. For all $m\geq 0$ and all $R>0$,

$$\left\langle\eta_m,\mathbb{1}_{U_m\setminus K}\eta_m\right\rangle_{\sigma,m}\leq\frac{1}{\overline{G}(R)}\left\langle\eta_m,\mathbb{1}_{|x|\geq R}G\eta_m\right\rangle_{\sigma,m}+\left\langle\eta_m,\mathbb{1}_{\{|x|<R\}\setminus K}\eta_m\right\rangle_{\sigma,m},\tag{2.28}$$

where G and $\overline{G}$ are defined in Hypothesis 2.5. Let us prove that $\langle \eta_m, G\eta_m \rangle_{\sigma,m}$ is uniformly bounded in $m \geq 0$. For all $x \in U_m$, (2.24) leads to

$$\frac{1}{2}G(x)\eta_m(x) = \lambda_m \eta_m(x) + \frac{1}{2}e^{V(x)}\Delta(\eta_m e^{-V})(x).$$

Then

$$\begin{aligned}\langle \eta_m, G\eta_m \rangle_{\sigma,m} &= \lambda_m \langle \eta_m, \eta_m \rangle_{\sigma,m} + \frac{1}{2}\int_{U_m} \eta_m(x) e^{-V(x)} \Delta(\eta_m e^{-V})(x) dx \\ &= \lambda_m - \int_{U_m} |\nabla \eta_m(x) e^{-V(x)}|^2 dx \\ &\leq \lambda_1,\end{aligned}$$

where the second equality is a consequence of the Green's formula (see [3, Corollary 3.2.4]). But $\overline{G}(R)$ goes to $+\infty$ when $R \to \infty$, then there exists $R_1 > 0$ such that $\frac{1}{\overline{G}(R_1)} \left\langle \eta_m, \mathbb{1}_{|x| \geq R_1} G\eta_m \right\rangle_{\sigma,m} \leq \frac{1}{4}$. Since $\kappa = \sup_{m \geq 0} \|\eta_m e^{-V}\|_\infty < \infty$, we deduce from (2.28) that

$$\left\langle \eta_m, \mathbb{1}_{U_m \setminus K} \eta_m \right\rangle_{\sigma,m} \leq \frac{1}{4} + \kappa^2 \int_{U_\infty} \mathbb{1}_{\{|x| < R_1\} \setminus K} dx.$$

But one can find a compact subset $K_1 \subset U_\infty$ such that $\int_{U_\infty} \mathbb{1}_{\{|x| < R_1\} \setminus K_1} dx \leq \frac{1}{4\kappa^2}$, then we have from (2.27)

$$\beta \left\langle \eta_0, \mathbb{1}_K \right\rangle_\sigma \geq \frac{1}{2\kappa}.$$

It yields that $\beta > 0$ and Proposition 2.6 follows. □

2.3.3 Numerical simulations

The Wright-Fisher case

The Wright-Fisher with values in $]0,1[$ conditioned to be absorbed at 0 is the diffusion process driven by the SDE

$$dZ_t = \sqrt{Z_t(1 - Z_t)} dB_t - Z_t dt,\ Z_0 = z \in]0,1[,$$

and absorbed when it hits 0 (1 is never reached). Huillet proved in [46] that the Yaglom limit of this process exists and has the density $2 - 2x$ with respect to the Lebesgue measure. In order to apply Theorem 2.4, we define $\mathbb{P}^\infty$ as the law of $X^\infty_. = \arccos(1 - 2Z_.)$. Then $\mathbb{P}^\infty$ is the law of the diffusion process with values in $U_\infty =]0,\pi[$, driven by the SDE

$$dX^\infty_t = dB_t - \frac{1 - 2\cos X^\infty_t}{2\sin X^\infty_t} dt,\ X^\infty_0 = x \in]0,\pi[,$$

absorbed when it hits 0 (π is never reached). One can easily check that this diffusion process fulfills Hypothesis 2.5. We denote by ν_∞ its Yaglom limit.

For all $m \geq 1$, we define $U_m =]\frac{1}{m}, \pi - \frac{1}{m}[$. Let $\mathbb{P}^m$ and ν_m be as in Section 2.3. We proceed to the numerical simulation of the N-interacting particle system $(X^{m,1},\ldots,X^{m,N})$ with $m = 1000$ and $N = 1000$. This leads us to the computation of $E(\mathcal{X}^{m,N})$, which is an approximation of ν_∞. After the change of variable $Z_. = 2\cos(X_.)$, we see on Figure 2.3 that the simulation is very close to the expected result $(2-2x)dx$, which shows the efficiency of the method.

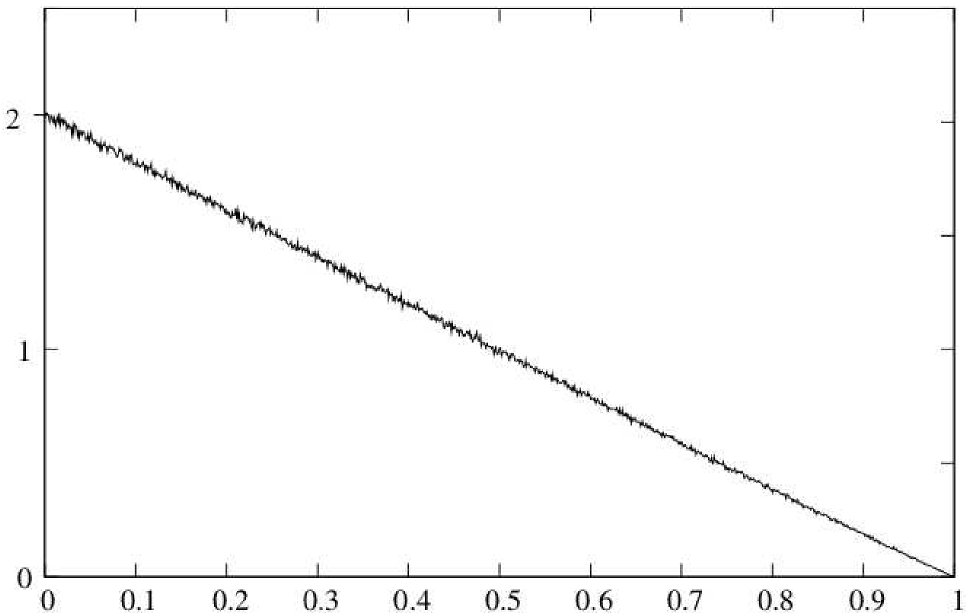

Figure 2.3: $E(\mathcal{X}^{m,N})$ in the Wright-Fisher case

The logistic case

The logistic Feller diffusion with values in $]0,+\infty[$ is defined by the stochastic differential equation

$$dZ_t = \sqrt{Z_t}dB_t + (rZ_t - cZ_t^2)dt,\ Z_0 = z > 0, \tag{2.29}$$

and absorbed when it hits 0. Here B is a 1-dimensional Brownian motion and r,c are two positive constants. In order to use Theorem 2.4, we make the change of variable $X_. = 2\sqrt{Z_.}$. This leads us to the study of the diffusion process with values in $U_\infty =]0,+\infty[$, which is absorbed at 0 and satisfies the SDE

$$dX_t^\infty = dB_t - \left(\frac{1}{2X_t^\infty} - \frac{rX_t^\infty}{2} + \frac{c(X_t^\infty)^3}{4}\right)dt,\ X_0^\infty = x \in]0,+\infty[.$$

We denote by $\mathbb{P}^\infty$ its law. Cattiaux et al. proved in [16] that Hypothesis 2.5 is fulfilled in this case. Then the Yaglom limit ν_∞ associated with $\mathbb{P}^\infty$ exists and one can apply Theorem 2.4 with $U_m =]\frac{1}{m}, m[$ for all $m \geq 1$. For all $N \geq 2$, we denote by $\mathbb{P}^m$ the law of the diffusion process restricted to U_m and by $\mathcal{X}^{m,N}$ the empirical stationary distribution of the N-interacting particle process associated with $\mathbb{P}^m$.

We've proceeded to the numerical simulation of the interacting particle process for a large number of particles and a large value of m. This allows us to compute $E(\mathcal{X}^{m,N})$, which gives us a numerical approximation of ν_∞, thanks to Theorem 2.4.

In the numerical simulations below, we set $m = 10000$ and $N = 10000$. We compute $E(\mathcal{X}^{m,N})$ for different values of the parameters r and c in (2.29). The results are graphically represented in Figure 2.4. As it could be wanted for, greater is c, closer is the support of the QSD to 0. We thus numerically describe the impact of the linear and quadratic terms on the Yaglom limit.

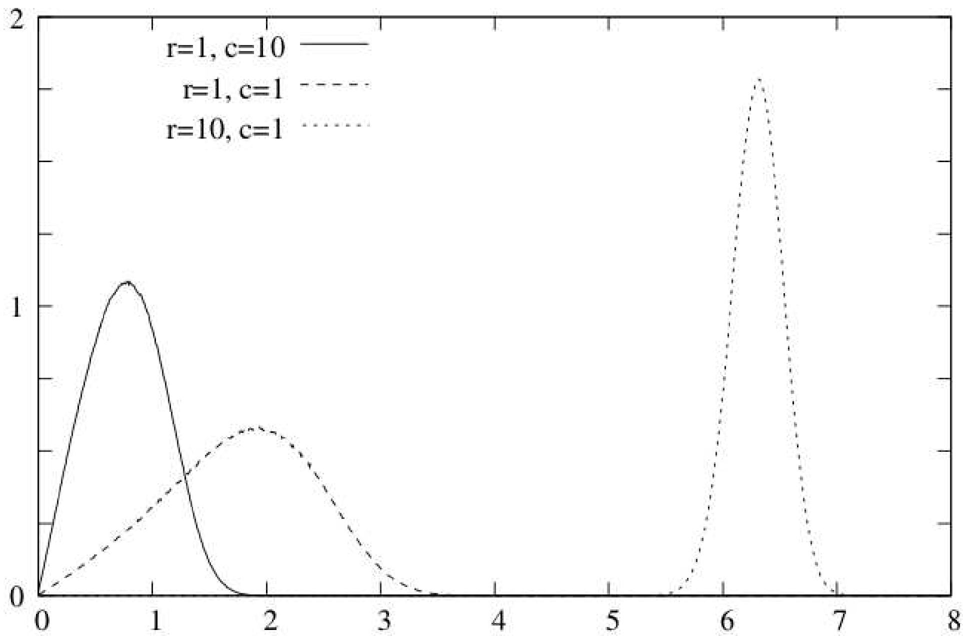

Figure 2.4: $E(\mathcal{X}^{m,N})$ for the diffusion process (2.29), with different values of r and c

Stochastic Lotka-Volterra Model

We apply our results to the stochastic Lotka-Volterra system with values in $D = \mathbb{R}_+^2$ studied in [17], which is defined by the following stochastic differential system

$$dZ_t^1 = \sqrt{\gamma_1 Z_t^1} dB_t^1 + \left(r_1 Z_t^1 - c_{11}(Z_t^1)^2 - c_{12} Z_t^1 Z_t^2\right) dt,$$
$$dZ_t^2 = \sqrt{\gamma_2 Z_t^2} dB_t^2 + \left(r_2 Z_t^2 - c_{21} Z_t^1 Z_t^2 - c_{22}(Z_t^2)^2\right) dt,$$

where (B^1,B^2) is a bi-dimensional Brownian motion. We are interested in the process absorbed at ∂D.

More precisely, we study the process $X^\infty = (Y^1,Y^2) = (2\sqrt{Z^1_\cdot/\gamma_1},2\sqrt{Z^2_\cdot/\gamma_2})$, with values in $U_\infty = \mathbb{R}^2_+$, which satisfies the SDE (2.14) and is absorbed at ∂U_∞. We denote its law by $\mathbb{P}^\infty$. The coefficients are supposed to satisfy

$$c_{11},c_{21} > 0,\ c_{12}\gamma_2 = c_{21}\gamma_1 < 0 \text{ and } c_{11}c_{22} - c_{12}c_{21} > 0. \tag{2.30}$$

In [17], this case was called *the weak cooperative case* and the authors proved that it is a sufficient condition for Hypothesis 2.5 to be fulfilled. Then the Yaglom limit $\nu_\infty = \lim_{t\to+\infty} \mathbb{P}^\infty_x (X^\infty \in .|t < \tau_\partial)$ is well defined and we are allowed to apply Theorem 2.4. For each $m \geq 1$, we define U_m as a rectangle whose angles have been rounded in a C^2 way, as it is described on Figure 2.5. With this definition, it is clear that all conditions of Theorems 2.1 and 2.4 are fulfilled.

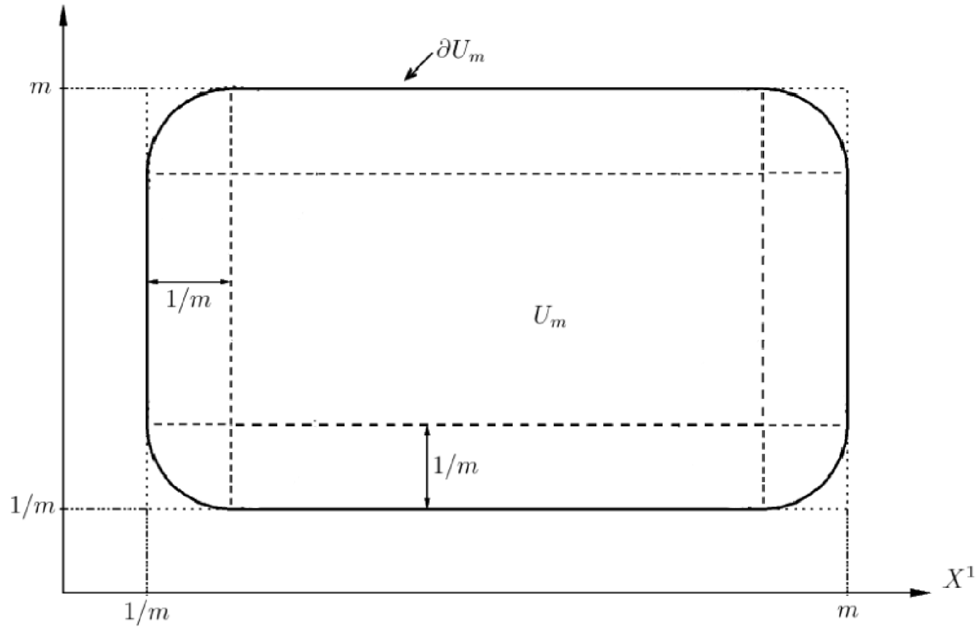

Figure 2.5: Definition of U_m

We choose $m = 10000$ and we simulate the long time behavior of the interacting particle process with $N = 10000$ particles for different values of c_{12} and c_{21}. The values of the other parameters are $r_1 = 1 = r_2 = 1$, $c_{11} = c_{22} = 1$, $\gamma_1 = \gamma_2 = 1$. The results are illustrated on Figure 2.6. One can observe that a greater value of the cooperating coefficients $-c_{12} = -c_{21}$ leads to a Yaglom limit whose support is further from the boundary and covers a smaller area. In other words, the more the two populations cooperate, the bigger the surviving populations are.

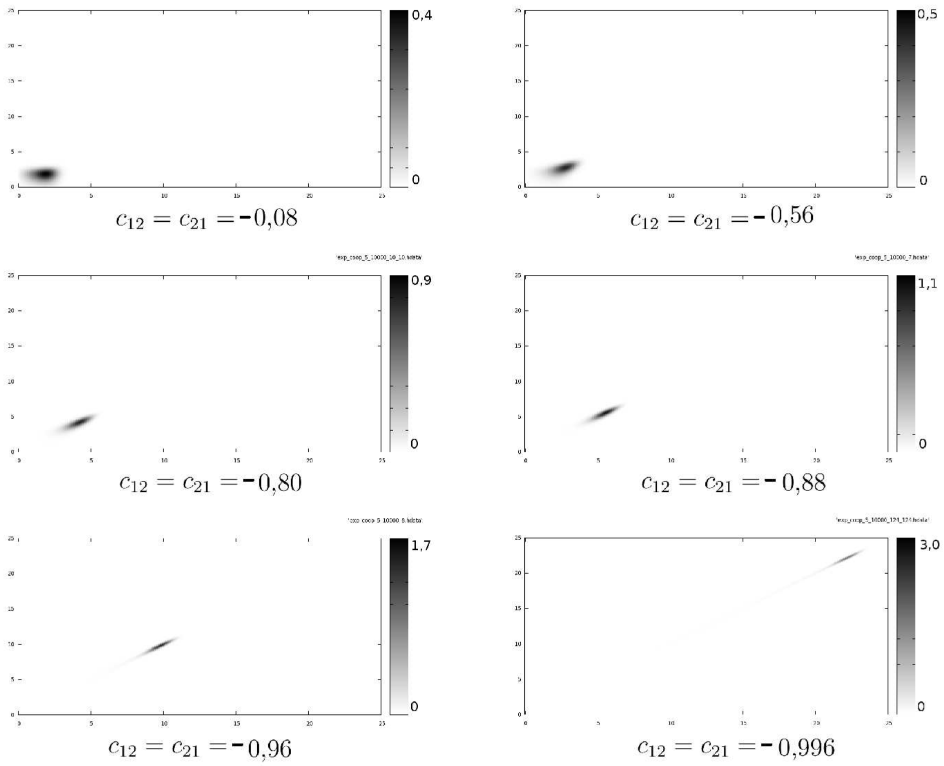

Figure 2.6: Empirical stationary distribution of the interacting particle process for different values of $c_{12} = c_{21}$

Chapter 3

Interacting particle processes and approximation of Markov processes conditioned to not be killed *

Abstract

We prove an approximation method for general strong Markov processes conditioned to not be killed. The method is based on a Fleming-Viot type interacting particle system, whose particles evolve as independent copies of the original strong Markov process and jump onto each others instead of being killed. We only assume that the number of jumps of the Fleming-Viot type system doesn't explode in finite time almost surely, and that the survival probability at fixed time of the original process is positive. We also give a speed of convergence for the approximation method.

A criterion for the non-explosion of the number of jumps is then given for general systems of time and environment dependent diffusion particles, which includes the case of the Fleming-Viot type system of the approximation method. The proof of the criterion uses an original non-attainability of (0,0) result for a pair of non-negative semi-martingales with positive jumps.

3.1 Introduction

Let F be a Banach space and ∂ be a point which doesn't belong to F. Let P be the semi-group of a strong Markov process $\mathcal{Z}$ which evolves in $F \cup \{\partial\}$ and

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denote by τ_∂ the hitting time of $\{\partial\}$. We assume that ∂ is a cemetery point for $\mathcal{Z}$, which means that $\mathcal{Z}_t = \partial$ for all $t \geq \tau_\partial$, and we call τ_∂ the *killing time* of $\mathcal{Z}$.

Killed Markov processes are commonly used in a large area of applications in biology, demography, chemistry or finance, where there is two natural ways of *killing* a Markov process, which correspond to different interpretations. The first way is to kill the process when it reaches a given set. For instance, a demographic's model is stopped when the size of the population hits 0, since it corresponds to the extinction of the population. The second way of killing a process is to stop it at an exponential time. For example, a chemical particle typically disappears by reacting with another one after an exponential time, whose rate depends on the concentration of reactant in the medium. If the killing time τ_∂ is given by the time at which the process reaches a set, we call it a *hard* killing time. If it is given by an exponential clock, we call it a *smooth* killing time. While the distribution of the process after its killing time is of poor interest, numerous studies concentrate on the behavior of the process conditioned to not be killed (see [16] and references therein). The main motivation of this paper is to provide an approximation method for the distribution of Markov processes evolving in a random/time dependent environment and conditioned to not be killed.

The main tool of the approximation method is given by a Fleming-Viot type interacting particle system introduced by Burdzy, Holyst, Ingermann and March in [13] and [14]: the N particles of the system evolve as independent Brownian motions in an open subset D of $\mathbb{R}^d$, and, when a particle hits the boundary ∂D, it jumps onto the position of an other particle chosen uniformly between the $N-1$ other ones; then the particles evolve as independent particles and so on. When N goes to infinity, the empirical measure of the process converges to the distribution of a standard multi-dimensional Brownian motion conditioned to not be killed at the current time. Such an approximation method has been proved by Grigorescu and Kang in [40] for a standard multi-dimensional Brownian motion, in [85] for Brownian motions with drift and by Del Moral and Miclo for smoothly killed Markov processes (see [23] and references therein). Let us also mention the work of Ferrari and Mariĉ [30], which regards continuous time Markov chains in discrete spaces.

In Section 3.2, we prove that this method works in a very general setting. Namely, let $(\mathcal{Z}^N)_{N\geq 2}$ be a sequence of strong Markov processes which evolve in $F\cup\{\partial\}$, where ∂ is the cemetery point for each $\mathcal{Z}^N$. We fix $T \geq 0$ and we assume that the sequence $(\mathcal{Z}^N_T)_N$ converges to $\mathcal{Z}_T$ in the sens of Hypothesis 3.1 . For each $N \geq 2$, we build a Fleming-Viot type system of N interacting particles as above: the particles evolve as independent copies of $\mathcal{Z}^N$ until one of them is killed; at this time, the killed particle jumps onto the position of another particle, chosen between the $N-1$ remaining ones. We assume that the number of jumps in the N particles system doesn't explode up to time T, and we prove in Theorem 3.1

that the associated sequence of empirical stationary distributions converges when $N \to \infty$ to the distribution of the process $\mathcal{Z}$ conditioned to not be killed at time T. We also give a speed of convergence for the method, which only depends on the survival probability of the Markov processes $\mathcal{Z}^N$, $N \geq 2$.

This result comes as an important generalization of the previously cited ones. Firstly, we allow both hard and soft killings, which is a natural setting in applications: typically, a species can disappear because of a lack of born of new specimens (which corresponds to a hard killing at 0) or because of a brutal natural catastrophe (which typically happens following an exponential time). Secondly, we implicitly allow time and environment dependency, which is also quite natural in applications, where individual paths are influenced by external stochastic factors (as the weather) whose distribution varies with time (because of the seasons by instance). Finally, we allow the process $\mathcal{Z}^N$ which drives the particles to depend on N, and we only require the non-explosion of the number of jumps of the Fleming-Viot type system build on $\mathcal{Z}^N$. As a consequence, one can apply the approximation method to a process $\mathcal{Z}$, without requiring that the Fleming-Viot process based on $\mathcal{Z}$ is well defined. This is typically the case for degenerate diffusions, or for diffusions with hard killing at the boundary of a non-regular domain, or for Markov processes with smooth killing given by an unbounded rate function. In our case, the three irregularities can be combined, by successive approximations of the coefficients, domain and rate of killing respectively.

Since the method works in a very general setting, it only remains us to prove the non-explosion of the number of jumps. This problem is studied in Section 3.3. Such non-explosion results have been recently obtained by Löbus in [62] and by Bienek, Burdzy and Finch in [11] for Brownian particles killed at the boundary of a given open set, by Grigorescu and Kang in [42] for time-homogeneous particles driven by a stochastic equation with regular coefficients killed at the boundary of a non-smooth domain (a survey of the previous results is done in the introduction of [42]) and in [85] for Brownian particles with drift. Other models of diffusions with jumps from a boundary have been introduced in [9], with a continuity condition on the jump measure that isn't fulfilled in our case, in [41], where fine properties of a Brownian motion with rebirth have been established, and in [55], [56], where Kolb and Wükber have studied the spectral properties of this model. In Section 3.3, we state the non-explosion of an interacting particle process, whose construction is a generalization of the previous ones. Indeed we consider particles which evolve as Itô diffusion processes in a random/time dependent environment with both hard and soft killings, with a different space of values for each particle. Moreover, at each killing time, we allow very general jump locations for the killed particle. In particular, this validates the approximation method described above for time/environment dependent diffusions with hard and soft killing.

The proof of the non-explosion is based on an original non-attainability of

(0,0) result for semi-martingales, which is stated in the last section of this paper.

3.2 Approximation of a Markov process conditioned to not be killed

Let F be a polish space and let $\mathcal{Z}$ be a càdlàg strong Markov process which evolves in F until it is killed. When it is killed, it jumps to a cemetery point $\partial \notin F$. The killing time is denoted by $\tau_\partial = \inf\{t \geq 0,\ \mathcal{Z}_t = \partial\}$. In this section, we fix $T \geq 0$ and we prove an approximation method for the distribution of the process $\mathcal{Z}_T$ starting with distribution $\mu_0 \in \mathcal{M}_1(F)$ and conditioned to the event $\{T < \tau_\partial\}$.

The approximation method is based on a sequence of Fleming-Viot type systems $\mathbb{X}^{(N)} = \left(X^{1,(N)},\ldots,X^{2,(N)}\right)$ with values in F^N, $N \geq 2$. A natural choice for the dynamic of $\mathbb{X}^{(N)}$, $N \geq 2$, should be the following: the particles evolve independently as N independent copies of $\mathcal{Z}$ until one of them is killed; at this time, the killed particle jumps from ∂ to the position of one of the $N-1$ remaining particles; then the particles evolve as N independent copies of $\mathcal{Z}$ until one of them is killed and so on. Unfortunately, for a general choice of $\mathcal{Z}$, the number of jumps of the system could explode in finite time, or the N particles could be killed at the same time (see [11, Example 5.3] for an example of explosion in a non-trivial setting). When this happens, the approximation method can no longer operate. In order to overcome this difficulty, we assume that we're given a sequence $\left(\mathcal{Z}^N\right)_{N\geq 2}$ of strong Markov processes which converges to $\mathcal{Z}$ at time T (Hypothesis 3.1 below) and such that, for all $N \geq 2$, the Fleming-Viot system with N particles driven by $\mathcal{Z}^N$ between the killings doesn't explode before time T (Hypothesis 3.2). Theorem 3.1 below states that the empirical measure at time T of the system $\mathbb{X}^{(N)}$ (whose particles are driven by $\mathcal{Z}^N$ between the killings) converges, when N goes to infinity, to the distribution of $\mathcal{Z}$ conditioned to $\{T < \tau_\partial\}$. A rate of convergence of the approximation method is also given, which only depends on the survival probability of $\mathcal{Z}^N$ at time $T \geq 0$.

Let $\left(\mathcal{Z}^N\right)_{N\geq 2}$ be a sequence of càdlàg strong Markov processes which evolve in $F \cup \{\partial\}$, where ∂ is a cemetery point for each $\mathcal{Z}^N$. We denote the killing time of $\mathcal{Z}^N$ by $\tau_\partial^N = \inf\{t \geq 0,\ \mathcal{Z}_t^N = \partial\}$. For each $N \geq 2$, we define the interacting particle system $\mathbb{X}^{(N)} = (X^{1,(N)},\ldots,X^{N,(N)})$ with values in F^N as follows:

- Let $m^{(N)} \in \mathcal{M}_1\left(F^N\right)$ be the initial distribution of the system.
- The N particles evolve as N independent copies of $\mathcal{Z}^N$ until one of them is killed. This killing time is denoted by $\tau_1^{(N)}$.
- At time $\tau_1^{(N)}$, the process is modified:
  - If there exists more than one particle which is killed at time $\tau_1^{(N)}$, we stop the interacting particle system itself and this time is denoted by

$\tau_{stop}^{(N)}$ (In fact, we will assume that this kind of event doesn't happen almost surely).

- Otherwise the unique killed particle jumps instantaneously onto the position of another particle, chosen uniformly between the $N-1$ remaining ones.

- At time $\tau_1^{(N)}$ and after proceeding to the jump, the process lies in F^N. Then the system evolves as N independent copies of $\mathcal{Z}^N$, until the next killing time, denoted by $\tau_2^{(N)}$.
- At this time, the process jumps with the same mechanism as above (and could be stopped at a time denoted by $\tau_{stop}^{(N)}$, as above).
- Then the particles evolve as N independent copies of $\mathcal{Z}^N$, and so on.

We set $\tau_{stop}^{(N)} = +\infty$ if $X^{i,(N)}$ and $X^{j,(N)}$ are never killed at the same time, for all $i \neq j$. On the event $\{\tau_{stop}^{(N)} = +\infty\}$, we denote by $\tau_1^{(N)} < \tau_2^{(N)} < ... < \tau_n^{(N)} < ...$ the sequence of jump times and we set

$$\tau_\infty^{(N)} = \lim_{n\to\infty} \tau_n^{(N)}. \tag{3.1}$$

If $\tau_{stop}^{(N)} < +\infty$, we set $\tau_\infty^{(N)} = +\infty$. The interacting particle system is then well defined for all time $t < \tau_{stop}^{(N)} \wedge \tau_\infty^{(N)}$.

