Variational approach to fracture mechanics with plasticity

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Variational Approach to Fracture Mechanics with Plasticity

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Abstract

english

In the framework of rate-independent systems, an elastic-plastic-damage model, aimed at the description of ductile fracture processes, is proposed and investigated through a variational formulation. A cohesive, or ductile, crack occurs when the displacement field suffers a discontinuity whilst still being associated to a non-vanishing tensile stress. To predict and effectively describe ductile fracture phenomena is a crucial task for many engineering materials (metals, polymers, ...), as testified by the great interest of the scientific community on the subject.

Gradient damage models have been fruitfully used for the description of brittle fractures: in such cases, once the damage level reaches its maximum value, a crack is created where the traction between the two opposite lips immediately drops to zero. On the other hand, the perfect plasticity model could describe the formation of plastic slips at constant stress level. Hence, in order to describe the typical effects of a cohesive fracture, the main idea is to couple, through a variational approach, the perfect plasticity model and a softening gradient damage model. The use of a variational approach results in a weak and derivative-free formulation, gives effective means to deal with the concepts of bifurcation and stability, is intrinsically discrete and indicates a natural and rational way to define efficient numerical algorithms.

Embedding damage effects in a plasticity model is not a new idea. Nevertheless the proposed model presents many original aspects as the coupling between plasticity and damage and the way the governing equations of the variables are found. The variational approach relies simply on three concepts: the irreversibility condition, a global, local or differential stability condition and the energy balance. The resulting model possesses a great flexibility in the possible coupled responses, depending on the constitutive parameters. These various responses are first considered by investigating in a one-dimensional quasi-static traction bar test a homogeneous evolution which highlights the main features of the model. The discussion about the stability of the homogeneous solutions leads to the existence of a critical bar length which in turn depends on the characteristic internal material length. For bars that are longer than this critical value the homogeneous response is proven to become unstable and a localization must appear. A construction of localization is then proposed which explicitly takes into account the irreversibility condition on the damage field. This allows the non-homogeneous evolution and the global response to be investigated. It turns out that in general a cohesive crack appears at the center of the damage zone before the complete rupture. At this point the plastic strain localises as a Dirac measure which becomes responsible for this cohesive crack. The associated cohe-

sive law is obtained in a closed form in terms of the parameters of the model and it recovers the cohesive fracture law postulated by Barenblatt.

Finally, a numeric resolution scheme is proposed, which is based on an alternate minimization algorithm, and implemented through a finite element library only for the one-dimensional traction bar test. Although the adopted finite element spaces do not embed discontinuities, the numeric results agree perfectly with the analytic solutions. This is due to a kind of numeric regularisation. Nevertheless, future developments aim to extend the simulations in a two/three-dimensional setting and test a generalized finite element method.

italian

Nell'ambito dei sistemi *rate-independent* si propone un modello di plasticità e danno basato su un approccio variazionale con lo scopo di descrivere il fenomeno della frattura duttile. In particolare si è in presenza di una frattura coesiva o duttile se il campo di spostamento presenta una discontinuità con una tensione all'interfaccia della discontinuità stessa non evanescente. Il fenomeno della frattura duttile riguarda numerosi materiali di interesse ingegneristico (metalli, polimeri, ecc.) ed è oggetto di numerosi studi da parte della comunità scientifica.

E' ben noto che modelli costitutivi di danno a gradiente con risposta incrudente negativamente sono in grado di descrivere la frattura fragile: in queste situazioni la frattura è rappresentata dall'insieme dei punti dove la variabile di danno raggiunge il suo valore massimo e che corrispondono a valori nulli della tensione. Il modello di plasticità perfetta invece è in grado di descrivere delle localizzazioni di deformazione nelle quali si ha uno scorrimento plastico a tensione costante. Con lo scopo di descrivere la frattura coesiva l'idea allora è di accoppiare, per mezzo di un approccio variazionale, il modello di plasticità perfetta con quello di danno a gradiente. L'approccio variazionale è giustificato da numerosi motivi. Esso è in grado di fornire una formulazione per l'evoluzione svincolata da operazioni di derivazione, permette di dare senso compiuto allo studi di stabilità e biforcazione materiale, nonchè, essendo la formulazione intrinsecamente discreta, di fornire un modo razionale per l'implementazione numerica.

Il modello così risultante risulta capace di cogliere le diverse risposte materiali con pochi parametri materiali e di descrivere risposte nelle quali plasticitá e danno evolvono contemporaneamente. La delocalizzazione della risposta é ottenuta per mezzo dell'introduzione nel funzionale energetico di un termine funzione del gradiente del danno. La formulazione astratta é in un contesto tri-dimensionale, mentre su un supporto monodimensionale si é eseguito uno studio piú approfondito della risposta con particolare riferimento alla stabilitá delle risposte omogenee e della costruzione di risposte con localizzazioni. In particolare, lo studio della evoluzione delle localizzazioni ha mostrato l'esistenza di una risposta di frattura coesiva rappresentativa di un modello di Barenblatt. Il modello è stato implementato numericamente dove la strategia risolutiva è basata su un algoritmo di minimizzazione alternata.

french

Dans le cadre des systèmes rate-independent, un modèle de plasticité avec endommagement, visant à la description des processus de rupture ductile, est proposé et étudié par une formulation variationnelle. Une fissure cohésive, ou ductile, se produit lorsque le champ de déplacement subit une discontinuité, tout en étant encore associé à une contrainte de traction non nulle. Prévoir et décrire efficacement les phénomènes de rupture ductile est une tâche cruciale pour de nombreux matériaux d'ingénierie (métaux, polymères, ...), comme en témoigne le grand intérêt de la communauté scientifique sur le sujet. Modèles d'endommagement à gradient ont été fructueusement utilisé pour la description des ruptures fragiles : dans ce cas, une fois que le niveau d'endommagement atteint sa valeur maximale, une fissure est créée lorsque la traction entre les deux faces opposées tombe immédiatement à zéro. D'autre part, le modèle de plasticité parfaite pourrait décrire la formation de la glisse plastique au niveau constant de contrainte. Par conséquent, afin de décrire les effets typiques d'une rupture cohésive, l'idée principale consiste à coupler, par une approche variationnelle, le modèle de plasticité parfaite et un modèle d'endommagement à gradient. L'utilisation d'une approche variationnelle se traduit par une formulation faible et sans dérivées, fournit des moyens efficaces pour traiter les notions de bifurcation et de stabilité, est intrinsèquement discret et indique une manière naturelle et rationnelle pour définir des algorithmes numériques efficaces.

L'incorporation des effets d'endommagement dans un modèle de plasticité n'est pas une idée nouvelle. Néanmoins, le modèle proposé présente de nombreux aspects originaux comme le couplage entre la plasticité et l'endommagement et la façon avec laquelle l'évolutions des variables se trouvent. L'approche variationnelle s'appuie simplement sur trois concepts : une condition d'irréversibilité, une condition de stabilité globale, locale ou différentielle et le bilan énergétique. Le modèle résultant possède une grande flexibilité dans les réponses possibles couplées, en fonction des paramètres constitutifs. Ces diverses réponses sont d'abord examinées avec un test d'une barre unidimensionnelle en traction quasi-statique en assumant une évolution homogène qui met en évidence les principales caractéristiques du modèle. La discussion sur la stabilité des solutions homogènes conduit à l'existence d'une longueur de la barre critique qui à son tour dépend de la longueur interne caractéristique du matériel. En considérant des barres plus longues par rapport à cette valeur critique, on démontre que la réponse homogène devient instable. Par conséquence une localisation doit apparaitre dans la barre.

Une construction de localisation est ensuite proposée, qui prend explicitement en compte la condition d'irréversibilité sur le champ d'endommagement. Ceci permet d'étudier l'évolution non homogène et la réponse globale. Il s'avère que, en général, une fissure cohésive apparaît au centre de la zone d'endommagement avant la rupture. A ce stade, la déformation plastique se localise comme une mesure de Dirac qui devient responsable de cette fissure cohésive. On obtient la loi cohésive associée en termes de paramètres du modèle et retrouve la loi de fracture cohésive postulée par Barenblatt. Enfin, un schéma de résolution numérique est proposé, qui est basé sur un algorithme de minimisation alternée, et mis en œuvre par une librairie d'éléments finis uniquement pour le test de barre en traction. Même si l'espace d'éléments finis adoptés ne peut pas incorporer les discontinuités, les résultats numériques s'accordent parfaitement avec les solutions analytiques. Néanmoins, les développements futurs visent à étendre les simulations dans un cadre à deux / trois dimensions et de tester une méthode d'éléments finis généralisée.

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Summary

Motivations and Ductile Fracture Phenomenology

Fracture is one of the most important key concepts in the field of Materials Science and Engineering, Rossmanith 1997. With the progress of technology and knowledge of materials, structures are more and more harnessed in the neighbourhood of their ultimate strength. It is precisely at failure conditions that the phenomenon of fracture becomes crucial for the efficiency and structural safety. Although not always noticeable, any material may be involved with fracture. Convincingly Fig. 1 shows several cracking examples in the various situations.

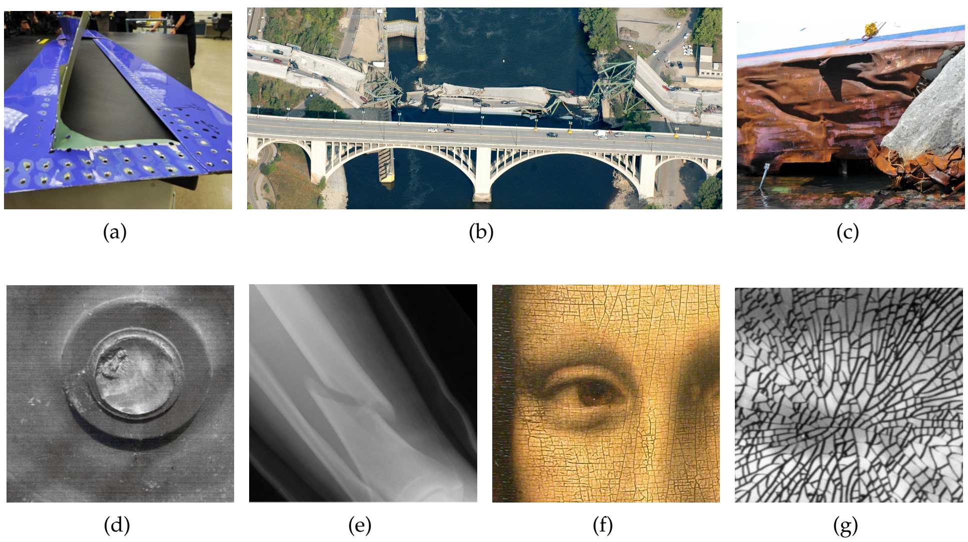
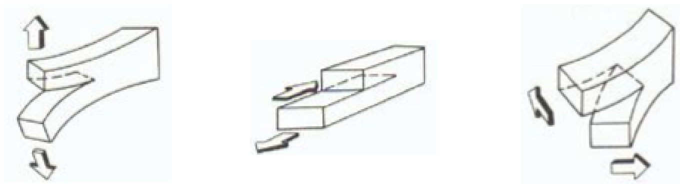


Figure 1: (a), The 5-foot-long fuselage skin section that split open on a Southwest Airlines Boeing 737-300 ; (b), bridge collapse in Minnesota on the Interstate 35W; (c), particular of the gash of the Costa Concordia accident; (d), detail of break down of the axle of the carriage for fatigue failure in the Viareggio train derailment; (e), X-ray of a broken radial bone; (f), detail of the crack patterns on the face of Mona Lisa portrait; (g), crack patterns in a broken glass sheet.

The science of fracture mechanics, as it is known today, was born and came to maturity in the 20th century. Nevertheless, the origin of this science dates back to past centuries. A brief but interesting historical overview can be found in Cotterell 2002.

As demonstrated in the examples of Fig. 1 understanding fracture is a tall order. The main issues concerning fracture are undoubtedly crack initiation, crack path and crack bifurcation. On a macro-scale fracture is characterized by surface discontinuities, that is discontinuities in the displacement field. The three fundamental fracture mechanisms are represented in Fig. 2. Mode I, II and III correspond respectively to an opening or tensile mode, Fig. 2a, to a sliding model, Fig. 2b, and to a tearing mode, Fig. 2c. Any other fracture mechanism is simply a combination

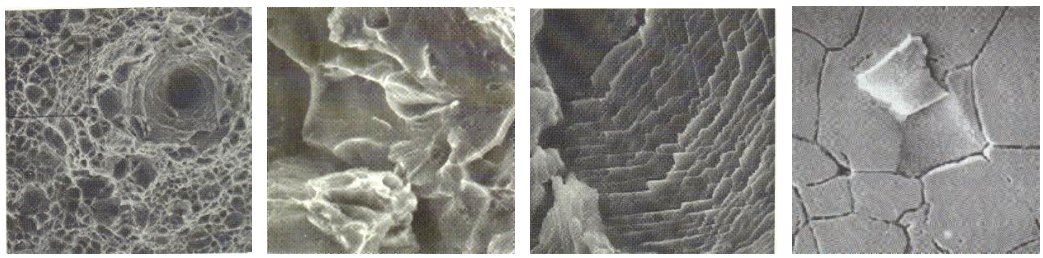


(a) *Mode I* (b) *Mode II* (c) *Mode III*

Figure 2: Elementary fracture mechanisms, Parton 1992

of these elementary modes.

On the other hand, on a micro-scale the fracture phenomenon becomes more complicated than it appears to the human eye. The best evidence is given by Fractography, Mills et al. 1987, which is the study of the fractured surfaces of materials. A microscopic investigation of the fracture surfaces and their neighbourhood reveals numerous different crack states. Some of the most important are represented in Fig. 3. Moreover, Fractography highlights how the inter-atomic structure as the crystalline structure and the material defects crucially affects the crack mechanism.



(a) *Microvoid Coalescence or Dimple Fracture* (b) *Transgranular Cleavage* (c) *Fatigue failure* (d) *Decohesive Rupture*

Figure 3: Different types of fracture surface characteristics on a micro-scale; (a)-(c) Bhattacharyya 1979 and (d) C. And Esaklul 1993

The intimate interrelation among the material, the fracture modes, the load conditions and the microscopic failure mechanisms give rise to the macroscopic fracture visible to the naked eye.

Despite the complexity of the phenomenon, fruitful theories exist to describe both qualitatively and quantitatively fracture based on the spirit of continuum mechanics where a continuous mass material is assumed as representative of the real discrete material.

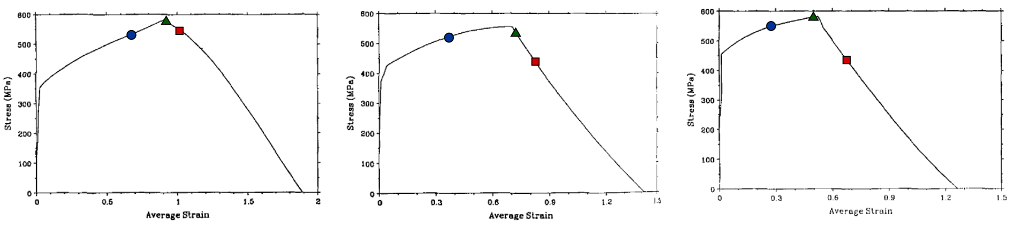
The microscopic investigation also unveils another important mechanism at the atomic scale which may eventually occur before and/or during cracking, namely dislocation (typical in metals).

It is then worth noting that in continuum mechanics two other phenomenological models play a crucial role in material modelling, namely plasticity and damage the first associated at the micro-scale to dislocations and/or crack slips while the latter mostly to void nucleations.

Once plasticity is taken into account, it is then possible to classify the macroscopic description of fracture into three main groups: brittle, ductile and fatigue. The brittle and ductile fracture involves a single load application while fatigue fracture occurs for a cyclic loading. Brittle fracture and ductile fracture are fairly general terms describing the two opposite extremes of the fracture spectrum which require some important considerations. In general, the main difference between brittle and ductile fracture can be attributed to the amount of plastic deformation that the material undergoes before fracture occurs. Ductile materials exhibits large amounts of plastic strains while brittle materials show little or no plastic strains before fracture.

However, in fracture mechanics there are many shades of gray. Fracture experimental tests on glass, Ferretti, Rossi, and Royer-Carfagni 2011, demonstrate the existence at the crack tip of a process zone associated to the formation of microcracks and microvoids with plastic strains although glass is typically considered a brittle material. It then becomes reasonable to acknowledge plasticity, damage and fracture to be often closely related to each other.

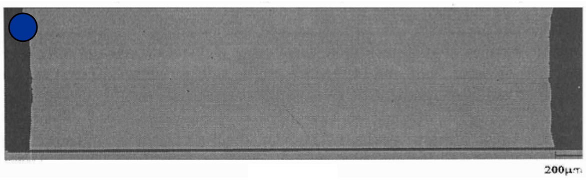
To give credit to this last point and to highlight the ductile fracture phenomenology it suffices to account a simple tensile test of a ductile material. For example, in **struges1997** cooper bars of different sizes are tested by means of a monotonic increasing displacement at one bar end. Fig. 4 shows the true stress-strain curves for different notched specimen. These responses are compared with optical mi-



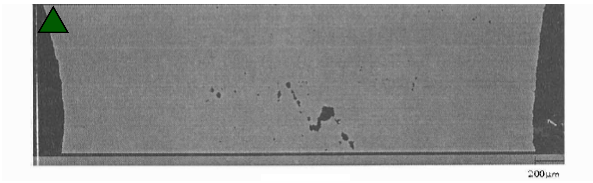
(a) *4mm notched specimen* (b) *2mm notched specimen* (c) *1mm notched specimen*

Figure 4: True stress–strain curves of cylindrical copper bars for different notched radii, **struges1997** Void nucleation: blue circle; Void coalescence; green triangle; crack formation: red square

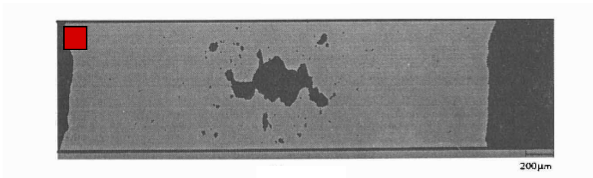
crographs in Fig. 5 which allow to identify the marked points (blue circle, green triangle, red square) with the internal material state. It is then clear that during the evolution damage (micro void nucleation Fig. 5b) and plastic strains (residual strains Fig. 5d) have occurred.



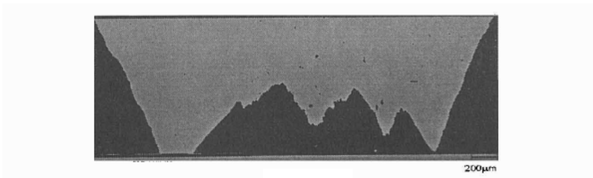
(a) *Void nucleation*



(b) *Void coalescence*



(c) *Crack formation*



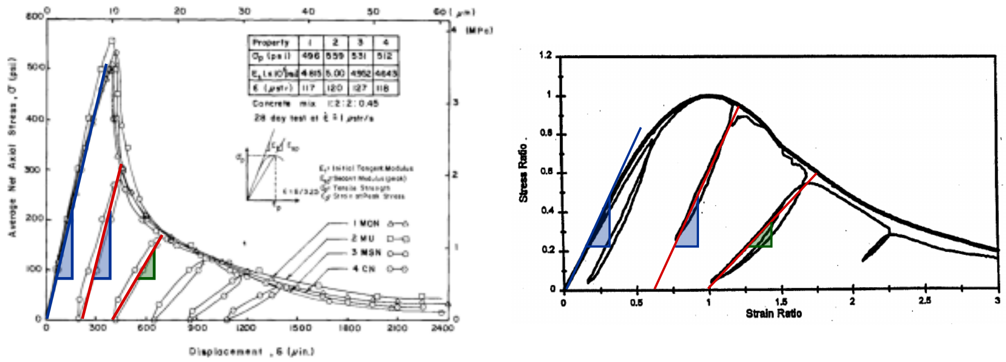
(d) *Final fracture*

Figure 5: Micrographs images corresponding to the different points of the stress-strain curves in Fig. 4

It is an unloading path in a stress-strain curve that could reveal the occurrence of plasticity and damage. Focusing for example on an uniaxial test of a concrete material, Fig. 6, the stress–strain curves in traction, Fig. 6a, and compression, Fig. 6b, although qualitatively different, both highlight the contemporary evolution of plasticity and damage by means of the unloading curves.

It can also be understood from Fig. 4 that a size effect and instabilities occur in the phenomenon. That is, the material is characterized by an internal length scale. The study of size effects dates back to ancient times. Leonardo da Vinci and later Galileo Galilei were the firsts to be interested in and to start to investigate on the relationship between the specimen strength and its actual size, see e.g. Bažant and Chen 1997; Bažant and Planas 1997.

With the term size effects or scaling a large number of phenomena is to be understood. The subject, as testified by a wide spread scientific literature, is very wide and complex, with different results depending on the material, the load conditions



(a) *Stress–strain curve for an uniaxial tension test* (b) *Stress–strain curve for an uniaxial compression test*

Figure 6: Stress–strain responses of concrete specimens with unloading curves. On the one hand the straight lines colours represent in blue the initial elastic response and in red the evolution of plastic strains while on the other hand the triangle colours represent in blue an elastic stiffness and in green a degraded stiffness due to damage

and geometrical dimensions, Barenblatt 2003.
Although all these phenomena may not have the same explanation, their interpretation requires the introduction of an internal length scale. Different approaches are possible, reviewed in Sec. 1.4, to take into account an internal length scale in the constitutive model. Another reason exists for the introduction of an internal length scale in a material model, which is related to the mathematical modelling of softening materials. Its aim is to avoid a localised response with vanishing dissipated work. This point will be explained in the next chapter.

Reassembling all the highlighted features, one can state that a mathematical model able to describe properly the ductile fracture phenomenon should:

- deal with initiation and crack propagation;
- take into account both damage and plasticity
- take into account size effects;
- rationally understand and describe material instabilities.

Cohesive fracture models and coupled plasticity-damage models are intimately linked to each other and represents two possible ways for the description of ductile fracture. Indeed, the former models explicitly embed a cohesive surface energy term while the latter can often be regarded as the regularisation of the first. Nevertheless, plasticity-damage models have their own merits and express wider capabilities.

A quick overview and discussion about the most important "classical" models is given in Sec. 1.4, each of them with its strengths and weaknesses. The undertaken direction is a recent variational approach which appears as the most promising for reasons that becomes clear in the next section.

Outline

The main objective of this thesis is to apply the variational approach for the description of ductile fracture, to investigate the features of the resulting model (responses, stability analyses, ...) and to define an efficient algorithm in order to perform numeric simulations.

The point of departure has been a gradient damage model, which is well known to be able to account for the behaviour of brittle and quasi-brittle materials. This model stems from the results of Ambrosio and Tortorelli 1990 and has been successfully used in the variational theory of fracture, Bourdin, Francfort, and Marigo 2008, as a regularisation of the revisited Griffith's law, Francfort and Marigo 1998. It turns out that it is possible to prove that (a family of) gradient damage models converge, in the sense of Gamma-convergence, to Griffith's model when the internal length, contained in those models and introduced by means of the gradient term, tends to zero, Braides 2002; Dal Maso and Toader 2002. Nevertheless, these models have their own merits and have been developed independently in a classical framework without invoking an energy minimisation, Benallal and Marigo 2007; Comi 1999; Comi et al. 2006; Lorentz, Cuvilliez, and Kazymyrenko 2011; Peerlings et al. 1998; Pham and Marigo 2010a; Pham and Marigo 2010c.

In this variational setting the elastic solution as well as the evolution of the crack is simply governed by three principles: irreversibility condition, stability condition and energy balance. These are the building blocks of Mielke's energetic formulation for rate independent systems, Mielke 2011. Accordingly, the process of crack nucleation in these gradient damage models is widely explored in Pham and Marigo 2011; Pham et al. 2011. First when the stress field reaches the critical level somewhere in the domain damage evolves. But because of the softening character of the material behaviour, damage localises. Due to the gradient term the localisation stems out and evolves in a non vanishing subregion and its characteristic width is controlled by the internal length introduced precisely by the gradient term. A crack (a discontinuity in the displacement field) appears at the center of the localisation zone where damage reaches its ultimate value and the material stiffness drops to zero. On the energetic point of view, during this crack nucleation process, some energy is dissipated inside the damage localisation zone. This dissipated energy is related to the effective surface energy of Griffith's theory, G_c.

A well-known limit of these models is that they are not able to account for residual strains and consequently cannot be used in ductile fracture modelling. In particular there is no discontinuity of the displacement in the localisation zone before damage has reached its maximum value and hence the loss of stiffness which is instead peculiar of a cohesive response. In other words these models cannot lead to the existence of surfaces where the displacement field is not continuous but the stress is non vanishing.

The natural way to include such effects is to introduce plastic strains into the model and to couple both plasticity and damage evolutions. Of course, this idea is not new and a great number of damage models coupled with plasticity have been developed from the eighties in the spirit of Lemaitre and Chaboche, Lemaitre and Chaboche 1985, see for instance Dimitrijevic and Hackl 2011.

But the purpose of this work is to construct such models in a softening framework with gradient of damage terms and to see how these models can account for the nucleation of cracks in presence of plasticity. To the writer knowledge, the previous works were not able to go so far.

What actually characterised the approach and allows the original contributions listed below is the variational framework. The adopted variational approach has been rigorously formalised by Mielke and successfully applied in many engineering problems. The main ingredients are:

1. the definition of two energy quantities, the elastic stored energy and the dissipation potential, which depend upon the state variables: the displacement field and the internal variable fields, namely damage and plasticity;

2. the three aforementioned principles, *irreversibility*, *stability* and *energy* balance which are assumed to be sufficient to govern the evolution of the system. In particular, the stability condition involves a functional minimisation and provides a rational way to construct and investigate the model.

These simple ingredients establish a flexible and at the same time robust ground over which this model has been developed. Specifically, the coupling among damage and plasticity has been achieved by assuming an additive decomposition of the dissipation potential, the former depending on the plastic strain rate and damage and the latter depending on the damage rate, damage and the accumulated plastic strain. Of course, in a particular case, aspects of the classical approach could be retrieved as a consequences of the variational approach like yield functions and yield stresses where the role of the various constitutive functions is revealed. With this choice of the dissipation potential the plastic (damage) yield stress results to be damage (plastic) dependent, that is it is possible to describe coupled effects. An interested investigated case corresponds to plastic and damage yield stresses which decrease with the evolution respectively of damage and the accumulated plastic strain. Moreover the damage yield stress is assumed to be damage-softening and damage-gradient dependent.
While mathematicians are often more interested in aspects like existence and uniqueness here the mechanical interpretation of the results plays a central role as well as the attention for numeric implementation.

The most important result of this thesis is the capability of the proposed coupled model to account for cohesive or ductile fracture, that is nucleation of cracks where the stress does not vanish.

Original Contributions

This work stems out after a historical overview from the origins of the internal state variable theory up to the recent variational formulation for rate-independent systems, Chap. 1. Moreover a critical analysis of the literature proposals for nonlocal damage-plasticity models with their main features is carried out, Sec. 1.4.6. The main original contributions to the existing literature are described in some detail at the end of each chapter but are here summarized:

- **A new variational constitutive model for the coupling of plasticity and damage in a three-dimensional setting** To this aim, as for the first order stability condition, the evaluation of the Gateaux derivative of the energy functionals deserved some non-trivial calculations, appendix A. This is due to the fact that the dissipated work results to be in the most common cases not a state function as it possibly depends upon the past history of the plastic strain trough the accumulated plastic strain. Moreover, the equations resulting from the first order stability condition and the energy balance give a deep mechanical insight to the proposed model unveiling the coupling mechanism. The yield functions and consistency conditions automatically descends from the variational formulation and have not to be postulated *a-priori*, Sec. 2.2 and Sec. 2.3.

- **The thorough and complete analysis of the homogeneous responses** In order to understand the model capabilities its homogeneous response has been investigated. Several conditions on the constitutive functions that determine different evolutions has been found. These conditions dictates whether plasticity, damage or both may evolve during an evolution and in which order, Sec. 3.2.

- **The material stability analysis of the homogeneous responses** The stability analysis in the one-dimensional setting of homogeneous responses allows the interpretation of the internal length scale and furnishes the condition for the occurrence of a non-homogeneous solution, Sec. 3.4.

- **The construction of a non-homogeneous solution** When the homogeneous solution becomes unstable a non-homogeneous solution must appear, based on a localisation. A method for the construction of a localisation is proposed which takes explicitly into account the irreversibility condition, Sec. 4.2. It is shown that an accumulated plastic strain localisation is associated to a discontinuity in the derivative of the damage field, Sec. 4.48. This key property allows a cohesive fracture. A straightforward analytical example of this construction can be found in the submitted paper, Chap. B.

- **The analysis of non-homogeneous responses in a one-dimensional setting** The construction of a localisation is necessary for the investigation of the global response. Within this context it is proven the capability of the resulting model to describe the nucleation of a cohesive fracture where a jump of the displacement field occurs with a non-vanishing stress. Indeed, the fracture energy may become a function of the jump of the displacement field by means of the accumulated plastic strain, $G(\llbracket u \rrbracket)$. Moreover, since G tends towards an asymptotic value, say G_c, when the displacement jump tends to infinity and the the plastic yield stress is finite, the model describes exactly the Barenblatt's cohesive fracture model, Sec. 4.26.

- **The development of a numerical algorithm and its application to the one-dimensional problem** One of the most important advantages of the variational formulation is that it leads to a natural and rational numeric implementation. Due to the convexity properties of the model an alternate minimisation algorithm, which looks for a global minimum, has been developed and successfully applied to a sample case in a one-dimensional setting, Sec. 5.1. Although the finite element spaces do not embed explicitly jumps the numerical results perfectly agree with the analytical results, Sec. 5.2.

Structure of the Thesis

Chapter 1. *Background and state of art*. This chapter is devoted to a historical review covering the internal state variable theory, the Standard Generalised Materials theory and the Energetic Formulation for rate independent systems. Several classical material models are briefly compared highlighting their peculiarities and limitations. Finally, the fracture, plasticity and damage models are singularly introduced as building blocks for the coupled plasticity-damage model proposed in Chap. 2.

Chapter 2. *A variational elastic-plastic-damage model*. This chapter is the core of the present work. Here the coupled plastic-gradient damage model is developed in a three-dimensional setting by means of the energetic formulation. The first order stability condition furnishes some (weak) necessary inequalities which the model has to be satisfied and from which it is possible to define in a classical sense the yield functions and disclose their (strong) expression. Nevertheless it is the weak form that reveals the capability of the model to describe a cohesive fracture. From the energy balance it is possible to derive the classical consistency conditions.

Chapter 3. *Homogeneous evolutions*. The great flexibility of the model is investigated for a homogeneous evolution. This simply highlights all the possible responses that could be expected from the model. Specifically, elastic, plastic and damage stages occurs singularly or coupled depending on the constitutive functions of the model and are understood by means of numerical examples. Moreover, a stability analysis is carried out which gives a condition for the occurrence of a localisation depending on the overall length of the bar and the internal length parameter.

Chapter 4. *Non-homogeneous evolutions*. Focusing on the one-dimensional setting, the construction of a localisation which accounts for the irreversibility condition is first presented. More in detail, the damage localisation profile is symmetric with the shape of a bell. The plastic yield criterion can be then attained in one point corresponding to the center of the localisation zone where instead plasticity localises but as a Dirac measure leading precisely to a cohesive crack *a la* Barenblatt. Once the construction of the localisation has been introduced, a one-dimensional traction bar test is investigated and the global response retrieved.

Chapter 5. *Numeric implementation and simulations*. The variational approach leads to a natural and rational way to define efficient numerical algorithms since its intrinsic discrete nature. The adopted numeric scheme is an alternate minimisation algorithm in a finite element framework. Numeric simulations in the one-dimensional setting have been carried out and compared with the analytical solutions. Despite the use of C^0-elements for the space approximation of the displacement field the results are in perfect agreement with the analytical solutions.

Chapter 1

Background and State of Art

In this chapter the main background theories leading to the adopted variational formulation are presented through both their logical and historical interrelation. More specifically the point of departure is the thermodynamics framework which almost all agree any material model should respect. After having presented some historical remarks, the attention is focused on a particular class of material models, namely the Standard Generalized Materials, where the dissipated forces are derived from a dissipation potential. Under the assumption of rate-independence the energetic (variational) formulation descends. The chapters finally ends with the reference to the existing fracture, plasticity and damage model which singularly establish the building blocks of the coupled model.

A substantial historical perspective covering several aspects of continuum thermodynamics and material modelling is given in Maugin 1992.

1.1 Continuum Mechanics with internal variables

1.1.1 Introduction

The presentation of this section is essentially driven by the works of Bataille and Kestin 1979; Maugin 1992. The topic, in its wholeness, is very complicated and still under debate. Many currents of thought exists and a well defined, unique presentation is impossible. Nevertheless, it is believed that a sketch of key-concepts is useful to understand the logical and formal developments of the present work.

The key-point of *classical thermodynamics*, despite the word "-dynamic", is that the theory is concerned about systems that are in equilibrium. Therefore, many authors prefer to refer to it with the name *thermostatics*. Any process is then considered as a sequence of equilibrium states. In particular, as stated in Germain, Nguyen, and Suquet 1983, classical thermodynamics is formally correct only if during the process all variables have a full physical meaning, that is, the irreversible system can be embedded in a larger one which is reversible and where the definition of an entropy-like function is possible.

What then can be said about states outside equilibrium, that is to the term thermodynamics in its full sense? Answers, for dealing with states in not-equilibrium, may come from three different point of view and approaches, as

- the thermodynamics of irreversible processes;
- the thermodynamics of Colemann and Noll or rational thermodynamics;
- the thermodynamics with internal variables.

Actually, the thermodynamics with internal variables leans against the first two approaches, although its development can be considered independent. Hence, in order to fully justify and understand the thermodynamics with internal variables, which is the ground for the construction of the considered constitutive problem, a brief introduction of the first two thermodynamics theories is necessary. A more extended survey about branches of thermodynamics can be found in the review article Muschik 2008.

In the sections that follow the setting is those of continuum mechanics. The work of Duhem 1911 deserves a special mention. This book highlight the role of thermodynamics in continuum physics. First the main results of continuum thermostatics are presented.

1.1.2 Fundamental laws in Continuum Mechanics

In this work it is assumed that deformations and gradient of deformations remain small during the whole evolution, *i.e.* no difference is considered between the reference and current configuration. The strain-displacement relation becomes

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}}\right). \tag{1.1}$$

The fundamental laws of continuum mechanics are

- balance of mass;
- balance of linear momentum;
- balance of angular momentum;
- balance of energy;
- etropy inequality principle.

In particular, the last two can be made to descend from the thermodynamics formulation introduced in the next subsections.

The conservation of mass for a continuum reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}(t)} \rho(\boldsymbol{x}, t)\ \mathrm{d}\Omega = \text{constant}, \tag{1.2}$$

where $\mathcal{P} \subset \mathcal{B}$ is a generic portion of the considered body and $(\boldsymbol{x}, t) \mapsto \rho(\boldsymbol{x}, t)$ is the mass density. Due to the assumption (1.1) and to the initial unitary mass density, the balance of mass (1.2) is always satisfied.

Once the *Cauchy stress tensor*, or simply *stress tensor*, $\boldsymbol{\sigma}$ is introduced, the balance of linear momentum leads to

$$\int_{\mathcal{P}} (\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} - \rho\ddot{\boldsymbol{x}})\ \mathrm{d}\Omega = 0 \tag{1.3}$$

or in local form

$$\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} - \rho\ddot{x} = 0, \quad \forall\, \boldsymbol{x}. \tag{1.4}$$

Moreover, from the balance of angular momentum, one obtains

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}. \tag{1.5}$$

The *virtual power principle* can be considered an equivalent formulation for (1.3) and is widely used.

Clearly, equations (1.1), (1.4) and (1.5) are insufficient to uniquelly determine the unknowns $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$. In particular, 6 equations are missing, namely the *constitutive equations*. While the afore-mentioned formulation is valid for any materials, the constitutive equations instead depend upon it.

As stated by Germain et al., Germain, Nguyen, and Suquet 1983, thermodynamics furnishes a satisfactory framework with the aim of formulating consistent constitutive equations,

> "Continuum thermodynamics impose new important restrictions on constitutive equations when a suitable generalization of the second law is formulated. The main objective of thermodynamics is ti provide a method to write constitutive equations that fulfil these new restrictions."

1.1.3 Thermostatics

Usually thermostatics refers equivalently to reversible processes. Indeed, in this case, all quantities are well defined in an equilibrium state. That is, temperature and entropy do not affect the evolution. However they become necessary to describe irreversible processes.

Thermostatics compares states in thermodynamic equilibrium, where a definition of temperature and entropy is possible. In order to present the first and second law of thermostatics some fundamentals definitions are called upon. A *system* is defined as a specific position of the physical universe which is a specific quantity of matter. In the present work, in particular, a system which does not exchange matter through its boundary is considered as a continuum with unitary density. *State variables* $\boldsymbol{\chi} = (\chi_1, \chi_2, \ldots, \chi_n)$ are introduced to characterize the state of the system. The change of those variables depends only on the initial and final state of the system. About this last point a clarification should be given. In general, the identification of the state variables of a given system is not trivial and not unique. The choice depends upon the physical system one is interested in, Maugin and Muschik 1994. On the other hand, having established the concept of state variables, the concept of state function follows. A *state function* is a function that only depends on the state of the system and not on the manner in which this state has been reached. It's worth noting that neither mechanical work nor heat are state functions, Ottosen and Ristinmaa 2005. In particular, a thermodynamic system is in equilibrium if the variable that describe its state $\boldsymbol{\chi}$ remains unchanged in time. If the evolution is sufficiently slow such that any state can be considered in thermodynamic equilibrium, two quantities can be attached to the system. These two quantities are the *internal energy* $\mathcal{E}$, which represents the sum of all the microscopic forms of energy, and the *kinetic energy* $\mathcal{K}$, due to the macroscopic velocity of the system constituents expressed in a Galilean reference. They are state functions satisfying for any transformation the equation

$$\mathrm{d}\mathcal{E} + \mathrm{d}\mathcal{K} = \delta\mathcal{W} + \delta\mathcal{H}. \tag{1.6}$$

These state functions are additive with respect to material sets. In (1.6), $\delta\mathcal{W}$ and $\delta\mathcal{H}$ represent respectively the total amount of mechanical work and the total amount of heat received from the external world. The sum $\mathcal{E} + \mathcal{K}$ represent the stored energy of the system while (1.6) expresses the balance of energy.

The *second law* of thermodynamics can be formulated in various ways, for example remarkable are the versions of Kelvin, of Clausius, see Clausius and Hirst 1867, and of Carathéodory, see Carathéodory 1909. By considering Clausius theorem, that is

$$\oint \frac{\delta\mathcal{H}}{T} \begin{cases} = 0 & \text{reversible process} \\ > 0 & \text{irreversible process,} \end{cases} \tag{1.7}$$

where T is the *absolute temperature* (or thermodynamic temperature) introduced by Carnot's theorem, another state function, the *entropy* S, can be introduced as

$$\mathrm{d}S \geq \frac{\delta\mathcal{H}}{T}. \tag{1.8}$$

Both the internal energy $\mathcal{E}$ and the entropy S are extensive quantities[1]. Particular attention should be paid to *normal* or *natural* state variables[2]. The state variables $(T, \boldsymbol{\chi})$ are *normal* if for

$$\mathrm{d}\boldsymbol{\chi} = 0 \quad \Rightarrow \quad \delta\mathcal{W} = \mathrm{d}\mathcal{K} = T\,\mathrm{d}S - \delta\mathcal{H} = 0. \tag{1.9}$$

Assumed normal state variables, it is possible to write then

$$T = \frac{\partial\mathcal{E}}{\partial S}(S, \boldsymbol{\chi})\,, \qquad \boldsymbol{X} = \frac{\partial\mathcal{E}}{\partial\boldsymbol{\chi}}(S, \boldsymbol{\chi})\,, \tag{1.10}$$

where T and $\boldsymbol{X}$ are respectively the dual variables associated with S and $\boldsymbol{\chi}$. $\mathcal{E}(S, \boldsymbol{\chi})$ can be regarded as a thermodynamic potential. Moreover, it can be proved that $\mathcal{E}(S, \boldsymbol{\chi})$ is convex with respect to S.

In continuum thermodynamics, one prefer to deal with the *Helmotz's free energy* or simply *free energy* Ψ which is given by the Legendre-Fenchel's transformation, *i.e.*

$$-\,\Psi(T, \boldsymbol{\chi}) := \max_S T\,S - \mathcal{E}(S, \boldsymbol{\chi}) = T\,S - \mathcal{E}(S, \boldsymbol{\chi})\,, \tag{1.11}$$

which is still a thermodynamic potential,

$$S = -\frac{\partial\Psi}{\partial T}(T, \boldsymbol{\chi})\,, \qquad \boldsymbol{X} = \frac{\partial\Psi}{\partial\boldsymbol{\chi}}(S, \boldsymbol{\chi})\,. \tag{1.12}$$

The equality in (1.11) follows from the normal state variables assumption.

To extend this formulation to continuum systems and to not-equilibrium states, modifications to the classical theory have to be done.

[1] An *extensive* quantity, in a continuum system, is a quantity proportional to the mass of the system as opposed to *intensive* quantity.

[2] The term *normal* was first introduced in Duhem 1911.

1.1.4 The theory of irreversible processes

The *theory of irreversible processes* is the most standard approach to non-equilibrium thermodynamics. An extended overview of this theory can be found in De Groot and Mazur 2011.

For the further developments, it suffices to emphasize the fundamental role played by the *axiom of the local (equilibrium) state* in the theory of irreversible processes, which states that each part $\mathcal{P}$ of a material system $\mathcal{B}$ ($\mathcal{P} \subset \mathcal{B}$) can be approximately considered, at any time t, being in thermal equilibrium. This postulate, is the basis of continuum thermodynamics. Indeed, at any material point $\boldsymbol{x}$, time t, temperature T and state variables χ, which in this context are assumed to be controllable and observable by an external observer, one can introduce a specific internal energy $e = e(\boldsymbol{x}, S, \chi)$, a specific free energy $w = w(\boldsymbol{x}, T, \chi)$ and a specific entropy $s = s(\boldsymbol{x})$, all *intensive variables*, such that for any $\mathcal{P}$,

$$\mathcal{E}_{\mathcal{P}} = \int_{\mathcal{P}} \rho\, e \; \mathrm{d}\Omega, \qquad \Psi_{\mathcal{P}} = \int_{\mathcal{P}} \rho\, w \; \mathrm{d}\Omega, \qquad S_{\mathcal{P}} = \int_{\mathcal{P}} \rho\, s \; \mathrm{d}\Omega, \tag{1.13}$$

since energy and entropy are additive quantities. Moreover, for a continuum, the kinetic energy is

$$\mathcal{K}_{\mathcal{P}} = \frac{1}{2} \int_{\mathcal{P}} \rho\, \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} \; \mathrm{d}\Omega. \tag{1.14}$$

In particular, this allows one to view a thermodynamic process close to equilibrium as a sequence of thermostatic equilibria and thus to grant to entropy and temperature their usual thermostatic definitions. That is, at each instant there still exists a set of normal state variables (s, χ), and a specific internal energy $e(s, \chi)$ such that temperature T and the laws of state are given by

$$T = \frac{\partial e}{\partial s}, \qquad \boldsymbol{X} = \frac{\partial e}{\partial \chi}. \tag{1.15}$$

The closeness to equilibrium is measured by the ratio of the characteristic response time τ_R, which allows the thermostatic system to recover a new state of thermostatic equilibrium, and the characteristic duration τ_D of the kinematic and dynamic evolution of the medium. This ratio, defined as the time of relaxation over the time of observation, should be small,

$$\mathscr{D}e = \frac{\tau_R}{\tau_D} \ll 1. \tag{Deborah number}$$

This will prove to be impracticable each time that the evolution of the system is too fast (e.g., in a shock-like evolution).

Under the assumptions of the local equilibrium state and through time derivatives, one may then write locally the first law of thermostatics. To this aim, it is sufficient to take the time derivative of (1.6). The *received heat power* from the body is then defined as

$$\frac{\delta \mathcal{H}}{\mathrm{d}t} = \int_{\mathcal{P}} \rho\, h \; \mathrm{d}\Omega - \int_{\partial\mathcal{P}} \boldsymbol{h} \cdot \boldsymbol{n} \; \mathrm{dS} = \int_{\mathcal{P}} \rho\, h \; \mathrm{d}\Omega - \int_{\mathcal{P}} \mathrm{div}\boldsymbol{h} \; \mathrm{d}\Omega \tag{1.16}$$

where h and $\boldsymbol{h}$ are respectively the *heat source* and the *heat flux vector*. The last equality follows form the Gauss's divergence theorem[3]. On the other hand, the

[3] $\int_{\Omega} \mathrm{div}\boldsymbol{v} \; \mathrm{d}\Omega = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n} \; \mathrm{dS}$, Spiegel and Lipschutz 2009.

received mechanical external power from the body is defined as

$$\frac{\delta\mathcal{L}}{\mathrm{d}t} = \int_{\mathcal{P}} \rho\, \boldsymbol{b} \cdot \dot{\boldsymbol{u}}\ \mathrm{d}\Omega + \int_{\partial\mathcal{P}} \boldsymbol{f} \cdot \dot{\boldsymbol{u}}\ \mathrm{dS} \tag{1.17}$$

$$= \int_{\mathcal{P}} \rho\, \boldsymbol{b} \cdot \dot{\boldsymbol{u}}\ \mathrm{d}\Omega + \int_{\mathcal{P}} \mathrm{div}\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\ \mathrm{d}\Omega + \int_{\mathcal{P}} \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}\ \mathrm{d}\Omega \tag{1.18}$$

with $\boldsymbol{b}$, $\boldsymbol{f}$ respectively the body force vector per unit mass and the traction vector acting on the boundary, where for the last equation the Gauss's divergence theorem and the equality $\boldsymbol{f} = \boldsymbol{\sigma}[\boldsymbol{n}]$ has been used.

Inserting eqs. (1.13), (1.14), (1.16) and (1.18) in the time-derivative of eq. (1.6), one obtains

$$\rho\,\dot{e} = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} + \rho\, h - \mathrm{div}\boldsymbol{h}, \tag{1.19}$$

which is the local form of the first law of thermodynamics and where eq. (1.4) has been used. Similarly, the second law of thermodynamics (1.8) becomes in local differential form:

$$\rho\,\dot{s} - \frac{\rho\, h}{T} + \frac{\mathrm{div}\boldsymbol{h}}{T} + \boldsymbol{h} \cdot \nabla(T^{-1}) \geq 0, \tag{1.20}$$

called also *Clausius-Duhem inequality*. Assuming that $e = e(s, \boldsymbol{\varepsilon}, \boldsymbol{\chi})$, where $\boldsymbol{\chi}$ are other observable state variables, one has

$$\dot{e} = T\,\dot{s} + \frac{1}{\rho}\sigma \cdot \dot{\boldsymbol{\varepsilon}} + \frac{1}{\rho}\boldsymbol{X} \cdot \dot{\boldsymbol{\chi}} \tag{1.21}$$

that replaced into (1.20) through (1.19), furnishes the so called *internal dissipated power d* as

$$d = -\boldsymbol{X} \cdot \dot{\boldsymbol{\chi}} + T\,\boldsymbol{h} \cdot \nabla(T^{-1}) \geq 0. \tag{1.22}$$

The variables $\dot{\boldsymbol{\chi}}$ are called *fluxes* while the variables $\boldsymbol{X}$ *forces*. A relation between the fluxes and the forces needs to be taken into account in order to solve the problem. The most simple relation was proposed by Onsager and Casimir, **osnager1931** justified by the hypothesis of small deviations from equilibrium. Once the Helmotz's free energy is introduced, (1.11), the Clausius-Duhem inequality becomes

$$d = -\rho\left(\dot{w} + s\dot{T}\right) + \sigma \cdot \dot{\boldsymbol{\varepsilon}} + T\,\boldsymbol{h} \cdot \nabla(T^{-1}) \geq 0. \tag{1.23}$$

1.1.5 The "Rational" Thermodynamics of Colemann and Noll

Rational thermodynamics, developed from 1960 to 1970, Coleman 1964; Coleman and Mizel 1967; Coleman and Noll 1963; Day 1972; Gurtin 1968; Truesdell 1984, deals with non-equilibrium systems and is based essentially on two strong assumptions:

- entropy and temperature are two primitive quantities, also existing in non-equilibrium states;
- the Clausius-Duhem inequality (1.20) is considered as an *a priori* postulate.

The implications of such axiomatic theory is a field theory of "elegant formalism" and "mathematical attractiveness", Bataille and Kestin 1979. The constitutive equations are formulated in a functional form, where the whole past history is involved, recovering the dynamic aspect of the process. This theory has proved capable only of describing materials with "memory" such that one can meet in viscoelastic media.

1.1.6 The theory of internal variables

The first formalization of the theory goes back to the watershed paper of Coleman and Gurtin, Coleman and Gurtin 1967. The main idea behind this theory is that a thermodynamic state is not only characterized by the usual observable state variables but also by *internal variables* $\boldsymbol{\alpha}$. These variables are hidden to the observer and are considered essential for the definition of the internal structure of the material. Actually, the fact that a variable is observable or not is mostly a matter of scale. According to Mandel 1980, one is often able to measure these variables but not to control them.

The definition of the dependent variables, like the stress $\boldsymbol{\sigma}$, rely on both the observable and internal variables. The crucial point is the definition of proper evolution laws for the internal variables. The following sections are devoted to an overview on how evolution laws can be properly defined.

Moreover, one can state that this theory is a synthesis of aspects of the two previous theories. Indeed, on one hand, the past history of the process is taken into account simply through the value of the finite number of internal variables, representing a kind of average effect. On the other hand, the general framework continues to remain dynamic. An extended version of the *local-state axiom* has to be formulated to encompass the internal variables provided that the characteristic times, that enter in the definition of the evolution laws, remains small compared to the macroscopic evolution. It is remarkable to note that for a wide range of material models, like plasticity and damage, time doesn't matter since no time derivatives appears in the constitutive equations. Hence, the process can be considered as rate independent. This point will be explored deeper in the next sections.

To summarize, the theory of internal variables postulates that the thermodynamic state of a material medium at a given point and instant is completely defined by the knowledge of the values of a certain number of variables both observable and hidden. Assuming that all further state variables χ are internal, the free energy density reads

$$w = w(T, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})\,. \tag{1.24}$$

The state function of state, or constitutive equations are

$$s = -\frac{\partial w}{\partial T}, \quad \boldsymbol{\sigma} = \rho \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{A} = -\rho \frac{\partial w}{\partial \boldsymbol{\alpha}}, \tag{1.25}$$

while the Clausius-Duhem inequality reads

$$d = \boldsymbol{A} \cdot \dot{\boldsymbol{\alpha}} + T\boldsymbol{h} \cdot \nabla(T^{-1}) \geq 0. \tag{1.26}$$

The choice of the evolution laws is a key point and the next section is devoted to their formulation. In general, two main approaches are noteworthy:

- direct approach;
- potential approach.

They differ essentially from the point of view with which the second law is considered.

Indeed, in the direct approach, the evolution laws for $\dot{\boldsymbol{\alpha}}$ are postulated and the satisfaction of (1.23) checked *a posteriori*. The first famous relation in this sense

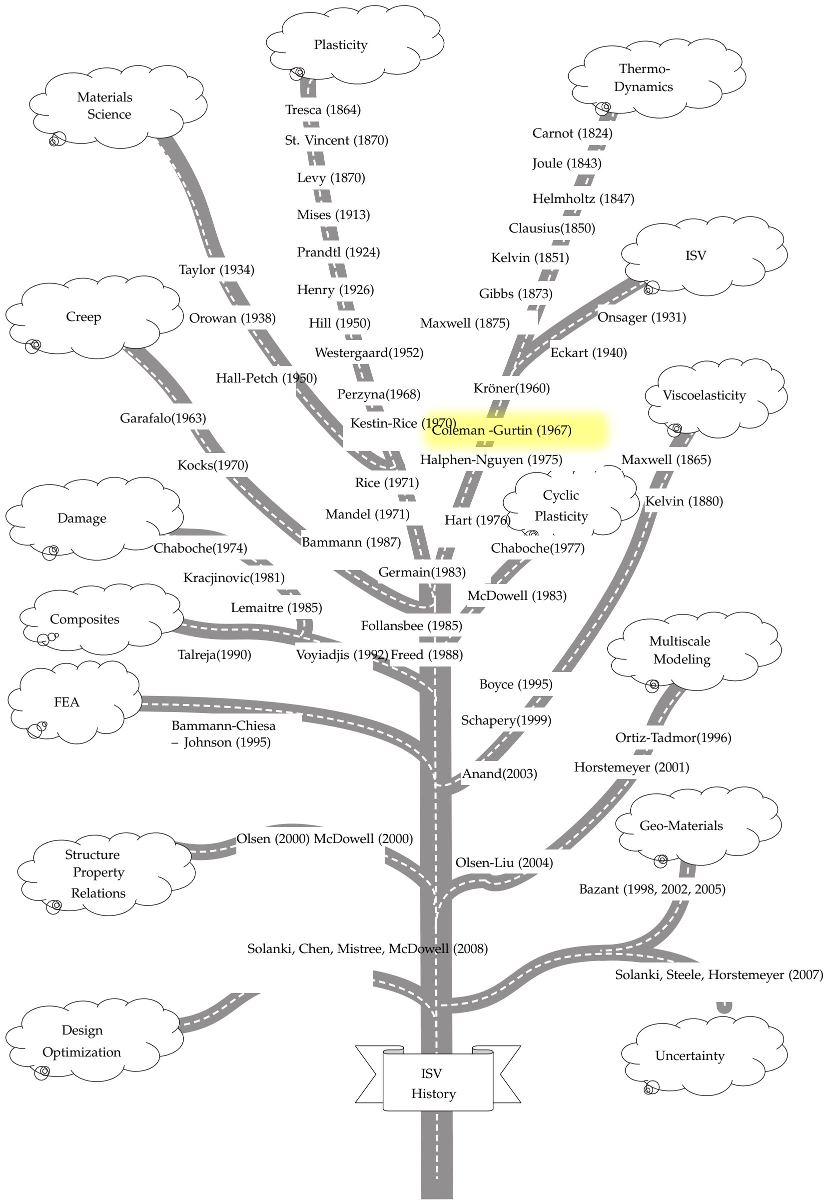


Figure 1.1: Historical summary of different roads that formulated the modern internal state variable theory, Horstemeyer and Bammann 2010

was proposed by Onsager,see Onsager 1931. It assumes a linear dependence among forces and fluxes and hence is the simplest. Furthermore, although it automatically fulfils the second law, it is unable to cover all physical phenomena.

The second approach assumes a relation for the evolution laws which satisfies *a priori* the second law of thermodynamics (1.23).

1.2 Generalized Standard Models

As mentioned before, once additional internal state variables are introduced, further state laws and evolution laws have to be formulated. To this aim, a rational and fruitful approach is given by the Generalized Standard Model, which is an example of the previously mentioned "potential approach". The name "Generalized Standard Model" was first introduced by Halpphen and Nguyen 1975. Other key-references are Germain, Nguyen, and Suquet 1983; Moreau 1970; Nguyen 2002. From now on isothermic conditions are assumed during all processes.

By definition in Nguyen 2000, p. 37:

> "A model of material behaviour is a generalized standard model if it is defined by two potentials, the energy potential and the dissipation potential. The energy is a function of state variables and the dissipation potential is a convex function of flux and may eventually depend on the present state."

The ingredients of such model then are:

- energy potential
- dissipation potential

In particular, the *energy potential* W (free-energy per unit-mass, elastic energy, stored-energy density, ecc. ecc.) is a function of state variables defined as

$$W(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) = \rho\, w(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})\,, \tag{1.27}$$

and allows the following constitutive equations

$$\boldsymbol{\sigma} = \frac{\partial}{\partial \boldsymbol{\varepsilon}} W(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})\,, \qquad \boldsymbol{A} = -\frac{\partial}{\partial \boldsymbol{\alpha}} W(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})\,. \tag{1.28}$$

On the other hand, the dissipation potential $\psi = \psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) \in \mathbb{R}^+$ is a convex function with respect to the flux $\dot{\boldsymbol{\alpha}}$ and may depend on the present state through the value of $\boldsymbol{\alpha}$, so that

$$\boldsymbol{A} = \frac{\partial}{\partial \dot{\boldsymbol{\alpha}}} \psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})\,. \tag{1.29}$$

Moreover, in order to satisfy the second principle of thermodynamics (1.26), the dissipation potential $\psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$ has to satisfy the following property:

$$\mathbf{0} \in S_{\dot{\boldsymbol{\alpha}}} := \left\{ \dot{\boldsymbol{\alpha}} : \psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) \leq m,\ \forall m \in \mathbb{R}^+ \right\}, \tag{1.30}$$

that is, the nul vector must belong to the convex set $S_{\dot{\boldsymbol{\alpha}}}$. The fact that energy and dissipation potentials leads to a general framework in the study of dissipative effects in materials and structures is explained in Germain 1973, Lemaitre 1985 and Maugin 1992.

- It's implementation is systematic;
- It can be generalized to any kind of thermomechanical behaviour;
- It does not disagree with well established thermodynamical principles and even permits to automatically satisfy the second law;

- It agrees with the most usual constitutive relations;
- It generally leads to "nice" initial-boundary value problems with "good" mathematical properties.

The issue about boundary conditions for the internal state variables is still under debate. Different assumptions in that sense can lead to very different solutions. This point will be further explored in the next sections.

1.2.1 Sub-differential formulation

In case of time-independent irreversible processes, like plasticity, the dissipation potential is no longer differentiable in the origin. A generalized differentiation in the sense of sub-gradient of a convex function can be introduced and the notion of potential for non-differentiable functions preserved, Ekeland and Téman 1999; Moreau 1970. In addition, assuming the positively 1-homogeneous and lower semi-continuity properties for the dissipation potential, the dependence between force and flux can be then written as

$$\boldsymbol{A} = \partial_{\dot{\boldsymbol{\alpha}}}\, \psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})\,. \tag{1.31}$$

Condition (1.31) was introduced by Moreau 1970 and is called *normal dissipation hypothesis*.

Sometimes it is straightforward to use the dual dissipation potential ψ^* obtained by the Legendre-Fenchel's transformation

$$\psi^*(\boldsymbol{\alpha}, \boldsymbol{A}) = \max_{\dot{\boldsymbol{\alpha}}} \boldsymbol{A}\,\dot{\boldsymbol{\alpha}} - \psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})\,, \tag{1.32}$$

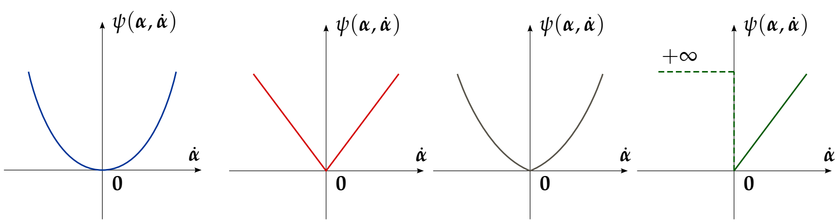
so that the complementary laws become

$$\dot{\boldsymbol{\alpha}} = \partial_{\boldsymbol{A}} \psi^*(\boldsymbol{\alpha}, \boldsymbol{A})\,. \tag{1.33}$$

Obviously the property (1.30) turns out to be

$$\mathbf{0} \in S_{\boldsymbol{A}} \quad = \left\{ \boldsymbol{A} \colon \psi^*(\boldsymbol{\alpha}, \boldsymbol{A}) \leq m,\ \forall m \in \mathbb{R}^+ \right\}. \tag{1.34}$$

Figure 3.2 shows some examples about common standard generalized models.



(a) *visco-elastic* (b) *plastic* (c) *visco-plastic* (d) *damage*

Figure 1.2: Examples of common dissipation potentials in mechanics

1.2.2 Standard dissipative system: Biot's equation

Once a standard generalized model is adopted the internal variable evolution can be obtained from a differential equation

$$\partial_{\boldsymbol{\alpha}} W + \partial_{\dot{\boldsymbol{\alpha}}} \psi = 0, \qquad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0 \tag{1.35}$$

known as *Biot's equation*, Biot 1965. Moreover, if inertia $J = J(\ddot{\boldsymbol{u}})$ is considered and the dissipation potential also depends on the displacement rate as in viscous materials $\psi = \psi(\dot{\boldsymbol{u}}, \boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$, then the equation

$$J + \frac{\partial W}{\partial \boldsymbol{u}} + \frac{\partial \psi}{\partial \dot{\boldsymbol{u}}} = 0 \tag{1.36}$$

together with (1.35) and the boundary conditions constitute a so called *standard dissipative system*.

1.2.3 Rate-independence

In this work the material response taken into account is rate-independent. This means that the response is linearly proportional to the adopted time scale. For example, plasticity, damage and dry-friction are rate-independent or non-viscous irreversible processes.

