Solitons and large time asymptotics of solutions for the Novikov-Veselov equation

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Solitons and large time asymptotics of solutions for the Novikov-Veselov equation

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Contents

Abstract

This work is concerned with the study of the Novikov-Veselov equation, a $(2+1)$-dimensional analog of the renowned Korteweg-de Vries equation, integrable via the inverse scattering transform for the 2-dimensional stationary Schrödinger equation at a fixed energy. We start by studying a special class of rational nonsingular algebraically localized solutions of the Novikov-Veselov equation at positive energy constructed by Grinevich and Zakharov and we demonstrate that these solutions are multisolitons. Grinevich-Zakharov solutions are localized as $O\left(|x|^{-2}\right)$, $|x| \to \infty$, and in the present work we prove that this localization is almost the strongest possible : we show that the Novikov-Veselov equation at nonzero energy does not possess solitons localized stronger than $O\left(|x|^{-3}\right)$, $|x| \to \infty$. For the case of zero energy we show that if the solitons of the Novikov-Veselov equation belong to the range of solutions of the modified Novikov-Veselov equation under Miura transform, then localization stronger than $O\left(|x|^{-2}\right)$ is not possible. In the present work we also study the question of the asymptotic behavior of solutions to the Cauchy problem for the Novikov-Veselov equation at nonzero energy (for the case of positive energy transparent or reflectionless solutions are considered). Under assumption that the scattering data for the solutions are nonsingular we obtain that these solutions decrease uniformly with time as $O\left(t^{-1}\right)$, $t \to +\infty$, in the case of positive energy and as $O\left(t^{-3/4}\right)$, $t \to +\infty$, in the case of negative energy ; in the latter case we also demonstrate that the obtained estimate is optimal.

Résumé

Ce travail est consacré à l'étude de l'équation de Novikov-Veselov, un analogue $(2+1)$-dimensionnel de l'équation renommée de Korteweg-de Vries, intégrable via la transformée de la diffusion inverse pour l'équation de Schrödinger stationnaire en dimension 2 à énergie fixe. Nous commençons par étudier une classe spéciale de solutions rationnelles non singulières de l'équation de Novikov-Veselov à énergie positive, construites par Grinevich et Zakharov, et nous démontrons que ces solutions sont multisolitons. Les solutions de Grinevich-Zakharov sont localisées comme $O\left(|x|^{-2}\right)$, $|x| \to \infty$, et dans le travail présent, nous prouvons que cette localisation est presque la plus forte possible pour les solitons de l'équation de Novikov-Veselov : nous montrons que l'équation de Novikov-Veselov à énergie non nulle ne possède pas de solitons localisés plus fort que $O\left(|x|^{-3}\right)$, $|x| \to \infty$. Pour le cas d'énergie zéro, nous montrons que si les solitons de l'équation de Novikov-Veselov apartiennent à l'image des solutions de l'équation de Novikov-Veselov modifiée sous la transformation de Miura, dans ce cas, la localisation plus forte que $O\left(|x|^{-2}\right)$ n'est pas possible. Dans le travail présent, nous étudions également la question du comportement asymptotique des solutions du problème de Cauchy pour l'équation de Novikov-Veselov à énergie non nulle (pour le cas d'énergie positive, les solutions transparentes ou *reflectionless* sont considérées). Sous l'hypothèse de non singularité des données de diffusion des solutions nous obtenons que ces solutions décroissent avec le temps de façon uniforme comme $O\left(t^{-1}\right)$, $t \to +\infty$, dans le cas d'énergie positive et comme $O\left(t^{-3/4}\right)$, $t \to +\infty$, dans le cas d'énergie negative ; dans ce dernier cas, nous démontrons également que l'estimation obtenue est optimale.

Introduction

In the present work we study solitons and large time behavior of solutions of a $(2+1)$–dimensional analog of the renowned Korteweg–de Vries equation (KdV) proposed in 1984 by S.P. Novikov and A.P. Veselov and integrable via the inverse scattering transform for the stationary 2–dimensional Schrödinger equation at fixed energy.

The Korteweg-de Vries equation

$$v_t + 6vv_x + v_{xxx} = 0,$$

derived at the end of the XIX century as an equation describing long waves on shallow water, attracted a broad interest of researchers in the second half of the XX century, when in 1967 C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura discovered that this equation can be intergrated via the scattering transform for the stationary Schrödinger equation ([17]). It was demonstrated that if KdV is written in terms of the scattering data for the stationary Schrödinger equation with potential $v(x,t)$, depending on an additional parameter t, then KdV is transformed into an ensemble of noninteracting linear oscillators. Thus the problem of integrating a nonlinear differential equation was reduced to a sequence of linear problems ; in addition, it was shown that the scattering transform for the solution of the KdV equation is an analog of the Fourier transform for the solutions of linear differential equations.

The exact method of integration allowed to study in detail the asymptotic behavior of solutions for the KdV equation (see [18], [58], [65]). It was shown that any sufficiently localized solution of KdV behaves at large times as a sum of N solitons (corresponding to N eigenvalues of the Schrödinger operator with potential $v(x,0)$), i.e. a sum of N solitary waves that propagate preserving their form and restoring it after interaction.

The algebraic nature of integrability of KdV was revealed by P.D. Lax ([42]), who showed that KdV is equivalent to the following equation describing the evolution of operator $\mathcal{L}$

$$\frac{\partial \mathcal{L}}{\partial t} = [\mathcal{L}, A], \text{ where} \tag{0.1}$$

$$\mathcal{L} = -\partial_x^2 + v(x,t), \quad A = \partial_x^3 - \frac{3}{4}(v(x,t)\partial_x + \partial_x v), \quad [\cdot,\cdot] \text{ is the commutator} \tag{0.2}$$

(the so-called Lax representation). This equation can be regarded as a compatibility condition for the following equations

$$\mathcal{L}\psi = \lambda\psi, \tag{0.3}$$

$$\psi_t = A\psi \tag{0.4}$$

for all values of spectral parameter λ.

Representation of the form (0.1) was later obtained for a large class of well-known nonlinear equations, including the cubic Schrödinger equation, sine-Gordon equation, Kadomtsev-Petviashvili equation and others (see, for example, references in [22] and [52]). These results led to a rapid development of an exact integration method for a certain class of nonlinear differential equations, called the inverse scattering transform (IST) method. In particular, it was found out that solutions of a large number of equations, integrable via IST, have properties similar to those of KdV : for example many of them possess N-soliton solutions.

An attempt to generalize KdV to higher space dimensions leads to different analogs of this equation. Thus the renowned Kadomtsev-Petviashvili (KP) equation is obtained if transverse perturbations of the line soliton solution of KdV are taken into account ([31]). Zakharov-Kuznetsov equation describing the propagation of nonlinear ion-acoustic waves in the magnetic field in plasma (see [64]) reduces to KdV if the solution depends only on one space coordinate. However, from the mathematical point of view the most natural $(2+1)$-dimensional generalization of KdV is the Novikov-Veselov equation

$$\partial_t v = 4\mathrm{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w), \tag{0.5a}$$

$$\partial_{\bar z} w = -3\partial_z v, \quad v = \bar v, \text{ i.e. } v \text{ is a real-valued function}, \quad E \in \mathbb{R}, \tag{0.5b}$$

$$v = v(x,t), \quad w = w(x,t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \tag{0.5c}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right). \tag{0.6}$$

First of all, this equation reduces to the classical KdV when $v = v(x_1, t)$, $w = w(x_1, t)$. Further, it is integrable via the inverse scattering transform for the two-dimensional stationary Schrödinger equation at fixed energy E :

$$\begin{aligned} &L\psi = E\psi, \quad L = -\Delta + v(x,t), \\ &x \in \mathbb{R}^2, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad E = E_{fixed}. \end{aligned} \tag{0.7}$$

In addition, when $E \to \pm\infty$, the Novikov-Veselov equation transforms into KPI and KPII correspondingly (see, for example, [22]). Indeed, let us put $E = \pm\varkappa^2$. Assume that the solution varies slowly in x_2 or, more precisely, that $x_2 = \varkappa Y$. Let us take $w = -3v + \frac{6i}{\varkappa}\partial_{x_1}^{-1}\partial_Y v \mp 3\varkappa^2$ (then condition for w in (0.5b) holds up to a member of order $\frac{1}{E}$). Now tending $\varkappa \to \infty$ we obtain

$$\partial_t v - 2\partial_{x_1}^3 v + 12v\partial_{x_1} v = \mp 12\partial_{x_1}^{-1}\partial_Y^2 v.$$

Substitution $x_1 \to -x_1$, $v \to -v$ yields

$$\partial_t v + 2\partial_{x_1}^3 v + 12v\partial_{x_1} v = \pm 12\partial_{x_1}^{-1}\partial_Y^2 v.$$

The Novikov-Veselov equation can also be viewed as a natural $(2+1)$-dimensional generalization of the Korteweg-de Vries equation since it permits to extend to the two-dimensional case the Miura transform relating the solutions of KdV and modified KdV equations : namely, in [2] it was shown that a two-dimensional analog of the Miura transform maps solutions of the modified Novikov-Veselov equation (a $(2+1)$-dimensional analog of mKdV and a member of the Davey-Stewartson II integrable hierarchy) to the solutions of the Novikov-Veselov equation.

The Novikov-Veselov equation was first obtained in an implicit form in [43]. It was found out that representation (0.1) holds in the two-dimensional case only if $\mathcal{L}$ and A are operators with constant coefficients or if operator $\mathcal{L}$ is a differential operator of order not higher than one with respect to one of the variables ([43]). However, S.V. Manakov in [43] showed that in the two-dimensional case instead of representation (0.1) it is more appropriate to consider the following representation

$$\frac{\partial \mathcal{L}}{\partial t} = [\mathcal{L}, A] + B\mathcal{L} \tag{0.8}$$

(where B is a certain differential operator depending on A and $\mathcal{L}$) or, in other words, to seek a compatibility condition of equations (0.3)-(0.4) for one fixed value of λ.

For the particular case of the two-dimensional Schrödinger operator (0.7) operators A and

B satisfying (0.8) with $\mathcal{L} = L - E$, namely

$$\begin{aligned} A &= -8\partial_z^3 - 2w\partial_z - 8\partial_{\bar z}^3 - 2\bar w\partial_{\bar z}, \\ B &= 2\partial_{\bar z} w + 2\partial_{\bar z}\bar w, \end{aligned} \quad \text{where } w \text{ is defined via (0.5b)}, \tag{0.9}$$

were found by S.P. Novikov and A.P. Veselov in [53], [54], along with a hierarchy of evolution equations integrable via the inverse scattering method for the two-dimensional Schrödinger equation at a fixed energy. This hierarchy includes, in particular, equation (0.5). Note also that the first-order differential equation from this hierarchy, the dispersionless Novikov-Veselov equation, was derived in a model of nonlinear geometrical optics ([39]).

The scattering theory for the two-dimensional Schrödinger equation was built in the series of works by P.G. Grinevich, S.V. Manakov, R.G. Novikov, S.P. Novikov (see, for example, [21–23, 25–27, 47–50]). Important contributions for the case of zero energy were made by M. Boiti et al in [3], T.-Y. Tsai in [60], A. I. Nachman in [46]. The development of this theory made it possible to study the asymptotic behavior of the Novikov-Veselov equation in detail.

The first rational solutions for the Novikov-Veselov equation on the positive energy level E were constructed in [21]. These solutions are, de facto, multisolitons (see [37]) decreasing as $O\left(\frac{1}{|x|^2}\right)$ as $|x| \to \infty$. In [51] it was shown that, similarly to the one-dimensional case, the solitons of the Novikov-Veselov equation at positive energy are transparent potentials. On the other hand the same paper demonstrated an essential feature which differs the Novikov-Veselov equation from its $(1+1)$-dimensional counterpart : the Novikov-Veselov equation at $E > 0$ does not possess exponentially localized solitons.

It is necessary to note that, despite its evident mathematical significance, the Novikov-Veselov equation has been relatively poorly studied in comparison with several other analogs of the renowned KdV equation. The aim of the present work is to attempt to fill in partially this gap.

In the present work we consider solutions (v, w) of the Novikov-Veselov equation (0.5) such that

$$\begin{aligned} &v \text{ is sufficiently regular and has sufficient decay as } |x| \to \infty, \\ &w \text{ is decaying as } |x| \to \infty. \end{aligned}$$

According to an established tradition in the theory of integrable nonlinear equations one of our main concerns is devoted to the question of existence or absence of soliton solutions for the Novikov-Veselov equation.

Definition. *We say that a solution* (v, w) *of (0.5) is a soliton (a traveling wave) iff*

$$v(x,t) = V(x - ct), \quad w(x,t) = W(x - ct), \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}, \tag{0.10}$$

for some functions V, W *on* $\mathbb{R}^2$ *and some velocity* $c \in \mathbb{R}^2$. *In addition, we identify* $c = (c_1, c_2) \in \mathbb{R}^2$ *with* $c = c_1 + ic_2 \in \mathbb{C}$.

The present text is organized as follows. In Sections 1, 2 we recall some known notions and statements from the theory of direct scattering for the two-dimensional stationary Schrödinger equation. For the case of the Schrödinger equation at nonzero energy we introduce, in particular, its classical and Faddeev generalized scattering data and we present briefly the scheme of reconstruction of potential from these data. For the case of the Schrödinger equation at zero energy we introduce, in particular, its Faddeev scattering data and discuss their properties when the potential in the Schrödinger equation is of the so-called conductivity type.

We begin our studies of the Novikov-Veselov equation in Section 3 by analyzing the family of solutions for equation (0.5) at positive energy constructed by P. G. Grinevich and V. E. Zakharov (see [21, 22]). We show that this family contains soliton solutions and we find the range of possible soliton velocities. Further, we demonstrate that each Grinevich-Zakharov solution behaves at large times as a finite sum of localized traveling waves. The results of this Section are based on our publication [37]. Note that for KPI equation a prototype of these results was given in [45]. The reader may also refer to [9] for another class of rational nonsingular soliton solutions of the Novikov-Veselov equation at positive energy and to [24] for analogs of multisoliton solutions. For the Novikov-Veselov equation at zero energy an example of rational nonsingular soliton solution, localized as $O(|x|^{-2})$ as $|x| \to \infty$, was given in [6].

As it was mentioned above, Grinevich-Zakharov solitons are rational functions localized as $O\left(\frac{1}{|x|^2}\right)$, $|x| \to \infty$. In the present work we show that this is almost the strongest localization that one can obtain for the soliton solutions of the Novikov-Veselov equation. More precisely, in Section 4 we demonstrate that the Novikov-Veselov equation at nonzero energy does not admit sufficiently regular solitons localized as $O\left(\frac{1}{|x|^{3+\varepsilon}}\right)$, $|x| \to \infty$, for any $\varepsilon > 0$. This result was presented in our paper [33] and is a continuation of work [51] and our paper [36] where, via a different method of analysis, the absence of exponentially localized soliton solutions for (0.5) at $E \neq 0$ was proved.

For the Novikov-Veselov equation at zero energy there exists a special interesting class of solutions, the so-called solutions of conductivity type. Potentials of conductivity type arise naturally when Calderón conductivity problem is studied via the Gelfand inverse problem for the stationary Schrödinger equation at zero energy ([5, 19, 46, 57]). Solutions of equation (0.5) of conductivity type are the image of solutions of the modified Novikov-Veselov equation under the Miura transformation (see [55]). In Section 5 we show that the Novikov-Veselov equation does not possess sufficiently localized and regular solitons of conductivity type. This result was presented in our paper [34].

Note that KPI equation possesses localized soliton solutions and these solutions decay as $O\left(|x|^{-2}\right)$ when $|x| \to \infty$. By contrast, KPII does not possess sufficiently localized soliton solutions. For the results on existence and nonexistence of localized soliton solutions of KPI, KPII and their generalized versions see [7]; the symmetry properties and the decay rates of these solutions were derived in [8].

For more results on integrable $(2+1)$-dimensional systems admitting localized soliton solutions, see [1, 4, 13, 14] and references therein. In particular, for more results on the behavior of solutions for the Novikov-Veselov equation at zero energy see [6, 40, 55, 59].

In Sections 6, 7 we study the large time asymptotics of solutions to the Novikov-Veselov equation constructed from the scattering data that are everywhere well-defined, sufficiently regular and localized. It is shown that such solutions decrease uniformly with time; the velocity of this decrease is estimated. Results of Sections 6, 7 imply, in particular, that solitons of the Novikov-Veselov equation may only arise if the scattering data of the associated inverse problem have singularities. The results of these Sections were published in our articles [32, 35]. Note that a similar problem for the Kadomtsev-Petviashvili equation was studied in [28, 38, 44].

Chapter I

Scattering for the stationary Schrödinger equation on the plane

1 Direct scattering for the two-dimensional Schrödinger equation at $E \neq 0$

1.1 Classical scattering functions, scattering amplitude and Faddeev exponentially growing solutions

In the present Section we present some known notions and results from the scattering theory for the two-dimensional Schrödinger equation at nonzero energy, summarized, for instance, in [22, 25, 27, 49].

We are concerned with the the two-dimensional Schrödinger equation at fixed nonzero energy

$$
\begin{aligned}
L\psi = E\psi, \quad E = E_{fixed} \in \mathbb{R}\backslash 0,\\
L = -\Delta + v, \quad \Delta = 4\partial_z\partial_{\bar z}, \quad v = v(x), \quad x \in \mathbb{R}^2,
\end{aligned}
\tag{1.1}
$$

where ∂_z, $\partial_{\bar z}$ are defined in (0.6), with a potential v satisfying the following conditions

$$v(x) = \overline{v(x)}, \quad v(x) \in L^\infty(\mathbb{R}^2), \tag{1.2a}$$

$$|\partial_{x_1}^{j_1}\partial_{x_2}^{j_2} v(x)| < q(1+|x|)^{-2-\varepsilon} \text{ for some } q > 0,\ \varepsilon > 0,\ \text{where } j_1, j_2 \in \mathbb{N}\cup 0,\ j_1 + j_2 \leqslant 3. \tag{1.2b}$$

It is known that for any $k \in \mathbb{R}^2$, $k^2 = E > 0$, there exists a unique continuous solution $\psi^+(x,k)$ of (1.1) with $E > 0$ with the following asymptotics

$$\psi^+(x,k) = e^{ikx} - i\pi\sqrt{2\pi}e^{-\frac{i\pi}{4}} f\left(k, |k|\frac{x}{|x|}\right)\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{\sqrt{|x|}}\right), \quad |x| \to \infty, \tag{1.3}$$

with some a priori unknown function f. This solution $\psi^+(x,k)$ describes the scattering of an incident plane wave e^{ikx} on the potential $v(x)$ and is the classical scattering eigenfunction of equation (1.1) at $E>0$. Function $f=f(k,l)$, $k\in\mathbb{R}^2$, $l\in\mathbb{R}^2$, $k^2=l^2=E$, arising in (1.3), is the scattering amplitude for $v(x)$ in the framework of equation (1.1).

In the present work a particular attention is devoted to transparent potentials v for equation (1.1). We say that v is transparent if its scattering amplitude f is identically zero at fixed energy E. The notion of transparency is closely related to the notion of invisibility and, in fact, goes back to [56]. We consider transparent potentials for equation (1.1) as analogs of reflectionless potentials for the one–dimensional Schrödinger equation at all positive energies; see, for example, [58], [65] as regards reflectionless potentials in dimension one.

In [25] in connection with transparent potentials it was shown that

1. There are no nonzero transparent potentials for equation (1.1), where

$$v(x)=\overline{v(x)},\quad v\in L^\infty(\mathbb{R}^2),\quad |v(x)|<\alpha e^{\beta|x|},\quad \alpha>0,\ \beta>0.$$

2. There is a family of nonzero transparent potentials for equation (1.1), where

$$v(x)=\overline{v(x)},\quad v\in\mathcal{S}(\mathbb{R}^2),$$

   and $\mathcal{S}$ denotes the Schwartz class.

In [51] it was shown that

3. Solitons of the Novikov-Veselov equation at $E>0$ are transparent potentials (see also Lemma 4.3).

For $k\in\Sigma_E$, where

$$\Sigma_E=\{k\in\mathbb{C}^2\colon k^2=E,\ \mathrm{Im}k\neq 0\},\ \text{if } E>0,$$
$$\Sigma_E=\{k\in\mathbb{C}^2\colon k^2=E\},\ \text{if } E<0,$$

we consider Faddeev eigenfunctions $\psi(x,k)$ of equation (1.1) with $E\in\mathbb{R}\backslash 0$ specified by the following asymptotics

$$\psi(x,k)=e^{ikx}(1+o(1)),\quad |x|\to\infty \tag{1.4}$$

(see [10], [47], [22]).

If potential v in equation (1.1) satisfies condition

$$|\partial_{x_1}^{j_1}\partial_{x_2}^{j_2}v(x)| < q(1+|x|)^{-3-\varepsilon} \text{ for some } q>0,\ \varepsilon>0,\ \text{where } j_1, j_2 \in \mathbb{N}\cup 0,\ j_1+j_2 \leqslant 3, \quad (1.5)$$

then we will also consider eigenfunctions $\varphi(x,k)$ of equation (1.1) with $E \in \mathbb{R}\backslash 0$, defined for $k \in \Sigma_E$ and specified by

$$\varphi(x,k) = e^{ikx}(k_1x_2 - k_2x_1 + o(1)), \quad |x| \to \infty. \quad (1.6)$$

These functions are the analogs of solutions of (1.1) introduced in [3] for the case of zero energy.

Note that the classical scattering function $\psi^+(x,k)$ is the solution of the Lippmann-Schwinger integral equation

$$\psi^+(x,k) = e^{ikx} + \iint\limits_{\mathbb{R}^2} G^+(x-y,|k|)v(y)\psi^+(y,k)dy, \quad (1.7)$$

where

$$G^+(x,|k|) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{R}^2} \frac{e^{i\zeta x}}{k^2-\zeta^2+i0}d\zeta, \quad k \in \mathbb{R}^2.$$

It is known that

for the real-valued potential v satisfying (1.2)
equation (1.7) is uniquely solvable for $k \in \mathbb{R}^2$, $|k| \neq 0$ (1.8)

(see, for example, [11] and references therein).

Solutions $\psi(x,k)$, $\varphi(x,k)$ of (1.1) with asymptotics (1.4), (1.6) can be defined as solutions of the following integral equations

$$\psi(x,k) = e^{ikx} + \iint\limits_{\mathbb{R}^2} G(x-y,k)v(y)\psi(y,k)dy, \quad (1.9)$$

$$\varphi(x,k) = e^{ikx}(k_1x_2 - k_2x_1) + \iint\limits_{\mathbb{R}^2} G(x-y,k)v(y)\varphi(y,k)dy, \quad (1.10)$$

where

$$G(x,k) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{R}^2} \frac{e^{i\zeta x}}{k^2-\zeta^2}d\zeta, \quad k \in \Sigma_E. \quad (1.11)$$

In terms of functions

$$m(x,k) = (1+|x|)^{-(2+\varepsilon/2)}e^{-ikx}\psi(x,k), \quad n(x,k) = (1+|x|)^{-(2+\varepsilon/2)}e^{-ikx}\varphi(x,k)$$

equations (1.9) and (1.10) can be written

$$m(x,k) = (1+|x|)^{-(2+\varepsilon/2)} + \\ + \iint\limits_{\mathbb{R}^2} (1+|x|)^{-(2+\varepsilon/2)} g(x-y,k) \frac{v(y)}{(1+|y|)^{-(2+\varepsilon/2)}} m(y,k)dy, \quad (1.12)$$

$$n(x,k) = (k_1x_2 - k_2x_1)(1+|x|)^{-(2+\varepsilon/2)} + \\ + \iint\limits_{\mathbb{R}^2} (1+|x|)^{-(2+\varepsilon/2)} g(x-y,k) \frac{v(y)}{(1+|y|)^{-(2+\varepsilon/2)}} n(y,k)dy, \quad (1.13)$$

where

$$g(x,k) = -\left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{R}^2} \frac{e^{i\zeta x}}{\zeta^2 + 2k\zeta} d\zeta,$$

and $x \in \mathbb{R}^2$, $k \in \Sigma_E$.

The integral operator $H(k)$ of the integral equations (1.12), (1.13) is a Hilbert-Schmidt operator : more precisely, $H(\cdot,\cdot,k) \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$, where $H(x,y,k)$ is the Schwartz kernel of the integral operator $H(k)$, and $|\mathrm{Tr} H^2(k)| < \infty$. Thus, the modified Fredholm determinant $\Delta(k)$ for (1.12) and (1.13) can be defined by means of the formula :

$$\ln \Delta(k) = \mathrm{Tr}(\ln(I - H(k)) + H(k)). \quad (1.14)$$

For the precise sense of this definition see [20]. Considerations of Δ go back to [11].

Note that for $\mathrm{Im} k = 0$ formulas (1.9)–(1.11) do not make sense ; however the following limits can be considered :

$$G_\gamma(x,k) = G(x, k + i0\gamma), \quad \psi_\gamma(x,k) = \psi(x, k+i0\gamma) \quad \Delta_\gamma(k) = \Delta(k + i0\gamma), \quad (1.15)$$

where $k, \gamma \in \mathbb{R}^2$, $k^2 = E$, $\gamma^2 = 1$. In particular, in [11] the following relations were established

$$G_{k/|k|}(x,k) = G^+(x,|k|), \quad \psi_{k/|k|}(x,k) = \psi^+(x,k), \quad \Delta_{k/|k|}(k) = \Delta^+(|k|), \quad (1.16)$$

where $\Delta^+(|k|)$ is the modified Fredholm determinant for equation (1.12) with $g(x,k)$ replaced by $g^+(x,k) = e^{-ikx} G^+(x,|k|)$ (the modified Fredholm determinant for the Lippmann-Schwinger equation).

Define

$$h(k,l) = \iint\limits_{\mathbb{R}^2} e^{-ilx}\psi(x,k)v(x)dx,$$

where $k, l \in \mathbb{C}^2$, $\mathrm{Im}k = \mathrm{Im}l \neq 0$, $k^2 = l^2 = E$. Then in [11] it was shown that $h_\gamma(k,l) = h(k+i0\gamma, l+i0\gamma)$ satisfies the following integral equation

$$h_\gamma(k,l) = (2\pi)^2 f(k,l) + 2\pi i \iint\limits_{\mathbb{R}^2} h_\gamma(k,m)\theta((m-k,\gamma))\delta(m^2-l^2)f(m,l)dm, \tag{1.17}$$

where f is the scattering amplitude of potential v in the framework of equation (1.1). Denote $\tilde{\Delta}_\gamma(k)$ the Fredholm determinant for equation (1.17) defined by

$$\ln \tilde{\Delta}_\gamma(k) = \mathrm{Tr}(\ln(I - \tilde{H}_\gamma(k))),$$

where $\tilde{H}_\gamma$ is the integral operator of (1.17). In [11] the following important formula relating $\Delta_\gamma(k)$, $\Delta^+(|k|)$ and $\tilde{\Delta}_\gamma(k)$ was established :

$$\Delta_\gamma(k) = \Delta^+(|k|)\tilde{\Delta}_\gamma(k). \tag{1.18}$$

In what follows it will be convenient for us to introduce the following new variables :

$$z = x_1 + ix_2, \quad \lambda = \frac{k_1 + ik_2}{\sqrt{E}}.$$

Note that $k_1 = \frac{\sqrt{E}}{2}\left(\lambda + \frac{1}{\lambda}\right)$, $k_2 = \frac{i\sqrt{E}}{2}\left(\frac{1}{\lambda} - \lambda\right)$.

In the new variables $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$ functions ψ and φ are solutions of (1.1) with the following asymptotic behavior

$$\psi(z,\lambda) = e^{\frac{i\sqrt{E}}{2}(\lambda\bar{z} + z/\lambda)}\mu(z,\lambda), \quad \mu(z,\lambda) = 1 + o(1), \text{ as } |z| \to \infty, \tag{1.19}$$

$$\varphi(z,\lambda) = e^{\frac{i\sqrt{E}}{2}(\lambda\bar{z} + z/\lambda)}\nu(z,\lambda), \quad \nu(z,\lambda) = \frac{i\sqrt{E}}{2}\left(\lambda\bar{z} - \frac{1}{\lambda}z\right) + o(1), \text{ as } |z| \to \infty. \tag{1.20}$$

Functions $\mu(z,\lambda)$ and $\nu(z,\lambda)$ arising in the above formulas can also be defined as solutions of the following integral equations

$$\mu(z,\lambda) = 1 + \iint\limits_{\mathbb{C}} g(z-\zeta,\lambda)v(\zeta)\mu(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{1.21}$$

$$\nu(z,\lambda) = \frac{i\sqrt{E}}{2}\left(\lambda\bar{z} - \frac{1}{\lambda}z\right) + \iint\limits_{\mathbb{C}} g(z-\zeta,\lambda)v(\zeta)\nu(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \text{ where} \tag{1.22}$$

$$g(z,\lambda) = -\left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{C}} \frac{e^{\frac{i}{2}(p\bar{z}+\bar{p}z)}}{p\bar{p} + \sqrt{E}(\lambda\bar{p} + p/\lambda)} d\mathrm{Re}p d\mathrm{Im}p, \tag{1.23}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$ and if $E > 0$, then $|\lambda| \neq 1$.

In terms of $m(z,\lambda) = (1+|z|)^{-(2+\varepsilon/2)}\mu(z,\lambda)$ and $n(z,\lambda) = (1+|z|)^{-(2+\varepsilon/2)}\nu(z,\lambda)$ equations (1.21) and (1.22), respectively, take the forms

$$m(z,\lambda) = (1+|z|)^{-(2+\varepsilon/2)} + \\ + \iint\limits_{\mathbb{C}} (1+|z|)^{-(2+\varepsilon/2)} g(z-\zeta,\lambda) \frac{v(\zeta)}{(1+|\zeta|)^{-(2+\varepsilon/2)}} m(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{1.24}$$

$$n(z,\lambda) = \frac{i\sqrt{E}}{2}\left(\lambda\bar{z} - \frac{1}{\lambda}z\right)(1+|z|)^{-(2+\varepsilon/2)} + \\ + \iint\limits_{\mathbb{C}} (1+|z|)^{-(2+\varepsilon/2)} g(z-\zeta,\lambda) \frac{v(\zeta)}{(1+|\zeta|)^{-(2+\varepsilon/2)}} n(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \tag{1.25}$$

We will denote by $\Delta(\lambda)$ the modified Fredholm determinant for equations (1.24), (1.25) defined via

$$\ln \Delta(\lambda) = \mathrm{Tr}(\ln(I - H(\lambda)) + H(\lambda)), \tag{1.26}$$

where $H(\lambda)$ is the Hilbert-Schmidt integral operator of equations (1.24), (1.25).

We will also define

$$\mathcal{E} = \{\lambda \in \Sigma \colon \Delta(\lambda) = 0\}, \tag{1.27}$$

where

$$\Sigma = \mathbb{C}\backslash(0 \cup T) \text{ if } E > 0 \quad \text{and} \quad \Sigma = \mathbb{C}\backslash 0 \text{ if } E < 0, \quad T = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}.$$

In this notation $\mathcal{E}$ represents the set of λ for which either the existence or the uniqueness of the solution of (1.1) with asymptotics (1.19) (or, similarly, of the solution of (1.1) with asymptotics (1.20)) fails. The theory of inverse scattering has been well developed under the assumption that $\mathcal{E} = \varnothing$, i.e. Faddeev eigenfunctions have no singularities. However, this is not a typical situation. In [27] it was shown that at negative energy E above the ground state one can expect contour singularities in the complex plane of spectral parameter. In [26] explicit examples of potentials and related Faddeev eigenfunctions having contour singularities in spectral parameter were constructed. Note also that if v is sufficiently small, then $\mathcal{E} = \varnothing$ and, thus, equations of direct scattering are everywhere uniquely solvable.

1.2 Faddeev scattering data and their properties

For $\lambda \in \mathbb{C} \backslash (\mathcal{E} \cup 0)$ we define the following "scattering data" for the potential v :

$$a(\lambda) = \iint\limits_{\mathbb{C}} v(\zeta)\mu(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{1.28}$$

$$b(\lambda) = \iint\limits_{\mathbb{C}} \exp\left(\frac{i\sqrt{E}}{2}\left(1 + (\mathrm{sgn}E)\frac{1}{\lambda\bar{\lambda}}\right)((\mathrm{sgn}E)\zeta\bar{\lambda} + \lambda\bar{\zeta})\right) v(\zeta)\mu(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{1.29}$$

$$\alpha(\lambda) = \iint\limits_{\mathbb{C}} v(\zeta)\nu(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{1.30}$$

$$\beta(\lambda) = \iint\limits_{\mathbb{C}} \exp\left(\frac{i\sqrt{E}}{2}\left(1 + (\mathrm{sgn}E)\frac{1}{\lambda\bar{\lambda}}\right)((\mathrm{sgn}E)\zeta\bar{\lambda} + \lambda\bar{\zeta})\right) v(\zeta)\nu(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \tag{1.31}$$

Functions a, b are Faddeev "scattering data" for the 2-dimensional Schrödinger equation. They also arise in a more precise version of expansion (1.4).

For $E > 0$ function b is considered to be scattering data for (1.1) in addition to the scattering amplitude f arising in (1.3). Suppose that $\mathcal{E} = \varnothing$. Then it was shown in [23, 47, 49] that at fixed positive energy f and b uniquely determine the potential v satisfying (1.2) (while f alone is insufficient for this purpose).

For $E < 0$ function b alone constitutes the scattering data for (1.1) (recall, that for $E < 0$ the scattering amplitude is not defined). If $\mathcal{E} = \varnothing$, then b determines potential v satisfying (1.2) uniquely (see [27]).

The "scattering data" α, β for the case of the Schrödinger equation at zero energy were introduced in [3]. We will use them in Section 4 to prove the absence of sufficiently localized solitons for equation (0.5).

Consider the following function

$$h(\lambda, \lambda') = \iint\limits_{\mathbb{C}} \exp\left(-\frac{i\sqrt{E}}{2}(\lambda'\bar{z} + \frac{z}{\lambda'})\right) \psi(z,\lambda) v(z) d\mathrm{Re}z d\mathrm{Im}z \tag{1.32}$$

for $\lambda \in \mathbb{C}\backslash(\mathcal{E} \cup 0)$, $\lambda' \in \mathbb{C}\backslash 0$. Under assumption (1.2) the integral in (1.32) is well-defined if $\lambda' = \lambda$ or if $\lambda' = 1/\bar{\lambda}$, but is not well-defined in general. Note that if $\lambda \in \mathbb{C}\backslash(\mathcal{E} \cup 0)$, then formulas (1.28), (1.29) can be written in the following form

$$a(\lambda) = h(\lambda, \lambda), \quad b(\lambda) = h\left(\lambda, \frac{1}{\bar{\lambda}}\right). \tag{1.33}$$

For $E > 0$ and $|\lambda| = |\lambda'| = 1$ denote

$$h_{\pm}(\lambda, \lambda') = \iint\limits_{\mathbb{C}} \exp\left(\frac{-i\sqrt{E}}{2}(\lambda'\bar{z} + \frac{z}{\lambda'})\right) \psi(z,\lambda(1\mp 0)) v(z) d\mathrm{Re}z d\mathrm{Im}z. \tag{1.34}$$

In [49] it was shown, in particular, that functions $h_{\pm}$ satisfy the following integral equations

$$h_+(\lambda,\lambda') - \pi i \int\limits_{|\lambda''|=1} h_+(\lambda,\lambda'')\theta\left[+\frac{1}{i}\left(\frac{\lambda''}{\lambda} - \frac{\lambda}{\lambda''}\right)\right] f(\lambda'',\lambda')|d\lambda''| = (2\pi)^2 f(\lambda,\lambda'), \tag{1.35}$$

$$h_-(\lambda,\lambda') - \pi i \int\limits_{|\lambda''|=1} h_-(\lambda,\lambda'')\theta\left[-\frac{1}{i}\left(\frac{\lambda''}{\lambda} - \frac{\lambda}{\lambda''}\right)\right] f(\lambda'',\lambda')|d\lambda''| = (2\pi)^2 f(\lambda,\lambda'), \tag{1.36}$$

where $\theta(\cdot)$ is the Heaviside function.