We denote by $A_t^{i,(N)}$ the number of jumps of the i^{th} particle up to time t, $t < \tau_{stop}^{(N)} \wedge \tau_\infty^{(N)}$. We denote the total number of jumps of the system by $A_t^{(N)}$:

$$A_t^{(N)} = \sum_{i=1}^N A_t^{i,(N)},$$

and by $\mu_t^{(N)}$ the empirical distribution of $\mathbb{X}_t^{(N)}$:

$$\mu_t^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,(N)}} \in \mathcal{M}_1(F),$$

where $\mathcal{M}_1(F)$ denotes the space of probability measures on F.

The first assumption concerns the convergence of $\mathcal{Z}_T^N$ starting with initial random distribution $\mu_0^{(N)}$ to $\mathcal{Z}_T$ starting with (possibly random) distribution μ_0.

Hypothesis 3.1. *We assume that, for all bounded and continuous functions $f : F \cup \{\partial\} \mapsto \mathbb{R}_+$ such that $f(\partial) = 0$,*

$$\mu_0^{(N)}\left(P_T^N f\right) \xrightarrow[N\to\infty]{law} \mu_0\left(P_T f\right).$$

where $P_\cdot^N$ (respectively $P_\cdot$) denotes the semi-group of the process with killing $\mathcal{Z}^N$ (respectively $\mathcal{Z}$).

Remark 3.1. A typical situation where Hypothesis 3.1 is fulfilled is the following: we're given μ_0, $\mathcal{Z}$, and a sequence $\mathcal{Z}^N$ such that, for all $x \in F$ and all continuous and bounded function $f : F \mapsto \mathbb{R}_+$,

$$P_T^N f(x) \xrightarrow[N\to\infty]{} P_T f(x). \tag{3.2}$$

If we assume that $m^{(N)} = \mu_0^{\otimes N}$, then Hypothesis 3.1 is fulfilled. Indeed, we have

$$\mu_0^{(N)}\left(P_T^N f\right) \overset{law}{=} \frac{1}{N}\sum_{i=1}^{N}\left[P_T^N f(x_i) - \mu_0\left(P_T^{(N)} f\right)\right] + \mu_0\left(P_T^{(N)} f\right),$$

where $(x_i)_{i\geq 1}$ is an *iid* sequence of random variables with law μ_0. By the law of large numbers, the first right term converges to 0 almost surely. By the convergence assumption (3.2) and by dominated convergence, the second right term converges almost surely to $\mu_0\left(P_T f\right)$, so that Hypothesis 3.1 is fulfilled.

The second assumption concerns the non-explosion of the number of jumps for the system with N particles driven by $\mathcal{Z}^N$ between the killings.

Hypothesis 3.2. *We assume that, for all $N \geq 2$, the process $\mathbb{X}^N$ is well defined up to time T, which means that*

$$P_{m^{(N)}}\left(T < \tau_{stop} \wedge \tau_\infty^{(N)}\right) = 1.$$

Hypothesis 3.2 is clearly fulfilled if $\mathcal{Z}^N$ is only subject to smooth killing events happening with uniformly bounded killing rates (the question has not been answered to in the case of unbounded killing rates). In the case of an Itô'S diffusion driven by time-homogeneous stochastic differential equations and hardly killed when it hits the boundary of an open set, the problem is much harder and has been extensively studied recently (see [85], [42] and references therein for different and quite general criteria of non-explosion). The case of Itô diffusions driven by stochastic differential equations with time/environment dependent coefficients subject to soft and hard killings is treated in Section 3.3 of this paper.

Theorem 3.1. *We assume that the survival probability of $\mathcal{Z}$ at time T is strictly positive, which means that*

$$\mu_0\left(P_T \mathbf{1}_F\right) > 0, \text{ almost surely.} \tag{3.3}$$

Assume that Hypotheses 3.1 and 3.2 are fulfilled. Then, for any continuous and bounded function $f : F \mapsto \mathbb{R}_+$,

$$\mu_T^{(N)}(f) \xrightarrow[N\to\infty]{law} \frac{\mu_0\left(P_T f\right)}{\mu_0\left(P_T \mathbf{1}_F\right)}.$$

Moreover, for any bounded measurable function $f : F \mapsto \mathbb{R}_+$, we have the inequality

$$E\left(\left|\mu_T^{(N)}(f) - \frac{\mu_0^{(N)}\left(P_T^{(N)} f\right)}{\mu_0^{(N)}\left(P_T^{(N)}\mathbf{1}_F\right)}\right|\right) \leq \frac{4\|f\|_\infty}{\sqrt{N}}\sqrt{E\left(\frac{1}{\left(\mu_0^{(N)}\left(P_T^N \mathbb{1}_F\right)\right)^2}\right)}.$$

Remark 3.2. In Section 3.3, we give a non-explosion criterion for systems whose particles are driven by diffusions evolving in a random/time dependent environment, killed after exponential times or when they hit the boundary of a given open set. In particular, this criterion requires that the rate of killing is bounded and that the killing boundary and the coefficients of the diffusions are smooth. If $\mathcal{Z}$ is a diffusion in random environment, with unbounded killing rate, irregular coefficients and non-smooth killing boundary, one can define a sequence of strong Markov processes $(\mathcal{Z}^N)_{N\geq 2}$ which approximates $\mathcal{Z}$ and fulfills the criterion of Section 3.3 for all $N \geq 2$, proceeding by successive approximations of the rate of killing, the killing boundary and the coefficients of the diffusion $\mathcal{Z}$. It yields that Theorem 3.1 gives an approximation method for $\mathcal{Z}$ conditioned to $\{T < \tau_\partial\}$, while $\mathcal{Z}$ is degenerate. This example illustrates that allowing an approximating sequence $\mathcal{Z}^N$ for $\mathcal{Z}$ gives a great generality to the approximation method of Theorem 3.1.

Remark 3.3. In the particular case of a process $\mathcal{Z}$ with a uniformly bounded killing rate and without hard killing, a uniform rate of convergence over all times T can be obtained, using the stability of the underlying Feynman-Kac semi-group (we refer the reader to Rousset's work [75] and references therein).

Proof of Theorem 3.1. The proof consists of three steps. In a first step, we fix $N \geq 2$ and we prove that, for any bounded and measurable function $f : F \cup \{\partial\}$ such that $f(\partial) = 0$, there exists a martingale $M_t^{(N)}$ such that

$$\mu_t^{(N)}\left(P_{T-t}^N f\right) = \mu_0^{(N)}\left(P_T^N f\right) + M_t^{(N)} + \frac{1}{N}\sum_{i=1}^{N}\sum_{n=1}^{A_t^{i,(N)}}\left[\frac{1}{N-1}\sum_{j\neq i} P^N_{T-\tau_n^{i,(N)}} f(X^{j,(N)}_{\tau_n^{i,(N)}})\right] \tag{3.4}$$

where $\tau_n^{i,(N)}$ is the n^{th} killing time of the i^{th} particle. In a second step, we define the measure $\nu_t^{(N)}$ on F by

$$\nu_t^{(N)}(dx) = \left(\frac{N-1}{N}\right)^{A_t^{(N)}} \mu_t^{(N)}(dx),$$

where a loss of mass is introduced at each jump, in order to compensate the last right term in (3.4): we prove that $\nu_T^{(N)}(f) - \mu_0^{(N)}\left(P_T^N f\right)$ is the sum of two martingales. Then we prove that the L^2 norm of each of these martingales is bounded by $\|f\|_\infty/\sqrt{N}$, which yields us to

$$\sqrt{E\left(\left|\nu_T^{(N)}(f)-\mu_0^{(N)}(P_T^N f)\right|^2\right)}\leq\frac{2\|f\|_\infty}{\sqrt{N}}.$$

In the third step of the proof, we remark that $\nu_T^{(N)}$ and $\mu_T^{(N)}$ are proportional measures, which allows us to conclude the proof of Theorem 3.1 by renormalizing $\nu_T^{(N)}$ and $\mu_0^{(N)}\left(P_T^N.\right)$.

Step 1: Fix $N\geq 2$ and let $f:F\cup\{\partial\}\mapsto\mathbb{R}_+$ be a measurable bounded function such that $f(\partial)=0$. Let us prove (3.4). We define, for all $t\in[0,T]$ and $z\in F\cup\{\partial\}$,

$$\psi_t^N(z)=P_{T-t}^N f(z).$$

The process $\left(\psi_t^N(\mathcal{Z}_t^N)\right)_{t\in[0,T]}$ is a martingale which is equal to 0 at time τ_∂^N almost surely, as soon as $\tau_\partial^N\leq T$. Indeed, for all $s,t\geq 0$ such that $s+t\leq T$, we have by the Markov property and the fact that P^N is a semi-group:

$$E\left(\psi_{t+s}^N(\mathcal{Z}_{t+s}^N)|\left(\mathcal{Z}_u^N\right)_{u\in[0,t]}\right)=P_s^N\psi_{t+s}^N(\mathcal{Z}_t^N)=\psi_t^N(\mathcal{Z}_t^N).$$

Moreover ∂ is an absorbing state and $f(\partial)=0$, then

$$\psi^N_{\tau_\partial^N\wedge T}(\mathcal{Z}^N_{\tau_\partial^N\wedge T})=\psi^N_{\tau_\partial^N}(\partial)\mathbb{1}_{\tau_\partial\leq T}+\psi^N_{\tau_\partial^N}(\mathcal{Z}_T^N)\mathbb{1}_{\tau_\partial>T}=\psi^N_{\tau_\partial^N}(\mathcal{Z}_T^N)\mathbb{1}_{\tau_\partial>T}.$$

Fix $i\in\{1,\ldots,N\}$ and denote by $\tau_n^{i,(N)}$ the n^{th} jump time of the particle i. For all $n\geq 0$, we define the process $\left(\mathbb{M}_t^{i,n,(N)}\right)_{t\in[0,T]}$ by

$$\mathbb{M}_t^{i,n,(N)}=\mathbb{1}_{t<\tau_{n+1}^{i,(N)}}\psi^N_{t\wedge\tau_{n+1}^{i,(N)}}(X^{i,(N)}_{t\wedge\tau_{n+1}^{i,(N)}})-\psi^N_{t\wedge\tau_n^{i,(N)}}(X^{i,(N)}_{t\wedge\tau_n^{i,(N)}})\ (\text{with }\tau_0^{i,(N)}=0\ ).$$

Since $X^{i,(N)}$ evolves as $\mathcal{Z}^N$ in the time interval $[\tau_n^{i,(N)},\tau_{n+1}^{i,(N)}[$, $\mathbb{M}_t^{i,(N),n}$ is a martingale which fulfills almost surely

$$\mathbb{M}_t^{i,n,(N)}=\begin{cases}-\psi^N_{\tau_n^{i,(N)}}(X^{i,(N)}_{\tau_n^{i,(N)}}),\text{ if }n<A_t^{i,(N)},\\ \psi_t^N(X_t^{i,(N)})-\psi^N_{\tau_n^{i,(N)}}(X^{i,(N)}_{\tau_n^{i,(N)}}),\text{ if }n=A_t^{i,(N)},\\ 0,\text{ if }n>A_t^{i,(N)},\end{cases}$$

since $n<A_t^{i,(N)}$ is equivalent to $\tau_{n+1}^{i,(N)}<t$, while $n>A_t^{i,(N)}$ is equivalent to $\tau_n^{i,(N)}>t$. Summing over all jumps, we get

$$\psi_t^N(X_t^{i,(N)})=\psi_0(X_0^{i,(N)})+\sum_{n=0}^{A_t^{i,(N)}}\mathbb{M}_t^{i,n,(N)}+\sum_{n=1}^{A_t^{i,(N)}}\psi^N_{\tau_n^{i,(N)}}(X^{i,(N)}_{\tau_n^{i,(N)}}).\tag{3.5}$$

Defining

$$\mathbb{M}_t^{i,(N)} = \sum_{n=0}^{A_t^{i,(N)}} \mathbb{M}_t^{i,n,(N)} \text{ and } \mathbb{M}_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{M}_t^{i,(N)}$$

and summing over $i \in \{1,...,N\}$, we get

$$\mu_t^{(N)}(\psi_t^N) = \mu_0^{(N)}(\psi_0^N) + \mathbb{M}_t^{(N)} + \frac{1}{N}\sum_{i=1}^{N}\sum_{n=1}^{A_t^{i,(N)}} \psi^N_{\tau_n^{i,(N)}}(X^{i,(N)}_{\tau_n^{i,(N)}}).$$

At each jump time $\tau_n^{i,(N)}$, the position of the particle $X^{i,(N)}$ after the jump is chosen with respect to the empirical measure of the other particles. The expectation of $\psi^N_{\tau_n^{i,(N)}}(X^{i,(N)}_{\tau_n^{i,(N)}})$ conditionally to the position of the other particles at the jump time is then the average value $\frac{1}{N-1}\sum_{j\neq i}\psi^N_{\tau_n^{i,(N)}\text{-}}(X^{j,(N)}_{\tau_n^{i,(N)}\text{-}})$. We deduce that

$$\mathcal{M}_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N}\sum_{n=1}^{A_t^{i,(N)}}\left(\psi^N_{\tau_n^{i,(N)}}(X^{i,(N)}_{\tau_n^{i,(N)}}) - \frac{1}{N-1}\sum_{j\neq i}\psi^N_{\tau_n^{i,(N)}\text{-}}(X^{j,(N)}_{\tau_n^{i,(N)}\text{-}})\right).$$

is a local martingale. We finally get

$$\mu_t^{(N)}(\psi_t^N) = \mu_0^{(N)}(\psi_0^N) + \mathbb{M}_t^{(N)} + \mathcal{M}_t^{(N)} + \frac{1}{N}\sum_{i=1}^{N}\sum_{n=1}^{A_t^{i,(N)}}\left[\frac{1}{N-1}\sum_{j\neq i}\psi^N_{\tau_n^{i,(N)}\text{-}}(X^{j,(N)}_{\tau_n^{i,(N)}\text{-}})\right], \tag{3.6}$$

which is exactly (3.4).

Step 2: Let us now explain why $\nu^{(N)}_{T\wedge\tau_\alpha^{(N)}}(\psi^N_{T\wedge\tau_\alpha^{(N)}}) - \nu_0^{(N)}(\psi_0^N)$ is the sum of two martingales. Since N is fixed and in order to clarify the calculus, we remove the superscripts N and (N) when there is no risk of confusion. Denoting by $\mathbb{M}^C$ the continuous part of $\mathbb{M} = \mathbb{M}^{(N)}$, we deduce from (3.6) that

$$\nu_T(\psi_T) - \nu_0(\psi_0) = \int_0^T \left(\frac{N-1}{N}\right)^{A_{t\text{-}}} d\mathbb{M}_t^C + \sum_{n=1}^{A_T} \nu_{\tau_n}(\psi_{\tau_n}) - \nu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}).$$

Let us compute each term in the right side sum. For all $n \geq 1$,

$$\begin{aligned}\nu_{\tau_n}(\psi_{\tau_n}) - \nu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) = &\left(\frac{N-1}{N}\right)^{A_{\tau_n}}(\mu_{\tau_n}(\psi_{\tau_n}) - \mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}))\\ &+ \mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}})\left(\left(\frac{N-1}{N}\right)^{A_{\tau_n}} - \left(\frac{N-1}{N}\right)^{A_{\tau_n\text{-}}}\right).\end{aligned}$$

On the one hand, we have

$$\left(\frac{N-1}{N}\right)^{A_{\tau_n}} - \left(\frac{N-1}{N}\right)^{A_{\tau_n\text{-}}} = -\frac{1}{N-1}\left(\frac{N-1}{N}\right)^{A_{\tau_n}}.$$

On the other hand, denoting by i the index of the killed particle at time τ_n, we have

$$\mu_{\tau_n}(\psi_{\tau_n}) - \mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) \quad = \quad \frac{1}{N(N-1)}\sum_{j\neq i}\psi_{\tau_n^i\text{-}}(X^j_{\tau_n^i\text{-}}) + \mathbb{M}_{\tau_n} - \mathbb{M}_{\tau_n\text{-}} + \mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}},$$

where

$$\frac{1}{N(N-1)}\sum_{j\neq i}\psi_{\tau_n^i\text{-}}(X^j_{\tau_n^i\text{-}}) = \frac{1}{N-1}\mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) - \frac{1}{N(N-1)}\psi_{\tau_n\text{-}}(X^i_{\tau_n\text{-}})$$

and, by the definition of $\mathbb{M} = \mathbb{M}^{(N)}$,

$$-\frac{1}{N(N-1)}\psi_{\tau_n\text{-}}(X^i_{\tau_n\text{-}}) = \frac{1}{N-1}\left(\mathbb{M}_{\tau_n} - \mathbb{M}_{\tau_n\text{-}}\right).$$

We then have

$$\mu_{\tau_n}(\psi_{\tau_n}) - \mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) \quad = \quad \frac{1}{N-1}\mu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) + \frac{N}{N-1}\left(\mathbb{M}_{\tau_n} - \mathbb{M}_{\tau_n\text{-}}\right) + \mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}},$$

Finally, we get

$$\nu_{\tau_n}(\psi_{\tau_n}) - \nu_{\tau_n\text{-}}(\psi_{\tau_n\text{-}}) \quad = \quad \left(\frac{N-1}{N}\right)^{A_{\tau_n\text{-}}}\left(\mathbb{M}_{\tau_n} - \mathbb{M}_{\tau_n\text{-}}\right) + \left(\frac{N-1}{N}\right)^{A_{\tau_n}}\left(\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}}\right).$$

The process $\nu_t(\psi_t) - \nu_0(\psi_0)$ is then the sum of two local martingales and we have

$$\nu_T(\psi_T) - \nu_0(\psi_0) = \int_0^T \left(\frac{N-1}{N}\right)^{A_{t\text{-}}} d\mathbb{M}_t + \frac{N-1}{N}\int_0^T \left(\frac{N-1}{N}\right)^{A_{t\text{-}}} d\mathcal{M}_t \quad (3.7)$$

Let us bound both terms on the right-hand side (where N is still fixed). We do not have any control on the moments of the number of jumps, while we would like to deal with real martingales instead of local ones. In order to do this, we fix an integer $\alpha \geq 1$ and we stop the interacting particle system when the number of jumps A_t reaches α, which is equivalent to stop the process at time $\tau_\alpha = \tau_\alpha^{(N)}$. By the optional stopping time theorem, the processes $\mathbb{M}$ and $\mathcal{M}$ stopped at time $\tau_\alpha^{(N)}$ are true martingales, almost surely bounded by $\alpha\|f\|_\infty$.

On the one hand, the martingale jumps $\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}}$ are bounded by $\|f\|_\infty/N$, while the martingale is constant between the jumps, then

$$\begin{aligned} E\left(\left|\frac{N-1}{N}\int_0^{T\wedge\tau_\alpha}\left(\frac{N-1}{N}\right)^{A_{t\text{-}}} d\mathcal{M}_t\right|^2\right) \quad &= \quad E\left[\sum_{n=1}^{A_T\wedge\alpha}\left(\frac{N-1}{N}\right)^{2A_{\tau_n\text{-}}}\left(\mathcal{M}_{\tau_n} - \mathcal{M}_{\tau_n\text{-}}\right)^2\right] \\ &\leq \quad \frac{\|f\|_\infty^2}{N}. \end{aligned} \quad (3.8)$$

On the other hand, we have

$$\begin{aligned} E\left(\left(\int_0^{T\wedge\tau_\alpha}\left(\frac{N-1}{N}\right)^{A_{t-}} d\mathbb{M}_t\right)^2\right) &\leq E\left(\left(\mathbb{M}_{T\wedge\tau_\alpha}\right)^2\right) \\ &= \frac{1}{N^2}\sum_{i,j=1}^N E\left(\mathbb{M}^i_{T\wedge\tau_\alpha}\mathbb{M}^j_{T\wedge\tau_\alpha}\right) \end{aligned}$$

where

$$E\left(\mathbb{M}^i_{T\wedge\tau_\alpha}\mathbb{M}^j_{T\wedge\tau_\alpha}\right) = \sum_{m=0,n=0}^{\alpha} E\left(\mathbb{M}^{i,m}_{T\wedge\tau_\alpha}\mathbb{M}^{j,n}_{T\wedge\tau_\alpha}\right).$$

If $i \neq j$, then the expectation of the product of the martingales $\mathbb{M}^{i,n}$ and $\mathbb{M}^{j,m}$ is 0, since the particles are independent between the jumps and do not jump simultaneously. Assume $i = j$ and fix $m < n$. By definition, we have

$$M^{i,m}_{T\wedge\tau_\alpha} = M^{i,m}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}},$$

which is measurable with respect to $\mathbb{X}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}}$, then

$$\begin{aligned} E\left(M^{i,m}_{T\wedge\tau_\alpha}M^{i,n}_{T\wedge\tau_\alpha}|\mathbb{X}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}}\right) &= M^{i,m}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}} E\left(M^{i,n}_{T\wedge\tau_\alpha}|\mathbb{X}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}}\right) \\ &= M^{i,m}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}} M^{i,n}_{T\wedge\tau_\alpha\wedge\tau^i_{m+1}} = 0, \end{aligned}$$

using the optional sampling theorem with the martingale $M^{i,n}_{T\wedge\tau_\alpha}$ and the uniformly bounded stopping time $T\wedge\tau_\alpha\wedge\tau^i_n$. We deduce that

$$\begin{aligned} E\left(\left(M^i_{T\wedge\tau_\alpha}\right)^2\right) &= E\left(\sum_{n=0}^{\alpha}\left(M^{i,n}_{T\wedge\tau_\alpha}\right)^2\right) \\ &\leq E\left(\sum_{n=0}^{\alpha}\psi_{T\wedge\tau^i_n}(X^i_{T\wedge\tau^i_n})^2\right) \\ &\leq \|f\|_\infty E\left(\sum_{n=0}^{\alpha}\psi^N_{T\wedge\tau^i_n}(X^i_{T\wedge\tau^i_n})\right). \end{aligned}$$

By (3.5), we have

$$E\left(\sum_{n=0}^{\alpha}\psi_{T\wedge\tau^i_n}(X^i_{T\wedge\tau^i_n})\right) \leq \|f\|_\infty,$$

and we deduce that

$$E\left(\left(M^i_{T\wedge\tau_\alpha}\right)^2\right) \leq \|f\|^2_\infty.$$

Finally, we have

$$E\left(\left(\int_0^{T\wedge\tau_\alpha}\left(\frac{N-1}{N}\right)^{A_{t-}} d\mathbb{M}_t\right)^2\right) \leq \frac{\|f\|^2_\infty}{N}. \tag{3.9}$$

The formula (3.7) and inequalities (3.8) and (3.9) lead us to

$$\sqrt{E\left(\left|\nu^{(N)}_{T\wedge\tau^{(N)}_\alpha}(P^N_{T-T\wedge\tau^{(N)}_\alpha}f) - \mu^{(N)}_0(P^N_{T\wedge\tau^{(N)}_\alpha}f)\right|^2\right)} \leq \frac{2\|f\|_\infty}{\sqrt{N}}.$$

The number of jumps of the interacting particle system remains bounded up to time T by Hypothesis 3.2, so that $T\wedge\tau^{(N)}_\alpha$ is equal to T for α big enough almost surely. As a consequence, making α go to infinity in the inequality above, we get by the dominated convergence theorem

$$\sqrt{E\left(\left|\nu_T^{(N)}(f)-\mu_0^{(N)}(P_T^N f)\right|^2\right)}\leq\frac{\sqrt{2}\|f\|_\infty}{\sqrt{N}}. \tag{3.10}$$

Step 3: Let us now conclude the proof of Theorem 3.1. By Hypothesis 3.1, $\mu_0^{(N)}(P_T^N.)$ converges in distribution to $\mu_0(P_T^N.)$. It yields that, for each continuous and bounded function $f:F\to\mathbb{R}_+$, the sequence of random variables $\left(\mu_0^{(N)}(P_T^N\mathbf{1}_F),\mu_0^{(N)}(P_T^N f)\right)$ converges in distribution to the random variable $(\mu_0(P_T\mathbf{1}_F),\mu_0(P_T f))$. By (3.10), we deduce that the sequence of random variables $\left(\nu_T^{(N)}(\mathbf{1}_F),\nu_T^{(N)}(f)\right)$ converges in distribution to the random variable $(\mu_0(P_T\mathbf{1}_F),\mu_0(P_T f))$. Finally, using that $\mu_0(P_T\mathbf{1}_F)$ never vanishes almost surely, we get

$$\mu_T^{(N)}(f)=\frac{\nu_T^{(N)}(f)}{\nu_T^{(N)}(\mathbf{1}_F)}\xrightarrow[N\to\infty]{law}\frac{\mu_0(P_T f)}{\mu_0(P_T\mathbf{1}_F)},$$

for any continuous and bounded function $f:F\to\mathbb{R}_+$, which implies the first part of Theorem 3.1.

We can also deduce from (3.10) that

$$\sqrt{E\left(\left|\left(\frac{N-1}{N}\right)^{A_T^{(N)}}-\mu_0^{(N)}\left(P_T^N\mathbb{1}_F\right)\right|^2\right)}\leq\frac{2}{\sqrt{N}},$$

then

$$\sqrt{E\left(\left|\mu_0^{(N)}(P_T^N\mathbb{1}_F)\mu_T^{(N)}(f)-\mu_0^{(N)}(P_T^N f)\right|^2\right)}\leq\frac{4\|f\|_\infty}{\sqrt{N}}.$$

Using the Cauchy Schwartz inequality, we deduce that

$$E\left(\left|\mu_T^{(N)}(f)-\frac{\mu_0^{(N)}(P_T^N f)}{\mu_0^{(N)}P_T^N\mathbb{1}_F}\right|\right)\leq\sqrt{E\left(\frac{1}{\left(\mu_0^{(N)}\left(P_T^N\mathbb{1}_F\right)\right)^2}\right)}\frac{4\|f\|_\infty}{\sqrt{N}},$$

which concludes the proof of Theorem 3.1 . □

3.3 Criterion for the non-explosion of the number of jumps

Fix $N\geq 2$. The aim of this section is to give a criterion for the non-explosion assumption of Hypothesis 3.2 (Section 3.2) when the process $\mathcal{Z}^N$ is driven by a stochastic differential equation in a random time/dependent environment, with a uniformly bounded smooth killing rate and a hard killing set given by the boundary of an open set. While this problem is the main motivation for proving our non-explosion result, Theorem 3.2 below is stated in a far more general setting. Firstly, we do not require that the particles follow the same dynamic between the killings: the i^{th} particle will be driven by the dynamic of a strong Markov process $\mathcal{Z}^{i,N}$, *a priori* different for each $i \in \{1,...,N\}$. Secondly, the jump position of the killed particle is chosen with respect to a general jump measure, not necessarily supported by the positions of the $N-1$ remaining particles.