In the framework of standard generalized materials, *rate-independence* for irreversible processes means that the dissipation potential is positively homogeneous of degree 1 with respect to $\dot{\boldsymbol{\alpha}}$,

$$\psi(\boldsymbol{\alpha}, \lambda\, \dot{\boldsymbol{\alpha}}) = \lambda\, \psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) \qquad \forall\, \lambda > 0. \tag{1.37}$$

In the models that follow, the dissipation potential is always considered as a convex, positively homogeneous of degree 1 function such that condition (1.30) is satisfied. Under the hypothesis of rate-independence, it is possible to consider equivalent formulations for describing the constitutive equations for standard generalized materials. A different approach is given by the variational formulation or the energetic formulation explored in Sec. 1.3 which results to be more flexible and less demanding.

1.2.4 Equivalent formulations

Focusing only on rate independent phenomena, like plasticity or damage, equivalent constitutive formulations for standard generalalized materials exists beyond the sub-differential formulation 1.2.1. These are summarized in the following.

1.2.4.1 KKT - yield surface

One of the most widely used methods is based on the definition of a yield function f, depending on invariants of the dual internal variables and on the present state by means of $\boldsymbol{\alpha}$, which defines an elastic region $S_{\boldsymbol{A}}$ and a yield surface $\partial S_{\boldsymbol{A}}$,

$$S_{\boldsymbol{A}} = S_{\boldsymbol{A}}(\boldsymbol{\alpha}, \boldsymbol{A}) = \{\boldsymbol{A} \colon f(\boldsymbol{\alpha}, \boldsymbol{A}) \leq 0\}\,, \quad \partial S_{\boldsymbol{A}} = \partial S_{\boldsymbol{A}}(\boldsymbol{\alpha}, \boldsymbol{A}) = \{\boldsymbol{A} \colon f(\boldsymbol{\alpha}, \boldsymbol{A}) = 0\}\,. \tag{1.38}$$

The equivalent formulation is obtained once an *associated flow rule* is adopted, that is the considered yield surface is convex and the flux collinear to the unit exterior normal to $\partial S_{\boldsymbol{A}}$, namely

$$\begin{cases} \dot{\boldsymbol{\alpha}} = \mathbf{0}, & f(\boldsymbol{\alpha}, \boldsymbol{A}) < 0 \\ \dot{\boldsymbol{\alpha}} = \dot{\lambda} \, \dfrac{\partial}{\partial \boldsymbol{A}} f(\boldsymbol{\alpha}, \boldsymbol{A}) \,, & f(\boldsymbol{\alpha}, \boldsymbol{A}) = 0, \end{cases} \tag{1.39}$$

where $\dot{\lambda} \geq 0$ expresses essentially the flux rate. In expression (1.39) is implicitly assumed smoothness of the yield surface. Indeed, if the yield surface has corner points, normal cones have to be considered at the same points.

The system (1.39) can then be rewritten through a set of *Karush-Kuhn-Tucker condition* [4] , that is,

$$\dot{\lambda} \geq 0, \qquad f \leq 0, \qquad \dot{\lambda} \, f = 0 \tag{1.40}$$

with the normal flow rule

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda} \, \frac{\partial}{\partial \boldsymbol{A}} f(\boldsymbol{\alpha}, \boldsymbol{A}) \,. \tag{1.41}$$

It is worth mentioning a more general choice for (1.41) which does not fit the generalized standard framework, namely the *non-associated flow rule*. A non-associated flow rule is obtained by replacing the yield function f in (1.41) by another function g, still a potential, such that

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda} \, \frac{\partial}{\partial \boldsymbol{A}} g(\boldsymbol{\alpha}, \boldsymbol{A}) \,. \tag{1.42}$$

A non-associated flow rule does not respect the normality rule and hence does not fall into a generalized standard model. First applications of this method can be traced in Melan 1938 and later by Prager 1949.

1.2.4.2 Normality rule

The following approach, based on the normality rule, has already been mentioned in the sub-differential formulation 1.2.1. The origin can be found in the works of Moreau, see Moreau 1970, and is thoroughly detailed in Han and Reddy 1999; Nguyen 2000. This theory has stemmed out from the mathematical convex analysis framework. The normality rule was introduced in the setting of plasticity and can be extended to rate-independent processes.

With (1.37) in mind, a convex domain with the null element of admissible forces is introduced through the sub-gradient of the dissipation potential,

$$E_{\boldsymbol{A}} = \partial_{\dot{\boldsymbol{\alpha}}} \psi(\boldsymbol{\alpha}, 0) \,. \tag{1.43}$$

In particular, $E_{\boldsymbol{A}}$ represents the *elastic domain*. The dual dissipation potential ψ^* is in this case the indicator function of $E_{\boldsymbol{A}}$,

$$\psi^*(\boldsymbol{\alpha}, \boldsymbol{A}) = I_{E_{\boldsymbol{A}}}(\boldsymbol{\alpha}, \boldsymbol{A}) \,. \tag{1.44}$$

[4]The *Karush-Kuhn-Tucker condition*, Karush 1939; Kuhn 1982, arise in seeking an optimal solution in the field of nonlinear programming. The KKT approach can be regarded as an extension of the method of Lagrange multipliers in presence of inequality constraints.

Hence, the force-flux relationship $\dot{\boldsymbol{\alpha}} = \partial_{\boldsymbol{A}} \psi^*(\boldsymbol{\alpha}, \boldsymbol{A})$ can be written as

$$\dot{\boldsymbol{\alpha}} \in N_{E_A}(\boldsymbol{\alpha}, \boldsymbol{A}) \tag{1.45}$$

which states that $\dot{\boldsymbol{\alpha}}$ must be an external normal to the admissible domain at the present state of the force $\boldsymbol{A}$.

1.2.4.3 Principle of maximum dissipation

The principle of *maximum dissipation*, also known by different names such as *Hill-Mandel maximal-dissipation principle* or simply *Hill's principle*, was first introduced by Onsager 1931 for heat conduction and Onsager 1945 for diffusion. Later on this principle was introduced in the plasticity context by Hill in Hill 1948 and properly formalised by Rice 1970 or Eve, Reddy, and Rockafellar 1990. The principle of maximum dissipation states that

$$(\boldsymbol{A} - \boldsymbol{A}^*) : \dot{\boldsymbol{\alpha}} \geq 0 \quad \forall \boldsymbol{A}^* \in E_{\boldsymbol{A}}. \tag{1.46}$$

Then the associated dissipation potential nothing but

$$\psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) = \max_{\boldsymbol{A}^* \in E_{\boldsymbol{A}}} \boldsymbol{A}^* : \dot{\boldsymbol{\alpha}}. \tag{1.47}$$

The equivalence between this principle and the normality rule (1.45) is straightforward.

1.2.5 Drucker-Ilyushin's postulate

The Drucker-Ilyushin postulate also deserves to be mentioned, although it does not fit the generalized standard material framework, at least for some models. As pointed out in Marigo 2002, the restriction given by the generalized standard theory is too strong with respect to usual universal principles, like material frame indifference or the second law of thermodynamics. Between the two extremes stands the Drucker-Ilyushin's postulate.

In the fifties Drucker Drucker 1959, 1951 formulated this postulate in order to restrict the possible constitutive relations in plastic theory. He introduced his stability postulate in terms of stress cycles and of strain work. Specifically denoting by $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}$ the initial and current stresses of the material element, the postulate requires that

$$\oint_{\boldsymbol{\sigma}} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) : \, \mathrm{d}\boldsymbol{\varepsilon} \geq 0 \tag{1.48}$$

in any stress cycle, that is along a path starting and ending at $\boldsymbol{\sigma}_0$. Some years later, Ilyushin Ilyushin 1961 formulated the stability postulate in terms of strain cycles and the total strain work by requiring that

$$\oint_{\boldsymbol{\varepsilon}} \boldsymbol{\sigma} : \, \mathrm{d}\boldsymbol{\varepsilon} \geq 0. \tag{1.49}$$

It has been shown, Marigo 2002, that Ilyushin's version is more easily extended in general thermodynamical systems and that Drucker's version would lead to unsatisfactory results for some classes of materials and behaviours like stress softening, see Ottosen and Ristinmaa 2005. Moreover, an interesting appraisal about Ilyushin's work is given in Maugin 2010.

The relationship between Ilyushin's, Drucker's and the above formulations was investigated by several authors, for example Marigo 2002, 2001; Maugin 1992. In the lengthy and detailed analysis of Manville Manville 1927, the thougths of Duhem where transcribed mathematically in the inequality

$$\oint \boldsymbol{A} : \dot{\boldsymbol{\alpha}} \, \mathrm{d}t \geq 0 \tag{1.50}$$

which is a kind of extension of (1.49). Actually the names of Drucker and Ilyushin should then be put side by side with Duhem's name.

1.3 Energetic Formulation (Variational Formulation)

The previous formulations summarised in section 1.2 implies in a certain sense the explicit integration of the evolution laws (1.29) and (1.33). Beside these another possible formulation of the constitutive problem, called hereafter *energetic formulation*, is possible set in a variational framework. That is, the solution (*energetic solution*) of the problem is obtained through a minimum principle.

In the evolution of rate-independent systems solutions are expected to develop jumps. Hence it is desirable to find a weaker formulation avoiding derivatives. The energetic formulation provides a very weak, derivative-free form for the determination of the evolution. It is based on a global stability condition and an energy balance. Using time-incremental minimization problems, which allow for the usage of the rich theory in the direct method of the calculus of variations, it is possible to establish general and abstract existence results as well as convergence properties for numerical approximations.

A mathematical formalization of the energetic formulation about rate-independent systems is essentially due to Mielke, see Mielke 2011; Mielke 2006a,b. In particular, the lecture notes in Mielke 2011 furnish a straightforward classification about possible solutions compared to regularity assumptions for rate-independent processes. Without lose of generality one can state that, once state variables, energy functionals and load conditions are defined, the energetic formulation in its weakest form is essentially based on two concepts:

- **(global) stability condition (ST)**
- **energy balance (EB)**.

The mathematical setup of the problem is given in Mielke 2011.

It is worth preliminary to highlight strengths and virtues of such a formulation, compared to those previously explored:

- the energetic formulation is totally derivative free. No derivatives are involved in its most general setting, neither for the constitutive equations;
- everything is based on functionals defined through domain integrals so that powerful mathematical tools as the direct method in calculus of variations can be used in order to face the issues of existence and uniqueness;

- boundary conditions for the internal variables descend automatically from the formulation;

- it provides an insight for the concepts of bifurcation and stability and gives tools to tackle them;

- it furnishes a natural and rational way to define efficient numerical algorithms.

Nevertheless, also some drawbacks are present, Mielke 2006a:

- the energetic formulation offers only few results on uniqueness of solutions, see Brokate, Krej, and Schnabel 2004; Mielke and Theil 2004. The main difficulties depends on the lack of mathematical tools belonging to the field of functional analysis and calculus of variation;

- the stability condition involves global minimisation although for a better physical modelling and for numeric implementation it would be desirable to be able to consider local stability. In this sense, first attempts have been made even if a general theory is still missing.

1.3.1 State variables and energy functionals

The body $\Omega \subset \mathbb{R}^d$, with $d \in \{1,2,3\}$ is assumed to be open, bounded and with a Lipschitz boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ such that integration by parts and Sobolev embeddings are available. On the boundaries $\partial\Omega_D$ and $\partial\Omega_N$ are prescribed respectively Dirichlet and Neumann boundary conditions.

Being interested in continuum mechanics applications, the only variables that are considered in this general presentation of the energetic formulation are

$\boldsymbol{u}$	displacement field[5]
$\boldsymbol{\alpha}$	set of generic internal variables

where the displacement $\boldsymbol{u} \in \mathcal{F}(\Omega, F)$ and $\boldsymbol{u}(\boldsymbol{x}, t) : (\Omega, [0,T]) \to F \in \mathbb{R}^d$ while the internal variables $\boldsymbol{\alpha} \in \mathcal{Z}(\Omega, Z)$ with $\boldsymbol{\alpha}(\boldsymbol{x}, t) : (\Omega, [0,T]) \to Z \in \mathbb{R}^m$. Both $\mathcal{F}$ and $\mathcal{Z}$ can be assumed to be simply Banach spaces although topological vector spaces suffices for the general theory. The prescribed displacements $\bar{\boldsymbol{u}}$ are applied on $\partial\Omega_D$. Once the states space has been introduced, two functionals have to be defined,

- the *energy functional*,

- the *dissipation distance*,

global quantities which are involved in the next developments. In the setting of continuum mechanics, the energy functional equals the total potential energy $\mathcal{P} : [0,T] \times \mathcal{F} \times \mathcal{Z} \to \mathbb{R}$, defined by means of an energy density $e : [0,T] \times F \times Z \to \mathbb{R}_\infty$, which has the same meaning as the *stored-energy density* introduced in (1.27), while

[5]One could assume as state variable the deformation gradient F which relies on the deformation φ, but since the theory is applied in case of infinitesimal strains, the kinematic description through the displacement field is preferred.

the global dissipation distance $D : \mathcal{Z}\times\mathcal{Z} \to \mathbb{R}$, whose definition is less obvious, is defined by means of the *dissipation distance density* $d : Z\times Z \to \mathbb{R}^+_\infty$.[6]

The total potential energy is the sum of two contributions. The stored internal energy $\mathcal{E}$ and the potential energy of external forces $-\mathcal{L}$, which respectively reads as

$$\mathcal{E} := \mathcal{E}(t, \boldsymbol{u}, \boldsymbol{\alpha}) = \int_\Omega e(\nabla \boldsymbol{u}, \boldsymbol{\alpha})\ \mathrm{d}\Omega, \tag{1.51}$$

$$\mathcal{L} := \mathcal{L}(t, \boldsymbol{u}) = \int_\Omega \boldsymbol{b}(t) \cdot \boldsymbol{u}\ \mathrm{d}\Omega + \int_{\partial\Omega} \boldsymbol{f}(t) \cdot \boldsymbol{u}\, \mathrm{dS}. \tag{1.52}$$

Obviously, in the definition (1.52) the internal product $\cdot$ has been introduced. The vector functions $\boldsymbol{b}$ and $\boldsymbol{f}$ are respectively the *body force density* and the *traction forces* applied on the boundary. The traction forces are in turn divided into $\boldsymbol{f}_a$, the active forces on $\partial\Omega_N$, and $\boldsymbol{f}_r$, the reactive forces $\partial\Omega_D$. Henceforth this distinction will be explicitly omitted but still considered.
A common classification of external inputs is given in Tab. 1.1.

choices	description
$\begin{cases} \boldsymbol{b} \neq 0 & \text{in } \Omega \\ \boldsymbol{f} \neq 0 & \text{on } \partial\Omega_N \\ \bar{\boldsymbol{u}} = 0 & \text{on } \partial\Omega_D \end{cases}$	soft devices
$\begin{cases} \boldsymbol{b} = 0 & \text{in } \Omega \\ \boldsymbol{f} = 0 & \text{on } \partial\Omega_N \\ \bar{\boldsymbol{u}} \neq 0 & \text{on } \partial\Omega_D \end{cases}$	hard devices
$\begin{cases} \boldsymbol{b} \neq 0 & \text{in } \Omega \\ \boldsymbol{f} \neq 0 & \text{on } \partial\Omega_N \\ \bar{\boldsymbol{u}} \neq 0 & \text{on } \partial\Omega_D \end{cases}$	mixed devices

Table 1.1: Classification of the possible external imputs.

The total potential energy then reads

$$\mathcal{P} = \mathcal{P}(t, \boldsymbol{u}, \boldsymbol{\alpha}) = \mathcal{E}(t, \boldsymbol{u}, \boldsymbol{\alpha}) - \mathcal{L}(t, \boldsymbol{u}) \tag{1.53}$$

For what concerns the dissipation distance density d, one assumes:

(i) $\forall\, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3 \in Z : d(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_3) \leq d(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) + d(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \quad \forall \boldsymbol{x} \in \Omega;$

(ii) $\forall\, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in Z : d(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = 0 \Longleftrightarrow \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2, \quad \forall \boldsymbol{x} \in \Omega;$

(iii) $d : Z\times Z \to \mathbb{R}^+_\infty$ is lower semicontinuous.

Condition (i) is simply the subadditivity property or triangle inequality, condition (ii) is the identity of indiscernibles, or coincidence axiom while condition (iii) allow the dissipation distance to achieve the value ∞. Assumptions (i) and (ii) together

[6] It is worth nothing the duality with the standard generalized materials framework in the introduction of section 1.2.

assert that the dissipation distance d is non-symmetric, $\forall\, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in Z \quad d(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \neq d(\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1)$. The non-symmetric property permits to consider variables which are constrained to an irreversibility condition, such as damage. The dissipation distance is hence an extended quasi-distance.[7] It is convenient sometimes to explicitly consider the irreversibility condition and in addition to assume the symmetry property for the dissipation distance.

The dissipation distance d can be as well introduced through a *dissipation potential* $\psi : TZ \to \mathbb{R}^+_\infty$, which is a Finslerian metric[8], (Finsler 1918); (Bao, Chern, and Shen 2000), such that

$$d(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1) = \inf \left\{ \int_0^1 \psi(\boldsymbol{\beta}(s), \dot{\boldsymbol{\beta}}(s))\, \mathrm{d}s : \boldsymbol{\beta} \in \mathrm{C}^1(\mathcal{Z}, [0,1]), \boldsymbol{\beta}(0) = \boldsymbol{\alpha}_0, \boldsymbol{\beta}(1) = \boldsymbol{\alpha}_1 \right\}. \tag{1.54}$$

In the following, the formulation through the dissipation potential will be preferred for its connection with the classical approaches. The assumptions for the dissipation potential are then

(a) $\forall\, \lambda \in [0,1], \forall\, (\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}_1), (\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}_2) \in TZ$:

$$\psi(\boldsymbol{\alpha}, \lambda\dot{\boldsymbol{\alpha}}_1 + (1-\lambda)\dot{\boldsymbol{\alpha}}_2) \leq \lambda\psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}_1) + (1-\lambda)\psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}_2);$$

(b) $\forall\, (\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) \in T\mathcal{Z}$, $\psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) \geq 0$ and $\psi(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) = 0 \Longleftrightarrow \dot{\boldsymbol{\alpha}} = 0$;

(c) $\psi : TZ \to \mathbb{R}^+_\infty$ is lower semicontinuous.

Condition (a) expresses the convexity property. Instead conditions (b) expresses the non-negativeness of the dissipation potential and property (1.30). It is straightforward to check that such hypotheses satisfy the requirements (i), (ii) and (iii) for the dissipation distance density mentioned above.

The dissipation distance D defined over all the domain Ω is hence given by

$$D = D(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1) = \int_\Omega d(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1)\, \mathrm{d}\Omega. \tag{1.55}$$

Moreover, one is now able to define the *dissipated work* or *dissipated energy* D along an arbitrary path $\boldsymbol{\alpha} : [0, T] \to \mathcal{Z}$ as

$$\mathcal{D}_D(\boldsymbol{\alpha}, [s,t]) = \sup \left\{ \sum_{j=1}^{n} D(\boldsymbol{\alpha}(t_{j-1}), \boldsymbol{\alpha}(t_j)) \mid n \in \mathbb{N}, s \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq t \right\}. \tag{1.56}$$

Equation 1.56 can be regarded as the length needed to reach a state $\boldsymbol{\alpha}(t)$ from an initial state $\boldsymbol{\alpha}(0)$ through the curve $\boldsymbol{\alpha}(s)$.

For further developments, in case of local models, the introduction of a dissipated energy density δ could be useful. In particular

$$\delta(\boldsymbol{\alpha}, [s,t]) = \sup \left\{ \sum_{j=1}^{n} d(\boldsymbol{\alpha}(t_{j-1}), \boldsymbol{\alpha}(t_j)) \mid n \in \mathbb{N}, s \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq t \right\} \tag{1.57}$$

[7] *Extended* because infinite value can attained and *quasi* because of the lack of symmetry.

[8] TZ stands for the tangent space

so that

$$\mathcal{D}_D(\boldsymbol{\alpha}, [s,t]) = \int_\Omega \delta(\boldsymbol{\alpha}, [s,t])\ \mathrm{d}\Omega. \tag{1.58}$$

In case of smooth evolutions, the dissipated energy turns to be equal to

$$\mathcal{D}_D(\boldsymbol{\alpha}, [s,t]) = \int_\Omega \int_s^t \psi(\boldsymbol{x}, \boldsymbol{\alpha}(\tau), \dot{\boldsymbol{\alpha}}(\tau))\, \mathrm{d}\tau\ \mathrm{d}\Omega. \tag{1.59}$$

Observe that the dissipated energy owns the additive property, that is

$$\mathcal{D}_D(\boldsymbol{\alpha}, [r,t]) = \mathcal{D}_D(\boldsymbol{\alpha}, [r,s]) + \mathcal{D}_D(\boldsymbol{\alpha}, [s,t]), \quad \forall\, r < s < t. \tag{1.60}$$

Now that all ingredients of the energetic formulation has been established, one is able to introduce the two key conditions on which the energetic formulation relies on: the stability condition and the energy balance.

1.3.2 Stability condition

Different versions of the Stability condition could be given depending on the topology richness and smoothness properties of both the functionals and evolution, namely

- global stability condition;
- local stability condition through the introduction of a convenient topology;
- differential stability condition.

Of course under suitable convex and regularity properties these different versions equals each other.

1.3.2.1 Global stability condition

A process $(\boldsymbol{u}, \boldsymbol{\alpha}) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ satisfies the *global stability condition* if, for all $t \in [0,T]$, the following condition holds:

$$\mathcal{P}(t, \boldsymbol{u}, \boldsymbol{\alpha}) \leq \mathcal{P}(t, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) + D(\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}}) \qquad \forall\, (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) \in \mathcal{F} \times \mathcal{Z} \tag{\textbf{ST}}$$

where $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}})$ is a different state. Clearly, the global stability condition (**ST**) does not request neither a topology over the state spaces nor regularity properties of the energy functional. Therefore, the global stability is the weakest and more general condition, able to deal with discontinuous evolutions and irregular functional.

1.3.2.2 Local stability condition

As pointed out in Mielke 2006a, p. 354, although (**ST**) results in great flexibility, an inconvenience is that it involves a global minimisation while a local minimisation would be more physical. Nevertheless, in order to introduce a local condition and to express "locality" one needs to specify a topology such that a neighbourhood of the candidate solution state can be defined. From (**ST**), a local minimization can be expressed as

$$\mathcal{P}(t, \boldsymbol{u}, \boldsymbol{\alpha}) \leq \mathcal{P}(t, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) + D(\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}}), \quad \forall\, (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) \in \mathcal{F} \times \mathcal{Z} \colon \|\, (\boldsymbol{u}, \boldsymbol{\alpha}) - (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}})\, \| < \delta \tag{\textbf{st}}$$

with $\delta > 0$ and where $\|\cdot\|$ denotes a suitable norm.
One could observe that in an infinite vector space, not all norms are equivalents. Hence, the local states of (**st**) could be different, depending on the choice of the norm $\|\cdot\|$.

Motivated by the previous remark and for further developments, a *directional local stability* condition is defined as follows. At every instant t, a state $(\boldsymbol{u}, \boldsymbol{\alpha})$ is *directionally stable*, if for every available direction[9] exists a different state with an equal or greater energy level. To put it better, the state $(\boldsymbol{u}, \boldsymbol{\alpha}) \in \mathcal{F} \times \mathcal{Z}$ is directionally stable if

$$\forall\ (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) \in \widetilde{\mathcal{F}} \times \widetilde{\mathcal{Z}},\ \exists\, \bar{h} > 0 \ \mid\ \forall\, h \in \left[0, \bar{h}\right],$$

$$\mathcal{P}(t, \boldsymbol{u}, \boldsymbol{\alpha}) \leq \mathcal{P}(t, \boldsymbol{u} + h\widetilde{\boldsymbol{u}}, \boldsymbol{\alpha} + h\widetilde{\boldsymbol{\alpha}}) + D(\boldsymbol{\alpha}, \boldsymbol{\alpha} + h\widetilde{\boldsymbol{\alpha}})\,. \qquad \textbf{(st-d)}$$

which relies on the usual $\mathbb{R}$-norm. In (**st-d**) $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{Z}}$ are respectively the function space of the displacement variations defined as $\widetilde{\mathcal{F}} := \{\boldsymbol{a} \in \mathcal{F} : \boldsymbol{a} = \boldsymbol{0} \text{ on } \partial\Omega_D\}$ while $\widetilde{\mathcal{Z}}$ is the function space of internal variables variations which may embed irreversibility conditions.
The directional stability condition (**st-d**) is more suitable compared to (**st**) when an explicit irreversibility condition and a symmetric dissipation distance is given. Indeed, in that case, it is simpler to define available directions as shown in (1.4.3).

1.3.2.3 Differential or n-order stability condition

For sufficiently regular energy functionals, namely directional derivable or Gateaux differentiable, the condition (**st-d**) and (**st**) becomes equivalent to n-th differential stability conditions.

Indeed, by expanding the right-hand side of (**st-d**) around $h = 0$ and without expressing explicitly the evolution variable t, one obtains

$$\begin{aligned}\mathcal{P}(\boldsymbol{u} + h\widetilde{\boldsymbol{u}}, \boldsymbol{\alpha} + h\widetilde{\boldsymbol{\alpha}}) + D(\boldsymbol{\alpha}, \boldsymbol{\alpha} + h\widetilde{\boldsymbol{\alpha}}) = \\ \mathcal{P}(\boldsymbol{u}, \boldsymbol{\alpha}) + \left(\mathcal{P}'(\boldsymbol{u}, \boldsymbol{\alpha})(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) + D'(\boldsymbol{\alpha})(\widetilde{\boldsymbol{\alpha}})\right) h + \\ + \frac{1}{2}\left(\mathcal{P}''(\boldsymbol{u}, \boldsymbol{\alpha})(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) + D''(\boldsymbol{\alpha})(\widetilde{\boldsymbol{\alpha}})\right) h^2 + \mathrm{w}(h^2).\end{aligned} \qquad (1.61)$$

Condition (**st-d**) then becomes

$$\begin{aligned}0 \leq \left(\mathcal{P}'(\boldsymbol{u}, \boldsymbol{\alpha})(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) + D'(\boldsymbol{\alpha})(\widetilde{\boldsymbol{\alpha}})\right) h + \\ + \frac{1}{2}\left(\mathcal{P}''(\boldsymbol{u}, \boldsymbol{\alpha})(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) + D''(\boldsymbol{\alpha})(\widetilde{\boldsymbol{\alpha}})\right) h^2 + \mathrm{w}(h^2)\end{aligned} \qquad (1.62)$$

for every sufficiently small h. In (1.62), $\mathcal{P}'(\alpha)(\beta)$ is the Gateaux derivative of $\mathcal{P}'$ at α with respect to the direction β while $D'(\alpha)(\beta) = \lim_{h \to 0}(1/h)\left(D(\alpha, \alpha + h\beta)\right)$.
Condition (1.62) expresses two differential inequalities since the expansion has ben taken up to the second order.
Then, from (1.62), a necessary (sufficient) condition for the stability of the state $(\boldsymbol{u}, \boldsymbol{\alpha}) \in \mathcal{F} \times \mathcal{Z}$, in sense of (**st-d**), is the *first order stability* condition

$$\mathcal{P}'(\boldsymbol{u}, \boldsymbol{\alpha})(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}) + D'(\boldsymbol{\alpha})(\widetilde{\boldsymbol{\alpha}}) \geq (>)\, 0. \qquad \textbf{(st-1D)}$$

[9] The available directions embed explicitly the irreversibility condition.

Clearly, the condition (**st-1D**) becomes sufficient for the global stability (**ST**) in case of opportune convex properties.
When (**st-1D**) occurs with an equality, the sign of the second order term in (1.62) has to be inspected. In this case, the *second order stability* condition states that a necessary (sufficient) condition for stability becomes

$$\mathcal{P}''(\boldsymbol{u},\boldsymbol{\alpha})(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{\alpha}}) + D''(\boldsymbol{\alpha})(\widetilde{\boldsymbol{\alpha}}) \geq (>)\, 0. \qquad \textbf{(st-2D)}$$

Furthermore, if condition (**st-2D**) is satisfied as an equality, the sign of higher order terms, descending from (1.61), have to be taken into account. One then has to consider n-differential inequalities up to the first strictly positive inequality.

As one will appreciate later, the second order stability condition becomes fundamental for ascertain bifurcations in the response of the system. From (**st-2D**) the model internal length and its role in the definition of stable bifurcated branches will after emerge.

With opportune caution it can be stated that these stability conditions are increasingly weak, in sense that

$$\text{global stability} \Longrightarrow \text{local stability} \Longrightarrow \text{first order stability}. \qquad (1.63)$$

The converse is of course not always true.

1.3.3 Energy balance

The other key concept in the energetic formulation is the energy balance. As for the stability condition, two versions of the energy balance could be given depending on the regularity assumption:

- (global) energy balance;
- regular (or local or differential) energy balance.

1.3.3.1 Weak energy balance

Recalling the definition of the dissipated work (1.56) or (1.59), the *energy balance* in its most general form states

$$\mathcal{P}(t,\boldsymbol{u},\boldsymbol{\alpha}) + \mathcal{D}_D(\boldsymbol{\alpha},[0,t]) = \mathcal{P}(t,\boldsymbol{u}_0,\boldsymbol{\alpha}_0) + \int_0^t \partial_t \mathcal{P}(t,\boldsymbol{u})\ \mathrm{d}\tau \qquad \textbf{(EB)}$$

where

$$\partial_t \mathcal{P}(t,\boldsymbol{u}) = \int_\Omega \dot{\boldsymbol{b}}\cdot\boldsymbol{u}\ \mathrm{d}\Omega + \int_{\partial\Omega_N} \dot{\boldsymbol{f}}\cdot\boldsymbol{u}\,\mathrm{dS}. \qquad (1.64)$$

Condition (**EB**) is essentially the mechanical form of the first law of thermodynamics with internal variable (1.19) descending from (1.6).

1.3.3.2 Regular energy balance

Under regularity assumptions on both the evolution of the state variables and the energy functionals, the energy balance condition **EB** turns into the following differential form. Differentiating **EB** with respect to time

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{P}(t,\boldsymbol{u},\boldsymbol{\alpha}) + \mathcal{D}_D(\boldsymbol{\alpha},[0,t]) - \mathcal{P}(t,\boldsymbol{u}_0,\boldsymbol{\alpha}_0) - \int_0^t \partial_t \mathcal{P}(t,\boldsymbol{u},\boldsymbol{\alpha})\ \mathrm{d}\tau\right) = 0, \qquad (1.65)$$

one obtains

$$\frac{\partial \mathcal{P}(t,\boldsymbol{u},\boldsymbol{\alpha})}{\partial \boldsymbol{u}} \cdot \dot{\boldsymbol{u}} + \frac{\partial \mathcal{P}(t,\boldsymbol{u},\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \cdot \dot{\boldsymbol{\alpha}} + \int_\Omega \psi(\boldsymbol{\alpha},\dot{\boldsymbol{\alpha}})\ \mathrm{d}\Omega + \partial_t \mathcal{P}(t,\boldsymbol{u}) = 0, \quad \forall\, t. \tag{1.66}$$

1.3.4 Irreversible phenomena

For irreversible phenomena, irreversibility can be moved from the definition of the dissipation potential to the definition of the admissible function spaces as a restriction of the accessible states.
A typical dissipation potential for a homogeneous material and for a process with a kind of irreversibility is defined as $\psi(\boldsymbol{\alpha},\dot{\boldsymbol{\alpha}}) : Z \times Z \to \mathbb{R}^+_\infty$,

$$\psi(\boldsymbol{\alpha},\dot{\boldsymbol{\alpha}}) = \begin{cases} < +\infty & \text{if } \ \dot{\alpha}_i \geq 0 \\ +\infty & \text{if } \ \dot{\alpha}_i < 0 \end{cases} \qquad \forall\ \boldsymbol{\alpha},\dot{\boldsymbol{\alpha}}\ \in\ Z \tag{1.67}$$

where the reference to $\dot{\alpha}_i$ with $i \leq m$ means that irreversibility is prescribed only on some components of the internal variable vector. If for an evolution there exists an instant where $\dot{\alpha}_i < 0$, then the energy balance could not be satisfied since the dissipated work goes to infinity.

The same result can be achieved moving the irreversibility condition on the function space and the space of admissible variations, namely (1.67) can be replaced by

$$\psi(\boldsymbol{\alpha},\dot{\boldsymbol{\alpha}}) : Z \times Z_+ \to \mathbb{R}^+ \tag{1.68}$$

where $Z_+ = \{\boldsymbol{\beta} \in Z : \beta_i \geq 0\}$ becomes the space of available rates or admissible variations for the internal states variables.

Henceforth this kind of approach will be followed being more suitable for damage problems as shown in subsection 1.4.3.

For gradient internal variables the presented energetic formulation changes only slightly in its formalism but not in its intimate meaning.

A proof of the equivalence between energetic formulation and classic formulation could be found in Mielke 2003 and Engelen, Geers, and Baaijens 2003 at least for some meaningful material models.

1.4 Fracture, Plasticity and Damage: local and non-local models

This section is devoted to a brief introduction to fracture, plasticity and damage models. Particular emphasis is given on non-local models. These models are the building blocks and the main ingredients of the next developed coupled model, object of this survey.

In order to be able to take into account size-effects in phenomena like fracture, plasticity and damage, an internal length has to be introduced in the constitutive model as already discussed in the Summary. To this aim, several approaches are possible. Moreover, in case of stress softening materials, the introduction of an internal length scale becomes also necessary to overcome the ill-posedeness of the associated boundary value problem.

1.4.1 Fracture models

In continuum mechanics, *fracture* means a (strong[10]) discontinuity of the displacement field in some parts of the domain.

A proper description of the complex cracking phenomena observed in a wide range of materials and structures demands to correctly deal with complex crack geometries, crack nucleation, and evolution, which eventually may turn out to be non-smooth in space-time, Summary.

1.4.1.1 Classical formulations

These aforementioned issues (crack geometries, crack nucleation, crack evolution) are known to be hard difficulties for the classical approach to fracture. Mostly, classical theory relies on the concepts of stress singularity and stress intensity factors K_I, K_{II} and K_{III}, see Maugin 1992. Several efforts have been made in the framework of the classical approach since the pioneering works of Griffith without a complete success.

Few are the models which attempt to face the issue of the crack path. Generally, the crack path is prescribed and the crack evolution is governed by a scalar parameter, say ℓ, the length of the crack.

brittle fracture The most simple fracture propagation criterion is due to Irwin and Kries Irwin and Kries 1951 which reads

$$\begin{cases} \text{if} \quad K < K_c, & \dot{\ell} = 0 \quad \text{no propagation} \\ \text{if} \quad K = K_c, & \dot{\ell} \geq 0 \quad \text{the crack might evolve (also } \dot{\ell} = 0 \text{ is admissible)} \end{cases} \tag{1.69}$$

where K_c represent the *tenacity* or *fracture toughness* of the material and K a stress intensity factor.

The more important criterion for the developments of the fracture theory is undoubtedly Griffith's criterion, Griffith 1921. Once the *energy release rate* has been defined, Maugin 1992, as

$$G = -\frac{\partial \mathcal{P}}{\partial \ell} \tag{1.70}$$

the criterion states

$$\begin{cases} \text{if} \quad G < G_c, & \dot{\ell} = 0 \quad \text{no propagation} \\ \text{if} \quad G = G_c, & \dot{\ell} \geq= 0 \quad \text{the crack might evolve (also } \dot{\ell} = 0 \text{ is admissible)} \end{cases} . \tag{1.71}$$

where G_c represents the surface fracture energy density. A relation between the energy release rate and the stress intensity factors can be established. A well known drawback of Griffith's theory is the inability to describe crack initiation. This drawback can be overcome by considering a cohesive fracture model.

[10]The word strong is opposed to the word weak, the former refers to a discontinuity of a function while the latter to its derivative

cohesive fracture In a cohesive fracture model, the surface fracture energy ϕ is not immediately all released at the occurrence of a crack but it depends on the amplitude of the crack itself, $\phi = \phi(\llbracket u \rrbracket)$. Two models are remarkable, Barenblatt's model, see Barenblatt 1962, and Dugdale's model, see Dugdale 1960. In Barenblatt's model the surface fracture energy monotonically increases with the crack opening up to an asymptotic finite value,

$$\phi(0) = 0, \qquad \phi(\llbracket u \rrbracket) \geq 0 \quad \text{when} \quad \llbracket u \rrbracket > 0, \qquad \phi(\infty) = G_c. \tag{1.72}$$

Instead the surface fracture energy in Dugdale's model is a piecewise linear function where the second line starts for a finite value of the crack opening $\llbracket u \rrbracket_c$ and is constant with value G_c. In formulas

$$\phi(\llbracket u \rrbracket) = \begin{cases} \dfrac{G_c}{\llbracket u \rrbracket_c} \llbracket u \rrbracket, & \llbracket u \rrbracket \leq \llbracket u \rrbracket_c \\ G_c, & \llbracket u \rrbracket > \llbracket u \rrbracket_c \end{cases} \tag{1.73}$$

1.4.1.2 Variational formulations

The advantages of a variational formulation has been already discussed. The translation of the brittle fracture problem into a variational setting is due to Francfort and Marigo, see Francfort and Marigo 1998. A main convenience with respect to Griffith's theory is that in the variational approach the fracture onset and evolution are obtained without assuming any pre-existing defect, any pre-destined rupture surface in the body, and without any ad hoc evolution criterion for the fracture. The core of such formulation is the capability to regularise the fracture problem, a mathematical free-discontinuity problem, into a more simple one where a new variable, damage like, is introduced but without the occurrences of discontinuities in its trend. In the following the main steps leading to the regularised formulation are summarised. Although a wide literature exists an exhaustive survey can be found in Bourdin 2000; Bourdin, Francfort, and Marigo 2008, the approximation idea in Ambrosio and Tortorelli 1990, the discretisation and quasi-static evolution in Giacomini 2005. Several details are omitted since the aim is only to give an overview and the main ideas of the subject.

Griffith revisited The first step is a revisitation of Griffith's criterion in a three-dimensional variational framework based on a irreversibility condition, a stability condition and a energy balance. Having introduced Griffith's criterion (1.71), the variational formulation reads

$$\begin{aligned} \text{irreversibility:} \quad & \dot{\ell}(t) \geq 0, \\ \text{stability:} \quad & G(t) \leq G_c, \\ \text{energy balance:} \quad & (G(t) - G_c)\, \dot{\ell}(t) = 0. \end{aligned} \tag{1.74}$$

Fracture as free-discontinuity problem The problem was regarded in Francfort and Marigo 1998 as a mathematically so called free-discontinuity problem. The main idea was to abolish the path constraint of Griffith's formulation and to reformulate the problem as a time discrete minimization problem where both the displacement field $\boldsymbol{u}$ and the fracture path Γ are unknowns. Without entering into details, the problem becomes to find, for a time step t, $\boldsymbol{u}_t$ and Γ_t as minimizer $(\boldsymbol{u}, \Gamma)$ of

$$\min \left\{ \int_{\Omega\setminus\Gamma} e(\nabla \boldsymbol{u})\,\mathrm{d}\Omega - \mathcal{L}(t,\boldsymbol{u}) + \mathcal{H}^{n-1}(\Gamma) : \Gamma \subset \Gamma_{t-1} = \overline{\Gamma}_{t-1} \in \overline{\Omega}, \right.$$
$$\left. \boldsymbol{u} \in \mathrm{H}^1(\Omega\setminus\Gamma)\,,\ \boldsymbol{u} = g(t) \ \text{ on } \partial\Omega_D \setminus \Gamma \right\}. \tag{1.75}$$

This problem is often called *strong variational evolution*, Bourdin, Francfort, and Marigo 2008.

Change of function space (SBD) The strong variational evolution is still too hard to be resolved with ease. Through various contributions, Ambrosio, Fusco, and Pallara 2000 for a review, it has been proven that the the strong variational evolution problem is equivalent to a *weak variational evolution* problem once the function space SBV[11] has been introduced . That is, $\boldsymbol{u}_t$ is a $\boldsymbol{u}$ of

$$\min \left\{ \int_{\Omega\setminus\Gamma_{t-1}} e(\nabla \boldsymbol{u})\,\mathrm{d}\Omega - \mathcal{L}(t,\boldsymbol{u}) + \mathcal{H}^{n-1}(S(\boldsymbol{u}) \cap (\Omega\setminus\Gamma_{t-1})) \right.$$
$$\left. \boldsymbol{u} \in \mathrm{SBD}(\Omega\setminus\Gamma_{t-1})\,,\ \boldsymbol{u} = g(t) \ \text{ on } \partial\Omega_D \setminus (S(\boldsymbol{u}) \cup \Gamma_{t-1}) \setminus \Gamma \right\} \tag{1.76}$$

where $S(\boldsymbol{u})$ is the set of Lebesgue points of $\boldsymbol{u}$ and $\Gamma_t = \Gamma_{t-1} \cup \overline{S(\boldsymbol{u})}$. Roughly speaking the main difference between the strong and the weak problem is to have transfer the singularity from the domain to the displacement function. Apparently this effort seems not of value. Actually it is crucial since it allows the use an important regularization result.

Regularization of the problem through Γ-convergence The weak problem (1.75) can be then approximate[12] by an Γ-converging elliptic functional, say $\mathcal{F}_\varepsilon(\boldsymbol{u}, \alpha)$. The involved elliptic functional is

$$\mathcal{F}_\varepsilon(\boldsymbol{u},\alpha) = \int_\Omega \left(\left(1-\alpha^2\right) + k_\varepsilon \right) e(\nabla \boldsymbol{u})\,\mathrm{d}\Omega - \mathcal{L}(t,\boldsymbol{u}) + G \int_\Omega \left(\frac{\alpha^2}{4\varepsilon} + \varepsilon \|\nabla\alpha\| \right) \mathrm{d}\Omega. \tag{1.77}$$

This form allows for efficient numerical implementation and an interesting mechanical interpretation, that is α can be regarded as a damage variable, Sicsic and Marigo 2012.

1.4.2 Plasticity models

Plasticity has been in the past, (and is still often today) the driving force and source of mathematical material modelling. For this reason an extreme rich and a wide literature as well as numerous monographs exists about this topic. For the aim of this work the worth noting references are:

- Lubliner 2008 for plasticity phenomenology and its modelling;

[11] The space of bounded variations

[12] The approximation stands for the convergence of minima and minimizers as a small parameter ε tends towards 0

- Fuchs and Seregin 2000; Han and Reddy 1999; Nguyen 2000; Temam 1985 for a mathematical overview of the problem;
- Simo and Hughes 1998 for the numerical implementation.

From the didactic point of view, plasticity is often presented following Sec. 1.2.4.1 where a yield surface is first introduce. Of course, different choices are possible. Nevertheless, for the developments of the next chapters it suffices to introduce only the well-known Von Mises yield criterion and some energetic quantities.

The *Von Mises* yield function depends on the deviatoric part of the stress and reads for a perfect plastic model

$$f_p = f_p(\boldsymbol{\sigma}_d) := \|\boldsymbol{\sigma}_d\| - \sqrt{\frac{2}{3}}\sigma_{\mathsf{P}} \leq 0. \tag{1.78}$$

where σ_{P} is a positive constant value which represents the plastic yield stress. Moreover, in order to take isotropic and kinematic linear hardening effects into account it suffices respectively to add the term $-\mathsf{K}\,\boldsymbol{\varepsilon}_p$ in the norm and $-\sqrt{2}\mathsf{I}\,p/\sqrt{3}$ in the expression (1.78). The yield function then becomes $f_p = f_p\big(\boldsymbol{\varepsilon}_p, p, \boldsymbol{\sigma}_d\big)$. In this last expression $\mathbb{R}^+ \ni p = \int_0^t \|\dot{\boldsymbol{\varepsilon}}_p\| \,\mathrm{d}\tau$ represents the accumulated plastic strain.

In the perfect plasticity model, the displacement function space belongs to the special bounded deformations space, namely

$$\boldsymbol{u}(\boldsymbol{x},t) \in \mathcal{F} \equiv \mathrm{SBD}((\Omega,[0,T])\,,F)\,. \tag{1.79}$$

The space of special bounded deformations SBD coincides, for $d = 1$, with SBV. More details about this function space can be found in Simo, Oliver, and Armero 1993; Suquet 1981.
Focusing on the infinitesimal strain gradient theory, the additive decomposition of the total strain is commonly accepted,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{el} + \boldsymbol{\varepsilon}_p. \tag{1.80}$$

The next chapter extends the discussion about the function spaces.

The stored elastic energy, (1.51), is defined as

$$\mathcal{E}\big(\boldsymbol{u},\boldsymbol{\varepsilon}_p\big) = \int_0^{\mathrm{L}} e\big(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}_p\big)\,\mathrm{d}x = \int_0^{\mathrm{L}} \frac{1}{2}\mathbb{C}_0\big[\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_p\big] : \big(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_p\big)\,\mathrm{d}x + \mathsf{H}\big(\boldsymbol{\varepsilon}_p,p\big)\,. \tag{1.81}$$

where $\mathsf{H}\big(\boldsymbol{\varepsilon}_p,p\big)$ corresponds to hardening (or softening) effects. In particular, a *linear isotropic hardening* and a *linear kinematic hardening* correspond respectively to $\mathsf{H}\big(\boldsymbol{\varepsilon}_p,p\big) = \frac{1}{2}\,\mathsf{I}\,p^2$ and $\mathsf{H}\big(\boldsymbol{\varepsilon}_p,p\big) = \frac{1}{2}\,\mathsf{K}\,\boldsymbol{\varepsilon}_p : \boldsymbol{\varepsilon}_p$. The tensor $\mathbb{C}_0$ is the 4-th order elastic tensor which for isotropic linear elastic materials reads

$$\mathbb{C}_0[\boldsymbol{\varepsilon}] = \lambda\,\mathrm{tr}\,\boldsymbol{\varepsilon}\,\mathbb{I} + 2\mu\,\boldsymbol{\varepsilon} \tag{1.82}$$

with λ and μ the Lamé parameters. On the other hand, an example of the dissipation potential (Von Misses model) is

$$\psi\big(\dot{\boldsymbol{\varepsilon}}_p\big) = \sqrt{\frac{2}{3}}\sigma_{\mathsf{P}}\|\dot{\boldsymbol{\varepsilon}}_p\| \tag{1.83}$$

where no hardening effects has been taken into account. The dissipated work in a time interval $[0,t]$ is then given, in case of sufficiently continuous evolutions, by (1.59):

$$\mathcal{D}_D(\varepsilon_p,[0,t]) = \sqrt{\frac{2}{3}}\sigma_{\mathrm{P}} \int_0^t \int_\Omega \|\dot{\varepsilon}_p\| \;\mathrm{d}\Omega\;\mathrm{d}\tau = \sqrt{\frac{2}{3}}\sigma_{\mathrm{P}} \int_\Omega p \;\mathrm{d}\Omega. \tag{1.84}$$

In the framework of the energetic formulation plasticity has been widely exploited, Dal Maso, De Simone, and Mora 2006; Francfort and Giacomini 2011; Mielke, Rossi, and Savaré 2008; Mielke, Rossi, and Savaré 2012.

Besides local models, many non-local models exits. Some references are given in Sec. 1.4.5 and in Tab. 1.2.

1.4.3 Damage models

As well as plasticity many research efforts has been spent in the understanding and modelling of damage. As already discussed in the Summary the damage phenomenology is mostly related to the nucleation of microvoids and microcracks. Kachanov 1986; Krajcinovic 1996; Lemaitre and Chaboche 1985 represent surveys about damage mechanics while Krajcinovic and Mastilovic 1995 highlights some fundamental issues as the parameter choice (scalar, second-, fourth-, ecc. order tensor) for the damage modelling.
Moreover, Lemaitre's description and point of view has been determinant for the interpretation of the damage phenomenon. The *strain equivalence hypothesis* play a crucial role in the definition of the damage variable. The strain equivalence hypothesis states that the strain associated with a damage state under the applied stress is equivalent to the strain associated with its undamaged state under the effective stress. This means that the constitutive equations of a damaged material are the same of those of the virgin one with no damage where the stress is simply replaced by the effective stress, Bonora 1997.

Hereafter damage is described only through a single scalar variable just for simplicity and Kachanov's phenomenlogical approach is followed. An extension to more complex physical descriptions do not represent a conceptual obstacle.
Different choices are possible for the range of the damage variable even though all choices become equivalent, through a change of variable, see Pham and Marigo 2010a. Hence, the scalar damage variable α is here chosen such that

$$\alpha \in [0,1]\,. \tag{1.85}$$

The state $\alpha = 0$ corresponds to a sound material while the state $\alpha = 1$ corresponds to a full damaged material where a loss of stiffness occurs. Similarly to plasticity, in a classical approach a yield stress function is often postulated,

$$f_d = f_d(\sigma,\alpha) := g(\sigma) - \sigma_{\mathrm{D}}(\alpha) \leq 0 \tag{1.86}$$

where $g(\sigma)$ is the scalar equivalent stress and $\sigma_{\mathrm{D}}(\alpha)$ the yield stress possibly depending on damage. Since a material capable to restore its internal structure during the evolution is excluded, an irreversibility condition is explicitly assumed for the damage variable, that is

$$\alpha(t_1) \leq \alpha(t_2)\,, \qquad \forall\, t_1 < t_2. \tag{1.87}$$

The stored elastic energy, (1.51), is defined as

$$\mathcal{E}(\boldsymbol{u},\alpha) = \int_0^{\mathrm{L}} e(\boldsymbol{\varepsilon},\alpha)\ \mathrm{d}x = \int_0^{\mathrm{L}} \frac{1}{2}\mathbb{C}(\alpha)[\boldsymbol{\varepsilon}] : \boldsymbol{\varepsilon}\ \mathrm{d}x, \tag{1.88}$$

where $\mathbb{C}(\alpha)$ is the 4-th order elastic tensor whose components depends on the damage variable. Different damage models are possible depending on the degradation of the elastic tensor. In the most simple case, that is of an isotropic material with an isotropic damage,

$$\mathbb{C}(\alpha) = f(\alpha)\,\mathbb{C}_0, \tag{1.89}$$

where $f(\alpha) : C \to [0,1]$. It is then reasonable to describe a typical mechanical behaviour of a damaging material by requiring

$$f(0) > 0, \qquad f'(\alpha) < 0,\ \forall\, \alpha \in [0,1)\,, \qquad f(1) = f'(1) = 0. \tag{1.90}$$

On the other hand, a simple example of dissipation potential is

$$\psi(\alpha,\dot{\alpha}) = \partial_t\, \mathsf{w}(\alpha)\,. \tag{1.91}$$

In (1.91), the function $\mathsf{w}(\alpha)$ is defined as $\mathsf{w}(\alpha) : C \to \mathbb{R}^+_\infty$. Motivated by physical reasons as the fact for a positive dissipated power, it is assumed

$$\mathsf{w}(0) = 0, \qquad \mathsf{w}'(\alpha) > 0,\ \forall\, \alpha \in [0,1)\,, \qquad \mathsf{w}(1) < +\infty. \tag{1.92}$$

This allows the dissipated work (1.59) in a time interval $[0,t]$, in case of sufficiently continuous evolutions, to simply be

$$\mathcal{D}_D(\alpha,[0,t]) = \int_0^t \int_\Omega \partial_t\, \mathsf{w}(\alpha)\ \mathrm{d}\Omega\ \mathrm{d}\tau = \int_\Omega \mathsf{w}(\alpha(\boldsymbol{x},t))\ \mathrm{d}\Omega \tag{1.93}$$

where one has assumed $\mathsf{w}(\alpha(\boldsymbol{x},0)) = 0$.

In the model proposed by Marigo, see Pham and Marigo 2010c, which is part of the variational formulation, the explicit dependence of the damage gradient is assumed for the dissipation potential,

$$\psi(\alpha,\nabla\alpha,\dot{\alpha},\nabla\dot{\alpha}) = \partial_t\, \mathsf{w}(\alpha) + \partial_t \left(\frac{1}{2}\eta^2(\alpha)\,\nabla\alpha\cdot\nabla\alpha\right). \tag{1.94}$$

1.4.4 Uniqueness, stability and bifurcation

Before the introduction of non-local models, it is worth to introduce the key concepts of uniqueness, stability and bifurcation in the infinitesimal strain theory. The matter is very complex and detailed. A general discussion can be found in Bigoni and Zaccaria 1992; Ottosen and Ristinmaa 2005 concerning elasto-plasticity. The global *uniqueness* condition is

$$\int_\Omega \Delta\dot{\boldsymbol{\varepsilon}} : \Delta\dot{\boldsymbol{\sigma}}\ \mathrm{d}\Omega > 0 \tag{1.95}$$

Clearly, a necessary condition for a unique response is that the constitutive tensor $\mathcal{C}$, possibly inelastic, is positive definite, that is

$$\dot{\boldsymbol{\varepsilon}} : \mathcal{C}[\dot{\boldsymbol{\varepsilon}}] > 0, \qquad \forall\dot{\varepsilon}\ \text{adm.} \tag{1.96}$$

which is equivalent to requiring that the determinant of the symmetric part of the constitutive tensor is positive,

$$\det \mathcal{C}_{\text{sym}} > 0. \tag{1.97}$$

On the other hand, when uniqueness is lost, more solutions are possible and apart from the fundamental solution, *bifurcations* may occur. These bifurcations can manifest themselves in terms of *continuous bifurcations* or *discontinuous bifurcations* (*strain localizations*).

Continuous bifurcations, like necking in plasticity, are associated to loss of *strong ellipticity* of the boundary value problem. Strong ellipticity means

$$\boldsymbol{n} \cdot \mathcal{C}[\boldsymbol{g} \otimes \boldsymbol{n}]\, \boldsymbol{g} > 0, \qquad \forall\, \boldsymbol{n}, \boldsymbol{g} \tag{1.98}$$

where $\boldsymbol{n}$ denotes the normal to the surface of discontinuity and $\boldsymbol{g} = [\![\dot{\boldsymbol{u}}_{,\boldsymbol{n}}]\!]$ the jump of the normal derivative of $\dot{\boldsymbol{u}}$. Condition (1.98) equivalently corresponds to the loss of positive definiteness of the symmetric part of the acoustic tensor, namely

$$\det \mathcal{A}_{\text{sym}} > 0 \tag{1.99}$$

where $\mathcal{A}_{\text{sym}}$ is the symmetric part of the *acoustic tensor* $\mathcal{A} = \boldsymbol{n} \cdot \mathcal{C}[\boldsymbol{n}]$. This conditions are obtained trough Hadamard's compatibility condition.

Discontinuous bifurcations or strain localizations, like shear bands, are understood as the appearance of a discontinuities in strain rates which mark the onset of non-uniform responses. They are also related to stationary acceleration waves. Discontinuous bifurcations associated to loss of *ellipticity* of the boundary value problem. Ellipticity means

$$\mathcal{C}[\boldsymbol{g} \otimes \boldsymbol{n}]\, \boldsymbol{n} \neq 0, \qquad \forall\, \boldsymbol{n}, \boldsymbol{g} \tag{1.100}$$

which is equivalent to the condition

$$\det \mathcal{A} \neq 0 \tag{1.101}$$

in the direction of the discontinuity. This criterion is also known as *Rice criterion*. Physically, condition (1.101) indicates the existence of a discontinuity in the velocity gradient in direction $\boldsymbol{n}$. Ellipticity is a necessary condition for well-posedness of the rate boundary value problem, in the sense that a finite number of linearly independent solutions are admitted, continuously depending on the data and not involving discontinuities.

For the most general materials, the loss of uniqueness precedes continuous bifurcations (loss of strong ellipticity) which in turn precedes discontinuous bifurcation (loss of ellipticity).

1.4.5 Non-local models overview

Generally speaking, constitutive models are mainly divided into local and non-local. While local models are good for hardening behaviours, they result to fail in case of softening behaviours. The resulting boundary value problem is mathematically ill-posed Comi and Perego 1996; Lasry and Belytschko 1988 in the sense that it admits as infinite number of linearly independent solutions, Benallal and Marigo

2007 for more details. This correspond equivalently to the loss of ellipticity of the boundary value problem, to the singularity of the acoustic tensor in some direction and some points or to the dynamic problem where the wave speeds becomes imaginary and the dynamic problem from elliptic to hyperbolic. The response is associated with a strain localization of an arbitrarily narrow zone which result in a vanishing dissipated work, clearly a physically wrong result. Furthermore numerical simulation with local models using the finite element method are strongly mesh sensitive Bažant, Belytschke, and Chang 1984; De Borst and Mahlhaus 1992.

To overcome these limits several approaches exist all based more or less evidently on the introduction of an internal length in the constitutive model and can be considered as a regularization of local models. The regularisation effect in non-local models is mostly to stretch the localisation bands up to a finite region. Moreover the regularisation may involve either the primal variables as the strain or the dual variables as the internal variables. An extended survey about non-local theories of material media can be found in Rogula 1982.
The most adopted classical formulations are summarised in the following:

nonlocal integral formulation Generally speaking, the nonlocal integral approach consists in replacing a certain variable by its nonlocal counterpart obtained by weighted averaging over a spatial neighbourhood of each point under consideration. If $f(\boldsymbol{x})$ is some "local" field in a solid body occupying a domain Ω, the corresponding nonlocal field, $\bar{f}(\boldsymbol{x})$, is defined by

$$\bar{f}(\boldsymbol{x}) = \int_{\Omega} f(\boldsymbol{\zeta}) a(\boldsymbol{x}, \boldsymbol{\zeta}) \, \mathrm{d}\boldsymbol{\zeta} \tag{1.102}$$

where $a(\boldsymbol{x}, \boldsymbol{\zeta})$ is nonlocal weight function. In applications to softening materials, it is often required that the nonlocal operator should not alter a uniform field, which means that the weight function must satisfy the normalising condition

$$\int_{\Omega} a(\boldsymbol{x}, \boldsymbol{\zeta}) \, \mathrm{d}\boldsymbol{\zeta} = 1, \quad \forall \boldsymbol{x} \in \Omega. \tag{1.103}$$

Clearly, this approach discards the principle of local action of the classical continuum mechanics theory. For a complete survey about integral-type nonlocal models one can refer to Bažant and Jirasek 2002.

additional variables In this approach additional primal variables are considered. Typical example are the micropolar continua of Cosserat's kind.

implicit gradient (or strongly nonlocal) formulation Implicit gradient models includes in the formulation, for example in the yield stress function, a non-local field $\bar{f}(\boldsymbol{x})$ deduced from the dual local field $f(\boldsymbol{x})$ by means of the following differential relation of Helmotz type,

$$\bar{f}(\boldsymbol{x}) - c(\ell) \nabla^2 \bar{f}(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \text{in } \Omega \tag{1.104}$$

where ∇^2 is the Laplace operator and ℓ the internal material length. The implicit gradient formulation can be show to be equivalent to the integral-type formulation with special weight functions used for the weighted averaging.

To uniquely specify $\bar{f}$, (1.104) must be supplemented by appropriate boundary conditions. The precise form of these conditions is not obvious, but seems reasonable to require that the transformation should not alter a constant field. If $f(x) = f_0 = \text{const.}$, then $\bar{f}(x) = f_0$ satisfy the differential equation (1.104), and it should also satisfy the boundary conditions, indipendently of the value of f_0. Clearly it is not possible to use the Dirichlet boundary conditions, but every constant field satisfies the homogeneous Neumann boundary conditions

$$\boldsymbol{n} \cdot \nabla \bar{f} = 0, \quad \text{on } \partial\Omega. \tag{1.105}$$

explicit gradient (or weakly nonlocal) formulation Similar to the implicit gradient formulation a non-local field $\bar{f}(\boldsymbol{x})$ is used in the constitutive model instead of the dual local field $f(\boldsymbol{x})$. The non local field is simply obtained by considering gradient terms of the local field. An example is

$$\bar{f}(\boldsymbol{x}) = f(\boldsymbol{x}) + c(\ell)\nabla^2 f(\boldsymbol{x}) + \mathrm{w}\left(\nabla^2 f\right), \quad \text{in } \Omega. \tag{1.106}$$

In the explicit gradient formulation the non-local variable can depend only upon even powers of the gradient of the local variable, Engelen, Geers, and Baaijens 2003. Here, however, with explicit gradient formulations are meant models which includes directly gradient terms of some variables not necessarily descending from (1.104).

Tab. 1.2 gives examples of the aforementioned formulations separately for plasticity and damage models and with respect to the involved variable in the regularization (primal or dual).

	primal variables	internal variables
integral type	• Eringen 1983 • Bažant and Lin 1988a	• Bažant and Lin 1988b • Bažant and Pijaudier-Cabot 1988
additional	• Cosserat and Cosserat 1909 • Eringen 1964 • dell'Isola, Sciarra, and Vidoli 2009	-
implicit gradient	• Pamin, Askes, and Borst 2003 • Peerlings et al. 1996	• Geers 2004 • -
explicit gradient	• Qiu et al. 2003 • -	• Comi and Perego 1996 • Comi 1999

Table 1.2: Non-local formulations with references to plasticity (red) and damage (green) models

1.4.6 Coupled plasticity-damage models

A large quantity of materials exhibit a strong interaction between plastic flows and microcracks or microvoids growth at failure conditions. Several works aimed to combine plasticity and damage in a constitutive framework.

Usually the problem is formulated in perfect analogy to plasticity models, with the introduction of a damage yield function which is allowed to vary during the damage evolution. A common feature of these elastic-plastic-damage constitutive models is the concept of effective stress (or effective strain), Lemaitre and Chaboche 1985, which enables the plastic and damage internal mechanisms to be coupled with each other. However a drawback is that in general this approach leads to non-symmetric tangent stiffness matrices.

An alternative approach to this coupling problem stems from the consideration that the coupling between the internal plastic and damage mechanisms should more consistently be realized through the internal variables themselves, rather than the external variables like stresses, or strains.