Now we formulate some properties of the introduced functions that will play a substantial role in the proof of our results. In what follows we denote by T the unit circle on the complex plane :

$$T = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}.$$

Statement 1.1 (see [29, 36, 49]). *Let v satisfy conditions (1.2). Then function $\Delta(\lambda)$ satisfies the following properties :*

1. *$\Delta \in C(\bar{D}_+)$, $\Delta \in C(\bar{D}_-)$, where $\bar{D}_+ = D_+ \cup \partial D_+$, $D_+ = \{\lambda \in \mathbb{C} \colon |\lambda| < 1\}$, $\bar{D}_- = D_- \cup \partial D_-$, $D_- = \{\lambda \in \mathbb{C} \colon |\lambda| > 1\}$;*

   if $E < 0$ or if v is transparent at $E > 0$, then $\Delta \in C(\mathbb{C})$;
2. *$\Delta(\lambda) \to 1$ as $|\lambda| \to \infty$, $|\lambda| \to 0$;*
3. *Δ is real-valued ;*

4. *$\Delta(\lambda)$ satisfies the following $\bar\partial$-equation*

$$\frac{\partial \Delta}{\partial \bar\lambda} = -\frac{\operatorname{sgn}(\lambda\bar\lambda - 1)}{4\pi\bar\lambda}\left(a\left(-(\operatorname{sgn}E)\frac{1}{\bar\lambda}\right) - \hat v(0)\right)\Delta, \tag{1.37}$$

where $\hat v(0) = \iint\limits_{\mathbb{C}} v(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta$, $\lambda \in \mathbb{C}\backslash(T \cup \mathcal{E} \cup 0)$ *;*

5. $\Delta(\lambda) = \Delta\left(-(\operatorname{sgn}E)\frac{1}{\bar\lambda}\right)$, $\lambda \in \mathbb{C}\backslash 0$ *;*

6. *if* $E < 0$, *then* $\Delta(\lambda) \equiv \mathrm{const}$ *for* $\lambda \in T$ *and if* $E > 0$, *then* $\Delta(\lambda(1 \mp 0)) = \Delta_{\pm n_\perp}(k)$ *for* $\lambda \in T$, *where* $k \in \mathbb{R}^2$, $k^2 = E$, $\lambda = \frac{k_1 + ik_2}{\sqrt{E}}$, $n_\perp = \frac{(-k_2, k_1)}{|k|}$ *and* $\Delta_\gamma(k)$, *for* $\gamma \in \mathbb{R}^2$, $\gamma^2 = 1$, *was introduced in (1.15).*

Note that $\Delta \notin C(\mathbb{C})$ for $E > 0$, in general. In this case Δ on $\bar D_\pm$ is considered as an extension from $D_\pm$.

Items 1–3 of Statement 1.1 are either known or follow from results mentioned in [29], [49] (see page 129 of [29] and pages 419, 420, 423, 429 of [49]). In particular, item 1 of Statement 1.1 is a consequence of continuous dependency of $g(z,\lambda)$ from (1.23) on λ, $\lambda \in \mathbb{C}\backslash(0 \cup T)$ if $E > 0$ and $\lambda \in \mathbb{C}\backslash 0$ if $E < 0$. In addition, item 5 of Statement 1.1 follows from item 3 of this statement and from symmetry $\overline{G(z,\lambda)} = G(z, -\operatorname{sgn}E/\bar\lambda)$, $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$, where $G(z,\lambda) = e^{\frac{i\sqrt{E}}{2}(\lambda\bar z + z/\lambda)} g(z,\lambda)$. Item 6 of Statement 1.1 for $E < 0$ is a consequence of (1.26) and of the formula (see pages 420, 423 of [49]) $G(z,\lambda) = (-i/4)H_0^1(\sqrt{E}|z|)$, $z \in \mathbb{C}$, $\lambda \in T$, where G is defined above and H_0^1 is the Hankel function of the first type. For $E > 0$ item 6 of Statement 1.1 is a consequence of the property (see page 419 of [49]) $G(z, \lambda(1 \mp 0)) = G_{\pm n_\perp}(x,k)$, where $G_\gamma(x,k)$ is defined in (1.15), $z = x_1 + ix_2$ and λ, k, $n_\perp$ are related as in item 6 of Statement 1.1.

Statement 1.2 (see [22, 25, 27, 29, 49])**.** *Let v satisfy conditions (1.2) and if $E > 0$ let v be transparent at given energy E. Then*

1. *$\mu(z,\lambda)$ is a continuous function of λ on $\mathbb{C}\backslash\mathcal{E}$;*

2. *$\mu(\lambda)$ satisfies the following $\bar\partial$-equation on $\lambda \in \mathbb{C}\backslash(0 \cup T \cup \mathcal{E})$:*

$$\frac{\partial \mu(z,\lambda)}{\partial \bar\lambda} = r(z,\lambda)\overline{\mu(z,\lambda)}, \quad \textit{where} \tag{1.38a}$$

$$r(z,\lambda) = r(\lambda)\exp\left\{\frac{-i\sqrt{E}}{2}\left(1 + (\operatorname{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\operatorname{sgn}E)z\bar\lambda + \bar z\lambda\right)\right\}, \tag{1.38b}$$

$$r(\lambda) = \frac{1}{4\pi\bar\lambda}\operatorname{sgn}(\lambda\bar\lambda - 1)b(\lambda); \tag{1.38c}$$

3. *$\mu \to 1$, as $\lambda \to 0$, $\lambda \to \infty$;*

4. $\mu(z,\lambda) = \overline{\mu\left(z, -(\mathrm{sgn}E)\frac{1}{\bar\lambda}\right)}$;

5. *the scattering data* $a(\lambda)$, $b(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$ *of* (1.28)–(1.31) *are continuous on* $\mathbb{C}\backslash\mathcal{E}$;

6. *if, in addition,* $v \in \mathcal{S}(\mathbb{R}^2)$*, where* $\mathcal{S}$ *denotes the Schwartz class, then the scattering data* $a(\lambda)$, $b(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$ *of* (1.28)–(1.31) *belong to* $\mathcal{S}(\mathbb{C}\backslash\mathcal{E})$;

7. $\hat v(0) = \lim\limits_{\lambda\to 0,\infty} a(\lambda)$*, where* $\hat v(0) = \iint\limits_{\mathbb{C}} v(z) d\mathrm{Re}z d\mathrm{Im}z$;

8. *for the scattering data* b *the following symmetries hold*

$$b\left(\mathrm{sgn}E\frac{1}{\bar\lambda}\right) = b(\lambda), \quad b\left(-\mathrm{sgn}E\frac{1}{\bar\lambda}\right) = \overline{b(\lambda)}, \quad \lambda\in\mathbb{C}\backslash 0;$$

9. *if* $E > 0$*, then*

$$\partial_\lambda^m \partial_{\bar\lambda}^n b(\lambda)|_{|\lambda|=1} = 0 \text{ for all } m,n > 0;$$

10. *if* $E < 0$*, then* $a(\lambda) = b(\lambda)$ *for* $\lambda \in T\backslash\mathcal{E}$;

11. $a(\lambda)$ *satisfies the following* $\bar\partial$*-equation on* $\lambda\in\mathbb{C}\backslash(0\cup T\cup\mathcal{E})$:

$$\frac{\partial a(\lambda)}{\partial\bar\lambda} = \frac{1}{4\pi\bar\lambda}\mathrm{sgn}(\lambda\bar\lambda - 1) b(\lambda)\overline{b(\lambda)}.$$

Note that if $\mathcal{E}\neq 0$, then properties 1, 2, 3 of Statement 1.2 define function $\mu(z,\lambda)$ from the nonsingular scattering data $b(\lambda)$ uniquely. Function μ can be reconstructed as the solution of the following integral equation

$$\mu(z,\lambda) = 1 - \frac{1}{\pi}\iint\limits_{\mathbb{C}} r(z,\zeta)\overline{\mu(z,\zeta)}\frac{d\mathrm{Re}\zeta d\mathrm{Im}\zeta}{\zeta-\lambda}. \tag{1.39}$$

Then potential v (transparent in the case of positive energy E) can be found from the following formula

$$v(z) = 2i\sqrt{E}\frac{\partial\mu_{-1}(z)}{\partial z}, \tag{1.40}$$

where $\mu_{-1}(z)$ is defined via the following expansion

$$\mu(z,\lambda) = 1 + \frac{\mu_{-1}(z)}{\lambda} + o\left(\frac{1}{|\lambda|}\right), \text{ as } \lambda\to\infty. \tag{1.41}$$

2 Direct scattering for the two-dimensional Schrödinger equation at $E=0$

In the present Section we present some notions from the scattering theory for the two-dimensional Schrödinger equation at zero energy introduced in [3] and we state some results of this theory obtained in [46] and concerning the conductivity type potentials. For a further discussion and results on the zero energy case the reader may refer to [27, 40, 41, 60].

We are concerned with the the two-dimensional Schrödinger equation at fixed zero energy

$$\begin{aligned} &L\psi = 0, \\ &L = -\Delta + v, \quad \Delta = 4\partial_z\partial_{\bar z}, \quad v = v(x), \quad x \in \mathbb{R}^2, \end{aligned} \tag{2.1}$$

where ∂_z, $\partial_{\bar z}$ are defined in (0.6), with a potential v satisfying conditions (1.2).

For $\lambda \in \mathbb{C}$ we consider solution $\psi(z,\lambda)$ of (2.1) having the following asymptotics

$$\psi(z,\lambda) = e^{i\lambda z}\mu(z,\lambda), \quad \mu(z,\lambda) = 1 + \overline{o}(1), \text{ as } |z| \to \infty. \tag{2.2}$$

Solution of (2.1) with asymptotics (2.2) is a Faddeev exponentially growing solution (see [3, 10]).

Function $\mu(z,\lambda)$, defined by (2.2), can be also represented as the solution of the following integral equation

$$\mu(z,\lambda) = 1 + \iint\limits_{\mathbb{C}} g(z-\xi,\lambda)v(\xi)\mu(\xi,\lambda)d\mathrm{Re}\xi d\mathrm{Im}\xi, \text{ where} \tag{2.3}$$

$$g(z,\lambda) = -\left(\frac{1}{4\pi}\right)^2 \iint\limits_{\mathbb{C}} \frac{e^{\frac{i}{2}(p\bar z + \bar p z)}}{p\bar p + 2p\lambda} d\mathrm{Re}p d\mathrm{Im}p, \tag{2.4}$$

for $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$.

The integral in (2.4) can be computed explicitly (see [3]) :

$$g(z,\lambda) = \frac{1}{16\pi}\exp(-i\lambda z)\left(\mathrm{Ei}(i\lambda z) + \mathrm{Ei}(-i\bar\lambda\bar z)\right).$$

Here $\mathrm{Ei}(z)$ is the exponential-integral function defined as follows :

$$\mathrm{Ei}(z) - \gamma - \ln(-z) = \sum_{n=1}^{\infty}\frac{z^n}{nn!}, \quad z \in \mathbb{C}\backslash(\mathbb{R}_+ \cup 0), \tag{2.5}$$

where γ is the Euler-Mascheroni constant

$$\gamma = -\int_0^\infty e^{-x} \ln x dx,$$

the branch cut for the logarithm function is taken on the negative real axis : $\ln(z) = \ln|z| + i\mathrm{Arg}z$, $|\mathrm{Arg}z| < \pi$, and the series in the right part of (2.5) converges on the whole complex plane.

It is easy to see that function $g(z,\lambda)$, defined in (2.4), has a logarithmic singularity at $\lambda = 0$ and thus function $\mu(z,\lambda)$ is not generally defined for $\lambda = 0$ even for arbitrarily small values of v. However, there exists a class of potentials, namely, potentials of conductivity type, for which function μ is defined for $\lambda \in \mathbb{C}\backslash 0$ and is bounded on the whole complex plane (see [46]).

Definition. *A potential $v \in L^p(\mathbb{R}^2)$, $1 < p < 2$, is of conductivity type if $v = \gamma^{-1/2}\Delta\gamma^{1/2}$ for some real-valued positive $\gamma \in L^\infty(\mathbb{R}^2)$, such that $\gamma \geqslant \delta_0 > 0$ and $\nabla\gamma^{1/2} \in L^p(\mathbb{R}^2)$.*

The physical meaning and some mathematical properties of the conductivity type potentials are discussed in detail in Section 5. Here we present some properties of the scattering data for this class of potentials obtained in [46].

For $\lambda \in \mathbb{C}\backslash 0$ we define the following "scattering data" for the potential v of conductivity type :

$$a(\lambda) = \iint_{\mathbb{C}} v(z)\mu(z,\lambda) d\mathrm{Re}z d\mathrm{Im}z, \tag{2.6}$$

$$b(\lambda) = \iint_{\mathbb{C}} e^{i\lambda z + i\bar\lambda\bar z} v(z)\mu(z,\lambda) d\mathrm{Re}z d\mathrm{Im}z. \tag{2.7}$$

Functions a, b are the Faddeev generalized scattering data for the 2-dimensional Schrödinger equation at zero energy.

Statement 2.1 (see [46]). *Let v satisfy conditions (1.2) and let v be of conductivity type. Then*

1. *$\mu(z,\lambda)$ is a well-defined and continuous function of λ on $\mathbb{C}\backslash 0$;*
2. *$\mu(\lambda)$ satisfies the following $\bar\partial$-equation :*

$$\frac{\partial\mu(z,\lambda)}{\partial\bar\lambda} = \frac{1}{4\pi\bar\lambda} e^{-i(\lambda z + \bar\lambda\bar z)} b(\lambda)\overline{\mu(z,\lambda)}, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{C}\backslash 0; \tag{2.8}$$

3. *$\mu(z,\lambda)$ is bounded in the neighborhood of $\lambda = 0$;*

4. *$\mu(z, \lambda) \to 1$, as $\lambda \to \infty$;*
5. *the scattering data $a(\lambda)$, $b(\lambda)$ are continuous on $\mathbb{C}\backslash 0$.*

Chapter II

Existence/absence of solitons for the Novikov-Veselov equation

3 Soliton solutions of the Novikov-Veselov equation

3.1 Grinevich-Zakharov solutions

In the present Section we are focused on a family of solutions for equation (0.5) for $E = E_{fixed} > 0$ constructed by P.G. Grinevich and V.E. Zakharov, see [21, 22] (containing also a reference to private communication from V.E. Zakharov). The solutions of this family are given by

$$\begin{aligned} v(x,t) &= -4\partial_z \partial_{\bar z} \ln \det A, \\ w(x,t) &= 12\partial_z^2 \ln \det A, \end{aligned} \tag{3.1}$$

where $A = (A_{lm})$ is $4N \times 4N$–matrix,

$$\begin{aligned} A_{ll} &= \frac{iE^{1/2}}{2}\left(\bar z - \frac{z}{\lambda_l^2}\right) - 3iE^{3/2}t\left(\lambda_l^2 - \frac{1}{\lambda_l^4}\right) - \gamma_l, \\ A_{lm} &= \frac{1}{\lambda_l - \lambda_m} \text{ for } l \neq m, \end{aligned} \tag{3.2}$$

$E^{1/2} > 0$, $z = x_1 + ix_2$, $\bar z = x_1 - ix_2$, ∂_z, $\partial_{\bar z}$ are defined in (0.6), and $\lambda_1, \ldots, \lambda_{4N}$, $\gamma_1, \ldots, \gamma_{4N}$ are complex numbers such that

$$\begin{aligned} &\lambda_j \neq 0, \quad |\lambda_j| \neq 1, \quad j = 1, \ldots, 4N, \quad \lambda_l \neq \lambda_m \text{ for } l \neq m, \\ &\lambda_{2j} = -\lambda_{2j-1}, \quad \gamma_{2j-1} - \gamma_{2j} = \frac{1}{\lambda_{2j-1}}, \quad j = 1, \ldots, 2N, \\ &\lambda_{4j-1} = \frac{1}{\bar\lambda_{4j-3}}, \quad \gamma_{4j-1} = \bar\lambda_{4j-3}^2 \bar\gamma_{4j-3}, \quad j = 1, \ldots, N. \end{aligned} \tag{3.3}$$

Functions v, w of (3.1)–(3.3) satisfy the Novikov–Veselov equation (0.5) for positive E of (3.2) and have also, in particular, the following properties (see [21, 22]) :

$$
\begin{aligned}
&v = \bar{v}, \quad v, w \in C^{\infty}(\mathbb{R}^2 \times \mathbb{R});\\
&v(x,t), w(x,t) \text{ are rational functions of } x \text{ and } t;\\
&v(x,t) = O\left(|x|^{-2}\right), w(x,t) = O\left(|x|^{-2}\right), |x| \to \infty, \text{ for each } t \in \mathbb{R};\\
&v(x,t) \text{ has zero scattering amplitude for fixed } E > 0 \text{ and } t \in \mathbb{R} \text{ in the framework}\\
&\quad \text{of equation (1.1) (or, in other words, } v \text{ is a transparent potential).}
\end{aligned}
\tag{3.4}
$$

Note that in many respects the solutions (3.1)–(3.3) of (0.5) are similar to related solutions of the KPI equation, see [13, 21, 22].

Note that potentials v of (3.1)–(3.3) play an important role in the direct and inverse scattering theory for the Schrödinger equation (1.1) at fixed energy $E > 0$ and in the theory of integrable systems, see [15, 22, 30, 46, 61, 63] and references therein. However, the properties of these potentials were not yet studied sufficiently in literature.

The main results of the present Section consist of the following :

(1) We show that (v, w) of the form (3.1)–(3.3) is a traveling wave iff $N = 1$. See Lemma 3.1 of the present Section.

(2) We show that there are no traveling waves of the form (3.1)–(3.3), $N = 1$, for $c \in \mathbb{U}_E$, and that there is an unique (modulo translations) traveling wave of the form (3.1)–(3.3), $N = 1$, for $c \in \mathbb{C}\backslash\mathbb{U}_E$, where c denotes traveling wave velocity and $\mathbb{U}_E$ is defined by formula (3.6). In addition we show that there is one–to–one correspondence between permitted velocities $c \in \mathbb{C}\backslash\mathbb{U}_E$ and λ–sets $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of (3.1)–(3.3), $N = 1$. See Lemma 3.2 of the present Section.

(3) We show that the large time asymptotics for the Grinevich–Zakharov potentials, that is for (v, w) defined by (3.1)–(3.3), is described by a sum of N localized traveling waves propagating with different velocities. See Theorem 3.1 of the present Section.

In Lemmas 3.1, 3.2 and Theorem 3.1 we present a more rigorous formulation of the results presented above.

Lemma 3.1. *Let (v, w) be defined by (3.1)–(3.3). Then (v, w) admits representation (0.10) (and is a traveling wave solution for (0.5)) if and only if $N = 1$. In addition,*

$$c = 6E\left(\bar{\lambda}^2 + \frac{1}{\lambda^2} + \frac{\lambda^2}{\bar{\lambda}^2}\right), \tag{3.5}$$

where c is the traveling wave velocity and λ is any of λ_j, $j = 1, 2, 3, 4$, which, in virtue of (3.3), determines uniquely the λ set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ for $E > 0$.

Lemma 3.1 is proved in subsection 3.2.

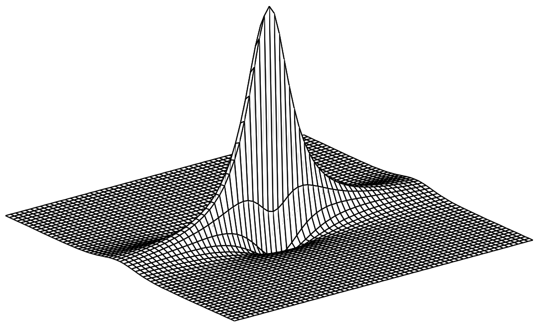

Figure II.1 – Grinevich-Zakharov soliton

Let

$$\begin{aligned} &\mathbb{U} = \{u \in \mathbb{C}\colon u = re^{i\varphi}, r \le |6(2e^{-i\varphi} + e^{2i\varphi})|, \varphi \in [0, 2\pi]\}, \\ &\mathbb{U}_E = \{u \in \mathbb{C}\colon u/E \in \mathbb{U}\}. \end{aligned} \tag{3.6}$$

One can see that $\mathbb{U}_1 = \mathbb{U}$.

Lemma 3.2. *(a) Let $c \in \mathbb{U}_E$. Then there is no traveling wave solution of (0.5) of the form (3.1)–(3.3) with $N = 1$ and the given traveling wave velocity c.*

(b) Let $c \in \mathbb{C}\backslash\mathbb{U}_E$. Then there exists unique (modulo translations) solution of (0.5) of the form (3.1)–(3.3) with $N = 1$ and the given traveling wave velocity c.

(c) There is a one–to–one correspondence between $c \in \mathbb{C}\backslash\mathbb{U}_E$ and the sets $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ satisfying (3.3).

The proof of this Lemma is given in subsection 3.2 and is based principally on the following auxiliary lemma.

Lemma 3.3. *(a) Let $c \in \mathbb{U}_E$. Then equation (3.5) has no solution λ satisfying $|\lambda| \neq 1$.*

(b) Let $c \in \mathbb{C}\backslash\mathbb{U}_E$, then equation (3.5) has exactly four solutions λ_1, λ_2, λ_3, λ_4 satisfying the conditions indicated in (3.3) for $N = 1$.

This Lemma is a corollary of Lemma 6.1 from Section 6 and is based on an analysis of the following algebraic equation

$$\lambda^6 - \frac{\bar{c}}{6E}\lambda^4 + \frac{c}{6E}\lambda^2 - 1 = 0$$

depending on the complex parameter $u = c/E$.

Theorem 3.1. *Let (v, w) be a solution of (0.5) constructed via (3.1)–(3.3). Then the asymptotical behavior of (v, w) can be described as follows :*

$$v \sim \sum_{k=1}^{N} \nu_k(\xi_k), \quad w \sim \sum_{k=1}^{N} \omega_k(\xi_k) \quad \text{as } t \to \infty, \tag{3.7}$$

where $\xi_k = z - c_{4k}t$ and

$$c_l = 6E\left(\bar{\lambda}_l^2 + \frac{1}{\lambda_l^2} + \frac{\lambda_l^2}{\bar{\lambda}_l^2}\right). \tag{3.8}$$

Functions ν_k, ω_k are defined by formulas

$$\begin{aligned} \nu_k &= -4\partial_z\partial_{\bar{z}} \ln \det A^{(k)}, \\ \omega_k &= 12\partial_z^2 \ln \det A^{(k)}, \end{aligned} \tag{3.9}$$

where matrix $A^{(k)}$ is a 4×4 submatrix of matrix A, defined by formulas (3.2), such that

$$A^{(k)} = \{A_{lm}\}_{l,m=4(k-1)+1}^{4k}. \tag{3.10}$$

Remark. *Relation (3.7) is understood in the following sense :*

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \sum_{k=1}^{N} \nu_k(\xi_k) \quad \text{for fixed } \xi = z - ct, \tag{3.11}$$

where

$$\lim_{t\to\infty} \nu_k(\xi_k) = \begin{cases} 0, & \text{for fixed } \xi = z - ct,\ c \neq c_{4k}, \\ \nu_k(\xi), & \text{for fixed } \xi = z - c_{4k}t. \end{cases} \tag{3.12}$$

One can see that the asymptotics (3.7) are the same for ∞ specified as $+\infty$ or as $-\infty$. Thus the solitons of (3.1)–(3.3) for the Novikov–Veselov equation (0.5) do not interact at all in a similar way with the result of [45] for a family of rational soliton solutions for the KPI equation.

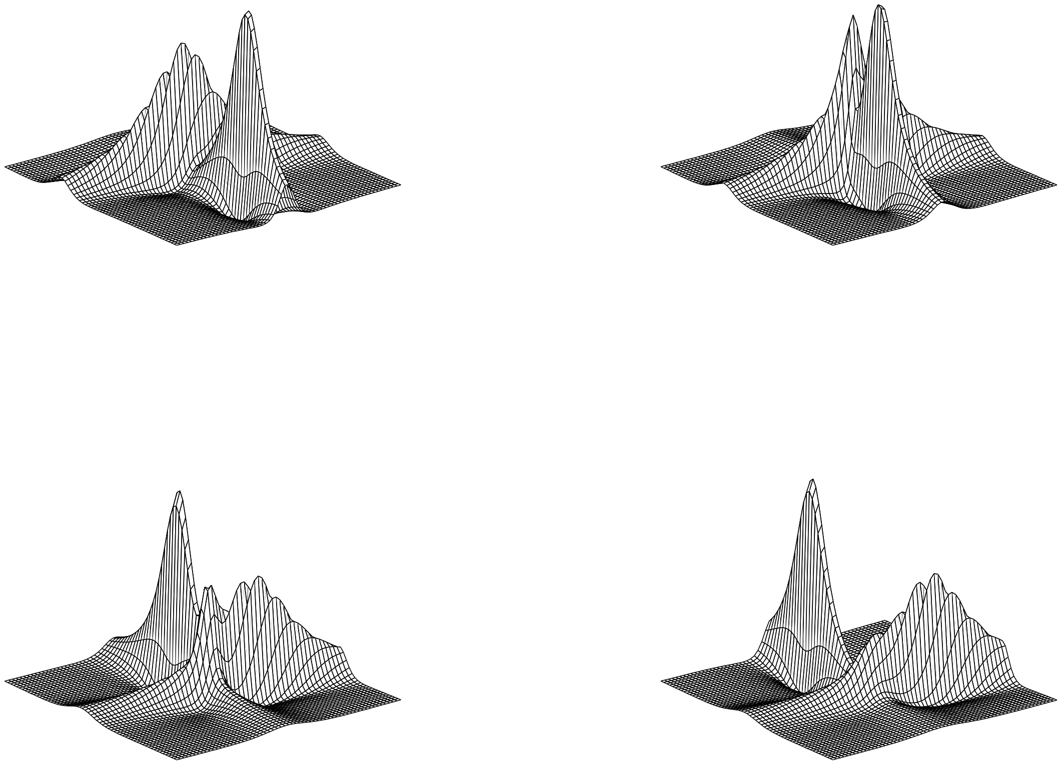

Figure II.2 – Interaction of Grinevich-Zakharov solitons

Theorem 3.1 is proved in subsection 3.2. The scheme of the proof of this theorem follows principally the scheme of the derivation of the large time asymptotics for the multisoliton solutions of the classic KdV equation (see, for example, [16]).

3.2 Dynamical properties of Grinevich-Zakharov solutions (proofs of Lemmas 3.1, 3.2 and Theorem 3.1)

The text of the proofs presented below does not completely follow the order of statements in subsection 3.1 as it was constructed to form a whole logical unit. However, we specify in due course which statement is being proved.

3.2.1 Proof of the sufficiency part of Lemma 3.1

Let us first consider the Grinevich–Zakharov potentials defined by (3.1)–(3.3) with $N = 1$. Then A is a 4×4 matrix and, in virtue of (3.3), the choice of any of λ_j, $j = 1, 2, 3, 4$, uniquely determines the set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Let us find c_j such that $A_{jj} = A_{jj}(z - c_j t)$. Such value c_j is a solution of the following equation

$$\frac{i}{2}E^{1/2}\left(\bar{z} - \bar{c}t - \frac{1}{\lambda_j^2}(z - ct)\right) = \frac{i}{2}E^{1/2}\left(\bar{z} - \frac{z}{\lambda_j^2}\right) - 3iE^{3/2}t\left(\lambda_j^2 - \frac{1}{\lambda_j^4}\right). \tag{3.13}$$

If $|\lambda_j| \neq 1$, then this equation is uniquely solvable and its solution is given by

$$c_j = 6E\left(\bar{\lambda}_j^2 + \frac{1}{\lambda_j^2} + \frac{\lambda_j^2}{\bar{\lambda}_j^2}\right). \tag{3.14}$$

It is easy to see that due to (3.3) $c_1 = c_2 = c_3 = c_4$. Thus $A = A(z - ct)$, and representation (0.10) with c defined by (3.5) holds. Thus sufficiency in Lemma 3.1 is proved.

3.2.2 Proof of Lemma 3.2

If $c \in \mathbb{U}_E$, then, as follows from item (a) of Lemma 3.3, $c \neq c_j$, defined by (3.14), $j = 1, 2, 3, 4$ for any λ_1, λ_2, λ_3, λ_4 satisfying conditions indicated in (3.3) for $N = 1$. This and the sufficiency part of Lemma 3.1 imply item (a) of Lemma 3.2.

If $c \in \mathbb{C}\backslash\mathbb{U}_E$, then, as follows from item (b) of Lemma 3.3, it determines via (3.14) uniquely the set of λ_1, λ_2, λ_3, λ_4 satisfying conditions indicated in (3.3) for $N = 1$. Then the solution (v, w), constructed according to formulas (3.1)–(3.3) with $N = 1$, constitutes a traveling wave solution of equation (0.5) with the given velocity c. In the construction procedure one of the parameters γ_j can be chosen arbitrarily and it determines uniquely the whole set $\{\gamma_1, \ldots, \gamma_4\}$.

One can see that the transform

$$z \to z + \zeta,$$
$$t \to t + \tau$$

turns the potential (v, w) into another Grinevich–Zakharov potential $(\tilde{v}, \tilde{w})$ with the parameters

$\{\lambda_1, \ldots, \lambda_4, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_4\}$, where

$$\gamma_j - \tilde{\gamma}_j = \frac{iE}{2}\left(\bar{\zeta} - \frac{\zeta}{\lambda_j^2}\right) - 3iE^{3/2}\tau\left(\lambda_j^2 - \frac{1}{\lambda_j^4}\right) \tag{3.15}$$

for $j = 1, 2, 3, 4$.

On the other hand, if $(\tilde{v}, \tilde{w})$ is a Grinevich–Zakharov potential with the set of parameters $\{\lambda_1, \ldots, \lambda_4, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_4\}$, then it can be obtained from (v, w) by a translation, i.e. $\tilde{v}(z,t) = v(z+\zeta, t+\tau)$, $\tilde{w}(z,t) = w(z+\zeta, t+\tau)$ for appropriate $\zeta \in \mathbb{C}$ and $t \in \mathbb{R}$ such that (3.15) holds for some j (equations (3.15) are equivalent for $j = 1, 2, 3, 4$ in virtue of (3.3)). In addition, one can assume, for example, that $\tau = 0$ in this translation.

Thus we have proved that any $c \in \mathbb{C}\backslash\mathbb{U}_E$ determines uniquely, modulo translations, the solution of (0.5) of the form (3.1)–(3.3) with $N = 1$ and the given traveling velocity c. This proves item (b) of Lemma 3.2.

Item (c) of Lemma 3.2 follows immediately from Lemma 3.3. Lemma 3.2 is proved.

3.2.3 Proof of Theorem 3.1

Let us consider a more convenient representation of (v, w) defined by (3.1)–(3.3). For this purpose we first perform the differentiation with respect to $\bar{z}$ in the right–hand side of formula for v in (3.1) :

$$v = -4\partial_z\left[(\det A)^{-1}\partial_{\bar{z}}(\det A)\right] = -4\partial_z\left[(\det A)^{-1}\sum_{i,j=1}^{4N}\frac{\partial A_{ij}}{\partial \bar{z}}\hat{A}_{ij}\right], \tag{3.16}$$

where $\hat{A}_{ij}$ is the (i,j) cofactor of the matrix A. Similarly,

$$w = 12\partial_z\left[(\det A)^{-1}\sum_{i,j=1}^{4N}\frac{\partial A_{ij}}{\partial z}\hat{A}_{ij}\right]. \tag{3.17}$$

In matrix A only diagonal elements depend on z, $\bar{z}$, thus

$$v = -2iE^{1/2}\partial_z\left[(\det A)^{-1}\sum_{j=1}^{4N}\hat{A}_{jj}\right], \quad w = -6iE^{1/2}\partial_z\left[(\det A)^{-1}\sum_{j=1}^{4N}\frac{1}{\lambda_j^2}\hat{A}_{jj}\right]. \tag{3.18}$$

Let us consider the following families $V^{(j)}$ and $W^{(j)}$ of systems of linear algebraic equations for functions $\psi_k^{(j)}$, $\eta_k^{(j)}$, $j, k = 1, \ldots 4N$:

$$V^{(j)}: \quad \sum_{k=1}^{4N} A_{mk}\psi_k^{(j)} = -2iE^{1/2}\delta_{mj}, \quad m = 1, \ldots, 4N, \tag{3.19}$$

$$W^{(j)}: \quad \sum_{k=1}^{4N} A_{mk}\eta_k^{(j)} = -6iE^{1/2}\frac{1}{\lambda_j^2}\delta_{mj}, \quad m = 1, \ldots, 4N. \tag{3.20}$$

Then functions v, w can be represented in the following form

$$v = \sum_{j=1}^{4N} \frac{\partial\psi_j^{(j)}}{\partial z}, \quad w = \sum_{j=1}^{4N} \frac{\partial\eta_j^{(j)}}{\partial z}. \tag{3.21}$$

In order to write a system of linear algebraic equations for $\frac{\partial\psi_k^{(j)}}{\partial z} = \left(\psi_k^{(j)}\right)_z$, we differentiate (3.19) with respect to z :

$$\sum_{k=1}^{4N} (A_{mk})_z \psi_k^{(j)} + \sum_{k=1}^{4N} A_{mk}\left(\psi_k^{(j)}\right)_z = 0, \quad m = 1, \ldots, 4N,$$

and thus obtain

$$\sum_{k=1}^{4N} A_{mk}\left(\psi_k^{(j)}\right)_z = \frac{iE^{1/2}}{2\lambda_m^2}\psi_m^{(j)}, \quad m = 1, \ldots, 4N. \tag{3.22}$$

Now let us note that A_{jj} can be represented in the form

$$A_{jj} = \frac{iE^{1/2}}{2}\left[(\bar z - \bar c_j t) - \frac{1}{\lambda_j^2}(z - c_j t)\right] - \gamma_j,$$

where c_j is given by formula (3.8). As follows from item (c) of Lemma 3.2 $c_j = c_k$ iff $\lfloor (j-1)/4 \rfloor = \lfloor (k-1)/4 \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x.

Now let us fix

$$\xi = z - ct$$

and find the limits $\psi_k^{(j)}\Big|_{\substack{t\to\infty\\ \xi \text{ fixed}}}$, $\left(\psi_k^{(j)}\right)_z\Big|_{\substack{t\to\infty\\ \xi \text{ fixed}}}$. We note that

$$A_{jj} = \frac{iE^{1/2}}{2}\left[\{\bar\xi + (\bar c - \bar c_j)t\} - \frac{1}{\lambda_j^2}\{\xi + (c - c_j)t\}\right].$$

If $c = c_j$, then $A_{jj} = \frac{iE^{1/2}}{2}\left[\bar\xi - \frac{1}{\lambda_j^2}\xi\right]$ and is independent of t. Otherwise, $|A_{jj}| \to \infty$ as $t \to \infty$

at fixed ξ. We substitute this into (3.19) and consider the leading term in the Cramer formula for $\psi_k^{(j)}$ as $t \to \infty$. Thus we obtain

$$\psi_k^{(j)}\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = \hat{\psi}_k^{(j)}(\xi), \quad k, j\colon c_k = c_j = c,$$

$$\psi_k^{(j)}\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = 0, \quad k\colon c_k \neq c \text{ or } j\colon c_j \neq c.$$

Here $\hat{\psi}_k^{(j)}(\xi)$ denotes some function of ξ independent of t at fixed ξ.