For all $i \in \{1,...,N\}$, we assume that the process $\mathcal{Z}^{i,N}$ is a strong Markov process equal to a 3-tuple $(t,e^i_t,Z^i_t)_{t\in[0,\tau_\partial[}$ up to its killing time, where t is the time, e^i_t is the environment and Z^i_t is the actual position of the diffusion. The environment e^i_t evolves in an open set $E_i \subset \mathbb{R}^{d_i}$ ($d_i \geq 1$), the position Z^i_t evolves in an open set $D_i \subset \mathbb{R}^{d'_i}$ ($d'_i \geq 1$), and we assume that there exist four measurable functions

$$
\begin{aligned}
s_i &: [0,T] \times E_i \times D_i \mapsto \mathbb{R}^{d_i} \times \mathbb{R}^{d_i}\\
m_i &: [0,T] \times E_i \times D_i \mapsto \mathbb{R}^{d_i}\\
\sigma_i &: [0,T] \times E_i \times D_i \mapsto \mathbb{R}^{d'_i} \times \mathbb{R}^{d'_i}\\
\mu_i &: [0,T] \times E_i \times D_i \mapsto \mathbb{R}^{d'_i},
\end{aligned}
$$

such that $\mathcal{Z}^{i,N} = (.,e^i,Z^i)$ fulfills the stochastic differential equation

$$
\begin{aligned}
de^i_t &= s_i(t,e^i_t,Z^i_t)d\beta^i_t + m_i(t,e^i_t,Z^i_t)dt,\ e^i_0 \in E_i,\\
dZ^i_t &= \sigma^i(t,e^i_t,Z^i_t)dB^i_t + \mu^i(t,e^i_t,Z^i_t)dt,\ Z^i_0 \in D_i,
\end{aligned}
$$

where (β^i,B^i) is a standard $d_i + d'_i$ Brownian motion. We also assume that the process $\mathcal{Z}^{i,N}$ is hardly killed when Z^i_t hits ∂D_i and smoothly killed with a rate of killing $\kappa_i(t,e^i_t,Z^i_t) \geq 0$, where

$$
\kappa_i : [0,+\infty[\times E_i \times D_i \mapsto \mathbb{R}_+
$$

is a measurable function. We recall that the distribution of the smooth killing time produced by the rate of killing κ_i is given by

$$
P\left(\tau_\partial^{smooth} > t\right) = E\left(e^{-\int_0^t \kappa_i(\mathcal{Z}^{i,N}_s)ds}\right).
$$

Each particle in the system is a 3-tuple $(t,o^i_t,X^i_t) \in [0,+\infty[\times E_i \times D_i$ and we denote the whole system by $(t,\mathbb{O}_t,\mathbb{X}_t)$, where

$$
\begin{aligned}
\mathbb{O}_t &= (o^1_t,...,o^N_t) \in E \stackrel{def}{=} E_1 \times ... \times E_N \text{ and}\\
\mathbb{X}_t &= (X^1_t,...,X^N_t) \in D \stackrel{def}{=} D_1 \times ... \times D_N,
\end{aligned}
$$

denote respectively the vector of environments and the vector of positions. Let $S : [0,+\infty[\times E^N \times D^N \to \mathcal{M}_1(E^N \times D^N)$ and $\mathcal{H} : [0,+\infty[\times E^N \times \partial(D^N) \to$

$\mathcal{M}_1(E^N \times D^N)$ be two given measurable jump measures, which will be used to choose the jump location after the smooth killing and hard killing respectively. We define the dynamics of the system $(t,\mathbb{O}_t,\mathbb{X}_t)$ starting from $(0,\mathbb{O}_0,\mathbb{X}_0)$ as follows:

- For all $i \in \{1,...,N\}$, the 3-tuple (t,o^i_t,X^i_t) starts from $(0,o^i_0,X^i_0)$ and evolves as $\mathcal{Z}^{i,N} = (.,e^i,Z^i)$ independently of the rest of the system until one of the particles is killed. This first killing time is denoted by τ_1.

- At time τ_1, the process jumps to a new position, whose choice depends on the kind of killing (the time component isn't changed):

  - if it is a smooth killing event, then the process jumps to a position chosen with respect to the jump measure $\mathcal{S}(\tau,\mathbb{O}_{\tau\text{-}},\mathbb{X}_{\tau\text{-}})$,
  - if it is a hard killing event and there exists one and only one element $i_1 \in \{1,...,N\}$ such that $X^{i_1}_{\tau_1\text{-}}$ belongs to ∂D_{i_1}, then the position of $(\mathbb{O},\mathbb{X})$ at time τ_1 is chosen with respect to the probability measure $\mathcal{H}(\tau,\mathbb{O}_{\tau\text{-}},\mathbb{X}_{\tau\text{-}})$.
  - if it is a hard killing event and there exist more than one element which hits its corresponding boundary ∂D_i, we stop the process and this time is denoted by τ_{stop} (in fact, we will prove that this kind of event doesn't happen almost surely under our hypotheses).

- At time τ_1 and after proceeding to the jump, the process lies in $\{\tau_1\}\times E\times D$. Then each 3-tuple (t,o^i,X^i) evolves as $(.,e^i,Z^i)$ starting from $(\tau_1,o^i_{\tau_1},X^i_{\tau_1})$, independently of the rest of the system and until one of them is killed. This second killing time is denoted by τ_2.

- At this time, the process jumps with the same mechanism as above (and could be stopped at a time denoted by τ_{stop}, as above).

- Then each 3-tuple (t,o^i,X^i) evolves as $(.,e^i,Z^i)$ starting from $(\tau_2,o^i_{\tau_2},X^i_{\tau_2})$, independently of the rest of the system, and so on.

We set $\tau_{stop} = +\infty$ if (X^i,X^j) never reaches $\partial D_i \times \partial D_j$, for all $i \neq j$. On the event $\{\tau_{stop} = +\infty\}$, we denote by $\tau_1 < \tau_2 < ... < \tau_n < ...$ the sequence of jump times and we set

$$\tau_\infty = \lim_{n\to\infty} \tau_n. \tag{3.11}$$

The number of jumps of the system explodes in finite time if and only if $\tau_\infty < +\infty$. We prove in Theorem 3.2 below, that this doesn't happen almost surely under the two following conditions: Hypothesis 3 and Hypothesis 4.

In the following hypothesis, the function ϕ_i is the Euclidean distance from the boundary ∂D_i, which means that

$$\phi_i(x) = \min_{z\in\partial D_i} |x-z|,\ \forall x \in D_i,$$

where $|.|$ denotes the Euclidean distance. For all $a > 0$, D_i^a will denote the boundary's neighborhood

$$D_i^a = \{x \in D_i,\, \phi_i(x) < a\}.$$

Hypothesis 3.3. *We assume that, for all $i \in \{1,...,N\}$ and all $T \geq 0$, there exists $a > 0$ such that*

1. *ϕ_i is of class C_b^2 on D_i^a,*
2. *the smooth killing rate κ_i is uniformly bounded on $[0,T] \times E_i \times D_i$*
3. *s_i,σ_i,m_i and μ_i are uniformly bounded on $[0,T] \times E_i \times D_i^a$,*
4. *there exist two measurable functions $f_i : [0,T] \times E_i \times D_i^a \mapsto \mathbb{R}_+$ and $g_i : [0,T] \times E_i \times D_i^a \mapsto \mathbb{R}$ such that $\forall (t,\epsilon,z) \in [0,T] \times E \times D_i^a$,*

$$\sum_{k,l} \frac{\partial \phi_i}{\partial x_k}(z)\frac{\partial \phi_i}{\partial x_l}(z)[\sigma_i\sigma_i^*]_{kl}(t,\epsilon,z) = f_i(t,\epsilon,z) + g_i(t,\epsilon,z), \tag{3.12}$$

   and such that

   a) f_i is of class C^1 in time and of class C^2 in environment/space, and the derivatives of f_i are uniformly bounded,

   b) there exists a positive constant $k_g > 0$ such that, for all $(t,\epsilon,z) \in [0,T] \times E_i \times D_i^a$,

$$|g_i(t,\epsilon,z)| \leq k_g \phi_i(z),$$

   c) there exists two positive constants $0 < c_\pi < C_\pi$ such that, for all $(t,\epsilon,z) \in [0,T] \times E_i \times D_i^a$,

$$c_\pi < f_i(t,\epsilon,z) + g_i(t,\epsilon,z) < C_\pi.$$

The last point of Hypothesis 3.3 says that the term (3.12), which naturally appears in the quadratic variation of $\phi_i(Z_t^i)$, is well approximated by a smooth positive function f_i near the boundary ∂D_i. However, we do not require any strict regularity assumption on σ_i, since g_i is only required to be measurable.

Remark 3.4.

1. We recall that the C^k regularity of ϕ_i near the boundary is equivalent to the C^k regularity of the boundary ∂D_i itself, for all $k \geq 2$ (see [25, Chapter 5, Section 4]).
2. In particular, if each D_i is bounded and has a boundary of class C^3, and if σ_i is of class C^2, then the first point and the last point of Hypothesis 3.3 are fulfilled. Indeed, the regularity of D_i implies that ϕ_i is of class C^3 on a neighborhood of ∂D_i, and the regularity of σ_i implies that (3.12) happens, with $g_i = 0$.

We introduce now a condition on the jump measure $\mathcal{H}$, which will ensure that $\tau_\infty < +\infty$ implies that at least two particles converge to the boundary when the time goes to τ_∞. we denote by $\mathcal{D}_i$ the set

$$\mathcal{D}_i = D_1 \times ... \times D_{i-1} \times \partial D_i \times D_{i+1} \times ... \times D_N.$$

Since we decide to stop the process when more than two particles hit simultaneously their corresponding boundaries, it is sufficient to define the jump measure $\mathcal{H}$ on $\cup_{i=1}^N \mathcal{D}_i$.

Hypothesis 3.4. *1. There exists a non-decreasing continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ vanishing only at 0 such that, $\forall i \in \{1,...,N\}$,*

$$\inf_{(t,e,(x_1,...,x_N))\in[0,+\infty[\times E\times\mathcal{D}_i} \mathcal{H}(t,e,x_1,...,x_N)(E \times A_i) \geq p_0,$$

where $p_0 > 0$ is a positive constant and $A_i \subset D$ is the set defined by

$$A_i = \{(y_1,...,y_N) \in D \,|\, \exists j \neq i \text{ such that } \phi_i(y_i) \geq h(\phi_j(y_j))\}.$$

2. We have

$$\inf_{(t,e,(x_1,...,x_N))\in[0,+\infty[\times E\times\mathcal{D}_i} \mathcal{H}(t,e,x_1,...,x_N)(E \times B_{x_1,...,x_n}) = 1,$$

where

$$B_{x_1,...,x_n} = \{(y_1,...,y_N) \in D \,|\, \forall i,\ \phi_i(y_i) \geq \phi_i(x_i)\}$$

Informally, $h(\phi_j)$ is a kind of distance from the boundary and we assume in the first point that, if all the not-killed particles are far from their respective boundaries at time τ_n, then the jump position $X^i_{\tau_n}$ is chosen far from ∂D_i with probability $p_0 > 0$. The second point ensures that each particle lies farther from its boundary after than before a hard killing jump.

Remark 3.5. 1. The model of interacting particles system introduced above is very general, even if e^i is required to be continuous up to the killing time. Indeed, it also includes the case of a diffusion evolving in an environment given by a continuous time Markov Chain. By instance, if one set s_i and m_i equal to 0, κ_i equal to 1 and $\mathcal{S} = \frac{1}{2}\left(\delta_{(t,\epsilon_i+1,z_i)} + \delta_{(t,\epsilon_i-1,z_i)}\right)$, then the particle X^i will evolve as a diffusion with an environment o^i defined as a simple continuous time random walk.

2. Hypothesis 3.4 is very general and allows a lot of choices for $\mathcal{H}$. For instance:

   a) For all $\mu \in \mathcal{M}_1(E \times D)$, one can find a compact set $K \subset E \times D$ such that $\mu(K) > 0$. Then $\mathcal{H} = \mu$ fulfills the assumption with $p_0 = \mu(K)$ and $h(\phi_j) = \phi_j \wedge d(K, E \times \partial D)$.

b) Hypothesis 3.4 also includes the case studied by Grigorescu and Kang in [42], where

$$\mathcal{H} = \sum_{j \neq i} p_{ij}(x_i)\delta_{x_j},\ \forall (x_1,\ldots,x_N) \in \mathcal{D}_i.$$

with $\sum_{j\neq i} p_{ij}(x_i) = 1$ and $\inf_{i\in\{1,\ldots,N\},j\neq i,x_i\in\partial D} p_{ij}(x_i) > 0$. In that case, the particle on the boundary jumps to the position of another one, with positive weights. It yields that Hypothesis 3.4 is fulfilled with $p_0 = 1$ and $h(\phi_j) = \phi_j$. This is also the case for the Fleming-Viot type system used in the approximation method proved in Section 3.2.

We're now able to state the main result of this section:

Theorem 3.2. *Assume that Hypotheses 3.3 and 3.4 are satisfied. Then $\tau_\infty = +\infty$ almost surely.*

Remark 3.6. Another model of diffusions killed at the boundary of an open set can be defined as follows: the particle is reflected on the boundary until its local time on this boundary reaches an independent exponentially distributed random variable, then it is killed. We emphasize that the statement of Theorem 3.2 is still valid if the particles are driven by such diffusions with reflecting/killing boundaries. Indeed, the only difference with our proof is that the reflection on the boundary makes appear an additional increasing local time in the decomposition of the semi-martingale Y^i (see (3.14) in the proof).

The long-time behavior of diffusions with reflecting/killing boundaries conditioned to not be killed has been studied in [54] by Kolb and Steinsaltz and in [79] by Evans and Steinsaltz. The approximation method proved in Section 3.2 can be used to compute the distribution of diffusions with reflecting/killing boundaries conditioned to not be killed.

Proof of Theorem 3.2. Since κ_i is uniformly bounded for all $i \in \{1,\ldots,N\}$ in finite time almost surely, there is no accumulation of soft killing events almost surely. As a consequence, we only have to prove the non-accumulation of hard killing events and we assume until the end of the proof that $\kappa_i = 0$ for all $i \in \{1,\ldots,N\}$.

The proof is organised as follows. For each particle X^i, we compute the Itô's decomposition of the semi-martingale $\phi_i(X^i)$ when X^i is in D_i^a. Then we prove that $\tau_{stop} \wedge \tau_\infty < +\infty$ implies that at least two particles X_t^i and X_t^j converge to their respective boundaries when $t \to \tau_{stop} \wedge \tau_\infty$. Denoting by

$$T_0^{ij} = \inf\{t \geq 0,\ \phi_i(X_{t\text{-}}^i) = \phi_j(X_{t\text{-}}^j) = 0\},$$

we deduce that

$$P\left(\tau_{stop} \wedge \tau_\infty < +\infty\right) \leq \sum_{1\leq i<j\leq N} P\left(T_0^{ij} < +\infty\right)$$

This allows us to reduce the problem of non-explosion of the number of jumps to a problem of non-attainability of (0,0) for a pair of semi-martingales. $(\phi_i(X^i),\phi_j(X^j))$ fulfills a criterion which implies its non-attainability of (0,0) in finite time almost surely, concluding the proof of Theorem 3.2. The above-mentioned criterion of non-attainability is proved in the last section of the present paper (Proposition 3.3).

By definition, if $\tau_{stop} < +\infty$, then at least two particles X^{i_0} and X^{j_0} hit their respective boundaries at time τ_{stop}. It yields that $\phi_i(X^i_{\tau_{stop}\text{-}}) = \phi_j(X^j_{\tau_{stop}\text{-}}) = 0$. Now, we define the event

$$\mathcal{E} = \{\tau_\infty < T \text{ and } \tau_{stop} = +\infty\}.$$

Conditionally to $\mathcal{E}$, the total number of jumps of the system goes to $+\infty$ up to time τ_∞. Since there is only a finite number of particles, at least one of them, say i_0, jumps infinitely many times up to time τ_∞. For each jumping time τ_n, we denote by $\sigma_n^{i_0}$ the next jump time of i_0, with $\tau_n < \sigma_n^{i_0} < \tau_\infty$. Conditionally to the event $\mathcal{E}$, we get $\sigma_n^{i_0} - \tau_n \to 0$ when $n \to \infty$. Let $\gamma :]0,a[\mapsto \mathbb{R}_+$ be a C^2 function with compact support in $]0,a[$. The Itô's formula applied to the semi-martingale $\gamma(\phi_i(X^{i_0}))$ and Hypothesis 3.3 immediately imply that $\gamma(\phi_i(X^{i_0}))$ is a continuous diffusion process with bounded coefficients between τ_n and $\sigma_n^{i_0}$-. Moreover, $\phi_i(X_t^{i_0})$ goes to 0 when t goes to $\sigma_n^{i_0}$, then $\gamma(\phi_i(X^{i_0}_{\sigma_n^{i_0}\text{-}})) = 0$. We deduce that

$$\sup_{t\in[\tau_n,\sigma_n^{i_0}[} \gamma(\phi_i(X_t^{i_0})) = \sup_{t\in[\tau_n,\sigma_n^{i_0}[} \gamma(\phi_i(X_t^{i_0})) - \gamma(\phi_i(X^{i_0}_{\sigma_n^{i_0}\text{-}})) \xrightarrow[n\to\infty]{} 0,\ a.s.$$

Since the process $\phi_{i_0}(X^{i_0})$ is continuous between τ_n and $\sigma_n^{i_0}-$, we conclude that $\phi_{i_0}(X^{i_0}_{\tau_n})$ doesn't lie above the support of γ, for n big enough, almost surely. But the support of γ can be chosen arbitrarily close to 0, it yields that $\phi_{i_0}(X^{i_0}_{\tau_n})$ goes to 0 almost surely conditionally to $\mathcal{E}$. Let us denote by $(\tau_n^{i_0})_n$ the sequence of jumping times of the particle i_0. We denote by $\mathcal{A}_n$ the event

$$\mathcal{A}_n = \left\{\exists j \neq i_0 \,|\, \phi_{i_0}(X^{i_0}_{\tau_n^{i_0}}) \geq h(\phi_j(X^j_{\tau_n^{i_0}}))\right\},$$

where h is the function of Hypothesis 3.4. We have, for all $1 \leq k \leq l$,

$$\begin{aligned} P\left(\bigcap_{n=k}^{l+1} \mathcal{A}_n^c\right) &= E\left(E\left(\prod_{n=k}^{l+1} \mathbb{1}_{\mathcal{A}_n^c} \,|\, (X_t^1,\ldots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right)\right) \\ &= E\left(\prod_{n=k}^{l} \mathbb{1}_{\mathcal{A}_n^c} E\left(\mathbb{1}_{\mathcal{A}_{l+1}^c} \,|\, (X_t^1,\ldots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right)\right). \end{aligned}$$

By definition of the jump mechanism of the interacting particle system and by the first point of Hypothesis 3.4,

$$\begin{aligned} E\left(\mathbb{1}_{\mathcal{A}_{l+1}^c} \,|\, (X_t^1,\ldots X_t^N)_{0\leq t<\tau_{l+1}^{i_0}}\right) &= \mathcal{H}(t,\mathbb{O}_{\tau_{l+1}^{i_0}},\mathbb{X}_{\tau_{l+1}^{i_0}})\left(A_{i_0}^c\right) \\ &\leq 1 - p_0, \end{aligned}$$

where A_{i_0} and p_0 are defined in Hypothesis 3.4. By induction on l, we get

$$P\left(\bigcap_{n=k}^{l} \mathcal{A}_n^c\right) \leq (1-p_0)^{l-k},\ \forall 1 \leq k \leq l.$$

Since $p_0 > 0$, it yields that

$$P\left(\bigcup_{k\geq 1}\bigcap_{n=k}^{\infty} \mathcal{A}_n^c\right) = 0.$$

It means that, for infinitely many jumps τ_n almost surely, one can find a particle j such that $\phi_{i_0}(X^{i_0}_{\tau_n}) \geq h(\phi_j(X^j_{\tau_n}))$. Because there is only a finite number of other particles, one can find a particle, say j_0 (which is a random variable), such that

$$\phi_{i_0}(X^{i_0}_{\tau_n}) \geq h(\phi_{j_0}(X^{j_0}_{\tau_n})), \text{ for infinitely many } n \geq 1.$$

In particular, $\lim_{n\to\infty} \phi_{j_0}(X^{j_0}_{\tau_n}) = 0$ almost surely. We deduce that

$$\lim_{n\to\infty} (\phi_{i_0}(X^{i_0}_{\tau_n}),\phi_{j_0}(X^{j_0}_{\tau_n})) = (0,0).$$

This immediately imply that

$$\left(\phi_{i_0}(X^{i_0}_{\tau_\infty\text{-}}),\phi_{j_0}(X^{j_0}_{\tau_\infty\text{-}})\right) = (0,0).$$

We finally conclude that

$$P\left(\tau_{stop}\wedge\tau_\infty < +\infty\right) \leq \sum_{1\leq i<j\leq N} P\left(T_0^{ij} < +\infty\right). \tag{3.13}$$

Fix $i \neq j \in \{1,...,N\}$ and let us prove that $P\left(T_0^{ij} < +\infty\right) = 0$. We begin to divide the time into a sequence of intervals $[t_n,t_{n+1}[$ such that, for each interval, or the pair $(\phi_i(X^i),\phi_j(X^j))$ is far from $(0,0)$, or the distance functions ϕ_i and ϕ_j are of class C^2 (which will allow us to use the Itô's formula). Let $(t_n)_{n\geq 0}$ be the sequence of stopping times defined by

$$t_0 = \inf\{t \in [0,\tau_{stop}\wedge\tau_\infty[,\ \phi^i(X^i_t) + \phi^j(X^j_t) \leq a/2\}$$

and, for all $n \geq 0$,

$$\begin{aligned} t_{2n+1} &= \inf\{t \in [t_{2n},\tau_{stop}\wedge\tau_\infty[,\ \phi_i(X^i_t) + \phi_j(X^j_t) \geq a\} \\ t_{2n+2} &= \inf\{t \in [t_{2n+1},\tau_{stop}\wedge\tau_\infty],\ \phi_i(X^i_t) + \phi_j(X^j_t) \leq a/2\}. \end{aligned}$$

By construction, we have for all $n \geq 0$,

$$\begin{cases} \phi_i(X^i_t) < a \text{ and } \phi_j(X^j_t) < a,\ \forall t \in [t_{2n},t_{2n+1}[, \\ \phi_i(X^i_t) \geq a/2 \text{ or } \phi_j(X^j_t) \geq a/2 \text{ otherwise.} \end{cases}$$

We emphasize that $T_0^{ij} \notin [t_{2n+1},t_{2n+2}[$ almost surely, while, for all $t \in [t_{2n},t_{2n+1}[$, ϕ_i and ϕ_j are of class C^2 at X_t^i and X_t^j, which will allow us to use the Itô's formula during these intervals of time. In particular, Hypothesis 3.3 and the Itô's formula immediately implies that $\phi_i(X^i)+\phi_j(X^j)$ is an Itô diffusion process with bounded coefficients between times t_{2n} and t_{2n+1} for all $n \geq 0$. Since $\phi_i(X^i)+\phi_j(X^j)$ goes from $a/2$ to a between times t_{2n} and t_{2n+1}, we deduce that $(t_n)_{n\geq 0}$ converges to $+\infty$ almost surely. We deduce that

$$P\left(T_0^{ij} < +\infty\right) \leq \sum_{n=0}^{+\infty} P\left(T_0^{ij} \in [t_{2n},t_{2n+1}[\right).$$

It remains us to prove that $P\left(T_0^{ij} \in [t_{2n},t_{2n+1}[\right) = 0$ for all $n \geq 0$.

Fix $n \geq 0$. We define the positive semi-martingale Y^i by

$$Y_t^i = \begin{cases} \phi_i(X^i_{t_{2n}+t}) \text{ if } t < t_{2n+1} - t_{2n}, \\ a/2 + |W_t^i| \text{ if } t \geq t_{2n+1} - t_{2n}, \end{cases} \tag{3.14}$$

where W^i is a standard one dimensional Brownian motion, which allows us to define Y_t^i for all time $t \in [0,+\infty[$. We define similarly the semi-martingale Y^j. It is clear that

$$P(T_0^{ij} \in [t_{2n},t_{2n+1}[) \leq P(\exists t \geq 0,\, (Y^i_{t\text{-}},Y^j_{t\text{-}}) = (0,0)).$$

The problem of non-explosion of our interacting process is then reduced to the problem of the attainability of (0,0) by a given semi-martingale. In order to prove the non-attainability of (0,0) by (Y^i,Y^j), we need to compute the Itô's decomposition of Y^i and Y^j.

Let us set

$$\pi_t^i = \begin{cases} f_i(t,o_t^i,X_t^i), & \text{if } t < t_{2n+1} - t_{2n}, \\ 1, & \text{if } t \geq t_{2n+1} - t_{2n} \end{cases} \text{ and } \rho_t^i = \begin{cases} g(t,o_t^i,X_t^i), & \text{if } t < t_{2n+1} - t_{2n}, \\ 0, & \text{if } t \geq t_{2n+1} - t_{2n}, \end{cases}$$

where f_i and g_i are given by Hypothesis 3.3. By the Itô's formulas applied to Y^i, we have

$$dY_t^i = dM_t^i + b_t^i dt + dK_t^i + Y_t^i - Y^i_{t\text{-}},$$

where M^i is a local martingale such that

$$d\langle M^i\rangle_t = (\pi_t^i + \rho_t^i)dt,$$

b^i is the adapted process given by

$$b_t^i = \begin{cases} \sum_{k=1}^{d'i} \frac{\partial \phi_i}{\partial x_k}(X_t^i)[\mu_i]_k(t,o_t^i,X_t^i) + \frac{1}{2}\sum_{k,l=1}^{d_i} \frac{\partial^2 \phi}{\partial x_k \partial x_l}(X_t^i)[\sigma_i\sigma_i^*]_{kl}(t,o_t^i,X_t^i), & \text{if } t < t_{2n+1} - t_{2n}, \\ 0 & \text{if } t \geq t_{2n+1} - t_{2n}, \end{cases}$$

and K^i is a non-decreasing process given by the local time of $|W_t|$ at 0 after time $t_{2n+1} - t_{2n}$. By the 4^{th} point of Hypothesis 3.3, we have, for all $t \geq 0$,

$$c_\pi \wedge 1 \leq \pi_t^i + \rho_t^i \leq C_\pi \vee 1, \text{ and } |\rho_t^i| \leq k_0 Y_t^i \tag{3.15}$$

The regularity of ϕ_i in D_a (1^{st} point of Hypothesis 3.3) and the boundedness of μ_i,σ_i (3^{rd} point of Hypothesis 3.3), implies that there exists $b_\infty > 0$ such that, for all $t \geq 0$,

$$b_t^i \geq -b_\infty. \tag{3.16}$$

Similarly, we get the decomposition of Y^j, with π^j, ρ^j and b^j fulfilling inequalities (3.15) and (3.16) (without loss of generality, we keep the same constants c_π, C_π, k_0 and b_∞).

The previous decomposition isn't *a priori* sufficient to prove the non-attainability of (0,0) by (Y^i,Y^j): we also need to compute the decomposition of π^i and π^j. We deduce from the Itô's formula that there exists a local martingale N^i and a finite variational process L^i such that, for all $t \geq 0$,

$$d\pi_t^i = dN_t^i + dL_t^i + \pi_t^i - \pi_{t\text{-}}^i.$$

We emphasize that we do not need the explicit computation of L^i. Let us set, for all $t < t_{2n+1} - t_{2n}$,

$$\begin{aligned}\xi_t^i = &\sum_{k=1,l}^{d_i} \frac{\partial f_i}{\partial e_k}(t,o_t^i,X_t^i)\frac{\partial f_i}{\partial e_l}(t,o_t^i,X_t^i)[s_i s_i^*]_{kl}(t,o_t^i,X_t^i) \\ &+ \sum_{k=1,l}^{d_i'} \frac{\partial f_i}{\partial x_k}(t,o_t^i,X_t^i)\frac{\partial f_i}{\partial x_l}(t,o_t^i,X_t^i)[\sigma_i \sigma_i^*]_{kl}(t,o_t^i,X_t^i)\end{aligned}$$

and, for all $t \geq t_{2n+1} - t_{2n}$, $\xi_t^i = 0$. Then we have

$$\langle N^i \rangle_t = \xi_t^i dt.$$

Thanks to the regularity assumptions on f_i and the boundedness of s_i,σ_i, there exists $C_\xi > 0$ such that

$$\xi_t^i \leq C_\xi. \tag{3.17}$$

Of course, the same holds for π^j.

Since the particles are independent between the jumps, we have for all $i \neq j$,

$$\langle M^i, M^j \rangle = 0 \text{ and } \langle N^i, N^j \rangle = 0 \text{ a.s.} \tag{3.18}$$

We claim that the decompositions of Y^i, Y^j, π^i, π^j, together with the inequalities (3.15), (3.16), (3.17) and equation (3.18), imply that (Y^1,Y^2) never converges to (0,0) almost surely. This is proved in the next section, where a criterion for non-attainability of (0,0) for semi-martingales is given (Hypothesis 3.5

and Proposition 3.3 of Section 3.4). In particular, we deduce that $T_0^{ij} \notin [t_{2n},t_{2n+1}[$ almost surely, for all $n \geq 0$.

We then have $T_0^{ij} = +\infty$ almost surely, for all $i \neq j \in \{1,\dots,N\}$, which imply, by (3.13), that $\tau_{stop} \wedge \tau_{infty} = +\infty$. This concludes the proof of Theorem 3.2.