One way to achieve this goal within continuum thermodynamics is to let the a convex damage and plastic yield surface being affected by all the internal state variables. This has several beneficial consequences on the description of the material behaviour; namely, i) the tangent stiffness matrix is symmetric; ii) the softening behaviour can be reproduced as the consequence of the material degradation due to damage, rather than a "negative" hardening (often introduced in the realm of plasticity), which is more adherent to the physics of the material; iii) it is possible to reproduce experimental diagrams and in particular to account for the stress relaxation phenomenon exhibited by a bar specimen subjected to strain-driven load cycles; iv) anisotropic elastic degradation can be accounted for by properly defining the material free energy parameters. It is then clear that a great variability in the modelling exists.

In the following a short literature review about coupled plasticity-damage models is reported. The purpose is not to be exhaustive since the argument is boundless but only to highlight some features of classical approaches. Within local continuum formulations several theories coupling damage and plasticity exists. They differs for numerous aspects. Any model is hereafter shortly described and labelled by means of the differences in Tab. 1.3.

strain amplitude	constitutive variables	constitutive modelling	regularization technique
• small strains (SS) • large strains (LS)	• effective stress/strain (ES) • dependence on all internal state variables (IV)	• postulate of yield functions (YF) • postulate of potentials (P)	• any (NR) • integral (IN) • gradient of external variables (GE) • gradient of internal variables (GI)

Table 1.3: Differences and underlying approaches of coupled plasticity-damage models

Lemaitre and Chaboche 1985 - (SS), (ES), (YF/P), (NR) The concept of effective stress is introduced (Lemaitre 1984) together with the normal dissipation hypothesis. This model was presented in Lemaitre 1985.

Mazars and Pijaudier-Cabot 1989 - (SS), (ES), (YF), (NR) It covers different damage models like one-parameter, two-parameters and anisotropic models;

Simo and Ju 1987, (SS), (ES), (NR) ;

Pamin 1994, (SS), (ES), (YF), (GE) Regularization is achieved by an implicit second-order scheme over the total strain;

Hansen and Schreyer 1994 - (SS), (ES), (YF), (NR) An associative plastic law is considered while the principle of maximum entropy provides the evolutionary relations;

Doghri 1995 - (SS) Several non-linear hardening effects are taken into account by means of Chaboche-Marquis plastic model and Lemaitre-Chaboche ductile damage model;

Bonora 1997 - (SS), (ES), (P), (NR) ;

De Sciarra 1997 - (ES), (P) The paper aims to develop some of the most common models of coupled elasto-plasticity with damage into a unitary framework. The coupling between plasticity and damage is based on the concept of effective stress and on the definition of a convex elastic-damage domain and a convex damage domain;

Borst, Pamin, and Geers 1999 - (SS), (ES), (YF), (GI) This work covers several coupled models where regularization is achieved either by considering the gradient of some plastic variables or the gradient of damage variables;

Al-Rub and Voyiadjis 2003 - (SS), (ES), (P), (NR) In this model both damage and plastic hardening effects are taken into account. Damage is considered anisotropic;

Grassl and Jirásek 2006 - (SS), (ES), (P), (IN) A local-nonlocal damage variable is considered in the coupling with plasticity;

Einav, Houlsby, and Nguyen 2007 - (SS), (IV), (P), (NR) It is shown that the entire constitutive knowledge of the coupled model can be expressed by means of the definition of two potentials dependent on all the state variables. Non-locality is not faced;

Belnoue et al. 2010 - (SS), (IV), (YF), (IN) Regularization effects are obtained by considering in the yield functions a local-nonlocal equivalent plastic strain by a weighted integral;

Dimitrijevic and Hackl 2009 - (SS), (IV), (P), (GI) ;

Simo and Ju 1989 - (ES) The basic feature of this strain-based constitutive model is an additive split of the stress tensor into the initial elastic and the plastic relaxation parts. Such a split enables one to incorporate a simple strain-based continuum damage model capable of predicting so-called splitting modes (see also Ortiz 1985). Additive splits of the stress tensor has been proposed in Simo (1986) for finite strain viscoelasticity, and in Simo and Ju 1987 for infinitesimal damage-elastoplasticity.

Lubarda 1994; Voyiadjis and Kattan 1992; Zysset and Curnier 1996 - (LS) ;

Areias, César de Sá, and Conceição António 2003 - (ES), (IV), (P), (GE) :

Soyarslan and Tekkaya 2010 - (FS), (EF), (YF), (NR) The formulation framework is in the principal stress space.

Articles which cover comparisons about the previous articles are:

Lämmer and Tsakmakis 2000

Nguyen 2005 - (SS), (ES), (YF) It resumes several coupled models for the description of concrete.

Interesting models able to describe ductile fracture are those where coupling between plasticity and damage is obtained by considering only a single yield function, see Gurson 1977; Hesebeck 2001.

Some observations In order to achieve a coupled response in the constitutive equations it is possible to

- consider the concept of effective stress (or effective strain) in the single yield functions
- let depend the yield functions on all the internal variables rather than external variables (like effective stresses)

Chapter 2

A variational plastic-damage model

This chapter is the core of the present work. Here the coupled plastic-gradient damage model is developed in a three-dimensional setting by means of the energetic formulation introduced in the previous Chapter, in particular Sec. 1.3. The first order stability condition furnishes some (weak) necessary inequalities which the model has to be satisfied and from which it is possible to define in a classical sense the yield functions and disclose their (strong) expression. Nevertheless it is the weak form that reveals the capability of the model to describe a cohesive fracture. From the energy balance it is possible to derive the classical consistency conditions.

2.1 Model assumptions

Both plasticity and damage are here considered as rate-independent processes. Hence, the thermodynamic framework with internal variables, Sec. 1.1.6, could apply motivated by the fact that in each instant equilibrium is assured. Moreover, in the context of isothermal processes, the energetic formulation is adopted in order to describe the evolution of the system. A continuous body $\mathcal{B}$ is considered, embedded in an Euclidean space of dimension d $(1, 2, 3)$ and represented by its geometrical domain Ω with a sufficiently smooth[1] boundary $\partial\Omega$. For simplicity, deformations gradient is assumed "small" so that the infinitesimal elasticity theory can be applied. The evolution is governed by a "time" parameter t starting from $t = 0$. The dissipation is only due to two phenomena, plasticity and damage while regular energy functionals are also assumed in order to obtain results in closed form. Although not necessary, the material will be always considered initially unstretched, not plasticised and undamaged.

2.1.1 State variables and Function Spaces

Throughout the following the definitions of the state variables together with their function spaces are given. A list of the considered state variables with a brief de-

[1] To be more precise: Lipschitz continuous

	state variable	type
$\boldsymbol{u}(\boldsymbol{x},t)$	displacement	observable
$\boldsymbol{\varepsilon}(\boldsymbol{x},t)$	total strain	observable, dependent
$\boldsymbol{\varepsilon}_p(\boldsymbol{x},t)$	plastic strain	internal, reversible
$p(\boldsymbol{x},t)$	accumulated plastic strain	internal, irreversible, dependent
$\alpha(\boldsymbol{x},t)$	damage	internal, irreversible
$\nabla\alpha(\boldsymbol{x},t)$	gradient of damage	internal, dependent

Table 2.1: State variables. The dependence on the position $\boldsymbol{x}$ is not formally correct for some variables since measures are involved as explained in the sequel

scription is given in Tab. 2.1. Moreover also the space of accessible states and the space of the variations will be specified for any state variable. These definitions play a crucial role respectively in the expression of the global stability condition and of the local stability condition, Sec. 1.3.2. For what concerns the global stability condition, while for the displacement and plastic strain fields the function spaces are the same as the one for the accessible states this holds not more true for the damage field because of irreversibility, Sec. 1.3.4. The accessible states then has to be explicitly defined for the damage field. On the other hand for the local stability condition, while the function space of the plastic strain field is the same as its variation space this holds not more true for the displacement and damage fields because respectively of the boundary conditions and irreversibility. The variations space then has to be explicitly defined for the displacement and damage field.

The displacement and the total strain are maps defined as

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &: (\Omega,[0,T]) \to F \equiv \mathbb{R}^d, \\ \boldsymbol{\varepsilon}(\boldsymbol{x},t) &: (\Omega,[0,T]) \to S := \left\{a \in \mathbb{R}^{d\times d} \colon \boldsymbol{\varepsilon}^{\mathsf{T}} = \boldsymbol{\varepsilon}\right\}, \end{aligned} \tag{2.1}$$

with $\boldsymbol{\varepsilon}(\boldsymbol{x},t) = \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},t))$ by means of the strain-diplacement relation (1.1). The proper function space of the displacement field is

$$\boldsymbol{u} \in \mathcal{F} \equiv \mathrm{SBD}((\Omega,[0,T]),F) \tag{2.2}$$

where SBD is the space of special bounded deformations, the proper space for the description of fracture or perfect plasticity problems[2], Temam and Strang 1980 for details. The displacement variations are then defined as

$$\widetilde{\boldsymbol{u}} \in \widetilde{\mathcal{F}} \equiv \{\boldsymbol{a} \in \mathcal{F} \colon \boldsymbol{a} = \boldsymbol{0} \,\mathrm{on}\, \partial\Omega_D\}\,. \tag{2.3}$$

The space $\mathcal{F}$ coincides also with the space of accessible states needed in the global stability condition.

The plastic and accumulated plastic strains are maps defined as

$$\begin{aligned} \boldsymbol{\varepsilon}_p(\boldsymbol{x},t) &: (\Omega,[0,T]) \to Q := \left\{\boldsymbol{a} \in \mathbb{R}^{d\times d} \colon \operatorname{tr}\boldsymbol{a} = 0 \,\text{and}\, \boldsymbol{a}^{\mathsf{T}} = \boldsymbol{a}\right\}, \\ p(\boldsymbol{x},t) &: (\Omega,[0,T]) \to R \equiv \mathbb{R}^+_\infty, \end{aligned} \tag{2.4}$$

[2]Clearly $\mathrm{SBV}(\Omega) \supseteq \mathrm{SBD}(\Omega)$ where the equality holds only for one-dimensional domains.

with

$$p(\boldsymbol{x}, t) = \int_0^t \|\dot{\boldsymbol{\varepsilon}}_p(\boldsymbol{x}, \tau)\| \, \mathrm{d}\tau. \tag{2.5}$$

The proper function space of the plastic strain field is

$$\boldsymbol{\varepsilon}_p \in \mathcal{Q} \equiv \mathrm{SBM}((\Omega, [0, T]), Q), \tag{2.6}$$

where SBM is the space of special bounded measures[3]. The function space for the accumulated plastic strain p is consequently defined. Both the accessible states and the variations space of the plastic strain coincides with the function space $\mathcal{Q}$. Therefore

$$\widetilde{\boldsymbol{\varepsilon}}_p \in \widetilde{\mathcal{Q}} \equiv \mathcal{Q}. \tag{2.7}$$

Moreover, in order to have a finite elastic energy, one assumes

$$\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p \in \mathrm{L}^2\Big((\Omega, [0, T]), \mathbb{R}^{d \times d}\Big). \tag{2.8}$$

The damage and gradient of damage are maps defined as

$$\begin{aligned} \alpha(\boldsymbol{x}, t) &: (\Omega, [0, T]) \rightarrow C \equiv [0, 1] \\ \nabla\alpha(\boldsymbol{x}, t) &: (\Omega, [0, T]) \rightarrow G \equiv \mathbb{R}^d \end{aligned} \tag{2.9}$$

while the proper function space of the damage field is

$$\alpha \in \mathcal{C} \equiv \mathrm{H}^1((\Omega, [0, T]), C). \tag{2.10}$$

The damage variations are then defined as

$$\widetilde{\alpha} \in \mathcal{C}_+ \equiv \{a \in \mathcal{C} : \ a \geq 0\} \tag{2.11}$$

which embed the irreversibility condition (1.87) that impose

$$\alpha(t_1) \leq \alpha(t_2), \qquad \forall \, t_1 < t_2. \tag{2.12}$$

When dealing with the global stability condition the accessible states for damage are state dependent and needs to be specified due to irreversibility. The accessible states then are state dependent,

$$\bar{\alpha} \in \overline{\mathcal{C}}(\alpha) \equiv \{a \in \mathcal{C} : \ \alpha \leq a \leq 1\}. \tag{2.13}$$

The explicit dependence of the state variables on both the position $\boldsymbol{x}$ and the time t will be omitted in the sequel for clarity of exposition if no source of confusion occurs. Eventually the time dependence is made explicit through a subscript.

2.1.2 Energy functionals and constitutive assumptions

According to the energetic formulation of Sec. 1.3.1 the total potential energy and the dissipation distance, are here defined as well as some constitutive assumptions.

[3] An example of a special bounded measure is the Dirac measure.

To this aim, it is reasonable to assume the stored elastic energy (1.51) as:

$$\mathcal{E}(\boldsymbol{u}, \boldsymbol{\varepsilon}_p, \alpha) = \int_\Omega e(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p, \alpha)\ \mathrm{d}\Omega = \int_\Omega \frac{1}{2}\mathbb{C}(\alpha)\left[\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p\right] : \left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p\right)\ \mathrm{d}\Omega. \tag{2.14}$$

This is motivated by the fact that (2.14) simply merges the stored elastic energies of the perfect plastic and the damage models already introduced respectively in (1.81) and (1.88).n Once (2.14) has been introduced, the stress tensor $\boldsymbol{\sigma}$ is given by the state law (1.28), hence

$$\sigma = \frac{\partial e(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p, \alpha)}{\partial \boldsymbol{\varepsilon}} = \mathbb{C}(\alpha)\left[\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p\right]. \tag{2.15}$$

The potential energy of external forces is simply given by minus (1.52).

In order to define the dissipation distance, an appropriate dissipation potential is first introduced. For the dissipation potential, an additive decomposition is assumed such that

$$\psi(p, \alpha, \nabla\alpha, \dot{p}, \dot{\alpha}, \nabla\dot{\alpha}) := \psi_p(\alpha, \dot{p}) + \psi_d(p, \alpha, \nabla\alpha, \dot{\alpha}, \nabla\dot{\alpha}) \tag{2.16}$$

where

$$\psi_p(\alpha, \dot{p}) := \sigma_{\mathsf{P}}(\alpha)\,\|\dot{\boldsymbol{\varepsilon}}_p\|, \tag{2.17}$$

$$\psi_d(p, \alpha, \nabla\alpha, \dot{\alpha}, \nabla\dot{\alpha}) := \partial_t\,\mathsf{w}(\alpha) + \partial_t\left(\frac{1}{2}\left(\eta(\alpha)\right)^2 \nabla\alpha \cdot \nabla\alpha\right) - \mathsf{q}(p)\,\mathsf{m}(\alpha)\,\dot{\alpha}. \tag{2.18}$$

The dissipation potential involves coupled terms. The coupling is achieved by letting the plastic (damage) dissipation potential (1.83) ((1.91)) depend explicitly on damage (plasticity) by means of α (p). In particular, the plastic yield stress becomes a function of the damage level in the plastic dissipation potential while a term depending on the accumulated plastic strain is added to the damage dissipation potential. Of course different choices could have been done for the dissipation potential. Nevertheless, as will be revealed in the sequel, the choice (2.16) turned out to be worthy of consideration.

In the proposed model, the material is identified by the choice of six constitutive functions,

$$\mathbb{C}(\alpha)\,, \qquad \sigma_{\mathsf{P}}(\alpha)\,, \qquad \mathsf{w}(\alpha)\,, \qquad \eta(\alpha)\,, \qquad \mathsf{m}(\alpha)\,, \qquad \mathsf{q}(p)\,. \tag{2.19}$$

Hereafter, the following assumptions are made, motivated either by physical reasons or for simplicity. In the elastic energy (2.14), the simplest choice for the elastic tensor corresponds to an isotropic medium such that

$$\mathbb{C}(\alpha) = f(\alpha)\,\mathbb{C}_0, \tag{2.20}$$

with $f(\alpha) : C \to [0, 1]$ and where the function f satisfies the same assumptions as in (1.90). $\mathbb{C}_0$ is the isotropic elastic tensor already defined in (1.82). In the following the compliance function, defined as $\mathbb{S}(\alpha) = \left(\mathbb{C}(\alpha)\right)^{-1}$, will be also considered. The function $\sigma_{\mathsf{P}}(\alpha)$ is identified with the plastic yield stress. Hence, one requires

$$\sigma_{\mathsf{P}}(\alpha) : C \to \mathbb{R}^+, \qquad \sigma'_{\mathsf{P}}(\alpha) < 0,\ \forall\,\alpha \in [0, 1) \qquad \sigma_{\mathsf{P}}(1) = 0. \tag{2.21}$$

The function definition $(2.21)_1$ allows the plastic dissipation potential to be positive. Condition $(2.21)_2$ is necessary to trigger plasticity in a damaging material while condition $(2.21)_3$ is a natural request. Instead, the function $\mathsf{w}(\alpha)$ is associated to a dissipated work contribution which depends only on the level of damage. The same conditions expressed and motivated in (2.22) are assumed to be valid. For simplicity, the function $\eta(\alpha)$ related to the internal length of the material model is assumed to be constant, $\eta \geq 0$. A vanishing η let to recover a homogeneous model. Finally for the last two functions in (2.19), defined as $\mathsf{m}(\alpha) : C \to \mathbb{R}^+$ and $\mathsf{q}(p) : R \to \mathbb{R}^+$ one requires

$$\mathsf{m}(\alpha) > 0,\ \forall\, \alpha \in [0,1) \qquad \text{and} \qquad \mathsf{q}(p) > 0,\ \forall\, p \in (0,\infty), \quad \mathsf{q}(0) = 0 \tag{2.22}$$

As will become clearer in the following, these are necessary to trigger damage once plasticity occurs. A very important remark has be done about the function $\mathsf{q}(p)$. In order to be the dissipation distance finite and since p belongs to the space of bounded measures, $\mathsf{q}(p)$ must be a sublinear function, that is $\mathsf{q}(x) = O(x)$.

A stress-softening behaviour is achieved if

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\mathbb{S}'(\alpha)}{2\left(\mathsf{w}'(\alpha) - \mathsf{q}(p)\,\mathsf{m}(\alpha)\right)}\right) \geq 0, \qquad \lim_{\alpha\to 1}\left(\frac{\mathbb{S}'(\alpha)}{2\left(\mathsf{w}'(\alpha) - \mathsf{q}(p)\,\mathsf{m}(\alpha)\right)}\right) = +\infty. \tag{2.23}$$

Another strong requirement would be to avoid snap-back phenomena in the response. To obtain a strain-hardening behaviour in the homogeneous response with respect to damage one must assume

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{-\mathbb{C}'(\alpha)}{2\left(\mathsf{w}'(\alpha) - \mathsf{q}(p)\,\mathsf{m}(\alpha)\right)}\right) \leq 0. \tag{2.24}$$

but this be not always assumed. Both conditions (2.24) and (2.23) will be justified in Sec. 2.2.

With this preliminary statements the total potential energy then reads

$$\mathcal{P}\left(t, \boldsymbol{u}, \boldsymbol{\varepsilon}_p, \alpha\right) = \mathcal{E}\left(\boldsymbol{u}, \boldsymbol{\varepsilon}_p, \alpha\right) - \mathcal{L}(t, \boldsymbol{u}) \tag{2.25}$$

where $\mathcal{E}$ and $\mathcal{L}$ are respectively given by (2.14) and (1.52). Once the dissipation potential (2.16) has been introduced the dissipation distance D is given. For the investigated model, the dissipated distance density (1.54) reads

$$\begin{aligned} d\big(\left(\boldsymbol{\varepsilon}_{p_0}, \alpha_0, \nabla\alpha_0\right), \left(\boldsymbol{\varepsilon}_{p_1}, \alpha_1, \nabla\alpha_1\right)\big) &= \left(\mathsf{w}(\alpha_1) - \mathsf{w}(\alpha_0)\right) + \frac{1}{2}\eta^2\left(\nabla\alpha_1\cdot\nabla\alpha_1 - \nabla\alpha_0\cdot\nabla\alpha_0\right) \\ &+ \inf\Bigg\{\int_0^1 \left(\sigma_{\mathsf{P}}(\beta)\,\dot{q} - \mathsf{q}(q)\,\mathsf{m}(\beta)\,\dot{\beta}\right)\mathrm{d}s : (\boldsymbol{r}, \beta) \in \mathrm{C}^1(Q\times C, [0,1]), \\ q = p(t) + \int_0^s \|\boldsymbol{r}(\tau)\|\,\mathrm{d}\tau, \dot{\beta} \geq 0, &\left(\boldsymbol{r}(0), \beta(0)\right) = \left(\boldsymbol{\varepsilon}_{p_0}, \alpha_0\right), \left(\boldsymbol{r}(1), \beta(1)\right) = \left(\boldsymbol{\varepsilon}_{p_1}, \alpha_1\right)\Bigg\} \end{aligned} \tag{2.26}$$

where, for simplicity, the variable s has been omitted in the functions q and β in the integral and $p(t) = \int_0^t \|\boldsymbol{\varepsilon}_{p_0}\|\,\mathrm{d}\tau$. It is worth noting that in (2.26) the first two terms are path independent while the third term needs the evaluation of the infimum of the integral, because of the particular definition of the dissipation potential. Nevertheless, for some particular constitutive choices as in the particular model of Sec. 2.4

all the terms in the dissipation distance can be considered path independent. It is then possible to introduce a total energy as a state function and to consider the stability condition only on such functional, Nguyen 2000. In this case, the crucial issue is to define how the accumulated plastic strain p varies as a result of the variation of the plastic strain $\boldsymbol{\varepsilon}_p$.

Once the dissipation distance (1.55) has been introduced, the dissipated work (1.56) along a process is immediately defined and reads, for smooth evolutions (1.59), as

$$\mathcal{D}_D\big((\boldsymbol{\varepsilon}_p,\alpha)\,,[0,t]\big) = \int_\Omega \left(\mathsf{w}(\alpha(t)) + \frac{1}{2}\eta^2\,\nabla\alpha(t)\cdot\nabla\alpha(t)\right)\,\mathrm{d}\Omega + \\ + \int_\Omega\int_0^t \left(\sigma_{\mathsf{P}}(\alpha(\tau))\,\|\dot{\boldsymbol{\varepsilon}}_p(\tau)\,\| - \mathsf{q}(p(\tau))\,\mathsf{m}(\alpha(\tau))\,\dot{\alpha}(\tau)\right)\mathrm{d}\tau\;\mathrm{d}\Omega. \quad (2.27)$$

An important condition is the dissipated power being non-negative at any instant of the evolution, that is

$$\mathcal{D}_D\big((\boldsymbol{\varepsilon}_p,\alpha)\,,[0,t_1]\big) \le \mathcal{D}_D\big((\boldsymbol{\varepsilon}_p,\alpha)\,,[0,t_2]\big) \quad \forall\, t_1 < t_2, \quad (2.28)$$

or for smooth evolutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}_D\big((\boldsymbol{\varepsilon}_p,\alpha)\,,[0,t]\big) \ge 0. \quad (2.29)$$

which in local form becomes

$$\psi(p,\alpha,\nabla\alpha,\dot{p},\dot{\alpha},\nabla\dot{\alpha}) \ge 0, \qquad \forall\,\boldsymbol{x}\in\Omega. \quad (2.30)$$

This last condition will be proven, at least for continuous evolutions, to be always true for the particular constitutive functions of a specific model assumed at the end of Sec. 2.2.1.

2.2 Stability condition

In this and forth coming sections the theoretical results introduced in Sec. 1.3 are applied to the plastic-damage model. To highlight the capabilities of the model only the differential stability condition Sec. 1.3.2.3 is considered. On the other hand, the global stability condition is in a certain sense preferred for numeric implementations as discussed in Chap. 5.

2.2.1 First order stability condition

First of all, the first-order stability condition (**st-1D**) is considered for the investigated plastic-damage model. To this aim, the first variation of the potential energy in direction $(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{\varepsilon}}_p,\widetilde{\alpha})$ reads by standard arguments as

$$\begin{aligned}\mathcal{P}'\big(\boldsymbol{u},\boldsymbol{\varepsilon}_p,\alpha\big)\big(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{\varepsilon}}_p,\widetilde{\alpha}\big) &= \int_\Omega \boldsymbol{\sigma} : \ \big(\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}) - \widetilde{\boldsymbol{\varepsilon}}_p\big)\ \mathrm{d}\Omega - \int_\Omega \boldsymbol{b}\cdot\widetilde{\boldsymbol{u}}\ \mathrm{d}\Omega - \int_{\partial\Omega_N} \boldsymbol{f}_a\cdot\widetilde{\boldsymbol{u}}\ \mathrm{dS}+\\ &+\frac{1}{2}\int_\Omega \mathbb{S}'(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma}\,\widetilde{\alpha}\ \mathrm{d}\Omega\\ &= -\int_\Omega (\mathrm{div}\boldsymbol{\sigma}+\boldsymbol{b})\cdot\widetilde{\boldsymbol{u}}\ \mathrm{d}\Omega + \int_{\partial\Omega_N} (\boldsymbol{\sigma}[\boldsymbol{n}]-\boldsymbol{f}_a)\cdot\widetilde{\boldsymbol{u}}\ \mathrm{dS}+\\ &-\int_\Omega \boldsymbol{\sigma}:\widetilde{\boldsymbol{\varepsilon}}_p\ \mathrm{d}\Omega + \frac{1}{2}\int_\Omega \mathbb{S}'(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma}\,\widetilde{\alpha}\ \mathrm{d}\Omega.\end{aligned} \tag{2.31}$$

The evaluation of the first variation of the dissipated distance (1.55) requires a more careful treatment. The delicate issue is due to the variation with respect to plasticity because of the path dependent state variable p. In order to evaluate the variation of terms containing the accumulated plastic strain p, which depends on the past history of $\boldsymbol{\varepsilon}_p$, the non-trivial definition of (1.54) has to be taken into account, leading to

$$\begin{aligned}D'\big(\boldsymbol{\varepsilon}_p,p,\alpha\big)\ \big(\widetilde{\boldsymbol{\varepsilon}}_p,\widetilde{p}\big(\widetilde{\boldsymbol{\varepsilon}}_p\big),\widetilde{\alpha}\big) &= \int_\Omega \sigma_{\mathsf{P}}(\alpha)\,\|\widetilde{\boldsymbol{\varepsilon}}_p\|\ \mathrm{d}\Omega\\ &+\int_\Omega \Big(\big(\mathsf{w}'(\alpha)-\mathsf{q}(p)\ \mathsf{m}(\alpha)\big)\,\widetilde{\alpha}+\eta^2\,\nabla\alpha\cdot\nabla\widetilde{\alpha}\Big)\ \mathrm{d}\Omega\\ &=\int_\Omega \psi\big(p,\alpha,\nabla\alpha,\widetilde{\boldsymbol{\varepsilon}}_p,\widetilde{\alpha},\nabla\widetilde{\alpha}\big)\ \mathrm{d}\Omega.\end{aligned} \tag{2.32}$$

The proof of (2.32) can be found in appendix A.

Since (**st-1D**) must be valid for all $(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{\varepsilon}}_p,\widetilde{\alpha}) \in \widetilde{\mathcal{F}}\times\widetilde{\mathcal{Q}}\times\mathcal{C}_+$ the following results hold:

For $\widetilde{\boldsymbol{\varepsilon}}_p=\boldsymbol{0}$ and $\widetilde{\alpha}=0$, one obtains

$$-\int_\Omega (\mathrm{div}\boldsymbol{\sigma}+\boldsymbol{b})\cdot\widetilde{\boldsymbol{u}}\ \mathrm{d}\Omega + \int_{\partial\Omega_N} (\boldsymbol{\sigma}[\boldsymbol{n}]-\boldsymbol{f}_a)\cdot\widetilde{\boldsymbol{u}}\ \mathrm{dS}\ge 0. \tag{2.33}$$

Since $\widetilde{\boldsymbol{u}}$ is totally arbitrary, the inequality must hold as an equality. Hence,

$$\mathrm{div}\boldsymbol{\sigma}+\boldsymbol{b}=0, \qquad \forall\,\boldsymbol{x}\in\Omega \tag{2.34}$$

are the strong equilibrium equations while

$$\boldsymbol{\sigma}[\boldsymbol{n}]=\boldsymbol{f}_a, \qquad \forall\,\boldsymbol{x}\in\partial\Omega_N, \tag{2.35}$$

are the associated Neumann boundary conditions.

For $\widetilde{\boldsymbol{u}}=\boldsymbol{0}$ and $\widetilde{\alpha}=0$, one gains

$$\int_\Omega \big(\sigma_{\mathsf{P}}(\alpha)\,\|\widetilde{\boldsymbol{\varepsilon}}_p\| - \boldsymbol{\sigma}:\widetilde{\boldsymbol{\varepsilon}}_p\big)\ \mathrm{d}\Omega\ge 0. \tag{2.36}$$

Noting that $\sigma_{\mathsf{P}}(\alpha)\,\|\widetilde{\boldsymbol{\varepsilon}}_p\| > 0$, the maximum positive value admissible for $\boldsymbol{\sigma}:\widetilde{\boldsymbol{\varepsilon}}_p$ is clearly equal to $\|\boldsymbol{\sigma}_d\|\|\widetilde{\boldsymbol{\varepsilon}}_p\|$. The minimum value in (2.36) is achieved once the deviatoric part of the stress is taken collinear to the variation of the plastic strain. Therefore, the condition

$$f_p(\alpha,\boldsymbol{\sigma}_d)=\|\boldsymbol{\sigma}_d\|-\sigma_{\mathsf{P}}(\alpha)\le 0, \qquad \forall\,\boldsymbol{x}\in\Omega \tag{2.37}$$

follows and can be interpreted as a Von Mises yield criterion (1.78). The function $f_p(\alpha, \boldsymbol{\sigma}_d)$ has then the usual meaning of a plastic yield function.

Finally, for $\widetilde{\boldsymbol{u}} = \mathbf{0}$ and $\widetilde{\boldsymbol{\varepsilon}}_p = 0$, one obtains

$$\int_\Omega \left(-\frac{1}{2}\mathbb{S}'(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma} + \mathsf{w}'(\alpha) - \mathsf{q}(p)\,\mathsf{m}(\alpha) \right) \widetilde{\alpha} + \eta^2\, \nabla\alpha \cdot \nabla\widetilde{\alpha}\ \mathrm{d}\Omega \geq 0, \tag{2.38}$$

which represents the weak form of the damage yield criterion. As better explained later for the 1D-model in Ch. 4, a strong form of (2.38) could be obtained once the damage field is assumed to belong to a more regular space, say

$$\mathcal{C} \equiv \mathrm{H}^2((\Omega, [0,T]), C)\,. \tag{2.39}$$

For the pure damage model this last (2.39) becomes a necessary condition for the stability, Pham and Marigo 2011. On the other hand, for the coupled plasticity-damage model this is no longer true since plasticity could allow the damage profile to suffer jumps in its derivatives.

Integrating by parts (2.38) with respect to α one finds

$$\int_\Omega \left(-\frac{1}{2}\mathbb{S}'(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma} + \mathsf{w}'(\alpha) - \mathsf{q}(p)\,\mathsf{m}(\alpha) - \eta^2 \mathrm{div}\nabla\alpha \right) \widetilde{\alpha}\ \mathrm{d}\Omega \\ + \int_{\partial\Omega} \eta^2(\alpha)\, \nabla\alpha \cdot \boldsymbol{n}\,\widetilde{\alpha}\ \mathrm{dS} \geq 0, \tag{2.40}$$

where $\boldsymbol{n}$ is the outward normal to the boundary $\partial\Omega$. The inequality (2.40) leads to the bulk condition

$$f_d(p, \alpha, \nabla\alpha, \boldsymbol{\sigma}) = \frac{1}{2}\mathbb{S}'(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma} - \mathsf{w}'(\alpha) + \mathsf{q}(p)\,\mathsf{m}(\alpha) + \eta^2(\alpha)\,\mathrm{div}\nabla\alpha \leq 0, \\ \forall\, \boldsymbol{x} \in \Omega \tag{2.41}$$

which again can be understand as a classical damage yield condition (1.86). The function $f_d(p, \alpha, \nabla\alpha, \boldsymbol{\sigma})$ then has the usual meaning of a damage yield function. Moreover, the first order stability condition also furnishes the damage boundary conditions

$$\nabla\alpha \cdot \boldsymbol{n} \geq 0. \tag{2.42}$$

Here, it is remarkable to notice that the boundary conditions for the internal variables descend naturally from the variational approach while for classical approaches they have in a certain sense to be postulated.

The fact that the dissipated power is not negative is here proven for time continuous evolutions. The total time derivative of (2.27) reads

$$\dot{\mathcal{D}}\big((\boldsymbol{\varepsilon}_p, \alpha), [0,t]\big) = \\ \int_\Omega \left(\mathsf{w}'(\alpha)\,\dot{\alpha} - \mathsf{q}(p)\,\mathsf{m}(\alpha)\,\dot{\alpha} + \frac{1}{2}\eta^2\,\nabla\alpha \cdot \nabla\dot{\alpha} \right) \mathrm{d}\Omega + \int_\Omega \sigma_{\mathsf{P}}(\alpha)\,\|\dot{\boldsymbol{\varepsilon}}_p(\tau)\|\ \mathrm{d}\Omega. \tag{2.43}$$

Since $\dot{\alpha} \in \mathcal{C}_+$ and $\boldsymbol{\varepsilon}_p \in \widetilde{\mathcal{Q}}$ the two terms in (2.43) are positive, the first because of the result (2.38) descending from the first order stability condition, the latter because of the constitutive assumptions (2.21).

2.2.2 Second order stability condition

Clearly the first order stability condition is only a necessary condition for stability when it is satisfied as an equality. In this section it is assumed that both (2.36) and (2.38) are satisfied as an equality. The case when one between the plasticity yield condition and the damage yield condition is still a strictly inequality is not considered since the study becomes simpler then in the presented general case. More specifically, the elastic-perfect plastic problem is always indefinite stable, see Fuchs and Seregin 2000; Temam 1985, while the stability of the elastic-damage model has been already extensively studied, for example in Pham 2010.

In order to achieve a sufficient condition it is essential to investigate the sign of the higher order terms in (1.61) as the second order one. To this aim, refering to Sec. 1.3.2.3, the second variation of the potential energy and the dissipation distance are necessary to express the second-order stability condition (**st-2D**). The second variation of the potential energy then reads

$$
\begin{aligned}
\mathcal{P}''\big(\boldsymbol{u},\boldsymbol{\varepsilon}_p,\alpha\big)\big(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{\varepsilon}}_p,\widetilde{\alpha}\big) &= \int_\Omega \mathbb{C}(\alpha)\big[\widetilde{\boldsymbol{\varepsilon}}-\widetilde{\boldsymbol{\varepsilon}}_p\big] : \big(\widetilde{\boldsymbol{\varepsilon}}-\widetilde{\boldsymbol{\varepsilon}_p}\big)\ \mathrm{d}\Omega \\
&+ \int_\Omega 2\,\mathbb{C}'(\alpha)\big[\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_p\big] : \big(\widetilde{\boldsymbol{\varepsilon}}-\widetilde{\boldsymbol{\varepsilon}_p}\big)\,\widetilde{\alpha}\ \mathrm{d}\Omega \\
&+ \int_\Omega \frac{1}{2}\mathbb{C}''(\alpha)\big[\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_p\big] : \big(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_p\big)\,\widetilde{\alpha}^2\ \mathrm{d}\Omega \\
&= \int_\Omega \mathbb{C}(\alpha)\big[\big(\widetilde{\boldsymbol{\varepsilon}}-\widetilde{\boldsymbol{\varepsilon}}_p\big)-\mathbb{S}(\alpha)[\boldsymbol{\sigma}]\,\widetilde{\alpha}\big] : \big(\big(\widetilde{\boldsymbol{\varepsilon}}-\widetilde{\boldsymbol{\varepsilon}}_p\big)-\mathbb{S}'(\alpha)[\boldsymbol{\sigma}]\,\widetilde{\alpha}\big)\ \mathrm{d}\Omega \\
&- \int_\Omega \frac{1}{2}\mathbb{S}''(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma}\,\widetilde{\alpha}^2\ \mathrm{d}\Omega
\end{aligned}
\tag{2.44}
$$

while the second variation of the total dissipation distance, through the same reasonings adopted for (2.32), is

$$
\begin{aligned}
D''\big(\boldsymbol{\varepsilon}_p,p,\alpha\big)\ \big(\widetilde{\boldsymbol{\varepsilon}}_p,\widetilde{p}(\widetilde{\boldsymbol{\varepsilon}}_p)\,,\widetilde{\alpha}\big) &= \int_\Omega \big(\sigma_{\mathsf{P}}'(\alpha)-\mathsf{q}'(p)\ \mathsf{m}(\alpha)\big)\,\|\widetilde{\boldsymbol{\varepsilon}}_p\|\,\widetilde{\alpha}\ \mathrm{d}\Omega \\
&+ \int_\Omega \big(\mathsf{w}''(\alpha)-\mathsf{q}(p)\ \mathsf{m}'(\alpha)\big)\,\widetilde{\alpha}^2\ \mathrm{d}\Omega \\
&+ \int_\Omega \eta^2\,\nabla\widetilde{\alpha}\cdot\nabla\widetilde{\alpha}\ \mathrm{d}\Omega.
\end{aligned}
\tag{2.45}
$$

The study of the sign of $\mathcal{P}''+D''$ becomes equivalent to the study of the minimum of the associated *Raiyleigh ratio* $\mathcal{R}\colon \widetilde{\mathcal{F}}\times\widetilde{\mathcal{Q}}\times\mathcal{C}_+ \to \mathbb{R}_0^+$ defined as

$$
\mathcal{R}\big(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{\varepsilon}}_p,\widetilde{\alpha}\big) = \frac{\mathcal{R}_N}{\mathcal{R}_D}
\tag{2.46}
$$

where $\mathcal{R}_N$ and $-\mathcal{R}_D$ collect respectively the positive and negative contributions of $\mathcal{P}''+D''$, namely (2.44) and (3.65). To consider the Rayleigh ratio is more convenient since the sum $\mathcal{P}''+D''$ could not be bounded from below. The explicit dependence of the present state $\big(\boldsymbol{u},\boldsymbol{\varepsilon}_p,p,\alpha\big)$ in the Rayleigh ratio, (2.46), has been omitted just for readability. In particular,

$$
\begin{aligned}
\mathcal{R}_N = \int_\Omega \mathbb{C}(\alpha)\big[\big(\widetilde{\boldsymbol{\varepsilon}}-\widetilde{\boldsymbol{\varepsilon}}_p\big)-\mathbb{S}(\alpha)[\boldsymbol{\sigma}]\,\widetilde{\alpha}\big] : \big(\big(\widetilde{\boldsymbol{\varepsilon}}-\widetilde{\boldsymbol{\varepsilon}}_p\big)-\mathbb{S}'(\alpha)[\boldsymbol{\sigma}]\,\widetilde{\alpha}\big)\ \mathrm{d}\Omega \\
+ \int_\Omega \eta^2\,\nabla\widetilde{\alpha}\cdot\nabla\widetilde{\alpha}\ \mathrm{d}\Omega
\end{aligned}
\tag{2.47}
$$

while

$$\mathcal{R}_D = \int_\Omega \left(\frac{1}{2} \mathsf{S}''(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma} - \mathsf{w}''(\alpha) + \mathsf{q}(p)\ \mathsf{m}'(\alpha) \right) \widetilde{\alpha}^2 \, \mathrm{d}\Omega \\ - \int_\Omega \left(\sigma'_\mathsf{P}(\alpha) - \mathsf{q}'(p)\ \mathsf{m}(\alpha) \right) \|\widetilde{\boldsymbol{\varepsilon}}_p\| \, \widetilde{\alpha} \, \mathrm{d}\Omega. \quad (2.48)$$

Here, the last term in (2.44) is considered negative, hence belonging to (2.48), because of assumptions (2.23).

In the most general case the second order stability condition then becomes a global minimization problem of the Rayleigh ratio. The state $(\boldsymbol{u}, \boldsymbol{\varepsilon}_p, p, \alpha)$ is stable if (only if)

$$\min_{\widetilde{\mathcal{F}} \times \widetilde{\mathcal{Q}} \times \mathcal{C}_+} \mathcal{R}(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\varepsilon}}_p, \widetilde{\alpha}) > (\geq)\, 1. \quad (2.49)$$

The proof of the existence of a minimum is assured by compactness and lower semi-continuity properties. A complete proof requires technical mathematical tools of functional analyses and therefore it is left out here.

An efficient procedure for the minimization problem (3.68) in case of an homogeneous state is suggested in Kohn 1991 and successfully applied for example in Pham 2010 for an elastic-damage model.

2.3 Energy balance

Once the total potential energy (2.25) and the dissipated work (2.27) have been introduced, the energy balance is immediately defined by (**EB**). The energy balance express the fact that the total energy must remain constant along the evolution.

In particular, by assuming smooth evolutions, equ. (1.66) gives

$$- \int_\Omega (\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b}) \cdot \dot{\boldsymbol{u}} \, \mathrm{d}\Omega + \int_{\partial\Omega} (\boldsymbol{\sigma}[\boldsymbol{n}] - \boldsymbol{f}) \cdot \dot{\boldsymbol{u}} \, \mathrm{dS} + \int_\Omega (\sigma_\mathsf{P}(\alpha) - \|\boldsymbol{\sigma}_d\|) \, \|\dot{\boldsymbol{\varepsilon}}_p\| \, \mathrm{d}\Omega \\ + \int_\Omega \left(-\frac{1}{2} \mathsf{S}'(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma} + \mathsf{w}'(\alpha) - \mathsf{q}(p)\, \mathsf{m}(\alpha) \right) \dot{\alpha} \, \mathrm{d}\Omega \\ + \int_\Omega \eta^2(\alpha) \, \nabla\alpha \cdot \nabla\dot{\alpha} \, \mathrm{d}\Omega = 0, \quad (2.50)$$

where the same remarks as in Sec. 2.2 has been adopted. Clearly, because of the first order stability condition, the equality (2.50) must hold for any $(\dot{\boldsymbol{u}}, \dot{\boldsymbol{\varepsilon}}_p, \dot{\alpha})$.

Hence, for $\dot{\boldsymbol{\varepsilon}}_p = \boldsymbol{0}$ and $\dot{\alpha} = 0$ and through the results (2.66) and (2.35), one simply obtains

$$\boldsymbol{\sigma}[\boldsymbol{n}] = \boldsymbol{f}_r(t), \qquad \forall\, \boldsymbol{x} \in \partial\Omega_D \quad (2.51)$$

which states the equilibrium of the internal forces with the external reactions at the constrained Dirichlet boundary $\partial\Omega_D$.

For $\dot{\boldsymbol{u}} = \boldsymbol{0}$ and $\dot{\alpha} = 0$ one gains the local condition

$$(\sigma_\mathsf{P}(\alpha) - \|\boldsymbol{\sigma}_d\|) \, \|\dot{\boldsymbol{\varepsilon}}_p\| = 0, \qquad \forall\, \boldsymbol{x} \in \Omega \quad (2.52)$$

and $\boldsymbol{\sigma}_d$ collinear with $\dot{\boldsymbol{\varepsilon}}_p$ which essentially expresses the associative character of the underlying plasticity model.

Finally, for $\dot{\boldsymbol{u}} = \mathbf{0}$ and $\dot{\varepsilon}_p = 0$ one finds either the weak condition

$$\int_\Omega \left(-\frac{1}{2}\mathbb{S}'(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma} + \mathsf{w}'(\alpha) - \mathsf{q}(p)\,\mathsf{m}(\alpha) \right) \dot{\alpha} + \eta^2\, \nabla\alpha \cdot \nabla\dot{\alpha}\ \mathrm{d}\Omega = 0, \quad \forall\, \boldsymbol{x} \in \Omega \tag{2.53}$$

or, in the case that $\alpha \in \mathrm{H}^2$, the strong local condition

$$\left(-\frac{1}{2}\mathbb{S}'(\alpha)[\boldsymbol{\sigma}] : \boldsymbol{\sigma} + \mathsf{w}'(\alpha) - \mathsf{q}(p)\,\mathsf{m}(\alpha) - \eta^2(\alpha)\,\mathrm{div}\nabla\alpha \right) \dot{\alpha} = 0, \quad \forall\, \boldsymbol{x} \in \Omega. \tag{2.54}$$

Moreover further damage boundary conditions also follows,

$$\nabla\alpha \cdot \boldsymbol{n}\,\dot{\alpha} = 0, \qquad \forall\, \boldsymbol{x} \in \partial\Omega. \tag{2.55}$$

Merging together (2.42) and (2.55) one obtains the final boundary conditions for the damage field which read

$$\nabla\alpha \cdot \boldsymbol{n} = 0, \qquad \forall\, \boldsymbol{x} \in \partial\Omega. \tag{2.56}$$

2.4 A particular model

In this section a particular elastic-plastic-damage model is considered descending from the more general one of the previous sections. This model is simply obtained by considering

$$\mathsf{m}(\alpha) = -\sigma_{\mathsf{P}}'(\alpha) \quad \text{and} \quad \mathsf{q}(p) = p, \tag{2.57}$$

which clearly satisfy the two assumptions (2.22). The dissipation potential (2.16) can then be rewritten as

$$\psi(p, \alpha, \nabla\alpha, \dot{p}, \dot{\alpha}, \nabla\dot{\alpha}) = \partial_t \left(\sigma_{\mathsf{P}}(\alpha)\, p + \mathsf{w}(\alpha) + \frac{1}{2}\eta^2 \nabla\alpha \cdot \nabla\alpha \right) \tag{2.58}$$

being $\partial_t\, (\sigma_{\mathsf{P}}(\alpha)\, p) = \sigma_{\mathsf{P}}(\alpha)\, \|\dot{\boldsymbol{\varepsilon}}_p\| + p\, \sigma_{\mathsf{P}}'(\alpha)\ \dot{\alpha}$.

The main advantage of this choice consists that the resulting dissipation distance allows the dissipated work to be a state function. Indeed, the dissipation distance density d becomes

$$d((p_0, \alpha_0, \nabla\alpha_0), (p_1, \alpha_1, \nabla\alpha_1)) = d((0,0,0), (p_1, \alpha_1, \nabla\alpha_1)) - d((0,0,0), (p_0, \alpha_0, \nabla\alpha_0)) \tag{2.59}$$

where $d((0,0,0), (p, \alpha, \nabla\alpha))$ is simply given by

$$d((0,0,0), (p, \alpha, \nabla\alpha)) = \sigma_{\mathsf{P}}(\alpha)\, p + \mathsf{w}(\alpha) + \frac{1}{2}\eta^2 \nabla\alpha \cdot \nabla\alpha. \tag{2.60}$$

The dissipated work then becomes a state function defined as

$$\mathcal{D}_D(p, \alpha) = \int_\Omega \left(\sigma_{\mathsf{P}}(\alpha)\, p + \mathsf{w}(\alpha) + \frac{1}{2}\eta^2 \nabla\alpha \cdot \nabla\alpha \right) \mathrm{d}\Omega. \tag{2.61}$$

In such framework it is possible to reformulate the variational problem by simply introducing a unique energy functional, namely the *total work* (or total energy) $\mathcal{T}$ of the system which turns out to be a state function. The total work then reads

$$\mathcal{T}(t, \boldsymbol{u}, \boldsymbol{\varepsilon}_p, p, \alpha) = \mathcal{P}(t, \boldsymbol{u}, \boldsymbol{\varepsilon}_p, \alpha) + \mathcal{D}_D(p, \alpha) \tag{2.62}$$

For this particular model, one could have postulated directly the total energy functional. The global stability condition (**ST**) changes in

$$\mathcal{T}\big(t, \boldsymbol{u}, \boldsymbol{\varepsilon}_p, p, \alpha\big) \leq \mathcal{T}\big(t, \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\varepsilon}}_p, p + \|\widetilde{\boldsymbol{\varepsilon}}_p\|, \widetilde{\alpha}\big)\,, \quad \forall\,\big(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\varepsilon}}_p, \widetilde{\alpha}\big) \in \mathcal{F} \times \mathcal{Q} \times \overline{\mathcal{C}}(\alpha)\,. \tag{2.63}$$

It is worth noting in (2.63) that one has assumed as variation for the accumulated plastic strain the state corresponding to a linear monotonic path,

$$\widetilde{p} = p + \|\widetilde{\boldsymbol{\varepsilon}}_p\|. \tag{2.64}$$

While this last choice is fully justified for the general model where a dissipation metric has been introduced, in this model and approach it becomes an assumption. Both the local and differential stability condition follows from the global stability condition. On the other hand the energy balance (**EB**) becomes

$$\mathcal{T}\big(t, \boldsymbol{u}, \boldsymbol{\varepsilon}_p, p, \alpha\big) - \int_0^t \partial_\tau \mathcal{T}\big(\tau, \boldsymbol{u}, \boldsymbol{\varepsilon}_p, p, \alpha\big)\, \mathrm{d}\tau = \text{constant}. \tag{2.65}$$

2.5 Conclusions and perspectives

Let here briefly recall the main results of the previous sections obtained through the energetic formulation. Results are divided with respect to the state variables. For what concerns the variable $\boldsymbol{u}$ one has found

$$\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} = 0, \quad \forall\, \boldsymbol{x} \in \Omega; \qquad \boldsymbol{\sigma}[\boldsymbol{n}] = \boldsymbol{f}, \quad \forall\, \boldsymbol{x} \in \partial\Omega, \tag{2.66}$$

which corresponds to the local equilibrium equations in classical Continuum Mechanics for small strains.
With regards to plasticity, one has recovered the classical local KKT system (1.40) with the plastic strain rate collinear to the deviatoric part of the stress (associative plasticity), namely

$$f_p(\alpha, \boldsymbol{\sigma}_d) \leq 0, \quad \lambda \geq 0, \quad f_p(\alpha, \boldsymbol{\sigma}_d)\, \lambda = 0, \qquad \forall\, \boldsymbol{x} \in \Omega, \tag{2.67}$$

with

$$\dot{\boldsymbol{\varepsilon}}_p = \lambda \frac{\partial}{\partial \boldsymbol{\sigma}} f_p(\alpha, \boldsymbol{\sigma}_d)\,. \tag{2.68}$$

Finally, for what concerns damage, in case of regular fields, say $\alpha \in \mathrm{H}^2$, one has recovered the classical local KKT system (1.40) as

$$f_d(p, \alpha, \nabla\alpha, \boldsymbol{\sigma}) \leq 0, \quad \dot{\alpha} \geq 0, \quad f_d(p, \alpha, \nabla\alpha, \boldsymbol{\sigma})\, \dot{\alpha} = 0, \qquad \forall\, \boldsymbol{x} \in \Omega, \tag{2.69}$$

with the following boundary conditions

$$\nabla\alpha \cdot \boldsymbol{n} = 0, \qquad \forall\, \boldsymbol{x} \in \partial\Omega. \tag{2.70}$$

Moreover, if $\alpha \in \mathrm{H}^1$, then the damage yield condition $f_d \leq 0$ has to be understood in its weak form (2.38).

The variational formulation allows rationally to deal with weak solutions. This will be crucial to catch particular material responses like a cohesive fracture, otherwise difficult to be described.

This chapter opens several directions for further investigations:

Enrichment of the dissipation potential Letting the dissipation potential (2.16) depending on the plastic strain ε_p would extend the range of possible models. This choice has not been taken into account here but could be an interesting point of departure for models whose behaviours depends on the plastic strain;

Plasticity hardening or softening effects The introduction of hardening or softening effects in the elastic potential energy $\mathcal{E}$ is worthy of being deepened;

Different choices for the regularization Different regularisation techniques are worthy to be investigated. Motivated by Chap. 4 the most appealing different choice seems to be to include terms depending on the gradient of plastic terms in the dissipation potential. As example on could add the the accumulated plastic strain gradient term ($\partial_t \left(\frac{1}{2}\ell^2 \nabla p \cdot \nabla p\right)$) to the dissipation potential (2.16) by eventually replacing it with the damage gradient term.

Different models in the variational setting Another straightforward step would be to consider, in the variational setting, different underlying models for plasticity and damage as for example the Gurson-Tvergaard model, often use in the description of ductile fracture, see Gurson 1977; Tvergaard 1981, 1982, and anysotropic damage models.

Chapter 3

Homogeneous evolution

In this chapter a one-dimensional model is investigated for a homogeneous evolution. The model descends from the general three-dimensional model of the previous chapter. After the introduction of the reduced model, a one-dimensional traction bar test is first considered for a prescribed monotonic increasing displacement at the right bar end which is equivalent to consider a material point response under an imposed monotonic increasing strain. Different responses are investigated which are able to highlight the main features of the proposed model. Then, numeric examples for the particular model are shown. Different examples can be found in the submitted article in Chap. B. Finally, attention is given to the stability of the homogeneous responses. Once the gradient model is recovered, a stability analysis is carried on leading to conditions for which the homogeneous solutions become unstable and hence localizations may appear with respect to plasticity and/or damage.

3.1 Introduction to the 1D model

In this section the one-dimensional plastic-damage model is introduced descending from the more general three-dimensional model of the previous chapter.
In this context, the physical domain is a closed subset of the real line of length L,

$$\Omega = [0, L]. \tag{3.1}$$

For simplicity, only hard devices, Tab. 1.1, are taken into account. Hereafter one can assume, without lose of generality, that

$$u(0) = 0, \quad u(L) = U(t)\,, \tag{3.2}$$

where $t \in [0, T]$ is the time-evolution parameter. The statements (3.1) and (3.2) represent nothing but a 1D traction bar test, Fig. 3.1. Throughout the following, an

Figure 3.1: 1D bar traction test

initially unstretched, unplastified and undamaged state is always considered,

$$u(x,0)=0, \quad \varepsilon_p(x,0)=0, \quad p(x,0)=0, \quad \alpha(x,0)=0. \tag{3.3}$$

The stored elastic energy reads

$$\mathcal{E}\big(u,\varepsilon_p,\alpha\big)=\int_0^{\mathrm{L}} e\big(\varepsilon,\varepsilon_p,\alpha\big)\;\mathrm{d}x=\int_0^{\mathrm{L}}\frac{1}{2}\mathsf{E}(\alpha)\,\big(\varepsilon-\varepsilon_p\big)^2\;\mathrm{d}x, \tag{3.4}$$

from which the stress σ is simply given by

$$\sigma=\mathsf{E}(\alpha)\,\big(\varepsilon-\varepsilon_p\big)\,. \tag{3.5}$$

The elastic tensor now reduces to a scalar function $\mathsf{E}\colon [0,1]\to\mathbb{R}^+$ with the following properties,

$$\mathsf{E}(0)=\mathsf{E}_0, \qquad \mathsf{E}(1)=0, \qquad \mathsf{E}'(1)=0 \tag{3.6}$$

and

$$\mathsf{E}(\alpha)>0, \qquad \mathsf{E}'(\alpha)<0, \quad \forall\,\alpha\in[0,1). \tag{3.7}$$

Therefore, the compliance modulus $\mathsf{S}(\alpha)\,:\,[0,1]\to\mathbb{R}_+$, is defined as

$$\mathsf{S}(\alpha)=\frac{1}{\mathsf{E}(\alpha)}, \qquad \mathsf{S}'(\alpha)=-\frac{\mathsf{E}'(\alpha)}{\mathsf{E}(\alpha)^2}, \qquad \mathsf{S}''(\alpha)=2\frac{\mathsf{E}'(\alpha)}{\mathsf{E}(\alpha)^3}-\frac{\mathsf{E}''(\alpha)}{\mathsf{E}(\alpha)^4}. \tag{3.8}$$

Assumptions (3.6) and (3.7) are a consequence of (1.90). Moreover, all the conditions in Sec. 2.1.2 about the three-dimensional model are properly still considered valid for the 1D model. The stress-displacement relation becomes

$$\sigma=\frac{1}{\displaystyle\int_0^{\mathrm{L}}\mathsf{S}(\alpha)\;\mathrm{d}x}\left(U(t)-\int_0^{\mathrm{L}}\varepsilon_p\;\mathrm{d}x\right) \tag{3.9}$$

The potential of external work reads

$$\mathcal{L}(t,u)=f_r(L)\;U(t)\,. \tag{3.10}$$

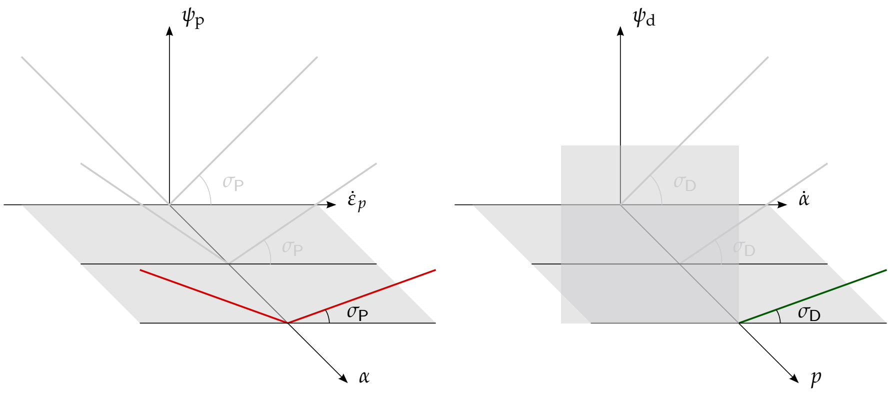
For what concerns the dissipation potential, the further assumptions are made

$$\mathsf{q}(p)=p, \quad \eta(\alpha)=\eta, \tag{3.11}$$

where $(3.11)_1$ has been already motivated about the particular model in Sec. 2.4. A clearer explanation of (3.11) will be given in the next section about the homogeneous responses. The expression of the dissipation potential follows from (2.16) and becomes

$$\begin{aligned}\psi\big(p,\alpha,\alpha',\dot p,\dot\alpha,\dot\alpha'\big)&=\psi_p(\alpha,\dot p)+\psi_d\big(p,\alpha,\dot\alpha,\dot\alpha'\big)\\&:=\sigma_{\mathsf{P}}(\alpha)\,\dot p+\partial_t\,\mathsf{w}(\alpha)-p\,\mathsf{m}(\alpha)\,\dot\alpha+\partial_t\left(\frac{1}{2}\eta^2\alpha'^2\right).\end{aligned} \tag{3.12}$$

Once the dissipation potential is given both the dissipation distance and the dissipated work are defined. Qualitative trends of the plastic and damage dissipation potentials are illustrated in Fig. 3.2. The yield stresses are related to the slope of



(a) *plastic dissipation potential* (b) *damage dissipation potential*

Figure 3.2: Qualitative admissible trends for the plastic and damage dissipation potentials with respect to the internal variables

the curves. For the damage dissipation potential only positive damage rates are accessible due to irreversibility. Not that the trends are exacly those of a positively one-degree homogeneous functions.

Having assumed sufficiently regular evolutions the results of Sec. 2.5 can be used.
Hence the governing equations become:

$$\begin{array}{c} \sigma' = 0, \\ \displaystyle\int_0^L u' \,\mathrm{d}x = U(t) \end{array} \quad ; \quad \begin{cases} f_p(\sigma, \alpha) \leq 0 \\ \lambda \geq 0 \\ f_p(\sigma, \alpha)\, \lambda = 0 \\ \dot{\varepsilon}_p = \operatorname{sign}(\sigma)\, \lambda \end{cases} \quad ; \quad \begin{cases} f_d(\sigma, p, \alpha, \alpha'') \leq 0 \\ \dot{\alpha} \geq 0 \\ f_d(\sigma, p, \alpha, \alpha'')\, \dot{\alpha} = 0 \end{cases} \tag{3.13}$$

where f_p in (2.37) and f_d in (2.41) change as follows

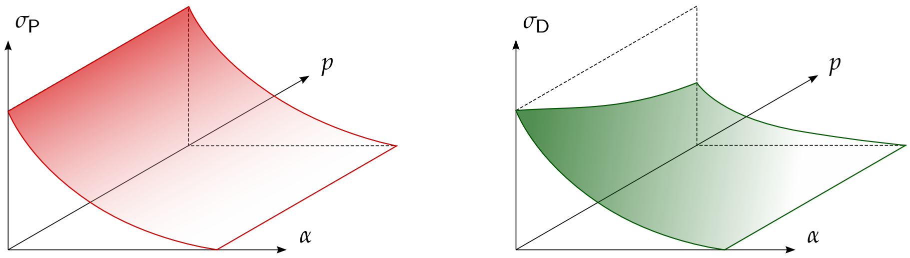
$$f_p(\sigma, \alpha) := |\sigma| - \sigma_\mathrm{P}(\alpha) \leq 0, \tag{3.14}$$

$$f_d(\sigma, p, \alpha, \alpha') := \frac{1}{2}\mathsf{S}'(\alpha)\, \sigma^2 + p\, \mathsf{m}(\alpha) - \mathsf{w}'(\alpha) + \eta^2 \alpha'' \leq 0. \tag{3.15}$$

Clearly, as previously mentioned, conditions $(3.13)_3$ must to be replaced by their weak form in cases where the derivative of the damage field suffers jumps, $\alpha \in \mathrm{H}^1$.
The yield functions (3.14) and (3.15) allow for the following yield stresses

$$\sigma_\mathrm{P}(\alpha), \quad \sigma_\mathrm{D}(p, \alpha, \alpha'') := \sqrt{\frac{2\,(\mathsf{w}'(\alpha) - p\,\mathsf{m}(\alpha) - \eta^2 \alpha'')}{\mathsf{S}'(\alpha)}}, \tag{3.16}$$

where the coupling between the two phenomena is clearly noticeable, Fig. 3.3.