Similarly, from (3.22) we obtain that

$$\left(\psi_k^{(j)}\right)_z\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = \bar{\psi}_k^{(j)}(\xi), \quad k, j\colon c_k = c_j = c,$$

$$\left(\psi_k^{(j)}\right)_z\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = 0, \quad k\colon c_k \neq c \text{ or } j\colon c_j \neq c,$$

and, as previously, $\bar{\psi}_k^{(j)}(\xi)$ denotes some function of ξ independent of t at fixed ξ.

In addition, one can see that if there exists k such that $c = c_{4(k-1)+1} = \ldots = c_{4k}$, then

$$v\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = \sum_{j=4(k-1)+1}^{4k} \bar{\psi}_j^{(j)}(\xi) = \nu_k(\xi), \tag{3.23}$$

where ν_k is defined by formula

$$\nu_k = -4\partial_z\partial_{\bar{z}} \ln \det A^{(k)}, \tag{3.24}$$

matrix $A^{(k)}$ is a 4×4 submatrix of matrix A from (3.2), such that $A^{(k)} = \{A_{lm}\}_{l,m=4(k-1)+1}^{4k}$.

Similarly, for the case of function w we have

$$\eta_k^{(j)}\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = \hat{\eta}_k^{(j)}(\xi), \quad \left(\eta_k^{(j)}\right)_z\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = \bar{\eta}_k^{(j)}(\xi), \quad k, j\colon c_k = c_j = c,$$

$$\eta_k^{(j)}\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = 0, \quad \left(\eta_k^{(j)}\right)_z\Big|_{\substack{t\to\infty\\ \xi\ \text{fixed}}} = 0, \quad k\colon c_k \neq c \text{ or } j\colon c_j \neq c,$$

where $\hat{\eta}_k^j(\xi)$, $\bar{\eta}_k^j(\xi)$ are some functions of ξ independent of t at fixed ξ.

If there exists k such that $c = c_{4(k-1)+1} = \ldots = c_{4k}$, then

$$w\Big|_{\substack{t\to\infty\\ \xi \text{ fixed}}} = \sum_{j=4(k-1)+1}^{4k} \bar{\eta}_j^{(j)}(\xi) = \omega_k(\xi), \tag{3.25}$$

where ω_k is defined by formula

$$\omega_k = 12\partial_z^2 \ln \det A^{(k)} \tag{3.26}$$

and matrix $A^{(k)}$ is the same as in (3.24).

From (3.21), (3.23)–(3.26) it follows that

$$v \sim \sum_{k=1}^{N} \nu_k(\xi_k), \quad w \sim \sum_{k=1}^{N} \omega_k(\xi_k), \quad t \to \infty. \tag{3.27}$$

Here $\xi_k = z - c_{4k}t$, ν_k, ω_k are defined by (3.9)–(3.10) and the meaning of the relation (3.27) is specified by (3.11)–(3.12). Theorem 3.1 is proved.

3.2.4 Proof of the necessity part of Lemma 3.1

From (3.27), taking into account (3.11)–(3.12), one can see that (v, w) can be a traveling wave only if $N = 1$. This completes the proof of Lemma 3.1.

4 Absence of sufficiently localized solitons at $E \neq 0$

Solitons from the family of Grinevich-Zakharov potentials of [21] and solitons constructed in [9] are solutions of the Novikov-Veselov equation localized in space as $O(|x|^{-2})$, $|x| \to \infty$. In this Section we prove that this is almost the optimal localization that can be obtained for soliton solutions of the Novikov-Veselov equation at nonzero energy, i.e. localization stronger than $O(|x|^{-3})$, $|x| \to \infty$, is already not possible. These results are a continuation of work started in [51], [36] where the absence of exponentially localized solitons for the Novikov-Veselov equation at nonzero energy was proved.

4.1 Formulation and proof of the result

In this subsection we are concerned with regular, sufficiently localized solutions of (0.5) at nonzero energy $E \neq 0$ satisfying the following conditions

$$\bullet\ v, w \in C(\mathbb{R}^2 \times \mathbb{R}),\ v(\cdot, t) \in C^4(\mathbb{R}^2) \quad \forall t \in \mathbb{R}; \tag{4.1}$$

$$\bullet\ |\partial_x^j v(x,t)| \leqslant \frac{q(t)}{(1+|x|)^{3+j+\varepsilon}},\ j = (j_1, j_2) \in (\mathbb{N} \cup 0)^2,\ j_1 + j_2 \leqslant 4,\ \text{for some } q(t) > 0, \varepsilon > 0; \tag{4.2}$$

$$\bullet\ |w(x,t)| \to 0,\ \text{when } |x| \to \infty, \quad t \in \mathbb{R}. \tag{4.3}$$

The main result of this Section consists in the following theorem.

Theorem 4.1. *Let (v, w) be a soliton solution of (0.5) with $E \neq 0$ satisfying properties (4.1)-(4.3). Then $v \equiv 0$, $w \equiv 0$.*

In order to prove this result we use methods of the scattering theory for the 2-dimensional Schrödinger equation. Note, however, that we do not impose a priori the small norm assumption or any condition that guarantees the unique solvability of the scattering equations. A priori we allow for the Faddeev eigenfunctions to have singularities. Then, by using special eigenfunctions of the 2-dimensional Schrödinger operator introduced in [3] and ideas proposed in [51], we are able to prove that such singularities are absent. This result justifies the use of the inverse scattering method which implies that the only possible soliton solution of (0.5) with $E \neq 0$ satisfying properties (4.1)-(4.3) is zero.

We will start the proof of Theorem 4.1 by formulating some preliminary lemmas. The proofs of Lemmas 4.1 and 4.2 are given in subsection 4.3. The proof of Lemma 4.4 is given is subsection 4.2.

Let us denote by

$$S(\lambda) = \{a(\lambda), b(\lambda), \alpha(\lambda), \beta(\lambda)\}, \quad \lambda \in \mathbb{C} \backslash (\mathcal{E} \cup 0), \tag{4.4}$$

the scattering data for a potential v, defined by (1.28)-(1.31) in the framework of equation (1.1).

Lemma 4.1. *Let $v(z)$ be a potential satisfying (1.2a), (1.5) with the Fredholm determinant $\Delta(\lambda)$, defined by (1.26) in the framework of equation (1.1), and the scattering data $S(\lambda)$, $\lambda \in \mathbb{C} \backslash (\mathcal{E} \cup 0)$. Then the Fredholm determinant $\Delta_\eta(\lambda)$ and the scattering data $S_\eta(\lambda)$ for the potential $v_\eta(z) = v(z - \eta)$ have the following properties :*

1. $\Delta_\eta(\lambda) = \Delta(\lambda)$;
2. *scattering data* $S_\eta(\lambda)$ *are defined for* $\lambda \in \mathbb{C}\backslash(\mathcal{E} \cup 0)$ *and are related to* $S(\lambda)$ *by the following formulas*

$$a_\eta(\lambda) = a(\lambda), \tag{4.5}$$

$$b_\eta(\lambda) = \exp\left\{\frac{i\sqrt{E}}{2}\left(1 + (\mathrm{sgn}E)\frac{1}{\lambda\bar{\lambda}}\right)\left((\mathrm{sgn}E)\bar{\lambda}\eta + \lambda\bar{\eta}\right)\right\} b(\lambda), \tag{4.6}$$

$$\alpha_\eta(\lambda) = \alpha(\lambda) + \frac{i\sqrt{E}}{2}\left(\lambda\bar{\eta} - \frac{1}{\lambda}\eta\right) a(\lambda), \tag{4.7}$$

$$\beta_\eta(\lambda) = \exp\left\{\frac{i\sqrt{E}}{2}\left(1 + (\mathrm{sgn}E)\frac{1}{\lambda\bar{\lambda}}\right)\left((\mathrm{sgn}E)\bar{\lambda}\eta + \lambda\bar{\eta}\right)\right\}\left(\beta(\lambda) + \frac{i\sqrt{E}}{2}\left(\lambda\bar{\eta} - \frac{1}{\lambda}\eta\right) b(\lambda)\right). \tag{4.8}$$

Lemma 4.2. *Let* (v, w) *satisfy equation (0.5) for some* $E \neq 0$ *and conditions (4.1)-(4.3). Let* $S(\lambda, t)$ *be the scattering data for* v *defined by (4.4) for a certain* $\lambda \in \mathbb{C}\backslash(\mathcal{E} \cup 0)$ *and all* $t \in \mathbb{R}$. *Then the evolution of these scattering data is described as follows :*

$$a(\lambda, t) = a(\lambda, 0), \tag{4.9}$$

$$b(\lambda, t) = \exp\left\{i(\sqrt{E})^3\left(\lambda^3 + \frac{1}{\lambda^3} + (\mathrm{sgn}E)\left(\bar{\lambda}^3 + \frac{1}{\bar{\lambda}^3}\right)\right)t\right\} b(\lambda, 0), \tag{4.10}$$

$$\alpha(\lambda, t) = \alpha(\lambda, 0) + 3i(\sqrt{E})^3\left(\lambda^3 - \frac{1}{\lambda^3}\right)(a(\lambda, 0) - \hat{v}(0))t, \tag{4.11}$$

$$\beta(\lambda, t) = \exp\left\{i(\sqrt{E})^3\left(\lambda^3 + \frac{1}{\lambda^3} + (\mathrm{sgn}E)\left(\bar{\lambda}^3 + \frac{1}{\bar{\lambda}^3}\right)\right)t\right\}\left(\beta(\lambda, 0) + 3i(\sqrt{E})^3\left(\lambda^3 - \frac{1}{\lambda^3}\right) b(\lambda, 0)t\right), \tag{4.12}$$

where $\hat{v}(0) = \iint\limits_{\mathbb{C}} v(\zeta) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta$

Note that formulas (4.8), (4.12) will not be used in the proof of the main result ; however we present them here for consistency of exposition.

Lemma 4.3 (see [51])**.** *Let* (v, w) *satisfy equation (0.5) for some* $E > 0$, *conditions (4.1) and*

$$|\partial_x^j v(x, t)| \leqslant \frac{q(t)}{(1 + |x|)^{2+\varepsilon}},\ j = (j_1, j_2) \in (\mathbb{N} \cup 0)^2,\ j_1 + j_2 \leqslant 3,\ \text{for some } q(t) > 0, \varepsilon > 0,$$

$$|w(x, t)| \to 0,\ \text{as } |x| \to \infty, \quad \forall t \in \mathbb{R}.$$

In addition, let v *be a soliton, i.e.* $v(x, t) = V(x - ct)$ *for some* $c = (c_1, c_2) \in \mathbb{R}^2$. *Then* $f(k, l) \equiv 0$, $k, l \in \mathbb{R}^2$, $k^2 = l^2 = E > 0$, *where* f *is the scattering amplitude for the potential* v

in the framework of the Schrödinger equation (1.1).

Lemma 4.4. *Let (v, w) be a soliton solution of equation (0.5) for some $E \neq 0$ satisfying conditions (4.1)-(4.3). Let a, b be the scattering data for v defined by (1.28), (1.29), correspondingly. Then $a(\lambda, t) \equiv 0$, $b(\lambda, t) \equiv 0$ for $\lambda \in \mathbb{C} \backslash \mathcal{E}$.*

Proof of Theorem 4.1. In view of Lemmas 4.1–4.4 the concluding part of the proof of Theorem 4.1 consists in the following. First of all, if (v, w) is a soliton solution of equation (0.5) for some $E > 0$, then v is transparent due to Lemma 4.3.

Further, from item 1 of Lemma 4.1 it follows that the modified Fredholm determinant Δ and, as a consequence, the set $\mathcal{E}$ of its zeros do not depend on t.

Lemma 4.4 together with item 7 of Statement 1.2 and equation (1.37) imply that Δ is holomorphic on $\mathbb{C} \backslash (\mathcal{E} \cup T \cup 0)$, where $T = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$. From items 1, 2 of Statement 1.1 it follows that Δ is holomorphic on $\mathbb{C} \backslash (\mathcal{E} \cup T)$.

Suppose now that $\mathcal{E} \neq \varnothing$. Since $\mathcal{E}$ is a closed set, then there exists $\lambda_* \in \mathcal{E}$ such that $|\lambda_*| = \min\limits_{\lambda \in \mathcal{E}} |\lambda|$. Note that item 2 of Statement 1.1 implies that $|\lambda_*| > 0$.

If $|\lambda_*| \geqslant 1$, then $\Delta(\lambda)$ is holomorphic on $D_+ = \{\lambda \in \mathbb{C} \colon |\lambda| < 1\}$ and items 2, 3 of Statement 1.1 imply that $\Delta \equiv 1$ on D_+. If $|\lambda_*| < 1$, then $\Delta(\lambda)$ is holomorphic on the set $D_+^h = \{\lambda \in \mathbb{C} \colon |\lambda| < \lambda_*\}$ and items 2, 3 of Statement 1.1 imply that $\Delta \equiv 1$ on D_+^h. On the other hand, $\Delta(\lambda_*) = 0$, which contradicts item 1 of Statement 1.1. Thus we have proved that $\Delta(\lambda) \equiv 1$ on D_+. Item 5 of Statement 1.1 implies that $\Delta(\lambda) \equiv 1$ on $D_- = \{\lambda \in \mathbb{C} \colon |\lambda| > 1\}$. Finally, from item 1 of Statement 1.1 it follows that $\Delta \equiv 1$ on $\mathbb{C}$.

Function μ is holomorphic on $\mathbb{C}$ as follows from items 2, 3 of Statement 1.2 and the established facts that $\mathcal{E} = \varnothing$, $b \equiv 0$. Function μ is also bounded due to item 3 of Statement 1.2. From Liouville theorem it follows that $\mu \equiv 1$. Then, finally, from (1.40), (1.41) we obtain that $v \equiv 0$. □

4.2 Vanishing scattering data for the localized soliton solutions of the Novikov-Veselov equation at $E \neq 0$ (proof of Lemma 4.4)

Proof. Since (v, w) is a soliton, from item 1 of Lemma 4.1 it follows that the set $\mathcal{E}$ of values of $\lambda \in \mathbb{C}$ for which the scattering data $a(\lambda, t)$, $b(\lambda, t)$ are not well-defined does not depend on t. Lemma 4.3 also implies that if (v, w) is a soliton solution of equation (0.5) for some $E > 0$, then v is transparent.

We will start by proving that the scattering data $b(\lambda, t)$ vanish everywhere where they are well-defined. Since (v, w) is a soliton, the time dynamics of b is described by the formula

$$b(\lambda, t) = \exp\left\{\frac{i\sqrt{E}}{2}\left(\left(\lambda + (\mathrm{sgn}E)\frac{1}{\bar\lambda}\right)\bar c t + \left((\mathrm{sgn}E)\bar\lambda + \frac{1}{\lambda}\right)ct\right)\right\} b(\lambda, 0)$$

(see formula (4.6) of Lemma 4.1). Combining this with (4.10) from Lemma 4.2 gives

$$\exp\left\{i(\sqrt{E})^3\left(\lambda^3 + \frac{1}{\lambda^3} + (\mathrm{sgn}E)\left(\bar\lambda^3 + \frac{1}{\bar\lambda^3}\right)\right)t\right\} b(\lambda, 0) = \\ = \exp\left\{\frac{i\sqrt{E}}{2}\left(\left(\lambda + (\mathrm{sgn}E)\frac{1}{\bar\lambda}\right)\bar c t + \left((\mathrm{sgn}E)\bar\lambda + \frac{1}{\lambda}\right)ct\right)\right\} b(\lambda, 0).$$

Since functions λ, $\bar\lambda$, λ^3, $\bar\lambda^3$, $\frac{1}{\lambda}$, $\frac{1}{\bar\lambda}$, $\frac{1}{\lambda^3}$, $\frac{1}{\bar\lambda^3}$, 1 are linearly independent in any open nonempty neighborhood of any point in $\mathbb{C}\backslash 0$ and $b(\lambda, 0)$ is continuous on $\mathbb{C}\backslash(\mathcal{E} \cup 0)$, we obtain that $b(\lambda, 0) \equiv 0$ on $\mathbb{C}\backslash(\mathcal{E} \cup 0)$.

Now we will use a similar reasoning to prove that the scattering data $a(\lambda, t)$ vanish everywhere where they are well-defined. From (4.7), (4.11) we get that

$$a(\lambda, 0) + \frac{i\sqrt{E}}{2}\left(\lambda\bar c - \frac{1}{\lambda}c\right)ta(\lambda, 0) = a(\lambda, 0) + 3i(\sqrt{E})^3\left(\lambda^3 - \frac{1}{\lambda^3}\right)t(a(\lambda, 0) - \hat v(0)),$$

or

$$a(\lambda, 0) = \frac{3i(\sqrt{E})^3\left(\lambda^3 - \frac{1}{\lambda^3}\right)\hat v(0)}{3i(\sqrt{E})^3\left(\lambda^3 - \frac{1}{\lambda^3}\right) - \frac{i\sqrt{E}}{2}\left(\lambda\bar c - \frac{1}{\lambda}c\right)}. \tag{4.13}$$

We will prove that expression (4.13) implies that $\hat v(0) = 0$ and thus $a(\lambda, 0) \equiv 0$ for $\lambda \in \mathbb{C}\backslash(\mathcal{E} \cup 0)$.

First of all, let us consider the right-hand side of expression (4.13) :

$$\tilde a(\lambda) \stackrel{\mathrm{def}}{=} \frac{3i(\sqrt{E})^3\left(\lambda^3 - \frac{1}{\lambda^3}\right)\hat v(0)}{3i(\sqrt{E})^3\left(\lambda^3 - \frac{1}{\lambda^3}\right) - \frac{i\sqrt{E}}{2}\left(\lambda\bar c - \frac{1}{\lambda}c\right)}. \tag{4.14}$$

It is well-defined everywhere except for the points which are the roots of equation

$$3i(\sqrt{E})^3\left(\lambda^3 - \frac{1}{\lambda^3}\right) - \frac{i\sqrt{E}}{2}\left(\lambda\bar c - \frac{1}{\lambda}c\right) = 0. \tag{4.15}$$

Evidently, equation (4.15) has six roots λ_j, $j = 1, \ldots, 6$, counted with multiplicity. These roots are studied in detail in Lemma 6.1 of Section 6. In particular, it is shown that for any value of parameter c

$$\text{equation (4.15) has at least two roots on } T = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}. \tag{4.16}$$

Now the proof is carried out separately for cases $E > 0$ and $E < 0$.

I. $E > 0$

Consider equation (1.17). From transparency of potential it follows that $h_\gamma = 0$. For a transparent potential relation (1.18) transforms into

$$\Delta_\gamma(k) = \Delta^+(|k|), \quad k, \gamma \in \mathbb{R}^2, \quad k^2 = E, \quad \gamma^2 = 1. \tag{4.17}$$

From (1.8) it follows that $\Delta^+(|k|) \neq 0$ and, thus, $\Delta_\gamma(k) \neq 0$ $\forall k \in \mathbb{R}^2$, $k^2 = E$. From item 6 of Statement 1.1 we obtain that $\Delta(\lambda(1 \mp 0)) = \Delta^+(|k|) \neq 0$, $\lambda = \frac{k_1 + ik_2}{\sqrt{E}} \in T$. Consequently, $\mathcal{E} \cap T = \varnothing$, which implies, in particular, that functions $h_\pm(\lambda, \lambda')$ of (1.34) are well-defined for $\lambda \in T$. From equations (1.35), (1.36) for a transparent potential we obtain that $h_\pm(\lambda, \lambda') = 0$. Then from (1.33) we obtain that $a(\lambda) \equiv 0$ on T. In view of formula (4.13) and property (4.16) this can only be true if

$$\hat{v}(0) = 0. \tag{4.18}$$

II. $E < 0$

For the negative energy level we will consider two subcases depending on whether Δ vanishes on T or not. Note that item 6 of Statement 1.1 implies that if Δ vanishes in one point of T, then it vanishes in every point of T.

(i) $\Delta \neq 0$ on T

In this subcase $a(\lambda, 0)$ and $b(\lambda, 0)$ are well-defined on T. Thus $b(\lambda, 0) \equiv 0$ on T. From item 10 of Statement 1.2 we obtain that $a(\lambda, 0) \equiv 0$ on T. In view of representation (4.13) and property (4.16) this can only hold if $\hat{v}(0) = 0$.

(ii) $\Delta \equiv 0$ on T

Denote by $\lambda_{\min}$ the minimal in absolute value root of equation (4.15) and $B_0 = \{\lambda \in \mathbb{C} \colon |\lambda| < |\lambda_{\min}|\}$. Let us also define

$$\gamma_j = \{re^{i\varphi},\ r > 0,\ \varphi_j \in [0, 2\pi) \colon \varphi_j = \mathrm{Arg}\lambda_j\}, \quad j = 1, \ldots, 6, \tag{4.19}$$

i.e. γ_j are rays starting at the origin and passing through the roots of polynomial (4.13) (some of them might coincide). Finally denote

$$\Omega = \left(\mathbb{C}\backslash \bigcup_{j=1}^{6} \gamma_j\right) \bigcup B_0. \tag{4.20}$$

Note that Ω is simply connected and $\tilde{a}$ of (4.14) is well-defined and holomorphic on Ω.

Consider function $\tilde{\Delta}$ defined on Ω by

$$\tilde{\Delta} = \tilde{\Phi}(\bar{\lambda})\overline{\tilde{\Phi}(\bar{\lambda})}, \tag{4.21}$$

$$\tilde{\Phi}(\bar{\lambda}) = \exp \tilde{U}(\bar{\lambda}), \quad \tilde{U}(\bar{\lambda}) = \int_0^{\bar{\lambda}} \tilde{u}(\bar{\zeta})d\bar{\zeta}, \tag{4.22}$$

$$\tilde{u}(\bar{\lambda}) = -\frac{\operatorname{sgn}(\lambda\bar{\lambda}-1)}{4\pi\bar{\lambda}} \left(\tilde{a}\left(-(\operatorname{sgn}E)\frac{1}{\bar{\lambda}}\right) - \hat{v}(0)\right).$$

Note that from expression (4.13) it follows that $\tilde{u}(\bar{\lambda})$ is well-defined at $\lambda = 0$ and thus the integral in (4.22) makes sense.

For $\tilde{\Delta}$ defined via (4.21) we have that

$$\forall \lambda \in \Omega \quad \tilde{\Delta}(\lambda) \neq 0. \tag{4.23}$$

Now we will obtain an expression for the modified Fredholm determinant Δ similar to (4.21). Recall that due to item 3 of Statement 1.1 $\Delta \in \mathbb{R}$ and due to item 2 of Statement 1.1 $\Delta(0) > 0$.

Consider

$$D = \bigcup_{\varphi\in[0,2\pi)} l_\varphi, \quad l_\varphi = \{re^{i\varphi}, 0 \leqslant r < r' \leqslant 1\colon \Delta(re^{i\varphi}) > 0, \Delta(r'e^{i\varphi}) = 0\},$$

$$\tilde{D} = \bigcup_{\varphi\in[0,2\pi)} \tilde{l}_\varphi, \quad \tilde{l}_\varphi = \{re^{i\varphi}, 0 \leqslant r \leqslant r' \leqslant 1\colon \Delta(re^{i\varphi}) > 0, \Delta(r'e^{i\varphi}) = 0\},$$

$$\Gamma = \bigcup_{\varphi\in[0,2\pi)} z_\varphi, \quad z_\varphi = r'e^{i\varphi},\ 0 < r' \leqslant 1,\ \text{such that } \Delta(r'e^{i\varphi}) = 0,\ \Delta(re^{i\varphi}) > 0 \quad \forall\, 0 \leqslant r < r'.$$

Note that $\tilde{D} = D \cup \Gamma$. Domain D is simply connected and $\Delta > 0$ on D. Thus $a(\lambda, 0)$ is well-defined on D and, in particular, (4.13) holds on D.

Denote

$$u(\bar{\lambda}) = -\frac{\operatorname{sgn}(\lambda\bar{\lambda}-1)}{4\pi\bar{\lambda}} \left(a\left(-(\operatorname{sgn}E)\frac{1}{\bar{\lambda}}\right) - \hat{v}(0)\right), \quad \lambda \in D\backslash 0. \tag{4.24}$$

From definition (1.28), asymptotic formula (1.41) and symmetry property stated in item 4 of

Statement 1.2 it follows that u is bounded as $\lambda \to 0$. From item 11 of Statement 1.2 and the established fact that $b \equiv 0$ on $\mathbb{C}\backslash\mathcal{E}$ it follows that a is a holomorphic function on $\mathbb{C}\backslash\mathcal{E}$. Thus $u(\bar\lambda)$ defines an antiholomorphic function on D. Then using also (1.37) we obtain that $\ln\Delta = U(\bar\lambda) + f(\lambda)$ on D, where $U(\bar\lambda) = \int\limits_0^{\bar\lambda} u(\bar\zeta)d\bar\zeta$ and $f(\lambda)$ is some holomorphic function on D. From item 3 of Statement 1.1 it follows that $f(\lambda) = \overline{U(\bar\lambda)}$. In other words,

$$\Delta(\lambda) = \Phi(\bar\lambda)\overline{\Phi(\bar\lambda)} \text{ on } D, \text{ where } \Phi(\bar\lambda) = \exp U(\bar\lambda).$$

Note that Δ is defined on $\tilde D$ since $\tilde D = D \cup \Gamma$ and $\Delta = 0$ on Γ. From item 1 of Statement 1.1 it also follows that Δ is continuous on $\tilde D$. We also have that

$$\Delta = \tilde\Delta \text{ on } \tilde D \cap \Omega, \tag{4.25}$$

where $\tilde\Delta$ is defined in (4.21) and Ω is defined in (4.20).

Now choose some φ, such that $\tilde l_\varphi \in \tilde D \cap \Omega$. Then $z_\varphi \in \tilde D \cap \Omega$ and $\Delta(z_\varphi) = 0$. However, from (4.23) we get that $\tilde\Delta(z_\varphi) \neq 0$ which contradicts (4.25). Thus we have shown that subcase (ii), when $\Delta \equiv 0$ on T, cannot hold.

We have shown that $\hat v(0) = 0$ and, consequently, $a(\lambda, 0) \equiv 0$ on $\lambda \in \mathbb{C}\backslash(\mathcal{E} \cup 0)$. $\square$

4.3 Dynamics of scattering data (proofs of Lemmas 4.1, 4.2)

Proof of Lemma 4.1.

1. Item 1 of Lemma 4.1 can be proved by considering the right-hand side of (1.26) as a sum of convergent series; then it can be checked that every member of this series for $\Delta_\eta(\lambda)$ coincides with the corresponding member of the series for $\Delta(\lambda)$.

2. We note that $\psi(z-\eta,\lambda)$ satisfies (1.1) with $v_\eta(z)$ and has the asymptotics

$$\psi(z-\eta,\lambda) = e^{\frac{i\sqrt{E}}{2}(\lambda(\bar z - \bar\eta) + (z-\eta)/\lambda)}(1 + o(1)),$$

as $|z| \to \infty$. Thus for Faddeev eigenfunction $\psi_\eta(z,\lambda)$ corresponding to potential $v_\eta(z)$ we obtain the following representation : $\psi_\eta(z,\lambda) = e^{\frac{i\sqrt{E}}{2}(\lambda\bar\eta + \eta/\lambda)}\psi(z-\eta,\lambda)$. Consequently, for function $\mu_\eta(z,\lambda)$ corresponding to function $\psi_\eta(z,\lambda)$ and defined via (1.19) we have $\mu_\eta(z,\lambda) = \mu(z-\eta,\lambda)$.

For the scattering data we have

$$a_\eta(\lambda) = \iint\limits_{\mathbb{C}} v_\eta(\zeta)\mu_\eta(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = \iint\limits_{\mathbb{C}} v(\zeta-\eta)\mu(\zeta-\eta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = a(\lambda)$$

and

$$b_\eta(\lambda) = \iint\limits_{\mathbb{C}} \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\zeta + \lambda\bar\zeta\right)\right\} v_\eta(\zeta)\mu_\eta(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \iint\limits_{\mathbb{C}} \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\zeta + \lambda\bar\zeta\right)\right\} v(\zeta-\eta)\mu(\zeta-\eta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\eta + \lambda\bar\eta\right)\right\} b(\lambda).$$

Similarly, to derive formulas (4.7), (4.8), we note that $\varphi(z-\eta,\lambda)$ satisfies (1.1) with $v_\eta(z)$ and has the asymptotics

$$\varphi(z-\eta,\lambda) = e^{\frac{i\sqrt{E}}{2}(\lambda(\bar z-\bar\eta)+(z-\eta)/\lambda)}\left(\frac{i\sqrt{E}}{2}\left(\lambda(\bar z-\bar\eta) - \frac{1}{\lambda}(z-\eta)\right) + o(1)\right),$$

as $|z| \to \infty$. Thus for eigenfunction $\varphi_\eta(z,\lambda)$ of equation (1.1) with potential $v_\eta(z)$ satisfying asymptotics (1.20) we obtain the following representation : $\varphi_\eta(z,\lambda) = e^{\frac{i\sqrt{E}}{2}(\lambda\bar\eta+\eta/\lambda)}(\varphi(z-\eta,\lambda) + \frac{i\sqrt{E}}{2}\left(\lambda\bar\eta - \frac{1}{\lambda}\eta\right)\psi(z-\eta,\lambda))$. Consequently, for function $\nu_\eta(z,\lambda)$ corresponding to function $\varphi_\eta(z,\lambda)$ and defined via (1.20) we have $\nu_\eta(z,\lambda) = \nu(z-\eta,\lambda) + \frac{i\sqrt{E}}{2}\left(\lambda\bar\eta - \frac{1}{\lambda}\eta\right)\mu(z-\eta,\lambda)$.

For the scattering data we have

$$\alpha_\eta(\lambda) = \iint\limits_{\mathbb{C}} v_\eta(\zeta)\nu_\eta(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \iint\limits_{\mathbb{C}} v(\zeta-\eta)\nu(\zeta-\eta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \frac{i\sqrt{E}}{2}\left(\lambda\bar\eta - \frac{1}{\lambda}\eta\right)\iint\limits_{\mathbb{C}} v(\zeta-\eta)\mu(\zeta-\eta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \alpha(\lambda) + \frac{i\sqrt{E}}{2}\left(\lambda\bar\eta - \frac{1}{\lambda}\eta\right) a(\lambda)$$

and

$$\beta_\eta(\lambda) = \iint\limits_{\mathbb{C}} \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\zeta+\lambda\bar\zeta\right)\right\} v_\eta(\zeta)\nu_\eta(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \iint\limits_{\mathbb{C}} \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\zeta+\lambda\bar\zeta\right)\right\} v(\zeta-\eta)\nu(\zeta-\eta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$

$$+\frac{i\sqrt{E}}{2}\left(\lambda\bar\eta-\frac{1}{\lambda}\eta\right)\times$$

$$\times \iint\limits_{\mathbb{C}} \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\zeta+\lambda\bar\zeta\right)\right\} v(\zeta-\eta)\mu(\zeta-\eta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\eta+\lambda\bar\eta\right)\right\}\left(\beta(\lambda)+\frac{i\sqrt{E}}{2}\left(\lambda\bar\eta-\frac{1}{\lambda}\eta\right)b(\lambda)\right).$$

$\square$

In order to prove Lemma 4.2 we introduce the following operator

$$T = \partial_t - 8\partial_z^3 - 2w\partial_z - 8\partial_{\bar z}^3 - 2\bar w\partial_{\bar z}, \tag{4.26}$$

where w is defined via (0.5b) for some potential v and $|w|\to 0$ as $|x|\to\infty$ $\forall t\in\mathbb{R}$. Note that $T=\partial_t + A$, where A is the operator of (0.9).

We will need the following auxiliary lemma describing how T acts on the spectral solutions of the two-dimensional Schrödinger equation.

Lemma 4.5. *Let v satisfy conditions (4.1)-(4.2) and*

$$|\partial_t v(x,t)| \leqslant \frac{\tilde q(t)}{(1+|x|)^{3+\varepsilon}},\ \textit{for some } \tilde q(t)>0.$$

Suppose that for a certain $\lambda\in\mathbb{C}$, $|\lambda|\neq 1$, and t belonging to a certain interval $t\in(t_1,t_2)$ solution $\psi(z,\lambda,t)$ of (1.1) with asymptotics (1.19) exists and is unique. Similarly, suppose that solution $\varphi(z,\lambda,t)$ of (1.1) with asymptotics (1.20) exists and is unique. Then

$$T\psi = i(\sqrt{E})^3 e^{\frac{i\sqrt{E}}{2}(\lambda\bar z+z/\lambda)}\left(\left(\lambda^3+\frac{1}{\lambda^3}\right)+o(1)\right),\ \textit{as } |z|\to\infty, \tag{4.27}$$

$$T\varphi = i(\sqrt{E})^3 e^{\frac{i\sqrt{E}}{2}(\lambda\bar z+z/\lambda)}\left(\frac{i\sqrt{E}}{2}\left(\lambda^3+\frac{1}{\lambda^3}\right)\left(\lambda\bar z-\frac{1}{\lambda}z\right)+3\left(\lambda^3-\frac{1}{\lambda^3}\right)+o(1)\right),\quad \textit{as } |z|\to\infty. \tag{4.28}$$

Proof. First of all, since we assumed that $w \to 0$ as $|z| \to \infty$, from Statement A.1 it follows that in order to demonstrate (4.27), (4.28) it is sufficient to show that

$$\partial_t \mu \to 0, \quad \partial_z^j \mu \to 0, \quad \partial_{\bar z}^j \mu \to 0, \quad j = 1, 2, 3, \text{ as } |z| \to \infty, \tag{4.29}$$

$$\partial_t \nu \to 0, \quad \partial_z \nu \to -\frac{i\sqrt{E}}{2\lambda}, \quad \partial_{\bar z} \nu \to \frac{i\sqrt{E}}{2}\lambda, \quad \partial_z^k \nu \to 0, \quad \partial_{\bar z}^k \nu \to 0, \quad k = 2, 3, \text{ as } |z| \to \infty, \tag{4.30}$$

where $\mu(z,\lambda,t) = e^{-\frac{i\sqrt{E}}{2}(\lambda\bar z + z/\lambda)}\psi(z,\lambda,t)$, $\nu(z,\lambda,t) = e^{-\frac{i\sqrt{E}}{2}(\lambda\bar z + z/\lambda)}\varphi(z,\lambda,t)$. We will only prove properties (4.29). Properties (4.30) are proved similarly.

Function μ is defined as the solution of the integral equation (1.21), where notation (1.23) is used. Differentiating (1.21) with respect to t yields the following integral equation for $\partial_t \mu$:

$$\partial_t \mu(z,\lambda,t) = \iint\limits_{\mathbb{C}} g(z-\zeta,\lambda)\partial_t v(\zeta,t)\mu(\zeta,\lambda,t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \\ + \iint\limits_{\mathbb{C}} g(z-\zeta,\lambda) v(\zeta,t)\partial_t \mu(\zeta,\lambda,t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \tag{4.31}$$

Differentiating (1.21) j times with respect to z yields the following integral equation for $\partial_z^j \mu$:

$$\partial_z^j \mu(z,\lambda,t) = \iint\limits_{\mathbb{C}} g(z-\zeta,\lambda)\partial_\zeta^j (v(\zeta,t)\mu(\zeta,\lambda,t)) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \quad j = 1, 2, 3. \tag{4.32}$$

Similarly, $\partial_{\bar z}^j \mu$ satisfies the following integral equation

$$\partial_{\bar z}^j \mu(z,\lambda,t) = \iint\limits_{\mathbb{C}} g(z-\zeta,\lambda)\partial_{\bar\zeta}^j (v(\zeta,t)\mu(\zeta,\lambda,t)) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \quad j = 1, 2, 3. \tag{4.33}$$

Equation (4.32) is an equation on the unknown function $\partial_z^j \mu$, where it is assumed that functions $\partial_z^k \mu$, $k < j$, are already defined. Similarly equation (4.33) is an equation on the unknown function $\partial_{\bar z}^j \mu$, where it is assumed that functions $\partial_{\bar z}^k \mu$, $k < j$, are already defined. The assumptions of Lemma 4.5 imply that for each of the equations (4.31), (4.32), (4.33) its solution exists, is unique and can be represented as $(1+|z|)^{2+\varepsilon/2} u(z,\lambda,t)$ with some corresponding $u(\cdot,\lambda,t) \in L^2(\mathbb{C})$.