□

3.4 Non-attainability of $(0,0)$ for semi-martingales

Fix $T > 0$ and let $(Y_t^i)_{t\in[0,T]}$, $i = 1,2$, be two non-negative one-dimensional semi-martingales such that,

$$dY_t^i = dM_t^i + b_t^i dt + dK_t^i + I_t^i - I_{t\text{-}}^i,\ Y_0^i > 0,$$

where $(M_t^i)_{t\in[0,T]}$ is a continuous local martingale , $(b_t^i)_{t\in[0,T]}$ is an adapted process, $(K_t^i)_{t\in[0,T]}$ is a continuous and non-decreasing adapted process, and I_t^i is a pure-jump càdlàg process. The aim of this section is to give some conditions, which ensure that (Y^1,Y^2) doesn't hit $(0,0)$ up to time T. The problem has been solved for time homogeneous stochastic differential equations by Friedman [32], Ramasubramanian [72] and the proof of Proposition 3.3 below is inspired by the recent work of Delarue [24], which obtains lower and higher bound for the hitting time of a corner for a diffusion driven by a time homogeneous SDE reflected in the square. In our case, time-dependency is allowed and we don't require any Markovian property. This generalization finds an important application in the previous section, where the non-explosion of a very general interacting particle system with jumps from a boundary is proved.

Hypothesis 3.5. *For each $i = 1,2$, there exists a non-negative local semi-martingale π^i such that*

$$d\pi_t^i = dN_t^i + dL_t^i + J_t^i - J_{t\text{-}}^i,$$

where N^i is a continuous local martingale and L^i is a continuous finite variational adapted process and J_t^i is a pure-jump càdlàg process. Moreover, there exist two adapted processes ρ_t^i and ξ_t^i, and some positive constants $b_\infty,k_0,c_\pi,C_\pi, C_\xi$ such that, almost surely,

1. *$d\left\langle M^i\right\rangle_t = (\pi_{t\text{-}}^i + \rho_{t\text{-}}^i)dt$ and $d\left\langle N^i\right\rangle_t = \xi_{t\text{-}}^i dt$,*
2. *$c_\pi \leq \pi_t^i + \rho_t^i \leq C_\pi$, $|\rho_t^i| \leq k_0 Y_t^i$, $\xi_t \leq C_\xi$ and $b_t^i \geq -b_\infty$ for all $t \in [0,T]$*
3. *$\left\langle M^1,M^2\right\rangle$ and $\left\langle N^1,N^2\right\rangle$ are non-increasing processes.*
4. *I^i and J^i are such that, for all jump time t of the processes I and J,*

$$\frac{Y_t^i}{\sqrt{\pi_t^i}} - \frac{Y_{t\text{-}}^i}{\sqrt{\pi_{t\text{-}}^i}} \geq 0.$$

The third point of Hypothesis 3.5 has the following geometrical interpretation: when an increment of M^1 is non-positive (that is when M^1 goes closer to 0), the increment of M^2 is non-negative (so that M^2 goes farther from 0), as a consequence (M^1,M^2) remains away from 0. A nice graphic representation of this phenomenon is given by Delarue's [24, Figure 1].

Remark 3.7. An example of a pair of semi-martingales which fulfills Hypothesis 3.5 is given in the proof of Theorem 3.2 in Section 3.3, where (Y^1,Y^2) is given by a smooth function of a pair of diffusion processes. In this typical case, checking the validity of our assumption is a simple application of the Itô's formula.

The process

$$\Phi_t \stackrel{def}{=} -\frac{1}{2}\log\left(\frac{(Y_t^1)^2}{\pi_t^1}+\frac{(Y_t^2)^2}{\pi_t^2}\right) \tag{3.19}$$

goes to infinity when (Y_t^1,Y_t^2) goes to (0,0), since π_t^i is uniformly bounded below by c_π. For all $\epsilon>0$, we define the stopping time $T_\epsilon=\inf\{t\in[0,T],\ \Phi_t\geq\epsilon^{-1}\}$. We denote the hitting time of (0,0) by $T_0=\inf\{t\in[0,T],\ (Y_{t\text{-}}^1,Y_{t\text{-}}^2)=(0,0)$ or $(Y_t^1,Y_t^2)=(0,0)\}$. In particular, we have

$$T_0=\lim_{\epsilon\to 0}T_\epsilon,\text{ almost surely.}$$

We are now able to state our non-attainability result.

Proposition 3.3. *Assume that Hypothesis 3.5 is fulfilled. Then (Y^1,Y^2) doesn't go to (0,0) in $[0,T]$ almost surely, which means that T_0 is equal to $+\infty$ almost surely.*

Moreover, there exists a positive constant C which only depends on $b_\infty,k_0,c_\pi,C_\pi,C_\xi$ such that, for all $\epsilon^{-1}>\Phi_0$,

$$P\left(T_\epsilon\leq T\right)\leq\frac{1}{\epsilon^{-1}-\Phi_0}C\left(E(|L^1|_T+|L^2|_T)+T\right),$$

where $|L^i|_T$ is the total variation of L^i at time T and Φ_0 is defined in (3.19).

Proof of Proposition 3.3: Let $(\theta'_n)_{n\in\mathbb{N}}$ and $(\theta''_n)_{n\in\mathbb{N}}$ be two increasing sequences of stopping times which converge to T such that $(M_t^i)_{t\in[0,\theta'_n]}$ and $(N_t^i)_{t\in[0,\theta'_n]}$ are true martingales and such that $\theta''_n=\inf\{t\in[0,T],\ \int_0^{\theta''_n}d|L^i|_t\geq n\}\wedge T$. The whole proof is based on an application of the Itô's formula to the semi-martingale

$$\left(\int_0^{\Phi_t}\exp(e^{C_F}e^{-u})du\right)_{t\in[0,T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}]},\ n',n''\in\mathbb{N},$$

where $C_F>0$ is a constant which only depends on the parameters $b_\infty,k_0,c_\pi,C_\pi,C_\xi$. We prove that, for a good choice of C_F, there exists a constant C which doesn't depend on ϵ, n' and n'' such that

$$E\left(\int_{\Phi_0}^{\Phi_{T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}}}\exp(e^{C_F}e^{-u})du\right)\leq C(E(|L^1|_{\theta''_{n''}}+|L^2|_{\theta''_{n''}})+T). \tag{3.20}$$

Assume that this inequality has been proved. We notice that $\Phi_{t\wedge T_\epsilon \wedge \theta'_{n'}\wedge\theta''_{n''}}$ reaches ϵ^{-1} if and only if $T_\epsilon \leq \theta'_{n'}\wedge\theta''_{n''}$, then, by the right continuity of Y^1, Y^2, π^1 and π^2,

$$\begin{aligned} P\left(T_\epsilon \leq \theta'_{n'}\wedge\theta''_{n''}\right) &= P\left(\Phi_{T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}} - \Phi_0 \geq \epsilon^{-1} - \Phi_0\right) \\ &\leq P\left(\int_{\Phi_0}^{\Phi_{T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}}} \exp(e^{C_F}e^{-u})du \geq \epsilon^{-1}-\Phi_0\right), \end{aligned}$$

since $r-q \leq \int_q^r \exp(e^{C_F}e^{-u})du$ for all $0\leq q\leq r$. Finally, using the Markov inequality and (3.20), we get, for all $\epsilon^{-1} > \Phi_0$,

$$P\left(T_\epsilon \leq \theta'_{n'}\wedge\theta''_{n''}\right) \leq \frac{1}{\epsilon^{-1}-\Phi_0} C\left(E(|L^1|_{\theta''_{n''}} + |L^2|_{\theta''_{n''}}) + T\right).$$

Letting n' go to ∞, then ϵ go to 0 and finally n'' go to ∞, we deduce that $P(T_0 \leq T) = 0$, which is the first point of Proposition 3.3. Since $\theta'_{n'}$ and $\theta''_{n''}$ converge to T almost surely, letting n' and n'' go to ∞ implies the second part of Proposition 3.3, which concludes the proof.

It remains us to prove inequality (3.20). We assume in a first time that $\langle M^1,M^2\rangle = \langle N^1,N^2\rangle = 0$. We define the function

$$\begin{aligned} \Phi:\quad \mathbb{R}_+^*\times\mathbb{R}_+^*\times\mathbb{R}_+\times\mathbb{R}_+^* &\to \mathbb{R} \\ (\alpha_1,\alpha_2,x_1,x_2) &\mapsto -\log\left(\tfrac{x_1^2}{\alpha_1}+\tfrac{x_2^2}{\alpha_2}\right). \end{aligned}$$

We have $\Phi_t = \Phi(\pi_t^1,\pi_t^2,Y_t^1,Y_t^2)$. We will apply the Itô's formula to the semi-martingale $(\Phi_t)_{t\in[0,T_\epsilon\wedge\theta'_{n'}\wedge\theta''_{n''}[}$. The successive derivatives of the function Φ are

$$\begin{aligned} \frac{\partial\Phi}{\partial x_i} &= -\alpha_i^{-1}x_ie^{2\Phi},\ \frac{\partial^2\Phi}{\partial x_i^2} = -\alpha_i^{-1}e^{2\Phi}+2\alpha_i^{-2}x_i^2e^{4\Phi}, \\ \frac{\partial\Phi}{\partial\alpha_i} &= \frac{1}{2}\alpha_i^{-2}x_i^2e^{2\Phi},\ \frac{\partial^2\Phi}{\partial\alpha_i^2} = -\alpha_i^{-3}x_i^2e^{2\Phi}+\alpha_i^{-4}x_i^4e^{4\Phi}, \\ \frac{\partial^2\Phi}{\partial x_i\alpha_i} &= \alpha_i^{-2}x_ie^{2\Phi}-\alpha_i^{-3}x_i^3e^{4\Phi},\ \frac{\partial^2\Phi}{\partial x_i\alpha_j} = -\alpha_i^{-1}\alpha_j^{-2}x_ix_j^2e^{4\Phi} \text{ with } i\neq j. \end{aligned}$$

In particular, one can check that

$$\sum_{i=1,2}\frac{\partial^2\Phi}{\partial x_i^2}(\pi_{t\text{-}}^1,\pi_{t\text{-}}^2,Y_{t\text{-}}^1,Y_{t\text{-}}^2)\pi_{t\text{-}}^i = 0, \text{ almost surely.}$$

Using the previous equalities and the Itô's formula, we get

$$\begin{aligned}
d\Phi_t = &-\sum_{i=1,2} \frac{Y^i_t}{\pi^i_t} e^{2\Phi_t} dM^i_t + \sum_{i=1,2} \frac{(Y^i_t)^2}{2(\pi^i_t)^2} e^{2\Phi_t} dN^i_t - \sum_{i=1,2} \frac{Y^i_t}{\pi^i_t} e^{2\Phi_t} dK^i_t \\
&- \sum_{i=1,2} \frac{Y^i_t}{\pi^i_t} e^{2\Phi_t} b^i_t dt + \sum_{i=1,2} \frac{(Y^i_t)^2}{2(\pi^i_t)^2} e^{2\Phi_t} dL^i_t \\
&+ \frac{1}{2} \sum_{i=1,2} \left(-\frac{1}{\pi^i_t} e^{2\Phi_t} + 2\frac{(Y^i_t)^2}{(\pi^i_t)^2} e^{4\Phi_t} \right) \rho^i_t dt \\
&+ \frac{1}{2} \sum_{i=1,2} \left(-\frac{(Y^i_t)^2}{(\pi^i_t)^3} e^{2\Phi_t} + \frac{(Y^i_t)^4}{(\pi^i_t)^4} e^{4\Phi_t} \right) d\left\langle N^i \right\rangle_t \\
&+ \frac{1}{2} \sum_{i=1,2} \left(\frac{Y^i_t}{(\pi^i_t)^2} e^{2\Phi_t} - \frac{(Y^i_t)^3}{(\pi^i_t)^3} e^{4\Phi_t} \right) d\left\langle M^i, N^i \right\rangle_t \\
&- \frac{1}{2} \sum_{i \neq j \in \{1,2\}} \frac{Y^i_t (Y^j_t)^2}{\pi^i_t (\pi^j_t)^2} e^{4\Phi_t} d\left\langle M^i, N^j \right\rangle_t + \Phi_t - \Phi_{t\text{-}}
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
d\left\langle \Phi \right\rangle_t = &\sum_{i=1,2} \frac{(Y^i_t)^2}{(\pi^i_t)^2} e^{4\Phi_t} (\rho^i_t + \pi^i_t) dt + \sum_{i=1,2} \frac{(Y^i_t)^4}{4(\pi^i_t)^4} e^{4\Phi_t} d\left\langle N^i \right\rangle_t \\
&- \sum_{i=1,2} \frac{(Y^i_t)^3}{2(\pi^i_t)^3} e^{4\Phi_t} d\left\langle M^i, N^i \right\rangle_t - \sum_{i \neq j \in \{1,2\}} \frac{Y^i_t (Y^j_t)^2}{2\pi^i_t (\pi^j_t)^2} e^{4\Phi_t} d\left\langle M^i, N^j \right\rangle_t .
\end{aligned}$$

Let $C_F > 0$ be a positive constant that will be fixed later in the proof and define the function $F : \mathbb{R} \mapsto \mathbb{R}$ by

$$F(r) = \int_0^r \exp\left(C_F e^{-s} \right) ds.$$

We check that

$$r \leq F(r),\ 1 \leq F'(r) \leq e^{C_F} \text{ and } F''(r) = -C_F e^{-r} F'(r),\ \forall r \in \mathbb{R}_+.$$

We deduce from Itô's formula that

$$F(\Phi_t) - F(\Phi_0) = \int_0^t F'(\Phi_s) d\Phi^c_s - \frac{C_F}{2} \int_0^t e^{-\Phi_s} F'(\Phi_s) d\left\langle \Phi \right\rangle_s + \sum_{0 \leq s \leq t} F(\Phi_s) - F(\Phi_{s\text{-}}), \tag{3.22}$$

where $d\Phi^c_s$ is the continuous part of $d\Phi_s$.

Using equation (3.21), we begin to prove a higher bound for $\int_0^t F'(\Phi_s) d\Phi^c_s$. We define the local martingale

$$M_t = -\sum_{i=1,2} \int_0^t \frac{Y^i_s}{\pi^i_s} e^{2\Phi_s} F'(\Phi_s) dM^i_s + \sum_{i=1,2} \int_0^t \frac{(Y^i_s)^2}{2(\pi^i_s)^2} e^{2\Phi_s} F'(\Phi_s) dN^i_s.$$

Since K^i is non-decreasing, we have

$$-\sum_{i=1,2}\int_0^t \frac{Y_s^i}{\pi_s^i} e^{2\Phi_s} F'(\Phi_s) dK_s^i \leq 0.$$

One can easily check that, for all $t \in [0,T_0[$, $Y_t^i e^{\Phi_t} \leq \sqrt{\pi_t^i}$, then

$$\frac{Y_t^i}{\pi_t^i} e^{\Phi_t} \leq \frac{1}{\sqrt{c_\pi}}.$$

Since $b_t^i \geq -b_\infty$ for all $t \in [0,T_0[$, we have

$$-\sum_{i=1,2}\int_0^t \frac{Y_s^i}{\pi_s^i} F'(\Phi_s) e^{2\Phi_s} b_s^i ds \leq \frac{2b_\infty}{\sqrt{c_\pi}} \int_0^t e^{\Phi_s} F'(\Phi_s) ds.$$

The inequality $F'(\Phi_s) \leq e^{C_F}$ yields to

$$\sum_{i=1,2}\int_0^t \frac{(Y_s^i)^2}{2(\pi_s^i)^2} e^{2\Phi_s} F'(\Phi_s) dL_s^i \leq \frac{e^{C_F}}{2c_\pi}\left(|L^1|_t + |L^2|_t\right).$$

We deduce from

$$|\rho_t^i| e^{\Phi_t} \leq k_0 Y_t^i e^{\Phi_t} \leq k_0 \sqrt{\pi_t^i} \leq k_0 \sqrt{C_\pi}$$

that

$$\frac{1}{2}\sum_{i=1,2}\int_0^t \left(-\frac{1}{\pi_s^i} e^{2\Phi_s} + 2\frac{(Y_s^i)^2}{(\pi_s^i)^2} e^{4\Phi_s}\right) \rho_s^i F'(\Phi_s) ds \leq \frac{3e^{C_F} k_0 \sqrt{C_\pi}}{c_\pi} \int_0^t e^{\Phi_s} F'(\Phi_s) ds.$$

Since $d\langle N^i\rangle_t = \xi_t^i dt$, with $0 \leq \xi_t^i \leq C_\xi$, we have

$$\frac{1}{2}\sum_{i=1,2}\int_0^t \left(-\frac{(Y_s^i)^2}{(\pi_s^i)^3} e^{2\Phi_s} + \frac{(Y_s^i)^4}{(\pi_s^i)^4} e^{4\Phi_s}\right) F'(\Phi_s) d\left\langle N^i\right\rangle_s \leq \frac{e^{C_F} C_\xi t}{c_\pi^2}.$$

By the Kunita-Watanabe inequality (see [74, Corollary 1.16 of Chapter IV]), we get, for all predictable process h_s,

$$\left|\int_0^t h_s \left\langle M^i, N^j\right\rangle_s\right| \leq \sqrt{\int_0^t h_s \left\langle M^i\right\rangle_s} \sqrt{\int_0^t h_s \left\langle N^j\right\rangle_s} \leq \sqrt{C_\pi C_\xi} \int_0^t h_s ds,$$

so that

$$\frac{1}{2}\sum_{i=1,2}\int_0^t \left(\frac{Y_s^i}{(\pi_s^i)^2} e^{2\Phi_s} - \frac{(Y_s^i)^3}{(\pi_s^i)^3} e^{4\Phi_s}\right) F'(\Phi_s) d\left\langle M^i, N^i\right\rangle_s \leq \frac{2\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}} \int_0^t e^{\Phi_s} F'(\Phi_s) ds$$

and

$$-\frac{1}{2}\sum_{i\neq j\in\{1,2\}}\int_0^t \frac{Y_s^i (Y_s^j)^2}{\pi_s^i (\pi_s^j)^2} e^{4\Phi_s} F'(\Phi_s) d\left\langle M^i, N^j\right\rangle_s \leq \frac{\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}} \int_0^t e^{\Phi_s} F'(\Phi_s) ds.$$

We finally get

$$\int_0^t F'(\Phi_s)d\Phi_s^c \leq M_t + C'\int_0^t e^{\Phi_s}F'(\Phi_s)ds + \frac{e^{C_F}}{2c_\pi}\left(|L^1|_t + |L^2|_t\right) + \frac{e^{C_F}C_\xi t}{c_\pi^2}. \quad (3.23)$$

where

$$C' = \frac{2b_\infty}{\sqrt{c_\pi}} + \frac{3k_0\sqrt{C_\pi}}{c_\pi} + \frac{3\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}} > 0.$$

We prove now a lower bound for $\int_0^t e^{-\Phi_s}F'(\Phi_s)d\langle\Phi\rangle_s$. We have

$$\frac{e^{2\Phi_s}}{C_\pi} \leq \sum_{i=1,2}\frac{(Y_s^i)^2}{(\pi_s^i)^2}e^{4\Phi_s} \leq \frac{2e^{2\Phi_s}}{c_\pi},\ \pi_s^i \geq c_\pi \text{ and } \rho_s^i \geq -k_0\sqrt{C_\pi}e^{-\Phi_s}$$

then

$$\int_0^t \sum_{i=1,2}\frac{(Y_s^i)^2}{(\pi_s^i)^2}e^{4\Phi_s}(\pi_s^i+\rho_s^i)e^{-\Phi_s}F'(\Phi_s)ds \geq \frac{c_\pi}{C_\pi}\int_0^t e^{\Phi_s}F'(\Phi_s)ds - \frac{2k_0\sqrt{C_\pi}}{c_\pi}e^{C_F}t.$$

The process $\langle N^i\rangle$ being non-decreasing, we have

$$\sum_{i=1,2}\int_0^t \frac{(Y_s^i)^4}{4(\pi_s^i)^4}e^{4\Phi_s}F'(\Phi_s)e^{-\Phi_s}d\left\langle N^i\right\rangle_s \geq 0.$$

The same argument as above leads us to

$$-\sum_{i=1,2}\int_0^t \frac{(Y_s^i)^3}{2(\pi_s^i)^3}e^{4\Phi_s}F'(\Phi_s)e^{-\Phi_s}d\left\langle M^i,N^i\right\rangle_s \geq -\frac{\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}e^{C_F}t$$

and

$$-\sum_{i\neq j\in\{1,2\}}\int_0^t \frac{Y_s^i(Y_s^j)^2}{2\pi_s^i(\pi_s^j)^2}e^{4\Phi_s}F'(\Phi_s)e^{-\Phi_s}d\left\langle M^i,N^j\right\rangle_s \geq -\frac{\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}e^{C_F}t.$$

We finally deduce that

$$\int_0^t e^{-\Phi_s}F'(\Phi_s)d\langle\Phi\rangle_s \geq \frac{c_\pi}{C_\pi}\int_0^t e^{\Phi_s}F'(\Phi_s)ds - \left(\frac{2k_0\sqrt{C_\pi}}{c_\pi} + \frac{2\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}\right)e^{C_F}t \quad (3.24)$$

Since the jumps of Φ_t are negative and F is non-decreasing, we get

$$\sum_{0\leq s\leq t} F(\Phi_s) - F(\Phi_{s\text{-}}) \leq 0. \quad (3.25)$$

By (3.23), (3.24) and (3.25), we deduce from (3.22) that

$$F(\Phi_t) - F(\Phi_0) \leq M_t + \left(C' - \frac{C_F c_\pi}{2C_\pi}\right)\int_0^t e^{\Phi_s}F'(\Phi_s)ds + \frac{e^{C_F}}{2c_\pi}\left(|L^1|_t + |L^2|_t\right)$$
$$+ \left(\frac{C_\xi}{c_\pi^2} - \frac{2k_0C_F\sqrt{C_\pi}}{2c_\pi} - \frac{C_F\sqrt{C_\pi C_\xi}}{c_\pi^{3/2}}\right)e^{C_F}t.$$

Choosing $C_F = 2C_\pi C'/c_\pi$, we've proved that there exists $C > 0$ such that

$$F(\Phi_t) - F(\Phi_0) \leq M_t + C\left(|L^1|_t + |L^2|_t + t\right).$$

This yields to (3.20), since the process M_t stopped at $T_\epsilon \wedge \theta'_n \wedge \theta''_n$ is a true martingale. The proposition is then proved when $\langle M^1, M^2 \rangle = \langle N^1, N^2 \rangle = 0$.

Assume now that $\langle M^1, M^2 \rangle$ and $\langle N^1, N^2 \rangle$ are non-increasing. We define Φ'_t as the process starting from Φ_0 and whose increments are defined by the right term of (3.21). On the one hand, the same calculation as above leads to

$$F(\Phi'_t) \leq M_t + \frac{e^{C_F}}{c_\pi}\left(|L^1|_{\theta''_{n''}} + |L^2|_{\theta''_{n''}}\right) + \left(e^{C_F}C' + \frac{C_F}{2}C'''\right)t. \tag{3.26}$$

On the other hand,

$$d\Phi_t = d\Phi'_t + \frac{\partial^2\Phi}{\partial x_1 \partial x_2}(\pi^1_t, \pi^2_t, Y^1_t, Y^2_t)\, d\left\langle M^1, M^2\right\rangle_t + \frac{\partial^2\Phi}{\partial \alpha_1 \partial \alpha_2}(\pi^1_t, \pi^2_t, Y^1_t, Y^2_t)\, d\left\langle N^1, N^2\right\rangle_t,$$

and we can check that $\frac{\partial^2\Phi}{\partial x_1\partial x_2}$ and $\frac{\partial^2\Phi}{\partial \alpha_1\partial \alpha_2}$ are non-negative functions. We deduce from the third point of Hypothesis 3.5 that $\Phi_t \leq \Phi'_t$. But F is increasing, so that (3.26) leads us to (3.20) in the general case. □

Chapter 4

Uniform tightness for time inhomogeneous particle systems

Abstract

In the previous chapter, we proved an existence result for a general class of interacting particle systems, whose particles evolve as diffusion processes in a random environment and jump when they hit the boundary of an open set. In the present section, we consider sequences of such interacting particle systems and we prove a criterion for the uniform tightness of the family of laws of their empirical distributions at any time $t > 0$.

Let E_0 and D_0 be two bounded open subsets of $\mathbb{R}^{d_0}$ and $\mathbb{R}^{d'_0}$. For all $N \geq 2$, let $\mathcal{Z}^{1,N}$,...,$\mathcal{Z}^{N,N}$ be a family of N strong Markov processes , each of them being equal to a 3-tuple $(t,e_t^{i,N},Z_t^{i,N})_{t\in[0,\tau_\partial[}$ which evolves in $\mathbb{R} \times E_0 \times D_0$ as a time inhomogeneous environment dependent diffusion process. In the 3-tuple $(t,e_t^{i,N},Z_t^{i,N})$, the parameter t denotes the time, $e_t^{i,N}$ denotes the environmental dynamics and $Z_t^{i,N}$ denotes the actual position of the diffusion.

By a *time inhomogeneous environment dependent diffusion process*, we mean that, for any $N \geq 2$ and any $i \in \{1,...,N\}$, there exist four measurable functions

$$
\begin{aligned}
s_i^N &: [0,T] \times E_0 \times D_0 \mapsto \mathbb{R}^{d_0} \times \mathbb{R}^{d_0}\\
m_i^N &: [0,T] \times E_0 \times D_0 \mapsto \mathbb{R}^{d_0}\\
\sigma_i^N &: [0,T] \times E_0 \times D_0 \mapsto \mathbb{R}^{d'_0} \times \mathbb{R}^{d'_0}\\
\eta_i^N &: [0,T] \times E_0 \times D_0 \mapsto \mathbb{R}^{d'_0},
\end{aligned}
$$

such that $\mathcal{Z}^{i,N} = (.,e^{i,N},Z^{i,N})$ fulfills the stochastic differential system

$$\begin{aligned} de_t^{i,N} &= s_i^N(t,e_t^{i,N},Z_t^{i,N})d\beta_t^{i,N} + m_i(t,e_t^{i,N},Z_t^{i,N})dt, \quad e_0^{i,N} \in E_0,\\ dZ_t^{i,N} &= \sigma_i^N(t,e_t^{i,N},Z_t^i)dB_t^{i,N} + \eta^{i,N}(t,e_t^{i,N},Z_t^{i,N})dt, \quad Z_0^{i,N} \in D_0, \end{aligned}$$

where $(\beta^{i,N},B^{i,N})$ is a standard $d_0 + d_0'$ Brownian motion. Each process $\mathcal{Z}^{i,N}$ is hardly killed when $Z_t^{i,N}$ hits ∂D_0 and smoothly killed with a rate of killing $\kappa_i^N(t,e_t^{i,N},Z_t^{i,N}) \geq 0$, where

$$\kappa_i^N : [0,+\infty[\times E_0 \times D_0 \mapsto \mathbb{R}_+$$

is a uniformly bounded measurable function. We emphasize that, contrarily to the case of the previous chapter, each process $\mathcal{Z}^{i,N}$ evolves in the same state space $\mathbb{R} \times E_0 \times D_0$, for any i,N.

In order to build an interacting particle system with N particles as in Section 3 of Chapter 3, we assume that, for any $N \geq 2$, we're given the measurable jump measures

$$\mathcal{S}^N : [0,+\infty[\times E_0^N \times D_0^N \to \mathcal{M}_1(E_0^N \times D_0^N)$$

and

$$\mathcal{H}^N : [0,+\infty[\times E_0^N \times \partial(D_0^N) \to \mathcal{M}_1(E_0^N \times D_0^N).$$

As in Section 3 of Chapter 3, we define, for any $N \geq 2$, the interacting particle system $\left(.,\mathbb{O}^{(N)},\mathbb{X}^{(N)}\right)$ whose particles evolve as independent copies of $\mathcal{Z}^{i,N}$, $i = 1,...,N$, jump with respect to $\mathcal{S}^N$ when one of them is smoothly killed and jump with respect to $\mathcal{H}^N$ when one of them is hardly killed. For any $N \geq 2$ and any $i \in \{1,...,N\}$, we denote by $(.,o^{i,N},X^{i,N})$ the i^{th} particle of the system $\left(.,\mathbb{O}^{(N)},\mathbb{X}^{(N)}\right)$ and by μ_t^N the empirical distribution of $\mathbb{X}_t^{(N)}$, defined by

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}} \in \mathcal{M}_1(D_0),$$

where $\mathcal{M}_1(D_0)$ denotes the set of probability measures on D_0. In Theorem 4.1, we give a sufficient criterion for the family of laws of the random probability measures $(\mu_t^N)_{N\geq 2}$ to be uniformly tight.