(a) *plastic dissipation potential* (b) *damage dissipation potential*

Figure 3.3: Qualitative admissible trends for the plastic and damage yield stresses

3.2 The abstract homogeneous evolutions

In this section the homogeneous response is investigated. The response is considered homogeneous if the state variables are constant in space at any instant t. Clearly in this setting

$$\alpha'(x) = 0, \tag{3.17}$$

and the dependence α' or α'' has to be omitted in all the defined quantities, like the yield functions or yield stresses. Moreover, the non-local model is reduced to a local one where all variables becomes independent with respect to the abscissa x.

The local model is able to describe some key behaviours of the plastic-damage model like the conditions under which one or both phenomena are triggered or when coupled responses occur. To highlight such behaviours it is sufficient to analyze the response of a single material point subjected to a prescribed strain, which assumes the meaning of an external action. The applied strain is assumed to be linearly increasing with t, Fig. 3.4, since the process is rate independent. For simplicity

$$\varepsilon(t) = t \tag{3.18}$$

is assumed.

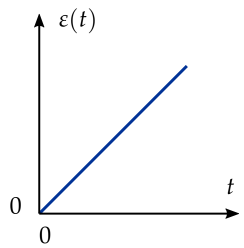


Figure 3.4: The strain load history for the homogeneous evolution

The most simple but meaningful qualitative evolutions expected from the response are summarized in Fig. 3.5. A deep analysis of these is the main subject of this chapter. Each of them owns a purely elastic phase. After one yield limit is reached, evolution may continue either with a plastic phase or a damage phase. If a second yield point occurs, then a coupled evolution of plasticity and damage is possible[1]. All these scenarios will be now investigated. More complicated evolutions, like an alternate of dissipation phenomena, are here not taken into account since they do not add significant aspects to the response. Hereafter the first instant where plasticity (damage) occurs is called t_{P} (t_{D}). On the other hand, t_I denotes the first yield instant, hence $t_I = \min(t_{\mathrm{P}}, t_{\mathrm{D}})$, while t_{II} denotes the second yield instant with $t_{II} = \max(t_{\mathrm{P}}, t_{\mathrm{D}})$.

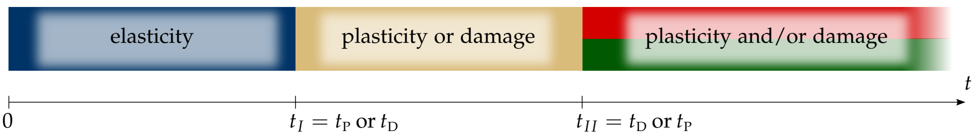


Figure 3.5: The main model responses for the general elastic-plastic-damage evolution

In general, the possible evolutions can be divided in two groups depending on whether plasticity or damage occurs first:

- Plasticity as first dissipation phenomenon (*E-P-** model)[2];
- Damage as first dissipation phenomenon (*E-D-** model).

Clearly they depend on the initial yield stresses:

$$\begin{aligned} \textit{E-P-*} \quad &\Longleftrightarrow \quad \sigma_{\mathsf{P}}(0) < \sigma_{\mathsf{D}}(0,0)\,, \\ \textit{E-D-*} \quad &\Longleftrightarrow \quad \sigma_{\mathsf{P}}(0) > \sigma_{\mathsf{D}}(0,0)\,. \end{aligned}$$

Hereafter the exposition proceeds for each phase (elastic, first dissipation phenomenon, second dissipation phenomenon) and bifurcates in parallel when differences between the *E-P-** and *E-D-** sequences arise.
The following passages rely on the explicit evaluation of the consistency condition both for plasticity and damage. Hence, assuming here without loss of generality, $\sigma \geq 0$ and as state variables ε, ε_p, p and α on has:

$$\dot{f}_p(\sigma,\alpha) = \mathsf{E}(\alpha)\,(1-\dot{\varepsilon}_p) + \left(\mathsf{E}'(\alpha)\,(t-\varepsilon_p) - \sigma'_{\mathsf{P}}(\alpha)\right)\dot{\alpha} \tag{3.19}$$

$$\begin{aligned} \dot{f}_d(\sigma,p,\alpha) = &-\mathsf{E}'(\alpha)\,(t-\varepsilon_p)(1-\dot{\varepsilon}_p) + \mathsf{m}(\alpha)\,|\dot{\varepsilon}_p| \\ &- \left(\frac{1}{2}\mathsf{E}''(\alpha)\,(t-\varepsilon_p)^2 - p\,\mathsf{m}'(\alpha) + \mathsf{w}''(\alpha)\right)\dot{\alpha}. \end{aligned} \tag{3.20}$$

3.2.1 Elastic phase ($t < t_I$)

The elastic phase is common to any response. In such phase,

$$\sigma < \min\left(\sigma_{\mathsf{P}}(0)\,, \sigma_{\mathsf{D}}(0,0)\right),$$

[1] Clearly considering further yield points where the coupled response changes is of any interest under the mathematical point of view.

[2] From here on the "*" symbol stands for the special character in regular expressions. In *information technology* it is used to match any character(s)

or equivalently

$$f_p(\sigma, 0) < 0, \qquad f_d(\sigma, 0, 0) < 0,$$

with

$$\sigma = \mathsf{E}_0\, \varepsilon(t)\,.$$

Since the stress is linearly increasing and since the yield functions tends to infinity as long as the stress tends to infinity, admitting a sufficiently large T, an instant t_I clearly always exists.
The elastic phase then ended when the instant t_I is reached. Whether the plastic or damage yield limit is reached first the response falls into a *E-P-** sequence or a *E-D-** sequence:

E-P-* sequence:

The first yield instant corresponds to

$$t_I = t_{\mathrm{P}},$$

where

$$\sigma = \sigma_{\mathsf{P}}(0) < \sigma_{\mathsf{D}}(0, 0)$$

and

$$t_I = \frac{\sigma_{\mathsf{P}}(0)}{\mathsf{E}_0}. \tag{3.21}$$

E-D-* sequence:

The first yield instant corresponds to

$$t_I = t_{\mathrm{D}},$$

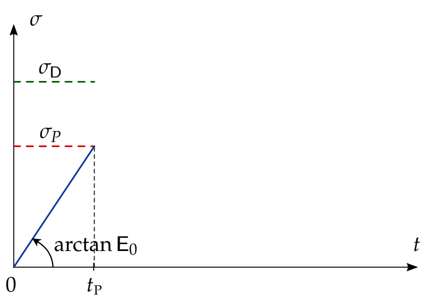
where

$$\sigma = \sigma_{\mathsf{D}}(0, 0) < \sigma_{\mathsf{P}}(0)\,,$$

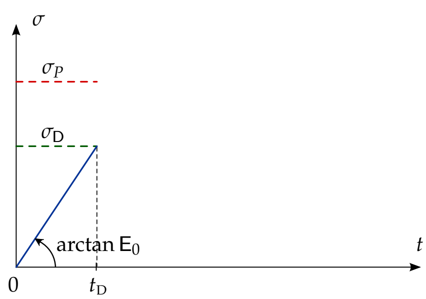
and

$$t_I = \sqrt{\frac{2\mathsf{w}'(0)}{-\mathsf{E}'(0)}}. \tag{3.22}$$

The elastic response for both sequences is represented with the trends of the stress yield limits in Fig. 3.6.



(a) *E-P-* sequence*



(b) *E-D-* sequence*

Figure 3.6: Elastic response

3.2.2 First dissipation phase ($t_I \leq t < t_{II}$)

At this point it is necessary to remark that the existence of a finite value for t_{II} is not always ensured. So, at this stage, it may be infinite.

In the following the analysis is carried on first for the *E-P-** responses and then for the *E-D-** responses.

E-P-* **sequence** ($t_{\mathrm{P}} \leq t < t_{\mathrm{D}}$)

All the possible evolutions in such phase have to be investigated. Observing that at $t = t_{\mathrm{P}}$

$$\begin{cases} f_p(\sigma, 0) = 0 \\ f_d(\sigma, 0, 0) < 0 \end{cases}, \tag{3.23}$$

the evolutions to be analysed, dictated by the KKT conditions (3.13), are:

$\begin{cases} \dot{\varepsilon}_p = 0 \\ \dot{\alpha} = 0 \end{cases}$	1	elastic evolution	$\Rightarrow \quad \dot{f}_p\|_{t_{\mathrm{P}}<t<t_{\mathrm{D}}} \leq 0$
$\begin{cases} \dot{\varepsilon}_p \neq 0 \\ \dot{\alpha} = 0 \end{cases}$	2	plastic evolution	$\Rightarrow \quad \dot{f}_p\|_{t_{\mathrm{P}}<t<t_{\mathrm{D}}} = 0$

To find out the right evolution one observes that

1 **- elastic evolution** An elastic evolution occurs if

$$\dot{f}_p|_{t_{\mathrm{P}} \leq t < t_{\mathrm{D}}} = \mathsf{E}_0 \leq 0 \tag{3.24}$$

which is impossible;

2 **- plastic evolution** A plastic evolution occurs if

$$\dot{f}_p|_{t_{\mathrm{P}} \leq t < t_{\mathrm{D}}} = \mathsf{E}_0 \left(1 - \dot{\varepsilon}_p\right) = 0 \tag{3.25}$$

which is admissible, leading to

$$\varepsilon_p = t - \frac{\sigma_{\mathsf{P}}(0)}{\mathsf{E}_0}, \quad \forall\, t > t_{\mathrm{P}}. \tag{3.26}$$

E-D-* **sequence** ($t_{\mathrm{D}} \leq t < t_{\mathrm{P}}$)

All the possible evolutions in such phase have to be investigated. Observing that at $t = t_{\mathrm{D}}$

$$\begin{cases} f_p(\sigma, 0) < 0 \\ f_d(\sigma, 0, 0) = 0 \end{cases}, \tag{3.27}$$

the evolutions to investigate, dictated by the KKT conditions (3.13) are:

$\begin{cases} \dot{\varepsilon}_p = 0 \\ \dot{\alpha} = 0 \end{cases}$	1	elastic evolution	$\Rightarrow \quad \dot{f}_d\|_{t_{\mathrm{D}}<t<t_{\mathrm{P}}} \leq 0$
$\begin{cases} \dot{\varepsilon}_p = 0 \\ \dot{\alpha} > 0 \end{cases}$	2	damage evolution	$\Rightarrow \quad \dot{f}_d\|_{t_{\mathrm{D}}<t<t_{\mathrm{P}}} = 0$

To find out the right evolution one observes that

1 **- elastic evolution** An elastic evolution occurs if

$$\dot{f}_d|_{t_{\mathrm{D}} \leq t < t_{\mathrm{P}}} = -\mathsf{E}'(\alpha)\, t \leq 0 \tag{3.28}$$

which is impossible;

(2) - damage evolution A damage evolution occurs if

$$\dot{f}_d|_{t_{\mathrm{D}} \le t < t_{\mathrm{P}}} = -\mathsf{E}'(\alpha)\, t - \Big(\mathsf{E}''(\alpha)\, t^2 + \mathsf{w}''(\alpha)\Big)\, \dot{\alpha} = 0 \tag{3.29}$$

which is admissible provided that

$$\mathsf{E}''(\alpha)\, t^2 + \mathsf{w}''(\alpha) > 0 \tag{3.30}$$

which is satisfied since it correspond to the strain-hardening property assumed in (2.24). In such case $\alpha = \alpha(t)$ is the solution of the ordinary differential equation 3.29.

The evolution is the solution of the following equations,

E-P-* sequence:

Evolution of the plastic dissipation phase:

$$\begin{cases} \varepsilon_p = p = t - \dfrac{\sigma_{\mathrm{P}}(\alpha_0)}{\mathsf{E}(\alpha_0)} \\ \alpha = 0, \end{cases}$$

with

$$\sigma = \sigma_{\mathrm{P}}(0) < \sigma_{\mathrm{D}}(p, 0)\,. \tag{3.31}$$

E-D-* sequence:

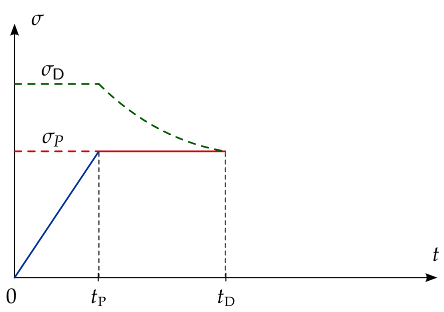
Evolution of the damage dissipation phase:

$$\begin{cases} \varepsilon_p = p = 0, \\ \alpha = \text{given by 3.29.} \end{cases}$$

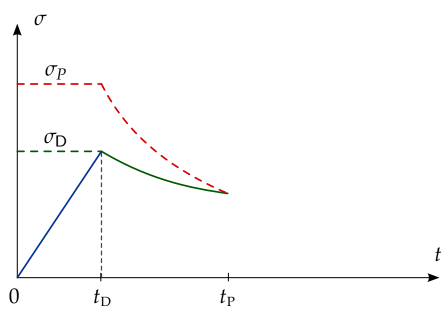
with

$$\sigma = \sigma_{\mathrm{D}}(0, \alpha) < \sigma_{\mathrm{P}}(\alpha)\,. \tag{3.32}$$

which are respectively represented in Fig. 3.7.



(a) *The E-P-* sequence*



(b) *The E-D-* sequence*

Figure 3.7: First dissipation reponse

Whether a second yield point exists depends on the following conditions.

E-P-* **sequence:**

The second yield instant, which always exists, corresponds to

$$t_{II} = t_{\text{D}}$$

where:

$$\sigma = \sigma_{\mathsf{P}}(0) = \sigma_{\mathsf{D}}(\varepsilon_p, 0)$$

and

$$t_{II} = \sqrt{\frac{2\left(\mathsf{w}'(0) - p\,\mathsf{m}(0)\right)}{-\mathsf{E}'(0)}} + \varepsilon_p. \quad (3.33)$$

E-D-* **sequence:**

The second yield instant

$$t_{II} = t_{\text{P}}$$

exists if $t \mapsto \alpha(t)$ given by (3.29) is such that:

$$\sigma = \sigma_{\mathsf{D}}(0, \alpha) = \sigma_{\mathsf{P}}(\alpha)\,.$$

In this case

$$t_{II} = \frac{\sigma_{\mathsf{P}}(\alpha)}{\mathsf{E}(\alpha)}. \quad (3.34)$$

3.2.3 Second dissipation phase ($t_{II} \leq t$)

If a second dissipation instant exists, where the yield condition of the inactivated dissipation phenomenon occurs, $t = t_{II}$, then from that point different evolutions are possible. The analysis carried out in the following leads to different response sequences depending on the constitutive functions.

The instant $t = t_{II}$, if it exists, means for any sequence that

E-P-* **sequence** ($t = t_{\text{D}} = t_{II}$)

$$\begin{cases} f_p(\sigma(t_{II})\,, 0) = 0 \\ f_d(\sigma(t_{II})\,, p(t_{II})\,, 0) = 0 \end{cases} ; \quad (3.35)$$

E-D-* **sequence** ($t = t_{\text{P}} = t_{II}$)

$$\begin{cases} f_p(\sigma(t_{II})\,, \alpha(t_{II})) = 0 \\ f_d(\sigma(t_{II})\,, 0, \alpha(t_{II})) = 0 \end{cases} . \quad (3.36)$$

In principle from the instant t_{II}, regardless to the previous response, several different evolutions are possible,

$\begin{cases} \dot\varepsilon_p = 0 \\ \dot\alpha = 0 \end{cases}$	1	elastic evolution	$\Rightarrow$	$\begin{cases} \dot f_p\vert_{t_{II}\leq t} \leq 0 \\ \dot f_d\vert_{t_{II}\leq t} \leq 0 \end{cases}$
$\begin{cases} \dot\varepsilon_p \neq 0 \\ \dot\alpha = 0 \end{cases}$	2	plastic evolution	$\Rightarrow$	$\begin{cases} \dot f_p\vert_{t_{II}\leq t} = 0 \\ \dot f_d\vert_{t_{II}\leq t} \leq 0 \end{cases}$
$\begin{cases} \dot\varepsilon_p = 0 \\ \dot\alpha \neq 0 \end{cases}$	3	damage evolution	$\Rightarrow$	$\begin{cases} \dot f_p\vert_{t_{II}\leq t} \leq 0 \\ \dot f_d\vert_{t_{II}\leq t} = 0 \end{cases}$
$\begin{cases} \dot\varepsilon_p \neq 0 \\ \dot\alpha \neq 0 \end{cases}$	4	pl.-dam. evolution	$\Rightarrow$	$\begin{cases} \dot f_p\vert_{t_{II}\leq t} = 0 \\ \dot f_d\vert_{t_{II}\leq t} = 0 \end{cases}$

Which type of evolution may appear is discussed in the following by assuming, without loss of generality and for simplicity, that the present state is given by the state variables ε, ε_p, p and α regardless to the previously considered sequence *E-P-** or *E-D-**:

① **- elastic evolution** An elastic evolution occurs if

$$\dot{f}_p|_{t_{II}\leq t} = \mathsf{E}(\alpha) \leq 0, \tag{3.37}$$
$$\dot{f}_d|_{t_{II}\leq t} = -\mathsf{E}'(\alpha)\left(t - \varepsilon_p\right) \leq 0, \tag{3.38}$$

which is impossible because $\mathsf{E}(\alpha) > 0$, $\forall\, \alpha < 1$.

② **- plastic evolution** A plastic evolution occurs if

$$\dot{f}_p|_{t_{II}\leq t} = \mathsf{E}(\alpha)\left(1 - \dot{\varepsilon}_p\right) = 0, \tag{3.39}$$
$$\dot{f}_d|_{t_{II}\leq t} = -\mathsf{E}'(\alpha)\left(t - \varepsilon_p\right)\left(1 - \dot{\varepsilon}_p\right) + \mathsf{m}(\alpha)\,|\dot{\varepsilon}_p| \leq 0. \tag{3.40}$$

Condition (3.39) gives $\dot{\varepsilon}_p = 1$. Hence, the inequality 3.40 is valid if $\mathsf{m}(\alpha) \leq 0$, which is in contrast with the assumption (2.22). In fact, the condition for reaching the second yield point states $\mathsf{m}(\alpha) \geq 0$, as evident in (3.33).
Then a plastic evolution without damaging is impossible once the damage criterion has been attained at least in a past instant.

③ **- damage evolution** A damage evolution occurs if

$$\dot{f}_p|_{t_{II}\leq t} = \mathsf{E}(\alpha) + \left(\mathsf{E}'(\alpha)\left(t - \varepsilon_p\right) - \sigma_{\mathsf{P}}'(\alpha)\right)\dot{\alpha} \leq 0, \tag{3.41}$$
$$\dot{f}_d|_{t_{II}\leq t} = -\mathsf{E}'(\alpha)\left(t - \varepsilon_p\right) - \left(\frac{1}{2}\mathsf{E}''(\alpha)\left(t - \varepsilon_p\right)^2 - p\,\mathsf{m}'(\alpha) + \mathsf{w}''(\alpha)\right)\dot{\alpha} = 0. \tag{3.42}$$

In such case the discussion becomes straightforward. It is useful to preliminary define from both relations (3.41) and (3.42) the quantities

$$R := -\,\mathsf{E}'(\alpha)\left(t - \varepsilon_p\right) + \sigma_{\mathsf{P}}'(\alpha) > 0, \tag{3.43}$$
$$S := \frac{1}{2}\mathsf{E}''(\alpha)\left(t - \varepsilon_p\right)^2 - p\,\mathsf{m}'(\alpha) + \mathsf{w}''(\alpha) > 0. \tag{3.44}$$

The functions R and S are introduced because several properties, like stability, rely on them. In particular, the meaning of $R = R(\alpha)$ is related to the decreasing rate with respect to damage of the stress σ and the plastic yield stress σ_{P}. Indeed, considering $\sigma = \sigma(\alpha) = \mathsf{E}(\alpha)\left(t - \varepsilon_p\right)$ one has

$$\begin{cases} \sigma'(\alpha) < \sigma_{\mathsf{P}}'(\alpha)\,, & R > 0, \\ \sigma'(\alpha) > \sigma_{\mathsf{P}}'(\alpha)\,, & R < 0. \end{cases} \tag{3.45}$$

Figure 3.8 represents condition (3.41).

From equation (3.42)

$$\dot{\alpha} = \frac{-\mathsf{E}'(\alpha)\left(t - \varepsilon_p\right)}{H}. \tag{3.46}$$

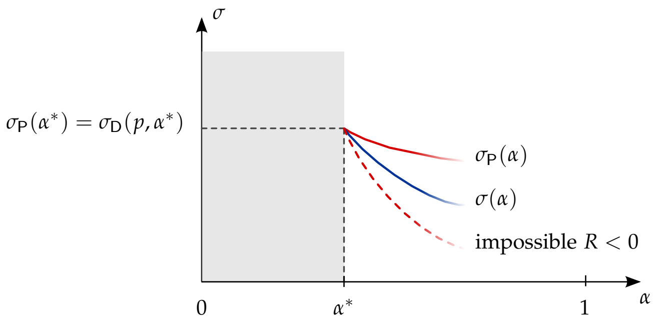


Figure 3.8: Condition $R > 0$ for allowing a damage evolution without plastic evolution from the second yield instant on

Replacing the obtained damage rate $\dot{\alpha}$ in 3.41 gives:

$$\mathsf{E}(\alpha)\; S + \mathsf{E}'(\alpha)\, \big(t - \varepsilon_p\big)\; R \leq 0. \tag{3.47}$$

Finally, the previous condition can be posed as

$$0 < S \leq \frac{-\mathsf{E}'(\alpha)\, \big(t - \varepsilon_p\big)}{\mathsf{E}(\alpha)} R =: T \tag{3.48}$$

where the first inequality descends from the strain-hardening assumption (2.24) and where the last term is clearly always positive .

4 **- plastic and damage evolution** A coupled plasticity-damage evolution occurs, if

$$\dot{f}_p|_{t_{II} \leq t} = \mathsf{E}(\alpha)\,(1 - \dot{\varepsilon}_p) + \big(\mathsf{E}'(\alpha)\,(t - \varepsilon_p) - \sigma_\mathsf{P}'(\alpha)\big)\,\dot{\alpha} = 0, \tag{3.49}$$

$$\begin{aligned} \dot{f}_d|_{t_{II} \leq t} = & -\mathsf{E}'(\alpha)\,(t - \varepsilon_p)(1 - \dot{\varepsilon}_p) + \mathsf{m}(\alpha)\,|\dot{\varepsilon}_p| \\ & - \left(\frac{1}{2}\mathsf{E}''(\alpha)\,(t - \varepsilon_p)^2 - p\,\mathsf{m}'(\alpha) + \mathsf{w}''(\alpha)\right)\dot{\alpha} = 0. \end{aligned} \tag{3.50}$$

Through (3.49) one has

$$1 - \dot{\varepsilon}_p = \frac{R}{\mathsf{E}(\alpha)}\dot{\alpha} \tag{3.51}$$

that replaced in (3.50) gives

$$\left(\frac{-\mathsf{E}'(\alpha)\,\big(t - \varepsilon_p\big)}{\mathsf{E}(\alpha)} R - S\right)\dot{\alpha} - \mathsf{m}(\alpha)\,|\dot{\varepsilon}_p| = 0 \tag{3.52}$$

and finally

$$\frac{-\mathsf{E}'(\alpha)\,\big(t - \varepsilon_p\big)}{\mathsf{E}(\alpha)} R - S < 0. \tag{3.53}$$

The evolution condition then becomes

$$S > T > 0. \tag{3.54}$$

	sequence label	condition	figure
*E-P-** $\Rightarrow$	*E-P-D*,	$S \leq T$	3.9a
	E-P-DP,	$S > T$	3.10a
*E-D-** $\Rightarrow$	*E-D*,	$S \leq T$	3.9b
	E-D-PD,	$S > T$	3.10b

Table 3.1: The four simplest and most meaningful considered evolutions

Tab. 3.1 summarise the simplest but most meaningful possible evolutions.

The state variables evolutions for responses where plasticity and damage never evolve together are found as follows,

E-P-D sequence:

For $t > t_{II} = t_P$ the plastic and damage evolutions correspond to

$$\begin{cases} \varepsilon_p & = t_{II} - t_I \\ \alpha & = \left(f_d(\sigma, p(t_{II}), \alpha)\right)^{-1} \end{cases}$$

where t_I and t_{II} are respectively given by (3.21) and (3.33) with

$$\sigma = \sigma_\text{D}(p(t_{II}), \alpha) < \sigma_\text{P}(\alpha) \tag{3.55}$$

and

$$\sigma = \mathsf{E}(\alpha)\left(t - \varepsilon_p(t_{II})\right).$$

E-D sequence:

For $t > t_{II} = t_P$ the plastic and damage evolutions correspond to

$$\begin{cases} \varepsilon_p & = p = 0, \\ \alpha & = \left(f_d(\sigma(t), 0, \alpha)\right)^{-1} \end{cases}$$

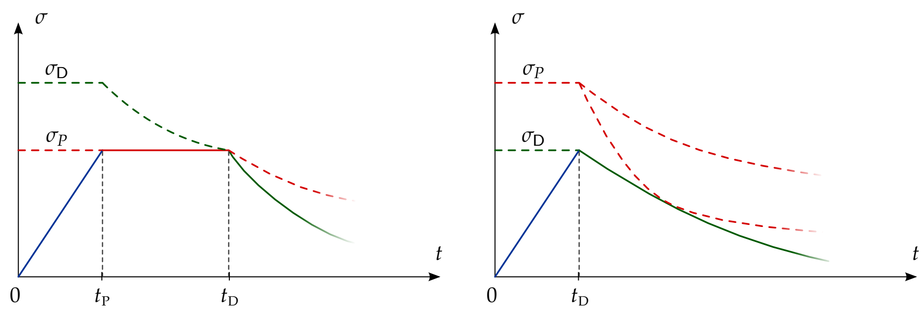
with

$$\sigma = \sigma_\text{D}(0, \alpha) < \sigma_\text{P}(\alpha) \tag{3.56}$$

and

$$\sigma = \mathsf{E}(\alpha)\ t.$$

Both these responses are qualitatively represented in Fig. 3.9.



(a) *E-P-D sequence.* (b) *E-D sequence.*

Figure 3.9: Evolutions where at each instant only a dissipation phenomenon occurs.

The state variables evolutions for responses where a coupling between plasticity and damage occurs are found as follows,

***E-P-DP* sequence** and ***E-D-PD* sequence**

$$\varepsilon_p,\ \alpha \rightarrow \text{given by (3.51) and (3.52)}$$

where the proper boundary condition, that is $p(t_{II})$, $\varepsilon_p(t_{II})$ and $\alpha(t_{II})$ has to be considered. For $t \geq t_{II}$

$$f_d(\sigma, p, \alpha) = f_p(\sigma, \alpha) = 0$$

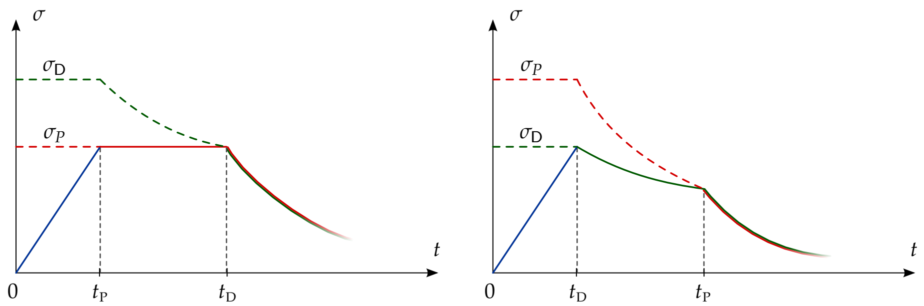
with

$$\sigma = \sigma_{\mathsf{P}}(\alpha) = \sigma_{\mathsf{D}}(p, \alpha) \tag{3.57}$$

and

$$\sigma = \mathsf{E}(\alpha)\left(t - \varepsilon_p\right).$$

Both these responses are represented in Fig. 3.10.



(a) *E-P-* sequence.* (b) *E-D-* sequence.*

Figure 3.10: Second dissipation phenomenon with coupled evolution

The assumptions (2.21) and (2.22) should now be clear and motivated respectively in Tab. 3.3 and Tab. **??**.

$\sigma'_{\mathsf{P}}(\alpha) > 0$	equivalent plastic stress increasing with damage (unrealistic)
$\sigma'_{\mathsf{P}}(\alpha) = 0$	damage that doe not affect the plasitc yield criterion
$\sigma'_{\mathsf{P}}(\alpha) < 0$	equivalent plastic stress decreasing with damage (realistic)

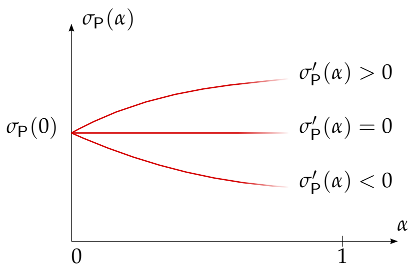


Table 3.2: Motivation for the assumption for the function $\alpha \mapsto \sigma_{\mathsf{P}}(\alpha)$

3.3 Analytic examples

In this section a simple and straightforward analytical homogeneous 1D model is presented capable of highlighting all meaningful sequences already introduced in Tab. 3.1.

$\mathsf{m}(\alpha) > 0$	damage yield stress increasing with plasticity (unrealistic)
$\mathsf{m}(\alpha) = 0$	plasticity that does not affect the damage yield criterion
$\mathsf{m}(\alpha) < 0$	damage yield stress decreasing with plasticity (realistic)

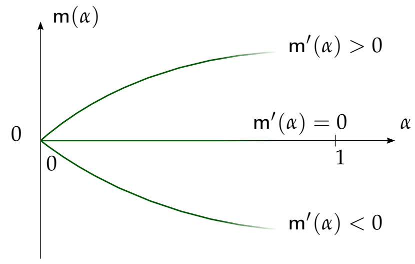


Table 3.3: Motivation for the assumption for the function $\alpha \mapsto \mathsf{m}(\alpha)$

Simple analytical expressions are chosen for the constitutive functions in (3.4) and (3.12), namely

$$\mathsf{E}(\alpha) = \mathsf{E}_0\,(1-\alpha)^2\,, \quad \mathsf{w}(\alpha) = \mathsf{w}_0\,\alpha, \quad \sigma_{\mathsf{P}}(\alpha) = \sigma_{\mathsf{P}_0}\,(1-\alpha)^{\gamma}\,, \quad \mathsf{m}(\alpha) = -\sigma'_{\mathsf{P}}(\alpha)\,, \tag{3.58}$$

which depends upon three real positive parameters and one exponent, that is

$$\mathsf{E}_0 > 0, \quad \mathsf{w}_0 > 0, \quad \sigma_{\mathsf{P}_0} > 0, \quad \gamma \geq 1. \tag{3.59}$$

The initial yield stresses for plasticity and damage are respectively

$$\sigma_{\mathsf{D}0} = \sigma_{\mathsf{D}}(0) = \sqrt{\mathsf{E}_0\,\mathsf{w}_0}, \quad \text{and} \quad \sigma_{\mathsf{P}0}, \tag{3.60}$$

while the dissipated work density (1.57) is given by

$$\delta(p,\alpha) = \mathsf{w}(\alpha) + p\,\sigma_{\mathsf{P}}(\alpha)\,. \tag{3.61}$$

Tab. 3.4 shows different constitutive parameters able to describe the possible responses of Tab. 3.1. Moreover the trends of the constitutive functions are shown in Fig. 3.11.

sequence	E_0	w_0	σ_{P_0}	γ
E-D	1	1	1.5	1
E-P-D	1	1.7	1	1
E-P-DP	1	2	1	2
E-D-PD	1	2	2	2

Table 3.4: Constitutive parameters for the homogeneous evolution examples

3.3.1 Example: *E-D* sequence

For such a model, the yield stresses becomes

$$\sigma_{\mathsf{P}}(\alpha) = \sigma_{\mathsf{P}_0}\,(1-\alpha)\,, \quad \sigma_{\mathsf{D}}(p,\alpha) = \sqrt{\mathsf{E}_0\,\left(\mathsf{w}_0 - \sigma_{\mathsf{P}_0}\,p\right)}\sqrt{(1-\alpha)^3}. \tag{3.62}$$

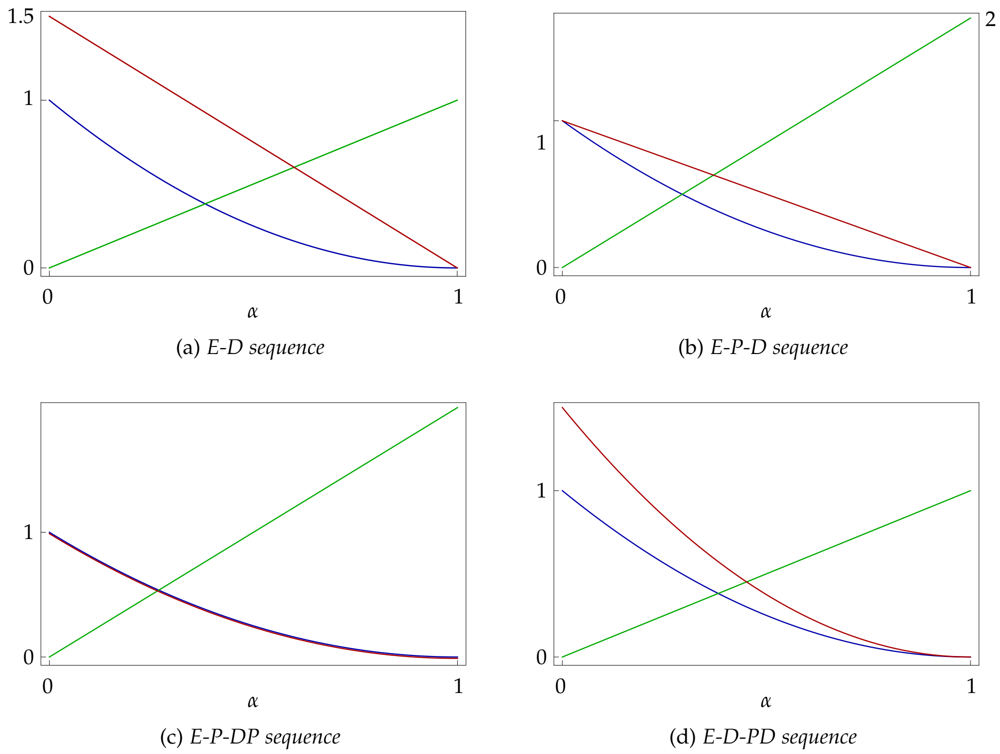


Figure 3.11: Consitutive functions trends for the considered examples: $\mathsf{E}(\alpha)$ (solid blue line); $\mathsf{w}(\alpha)$ (solid red line); $\sigma_{\mathsf{P}}(\alpha)$ (solid green line)

Since $\sigma_{\mathsf{P}}(0) > \sigma_{\mathsf{D}}(0,0)$, after the elastic phase a damaging phase occurs. Moreover for any damage level one has $\sigma_{\mathsf{P}}(\alpha) > \sigma_{\mathsf{D}}(0,\alpha)$ and hence a second yield instant does not exist. Then the response is an *E-D* sequence.
The complete evolution is simply given by

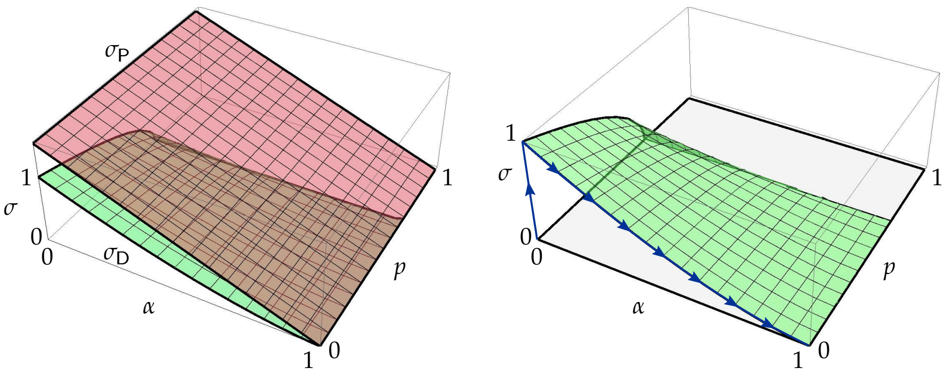
- *E* stage, $t \in [0, \sqrt{\mathsf{w}_0/\mathsf{E}_0})$:

$$p(t) = 0, \quad \alpha(t) = 0 \quad \sigma(t) = \mathsf{E}_0\, t;$$

- *D* stage, $t \in [\sqrt{\mathsf{w}_0/\mathsf{E}_0}, \infty)$:

$$p(t) = 0, \quad \alpha(t) = 1 - \frac{\mathsf{w}_0}{\mathsf{E}_0}\frac{1}{t^2} \quad \sigma(t) = \frac{\mathsf{w}_0^2}{\mathsf{E}_0}\frac{1}{t^3};$$

and shown in Fig. 3.16. Specifically in Fig. 3.16a the stress strain response is represented with the trends of both the yield stress limits and the internal state variables. One could appreciate the nonexistence of a second yield point since the yield stresses never intersect. Fig. 3.16b highlights through the unloading curves the evolution of damage as decrease of stiffness. Any residual strain appears by unloading since plasticity does not evolve. Finally, Fig. 3.16c represents the energy density contributions where the dissipation is only due to damage. Fig. 3.12 shows the stress yield limits 3.12a and the load path 3.12b in the state variable space (σ, p, α).



(a) *yield stress limits* (b) *load path*

Figure 3.12: *E-D* sequence: the green and red surfaces represent respectively the damage and plastic yield stress while the blue path represents the material response

3.3.2 Example: *E-P-D* sequence

For such model, the yield stresses are the same as in (3.62). Since $\sigma_{\mathsf{P}}(0) < \sigma_{\mathsf{D}}(0,0)$, a plastic phase occurs after the elastic phase.
When $p(t_{II}) = \mathsf{w}_0/\sigma_{\mathsf{P}_0} - \sigma_{\mathsf{P}_0}/\mathsf{E}_0$ a second yield point exists. From that instant on, one has $\sigma_{\mathsf{P}}(\alpha) > \sigma_{\mathsf{D}}(p(t_{II}), \alpha)$ and therefore only damage evolves leading to a *E-P-D* sequence in the response.
The complete evolution is given by

- *E* stage, $t \in [0, \sigma_{\mathsf{P}_0}/\mathsf{E}_0)$:

$$p(t) = 0, \quad \alpha(t) = 0 \quad \sigma(t) = \mathsf{E}_0\, t;$$

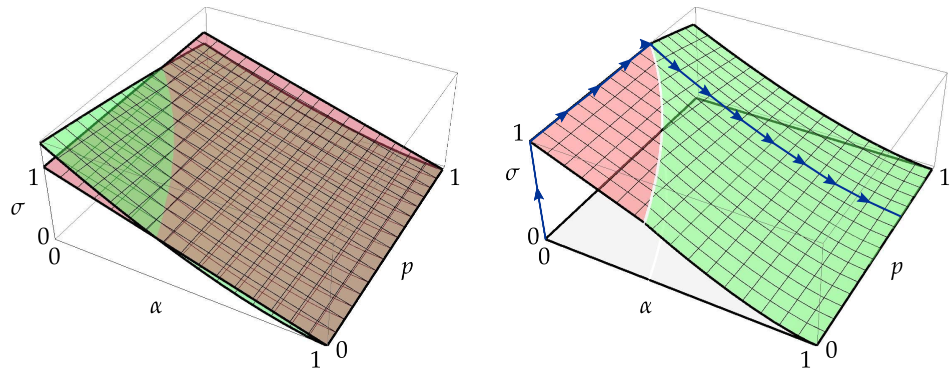
- *P* stage, $t \in [\sigma_{\mathsf{P}_0}/\mathsf{E}_0, \mathsf{w}_0/\sigma_{\mathsf{P}_0})$:

$$p(t) = t - \frac{\sigma_{\mathsf{P}_0}}{\mathsf{E}_0}, \quad \alpha(t) = 0 \quad \sigma(t) = \sigma_{\mathsf{P}_0};$$

- *D* stage, $t \in [\mathsf{w}_0/\sigma_{\mathsf{P}_0}, \infty)$:

$$p(t) = \bar{p} = \frac{\mathsf{w}_0}{\sigma_{\mathsf{P}_0}} - \frac{\sigma_{\mathsf{P}_0}}{\mathsf{E}_0}, \quad \alpha(t) = 1 - \frac{\sigma_{\mathsf{P}_0}^2}{\mathsf{E}_0^2}\frac{1}{(t-\bar{p})^2} \quad \sigma(t) = \frac{\sigma_{\mathsf{P}_0}^4}{\mathsf{E}_0^3}\frac{1}{(t-\bar{p})^3};$$

and shown in Fig. 3.17. Specifically in Fig. 3.17a the stress strain response is represented with the trends of both the yield stress limits and the internal state variables. Fig. 3.17b highlights the evolution of both plasticity as residual strain and damage as decrease of stiffness through the unloading curves. Finally, Fig. 3.17c represents the energy density contributions. Fig. 3.13 shows the stress yield limits, Fig. 3.13a, and the load path ,Fig. 3.13b, in the state variable space (σ, p, α).



(a) *yield stress limits*

(b) *load path*

Figure 3.13: *E-P-D* sequence: the green and red surfaces represent respectively the damage and plastic yield stress while the blue path represents the material response

3.3.3 Example: *E-P-DP* sequence

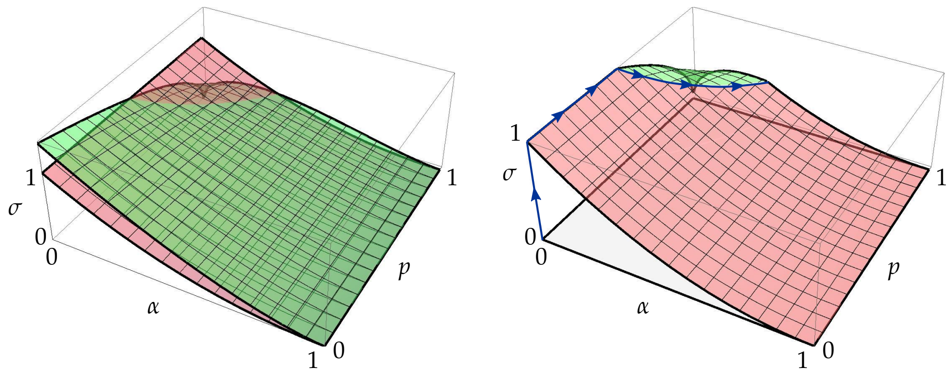
For such model, the yield stresses are

$$\sigma_{\mathsf{P}}(\alpha) = \sigma_{\mathsf{P}_0}(1-\alpha)^2, \quad \sigma_{\mathsf{D}}(p,\alpha) = \sqrt{\mathsf{E}_0\left(\mathsf{w}_0 - \sigma_{\mathsf{P}_0}\,p\right)}\sqrt{(1-\alpha)^3}. \tag{3.63}$$

Since $\sigma_{\mathsf{P}}(0) < \sigma_{\mathsf{D}}(0,0)$, after the elastic phase a plastic phase occurs.
When $p(t_{II}) = \mathsf{w}_0/\sigma_{\mathsf{P}_0} - \sigma_{\mathsf{P}_0}/\mathsf{E}_0$ the second yield point exists. Since, in that instant, $\sigma_{\mathsf{P}}(\alpha) < \sigma_{\mathsf{D}}(p(t_{II}),\alpha)$ both plasticity and damage evolves. Then the response is an *E-P-DP* sequence.
The complete evolution is given by

- *E* stage, $t \in [0, \sigma_{\mathsf{P}_0}/\mathsf{E}_0)$:
$$p(t) = 0, \quad \alpha(t) = 0 \quad \sigma(t) = \mathsf{E}_0\,t;$$
- *P* stage, $t \in [\sigma_{\mathsf{P}_0}/\mathsf{E}_0, \mathsf{w}_0/\sigma_{\mathsf{P}_0})$:
$$p(t) = t - \frac{\sigma_{\mathsf{P}_0}}{\mathsf{E}_0}, \quad \alpha(t) = 0 \quad \sigma(t) = \sigma_{\mathsf{P}_0};$$
- *PD* stage, $t \in [\mathsf{w}_0/\sigma_{\mathsf{P}_0}, \infty)$:
$$p(t) = t - \frac{\sigma_{\mathsf{P}_0}}{\mathsf{E}_0}, \quad \alpha(t) = 1 - \frac{\mathsf{w}_0}{\sigma_{\mathsf{P}_0}}\frac{1}{\left(2\,t - \sigma_{\mathsf{P}_0}/\mathsf{E}_0\right)} \quad \sigma(t) = \frac{\mathsf{w}_0^2}{\sigma_{\mathsf{P}_0}}\frac{1}{\left(2\,t - \sigma_{\mathsf{P}_0}/\mathsf{E}_0\right)^2};$$

and is shown in Fig. 3.18. Specifically in Fig. 3.18a the stress strain response is represented with the trends of both the yield stress limits and the internal state variables. Fig. 3.18b highlights the evolution of both plasticity as residual strain and damage as decrease of stiffness through the unloading curves. After the second yield point, the contemporary evolution of plasticity and damage is remarkable. Finally, Fig. 3.18c represents the energy density contributions. Fig. 3.14 shows the stress yield limits, Fig. 3.14a, and the load path, Fig. 3.14b, in the state variable space (σ, p, α).



(a) *yield stress limits* (b) *load path*

Figure 3.14: *E-P-DP* sequence: the green and red surfaces represent respectively the damage and plastic yield stress while the blue path represents the material response

3.3.4 Example: *E-D-PD* sequence

For such model, the yield stresses are the same as in (3.63). Since $\sigma_\mathsf{P}(0) > \sigma_\mathsf{D}(0,0)$, a damaging phase occurs after the elastic phase.
When $\alpha(t_{II}) = 1 - \mathsf{w}_0\mathsf{E}_0/\sigma^2_{\mathsf{P}_0}$ the second yield point exists. Since, from that instant, $t_{II} = \sigma_{\mathsf{P}_0}/\mathsf{E}_0$, $\sigma_\mathsf{P}(\alpha) < \sigma_\mathsf{D}(0,\alpha)$ both plasticity and damage evolves. The response is then an *E-D-PD* sequence.
The complete evolution is given by

- *E* stage, $t \in [0, \sqrt{\mathsf{w}_0/\mathsf{E}_0})$:

$$p(t) = 0, \quad \alpha(t) = 0 \quad \sigma(t) = \mathsf{E}_0\, t;$$

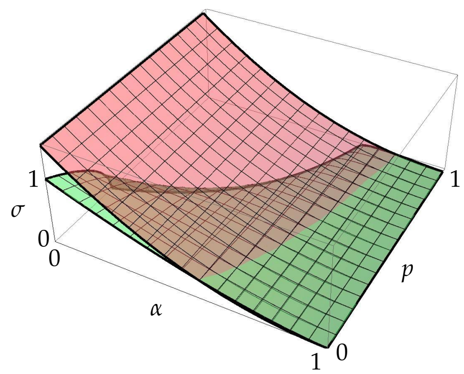
- *D* stage, $t \in [\sqrt{\mathsf{w}_0/\mathsf{E}_0}, \sigma_{\mathsf{P}_0}/\mathsf{E}_0)$:

$$p(t) = 0, \quad \alpha(t) = 1 - \frac{\mathsf{w}_0}{\mathsf{E}_0}\frac{1}{t^2} \quad \sigma(t) = \frac{\mathsf{w}_0^2}{\mathsf{E}_0}\frac{1}{t^3};$$

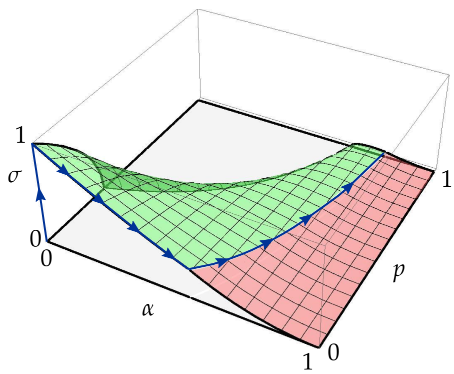
- *PD* stage, $t \in [\sigma_{\mathsf{P}_0}/\mathsf{E}_0, \infty)$:

$$p(t) = t - \frac{\sigma_{\mathsf{P}_0}}{\mathsf{E}_0}, \quad \alpha(t) = 1 - \frac{\mathsf{w}_0}{\sigma_{\mathsf{P}_0}}\frac{1}{(2\,t - \sigma_{\mathsf{P}_0}/\mathsf{E}_0)} \quad \sigma(t) = \frac{\mathsf{w}_0^2}{\sigma_{\mathsf{P}_0}}\frac{1}{\left(2\,t - \sigma_{\mathsf{P}_0}/\mathsf{E}_0\right)^2};$$

and represented in Fig. 3.19. Specifically in Fig. 3.19a the stress strain response is represented with the trends of both the yield stress limits and the internal state variables. Fig. 3.19b highlights the evolution of both plasticity as residual strain and damage as decrease of stiffness through the unloading curves. After the second yield point, the contemporary evolution of plasticity and damage is remarkable. Finally, Fig. 3.19c represents the energy density contributions. Fig. 3.15 shows the stress yield limits, Fig. 3.15a, and the load path, Fig. 3.15b, in the state variable space (σ, p, α).

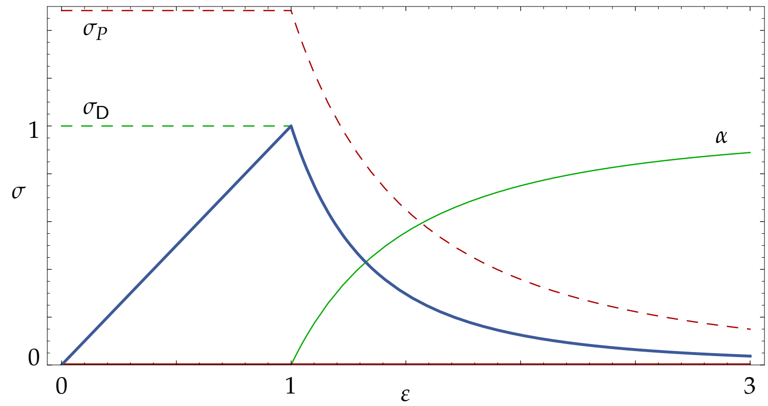


(a) *yield stress limits*

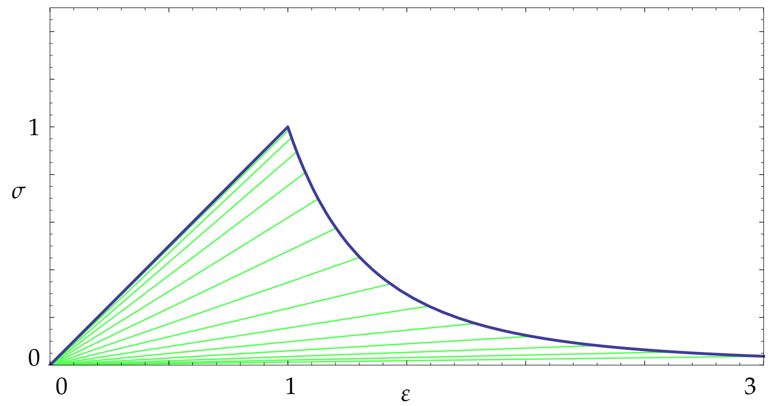


(b) *load path*

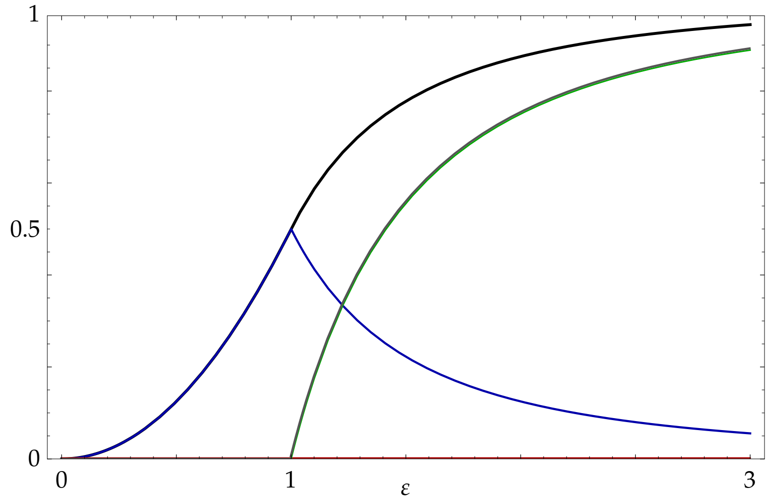
Figure 3.15: *E-D-PD* sequence: the green and red surfaces represent respectively the damage and plastic yield stress while the blue path represents the material response



(a) *Stress-strain curve (solid blue) with yield stresses and state variables evolutions*

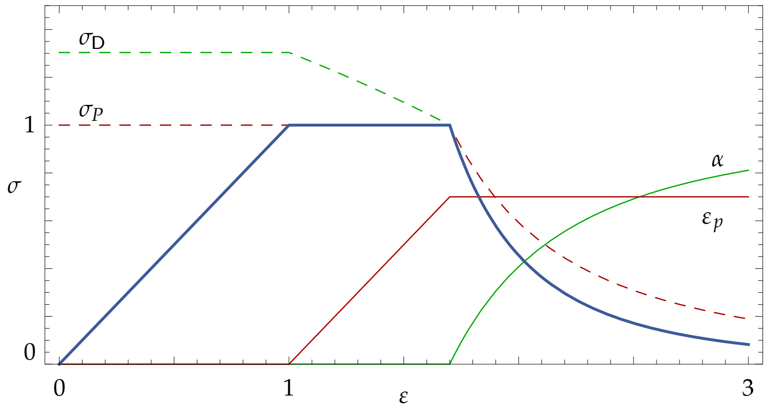


(b) *Particular of the unloading paths*

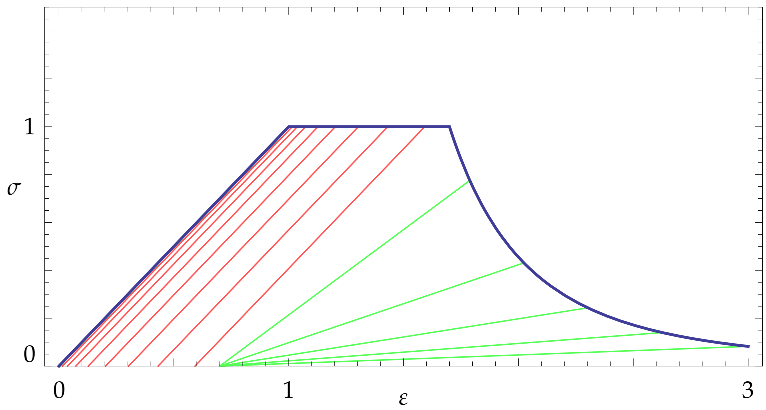


(c) *Energy density contributions: black line, total energy; blue line, elastic energy; gray line, dissipated work; thin red line, dissipated plastic work; thin green line, dissipated damage work*

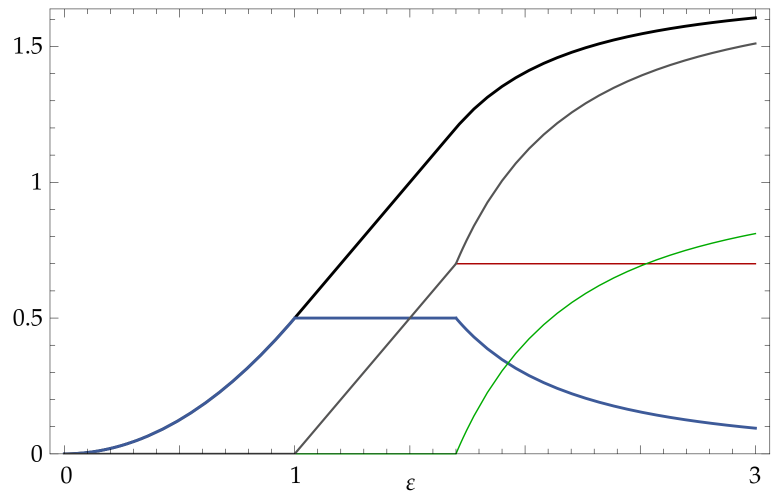
Figure 3.16: Homogeneous response of the *E-D* sequence



(a) *Stress-strain curve (solid blue) with yield stresses and state variables evolutions*

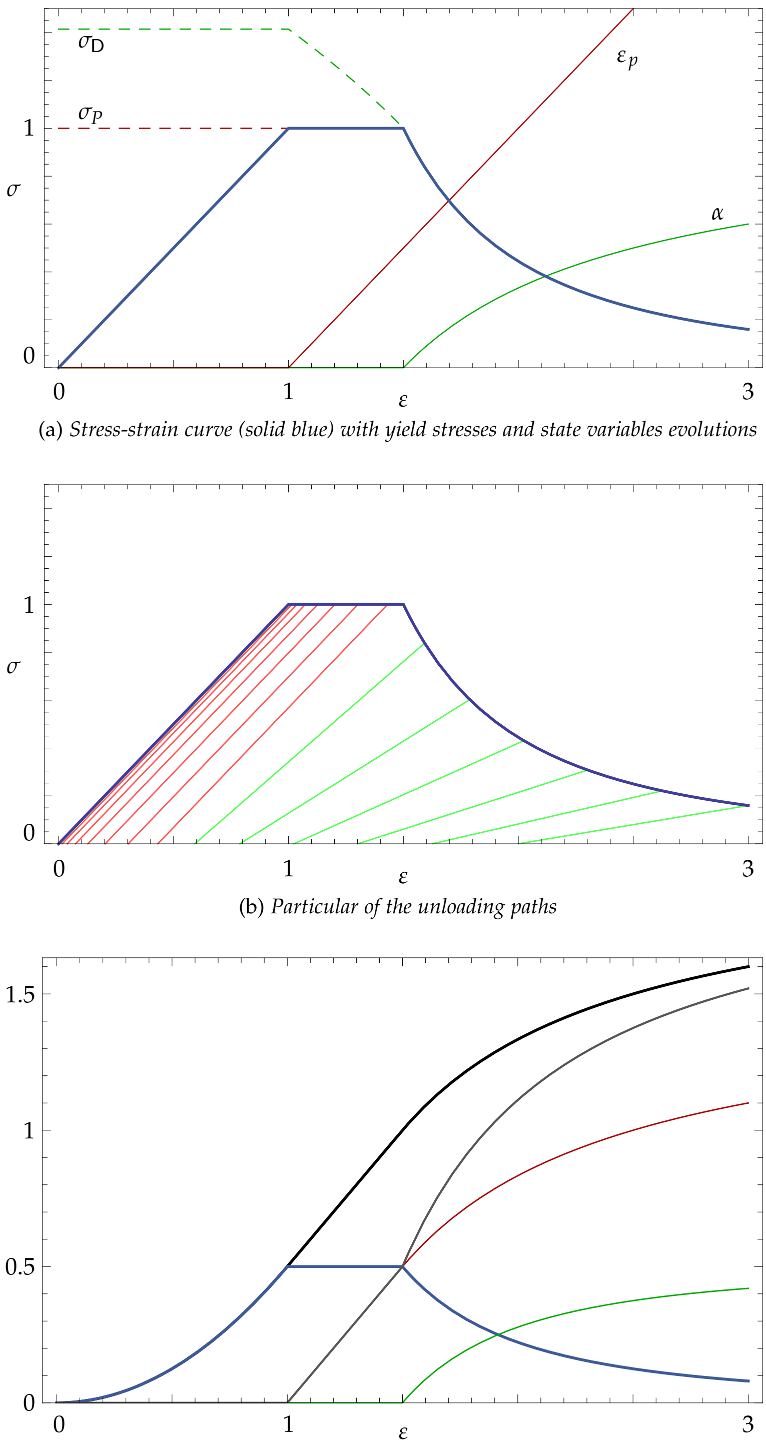


(b) *Particular of the unloading paths*



(c) *Energy density contributions: black line, total energy; blue line, elastic energy; gray line, dissipated work; thin red line, dissipated plastic work; thin green line, dissipated damage work*

Figure 3.17: Homogeneous response of the *E-P-D* sequence

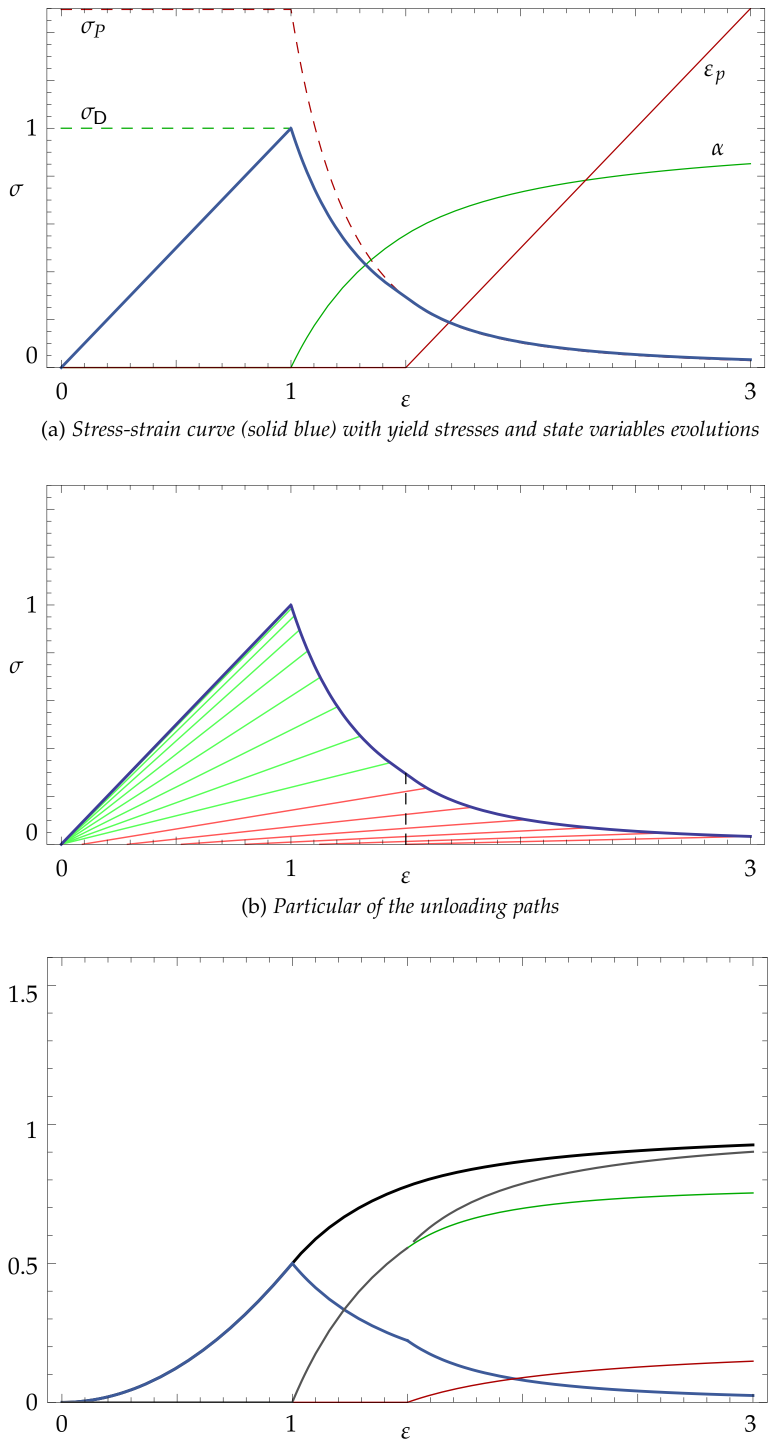


(a) *Stress-strain curve (solid blue) with yield stresses and state variables evolutions*

(b) *Particular of the unloading paths*

(c) *Energy density contributions: black line, total energy; blue line, elastic energy; gray line, dissipated work; thin red line, dissipated plastic work; thin green line, dissipated damage work*

Figure 3.18: Homogeneous response of the *E-P-DP* sequence



(a) *Stress-strain curve (solid blue) with yield stresses and state variables evolutions*

(b) *Particular of the unloading paths*

(c) *Energy density contributions: black line, total energy; blue line, elastic energy; gray line, dissipated work; thin red line, dissipated plastic work; thin green line, dissipated damage work*

Figure 3.19: Homogeneous response of the *E-D-PD* sequence

3.4 The stability of homogeneous states

This section is devoted to the issue of stability, crucial in softening responses. In order to study the stability of the homogeneous response, the 1D domain is again considered. The stability analysis involves exactly the same response sequences analyzed in the preceeding section about the homogeneous solutions. Since any state satisfies the first-order stability condition, the investigation of the second-order stability condition becomes necessary by means of the study of the sign of the second variation of the potential energy and the dissipation distance or of the minimization of the Rayleigh-ratio introduced in (2.46). For the homogeneous model, the second variation of the total potential energy reads

$$
\begin{aligned}
&\mathcal{P}''\left(u,\varepsilon_p,\alpha\right)\left(\widetilde{u},\widetilde{\varepsilon}_p,\widetilde{\alpha}\right) = \\
&= \int_0^{\mathrm{L}} \left(\mathsf{E}(\alpha)\left(\widetilde{u}'-\widetilde{\varepsilon}_p\right)^2 + 2\,\mathsf{E}'(\alpha)\left(u'-\varepsilon_p\right)\left(\widetilde{u}'-\widetilde{\varepsilon}_p\right)\widetilde{\alpha} + \frac{1}{2}\mathsf{E}''(\alpha)\left(u'-\varepsilon_p\right)^2\widetilde{\alpha}^2\right)\,\mathrm{d}x = \\
&\qquad = \int_0^{\mathrm{L}} \left(\mathsf{E}(\alpha)\left(\left(\widetilde{u}'-\widetilde{\varepsilon}_p\right)-\mathsf{S}(\alpha)\,\sigma\,\widetilde{\alpha}\right)^2 - \frac{1}{2}\mathsf{S}''(\alpha)\,\sigma^2\,\widetilde{\alpha}^2\right)\,\mathrm{d}x
\end{aligned}
\tag{3.64}
$$

while the second variation of the total dissipation distance, through the same reasonings as for (2.32), is

$$
\begin{aligned}
D''\left(\varepsilon_p,\alpha\right)\left(\widetilde{\varepsilon}_p,\widetilde{\alpha}\right) &= \int_0^{\mathrm{L}} \left(\sigma_{\mathsf{P}}'(\alpha)-\mathsf{m}(\alpha)\right)|\widetilde{\varepsilon}_p|\,\widetilde{\alpha}\,\mathrm{d}x \\
&\quad + \int_0^{\mathrm{L}} \left(\mathsf{w}''(\alpha)-p\,\mathsf{m}'(\alpha)\right)\widetilde{\alpha}^2\,\mathrm{d}x + \int_0^{\mathrm{L}} \eta^2\,\widetilde{\alpha}'^2\,\mathrm{d}x.
\end{aligned}
\tag{3.65}
$$

3.4.1 Elastic phase ($t < t_I$)

In this phase, if $\widetilde{\varepsilon}_p \neq 0$ or $\widetilde{\alpha} \neq 0$ then the first order-stability condition is satisfied as an inequality. Clearly, the study of the sign of the second variation of the total energy is necessary only when $\widetilde{\varepsilon}_p = 0$ and $\widetilde{\alpha} = 0$. Since

$$
\mathcal{P}''(u,0,0)(\widetilde{u},0,0) = \mathsf{E}_0\,\widetilde{u}'^2 > 0 \tag{3.66}
$$

the elastic phase is always stable.