It was shown in [47] that function g defined by (1.23) possesses the following property : $|g(z,\lambda)| \leqslant \frac{\mathrm{const}}{|z|}$ for $\forall \lambda \in \mathbb{C}$, $|\lambda| \neq 1$ and sufficiently large z, and $|g(z,\lambda)| \leqslant \mathrm{const} \ln|z|$ for

$\forall\lambda \in \mathbb{C}$, $|\lambda| \neq 1$ and sufficiently small z. This property implies that

$$\left|\iint\limits_{\mathbb{C}} g(z-\zeta,\lambda)U(\zeta)d\mathrm{Re}\zeta d\mathrm{Im}\zeta\right| \to 0 \text{ as } |z| \to \infty \quad \text{for any } U \in L^1(\mathbb{C}) \cap L^2(\mathbb{C}). \tag{4.34}$$

Let us denote by $\xi(z,\lambda,t)$ the solution of any of the equations (1.21), (4.31)-(4.33). As noted before, $\xi(z,\lambda,t)$ can be represented in the form $\xi(z,\lambda,t) = (1+|z|)^{2+\varepsilon/2}u(z,\lambda,t)$ for some $u(\cdot,\lambda,t) \in L^2(\mathbb{C})$. Then from assumptions on v it follows that $\partial_z^j v(\cdot,t)\xi(\cdot,\lambda,t) \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$, $\partial_{\bar z}^j v(\cdot,t)\xi(\cdot,\lambda,t) \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$ for $j = 0,\dots,3$ and $\partial_t v(\cdot,t)\xi(\cdot,\lambda,t) \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$. Thus, from (4.34) with $U(\cdot) = \partial_t v(\cdot,t)\mu(\cdot,\lambda,t)$ and $U(\cdot) = v(\cdot,t)\partial_t\mu(\cdot,\lambda,t)$ it follows that the right part of (4.31) tends to zero as $|z| \to \infty$. Similarly, considering equations (4.32), (4.33) consecutively we obtain that the right part of each of these equations tends to zero as $|z| \to \infty$. Consequently, $\partial_t\mu \to 0$, $\partial_z^j\mu \to 0$, $\partial_{\bar z}^j\mu \to 0$, $j = 1,2,3$, as $|z| \to \infty$.

□

Proof of Lemma 4.2. Formulas (4.9), (4.10) have already been known in literature (see for example [22]). Since their derivation is similar to the derivation of formulas (4.11), (4.12), we confine ourselves to the derivation of the latter. The derivation of analogs of formulas (4.9)-(4.12) for the case of zero energy can be found in [3].

Equation (0.5) represents a condition under which the following is true

$$[T,L]\eta = ET\eta, \quad \forall\eta\colon L\eta = E\eta, \tag{4.35}$$

where L is defined in (1.1) and T is defined in (4.26) (see [43], [3]).

Let us take $\eta = \varphi$, where φ is the solution of $L\varphi = E\varphi$ with the asymptotics (1.20). Then (4.35) implies

$$LT\varphi = E\varphi.$$

From Lemma 4.5 we have that

$$T\varphi = i(\sqrt{E})^3 e^{\frac{i\sqrt{E}}{2}(\lambda\bar z + z/\lambda)}\left(\frac{i\sqrt{E}}{2}\left(\lambda^3+\frac{1}{\lambda^3}\right)\left(\lambda\bar z - \frac{1}{\lambda}z\right) + 3\left(\lambda^3 - \frac{1}{\lambda^3}\right) + o(1)\right), \quad |z| \to \infty.$$

The uniqueness of the solution of (1.1) with the asymptotics (1.19) at the considered value of λ implies that

$$T\varphi = i(\sqrt{E})^3\left(\lambda^3+\frac{1}{\lambda^3}\right)\varphi + 3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)\psi.$$

In other words,

$$\partial_t \varphi = 8\partial_z^3 \varphi + 2w\partial_z \varphi + 8\partial_{\bar z}^3 \varphi + 2\bar w \partial_{\bar z}\varphi + i(\sqrt{E})^3 \left(\lambda^3 + \frac{1}{\lambda^3}\right)\varphi + 3i(\sqrt{E})^3 \left(\lambda^3 - \frac{1}{\lambda^3}\right)\psi. \quad (4.36)$$

Now we write $\alpha(\lambda, t)$ in the form

$$\alpha(\lambda, t) = \iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta + \zeta/\lambda)} v(\zeta, t)\varphi(\zeta, \lambda, t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta$$

and compute its derivative with respect to time :

$$\partial_t \alpha(\lambda, t) = \iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta + \zeta/\lambda)} \partial_t v(\zeta, t)\varphi(\zeta, \lambda, t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \\ + \iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta + \zeta/\lambda)} v(\zeta, t)\partial_t \varphi(\zeta, \lambda, t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \quad (4.37)$$

Substituting (0.5) and (4.36) into (4.37), integrating the resulting expression by parts and taking into account that $-4\partial_\zeta \partial_{\bar\zeta}\varphi + v\varphi = E\varphi$, we obtain

$$\partial_t \alpha(\lambda, t) = 3i(\sqrt{E})^3 \left(\lambda^3 - \frac{1}{\lambda^3}\right)(a(\lambda, t) - \hat v(0)), \quad (4.38)$$

where $\hat v(0) = \iint\limits_{\mathbb{C}} v(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta$ (see subsection A.2 of Appendix for a detailed derivation of this formula). Formulas (4.9), (4.38) yield (4.11).

Similarly, we write $\beta(\lambda, t)$ in the form

$$\beta(\lambda, t) = \iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta + \bar\zeta/\bar\lambda)} v(\zeta, t)\varphi(\zeta, \lambda, t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta$$

and compute its derivative with respect to time :

$$\partial_t \beta(\lambda, t) = \iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta + \bar\zeta/\bar\lambda)} \partial_t v(\zeta, t)\varphi(\zeta, \lambda, t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \\ + \iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta + \bar\zeta/\bar\lambda)} v(\zeta, t)\partial_t \varphi(\zeta, \lambda, t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \quad (4.39)$$

Substituting (0.5) and (4.36) into (4.39), integrating the resulting expression by parts and taking into account that $-4\partial_\zeta\partial_{\bar\zeta}\varphi + v\varphi = E\varphi$, we obtain

$$\partial_t\beta(\lambda,t) = i(\sqrt{E})^3\left(\lambda^3+\frac{1}{\lambda^3}+(\mathrm{sgn}E)\left(\bar\lambda^3+\frac{1}{\bar\lambda^3}\right)\right)\beta(\lambda,t)+3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)b(\lambda,t) \quad (4.40)$$

(see subsection A.2 of Appendix for a detailed derivation of this formula). Using formula (4.10), we obtain (4.12). □

5 Absence of conductivity type solitons at $E = 0$

The study of the Novikov-Veselov equation at zero energy presents, in a certain sense, a more difficult case in comparison with the case of nonzero energy due to the fact that the scattering data of the associated two-dimensional stationary Schrödinger equation at zero energy have a logarithmic singularity at the origin. Thus even the small-norm assumption on the potential does not guarantee the solvability of the direct scattering equations (see Section 2). However, there exists a class of potentials, the so-called conductivity type potentials, for which the scattering data are everywhere well-defined. The importance of this class of potentials is two-fold : they arise in physical applications and they exhibit some interesting mathematical properties. In the present Section, using the property of nonsingularity of the scattering data for the conductivity type potentials, we prove that the Novikov-Veselov equation at zero energy does not possess solitons of conductivity type localized as $O\left(|x|^{-2-\varepsilon}\right)$, $|x|\to\infty$, for $\forall\varepsilon>0$. Note that there exist, however, localized nonsingular soliton solutions of the Novikov-Veselov equation at zero energy which are not of conductivity type. For example, in [6] a nonsingular rational stationary solution of (0.5) at $E=0$ is presented :

$$\begin{aligned}
&v(z,t) = U\left(\frac{1}{2}z,t\right), \quad w(z,t) = -3V\left(\frac{1}{2}z,t\right), \text{ where}\\
&U(z,t) = -\frac{240z\bar z}{(30\bar z^2z^2+1)^2},\\
&V(z,t) = \frac{3600\bar z^4z^2-120\bar z^2}{(30\bar z^2z^2+1)^2}.
\end{aligned}$$

Note that $|w(z)| = O\left(|z|^{-2}\right)$ as $|z|\to\infty$.

Recall the definition of the conductivity type potentials given in Section 2.

Definition. *A potential $v\in L^p(\mathbb{R}^2)$, $1<p<2$, is of conductivity type if $v=\gamma^{-1/2}\Delta\gamma^{1/2}$ for some real-valued positive $\gamma\in L^\infty(\mathbb{R}^2)$, such that $\gamma\geqslant\delta_0>0$ and $\nabla\gamma^{1/2}\in L^p(\mathbb{R}^2)$.*

Potentials of conductivity type arise naturally when the Calderón conductivity problem is studied in the context of the inverse scattering problem for the 2-dimensional Schrödinger equation at zero energy. The problem of recovering an electrical conductivity from boundary measurements, voltage and current proposed by A.P. Calderón in [5] consists in the following. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, and let γ be a real-valued function in $L^\infty(\Omega)$ with a positive lower bound. The corresponding Dirichlet-to-Neumann map is the operator $\Lambda_\gamma \colon H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$

$$\Lambda_\gamma f = \gamma \nabla u \cdot \nu|_{\partial\Omega},$$

where $f \in H^{1/2}(\partial\Omega)$, ν is the outer normal to $\partial\Omega$ and u is the unique $H^1(\Omega)$ solution of the Dirichlet problem

$$\nabla(\gamma\nabla u) = 0 \text{ in } \Omega, u|_{\partial\Omega} = f. \tag{5.1}$$

Equation (5.1) represents the conservation of the electrical charge on Ω if the voltage f is applied to $\partial\Omega$, and $\Lambda_\gamma f$ is the current flux at the boundary. The Calderón problem consists in finding γ in Ω given operator Λ_γ.

One of the first and most studied strategies to solve Calderón problem is to substitute $\tilde{u} = u\sqrt{\gamma}$ into (5.1) to obtain

$$(-\Delta + v)\tilde{u} = 0 \text{ in } \Omega \text{ with } v = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{5.2}$$

If we define the Dirichlet-to-Neumann operator

$$\Lambda_v \tilde{f} = \nabla\tilde{u} \cdot \nu|_{\partial\Omega}, \tag{5.3}$$

where $\tilde{f} \in H^{1/2}(\partial\Omega)$ and $\tilde{u}$ is the unique solution of the Dirichlet problem for (5.2) with value $\tilde{f}$ on the boundary, then it can be related to Λ_γ in the following way : $\Lambda_v = \gamma^{-1/2}\left(\Lambda_\gamma + \frac{1}{2}\frac{\partial\gamma}{\partial\nu}\right)\gamma^{-1/2}$. Thus the Calderón problem can be reduced to the inverse scattering problem for the two-dimensional Schrödinger equation (5.2), i.e. to finding the potential of the Schrödinger operator of (5.2) given the associated Dirichlet-to-Neumann operator (5.3). In this form the problem was first posed by I.M. Gelfand in [16].

There exists a vast range of works devoted to the Gelfand-Calderón problem (we note, in particular, paper [48], where the first reconstruction algorithm was proposed) ; the reader may refer, for example, to [57] for a review of related results. In this work we will be mostly interested in the result of [46] which states that the scattering data for the Schrödinger operator with potential v of conductivity type are nonsingular.

Potentials of conductivity type exhibit some interesting mathematical properties related to the Novikov-Veselov equation. In [41] it was shown that the Novikov-Veselov equation at zero energy preserves the conductivity property of solutions. In [55] it was noted that the solutions of the Novikov-Veselov equation of conductivity type are exactly the range of the solutions of the modified Novikov-Veselov equation (a member of the Davey-Stewartson II integrable hierarchy and a $(2+1)$–dimensional analog of mKdV) under a Miura-type transformation proposed by L.V. Bogdanov in [2].

In this Section we prove that the Novikov-Veselov equation does not have conductivity type solitons localized stronger than $O\left(|x|^{-2}\right)$, $|x| \to \infty$. More precisely, in this subsection we are concerned with the solutions of the Novikov-Veselov equation that satisfy the following conditions :

- $v, w \in C(\mathbb{R}^2 \times \mathbb{R}),\ v(\cdot, t) \in C^3(\mathbb{R}^2) \quad \forall t \in \mathbb{R};$ (5.4)
- $|\partial_x^j v(x,t)| \leqslant \dfrac{q(t)}{(1+|x|)^{2+\varepsilon}},\ j=(j_1,j_2) \in (\mathbb{N}\cup 0)^2,\ j_1+j_2 \leqslant 3,$ for some $q(t) > 0, \varepsilon > 0;$ (5.5)
- $|w(x,t)| \to 0,$ when $|x| \to \infty, \quad t \in \mathbb{R}.$ (5.6)

The main result of the present Section consists in the following theorem.

Theorem 5.1. *Let (v, w) be a traveling wave solution of (0.5) of conductivity type satisfying conditions (5.4)-(5.6). Then $v \equiv 0$, $w \equiv 0$.*

The proof of the theorem is based on the ideas proposed in [51]. First we present a lemma that describes the scattering data corresponding to a shifted potential.

Lemma 5.1. *Let $v(z)$ be a potential satisfying (1.2) with the scattering data $b(\lambda)$. The scattering data $b_\eta(\lambda)$ for the potential $v_\eta(z) = v(z-\eta)$ are related to $b(\lambda)$ by the following formula*

$$b_\eta(\lambda) = e^{i(\lambda\eta + \bar\lambda\bar\eta)} b(\lambda), \quad \lambda \in \mathbb{C}\backslash 0, \quad \eta \in \mathbb{C}. \tag{5.7}$$

Proof. We note that $\psi(z-\eta,\lambda)$ satisfies (2.1) with $v_\eta(z)$ and has the asymptotics $\psi(z-\eta,\lambda) = e^{i\lambda(z-\eta)}(1+o(1))$ as $|z| \to \infty$. Thus $\psi_\eta(z,\lambda) = e^{i\lambda\eta}\psi(z-\eta,\lambda)$ and $\mu_\eta(z,\lambda) = \mu(z-\eta,\lambda)$. Finally, we have

$$b_\eta(\lambda) = \iint\limits_{\mathbb{C}} e^{i(\lambda\zeta + \bar\lambda\bar\zeta)} v_\eta(\zeta)\mu_\eta(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$
$$= \iint\limits_{\mathbb{C}} e^{i(\lambda\zeta + \bar\lambda\bar\zeta)} v(\zeta-\eta)\mu_\zeta(\zeta-\eta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = e^{i(\lambda\eta+\bar\lambda\bar\eta)} b(\lambda).$$

$\square$

As for the time dynamics of the scattering data, in [3], [27] it was shown that if the solution (v, w) of (0.5) at $E = 0$ exists and the scattering data for this solution are well-defined, then the time evolution of these scattering data is described as follows :

$$b(\lambda, t) = e^{8i(\lambda^3 + \bar{\lambda}^3)t} b(\lambda, 0), \quad \lambda \in \mathbb{C}\backslash 0, \quad t \in \mathbb{R}. \tag{5.8}$$

Now we are ready to give the proof of Theorem 5.1.

Proof of Theorem 5.1. Since (v, w) is a soliton solution of (0.5) at $E = 0$, the dynamics of its scattering data b is described by (5.8) and by (5.7) with $\eta = ct$ for some $c \in \mathbb{C}$. From (5.7), (5.8), continuity of $b(\lambda)$ on $\mathbb{C}\backslash 0$ and the fact that the functions λ, $\bar{\lambda}$, λ^3, $\bar{\lambda}^3$, 1 are linearly independent in the neighborhood of any point, it follows that $b \equiv 0$. Equation (2.8) implies that in this case function $\mu(z, \lambda)$ is holomorphic on λ, $\lambda \in \mathbb{C}\backslash 0$. Using items 4 and 3 of Statement 2.1 we apply Liouville theorem to obtain that $\mu \equiv 1$. Then $\psi(z, \lambda) = e^{i\lambda z}$ and from (2.1) it follows that $v \equiv 0$. $\square$

Chapter III

Large time asymptotics for solutions of the Novikov-Veselov equation

6 Large time asymptotics for transparent potentials for the Novikov-Veselov equation at positive energy

In this Section we study the large time asymptotics of the solutions to the Cauchy problem for the Novikov-Veselov equation at positive energy. We consider the Cauchy problem for equation (0.5) at $E > 0$ with the initial data

$$v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x). \tag{6.1}$$

We will assume that function $v_0(x)$ satisfies the following conditions

- $v_0 = \bar{v}_0,$ i.e. v_0 is a real-valued function, (6.2a)
- the integral equations of direct scattering for the 2-dimensional Scrödinger equation (1.1) with the given potential $v_0(x)$ are everywhere uniquely solvable, (6.2b)
- $v_0 \in \mathcal{S}(\mathbb{R}^2),$ where $\mathcal{S}$ denotes the Schwartz class, (6.2c)
- v_0 is transparent for (1.1) . (6.2d)

As for function $w_0(x)$, we will assume that it is a continuous function decaying at infinity and determined using $\partial_{\bar{z}} w_0(x) = -3\partial_z v_0(x)$ from (0.5).

Condition (6.2b) is equivalent to non–singularity of scattering data for $v_0(x)$. Conditions (6.2) define the class of initial values for which the direct scattering equation (1.21) and the inverse scattering equation (1.38), together with items 1, 3 of Statement 1.2 and the time dynamics given by (4.10), are everywhere solvable and the corresponding solution v of (0.5) belongs to $C^{\infty}(\mathbb{R}^2, \mathbb{R})$. Note that the Novikov-Veselov equation does not preserve the property of strong space localization, since its solutions, satisfying conditions (6.2) at the initial moment of time, are localized as $O\left(|x|^{-3}\right)$, $|x| \to \infty$, at any subsequent moment of time (see [25]). Our choice to study transparent potentials is motivated, in particular, by our interest in the question of existence of solitons for the Novikov-Veselov equation and by the result of [51] which says that solitons of the Novikov-Veselov equation at positive energy are transparent potentials (see Lemma 4.3).

We will call a solution $(v(x,t), w(x,t))$, constructed from $(v_0(x), w_0(x))$ via the inverse scattering method, an "inverse scattering solution" of (0.5). The main result of this Section consists of the following (see also Theorem 6.1) : we show that for the "inverse scattering solution" $v(x,t)$ of (0.5), (6.1), where $E > 0$ and $v(x,0) = v_0(x)$ satisfies (6.2), the following estimate holds :

$$|v(x,t)| \leqslant \frac{\mathrm{const}(v_0)\ln(3+|t|)}{1+|t|}, \quad t \in \mathbb{R}, \text{ uniformly on } x \in \mathbb{R}^2. \tag{6.3}$$

Estimate (6.3) implies that there are no isolated soliton type waves in the large time asymptotics for $v(x,t)$, in contrast with large time asymptotics for solutions of the KdV equation with reflectionless initial data.

Estimate (6.3) is proved using the stationary phase method (see, for example, Chapter III of [12]), techniques developed in [25] and [38] and an analysis of some cubic algebraic equation depending on a complex parameter.

6.1 Preliminaries

Consider

$$\begin{aligned} I(t,z) &= \iint\limits_{\mathbb{C}} f(\zeta)\exp(iS(\zeta,z,t))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \\ J(t,z) &= -3\iint\limits_{\mathbb{C}} \frac{\bar{\zeta}}{\zeta} f(\zeta)\exp(iS(\zeta,z,t))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \end{aligned} \tag{6.4}$$

where $f(\zeta) \in L^1(\mathbb{C})$, S is defined by

$$S(\lambda,z,t) = -\frac{1}{2}\left(\left(\lambda + \frac{1}{\bar{\lambda}}\right)\bar{z} + \left(\bar{\lambda} + \frac{1}{\lambda}\right)z\right) + t\left(\lambda^3 + \frac{1}{\lambda^3} + \bar{\lambda}^3 + \frac{1}{\bar{\lambda}^3}\right), \tag{6.5}$$

and $z \in \mathbb{C}$, $t \in \mathbb{R}$.

If $v(z,t) = I(t,z)$, $w(z,t) = J(t,z)$, where

$$(|\zeta|^3 + |\zeta|^{-3}) f(\zeta) \in L^1(\mathbb{C})$$

as a function of ζ, and, in addition,

$$\overline{f(\zeta)} = f(-\zeta) \quad \text{and/or} \quad \overline{f(\zeta)} = -|\zeta|^{-4} f\left(-\frac{1}{\bar{\zeta}}\right),$$

then v, w satisfy the linearized Novikov–Veselov equation (0.5) with $E = 1$. In addition,

$$\hat{v}(p,t) \equiv 0 \quad \text{for} \quad |p| < 2, \quad t \in \mathbb{R},$$

where $\hat{v}(\cdot,t)$ is the Fourier transform of $v(\cdot,t)$, that is $v(\cdot,t)$ is transparent in the Born approximation at energy $E = 1$ for each $t \in \mathbb{R}$.

The goal of the following subsection will be to give, in particular, a uniform estimate of the large–time behavior of integral $I(t,z)$ of (6.4) under assumptions that

$$\begin{aligned}
&f \in C^\infty(\mathbb{C}),\\
&\partial_\zeta^m \partial_{\bar{\zeta}}^n f(\zeta) = \begin{cases} O\left(|\zeta|^{-\infty}\right) & \text{as } |\zeta| \to \infty,\\ O\left(|\zeta|^{\infty}\right) & \text{as } |\zeta| \to 0,\end{cases}\\
&\partial_\zeta^m \partial_{\bar{\zeta}}^n f(\zeta)|_{|\zeta|=1} = 0
\end{aligned} \tag{6.6}$$

for all $m, n \geqslant 0$.

We can integrate $I(t,z)$ by parts in a similar way as in Lemma 2.1 in Chapter III of [12] using (6.6) and the fact that

$$\frac{\partial}{\partial\lambda} R(\lambda) \neq 0, \quad \frac{\partial}{\partial\bar{\lambda}} R(\lambda) \neq 0, \text{ for } |\lambda| \neq 1, \quad \lambda \neq 0,$$

where R is

$$R(\lambda) = \lambda^3 + \bar{\lambda}^3 + \frac{1}{\lambda^3} + \frac{1}{\bar{\lambda}^3}.$$

Thus we obtain

$$|I(t,z)| = O\left(\frac{1}{|t|^n}\right), t \to \infty, \quad n \in \mathbb{N}, \tag{6.7}$$

uniformly on $z \in K$, where K is any compact set of the complex plane. Nevertheless, this is not sufficient to guarantee the absence of soliton–type waves in the large time asymptotics of the potential $v(z,t) = I(t,z)$. So our further reasoning will be devoted to obtaining an estimate like (6.7) for $n = 1$ uniformly on $z \in \mathbb{C}$.

For this purpose we introduce parameter $u = \frac{z}{t}$ and write the integral I in the following form

$$I(t,u) = \iint\limits_{\mathbb{C}} f(\zeta) \exp(itS(u,\zeta)) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{6.8}$$

where

$$S(u,\zeta) = -\frac{1}{2}\left(\bar{\zeta}u + \zeta\bar{u} + \frac{u}{\zeta} + \frac{\bar{u}}{\bar{\zeta}}\right) + \left(\zeta^3 + \bar{\zeta}^3 + \frac{1}{\zeta^3} + \frac{1}{\bar{\zeta}^3}\right). \tag{6.9}$$

We will start by studying the properties of the stationary points of the function $S(u,\zeta)$. These points satisfy the equation

$$S'_\zeta = -\frac{\bar{u}}{2} + \frac{u}{2\zeta^2} + 3\zeta^2 - \frac{3}{\zeta^4} = 0. \tag{6.10}$$

The degenerate stationary points obey additionally the equation

$$S''_{\zeta\zeta} = -\frac{u}{\zeta^3} + 6\zeta + \frac{12}{\zeta^5} = 0. \tag{6.11}$$

We denote $\xi = \zeta^2$ and

$$Q(u,\xi) = -\frac{\bar{u}}{2} + \frac{u}{2\xi} + 3\xi - \frac{3}{\xi^2}.$$

For each ξ, a root of the function $Q(u,\xi)$, there are two corresponding stationary points of $S(u,\zeta)$, $\zeta = \pm\sqrt{\xi}$.

Function $S'_\zeta(u,\zeta)$ can be represented in the following form

$$S'_\zeta(u,\zeta) = \frac{3}{\zeta^4}(\zeta^2 - \zeta_0^2(u))(\zeta^2 - \zeta_1^2(u))(\zeta^2 - \zeta_2^2(u)). \tag{6.12}$$

We will also use hereafter the following notations :

$$\mathfrak{U} = \{u = 6(2e^{-i\varphi} + e^{2i\varphi}),\ \varphi \in [0, 2\pi)\}$$

and

$$\mathbb{U} = \{u = re^{i\varphi} \colon r \le |6(2e^{-i\varphi} + e^{2i\varphi})|,\ \varphi \in [0, 2\pi)\},$$

the domain limited by the curve $\mathfrak{U}$ (see also Figure III.1).

Lemma 6.1.

1. *If* $u = 18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, *then*

$$\zeta_0(u) = \zeta_1(u) = \zeta_2(u) = e^{-\frac{\pi ik}{3}}$$

and $S(u,\zeta)$ *has two degenerate stationary points, corresponding to a third-order root of the function* $Q(u,\xi)$, $\xi_1 = e^{-\frac{2\pi ik}{3}}$.

2. *If* $u \in \mathfrak{U}$ (*i.e.* $u = 6(2e^{-i\varphi} + e^{2i\varphi})$) *and* $u \neq 18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, *then*

$$\zeta_0(u) = \zeta_1(u) = e^{i\varphi/2}, \quad \zeta_2(u) = e^{-i\varphi}.$$

Thus $S(u,\zeta)$ *has two degenerate stationary points, corresponding to a second-order root of the function* $Q(u,\xi)$, $\xi_1 = e^{i\varphi}$, *and two non–degenerate stationary points corresponding to a first-order root,* $\xi_2 = e^{-2i\varphi}$.

3. *If* $u \in \mathrm{int}\mathbb{U}$, *then*

$$\zeta_i(u) = e^{i\varphi_i}, \quad \textit{and} \quad \zeta_i(u) \neq \zeta_j(u) \quad \textit{for} \quad i \neq j.$$

In this case the stationary points of $S(u,\zeta)$ *are non-degenerate and correspond to the roots of the function* $Q(u,\xi)$ *with absolute values equal to 1.*

4. *If* $u \in \mathbb{C}\backslash\mathbb{U}$, *then*

$$\zeta_0(u) = (1+\omega)e^{i\varphi/2}, \quad \zeta_1(u) = e^{-i\varphi}, \quad \zeta_2(u) = (1+\omega)^{-1}e^{i\varphi/2}$$

for certain φ *and* $\omega > 0$.

In this case the stationary points of the function $S(u,\zeta)$ *are non-degenerate, and correspond to the roots of the function* $Q(u,\xi)$ *that can be expressed as* $\xi_0 = (1+\tau)e^{i\varphi}$, $\xi_1 = e^{-2i\varphi}$, $\xi_2 = (1+\tau)^{-1}e^{i\varphi}$, $(1+\tau) = (1+\omega)^2$.

Lemma 6.1 is proved in subsection 6.4.

Formula (6.12) and Lemma 6.1 give a complete description of the stationary points of the function $S(u,\zeta)$.

In order to estimate uniformly on $u, \lambda \in \mathbb{C}$ the large–time behavior of the integral

$$I(t,u,\lambda) = \iint\limits_{\mathbb{C}} f(\zeta,\lambda)\exp(itS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{6.13}$$

where f satisfies assumptions (6.6), in the following subsections we will use the following general scheme.

1. Consider D_ε, the union of disks with a radius of ε and centers in singular points of function $f(\zeta,\lambda)$ and stationary points of $S(u,\zeta)$.

2. Represent $I(t,u,\lambda)$ as the sum of integrals over D_ε and $\mathbb{C}\backslash D_\varepsilon$:

$$\begin{aligned} &I(t,u,\lambda) = I_{int} + I_{ext}, \qquad \text{where} \\ &I_{int} = \iint\limits_{D_\varepsilon} f(\zeta,\lambda)\exp(itS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \\ &I_{ext} = \iint\limits_{\mathbb{C}\backslash D_\varepsilon} f(\zeta,\lambda)\exp(itS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \end{aligned} \tag{6.14}$$

3. Find an estimate of the form

$$|I_{int}| = O\left(\varepsilon^\alpha\right), \quad \text{as} \quad \varepsilon \to 0 \quad (\alpha \geq 1)$$

uniformly on u, λ, t.

4. Integrate I_{ext} by parts using Stokes formula

$$\begin{aligned} I_{ext} = \frac{1}{2t}\int\limits_{\partial D_\varepsilon} \frac{f(\zeta,\lambda)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}d\bar{\zeta} - \frac{1}{it}\iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{f'_\zeta(\zeta,\lambda)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta - \\ - \frac{1}{it}\iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{f(\zeta,\lambda)\exp(itS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}d\mathrm{Re}\zeta d\mathrm{Im}\zeta = \\ = -\frac{1}{t}(I_1 - I_2 - I_3). \end{aligned} \tag{6.15}$$

5. For each I_i find an estimate of the form

$$(a)\ |I_i| = O\left(\ln\frac{1}{\varepsilon}\right) \quad \text{or} \quad (b)\ |I_i| = O\left(\frac{1}{\varepsilon^\beta}\right), \qquad \text{as } \varepsilon \to 0.$$

6. In case (a) set $\varepsilon = \dfrac{1}{|t|}$ which yields the overall estimate

$$|I(t,u,\lambda)| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad \text{as} \quad t \to \infty.$$

In case (b) set $\varepsilon = \dfrac{1}{|t|^k}$, where $k(\alpha+\beta) = 1$, which yields the overall estimate

$$|I(t,u,\lambda)| = O\left(\frac{1}{|t|^{\frac{\alpha}{\alpha+\beta}}}\right), \quad \text{as} \quad t \to \infty.$$

6.2 Estimate for the solution of the linearized equation

Using the scheme presented in the previous subsection we obtain, in particular, the following result

Lemma 6.2. *Under assumptions (6.6) the following estimate is valid for $I(t,u)$ defined by (6.8) :*

$$|I(t,u)| = O\left(\frac{\ln(3+|t|)}{1+|t|}\right) \quad \textit{for} \quad t \in \mathbb{R}$$

uniformly on $u \in \mathbb{C}$.

Proof. We take D_ε to be the union of disks with a radius of ε centered in the stationary points of $S(u,\zeta)$. The integral I_{int} (as in (6.14) with $f(\zeta,\lambda) = f(\zeta)$ satisfying properties (6.6)) is estimated as

$$|I_{int}| = \left|\iint\limits_{D_\varepsilon} f(\zeta)\exp(itS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta\right| \leqslant \mathrm{const}\left|\iint\limits_{D_\varepsilon} d\mathrm{Re}\zeta d\mathrm{Im}\zeta\right| = O(\varepsilon^2).$$

The estimate for I_{ext} (as in (6.14) with $f(\zeta,\lambda) = f(\zeta)$ satisfying properties (6.6)) is proved separately for $u \in \mathbb{U}$ and $u \in \mathbb{C}\backslash\mathbb{U}$.

I. $u \in \mathbb{U}$:
In this case all $\zeta \in D_\varepsilon$ lie in the ε–neighborhood of the unit circle. Consequently, from the fact that $b(\cdot,t) \in \mathcal{S}(\mathbb{C})$ and from property 9 of Statement 1.2 it follows that for any $N \in \mathbb{N}$ and any $\varepsilon_0 \in (0,1/2]$ there exists $C = C(\varepsilon_0, N)$ such that

$$|f(\zeta)| \leqslant C\rho^N, \quad |f'_\zeta(\zeta)| \leqslant C\rho^N$$

for all $\zeta \in D_\rho$, $\rho \leqslant \varepsilon_0$.

The function $S'_\zeta(u,\zeta)$ can be estimated as

$$|S'_\zeta(u,\zeta)| \geqslant 3\frac{\varepsilon_0^6}{|\zeta|^4} \quad \text{for } \zeta\in\mathbb{C}\backslash D_{\varepsilon_0}, \quad \text{and}$$
$$|S'_\zeta(u,\zeta)| \geqslant 3\frac{\rho^6}{|\zeta|^4} \quad \text{for } \zeta\in\partial D_\rho, \quad \varepsilon\leqslant\rho\leqslant\varepsilon_0.$$

Taking $N=5$ results in the following estimate for I_1

$$|I_1| \leqslant \frac{1}{2}\int\limits_{\partial D_\varepsilon}\frac{|f(\zeta)|}{|S'_\zeta(u,\zeta)|}d\bar\zeta \leqslant \text{const}\frac{\varepsilon^N}{\varepsilon^6}\int\limits_{\partial D_\varepsilon}|\zeta|^4 d\bar\zeta \leqslant$$
$$\leqslant \text{const}\frac{\varepsilon^N\varepsilon}{\varepsilon^6}(1+\varepsilon)^4 = O(1), \quad \text{as}\, \varepsilon\to 0.$$

When estimating I_2 and I_3 we integrate separately over $D_{\varepsilon_0}\backslash D_\varepsilon$ and $\mathbb{C}\backslash D_\varepsilon$:

$$|I_2| \leqslant \iint\limits_{D_{\varepsilon_0}\backslash D_\varepsilon}\left|\frac{f'_\zeta(\zeta)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}}\left|\frac{f'_\zeta(\zeta)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant$$
$$\leqslant \text{const}\int\limits_\varepsilon^{\varepsilon_0}\frac{\rho^N\rho}{\rho^6}d\rho + \text{const}\iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}}|f(\zeta)||\zeta^4| d\mathrm{Re}\zeta d\mathrm{Im}\zeta = O(1), \quad \text{as} \quad \varepsilon\to 0,$$

$$|I_3| \leqslant \iint\limits_{D_{\varepsilon_0}\backslash D_\varepsilon}\left|\frac{f(\zeta)\exp(itS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$
$$+\iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}}\left|\frac{f(\zeta)\exp(itS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant$$
$$\leqslant \text{const}\int\limits_\varepsilon^{\varepsilon_0}\frac{\rho^N\rho}{\rho^{12}}d\rho + \text{const}\iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}}|f(\zeta)||\zeta^3| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \overset{N=11}{=} O(1), \quad \text{as} \quad \varepsilon\to 0.$$

Setting finally $\varepsilon=\frac{1}{|t|}$ yields

$$I(t,u) = O\left(\frac{1}{|t|}\right), \quad \text{as} \quad t\to\infty$$

uniformly on $u\in\mathbb{U}$.

II. $u\in\mathbb{C}\backslash\mathbb{U}$:

Let us divide the complex plane into six sets, each containing one and only one stationary point of $S(u,\zeta)$: $\mathbb{C} = \bigcup\limits_{k=0}^{2} \left(Z_k^+ \cup Z_k^- \right)$. We define the set $Z_k^{\pm}$ as the set of points of the complex plane to which the stationary point $\pm\zeta_k$ is the closest :

$$\begin{aligned} Z_k^+ &= \{\zeta \in \mathbb{C} \colon |\zeta - \zeta_i| \geqslant |\zeta - \zeta_k|, \quad |\zeta + \zeta_j| \geqslant |\zeta - \zeta_k|, \quad i,j \in \{0,1,2\}\}, \\ Z_k^- &= \{\zeta \in \mathbb{C} \colon |\zeta - \zeta_i| \geqslant |\zeta + \zeta_k|, \quad |\zeta + \zeta_j| \geqslant |\zeta + \zeta_k|, \quad i,j \in \{0,1,2\}\} \end{aligned}$$

where $k \in \{0,1,2\}$. We will estimate the integral over each Z_k^+ separately. The integrals over Z_k^- are treated similarly.