Before turning to the statement of the theorem, let us define our main assumption. In particular, it will ensure that, for any $N \geq 2$, the number of killing/jumps of $(\cdot,\mathbb{O}^{(N)},\mathbb{X}^{(N)})$ remains finite in finite time almost surely, which is necessary for $(t,\mathbb{O}_t^{(N)},\mathbb{X}_t^{(N)})$ to be well defined at any time $t \geq 0$. In what follows, ϕ_0 denotes the Euclidean distance to the boundary ∂D_0, defined for all $x \in \mathbb{R}^{d_0'}$ by

$$\phi_0(x) = \inf_{y\in\partial D_0} \|x-y\|_2,$$

where $\|.\|_2$ denotes the Euclidean norm of $\mathbb{R}^{d_0'}$.

Hypothesis 4.1. *We assume that there exists $a > 0$ such that*

1. *ϕ_0 is of class C^2 on $D_0^a = \{x \in D_0,\ \phi_0(x) < a\}$,*
2. *for all $N \geq 2$, $\mathcal{S}^N$ and $\mathcal{H}^N$ fulfill Hypothesis 3.2 of Chapter 3.*
3. *κ_i^N is uniformly bounded on $[0,+\infty[\times E_0 \times D_0$ and s_i^N,σ_i^N,m_i^N and η_i^N are uniformly bounded on $[0,+\infty[\times E_0 \times D_0^a$. These uniform bounds are also supposed to be uniform in $N \geq 2$ and $i \in \{1,\dots,N\}$,*
4. *for all $N \geq 2$ and $i \in \{1,\dots,N\}$, there exist two measurable functions $f_i^N : [0,+\infty[\times E_0 \times D_0^a \to \mathbb{R}_+$ and $g_i^N : [0,+\infty[\times E_0 \times D_0^a \to \mathbb{R}$ such that $\forall (t,e,z) \in [0,+\infty[\times E_0 \times D_0^a$,*
$$\sum_{k,l} \frac{\partial \phi_0}{\partial x_k}(z) \frac{\partial \phi_0}{\partial x_l}(z) [\sigma_i^N \sigma_i^{N*}]_{kl}(t,e,z) = f_i^N(t,e,z) + g_i^N(t,e,z), \tag{4.1}$$
and such that
   a) *f_i^N is of class C^1 in time and of class C^2 in environment/space, and the derivatives of f_i^N are uniformly bounded in $[0,+\infty[\times E_0 \times D_0^a$, uniformly in $N \geq 2$ and $i \in \{1,\dots,N\}$,*
   b) *there exists a positive constant $k_g > 0$ such that, for all $(t,e,z) \in [0,+\infty[\times E_0 \times D_0^a$,*
$$|g_i^N(t,e,z)| \leq k_g \phi_0(z),$$
   c) *there exists two positive constants $0 < c_0 < C_0$ such that, for all $(t,e,z) \in [0,+\infty[\times E_0 \times D_0^a$,*
$$c_0 < f_i^N(t,e,z) + g_i^N(t,e,z) < C_0.$$

We emphasize that k_g, c_0 and C_0 are required to not depend on i,N.

We're now able to state our tightness result. In the next chapter, Theorem 1 will be used in the proof of a strong mixing property for time inhomogeneous diffusions conditioned to not be killed.

Theorem 4.1. *Assume that Hypothesis 4.1 is fulfilled. Then, for all $\epsilon > 0$ and all $t_0 > 0$, there exists $\alpha_{\epsilon,t_0} > 0$ and $N_\epsilon \geq 2$ such that*
$$E\left(\mu_t^N\left(D_0^{\alpha_\epsilon}\right)\right) \leq \epsilon,\ \forall t \geq t_0\ \forall N \geq N_\epsilon,$$
independently of the sequence of initial empirical distributions $(\mu_0^N)_{N\geq 2}$.

As a consequence, for any sequence of initial distributions $m^N \in \mathcal{M}_1(D^N)$ and all $t > 0$, the family of laws of the random measures $\left(\mu^N(t,dx)\right)_{N\geq 2}$ is uniformly tight.

Proof of Theorem 4.1. By Theorem 3.1 of Chapter 3 and by Hypothesis 4.1, the interacting particle process $(\cdot,\mathbb{O}_\cdot^{(N)},\mathbb{X}_\cdot^{(N)})$ is well defined at any time for all $N \geq 2$.

Fix $\epsilon > 0$ and $T > 0$. Assume in a first time that the killing rate κ_i^N is equal to 0, for all $N \geq 2$ and $i \in \{1,\dots,N\}$. For any $N \geq 2$ and any $\alpha > 0$, we have

$$\begin{aligned}
E\left(\mu_T^N(D_0^\alpha)\right)^2 &\leq E\left(\mu_T^N(D_0^\alpha)^2\right),\\
&\leq E\left(\frac{1}{N^2}\sum_{i=1}^N \left(\delta_{X_T^{i,N}}(D_0^\alpha)\right)^2 + \frac{1}{N^2}\sum_{1\leq i\neq j\leq N}\delta_{X_T^{i,N}}(D_0^\alpha)\delta_{X_T^{j,N}}(D_0^\alpha)\right)\\
&\leq \frac{1}{N}E\left(\mu_T^N(D_0^\alpha)\right) + \frac{1}{N^2}\sum_{1\leq i\neq j\leq N} E\left(\delta_{X_T^{i,N}}(D_0^\alpha)\delta_{X_T^{j,N}}(D_0^\alpha)\right)\\
&\leq \frac{1}{N}E\left(\mu_T^N(D_0^\alpha)\right) + \max_{1\leq i\neq j\leq N} P\left(\phi_0(X_T^{i,N}) \leq \alpha \text{ and } \phi_0(X_T^{j,N}) \leq \alpha\right).
\end{aligned}$$

Fix $N \geq 2$ and $i \neq j \in \{1,\dots,N\}$. For all $\gamma \in [0,\frac{a}{2}]$, we define the stopping time

$$S_\gamma^{i,N} = \inf\{t \geq 0,\ \phi_0(X_t^{i,N}) \geq \gamma\}.$$

We need the following Lemma, whose proof uses Proposition 4.1 of Chapter 3 and Itô's calculus and is postponed to the end of this section.

Lemma 4.2. *There exists a positive constant $C > 0$, independent of i,j,N and γ, such that for all $\alpha \in [0,\gamma[$,*

$$P\left(\exists t \in [S_\gamma^{i,N},T], \phi_0(X_t^{i,N}) + \phi_0(X_t^{j,N}) \leq \alpha\right) \leq \frac{CT}{\log\left(\frac{\gamma}{\alpha}\right)}.$$

We immediately deduce from Lemma 4.2 that, for all $\alpha \in [0,\gamma/2[$,

$$P\left(\phi_0(X_T^{i,N}) \leq \alpha \text{ and } \phi_0(X_T^{j,N}) \leq \alpha\right) \leq \frac{CT}{\log\left(\frac{\gamma}{2\alpha}\right)} + P(S_\gamma^{i,N} > T).$$

In particular, we deduce that

$$E\left(\mu_T^N(D_0^\alpha)\right)^2 \leq \frac{1}{N}E\left(\mu_T^N(D_0^\alpha)\right) + \frac{CT}{\log\left(\frac{\gamma}{2\alpha}\right)} + \max_{1\leq i\leq N} P\left(S_\gamma^{i,N} > T\right). \tag{4.2}$$

Now, we use the following lemma, whose proof uses a coupling argument and is postponed to the end of this section.

Lemma 4.3. *There exists a constant $\gamma_\epsilon > 0$ such that, for all $N \geq 2$ and all $i \in \{1,\dots,N\}$, we have*

$$P\left(S_{\gamma_\epsilon}^{i,N} > T\right) \leq \epsilon/3, \tag{4.3}$$

independently of the sequence of initial distributions.

By (4.2), we deduce that, for all $\alpha \in [0,\gamma/2[$,

$$E\left(\mu_T^N(D_0^\alpha)\right)^2 \leq \frac{1}{N}E\left(\mu_T^N(D_0^\alpha)\right) + \frac{CT}{\log\left(\frac{\gamma_\epsilon}{2\alpha}\right)} + \frac{\epsilon}{3}.$$

Fix an integer $N_\epsilon \geq 2 \vee \frac{3}{\epsilon}$. We thus have, for all $N \geq N_\epsilon$,

$$\frac{1}{N}E\left(\mu_T^N(D_0^\alpha)\right) \leq \frac{\epsilon}{3}.$$

Let $\alpha_\epsilon > 0$ be a positive constant such that $\log\left(\frac{\gamma_\epsilon}{2\alpha_\epsilon}\right) \leq 3(CT\epsilon)^{-1}$. We then have

$$E\left(\mu_T^N(D_0^{\alpha_\epsilon})\right)^2 \leq \epsilon,\ \forall N \geq N_\epsilon,$$

independently of the sequence of initial distributions.

Fix $t > T$. Since the previous inequality doesn't depend on the distribution of the initial position $(X_0^{1,N},...X_0^{N,N})$, it can be applied to the process initially distributed with the same distribution as $(X_{t-T}^{1,N},...X_{t-T}^{2,N})$. By the Markov property of the interacting particle system, we thus obtain

$$E\left(\mu_t^N(D_0^{\alpha_\epsilon})\right)^2 \leq \epsilon,\ \forall N \geq N_\epsilon.$$

This allows us to conclude the proof of the first part of Theorem 4.1 when $\kappa_i^N = 0$ for all $N \geq 2$ and $i \in \{1,...,N\}$.

Fix $N \geq 2$ and assume now that $(\kappa_i^N)_{i\in\{1,...,N\}}$ isn't equal to 0. Fix $i,j \in \{1,...,N\}$. For any $\gamma > 0$, we define the stopping time $S_\gamma^{i,N}$ as above. We also denote by τ_γ^{smooth} the first smooth killing time of $X^{i,N}$ or $X^{j,N}$ after $S_\gamma^{i,N}$:

$$\tau_\gamma^{smooth} = \inf\{t \geq S_\gamma^{i,N},\ X^{i,N} \text{ or } X^{j,N} \text{ is smoothly killed at time } t\}.$$

The same proof as the proof of Lemma 4.2 leads us to the following inequality

$$P\left(\exists t \in [S_\gamma,\tau_\gamma^{smooth} \wedge T[, \phi_0(X_t^{i,N}) + \phi_0(X_t^{j,N}) \leq \alpha\right) \leq \frac{C}{\log\left(\frac{\gamma}{\alpha}\right)}.$$

In particular, we deduce that

$$\begin{aligned} &P\left(\phi_0(X_T^{i,N}) \leq \alpha \text{ and } \phi_0(X_T^{j,N}) \leq \alpha\right) \\ &\qquad\qquad \leq \frac{C}{\log\left(\frac{\gamma}{\alpha}\right)} + P(S_\gamma > T \text{ and } \tau_\gamma^{smooth} \geq T) + P(\tau_\gamma^{smooth} < T). \end{aligned}$$

By Hypothesis 4.1, the killing rates κ_i^N and κ_j^N are uniformly bounded by a constant $\kappa_\infty > 0$. As a consequence, there exists $T_0 > 0$ such that, for all $T \leq T_0$ and all $\gamma > 0$,

$$P(\tau_\gamma^{smooth} < T) \leq \frac{\epsilon}{4}.$$

We emphasize that T_0 is chosen so that it only depends on the uniform bound κ_∞. Assume that $T \leq T_0$. By the same arguments as in the proof of Lemma 4.3, we can find $\gamma_\epsilon > 0$ such that

$$P(S^{i,N}_\gamma > T \text{ and } \tau^{smooth}_\gamma \geq T) \leq \frac{\epsilon}{4}.$$

Finally, choosing α_ϵ small enough, we deduce that

$$P\left(\phi_0(X^{i,N}_T) \leq \alpha_\epsilon \text{ and } \phi_0(X^{j,N}_T) \leq \alpha_\epsilon\right) \leq \frac{3\epsilon}{4}.$$

Proceeding as in the first part of the proof, we deduce that the first part of Theorem 4.1 holds for all $T \in]0,T_0[$. Thus it clearly holds for any $T > 0$.

The uniform tightness property is immediately deduced from the tightness criterion proved by Jakubowski in [48]: the family of laws of the random measures μ^N_t is uniformly tight, if, $\forall \epsilon > 0$, there exists a compact set $K_\epsilon \subset D_0$ such that $E(\mu^N_t(D_0 \setminus K_\epsilon)) \leq \epsilon$ for any $N \geq 2$. This conclude the proof of Theorem 4.1. □

Proof of Lemma 4.2. Fix $\gamma \in [0,a/2]$ and let us prove that, for all $\alpha \in [0,\gamma[$,

$$P\left(\exists t \in [S^{i,N}_\gamma,T], \phi_0(X^{i,N}_t) + \phi_0(X^{j,N}_t) \leq \alpha\right) \leq \frac{CT}{\log\left(\frac{\gamma}{\alpha}\right)}.$$

Let $(t_n)_{n\geq 0}$ be the sequence of stopping times defined by

$$t_0 = \inf\{t \in [S^{i,N}_\gamma,T],\ \phi_0(X^{i,N}_t) + \phi_0(X^{j,N}_t) \leq a/2\} \wedge T$$

and, for all $n \geq 0$,

$$\begin{aligned} t_{2n+1} &= \inf\{t \in [t_{2n},T],\ \phi_0(X^{i,N}_t) + \phi_0(X^{j,N}_t) \geq a\} \wedge T \\ t_{2n+2} &= \inf\{t \in [t_{2n+1},T],\ \phi_0(X^{i,N}_t) + \phi_0(X^{j,N}_t) \leq a/2\} \wedge T. \end{aligned}$$

It is immediate that t_n converges almost surely to T. By construction, we have for all $n \geq 0$,

$$\begin{cases} \phi_0(X^{i,N}_t) < a \text{ and } \phi_0(X^{j,N}_t) < a,\ \forall t \in [t_{2n},t_{2n+1}[, \\ \phi_0(X^{i,N}_t) \geq a/2 \text{ or } \phi_0(X^{j,N}_t) \geq a/2 \text{ otherwise.} \end{cases}$$

In particular, by the first point of Hypothesis 4.1, ϕ_0 is of class C^2 at $X^{i,N}_t$ and $X^{j,N}_t$, for all $t \in [t_{2n},t_{2n+1}[$ almost surely. This will allow us to compute the Itô's decomposition of $\phi_0(X^{i,N})$ and $\phi_0(X^{j,N})$ during this interval of time, at any time $t \in [t_{2n},t_{2n+1}[$, using Itô's formula.

For all $n \geq 0$, we have

$$P\left(\exists t \in [t_{2n+1},t_{2n+2}[,\ \phi_0(X^{i,N}_t) \leq \alpha \text{ and } \phi_0(X^{j,N}_t) \leq \alpha\right) = 0,\ \forall \alpha < a/2. \quad (4.4)$$

Fix $n \geq 0$ and let us now prove that there exists a constant $C > 0$ such that

$$P\left(\exists t \in [t_{2n},t_{2n+1}[,\ \phi_0(X_t^{i,N}) \leq \alpha \text{ and } \phi_0(X_t^{j,N}) \leq \alpha\right) \leq \frac{C}{\log\left(\frac{\gamma}{\alpha}\right)} E\left(t_{2n+1} - t_{2n}\right).$$

We define the positive semi-martingale Y^i by

$$Y_t^i = \begin{cases} \phi_0(X_{t_{2n}+t}^{i,N}) \text{ if } t < t_{2n+1} - t_{2n}, \\ a/2 + |W_t^i| \text{ if } t \geq t_{2n+1} - t_{2n}, \end{cases} \tag{4.5}$$

where W^i is a standard one dimensional Brownian motion independent of the rest of the process. The extension after time $t_{2n+1} - t_{2n}$ allows us to define Y_t^i at any time $t \in [0,+\infty[$. We define similarly the semi-martingale Y^j. In order to apply Proposition 4.1 of Chapter 3 to the pair of semi-martingales (Y^i,Y^j), we need the Itô's decompositions of Y^i and Y^j. Let us set

$$\pi_t^i = \begin{cases} f_i^N(t_{2n}+t,o_{t_{2n}+t}^{i,N},X_{t_{2n}+t}^{i,N}), & \text{if } 0 \leq t < t_{2n+1} - t_{2n}, \\ 1, & \text{if } t \geq t_{2n+1} - t_{2n} \end{cases}$$

and

$$\rho_t^i = \begin{cases} g_i^N(t_{2n}+t,o_{t_{2n}+t}^{i,N},X_{t_{2n}+t}^{i,N}), & \text{if } t < t_{2n+1} - t_{2n}, \\ 0, & \text{if } t \geq t_{2n+1} - t_{2n}, \end{cases}$$

where f_i^N and g_i^N are given by Hypothesis 4.1. By the Itô's formula applied to Y^i, we have

$$dY_t^i = dM_t^i + b_t^i dt + dK_t^i + Y_t^i - Y_{t\text{-}}^i,$$

where M^i is a local martingale such that

$$d\langle M^i\rangle_t = (\pi_t^i + \rho_t^i)dt;$$

b^i is the adapted process given, if $t < t_{2n+1} - t_{2n}$, by

$$b_t^i = \sum_{k=1}^{d'i} \frac{\partial \phi_i}{\partial x_k}(X_{t_{2n}+t}^{i,N})[\eta_i^N]_k(t_{2n}+t,o_{t_{2n}+t}^{i,N},X_{t_{2n}+t}^{i,N})$$
$$+ \frac{1}{2}\sum_{k,l=1}^{d_i} \frac{\partial^2 \phi}{\partial x_k \partial x_l}(X_{t_{2n}+t}^{i,N})[\sigma_i^N \sigma_i^{N*}]_{kl}(t_{2n}+t,o_{t_{2n}+t}^{i,N},X_{t_{2n}+t}^{i,N}),$$

and, if $t \geq t_{2n+1} - t_{2n}$, by $b_t^i = 0$; K^i is a non-decreasing process given by the local time of $|W_t|$ at 0 after time $t_{2n+1} - t_{2n}$. By the 4^{th} point of Hypothesis 4.1, we have, for all $t \geq 0$,

$$c_0 \wedge 1 \leq \pi_t^i + \rho_t^i \leq C_0 \vee 1, \text{ and } |\rho_t^i| \leq k_g Y_t^i \tag{4.6}$$

By Hypothesis 4.1, ϕ_0 is of class C^2 on D_0^a and η_i^N,σ_i^N are uniformly bounded. This implies that there exists $b_\infty > 0$ such that, for all $t \geq 0$,

$$b_t^i \geq -b_\infty. \tag{4.7}$$

Similarly, we get the decomposition of Y^j, with π^j, ρ^j and b^j fulfilling inequalities (4.6) and (4.7) (without loss of generality, we keep the same constants c_0, C_0, k_g and b_∞).

Let us now compute the Itô's decompositions of π^i and π^j. We deduce from the Itô's formula that there exist a local martingale N^i and a finite variational process L^i such that, for all $t \geq 0$,

$$d\pi_t^i = dN_t^i + dL_t^i + \pi_t^i - \pi_{t\text{-}}^i,$$

where, for all $t \in [0,t_{2n+1} - t_{2n}[$,

$$L_t^i = \int_0^t \left(\sum_{k=1}^{d_0} \frac{\partial f_i^N}{\partial e^k}(t_{2n}+s,o_{t_{2n}+s}^i,X_{t_{2n}+s}^i) + \sum_{k=1}^{d_0'} \frac{\partial f_i^N}{\partial x^k}(t_{2n}+s,o_{t_{2n}+s}^i,X_{t_{2n}+s}^i) \right) ds.$$

By Hypothesis 4.1, the derivatives of f_i^N are uniformly bounded, so that there exists a constant C_L such that

$$E\left(|L^i|_{t_{2n+1}-t_{2n}}\right) \leq C_L E\left(t_{2n+1} - t_{2n}\right). \tag{4.8}$$

Let us set, for all $t < t_{2n+1} - t_{2n}$,

$$\begin{aligned}\xi_t^i = &\sum_{k=1,l}^{d_i} \frac{\partial f_i^N}{\partial e_k}(t,o_t^{i,N},X_t^{i,N}) \frac{\partial f_i^N}{\partial e_l}(t,o_t^{i,N},X_t^{i,N}) [s_i^N s_i^{N*}]_{kl}(t,o_t^{i,N},X_t^{i,N}) \\ &+ \sum_{k=1,l}^{d_i'} \frac{\partial f_i^N}{\partial x_k}(t,o_t^{i,N},X_t^{i,N}) \frac{\partial f_i^N}{\partial x_l}(t,o_t^{i,N},X_t^{i,N}) [\sigma_i^N \sigma_i^{N*}]_{kl}(t,o_t^{i,N},X_t^{i,N})\end{aligned}$$

and, for all $t \geq t_{2n+1} - t_{2n}$, $\xi_t^i = 0$. Then we have

$$\langle N^i \rangle_t = \xi_t^i dt.$$

Thanks to the regularity assumptions on f_i^N and the boundedness of s_i^N,σ_i^N, there exists $C_\xi > 0$ such that

$$\xi_t^i \leq C_\xi. \tag{4.9}$$

The same decomposition and inequalities hold for π^j, with the same constants C_L and C_ξ. We emphasize that these constants are chosen independently of i, j and N, since the bounds that we used are by assumption uniform in i,j,N.

We define the process

$$\Phi_t \stackrel{def}{=} -\frac{1}{2} \log \left(\frac{(Y_t^i)^2}{\pi_t^1} + \frac{(Y_t^j)^2}{\pi_t^2} \right)$$

and we set, for all $\epsilon > 0$, $T_\epsilon = \inf\{t \in [0,T],\ \Phi_t \geq \epsilon^{-1}\}$. By the previous Itô's decompositions, one can apply Proposition 4.1 of Chapter 3 to the pair of semi-martingales Y^1,Y^2. Applying minor changes in the proof of Proposition 4.1, one can replace T by any stopping time θ in the statement of the proposition, which yields to

$$P\left(T_\epsilon \leq \theta\right) \leq \frac{1}{\epsilon^{-1} - \Phi_0} C\left(E(|L^i|_\theta + |L^j|_\theta) + E(\theta)\right).$$

Applying this result to $\theta = t_{2n+1} - t_{2n}$ (which is a stopping time for the filtration of the process $(X^{i,N},X^{j,N})$ after time t_{2n}) and using (4.8), we deduce that there exists a constant $C' > 0$, which only depend on the constants b_∞,k_g,c_0,C_0,C_ξ, such that

$$P\left(T_\epsilon \in [0,t_{2n+1} - t_{2n}]\right) \leq \frac{1}{\epsilon^{-1} - \Phi_0} C'(2\,C_L + 1) E\left(t_{2n+1} - t_{2n}\right). \tag{4.10}$$

We have, for all $t \in [0,t_{2n+1} - t_{2n}[$ and by definition of Φ_t,

$$\Phi_t \geq -\log\left(\frac{(\Phi_t)^2}{c_0} + \frac{(\Phi_t)^2}{c_0}\right)$$

In particular we have

$$\phi_0 \geq -\log\left(\frac{(\Phi_0)^2}{c_0} + \frac{(\Phi_0)^2}{c_0}\right).$$

If $t_{2n} > S^{i,N}_\gamma$, then by definition of t_{2n}, we have $\phi_0(X^{i,N}_{t_{2n}\text{-}}) \geq a/2$. But $\phi_0(X^{i,N})$ is discontinuous only at times $t > 0$ such that $\phi_0(X^{i,N}_{t\text{-}}) = 0$, so that

$$\Phi_0 = \phi_0(X^{i,N}_{t_{2n}}) = \phi_0(X^{i,N}_{t_{2n}\text{-}}) \geq a/2 \geq \gamma.$$

If $t_{2n} = S^{i,N}_\gamma$, then, by right continuity of $\phi_0(X^i)$ and by the definition of $S^{i,N}_\gamma$, we deduce that $\Phi_0 = \phi_0(X^{i,N}_{t_{2n}}) \geq \gamma$. Finally, we have

$$\Phi_0 \leq -\log\left(\frac{\gamma^2}{c_0}\right).$$

Thus we deduce from (4.10) that

$$P\left(T_\epsilon \in [0,t_{2n+1} - t_{2n}]\right) \leq \frac{1}{\epsilon^{-1} + \log\left(\frac{\gamma^2}{c_0}\right)} C(2\,C_L + 1) E\left(t_{2n+1} - t_{2n}\right).$$

which is by definition equivalent to

$$\begin{aligned} P\left(\exists t \in [t_{2n},t_{2n+1}], \phi_0(X^{i,N}_t)^2 + \phi_0(X^{j,N}_t)^2 \leq c_0 e^{-\frac{1}{\epsilon}}\right) \\ \leq \frac{1}{\epsilon^{-1} + \log\left(\frac{\gamma^2}{c_0}\right)} C(2\,C_L + 1) E\left(t_{2n+1} - t_{2n}\right). \end{aligned}$$

For all $\alpha > 0$ small enough, replacing ϵ^{-1} by $-\log(\alpha^2/c_0)$, we deduce that

$$P\left(\exists t \in [t_{2n},t_{2n+1}], \phi_0(X^{i,N}_t)^2 + \phi_0(X^{j,N}_t)^2 \leq \alpha^2\right) \leq \frac{C(2\,C_L+1)}{2\log\left(\frac{\gamma}{\alpha}\right)} E\left(t_{2n+1} - t_{2n}\right).$$

Summing over $n \geq 0$ and using equality (4.4), we deduce that

$$P\left(\exists t \in [S_\gamma,T], \phi_0(X_t^{i,N})^2 + \phi_0(X_t^{j,N})^2 \leq \alpha^2\right) \leq \frac{C(2C_L+1)T}{2\log\left(\frac{\gamma}{\alpha}\right)}.$$

By equivalence of the norms $(x,y) \mapsto \sqrt{x^2+y^2}$ and $(x,y) \mapsto |x|+|y|$, we deduce Lemma 4.2. □

Proof of Lemma 4.3. In order to prove Lemma 4.3, we build a coupling between $\phi_0(X^i)$ and a time changed reflected Brownian motion with drift.