3.4.2 First dissipation phase ($t_I \leq t < t_{II}$)

A distintion between the *E-P-** or *E-D-** sequence have to be made.

*E-P-** sequence

For any variation with $\widetilde{\alpha} \neq 0$ the state is stable because of the first-order stability condition. Then the study of the sign of the second variation of the sum of the potential energy and the dissipation distance becomes necessary only when $\widetilde{\alpha} = 0$. Since

$$
\mathcal{P}''\left(u,\varepsilon_p,0\right)\left(\widetilde{u},\widetilde{\varepsilon}_p,0\right) = \mathsf{E}_0\left(\widetilde{u}'-\widetilde{\varepsilon}_p\right)^2 \geq 0 \tag{3.67}
$$

the elastic-plastic response is stable. To be more precise, the state is indefinitely stable, since any higher order variation of the total potential energy vanishes. Moreover, it is easy to proof that in reality not only the homogeneous response is stable but also a non-homogeneous one, including solutions with plastic localizations (slips). This indeterminate behaviour is typical of the perfect-plasticity model and disappears once hardening effects are taken into account. This point will be deeper examined in the next chapter about non-homogeneous evolutions.

*E-D-** sequence

For any variation such that $\widetilde{\varepsilon}_p \neq 0$, the state is stable because of the first-order stability condition. Clearly, the study of the sign of the second variation of the total energy is necessary only when $\widetilde{\varepsilon}_p = 0$. In this case, following the steps suggested in Pham 2010, the minimization of the Rayleigh ratio (3.68) for the 1D model reduces to

$$\min_{\widetilde{\mathcal{F}}\times\mathcal{C}_+} \mathcal{R}(\widetilde{u},\widetilde{\alpha}) \tag{3.68}$$

where $\mathcal{R}(\widetilde{u},\widetilde{\alpha}) = \mathcal{R}(\widetilde{u},0,\widetilde{\alpha})$ and

$$\mathcal{R}(\widetilde{u},\widetilde{\alpha}) = \frac{\int_0^L \mathsf{E}(\alpha)\,(\widetilde{u}' - \mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha})^2\,\mathrm{d}x + \eta^2 \int_0^L \widetilde{\alpha}'^2\,\mathrm{d}x}{\int_0^L \left(\frac{1}{2}\mathsf{S}''(\alpha)\,\sigma^2 - \mathsf{w}''(\alpha)\right)\widetilde{\alpha}^2\,\mathrm{d}x} \tag{3.69}$$

One can eliminate the dependence of the field $\widetilde{u}$ by minimizing the Rayleigh ratio $\mathcal{R}$ with respect to $\widetilde{u}$ at fixed ε_p and $\widetilde{\alpha}$. The associated optimality condition reads then $\mathcal{R}'(\widetilde{u},\widetilde{\alpha})\left(\widetilde{\widetilde{u}},0\right) = 0$, namely

$$\int_0^L \mathsf{E}(\alpha)\,(\widetilde{u}' - \mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha})\,\widetilde{\widetilde{u}}'\,\mathrm{d}x = 0 \qquad \forall\,\widetilde{\widetilde{u}} \in \mathcal{F}_0. \tag{3.70}$$

Integrating with respect to $\widetilde{u}$ and considering the boundary conditions $\widetilde{\widetilde{u}}(0) = \widetilde{\widetilde{u}}(L) = 0$, one obtains

$$\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathsf{E}(\alpha)\,(\widetilde{u}' - \mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha})\right)\widetilde{\widetilde{u}}\,\mathrm{d}x + \Big[\left(\mathsf{E}(\alpha)\,(\widetilde{u} - \mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha})\right)\widetilde{\widetilde{u}}\Big]_0^L = 0 \tag{3.71}$$

Hence

$$\mathsf{E}(\alpha)\,(\widetilde{u}' - \mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha}) = \text{constant} \tag{3.72}$$

Integrating (3.85) and considering the boundary conditions, one obtains

$$\widetilde{u}'(x) = \mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha} - \frac{\int_0^L (\mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha})\,\mathrm{d}x}{\int_0^L \mathsf{S}(\alpha)\,\mathrm{d}x}\mathsf{S}(\alpha) \tag{3.73}$$

which furnishes the optimal displacement variation profile

$$\widetilde{u}(x) = \int_0^x \left(\mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha}\right)\mathrm{d}x - \frac{\int_0^L (\mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha})\,\mathrm{d}x}{\int_0^L \mathsf{S}(\alpha)\,\mathrm{d}x}\int_0^x \mathsf{S}(\alpha)\,\mathrm{d}x. \tag{3.74}$$

Eq. (3.87) is then an optimality condition for the Rayleigh ratio. Inserting equation (3.74) in (3.69) leads to

$$\begin{aligned}\mathcal{R}^*(\widetilde{\alpha}) &= \frac{\int_0^L \mathsf{E}(\alpha)\left(-\frac{\int_0^L (\mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha})\,\mathrm{d}x}{\int_0^L \mathsf{S}(\alpha)\,\mathrm{d}x}\mathsf{S}(\alpha)\right)^2 \mathrm{d}x + \eta^2 \int_0^L \widetilde{\alpha}'^2\,\mathrm{d}x}{\int_0^L \left(\frac{1}{2}\mathsf{S}''(\alpha)\,\sigma^2 - \mathsf{w}''(\alpha)\right)\widetilde{\alpha}^2\,\mathrm{d}x} \\ &= \frac{\frac{1}{\int_0^L \mathsf{S}(\alpha)\,\mathrm{d}x}\left(\int_0^L (\mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha})\,\mathrm{d}x\right)^2 + \eta^2 \int_0^L \widetilde{\alpha}'^2\,\mathrm{d}x}{\int_0^L \left(\frac{1}{2}\mathsf{S}''(\alpha)\,\sigma^2 - \mathsf{w}''(\alpha)\right)\widetilde{\alpha}^2\,\mathrm{d}x}\end{aligned} \tag{3.75}$$

It is convenient for the next developments to normalise the position variable as

$$\hat{x} = x/L, \quad \mathrm{d}\hat{x} = \mathrm{d}x/L, \quad \text{with} \quad \hat{x} \in [0,1]\,, \tag{3.76}$$

and to rename some quantities as follows:

$$x = \hat{x}, \quad \mathsf{a} = \eta^2/L^2, \quad \mathsf{b} = \frac{1}{\int_0^1 \mathsf{S}(\alpha)\,\mathrm{d}x}, \quad \mathsf{c} = \mathsf{S}'(\alpha)\,\sigma, \quad \mathsf{m} = \frac{1}{2}\mathsf{S}''(\alpha)\,\sigma^2 - \mathsf{w}''(\alpha)\,. \tag{3.77}$$

Then the Rayleigh ratio gets the following expression

$$\mathcal{R}^*(\alpha) = \frac{\mathsf{a}\int_0^1 \alpha'^2\,\mathrm{d}x + \mathsf{b}\,\mathsf{c}^2\left(\int_0^1 \alpha\,\mathrm{d}x\right)^2}{\mathsf{m}\int_0^1 \alpha^2\,\mathrm{d}x}. \tag{3.78}$$

The condition for which the homogeneous evolution becomes unstable and hence allows for a non-homogeneous solution is given by the minimization of the Rayleigh ratio (3.78), specifically

$$\begin{cases} \min\limits_{\mathcal{C}_+} \mathcal{R}^*(\alpha) > 1, & \text{stable state} \\ \min\limits_{\mathcal{C}_+} \mathcal{R}^*(\alpha) < 1, & \text{unstable state} \end{cases} \tag{3.79}$$

The exact solution of the minimization of (3.78), as well the procedure to obtain it, is given in the appendix of Pham 2010 and reads

$$\min_{\mathcal{F}\times\mathcal{C}_+} \mathcal{R}(\widetilde{u},\widetilde{\alpha}) = \min_{\mathcal{C}_+} \mathcal{R}^*(\alpha) = \min\left(\frac{\mathsf{b}\,\mathsf{c}^2}{\mathsf{m}}, \left(\frac{\mathsf{a}\,\mathsf{b}^2\,\mathsf{c}^4\pi^2}{\mathsf{m}^3}\right)^{1/3}\right). \tag{3.80}$$

This last equation involves the length of the bar since a depends upon the ratio between the parameter η and the length of the bar L itself. Accordingly, each homogeneous state is stable if the length of the bar is lesser than a critical value and unstable otherwise. More precisely the state is stable if (only if)

$$L^2 < (\leq) \frac{\mathsf{b}^2\,\mathsf{c}^4\pi^2}{\mathsf{m}^3}\eta^2. \tag{3.81}$$

This last is obtained by observing that the first term in the minimum (3.93) is greater than one.

3.4.3 Second dissipation phase ($t_{II} \geq t$)

It is of interest to establish the condition wheter the homogeneous evolution is stable during a plastic-damaging phase (*P-D* stage).
The study consists then in seeking a variated state $(\widetilde{u}, \widetilde{\varepsilon}_p, \widetilde{\alpha})$ such that $\mathcal{P}'' + D''$ is minimum.

It is worth to remember that due to (2.36) only positive values for the plastic strain variations lead to an indetermination of the first order stability condition since the stress is positive. That is only variations where $\widetilde{\varepsilon}_p \geq 0$ regardless to $\widetilde{u}$ and $\widetilde{\alpha}$ needed to be studied. Hereinafter one defines $\mathcal{Q}_+ := \{a \in \mathcal{Q} \colon a \geq 0\}$ such that $\widetilde{\varepsilon}_p \in \mathcal{Q}_+$. With this remark in mind the minimum value of $\mathcal{P}'' + D''$ is explored by considering again the Rayleigh (2.46) ratio $\mathcal{R}$, which reads in this case

$$\mathcal{R}(\widetilde{u}, \widetilde{\varepsilon}_p, \widetilde{\alpha}) = \frac{\int_0^L \mathsf{E}(\alpha) \left((\widetilde{u}' - \widetilde{\varepsilon}_p) - \mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} \right)^2 \mathrm{d}x + \eta^2 \int_0^L \widetilde{\alpha}'^2 \, \mathrm{d}x}{\int_0^L \left(\frac{1}{2} \mathsf{S}''(\alpha)\, \sigma^2 - \mathsf{w}''(\alpha) + p\mathsf{m}'(\alpha) \right) \widetilde{\alpha}^2 \, \mathrm{d}x - \int_0^L \left(\sigma_{\mathsf{P}}'(\alpha) - \mathsf{m}(\alpha) \right) \widetilde{\varepsilon}_p \, \widetilde{\alpha} \, \mathrm{d}x} \tag{3.82}$$

One first shows that $\mathcal{R}$ is positive on $\widetilde{\mathcal{F}} \times \mathcal{Q}_+ \times \mathcal{C}_+$, the space of admissible variations. The Rayleigh ratio is clearly non negative. In order to $\mathcal{R} = 0$, one should have $\alpha(x) = \alpha = \text{const}$ and $u'(x) = \varepsilon_p(x) + \mathsf{S}'(\alpha)\, \sigma\, \alpha(x)$. Then in such case, since $u(0) = u(1) = 0$, one obtains both $\varepsilon_p(x) = 0$ and $\alpha = 0$ which is ruled-out. Therefore $\mathcal{R}(u, \varepsilon_p, \alpha) > 0$ for every $(u, \varepsilon_p, \alpha) \in \widetilde{\mathcal{F}} \times \mathcal{Q}_+ \times \mathcal{C}_+$.

Going ahead as the previous section, one can eliminate the dependence of the field $\widetilde{u}$ by minimizing the Rayleigh ration $\mathcal{R}$ with respect to $\widetilde{u}$ at fixed $\widetilde{\varepsilon}_p$ and $\widetilde{\alpha}$. The associated optimality condition is then $\mathcal{R}'(\widetilde{u}, \widetilde{\varepsilon}_p, \widetilde{\alpha}) \left(\widetilde{\widetilde{u}}, 0, 0 \right) = 0$, namely

$$\int_0^L \mathsf{E}(\alpha) \left((\widetilde{u}' - \widetilde{\varepsilon}_p) - \mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} \right) \widetilde{\widetilde{u}}' \, \mathrm{d}x = 0 \qquad \forall\, \widetilde{\widetilde{u}} \in \mathcal{F}_0, \tag{3.83}$$

which integrated with respect to $\widetilde{u}$ leads to

$$\int_0^L \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathsf{E}(\alpha) \left((\widetilde{u}' - \widetilde{\varepsilon}_p) - \mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} \right) \right) \widetilde{\widetilde{u}} \, \mathrm{d}x + \Big[\left(\mathsf{E}(\alpha) \left((\widetilde{u}' - \widetilde{\varepsilon}_p) - \mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} \right) \right) \widetilde{\widetilde{u}} \Big]_0^L = 0. \tag{3.84}$$

and finally to the condition

$$\mathsf{E}(\alpha) \left((\widetilde{u}' - \widetilde{\varepsilon}_p) - \mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} \right) = \text{constant}. \tag{3.85}$$

Integrating (3.85) and considering the boundary conditions $\widetilde{u}(0) = \widetilde{u}(L) = 0$, one obtains

$$\widetilde{u}(x) = \mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} + \widetilde{\varepsilon}_p - \frac{\int_0^L \left(\mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} + \widetilde{\varepsilon}_p \right) \mathrm{d}x}{\int_0^L \mathsf{S}(\alpha) \, \mathrm{d}x} \mathsf{S}(\alpha) \tag{3.86}$$

which furnishes the optimal displacement variation profile

$$\widetilde{u}(x) = \int_0^x \left(\mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} + \widetilde{\varepsilon}_p \right) \mathrm{d}x - \frac{\int_0^L \left(\mathsf{S}'(\alpha)\, \sigma \widetilde{\alpha} + \widetilde{\varepsilon}_p \right) \mathrm{d}x}{\int_0^L \mathsf{S}(\alpha) \, \mathrm{d}x} \int_0^x \mathsf{S}(\alpha) \, \mathrm{d}x. \tag{3.87}$$

Inserting equation (3.87) in (3.82) leads to

$$\mathcal{R}^*\left(\widetilde{\varepsilon}_p,\widetilde{\alpha}\right) = \frac{\int_0^L \mathsf{E}(\alpha)\left(-\frac{\int_0^L\left(\mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha}+\widetilde{\varepsilon}_p\right)\,dx}{\int_0^L \mathsf{S}(\alpha)\,dx}\mathsf{S}(\alpha)\right)^2 dx + \eta^2\int_0^L \widetilde{\alpha}'^2\,dx}{\int_0^L\left(\frac{1}{2}\mathsf{S}''(\alpha)\,\sigma^2 - \mathsf{w}''(\alpha) + p\mathsf{m}'(\alpha)\right)\widetilde{\alpha}^2\,dx - \int_0^L\left(\sigma_{\mathsf{P}}'(\alpha) - \mathsf{m}(\alpha)\right)\widetilde{\varepsilon}_p\,\widetilde{\alpha}\,dx}$$
$$= \frac{\frac{1}{\int_0^L \mathsf{S}(\alpha)\,dx}\left(\int_0^L\left(\mathsf{S}'(\alpha)\,\sigma\,\widetilde{\alpha}+\widetilde{\varepsilon}_p\right)\,dx\right)^2 + \eta^2\int_0^L \widetilde{\alpha}'^2\,dx}{\int_0^L\left(\frac{1}{2}\mathsf{S}''(\alpha)\,\sigma^2 - \mathsf{w}''(\alpha) + p\mathsf{m}'(\alpha)\right)\widetilde{\alpha}^2\,dx - \int_0^L\left(\sigma_{\mathsf{P}}'(\alpha) - \mathsf{m}(\alpha)\right)\widetilde{\varepsilon}_p\,\widetilde{\alpha}\,dx}. \tag{3.88}$$

As done in the previous section it is convenient for the next developments to normalise the position variable as

$$\hat{x} = x/L, \quad d\hat{x} = dx/L, \quad \text{with} \quad \hat{x}\in[0,1], \tag{3.89}$$

and to rename some quantities as follows:

$$x = \hat{x}, \qquad \mathsf{a} = \eta^2/L^2, \qquad \mathsf{b} = \frac{1}{\int_0^1 \mathsf{S}(\alpha)\,dx}, \qquad \mathsf{c} = \mathsf{S}'(\alpha)\,\sigma,$$
$$\mathsf{m} = \frac{1}{2}\mathsf{S}''(\alpha)\,\sigma^2 - \mathsf{w}''(\alpha) + p\mathsf{m}'(\alpha), \qquad \mathsf{n} = -\frac{1}{2}\left(\sigma_{\mathsf{P}}'(\alpha) - \mathsf{m}(\alpha)\right). \tag{3.90}$$

The Rayleigh ration (3.88) changes to

$$\mathcal{R}^*\left(\varepsilon_p,\alpha\right) = \frac{\mathsf{a}\int_0^1 \alpha'^2\,dx + \mathsf{b}\left(\int_0^1 \varepsilon_p\,dx\right)^2 + 2\,\mathsf{b}\,\mathsf{c}\int_0^1 \varepsilon_p\,dx\int_0^1 \alpha\,dx + \mathsf{b}\,\mathsf{c}^2\left(\int_0^1 \alpha\,dx\right)^2}{\int_0^1\left(\mathsf{m}\,\alpha^2 + 2\,\mathsf{n}\,\varepsilon_p\,\alpha\right)dx}. \tag{3.91}$$

From (3.88) or (3.91) and the calculations in the appendix A.2, one deduces that the minimum of the Rayleigh ratio is attained if $\left(\widetilde{\varepsilon}_p,\widetilde{\alpha}\right)\in \mathrm{L}^2\times\mathrm{H}^2(0,1)\subset\widetilde{\mathcal{Q}}\times\mathcal{C}_+$. The condition for which the homogeneous evolution becomes unstable and hence allows for a non-homogeneous solution is given by the minimization of the Rayleigh ratio (3.91), specifically

$$\begin{cases}\min_{\widetilde{\mathcal{Q}}\times\mathcal{C}_+}\mathcal{R}^*\left(\varepsilon_p,\alpha\right) > 1, & \text{stable state}\\ \min_{\widetilde{\mathcal{Q}}\times\mathcal{C}_+}\mathcal{R}^*\left(\varepsilon_p,\alpha\right) < 1, & \text{unstable state}\end{cases} \tag{3.92}$$

As a result of the minimization (3.78) one obtains

$$\min_{\widetilde{\mathcal{F}}\times\widetilde{\mathcal{Q}}\times\mathcal{C}_+}\mathcal{R}\left(u,\varepsilon_p,\alpha\right) = \min_{\widetilde{\mathcal{Q}}\times\mathcal{C}_+}\mathcal{R}^*\left(\varepsilon_p,\alpha\right) = \min\left(\frac{\mathsf{b}\,\mathsf{c}^2}{\mathsf{m}}, 4\mathsf{b}\frac{\mathsf{cn}-\mathsf{m}}{\mathsf{n}^2}, \left(\frac{\mathsf{a}\,\mathsf{b}^2\,\mathsf{c}^4\pi^2}{\mathsf{m}^3}\right)^{1/3}\right). \tag{3.93}$$

The non-trivial proof with all passages of this last result could be found in appendix A.2. The result is that with respect to solution (3.93) an additional term $4\mathsf{b}\left(\mathsf{cn}-\mathsf{m}\right)/\mathsf{n}^2$ which is not length dependent appears.

3.5 Conclusions and perspectives

In this section a fruitful application of the energetic formulation has been shown. After a deep investigation of the homogeneous response it has been given the guidelines for dealing with the issue of stability and the arise of multiple solution, extremely important in softening materials. Indeed in case of smooth evolutions the first-order stability condition allows to identify different possible solution while the second- or higher-order stability condition provides a selection criterion of these different solutions.

The analysis has been here limited to the study of the stability homogeneous solution. It has been found the conditions for which the homogeneous response becomes unstable and non-homogeneous solutions may appear. This condition depends as expected on the length of the bar and the characteristic internal length proportional to the parameter η. It is worth to remark that the presented analyses do not rely on the monotonic character of the loading. Cycles or different load path can be taken into account with the same ease.

The proposed model has highlighted several features and opened different perspectives:

Plasticity and damage coupling The order with which plasticity and damage succeeded allows to describe a high variety of material responses. More in detail, the *E-P* * response is interesting for different reasons: (i) it seems to be a good point of departure to the understanding and modelling of plastic cyclic failure; (ii) it resemble qualitatively the response of ductile fracture. The *E-D* response is strongly related to the brittle fracture problem. Nevertheless the similar response *E-D-PD* owns a small difference, the contemporary evolution of plasticity with damage, that is crucial, as will be clear in the next section, for the description of cohesive fracture.

Stability of a three-dimensional model An interesting and important development would be to investigate the stability of an homogeneous response in a three-dimensional setting. Such studies has been already accomplished for an elastic-damage model with a gradient regularization in Pham and Marigo 2012, 2010b; Pham 2010.

Parameters identification An essential aspect is to figure out how the constitutive parameters could be identified for a given material through simple experimental tests. While for some of them like the elastic modulus this should be a simple task the same cannot be said for the other parameters. To this aim it is fundamental to be able to reproduce an homogeneous response in a given material specimen.

The next chapter is devoted to the construction of non-homogeneous solutions. The issue of stability of non-homogeneous responses is not trivial and can be faced only through numerical approaches, see Beaurain 2011.

Chapter 4

Non-homogeneous evolutions

In the previous chapter it has been proven that the homogeneous response could become unstable and hence not admissible for a sufficiently long bar. This means that a different solution exists, clearly non-homogeneous. This chapter is exactly devoted to the description of non-homogeneous evolutions in the one dimensional setting where localizations arise. Moreover different aspects will be covered like: (i) a procedure for the construction of localizations; (ii) the global response; (iii) analytical examples. Moreover it is proven that the global response is able to describe a cohesive fracture *a la* Barenblatt.

4.1 Introduction to non-homogeneous evolutions

Throughout the following plasticity is mostly supposed to evolve uniformly in space, when not coupled with damage. The case where a damage localization arises in a support with a non uniform distribution of accumulated plastic strain, is only outlined at the end of the Sec. 4.2.

4.1.1 The setting of the problem

The mathematical setting of the problem in which non-homogeneous evolutions are considered is the same as Sec. 3.1. The domain is the closet subset of the real line of length L,

$$\Omega = [0, L] \tag{4.1}$$

while

$$u(0) = 0, \quad u(L) = U(t)\,. \tag{4.2}$$

To simplify the presentation but without loss of generality one assumes that the global state field $\xi = \left(u, \varepsilon_p, p, \alpha\right)$ is piecewise smooth and its singular part is localized on a ξ-dependent set $S(\xi)$ which contains a finite number of points in Ω. Further, singular points are not allowed on the boundary $\partial\Omega$.

Referring to the function spaces of the state variables in Chap. 2, in particular both to the space of special bounded deformation SBD for the displacement field (2.2) and the space of special bounded measure SBM for the plastic strain field (2.6),

an additive decomposition of the the associated state variables can be done splitting the regular part from the singular part.
More specifically for the one-dimensional setting, a variable m belonging to the SBD space like the plastic or accumulated plastic strain can be considered as a measure for which one assumes to be decomposed into its regular and singular parts where: (i) the regular part is denoted $m(x)\,\mathrm{d}x$, where $x \mapsto m(x)$ is at least an integrable function (in practise, a piecewise continuous function) and $\mathrm{d}x$ is the Lebesgue measure; (ii) the singular part is a linear combination of Dirac measures centered on a finite number of singular points with some weight $M(x_i)$ at point x_i. This leads to the following notation:

$$m = m(x)\,\mathrm{d}x + \sum_{i \in S(m)} M(x_i)\,\delta_{x_i}. \tag{4.3}$$

Considering the previous discussion, the displacement field can then be represented as

$$u = \bar{u}(x) + \sum_{x_i \in S(\xi)} [\![u]\!](x_i)\,H_{x_i} \tag{4.4}$$

where the first and second term are respectively the continuous and discontinuous part. More specifically in (4.4) $\bar{u} \in \mathrm{C}^1([0, L])$ and H_{x_i} is the Heaviside function. Here it is assumed that a jump of a function $f(x_i)$ is evaluated as $[\![f]\!](x_i) = \lim_{x \to x_i^+} f(x) - \lim_{x \to x_i^-} f(x)$ which defines the term $[\![u]\!](x_i)$.

Since $u' \in$ SBM, the (total) strain field can be decomposed by means of (4.3) as

$$u' = \varepsilon = \bar{u}'(x)\,\mathrm{d}x + \sum_{x_i \in S(\xi)} [\![u]\!](x_i)\,\delta_{x_i} \tag{4.5}$$

where the first and second term are respectively the regular and singular part of the total strain.

In order to let the elastic energy being finite, the plastic strain field has the same singular part as the strain field and hence can be seen as the following measure

$$\varepsilon_p = \bar{\varepsilon}_p(x)\,\mathrm{d}x + \sum_{x_i \in S(\xi)} [\![u]\!](x_i)\,\delta_{x_i} \tag{4.6}$$

where the first and second term are respectively the regular and singular part of the plastic strain as in (4.3).

Similarly by means of (4.3) the accumulated plastic strain field is decomposed in a regular and singular part as follows

$$p = \bar{p}(x) + \sum_{x_i \in S(\xi)} P(x_i)\,\delta_{x_i}. \tag{4.7}$$

Moreover one has

$$P(x_i) = \int_0^t |[\![\dot{u}]\!](x_i)|\,\mathrm{d}\tau. \tag{4.8}$$

Referring to the general plastic-damage gradient model (2) with the constitutive assumption (3.11) and once the previous state variable decomposition has been established one can introduce the energy functionals and the dissipation potential, the ingredients of the variational formulation.

The elastic stored energy then reads

$$\mathcal{E}\left(u,\varepsilon_p,\alpha\right)=\int_0^{\mathrm{L}} e\left(\varepsilon,\varepsilon_p,\alpha\right)\,\mathrm{d}x=\int_0^{\mathrm{L}}\frac{1}{2}\mathsf{E}(\alpha)\left(\bar{u}'(x)-\bar{\varepsilon}_p(x)\right)^2\,\mathrm{d}x \tag{4.9}$$

while the dissipation potential

$$\begin{aligned}\psi\left(p,\alpha,\alpha',\dot{p},\dot{\alpha},\dot{\alpha}'\right)=&\partial_t\,\mathsf{w}(\alpha(x))+\partial_t\left(\frac{1}{2}\eta^2\alpha'(x)^2\right)+\sigma_{\mathsf{P}}(\alpha(x))\,\dot{p}-p\,\mathsf{m}(\alpha(x))\,\dot{\alpha}(x)\\=&\partial_t\,\mathsf{w}(\alpha(x))+\partial_t\left(\frac{1}{2}\eta^2\alpha'(x)^2\right)+\sigma_{\mathsf{P}}(\alpha(x))\,\dot{\bar{p}}(x)-\bar{p}(x)\,\mathsf{m}(\alpha(x))\,\dot{\alpha}(x)\\&+\sum_{x_i\in S(\zeta)}\left(\sigma_{\mathsf{P}}(\alpha(x))\,\dot{P}(x_i)-P(x_i)\,\mathsf{m}(\alpha(x))\,\dot{\alpha}(x)\right)\end{aligned} \tag{4.10}$$

The dissipation distance D and dissipated work $\mathcal{D}_D$ are consequently defined.

4.1.2 The governing equations

4.1.2.1 Irreversibility

The irreversibility condition is given in (2.12). It involves only the damage field and can be expressed as

$$\alpha(x,t_1)\leq\alpha(x,t_2)\,,\qquad t_1\leq t_2,\quad\forall x\in[0,L] \tag{4.11}$$

for discrete evolutions or as

$$\dot{\alpha}(x)\geq 0,\qquad\forall x\in[0,L]\,. \tag{4.12}$$

for damage smooth evolutions.

4.1.2.2 Stability condition

In order to define the stable states of the system, the variations of the state variables have to be defined. According to Sec. 4.1.1 the variated global state is

$$\widetilde{\zeta}=\left(\widetilde{u},\widetilde{\varepsilon}_p,|\widetilde{\varepsilon}_p|,\widetilde{\alpha}\right)\in\widetilde{\mathcal{F}}\times\mathcal{Q}\times\mathcal{Q}_+\times\mathcal{C}_+. \tag{4.13}$$

Non-homogeneous solutions will be constructed starting from the first order stability condition. To achieve this aim, it is important to specify the variations for the displacement and plastic fields. Accordingly to (4.5), the displacement variation $\widetilde{u}\in\widetilde{\mathcal{F}}$ kinematically admissible is

$$\widetilde{u}=\overline{\widetilde{u}}+h\sum_{x_i\in S(\widetilde{u})}[\![\widetilde{u}]\!](x_i)\,H_{x_i}. \tag{4.14}$$

where $S\left(\widetilde{u}\right)$ is the set of points where $\widetilde{u}$ is discontinuous. The weak derivative of the displacement variation, that is the variated total strain, follows from (4.14) consequently.
Similarly, the plastic strain variations becomes

$$\widetilde{\varepsilon}_p=\overline{\widetilde{\varepsilon}_p}+h\sum_{x_i\in S(\widetilde{u})}[\![\widetilde{u}]\!](x_i)\,\delta_{x_i}. \tag{4.15}$$

Clearly in general $S\left(\xi\right) \neq S\left(\widetilde{u}\right)$.

Referring to the previous definitions, the first order stability condition (**st-1D**) reads

$$0 \leq \int_{\Omega \backslash S(\widetilde{u})} \left(\sigma \left(\widetilde{u}' - \overline{\widetilde{\varepsilon}_p}\right) + \sigma_{\mathsf{P}}(\alpha)\,|\overline{\widetilde{\varepsilon}_p}|\right)\,\mathrm{d}x + \sum_{x_i \in S(\widetilde{u})} \sigma_{\mathsf{P}}(\alpha)\,|[\![\widetilde{u}]\!](x_i)|$$
$$+ \int_{\Omega \backslash S(\widetilde{u})} \left(\left(-\frac{1}{2}\mathsf{S}'(\alpha)\,\sigma^2 + \mathsf{w}'(\alpha) - p\,\mathsf{m}(\alpha)\right)\widetilde{\alpha} + \eta^2\,\alpha'\,\widetilde{\alpha}'\right)\,\mathrm{d}x$$
$$- \sum_{x_i \in S(\widetilde{u})} \mathsf{m}(\alpha)\,P(x_i)\,\widetilde{\alpha}(x_i) \quad (4.16)$$

which must valid for any admissibele $\widetilde{\zeta}$. In (4.16) the new definitions for the variations of the displacement (4.14) and plastic strain (4.15) fields has been adopted. Since this last inequality (4.16) must hold for all admissible $\widetilde{\xi}$ the following results state:

1. *Equilibrium equation.* Taking $\widetilde{\varepsilon}_p = 0$, $\widetilde{\alpha} = 0$ and $S\left(\widetilde{u}\right) = \emptyset$ one easily obtains that the stress field is constant in space *i.e.*
$$\sigma(x) = \sigma, \qquad \forall\, x \in \Omega. \quad (4.17)$$

2. *Plasticity yield condition.* Taking $\widetilde{u} = 0$ and $\widetilde{\alpha} = 0$ the first order stability condition (4.16) results in
$$\int_{\Omega \backslash S(\widetilde{u})} \left(\sigma \left(\widetilde{u}' - \overline{\widetilde{\varepsilon}_p}\right) + \sigma_{\mathsf{P}}(\alpha)\,|\overline{\widetilde{\varepsilon}_p}|\right)\,\mathrm{d}x + \sum_{x_i \in S(\widetilde{u})} \sigma_{\mathsf{P}}(\alpha)\,|[\![\widetilde{u}]\!](x_i)| \geq 0 \quad (4.18)$$
leading to
$$\sigma \leq \sigma_{\mathsf{P}}(\alpha(x))\,, \qquad \forall\, x \in \Omega \quad (4.19)$$
regardless to the domain $S(\widetilde{u})$.

3. *Damage yield condition.* Finally, taking $\widetilde{u} = 0$ and $\widetilde{\varepsilon}_p = 0$ the first order stability condition (4.16) becomes
$$\int_{\Omega \backslash S(\widetilde{u})} \left(\left(-\frac{1}{2}\mathsf{S}'(\alpha)\,\sigma^2 + \mathsf{w}'(\alpha) - p\,\mathsf{m}(\alpha)\right)\widetilde{\alpha} + \eta^2\,\alpha'\,\widetilde{\alpha}'\right)\,\mathrm{d}x$$
$$- \sum_{x_i \in S(\widetilde{u})} \mathsf{m}(\alpha)\,P(x_i)\,\widetilde{\alpha}(x_i) \geq 0 \quad (4.20)$$
Integrating by parts the term $\alpha'\,\widetilde{\alpha}'$ one obtains by standard arguments of Calculus of Variations the following conditions on both the regular part of the domain $\Omega \setminus S(\xi)$ and the singular parts $S(\xi)$,
$$-\frac{1}{2}\mathsf{S}'(\alpha)\,\sigma^2 + \mathsf{w}'(\alpha) - p\,\mathsf{m}(\alpha) - \eta^2\,\alpha'' \geq 0, \qquad \forall\, x \in \Omega \setminus S(\xi) \quad (4.21a)$$
$$-\mathsf{m}(\alpha)\,P(x_i) - \eta^2[\![\alpha']\!] \geq 0, \qquad \forall\, x \in S(\xi) \quad (4.21b)$$
$$\alpha'(0) \leq 0, \quad \alpha'(L) \geq 0. \quad (4.21c)$$

It is worth noting that the condition (4.21b) allows the derivative of the damage field to suffer jumps if and only if plasticity occurs with a singularity. Clearly, since the bulk damage criterion (4.21a) relies on the accumulated plastic strain p, if the damage field is continuous in a point at a certain instant, by virtue of (4.8), this point could never have been a singular point before.

Noting that $\widetilde{u}(0) = \widetilde{u}(L) = 0$ implies

$$\int_{\Omega\setminus S(\widetilde{u})} \overline{\widetilde{u}'}\,\mathrm{d}x + \sum_{x_i\in S(\widetilde{u})} [\![\widetilde{u}]\!](x_i) = 0, \tag{4.22}$$

one easily verifies that the equilibrium equation (4.17), the plasticity yield condition (4.19) and the damage yield criterion represented by (4.21a)-(4.21c) are sufficient to satisfy local stability (4.16) at first order.

4.1.2.3 Energy balance

Referring to the general definition (2.50) the regular energy balance for the investigated model reads

$$\begin{aligned}0 = &\int_{\Omega\setminus S(\widetilde{u})} \left(\sigma\left(\dot{u}' - \dot{\overline{\varepsilon_p}}\right) + \sigma_{\mathsf{P}}(\alpha)\,|\dot{\overline{\varepsilon_p}}|\right)\,\mathrm{d}x\\ &+ \int_{\Omega\setminus S(\widetilde{u})} \left(-\frac{1}{2}\mathsf{S}'(\alpha)\,\sigma^2 + \mathsf{w}'(\alpha) - p\,\mathsf{m}(\alpha)\right)\dot{\alpha} + \eta^2\,\alpha'\,\dot{\alpha}'\,\mathrm{d}x\\ &+ \sum_{x_i\in S(\widetilde{u})} \left(\sigma_{\mathsf{P}}(\alpha)\,|[\![\dot{u}]\!](x_i)| - \mathsf{m}(\alpha)\,P(x_i)\,\dot{\alpha}(x_i)\right) - \sigma(L)\,\dot{U}\end{aligned} \tag{4.23}$$

Using the identity (4.22) given by the boundary conditions and integrating by parts the term $\alpha'\,\dot{\alpha}'$, (4.23) leads to

$$\begin{aligned}0 = &\int_{\Omega\setminus S(\widetilde{u})} \left(\sigma_{\mathsf{P}}(\alpha)\,|\dot{\overline{\varepsilon_p}}| - \sigma\,\dot{\overline{\varepsilon_p}}\right)\,\mathrm{d}x\\ &+ \int_{\Omega\setminus S(\widetilde{u})} \left(-\frac{1}{2}\mathsf{S}'(\alpha)\,\sigma^2 + \mathsf{w}'(\alpha) - p\,\mathsf{m}(\alpha) - \eta^2\,\alpha''\right)\dot{\alpha}\,\mathrm{d}x\\ &+ \sum_{x_i\in S(\widetilde{u})} \left(-\mathsf{m}(\alpha)\,P(x_i) - \eta^2\,[\![\alpha']\!]\right)\dot{\alpha}(x_i) + \sum_{x_i\in S(\widetilde{u})} \left(\sigma_{\mathsf{P}}(\alpha)\,|[\![\dot{u}]\!](x_i)| - \sigma\,[\![\dot{u}]\!](x_i)\right)\\ &+ \eta^2\left(\alpha'(L)\,\dot{\alpha}(L) - \alpha'(0)\,\dot{\alpha}(0)\right).\end{aligned} \tag{4.24}$$

Using the plasticity yield criterion (4.19), the damage yield criteria (4.21a)–(4.21c) and the irreversibility condition (4.12), one finally obtains the consistency relations and the plastic flow rules:

$$\left(-\frac{1}{2}\mathsf{S}'(\alpha)\,\sigma^2+\mathsf{w}'(\alpha)-p\,\mathsf{m}(\alpha)-\eta^2\,\alpha''\right)\dot{\alpha}=0,\qquad \forall\, x\in\Omega\setminus S(\xi) \tag{4.25a}$$

$$\begin{cases}\dot{\overline{\varepsilon_p}}\geq 0 \text{ if } \sigma=\sigma_{\mathsf{P}}(\alpha)\\ \dot{\overline{\varepsilon_p}}\leq 0 \text{ if } \sigma=-\sigma_{\mathsf{P}}(\alpha)\\ \dot{\overline{\varepsilon_p}}=0 \text{ if } |\sigma|<\sigma_{\mathsf{P}}(\alpha)\end{cases},\qquad \forall\, x\in\Omega\setminus S(\xi) \tag{4.25b}$$

$$\left(-\mathsf{m}(\alpha)\,P(x_i)-\eta^2\,[\![\alpha']\!]\right)\dot{\alpha}(x_i)=0,\qquad \forall\, x\in S(\xi) \tag{4.25c}$$

$$\begin{cases}[\![\dot{u}]\!]\geq 0 \text{ if } \sigma=\sigma_{\mathsf{P}}(\alpha)\\ [\![\dot{u}]\!]\leq 0 \text{ if } \sigma=-\sigma_{\mathsf{P}}(\alpha)\\ [\![\dot{u}]\!]=0 \text{ if } |\sigma|<\sigma_{\mathsf{P}}(\alpha)\end{cases},\qquad \forall\, x\in S(\xi) \tag{4.25d}$$

$$\alpha'\,\dot{\alpha}=0,\qquad x=\{0,L\}\,. \tag{4.25e}$$

Note that the consistency equation and the plasticity flow rule hold on the singular set too.

4.1.3 The general assumptions

With regard to the previous Chapter when the homogeneous solution is (or becomes) not stable a non-homogeneous solution must appear. Nevertheless, limited to the first order stability condition, the homogeneous solution is still admissible. Obviously, non homogeneous solutions involve localizations of the state functions and must satisfy at least the first order stability condition (4.16) too. In the next section attention is given to the construction of a single localization regardless of the global response which is investigated in a later section.

4.2 The construction of localizations

In this section a procedure for the construction of a single localization zone is shown for several different cases depending on the constitutive parameters and reflecting the order in which plasticity and damage occur. The global response is instead obtained in the consecutive section. Nevertheless, one attributes the study to four meaningful cases, the same already introduced in the previous chapter. All other possible localized evolutions simply become a combination of such fundamental cases. These can be classified depending on whether an initial plastic or damage phase occurs:

- damage localization zone with initial vanishing plasticity (*E-D-** phase);
- damage localization zone with initial non vanishing plasticity (*E-P-** phase);

or whether plasticity evolves during the evolution of the localization zone:

- evolution of the damage localization without a plastic evolution (*-*D*);
- evolution of the damage localization with a plastic evolution (*-*PD*);

The attention is here limited on the formation and evolution of only one internal localized damaged zone, thus excluding from the analysis damage localizations near the boundaries and the interactions between multiple damage profiles. Obviously, with respect to the homogeneous case, the damage gradient term in the dissipation potential plays a crucial role since the damage yield condition becomes a differential equation in α from which the damage profile is constructed.

The analysis starts when the damage yield criterion is satisfied somewhere in the bar as an equality. This instant t_{D} corresponds either to the end of an elastic phase if $\sigma_{\mathsf{D}}(0,0) < \sigma_{\mathsf{P}}(0)$ or at the end of a plastic phase $\sigma_{\mathsf{D}}(0,0) \geq \sigma_{\mathsf{P}}(0,0)$. When $t > t_{\mathrm{D}}$ it is assumed that the stress σ_t is monotonically decreasing from $\sigma_{t_{\mathrm{D}}}$ to 0 because of the softening properties of the model.

Starting from an undamaged and not plasticised state, the purpose is to determine the damaged localization profile $\alpha(x)$ and the corresponding plastic strain fields $\varepsilon_p(x)$ and $p(x)$. This aim is achieved by controlling the stress σ: once the critical value $\sigma = \sigma_{\mathsf{D}}(p,0)$ is reached, the material softening response is investigated by decreasing the stress continuously from $\sigma_{\mathsf{D}}(p,0)$ to 0.
In particular, an abstract sequence of the necessary steps for the construction of a single localization is listened below:

1. A homogeneous undamaged state, where $\sigma = \sigma_{\mathsf{D}}$, is chosen to start with;

2. The strong form (4.21a) of the the damage yield condition, a differential equation in α, is initially used to determine the damage profile as σ is decreased. At this stage no singular points are assumed, $S(\xi) = \emptyset$. This analysis produces a function $\sigma \to \alpha_\sigma(x)$ mapping any stress level in the damage profile at constant plasticity ($\dot{p} = 0$) . The damage irreversibility condition is then explicitly considered.

3. The violation of the plastic yield condition is checked for all σ. A violation may occur if in some point $\sigma > \sigma_P(\alpha_\sigma)$. In this case, singular points may appear, $S(\xi) \neq \emptyset$, and all conditions (4.21a)-(4.21b) must be used together with the plastic criterion (4.19) to modify the construction and fulfil all the conditions.

Thus, the problem reduces at this stage to the study of one localized zone only, say the fields $(u, \varepsilon_p, p, \alpha)$ in the domain $\mathcal{S} = (\bar{x} - D, \bar{x} + D)$. Here and in the following, D and $\bar{x}$ denote respectively half the size of the localization zone and the position of its center.
Hereafter, if there is no risk of confusion, the explicit dependence on time t of the state fields will be omitted.

4.2.1 *E-D-** case

In this case, the instant where a localization may appear corresponds to the end of the elastic phase. Since $\sigma_{\mathsf{D}}(0,0) < \sigma_{\mathsf{P}}(0)$ no plasticity has occurred before, hence $t_{\mathrm{yI}} = t_{\mathrm{D}}$. The starting state for the construction of a localisation is homogeneous,

$$u(x) = \frac{\sigma_{\mathsf{D}}(0,0)}{\mathsf{E}_0} x, \quad \varepsilon_p = p = \alpha = 0, \qquad \text{at } t = t_{\mathrm{yI}}. \tag{4.26}$$

The analysis is traced by the studies carried out in Pham, Marigo, and Maurini 2011 and Pham and Marigo 2011. Here, in addition, the irreversibility condition will also be explicitly considered. Initially no singular points are considered inasmuch the plastic criterion is not attained. The localization process is then governed by the damage yield criterion,

$$-\frac{1}{2}\mathsf{S}'(\alpha)\,\sigma^2+\mathsf{w}'(\alpha)-\eta^2\,\alpha''=0,\quad \text{in}\quad \mathcal{S}. \tag{4.27}$$

This last autonomous second order differential equation is an Euler-Lagrange equation which admits a first integral,

$$-\,\mathsf{S}(\alpha)\,\sigma^2+2\mathsf{w}(\alpha)-\eta^2\,\alpha'^2=C,\quad \forall\, x \text{ in } \mathcal{S}. \tag{4.28}$$

The constant C is determined by the boundary conditions which depend on the irreversibility condition. In the most general case one has

$$\eta^2\,\alpha'(x)^2=\mathsf{H}(\sigma,\alpha(x),C)\,,\quad \forall\, x \text{ in } \mathcal{S} \tag{4.29}$$

where $\mathsf{H}\colon\ [0,\sigma_{\mathsf{D}}(0)]\times[0,1)\times\mathbb{R}\to\mathbb{R}$ and

$$\mathsf{H}(\sigma,\alpha,C):=-\mathsf{S}(\alpha)\,\sigma^2+2\mathsf{w}(\alpha)-C. \tag{4.30}$$

For the determination of the constant C two different situations are possible.

Regardless of irreversibility if one assumes initially as boundary conditions for the damage profile the matching conditions with the elastic zone

$$\alpha(\bar{x}\pm D)=\alpha'(\bar{x}\pm D)=0 \tag{4.31}$$

then obtains

$$C(\sigma)=-\mathsf{S}(0)\,\sigma^2+2\mathsf{w}(0)\,. \tag{4.32}$$

In this case, for assigned values of σ, one notes that $\mathsf{H}(\sigma,0,C(\sigma))=0$. If the function $\mathsf{H}(\sigma,\cdot,C(\sigma))$ does not have any other zero, then, considering the boundary conditions, equ. (4.29) has only the trivial solution $\alpha(x)=0$. However, the assumptions (2.23) ensure that $\mathsf{H}(\sigma,\cdot,C(\sigma))$ has at least one non-vanishing zero, say $0<\alpha^*(\sigma)<1$. Since $\alpha\geq 0$ and $\eta\alpha'=\pm\sqrt{\mathsf{H}(\sigma,\alpha,C(\sigma))}$, by standard arguments the damage profile has a maximum in $\bar{x}$ such that $\alpha(\bar{x})=\alpha^*(\sigma)$ and $\alpha'(\bar{x})=0$. It is then possible to construct the damage profile in the localization zone by means of

$$\eta\alpha'=\ \text{sign}(x-\bar{x})\,\sqrt{\mathsf{H}(\sigma,\alpha,C(\sigma))},\quad \forall\, x \text{ in } \mathcal{S}. \tag{4.33}$$

The damage field is symmetric with respect to $\bar{x}$ where damage attains its maximum value, $\alpha^*(\sigma)$. By separating the variables x and α in (4.33) one gets the damage field $\alpha_\sigma(x)$ in an implicit form,

$$x-\bar{x}=\eta\int_{\alpha}^{\alpha^*}\frac{1}{\sqrt{\mathsf{H}(\sigma,\beta,C(\sigma))}}\,\mathrm{d}\beta. \tag{4.34}$$

Thus one has drawn the damage field with respect to the stress level. The obtained damage field $\alpha_\sigma(x)$ depends on the stress. The half-damage support D becomes a function of σ an from (4.58) simply reads

$$D(\sigma)=\eta\int_{0}^{\alpha^*}\frac{1}{\sqrt{\mathsf{H}(\sigma,\beta,C(\sigma))}}\,\mathrm{d}\beta. \tag{4.35}$$

The question is now whether the constructed map $\sigma\to\alpha_\sigma(x)$ satisfies at any point $x\in\mathcal{S}$ the irreversibility condition or not:

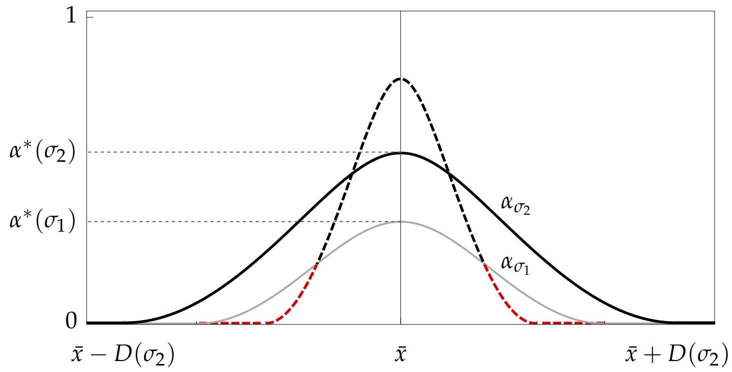


Figure 4.1: Admissible and not-admissible evolutions for the damage profile. The black, thick curve represents an admissible evolution while the dashed curve violates the irreversibility condition in the red part and hence represents a not admissible evolution

1. If for any σ the damage field $\sigma \to \alpha_\sigma(x)$ satisfies the irreversibility condition, that is
$$\forall\, \sigma_2 < \sigma_1, \qquad \alpha_{\sigma_2}(x) \geq \alpha_{\sigma_1}(x)\,, \qquad \forall x \in \mathcal{S}_{\sigma_1}, \tag{4.36}$$
the construction is admissible. In this last relation $\mathcal{S}_{\sigma_1} = (\bar{x} - D(\sigma_1)\,, \bar{x} + D(\sigma_1))$;

2. On the contrary, if a $\sigma_2 < \sigma_1$ exists such that in a subset of $\mathcal{S}$ the damage field decreases, that is
$$\exists\, \sigma_2 < \sigma_1, \quad \Rightarrow \qquad \alpha_{\sigma_2}(x) < \alpha_{\sigma_1}(x)\,, \qquad \text{for some } x \in \mathcal{S}_{\sigma_1}, \tag{4.37}$$
the solution is not admissible and a different construction is proposed.

Both these situations are represented in Fig. 4.1.

If the irreversibility is no longer satisfied from a certain stress level on, a different construction is proposed. Let $\alpha_0(x)$ be the last damage profile for the given stress σ_0 which satisfies the irreversibility condition. The aim becomes to find for a $\sigma < \sigma_0$ a new damage profile which satisfies the irreversibility condition and the first order stability condition, and hence is continuous with a continuous space derivative. This can only be achieved if one admits that only a subset $\mathcal{S} \supset \hat{\mathcal{S}} = \left(\bar{x} - \hat{D}, \bar{x} + \hat{D}\right)$ with $\hat{D} < D(\sigma_0)$ continues to evolve while the remainder of the damage field is subjected to an elastic unloading.

To obtain the new profile, a different constant C in (4.30) has to be found through different boundary conditions. The problem is to find $\hat{\alpha}_\sigma(x)$, the damage profile that continues to evolve and $\hat{D}$ where the profiles $\hat{\alpha}$ and α_0 match.

The new constant $\hat{C}$ is given by
$$\hat{\mathsf{C}}(\sigma, \hat{D}) = -\mathsf{S}\big(\hat{\alpha}_0(\bar{x} - \hat{D})\big)\,\sigma^2 + 2\mathsf{w}\big(\hat{\alpha}_0(\bar{x} - \hat{D})\big) - \eta^2\,\hat{\alpha}_0^2(\bar{x} - \hat{D}) \tag{4.38}$$
so that the new maximum damage level reads
$$\alpha^*(\sigma, \hat{D}) : \mathsf{H}\big(\sigma, \alpha, \hat{\mathsf{C}}(\sigma, \hat{D})\big) = 0. \tag{4.39}$$

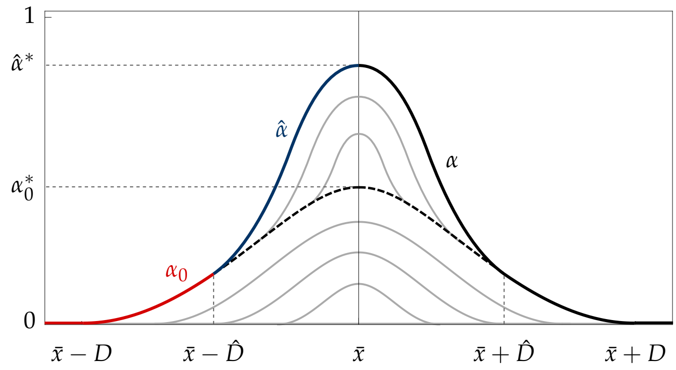


Figure 4.2: The solid black thick line represent the modified construction that satisfies irreversibility. The solid red branch is the profile which does not evolve because subjected to an elastic unloading while the blue branch represents a still evolving profile. The solid gray curves represent the past damage evolution for a decreasing stress.

The unknown $\hat{D}$ is found forcing the center of the localization to remain the same for symmetric reasons, namely

$$\hat{D}\colon \hat{D}-\bar{x}=\eta\int_{\alpha(\bar{x}-\hat{D})}^{\alpha^*(\sigma,\hat{D})}\frac{1}{\sqrt{\mathsf{H}(\sigma,\beta,\hat{C}(\sigma,\hat{D}))}}\,\mathrm{d}\beta. \tag{4.40}$$

Most of the time this last relation cannot be resolved explicitly and a numeric scheme has to be adopted. A numeric implementation has been carried out though the software *Mathematica*®. Once the new damage support $\hat{D}$ has been found the damage profile in the evolution domain $\hat{\mathcal{S}}$ is given by the implicit relation

$$\hat{\alpha}_\sigma(x) : x-\bar{x}=\eta\int_{\hat{\alpha}}^{\hat{\alpha}^*}\frac{1}{\sqrt{\mathsf{H}(\sigma,\beta,\hat{C})}}\,\mathrm{d}\beta, \tag{4.41}$$

that combined with the damage profile subject to the elastic unloading α_0 gives the overall damage evolution,

$$\alpha_\sigma(x)=\begin{cases}\hat{\alpha}_\sigma(x), & \forall\, x\in\hat{\mathcal{S}}\\ \alpha_0(x), & \forall\, x\in\mathcal{S}\setminus\hat{\mathcal{S}}.\end{cases} \tag{4.42}$$

This last construction is depicted in Fig. 4.2.

Now that one is able to construct damage evolutions which can satisfy the irreversibility and are based on the first order stability condition another question arise. Namely if during such evolution $\sigma\to\alpha_\sigma$ the plastic yield criterion is attained or not. Two cases are possible:

1. During all the evolution the plastic criterion is never attained,

$$\sigma<\sigma_{\mathsf{P}}(\alpha_\sigma(x)),\qquad \forall x\in\mathcal{S}; \tag{4.43}$$

2. During the evolution it exist a stress level σ and at least one point x where the plastic yield condition is violated,

$$\exists\, x \in \mathcal{S},\, \sigma > 0, \quad \text{such that} \quad \sigma \geq \sigma_{\mathsf{P}}(\alpha_\sigma(x))\,; \tag{4.44}$$

In case 1 the solution is admissible and the damage process becomes a candidate for the global response since it fulfils all the requirements of the first order stability. On the other hand, case 2 realises a process which at a certain stress level violates the plastic criterion. Hence the process becomes no more stable and a new solution has to be found. In this last case, a different construction is proposed. More specifically all conditions (4.21a)-(4.21b) has to be taken into account and singular points may arise leading to an evolution in the process zone of both damage and plasticity.

One can preliminary identify the evolutions corresponding to the case 1 with an *E-D* response while the case 2 with *E-D-PD* response.

4.2.1.1 *E-D-PD* case

If during the construction of the localization exposed in the previous section the plastic criterion is violated, say at $t = t_{\mathrm{P}} = t_{\mathrm{y_{II}}}$, a singular point appears exactly in the center of the damage profile which corresponds to its maximum value. In this section it is assumed for sake of simplicity that the presence of this singular points is ensured for any $t > t_{\mathrm{P}}$.