Let us first take $\zeta \in Z_1^+$. Using the definition of Z_1^+ and the property that all $\zeta \in D_\varepsilon \bigcap Z_1^+$ lie in the ε–neighborhood of the unit circle one can see that the scheme of reasoning for the case I is applicable.

Now let us consider $\zeta \in Z_0^+ \bigcup Z_2^+$.

(A) First, we will study the set of values of parameter ω for which $\zeta_2(u)$, $-\zeta_2(u)$ lie outside the $2\varepsilon_0$–neighborhood of zero, i.e. $\frac{1}{1+\omega} > 2\varepsilon_0$.

We will consider $\zeta \in Z_0^+$ (the case $\zeta \in Z_2^+$ is treated similarly). If $\zeta \in Z_0^+ \bigcap D_{\varepsilon_0}$ (for a certain ε_0), then it can be represented as

$$\zeta = (1+\omega)e^{i\varphi/2} + \rho e^{i\theta}, \quad \rho \leqslant \varepsilon_0.$$

Let us estimate the ratio $\dfrac{f(\zeta)}{S'_\zeta(\zeta,u)}$ in D_{ε_0}. If ζ_0 belongs to the $2\varepsilon_0$–neighborhood of ζ_1, then all $\zeta \in Z_0^+ \bigcap D_{\varepsilon_0}$ belong to the $3\varepsilon_0$–neighborhood of ζ_1. The following estimates hold :

$$|\zeta + \zeta_1| \geqslant 2 - 3\varepsilon_0, \quad |\zeta + \zeta_0| > 2 - \varepsilon_0, \quad |\zeta + \zeta_2| \geqslant 1 + \varepsilon_0, \quad \zeta \in Z_0^+ \bigcap D_{\varepsilon_0}. \tag{6.16}$$

Further, we note that for any $N \in \mathbb{N}$ there exists a function $\tilde{f}(\zeta)$ satisfying properties (6.6), such that $|f(\zeta)| \leqslant |\zeta - \zeta_1|^N |\tilde{f}(\zeta)|$ for ζ belonging to the $3\varepsilon_0$–neighborhood of ζ_1. This and (6.16) imply that

$$\left| \frac{f(\zeta)}{S'_\zeta(\zeta,u)} \right| \leqslant \text{const} \frac{|\tilde{f}(\zeta)||\zeta|^4}{|\zeta - \zeta_2||\zeta - \zeta_0|}. \tag{6.17}$$

A similar reasoning holds for the case when ζ_0 belongs to the $2\varepsilon_0$–neighborhood of $-\zeta_1$. Now if ζ_0 does not belong to the $2\varepsilon_0$–neighborhood of ζ_1 and $-\zeta_1$, then $|\zeta - \zeta_1| \geqslant \varepsilon_0$ and $|\zeta + \zeta_1| \geqslant \varepsilon_0$ for all $\zeta \in Z_0^+ \bigcap D_{\varepsilon_0}$. Two last estimates of (6.16) hold and thus (6.17)

holds with $\tilde{f}(\zeta) = f(\zeta)$.
The difference $|\zeta - \zeta_2|$ can be estimated

$$|\zeta - \zeta_2| \geqslant \frac{1}{2}|\zeta_0 - \zeta_2| = \frac{\omega(2+\omega)}{2(1+\omega)}, \quad \zeta \in Z_0^+. \tag{6.18}$$

In order to get rid of this member in the denominator, let us represent $\tilde{f}(\zeta)$ by the Taylor formula in the neighborhood of $(1+\omega)e^{i\varphi/2}$:

$$\tilde{f}(\zeta) = \tilde{f}(e^{i\varphi/2} + \omega e^{i\varphi/2}) + \tilde{f}'(e^{i\varphi/2} + \omega e^{i\varphi/2} + se^{i\theta})\rho, \quad s = \lambda\rho \text{ for some } \lambda \in [0,1],$$

where $'$ denotes the derivative with respect to s and where λ depends, in particular, on ρ.
For an arbitrary value of ω the following estimates hold :

$$|\tilde{f}(e^{i\varphi/2} + \omega e^{i\varphi/2})| \leqslant \mathrm{const}|\omega|^N, \tag{6.19}$$

$$|\tilde{f}'(e^{i\varphi/2} + \omega e^{i\varphi/2})| \leqslant \mathrm{const}|\omega|^N. \tag{6.20}$$

This finally yields

$$\frac{|\tilde{f}(\zeta)||\zeta|^4}{|\zeta-\zeta_2||\zeta-\zeta_0|} \leqslant \mathrm{const}\left(\frac{|\tilde{f}(e^{i\varphi/2} + \omega e^{i\varphi/2})|}{\frac{\omega(2+\omega)}{2(1+\omega)}|\zeta-\zeta_0|} + \frac{|\tilde{f}'(e^{i\varphi/2} + \omega e^{i\varphi/2} + \lambda\rho e^{i\theta})|\rho}{\rho|\zeta-\zeta_0|}\right) \leqslant$$

$$\leqslant \frac{\mathrm{const}}{|\zeta-\zeta_0|}, \quad \zeta \in Z_0^+ \bigcap D_{\varepsilon_0}.$$

Now we are ready to estimate I_1 :

$$\int\limits_{Z_0^+ \bigcap \partial D_\varepsilon} \left|\frac{f(\zeta)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\bar{\zeta} \leqslant \mathrm{const} \int\limits_{Z_0^+ \bigcap \partial D_\varepsilon} \frac{d\bar{\zeta}}{|\zeta-\zeta_0|} = O(1) \quad \text{as} \quad \varepsilon \to 0.$$

For I_2 we note that the estimate

$$\left|\frac{f'_\zeta(\zeta)}{S'_\zeta(u,\zeta)}\right| \leqslant \frac{\mathrm{const}}{|\zeta-\zeta_0|}, \quad \zeta \in Z_0^+ \bigcap D_{\varepsilon_0},$$

can be obtained using the same reasoning as for the ratio $\dfrac{f(\zeta)}{S'_\zeta(u,\zeta)}$. Thus

$$\iint\limits_{Z_0^+\setminus D_\varepsilon} \left|\frac{f'_\zeta(\zeta)\exp(itS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \mathrm{const}\Bigg(\iint\limits_{Z_0^+\setminus D_{\varepsilon_0}} |f'_\zeta(\zeta)||\zeta|^4 d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \\ + \iint\limits_{Z_0^+\bigcap(D_{\varepsilon_0}\setminus D_\varepsilon)} \frac{d\mathrm{Re}\zeta d\mathrm{Im}\zeta}{|\zeta-\zeta_0|}\Bigg) = O(1) \quad \text{as} \quad \varepsilon\to 0.$$

In order to estimate $\dfrac{f(\zeta)}{(S'_\zeta(u,\zeta))^2}$ in D_{ε_0} we take the members in the Taylor formula for $\tilde f(\zeta)$ up to the second order :

$$\tilde f(\zeta) = \tilde f(e^{i\varphi/2}+\omega e^{i\varphi/2}) + \tilde f'(e^{i\varphi/2}+\omega e^{i\varphi/2})\rho + \frac{1}{2}\tilde f''(e^{i\varphi/2}+\omega e^{i\varphi/2}+\lambda\rho e^{i\theta})\rho^2. \quad (6.21)$$

From formulas (6.17)–(6.21) it follows that

$$\left|\frac{f(\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{|\tilde f(\zeta)||\zeta|^8}{|\zeta-\zeta_2|^2|\zeta-\zeta_0|^2} \leqslant \mathrm{const}\Bigg(\frac{|\tilde f(e^{i\varphi/2}+\omega e^{i\varphi/2})|}{\left(\frac{\omega(2+\omega)}{2(1+\omega)}\right)^2|\zeta-\zeta_0|^2} + \\ + \frac{|\tilde f'(e^{i\varphi/2}+\omega e^{i\varphi/2})|\,\rho}{\left(\frac{\omega(2+\omega)}{2(1+\omega)}\right)^2|\zeta-\zeta_0|^2} + \frac{|\tilde f''(e^{i\varphi/2}+\omega e^{i\varphi/2}+\lambda\rho e^{i\theta})|\rho^2}{2\rho^2|\zeta-\zeta_0|^2}\Bigg) \leqslant \\ \leqslant \frac{\mathrm{const}}{|\zeta-\zeta_0|^2}, \quad \zeta\in Z_0^+\bigcap D_{\varepsilon_0}.$$

Thus

$$\iint\limits_{Z_0^+\bigcap(D_{\varepsilon_0}\setminus D_\varepsilon)} \left|\frac{f(\zeta)\exp(itS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \\ \leqslant \mathrm{const}\int\limits_\varepsilon^{\varepsilon_0}\frac{d\rho}{\rho} = \mathrm{const}\ln\frac{\varepsilon_0}{\varepsilon}.$$

Setting $\varepsilon = \frac{1}{|t|}$ yields $I_3 = O(\ln(|t|))$, as $t\to\infty$.

(B) Now let $\dfrac{1}{1+\omega} \leqslant 2\varepsilon_0$.

If $\zeta \in Z_0^+ \bigcap D_{\varepsilon_0}$, then the following estimates hold

$$|\zeta \pm \zeta_1| \geqslant \frac{1}{2\varepsilon_0} - 1 - \varepsilon_0, \quad |\zeta \pm \zeta_2| \geqslant \frac{1}{2\varepsilon_0} - 3\varepsilon_0, \quad |\zeta + \zeta_0| > 2 - \varepsilon_0.$$

Consequently,

$$|S'_\zeta(u,\zeta)| \geqslant \frac{\text{const}}{|\zeta|^4}|\zeta - \zeta_0|, \quad \zeta \in Z_0^+ \bigcap D_{\varepsilon_0}.$$

and the part of the integral I_{ext} over Z_0^+ for this case can be estimated, proceeding as in the previous item, as $O\left(\frac{\ln(|t|)}{|t|}\right)$, $t \to \infty$.
If $\zeta \in Z_2^+ \bigcap D_{\varepsilon_0}$, then the following estimates hold

$$|\zeta \pm \zeta_1| \geqslant 1 - 3\varepsilon_0, \quad |\zeta \pm \zeta_0| \geqslant 1 - 3\varepsilon_0,$$

and thus

$$|S'_\zeta(u,\zeta)| \geqslant \frac{\text{const}}{|\zeta|^4}|\zeta - \zeta_2||\zeta + \zeta_2|.$$

We can estimate $|\zeta + \zeta_2| \geqslant |\zeta_2| = \dfrac{1}{1+\omega}$. Now let us expand $f(\zeta)$ into Taylor formula in the neighborhood of $\frac{1}{1+\omega}e^{i\varphi/2}$:

$$f(\zeta) = f\left(\frac{1}{1+\omega}e^{i\varphi/2}\right) + f'\left(\frac{1}{1+\omega}e^{i\varphi/2} + \lambda\rho e^{i\theta}\right)\rho, \quad \lambda \in [0,1].$$

For an arbitrary value of $\omega > 0$ (satisfying $\frac{1}{1+\omega} \leqslant 2\varepsilon_0$) the following estimate holds :

$$\left|f\left(\frac{1}{1+\omega}e^{i\varphi/2}\right)\right| \leqslant \text{const}\left|\frac{1}{1+\omega}\right|^N.$$

This yields

$$\left|\frac{f(\zeta)}{S'_\zeta(u,\zeta)}\right| \leqslant \text{const}\left(\frac{|f(\frac{1}{1+\omega}e^{i\varphi/2})|}{\frac{1}{1+\omega}|\zeta - \zeta_2|} + \frac{|f'(\frac{1}{1+\omega}e^{i\varphi/2} + \lambda\rho e^{i\theta})|\rho}{\rho|\zeta - \zeta_2|}\right) \leqslant \frac{\text{const}}{|\zeta - \zeta_2|}.$$

In the same manner

$$\left|\frac{f(\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{|f(\zeta)||\zeta|^8}{|\zeta+\zeta_2|^2|\zeta-\zeta_2|^2} \leqslant \text{const}\left(\frac{|f(\frac{1}{1+\omega}e^{i\varphi/2})|}{\left(\frac{1}{1+\omega}\right)^2|\zeta-\zeta_2|^2}+\right.$$
$$\left. + \frac{|f'(\frac{1}{1+\omega}e^{i\varphi/2})|\,\rho}{\left(\frac{1}{1+\omega}\right)^2|\zeta-\zeta_2|^2} + \frac{|f''(\frac{1}{1+\omega}e^{i\varphi/2}+\lambda\rho e^{i\theta})|\rho^2}{2\rho^2|\zeta-\zeta_2|^2}\right) \leqslant \frac{\text{const}}{|\zeta-\zeta_2|^2}.$$

Following further the reasoning from the case (A) we obtain that

$$I_{ext} = O\left(\frac{\ln(|t|)}{|t|}\right), \quad \text{as} \quad t\to\infty$$

uniformly on $u\in\mathbb{C}$. $\square$

6.3 Estimate for the "inverse scattering solution"

In this subsection we prove estimate (6.3) for the solution $v(x,t)$ of the Cauchy problem for the Novikov–Veselov equation at positive energy with the initial data $v(x,0)$ satisfying properties (6.2) or, in other words, for $v(x,t)$ constructed by means of (1.39)–(1.41) with (1.38b), (1.38c).

We proceed from the formulas (1.40), (1.41) for the potential $v(z,t)$ and the integral equation (1.39) for $\mu(z,\lambda,t)$.

We write (1.39) as

$$\mu(z,\lambda,t) = 1 + (A_{z,t}\mu)(z,\lambda,t), \tag{6.22}$$

where

$$(A_{z,t}f)(\lambda) = \partial_{\bar\lambda}^{-1}(r(\lambda)\exp(itS(u,\lambda))\overline{f(\lambda)}) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}}\frac{r(\zeta)\exp(itS(u,\zeta))\overline{f(\zeta)}}{\zeta-\lambda}d\text{Re}\zeta\, d\text{Im}\zeta$$

and $S(u,\zeta)$ is defined by (6.9), $u=\dfrac{z}{t}$.

Equation (6.22) can be also written in the form

$$\mu(z,\lambda,t) = 1 + A_{z,t}\cdot 1 + (A^2_{z,t}\mu)(z,\lambda,t). \tag{6.23}$$

According to the theory of generalized analytic functions (see [62]), equations (6.22), (6.23) have a unique solution for all z, t. This solution can be written as

$$\mu(z, \lambda, t) = (I - A_{z,t}^2)^{-1}(1 + A_{z,t} \cdot 1). \tag{6.24}$$

Equation (6.24) possesses a formal asymptotic expansion

$$\mu(z, \lambda, t) = (I + A_{z,t}^2 + A_{z,t}^4 + \ldots)(1 + A_{z,t} \cdot 1). \tag{6.25}$$

From estimate (6.31) given below it follows that (6.25) converges uniformly for sufficiently large t. We will also write formula (6.25) in the form

$$\mu(z, \lambda, t) = 1 + A_{z,t} \cdot 1 + R, \tag{6.26}$$

where $R = \left(\sum\limits_{k=1}^{\infty} A_{z,t}^{2k}\right)(1 + A_{z,t} \cdot 1)$.

In addition to $A_{z,t}$ we introduce another integral operator $B_{z,t}$ defined as

$$B_{z,t} \cdot f = \iint\limits_{\mathbb{C}} r(\zeta)\exp(itS(u, \zeta))\overline{f(\zeta)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \tag{6.27}$$

To study (6.25) we will need some estimates on the values of operators $A_{z,t}$ and $B_{z,t}$.

Lemma 6.3. *Under assumption that $b(\cdot, t) \in \mathcal{S}(\mathbb{C})$ and for $b(\cdot, t)$ properties 8, 9 of Statement 1.2 hold, the following estimates are valid :*

(a)

$$|B_{z,t} \cdot 1| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad as \quad t \to \infty \tag{6.28}$$

uniformly on $u \in \mathbb{C}$;

(b)

$$|B_{z,t} \cdot A_{z,t} \cdot 1| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad as \quad t \to \infty \tag{6.29}$$

uniformly on $u \in \mathbb{C}$;

(c)

$$|(A_{z,t}^2 \cdot 1)(\lambda)| \leqslant \frac{\beta}{|t|^{1/14}}, \quad |t| \geqslant 1, \tag{6.30}$$

where β is a constant independent of u and λ ;

(d)

$$|(A_{z,t}^n \cdot 1)(\lambda)| \leqslant \frac{\beta^{n-1}}{|t|^{\lfloor n/2 \rfloor/14}}, \quad |t| \geqslant 1, \tag{6.31}$$

where $\lfloor s \rfloor$ denotes the integer part of s;

(e)
$$|B_{z,t} \cdot A_{z,t}^{n-1} \cdot 1| \leqslant \frac{\beta^{n-1} \ln(3+|t|)}{|t|^{1+\lfloor (n-2)/2 \rfloor /14}}, \quad |t| \geqslant 1. \tag{6.32}$$

Proof of Lemma 6.3. We proceed according to the scheme described in subsection 6.1 (page 51).

(a) This item follows from Lemma 6.2.

(b) As $(A_{z,t} \cdot 1)(\lambda) \in C(\mathbb{C})$, we take D_ε to be the union of disks of a radius ε centered in the stationary points of $S(u,\zeta)$. We note that $(A_{z,t} \cdot 1)(\lambda) = O(1)$ uniformly on u (or, equivalently, on z) and λ. Thus the integral I_{int} (as in (6.14) with $f(\zeta,\lambda) = r(\zeta)\overline{(A_{z,t} \cdot 1)(\zeta)}$) can be estimated as

$$I_{int} = \iint\limits_{D_\varepsilon} r(\zeta) \exp(itS(u,\zeta)) \overline{(A_{z,t} \cdot 1)(\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta = O(\varepsilon^2).$$

Now let us estimate the integral I_{ext} (as in (6.14) with $f(\zeta,\lambda) = r(\zeta)\overline{(A_{z,t} \cdot 1)(\zeta)}$). For this purpose we apply the Stokes formula (as in (6.15)) taking into consideration that $\partial_{\bar\lambda}(A_{z,t} \cdot f)(\lambda) = r(\lambda)\exp(itS(u,\lambda))\overline{f(\lambda)}$:

$$\begin{aligned}
&\iint\limits_{\mathbb{C}\backslash D_\varepsilon} r(\zeta) \exp(itS(u,\zeta)) \overline{(A_{z,t} \cdot 1)(\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta = \\
&= \int\limits_{\partial D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))\overline{(A_{z,t} \cdot 1)(\zeta)}}{2tS'_\zeta(u,\zeta)} d\bar\zeta - \\
&- \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r'_\zeta(\zeta)\exp(itS(u,\zeta))\overline{(A_{z,t} \cdot 1)(\zeta)}}{itS'_\zeta(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \\
&+ \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))\overline{(A_{z,t} \cdot 1)(\zeta)} S''_{\zeta\zeta}(u,\zeta)}{it(S'_\zeta(u,\zeta))^2} d\mathrm{Re}\zeta d\mathrm{Im}\zeta - \\
&- \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{|r(\zeta)|^2 |\exp(itS(u,\zeta))|^2}{itS'_\zeta(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta.
\end{aligned} \tag{6.33}$$

Now, proceeding as in the proof of Lemma 6.2, we obtain (6.29).

(c) In this case we build D_ε as the union of disks with a radius of ε and centers in λ and stationary points of $S(u,\zeta)$. The integral I_{int} over D_ε (as in (6.14) with $f(\zeta,\lambda) = -\frac{1}{\pi}\frac{r(\zeta)\overline{(A_{z,t} \cdot 1)(\zeta)}}{\zeta - \lambda}$) behaves asymptotically as $O(\varepsilon)$. When estimating the integral I_{ext}

over $\mathbb{C}\backslash D_\varepsilon$ we use (6.12), (6.15) and the following inequalities

$$|\zeta - \lambda| \geqslant \varepsilon, \quad |\zeta - \zeta_i| \geqslant \varepsilon$$

(ζ_i are stationary points of $S(u,\zeta)$) which hold for all $\zeta \in \mathbb{C}\backslash D_\varepsilon$. Thus we obtain that the asymptotical behavior of I_{ext} is at most $O\left(\dfrac{1}{|t|\varepsilon^{13}}\right)$. Then, as proposed by the scheme, we choose $\varepsilon = |t|^{-1/14}$ and obtain the required estimate.

(d) This item is proved by induction. As in item *(c)*, D_ε is the union of disks with a radius of ε and centers in λ and stationary points of $S(u,\zeta)$.

For the integral I_{int} we have

$$|I_{int}| = \left| \frac{1}{\pi} \iint\limits_{D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))}{\zeta - \lambda} \overline{(A_{z,t}^{n-1}\cdot 1)(\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta \right| \leqslant$$

$$\leqslant \frac{\beta^{n-2}}{|t|^{\lfloor (n-1)/2 \rfloor/14}} \iint\limits_{D_\varepsilon} \left| \frac{r(\zeta)\exp(itS(u,\zeta))}{\zeta-\lambda} \right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \frac{\beta^{n-1}\varepsilon}{|t|^{\lfloor (n-2)/2\rfloor/14}}. \quad (6.34)$$

To estimate I_{ext} we use the following representation

$$\iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))\overline{(A_{z,t}^{n-1}\cdot 1)(\zeta)}}{\zeta-\lambda} d\mathrm{Re}\zeta d\mathrm{Im}\zeta =$$

$$= \int\limits_{\partial D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))\overline{(A_{z,t}^{n-1}\cdot 1)(\zeta)}}{2t(\zeta-\lambda)S'_\zeta(u,\zeta)} d\bar\zeta -$$

$$- \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r'_\zeta(\zeta)\exp(itS(u,\zeta))\overline{(A_{z,t}^{n-1}\cdot 1)(\zeta)}}{it(\zeta-\lambda)S'_\zeta(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$

$$+ \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))\overline{(A_{z,t}^{n-1}\cdot 1)(\zeta)}S''_{\zeta\zeta}(u,\zeta)}{it(\zeta-\lambda)(S'_\zeta(u,\zeta))^2} d\mathrm{Re}\zeta d\mathrm{Im}\zeta -$$

$$- \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{|r(\zeta)|^2|\exp(itS(u,\zeta))|^2\overline{(A_{z,t}^{n-2}\cdot 1)(\zeta)}}{it(\zeta-\lambda)S'_\zeta(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta +$$

$$+ \iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{r(\zeta)\exp(itS(u,\zeta))\overline{(A_{z,t}^{n-1}\cdot 1)(\zeta)}}{it(\zeta-\lambda)^2 S'_\zeta(u,\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta = J_1 + J_2 + J_3 + J_4 + J_5. \quad (6.35)$$

The integrals J_i can be estimated in the following way

$$|J_1| \leqslant \frac{\beta^{n-2}}{|t|\cdot|t|^{\lfloor(n-1)/2\rfloor/14}}\frac{1}{\varepsilon^7}\int\limits_{\partial D_\varepsilon}|r(\zeta)|d\zeta \leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor(n-1)/2\rfloor/14}}\frac{1}{\varepsilon^7}.$$

Similarly,

$$\begin{aligned}|J_2| &\leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor(n-1)/2\rfloor/14}}\frac{1}{\varepsilon^7},\\ |J_3| &\leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor(n-1)/2\rfloor/14}}\frac{1}{\varepsilon^{13}},\\ |J_4| &\leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor(n-2)/2\rfloor/14}}\frac{1}{\varepsilon^7},\\ |J_5| &\leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor(n-2)/2\rfloor/14}}\frac{1}{\varepsilon^8}.\end{aligned}$$

Thus,

$$|I_{ext}| \leqslant \frac{\beta^{n-1}}{|t|\cdot|t|^{\lfloor(n-2)/2\rfloor/14}\cdot\varepsilon^{13}}.$$

Now we set $\varepsilon = |t|^{-1/14}$ and obtain the overall estimate

$$|(A^n_{z,t}\cdot 1)(\lambda)| \leqslant \frac{\beta^{n-1}}{|t|^{1/14}|t|^{\lfloor(n-2)/2\rfloor/14}} = \frac{\beta^{n-1}}{t^{\lfloor n/2\rfloor/14}}.$$

(e) Proceeding from *(d)* this item is proved similarly to *(b)*. □

Lemma 6.4. *Under assumptions of Lemma 6.3, we have that :*

(a) $A_{z,t}\cdot 1 = \frac{a_1(z,t)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right)$ *for* $\lambda\to\infty$, *where*

$$|\partial_z a_1(z,t)| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad \textit{as} \quad t\to\infty \tag{6.36}$$

uniformly on $z\in\mathbb{C}$.

(b) $A^2_{z,t}\cdot 1 = \frac{a_2(z,t)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right)$ *for* $\lambda\to\infty$, *where*

$$|\partial_z a_2(z,t)| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad \textit{as} \quad t\to\infty \tag{6.37}$$

uniformly on $z\in\mathbb{C}$.

(c) $R = \frac{q(z,t)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right)$ *as* $\lambda \to \infty$, *where*

$$|\partial_z q(z,t)| = O\left(\frac{\ln(|t|)}{|t|}\right), \quad as \quad t \to \infty \tag{6.38}$$

uniformly on $z \in \mathbb{C}$.

Proof of Lemma 6.4. The asymptotics for $A_{z,t} \cdot 1$, $A^2_{z,t} \cdot 1$ and R follow from the definitions of $A_{z,t}$ and R, formula (1.41), property $b(\cdot,t) \in \mathcal{S}(\mathbb{C})$ and item 8 of Statement 1.2. The rest of the proof consists in the following.

(a) Estimate (6.36) follows from (6.28) and the formula

$$a_1(z,t) = \frac{1}{\pi} \iint\limits_{\mathbb{C}} r(\zeta) \exp(itS(u,\zeta)) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = B_{z,t} \cdot 1.$$

(b) Estimate (6.37) follows from (6.29) and the formula

$$a_2(z,t) = \frac{1}{\pi} \iint\limits_{\mathbb{C}} r(\zeta) \exp(itS(u,\zeta)) \overline{(A_{z,t} \cdot 1)(\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta = B_{z,t} \cdot A_{z,t} \cdot 1.$$

(c) We note that $q(z,t) = \sum\limits_{k=2}^{\infty} a_k(z,t)$ where $a_k(z,t)$ is defined

$$(A^k_{z,t} \cdot 1)(\lambda) = \frac{a_k(z,t)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right).$$

Next,

$$a_k(z,t) = \frac{1}{\pi} \iint\limits_{\mathbb{C}} r(\zeta) \exp(itS(u,\zeta)) \overline{(A^{k-1}_{z,t} \cdot 1)(\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta.$$

Thus, proceeding as in item *(e)* of Lemma 6.3, we obtain that $a_2(z,t)+a_3(z,t) = O\left(\frac{\ln(|t|)}{|t|}\right)$ and the rest of the members form a geometric progression that converges to the sum of order $O\left(\frac{\ln(|t|)}{|t|^{1+1/14}}\right)$. □

Formulas (1.40), (1.41), (6.26) and Lemma 6.4 imply

Theorem 6.1. *Let $v(x,t)$ be a solution of the Novikov–Veselov equation (0.5) with $E = 1$, constructed via (1.39)–(1.41) with (1.38b), (1.38c) under assumption that $b(\cdot,0) \in \mathcal{S}(\mathbb{C})$ and for $b(\cdot,0)$ properties 8, 9 of Statement 1.2 hold. Then*

$$|v(x,t)| \leqslant \frac{\mathrm{const}(v)\ln(3+|t|)}{1+|t|}, \quad x \in \mathbb{R}^2, t \in \mathbb{R}. \tag{6.39}$$

Corollary. *Let (v, w) be a solution to the Cauchy problem for the Novikov-Veselov equation (0.5) at $E = 1$ with initial data (v_0, w_0) satisfying properties (6.2). Then estimate (6.39) holds.*

6.4 Properties of stationary points (proof of Lemma 6.1)

Proof. Under the additional assumption that $\xi \neq 0$ the system of equations (6.10)–(6.11) is equivalent to the following system

$$\begin{cases} \xi^3 - \frac{\bar u}{6}\xi^2 + \frac{u}{6}\xi - 1 = 0, \\ \xi^3 - \frac{u}{6}\xi + 2 = 0. \end{cases} \tag{6.40}$$

We claim that $\xi = \xi_1$ corresponds to a degenerate stationary point of (6.9), iff

$$\begin{aligned} &\text{the polynomial } P(\xi) = \xi^3 - \frac{\bar u}{6}\xi^2 + \frac{u}{6}\xi - 1 \text{ can be represented} \\ &\text{in the form } P(\xi) = (\xi - \xi_1)^2(\xi - \xi_2)\ . \end{aligned} \tag{6.41}$$

Indeed, if $\xi = \xi_1 \neq 0$ is a zero of the function $Q(u, \xi) = -\frac{\bar u}{2} + \frac{u}{2\xi} + 3\xi - \frac{3}{\xi^2} = 3P(\xi)/\xi^2$, then $Q(u, \xi)$ is holomorphic in a certain neighborhood of $\xi = \xi_1$ and can be expanded into the following Taylor series

$$Q(u, \xi) = c_1(\xi - \xi_1) + c_2(\xi - \xi_1)^2 + c_3(\xi - \xi_1)^3 + \dots$$

Thus S'_ζ can be represented as

$$S'_\zeta(u, \zeta) = c_1(\zeta^2 - \xi_1) + c_2(\zeta^2 - \xi_1)^2 + c_3(\zeta^2 - \xi_1)^3 + \dots$$

After differentiating with respect to ζ we obtain

$$S''_{\zeta\zeta}(u, \zeta) = 2c_1\zeta + 4c_2(\zeta^2 - \xi_1)\zeta + 6c_3(\zeta^2 - \xi_1)^2\zeta + \dots$$

The stationary point corresponding to $\xi = \xi_1$ can be degenerate if and only if $c_1 = 0$.

So for the polynomial $P(\xi)$ we get the following representation in the neighborhood of $\xi = \xi_1$

$$P(\xi) = c_2(\xi - \xi_1)^2\xi^2 + O(|\xi - \xi_1|^3).$$

As $P(\xi)$ is a third–order polynomial, it follows that $P(\xi) = (\xi - \xi_1)^2(\xi - \xi_2)$.

Expanding the expression for $P(\xi)$ and equating coefficients for the corresponding powers of ξ results in the following system

$$\begin{cases} 2\xi_1 + \xi_2 = \frac{\bar{u}}{6}, \\ 2\xi_1\xi_2 + \xi_1^2 = \frac{u}{6}, \\ \xi_1^2\xi_2 = 1. \end{cases} \tag{6.42}$$

Excluding u from the system yields

$$\begin{cases} 2\xi_1\xi_2 + \xi_1^2 = 2\bar{\xi}_1 + \bar{\xi}_2, \\ \xi_1^2\xi_2 = 1. \end{cases} \tag{6.43}$$

We represent ξ_1 in the form $\xi_1 = re^{i\varphi}$. Then from the second equation in (6.43) we get that $\xi_2 = r^{-2}e^{-2i\varphi}$. Substituting this into the first equation of (6.43) yields

$$(r^2 - r^{-2})e^{2i\varphi} = (r - r^{-1})e^{-i\varphi}. \tag{6.44}$$

For $r = 1$ equation (6.44) holds for all φ. In addition, if $r \neq 1$, $r > 0$, then equation (6.44) can be rewritten as

$$(r + r^{-1})e^{3i\varphi} = 2$$

and has no solutions for real φ.

Using now the second equation in (6.42) one can see that the set of u values, for which (6.43), (6.42) are solvable and (6.41) holds, is a curve on the complex plane described in the parametric form by

$$u = 6(2e^{-i\varphi} + e^{2i\varphi})$$

(see Figure III.1). This curve has three singular points corresponding to $\varphi = \frac{2\pi k}{3}$, $k = 0, 1, 2$. For these values of φ we have that $e^{i\varphi} = \xi_1 = \xi_2 = e^{-2i\varphi}$, and $P(\xi)$ can be represented in the form $P(\xi) = (\xi - \xi_1)^3$. For $\varphi \neq \frac{2\pi k}{3}$, $k = 0, 1, 2$, we have that $P(\xi) = (\xi - \xi_1)^2(\xi - \xi_2)$ where $e^{i\varphi} = \xi_1 \neq \xi_2 = e^{-2i\varphi}$. The first two statements of the Lemma 6.1 are proved.

Let us now fix $\xi = e^{i\varphi}$ and find the set of u for which this ξ is a root of the polynomial $P(\xi)$.

One can see that ξ is the root of $P(\xi)$ iff

$$\xi\frac{u}{6} - \xi^2\frac{\bar{u}}{6} = 1 - \xi^3. \tag{6.45}$$

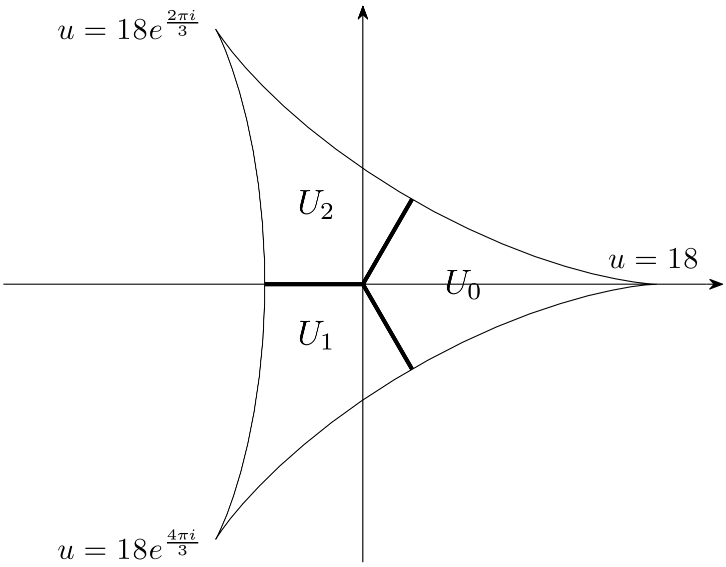

Figure III.1 – The curve $\mathcal{U}$ on the complex plane.

We now solve the homogeneous equation

$$e^{i\varphi}\frac{u}{6} - e^{2i\varphi}\frac{\bar{u}}{6} = 0 \tag{6.46}$$

with respect to u to find the plausible perturbations of u for which $\xi = e^{i\varphi}$ remains a root of $P(\xi)$.

From (6.46) we get $\frac{u}{\bar{u}} = e^{i\varphi}$, and thus $u = se^{i\varphi/2}$ for $s \in \mathbb{R}$. So for all u that belong to the line

$$u(\varphi, s) = 6(2e^{-i\varphi} + e^{2i\varphi}) + se^{i\varphi/2}, \quad s \in \mathbb{R} \tag{6.47}$$

one of the roots of $P(\xi)$ is equal to $e^{i\varphi}$.

Now we note that the tangent vector to $\mathcal{U}$

$$\frac{d}{d\varphi}(2e^{-i\varphi} + e^{2i\varphi}) = 2i(e^{2i\varphi} - e^{-i\varphi})$$

is collinear to the perturbation vector $se^{i\varphi/2}$ for all $\varphi \neq \frac{2\pi ik}{3}$, $k = 0, 1, 2$. Thus $u(\varphi, s)$ given by (6.47) is the tangent line to $\mathcal{U}$ passing through the point $6(2e^{-i\varphi} + e^{2i\varphi})$.