Let $(\theta_n)_{n\geq 0}$ be the sequence of stopping times defined by

$$\theta_0 = \inf\{t \in [0,T],\ \phi_0(X_t^i) + \phi_0(X_t^j) \leq a/2\} \wedge T$$

and, for all $n \geq 0$,

$$\begin{aligned}
\theta_{2n+1} &= \inf\{t \in [\theta_{2n},T],\ \phi_0(X_t^i) + \phi_0(X_t^j) \geq a\} \wedge T\\
\theta_{2n+2} &= \inf\{t \in [\theta_{2n+1},T],\ \phi_0(X_t^i) + \phi_0(X_t^j) \leq a/2\} \wedge T.
\end{aligned}$$

It is immediate that (θ_n) converges almost surely to T and that

$$\begin{aligned}
&\phi_0(X_t^{i,N}) \geq \frac{a}{2},\ \forall t \in [0,\theta_0] \text{ and } \forall t \in \cup_{n=0}^{\infty}[\theta_{2n+1},\theta_{2n+2}[\\
&\phi_0(X_t^{i,N}) < a,\ \forall t \in \cup_{n=0}^{\infty}[\theta_{2n},\theta_{2n+1}[.
\end{aligned}$$

Let Γ be a 1-dimensional Brownian motion independent of the process $(.,\mathbb{O}^{(N)},\mathbb{X}^{(N)})$. We set

$$M_t = \Gamma_t, \text{ for } t \in [0,\theta_0[,$$

and, for all $n \geq 0$,

$$\begin{aligned}
M_t &= M_{\theta_{2n}} + \int_{\theta_{2n}}^t \sum_{k=1}^{d_0'} \frac{\partial\phi_0}{\partial x_k}[\sigma_i]_{kl}(t,o_t^{i,N},X_t^{i,N})d[B_s^i]_l \text{ for } t \in [\theta_{2n},\theta_{2n+1}[,\\
M_t &= M_{\theta_{2n+1}} + (\Gamma_t - \Gamma_{\theta_{2n+1}}) \text{ for } t \in [\theta_{2n+1},\theta_{2n+2}[,
\end{aligned}$$

Informally, M is a square-integrable martingale which is parallel to the martingale part of $\phi_0(X^{i,N})$ when this one is near 0, and equal to an independent Brownian motion when $\phi_0(X^{i,N})$ is sufficiently far from 0. By [74, Theorem 1.9 (Knight)], M is a time changed Brownian motion. More precisely, there exists a 1-dimensional Brownian motion W such that, for all $t \geq 0$,

$$M_t = W_{\langle M\rangle_t}.$$

By Itô's formula, we have

$$\frac{\partial\langle M\rangle_t}{\partial t} = \begin{cases} f_i^N(t,o_t^{i,N},X_t^{i,N}) + g(t,o_t^{i,N},X_t^{i,N}) \text{ if } \exists n \geq 0 \text{ such that } t \in [\theta_{2n},\theta_{2n+1}[,\\ 1 \text{ if } \exists n \geq 0 \text{ such that } t \in [\theta_{2n+1},\theta_{2n+2}[.\end{cases}$$

By Hypothesis 4.1, we deduce that

$$c_0 \wedge 1 \leq \frac{\partial}{\partial t}\langle M\rangle_t \leq C_0 \vee 1. \tag{4.11}$$

By Hypothesis 4.1, there exists a positive constant $C_1 > 0$ such that, for all $t \in [\theta_{2n},\theta_{2n+1}[$,

$$-C_1 \leq \frac{1}{2}\sum_{k,l=1}^{d_0} \frac{\partial^2 \phi_0}{\partial x_k \partial x_l}(X_t)\left[\sigma_i^N\left(\sigma_i^N\right)^*\right]_{kl}(t,o_t^{i,N},X_t^{i,N}) + \sum_{k=1}^{d_0} \frac{\partial \phi_0}{\partial x_k}(X_t^{i,N})\left[\eta_i^N\right]_k(t,o_t^{i,N},X_t^{i,N}).$$

Let U be the diffusion process reflected on 0 and a, defined by

$$dU_t = dW_t - \frac{C_1}{c_0 \wedge 1}dt + dL_t^0 - dL_t^a,\ U_0 = 0,$$

where L^0 (resp. L^a) is the local time of U on 0 (resp. a). In particular, we have

$$dU_{\langle M\rangle_t} = dM_t - \frac{C_1}{c_0 \wedge 1}\frac{\partial}{\partial t}\langle M\rangle_t dt + dL^0_{\langle M\rangle_t} - dL^a_{\langle M\rangle_t},$$

where, by the fourth point of Hypothesis 4.1 and inequality (4.11),

$$-\frac{C_1}{c_0 \wedge 1}\frac{\partial}{\partial t}\langle M\rangle_t \leq \frac{1}{2}\sum_{k,l=1}^{d_0} \frac{\partial^2 \phi_0}{\partial x_k \partial x_l}(X_t)\left[\sigma_i^N\left(\sigma_i^N\right)^*\right]_{kl}(t,o_t^{i,N},X_t^{i,N}) + \sum_{k=1}^{d_0} \frac{\partial \phi_0}{\partial x_k}(X_t^{i,N})\left[\eta_i^N\right]_k(t,o_t^{i,N},X_t^{i,N}),$$

which is the drift part of the semi-martingale $\phi_0(X_t^{i,N})$. Informally, $U_{\langle M\rangle_t}$ evolves as $\phi_0(X_t^{i,N})$ but with a stronger drift toward 0 when it is away from the boundary; $U_{\langle M\rangle_t}$ is reflected on 0 while $\phi_0(X_t^{i,N})$ makes positive jumps when it hits 0; $U_{\langle M\rangle_t}$ is reflected on a while $\phi_0(X_t^{i,N})$ can become greater than a. As a consequence (see Proposition 2.2 of Chapter 2 for a rigorous proof), we have

$$0 \leq U_{\langle M\rangle_t} \leq \phi_0(X_t^{i,N}),\ \forall t \in [0,T].$$

Then, for all $\gamma > 0$,

$$\left\{\phi_0(X_t^{i,N}) \geq \gamma\right\} \supset \left\{U_{\langle M\rangle_t} \geq \gamma\right\},$$

where $\langle M\rangle_t \in [\frac{t}{C_0 \vee 1},\frac{t}{c_0 \wedge 1}]$ by inequality (4.11). It yields that

$$\left\{\exists t \in [0,T] | \phi_0(X_t^{i,N}) \geq \gamma\right\} \supset \left\{\exists t \in [0,\frac{T}{c_0 \wedge 1}] \text{ such that } U_t \geq \gamma, \right\},$$

which implies that

$$P(S_\gamma^{i,N} \leq T) \geq P\left(\exists t \in [0, \frac{T}{c_0 \wedge 1}] \text{ such that } U_t \geq \gamma\right).$$

The process $(U_t)_{t\geq 0}$ is a reflected Brownian motion with bounded drift, whose law doesn't depend on i,N. As a consequence, there exists $\gamma_\epsilon > 0$ independent of i,N such that $P\left(\exists t \in [0, \frac{T}{c_0 \wedge 1}] \text{ such that } U_t \geq \gamma_\epsilon\right) \geq 1 - \epsilon/3$. This implies inequality (4.3) of Lemma 4.3. □

Chapter 5

Strong mixing properties for time inhomogeneous diffusion processes with killing *

Abstract

We prove a strong mixing property for elliptic time inhomogeneous diffusion processes with periodic coefficients, with smooth and hard killings. We use the approximation method based on the Fleming-Viot type interacting particle system of the Chapter 3. We also use a generalization in the inhomogeneous setting of a coupling built by Enrico Priola and Feng-Yu Wang.

We also prove that the approximation method described in Chapter 3 converges uniformly in time for some time inhomogeneous diffusion processes with hard and soft killings.

5.1 Introduction

Let D be a bounded open subset of $\mathbb{R}^d$, $d \geq 1$, such that its boundary ∂D is of class C^2. Let

$$\sigma : \begin{array}{l} [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \\ (t,x) \mapsto \sigma(t,x) \end{array} \quad \text{and } b : \begin{array}{l} [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}^d \\ (t,x) \mapsto \sigma(t,x) \end{array}$$

be two bounded measurable functions. We consider the following stochastic differential equation:

$$dZ_t = \sigma(t,Z_t)dB_t + b(t,Z_t)dt,\ Z_0 \in D, \tag{5.1}$$

*In collaboration with Pierre Del Moral.

where B is a standard d dimensional Brownian motion.

Our first hypothesis concerns the regularity of the coefficients of the stochastic differential equation.

Hypothesis 5.1. *We assume that σ and b are continuous uniformly Lipschitz in x, uniformly in t. This means that there exists a constant $C_0 > 0$ such that*

$$\|\sigma(t,x) - \sigma(t,y)\| + |b(t,x) - b(t,y)| \leq C_0|x - y|.$$

We also assume that σ and b are periodic, with period Π.

Under this hypothesis, the above stochastic differential equation has a solution since σ and b are assumed to be continuous and bounded (see [28, Theorem 3.10, Chapter 5]). Moreover, the solution is pathwise unique up to time $\tau_D = \inf\{t \geq 0, Z_t \notin D\}$ (see [28, Theorem 3.7, Chapter 5]), which denotes the time when the solution hits the boundary.

For all $s > 0$ and any probability distribution μ on D, we denote by $(\mathcal{Z}^\mu_{s,t})_{t\geq s}$ the unique solution to this stochastic differential equation starting at time $s > 0$ with distribution μ, killed when it hits the boundary and killed with a rate $\kappa(t,\mathcal{Z}^x_{s,t}) \geq 0$, where

$$\kappa : [0, +\infty[\times D \to \mathbb{R}_+$$

is a bounded measurable function. By *"the process is killed"*, we mean that the process is sent to a cemetery point $\partial \notin D$, so that the killed process is càdlàg almost surely. If there exists $x \in D$ such that $\mu = \delta_x$, we set $(\mathcal{Z}^x_{s,t})_{t\geq s} = (\mathcal{Z}^{\delta_x}_{s,t})_{t\geq s}$. When the process is killed by hitting the boundary, we say that it has been *hardly* killed; when the process is killed strictly before reaching the boundary (because of the rate of killing κ), we say that it has been *smoothly* killed. We denote the killing time by $\tau_\partial = \inf\{t \geq s, \mathcal{Z}^{\delta_x}_{s,t} = \partial\}$.

Let $(Q_{s,t})_{0\leq s\leq t}$ be the semi-group defined, for all $0 \leq s \leq t$, $x \in D$ and any bounded measurable function $f : \mathbb{R}^d \cup \{\partial\} \mapsto \mathbb{R}$ which vanishes outside D, by

$$Q_{s,t}f(x) = \mathbb{E}\left(f(\mathcal{Z}^x_{s,t})\mathbf{1}_{t<\tau_\partial}\right) = \mathbb{E}\left(f(\mathcal{Z}^x_{s,t})\right).$$

We emphasize that, for any probability measure μ on D, the law of $\mathcal{Z}^\mu_{s,t}$ is given by the probability measure $\mu Q_{s,t}$, defined by

$$\mu Q_{s,t}(f) = \int_D Q_{s,t}f(x)d\mu(x).$$

In this chapter, we're interested in the long time behavior of the distribution of $\mathcal{Z}^\mu_{s,t}$ conditioned to $t < \tau_\partial$. For any initial distribution μ, the conditional distribution is given by

$$\mathbb{P}(\mathcal{Z}^\mu_{s,t} \in \cdot \,|\, t < \tau_\partial) = \frac{\mu Q_{s,t}(\cdot)}{\mu Q_{s,t}(\mathbf{1}_D)}. \tag{5.2}$$

As explained in Chapter 1, the long time behavior of such conditioned distributions has been widely studied when $\kappa = 0$ (*i.e.* there is no soft killing) and when the coefficients of the stochastic differential equation (5.1) are time-homogeneous and of class C^1. In that particular case, the conditioned distribution (5.2), which only depends on the difference $t-s$ and on μ, converges when t goes to $+\infty$ to a limiting distribution on D, called the Yaglom limit (see for instance Pinsky [68] and Gong, Qian and Zhao [37]). A recent result from Knobloch and Partzsch [53] shows that under additional assumptions, the convergence holds exponentially fast: there exists two constants $C > 0$ and $\gamma > 0$ such that, for any probability measure μ_1 and μ_2 on D,

$$\left\| \frac{\mu_1 Q_{s,t}}{\mu_1 Q_{s,t}(\mathbf{1}_D)} - \frac{\mu_2 Q_{s,t}}{\mu_2 Q_{s,t}(\mathbf{1}_D)} \right\|_{TV} \leq C e^{-\gamma t}. \tag{5.3}$$

As an immediate consequence of Section 1.5 of Chapter 1, this property is also fulfilled by logistic Feller diffusion processes, where $D =]0, +\infty[$ without soft killing, and whose conditional distribution has been originally studied by Cattiaux, Collet, Lambert, Martínez, Méléard and San Martín [16]. If $D = \mathbb{R}^d$, so that there is no hard killing, if $\|\kappa\|_\infty < +\infty$ and if the solutions to the stochastic differential equation (5.1) without killing fulfill some mixing properties, then the conditional distribution of the process with smooth killing also fulfills the exponential mixing property (5.3) (see for instance Del Moral and Miclo [23]).

The main result of this paper is Theorem 5.3, which generalizes the mixing property (5.3) to time inhomogeneous diffusion processes with both smooth and hard killings. The main tools of our proof are the approximation method proved in Chapter 3 and the coupling construction developed by Priola and Wang in [71].

Our mixing property criterion is composed of the two following assumptions, where $\phi_D : D \mapsto \mathbb{R}_+$ denotes the euclidean distance to the boundary ∂D:

$$\phi_D(x) = d(x, \partial D) = \inf_{z \in \partial D} |x - y|,$$

$\|\cdot\|$ being the Euclidean norm. For all $a > 0$, we define the open subset $D^a \subset D$ by

$$D^a = \{x \in D \text{ such that } \phi_D(x) < a\}.$$

Hypothesis 5.2. *We assume that D is bounded and that there exists $a > 0$ such that ϕ_D is of class C_b^2 on the boundary's neighborhood D^a.*

By [25, Chapter 5, Section 4], Hypothesis 5.2 is fulfilled if and only if D is a bounded open set whose boundary ∂D is of class C^2. The last hypothesis will allow us to use the approximation method of Chapter 3 and to build a coupling as in [71].

Hypothesis 5.3. *We assume that*

1. *κ is uniformly bounded in (x,t),*

2. *there exists a constant $c_0 > 0$ such that*

$$c_0|y| \leq |\sigma(t,x)y|,\ \forall(t,x,y) \in [0,+\infty[\times D \times \mathbb{R}^d.$$

3. *there exist two measurable functions $f : [0,+\infty[\times D^a \to \mathbb{R}_+$ and $g : [0,+\infty[\times D^a \to \mathbb{R}$ such that $\forall(t,x) \in [0,+\infty[\times D^a$,*

$$\sum_{k,l} \frac{\partial \phi_D}{\partial x_k}(z)\frac{\partial \phi_D}{\partial x_l}(z)[\sigma\sigma^*]_{kl}(t,x) = f(t,z) + g_i(t,x),$$

and such that

a) *f is of class C^1 in time and of class C^2 in space, and the derivatives of f are uniformly bounded,*

b) *there exists a positive constant $k_g > 0$ such that, for all $(t,x) \in [0,+\infty[\times D^a$,*

$$|g(t,x)| \leq k_g \phi_D(x),$$

We prove the mixing property in Section 5.2 and Section 5.3. In Section 5.2, we recall the approximation method of Chapter 3. We also deduce that the conditioned distribution (5.2) is uniformly tight. In Section 5.3, we generalize the coupling construction of [71] in our time-inhomogeneous setting and we state some estimates on the coupling time. Then we prove the mixing property (5.3) under Hypotheses 5.2, 5.1 and 5.3.

An interesting consequence of the mixing property is deduced in Section 5.4: we prove that the approximation method proved in Chapter 3 converges uniformly in time under Hypotheses 5.2, 5.1 and 5.3. This result generalizes a result obtained by Rousset [75] in a situation without hard killing.

5.2 Approximation method and uniform tightness of the conditioned distribution

We present in this section the approximation method proved in Chapter 3. This method has been introduced by Burdzy, Holyst, Ingerman and March [13] for standard Brownian motions and studied later by Burdzy, Hołyst and March [14] and by Grigorescu and Kang [40] for Brownian motions, by Del Moral and Miclo [23] and by Rousset [75] for processes with smooth killings, Ferrari and Marič [30] for Markov processes in countable state spaces, in [85] for diffusion processes and in [84] for general Markov processes.

The approximation method is based on a sequence of Fleming-Viot type interacting particle systems, whose associated sequence of empirical distributions converges to the conditioned distribution (5.2) when the number of particles tends to infinity. Fix $N \geq 2$ and let us define the Fleming-Viot type interacting particle system with N particles. The system of N particles $(X^{1,N}_{s,t},\ldots,X^{N,N}_{s,t})_{t\geq s}$ starts from an initial state $(X^{1,N}_{s,s},\ldots,X^{N,N}_{s,s}) \in D^N$, then:

- The particles evolve as N independent copies of $\mathcal{Z}^{X^{i,N}_{s,s}}_{s,\cdot}$ until one of them, say $X^{i_1,N}$, is killed. The first killing time is denoted by τ^N_1. We emphasize that under our hypotheses, the particle killed at time τ^N_1 is unique (Theorem 3.1, Chapter 3).
- At time τ^N_1, the particle $X^{i_1,N}$ jumps on the position of an other particle, chosen uniformly among the $N-1$ remaining ones. After this operation, the position $X^{i,N}_{s,\tau^N_1}$ is in D, for all $i \in \{1,\ldots,N\}$.
- Then the particles evolve as N independent copies of $\mathcal{Z}^{X^{i,N}_{s,\tau^N_1}}_{\tau^N_1,\cdot}$ until one of them, say $X^{i_2,N}$, is killed. This second killing time is denoted by τ^N_2. Once again, the killed particle is uniquely determined.
- At time τ^N_2, the particle $X^{i_2,N}$ jumps on the position of an other particle, chosen uniformly among the $N-1$ remaining ones.
- Then the particles evolve as independent copies of $\mathcal{Z}^{X^{i,N}_{s,\tau_2}}_{\tau^N_2,\cdot}$ and so on.

We denote by $0 < \tau^N_1 < \tau^N_2 < \ldots < \tau^N_n < \ldots$ the sequence of killing/jump times of the process. By Hypotheses 5.2, 5.3 and 5.1, the assumptions of Theorem 3.1 of Chapter 3 are fulfilled, so that

$$\lim_{n\to\infty} \tau^N_n = +\infty, \text{ almost surely.}$$

In particular, the above algorithm defines a Markov process $(X^{1,N}_{s,t},\ldots,X^{N,N}_{s,t})_{t\geq 0}$. For all $N \geq 2$ and all $0 \leq s \leq t$, we denote by $\mu^N_{s,t}$ the empirical distribution of $(X^{1,N}_{s,t},\ldots,X^{N,N}_{s,t})$, which means that

$$\mu^N_{s,t}(\cdot) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}_{s,t}}(\cdot) \in \mathcal{M}_1(D),$$

where $\mathcal{M}_1(D)$ denotes the set of probability measures on D.

By Theorem 2.1 of Chapter 3, we have, for any measurable bounded function $f : D \mapsto \mathbb{R}$,

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N} f(X^{i,N}_{s,t}) - \frac{\mu^N Q_{s,t}(f)}{\mu^N Q_{s,t}(\mathbf{1}_D)}\right| \leq \frac{4(t-s)\|f\|_\infty}{\sqrt{N}}\sqrt{\mathbb{E}\left(\frac{1}{\mu^N_{s,s} Q_{s,t}(\mathbf{1}_D)}\right)}.$$

If $\mu^N_{s,s}$ is deterministic, one can replace the expectation $\mathbb{E}\left(\frac{1}{\mu^N_{s,s}Q_{s,t}\mathbf{1}_D}\right)$ by $\frac{1}{\mu^N_{s,s}Q_{s,t}\mathbf{1}_D}$ in the right term. Thus we have the following result for non-deterministic values of $\mu^N_{s,s}$.

Proposition 5.1. *Under Hypotheses 5.2, 5.3 and 5.1, we have*

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(X^{i,N}_{s,t}) - \frac{\mu^N_{s,s}Q_{s,t}(f)}{\mu^N_{s,s}Q_{s,t}\mathbf{1}_D}\right| \;\middle|\; \mu^N_{s,s}\right) \leq \frac{4(t-s)\|f\|_\infty}{\sqrt{N\mu^N_{s,s}Q_{s,t}(\mathbf{1}_D)}}, \text{ almost surely.} \tag{5.4}$$

In order to prove the strong mixing property announced in the introduction of this paper, we also need the following uniform tightness result.

Proposition 5.2. *For all $\epsilon > 0$ and all $t_0 \geq 0$, there exist a positive constant $\alpha_\epsilon > 0$ and a number $N_\epsilon \geq 2$ such that, $\forall 0 \leq s \leq s + t_0 \leq t$,*

$$\mathbb{E}\left(\mu^N_{s,t}(D^{\alpha_\epsilon})\right) \leq \epsilon, \; \forall N \geq N_\epsilon, s + t_0 \leq t,$$

independently of the initial distribution $\mu^N_{s,s}$.

Moreover, if $\mu^N_{s,s}$ converges to a probability measure μ on D, then we have, for all $0 \leq s \leq s + t_0 \leq t$ and any probability measure μ on D,

$$\frac{\mu Q_{s,t}(D^{\alpha_\epsilon})}{\mu Q_{s,t}(\mathbf{1}_D)} \leq \epsilon.$$

Proof. The first part of the proposition is exactly Theorem 1 of Chapter 4.

We assume that $\mu^{(N)}_{s,s}$ converges almost surely to a probability measure μ. By the convergence result of Proposition 5.1,

$$\frac{\mu Q_{s,t}(D^{\alpha_\epsilon})}{\mu Q_{s,t}(\mathbf{1}_D)} = \lim_{N\to\infty} \mathbb{E}\left(\mu^N_{s,s}(D^{\alpha_\epsilon})\right).$$

This and the first part of Proposition 5.2 allows us to conclude the proof. □

5.3 Strong mixing property

Let us now state our main result.

Theorem 5.3. *Assume that Hypotheses 5.1, 5.2 and 5.3 are fulfilled. Then there exist two constants $C > 0$ and $\gamma > 0$ such that*

$$\sup_{\mu_1,\mu_2 \in \mathcal{M}_1(D)} \left\| \frac{\mu_1 Q_{0,T}}{\mu_1 Q_{0,T}\mathbf{1}_D} - \frac{\mu_2 Q_{0,T}}{\mu_2 Q_{0,T}\mathbf{1}_D} \right\|_{TV} \leq Ce^{-\gamma T}.$$

We prove Theorem 5.3 in the following subsections. In Subsection 5.3.1, we present the coupling for multi-dimensional time-inhomogeneous diffusion processes. In Subsection 5.3.2, we derive from this coupling two intermediate results which are key steps for the proof of Theorem 5.3. The first result (Lemma 5.5) concerns the existence of a family $(x_{s,t})_{0\leq s\leq t}$ and a constant $r_0>0$ such that

$$\inf_{x\in B(x_{s,t},r_0)} Q_{s,t}\mathbf{1}_D(x)\geq \frac{1}{2}\|Q_{s,t}\mathbf{1}_D\|_\infty,\ \forall 0\leq s+\Pi+1\leq t,$$

where we recall that Π is the time-period of the coefficients σ and b. The second result (Lemma 5.6) states the existence of a constant $\beta>0$ and a family of probability measures $(\nu_s^{x_1,x_2})_{s\geq 0,(x_1,x_2)\in D\times D}$ such that, for all $s\geq 0$, all $(x_1,x_2)\in D\times D$ and any non-negative measurable function f,

$$\frac{Q_{s,s+1}f(x_i)}{Q_{s,s+1}\mathbf{1}_D(x_i)}\geq \beta\nu_s^{x_1,x_2}(f),\ \ i=1,2.$$

We conclude the proof of Theorem 5.3 in Subsection 5.3.3, showing that Lemma 5.5 and Lemma 5.6 imply the strong mixing property.

5.3.1 Coupling

In this section, we present a coupling for multi-dimensional time-inhomogeneous diffusion processes.

Proposition 5.4. *For all $s\geq 0$ and all $(y^1,y^2)\in D\times D$, there exists a diffusion process $(Y^1_{s,t},Y^2_{s,t})_{t\geq s}$ such that*

1. *$(Y^1_{s,t})_{t\geq s}$ has the same law as $(\mathcal{Z}^{y^1}_{s,t})_{t\geq s}$,*
2. *$(Y^2_{s,t})_{t\geq s}$ has the same law as $(\mathcal{Z}^{y^2}_{s,t})_{t\geq s}$;*
3. *$Y^1_{s,t}$ and $Y^2_{s,t}$ are equal almost surely after the coupling time*

$$T_c^s=\inf\{t\geq 0, Y^1_{s,t}=Y^2_{s,t}\},$$

   where $\inf\emptyset=+\infty$ by convention.

4. *There exists a constant $c>0$ which doesn't depend on s,t such that*

$$\mathbb{P}(t<\tau_\partial^1\vee\tau_\partial^2 \text{ and } T_c^s>t\wedge\tau_\partial^1\wedge\tau_\partial^2)\leq \frac{c|y_1-y_2|}{\sqrt{1\wedge(t-s)}},$$

   where τ^1_∂ and τ^2_∂ denote the killing times of Y^1 and Y^2 respectively.

The proof of the existence of the coupling essentially follows the ideas introduced in [71] adapted to the time-inhomogeneous case. So we do not write the proof in details. Nevertheless, let us recall the idea behind the coupling construction.

By Hypothesis 5.3, there exists $\lambda_0>0$ such that $\sigma\sigma^*-\lambda_0 I$ is definite positive for all t,x. Let $\sigma_0:=\sqrt{\sigma\sigma^*-\lambda_0 I}$ be the unique symmetric definite positive matrix function such that $\sigma_0^2 = \sigma\sigma^* - \lambda_0 I$. Without loss of generality, one can choose λ_0 small enough so that σ_0 is uniformly positive definite. We define

$$u(x,y) = \frac{k(|x-y|)(x-y)}{(k(|x-y|)+1)|x-y|} \text{ and } C_t(x,y) = \lambda_0\left(I - 2u(x,y)u(x,y)^*\right) + \sigma_0(t,x)\sigma_0(t,y)^*,$$

where $k(r) = (k_0 r^2/2 \vee r)^{\frac{1}{4}}$. Before the coupling time, the coupling process is generated by

$$L_t(x,y) = \frac{1}{2}\sum_{i,j=1}^{d}\left\{[\sigma(t,x)\sigma(t,x)^*]_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + [\sigma(t,y)\sigma(t,y)^*]_{i,j}\frac{\partial^2}{\partial y_i \partial y_j}\right.$$
$$\left. +2[C_t(x,y)]_{i,j}\frac{\partial^2}{\partial x_i}\partial y_j\right\} + \sum_{i=1}^{d}\left\{b_i(t,x)\frac{\partial}{\partial x_i} + b_i(t,y)\frac{\partial}{\partial y_i}\right\}.$$

The coefficients of L_t are continuous and bounded over $\mathbb{R}^d$, then, for all $s \geq 0$ and $x = (y^1,y^2) \in D \times D$, there exists a not necessarily unique process $(X^x_{s,t})_{t\geq 0}$ with values in $\mathbb{R}^{2d}$ to the martingale problem associated with $(L_t)_{t\geq s}$ (see [47, Theorem 2.2, Chapter IV]). We set $X^x_{s,t} = (Y'^1_{s,t}, Y'^2_{s,t})$ and we consider the coupling time T'^s_c of $X^x_{s,\cdot}$, which is defined by

$$T'^s_c = \inf\{t \geq s, \text{ such that } Y'^1_{s,t} = Y'^2_{s,t}\}.$$

We define Y^1 and Y^2 as follows:

$$Y^i_t = \begin{cases} Y'^i_t,\ t \leq T'^s_c, \\ Y'^1_t,\ t > T'^s_c. \end{cases}$$

Moreover, each marginal process Y^i, $i = 1,2$, is killed either when it hits the boundary ∂D or with a rate κ.

The proof of the 4^{th} statement of Proposition 5.4 requires fine estimates and calculus which are plainly detailed in [71].

5.3.2 Intermediate results

In this section, we prove the two following lemmas, which are essential for the proof of Theorem 5.3. We recall that Π denotes the period of the coefficients and we emphasize that Lemma 5.5 is the only step which requires the periodicity assumption.

Lemma 5.5. *Let us denote by $x_{s,t}$ the point at which $Q_{s,t}\mathbf{1}_D$ is maximal. There exists a positive constant $r_0 > 0$, such that, denoting by $B(x_{s,t},r_0)$ the ball of radius r_0 centered on $x_{s,t}$, we have*

$$\inf_{x\in B(x_{s,t},r_0)} Q_{s,t}\mathbf{1}_D(x) \geq \frac{1}{2}\|Q_{s,t}\mathbf{1}_D\|_\infty,\ \forall 0 \leq s \leq s+\Pi+1 \leq t,$$

Proof of Lemma 5.5. Fix $s \geq 0$ and let $(Y^1_{s,\cdot}, Y^2_{s,\cdot})$ be the coupling of Proposition 5.4, starting from to points y_1 and y_2 in D. From the properties (1) and (2) of the proposition, we deduce that, for any measurable bounded function f which vanishes outside D, we have

$$\left|Q_{s,s+\Pi} f(y^1) - Q_{s,s+\Pi} f(y^2)\right| \leq \mathbb{E}\left|f(Y^1_{s,s+\Pi}) - f(Y^2_{s,s+\Pi})\right|$$

where $f(Y^1_{s,s+\Pi}) = f(Y^2_{s,s+\Pi}) = 0$ if $s + \Pi \geq \tau^1_\partial \vee \tau^2_\partial$ and, by property (3) of Proposition 5.4, $Y^2_{s,s+\Pi} = Y^2_{s,s+\Pi}$ if $T^s_c \leq s + \Pi \wedge \tau^1_\partial \wedge \tau^2_\partial$. Thus we have

$$\begin{aligned}\left|Q_{s,s+\Pi} f(y^1) - Q_{s,s+\Pi} f(y^2)\right| &\leq \|f\|_\infty \mathbb{P}\left(s+\Pi < \tau^1_\partial \vee \tau^2_\partial \text{ and } T^s_c > s + \Pi \wedge \tau^1_\partial \wedge \tau^2_\partial\right) \\ &\leq \|f\|_\infty \frac{c|y_1 - y_2|}{\sqrt{1 \wedge \Pi}}, \end{aligned} \tag{5.5}$$

by property (4) of Proposition 5.4.