More specifically the plastic yield condition could be attained only in $\bar{x}$ since the damage profile has only one maximum in $x = \bar{x}$ and $\alpha \mapsto \sigma_P(\alpha)$ is monotonically decreasing. The critical stress level where the plastic criterion is violated is $\bar{\sigma} = \sigma_{\mathsf{P}}(\alpha_\sigma(\bar{x}))$. This means that in the construction of a localization the maximum value of the damage field is not anymore dictated by the condition $\alpha'(\bar{x}) = 0$ where $\mathsf{H} = 0$ but by

$$\alpha_\sigma(\bar{x}) = \bar{\alpha}(\sigma) = \sigma_{\mathsf{P}}^{-1}(\sigma)\,. \tag{4.45}$$

Moreover the derivative of the damage profile must suffer a jump in the same point; Indeed, for any $\sigma \leq \bar{\sigma}$ the center of the localization becomes a singular point $S(\xi) = \{\bar{x}\}$. Since the damage profile in the connected sub-regions $\Omega \setminus S(\xi)$ is still governed by (4.21a) and the accumulated plastic strain is zero in $\mathcal{S} \setminus \{\bar{x}\}$ the damage field is given by

$$x - \bar{x} = \eta \int_{\alpha}^{\bar{\alpha}(\sigma)} \frac{1}{\sqrt{\mathsf{H}(\sigma, \beta, \mathsf{C}(\sigma))}}\, \mathrm{d}\beta, \tag{4.46}$$

where $\bar{\alpha}$ is given by (4.45). Clearly the damage profile is still symmetric with respect to the center $\bar{x}$ but its derivative is not anymore continuous. A jump $[\![\alpha']\!]$ must occur in $\bar{x}$. Its value reads by means of (4.29)

$$[\![\alpha']\!](\bar{x}) = -\frac{2}{\eta}\sqrt{\mathsf{H}(\sigma, \alpha^*, \mathsf{C}(\sigma))}, \tag{4.47}$$

from which one deduces through (4.21b) the coefficient (4.8) of the accumulated plastic strain Dirac measure as

$$P(\bar{x}) = \frac{\sigma_{\mathsf{P}}'(\alpha)\,[\![\alpha']\!](\bar{x})}{\eta^2}. \tag{4.48}$$

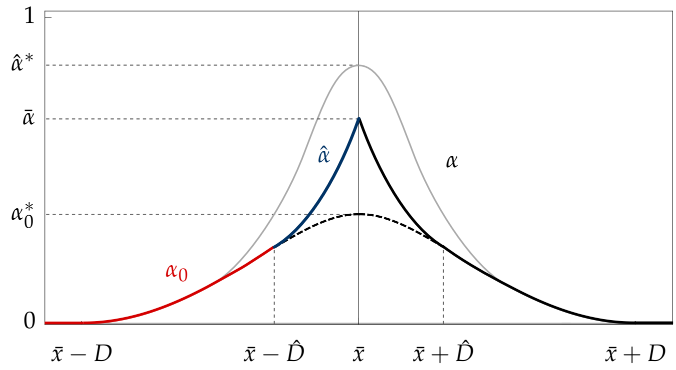


Figure 4.3: The solid black thick line represent the modified construction that satisfies irreversibility. The solid red branch is the profile which does not evolve because subjected to an elastic unloading while the blue branch represents a still evolving profile with a jump in its derivative in $\bar{x}$. The solid gray curve represent the damage evolution if there was not plasticity.

In the most general case, where irreversibility has to be taken into account, the problem becomes for an assigned stress $\sigma < \bar{\sigma}$ to find $\hat{D}$ given by

$$\hat{D}\colon \hat{D} - \bar{x} = \eta \int_{\alpha(\bar{x}-\hat{D})}^{\bar{\alpha}} \frac{1}{\sqrt{\mathsf{H}(\sigma, \beta, \hat{C}(\sigma, \hat{D}))}} \, d\beta. \tag{4.49}$$

The constant $\hat{C}(\sigma, \hat{D})$ is the same as (4.38).
The damage profile in $\hat{\mathcal{S}}$ is given in this case by

$$\hat{\alpha}_\sigma(x) : x - \bar{x} = \eta \int_{\hat{\alpha}}^{\bar{\alpha}} \frac{1}{\sqrt{\mathsf{H}(\sigma, \beta, \hat{C})}} \, d\beta, \tag{4.50}$$

and the overall damage function reads

$$\alpha_\sigma(x) = \begin{cases} \hat{\alpha}_\sigma(x), & \forall\, x \in \hat{\mathcal{S}} \\ \alpha_0(x), & \forall\, x \in \mathcal{S} \setminus \hat{\mathcal{S}}. \end{cases} \tag{4.51}$$

The damage construction for a *E-D-PD* evolution is shown in Fig. 4.3.
The global evolution of the localization is

$\sigma \geq \bar{\sigma}$:

$$\begin{cases} \alpha_\sigma(x) & \text{given in (4.42);} \\ p = \varepsilon_p = 0 \end{cases} \tag{4.52}$$

$\sigma < \bar{\sigma}$:

$$\begin{cases} \alpha_\sigma(x) & \text{given in (4.51);} \\ p = \varepsilon_p = & \begin{cases} 0, & \forall\, x \in \mathcal{S} \setminus \{\bar{x}\} \\ P(\bar{x})\, \delta_{\bar{x}} & \text{given in (4.48);} \end{cases} \end{cases} \tag{4.53}$$

4.2.2 *E-P-** case

The instant t_{yI}, when the first dissipation phenomenon is triggered, corresponds in this section to the instant when the plastic yield criterion is attained as an equality, $t_{\text{yI}} = t_{\text{P}}$. From that instant on $t \geq t_{\text{yI}}$, since the underlying plasticity model is a perfect plasticity model, a spatially indefinite plastic strain distribution could be expected varying from a uniform plastic distribution to a single point localization. For sake of simplicity it is assumed first that plasticity evolves for $t_{\text{yI}} \leq t \leq t_{\text{yII}}$ uniformly along the bar, say $p(x) = \bar{p}\left(t - t_{\text{yI}}\right) / \left(t_{\text{yII}} - t_{\text{yI}}\right)$. The case with a non uniform plastic evolution deserves a more careful treatment although the difficulty is not conceptual but only in the calculations.

Since $p \mapsto \sigma_{\mathsf{D}}(\alpha, p)$ is a decreasing function with respect to plasticity and $\lim_{p\to\infty} \sigma_{\mathsf{D}}(\alpha, p) = 0$ it exists necessarily a second instant t_{yII} where the damage criterion is reached, $t = t_{\text{yII}}$. Clearly a pure plastic evolution for $t \geq t_{\text{yII}}$ is impossible and either the evolution continues only with a damaging phase corresponding to an *E-P-D* response or with a coupled plasticity-damaging phase corresponding to an *E-P-DP* response. Which evolution will actually take place depends on the constitutive functions. In the following both cases are investigated and the relative localization constructions explained.

Moreover it is assumed that once a response is triggered it persists during all the evolution, that is alternate phases of a pure damage evolution and coupled plasticity-damage evolutions are excluded.

Focusing on the construction of localized solutions, for both cases the starting state is homogeneous while the state variables read

$$u(x) = \frac{\sigma_{\mathsf{D}}(0)}{\mathsf{E}_0} x, \quad \varepsilon_p(x) = p(x) = \frac{\mathsf{w}'(0) - \mathsf{S}'(0)\left(\sigma_{\mathsf{P}}(0)\right)^2}{2\,\mathsf{m}(0)}, \quad \alpha(x) = 0, \tag{4.54}$$

at $t = t_{\text{yII}}$ and $\forall x \in [0, L]$.

4.2.2.1 *E-P-D* case

In this case during the localization process plasticity stops and does not evolve anymore. No singular points occurs, $S(\zeta) = \emptyset$, and the damage evolution is dictated as for the *E-D* response by the damage yield criterion (4.21a) where in this case an initial uniform plastic distribution has to be taken into account,

$$-\frac{1}{2}\mathsf{S}'(\alpha)\,\sigma^2 + \mathsf{w}'(\alpha) + p\,\mathsf{m}(\alpha) - \eta^2\,\alpha'' = 0, \quad \forall\, x \text{ in } \mathcal{S}, \tag{4.55}$$

From now on, the procedure for the construction of the damage curves is the same as Sec. 4.2.1 except the presence of the term $p\,\mathsf{m}(\alpha)$. The first integral (4.28) changes into

$$-\,\mathsf{S}(\alpha)\,\sigma^2 + 2\left(\mathsf{w}(\alpha) + p\,\mathsf{M}(\alpha)\right) - \eta^2\,\alpha'^2 = C, \quad \forall\, x \text{ in } \mathcal{S}., \tag{4.56}$$

while the function H in (4.29) becomes also a function of the accumulated plastic strain such that $\mathsf{H}\colon\ [0, \sigma_{\mathsf{D}}(0)] \times [0, \infty) \times [0, 1) \times \mathbb{R} \to \mathbb{R}$ and

$$\mathsf{H}(\sigma, p, \alpha, C) := -\mathsf{S}(\alpha)\,\sigma^2 + 2\left(\mathsf{w}(\alpha) + p\,\mathsf{M}(\alpha)\right) - C. \tag{4.57}$$

In this last definition the only difference with Pham and Marigo 2011 is represented by the additional term $p\,\mathsf{M}(\alpha)$ which however is decisive in the characterization of the response. With such a change the damage field $\alpha_\sigma(x)$ is given in the implicit form

$$x - \bar{x} = \eta \int_{\alpha}^{\alpha^*} \frac{1}{\sqrt{\mathsf{H}(\sigma, p, \beta, C)}} \, \mathrm{d}\beta. \tag{4.58}$$

Clearly (4.58) refers to the case where irreversibility is automatically fulfil. On the contrary one has to modify the construction as in case 2 on page 87 where a construction that accounts for irreversibility is investigated and proposed. In this last case the constant C and the width D of half a localization change respectively to $\hat{C}$ and $\hat{D}$. The damage profile evolution is given by (4.42) coherently changed with the just introduced new definitions and the plastic strain remains constant both in time and space, that is equal to p given by (4.54). Fig 4.2 is still representative of the *E-P-D* evolution.

It is worth noting that the initial width D of the localization is greater for the *E-P-D* response compared to the *E-D* response. Indeed, for equal damage terms, $\mathsf{H}(\sigma, p, \alpha, C) \geq \mathsf{H}(\sigma, 0, \alpha, C) = \mathsf{H}(\sigma, \alpha, C)$.

4.2.2.2 *E-P-DP* case

With respect to the previous analyses, in this case after the instant $t_{\mathrm{y_{II}}} = t_{\mathrm{D}}$ has been reached, plasticity must continue to evolve since the damage profile, constructed by assuming no singular points, violates the plastic criterion. That is, $\alpha_\sigma(x)$ in (4.58) is such that $\sigma_{\mathsf{P}}(\alpha) < 0$ somewhere. Clearly the first point where the plastic criterion is attained is the center of the localization zone. Then as candidate for a stable solution is to take the center of the localization zone as a singular point where plasticity can localise. Besides, it is assumed for sake of simplicity that plasticity continues to evolve during all the process so that intervals where plasticity alternatively evolves or not are excluded.

The passages that follows are the same as in Sec. 4.2.1.1 except for the presence of an initial uniform accumulated plastic strain in the governing equations which slightly modifies the results. The center of the localization zone $\bar{x}$ becomes a singular point, $S(\zeta) = \{\bar{x}\}$, and the maximum damage level, attained in $\bar{x}$, is dictated for a given stress by (4.45). In the region $\mathcal{S} \setminus \{\bar{x}\}$ the damage profile descends from (4.21a) leading to the implicit definition

$$x - \bar{x} = \eta \int_{\alpha}^{\bar{\alpha}(\sigma)} \frac{1}{\sqrt{\mathsf{H}(\sigma, p, \beta, C)}} \, \mathrm{d}\beta, \tag{4.59}$$

where $\mathsf{H}(\sigma, p, \beta, C)$ is the same function and the same meaning as (4.57). By the same line of reasoning of Sec. 4.2.1.1 the damage profile suffers a jump in its derivative in $\bar{x}$ given in (4.47) while the coefficient (4.8) of the accumulated plastic strain Dirac measure is (4.48). In case where during the localization process irreversibility is not satisfied, the same changes in the construction of the damage profile of Sec. 4.2.1.1 have to be adopted. The damage profile evolution is given by (4.51) coherently modified with the just introduced new definitions while the global evolution of the localization reads

$\sigma = \bar{\sigma}$**:**

$$\begin{cases} \alpha_\sigma(x) = 0; \\ p = \varepsilon_p = \bar{p} \end{cases} \tag{4.60}$$

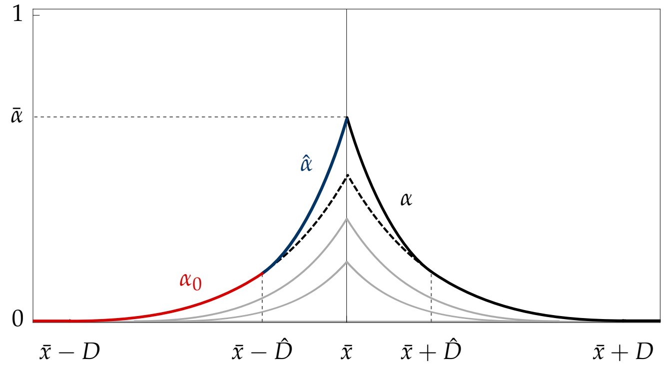


Figure 4.4: The solid black thick line represent the modified construction that satisfies irreversibility. The solid red branch is the profile which does not evolve because subjected to an elastic unloading while the blue branch represents a still evolving profile. The solid gray curves represent the past damage evolution for a decreasing stress. It is worth to remark the damage derivative jumps at the tip of the profiles from the very beginning of the evolution and the initial finite support.

$\sigma < \bar{\sigma}$**:**

$$\begin{cases} \alpha_\sigma(x) & \text{given in (4.51);} \\ p = \varepsilon_p = & \begin{cases} \bar{p}, & \forall\, x \in \mathcal{S} \setminus \{\bar{x}\} \\ \bar{p} + P(\bar{x})\,\delta_{\bar{x}} & \text{given in (4.48);} \end{cases} \end{cases} . \tag{4.61}$$

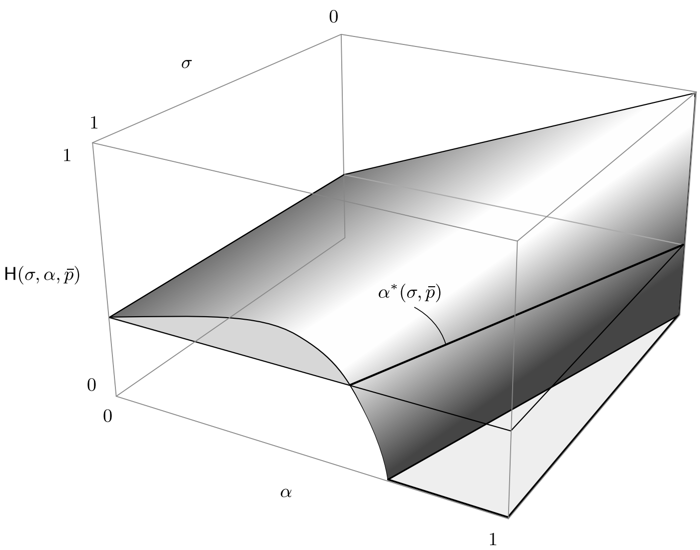


Figure 4.5: Example of the function H.

4.2.3 Incipient damage phase

With regard to the introduced responses, it is interesting and important to study the limit behaviour of the damage profile (4.58) and its support $2\,D$ for an incipient localisation, namely for $\bar{\alpha} \to 0$ and $\sigma \to \sigma_{\mathsf{D}}(0, \bar{p})$. To this aim, one expands up to the second order the function $\mathsf{H}(\sigma, \alpha, \bar{p})$ near $\alpha = 0$:

$$\mathsf{H}(\sigma, \alpha, \bar{p}) = \mathsf{A}_1\,\alpha - \mathsf{A}_2\,\alpha^2 + o\left(\alpha^2\right) \simeq \mathsf{A}_2\,\alpha\,(\alpha^* - \alpha)\,, \tag{4.62}$$

where

$$\alpha^* = \mathsf{A}_1/\mathsf{A}_2, \quad \mathsf{A}_1 := \mathsf{S}'(0)\,(\sigma_{\mathsf{D}}(0, \bar{p})^2 - \sigma^2), \tag{4.63}$$

$$\mathsf{A}_2 := \mathsf{S}''(0)\;\sigma^2_{\mathsf{D}}(0, \bar{p})\,/2 - \mathsf{w}''(0) - \bar{p}\,\mathsf{m}''(0) > 0. \tag{4.64}$$

Clearly the constant C in H is given in the most general case by (4.56). The condition for a positive A_2 follows from $(2.23)_1$. Consequently, the limit value of the half support $D_0(\bar{p})$ is explicitly evaluated to be

$$\begin{aligned} D_0(\bar{p}) = \lim_{\bar{\alpha}\to 0} D(\bar{\alpha}, \sigma_{\mathsf{D}}(0, \bar{p})\,, \bar{p}) &\simeq \sqrt{\frac{\eta^2}{\mathsf{A}_2}} \lim_{\bar{\alpha}\to 0} \int_0^{\bar{\alpha}} \frac{1}{\sqrt{\beta\,(\alpha^* - \beta)}}\,\mathrm{d}\beta \\ &= \sqrt{\frac{\eta^2}{\mathsf{A}_2}} \lim_{\bar{\alpha}\to 0} \left[2\arctan\left(\sqrt{\frac{\beta}{\alpha^* - \beta}}\right)\right]_0^{\bar{\alpha}}. \end{aligned} \tag{4.65}$$

For any examined response one then obtains:

***E-D-** response** In this case

$$2\,D_0 = D_0(0) = \pi\,\eta\left(\frac{1}{2}\mathsf{S}''(0)\;\sigma^2_{\mathsf{D}}(0,0) - \mathsf{w}''(0)\right)^{-1/2} \tag{4.66}$$

since the the zero α^* of the H function does coincide with the maximum value $\bar{\alpha}$ of the damage profile. Solution (4.66) corresponds to the one given in Pham and Marigo 2011;

***E-P-D* response** In this case

$$D_0 = D_0(\bar{p}) = \pi\,\eta\left(\frac{1}{2}\mathsf{S}''(0)\;\sigma^2_{\mathsf{D}}(0, \bar{p}) - \mathsf{w}''(0) - \bar{p}\,\mathsf{m}''(0)\right)^{-1/2} \tag{4.67}$$

for the same reasonings as the previous cases. A remark is that the support $D_0(\bar{p})$ of the localization profile of a plasticized bar can be bigger or smaller than $D_0(0)$ depending on the assumed sign of $\bar{p}\,\mathsf{m}''(0)$ in A_2.

***E-P-DP* response** In this case the limit in (4.65) becomes

$$\lim_{\bar{\alpha}\to 0}\left[2\arctan\left(\sqrt{\frac{\beta}{\alpha^* - \beta}}\right)\right]_0^{\sigma_{\mathsf{P}}^{-1}(\sigma_{\mathsf{D}}(\bar{\alpha}, \bar{p}))} = \chi\pi \tag{4.68}$$

where $\chi \in [0,1]$. The maximum damage value is not α^* but a fraction of it dictated by the plastic yield criterion. Hence

$$D_0 = D_0(\bar{p}) = \chi\pi\,\eta \left(\frac{1}{2}\mathsf{S}''(0)\;\sigma_{\mathsf{D}}^2(0,\bar{p}) - \mathsf{w}''(0) - \bar{p}\,\mathsf{m}''(0)\right)^{-1/2} \tag{4.69}$$

for the same reasonings as the previous cases.

Twice the length D_0 physically represents the minimal length of the bar in order to develop a damage localization profile provided the boundary conditions $\alpha(0) = \alpha(L) = 0$.

Finally note that, under the same hypothesis of incipient damage allowing the Taylor expansion of the H function near $\alpha = 0$, the dependence of $\alpha(x)$ in (4.58) can be made explicit:

$$\alpha(x) = \frac{\mathsf{S}'(0)\,(\sigma_{\mathsf{D}}^2 - \sigma^2)}{\mathsf{A}_2}\cos^2\frac{\pi\,(x-\bar{x})}{2\,D_0(\bar{p})}. \tag{4.70}$$

4.2.4 The case of non-uniform plastic strains

In the previous two subsections, the analysis has been limited to the study of damage profiles in presence of an initially uniform accumulated plastic strain. As only perfect plasticity is considered, the occurrence of such a uniform plastic field $p\,(x) = \bar{p}$ is only one out of infinite admissible solutions for $p(x)$. Indeed it is well-known the non-uniqueness of the problem of a perfectly plastic bar under traction. In this section, is is briefly discuss how such loss of uniqueness affects the contemporary evolution of damage and plasticity.

A first important consideration is the following: the presence of a damage yield stress, $\sigma_D(p,\alpha)$ depending on both the accumulated plastic strain and the damage level, induces the existence of a critical value p_c such that

$$\sigma_P(\alpha = 0) = \sigma_D(p_c, \alpha = 0)\,. \tag{4.71}$$

Indeed, if in some point p reaches the critical value p_c, a further evolution of plasticity alone violates the damage yield condition (4.21b)$_1$ being $p \mapsto \sigma_D(p)$ strictly decreasing, the damage yield stress, with α remaining vanishing, becomes bigger than the current stress value. Hence, if one has an initial non uniform accumulated plastic distribution $p(x) < p_c$, clearly the admissible evolutions allow only the growth of p. However, if the accumulated plasticity reaches the critical value in a set of points, say $p(x) = p_c$ for $x \in \mathcal{S}_c$, then several bifurcations of the equilibria are possible.

A first bunch of possibilities arises if within the set $\mathcal{S}_c$ there is a connected region of diameter larger than $2\,D_0(p_c)$; in this case the analyses carried on in the previous sections are valid as the critically plasticized zone is wider than the minimal support requested for a stemming damage profile. Nevertheless one could also increment the accumulated plastic strain outside $\mathcal{S}_c$: only when the whole bar has been plasticised, *i.e.* $\mathcal{S}_c \to [0,L]$ the evolution of damage evolution becomes necessary, Fig. 4.6.

Another set of different equilibria, satisfying the yield conditions, are possible if the set $\mathcal{S}_c$ cannot contain a whole damage profile. Indeed one could again increment the accumulated plastic strain outside $\mathcal{S}_c$ without allowing damage evolution; in this case the set $\mathcal{S}_c$ can only grow until containing the minimal support $2\,D_0(p_c)$. However it is remarkable that the damage yield condition (4.21a), dictating the shape of the damage profile when $p(x)$ is considered spatially constant, could still be solved as an equation for a non-uniform accumulated plastic strain. Clearly, this involves the solution of an ordinary differential equation with variable coefficients.

All these equilibria are not necessarily local minima; checking the associated second-order stability conditions is however beyond the current aims.

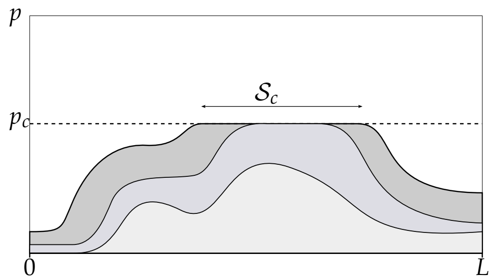


Figure 4.6: Non-uniform plastic distribution example and possibly its evolution

4.3 The global response

Starting from the construction of the various possible localizations, this section focuses on aspects for recovering the global response. It turns out that a cohesive fracture can be described. Roughly speaking a *cohesive fracture response* is achieved in the bar if a jump of the displacement occurs and, nevertheless, the bar still sustains a non vanishing stress. In the presented model, this kind of response relies on the coupling between the evolutions of damage and plasticity driven by the equations of Sec. 4.1.

The global response relies on the construction of a localisation. Due to the complexity of the model this last step has mostly to be solved numerically. Nevertheless a special case allows an analytical solution that gives a deep insight in the model. Examples based on the numeric construction of the localisation are investigated in the following section while a the analytic solution is presented in the submitted paper, Chap. B.

Having in the previous section constructed the damage profiles and examined the evolution of the accumulated plastic strain for all the values of σ during the localisation process, one is finally able to recover the global response in terms of stress-displacement and to discern the response through the energy contributions.

It is important to note that once a localized damage profile appears, the rest of the bar is elastically unloaded; this is due to σ being constant along the bar and to the softening material behaviour. Hence the global response for given constitutive parameters depends upon the length of the bar.

Since in the one dimensional setting the position where a localization appears is arbitrary and since no localizations at the boundaries are taken into account the localization zone is assumed to be placed at the center of the bar. Such choice clearly does not affect the global response of the bar. Hereafter only a single localization is assumed in the bar for simplicity.

The only load conditions considered in this section are those of a tearing test (soft devices). Nevertheless in case where snap-back phenomena occurs the response becomes stress driven. Hence, the global response referring to a traction bar test is assembled in two steps: first the right bar end is monotonically teared up to the instant $t \leq t_{\mathrm{D}}$; then from that instant on, since a snap-back phenomenon may occur and due to the softening behaviour of the damage model, the response is obtained by decreasing the stress and assuming the evolution of a localized zone.

First the stress-displacement response is investigated. The response is after characterized by means of the energy contributions. In case where a singular point appears a cohesive fracture model is retrieved.

4.3.1 Stress-displacement response

Under the aformentioned assumptions, the stress-displacement relation simply reads

$$u(x) = \begin{cases} \displaystyle\int_0^x \frac{\sigma}{\mathsf{E}(\alpha(s))}\,\mathrm{d}s, & \text{for } x < \bar{x}, \\ \displaystyle\int_0^x \frac{\sigma}{\mathsf{E}(\alpha(s))}\,\mathrm{d}s + [\![u]\!]_{\bar{x}}, & \text{for } x > \bar{x}. \end{cases} \tag{4.72}$$

Moreover, the cohesive fracture response is described by the stress depending on the displacement jump amplitude by means of the accumulated plastic strain, namely

$$\sigma = \sigma([\![u]\!])\,. \tag{4.73}$$

An analytical example of this relation can be found in equations (71) and (75) of the submitted paper, Chap. B, while is numerically retrieved in the forthcoming sections.

4.3.2 Cohesive fracture and energy contributions

The elastic stored energy is given for the one dimensional setting by (3.4). The total dissipated work instead descends from (2.27). For the particular choice of the constitutive functions the dissipated $\mathcal{D}_D$ is a function of state and reads

$$\mathcal{D}_D(p,\alpha,\alpha') = \int_0^{\mathrm{L}} \left(\mathsf{w}(\alpha(x)) + \bar{p}(x)\,\sigma_{\mathsf{P}}(\alpha(x)) + \frac{1}{2}\eta^2\,\left(\alpha'(x)\right)^2 \right)\,\mathrm{d}x + P(\bar{x})\sigma_{\mathsf{P}}(\alpha(\bar{x}))\,. \tag{4.74}$$

The total dissipated work could be split into more contributions without an univocal partition. It is reasonable for example to assume $\mathcal{D}_D = \mathcal{D}_p + \mathcal{D}_d$ corresponding respectively to the work dissipated by plasticity and the work dissipated by damage. More specifically

$$\mathcal{D}_p(\bar{p}) = \int_0^{\mathrm{L}} \bar{p}(x)\,\sigma_{\mathsf{P}}(0)\,\mathrm{d}x \tag{4.75}$$

which corresponds to the plastic dissipated work accomplished by the regular part of the accumulated plastic strain before damage is triggered. Instead the value

$$G = \mathcal{D}_D(p,\alpha,\alpha') - \mathcal{D}_p(\bar{p}) \tag{4.76}$$

may be associated to the work dissipated by plasticity and damage in the localization zone. Such contribution can be identified with the fracture energy, Pham et al. 2011.

4.4 Non-homogeneous evolution examples

The aim of this section is to highlight through simple examples the virtues of the proposed model and to give an overview on the wide range of possible responses capable to be described. Both these analytical and numerical examples, which rely on the construction of the localisation proposed in Sec. 4.2, are based on simple but rather general constitutive choices. While the former examples are extensively presented in the submitted article of Chap. B, the latter are hereafter investigated with the following constitutive assumptions

$$\mathsf{E}(\alpha) = \mathsf{E}_0\,(1-\alpha)^2\,, \quad \mathsf{w}(\alpha) = \mathsf{w}_0\,\alpha, \quad \sigma_\mathsf{P}(\alpha) = \sigma_{\mathsf{P}_0}\,(1-\alpha)^\gamma\,, \quad \mathsf{m}(\alpha) = -\sigma_\mathsf{P}'(\alpha)\,, \tag{4.77}$$

which has already introduced for the homogeneous evolution, (3.58). The different responses are obtained by considering slightly different numerical values for the constants as shown in Tab. 4.1.

sequence	$\mathsf{w}_0/\mathsf{E}_0$	$\sigma_{\mathsf{P}_0}/\mathsf{E}_0$	γ
E-P-D	$\sqrt{2}$	1	1
E-P-DP	2	1	2
E-D-PD	0.5	2	2

Table 4.1: Constitutive parameters for the non-homogeneous evolution examples

First, the construction of the damage profiles for the coupled model is compared with the associated damage model where no plasticity has been taken into account. Then, the global response of the various models is obtained by considering a bar of length one and half the width of the first damage profile. The position of the localization along the bar is indifferent with respect to the global response and hence is assumed to be always centred in the middle.

In the following diagrams the gradient temperature colours correspond to a decreasing stress level from the maximum value (blue) down to the minimum value (red).

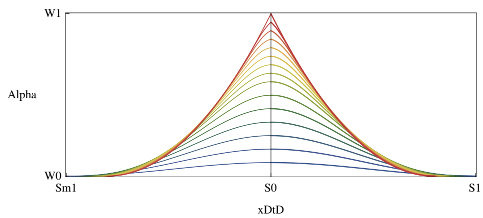
Figure 4.7: Legend for the stress values. Blue correspond to the maximum stress level sustained by the bar while red stands for a vanishing stress

4.4.1 *E-D* response

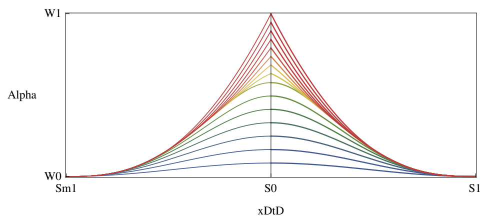
This response corresponds to the pure damage model and has been already extensively detailed in several works, as Benallal and Marigo 2007; Pham 2010; Pham and Marigo 2011. Moreover, it has been demonstrated the inability of this response to describe cohesive fracture, Sicsic and Marigo 2012.

4.4.2 *E-D-PD* response

This response involves the nucleation of a cohesive crack (*PD* stage) after an initial purely damaging phase (*D* stage). Fig. 4.8 focuses on the construction of the damage profiles and compares the coupled model Fig. 4.8b with the associated damage model Fig. 4.8a. Several comments could be done about the construction.
First, irreversibility has been taken into account only for the coupled response using the procedure exposed in (4.2) and not for the "pure" damage model. The initial half-width D of the localization support is clearly the same for both models. Moreover, the trends in Fig. 4.8b highlights the nucleation of a cohesive crack that arises when a non-homogeneous damaging phase has already occurred. The initial smoothness of the damage profiles is witness of the absence of plastic strain localizations. The point where the damage profile starts to suffer a jump in its derivative corresponds to a singular point and hence to a cohesive crack. Furthermore, for a given stress level, the damaging state is sensible lesser in the coupled model than in the "pure" damage model since plasticity dictates the maximum damage level.



(a) *Associated E-D response without damage irreversibility and plasticity*



(b) *E-D-PD response with damage irreversibility. From a damage level on, a jump of the damage derivative occurs*

Figure 4.8: The construction of the damage profiles

The localization evolution can be also appreciate in the phase diagram $\alpha - \alpha'$, Fig. 4.9. The phase diagram stands out the merging points of the evolving damage curves with the unloading zones, necessary to guarantee irreversibility, as well as the derivative jumps at the damage profiles tip.
Fig. 4.10 is representative of the spatial distribution of the damage profiles in the one-dimensional bar.
Since the damage profiles suffer a jump in their derivative from a given stress level on, a jump in the displacement field occurs which is represented with respect to the

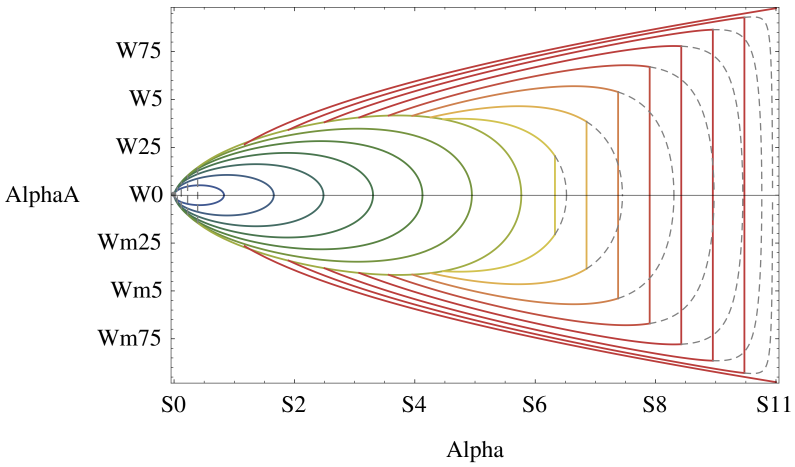


Figure 4.9: The phase diagrams $\alpha - \alpha'$ associated to the damage profiles in Fig. 4.8b. The dashed gray curves represent the hypothetical damage profiles not constrained partially excluded by the plastic limit

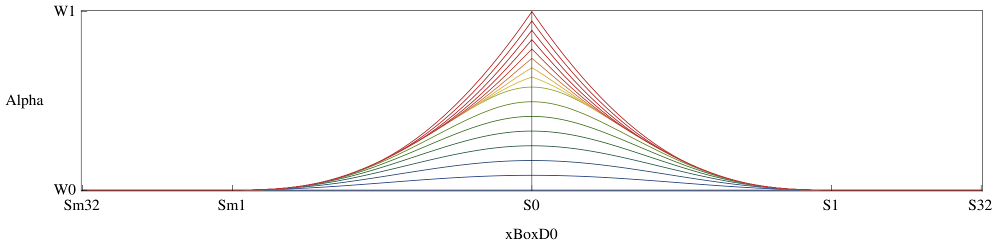


Figure 4.10: The damage profiles evolution in a one-dimensional bar

stress level in Fig. 4.11. The stress vanishes only for a infinite jump of the displacements. As a result this behaviour describes a cohesive fracture of Barenblatt's type, Barenblatt 1959.

Fig. 4.12 shows the evolution of the displacement fields along the bar. It is worth noting the non vanishing elastic energy with the contemporary presence of a fracture represented respectively by a non vanishing slope in the curves and a jump in the displacement fields.

The global stress-displacement response is shown in Fig. 4.13 together with the plastic and damage stress limits. The snap-back behaviour, typical for the associated damage model, Benallal and Marigo 2007, stops when plasticity is triggered.

The energy contributions to the fracture energy are represented in Fig. 4.14 with respect to the displacement jump. An initial fracture energy barrier occurs associated with the evolution of only damage. Except for this barrier, the response represents exactly a Barenblatt's fracture model, Sec. 1.4.1 and (1.72). This last property will be deepen after the next section in Sec. 4.4.4.

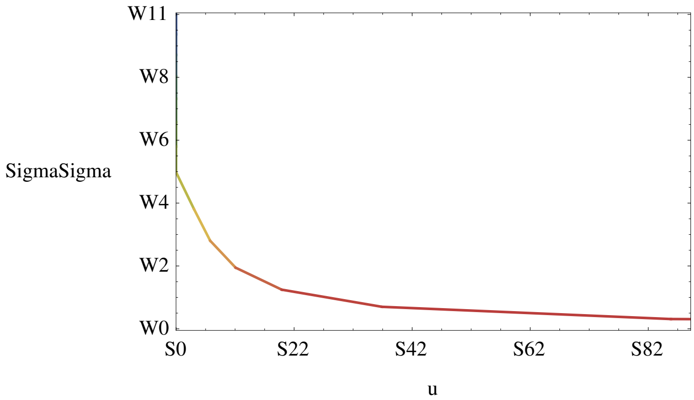


Figure 4.11: Amplitude of the displacement jump at the damage profile tip with respect to the stress level

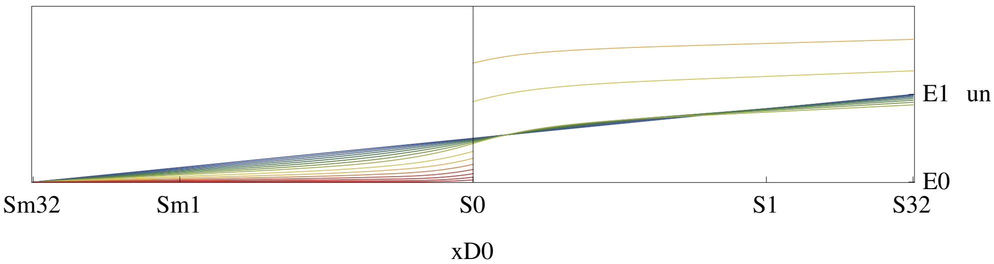


Figure 4.12: The displacement field evolution for different stress levels with a cohesive fracture in the middle of the bar

4.4.3 *E-P-D* response

In the *E-P-D* response plasticity occurs only before the localization process. A part the initial plastic phase the behaviour is quite similar to the *E-D* response, at least qualitatively. Fracture occurs only at $\alpha = 1$ and hence the can be considered brittle. Nevertheless plasticity changes slightly the response compared to an equivalent damage model. For example Fig. 4.15 shows the evolution of damage profiles for the coupled plastic-damage model and the associated damage model where the abscissa has been normalised with respect to the half width of the initial profile of the damage model. In this case irreversibility has not affected in both cases the construction. The coupled model shows a larger support of the damage profiles from the very beginning of the evolution . In this case, plasticity increases the damaging zone.

The phase diagram Fig. 4.16 is smooth since the plastic criterion is never attained during the damage evolution. No singular points exits along the bar at any instant. Fig. 4.17 and Fig. 4.18 represent respectively the damage and displacement evolution in the bar. Clearly, since no plastic localizations occur the displacement field is smooth without jumps.

In the stress-displacement response, Fig. 4.19, one can observe the interruption of plasticity when damage starts to evolve since from that point on the plastic yield

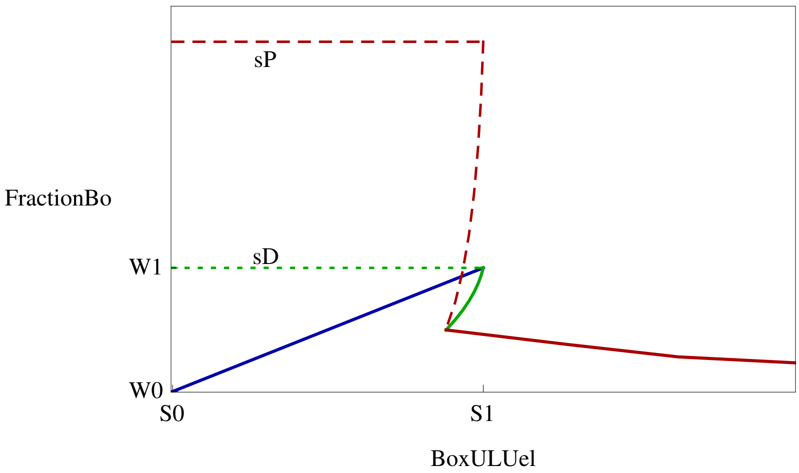


Figure 4.13: The stress-displacement diagram for the *E-D-PD* response: the elastic phase (blue); the damaging phase (solid green); the coupled plasticity-damaging phase (solid red); the damage yield stress (dashed green) and plastic yield stress (dashed red) at the center of the localization zone

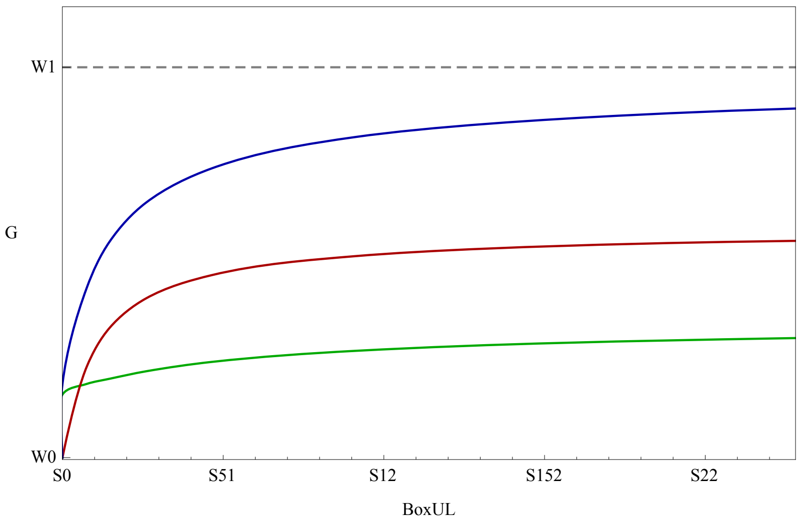
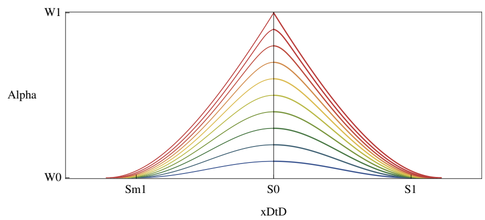
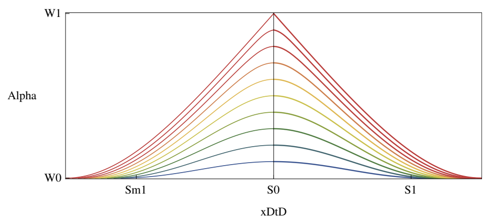


Figure 4.14: The total fracture energy (blue) with the contribution of the damage terms (green) and plasticity term (red) in (4.76)

criterion is never attained anymore. During the damaging phase a snap-back phenomenon appears due to the instability of such model at $t = t_{y_{II}}$.



(a) *The construction of the damage profiles in case where plasticity has not been taken into account*



(b) *The construction of the damage profiles for the coupled model*

Figure 4.15: The construction of the damage profiles. For both cases irreversibility is automatically satisfied

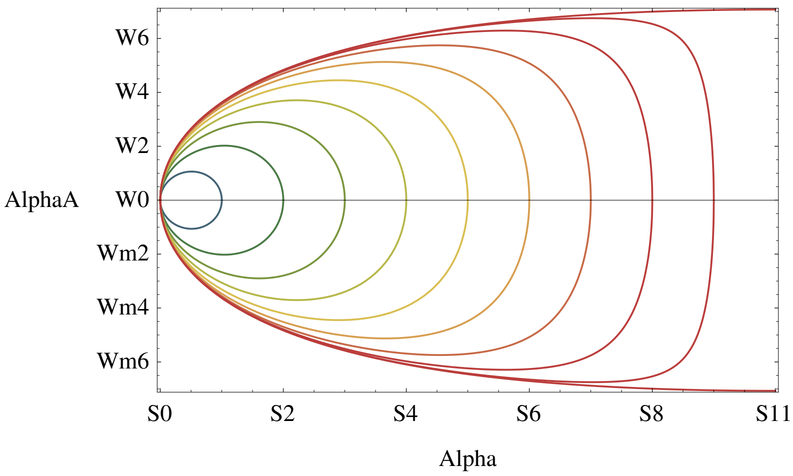


Figure 4.16: The phase diagrams $\alpha - \alpha'$ associated to the damage profiles in Fig. 4.15b

4.4.4 *E-P-DP* response

Like the *E-D-PD* response in Sec. 4.4.2 also the current response involves the nucleation of a cohesive crack (*PD* stage) although it is preceded by a homogeneous plastic phase. The ductile fracture starts from the very beginning of the localization process. This model seems to own all the feature for describing ductile fracture in metals where first a plastic phase occurs.

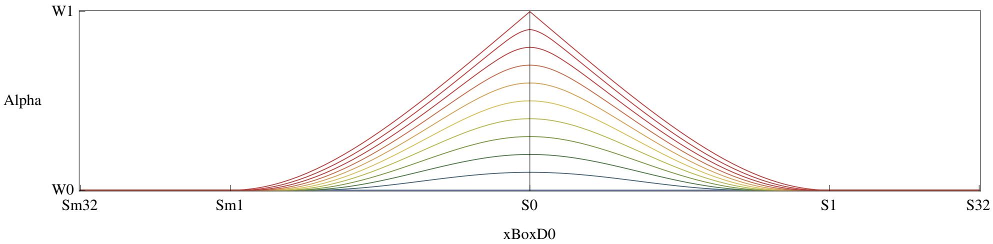


Figure 4.17: The damage profiles evolution in a one-dimensional bar

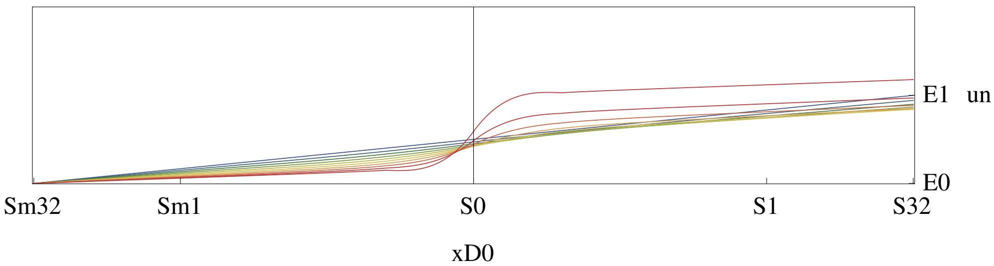


Figure 4.18: The displacement field evolution for different stress levels without fracture but only damage

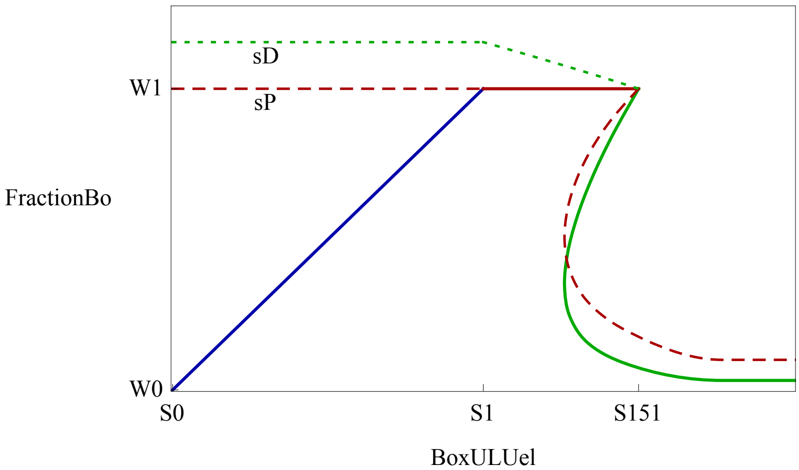
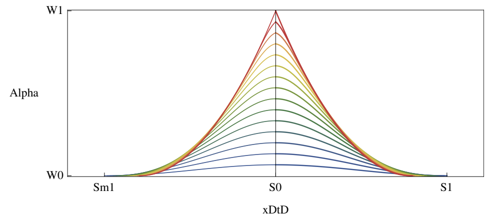
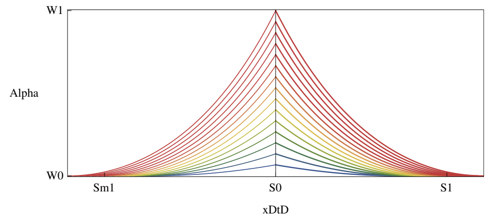


Figure 4.19: The stress-displacement diagram for the *E-P-D* response: the elastic phase (blue); the damaging phase (solid green); the coupled plasticity-damaging phase (solid red); the damage yield stress (dashed green) and plastic yield stress (dashed red) at the center of the localization zone

Fig. 4.20 compares the construction of the damage profiles of the coupled model Fig. 4.20b with the associated damage model Fig. 4.20a. While for the damage model the irreversibility condition is violated during the evolution, in the coupled model irreversibility is automatically fulfilled. Moreover, in this case, the nucleation of a cohesive fracture starts from the very beginning testified by the discontinuities of



(a) *The construction of the damage profiles in case where plasticity has not been taken into account*



(b) *The construction of the damage profiles where irreversibility is automatically fulfilled and a jump of the derivatives occurs from the very beginning of the evolution*

Figure 4.20: The construction of the damage profiles

the damage profiles during all the evolution of the process zone.

The evolution of the damage profiles and their derivative jumps at the center of the localization zone are more appreciable in the phase diagram $\alpha - \alpha'$, Fig. 4.21. Fig. 4.22 and Fig. 4.24 represent respectively the damage and displacement evolution in the bar. Accordingly to the plastic localization, the displacement field suffers a jump from the very beginning of the localization process. It is worth noting the non vanishing elastic energy with the contemporary presence of a fracture represented respectively by a non vanishing slope in the curves and a jump in the displacement field.

Since the damage profiles suffer a jump in their derivatives from the maximum admissible stress level on, a jump in the displacement field occurs which is represented with respect to the stress in Fig. 4.23. The stress vanishes only for a infinite jump of the displacements. Hence this behaviour describes a cohesive fracture of Barenblatt's type, Sec. 1.4.1 and (1.72).

Fig. 4.25 represents the stress-displacement response. After the second yield instant $t_{y_{\text{II}}}$ both plasticity and damage evolves together at the center of the localization zone. The fracture energy can be expressed in terms of the displacement jump $[\![u]\!]$, Fig. 4.26. The response is exactly the same as the Barenblatt model for cohesive fracture where all the fracture energy is spent only for an infinite value of the displacement jump as already pointed out by Fig. 4.23.

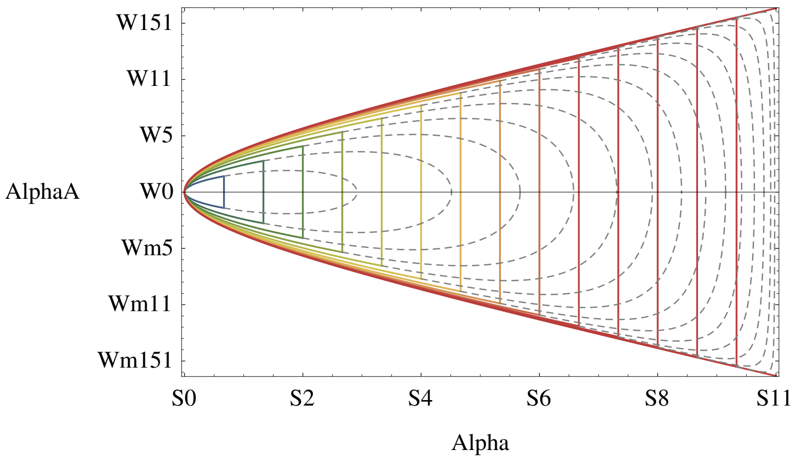


Figure 4.21: The phase diagrams $\alpha - \alpha'$ associated to the damage profiles in Fig. 4.20b. The dashed gray curves represent the hypothetical damage profiles not constrained by the plastic limit

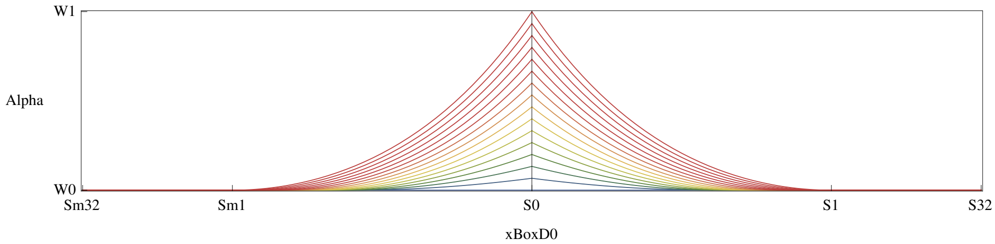


Figure 4.22: The damage profiles evolution in a one-dimensional bar

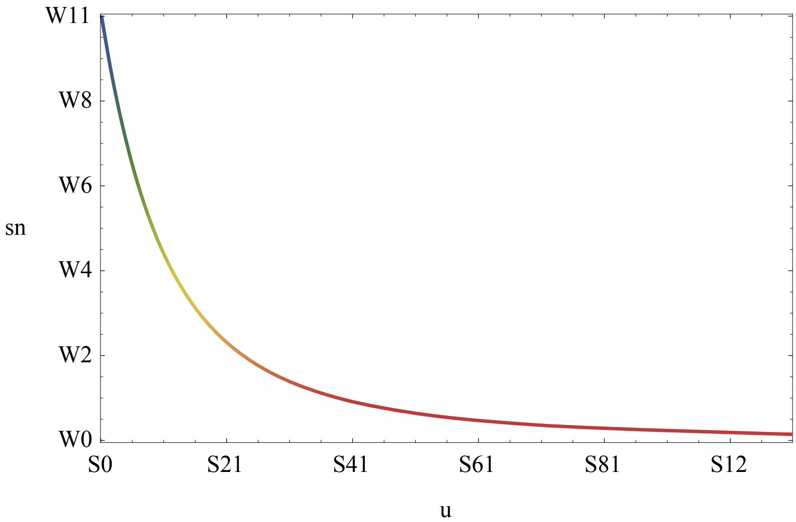


Figure 4.23: Amplitude of the displacement jump at the damage profile tip with respect to the stress level

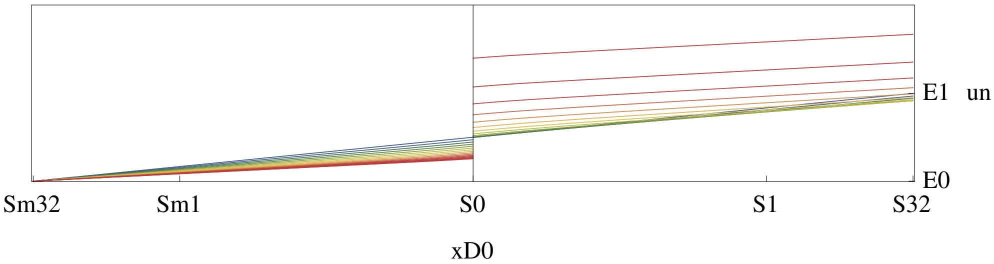


Figure 4.24: The displacement field evolution for different stress levels with a cohesive fracture in the middle of the bar

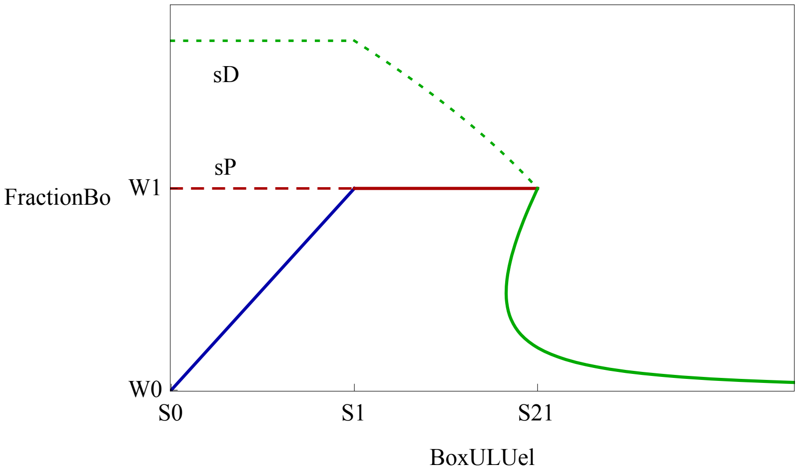


Figure 4.25: The stress-displacement response for the *E-P-DP* response: the elastic phase (blue); the damaging phase (solid green); the coupled plasticity-damaging phase (solid red); the damage yield stress (dashed green) and plastic yield stress (dashed red) at the center of the localization zone

4.5 Conclusions and perspectives

Having given a deep insight on the potentiality of the proposed plastic-gradient damage model, one is in position to draw some conclusions:

Barenblatt's cohesive fracture response The proposed model allows to recover the Barenblatt's cohesive fracture response for a simple and robust choice of the constitutive parameters. While damage is responsible for the localization by means of its gradient and for the degradation of the material stiffness, plasticity is responsible for the cohesive behaviour since plastic strains localise as a Dirac measure leading to a jump in the displacement field with a non vanishing stress;

Wide range of possible responses By tuning conveniently the constitutive parameters it is possible to obtain a wide range of material responses depending on how plasticity, damage or both succeeds.

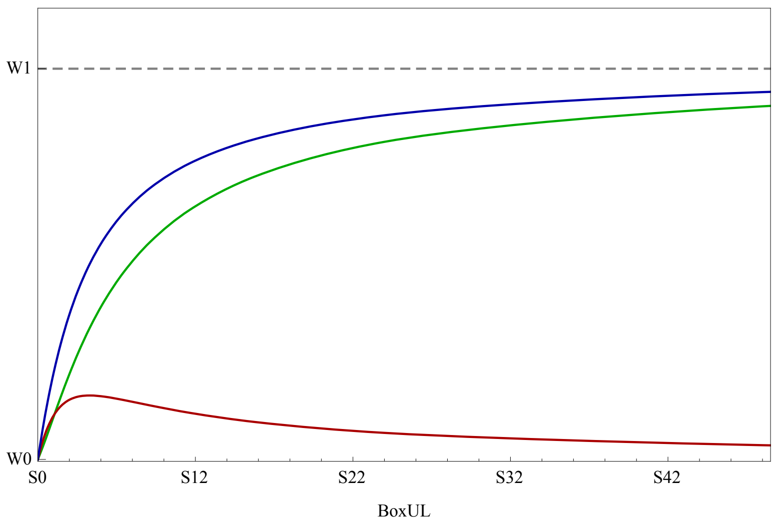


Figure 4.26: The total fracture energy (blue) with the contribution of the damage terms (green) and plasticity term (red) in (4.76)

Of course the current analysis opens many perspectives:

Non-uniform plasticity It could be worth to investigate deeper the case of a non-uniform plastic distribution at the stemming instant of a localization zone. The limit behaviour of a localized plastic response before damage is also of interest. It must be said however that although the perfect plastic model allows in the one-dimensional setting infinite solutions, this is not often the case in many higher dimensional problems since a structural positive hardening effect may occur, Sec. 1.4.1.

Dugdale fracture model It has been proven that the proposed model is able to describe Barenblatt's fracture model. It would be of interest to try to find appropriate constitutive functions, evidently less regular, in order to reproduce a Dugdale's cohesive fracture which has a bilinear curve representing the released energy with respect to the crack amplitude.

Stability of the localized solutions An open and interesting question would also be whether the constructed localized solutions are stable in sense of the second order stability condition. Clearly the appropriate ground to face this task seems to be only the numeric one since the problem becomes mathematically extreme difficult.

Chapter 5

Numeric implementation and simulations

In this section a numeric implementation of the non-homogeneous coupled model introduced in Chap. 4 is proposed taking advantage of the energetic formulation which is in and of itself discrete. The numeric simulations cover the most delicate aspect of the problem, that is the ability of the algorithm to understand and describe the plastic localization and the cohesive response. Although the finite element function spaces do not embed the capability to describe jumps, surprisingly a numerical delocalisation effect appears where the mesh size h owns the role of a regularization parameter. The adopted numerical approach is derived from the one detailed in depth in Bourdin 2000; Bourdin, Francfort, and Marigo 2008 which stems out from the regularization of the brittle fracture model.

Although numerical simulations are presented only for the one-dimensional traction bar test, the algorithm is also suited for further extensions in two-dimensional or three-dimensional settings and different load scenarios.

These first numerical attempts fave been performed through the scalable finite element library *FENICS*[1].

5.1 The implementation

The numerical implementation takes its advantage from the variational formulation. Indeed, the stability condition is reflected into the numerical strategy which is essentially based on looking for (global) minimzizer of an appropriate energy functional. There is always a debate surrounding global minimality as the proper framework. In short, global minimality may sometimes lead to nonphysical evolutions, but local minimality typically forbids crack initiation without singularities, the essence of the one-dimensional model, see Bourdin, Francfort, and Marigo 2008.

For simplicity no external loads are taken into account in the numeric simula-

[1] `http://fenicsproject.org/`

tions. Hence, instead of the total energy $\mathcal{T}$, the internal work $\mathcal{W}$ is here considered,

$$\begin{aligned}\mathcal{W}(u, \varepsilon_p, p, \alpha) = &\int_0^{\mathrm{L}} \frac{1}{2}\mathsf{E}_0 \left((1-\alpha(x))^2 + r_s\right)\left(u'(x) - \varepsilon_p(x)\right)^2 \,\mathrm{d}x \\ &+ \int_0^{\mathrm{L}} \left(\sigma_{\mathrm{P}_0}\, p(x)\,(1-\alpha(x))^2 + \mathsf{w}_0\,\alpha(x) + \frac{\eta^2}{2}\left(\alpha'(x)\right)^2\right) \mathrm{d}x\end{aligned} \tag{5.1}$$

which descends from assumptions (4.77) and is close to the one used in Bourdin 2000 apart the presence of plasticity.