We note that for each $u \in \mathrm{int}\mathbb{U}$ there exist two different tangents to the curve $\mathcal{U}$ passing through this u. Indeed, note that the tangent lines to the curve $\mathcal{U}$ passing through the points $u = 18e^{\frac{2\pi i k}{3}}$, $k = 0, 1, 2$, divide the domain $\mathrm{int}\mathbb{U}$ into three parts

$$U_0 = \{u = re^{i\varphi},\ 0 \leqslant r < 6(2e^{-i\varphi} + e^{2i\varphi}),\ \frac{-\pi}{3} \leq \varphi \leq \frac{\pi}{3}\},$$
$$U_1 = \{u = re^{i\varphi},\ 0 < r < 6(2e^{-i\varphi} + e^{2i\varphi}),\ \frac{\pi}{3} < \varphi \leq \pi\},$$
$$U_2 = \{u = re^{i\varphi},\ 0 < r < 6(2e^{-i\varphi} + e^{2i\varphi}),\ \pi < \varphi < \frac{5\pi}{3}\}$$

(see Figure III.1). We first study U_0. Let us consider K_{01} and K_{02} which are the sets of points of the following pencils of tangent lines :

$$K_{01} = \left\{u \in \mathbb{C}\colon u = 6(2e^{-i\varphi} + e^{2i\varphi}) + se^{i\varphi/2}, \quad s \in \mathbb{R}, \quad -\frac{2\pi}{3} \leqslant \varphi < 0\right\},$$
$$K_{02} = \left\{u \in \mathbb{C}\colon u = 6(2e^{-i\varphi} + e^{2i\varphi}) + se^{i\varphi/2}, \quad s \in \mathbb{R}, \quad 0 < \varphi \leqslant \frac{2\pi}{3}\right\}.$$

It is easily seen that $U_0 \subset K_{01}$, $U_0 \subset K_{02}$, i.e. each point from U_0 is covered by a certain tangent line from both pencils K_{01} and K_{02}. It can be shown similarly that each point of U_j, $j = 1, 2$, is covered twice by the corresponding tangent lines.

Thus every point from $\mathrm{int}\mathbb{U}$ is covered twice which means that for each u in the domain limited by the curve $\mathcal{U}$ there exist two different roots of $P(\xi)$ equal to 1 in absolute value. As the product of the roots of the polynomial $P(\xi)$ is equal to 1, the third root of the polynomial is also equal to 1 in absolute value. We do not consider the values u from the boundary of $\mathbb{U}$ which means that the above–mentioned roots correspond to non–degenerate stationary points. Thus the third statement of Lemma 6.1 is proved.

Now suppose $u \in \mathbb{C}\backslash\mathbb{U}$. We note that every such point u belongs to one and only one tangent line to the curve $\mathcal{U}$. That means that for every $u \in \mathbb{C}\backslash\mathbb{U}$ one of the roots of the polynomial $P(\xi)$, ξ_1, is such that $|\xi_1| = 1$. As $u \notin \mathcal{U}$, $\xi_0 \neq \xi_1$, $\xi_2 \neq \xi_1$. Besides $|\xi_0| \neq 1$ because otherwise there would be two different tangent lines passing through the corresponding point u. That means that for every $u \in \mathbb{C}\backslash\mathbb{U}$ there exists a root of the polynomial $P(\xi)$, namely $\xi = \xi_0$, such that $|\xi_0| \neq 1$.

Considering equation (6.45) and its conjugate yields the following system of linear equations for u and $\bar{u}$

$$\begin{cases} au + b\bar{u} = c, \\ \bar{b}\bar{u} + \bar{a}\bar{u} = \bar{c}, \end{cases}$$

where $a = \frac{\xi}{6}$, $b = -\frac{\xi^2}{6}$, $c = 1 - \xi^3$ for each $\xi \in \mathbb{C}$, $|\xi| \neq 1$, $\xi \neq 0$. Thus

$$u = \frac{c\bar{a} - \bar{c}b}{a\bar{a} - b\bar{b}} = 6\frac{\bar{\xi} - \bar{\xi}\xi^3 + \xi^2 - \xi^2\bar{\xi}^3}{\xi\bar{\xi}(1 - \xi\bar{\xi})} = 6\left(\frac{1}{\xi} + \bar{\xi} + \frac{\xi}{\bar{\xi}}\right)$$

for each $\xi \in \mathbb{C}$, $|\xi| \neq 1$, $\xi \neq 0$.

Now let us consider $\xi = \xi_0 = (1+\tau)e^{i\varphi}$, $0 < \tau < +\infty$. Then the corresponding value of the parameter u is

$$u = 6\left(2e^{-i\varphi} + e^{2i\varphi} + \frac{\tau^2}{1+\tau}e^{-i\varphi}\right).$$

In addition to $\xi = \xi_0$, for this value of the parameter u polynomial $P(\xi)$ has also a root $\xi = \xi_1 = e^{-2i\varphi}$. Indeed, if $u = 6(2e^{-i\varphi} + e^{2i\varphi})$, then $\xi_1 = e^{-2i\varphi}$ is a root of the polynomial $P(\xi)$, and the plausible perturbation of this u for which ξ_1 remains a root of $P(\xi)$ is equal to $se^{-i\varphi}$.

In addition, as the product of the roots of P is equal to 1, the third root is $\xi_2 = (1+\tau)^{-1}e^{i\varphi}$. The fourth statement of Lemma 6.1 is proved. □

7 Large time asymptotics for solutions of Cauchy problem for the Novikov-Veselov equation at negative energy

In this Section we consider a problem similar to that studied in the previous Section, but for the case of negative energy. More precisely, we consider the Cauchy problem for equation (0.5) at $E < 0$ with the initial data

$$v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \tag{7.1}$$

where function $v_0(x)$ satisfies the following conditions

- $v_0 = \bar{v}_0$, i.e. v_0 is a real-valued function, (7.2a)
- the integral equations of direct scattering for the 2-dimensional Scrödinger equation (1.1) with the given potential $v_0(x)$ are everywhere uniquely solvable, (7.2b)
- $v_0 \in \mathcal{S}(\mathbb{R}^2)$, where $\mathcal{S}$ denotes the Schwartz class. (7.2c)

As for function $w_0(x)$, we will assume that it is a continuous function decaying at infinity and determined using $\partial_{\bar z} w_0(x) = -3\partial_z v_0(x)$ from (0.5).

The main result of this Section consists in the following (see also Theorem 7.1) : we show that for the "inverse scattering solution" $v(x,t)$ of (0.5), (7.1), where $E<0$ and $v(x,0)=v_0(x)$ satisfies (7.2), the following estimate holds

$$|v(x,t)| \leqslant \frac{\mathrm{const}(v_0)\ln(3+|t|)}{(1+|t|)^{3/4}}, \quad t\in\mathbb{R}, \text{ uniformly on } x\in\mathbb{R}^2. \tag{7.3}$$

The proof of estimate (7.3) is based on the scheme developed in the previous Section. The main difference arises from the fact that item 9 of Statement 1.2, essentially used in the proofs of the previous Section, does not hold in the case of negative energy. This property also results in the slower convergence of the "inverse scattering solution" to zero in the case of negative energy. We show that for $E<0$ estimate (7.3) of the rate of convergence is optimal in the sense that for some initial values $v(x,0)$ and for some lines $x=\omega t$, $\omega\in\mathbb{S}^1$, the exact asymptotics of $v(x,t)$ along these lines is $\frac{\mathrm{const}}{(1+|t|)^{3/4}}$ as $|t|\to\infty$ (where the constant is nonzero).

7.1 Estimate for the solution of the linearized equation

Consider

$$\begin{aligned} I(t,z) &= \iint\limits_{\mathbb{C}} f(\zeta)\exp(S(\zeta,z,t))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \\ J(t,z) &= 3\iint\limits_{\mathbb{C}} \frac{\bar\zeta}{\zeta} f(\zeta)\exp(S(\zeta,z,t))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \end{aligned} \tag{7.4}$$

where $z\in\mathbb{C}$, $t\in\mathbb{R}$, $f(\zeta)\in L^1(\mathbb{C})$, S is defined by

$$S(\lambda,z,t) = \frac{1}{2}\left(\left(\lambda-\frac{1}{\bar\lambda}\right)\bar z - \left(\bar\lambda-\frac{1}{\lambda}\right)z\right) + t\left(\lambda^3+\frac{1}{\lambda^3}-\bar\lambda^3-\frac{1}{\bar\lambda^3}\right). \tag{7.5}$$

We will also assume that $f(\zeta)$ satisfies the following conditions

$$f\in C^\infty(\bar D_+), \quad f\in C^\infty(\bar D_-), \tag{7.6a}$$

$$\partial_\zeta^m\partial_{\bar\zeta}^n f(\zeta) = \begin{cases} O\left(|\zeta|^{-\infty}\right) & \text{as } |\zeta|\to\infty, \\ O\left(|\zeta|^{\infty}\right) & \text{as } |\zeta|\to 0, \end{cases} \quad \text{for all } m,n\geqslant 0, \tag{7.6b}$$

where

$$D_+ = \{\zeta\in\mathbb{C}\colon 0<|\zeta|\leqslant 1\}, \quad D_- = \{\zeta\in\mathbb{C}\colon |\zeta|\geqslant 1\}, \tag{7.7}$$

and $\bar{D}_+ = D_+ \cup T$, $\bar{D}_- = D_- \cup T$ with $T = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$.

Note that if $v(z,t) = I(t,z)$, $w(z,t) = J(t,z)$, where

$$(|\zeta|^3 + |\zeta|^{-3}) f(\zeta) \in L^1(\mathbb{C}) \text{ as a function of } \zeta,$$

and, in addition,

$$\overline{f(\zeta)} = f(-\zeta) \quad \text{and/or} \quad \overline{f(\zeta)} = -|\zeta|^{-4} f\left(\frac{1}{\bar{\zeta}}\right),$$

then v, w satisfy the linearized Novikov–Veselov equation (0.5) with $E = -1$.

Let us introduce parameter $u = \frac{z}{t}$ and write the integral I in the following form

$$I(t,u) = \iint\limits_{\mathbb{C}} f(\zeta) \exp(tS(u,\zeta)) d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{7.8}$$

where

$$S(u,\zeta) = \frac{1}{2}\left(\left(\zeta - \frac{1}{\bar{\zeta}}\right)\bar{u} - \left(\bar{\zeta} - \frac{1}{\zeta}\right)u\right) + \left(\zeta^3 - \bar{\zeta}^3 + \frac{1}{\zeta^3} - \frac{1}{\bar{\zeta}^3}\right). \tag{7.9}$$

We will start by studying the properties of the stationary points of the function $S(u,\zeta)$ with respect to ζ. These points satisfy the equation

$$S'_\zeta = \frac{\bar{u}}{2} - \frac{u}{2\zeta^2} + 3\zeta^2 - \frac{3}{\zeta^4} = 0. \tag{7.10}$$

The degenerate stationary points obey additionally the equation

$$S''_{\zeta\zeta} = \frac{u}{\zeta^3} + 6\zeta + \frac{12}{\zeta^5} = 0. \tag{7.11}$$

We denote $\xi = \zeta^2$ and

$$Q(u,\xi) = \frac{u}{2} - \frac{\bar{u}}{2\xi} + 3\xi - \frac{3}{\xi^2}.$$

For each ξ, a root of the function $Q(u,\xi)$, there are two corresponding stationary points of $S(u,\zeta)$, $\zeta = \pm\sqrt{\xi}$.

The function $S'_\zeta(u,\zeta)$ can be represented in the following form

$$S'_\zeta(u,\zeta) = \frac{3}{\zeta^4}(\zeta^2 - \zeta_0^2(u))(\zeta^2 - \zeta_1^2(u))(\zeta^2 - \zeta_2^2(u)). \tag{7.12}$$

We will also use hereafter the following notations :

$$\mathcal{U} = \{u = -6(2e^{-i\varphi} + e^{2i\varphi})\colon \varphi \in [0, 2\pi)\}$$

and

$$\mathbb{U} = \{u = re^{i\psi}\colon \psi = \mathrm{Arg}(-6(2e^{-i\varphi} + e^{2i\varphi})),\ 0 \leqslant r \leqslant |6(2e^{i\varphi} + e^{-2i\varphi})|,\ \varphi \in [0, 2\pi)\},$$

the domain limited by the curve $\mathcal{U}$.

Lemma 7.1 (see Lemma 6.1)**.**

1. *If $u = -18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, then*

$$\zeta_0(u) = \zeta_1(u) = \zeta_2(u) = e^{\frac{\pi ik}{3}} \tag{7.13}$$

and $S(u, \zeta)$ has two degenerate stationary points, corresponding to a third-order root of the function $Q(u, \xi)$, $\xi_1 = e^{\frac{2\pi ik}{3}}$.

2. *If $u \in \mathcal{U}$ (i.e. $u = -6(2e^{-i\varphi} + e^{2i\varphi})$) and $u \neq -18e^{\frac{2\pi ik}{3}}$, $k = 0, 1, 2$, then*

$$\zeta_0(u) = \zeta_1(u) = e^{-i\varphi/2}, \quad \zeta_2(u) = e^{i\varphi}. \tag{7.14}$$

Thus $S(u, \zeta)$ has two degenerate stationary points, corresponding to a second-order root of the function $Q(u, \xi)$, $\xi_1 = e^{-i\varphi}$, and two non–degenerate stationary points corresponding to a first-order root, $\xi_2 = e^{2i\varphi}$.

3. *If $u \in \mathrm{int}\mathbb{U} = \mathbb{U}\backslash\partial\mathbb{U}$, then*

$$\zeta_i(u) = e^{-i\varphi_i} \text{ for some real } \varphi_i, \quad \text{and} \quad \zeta_i(u) \neq \zeta_j(u) \quad \text{for} \quad i \neq j. \tag{7.15}$$

In this case the stationary points of $S(u, \zeta)$ are non-degenerate and correspond to the roots of the function $Q(u, \xi)$ with absolute values equal to 1.

4. *If $u \in \mathbb{C}\backslash\mathbb{U}$, then*

$$\zeta_0(u) = (1 + \omega)e^{-i\varphi/2}, \quad \zeta_1(u) = e^{i\varphi}, \quad \zeta_2(u) = (1 + \omega)^{-1}e^{-i\varphi/2} \tag{7.16}$$

for certain real values φ and $\omega > 0$.

In this case the stationary points of the function $S(u, \zeta)$ are non-degenerate, and correspond to the roots of the function $Q(u, \xi)$ that can be expressed as $\xi_0 = (1 + \tau)e^{-i\varphi}$,

$$\xi_1 = e^{2i\varphi},\ \xi_2 = (1+\tau)^{-1}e^{-i\varphi},\ (1+\tau) = (1+\omega)^2.$$

Formula (7.12) and Lemma 7.1 give a complete description of the stationary points of function $S(u,\zeta)$.

In order to estimate uniformly on $u, \lambda \in \mathbb{C}$ the large–time behavior of the integral

$$I(t,u,\lambda) = \iint\limits_{\mathbb{C}} f(\zeta,\lambda)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{7.17}$$

where f satisfies assumptions (7.6), in the following subsections we will use the following general scheme.

1. Consider D_ε, the union of disks with a radius of ε and centers in singular points of function $f(\zeta,\lambda)$ and stationary points of $S(u,\zeta)$ with respect to ζ.
2. Represent $I(t,u,\lambda)$ as the sum of integrals over D_ε and $\mathbb{C}\backslash D_\varepsilon$:

$$\begin{aligned} &I(t,u,\lambda) = I_{int} + I_{ext}, \quad \text{where} \\ &I_{int} = \iint\limits_{D_\varepsilon} f(\zeta,\lambda)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \\ &I_{ext} = \iint\limits_{\mathbb{C}\backslash D_\varepsilon} f(\zeta,\lambda)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \end{aligned} \tag{7.18}$$

3. Find an estimate of the form

$$|I_{int}| = O\left(\varepsilon^\alpha\right), \quad \text{as} \quad \varepsilon \to 0 \quad (\text{for a certain } \alpha > 0)$$

   uniformly on u, λ, t.
4. Integrate I_{ext} by parts using Stokes formula

$$\begin{aligned} I_{ext} = &-\frac{1}{2it}\int\limits_{\partial D_\varepsilon} \frac{f(\zeta,\lambda)\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}d\bar\zeta - \\ &-\frac{1}{2it}\int\limits_{T\backslash D_\varepsilon} \frac{(f_+(\zeta,\lambda)-f_-(\zeta,\lambda))\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}d\bar\zeta - \frac{1}{t}\iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{f'_\zeta(\zeta,\lambda)\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \\ &+\frac{1}{t}\iint\limits_{\mathbb{C}\backslash D_\varepsilon} \frac{f(\zeta,\lambda)\exp(tS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}d\mathrm{Re}\zeta d\mathrm{Im}\zeta = -\frac{1}{t}(I_1+I_2+I_3-I_4), \end{aligned} \tag{7.19}$$

   where $f_\pm(\zeta,\lambda) = \lim\limits_{\delta\to 0} f(\zeta(1\mp\delta),\lambda)$, $\zeta \in T$ and T is defined by $T = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$.

5. For each I_i find an estimate of the form

$$|I_i| = O\left(\frac{1}{\varepsilon^{\beta}}\right), \qquad \text{as } \varepsilon \to 0 \qquad (\text{for a certain } \beta > 0).$$

6. Set $\varepsilon = \dfrac{1}{(1+|t|)^k}$, where $k(\alpha+\beta) = 1$, which yields the overall estimate

$$|I(t,u,\lambda)| = O\left(\frac{1}{(1+|t|)^{\frac{\alpha}{\alpha+\beta}}}\right), \quad \text{as} \quad |t| \to \infty.$$

Using this scheme we obtain, in particular, the following result

Lemma 7.2. *Let a function f satisfy assumptions (7.6) and, additionally,*

$$f|_T \equiv 0, \quad T = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}. \tag{7.20}$$

Then the integral $I(t,u)$ of (7.8) can be estimated

$$|I(t,u)| \leqslant \frac{\mathrm{const}(f)\ln(3+|t|)}{(1+|t|)^{3/4}} \quad \textit{for} \quad t \in \mathbb{R} \tag{7.21}$$

uniformly on $u \in \mathbb{C}$.

A detailed proof of Lemma 7.2 is given in subsection 7.4.

7.2 Estimate for the "inverse scattering solution"

In this subsection we prove estimate (7.3) for the solution $v(x,t)$ of the Cauchy problem for the Novikov–Veselov equation at negative energy with the initial data $v(x,0)$ satisfying properties (7.2).

We proceed from formulas (1.40), (1.41) for potential $v(z,t)$ and the integral equation (1.39) for $\mu(z,\lambda,t)$.

We write (1.39) as

$$\mu(z,\lambda,t) = 1 + (A_{z,t}\mu)(z,\lambda,t), \tag{7.22}$$

where

$$(A_{z,t}f)(\lambda) = \partial_{\bar\lambda}^{-1}\big(r(\lambda)\exp(tS(u,\lambda))\overline{f(\lambda)}\big) = -\frac{1}{\pi}\iint\limits_{\mathbb{C}} \frac{r(\zeta)\exp(tS(u,\zeta))\overline{f(\zeta)}}{\zeta-\lambda} d\mathrm{Re}\zeta d\mathrm{Im}\zeta$$

and $S(u,\zeta)$ is defined by (7.9), $u = \dfrac{z}{t}$.

Equation (7.22) can be also written in the form

$$\mu(z,\lambda,t) = 1 + A_{z,t}\cdot 1 + (A_{z,t}^2\mu)(z,\lambda,t). \tag{7.23}$$

According to the theory of generalized analytic functions (see [62]), equations (7.22), (7.23) have a unique bounded solution for all z, t. This solution can be written as

$$\mu(z,\lambda,t) = (I - A_{z,t}^2)^{-1}(1 + A_{z,t}\cdot 1). \tag{7.24}$$

Equation (7.24) implies the following formal representation

$$\mu(z,\lambda,t) = (I + A_{z,t}^2 + A_{z,t}^4 + \ldots)(1 + A_{z,t}\cdot 1). \tag{7.25}$$

From results of Lemma 7.3 it follows that the series in (7.25) converges for large values of t.

We also introduce functions $\nu(z,\lambda,t) = \partial_z\mu(z,\lambda,t)$ and $\eta(z,\lambda,t) = \partial_{\bar z}\mu(z,\lambda,t)$. In terms of these functions potential $v(z,t)$ is obtained by the formula

$$v(z,t) = -2\nu_{-1}(z,t), \tag{7.26}$$

where $\nu_{-1}(z,t)$ is defined by expanding $\nu(z,\lambda,t)$ as $|\lambda|\to\infty$:

$$\nu(z,\lambda,t) = \frac{\nu_{-1}(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right), \quad |\lambda|\to\infty.$$

The pair of functions $\nu(z,\lambda,t)$, $\eta(z,\lambda,t)$ satisfies the following system of differential equations :

$$\begin{cases} \dfrac{\partial\nu(z,\lambda,t)}{\partial\bar\lambda} = \partial_z r(z,\lambda,t)\overline{\mu(z,\lambda,t)} + r(z,\lambda,t)\overline{\eta(z,\lambda,t)}, \\ \dfrac{\partial\eta(z,\lambda,t)}{\partial\bar\lambda} = \partial_{\bar z} r(z,\lambda,t)\overline{\mu(z,\lambda,t)} + r(z,\lambda,t)\overline{\nu(z,\lambda,t)}. \end{cases} \tag{7.27}$$

Equations (7.27) can also be written in the integral form

$$\begin{cases} \nu(z,\lambda,t) = (B_{z,t}\mu)(z,\lambda,t) + (A_{z,t}\eta)(z,\lambda,t), \\ \eta(z,\lambda,t) = (B_{z,t}\mu)(z,\lambda,t) + (A_{z,t}\nu)(z,\lambda,t), \end{cases} \tag{7.28}$$

where operator $B_{z,t}$ is defined

$$(B_{z,t}f)(\lambda)=\partial_{\bar\lambda}^{-1}(\partial_z r(z,\lambda,t)\overline{f(\lambda)})=-\frac{1}{\pi}\iint\limits_{\mathbb{C}}\frac{\partial_z r(z,\zeta,t)\overline{f(\zeta)}}{\zeta-\lambda}d\mathrm{Re}\zeta d\mathrm{Im}\zeta. \tag{7.29}$$

Thus for function $\nu(z,\lambda,t)$ we obtain equation

$$\nu=(B_{z,t}+A_{z,t}B_{z,t})\mu+A_{z,t}^2\nu,$$

or the following formal representation

$$\nu=(I+A_{z,t}^2+A_{z,t}^4+\ldots)((B_{z,t}+A_{z,t}B_{z,t})(I+A_{z,t}^2+A_{z,t}^4+\ldots)(1+A_{z,t}\cdot 1)). \tag{7.30}$$

We will write this formula in the form $\nu=B_{z,t}\cdot 1+A_{z,t}B_{z,t}\cdot 1+R_{z,t}(\lambda)$. From results of Lemma 7.3 it follows that the series in (7.30) converges for large values of t.

Lemma 7.3. *Let $f(\lambda,z,t)$ be an arbitrary testing function such that*

$$|f|\leqslant\frac{c_f}{(1+|t|)^{\delta}},\quad |\partial_\lambda f|\leqslant\frac{c_f}{(1+|t|)^{\delta}}\quad \forall\lambda\in\mathbb{C},z\in\mathbb{C},t\in\mathbb{R}$$

with some positive constant c_f independent of λ, z, t and some $\delta\geqslant 0$. Then :

1. *The following estimates hold for $B_{z,t}\cdot f$:*

$$(B_{z,t}\cdot f)(\lambda)=\frac{\beta_1(z,t)}{\lambda}+o\left(\frac{1}{|\lambda|}\right)\ \text{for}\ \lambda\to\infty,$$

where

$$\beta_1(z,t)=\frac{1}{\pi}\iint_{\mathbb{C}}\partial_z r(z,\zeta,t)\overline{f(\zeta,z,t)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta,$$

and

$$|\beta_1(z,t)|\leqslant\frac{\hat\beta_1 c_f\ln(3+|t|)}{(1+|t|)^{3/4+\delta}}; \tag{7.31}$$

in addition,

$$|(B_{z,t}\cdot f)(\lambda)|\leqslant\frac{\hat\beta_2 c_f\ln(3+|t|)}{(1+|t|)^{1/2+\delta}}\ \text{for}\ \lambda\in T, \tag{7.32}$$

where $T=\{\lambda\in\mathbb{C}\colon |\lambda|=1\}$, and

$$|(B_{z,t}\cdot f)(\lambda)|\leqslant\frac{\hat\beta_3 c_f}{(1+|t|)^{1/4+\delta}}\quad\forall\lambda\in\mathbb{C}. \tag{7.33}$$

2. The following estimates hold for $A_{z,t} \cdot B_{z,t} \cdot f$ *:*

$$(A_{z,t} \cdot B_{z,t} \cdot f)(\lambda) = \frac{\alpha_1(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right) \text{ for } \lambda \to \infty,$$

where

$$\alpha_1(z,t) = -\frac{1}{\pi^2} \iint\limits_{\mathbb{C}} d\mathrm{Re}\zeta d\mathrm{Im}\zeta \, r(z,\zeta,t) \iint\limits_{\mathbb{C}} \frac{\partial_z r(z,\eta,t)}{\eta - \zeta} \overline{f(\eta,z,t)} d\mathrm{Re}\eta \, d\mathrm{Im}\eta, \tag{7.34}$$

and

$$|\alpha_1(z,t)| \leqslant \frac{\hat{\alpha}_1 c_f}{(1+|t|)^{3/4+\delta}}; \tag{7.35}$$

in addition,

$$|(A_{z,t} \cdot B_{z,t} \cdot f)(\lambda)| \leqslant \frac{\hat{\alpha}_2 c_f}{(1+|t|)^{1/2+\delta}} \quad \forall \lambda \in \mathbb{C}. \tag{7.36}$$

3. The following estimates for $A^n_{z,t} \cdot f$ *hold :*

$$(A^n_{z,t} \cdot f)(\lambda) = \frac{\gamma_n(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right) \text{ for } \lambda \to \infty,$$

where

$$\gamma_n(z,t) = \frac{1}{\pi} \iint\limits_{\mathbb{C}} r(z,\zeta,t) \overline{(A^{n-1}_{z,t} \cdot f)(\zeta)} d\mathrm{Re}\zeta d\mathrm{Im}\zeta$$

and

$$|\gamma_n(z,t)| \leqslant \frac{(\hat{\gamma}_1)^n c_f}{(1+|t|)^{\delta + \frac{1}{5}\lceil \frac{n-1}{2} \rceil + \frac{2}{5}}}, \tag{7.37}$$

where $\lceil s \rceil$ *denotes the smallest integer following* s*. In addition,*

$$|(A^n_{z,t} \cdot f)(\lambda)| \leqslant \frac{(\hat{\gamma}_2)^n c_f}{(1+|t|)^{\delta + \frac{1}{5}\lceil \frac{n}{2} \rceil}} \quad \forall \lambda \in \mathbb{C}. \tag{7.38}$$

4. The following estimate holds for $R_{z,t}$ *:*

$$R_{z,t}(\lambda) = \frac{q(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right) \text{ for } \lambda \to \infty, \tag{7.39}$$

and

$$|q(z,t)| \leqslant \frac{\hat{q}(c_f)}{(1+|t|)^{9/10}}. \tag{7.40}$$

A detailed proof of Lemma 7.3 is given in subection 7.4.

From formulas (7.26), (7.30) and Lemma 7.3 follows immediately the following theorem.

Theorem 7.1. *Let $v(x,t)$ be the "inverse scattering solution" of the Cauchy problem for the Novikov–Veselov equation (0.5) with $E = -1$ and the initial data $v(x,0) = v_0(x)$ satisfying (7.2). Then*

$$|\,v(x,t)| \leqslant \frac{\mathrm{const}(v_0)\ln(3+|t|)}{(1+|t|)^{3/4}}, \quad x \in \mathbb{R}^2, t \in \mathbb{R}.$$

7.3 Optimality of estimates for the velocity of decay

In this subsection we show that estimates (7.3) and (7.21) are optimal in the following sense : there exists such a line $z = \hat{u}t$ that along this line $I(z,t)$ from (7.21) behaves asymptotically as $\frac{\mathrm{const}}{(1+|t|)^{3/4}}$ with some nonzero constant ; there exist such initial data satisfying (7.2) that the corresponding solution $v(z,t)$ behaves asymptotically as $\frac{\mathrm{const}}{(1+|t|)^{3/4}}$, when $|t| \to \infty$, along the line $z = \hat{u}t$, where the constant is nonzero.

7.3.1 Optimality of the estimate in the linearized case

Let us consider the integral

$$I(t,u) = \iint\limits_{\mathbb{C}} f(\zeta)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \tag{7.41}$$

where

$$f(\zeta) = \frac{\pi|1-\zeta\bar{\zeta}|}{2|\zeta|^2}b(\zeta), \tag{7.42}$$

with $b(\zeta)$ satisfying (7.6), and $S(u,\zeta)$ is defined by (7.9), for $u = \hat{u} = -18$. As shown in Lemma 7.1 for this value of parameter u the phase $S(\hat{u},\zeta) = S(\zeta)$ has two degenerate stationary points $\zeta = \pm 1$ of the third order.

To calculate the exact asymptotic behavior of $I(t,\hat{u})$ we will use the classic stationary method as described in [12]. First of all, we note that $f(\zeta)$ is continuous, but not continuously differentiable on $\mathbb{C}$. Thus we will consider separately the integrals

$$I_+(t) = \iint\limits_{D_+} f(\zeta)\exp(tS(\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \quad I_-(t) = \iint\limits_{D_-} f(\zeta)\exp(tS(\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta,$$

where D_+ and D_- are defined in (7.7).

Let us introduce a partition of unity $\chi_+(\zeta) + \chi_0(\zeta) + \chi_-(\zeta) \equiv 1$, such that $0 \leqslant \chi_j \leqslant 1$, $\chi_j \in C^\infty(\mathbb{C})$, $j \in \{0,+,-\}$, $\chi_\pm(\zeta) \equiv 1$ in some neighborhood of $\zeta = \pm 1$, respectively, and

$\chi_{\pm}(\zeta) \equiv 0$ everywhere outside some neighborhood of $\zeta = \pm 1$, respectively.

Consider

$$I_+^j(t) = \iint\limits_{D_+} f(\zeta)\chi_j(\zeta)\exp(tS(\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \quad j \in \{0, +, -\}.$$

Applying the Stokes formula we obtain

$$I_+^0(t) = -\frac{1}{2it}\int\limits_T \frac{f(\zeta)\chi_0(\zeta)\exp(tS(\zeta))}{S'_\zeta(\zeta)}d\bar{\zeta} - \frac{1}{t}\iint\limits_{D_+}\frac{(f(\zeta)\chi_0(\zeta))'_\zeta\exp(tS(\zeta))}{S'_\zeta(\zeta)}d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \\ + \frac{1}{t}\iint\limits_{D_+}\frac{f(\zeta)\chi_0(\zeta)\exp(tS(\zeta))S''_{\zeta\zeta}(\zeta)}{(S'_\zeta(\zeta))^2}d\mathrm{Re}\zeta d\mathrm{Im}\zeta, \quad (7.43)$$

where $T = \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$.

From property (7.6) of function b it follows that all integrands in (7.43) are well-defined and bounded in $\bar{D} \cap \mathrm{supp}\chi_0$. Thus

$$I_+^0(t) = O\left(\frac{1}{|t|}\right), \text{ as } t \to \infty,$$

and

$$I_+(t) = I_+^+(t) + I_+^-(t) + O\left(\frac{1}{|t|}\right) \text{ as } |t| \to \infty.$$

First we will estimate $I_+^+(t)$. We note that for the phase $S(\zeta)$ the following representation is valid

$$S(\zeta) = P(\zeta) - P(\bar{\zeta}),$$

where $P(\zeta)$ is a holomorphic function defined by

$$P(\zeta) = -9\zeta - \frac{9}{\zeta} + \zeta^3 + \frac{1}{\zeta^3} + 16;$$

in addition, $P_\zeta(\zeta) = S_\zeta(\zeta) = \frac{3}{\zeta^4}(\zeta-1)^3(\zeta+1)^3$.

Note that $P(\zeta)$ can be written in the form $P(\zeta) = \rho(\zeta)(\zeta-1)^4$, where $\rho(\zeta) = \frac{\zeta^2+4\zeta+1}{\zeta^3}$ and $\lim\limits_{\zeta\to 1}\rho(\zeta) \neq 0$. For the function $\rho(\zeta)$ the expression $(\rho(\zeta))^{1/4}$ can be uniquely defined in some neighborhood of $\zeta = 1$. Further, we define the transformation $\zeta \to \eta$:

$$\eta = (\rho(\zeta))^{1/4}(\zeta - 1).$$

Since we have that

$$\left.\frac{\partial \eta}{\partial \zeta}\right|_{\zeta=1} = \sqrt[4]{6} \neq 0, \tag{7.44}$$

the inverse transformation $\zeta = \varphi(\eta)$ is defined in some small neighborhood of $\eta = 0$. In terms of the new variable η the phase can be represented

$$S(\zeta) = \eta^4 - \bar{\eta}^4.$$

Now if we denote $x = \mathrm{Re}\eta$, $y = \mathrm{Im}\eta$, the integral $I_+^+(t)$ becomes

$$I_+^+(t) = \iint\limits_{\Delta_+} \tilde{f}(x+iy)\tilde{\chi}_+(x+iy)\exp(3itxy(x^2-y^2))|\partial_\eta \varphi(x+iy)|^2 dxdy,$$

where $\tilde{f} = f \circ \varphi$, $\tilde{\chi}_+ = \chi_+ \circ \varphi$ and $\Delta_+ = \{(x,y) \in \mathbb{R}^2 \colon x < 0\}$, i.e. Δ_+ is the half-plane containing the image of $D_+ \cap B_\varepsilon(1)$ under the transformation $x + iy = \varphi^{-1}(\zeta)$.

The integral $I_+^+(t)$ can be written in the form

$$I_+^+(t) = \int\limits_{-\infty}^{+\infty} dc \exp(3itc) \int\limits_{\gamma_c \cap \Delta_+} \tilde{f}(x+iy)\tilde{\chi}_+(x+iy)|\partial_\eta \varphi(x+iy)|^2 d\omega_S,$$

where $d\omega_S$ is the Gelfand–Leray differential form, defined as

$$dS \wedge d\omega_S = dx \wedge dy \tag{7.45}$$

and in the particular case under study equal to

$$d\omega_S = \frac{-(x^3 - 3xy^2)dx + (3x^2y - y^3)dy}{(x^2+y^2)^3};$$

γ_c is an oriented contour consisting of points of the set $\{S(x,y) = c\}$ with the orientation chosen so that (7.45) holds.