By Proposition 5.2 with $\epsilon = 1/2$ and $t_0 = 1$, there exists $\alpha_0 > 0$ such that, for all $0 \leq s \leq s + \Pi + 1 \leq t$,

$$Q_{s+\Pi,t} \mathbf{1}_D(x) \leq 2 Q_{s+\Pi,t} \mathbf{1}_{(D^{\alpha_0})^c}(x),$$

where $(D^{\alpha_0})^c = \{x \in D,\, \phi_D(x) \geq \alpha_0\}$. We emphasize α_0 does not depend on s,t. Since the coefficients of the SDE (5.1) and the killing rate κ are assumed to be uniformly bounded on D, there exists a positive constant c_{α_0}, such that, for any $t \geq 0$,

$$\inf_{x \in (D^{\alpha_0})^c} Q_{t,t+\Pi} \mathbf{1}_D \geq c_{\alpha_0} > 0.$$

In particular, we have

$$\mathbf{1}_{(D^{\alpha_0})^c}(x) \leq \frac{Q_{t,t+\Pi} \mathbf{1}_D}{c_{\alpha_0}}(x).$$

We deduce that, for all $0 \leq s \leq s + \Pi + 1 \leq t$,

$$Q_{s+\Pi,t} \mathbf{1}_D(x) \leq 2 Q_{s+\Pi,t} \frac{Q_{t,t+\Pi} \mathbf{1}_D}{c_{\alpha_0}}(x),$$

so that

$$\begin{aligned}\|Q_{s+\Pi,t}(\mathbf{1}_D)\|_\infty &\leq \frac{2}{c_{\alpha_0}} \|Q_{s+\Pi,t+\Pi} \mathbf{1}_D\|_\infty \\ &\leq \frac{2}{c_{\alpha_0}} \|Q_{s,t} \mathbf{1}_D\|_\infty,\end{aligned}$$

by the periodicity assumption of Hypothesis 5.1. Applying inequality (5.5) to $f = Q_{s+\Pi,t}(\mathbf{1}_D)$ and using the semi-group property of $(Q_{s,t})_{s \leq t}$, we deduce that, for all $s \leq s + \Pi + 1 \leq t$,

$$\left|Q_{s,t} f(y^1) - Q_{s,t} f(y^2)\right| \leq \frac{2c|y^1 - y^2|}{c_{\alpha_0}\sqrt{1 \wedge \Pi}} \|Q_{s,t} \mathbf{1}_D\|_\infty.$$

For any $0 \leq s \leq s + \Pi + 1 \leq t$, let $x_{s,t}$ be such that $Q_{s,t}\mathbf{1}_D(x_{s,t}) = \|Q_{s,t}\mathbf{1}_D\|_\infty$. We have by the previous inequality,

$$Q_{s,t}f(y) \geq \|Q_{s,t}\mathbf{1}_D\|_\infty - \frac{2c|x_{s,t}-y|}{c_{\alpha_0}\sqrt{1\wedge(t-s)}\|Q_{s,t}\mathbf{1}_D\|_\infty},\ \forall y \in D.$$

Choosing $r_0 = \frac{c_{\alpha_0}}{4c}$, one obtains Lemma 5.5.

□

Lemma 5.6. *There exist a constant $\beta > 0$ and a family of probability measures denoted by $(\nu_s^{x_1,x_2})_{s\geq 0,(x_1,x_2)\in D\times D}$ such that, for all $s \geq 0$, for all $(x_1,x_2) \in D\times D$ and for any non-negative measurable function f,*

$$\frac{Q_{s,s+1}f(x_i)}{Q_{s,s+1}\mathbf{1}_D(x_i)} \geq \beta\nu_s^{x_1,x_2}(f).$$

Moreover, for any $r_1 > 0$, we have for all $x \in D$

$$\inf_{s\geq 0,\,(x_1,x_2)\in D\times D} \nu_s^{x_1,x_2}\left(B(x,r_1)\cap D\right) > 0.$$

Proof of Lemma 5.6. Let us first prove that there exist a constant $\rho_0 > 0$ and a fixed point $x_0 \in D$ such that, for any $(y_1,y_2) \in B(x_0,\rho_0)\times B(x_0,\rho_0)$ and any $s \geq 0$, there exists a probability measure $\mu_s^{y_1,y_2}$ which fulfills

$$\mathbb{E}\left(f(\mathcal{Z}^{y^i}_{s+\frac{2}{3},s+1})\right) \geq \frac{1}{2}\mu_s^{y_1,y_2}(f). \tag{5.6}$$

Fix $x_0 \in D$ and $s \geq 0$. Let ρ_0 be a positive constant which will be fixed later in the proof. Let y_1,y_2 be two elements of $B(x_0,\rho_0)$ and let $(Y^1_{s+\frac{2}{3},\cdot},Y^2_{s+\frac{2}{3},\cdot})$ be the coupling of Proposition 5.4 starting from $(y_1,y_2) \in D\times D$ at time $s+\frac{2}{3}$. We define the event $\mathcal{E}$ by

$$\mathcal{E} = \left\{s+1 \vee \tau^1_\partial \wedge \tau^2_\partial \text{ or } T^s_c \leq s+1\wedge\tau^1_\partial\wedge\tau^2_\partial\right\},$$

where T^s_c is the coupling time of Proposition 5.4, and τ^1_∂ and τ^2_∂ the killing times of Y^1 and Y^2 respectively. By definition of the killing time, $s+1 \geq \tau^1_\partial \vee \tau^2_\partial$ implies $Y^1_{s+\frac{2}{3},s+1} = Y^1_{s+\frac{2}{3},s+1} = \partial$. Moreover, by the coupling property (3) of Proposition 5.4, $T^s_c \leq s+1\wedge\tau^1_\partial\wedge\tau^2_\partial$ implies $Y^1_{s+\frac{2}{3},s+1} = Y^2_{s+\frac{2}{3},s+1}$. Finally,

$$\mathcal{E} \subset \left\{Y^1_{s+\frac{2}{3},s+1} = Y^2_{s+\frac{2}{3},s+1}\right\},$$

so that

$$\mathbb{E}\left(f(Y^2_{s+\frac{2}{3},s+1})|\mathcal{E}\right) = \mathbb{E}\left(f(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right).$$

We have then (the first equality being a consequence of Proposition 5.4 (1)), for any measurable function f which vanishes outside D,

$$\begin{aligned}\mathbb{E}\left(f(\mathcal{Z}^{y_i}_{s+\frac{2}{3},s+1})\right) &= \mathbb{E}\left(f(Y^i_{s+\frac{2}{3},s+1})\right)\\ &\geq \mathbb{E}\left(f(Y^i_{s+\frac{2}{3},s+1})|\mathcal{E}\right)\mathbb{P}(\mathcal{E})\\ &\geq \mathbb{E}\left(f(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right)\mathbb{P}(\mathcal{E}).\end{aligned}$$

But Proposition 5.4 (4) implies

$$\mathbb{P}(\mathcal{E}) \;\geq\; 1-3c|y_1-y_2| \;\geq\; 1-6c\rho_0$$

so that

$$\mathbb{E}\left(f(\mathcal{Z}^{y_i}_{s+\frac{2}{3},s+1})\right) \geq \mathbb{E}\left(\mathbf{1}_D(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right)(1-6c\rho_0)\mu^{y_1,y_2}_s(f),\ \forall i=1,2, \tag{5.7}$$

where the probability measure μ^{y_1,y_2}_s on D is defined by

$$\mu^{y_1,y_2}_s(f) = \frac{\mathbb{E}\left(f(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right)}{\mathbb{E}\left(\mathbf{1}_D(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right)}.$$

It remains to bounds by below $\mathbb{E}\left(\mathbf{1}_D(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right)$ to conclude that (5.6) holds for a well chosen ρ_0. We have

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_D(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right) &= \frac{1}{\mathbb{P}(\mathcal{E})}\mathbb{E}\left(\mathbf{1}_D(Y^1_{s+\frac{2}{3},s+1})\right) - \frac{1-\mathbb{P}(\mathcal{E})}{\mathbb{P}(\mathcal{E})}\mathbb{E}\left(\mathbf{1}_D(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}^c\right)\\ &\geq \frac{1}{\mathbb{P}(\mathcal{E})}\mathbb{E}\left(\mathbf{1}_D(\mathcal{Z}^{y_1}_{s+\frac{2}{3},s+1})\right) - \frac{1-\mathbb{P}(\mathcal{E})}{\mathbb{P}(\mathcal{E})}\\ &\geq \mathbb{E}\left(\mathbf{1}_D(\mathcal{Z}^{y_1}_{s+\frac{2}{3},s+1})\right) - \frac{6c\rho_0}{1-6c\rho_0}.\end{aligned}$$

The coefficients of the SDE (5.1) and the killing rate κ are uniformly bounded on D, thus, for a fixed point x_0 and $\rho_0>0$ small enough,

$$\epsilon_0 \stackrel{def}{=} \inf_{s\geq 0}\inf_{y_1\in B(x_0,\rho_0)}\mathbb{E}\left(\mathbf{1}_D(\mathcal{Z}^{y_1}_{s+\frac{2}{3},s+1})\right) > 0.$$

Finally, we deduce from (5.7) that

$$\mathbb{E}\left(f(\mathcal{Z}^{y_i}_{s+\frac{2}{3},s+1})\right) \geq \left(\epsilon_0 - \frac{6c\rho_0}{1-6c\rho_0}\right)(1-6c\rho_0)\mu^{y_1,y_2}_s(f),$$

for any non-negative measurable function which vanishes outside D. In particular, choosing ρ_0 small enough, we deduce that, for all $i\in\{1,2\}$,

$$\mathbb{E}\left(f(\mathcal{Z}^{y_i}_{s+\frac{2}{3},s+1})\right) \geq \frac{\epsilon_0}{2}\mu^{y_1,y_2}_s(f). \tag{5.8}$$

Let us now conclude the proof of Lemma 5.6. By Proposition 5.2 with $\mu = \delta_x$, there exists a constant $\alpha_1 > 0$ such that, for all $s \geq 0$ and all $x \in D$,

$$\begin{aligned} Q_{s,s+\frac{1}{3}}\mathbf{1}_{(D^{\alpha_1})^c}(x) &\geq \frac{1}{2}Q_{s,s+\frac{1}{3}}\mathbf{1}_D(x) \\ &\geq \frac{1}{2}Q_{s,s+1}\mathbf{1}_D(x). \end{aligned} \tag{5.9}$$

Since the coefficients of the SDE (5.1) and the killing rate κ are uniformly bounded, we clearly have

$$\epsilon_1 \overset{def}{=} \inf_{s\geq 0, x\in (D^{\alpha_1})^c} Q_{s+\frac{1}{3},s+\frac{2}{3}}\mathbf{1}_{B(x_0,\rho_0)}(x) > 0.$$

In particular, we deduce from (5.9) that, for all $x \in D$,

$$\begin{aligned} Q_{s,s+\frac{2}{3}}\mathbf{1}_{B(x_0,\rho_0)}(x) &= Q_{s,s+\frac{1}{3}}\left(Q_{s+\frac{1}{3},s+\frac{2}{3}}\mathbf{1}_{B(x_0,\rho_0)}\right)(x) \\ &\geq \epsilon_1 Q_{s,s+\frac{1}{3}}\mathbf{1}_{(D^{\alpha_\epsilon})^c}(x) \\ &\geq \frac{\epsilon_1}{2}Q_{s,s+1}\mathbf{1}_D(x) \end{aligned} \tag{5.10}$$

Finally, we have, for all $x_1,x_2 \in D \times D$,

$$\begin{aligned} Q_{s,s+1}f(x_i) &\geq \int_{B(x_0,\rho_0)} Q_{s+\frac{2}{3},s+1}f(y_1)\left[\delta_{x_1}Q_{s,s+\frac{2}{3}}\right](dy_1) \\ &\geq \frac{1}{Q_{s,s+\frac{2}{3}}\mathbf{1}_{B(x_0,\rho_0)}(x_2)} \times \\ &\quad \int_{B(x_0,\rho_0)}\int_{B(x_0,\rho_0)} Q_{s+\frac{2}{3},s+1}f(y_1)\left[\delta_{x_1}Q_{s,s+\frac{2}{3}} \otimes \delta_{x_2}Q_{s,s+\frac{2}{3}}\right](dy_1,dy_2) \\ &\geq \frac{\epsilon_0}{2Q_{s,s+\frac{2}{3}}\mathbf{1}_{B(x_0,\rho_0)}(x_2)} \times \\ &\quad \int_{B(x_0,\rho_0)}\int_{B(x_0,\rho_0)} \mu_s^{y_1,y_2}(f)\left[\delta_{x_1}Q_{s,s+\frac{2}{3}} \otimes \delta_{x_2}Q_{s,s+\frac{2}{3}}\right](dy_1,dy_2) \text{ by (5.8)} \\ &\geq \epsilon_0 \frac{Q_{s,s+\frac{2}{3}}\mathbf{1}_{B(x_0,\rho_0)}(x_1)}{2}\nu_s^{x_1,x_2}(f), \end{aligned}$$

where $\nu_s^{x_1,x_2}$ is the probability measure on D defined by

$$\nu_s^{x_1,x_2}(f) = \frac{\int_{B(x_0,\rho_0)}\int_{B(x_0,\rho_0)} \mu_s^{y_1,y_2}(f)\left[\delta_{x_1}Q_{s,s+\frac{2}{3}} \otimes \delta_{x_2}Q_{s,s+\frac{2}{3}}\right](dy_1,dy_2)}{Q_{s,s+\frac{2}{3}}\mathbf{1}_{B(x_0,\rho_0)}(x_1)Q_{s,s+\frac{2}{3}}\mathbf{1}_{B(x_0,\rho_0)}(x_2)}.$$

This and Inequality (5.10) allow us to conclude the proof of the first part of Lemma 5.6.

Fix $r_1 > 0$ and let us prove the second part of the lemma. We have, for all $(y_1,y_2) \in (B(x_0,\rho_0))^2$ and all $x \in D$,

$$\begin{aligned}
\mu_s^{y_1,y_2}(B(x,r_1)) &= \frac{\mathbb{E}\left(\mathbf{1}_{B(x,r_1)}(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right)}{\mathbb{E}\left(\mathbf{1}_D(Y^1_{s+\frac{2}{3},s+1})|\mathcal{E}\right)} \\
&\geq \mathbb{E}\left(\mathbf{1}_{B(x,r_1)}(Y^1_{s+\frac{2}{3},s+1})\right) - (1-\mathbb{P}(\mathcal{E})) \\
&\geq \delta_{y_1} Q_{s+\frac{2}{3},s+1}(B(x,r_1)) - 6c\rho_0.
\end{aligned}$$

Since the value of $\rho_0 > 0$ is arbitrarily determined, one can choose ρ_0 small enough so that $\phi_D(x_0) \geq 2\rho_0$. We emphasize that the the boundedness and the regularity of D force $B(x,r_1) \cap D$ to have a strictly positive minimal Lebesgue measure. Then, since the coefficients of the SDE (5.1) and the killing rate κ are assumed to be uniformly bounded, we clearly have

$$\epsilon_2 \stackrel{def}{=} \inf_{s\geq 0,\ x\in D,\ y_1\in B(x,\rho_0)} Q_{s+\frac{2}{3},s+1}\mathbf{1}_{B(x,r_1)}(y_1) > 0.$$

We deduce that

$$\mu_s^{y_1,y_2}(B(x,r_1)) \geq \epsilon_2/2.$$

The same construction as above (with our new and smaller $\rho_0 > 0$) implies that

$$\nu_s^{x,y}(B(x,r_1)) \geq \epsilon_2/2,\ \forall x \in D.$$

This concludes the proof of Lemma 5.6. $\square$

5.3.3 Conclusion of the proof of Theorem 5.3

In this section, we conclude the proof of Theorem 5.3. Let us define, for all $0 \leq s \leq t \leq T$ the linear operator $R^T_{s,t}$ by

$$R^T_{s,t}f(x) = \frac{Q_{s,t}(fQ_{t,T}\mathbf{1}_D)(x)}{Q_{s,T}\mathbf{1}_D(x)},$$

for all $x \in D$ and any bounded measurable function f. Let us remark that the value $R^T_{s,t}f(x)$ is the expectation of $f(\mathcal{Z}^x_{s,t})$ conditioned to $T < \tau_\partial$. Indeed we have

$$\begin{aligned}
\mathbb{E}\left(f(\mathcal{Z}^x_{s,t})\,\middle|\, T < \tau_\partial\right) &= \frac{\mathbb{E}\left(f(\mathcal{Z}^x_{s,t})\mathbf{1}_{\mathcal{Z}^x_{s,T}\in D}\right)}{\mathbb{E}\left(T < \tau_\partial\right)} \\
&= \frac{\mathbb{E}\left(f(\mathcal{Z}^x_{s,t})\mathbb{E}\left(\mathbf{1}_{\mathcal{Z}^x_{s,T}\in D}\,\middle|\, \mathcal{Z}^x_{s,t}\right)\right)}{Q_{s,T}\mathbf{1}_D(x)} \\
&= \frac{\mathbb{E}\left(f(\mathcal{Z}^x_{s,t})\mathbb{E}\left(\mathbf{1}_{\mathcal{Z}^{\mathcal{Z}^x_{s,t}}_{t,T}\in D}|\mathcal{Z}^x_{s,t}\right)\right)}{Q_{s,T}\mathbf{1}_D(x)},
\end{aligned}$$

by the Markov property. Finally, since $\mathbb{E}\left(\mathbf{1}_{\mathcal{Z}^{\mathcal{Z}^x_{s,t}}_{s,T}\in D}|\mathcal{Z}^x_{s,t}\right) = Q_{t,T}\mathbf{1}_D(\mathcal{Z}^x_{s,t})$, we get the announced result.

For any $T>0$, the family $(R^T_{s,t})_{0\leq s\leq t\leq T}$ is a semi-group. Indeed, we have for all $0\leq u\leq s\leq t\leq T$

$$R^T_{u,s}(R^T_{s,t}f)(x) = \frac{Q_{u,s}(R^T_{s,t}fQ_{s,T}\mathbf{1}_D)(x)}{Q_{u,T}\mathbf{1}_D(x)},$$

where, for all $y\in D$,

$$R^T_{s,t}f(y)Q_{s,T}\mathbf{1}_D(y) = Q_{s,t}(fQ_{t,T}\mathbf{1}_D)(y),$$

then

$$\begin{aligned} R^T_{u,s}R^T_{s,t}f(x) &= \frac{Q_{u,s}(Q_{s,t}(fQ_{t,T}\mathbf{1}_D))(x)}{Q_{u,T}\mathbf{1}_D(x)} \\ &= \frac{Q_{u,t}(fQ_{t,T}\mathbf{1}_D)(x)}{Q_{u,T}\mathbf{1}_D(x)} = R^T_{u,t}f(x), \end{aligned}$$

where we have used that $(Q_{s,t})_{s\leq t}$ is a semigroup.

In order to prove the strong mixing property of Theorem 5.3, we need the following lemma, whose proof is postponed to the end of this subsection.

Lemma 5.7. *There exists a constant $\beta'>0$ such that, for all $0\leq s\leq T-\Pi-2$, we have*

$$R^T_{s,s+1}f(x_i) \geq \beta'\eta^{x_1,x_2}_s(f),\ i=1,2,$$

for all $(x_1,x_2)\in D\times D$ and any non-negative measurable function f.

For any orthogonal probability measures μ_1,μ_2 on D, we have

$$\begin{aligned} \|\mu_1R^T_{s,s+1}-\mu_2R^T_{s,s+1}\|_{TV} &= \sup_{f\in\mathcal{B}_1(D)}|\mu_1R^T_{s,s+1}(f)-\mu_2R^T_{s,s+1}(f)| \\ &\leq \sup_{f\in\mathcal{B}_1(D)}\int_{D\times D}\left|R^T_{s,s+1}f(x)-R^T_{s,s+1}f(y)\right|\ \mu_1\otimes\mu_2(dx,dy), \end{aligned}$$

where $\mathcal{B}_1(D)$ denotes the set of measurable functions f such that $\|f\|_\infty\leq 1$, and $\|\cdot\|_{TV}$ the total variation norm for signed measures. For any $x,y\in D\times D$ and any $f\in\mathcal{B}_1(D)$, we have by Lemma 5.7, for all $s\leq T-\Pi-2$,

$$\begin{aligned} \left|R^T_{s,s+1}f(x)-R^T_{s,s+1}f(y)\right| &= \left|\left(R^T_{s,s+1}f(x)-\beta'\eta^{x,y}_s(f)\right)-\left(R^T_{s,s+1}f(y)-\beta'\eta^{x,y}_s(f)\right)\right| \\ &\leq 2(1-\beta'). \end{aligned}$$

Since μ_1 and μ_2 are assumed to be orthogonal probability measures, we have $\|\mu_1-\mu_2\|_{TV}=2$, so that

$$\|\mu_1R^T_{s,s+1}-\mu_2R^T_{s,s+1}\|_{TV}\leq(1-\beta')\|\mu_1-\mu_2\|_{TV}.$$

If μ_1 and μ_2 are two different but not orthogonal probability measures, one can apply the previous result to the orthogonal probability measures $\frac{(\mu_1-\mu_2)_+}{(\mu_1-\mu_2)_+(D)}$ and $\frac{(\mu_1-\mu_2)_-}{(\mu_1-\mu_2)_-(D)}$. Then

$$\begin{aligned}\|\frac{(\mu_1-\mu_2)_+}{(\mu_1-\mu_2)_+(D)}R^T_{s,s+1} - \frac{(\mu_1-\mu_2)_-}{(\mu_1-\mu_2)_-(D)}R^T_{s,s+1}\|_{TV} \\ \leq (1-\beta')\|\frac{(\mu_1-\mu_2)_+}{(\mu_1-\mu_2)_+(D)} - \frac{(\mu_1-\mu_2)_-}{(\mu_1-\mu_2)_-(D)}\|_{TV}.\end{aligned}$$

But $(\mu_1-\mu_2)_+(D) = (\mu_1-\mu_2)_-(D)$ since $\mu_1(D)=\mu_2(D)=1$, then, multiplying the obtained inequality by $(\mu_1-\mu_2)_+(D)$, we deduce that

$$\|(\mu_1-\mu_2)_+R^T_{s,s+1} - (\mu_1-\mu_2)_-R^T_{s,s+1}\|_{TV} \quad \leq \quad (1-\beta')\|(\mu_1-\mu_2)_+ - (\mu_1-\mu_2)_-\|_{TV}.$$

Since $(\mu_1-\mu_2)_+ - (\mu_1-\mu_2)_- = \mu_1-\mu_2$, we obtain

$$\|\mu_1 R^T_{s,s+1} - \mu_2 R^T_{s,s+1}\|_{TV} \leq (1-\beta')\|\mu_1-\mu_2\|_{TV}.$$

In particular, using the semigroup property of $(R^T_{s,t})_{s,t}$, we deduce that

$$\begin{aligned}\|\delta_x R^T_{0,T-\Pi-2} - \delta_y R^T_{0,T-\Pi-2}\|_{TV} &= \|\delta_x R_{0,T-\Pi-3}R^T_{T-\Pi-3,T-\Pi-2} - \delta_y R^T_{0,T-\Pi-3}R^T_{T-\Pi-3,T-\Pi-2}\|_{TV} \\ &\leq (1-\beta')\|\delta_x R^T_{0,T-\Pi-3} - \delta_y R^T_{0,T-\Pi-3}\|_{TV} \\ &\leq 2(1-\beta')^{[T-\Pi-2]},\end{aligned}$$

where $[T-\Pi-2]$ denotes the integer part of $T-\Pi-2$. Theorem 5.3 is thus proved for any pair of probability measures (δ_x,δ_y), with $(x,y)\in D\times D$, for a good choice of C and γ.

Let μ be a probability measure on D and $y\in D$. We have

$$\begin{aligned}\left|\mu Q_{0,T}(f) - \mu Q_{0,T}(\mathbf{1}_D)\frac{\delta_y Q_{0,T}(f)}{\delta_y Q_{0,T}(\mathbf{1}_D)}\right| &= \left|\int_D Q_{0,T}f(x) - \delta_x Q_{0,T}(\mathbf{1}_D)\frac{\delta_y Q_{0,T}(f)}{\delta_y Q_{0,T}(\mathbf{1}_D)}d\mu(x)\right| \\ &\leq \int_D C^{e^{-\gamma T}}\delta_x Q_{0,T}(\mathbf{1}_D)d\mu(x),\end{aligned}$$

by the mixing property proved above. Dividing by $\mu Q_{0,T}(\mathbf{1}_D) = \int_D \delta_x Q_{0,T}(\mathbf{1}_D)d\mu(x)$, we deduce that

$$\left|\frac{\mu Q_{0,T}(f)}{\mu Q_{0,T}(\mathbf{1}_D)} - \frac{\delta_y Q_{0,T}(f)}{\delta_y Q_{0,T}(\mathbf{1}_D)}\right| \leq 2(1-\beta')^{[T]},$$

for any $f\in\mathcal{B}_1$. The same procedure, replacing δ_y by any probability measure, leads us to Theorem 5.3.

Proof of Lemma 5.7. By Lemma 5.6, there exist $\beta>0$ and a family of probability measures denoted by $(\nu_s^{x_1,x_2})_{s\geq 0,\,(x_1,x_2)\in D\times D}$ such that, for any $(x_1,x_2)\in D\times D$ and any $s\geq 0$, we have for all $i\in\{1,2\}$

$$Q_{s,s+1}f(x_i) \geq Q_{s,s+1}\mathbf{1}_D(x_i)\beta\nu_s^{x_1,x_2}(f),$$

for any non-negative measurable function f. Then we have

$$\begin{aligned} R^T_{s,s+1}f(x_i) &= \frac{Q_{s,s+1}(fQ_{s+1,T}\mathbf{1}_D)(x_i)}{Q_{s,T}\mathbf{1}_D(x_i)} \\ &\geq \frac{\beta\nu_s^{x_1,x_2}(fQ_{s+1,T}\mathbf{1}_D)Q_{s,s+1}\mathbf{1}_D(x_i)}{Q_{s,T}\mathbf{1}_D(x_i)}. \end{aligned}$$

Since $s+1+\Pi+1 \leq T$ by assumption, we deduce from Lemma 5.5 that there exist $x_{s+1,T} \in D$ and $r_0 > 0$ such that

$$\inf_{x\in B(x_{s+1,T},r_0)} Q_{s+1,T}\mathbf{1}_D(x) \geq \frac{1}{2}\|Q_{s+1,T}\mathbf{1}_D\|_\infty. \tag{5.11}$$

By the second part of Lemma 5.6, $\nu_s^{x_1,x_2}(B(x_{s,T},r_0))$ is uniformly bounded below by a constant $\epsilon > 0$ which only depend on r_0. The probability measure $\eta_s^{x_1,x_2}$ is defined, for any measurable subset $A \subset D$, by

$$\eta_s^{x_1,x_2}(A) \stackrel{def}{=} \frac{\nu_s^{x_1,x_2}(A\cap B(x_{s,T},r_0))}{\nu_s^{x_1,x_2}(B(x_{s,T},r_0))}.$$

In particular we have, for any bounded measurable function f,

$$\nu_s^{x_1,x_2}(f) \geq \epsilon\epsilon\eta_s^{x_1,x_2}(f).$$

We deduce that

$$\begin{aligned} R^T_{s,s+1}f(x_1) &\geq \frac{\frac{\epsilon}{2}\eta_s^{x_1,x_2}(f)Q_{s,s+1}\mathbf{1}_D(x_1)\|Q_{s+1,T}\mathbf{1}_D\|_\infty}{Q_{s,T}\mathbf{1}_D(x_1)} \\ &\geq \frac{\epsilon}{2}\eta_s^{x_1,x_2}(f). \end{aligned}$$

This concludes the proof of Lemma 5.7.