The last two terms in (5.1) resemble the fracture regularization term in Bourdin, Francfort, and Marigo 2008

$$G_c\left(\frac{\alpha}{\ell} + \ell\,\left(\alpha'\right)^2\right), \tag{5.2}$$

with an appropriate change of the constants w_0 and η. The material parameter r_s has the meaning of a residual stiffness. It is justified by the regularisation process through Γ-convergence of the variational fracture problem, Sec. 1.4.1, and gains an important role also for numerical reasons. A $r_s = 0$ would lead to numerical instability while a too big r_s would add some artificial rigidity worsening the accuracy of the solution. On the other hand, the parameter η controls the width of the localization zone which tends towards zero as $\eta \to 0$. Its physical meaning can be associated to an internal material length, see Pham, Marigo, and Maurini 2011.

It is worth noting that the energy functional (5.1) is non-convex although it is separately convex in any variable, Bourdin 2000. This remark gives a hint to the numerical solution strategy. Indeed, seeking a global minimum for the solution and taking advantage of this last property, it seems reasonable to adopt as descending algorithm an alternate minimization respectively in the variables u, ε_p and α[2].

The minimization of $\mathcal{W}$ with respect u at fixed ε_p and α is a straightforward unconstrained optimization problem solved as an elastic problem with prescribed boundary conditions.

The minimization with respect ε_p at fixed u and α is a nonlinear constrained problem. Since no space derivatives involves the field ε_p the optimality condition is local although non linear. Parallelization of the solution algorithm follows naturally in this case since any point is independent from its neighbourhood. A solution procedure can be found in Simo and Hughes 1998 which involves a standard return mapping algorithm. Moreover this allows plasticity to evolve homogeneously along the bar before damage is triggered.

The minimization with respect α at fixed u and ε_p is a box constrained quadratic optimisation problem.

It is well known that such an algorithm applied to a non-convex energy may not converge to a global minimizer but only to a critical point. This can be alleviated by implementing a backtracking algorithm, relying on a necessary condition for optimality with respect to the time evolution, Bourdin, Francfort, and Marigo 2008. In the current situation, when the competition between the three variables in the energy takes place, a similar optimality condition can be imposed, but the construction of an evolution satisfying it is not as straightforward. Thus, in the sequel the backtracking algorithm is not used.

[2] The adopted alternate minimization method is similar to the relaxation algorithm in the domain of quadratic programming

Another drawback of the alternate minimization algorithm is that no convergence criteria are available. Nevertheless, a convergence criterion has to be adopted with some caution. The convergence criterion is far from unique and well determined. In this case, several choices are possible which influence the accuracy of the solution and the calculation speed. Different convergence criterions could be considered, depending:

- on the choice of the tested quantity: variables or energies;
- possibly on the norm adopted for the convergence check;
- on the kind of tolerance: relative or absolute;

Convergence criterions based on the energy lead to inaccurate solutions. Both the three variables u, p and α with their respectively L_2, L_2 and L_{inf} norms were taken into account for the convergence criterion. Moreover, the convergence has been considered passed at a given time step if the single variables has passed singularly a relative fixed tolerance at a given iteration step of the alternate minimisation algorithm [3]. In addition to force an absolute tolerance turned out to an unaffordable increase of alternate minimisation iterations for a given time step which turns out to an increasing of the calculation time. Indeed, as will be immediately clear, plasticity localises in few points and grows inversely proportional to the mesh size.

The numerical solution scheme for the one-dimensional traction bar test is sketched in Fig. 1 where a time discretisation, related to the right-end imposed displacement, has been assumed. Time and space discretisations are examined in the next section.

5.1.1 Time and space discretisation

While for the time discretisation the procedure is standard, the space discretisation of the variables fields requires a more delicate discussion.

The time analysis period $[0, T]$ is discretised into $N+1$ intervals of uniform or not-uniform amplitude. The i-th time step $t_i \in (t_0 = 0, t_1, \ldots, t_{N-1}, t_N = T)$, can be meant as the evolution parameter once related to the active action. For the considered one-dimensional traction bar test the right-end of the bar is prescribed as $u(L) = U(t) = t$.

The function spaces are discretised through finite elements over the domain. Both the displacement and damage fields are projected over a piecewise affine finite element space (1-Lagrange elements) over the same triangulation domain. Motivations for not using higher degree finite elements can be found for example in Bourdin 1998. Conversely, the plastic strain field ε_p is projected over a discrete discontinuous space (Quadrature elements) due to the locality of the minimisation. These points correspond to the Gauss integration points although different choices are possible. For the one-dimensional case these correspond to the center of the finite elements.

[3] Here, as absolute and relative tolerances λ_{abs} and λ_{rel} for a variable x, the below conditions are respectively meant,

$$|x_i - x_{i-1}| \leq \lambda_{\text{abs}}, \quad \left|\frac{x_i - x_{i-1}}{x_{i-1}}\right| \leq \lambda_{\text{rel}}$$

Data: the load steps t_i $(i = 0, \dots, N)$ with $t_0 = 0$, $t_N = T$ and the constitutive parameters;
Result: the fields u, ε_p and α at any load step t_i;
1 initialisation to an unstretched, not plasticised and undamaged state;
2
$$i = 0, \quad u_0 = 0, \quad \varepsilon_{p_0} = 0, \quad p_0 = 0, \quad \alpha_0 = 0;$$
for $i \in (1, \dots, N)$ **do**
3 *alternate minimization algorithm*: initialisation;
4
$$k = 0, \quad u_i^0 = u_{i-1}, \quad \varepsilon_{p_i}^{\,0} = \varepsilon_{p_{i-1}}, \quad p_i^0 = p_{i-1}, \quad \alpha_i^0 = \alpha_{i-1};$$
while
$$\|u_i^{k-1} - u_i^k\|_2 > \lambda \quad \textit{and} \quad \|\varepsilon_{p_i}^{\,k-1} - \varepsilon_{p_i}^{\,k}\|_2 > \lambda \quad \textit{and} \quad \|\alpha_i^{k-1} - \alpha_i^k\|_{\inf} > \lambda$$
(relative convergence criterion) **do**
5 $k = k + 1$;
6 compute $u_i^k := \arg\min\limits_u \mathcal{W}\left(u, \varepsilon_{p_i}^{\,k-1}, p_i^{k-1}, \alpha_i^{k-1}\right)$ with $u(L) = U(t_i)$;
7 compute $\varepsilon_{p_i}^{\,k} := \arg\min\limits_{\varepsilon_p} \mathcal{W}\left(u_i^k, \varepsilon_p, p_i^{k-1} + |\varepsilon_p - \varepsilon_{p_i}^{\,k-1}|, \alpha_i^{k-1}\right)$
8 and $p_i^k = p_i^{k-1} + |\varepsilon_{p_i}^{\,k} - \varepsilon_{p_i}^{\,k-1}|$;
9 compute $\alpha_i^k := \arg\min\limits_\alpha \mathcal{W}\left(u_i^k, \varepsilon_{p_i}^{\,k}, p_i^k, \alpha\right)$ with $\alpha(x) \geq \alpha_{i-1}$;
10 **end while**
11
$$u_i = u_i^k, \quad \varepsilon_{p_i} = \varepsilon_{p_i}^{\,k}, \quad p_i = p_i^k, \quad \alpha_i = \alpha_i^k;$$
12 **end for**

Algorithm 1: Numerical solution scheme. Lines 5–9 corresponds to the core of the procedure, the *alternate minimization algorithm*

Two aspects in the time and a finite element discretisation have to be taken into account in the given model:

1. the proof of the convergence of the minimizer and the minimum of a discretized energy functional towards the original energy (5.1);
2. the capability of the discretized fields to grab the plastic localisation predicted by the analytical model;

Regarding the first point, no attempt is made to prove rigorously the convergence of the discretized model towards the continuous evolution. This complicated task is out of the scope of the present work since high mathematical tools, still investigated, are needed.

As for the second point it would seem that no chance exists to describe displacement jumps and plastic singularities leading to a cohesive response since the chosen finite element spaces do not own the capability to describe such solution. Nevertheless and quite surprisingly this kind of response is picked up from the simulations. The reason depends on how plasticity is implemented. Although plasticity is defined over discrete points the contribution of any point is spread over the entire finite element of size h. That is, the numerical approximation has the effect to regularize the displacement and plastic field while the mesh size h plays the role of a convergent parameter. This aspect of the response is explored with the analyses of the next section where the responses are compared for meshes of different sizes.

5.2 Analyses

This section is devoted to some numerical simulations. The aim is only to give sense and glimpse the potentiality and conceptual simplicity of the implementation of the variational approach. Because one focuses on a one-dimensional traction-bar test, the numerical experiments do not illustrate one of the strength of the variational approach, that is the ability to handle complicated crack paths. For simplicity only analyses concerning the *E-P-DP* response are presented. Nevertheless, the *E-P-DP* response is the most complete and challanging material behaviour for a numeric simulation.

This first numerical attempts are addressed to (i) the comparison with the analytic solution; (ii) show the capability in catching and describing the plastic localisation with respect to the mesh size. Analyses for different convergence tolerances has also been done but are not presented. Also a full parametric investigation is missing.

5.2.1 Analyses setting

A one-dimensional traction bar test is considered where the left-end is fixed and on the right-end a monotonically increasing displacement is prescribed, $u(L) = t$. The bar is initially unstreched, not plasticized and undamaged, that is,

$$u_0(x) = 0, \qquad \varepsilon_{p_0} = 0, \qquad p_0 = 0, \qquad \alpha_0 = 0. \tag{5.3}$$

Moreover, the bar is assumed to be of unitary length, $L = 1$. The constitutive functions considered for the model are the same as (3.58) while the constant values are given in Tab. 5.1.

sequence	E_0	w_0	σ_{P_0}	γ	η
E-P-DP	1	2	1	2	0.3

Table 5.1: Constitutive parameters for the numeric examples

With the previous analytical results of Chap. 4 in mind and for the assumed constitutive functions, the response listed below is expected:

elastic phase (E): The elastic phase lasts until the plastic yield condition is reached, that is, at the time instant $t_{y_{\mathrm{I}}} = t_{\mathrm{P}} = 1$

$$u(x) = x, \quad \varepsilon_p(x) = p(x) = 0, \quad \sigma = \sigma_{\mathsf{P}}(0) = 1. \tag{5.4}$$

plastic phase (P): Subsequently, plasticity is triggered. The plastic phase lasts until the damage yield criterion is reached, that is, at the time instant $t_{y_{\mathrm{II}}} = t_{\mathrm{D}} = 1.5$

$$\varepsilon_p(x) = p(x) = 0.5, \quad u(x) = 1.5x, \quad \sigma = \sigma_{\mathsf{P}}(0) = \sigma_{\mathsf{D}}(0.5, 0) = 1. \tag{5.5}$$

It is worth noting that although infinite solutions exists for such phase, due to the returning mapping algorithm the plastic strain field evolves uniformly in space.

plastic-damage phase (PD): After the instant $t_{y_{\mathrm{II}}}$ damage must evolve together with plasticity. At the center of the localisation zone a singular point is expected where the plastic field suffers a singularity and the displacement field a jump of finite amplitude. The analytic evolution for such phase has been described in Sec. 4.2.2.2.

For what concern the time discretisation several partitions have been considered. The result is that for the one-dimensional case only around the second yield point $t_{y_{\mathrm{II}}}$ a fine discretisation is needed while in any other circumstance coarser partitions are sufficient and do not affect significantly the solution. The analyses shown below refers to the load step partition of Fig. 5.1 where the thick black points concerns the load steps of the further displayed graphics corresponding also to Tab. 5.2.

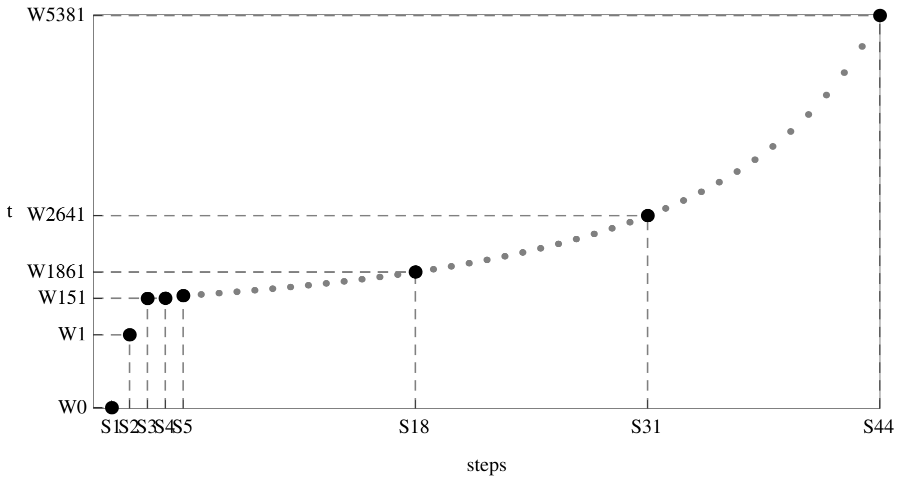


Figure 5.1: The load steps

With the purpose of focusing on the capability of the numeric implementation to catch and describe the localisation, different mesh sizes have been compared, Tab. 5.3. The compared meshes seems to be coarse but, as the results will testify, absolutely coherent for an accurate solution, also with a relative tolerance which has been assumed not too restrictive, $\lambda_{\mathrm{rel}} = 0.01$.

load step	1	2	3	4	5	18	31	44
load t	0	1	1.499	1.501	1.537	1.862	2.636	5.382

Table 5.2: Considered load steps for the displayed responses

n. elements	20	50	100	200	analytic
graphic stroke	dotted, gray	dot-dashed, gray	dashed, gray	solid, gray	solid, black

Table 5.3: Compared mesh sizes and respective graphic stroke

5.2.2 Analyses results

This set of analyses have the aim to compare the solution for different mesh sizes. Although the finite element spaces are not able to deal with singularities the solution nevertheless approximates quite well the predicted analytic response as shown by the following results.

Fig. 5.2 shows the evolution of the damage profiles of the discretized and analytic models for the last four load steps of Tab. 5.2. A more detailed comparison is shown in Fig. 5.3 where a striking matching and convergence towards the analytic solution is undeniable. Just not far away from the profiles tip (one element) the slope of the profiles of the numeric solution equals the slope of the analytic profile.

This agreement in the responses is confirmed by the comparison of the evolutions of the displacement profiles represented in Fig. 5.4 and more clearly from the detail of the center of the bar, Fig. 5.5. Although the displacement finite element function space does not embed jumps, the numeric profiles tends actually to the expected jump with more evidence as the mesh becomes finer. Nevertheless also the coarse mesh predicts the jump amplitude compatibly to the mesh size, that is with the distance of two finite elements.

The observations made on the damage and displacement profiles call for investigate the profiles of the plastic location from which the global response strongly depends. To this aim Fig. 5.6 represents the accumulated plastic profiles for different time steps. The analytic response predicts a Dirac measure at the center of the bar, which hence is not possible to represent in the figure. But what one can be observe is that the area included by the profiles and the x-axis is approximately constant at any load step, Fig. 5.6a–5.6d, regardless to the mesh size. Of course for the figures a linear interpolation has been adopted. The question that spontaneous arises is:

Does the the area below the accumulated plastic strain profile equal the coefficient of the Dirac measure (4.8) *predicted by the analytical solution?*

The answer is affirmative and to be convinced it is sufficient to pay the attention on the graphic of Fig. 5.7 which compares exactly these quantities.

The rate of convergence with a finer mesh can be appreciated in a detail, Fig. 5.8

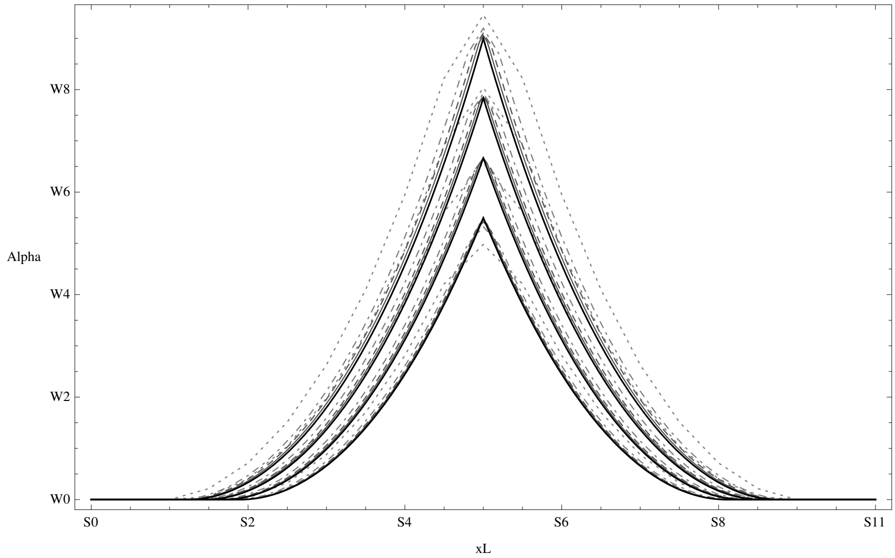


Figure 5.2: The damage field evolution along the bar corresponding to last four load steps of Tab. 5.2

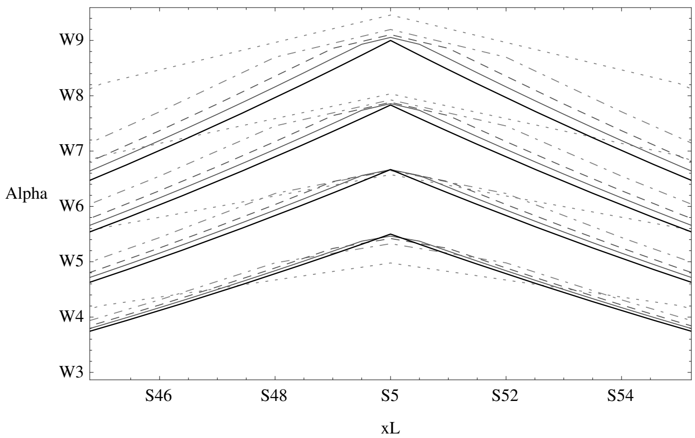


Figure 5.3: Detail of the damage profiles evolution at the center of the localisation zone for the same curves as in Fig.5.2

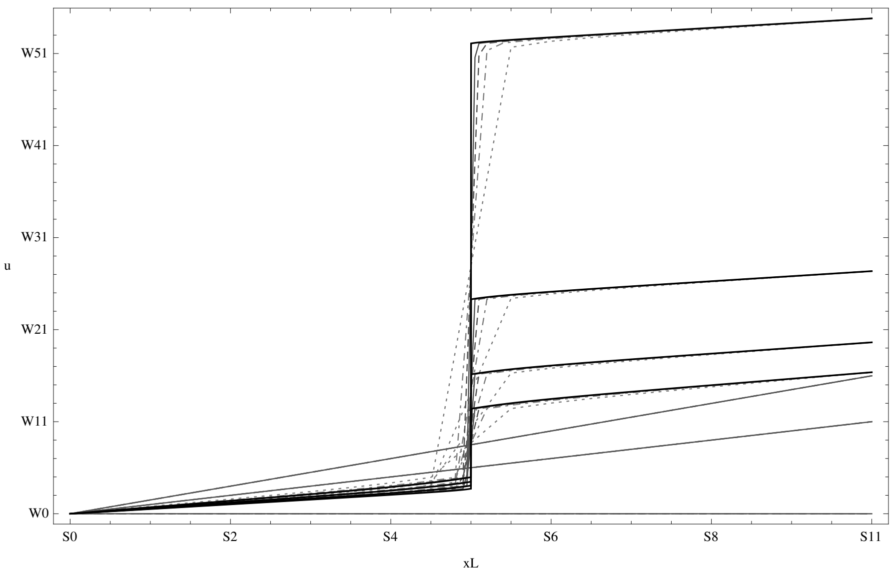


Figure 5.4: The displacement field evolutions along the corresponding to the load steps of Tab. 5.2

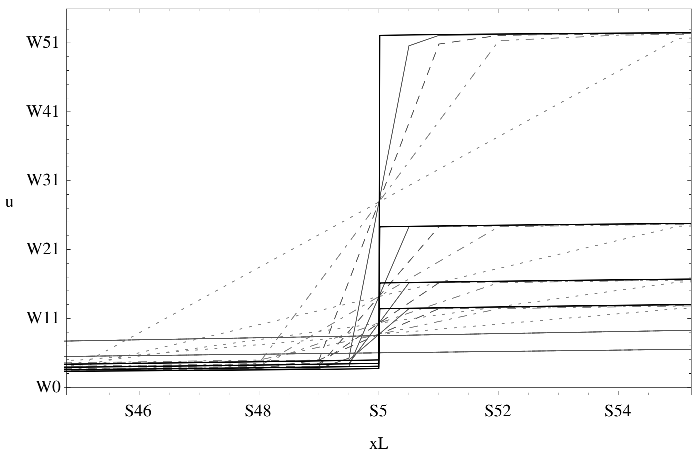
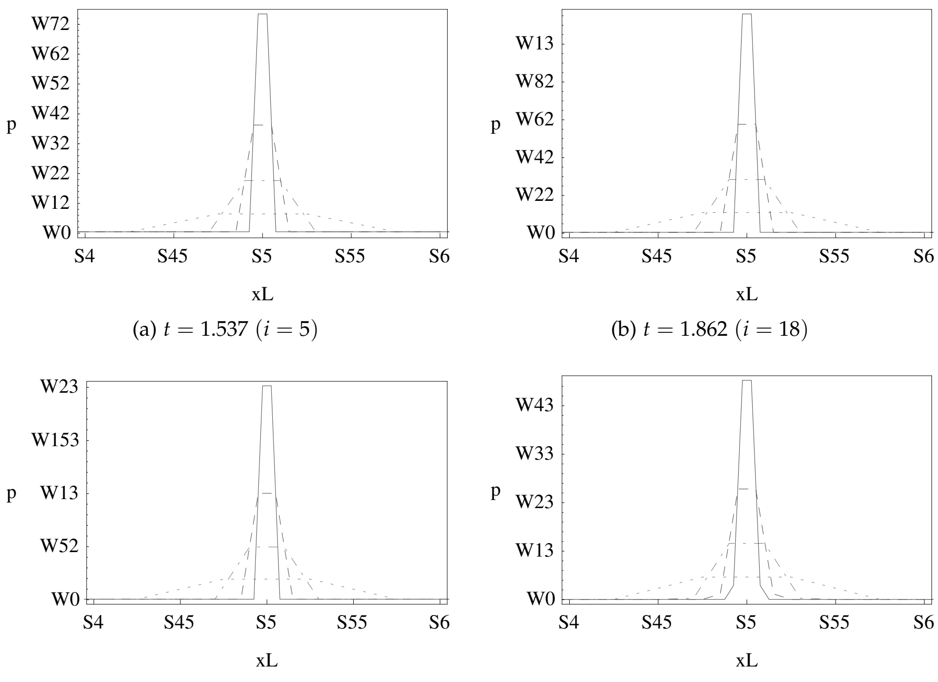


Figure 5.5: Particular of the displacement field evolution at the center of the localisation zone

A certain approximation of the response occurs for the stress profiles. As pointed out in Fig. 5.9, for higher load steps the stress tends to be no more constant so that the equilibrium equation is not satisfied. The reasons are double:

1. First the alternate minimisation algorithm stops with the optimisation with respect to the damage variable. Hence the equilibrium equation is not the last condition to be fulfilled;

2. Secondly this depends how plasticity is implemented. Indeed, the plastic cri-



(a) $t = 1.537$ $(i = 5)$

(b) $t = 1.862$ $(i = 18)$

(c) $t = 2.636$ $(i = 31)$

(d) $t = 5.382$ $(i = 44)$

Figure 5.6: The accumulated plastic strain profiles for different meshes and load steps. The stroke of the profiles pertinent to the different meshes is defined in Tab. 5.3 while load steps referred to are the last four of Tab. 5.2

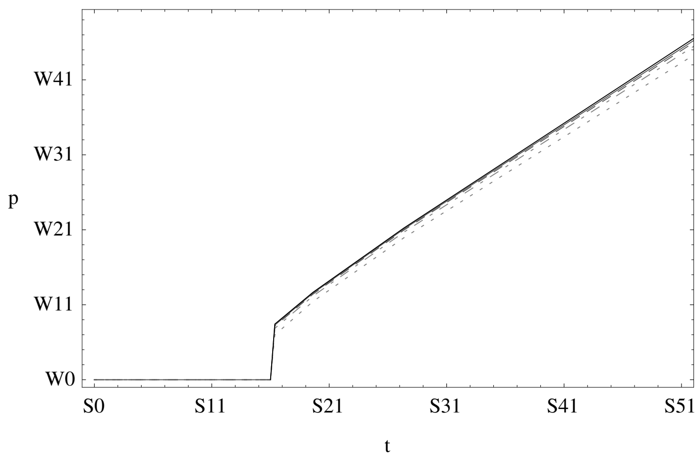


Figure 5.7: Comparison between the integral of the accumulated plastic strain for the different meshes net of the uniform plastic strain distribution occurred before $t_{y_{\text{II}}}$ and the plastic Dirac measure coefficient defined in (4.8)

terion is verified only at the Gauss points which are positioned, in this simu-

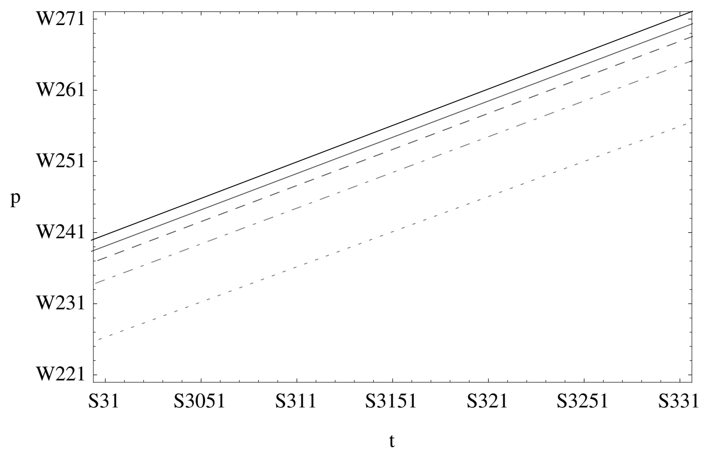


Figure 5.8: A detail of Fig. 5.7

lations, at the center of the elements. Since the damage profile attains its maximum in one node, at the Gauss point next to this node the value of damage pick from the plastic criterion is interpolated with the opposite node value of damage which is not maximum. This mismatch alters the stress limit. Several workaround are possible but has been not yet investigated.

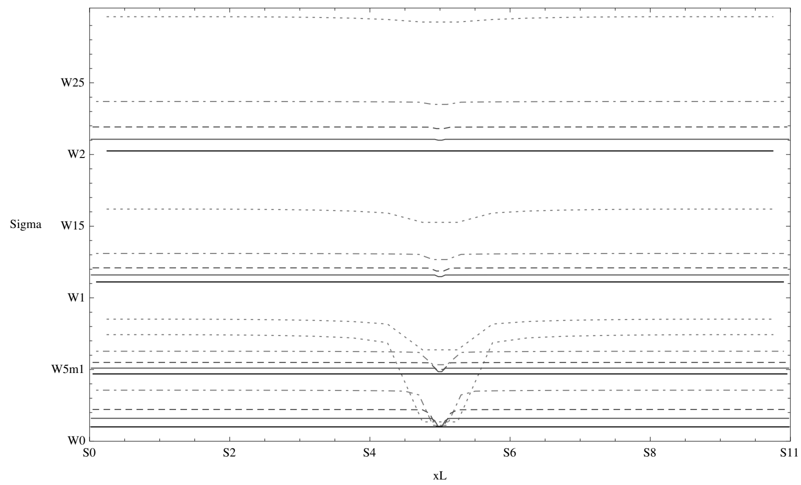


Figure 5.9: The stress field evolution along the bar corresponding to last four load steps of Tab. 5.2

Once the state variables fields have been investigated the global response can be retrieved. Fig. 5.10 and Fig. 5.11 represent respectively the global stress-displacement response and the energy contributions with respect to the load parameter and compare the analytic solution with the numeric ones for the four mesh sizes of Tab. 5.3. An impressive agreement already for coarse meshes (50 elements) confirms the effectiveness of this first numeric implementation attempt.

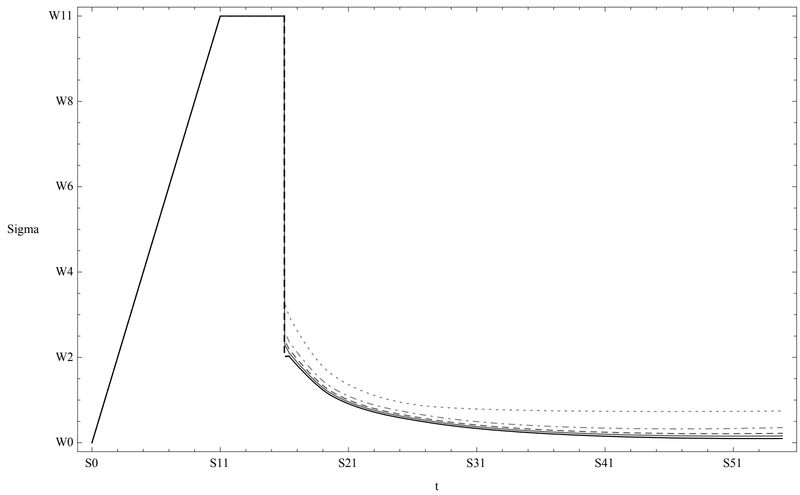


Figure 5.10: Stress displacement response for the numeric (different mesh sizes, Tab. 5.3) and analytic responses

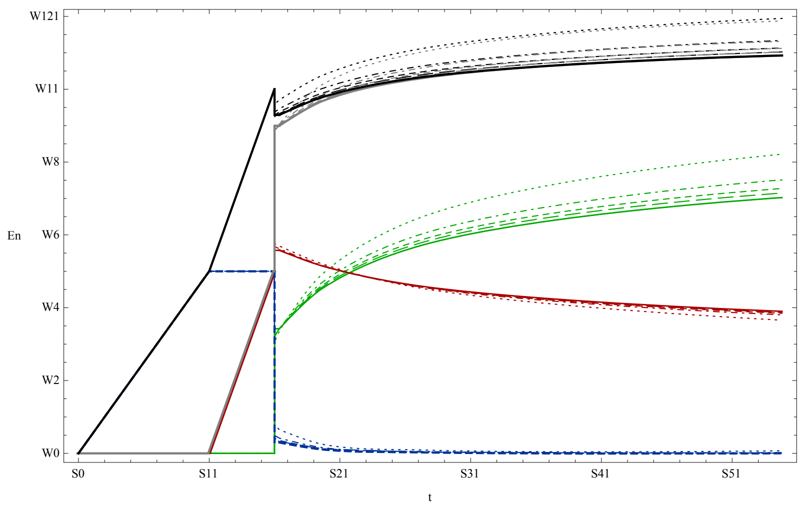


Figure 5.11: The various energy contributions: the total work (black) splited between the elastic energy (blue) and the dissipated work (gray) which in turn is the sum of the dissipated work repectively done by plasticity (red) and damage (green)

5.3 Conclusions and perspectives

An impressive agreement for still coarse meshes (50 elements) confirms the effectiveness of this first numeric implementation attempt although several improvements could be done. Some of them are listed below. One of the strength of the variational approach is that it leads to a natural and rational way for a numerical implementation. It overcomes easily obstacle like initiation, bifurcation and evolution paths. However this last feature has not been possible to highlight through the one dimensional model.

Several improvements could be done for the numerical simulations. Let focus on some of them:

2D and 3D simulations A natural and interesting extension of the numeric model would be to implement some 2D or 3D problems. No beforehand difficulties occur to this purpose except for the computational efficiency. But since the adopted finite element libraries relies on turn on parallel and scalable linear algebra libraries (PETSc)[4] acceptable efficiency should be preserved.

Plasticity implementation step Another improvement could derive by changing the way how plasticity is implemented like, for example, considering different points in the element where the plastic criterion is checked or by taking directly in the plastic criterion for an element the most strict damage value of the element nodes itself.

Extended finite elements What also could be done is to embed in the finite element function spaces functions capable to describe singularities and discontinuities of the response. That is, extending the displacement and plastic strain field to deal with singularities, for example by enriching the function space with discontinuous functions (XFEM).

[4] `http://www.mcs.anl.gov/petsc/index.html`

Appendix A

Partial results

A.1 Gateaux derivative of the dissipation distance

The first-order stability condition (**st-1D**) requires the evaluation of the Gateaux derivative of the dissipation distance D in direction $\left(\widetilde{\varepsilon}_p, \widetilde{\alpha}\right)$. This last, however, is path-dependent and is not a state functional as the stored elastic energy; hence, with respect to the model adopted for brittle fracture, Bourdin, Francfort, and Marigo 2008, or damage, Pham and Marigo 2011, non standard arguments are required.

To be more precise since the dissipation distance depends on the equivalent plastic strain and not directly on the plastic strain, the definition of the variated state deserves caution. In particular, one wants to evaluate explicitly

$$D'(p, \alpha)(\widetilde{p}, \widetilde{\alpha}) = \lim_{h \to 0} \frac{1}{h} D((p, \alpha), (p + h\,\widetilde{p}, \alpha + h\,\widetilde{\alpha})) . \tag{A.1}$$

But $\widetilde{p}$ is not unique since all the admissible paths from $\left(\varepsilon_p, \cdot\right)$ to the new variated state $\left(\varepsilon_p + h\,\widetilde{\varepsilon}_p, \cdot\right)$ lead to infinite possible values for the vairiated state $(p + h\widetilde{p})$. The non-trivial definition of the dissipation distance D in (1.54) becomes necessary to evaluat (A.1). One can state that

$$\widetilde{p} \in \left[\|\widetilde{\varepsilon}_p\|, +\infty\right) \tag{A.2}$$

and hence

$$D'(p, \alpha)(\widetilde{p}, \widetilde{\alpha}) = \lim_{h \to 0} \frac{1}{h} \int_\Omega \inf\left\{ \int_0^1 \left(\partial_s \left(\mathsf{w}(\beta) + \frac{1}{2}\eta^2\, \nabla\beta : \nabla\beta \right) + \sigma_{\mathsf{P}}(\beta)\, \dot{q} - \mathsf{q}(q)\, \mathsf{m}(\beta)\, \dot{\beta} \right) \mathrm{d}s \right\} \mathrm{d}\Omega \tag{A.3}$$

where the variables in the integral are constrained to start at $s = 0$ from the state (p, α), to arrive at $s = 1$ in the state $(p + h\widetilde{p}, \alpha + h\widetilde{\alpha})$ and to be sufficiently smooth, see (1.54) or (2.26). Clearly, the first and second addend in (A.3) are path independent and are easily evaluated to get

$$D'_{1,2} = \int_\Omega \left(\mathsf{w}'(\alpha)\, \widetilde{\alpha} + \eta^2\, \nabla\alpha : \nabla\widetilde{\alpha} \right) \mathrm{d}\Omega. \tag{A.4}$$

In order to evaluate D' one must add $D'_{3,4} = \lim_{h\to 0} D_{3,4}/h$ to $D'_{1,2}$, where

$$D_{3,4} = \int_\Omega \inf\left\{ \int_0^1 \left(\sigma_{\mathsf{P}}(\beta)\, \dot{q} - \mathsf{q}(q)\, \mathsf{m}(\beta)\, \dot{\beta} \right) \mathrm{d}s \right\} \mathrm{d}\Omega. \tag{A.5}$$

To estimate $D'_{3,4}$ one can find two functions $m(h)$ and $M(h)$ such that

$$\lim_{h\to 0} \frac{1}{h} m(h) \le D'_{3,4} \le \lim_{h\to 0} \frac{1}{h} M(h) \tag{A.6}$$

and show that the two limits, in (A.6), converge to the same value.

In particular, since $\beta \mapsto \sigma_{\mathsf{P}}(\beta)$ is decreasing, $\dot{\beta} \ge 0$ and $q \mapsto \mathsf{q}(q)\ \mathsf{m}(\beta)$ is increasing, the function $m(h)$ can be chosen as

$$\begin{aligned} m(h) &:= \int_\Omega \inf\left(\sigma_{\mathsf{P}}(\alpha + h\widetilde{\alpha}) \int_0^1 \dot{q}(s)\,\mathrm{d}s - \left(p + \int_0^1 \dot{q}(s)\,\mathrm{d}s \right) \int_0^1 \partial_s \mathsf{M}(\beta(s))\,\mathrm{d}s \right) \mathrm{d}\Omega = \\ &= \int_\Omega \inf\Big(\sigma_{\mathsf{P}}(\alpha + h\widetilde{\alpha})\, \bar{p} - (p + \bar{p})\,(\mathsf{M}(\alpha + h\widetilde{\alpha}) - \mathsf{M}(\alpha)) \Big)\, \mathrm{d}\Omega \le D_{3,4} \end{aligned} \tag{A.7}$$

where M is the indefinite integral of m. The background idea essentially correspond to have assumed for $m(h)$ the sum of the "inf" of the two terms in (A.5) singularly. Note that $\bar{p} \in \big[h\,\|\widetilde{\varepsilon}_p\|, +\infty\big)$ and hence can not be less than $h\,\|\widetilde{\varepsilon}_p\|$ since this lower bound corresponds to the shortest distance between the states ε_p and $\varepsilon_p + h\widetilde{\varepsilon}_p$.
The infimum in the definition of $m(h)$ can be evaluated once the derivative

$$\Delta(h) := \frac{\mathrm{d}F}{\mathrm{d}\bar{p}}(h) = \sigma_{\mathsf{P}}(\alpha + h\widetilde{\alpha}) - (\mathsf{M}(\alpha + h\widetilde{\alpha}) - \mathsf{M}(\alpha)) \tag{A.8}$$

is considered, where F corresponds to the argument of the infimum in (A.7). Indeed, if $\forall\, \bar{p} \in [h\,\|\widetilde{\varepsilon}_p\|, \infty)$ the derivative $\Delta(h)$ is non-negative then the infimum is attained in $\bar{p} = h\,\|\widetilde{\varepsilon}_p\|$ and its value is

$$\mu(h) := \sigma_{\mathsf{P}}(\alpha + h\,\widetilde{\alpha})\, h\,\|\widetilde{\varepsilon}_p\| - \mathsf{q}\big(p(t) + h\,\|\widetilde{\varepsilon}_p\|\big)\,(\mathsf{M}(\alpha + h\widetilde{\alpha}) - \mathsf{M}(\alpha))\,. \tag{A.9}$$

Instead, if the derivative $\Delta(h)$ can be negative, the infimum can be attained in $\bar{p} \ge h\,\|\widetilde{\varepsilon}_p\|$ and its value can be finite or not. Once estimated the behaviour of the infimum, one can observe that, in the limit for $h \to 0$, the derivative $\Delta(h)$ is non-negative $\forall\, \bar{p}$, as it tends to $\sigma_{\mathsf{P}}(\alpha)$. Therefore, the infimum is $\mu(h)$ and

$$\lim_{h\to 0} \frac{1}{h} m(h) = \lim_{h\to 0} \frac{1}{h} \int_\Omega \mu(h)\ \mathrm{d}\Omega = \int_\Omega \big(\sigma_{\mathsf{P}}(\alpha)\,\|\widetilde{\varepsilon}_p\| - \mathsf{q}(p)\,\mathsf{m}(\alpha)\,\widetilde{\alpha} \big)\ \mathrm{d}\Omega. \tag{A.10}$$

To define the function $M(h)$ one can choose, as admissible path, a linear interpolation, for $s \in [0,1]$, between the states, *i.e.* $q(s) = p(t) + s\,h\,\|\widetilde{\varepsilon}_p\|$ and $\beta(s) =$

$\alpha(t) + s\,h\,\widetilde{\alpha}$. With such a choice, $\dot{q} = h\,\|\widetilde{\varepsilon}_p\|$, $\dot{\beta} = h\,\widetilde{\alpha}$ and hence

$$
\begin{aligned}
D_{3,4} &\leq \int_\Omega \int_0^1 \left(\sigma_{\mathsf{P}}(\beta(s))\,\dot{q} - \left(p(t) + \int_0^s q(\tau)\,\mathrm{d}\tau \right) \partial_s \mathsf{M}(\beta(s)) \right) \mathrm{d}s \;\mathrm{d}\Omega \\
&= \int_\Omega \int_0^1 \left(\sigma_{\mathsf{P}}(\alpha(t) + s\,h\,\widetilde{\alpha})\,h\,\|\widetilde{\varepsilon}_p\| - \big(p(t) + s\,h\,\|\widetilde{\varepsilon}_p\|\big)\,\partial_s \mathsf{M}(\alpha(t) + s\,h\,\widetilde{\alpha}) \right) \mathrm{d}s \;\mathrm{d}\Omega \\
&\leq \int_\Omega \int_0^1 \left(\sigma_{\mathsf{P}}(\alpha(t))\,h\,\|\widetilde{\varepsilon}_p\| + \mathsf{q}(p(t))\,\partial_s \mathsf{M}(\alpha(t)) \right) \mathrm{d}s \;\mathrm{d}\Omega \\
&= \int_\Omega \Big(\sigma_{\mathsf{P}}(\alpha(t))\,h\,\|\widetilde{\varepsilon}_p\| - \mathsf{q}(p(t))\,\big(\mathsf{M}(\alpha(t) + h\,\widetilde{\alpha}) - \mathsf{M}(\alpha(t))\big) \Big)\; \mathrm{d}\Omega =: M(h)\,.
\end{aligned}
\tag{A.11}
$$

Passing to the limit, one obtains

$$
\lim_{h\to 0} \frac{1}{h} M(h) = \int_\Omega \big(\sigma_{\mathsf{P}}(\alpha)\,\|\widetilde{\varepsilon}_p\| - \mathsf{q}(p)\,\mathsf{m}(\alpha)\,\widetilde{\alpha} \big)\; \mathrm{d}\Omega. \tag{A.12}
$$

Recalling (A.5) and (A.6), the Gateaux derivative of the dissipation distance D then reads

$$
D'(p,\alpha)\big(\widetilde{p}(\widetilde{\varepsilon}_p)\,,\widetilde{\alpha}\big) = \int_0^{\mathrm{L}} \Big(\mathsf{w}'(\alpha)\,\widetilde{\alpha} + \eta^2\; \nabla\alpha : \nabla\widetilde{\alpha} + \sigma_{\mathsf{P}}(\alpha)\,\|\widetilde{\varepsilon}_p\| - \mathsf{q}(p)\,\mathsf{m}(\alpha)\,\widetilde{\alpha} \Big)\; \mathrm{d}x. \tag{A.13}
$$

A.2 Rayleigh ratio minimization: 1D case

The aim of this section is to evaluate the minimum of the Rayliegh ratio

$$\mathcal{R}(\varepsilon_p,\alpha) = \frac{\mathsf{a}\int_0^1 \alpha'^2\,\mathrm{d}x + \mathsf{b}\left(\int_0^1 \varepsilon_p\,\mathrm{d}x\right)^2 + 2\,\mathsf{b}\,\mathsf{c}\int_0^1 \varepsilon_p\,\mathrm{d}x\int_0^1 \alpha\,\mathrm{d}x + \mathsf{b}\,\mathsf{c}^2\left(\int_0^1 \alpha\,\mathrm{d}x\right)^2}{\int_0^1\left(\mathsf{m}\,\alpha^2 + 2\,\mathsf{n}\,\varepsilon_p\,\alpha\right)\mathrm{d}x} \tag{A.14}$$

over the domain $\mathcal{V} = \mathrm{SBV}(0,1)\times \mathrm{H}^1_+(0,1)$, where a, b, c, m and n are all positive constants.

Without going into details the minimum exists (in the sense that is reached by an admissible pair (ε_p,α) by virtue of the compactness $\mathcal{V}$ and by th weak lower-semicontinuity of a semi-norm.

Let $(\varepsilon_{p*},\alpha_*)$ then be a minimizer and $\mathcal{R}^* = \mathcal{R}^*(\varepsilon_{p*},\alpha_*)$ the minimum. Since $\mathcal{V}$ is not a linear space, but only a convex set, the minimizer has to satisfy the following variational inequality, Nguyen 2000,

$$\mathcal{R}'(\varepsilon_{p*},\alpha_*)(\widetilde{\varepsilon}_p-\varepsilon_{p*},\widetilde{\alpha}-\alpha_*)\geq 0,\qquad \forall\ (\widetilde{\varepsilon}_p,\widetilde{\alpha})\in\mathcal{V}. \tag{A.15}$$

The previous inequality, (A.15), is equivalent to imposing

$$\mathcal{R}'(\varepsilon_{p*},\alpha_*)(\varepsilon_{p*},\alpha_*) = 0 \tag{A.15a}$$
$$\mathcal{R}'(\varepsilon_{p*},\alpha_*)(\widetilde{\varepsilon}_p,\widetilde{\alpha})\geq 0,\qquad \forall\ (\widetilde{\varepsilon}_p,\widetilde{\alpha})\in\mathcal{V}. \tag{A.15b}$$

Condition (A.15a) equals

$$\int_0^1\left(\mathsf{bc}\int_0^1\alpha_*\,\mathrm{d}x + \mathsf{b}\int_0^1\varepsilon_{p*}\,\mathrm{d}x - \mathcal{R}_*\mathsf{n}\,\alpha_*\right)\varepsilon_{p*}\,\mathrm{d}x = 0 \tag{A.17a}$$
$$\int_0^1\left(\mathsf{bc}\int_0^1\varepsilon_{p*}\,\mathrm{d}x + \mathsf{bc}^2\int_0^1\alpha_*\,\mathrm{d}x - \mathcal{R}_*\left(\mathsf{m}\alpha_*+\mathsf{n}\varepsilon_{p*}\right)\right)\alpha_*\,\mathrm{d}x + \mathsf{a}\int_0^1\left(\alpha_*'\right)^2\mathrm{d}x = 0 \tag{A.17b}$$

while condition (A.15b) equals

$$\int_0^1\left(\mathsf{bc}\int_0^1\alpha_*\,\mathrm{d}x + \mathsf{b}\int_0^1\varepsilon_{p*}\,\mathrm{d}x - \mathcal{R}_*\mathsf{n}\,\alpha_*\right)\widetilde{\varepsilon}_p\,\mathrm{d}x \geq 0 \tag{A.18a}$$
$$\int_0^1\left(\mathsf{bc}\int_0^1\varepsilon_{p*}\,\mathrm{d}x + \mathsf{bc}^2\int_0^1\alpha_*\,\mathrm{d}x - \mathcal{R}_*\left(\mathsf{m}\alpha_*+\mathsf{n}\varepsilon_{p*}\right)\right)\widetilde{\alpha}\,\mathrm{d}x + \mathsf{a}\int_0^1\alpha_*'\,\widetilde{\alpha}'\,\mathrm{d}x \geq 0 \tag{A.18b}$$

It is then possible to prove that $\alpha_*\in\mathrm{H}^2(0,1)$ but the technical proof is omitted and it is assumed that this smoothness property holds. Accordingly, after an integration by parts, (A.17b) and (A.18b) become

$$\int_0^1\left(2\mathsf{bc}\int_0^1\varepsilon_{p*}\,\mathrm{d}x + 2\mathsf{bc}^2\int_0^1\alpha_*\,\mathrm{d}x - 2\mathsf{a}\alpha_*'' - \mathcal{R}_*\left(2\mathsf{m}\alpha_*+\mathsf{n}\varepsilon_{p*}\right)\right)\alpha_*\,\mathrm{d}x + 2\mathsf{a}\left[\alpha_*'\alpha_*\right]_0^1 = 0 \tag{A.19a}$$

$$\int_0^1\left(2\mathsf{bc}\int_0^1\varepsilon_{p*}\,\mathrm{d}x + 2\mathsf{bc}^2\int_0^1\alpha_*\,\mathrm{d}x - 2\mathsf{a}\alpha_*'' - \mathcal{R}_*\left(2\mathsf{m}\alpha_*+\mathsf{n}\varepsilon_{p*}\right)\right)\alpha\,\mathrm{d}x + 2\mathsf{a}\left[\alpha_*'\alpha\right]_0^1 \geq 0 \tag{A.19b}$$

By standard arguments, since $\varepsilon_{p*} \geq 0$ and $\alpha_* \geq 0$, we get the following strong field equations with the associated boundary conditions

$$\left(2\mathsf{bc}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x + 2\mathsf{bc}^2\int_0^1 \alpha_*\,\mathrm{d}x - 2\mathsf{a}\alpha_*'' - \mathcal{R}_*\left(2\mathsf{m}\alpha_* + \mathsf{n}\varepsilon_{p*}\right)\right)\alpha_* = 0 \qquad \text{in } (0,1), \tag{A.20}$$

$$\alpha_*'(1)\,\alpha_*(1) = \alpha_*'(0)\,\alpha_*(0) = 0, \tag{A.21}$$

$$2\mathsf{bc}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x + 2\mathsf{bc}^2\int_0^1 \alpha_*\,\mathrm{d}x - 2\mathsf{a}\alpha_*'' - \mathcal{R}_*\left(2\mathsf{m}\alpha_* + \mathsf{n}\varepsilon_{p*}\right) \geq 0 \qquad \text{in } (0,1), \tag{A.22}$$

$$\alpha_*'(1) \geq 0, \quad \alpha_*'(0) \leq 0. \tag{A.23}$$

Therefore, if $\alpha_*(0) > 0$, then $\alpha_*'(0) = 0$. If $\alpha_*(0) = 0$, since $\alpha_* \geq 0$, one must have necessary $\alpha_*'(0) \geq 0$. But since $\alpha_*'(0) \leq 0$ the fact that $\alpha_*'(0) = 0$ follows. Similary $\alpha_*'(1) = 0$. Accordingly, the minimizer is an element of $\mathrm{SBV}(0,1) \times \mathrm{H}^2(0,1)$ which satisfies

$$\left(2\mathsf{bc}\int_0^1 \alpha_*\,\mathrm{d}x + 2\mathsf{b}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x - \mathcal{R}_*\mathsf{n}\alpha_*\right)\varepsilon_{p*} = 0 \qquad \text{in } (0,1), \tag{A.24a}$$

$$\left(2\mathsf{bc}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x + 2\mathsf{bc}^2\int_0^1 \alpha_*\,\mathrm{d}x - 2\mathsf{a}\alpha_*'' - \mathcal{R}_*\left(2\mathsf{m}\alpha_* + \mathsf{n}\varepsilon_{p*}\right)\right)\alpha_* = 0 \quad \text{in } (0,1) \tag{A.24b}$$

$$2\mathsf{bc}\int_0^1 \alpha_*\,\mathrm{d}x + 2\mathsf{b}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x - \mathcal{R}_*\mathsf{n}\alpha_* \geq 0 \qquad \text{in } (0,1), \tag{A.24c}$$

$$\mathsf{bc}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x + 2\mathsf{bc}^2\int_0^1 \alpha_*\,\mathrm{d}x - 2\mathsf{a}\alpha_*'' - \mathcal{R}_*\left(2\mathsf{m}\alpha_* + \mathsf{n}\varepsilon_{p*}\right) \geq 0 \qquad \text{in } (0,1), \tag{A.24d}$$

$$\alpha_*'(1) = \alpha_*'(0) = 0. \tag{A.24e}$$

Figure A.1 shows an example qualitatively the admissible variations for $(\widetilde{\varepsilon}_p, \widetilde{\alpha})$.

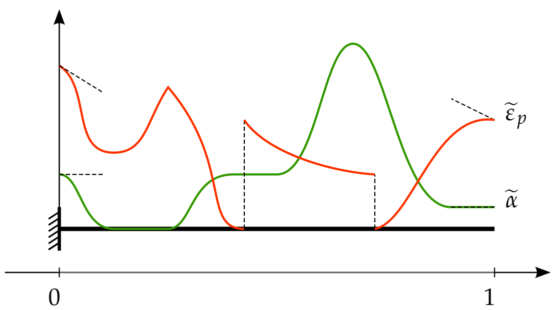


Figure A.1: Example of admissible variations

For sake of clarity the calculations that follow are divided into four steps. The point of departure are equ. (A.24a)–(A.24d).

step 1

From (A.24c) follows that α_* is bounded from above, namely

$$\alpha_*(x) \leq \alpha_m = \frac{\mathsf{bc} \int_0^1 \alpha_*(x)\, dx + \mathsf{b} \int_0^1 \varepsilon_{p*}(x)\, dx}{\mathsf{n}\mathcal{R}_*} \leq 1. \tag{A.25}$$

Condition (A.24c) gives

$$\mathcal{R}_* \leq \frac{\mathsf{bc} \int_0^1 \alpha_*\, dx + \mathsf{b} \int_0^1 \varepsilon_{p*}\, dx}{\mathsf{n}\, \alpha_*} \tag{A.26}$$

Before going on, it is convenient to identify some subspaces of the domain $\mathsf{L} = [0,1]$ as follows

$$\begin{cases} \mathsf{L}_0 = \{x \in [0,1] : \alpha_*(x) = 0\}\,, \\ \mathsf{L}_c = \{x \in [0,1] : \alpha_*'(x) = 0 \quad \text{and} \quad 0 < \alpha_*(x) < \alpha_m\}\,, \\ \mathsf{L}_m = \{x \in [0,1] : \alpha_*(x) = \alpha_m\}\,, \\ \mathsf{L}_v = \mathsf{L} \cap \mathsf{L}_0 \cap \mathsf{L}_c \cap \mathsf{L}_m. \end{cases} \tag{subdomains}$$

The various identified domains are shown as example in Fig. A.2.

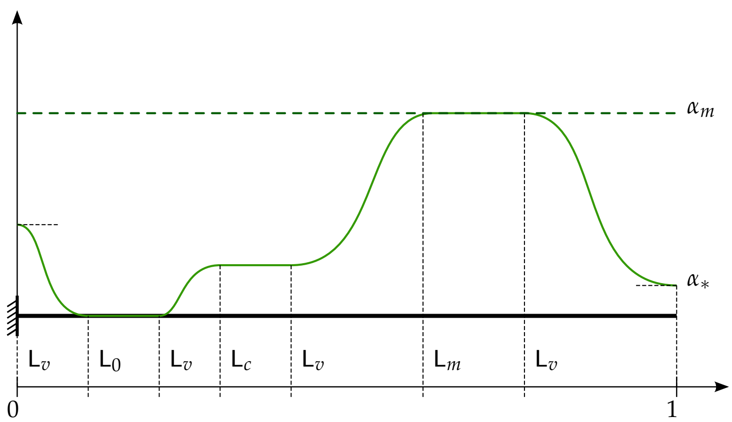


Figure A.2: The function α_* and the domain decomposition (example)

step 2

Through (A.24a) and refering to the introduced subdomains further specification on the plastic strain ε_{p*} field follows:

$$\begin{cases} \left.\begin{matrix} \mathsf{L}_0 \\ \mathsf{L}_c \\ \mathsf{L}_v \end{matrix}\right\} & \alpha < \alpha_m & \Longrightarrow & \varepsilon_{p*}(x) = 0 \\ \mathsf{L}_m & \alpha = \alpha_m & \Longrightarrow & \varepsilon_{p*}(x) \neq 0 \end{cases} \tag{A.27}$$

step 3

Considering now A.24b, the results of step2 and through simple calculations one can conclude that

$$\begin{cases} \mathsf{L}_0: \implies \alpha_*(x) = 0 \\ \mathsf{L}_c: \implies \alpha_*(x) = \alpha_{*c} = \dfrac{\mathsf{bc}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x + \mathsf{bc}^2\int_0^1 \alpha_*\,\mathrm{d}x}{\mathcal{R}_*\mathsf{m}} \quad \text{(constant)} \\ \mathsf{L}_v: \implies \alpha_*(x) = A\cos\sqrt{\dfrac{\mathsf{m}\mathcal{R}_*}{\mathsf{a}}}x + B\sin\sqrt{\dfrac{\mathsf{m}\mathcal{R}_*}{\mathsf{a}}}x + \dfrac{\mathsf{bc}^2}{\mathsf{m}\mathcal{R}_*}\int_0^1 \alpha_*(x)\,\mathrm{d}x \\ \mathsf{L}_m: \implies \varepsilon_{p*}(x) = \varepsilon_{p*_c} = \dfrac{2\mathsf{bc}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x + 2\mathsf{bc}^2\int_0^1 \alpha_*\,\mathrm{d}x - 2\mathcal{R}_*\mathsf{m}\alpha_m}{\mathcal{R}_*\mathsf{n}} \quad \text{(constant)} \end{cases} \tag{A.28}$$

In particular, the last equation becomes

$$\varepsilon_{p*_c} = \frac{2\mathsf{bcL}_m\varepsilon_{p*_c} + 2\mathsf{bc}^2\int_0^1 \alpha_*\,\mathrm{d}x - 2\mathcal{R}_*\mathsf{m}\alpha_m}{\mathcal{R}_*\mathsf{n}} \tag{A.29}$$

and hence

$$\varepsilon_{p*_c} = \frac{2\mathsf{bc}^2\int_0^1 \alpha_*\,\mathrm{d}x - 2\mathcal{R}_*\mathsf{m}\alpha_m}{\mathcal{R}_*\mathsf{n} - 2\mathsf{bcL}_m}. \tag{A.30}$$

Considering A.25 one has

$$\int_0^1 \alpha_*(x)\,\mathrm{d}x = \frac{\mathsf{n}\mathcal{R}_*\alpha_m - 2\mathsf{bL}_m\varepsilon_{p*_c}}{2\mathsf{bc}} \tag{A.31}$$

that inserted in the preceding equation leads to

$$\begin{aligned} \varepsilon_{p*_c} &= \frac{2\mathsf{bc}^2\frac{\mathsf{n}\mathcal{R}_*\alpha_m - 2\mathsf{bL}_m\varepsilon_{p*_c}}{2\mathsf{bc}} - 2\mathcal{R}_*\mathsf{m}\alpha_m}{\mathcal{R}_*\mathsf{n} - 2\mathsf{bcL}_m} \\ &= \frac{\mathsf{cn}\mathcal{R}_*\alpha_m - 2\mathsf{bcL}_m\varepsilon_{p*_c} - 2\mathcal{R}_*\mathsf{m}\alpha_m}{\mathcal{R}_*\mathsf{n} - 2\mathsf{bcL}_m} \end{aligned} \tag{A.32}$$

and hence to

$$\varepsilon_{p*_c} = \frac{\mathsf{cn} - 2\mathsf{m}}{\mathsf{n}}\alpha_*. \tag{A.33}$$

Clearly for this last result

$$\mathsf{cn} - 2\mathsf{m} > 0 \tag{A.34}$$

must hold. An example of ε_{p*} and α_* variations is shown in Fig. A.3.

step 4

In this step the last condition A.24d is taken into account leading to:

over L_0 :

$$\mathsf{bc}\int_0^1 \alpha_*\,\mathrm{d}x + \mathsf{b}\int_0^1 \varepsilon_{p*}\,\mathrm{d}x \geq 0 \tag{A.35}$$

trivially always satisfied;

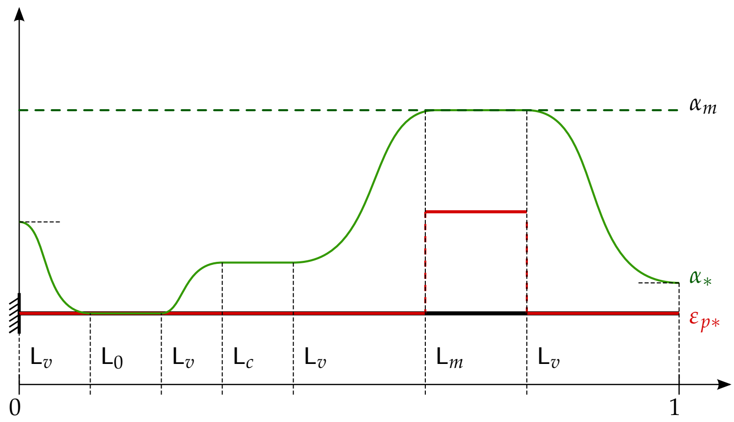


Figure A.3: α_* and ε_{p*} trends (example)

over L_c:

$$2bc \int_0^1 \alpha_* \, dx + 2b \int_0^1 \varepsilon_{p*} \, dx - \mathcal{R}_* n \alpha_{*c} \geq 0 \tag{A.36}$$

and hence to

$$\alpha_{*c} \leq \frac{2bc \int_0^1 \alpha_* \, dx + 2b \int_0^1 \varepsilon_{p*} \, dx}{\mathcal{R}_* n} = \alpha_m. \tag{A.37}$$

Comparing the obtained expressions for α_{*c} and α_m

$$\alpha_{*c} < \alpha_m \tag{A.38}$$

$$\frac{bc \int_0^1 \varepsilon_{p*} \, dx + bc^2 \int_0^1 \alpha_* \, dx}{\mathcal{R}_* m} < \frac{2bc \int_0^1 \alpha_* \, dx + 2b \int_0^1 \varepsilon_{p*} \, dx}{\mathcal{R}_* n} \tag{A.39}$$

which leads to the following inequality

$$c\, n - 2\, m < 0 \tag{A.40}$$

that compared to A.34 gives as result that $\mu(L_c)\,\mu(L_m) = 0$, namely that a constant damage variation cannot subsist with a constant maximum damage value.