As $\tilde{\chi}_+(x+iy)$ is equal to zero outside some $B_R(0)$, a disk of radius R centered in the origin, then there exists such $c_* > 0$ that the set $\{S(x,y) = c\}$ lies outside $B_R(0)$ for any $c < -c_*$, $c > c_*$. Thus the integral $I_+^+(t)$ can be written

$$I_+^+(t) = \int\limits_{-c_*}^{c_*} dc \exp(3itc) \int\limits_{\gamma_c \cap \Delta_+} \tilde{f}(x+iy)\tilde{\chi}_+(x+iy)|\partial_\eta \varphi(x+iy)|^2 d\omega_S.$$

Performing the change of variables

$$\begin{cases} x \mapsto c^{1/4}x, \quad y \mapsto c^{1/4}y \quad \text{for } c > 0, \\ x \mapsto (-c)^{1/4}x, \quad y \mapsto (-c)^{1/4}y \quad \text{for } c < 0 \end{cases}$$

yields

$$I_+^+(t) = \int\limits_0^{c_*} \frac{dc \exp(3itc)}{c^{1/2}} F_+(c) + \int\limits_{-c_*}^{0} \frac{dc \exp(3itc)}{(-c)^{1/2}} F_-(c),$$

where

$$F_+(c) = \int\limits_{\gamma_+ \cap \{c^{1/4}(x,y) \in \Delta_+\}} \mathcal{F}(c,x,y) d\omega_S, \quad F_-(c) = \int\limits_{\gamma_- \cap \{(-c)^{1/4}(x,y) \in \Delta_+\}} \mathcal{F}(-c,x,y) d\omega_S, \tag{7.46}$$

γ_+, γ_- are oriented contours consisting of points of the sets $\{S(x,y) = 1\}$, $\{S(x,y) = -1\}$ correspondingly with orientation chosen so that (7.45) holds and

$$\mathcal{F}(c,x,y) = \tilde{f}(c^{1/4}(x+iy))\tilde{\chi}_+(c^{1/4}(x+iy))|\partial_\eta \varphi(c^{1/4}(x+iy))|^2.$$

For any fixed positive c the integrals in (7.46) converge because the set $\{S(x,y) = 1\}$ is separated from zero and, consequently, the denominator does not vanish, and because $\tilde{\chi}_+$ is a function with a bounded support and thus the domains of integration in (7.46) are, in fact, bounded. Besides, since Δ_+ is a conic set, the functions $F_+(c)$ and $F_-(c)$ can be expressed as follows

$$F_+(c) = \int\limits_{\gamma_+ \cap \Delta_+} \mathcal{F}(c,x,y) d\omega_S, \quad F_-(c) = \int\limits_{\gamma_- \cap \Delta_+} \mathcal{F}(-c,x,y) d\omega_S, \tag{7.47}$$

In some neighborhood $\mathcal{U}_0$ of $c^{1/4}(x+iy) = 0$ containing the support of $\tilde{\chi}_+$ the function $\tilde{f}(c^{1/4}(x+iy))$ can be represented as

$$\tilde{f}(c^{1/4}(x+iy)) = f(1) + 6^{-1/4}[\partial_\zeta f_{D_+}(1)(x+iy) + \partial_{\bar{\zeta}} f_{D_+}(1)(x-iy)]c^{1/4} + g(c^{1/4}(x+iy)),$$

where $\partial_\zeta f_{D_+}(1) = \lim\limits_{\substack{\zeta \in D_+ \\ \zeta \to 1}} \partial_\zeta f$, $\partial_{\bar{\zeta}} f_{D_+}(1) = \lim\limits_{\substack{\zeta \in D_+ \\ \zeta \to 1}} \partial_{\bar{\zeta}} f$ and, in addition, g is a function that can be estimated

$$|g(c^{1/4}(x+iy))| \leqslant K|c^{1/4}(x+iy)|^{1+\alpha}, \quad c^{1/4}(x+iy) \in \mathcal{U}_0 \tag{7.48}$$

with some constants $\alpha > 0$, $K > 0$. Using (7.42), we note that $f(1) = 0$, $\partial_\zeta f_{D_+}(1) = \partial_{\bar{\zeta}} f_{D_+}(1) \stackrel{\text{def}}{=}$

$f'_{D_+}(1)$ and thus

$$\tilde{f}(c^{1/4}(x+iy)) = 6^{-1/4} f'_{D_+}(1) x c^{1/4} + g(c^{1/4}(x+iy)), \quad c^{1/4}(x+iy) \in \mathfrak{U}_0.$$

Taking into account (7.44) we obtain

$$\mathcal{F}(c,x,y) = \gamma f'_{D_+}(1) x c^{1/4} + \tilde{g}(c^{1/4}(x+iy)), \quad c^{1/4}(x+iy) \in \mathfrak{U}_0, \tag{7.49}$$

where $\gamma = 6^{-3/4}$ and $\tilde{g}$ is a function satisfying an estimate similar to (7.48).

It follows then that functions $F_\pm(c)$ behave asymptotically as

$$F_\pm(c) = \gamma f'_{D_+}(1) J^{\pm}_{\Delta_+} (\pm c)^{1/4} + R(c), \text{ when } c \to 0,$$

where

$$J^{+}_{\Delta_+} = \int\limits_{\gamma_+ \cap \Delta_+} x d\omega_S, \quad J^{-}_{\Delta_+} = \int\limits_{\gamma_- \cap \Delta_+} x d\omega_S$$

and $R(c)$ denotes the remainder.

The integrals $J^{\pm}_{\Delta_+}$ converge because the set $\{S(x,y) = \pm 1\}$ represents a combination of curves which do not pass through zero and converge either to lines $|y| = |x|$ or to the coordinate axes with velocities $|y| = \frac{1}{|x|^3}$ and $|x| = \frac{1}{|y|^3}$ correspondingly.

The remainder $R(c)$ behaves asymptotically as $o\left(c^{1/4}\right)$ because we can estimate

$$|R(c)| \leqslant \tilde{K} c^{1/4(1+\alpha)} \int\limits_{(\gamma_+ \cup \gamma_-) \cap \Delta_+} |x+iy|^{1+\alpha} |d\omega_S|.$$

and the integral converges due to the properties of the set $\{S(x,y) = \pm 1\}$ explained above.

Thus $I^{+}_{+}(t)$ behaves asymptotically as (see [12, Chapter III, §1])

$$I^{+}_{+}(t) = \frac{\gamma}{3^{3/4}} f'_{D_+}(1) \Gamma\left(\frac{3}{4}\right) \left[J^{+}_{\Delta_+} \exp\left(\frac{i\pi 3}{8}\right) + J^{-}_{\Delta_+} \exp\left(-\frac{i\pi 3}{8}\right)\right] \frac{1}{t^{3/4}} + o\left(\frac{1}{t^{3/4}}\right),$$

where Γ is the Gamma function.

Let us perform the same procedure for $I^{-}_{+}(t)$. In this case we will define $P(\zeta)$ as $P(\zeta) = -9\zeta - \frac{9}{\zeta} + \zeta^3 + \frac{1}{\zeta^3} - 16$ and obtain

$$P(\zeta) = \rho(\zeta)(\zeta+1)^4, \quad \rho(\zeta) = \frac{\zeta^2 - 4\zeta + 1}{\zeta^3}.$$

Since $\rho(-1) = -6$, we will define the following transformation $\zeta \to \eta$ in the neighborhood of $\zeta = -1 : \eta = (-\rho(\zeta))^{1/4}(\zeta+1)$ (note that $\frac{\partial \eta}{\partial \zeta}\Big|_{\zeta=-1} = \sqrt[4]{6}$). Then $S(\zeta)$ can be represented $S(\zeta) = -\eta^4 + \bar{\eta}^4$ and integral $I_+^-(t)$ becomes

$$I_+^-(t) = \iint_{\Delta_-} \tilde{f}(x+iy)\tilde{\chi}_-(x+iy)\exp(-3itxy(x^2-y^2))|\partial_\eta \varphi(x+iy)|^2 dxdy,$$

where $\Delta_- = \{(x,y) \in \mathbb{R}^2 \colon x > 0\}$ and functions $\tilde{f}$, $\tilde{\chi}_-$, φ are defined similarly to the case of $I_+^+(t)$. The integral $I_+^-(t)$ can also be written

$$I_+^-(t) = \int_{-\infty}^{+\infty} dc \exp(-3itc) \int_{\gamma_c \cap \Delta_-} \tilde{f}(x+iy)\tilde{\chi}_-(x+iy)|\partial_\eta \varphi(x+iy)|^2 d\omega_S,$$

where γ_c and $d\omega_S$ are the same as for the case of $I_+^+(t)$. Performing further the same procedure as for the case of $I_+^+(t)$ and taking into account that $J_{\Delta_+}^+ = -J_{\Delta_-}^+$, $J_{\Delta_+}^- = -J_{\Delta_-}^-$, we obtain the following asymptotic expansion for $I_+^-(t)$:

$$I_+^-(t) = -\frac{\gamma}{3^{3/4}} f'_{D_+}(-1)\Gamma\left(\frac{3}{4}\right)\left[J_{\Delta_+}^+ \exp\left(-\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(\frac{i\pi 3}{8}\right)\right]\frac{1}{t^{3/4}} + o\left(\frac{1}{t^{3/4}}\right).$$

Considering the case of $I_-(t)$ we note that in order to get an asymptotic representation for $I_-^+(t)$ and $I_-^-(t)$ we need to replace $D_+ \to D_-$, $\Delta_+ \to \Delta_-$, $\Delta_- \to \Delta_+$ in the formulas for $I_+^+(t)$ and $I_+^-(t)$ correspondingly. Taking into account that $f'_{D_+}(1) = -f'_{D_-}(1)$, $f'_{D_+}(-1) = -f'_{D_-}(-1)$, we obtain

$$I(t) = \frac{C}{t^{3/4}} + o\left(\frac{1}{t^{3/4}}\right),$$

where

$$C = \frac{2\gamma}{3^{3/4}}\Gamma\left(\frac{3}{4}\right)\left(f'_{D_+}(1)\left(J_{\Delta_+}^+ \exp\left(\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(-\frac{i\pi 3}{8}\right)\right) - \right.$$
$$\left. f'_{D_+}(-1)\left(J_{\Delta_+}^+ \exp\left(-\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(\frac{i\pi 3}{8}\right)\right)\right). \quad (7.50)$$

Thus we have shown that the linear approximation of the solution $v(z,t)$ of (0.5), when $z = -18t$, behaves asymptotically as $\frac{C}{t^{3/4}}$ when $t \to \infty$.

Note that on the set $\gamma_+ \cup \gamma_-$ the differential form $d\omega_S$ is positive. Thus $J_{\Delta_+}^\pm$ are some negative constants, and expressions $J_{\Delta_+}^+ \exp\left(\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(-\frac{i\pi 3}{8}\right)$, $J_{\Delta_+}^+ \exp\left(-\frac{i\pi 3}{8}\right) + J_{\Delta_+}^- \exp\left(\frac{i\pi 3}{8}\right)$ do not vanish. On the other hand, from property 8 of Statement 1.2 and (7.42) it follows that

$f'_{D_+}(1)/f'_{D_+}(-1) = b(1)/\overline{b(1)}$. Thus, in the general case the constant C from (7.50) is nonzero. We have proved the optimality of estimate (7.3) in the linear approximation.

7.3.2 Optimality of the estimate in the nonlinear case

Now we show that for certain initial values $v(z,0)$ the corresponding solution $v(z,t)$ of (0.5) behaves asymptotically as $\frac{c}{t^{3/4}}$ along the line $z = -18t$ for some $c \neq 0$. Let us show that the integral (7.34) with $f \equiv 1$ and $z = -18t$ behaves as $\frac{\text{const}}{t^{3/4}}$.

We can represent $\alpha_1(-18t,t)$ in the form (7.41) with $f(\zeta,t) = r(\zeta,t)\rho(\zeta,t)$, where

$$\rho(\zeta,t) = -\frac{1}{\pi^2}\iint\limits_{\mathbb{C}} \frac{\partial_z r(\eta,z,t)|_{z=-18t}}{\eta-\zeta} d\mathrm{Re}\eta d\mathrm{Re}\zeta.$$

In other terms,

$$f(\zeta) = f(\zeta,t) = \frac{\mathrm{sgn}(1-\zeta\bar\zeta)}{\bar\zeta} b(\zeta)\partial_{\bar\zeta}^{-1}\left[\frac{\pi|1-\zeta\bar\zeta|}{2|\zeta|^2} b(\zeta)\exp(tS(\zeta))\right].$$

We proceed following the scheme of estimate for $I(t)$ until formula (7.48). Then we represent

$$\tilde\chi_+(c^{1/4}(x+iy))|\partial_\eta\varphi(c^{1/4}(x+iy))|^2 = \frac{1}{\sqrt{6}} + kc^{1/4}(x+iy) + h(c^{1/4}(x+iy)),$$

where k is some coefficient and $h(c^{1/4}(x+iy))$ satisfies an estimate of type (7.48). Consequently, for $\mathcal{F}(c,x,y)$ we can write

$$\mathcal{F}(c,x,y) = f(1)(6^{-1/2}+kc^{1/4}(x+iy))+6^{-1/4}[\partial_\zeta f_{D_+}(1)(x+iy)+\partial_{\bar\zeta} f_{D_+}(1)(x-iy)]c^{1/4}+\tilde g(c^{1/4}(x+iy)),$$

where $\tilde g(c^{1/4}(x+iy))$ satisfies an estimate of type (7.48). This allows us to obtain in the end

$$\alpha_1(-18t,t) = \frac{l_1(f(\pm 1))}{t^{1/2}}\left(1+\frac{\text{const}}{t^{1/4}}\right) + \frac{l_2(f_\zeta(\pm1), f_{\bar\zeta}(\pm1))}{t^{3/4}} + o\left(\frac{1}{t^{3/4}}\right), \quad t\to\infty, \qquad (7.51)$$

where $l_1(f(\pm1))$ is a linear combination of the limit values of f as ζ tends to 1 and -1 from inside and outside of the unit circle, and $l_2(f_\zeta(\pm1), f_{\bar\zeta}(\pm1))$ is a linear combination of the limit values of f_ζ and $f_{\bar\zeta}$ as ζ tends to 1 and -1 from inside and outside of the unit circle.

Now let us consider the potential v_θ corresponding to the scattering data θb, where $\theta \in \mathbb{R}$

is some small parameter. In a way similar to which (7.51) was obtained it can be shown that

$$|f(\pm 1,t)| \leqslant \frac{c_1\theta^2}{t^{1/2}}, \quad |f_\zeta(\pm 1,t)| \leqslant c_2\theta^2, \quad |f_{\bar\zeta}(\pm 1,t)| \leqslant c_2\theta^2$$

for sufficiently large values of t, where c_1, c_2, c_3 are some constants independent of t and θ. When $\theta \to 0$ and $t \to \infty$, the linear approximation of v_θ behaves as $O\left(\frac{\theta}{t^{3/4}}\right)$, while the expression $\alpha_1(-18t,t)$ behaves as $O\left(\frac{\theta^2}{t^{3/4}}\right)$ (it can be shown that the member $o\left(\frac{1}{t^{3/4}}\right)$ in (7.51) depends quadratically on θ).

Finally, from (7.30) and Lemma 7.3 it follows that for θ small enough and for $z=-18t$,

$$v_\theta(z,t) = \frac{C_\theta}{t^{3/4}} + o\left(\frac{1}{t^{3/4}}\right), \quad t \to \infty,$$

where C_θ is some nonzero constant. Thus we have shown that the estimate (7.3) is optimal.

7.4 Estimates for intervening integrals (proofs of Lemmas 7.2, 7.3)

Proof of Lemma 7.2. The proof follows the scheme described in Section 7.1 and is carried out separately for four cases depending on the values of the parameter u. In all the reasonings that follow we denote by D_ε the union of disks with the radius ε centered in the stationary points of $S(u,\zeta)$ and we denote by T the unit circle on the complex plane :

$$T = \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}; \tag{7.52}$$

in addition, const will denote an independent constant and const(f) will denote a constant depending only on function f.

Case 1. $u \in \mathbb{U}$

In this case all the stationary points belong to T and due to assumptions (7.6) and (7.20) of Lemma 7.2 we can estimate

$$|f(\zeta)| \leqslant \mathrm{const}(f)\varepsilon \text{ for } \zeta \in D_\varepsilon. \tag{7.53}$$

Now we estimate the integral I_{int} (as in (7.18)) as follows

$$|I_{int}| = \left|\iint\limits_{D_\varepsilon} f(\zeta)\exp(tS(u,\zeta))d\mathrm{Re}\zeta d\mathrm{Im}\zeta\right| \leqslant \mathrm{const}(f)\cdot\varepsilon \iint\limits_{D_\varepsilon} d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \mathrm{const}(f)\varepsilon^3.$$

The estimate for I_{ext} (as in (7.18)) is proved as follows.

We note that the function $S'_\zeta(u,\zeta)$ can be estimated as

$$\begin{aligned} &|S'_\zeta(u,\zeta)| \geqslant \mathrm{const}\frac{\varepsilon_0^3}{|\zeta|^4} \quad \text{for } \zeta\in\mathbb{C}\backslash D_{\varepsilon_0}, \quad \text{and} \\ &|S'_\zeta(u,\zeta)| \geqslant \mathrm{const}\frac{\rho^3}{|\zeta|^4} \quad \text{for } \zeta\in\partial D_\rho, \quad \varepsilon\leqslant\rho\leqslant\varepsilon_0. \end{aligned} \tag{7.54}$$

Similarly, we can estimate

$$\begin{aligned} &\left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \mathrm{const}\frac{|\zeta|^4}{\varepsilon_0^4} \quad \text{for } \zeta\in\mathbb{C}\backslash D_{\varepsilon_0}, \quad \text{and} \\ &\left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \mathrm{const}\frac{|\zeta|^4}{\rho_0^4} \quad \text{for } \zeta\in\partial D_\rho, \quad \varepsilon\leqslant\rho\leqslant\varepsilon_0. \end{aligned} \tag{7.55}$$

Thus we obtain the following estimate for I_1 from (7.19)

$$|I_1| \leqslant \frac{1}{2}\int\limits_{\partial D_\varepsilon} \frac{|f(\zeta)|}{|S'_\zeta(u,\zeta)|}|d\bar\zeta| \leqslant \mathrm{const}\frac{\varepsilon}{\varepsilon^3}\int\limits_{\partial D_\varepsilon}|\zeta|^4|d\bar\zeta| \leqslant \mathrm{const}(f)\frac{\varepsilon}{\varepsilon^2}(1+\varepsilon)^4 \leqslant \frac{\mathrm{const}(f)}{\varepsilon}.$$

Due to assumption (7.20) of Lemma 7.2 the integral I_2 from (7.19) is equivalent to zero.

When estimating I_3 and I_4 from (7.19) we fix some independent $\varepsilon_0 > 0$ and integrate separately over $D_{\varepsilon_0}\backslash D_\varepsilon$ and $\mathbb{C}\backslash D_{\varepsilon_0}$:

$$\begin{aligned} |I_3| \leqslant \iint\limits_{D_{\varepsilon_0}\backslash D_\varepsilon}\left|\frac{f'_\zeta(\zeta)\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}}\left|\frac{f'_\zeta(\zeta)\exp(tS(u,\zeta))}{S'_\zeta(u,\zeta)}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \\ \leqslant \mathrm{const}(f)\int\limits_\varepsilon^{\varepsilon_0}\frac{\rho}{\rho^3}d\rho + \mathrm{const}\iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}}|f'_\zeta(\zeta)||\zeta^4| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \frac{\mathrm{const}(f)}{\varepsilon}, \end{aligned}$$

$$\begin{aligned} |I_4| \leqslant \iint\limits_{D_{\varepsilon_0}\backslash D_\varepsilon}\left|\frac{f(\zeta)\exp(tS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \\ + \iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}}\left|\frac{f(\zeta)\exp(tS(u,\zeta))S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \\ \leqslant \mathrm{const}(f)\int\limits_\varepsilon^{\varepsilon_0}\frac{\rho^2}{\rho^4}d\rho + \mathrm{const}\iint\limits_{\mathbb{C}\backslash D_{\varepsilon_0}}|f(\zeta)||\zeta^3| d\mathrm{Re}\zeta d\mathrm{Im}\zeta \leqslant \frac{\mathrm{const}(f)}{\varepsilon}. \end{aligned}$$

Setting finally $\varepsilon = \frac{1}{(1+|t|)^{1/4}}$ yields

$$I(t,u) \leqslant \frac{\mathrm{const}(f)}{(1+|t|)^{3/4}}$$

uniformly on $u \in \mathbb{U}$.

Case 2. $u \in \mathbb{C}\backslash\mathbb{U}$ and ω from (7.16) satisfies $\omega_0 < \frac{\omega}{1+\omega} < 1-\omega_1$ for some fixed independent positive constants ω_0 and ω_1 (i.e. the roots ζ_0, ζ_2 from (7.16) are separated from T, defined by (7.52), and the root ζ_2 is separated from the origin)

In this case the we can estimate

$$\begin{aligned} |S'_\zeta(u,\zeta)| &\geqslant \frac{\mathrm{const}\rho}{|\zeta|^4} \text{ for } \zeta \in \partial D_\rho, \\ \left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| &\leqslant \frac{\mathrm{const}|\zeta|^4}{\rho^2} \text{ for } \zeta \in \partial D_\rho. \end{aligned}$$

Using these estimates and proceeding as in case 1, we obtain

$$|I_{int}| \leqslant \mathrm{const}(f)\varepsilon^2, \quad |I_1| \leqslant \mathrm{const}(f), \quad I_2 \equiv 0, \quad |I_3| \leqslant \mathrm{const}(f), \quad |I_4| \leqslant \mathrm{const}(f)\ln\frac{1}{\varepsilon}.$$

Setting $\varepsilon = \frac{1}{1+|t|}$, we obtain that

$$I(t,u) \leqslant \mathrm{const}(f)\frac{\ln(3+|t|)}{1+|t|}$$

uniformly for the considered values of the parameter u.

Case 3. $u \in \mathbb{C}\backslash\mathbb{U}$ and $\frac{\omega}{1+\omega} < \omega_0$ (i.e. the roots ζ_0 and ζ_2 from (7.16) lie in some neighborhood of T from (7.52))

Lemma 7.4. *For any $t \geqslant t_0$ with some fixed $t_0 > 0$ and any $\omega > 0$ one of the following conditions holds*

(a) $0 < \omega < \frac{2}{(1+|t|)^{1/4}}$ *;*

(b) $\omega > \frac{1}{(1+|t|)^{1/8}}$ *;*

(c) $\exists n\colon \frac{1}{(1+|t|)^{\gamma_{n+1}/(2+2\gamma_{n+1})}} < \omega < \frac{2}{(1+|t|)^{\gamma_n/(2+2\gamma_{n+1})}}$, *where* $\gamma_{n+1} = \frac{2}{3}\gamma_n + \frac{1}{3}$, $\gamma_1 = \frac{1}{3}$.

Proof. We note that

$$\frac{\gamma_{n+1}}{2+2\gamma_{n+1}} \to \frac{1}{4}, \quad n \to \infty;$$
$$\frac{\gamma_n}{2+2\gamma_{n+1}} < \frac{\gamma_n}{2+2\gamma_n};$$
$$\frac{\gamma_1}{2+2\gamma_2} < \frac{1}{8}.$$

Thus the intervals from the cases (a), (b), (c) $\forall n \in \mathbb{N}$ cover the whole range $0 < \omega < +\infty$. $\square$

We will prove the result separately for three different cases depending on the value of parameter ω

(a) $0 < \omega < 2\varepsilon = \frac{2}{(1+|t|)^{1/4}}$

In this case estimates (7.53), (7.54), (7.55) hold and so the reasoning of the case 1 can be carried out to obtain that

$$I(t,u) \leqslant \frac{\text{const}(f)}{(1+|t|)^{3/4}}$$

uniformly for the considered values of the parameter u satisfying

$$0 < \omega < \frac{2}{(1+|t|)^{1/4}}. \tag{7.56}$$

(b) $\omega > \varepsilon^{1/3} = \frac{1}{(1+|t|)^{1/8}}$

In this case we estimate $|I_{int}| \leqslant \text{const}(f)\varepsilon^2$.

Further, we note that the derivative of the phase is estimated as

$$|S'_\zeta(u,\zeta)| \geqslant \frac{\text{const}\,\varepsilon\omega^2}{|\zeta|^4} \text{ for } \zeta \in \partial D_\varepsilon. \tag{7.57}$$

Thus for I_1 we obtain $|I_1| \leqslant \text{const}(f)\frac{1}{\varepsilon^{2/3}}$.

In order to estimate the integral I_3 we use the following estimate of the derivative S'_ζ for $\zeta \in \partial D_\rho$ when $\varepsilon \leqslant \rho \leqslant \varepsilon_0$:

$$\begin{cases} |S'_\zeta(u,\zeta)| \geqslant \frac{\text{const}\rho\omega^2}{|\zeta|^4}, \text{ if } \rho < \omega, \\ |S'_\zeta(u,\zeta)| \geqslant \frac{\text{const}\rho^3}{|\zeta|^4}, \text{ if } \rho > \omega. \end{cases} \tag{7.58}$$

It allows to derive $|I_3| \leqslant \text{const}(f)\frac{1}{\varepsilon^{2/3}}$.

Finally, we proceed to the study of the integral I_4. We use the following estimates

$$\left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{\mathrm{const}|\zeta|^4}{\rho^2\omega^2}$$

and

$$\begin{cases} |f(\zeta)| \leqslant \mathrm{const}(f)\,\omega, \text{ if } \rho < \omega, \\ |f(\zeta)| \leqslant \mathrm{const}(f)\,\rho, \text{ if } \rho > \omega. \end{cases} \tag{7.59}$$

After integration we obtain the estimate $|I_4| \leqslant \mathrm{const}(f)\frac{1}{\varepsilon^{2/3}}$.
Setting finally $\varepsilon = \frac{1}{(1+|t|)^{3/8}}$, we obtain

$$I(t,u) \leqslant \frac{\mathrm{const}(f)}{(1+|t|)^{3/4}}$$

uniformly for the considered values of the parameter u satisfying

$$\omega > \frac{1}{(1+|t|)^{1/8}}. \tag{7.60}$$

(c) $\varepsilon^{\gamma_{n+1}} < \omega < 2\varepsilon^{\gamma_n}$, where $\varepsilon = \frac{1}{(1+|t|)^{1/(2+2\gamma_{n+1})}}$ and $\gamma_{n+1} = \frac{2}{3}\gamma_n + \frac{1}{3}$, $\gamma_1 = \frac{1}{3}$ (note that $\gamma_n \to 1$)
We proceed similarly to the case (b). Evidently, I_{int} can be estimated $|I_{int}| \leqslant \mathrm{const}(f)\varepsilon^{2+\gamma_n}$.
Employing the estimate (7.57) we obtain $|I_1| \leqslant \mathrm{const}(f)\frac{\varepsilon^{\gamma_n}}{\varepsilon^{2\gamma_{n+1}}}$.
Using (7.58) in order to estimate I_3 we obtain $|I_3| \leqslant \mathrm{const}(f)\frac{1}{\varepsilon^{\gamma_{n+1}}}$.
Finally, to estimate I_4 we use (7.59) and

$$\left|\frac{S''_{\zeta\zeta}}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \begin{cases} \frac{\mathrm{const}|\zeta|^4}{\rho^2\omega^2}, & \rho < \omega, \\ \frac{\mathrm{const}|\zeta|^4}{\rho^3\omega}, & \rho > \omega \end{cases}$$

to obtain $|I_4| \leqslant \mathrm{const}(f)\frac{\varepsilon^{\gamma_n}\ln(1/\varepsilon)}{\varepsilon^{2\gamma_{n+1}}}$.
Setting $\varepsilon = \frac{1}{(1+|t|)^{1/(2+2\gamma_{n+1})}}$ yields

$$|I(t,u)| \leqslant \frac{\mathrm{const}(f)\ln(3+|t|))}{(1+|t|)^{3/4}}$$

uniformly for the considered values of the parameter u satisfying

$$\frac{1}{(1+|t|)^{\gamma_{n+1}/(2+2\gamma_{n+1})}} < \omega < \frac{2}{(1+|t|)^{\gamma_n/(2+2\gamma_{n+1})}}. \tag{7.61}$$

Finally, from Lemma 7.4 it follows that we have proved the required estimate uniformly on the values of parameter $u \in \mathbb{C}\backslash\mathbb{U}$ such that $\frac{\omega}{1+\omega} < \omega_0$.

Case 4. $u \in \mathbb{C}\backslash\mathbb{U}$ and $\frac{\omega}{1+\omega} > 1 - \omega_1$ (i.e. the roots ζ_2, $-\zeta_2$ lie in the ω_1–neighborhood of the origin)

This case is treated similarly to the previous one. We denote $\tilde{\omega} = \frac{1}{1+\omega}$. Then we use estimates

$$
\begin{aligned}
&|f(\zeta)| \leqslant c(f)|\tilde{\omega} + \rho|,\\
&|S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}\rho^2}{|\zeta|^4} \text{ or } |S'_\zeta(u,\zeta)| \geqslant \frac{\mathrm{const}\rho\tilde{\omega}}{|\zeta|^4},\\
&\left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{\mathrm{const}|\zeta|^4}{\rho^3} \text{ or } \left|\frac{S''_{\zeta\zeta}(u,\zeta)}{(S'_\zeta(u,\zeta))^2}\right| \leqslant \frac{\mathrm{const}|\zeta|^4}{\rho^2\tilde{\omega}},
\end{aligned}
$$

which hold for $\zeta \in D_\rho$, to obtain the necessary estimates. □

Proof of Lemma 7.3. 1. The proof of inequality (7.31) repeats the proof of Lemma 7.2. The proof of inequality (7.32) also follows the scheme of the proof of Lemma 7.2. In this case we take D_ε to be the union of disks of the radius ε with centers in the stationary points of $S(u,\zeta)$ and in the point λ.

For the case when $\lambda \notin T$, where T is defined by (7.52), an estimate weaker than (7.32) can be obtained via a simplified reasoning. Indeed, I_{int}, as in (7.18), can be estimated $|I_{int}| \leqslant O\left(\frac{\varepsilon}{(1+|t|)^\delta}\right)$. Using estimates (7.54), (7.55) and

$$|\zeta - \lambda| \geqslant \rho \text{ for } \zeta \in \partial D_\rho \tag{7.62}$$

we obtain that $|I_{ext}| \leqslant O\left(\frac{1}{(1+|t|)^{1+\delta}\varepsilon^3}\right)$. Setting $\varepsilon = \frac{1}{(1+|t|)^{1/4}}$ we get the estimate (7.33).

2. In order to obtain estimates (7.35), (7.36) we proceed according to the scheme outlined in Section 7.1. In this case the integral I_2 does not annul. On the other hand, when the variable of integration belongs to T, the estimate (7.32) on the integrand of I_2 is stronger than the estimate (7.33) for the general case. Thus we obtain for $\alpha_1(z,t)$

$$
\begin{gathered}
|I_{int}| \leqslant O\left(\frac{\varepsilon^2}{(1+|t|)^{\delta+\frac{1}{4}}}\right), \quad |I_1| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{4}}\varepsilon^2}\right), \quad |I_2| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{4}}\varepsilon^3}\right),\\
|I_3| \leqslant O\left(\frac{1}{(1+|t|)^{\delta}\varepsilon}\right), \quad |I_4| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{4}}\varepsilon^2}\right).
\end{gathered}
$$

Setting $\varepsilon = \frac{1}{(1+|t|)^{1/4}}$ yields the required estimate. The estimate (7.36) is obtained similarly.

3. We will give the scheme of the proof for estimate (7.37). The estimate (7.38) is obtained similarly.

   We will prove (7.37) by induction. Suppose that (7.37) holds for all $n = 1, 2, \ldots, N$. Then following the scheme of Section 7.1 and taking into account that $\partial_\lambda \overline{(A^n_{z,t} \cdot f)(\lambda)} = \overline{(A^{n-1}_{z,t} \cdot f)(\lambda)}$, we obtain for $n = N+1$:

$$|I_{int}| \leqslant O\left(\frac{\varepsilon}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil}}\right), \quad |I_1| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil}\varepsilon^3}\right),$$
$$|I_2| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil}\varepsilon^4}\right), \quad |I_3| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-2}{2}\rceil}\varepsilon^3}\right),$$
$$|I_4| \leqslant O\left(\frac{1}{(1+|t|)^{\delta+\frac{1}{5}\lceil\frac{n-1}{2}\rceil}\varepsilon^3}\right).$$

   Setting $\varepsilon = \frac{1}{(1+|t|)^{1/5}}$ we obtain the required estimate.

4. We represent $R_{z,t}(\lambda)$ as the sum of the following members

$$R_{z,t}(\lambda) = B(A + A^2 + A^3 + \ldots) \cdot 1 + AB(A + A^2 + A^3 + \ldots) \cdot 1 +$$
$$+ (A + A^2 + A^3 + \ldots)AB(I + A + A^2 + \ldots) \cdot 1 = R^1_{z,t}(\lambda) + R^2_{z,t}(\lambda) + R^3_{z,t}(\lambda).$$

   The convergence of the series at sufficiently large times follows from the estimate (7.38). Now let
$$R^i_{z,t}(\lambda) = \frac{q_i(z,t)}{\lambda} + o\left(\frac{1}{|\lambda|}\right), \text{ as } \lambda \to \infty.$$

   From (7.32) and (7.38) it follows that $|q_1(z,t)| \leqslant \frac{\hat{q}_1(c_f)\ln(3+|t|)}{(1+|t|)^{3/4+1/5}}$. From (7.35) and (7.38) we obtain that $|q_2(z,t)| \leqslant \frac{\hat{q}_2(c_f)}{(1+|t|)^{3/4+1/5}}$. Finally, from (7.36), (7.37) and (7.38) it follows that $|q_3(z,t)| \leqslant \frac{\hat{q}_3(c_f)}{(1+|t|)^{1/2+2/5}}$. This yields the required estimate. □

Conclusion

In the present work we investigated the Novikov-Veselov equation, a nonlinear integrable evolution equation which is a $(2+1)$-dimensional analog of the celebrated Korteweg-de Vries equation. The Novikov-Veselov equation is integrable via the inverse scattering transform method for the 2-dimensional stationary Schrödinger equation and we used this method to study the large time asymptotics of its solutions and the question of existence or absence of solitons for this equation.

We showed that in the nonsingular case, i.e. when the scattering data for the associated 2-dimensional Schrödinger equation are everywhere well-defined, the solution of the Cauchy problem for the Novikov-Veselov equation at nonzero energy from the Schwartz class (and simultaneously a transparent solution in the case of positive energy) decreases uniformly with time and thus, in particular, does not have solitons in the large time asymptotics. Thus the above result at $E > 0$ highlights a significant difference between the Novikov-Veselov equation and its (1+1)-dimensional analog : recall that KdV possesses well-known exponentially localized reflectionless solitons.

Results of the present work demonstrate that in the nonsingular case the Novikov-Veselov equation does not possess solitons. Thus solitons may arise only if the scattering data have some singularities. This is precisely what happens in the case of Grinevich-Zakharov potentials : the poles of the scattering function give rise to multisoliton solutions of the Novikov-Veselov equation at positive energy as was demonstrated in the present work.

The present work demonstrates that the space localization of Grinevich-Zakharov potentials (as $O\left(|x|^{-2}\right)$ when $|x| \to \infty$) is almost the optimal that can be obtained for the solitons of the Novikov-Veselov equation. We show that in the case of nonzero energy the space localization stronger than $O\left(|x|^{-3}\right)$ is already impossible. For the case of zero energy we have shown that if we restrain ourselves to solutions that are obtained from solutions of the modified Novikov-Veselov equation via Miura type transformation, then the existence of solitons localized stronger than $O\left(|x|^{-2}\right)$ is not possible.

The technique developed in the present work can be applied to other problems of the soliton theory in dimension $2+1$.

Appendix

A.1 Behavior of $w(z)$ as $z \to \infty$

Statement A.1. *Let $v \in C^4(\mathbb{C})$ and*

$$|\partial_z^{j_1}\partial_{\bar z}^{j_2}v(z)| < \frac{q}{(1+|z|)^{j+3+\varepsilon}}, \quad j_1, j_2 \in \{0 \cup \mathbb{N}\},\ j = j_1 + j_2 \leqslant 4 \tag{A.1}$$

for $\forall z \in \mathbb{C}$ and some $q > 0$, $\varepsilon > 0$. Let w be defined by

$$\begin{aligned} &\partial_{\bar z} w = -3\partial_z v, \\ &w(z) \to 0 \text{ as } z \to \infty. \end{aligned}$$

Then

$$w(z) = \frac{3\hat v(0)}{\pi z^2} + O\left(\frac{1}{|z|^3}\right), \ \text{ as } z \to \infty, \tag{A.2}$$

$$\partial_z w(z) = -\frac{6\hat v(0)}{\pi z^3} + O\left(\frac{1}{|z|^4}\right), \ \text{ as } z \to \infty, \tag{A.3}$$

where $\hat v(0) = \iint\limits_{\mathbb{C}} v(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta$.