□

5.4 Uniform convergence of the approximation method

In this section, we prove that the approximation method described in the introduction converges uniformly in time to the conditioned distribution, which is an improvement of Inequality (5.4). We use the same notation as in Section 5.2.

Theorem 5.8. *Assume that Hypotheses 5.1, 5.2 and 5.3 hold. Assume that the family of empirical distributions $(\mu^N_{s,s})_{s\geq 0,\,N\geq 2}$ of the initial distributions of $(\cdot,\mathbb{X}^N_{s,\cdot})_{N\geq 2}$ is uniformly tight. Then*

$$\lim_{N\to\infty}\sup_{s\leq t\in[0,+\infty[}\sup_{f\in\mathcal{B}_1(D)}\mathbb{E}\left|\mu^N_{s,t}(f) - \frac{\mu^N_0 Q_{s,t}(f)}{\mu^N_0 Q_{s,t}(\mathbf{1}_D)}\right| = 0.$$

Proof. Fix $\epsilon > 0$ and let us prove that there exists $N_\epsilon \geq 2$ such that, for all $N \geq N_\epsilon$ and all measurable function $f : D \to \mathbb{R}$ such that $\|f\|_\infty \leq 1$,

$$\sup_{t\in[0,+\infty[} \mathbb{E}\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| \leq \epsilon. \tag{5.12}$$

Let γ be the constant of Theorem 5.3 and fix $t_0 \geq 1$ such that $2e^{-t_0\gamma/2} \leq \epsilon/3$. Fix $\alpha'_\epsilon > 0$ such that $\mathbb{E}(\mu^N_{s,s}(D^{\alpha'_\epsilon})) \leq \epsilon$ for all $N \geq 2$ (this is feasible since the sequence of initial distributions is assumed to be uniformly tight). We set

$$\beta_\epsilon \stackrel{def}{=} \inf_{x\in\left(D^{\alpha'_\epsilon}\right)^c} Q_{s,s+t_0+1}\mathbf{1}_D(x).$$

Since the coefficients of the SDE (5.1) and the killing rate κ are uniformly bounded, we clearly have $\beta_\epsilon > 0$. We have for all $t \in [0,t_0+1]$

$$\mathbb{E}\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| = \mathbb{E}\left[\mathbb{E}\left(\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| \mid \mu^N_{s,s}\right)\right],$$

where, by Proposition 5.1,

$$\mathbb{E}\left(\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| \mid \mu^N_{s,s}\right) \leq \sqrt{\frac{1}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\frac{4t}{\sqrt{N}}}.$$

Since $\|f\|_\infty \leq 1$, we have $\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| \leq 2$ almost surely. We deduce that

$$\begin{aligned}
\mathbb{E}\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| &\leq \frac{\epsilon}{2} + 2\mathbb{P}\left(4t\sqrt{\frac{1}{N\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}} \geq \frac{\epsilon}{2}\right)\\
&\leq \frac{\epsilon}{2} + 2\mathbb{P}\left(\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D) \leq \frac{64t^2}{\epsilon^2 N}\right)
\end{aligned}$$

But $\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D) \geq \mu^N_{s,s}\left((D^{\alpha'_\epsilon})^c\right)\beta_\epsilon$, thus

$$\begin{aligned}
\mathbb{E}\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| &\leq \frac{\epsilon}{2} + 2\mathbb{P}\left(\mu^N_{s,s}\left((D^{\alpha'_\epsilon})^c\right) \leq \frac{64t^2}{\epsilon^2 N\beta_\epsilon}\right)\\
&\leq \frac{\epsilon}{2} + 2\mathbb{P}\left(\mu^N_{s,s}\left(D^{\alpha'_\epsilon}\right) \geq 1 - \frac{64(t_0+1)^2}{\epsilon^2 N\beta_\epsilon}\right)\\
&\leq \frac{\epsilon}{2} + 2\frac{1}{1-\frac{64(t_0+1)^2}{\epsilon^2 N\beta_\epsilon}}\mathbb{E}\left(\mu^N_{s,s}\left(D^{\alpha'_\epsilon}\right)\right)\\
&\leq \frac{\epsilon}{2} + 2\frac{1}{1-\frac{64(t_0+1)^2}{\epsilon^2 N\beta_\epsilon}}\epsilon,
\end{aligned}$$

where we have used $0 \leq t \leq t_0 + 1$ and Markov's inequality. Finally, there exists $N'_\epsilon \geq 2$ such that, $\forall N \geq N'_\epsilon$,

$$\sup_{s\geq 0}\sup_{t\in[s,s+t_0]} \mathbb{E}\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| \leq \epsilon.$$

Fix now $t \geq t_0 + 1$. We have

$$\begin{aligned}\mathbb{E}\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| &\leq \mathbb{E}\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s+t-t_0}Q_{s+t-t_0,s+t}(f)}{\mu^N_{s,s+t-t_0}Q_{s+t-t_0,s+t}(\mathbf{1}_D)}\right| \\ &\quad + \mathbb{E}\left|\frac{\mu^N_{s,s+t-t_0}Q_{s+t-t_0,s+t}(f)}{\mu^N_{s,s+t-t_0}Q_{s+t-t_0,s+t}(\mathbf{1}_D)} - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right|.\end{aligned}$$

By Proposition 5.2, there exists $\alpha''_\epsilon > 0$ and $N''_\epsilon > 0$ such that for all $t \geq t_0 + 1$ and all $N \geq N''_\epsilon$,

$$\mathbb{E}\left(\mu^N_{s,s+t-t_0}(D^{\alpha''_\epsilon})\right) \leq \epsilon.$$

Since α'_ϵ and α''_ϵ, one can assume without restriction that $\alpha'_\epsilon = \alpha''_\epsilon$. With the same calculation and using the Markov property, we deduce that, for all $N \geq 2$,

$$\mathbb{E}\left|\mu^N_{s,t}(f) - \frac{\mu^N_{s,s+t-t_0}Q_{s+t-t_0,s+t}(f)}{\mu^N_{s,s+t-t_0}Q_{s+t-t_0,s+t}(\mathbf{1}_D)}\right| \leq \frac{\epsilon}{2} + 2\frac{1}{1 - \frac{64(t_0+1)^2}{\epsilon^2 N\beta_\epsilon}}\epsilon.$$

By Theorem 5.3, we also have

$$\mathbb{E}\left|\frac{\mu^N_{s,s+t-t_0}Q_{s,s+t}(f)}{\mu_{s+t-t_0}Q_{s,s+t}(\mathbf{1}_D)} - \frac{\mu_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| \leq 2e^{-\gamma t_0} = \epsilon/3.$$

Then there exists $N'''_\epsilon \geq 2$ such that, $\forall N \geq N'''_\epsilon$,

$$\mathbb{E}\left|\mu^N_{s,s+t}(f) - \frac{\mu^N_{s,s}Q_{s,s+t}(f)}{\mu^N_{s,s}Q_{s,s+t}(\mathbf{1}_D)}\right| \leq \epsilon.$$

Setting $N_\epsilon = N'_\epsilon \vee N'''_\epsilon$, we have proved (5.12), which concludes the proof of Theorem 5.8. $\square$

Chapter 6

Simulations

Abstract

In this chapter, we focus on the simulation part of the thesis. In particular, we present in detail the general algorithm used in the different numerical illustrations of chapters 1 and 2. We also describe the C++ functions that have been created in order to implement this algorithm and present how to use it in some simple examples.

6.1 Algorithm

6.1.1 Preliminaries

Let $(\mathcal{Z})_{t\geq 0}$ be a pure jump strong Markov process which evolves on a state space E until it is absorbed, that is until it reaches a cemetery point $\partial \notin E$. The absorbing time of $\mathcal{Z}$ is

$$\tau_\partial \stackrel{def}{=} \inf\{t \geq 0,\, \mathcal{Z}_t = \partial\}.$$

We also assume that two independent copies of $\mathcal{Z}$ cannot be absorbed at the same time almost surely. For any probability measure μ on E, the law of $\mathcal{Z}$ starting with distribution μ will be denoted by $\mathbb{P}_\mu$ and its associated expectation by $\mathbb{E}_\mu$. If there exists $x \in E$ such that $\mu = \delta_x$, this objects are respectively denoted by $\mathbb{P}_x$ and $\mathbb{E}_x$.

In Section 2 of Chapter 3, we built a sequence of Fleming-Viot type interacting particle systems $(\mathbb{X}^N)_{N\geq 2}$ in order to approximate the conditional distribution

$$\mathbb{P}_\mu\left(\mathcal{Z}_t \in \cdot | t < \tau_\partial\right).$$

The value of N is fixed across the whole section and we show how to compute

$$(X_t^i)_{t\in[0,T]},\ \forall i \in \{1,\ldots,N\},$$

where T is also fixed.

6.1.2 Theoretical description

Let $(X_0^1,...,X_0^N) \in E^N$ be the initial state of the interacting particle system. This is an input of the algorithm.

We will build by iteration a sequence of indexes $(i_n)_{n\geq 0} \in \{1,...,N\}^{\mathbb{N}}$ and a sequence of families of times $(s_n^1,...,s_n^N)_{n\geq 0} \in ([0,+\infty[^N)^{\mathbb{N}}$ such that

- The sequence (s_n^i) is non-decreasing and converges to $+\infty$, for any $i \in \{1,...,N\}$.
- the index i_n is the smallest index for the relation $\prec^n$ defined by

$$i \prec^n j \;\Leftrightarrow\; \Big[(s_n^i < s_n^j) \text{ or } (s_n^i = s_n^j \text{ and } X^i_{s_n^i} = \partial) \\ \text{or } (s_n^i = s_n^j \text{ and } X^i_{s_n^i} \neq \partial \text{ and } i_n < i)\Big].$$

  By instance, $i_0 = 1$ is the smallest index for $\prec^0$.

Informally, s_n^i will be the *next* jump time of the particle X^i. As a consequence, X^{i_n} will be the *next* jumping particle and, if two or more particles jump simultaneously at time $s_n^{i_n}$, i_n is the index chosen so that:

- either its position after the jump is ∂ (which means that the i_n^{th} particle has been absorbed), such a particle being unique by assumption,
- or, if none of the jumping particles are absorbed at time $s_n^{i_n}$, X^{i_n} has the smallest index among the jumping particles (this choice is arbitrary).

Elementary step $n \geq 1$

Fix $n \geq 0$ and assume that the finite sequences $(i_l)_{0\leq l\leq n}$ and $(s_l^1,...,s_l^N)_{0\leq l\leq n}$ are built.

If $X^{i_n}_{s_n^{i_n}} = \partial$, we choose randomly and uniformly an index j_n among $\{1,...,N\}\setminus\{i_n\}$ and we set

$$X^{i_n}_{s_n^{i_n}} = X^{j_n}_{s_n^{i_n}}.$$

After this step, $X^{i_n}_{s_n^{i_n}}$ belongs to E, since $X^{i_n}_{s_n^{i_n}} = X^{j_n}_{s_n^{i_n}} = \partial$ could only happen if two independent copies of $\mathcal{Z}$ are absorbed simultaneously (at time $s_n^{i_n}$), which is forbidden by our assumptions.

Then we compute the next jump time of the i_n^{th} particle, which is denoted by $s_{n+1}^{i_n}$, and the position of the next jump, which is $X^{i_n}_{s_{n+1}^{i_n}}$. We also set $s_{n+1}^i = s_n^i$

for any $i \neq i_n$. We define i_{n+1} as the smallest element of $\{1,...,N\}$ for the relation $\prec^n$.

By induction, we have defined the sequences i_n and s^i_n, and it is clear that the sequence s^i_n converges to $+\infty$.

We repeat the previous elementary step many times, and we stop as soon as $s^i_n > T$ for any $i \in \{1,...,N\}$. Thus we've got computed

$$(X^i_t)_{t\in[0,T]},\ \forall i \in \{1,...,N\}.$$

One of the main interests of this algorithm is that we only have to keep in memory the position of each particle X^i at times s^i_n and s^i_{n+1}, in order to compute the position of the system at time s^i_{n+2}.

6.1.3 Container of the set of particles: a complexity issue

The main difficulty concerns the implementation of the vector of particles. At the beginning of an elementary step $n \geq 1$ (n is fixed in this section), this vector has to contain the position of each particle at times s^i_n and s^i_{n+1} and the times themselves. We denote by `V` this container and assume that, for all $i \in \{1,...,N\}$, `V[i]` is an object with elements `a`, `b`, `pa` and `pb` such that, at the beginning of the n^{th} step,

`V[i].a`$= s^i_{n-1}$,
`V[i].b`$= s^i_n$,
`V[i].pa`$= X^i_{s^i_{n-1}}$
`V[i].pb`$= X^i_{s^i_n}$.

At each elementary step, we need

1. the index `in`$= i_n$ of the smallest particle for $\prec^n$,
2. if the position `V[in].pb` is equal to ∂, we need to get the value of `V[jn].pa`, where `jn`$= j_n$ is the index of a randomly chosen particle, different from i_n.

The following requirements for the type of `V` are directly derived

1. we need an efficient method to access the element `V[in]`, that is to access the $\prec^n$-smallest element of the array `V`,
2. we need an efficient method for random memory access.

Efficient vector implementations already exist in the C++ standard library, which will be denoted by `std` (we refer the reader to the web site cplusplus.com [44], where a systematic description of this library is provided). The three main vector containers of the C++ standard library are `std::list`, `std::vector` and `std::set`.

The first one, `std::list`, is implemented so that it is easy to insert or remove elements; we won't use these features since the size of our container `V` will remain constant, equal to N. Moreover, lists lack direct access to the elements by their position, which is our second requirement.

The second one, `std::vector`, allows access to individual elements by their position index in constant time; we need this feature in the second requirement. However, in order to access the *first* element in the sense of $\prec^n$ (which is needed at each elementary step), one has to iterate along the elements of the vector, which is done in linear time. As a consequence, `std::vector` doesn't entirely fulfills our requirements.

The third one, `std::set`, contains elements which are always sorted from lower to higher following a specific strict ordering criterion; we will use this feature to satisfy our first requirement. However, random memory access in `std::set` is poorly efficient (linear time), and in particular doesn't fulfill our second requirement.

The original solution that is used here is to combine both the `std::vector` and `std::set` in one new container type, which will be called `o_vector` (as "ordered vector"). An ordered vector `V` will have two elements:

- a `std::set` of pointer to the particles ordered following $\prec^n$, called `V.set`,
- a `std::vector` of pointer to the particles, called `V.vector`.

At the end of the *early computation*, we force the pointer to the particles in the `V.set` to be ordered. We proceed as follows:

- At the beginning of the n^{th} *elementary step*, we choose the first pointer to a particle of `V.set`, which points to the i_n^{th} particle of the process by construction of `V.set`. This gives us access to i_n^{th} particle in constant time.
- If the particle has been killed, then we choose randomly and uniformly an index j_n between 0 and $N-1$, different from i_n. The corresponding particle is pointed to by `V.vector[j_n]`. This allows us to obtain the new position of the i_n^{th} particle in constant time.
- Then we proceed to the increment of the path of the i_n^{th} particle, still in constant time.
- At this moment, the first element of `V.set` points to the i_n^{th} particle, which is no more the first one in the sense of $\prec^n$. We need to re-order this element, which is done in $\ln N$ time.

An elementary step is thus achieved in $\ln N$ time.

The number of elementary steps required for the whole simulation is clearly proportional to the number of particles. As a consequence, time length of the whole algorithm is proportional to $N \ln N$ (for a fixed time T). Using one of the predefined vector containers of the `std` would lead us to a time length proportional to $O(N^2)$.

6.2 C++ functions and some examples

6.2.1 C++ functions

In this subsection, we describe the important functions implemented in $C++$ during the thesis. We assume that we're dealing with a pure jump Markov process which evolves in a state space E, the state space being numerically implemented in the type `STATE_SPACE`. By instance, if $E = \mathbb{R}$, one could choose `STATE_SPACE=double`; if $E = \mathbb{N}$, one could choose `STATE_SPACE=int`.

The container

We present the way of building an interacting particle `X` system of type Fleming-Viot with N particles, each of them starting from a point x_0. The type of `X` is `IPS_FV_pj<STATE_SPACE>`.

Assume for instance that $N = 1000$, `STATE_SPACE=int` and $x_0 = 7$, then `X` is built as follows:

```
IPS_FV_pj<int> X(1000,7);
```

The dynamic of the particles between the jumps

The process which drives each particle is an object, whose type has to be derived from the purely virtual class `dynamic<STATE_SPACE>`. The only purely virtual method of `dynamic<STATE_SPACE>` is

```
bool increment(STATE_SPACE& x, double& t)=0;
```

Given a position $x \in E$ and a time $t \geq 0$, which are passed by address in order to be modified during the execution of the function, the method `increment` gives the (random) position and the (random) time of the next jump of the process $\mathcal{Z}$ starting from x at time t. These values are returned by modifying x and t. If the new position x is ∂ (that is if the particle has been absorbed), then the function returns `true`. It returns `false` otherwise.

By instance, if the process is a linear birth and death process with birth and death parameters equal to `b=1` and `d=2` respectively (see the definition of the linear birth and death process in Section 4 of Chapter 1), then the process object is defined as follows (where `gsl_` is a reference to the GNU Scientific Library):

```
class BD_process : public dynamic<int>{
public:
  BD_process(){}
  ~BD_process(){}

  bool increment(int& x, double& t){
    double b_rate=1*((double) x);
    double d_rate=2*((double) x);
    // In the following line, we pick a random value whose distribution is
    // the exponential law with rate d_rate+b_rate, using the GSL library:
    double dt=gsl_ran_exponential(this->r,1/(b_rate+d_rate));
    t+=dt;
    if(x>= 0)
      {
      // In the following line, we pick a random value whose distribution is
      // a Bernoulli law with parameter b_rate/(b_rate+d_rate)
      double test(gsl_ran_bernoulli(this->r,b_rate/(d_rate+b_rate)));
      test==1 ? x++ : x--;
      }
      else
      {
      x=0;
      }
      return x==0;
      }

};

BD_process a_bd_process(); //this is a BD_process object
```

We tell the interacting particle system `X` to use `birth_and_death_process` during the increments by using

```
X.set_dynamic(&a_bd_process);
```

Simulation of the interacting particle system: a complete example

We now have all the elements to compute the position at time `10` of the interacting particle system `X` whose `1000` particles start from `7`, evolving as `a_bd_process`.

```
#include "IPS.h" //contains all the required classes

class BD_process : public dynamic<int>{
```

```
public:
  BD_process(){}
  ~BD_process(){}

  bool increment(int& x, double& t){
    double b_rate=1*((double) x);
    double d_rate=2*((double) x);
    double dt=gsl_ran_exponential(this->r,1/(b_rate+d_rate));
    t+=dt;
    if(x>= 0)
      {
       double test(gsl_ran_bernoulli(this->r,b_rate/(d_rate+b_rate)));
       test==1 ? x++ : x--;
      }
    else
      {
      x=0;
      }
    return x==0;
    }

};

/*********
* main function
**********/

int main(){
// Definition of the GSL random generator
  const gsl_rng_type * T;
  gsl_rng_env_setup();
  T=gsl_rng_default;
  gsl_rng* r=gsl_rng_alloc(T);
gsl_rng_set(r,1);
// Definition of the process
  BD_process a_bd_process();
  a_bd_process.set_rng(r); // the process uses the random generator r
// Definition of the interacting particle system
  IPS_FV_pj<int> X(1000,7);
  X.set_rng(r); // the process uses the random generator r
  X.set_dynamic(&a_bd_process);
```

```
// Simulation
// In what follows, X.current_time() is the time b
//         of the next jumping particle
  double T(10);
  while(X.current_time()<T)
    {
    X.increment(); //one elementary step
    }

// Display the result
// X[i] is the i-th particle accessed by address
// (X[i])[T] is the position of this particle at time T
  for(int i=0; i<1000; i++)
    {
    std::cout << "\n" << i << "-th particle at time T: " << (X[i])[T];
    }
return 0;
}
```

6.2.2 An example of program

Many programs have been written during this thesis in order to illustrate or understand the behavior of conditioned diffusions. The following example has been used to compute the Yaglom limits of the logistic birth and death processes studied in Section 4 of Chapter 1.

The job of the program is to compute the position of an interacting particle system whose particles evolve as a logistic birth and death process, to save the positions in a file and to produce an histogram that can be drawn using the program `gnuplot`.

Here is the code, where `logistic_BD_process_ct` denotes a predefined process class (which is derived from `dynamic<int>`) and `int` is the state space of the process $\mathcal{Z}$.

```
#include "IPS.h"

typedef logistic_BD_process_ct PROCESS;
typedef int STATE_SPACE;

int main(int argv, char** argc)
{
simulation_pj_1D<PROCESS,STATE_SPACE>(argv,argc);
```

```
}
```

The compilation of the program is made as follows in a Linux shell

```
$ g++ -o program main.cpp -lgsl -lgslcblas
```

The resulting program, called here `program` can be used in a Linux shell as follows:

```
$ ./program --process file.proc --init 10000 7 --duration 10
--output data --hist -0.5 100.5 101
```

where

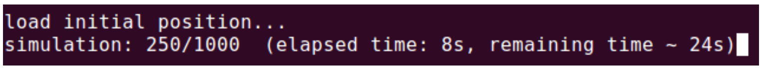
- `--process file.proc` means that we use the process described by `file.proc`, where `file.proc` is an existing file taken as an input and which described the parameters of logistic birth and death process to be used in the simulation. In our case, it is a common text-file which contains

  ```
  #############################
  # Name of the process
  #   logistic_BD_process_ct
  # Comment
  #   Process used as an illustration
  #############################

  #############################
  #
  # Number of parameters
  3
  # Birth parameter
  11
  # Death parameter
  1
  # Competition parameter
  1
  #
  ##############################
  ```

- `--init 10000 7` means that the system has 10000 particles, each of them starting from the position 7,
- `--duration 10` means that the simulation will last 10 units of time,
- `--output data` means that the final position of the particles will be saved in an output file `data`.

- `--hist -0.5 100.5 101` means that a file `data.hist` will be created, which will contain an histogram of the final positions of the particles. This histogram will have `101` uniform ranges between `-0.5` and `100.5`.

Once the program is launched, the following message appears in the terminal:

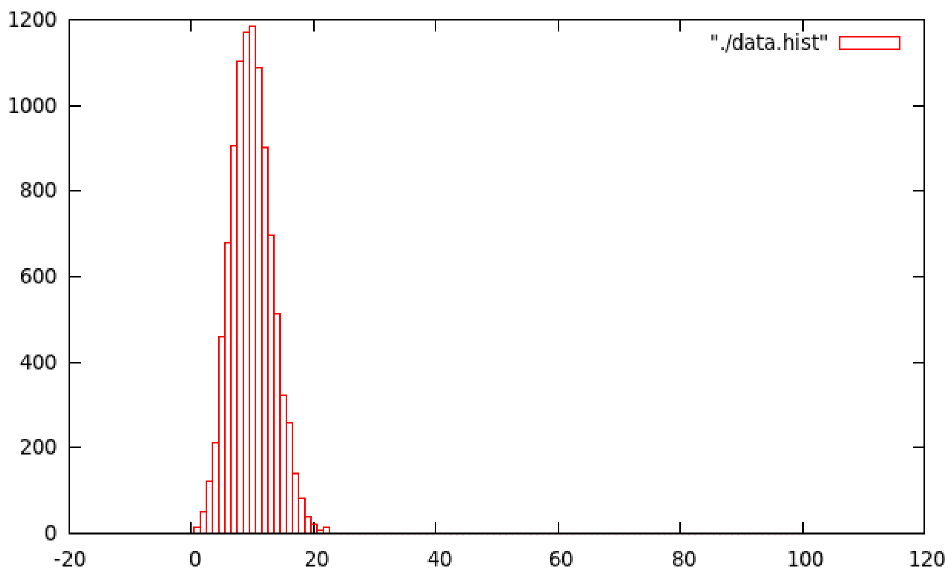


It indicates the proportion of the simulation that has been run up to now (`250/1000` here), the elapsed time and the estimated remaining time.

Once it is finished, one can represent graphically the result, using the software gnuplot on Linux, and then the `plot` function:

```
$ gnuplot
Terminal type set to 'wxt'
gnuplot> plot "./data.hist" with boxes
```

This produces the following graph:



An other example

Other examples can be immediately produced using pre-existing process types. By instance, here is a program for the simulation based on a logistic Feller diffusion (see Section 5 Chapter 1)

```
#include "IPS.h"

typedef logistic_Feller_diffusion_process PROCESS;
```

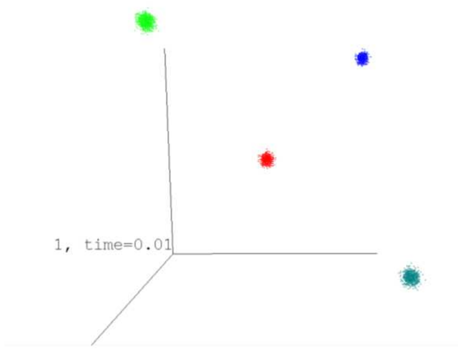
```
typedef double STATE_SPACE;

void main(int argv, char** argc)
{
simulation_pj_1D<PROCESS,STATE_SPACE>(argv,argc);
}
```

The same command lines as above could be used to compile and to run the program, with the only difference that the process file would be

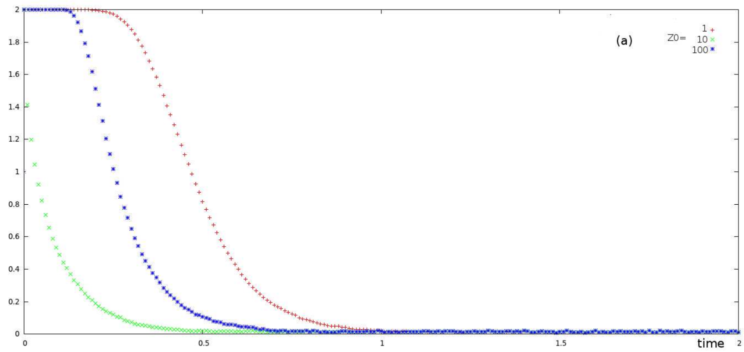
```
#############################
# Name of the process
#   logistic_Feller_diffusion_process
# Comment
#   Process used as an illustration
#############################

#############################
#
# Number of parameters
5
# maximal size of the time increment (dt)
0.001
# lower killing boundary
0.01
# upper killing boundary
100
# paremeter $r$
1
# parameter $c$
0.1
#
##############################
```

Other programs Other programs have been written during this thesis.

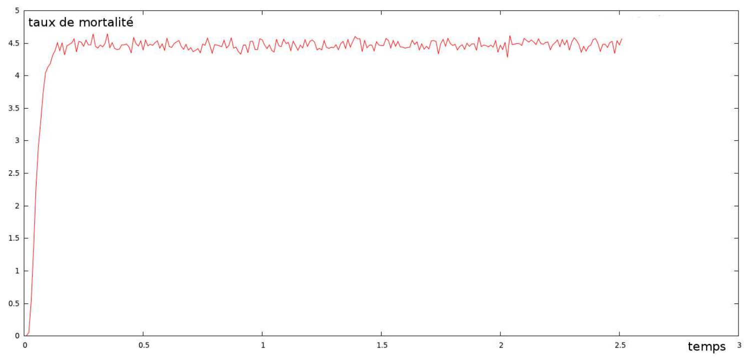
- A program which computes and represents graphically the animated particles of interacting particle system based on a 3-dimensional process $\mathcal{Z}$. Many interacting particle systems can be run simultaneously, starting from different positions. This program gives a great illustration of the convergence to the Yaglom limit, independently of the initial position. It uses the 3D-library Coin3D, which is based on OpenGL. Here is a screen shot of one simulation.



- A program to compute and represent graphically the evolution of the distance between the distributions of two conditioned pure jump Markov processes, using the approximation method studied in this thesis. This program can be used to illustrate the mixing property proved in chapter 5. It has been used to describe the convergence to the Yaglom limit of some Feller logistic diffusion processes in Chapter 1, with the following output.



- A program to compute and represent graphically the evolution of the absorption rate of an absorbed Markov process, using the interacting particle system studied in this thesis. It has been used in the introduction in order to compute the absorbing rate of a Brownian motion absorbed at 0 and 1, starting from 0.5.



Code source

The code source is too large to be reproduced here (more than 40 pages for a non-commented version of the code). However, latest versions of the whole project, pieces of code and functional programs can be obtained directly from the author, by e-mail or on his professional web page.

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