Proof. Let $z = \frac{1}{\xi}$ and $W(\xi) = w\left(\frac{1}{\xi}\right)$. If $W \in C^3(U)$, where U is some neighborhood of $\xi = 0$, then

$$W(\xi) = \partial_\xi W(0)\xi + \partial_{\bar\xi} W(0)\bar\xi + \frac{1}{2}\partial_\xi^2 W(0)\xi^2 + \partial_\xi\partial_{\bar\xi} W(0)\xi\bar\xi + \frac{1}{2}\partial_{\bar\xi}^2 W(0)\bar\xi^2 + O\left(|\xi|^3\right), \quad \xi \to 0.$$

First, let us prove that the first order derivatives of W vanish at $\xi = 0$. Indeed,

$$\partial_{\bar{\xi}} W(0) = -\lim_{\xi \to 0} \left(\partial_{\bar{z}} w \left(\frac{1}{\xi} \right) \frac{1}{\bar{\xi}^2} \right) = 3 \lim_{z \to \infty} \bar{z}^2 \partial_z v(z) = 0$$

due to property (A.1). Similarly,

$$\partial_{\xi} W(0) = \lim_{z \to \infty} z^2 \partial_z w(z) = \frac{3}{\pi} \lim_{z \to \infty} z^2 \iint\limits_{\mathbb{C}} \frac{\partial_{\zeta}^2 v(\zeta)}{\zeta - z} d\mathrm{Re}\zeta d\mathrm{Im}\zeta.$$

Using representation $\frac{z^2}{\zeta - z} = -\zeta - z + \frac{\zeta^2}{\zeta - z}$ and noting that

$$\iint\limits_{\mathbb{C}} \zeta \partial_{\zeta}^2 v(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = 0, \quad \iint\limits_{\mathbb{C}} \partial_{\zeta}^2 v(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = 0, \tag{A.4}$$

$$\lim_{z \to \infty} \iint\limits_{\mathbb{C}} \frac{\zeta^2 \partial_{\zeta}^2 v(\zeta)}{\zeta - z} d\mathrm{Re}\zeta d\mathrm{Im}\zeta = 0, \tag{A.5}$$

we obtain that $\partial_{\xi} W(0) = 0$.

Now we calculate the second order derivatives of W :

$$\begin{aligned} \partial_{\bar{\xi}}^2 W(0) &= \lim_{z \to \infty} (-3\bar{z}^4 \partial_z \partial_{\bar{z}} v - 6\bar{z}^3 \partial_z v) = 0, \\ \partial_{\xi} \partial_{\bar{\xi}} W(0) &= \lim_{z \to \infty} (-3|z|^4 \partial_z^2 v) = 0 \end{aligned}$$

due to assumption (A.1). In addition,

$$\partial_{\xi}^2 W(0) = \lim_{z \to \infty} (z^4 \partial_z^2 w + 2z^3 \partial_z w).$$

Using representations $\frac{z^3}{\zeta - z} = -z^2 - \zeta z - \zeta^2 + \frac{\zeta^3}{\zeta - z}$, $\frac{z^4}{\zeta - z} = -\zeta^3 - \zeta^2 z - \zeta z^2 - z^3 + \frac{\zeta^4}{\zeta - z}$ it is possible to show that

$$\lim_{z \to \infty} \frac{6}{\pi} \iint\limits_{\mathbb{C}} \frac{z^3 \partial_{\zeta}^2 v(\zeta)}{\zeta - z} d\mathrm{Re}\zeta d\mathrm{Im}\zeta = -\frac{6}{\pi} \iint\limits_{\mathbb{C}} \zeta^2 \partial_{\zeta}^2 v(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = -\frac{12}{\pi} \hat{v}(0)$$

and

$$\lim_{z \to \infty} \frac{3}{\pi} \iint\limits_{\mathbb{C}} \frac{z^4 \partial_{\zeta}^3 v(\zeta)}{\zeta - z} d\mathrm{Re}\zeta d\mathrm{Im}\zeta = -\frac{3}{\pi} \iint\limits_{\mathbb{C}} \zeta^3 \partial_{\zeta}^3 v(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta = \frac{18}{\pi} \hat{v}(0),$$

which yields

$$\partial_{\xi}^2 W(0) = \frac{6}{\pi} \hat{v}(0).$$

Finally, it is not difficult to verify that under assumption (A.1) the third order derivatives of W are well-defined in some neighborhood of $\xi = 0$ and thus the following representation holds :

$$W(\xi) = \frac{3\hat{v}(0)}{\pi}\xi^2 + O(|\xi|^3), \quad \xi \to 0,$$

which proves (A.2).

In order to prove (A.3) we note that $\partial_z w(z) = -\xi^2 \partial_\xi W$ and that under assumption (A.1) the following representation holds :

$$\partial_\xi W(\xi) = \partial_\xi^2 W(0)\xi + \partial_\xi \partial_{\bar{\xi}} W(0)\bar{\xi} + O(|\xi|^2), \quad \xi \to 0.$$

□

A.2 Derivation of formulas for $\partial_t\alpha$, $\partial_t\beta$

Here we present a detailed derivation of formulas (4.38), (4.40) proceeding from representations (4.37), (4.39), respectively.

Derivation of (4.38). Consider expression (4.37). For $\partial_t v$ and $\partial_t \varphi$, appearing in this expression, we will use representations (0.5) and (4.36), correspondingly. Note, however, that this produces the following integrals

$$-2E \iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar{\zeta}+\frac{\zeta}{\lambda})}\, \partial_\zeta w\, \varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \quad -2E \iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar{\zeta}+\frac{\zeta}{\lambda})}\, \partial_{\bar{\zeta}} \bar{w}\, \varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,$$

which are not, strictly speaking, convergent. Thus we will understand (4.37) in the following sense :

$$\partial_t \alpha(\lambda, t) = \lim_{R\to\infty} \left(\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar{\zeta}+\zeta/\lambda)} \partial_t v(\zeta,t)\varphi(\zeta,\lambda,t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta + \right.$$
$$\left. + \iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar{\zeta}+\zeta/\lambda)} v(\zeta,t)\partial_t\varphi(\zeta,\lambda,t) d\mathrm{Re}\zeta d\mathrm{Im}\zeta \right), \quad \text{(A.6)}$$

where B_R is a disk on the complex plane with radius R.

Substituting (0.5) and (4.36) into (A.6) yields

$$
\begin{aligned}
\partial_t \alpha(\lambda,t) = \lim_{R\to\infty}\Bigg(& 8\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta^3 v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + 8\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_{\bar\zeta}^3 v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \\
& + 2\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta v\, w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + 2\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\,\partial_\zeta w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \\
& + 2\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_{\bar\zeta} v\, \bar w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + 2\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\,\partial_{\bar\zeta} \bar w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta - \\
& - 2E\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta - 2E\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_{\bar\zeta} \bar w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \\
& + 8\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\,\partial_\zeta^3 \varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + 8\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\,\partial_{\bar\zeta}^3 \varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \\
& + 2\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\, w\,\partial_\zeta \varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + 2\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\, \bar w\,\partial_{\bar\zeta} \varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \\
& + i(\sqrt{E})^3\left(\lambda^3+\frac{1}{\lambda^3}\right)\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \\
& + 3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\,\psi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \Bigg) = \lim_{R\to\infty}\sum_{i=1}^{14} I_i. \qquad (A.7)
\end{aligned}
$$

We start by integrating I_9 by parts. Note that due to assumptions (4.2) the integrals over the boundary of B_R vanish as $R\to\infty$. Thus we obtain

$$
\begin{aligned}
\lim_{R\to\infty} I_9 = & -\frac{i(\sqrt{E})^3}{\lambda^3}\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \frac{6E}{\lambda^2}\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \\
& + \frac{12i\sqrt{E}}{\lambda}\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta^2 v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta - 8\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta^3 v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta.
\end{aligned}
$$

In this way it can be obtained that

$$
\begin{aligned}
& \lim_{R\to\infty}(I_1+I_2+I_9+I_{10}+I_{13}) = \\
& = \frac{6E}{\lambda^2}\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \frac{12i\sqrt{E}}{\lambda}\iint\limits_{\mathbb{C}} e^{\frac{-i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta^2 v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + \\
& + 6E\lambda^2\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_{\bar\zeta} v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta + 12i\sqrt{E}\lambda\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_{\bar\zeta}^2 v\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta. \qquad (A.8)
\end{aligned}
$$

Integrating I_{11} by parts we obtain

$$\lim_{R\to\infty}(I_{11}+I_7) = -2\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta v\, w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta+$$
$$+\lim_{R\to\infty}\left(\frac{i\sqrt{E}}{\lambda}\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\, w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta - 2E\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta\right)-$$
$$-2\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\,\partial_\zeta w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta.$$

Consider
$$\tilde I = \frac{i\sqrt{E}}{\lambda}\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, v\, w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta.$$

Taking into account that $-4\partial_\zeta\partial_{\bar\zeta}\varphi + v\varphi = E\varphi$ we obtain that

$$\tilde I = \frac{i(\sqrt{E})^3}{\lambda}\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, w\,\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta+\frac{4i\sqrt{E}}{\lambda}\iint\limits_{B_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, w\,\partial_\zeta\partial_{\bar\zeta}\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta = \tilde I_{(1)}+\tilde I_{(2)}.$$

Applying the Stokes theorem to $\tilde I_{(2)}$ we get that

$$\tilde I_{(2)} = -\frac{2\sqrt{E}}{\lambda}\int\limits_{C_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, w\,\partial_{\bar\zeta}\varphi\, d\bar\zeta - \frac{4i\sqrt{E}}{\lambda}\iint\limits_{B_R}\partial_\zeta\left(e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, w\right)\partial_{\bar\zeta}\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \quad \text{(A.9)}$$

where C_R is the boundary of B_R.

Note that

$$\partial_{\bar z}\varphi(z,\lambda) = \frac{i\sqrt{E}}{2}\lambda e^{\frac{i\sqrt{E}}{2}(\lambda\bar z+\frac{z}{\lambda})}\left(\frac{i\sqrt{E}}{2}\left(\lambda\bar z-\frac{1}{\lambda}z\right)+1+o(1)\right),\ \text{as } z\to\infty. \quad \text{(A.10)}$$

In addition, note that
$$\int\limits_{C_R}\frac{\zeta d\bar\zeta}{\zeta^2} = 0, \quad \int\limits_{C_R}\frac{\bar\zeta d\bar\zeta}{\zeta^2} = 0. \quad \text{(A.11)}$$

Thus from Statement A.1, (A.10), (A.11) it follows that as $R\to\infty$ the first summand in the right-hand side of (A.9) vanishes. We apply the Stokes theorem to the second summand :

$$-\frac{4i\sqrt{E}}{\lambda}\iint\limits_{B_R}\partial_\zeta\left(e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, w\right)\partial_{\bar\zeta}\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta = \frac{iE}{\lambda^2}\int\limits_{C_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, w\,\varphi\, d\zeta-$$
$$-\frac{2\sqrt{E}}{\lambda}\int\limits_{C_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta w\,\varphi\, d\zeta + \frac{4i\sqrt{E}}{\lambda}\iint\limits_{B_R}\partial_\zeta\partial_{\bar\zeta}\left(e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\, w\right)\varphi\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta. \quad \text{(A.12)}$$

From Statement A.1 and asymptotics of $\varphi(z,\lambda)$ as $z\to\infty$ it follows that the second summand in the right-hand side of (A.12) vanishes as $R\to\infty$. For the first summand we obtain

$$\frac{iE}{\lambda^2}\int\limits_{C_R} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,w\,\varphi\,d\zeta \sim \frac{(\sqrt{E})^3}{2\lambda^3}\frac{3\hat v(0)}{\pi}\int\limits_{C_R}\frac{d\zeta}{\zeta} = 3\frac{i(\sqrt{E})^3}{\lambda^3}\hat v(0) \text{ as } R\to\infty.$$

Finally, for $I_{11}+I_7$ we obtain that

$$\lim_{R\to\infty}(I_{11}+I_7) = -2\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta v\,w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta - \frac{6E}{\lambda^2}\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta -$$
$$-\frac{12i\sqrt{E}}{\lambda}\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta^2 v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta - 2\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,v\,\partial_\zeta w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta +$$
$$+3\frac{i(\sqrt{E})^3}{\lambda^3}\hat v(0).$$

Thus it can be obtained that

$$\lim_{R\to\infty}(I_3+I_4+I_5+I_6+I_7+I_8+I_{11}+I_{12}) =$$
$$= -\frac{6E}{\lambda^2}\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta - \frac{12i\sqrt{E}}{\lambda}\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_\zeta^2 v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta -$$
$$-6E\lambda^2\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_{\bar\zeta} v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta - 12i\sqrt{E}\lambda\iint\limits_{\mathbb{C}} e^{-\frac{i\sqrt{E}}{2}(\lambda\bar\zeta+\frac{\zeta}{\lambda})}\,\partial_{\bar\zeta}^2 v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta -$$
$$-3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)\hat v(0). \quad \text{(A.13)}$$

Finally,

$$\lim_{R\to\infty} I_{14} = 3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)a(\lambda,t) \quad \text{(A.14)}$$

and thus from (A.7)-(A.14) we obtain formula (4.38).

Derivation of (4.40). We proceed similarly to the derivation of formula (4.38). Substituting (0.5) and (4.36) into (4.39) yields

$$\partial_t\beta(\lambda,t) = \lim_{R\to\infty}\Bigg(8\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_\zeta^3 v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta + 8\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\bar\zeta}^3 v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta +$$
$$+2\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_\zeta v\,w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta + 2\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,\partial_\zeta w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta +$$

$$
\begin{aligned}
&+2\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\bar\zeta}v\,\bar w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+2\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,\partial_{\bar\zeta}\bar w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta-\\
&-2E\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\zeta}w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta-2E\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\bar\zeta}\bar w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+\\
&+8\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,\partial_{\zeta}^3\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+8\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,\partial_{\bar\zeta}^3\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+\\
&+2\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,w\,\partial_{\zeta}\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+2\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,\bar w\,\partial_{\bar\zeta}\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+\\
&+i(\sqrt{E})^3\left(\lambda^3+\frac{1}{\lambda^3}\right)\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+\\
&\left.+3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,\psi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta\right)=\lim_{R\to\infty}\sum_{i=1}^{14}J_i, \qquad \text{(A.15)}
\end{aligned}
$$

where B_R is a disk on the complex plane with radius R.

Integrating J_9 by parts yields

$$
\begin{aligned}
&\lim_{R\to\infty} J_9=\bar\lambda^3 i(\sqrt{E})^3(\operatorname{sgn}E)\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+\\
&+6\bar\lambda^2E\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\zeta}v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta-12\bar\lambda i\sqrt{E}(\operatorname{sgn}E)\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\zeta}^2v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta-\\
&-8\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\zeta}^3v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta.
\end{aligned}
$$

In this way it can be obtained that

$$
\begin{aligned}
&\lim_{R\to\infty}(J_1+J_2+J_9+J_{10}+J_{13})=i(\sqrt{E})^3\left(\lambda^3+\frac{1}{\lambda^3}+(\operatorname{sgn}E)\left(\bar\lambda^3+\frac{1}{\bar\lambda^3}\right)\right)\beta(\lambda,t)+\\
&+6E\bar\lambda^2\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\zeta}v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta-12i\sqrt{E}(\operatorname{sgn}E)\bar\lambda\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\zeta}^2v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+\\
&+\frac{6E}{\bar\lambda^2}\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\bar\zeta}v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta-\frac{12i\sqrt{E}}{\bar\lambda}(\operatorname{sgn}E)\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\bar\zeta}^2v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta.
\end{aligned}
\qquad \text{(A.16)}
$$

Integrating J_{11} by parts we obtain

$$
\lim_{R\to\infty}(J_{11}+J_7)=-2\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\operatorname{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\lambda})}\,\partial_{\zeta}v\,w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+
$$

$$+\lim_{R\to\infty}\left(-i\sqrt{E}(\mathrm{sgn}E)\bar\lambda\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\bar\lambda})}\,v\,w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta-\right.$$

$$\left.-2E\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\bar\lambda})}\,\partial_\zeta w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta\right)-2\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar\lambda\zeta+\frac{\bar\zeta}{\bar\lambda})}\,v\,\partial_\zeta w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta.$$

Consider

$$\tilde J=-i\sqrt{E}(\mathrm{sgn}E)\bar\lambda\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar\lambda\zeta+\frac{\bar\zeta}{\bar\lambda})}\,v\,w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta.$$

Taking into account that $-4\partial_\zeta\partial_{\bar\zeta}\varphi+v\varphi=E\varphi$ we obtain that

$$\tilde J=-i(\sqrt{E})^3(\mathrm{sgn}E)\bar\lambda\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar\lambda\zeta+\frac{\bar\zeta}{\bar\lambda})}\,w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta-$$

$$-4i\sqrt{E}(\mathrm{sgn}E)\bar\lambda\iint\limits_{B_R} e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar\lambda\zeta+\frac{\bar\zeta}{\bar\lambda})}\,w\,\partial_\zeta\partial_{\bar\zeta}\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta=\tilde J_{(1)}+\tilde J_{(2)}.$$

Applying the Stokes theorem to $\tilde J_{(2)}$ we get that

$$\tilde J_{(2)}=2\sqrt{E}(\mathrm{sgn}E)\bar\lambda\int\limits_{C_R} e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar\lambda\zeta+\frac{\bar\zeta}{\bar\lambda})}\,w\,\partial_{\bar\zeta}\varphi\,d\bar\zeta+$$

$$+4i\sqrt{E}(\mathrm{sgn}E)\bar\lambda\iint\limits_{B_R}\partial_\zeta\left(e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar\lambda\zeta+\frac{\bar\zeta}{\bar\lambda})}\,w\right)\partial_{\bar\zeta}\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta=\tilde J_{(21)}+\tilde J_{(22)},$$

where C_R is the boundary of B_R.

Transform $\tilde J_{(21)}$ into the following form

$$\tilde J_{(21)}=2\sqrt{E}(\mathrm{sgn}E)\bar\lambda\int\limits_0^{2\pi} e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)\left(\bar\lambda Re^{i\varphi}+\frac{Re^{-i\varphi}}{\bar\lambda}\right)}w(Re^{i\varphi})\partial_{\bar\zeta}\varphi(\zeta)|_{\zeta=Re^{i\varphi}}(-iRe^{-i\varphi})d\varphi=$$

$$=2\sqrt{E}(\mathrm{sgn}E)\bar\lambda\int\limits_0^{2\pi}q(R,\varphi)d\varphi.$$

Note that from Statement A.1 and asymptotics (A.10) it follows that

$$q(R,\varphi)\to\frac{3i\hat v(0)E}{4\pi}\left(\lambda^2e^{-2i\varphi}-1\right)\exp\left(\frac{i\sqrt{E}}{2}R\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda e^{i\varphi}+\lambda e^{-i\varphi}\right)\right),$$

as $R \to \infty$ uniformly on φ. Thus

$$\tilde{J}_{(21)} \to 2\sqrt{E}(\mathrm{sgn}E)\bar{\lambda}\int\limits_0^{2\pi} \exp\left(\frac{i\sqrt{E}}{2}R\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar{\lambda}}\right)\left((\mathrm{sgn}E)\bar{\lambda}e^{i\varphi}+\lambda e^{-i\varphi}\right)\right) f(\varphi)d\varphi, \text{ as } R \to \infty,$$

where $f(\varphi)$ is a smooth function of φ. Note that the resulting integral is an integral of stationary phase type and thus $\tilde{J}_{(21)} \to 0$ as $R \to \infty$.

Now let us apply the Stokes theorem to $\tilde{J}_{(22)}$:

$$\tilde{J}_{(22)} = 2\sqrt{E}(\mathrm{sgn}E)\bar{\lambda}\int\limits_{C_R} e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,\partial_\zeta w\,\varphi\,d\zeta +$$
$$+ iE\bar{\lambda}^2\int\limits_{C_R} e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,w\,\varphi\,d\zeta - 4i\sqrt{E}(\mathrm{sgn}E)\bar{\lambda}\iint\limits_{B_R}\partial_\zeta\partial_{\bar{\zeta}}\left(e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,w\right)\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta. \tag{A.17}$$

Treating the first two summands in the right-hand side of (A.17) in the similar way as we treated $\tilde{J}_{(21)}$, we obtain that

$$\tilde{J}_{(2)} \sim \tilde{J}_{(22)} \sim -4i\sqrt{E}(\mathrm{sgn}E)\bar{\lambda}\iint\limits_{B_R}\partial_\zeta\partial_{\bar{\zeta}}\left(e^{\frac{i\sqrt{E}}{2}(\mathrm{sgn}E)(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,w\right)\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta, \text{ as } R \to \infty.$$

Finally, for $J_{11}+J_7$ we obtain that

$$\lim_{R\to\infty}(J_{11}+J_7) = -2\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,\partial_\zeta v\,w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta -$$
$$-6E\bar{\lambda}^2\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,\partial_\zeta v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta + 12i\sqrt{E}(\mathrm{sgn}E)\bar{\lambda}\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,\partial_\zeta^2 v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta -$$
$$-2\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,v\,\partial_\zeta w\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta.$$

Thus it can be obtained that

$$\lim_{R\to\infty}(J_3+J_4+J_5+J_6+J_7+J_8+J_{11}+J_{12}) =$$
$$= -6E\bar{\lambda}^2\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,\partial_\zeta v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta + 12i\sqrt{E}(\mathrm{sgn}E)\bar{\lambda}\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,\partial_\zeta^2 v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta -$$

$$-\frac{6E}{\bar{\lambda}^2}\iint\limits_{\mathbb{C}} e^{\frac{i\sqrt{E}}{2}\mathrm{sgn}E(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,\partial_{\bar{\zeta}}v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta+\frac{12i\sqrt{E}}{\bar{\lambda}}(\mathrm{sgn}E)\iint\limits_{\mathbb{C}} e^{\frac{\sqrt{-E}}{2}(\bar{\lambda}\zeta+\frac{\bar{\zeta}}{\bar{\lambda}})}\,\partial^2_{\bar{\zeta}}v\,\varphi\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta. \tag{A.18}$$

Finally,

$$\lim_{R\to\infty} J_{14}=3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)b(\lambda,t) \tag{A.19}$$

and thus from (A.15)-(A.19) we obtain formula (4.40).

Bibliography

[1] Ablowitz, M. J., and Clarkson, P. A. *Solitons, evolution equations and inverse scattering.* Cambridge University Press, 1991.

[2] Bogdanov, L. V. The Veselov-Novikov equation as a natural generalization of the Korteweg-de Vries equation. *Teoret. Mat. Fiz. 70(2)* (1987), 309–314. Translation in *Theoret. and Math. Phys. 70(2)* (1987), 219–223.

[3] Boiti, M., Leon, J. J.-P., Manna, M., and Pempinelli, F. On a spectral transform of a KdV-like equation related to the Schrödinger operator in the plane. *Inverse Problems 3* (1987), 25–36.

[4] Boiti, M., Leon, J. J.-P., Martina, L., and Pempinelli, F. Scattering of localized solitons in the plane. *Phys. Lett. A 132* (1988), 432–439.

[5] Calderón, A. P. On an inverse boundary problem. *Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasiliera de Matematica, Rio de Janeiro* (1980), 61–73.

[6] Chang, J.-H. The Gould-Hopper polynomials in the Novikov-Veselov equation. *J. Math. Phys. 52(9)* (2011), 092703.

[7] de Bouard, A., and Saut, J. C. Solitary waves of generalized Kadomtsev-Petviashvili equations. *Ann. Inst. Henri Poincaré, Analyse Non Linéaire 14(2)* (1997), 211–236.

[8] de Bouard, A., and Saut, J. C. Symmetries and decay of the generalized Kadomtsev-Petviashvili solitary waves. *SIAM J. Math. Anal. 28(5)* (1997), 1064–1085.

[9] Dubrovsky, V. G., and Formusatik, I. B. New lumps of Veselov-Novikov equation and new exact rational potentials of two-dimensional Schrödinger equation via $\bar{\partial}$-dressing method. *Phys. Lett. 313(1-2)* (2003), 68–76.

[10] Faddeev, L. D. Growing solutions of the Schrödinger equation. *Dokl. Akad. Nauk SSSR 165(3)* (1965), 514–517. Translation in *Sov. Phys. Dokl. 10* (1966), 1033–1035.

[11] Faddeev, L. D. The inverse problem in the quantum theory of scattering. II. Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. *Dokl. Akad. Nauk SSSR 3* (1974), 93–180. Translation in *J. Math. Sciences. 5(3)* (1966), 1033–1035.

[12] Fedoryuk, M. V. *Asymptotics : integrals and series.* Moscow : Nauka, Mathematical Reference Library, 1987.

[13] Fokas, A. S., and Ablowitz, M. J. On the inverse scattering of the time–dependent Schrödinger equation and the associated Kadomtsev–Petviashvili (I) equation. *Studies in Appl. Math. 69* (1983), 211–228.

[14] Fokas, A. S., and Santini, P. M. Coherent structures in multidimensions. *Phys. Rev. Lett. 63* (1983), 1329–1333.

[15] Françoise, J.-P., and Novikov, R. G. Rational solutions of KdV–type equations in dimension $2 + 1$ and m–body problems on the line. *C.R. Acad. Sci. Paris Sér. I Math. 314(2)* (1992), 109–113. (in French).

[16] Gabov, S. A. *Introduction to the theory of non–linear waves.* Moscow State University Press, 1988. (in Russian).

[17] Gardner, C. S., Greene, J. M., Kruskal, M. D., and Miura, R. M. Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett. 19* (1967), 1095–1097.

[18] Gardner, C. S., Greene, J. M., Kruskal, M. D., and Miura, R. M. Korteweg-de Vries equation and generalizations. VI. Methods for exact solution. *Comm. Pure Appl. Math. 27* (1974), 97–133.

[19] Gelfand, I. M. Some aspects of functional analysis and algebra. In *Proceedings of the International Congress of Mathematicians* (1954), vol. 1, Amsterdam : Erven P. Noordhoff N.V., Groningen ; North-Holland Publishing Co., pp. 253–276.

[20] Gohberg, I. C., and Krein, M. G. *Introduction to the theory of linear nonselfadjoint operators.* Moscow : Nauka, 1965. [Volume 18 of Translations of Mathematical Monographs, American Mathematical Soc., 1969].

[21] Grinevich, P. G. Rational solitons of the Veselov–Novikov equation are reflectionless potentials at fixed energy. *Teoret. Mat. Fiz. 69(2)* (1986), 307–310. Translation in *Theor. Math. Phys. 69* (1986), 1170–1172.

[22] Grinevich, P. G. Scattering transformation at fixed non-zero energy for the two-dimensional Schrödinger operator with potential decaying at infinity. *Russ. Math. Surv. 55(6)* (2000), 1015–1083.

[23] Grinevich, P. G., and Manakov, S. V. Inverse problem of scattering theory for the two-dimensional Schrödinger operator, the $\bar{\partial}$-method and nonlinear equations. *Funkts. Anal. Prilozh. 20(2)* (1986), 14–24. Translation in *Funct. Anal. Appl. 20(2)* (1986), 103–113.

[24] Grinevich, P. G., and Novikov, R. G. Analogues of multisoliton potentials for the two-dimensional Schrödinger operator, and a nonlocal Riemann problem. *Dokl. Akad. Nauk SSSR 286(1)* (1986), 19–22. Translation in *Sov. Math. Dokl. 33(1)* (1986), 9–12.

[25] Grinevich, P. G., and Novikov, R. G. Transparent potentials at fixed energy in dimension two. Fixed energy dispersion relations for the fast decaying potentials. *Commun. Math. Phys. 174* (1995), 409–446.

[26] Grinevich, P. G., and Novikov, R. G. Faddeev eigenfunctions for point potentials in two dimensions. *Physics Letters A. 376(12-13)* (2012), 1102–1106.

[27] Grinevich, P. G., and Novikov, S. P. Two-dimensional "inverse scattering problem" for negative energies and generalized-analytic functions. I. Energies below the ground state. *Funkts. Anal. Prilozh. 22(1)* (1988), 23–33. Translation in *Funct. Anal. Appl. 22(1)* (1988), 19–27.

[28] Hayashi, N., Naumkin, P. I., and Saut, J.-C. Asymptotics for large time of global solutions to the generalized Kadomtsev-Petviashvili equation. *Commun. Math. Phys. 201(3)* (1999), 577–590.

[29] Henkin, G. M., and Novikov, R. G. The $\bar{\partial}$-equation in the multidimensional inverse scattering problem. *Uspekhi Mat. Nauk 42(3)* (1987), 93–152. Translation in *Russ. Math. Surv. 42(3)* (1987), 109–180.

[30] Joshi, M. S., and Sá Barreto, A. Recovering asymptotics of short range potentials. *Comm. Math. Phys. 193(1)* (1998), 197–208.

[31] Kadomtsev, B. B., and Petviashvili, V. I. On the stability of solitary waves in weakly dispersive media. *Dokl. Akad. Nauk SSSR. 192* (1970), 753–756. Translation in *Sov. Phys. Dokl. 15* (1970), 539–541.

[32] Kazeykina, A. V. A large time asymptotics for the solution of the Cauchy problem for the Novikov-Veselov equation at negative energy with non-singular scattering data. *Inverse Problems 28(5)* (2012), 055017.

[33] Kazeykina, A. V. Absence of solitons with sufficient algebraic localization for the Novikov-Veselov equation at nonzero energy. Funct. Anal. Appl. (to appear), 2012. arXiv :1201.2758.

[34] Kazeykina, A. V. Absence of traveling wave solutions of conductivity type for the Novikov-Veselov equation at zero energy. Funct. Anal. Appl. (to appear), 2012. arXiv :1106.5639.

[35] Kazeykina, A. V., and Novikov, R. G. A large time asymptotics for transparent potentials for the Novikov–Veselov equation at positive energy. *J. Nonlinear Math. Phys. 18(3)* (2011), 377–400.

[36] Kazeykina, A. V., and Novikov, R. G. Absence of exponentially localized solitons for the Novikov–Veselov equation at negative energy. *Nonlinearity 24* (2011), 1821–1830.

[37] Kazeykina, A. V., and Novikov, R. G. Large time asymptotics for the Grinevich–Zakharov potentials. *Bulletin des Sciences Mathématiques 135* (2011), 374–382.

[38] Kiselev, O. M. Asymptotics of a solution of the Kadomtsev–Petviashvili–2 equation. *Tr. Inst. Mat. Mekh. 7(1)* (2001), 105–134. Translation in *Proc. Inst. Math. Mech. suppl.1* (2001), S107–S139.

[39] Konopelchenko, B., and Moro, A. Integrable equations in nonlinear geometrical optics. *Studies in Applied Mathematics 113(4)* (2004), 325–352.

[40] Lassas, M., Mueller, J. L., and Siltanen, S. Mapping properties of the nonlinear Fourier transform in dimension two. *Commun. Part. Diff. Eq. 32* (2007), 591–610.

[41] Lassas, M., Mueller, J. L., Siltanen, S., and Stahel, A. The Novikov-Veselov Equation and the Inverse Scattering Method, Part I : Analysis. *Physica D. 241* (2012), 1322–1335.

[42] Lax, P. D. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math. 21* (1968), 467–490.

[43] Manakov, S. V. The inverse scattering method and two-dimensional evolution equations. *Uspekhi Mat. Nauk 31(5)* (1976), 245–246. (in Russian).

[44] Manakov, S. V., Santini, P. M., and Takchtadzyan, L. A. An asymptotic behavior of the solutions of the Kadomtsev-Petviashvili equations. *Phys. Lett. A. 75* (1988), 451–454.

[45] Manakov, S. V., Zakharov, V. E., Bordag, L. A., Its, A. R., and Matveev, V. B. Two–dimensional solitons of the Kadomtsev–Petviashvili equation and their interaction. *Phys. Lett. A. 63(3)* (1977), 205–206.

[46] Nachman, A. I. Global uniqueness for a two-dimensional inverse boundary value problem. *Annals of Mathematics. 143* (1995), 71–96.

[47] Novikov, R. G. Reconstruction of a two-dimensional Schrödinger operator from the scattering amplitude at fixed energy. *Funkt. Anal. i Pril. 20(3)* (1986), 90–91. Translation in *Funkt. Anal. and Appl. 20* (1986), 246–248.

[48] Novikov, R. G. Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. *Funkt. Anal. i Pril. 22(4)* (1988), 11–22. Translation in *Funkt. Anal. and Appl. 22* (1988), 263–272.

[49] Novikov, R. G. The inverse scattering problem on a fixed energy level for the two–dimensional Schrödinger operator. *Journal of Funct. Anal. 103* (1992), 409–463.

[50] Novikov, R. G. Approximate solution of the inverse problem of quantum scattering theory with fixed energy in dimension 2. *Tr. Mat. Inst. Steklova. 225* (1999), 301–318. Translation in *Proc. Steklov Inst. Math. 2(225)* (1999), 285–302.

[51] Novikov, R. G. Absence of exponentially localized solitons for the Novikov–Veselov equation at positive energy. *Phys. Lett. A 375* (2011), 1233–1235.

[52] Novikov, S. P., Manakov, S. V., Pitaevskii, L., and Zakharov, V. E. *Theory of solitons. The inverse scattering method.* New York - London : Plenum Publishing Corporation, Contemporary Soviet Mathematics (Transl. from Russian), 1984.

[53] Novikov, S. P., and Veselov, A. P. Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formula and evolutions equations. *Dokl. Akad. Nauk SSSR 279* (1984), 20–24. Translation in *Sov. Math. Dokl. 30* (1984), 588–591.

[54] Novikov, S. P., and Veselov, A. P. Finite-zone, two-dimensional, potential Schrödinger operators. Potential operators. *Dokl. Akad. Nauk SSSR 279* (1984), 784–788. Translation in *Sov. Math. Dokl. 30* (1984), 705–708.

[55] Perry, P. A. Miura maps and inverse scattering for the Novikov-Veselov equation, 2012. arXiv :1201.2385.

[56] Regge, T. Introduction to complex orbital momenta. *Nuovo Cimento 14* (1959), 951–976.

[57] Santacesaria, M. *Uniqeness, reconstruction, stability for the two-dimensional inverse problems.* PhD thesis, Ecole Polytechnique, France, 2012.

[58] Segur, H. The Korteweg-de Vries equation and waves. I. Solutions of the equation. *J. Fluid Mech. 59* (1973), 721–736.

[59] Taimanov, I. A., and Tsarev, S. P. On the Moutard transformation and its applications to spectral theory and soliton equations. *J. Math. Sci. 170(3)* (2010), 371–387.

[60] Tsai, T.-Y. The Schrödinger operator in the plane. *Inverse Problems. 9* (1993), 763–787.

[61] Vasy, A., and Wang, X.-P. Inverse scattering with fixed energy for dilation–analytic potentials. *Inverse Problems. 20(4)* (2004), 1349–1354.

[62] Vekua, I. N. *Generalized analytic functions.* Oxford : Pergamon Press, 1962.

[63] Weder, R., and Yafaev, D. On inverse scattering at a fixed energy for potentials with a regular behaviour at infinity. *Inverse Problems. 21(6)* (2005), 1937–1952.

[64] Zakharov, V. E., and Kuznetsov, E. A. Three-dimensional solitons. *JETP 66(2)* (1974), 594–597. Translation in *Sov. Phys. JETP 39(2)* (1974), 285–286.

[65] Zakharov, V. E., and Manakov, S. V. Asymptotic behaviour of non-linear wave systems integrated by the inverse scattering method. *JETP 71* (1976), 203–215. Translation in *Sov. Phys. JETP 44(1)* (1976), 106–112.