With these previous results in mind different possibilities are examined:

Case 1 ($\alpha_*(x) =$ constant): Two sub cases are then considered:

Case 1a $L = L_c$, ($\alpha_*(x) = \alpha_{*c}$): in this case, integrating the second equation in A.28 one obtains

$$\mathcal{R}^*(\alpha_*) = \frac{bc^2}{m}; \tag{A.41}$$

Case 1b $L = L_m$, ($\alpha_*(x) = \alpha_m$): In this case $\varepsilon_{p*} \neq 0$. Inserting A.25 into A.33 one obtains

$$\begin{aligned} \alpha_m &= \frac{2bc\, \alpha_m + 2b\, \frac{(cn-2m)}{n} \alpha_m}{n \mathcal{R}_*}, \\ n^2 \mathcal{R}_* \alpha_m &= 2bcn\, \alpha_m + 2b\, (cn - 2m)\, \alpha_m, \end{aligned} \tag{A.42}$$

and hence finally

$$\mathcal{R}_*(\alpha_*) = 4\mathsf{b}\frac{\mathsf{cn}-\mathsf{m}}{\mathsf{n}^2} \qquad \text{with} \qquad \mathsf{cn}-\mathsf{m} > 0. \tag{A.43}$$

Case 2 $L = \mathsf{L}_0 \cup \mathsf{L}_v$**:** the analysis is essentially the same of the case where of a damage model, Pham 2010. The minimum is

$$\mathcal{R}^*(\alpha_*) = \min\left(\frac{\mathsf{bc}^2}{\mathsf{m}}, \frac{1}{\mathsf{m}}\left(\pi^2\mathsf{a}\right)^{1/3}\left(\mathsf{bc}^2\right)^{2/3}\right) \tag{A.44}$$

and is attained for

$$\alpha_*(0) = \frac{2\mathsf{bc}^2}{\mathsf{m}\mathcal{R}_*}\int_0^1 \alpha_*\,\mathrm{d}x \leq \frac{2\mathsf{bc}}{\mathsf{n}\mathcal{R}_*}\int_0^1 \alpha_*\,\mathrm{d}x = \alpha_m \tag{A.45}$$

constrained to the following condition

$$\mathsf{cn}-\mathsf{m} \geq 0. \tag{A.46}$$

Case 3 $L = \mathsf{L}_0 \cup \mathsf{L}_v \cup \mathsf{L}_m$**:** trough translations of the single subdomains and their relative function trend it is always possible to bring back the problem to the situation where in a subdomain of L, the three considered subdomains appear in an arbitrary order with their relative solution. In this case

$$\alpha_*(x) = \quad A\cos\sqrt{\frac{\mathsf{m}\mathcal{R}_*}{\mathsf{a}}}\,x + \frac{\mathsf{bc}^2}{\mathsf{m}\mathcal{R}_*}\int_0^1 \alpha_*(x)\,\mathrm{d}x, \tag{A.47}$$

$$\alpha_*'(x) = -A\sqrt{\frac{\mathsf{m}\mathcal{R}_*}{\mathsf{a}}}\sin\sqrt{\frac{\mathsf{m}\mathcal{R}_*}{\mathsf{a}}}\,x \tag{A.48}$$

and by the same arguments of the case of a pure damage model, Pham 2010, one obtains

$$\begin{cases} \mathsf{L}_0 : \implies \quad \alpha_*(x) = 0 \\ \mathsf{L}_v : \implies \quad \alpha_*(x) = \dfrac{\mathsf{bc}^2\mathsf{D}^2}{\pi^2\mathsf{a}}\left(1+\cos\pi\dfrac{x}{\mathsf{D}}\right)\int_0^1 \alpha_*\,\mathrm{d}x = \dfrac{\mathsf{bc}^2}{\mathsf{m}\mathcal{R}_*}\left(1+\cos\pi\dfrac{x}{\mathsf{D}}\right)\int_0^1 \alpha_*\,\mathrm{d}x \\ \mathsf{L}_m : \implies \quad \alpha_*(0) = \alpha_m = \dfrac{2\mathsf{bc}^2\mathsf{D}^2}{\pi^2\mathsf{a}}\int_0^1 \alpha_*\,\mathrm{d}x = \dfrac{2\mathsf{bc}^2}{\mathsf{m}\mathcal{R}_*}\int_0^1 \alpha_*\,\mathrm{d}x \end{cases} \tag{A.49}$$

with $\mathsf{D} = \sqrt{\frac{\pi^2\mathsf{a}}{\mathsf{m}\mathcal{R}_*}}$. For sake of simplicity the discussion does not cover the issue of the alternation of different subdomains since the analysis leads to the same results.

From the integration of the damage profile α_* one obtains

$$\begin{aligned} \int_0^1 \alpha_*\,\mathrm{d}x &= n\int_0^{\mathsf{D}} \alpha_*(x)\,\mathrm{d}x + \mathsf{L}_m\,\alpha_m \\ &= n\frac{2\mathsf{bc}^2}{\mathsf{m}\mathcal{R}_*}\sqrt{\frac{\pi^2\mathsf{a}}{\mathsf{m}\mathcal{R}_*}}\int_0^1 \alpha_*\,\mathrm{d}x + \frac{2\mathsf{bc}^2\mathsf{L}_m}{\mathsf{m}\mathcal{R}_*}\int_0^1 \alpha_*\,\mathrm{d}x \end{aligned} \tag{A.50}$$

By expressing $\int_0^1 \alpha_* \, dx$ with respect to α_m

$$\int_0^1 \alpha_* \, dx = \left(\frac{\mathsf{n}\mathcal{R}_*}{2\mathsf{bc}} - \mathsf{L}_m + \mathsf{L}_m \frac{2\mathsf{m}}{\mathsf{cn}} \right) \alpha_m = \left(\frac{\mathsf{n}\mathcal{R}_*}{2\mathsf{bc}} - \mathsf{L}_m + \mathsf{L}_m \frac{2\mathsf{m}}{\mathsf{cn}} \right) \frac{2\mathsf{bc}^2}{\mathsf{m}\mathcal{R}_*} \int_0^1 \alpha_* \, dx \tag{A.51}$$

one can conclude with the following equations system

$$\begin{cases} \mathsf{m}\mathcal{R}_* = 2n\mathsf{bc}^2 \sqrt{\dfrac{\pi^2 \mathsf{a}}{\mathsf{m}\mathcal{R}_*}} + 2\mathsf{bc}^2 \mathsf{L}_m, \\ \mathsf{m}\mathcal{R}_* = 2\mathsf{bc}^2 \left(\dfrac{\mathsf{n}\mathcal{R}_*}{2\mathsf{bc}} - \mathsf{L}_m + \mathsf{L}_m \dfrac{2\mathsf{m}}{\mathsf{cn}} \right). \end{cases} \tag{A.52}$$

From the first equation one has

$$\mathsf{L}_m = \frac{\mathsf{m}\mathcal{R}_*}{2\mathsf{bc}^2} - n \sqrt{\frac{\pi^2 \mathsf{a}}{\mathsf{m}\mathcal{R}_*}} \tag{A.53}$$

which replaced in the second equation of (A.52) gives

$$\begin{aligned} \mathsf{m}\mathcal{R}_* &= 2\mathsf{bc}^2 \left(\frac{\mathsf{n}\mathcal{R}_*}{2\mathsf{bc}} - \frac{\mathsf{m}\mathcal{R}_*}{2\mathsf{bc}^2} + n \sqrt{\frac{\pi^2 \mathsf{a}}{\mathsf{m}\mathcal{R}_*}} + \left(\frac{\mathsf{m}\mathcal{R}_*}{2\mathsf{bc}^2} - n \sqrt{\frac{\pi^2 \mathsf{a}}{\mathsf{m}\mathcal{R}_*}} \right) \frac{2\mathsf{m}}{\mathsf{cn}} \right) \\ &= \mathsf{cn}\mathcal{R}_* - \mathsf{m}\mathcal{R}_* + 2\mathsf{bc}^2 \left(n \sqrt{\frac{\pi^2 \mathsf{a}}{\mathsf{m}\mathcal{R}_*}} + \left(\frac{\mathsf{m}\mathcal{R}_*}{2\mathsf{bc}^2} - n \sqrt{\frac{\pi^2 \mathsf{a}}{\mathsf{m}\mathcal{R}_*}} \right) \frac{2\mathsf{m}}{\mathsf{cn}} \right). \end{aligned} \tag{A.54}$$

Dividing this last equation by $\mathcal{R}^*$

$$2\mathsf{m} - \mathsf{cn} - \frac{2\mathsf{m}^2}{\mathsf{cn}} = 2\mathsf{bc}^2 n \sqrt{\frac{\pi \mathsf{a}}{\mathsf{m}}} \left(\mathcal{R}^* \right)^{-\frac{2}{3}} \left(1 - \frac{2\mathsf{m}}{\mathsf{cn}} \right) \tag{A.55}$$

and expressing $\mathcal{R}^*$ as

$$\left(\mathcal{R}^* \right)^{\frac{2}{3}} = \frac{2\mathsf{bc}^2 n \sqrt{\dfrac{\pi \mathsf{a}}{\mathsf{m}}} \left(1 - \dfrac{2\mathsf{m}}{\mathsf{cn}} \right)}{2\mathsf{m} - \mathsf{cn} - \dfrac{2\mathsf{m}^2}{\mathsf{cn}}} \tag{A.56}$$

finally follows

$$\mathsf{cn} - 2\mathsf{m} > 0 \qquad \text{and} \qquad 2\mathsf{m} - \mathsf{cn} - \frac{2\mathsf{m}^2}{\mathsf{cn}} > 0 \tag{A.57}$$

which of course is not possible since A.34 must hold. Then

$$\mu(\mathsf{L}_v) \, \mu(\mathsf{L}_m) = 0 \tag{A.58}$$

and a subdomain L_m cannot subsists with a subdomain L_v.

The last significant case becomes then

Case 4 $L = \mathsf{L}_0 \cup \mathsf{L}_c \cup \mathsf{L}_v$**:** with similar reasoning as the previous case, both L_c and L_v cannot subsist simultaneously,

$$\mu(\mathsf{L}_v)\,\mu(\mathsf{L}_c) = 0. \tag{A.59}$$

Summarising all previous results one can conclude that

$$\mathcal{R}_*\left(\varepsilon_{p*}, \alpha_*\right) = \min\left(\frac{\mathsf{bc}^2}{\mathsf{m}}, 4\mathsf{b}\frac{\mathsf{cn}-\mathsf{m}}{\mathsf{n}^2}, \frac{1}{\mathsf{m}}\left(\pi^2\mathsf{a}\right)^{1/3}\left(\mathsf{bc}^2\right)^{2/3}\right) \tag{A.60}$$

which is the sought result.

Appendix B

Related Pubblications

Nucleation of cohesive cracks in gradient damage models coupled with plasticity

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Abstract

In the framework of rate-independent systems, a family of elastic-plastic-damage models is proposed through a variational formulation. Since the goal is to account for softening behaviors until the total failure, the dissipated energy contains a gradient damage term in order to limit localization effects. The resulting model owns a great flexibility in the possible coupled responses, depending on the constitutive parameters. Moreover, considering the one-dimensional quasi-static problem of a bar under simple traction and constructing solutions with localization of damage, it turns out that in general a cohesive crack appears at the center of the damage zone before the rupture. The associated cohesive law is obtained in a closed form in terms of the parameters of the model.

Keywords: variational approach, ductile fracture, strain localization

1. Introduction

It is now well established that gradient damage models are very efficient to account for the behavior of brittle and quasi-brittle materials. Indeed, from the idea of Ambrosio and Tortorelli (1990), they have been used in the variational theory of fracture Bourdin et al. (2000, 2008) as a regularization of the revisited Griffith's law Francfort and Marigo (1998). In this approach, the evolution of cracks is governed by a principle of least energy (called global stability condition in the present paper) and it turns out that it is possible to prove that (a family of) gradient damage models converge (in the sense of Gamma-convergence) to Griffith's model when the internal length contained in those models goes to zero Braides (2002); Dal-Maso and Toader (2002). But these models have their own merit and have been developed independently Peerlings et al. (1998); Comi (1999); Comi et al. (2006); Benallal and Marigo (2007); Pham and Marigo (2010a,b); Lorentz et al. (2011). In fact, they are able to account for the nucleation of cracks without invoking global minimization. Their basic ingredients are: (i) a decreasing dependency of the stiffness $\mathsf{E}(\alpha)$ on the damage variable α; (ii) no more rigidity at the ultimate damage state (say $\mathsf{E}(1) = 0$); (iii) a critical stress σ_c; (iv) a softening behavior with a decrease of the stress from σ_c to 0 when the damage goes to 1; (v) a gradient damage term in the energy which necessarily contains an internal length ℓ and which limits the damage localization. Accordingly, the process of crack nucleation is as follows Pham et al. (2011a); Pham and Marigo (2012): (i) a first damage occurs when the stress field reaches the critical stress somewhere in the body; (ii) then, because of the softening character of the material behavior, damage localizes inside a strip the width of which is controlled by the internal length ℓ; (iii) the damage grows inside this strip, but not uniformly in space (the damage is maximal at the center of the strip and is continuously decreasing to 0 so that to match with the undamaged part of the body at the boundary of the strip); (iv) a crack appears

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at the center of the strip when the damage reaches there its ultimate value (say $\alpha = 1$). During this crack nucleation process, some energy is dissipated inside the damage strip and this dissipated energy involves a quantity $\mathsf{G_c}$ which can be considered as the effective surface energy of Griffith's theory. Therefore, $\mathsf{G_c}$ becomes a byproduct of the gradient damage model which can be expressed in terms of the parameters of the model (specifically, $\mathsf{G_c}$ is proportional to $\sigma_c^2\ell/\mathsf{E}(0)$ Pham and Marigo (2012)).

However, this type of "quasi-brittle" models are not able to account for residual strains and consequently cannot be used in ductile fracture. Moreover there is no discontinuity of the displacement in the damage strip before the loss of rigidity at its center, *i.e.* before the nucleation of a crack. In other words such a model cannot account for the nucleation of cohesive cracks, *i.e.* the existence of surface of discontinuity of the displacement with a non vanishing stress. The natural way to include such effects is to introduce plastic strains into the model and to couple their evolution with damage evolution. Of course, this idea is not new and a great number of damage models coupled with plasticity have been developed from the eighties in the spirit of Lemaitre and Chaboche (1985), see for instance Dimitrijevic and Hackl (2011). But our purpose is to construct such models in a softening framework with gradient of damage terms and to see how these models can account for the nucleation of cracks in presence of plasticity. In our knowledge, the previous works are not able to go so far. Here we will adopt a variational approach in the spirit of our previous works Bourdin et al. (2008); Pham and Marigo (2010a,b); Pham et al. (2011a); Sicsic and Marigo (2012). The main ingredients are the following ones: (i) one defines the total energy of the body in terms of the state fields which include the displacement field and the internal variable fields, namely the damage, the plastic strain and the cumulated plastic strain fields; (ii) one postulates that the evolution of the internal variables is governed by the three principles of *irreversibility*, *stability* and *energy balance*. In particular, the stability condition is essential as well for constructing the model in a rational and systematic way as for obtaining and proving general properties. Besides, we have the chance that the variational approach works and has been already developed both in plasticity and in damage mechanics, even though only separately up to now. So, it "suffices" to introduce the coupling by choosing the form of the total energy to obtain, by virtue of our plug and play device, a model of gradient damage coupled with plasticity. A part of our paper will be devoted to this task. Specifically, our model, presented here in a one-dimensional setting only, contains three state functions, namely $\mathsf{E}(\alpha)$, $\mathsf{d}(\alpha)$ and $\bar{\sigma}_{\mathsf{P}}(\alpha)$ which give the dependence of the stiffness, the local damage dissipated energy and the plastic yield stress on the damage variable. So, our choice of coupling is minimalist in the sense that it simply consists in introducing this dependence of the yield plastic stress $\sigma_{\mathsf{P}}(\alpha)$ on the damage variable (with the natural assumption that $\sigma_{\mathsf{P}}(\alpha)$ goes to 0 when the damages goes to 1). In turn, by virtue of the variational character of the model, the product $\sigma_{\mathsf{P}}'(\alpha)\bar{p}$ of the derivative of the state function $\sigma_{\mathsf{P}}(\alpha)$ by the cumulated plastic strain $\bar{p}$ enters in the damage criterion and this coupling plays a fundamental role in the nucleation of a cohesive crack.

Specifically, the paper is organized as follows. In Section 2, we first recall separately the perfect plasticity model and the gradient damage model. Then, using the fact that those classical models can also be formulated in a variational form, we construct the model which couples damage with plasticity by postulating the form of the energy. At this stage, the gradient of damage is not still introduced because one only considers the local behavior of the material. In Section 3, we study the response of the volume element submitted to a uniaxial monotonic stretching test. It is necessary to proceed in several steps because of the great variety of possible cases. This local analysis is finally illustrated by considering a family of models which contains two fundamental parameters and by plotting the associated local material behavior in a stress-strain diagram. In Section 4, we introduce the gradient of damage term into the energy and we set the problem which governs the evolution of the damage and the plastic strain in a one-dimensional bar under traction. Since we are interested by non homogeneous solutions with possible concentrations of the plastic strain, that needs to consider singular fields and hence to treat separately the singular points. Here the variational approach is particularly interesting for deriving in a rational way all the conditions that the evolution must satisfy. Section 5 is devoted to the construction of non homogeneous responses of the bar. We consider a particular family of models so that we can obtain the solution in a closed form. After distinguishing three different cases according to the parameters of the model, we are able to construct a solution with localization of damage up to the rupture of the bar in the spirit of what was already made for quasi-brittle materials. In two cases out of the three, we show that the coupling of damage with plasticity forces the plastic strain to be concentrated to the center of the damage zone. Consequently a cohesive crack is generated and we are able to obtain the cohesive law in a closed form. All the results are summarized and commented in a conclusion where we also present some natural extensions of the present work. The paper finishes by an appendix which contains the technical proofs of the main properties of the model presented in Sections 2 and 3. Throughout the paper, the following notations are used: the dependence on the time parameter t is indicated by a subscript whereas the dependence on the spatial coordinate x is indicated classically by parentheses, e.g. $x \mapsto u_t(x)$ stands for the displacement field at time t. In general, the state functions or the material parameters are represented by sans serif letters, like E, $\mathsf{E}(\alpha)$ or $\mathsf{S}(\alpha)$. The prime denotes either the derivative with respect to x or the derivative with respect to the damage parameter, the dot stands for the time derivative, e.g. $u_t'(x) = \partial u_t(x)/\partial x$, $\mathsf{E}'(\alpha) = d\mathsf{E}(\alpha)/d\alpha$, $\dot{u}_t(x) = \partial u_t(x)/\partial t$.

2. Construction of the local model

In this section we first present the basic ingredients of the model before to illustrate it by considering the response of the volume element under an uniaxial test. Since all the global analysis which follows in Sections 4-5, will be made in a one-dimensional setting of a bar under traction, we do not attempt to formulate the model in a general three-dimensional setting. Accordingly, all the usual mechanical quantities are scalar ones : the stress σ, the total strain ε, the plastic strain p and the cumulated plastic strain $\bar{p}$. Moreover, the damage variable α is also assumed to be a scalar. The model is constructed gradually: we first recall separately the standard models of perfect plasticity Germain et al. (1983) and of brittle damaging materials Marigo (1981) before to introduce the coupling between damage and plasticity. Moreover, the emphasis is laid on the variational character of such standard constitutive laws, this approach being fundamental when we will introduce in Section 4 the gradient damage regularizing terms. Throughout this section and the next one, we consider a volume element and describe its uniaxial behavior assuming that the time evolution of the strain, *i.e.* $t \mapsto \varepsilon_t$, is given. When there is no risk of confusion, the explicit reference to the current time t is omitted.

2.1. Perfect plasticity model

In a pure uniaxial setting, the standard model of perfect plasticity consists in the three following items

$$\text{Stress-strain relation} \quad : \quad \sigma = \mathsf{E}(\varepsilon - p), \tag{1}$$

$$\text{Plasticity yield criterion} \quad : \quad |\sigma| \leq \sigma_{\mathsf{P}}, \tag{2}$$

$$\text{Plasticity flow rule} \quad : \quad \dot{p} \begin{cases} \geq 0 & \text{if} \quad \sigma = +\sigma_{\mathsf{P}} \\ = 0 & \text{if} \quad |\sigma| < \sigma_{\mathsf{P}} \\ \leq 0 & \text{if} \quad \sigma = -\sigma_{\mathsf{P}} \end{cases}, \tag{3}$$

where σ, ε and p stand for the stress, the strain and the plastic strain, respectively. In (1) $\mathsf{E} > 0$ denotes the Young modulus and in (2) $\sigma_{\mathsf{P}} > 0$ represents the plastic yield stress, both are positive material constants. The last two items which govern the evolution of the plastic strain p can also be seen as a stability criterion and an energy balance principle. We reestablish below this variational property by following the presentation of Francfort and Giacomini (2012) which, in our simple context, does not require to use all the machinery of Mielke (2005).

Let us denote by $\bar{p}_t$ the cumulated plastic strain up to time t, *i.e.*

$$\bar{p}_t = \bar{p}_0 + \int_0^t |\dot{p}_s| \, ds, \tag{4}$$

where $\bar{p}_0$ represents the plastic strain cumulated before time 0. Defining the state variables of the volume element at time t as the triple $(\varepsilon_t, p_t, \bar{p}_t)$, then the current total energy density of the volume element W_t is given by the following function of state W_{P}:

$$W_t = \mathsf{W}_{\mathsf{P}}(\varepsilon_t, p_t, \bar{p}_t) \tag{5}$$

with

$$\mathsf{W}_\mathsf{P}(\varepsilon,p,\bar p):=\frac{1}{2}\mathsf{E}(\varepsilon-p)^2+\sigma_\mathsf{P}\bar p. \tag{6}$$

The first term on the hand right side of (6) is the elastic energy density while the second term can be considered as the density of energy dissipated during all the plastic process up to the current time. Accordingly, the perfect plastic model can be equivalently formulated in a variational form as it is stated in the following Proposition the proof of which is given in the appendix.

Proposition 1. *The three items* (1)–(3) *characterizing a perfect plasticity model are equivalent to the three following conditions*

$$\begin{aligned}
\textit{Stress-strain relation} \quad &: \quad \sigma=\frac{\partial \mathsf{W}_\mathsf{P}}{\partial\varepsilon}(\varepsilon,p,\bar p), && (7)\\
\textit{Stability condition} \quad &: \quad \mathsf{W}_\mathsf{P}(\varepsilon,p,\bar p)\le \mathsf{W}_\mathsf{P}(\varepsilon,p^*,\bar p+|p^*-p|),\quad \forall p^*\in\mathbb{R}, && (8)\\
\textit{Energy balance} \quad &: \quad \dot W=\sigma\dot\varepsilon. && (9)
\end{aligned}$$

By virtue of (9), the total energy density W_t represents the strain work up to time t. As far as the stability criterion is concerned, it consists in requiring that, at a given time t, the true total energy density W_t be less than that one obtained by changing instantaneously, at time t, the true plastic strain p_t by any arbitrary one p^*. (By virtue of (4), the cumulated plastic strain is then increased by $|p^*-p_t|$.) Let us note that the stability condition is *global* in the sense that it is true for arbitrary changes of the plastic strain and not only for small changes.

Remark 1. *In the case of an uniaxial monotonic test where the volume element starts from the state* $p_0=\bar p_0=0$ *and is submitted to an increasing strain,* ε *growing from* 0 *to infinity, the response is unique. That response is given by* $\bar p_\varepsilon=p_\varepsilon=0$ *for* $\varepsilon\in[0,\sigma_\mathsf{P}/\mathsf{E}]$ *and by* $\bar p_\varepsilon=p_\varepsilon=\varepsilon-\sigma_\mathsf{P}/\mathsf{E}$ *otherwise.*

2.2. Brittle damage model

Following the formulation first proposed by Marigo (1981), the standard model of brittle damage in a uniaxial setting consists in the four following items

$$\begin{aligned}
\text{Stress-strain relation} \quad &: \quad \sigma=\mathsf{E}(\alpha)\varepsilon, && (10)\\
\text{Irreversibility condition} \quad &: \quad 0\le\alpha\le 1,\quad \dot\alpha\ge 0, && (11)\\
\text{Damage yield criterion} \quad &: \quad -\frac{1}{2}\mathsf{E}'(\alpha)\varepsilon^2\le \mathsf{d}'(\alpha), && (12)\\
\text{Consistency equation} \quad &: \quad \left(\frac{1}{2}\mathsf{E}'(\alpha)\varepsilon^2+\mathsf{d}'(\alpha)\right)\dot\alpha=0. && (13)
\end{aligned}$$

Here α denotes the scalar damage variable which is chosen in such a manner that it grows from 0 to 1, $\alpha=0$ corresponding to the undamaged state and $\alpha=1$ to the completely damaged state. The smooth monotonically decreasing state function $\alpha\mapsto\mathsf{E}(\alpha)$ gives the evolution of the Young modulus of the material with its damage state. Specifically, we assume that

$$\mathsf{E}(0)=\mathsf{E}_0>0,\qquad \mathsf{E}'(\alpha)<0,\ \forall\alpha\in[0,1),\qquad \mathsf{E}(1)=\mathsf{E}'(1)=0. \tag{14}$$

Accordingly, we adopt the following hypotheses in terms of the compliance state function $\alpha\mapsto\mathsf{S}(\alpha)=1/\mathsf{E}(\alpha)$:

$$\mathsf{S}(0)=1/\mathsf{E}_0>0,\qquad \mathsf{S}'(\alpha)>0,\ \forall\alpha\in[0,1),\qquad \mathsf{S}(1)=\mathsf{S}'(1)=+\infty. \tag{15}$$

As we will see below, the smooth monotonically increasing state function $\alpha\mapsto\mathsf{d}(\alpha)$ gives the evolution of the dissipated energy density by the material with its damage state. Specifically, we assume that

$$\mathsf{d}(0)=0,\qquad \mathsf{d}'(\alpha)>0,\ \forall\alpha\in[0,1),\qquad \mathsf{d}(1)=\mathsf{d}_1<+\infty. \tag{16}$$

The damage law is standard in the sense that the damage yield criterion is stated in terms of the elastic energy release rate $-\frac{1}{2}\mathsf{E}'(\alpha)\varepsilon^2$. This type of criterion can be justified in a full three-dimensional setting by invoking Drucker-Ilyushin postulate, see Marigo (1989). In our uniaxial context, (12) can also read as

$$|\varepsilon| \leq \varepsilon_{\mathsf{D}}(\alpha) := \sqrt{\frac{2\mathsf{d}'(\alpha)}{|\mathsf{E}'(\alpha)|}}. \tag{17}$$

Accordingly, the set of admissible strain states is an interval which depends on the damage state. We will assume that this interval grows when the damage grows. That leads to the following

$$\text{Strain hardening condition :} \quad \alpha \mapsto \frac{\mathsf{d}'(\alpha)}{|\mathsf{E}'(\alpha)|} \text{ is monotonically increasing.} \tag{18}$$

This condition will allow us to obtain a unique response in uniaxial test under controlled strain path. Note that $\varepsilon_{\mathsf{D}}(1) := \lim_{\alpha\to 1} \varepsilon_{\mathsf{D}}(\alpha) = +\infty$ if $\mathsf{d}'(1) > 0$ but $\varepsilon_{\mathsf{D}}(1)$ can be finite when $\mathsf{d}'(1) = 0$. In the latter case the material element will be totally damaged when the strain will reach the finite value $\varepsilon_{\mathsf{D}}(1)$ whereas in the former case $\alpha = 1$ is not reached at finite strain.

The damage yield criterion (12) can be expressed in terms of the stress σ and read as

$$|\sigma| \leq \sigma_{\mathsf{D}}(\alpha) := \sqrt{\frac{2\mathsf{d}'(\alpha)}{\mathsf{S}'(\alpha)}}. \tag{19}$$

Since we are only interested by the case of softening behaviors, *i.e.* the case when $\sigma_{\mathsf{D}}(\alpha)$ is a monotonically decreasing function of α, we adopt the following

$$\text{Stress softening condition :} \quad \alpha \mapsto \frac{\mathsf{d}'(\alpha)}{\mathsf{S}'(\alpha)} \text{ is monotonically decreasing.} \tag{20}$$

Note that, by virtue of (15)-(16), $\sigma_{\mathsf{D}}(1) = 0$ and hence the material cannot sustain any stress when it is completely damaged.

Let us now show that the damage model can be also formulated in a variational form. Defining the state variables of the volume element at time t as the pair $(\varepsilon_t, \alpha_t)$, then the current total energy density of the volume element W_t is given by the following function of state W_{D}:

$$W_t = \mathsf{W}_{\mathsf{D}}(\varepsilon_t, \alpha_t) \tag{21}$$

with

$$\mathsf{W}_{\mathsf{D}}(\varepsilon, \alpha) := \frac{1}{2}\mathsf{E}(\alpha)\varepsilon^2 + \mathsf{d}(\alpha). \tag{22}$$

Hence, as for the perfect plastic model, the total energy density is the sum of the elastic energy and the dissipated energy. Accordingly, the brittle damage model can be equivalently formulated in a variational form as it is stated in the following Proposition the proof of which is given in the appendix.

Proposition 2. *Under the hypotheses* (14)–(18) *on the state functions* $\alpha \mapsto \mathsf{E}(\alpha)$ *and* $\alpha \mapsto \mathsf{d}(\alpha)$*, the four items* (10)–(13) *characterizing a brittle damage model are equivalent to the four following conditions*

$$\begin{aligned}
\textit{Stress-strain relation} \quad &: \quad \sigma = \frac{\partial \mathsf{W}_{\mathsf{D}}}{\partial \varepsilon}(\varepsilon, \alpha), && (23)\\
\textit{Irreversibility condition} \quad &: \quad 0 \leq \alpha \leq 1, \quad \dot{\alpha} \geq 0, && (24)\\
\textit{Stability condition} \quad &: \quad \mathsf{W}_{\mathsf{D}}(\varepsilon, \alpha) \leq \mathsf{W}_{\mathsf{D}}(\varepsilon, \alpha^*), \quad \forall \alpha^* \in [\alpha, 1], && (25)\\
\textit{Energy balance} \quad &: \quad \dot{W} = \sigma\dot{\varepsilon}. && (26)
\end{aligned}$$

Comparing with Proposition 1 shows that the variational formulation for the damage model has the same structure as for the plasticity model. Specifically, by virtue of (26), the total energy density W_t still represents the strain work up to time t. As far as the stability criterion is concerned, it consists in requiring that, at a given time t, the true total energy density W_t be less than the energy obtained by changing instantaneously, at time t, the true damage α_t by any arbitrary *greater* one α^*. The fact that we have only to compare with greater damage states comes from the irreversibility condition, condition that the plastic strain has not to satisfy. The stability condition is, here also, *global* in the sense that it is true for arbitrary amplitude of the virtual damage growth and not only for small changes. This global stability property is essentially due to the strain hardening condition (18) (see the proof in the appendix). Let us note that it remains valid in the case of softening behaviors.

Remark 2. *In the case of an uniaxial monotonic test where the volume element starts from the damage state* $\alpha_0 = 0$ *and is submitted to a strain increasing from* 0 *to infinity, the response is unique as long as* $\alpha_\varepsilon < 1$. *It is given by* $\alpha_\varepsilon = 0$ *for* $\varepsilon \in [0, \varepsilon_{\mathsf{D}}(0)]$ *and by* $\alpha_\varepsilon = \varepsilon_{\mathsf{D}}^{-1}(\varepsilon)$ *for* $\varepsilon \in (\varepsilon_{\mathsf{D}}(0), \varepsilon_{\mathsf{D}}(1))$.

2.3. Damage-plasticity coupled model

To construct the coupled model, the procedure is reversed in the sense that we start from the variational formulation to finally obtain the constitutive relations. Accordingly, we consider that the state variables of the volume element at time t is now the quadruple $(\varepsilon_t, \alpha_t, p_t, \bar{p}_t)$ and we assume that the current total energy density of the volume element W_t is given by the following function of state $\mathsf{W_{DP}}$:

$$W_t = \mathsf{W_{DP}}(\varepsilon_t, \alpha_t, p_t, \bar{p}_t) \tag{27}$$

with

$$\mathsf{W_{DP}}(\varepsilon, \alpha, p, \bar{p}) := \frac{1}{2}\mathsf{E}(\alpha)(\varepsilon - p)^2 + \mathsf{d}(\alpha) + \sigma_{\mathsf{P}}(\alpha)\bar{p}. \tag{28}$$

In (28), the state functions $\alpha \mapsto \mathsf{E}(\alpha)$ and $\alpha \mapsto \mathsf{d}(\alpha)$ are the same as in the brittle damage model and hence satisfy the properties (14)-(16) as well as the damage strain hardening condition (18) and the damage stress softening condition (20).

But the plastic yield stress now depends on the damage state and is given by the smooth state function $\alpha \mapsto \sigma_{\mathsf{P}}(\alpha)$. Since we focus on softening behaviors, we assume the following properties for $\alpha \mapsto \sigma_{\mathsf{P}}(\alpha)$:

$$\text{Plastic yield stress softening :} \quad \sigma_{\mathsf{P}}(0) = \bar{\sigma}_{\mathsf{P}} > 0, \quad \sigma_{\mathsf{P}}'(\alpha) < 0,\ \forall\alpha \in [0, 1), \quad \sigma_{\mathsf{P}}(1) = 0,\ \sigma_{\mathsf{P}}'(1) \le 0. \tag{29}$$

Hence the plastic yield stress is monotonically decreasing until 0 when the damage grows from 0 to 1. Accordingly, our model is quite different of Ambrosio et al. (2012); Del Piero et al. (2012) even if those models have also the goal for coupling fracture with plasticity by using a variational approach.

Let us now establish the constitutive relations which govern the behavior of the material point in the case of an uniaxial strain controlled test.

1. *Stress-strain relation.* The current stress σ is given in terms of the current state by

$$\sigma = \frac{\partial \mathsf{W_{DP}}}{\partial \varepsilon}(\varepsilon, \alpha, p, \bar{p}) = \mathsf{E}(\alpha)(\varepsilon - p). \tag{30}$$

2. *Irreversibility condition.* The damage variable is still a non decreasing function of time and must remain in the interval $[0, 1]$. In other words, α has still to satisfy (11).
3. *Stability conditions.* We distinguish three *degrees* of stability: *global stability*, *local stability* and *first order stability conditions.*

Definition 1. *A state* $(\varepsilon, \alpha, p, \bar{p}) \in \mathbb{R} \times [0, 1] \times \mathbb{R} \times \mathbb{R}^+$ *of the material point is said either globally stable or locally stable or stable at the first order according to that state satisfies the corresponding following condition:*

(a) Global stability condition:

$$\forall(\alpha^*, p^*) \in [\alpha, 1] \times \mathbb{R}, \qquad \mathsf{W_{DP}}(\varepsilon, \alpha, p, \bar{p}) \le \mathsf{W_{DP}}(\varepsilon, \alpha^*, p^*, \bar{p} + |p^* - p|). \tag{31}$$

(b) Local stability condition: *For all* $(\alpha^*, p^*) \in [\alpha, 1] \times \mathbb{R}$ *there exists* $\bar{h} > 0$ *such that*

$$\forall h \in [0, \bar{h}], \qquad \mathsf{W}_{\mathsf{DP}}(\varepsilon, \alpha, p, \bar{p}) \leq \mathsf{W}_{\mathsf{DP}}(\varepsilon, \alpha + h(\alpha^* - \alpha), p + h(p^* - p), \bar{p} + h\,|p^* - p|). \tag{32}$$

(c) First order stability conditions:

$$\begin{cases} \alpha = 1 \ \mathit{or} \ \dfrac{\partial \mathsf{W}_{\mathsf{DP}}}{\partial \alpha}(\varepsilon, \alpha, p, \bar{p}) \geq 0, \\[2ex] \dfrac{\partial \mathsf{W}_{\mathsf{DP}}}{\partial p}(\varepsilon, \alpha, p, \bar{p})q + \dfrac{\partial \mathsf{W}_{\mathsf{DP}}}{\partial \bar{p}}(\varepsilon, \alpha, p, \bar{p})\,|q| \geq 0, \quad \forall q \in \mathbb{R}. \end{cases} \tag{33}$$

One immediately sees that these stability conditions are increasingly weak in the sense that

$$\text{Global Stability} \Longrightarrow \text{Local stability} \Longrightarrow \text{First order stability}$$

Indeed, the first condition in (33) is deduced from (32) by choosing, when $\alpha < 1$, $p^* = p$ and $\alpha^* = \alpha + \beta$ with $\beta > 0$ and small enough, by then dividing by h and by passing to the limit when $h \to 0$. In the same way, the second condition in (33) is deduced from (32) by setting $\alpha^* = \alpha$ and $p^* = p + q$, by dividing then by h and by passing to the limit when $h \to 0$.
The converse implications are not always true but require that the state functions E, d and σ_{P} satisfy additional conditions that we will discuss in Section 3.4. Accordingly, we first consider the *first order stability conditions* which, by virtue of (28), can read as

$$\text{Damage yield criterion} \quad : \quad -\frac{1}{2}\mathsf{E}'(\alpha)(\varepsilon - p)^2 - \mathsf{d}'(\alpha) - \sigma_{\mathsf{P}}'(\alpha)\bar{p} \leq 0 \quad \text{if} \quad \alpha < 1, \tag{34}$$

$$\text{Plasticity yield criterion} \quad : \quad \mathsf{E}(\alpha)\,|\varepsilon - p| - \sigma_{\mathsf{P}}(\alpha) \leq 0. \tag{35}$$

4. *Energy balance.* The total energy density W_t still represents the strain work up to time t and hence (26), must hold true. As long as $\alpha < 1$, using (28) and the chain rule give

$$\left(\frac{1}{2}\mathsf{E}'(\alpha)\varepsilon^2 + \mathsf{d}'(\alpha) + \sigma_{\mathsf{P}}'(\alpha)\bar{p}\right)\dot{\alpha} + \sigma_{\mathsf{P}}(\alpha)\,|\dot{p}| - \sigma\dot{p} = 0.$$

Then, by virtue of (34) and (35), one obtains separately the damage consistency equation and the plasticity flow rule:

$$\text{Damage consistency equation} \quad : \quad \left(\frac{1}{2}\mathsf{E}'(\alpha)\varepsilon^2 + \mathsf{d}'(\alpha) + \sigma_{\mathsf{P}}'(\alpha)\bar{p}\right)\dot{\alpha} = 0; \tag{36}$$

$$\text{Plasticity flow rule} \quad : \quad \dot{p}\begin{cases} \geq 0 & \text{if} \quad \sigma = +\sigma_{\mathsf{P}}(\alpha) \\ = 0 & \text{if} \quad |\sigma| < \sigma_{\mathsf{P}}(\alpha) \\ \leq 0 & \text{if} \quad \sigma = -\sigma_{\mathsf{P}}(\alpha) \end{cases}. \tag{37}$$

Remark 3. *A direct consequence of the damage consistency equation is that the dissipated energy is actually a non decreasing function of time. Indeed, considering that the density of dissipated energy at time t, say D_t, is the complementary part of the elastic energy in the total energy W_t, D_t reads as*

$$D_t = \mathsf{d}(\alpha_t) + \sigma_{\mathsf{P}}(\alpha_t)\bar{p}_t.$$

Differentiating with respect to t and taking into account (36) *lead to*

$$\dot{D}_t = -\frac{1}{2}\mathsf{E}'(\alpha_t)\varepsilon_t^2\dot{\alpha}_t + \sigma_{\mathsf{P}}(\alpha_t)\,|\dot{p}_t| \tag{38}$$

where the first term on the hand right side is non negative by virtue of the decreasing of the rigidity and the irreversibility condition. Hence $\dot{D}_t \geq 0$. Moreover (38) *shows that the dissipated power is the sum of the elastic energy release rate and the plastic power.*

3. Uniaxial responses

3.1. General properties

Let us study the response predicted by the damage-plasticity coupled model when the material point is submitted to a monotonic uniaxial test. Specifically, we assume that the material point is at time 0 in the unstrained, unstressed and undamaged state, *i.e.* $(\varepsilon_0, \alpha_0, p_0, \bar{p}_0) = (0, 0, 0, 0)$, and then is submitted to an increasing uniaxial stretching where ε grows from 0 to $+\infty$. Accordingly, we can assimilate the time parameter with the strain, *i.e.* $\varepsilon = t$. The problem is to find the evolution of $(\alpha, p, \bar{p})$ with ε. That evolution is assumed to be smooth, in the sense that $\varepsilon \mapsto (\alpha_\varepsilon, p_\varepsilon, \bar{p}_\varepsilon)$ are at least absolutely continuous, and governed by the stress-strain relation (30), the damage irreversibility condition (24), the damage and plasticity yield criteria (34)-(35), the damage consistency equation (36) and the plasticity flow rule (37). While the study is trivial in the case of uncoupled models, see Remarks 1 and 2, it becomes much more difficult in the case of a coupling. In particular, the existence of the response is not ensured in the whole range of strains without introducing additional assumptions on the constitutive relations. Moreover, one can obtain a great variety of responses according to the values of the material parameters entering in the model. Accordingly, the analysis must be made in several steps. We first establish some general properties the proof of which are given in the appendix.

Proposition 3. *Under the condition that the states functions $\alpha \mapsto \mathsf{E}(\alpha)$, $\alpha \mapsto \mathsf{d}(\alpha)$ and $\alpha \mapsto \sigma_\mathsf{P}(\alpha)$ satisfy* (14), (16), (18), (20) *and* (29), *any response to a monotonically increasing uniaxial stretching test, i.e. any $\varepsilon \mapsto (\alpha_\varepsilon, p_\varepsilon, \bar{p}_\varepsilon)$ absolutely continuous which starts from* $(0, 0, 0)$ *and satisfies* (24), (30), (34)–(37), *enjoys the following properties:*

1. Elastic stage: *The response remains purely elastic at the beginning of the test, i.e.*

$$\forall \varepsilon \in [0, \bar{\varepsilon}_\mathsf{I}], \quad \alpha_\varepsilon = 0, \quad p_\varepsilon = \bar{p}_\varepsilon = 0,$$

   where

$$\bar{\varepsilon}_\mathsf{I} = \min\{\bar{\varepsilon}_\mathsf{D}, \bar{\varepsilon}_\mathsf{P}\}, \quad \bar{\varepsilon}_\mathsf{D} := \varepsilon_\mathsf{D}(0) = \sqrt{\frac{2\mathsf{d}'(0)}{|\mathsf{E}'(0)|}}, \quad \bar{\varepsilon}_\mathsf{P} := \frac{\sigma_\mathsf{P}(0)}{\mathsf{E}(0)} = \frac{\bar{\sigma}_\mathsf{P}}{\mathsf{E}_0}.$$

2. Plastic strain monotonicity: *As soon as $\varepsilon > 0$ and as long as $\alpha_\varepsilon < 1$, the stress is positive, $\sigma_\varepsilon > 0$. Hence the plastic strain p_ε cannot decrease and $0 \leq \bar{p}_\varepsilon = p_\varepsilon < \varepsilon$. Accordingly, the damage and plastic yield criteria can read as:*

$$f_\mathsf{D}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) := \frac{1}{2} |\mathsf{E}'(\alpha_\varepsilon)| (\varepsilon - p_\varepsilon)^2 + |\sigma_\mathsf{P}'(\alpha_\varepsilon)| p_\varepsilon - \mathsf{d}'(\alpha_\varepsilon) \leq 0 \quad \textit{if} \quad \alpha_\varepsilon < 1,$$

$$f_\mathsf{P}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) := \mathsf{E}(\alpha_\varepsilon)(\varepsilon - p_\varepsilon) - \sigma_\mathsf{P}(\alpha_\varepsilon) \leq 0,$$

3. Inelastic stage: *As soon as $\varepsilon > \bar{\varepsilon}_\mathsf{I}$ and as long as $\alpha_\varepsilon < 1$, the response is no more elastic and at least one of the two yield criteria are satisfied as an equality, i.e.*

$$\forall \varepsilon \geq \bar{\varepsilon}_\mathsf{I} \textit{ such that } \alpha_\varepsilon < 1, \quad f_\mathsf{P}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) = 0 \quad \textit{or} \quad f_\mathsf{D}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) = 0.$$

4. Onset of damage: *The damage yield criterion is necessarily reached at a finite strain $\varepsilon_d \geq \bar{\varepsilon}_\mathsf{I}$, i.e.*

$$\exists \varepsilon_d \in [\bar{\varepsilon}_\mathsf{I}, +\infty), \quad \forall \varepsilon \in [0, \varepsilon_d), \quad f_\mathsf{D}(\varepsilon, 0, p_\varepsilon) < 0, \qquad f_\mathsf{D}(\varepsilon_d, 0, p_{\varepsilon_d}) = 0.$$

5. Damage growth: *As soon as $\varepsilon \geq \varepsilon_d$ and as long as $\alpha < 1$, the damage yield criterion is satisfied as an equality, i.e.*

$$\forall \varepsilon \geq \varepsilon_d \textit{ such that } \alpha_\varepsilon < 1, \quad f_\mathsf{D}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) = 0.$$

6. Full damage state: *Damage will grow until $\alpha = 1$, i.e.* $\lim_{\varepsilon \to \infty} \alpha_\varepsilon = 1$.

3.2. The different possible stages

These properties allow us to distinguish the different possible stages of the response. That leads to the following definitions:

- Elastic stage: it is the interval $\mathbf{E} = [0, \bar{\varepsilon}_{\mathsf{I}})$ during which $f_{\mathsf{P}} < 0$ and $f_{\mathsf{D}} < 0$;
- Plastic stage: it is an interval of ε, denoted $\mathbf{P}$, during which $f_{\mathsf{P}} = 0$, $f_{\mathsf{D}} < 0$ and $\alpha < 1$;
- Damage stage: it is an interval of ε, denoted $\mathbf{D}$, during which $f_{\mathsf{P}} < 0$, $f_{\mathsf{D}} = 0$ and $\alpha < 1$;
- Damage-Plastic stage: it is an interval of ε, denoted $\mathbf{DP}$, during which $f_{\mathsf{P}} = f_{\mathsf{D}} = 0$ and $\alpha < 1$;
- Final stage: it is an interval of ε, denoted $\mathbf{F}$, during which $\alpha = 1$.

The $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{DP}$ stages are intervals of the form $(\varepsilon_0, \varepsilon_1)$ or $[\varepsilon_0, \varepsilon_1)$ or $[\varepsilon_0, \varepsilon_1]$ with $0 \leq \varepsilon_0 < \varepsilon_1 \leq +\infty$ and we will say that the stage starts at ε_0 and finishes at ε_1. By definition, α remains constant during the $\mathbf{E}$ and $\mathbf{P}$ stages while p remains constant during the $\mathbf{E}$ and $\mathbf{D}$ stages. By virtue of Proposition 3, if $\bar{\varepsilon}_{\mathsf{P}} < \bar{\varepsilon}_{\mathsf{D}}$, then the response starts by the sequence $\mathbf{E}$–$\mathbf{P}$ with $\mathbf{E} = [0, \bar{\varepsilon}_{\mathsf{P}})$ and will continue either by a $\mathbf{D}$ stage or by a $\mathbf{DP}$ stage or by a sequence of alternate $\mathbf{D}$ and $\mathbf{DP}$ stages, but a $\mathbf{P}$ stage will exist never again. On the other hand, if $\bar{\varepsilon}_{\mathsf{P}} > \bar{\varepsilon}_{\mathsf{D}}$, then $\mathbf{E} = [0, \bar{\varepsilon}_{\mathsf{D}})$ will be followed by a $\mathbf{D}$ stage or a $\mathbf{DP}$ stage or by a sequence of alternate $\mathbf{D}$ and $\mathbf{DP}$ stages, hence a $\mathbf{P}$ stage never exists. However, Proposition 3 gives only necessary conditions that a response must satisfy when it exists. It contains no existence and uniqueness result. To obtain such a result, we have first to study the $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{DP}$ stages. That leads to the following

Proposition 4. *Let $\alpha \mapsto \varepsilon_{\mathsf{P}}(\alpha)$, $\alpha \mapsto \pi_{\mathsf{D}}(\alpha)$, $\alpha \mapsto \pi_{\mathsf{DP}}(\alpha)$ and $\alpha \mapsto \varepsilon_{\mathsf{DP}}(\alpha)$ be defined for $\alpha \in [0, 1)$ by*

$$\varepsilon_{\mathsf{P}}(\alpha) = \frac{\sigma_{\mathsf{P}}(\alpha)}{\mathsf{E}(\alpha)}, \quad \pi_{\mathsf{D}}(\alpha) = \frac{\mathsf{d}'(\alpha)}{|\sigma'_{\mathsf{P}}(\alpha)|} \quad \pi_{\mathsf{DP}}(\alpha) = \pi_{\mathsf{D}}(\alpha)\left(1 - \frac{\varepsilon_{\mathsf{P}}(\alpha)^2}{\varepsilon_{\mathsf{D}}(\alpha)^2}\right), \quad \varepsilon_{\mathsf{DP}}(\alpha) = \pi_{\mathsf{DP}}(\alpha) + \varepsilon_{\mathsf{P}}(\alpha),$$

where $\varepsilon_{\mathsf{D}}(\alpha)$ is defined in (17)*. The* $\mathbf{P}$, $\mathbf{D}$ *and* $\mathbf{DP}$ *stages enjoy the following properties:*

1. $\mathbf{P}$ stage: *a plasticity stage exists if and only if $\bar{\varepsilon}_{\mathsf{P}} < \bar{\varepsilon}_{\mathsf{D}}$. In such a case, the* $\mathbf{P}$ *stage is the interval $[\bar{\varepsilon}_{\mathsf{P}}, \bar{\varepsilon}_{\mathsf{P}} + \pi_{\mathsf{DP}}(0))$ where the evolution is $\alpha_\varepsilon = 0$ and $p_\varepsilon = \varepsilon - \bar{\varepsilon}_{\mathsf{P}}$. The stress remains constant, $\sigma_\varepsilon = \bar{\sigma}_{\mathsf{P}}$.*
2. $\mathbf{DP}$ stage: *in such a stage starting at ε_0 and finishing at ε_1, the strain, the plastic strain and the damage are related by*
$$\varepsilon = \varepsilon_{\mathsf{DP}}(\alpha_\varepsilon), \qquad p_\varepsilon = \pi_{\mathsf{DP}}(\alpha_\varepsilon),$$
whereas the stress is given by $\sigma_\varepsilon = \sigma_{\mathsf{P}}(\alpha_\varepsilon)$ and hence is monotonically decreasing.
Accordingly, $\alpha \mapsto \varepsilon_{\mathsf{DP}}(\alpha)$ is necessarily monotonically increasing and $\alpha \mapsto \pi_{\mathsf{DP}}(\alpha)$ is necessarily non decreasing and non negative in the interval $[\alpha_0, \alpha_1]$ where α_0 and α_1 are such that $\varepsilon_0 = \varepsilon_{\mathsf{DP}}(\alpha_0)$ and $\varepsilon_1 = \varepsilon_{\mathsf{DP}}(\alpha_1)$.
Consequently, for a given damage state α_0 such that $\sigma_{\mathsf{P}}(\alpha_0) \leq \sigma_{\mathsf{D}}(\alpha_0)$, a unique $\mathbf{DP}$ *stage can starts at $\varepsilon_0 = \varepsilon_{\mathsf{DP}}(\alpha_0)$. This stage can continue as long as $\alpha \mapsto \varepsilon_{\mathsf{DP}}(\alpha)$ is increasing, $\alpha \mapsto \pi_{\mathsf{DP}}(\alpha)$ is non decreasing and $\alpha_\varepsilon < 1$.*
3. $\mathbf{D}$ stage: *a* $\mathbf{D}$ *stage necessarily starts at $\bar{\varepsilon}_{\mathsf{D}}$ when $\bar{\varepsilon}_{\mathsf{D}} < \bar{\varepsilon}_{\mathsf{P}}$. In such a case the plastic strain and the damage are given by*
$$\alpha_\varepsilon = \varepsilon_{\mathsf{D}}^{-1}(\varepsilon), \qquad p_\varepsilon = 0,$$
whereas the stress is given by $\sigma_\varepsilon = \sigma_{\mathsf{D}}(\alpha_\varepsilon)$ and hence is monotonically decreasing. This stage can continue as long as $\sigma_{\mathsf{D}}(\alpha) < \sigma_{\mathsf{P}}(\alpha)$.
Any other $\mathbf{D}$ *stage starts at $\varepsilon_0 \geq \varepsilon_{\mathsf{DP}}(0)$ with a initial plastic strain p_0 and a initial damage α_0 such that $p_0 = \pi_{\mathsf{DP}}(\alpha_0)$ and $\varepsilon_0 = \varepsilon_{\mathsf{DP}}(\alpha_0)$. The strain, the plastic strain and the damage are related by*
$$p_\varepsilon = \pi_{\mathsf{DP}}(\alpha_0), \quad \varepsilon = \pi_{\mathsf{DP}}(\alpha_0) + e_{\mathsf{D}}(\pi_{\mathsf{DP}}(\alpha_0), \alpha_\varepsilon)$$
with
$$e_{\mathsf{D}}(p, \alpha) = \varepsilon_{\mathsf{D}}(\alpha)\sqrt{1 - \frac{p}{\pi_{\mathsf{D}}(\alpha)}},$$

whereas the stress is given by $\sigma_\varepsilon = \mathsf{E}(\alpha_\varepsilon) e_{\mathsf{D}}(\pi_{\mathsf{DP}}(\alpha_0), \alpha_\varepsilon)$.
Accordingly, α_0 *is necessarily such that* $\sigma_{\mathsf{P}}(\alpha_0) \leq \sigma_{\mathsf{D}}(\alpha_0)$ *whereas* $\alpha \mapsto e_{\mathsf{D}}(\pi_{\mathsf{DP}}(\alpha_0), \alpha)$ *is necessarily monotonically increasing and* $\pi_{\mathsf{DP}}(\alpha) < \pi_{\mathsf{DP}}(\alpha_0) < \pi_{\mathsf{D}}(\alpha)$ *for* $\alpha \in (\alpha_0, \alpha_{\varepsilon_1})$.
Consequently, for a given damage state α_0*, a unique* **D** *stage can starts at* $\varepsilon_0 = \varepsilon_{\mathsf{DP}}(\alpha_0)$*. This stage can continue as long as* $\alpha \mapsto e_{\mathsf{D}}(\pi_{\mathsf{DP}}(\alpha_0), \alpha)$ *is increasing and* $\pi_{\mathsf{DP}}(\alpha) < \pi_{\mathsf{DP}}(\alpha_0) < \pi_{\mathsf{D}}(\alpha)$.

Proof. We only give a sketch of the proof of each property.

1. It is a direct consequence of the definition of a **P** stage and of the properties 3–5 of Proposition 3.
2. By definition, during a **DP** stage, $f_{\mathsf{D}}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) = f_{\mathsf{P}}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) = 0$. After some easy calculations, one obtains from the two criteria $\varepsilon = \varepsilon_{\mathsf{DP}}(\alpha_\varepsilon)$ and $p_\varepsilon = \pi_{\mathsf{DP}}(\alpha_\varepsilon)$. The monotonicity properties are direct consequences of the irreversibility condition for $\varepsilon \mapsto \alpha_\varepsilon$ and the monotonicity of $\varepsilon \mapsto p_\varepsilon$. The uniqueness of α_ε comes from the monotonicity of $\varepsilon_{\mathsf{DP}}$.
3. If $\bar{\varepsilon}_{\mathsf{D}} < \bar{\varepsilon}_{\mathsf{P}}$, then a **D** stage follows the **E** stage and during this stage p remains equal to 0. This stage can continue as long as $f_{\mathsf{P}}(\varepsilon_{\mathsf{D}}(\alpha), \alpha) < 0$ which is equivalent to $\sigma_{\mathsf{D}}(\alpha) < \sigma_{\mathsf{P}}(\alpha)$. Any other **D** stage will start at the end of the **P** stage or at the end of a **DP** stage. Therefore, the initial state of such a **D** stage is the one given in the statement. By definition and by continuity, p_ε remains equal to $\pi_{\mathsf{DP}}(\alpha_0)$ and $f_{\mathsf{D}}(\varepsilon, \alpha_\varepsilon, \pi_{\mathsf{DP}}(\alpha_0)) = 0$ for $\varepsilon \in \bar{\mathbf{D}}$. After easy calculations, one obtains the relation between ε and α_ε given in the statement. The inequality $\pi_{\mathsf{DP}}(\alpha_0) < \pi_{\mathsf{D}}(\alpha)$ is necessary in order that $e_{\mathsf{D}}(\pi_{\mathsf{DP}}(\alpha_0), \alpha)$ be defined and increasing with α. The inequality $\pi_{\mathsf{DP}}(\alpha_0) \geq \pi_{\mathsf{DP}}(\alpha)$ is equivalent to $f_{\mathsf{P}}(\varepsilon, \alpha, p_0) \leq 0$ and, by definition of a **D** stage, the equality can hold only at the ends of the stage.

□

Remark 4. *During a* **DP** *stage, the stress decreases since* σ_{P} *is decreasing. When there does not exist a* **P** *stage, the stress decreases during the* **D** *stage which starts after the* **E** *stage since* σ_{D} *is decreasing. However, during a* **D** *stage which starts after the* **P** *stage or a* **DP** *stage, the stress is not necessarily decreasing because of the presence of the initial plastic strain* $\pi_{\mathsf{DP}}(\alpha_0)$.

We are now in a position to obtain a uniqueness result.

Proposition 5. *As long as a response exists and* $\alpha < 1$*, this response is unique.*

Proof. The response is unique as long as $\varepsilon \leq \varepsilon_d$. Hence, if there exists several responses, then they must bifurcate at some $\varepsilon = \varepsilon_b \geq \varepsilon_d$. By continuity, all bifurcated branches start from the same state (α_b, p_b) at ε_b. For $h > 0$ small enough, the interval $(\varepsilon_b, \varepsilon_b + h)$ of any branch must belong to a **D** or a **DP** stage. But, by virtue of Proposition 4 (property 2), a unique **DP** stage can start at ε_b with $\alpha = \alpha_b$, and, by virtue of Proposition 4 (property 3), a unique **D** stage can start at ε_b with $\alpha = \alpha_b$ and $p = p_b$. So, the unique possibility is that one branch corresponds to a **DP** stage and the other to a **D** stage. But then, on one hand, one should have $p_b = \pi_{\mathsf{DP}}(\alpha_b) < \pi_{\mathsf{DP}}(\alpha)$ for α in some interval $(\alpha_b, \alpha_b + \eta)$ by virtue of the properties of the **DP** stages, while, on the other hand, one should have $p_b = \pi_{\mathsf{DP}}(\alpha_b) > \pi_{\mathsf{DP}}(\alpha)$ for α in some interval $(\alpha_b, \alpha_b + \eta)$ by virtue of the properties of the **D** stages. Hence, there is no bifurcation. □

3.3. Examples

We finish this section by some examples of damage-plasticity coupled models with the associated response in a monotonic uniaxial stretching test. All these models belong to the same family which contains three dimensionless parameters, namely $\mathsf{k} > 1$, $\theta > 0$ and $\mathsf{n} > 0$. Specifically, the state functions E, d and σ_{P} are given by

$$\mathsf{E}(\alpha) = \frac{1 - \mathsf{w}(\alpha)}{1 + (\mathsf{k} - 1)\mathsf{w}(\alpha)} \mathsf{E}_0, \qquad \mathsf{d}(\alpha) = \frac{\mathsf{k}\bar{\sigma}_{\mathsf{D}}^2}{2\mathsf{E}_0} \mathsf{w}(\alpha), \qquad \sigma_{\mathsf{P}}(\alpha) = (1 - \mathsf{w}(\alpha))^{\mathsf{n}} \theta \bar{\sigma}_{\mathsf{D}}, \tag{39}$$

where

$$\mathsf{w}(\alpha) = 1 - (1 - \alpha)^2. \tag{40}$$

In (39), $\bar{\sigma}_{\mathsf{D}}$ represents the critical stress $\sigma_{\mathsf{D}}(0)$, the associated critical strain is $\bar{\varepsilon}_{\mathsf{D}} = \bar{\sigma}_{\mathsf{D}}/\mathsf{E}_0$ where E_0 is the Young modulus of the sound material. Moreover,

$$\mathsf{d}_1 := \mathsf{d}(1) = \frac{\mathsf{k}\bar{\sigma}_{\mathsf{P}}^2}{2\mathsf{E}_0}.$$

Thus the state functions depend in fact on the variable $\omega = \mathsf{w}(\alpha)$ which grows from 0 to 1 as α does. The other state functions read as

$$\varepsilon_{\mathsf{D}}(\alpha) = (1 + (\mathsf{k}-1)\mathsf{w}(\alpha))\bar{\varepsilon}_{\mathsf{D}}, \quad \sigma_{\mathsf{D}}(\alpha) = (1-\mathsf{w}(\alpha))\bar{\sigma}_{\mathsf{D}}, \quad \varepsilon_{\mathsf{P}}(\alpha) = (1 + (\mathsf{k}-1)\mathsf{w}(\alpha))(1-\mathsf{w}(\alpha))^{\mathsf{n}-1}\theta\bar{\varepsilon}_{\mathsf{D}},$$

$$\pi_{\mathsf{D}}(\alpha) = \frac{\mathsf{k}\bar{\varepsilon}_{\mathsf{D}}}{2\mathsf{n}\theta(1-\mathsf{w}(\alpha))^{\mathsf{n}-1}}, \qquad \pi_{\mathsf{DP}}(\alpha) = \frac{\mathsf{k}\bar{\varepsilon}_{\mathsf{D}}}{2\mathsf{n}\theta}\left(\frac{1}{(1-\mathsf{w}(\alpha))^{\mathsf{n}-1}} - \theta^2(1-\mathsf{w}(\alpha))^{\mathsf{n}-1}\right),$$

$$e_{\mathsf{D}}(p,\alpha) = \bar{\varepsilon}_{\mathsf{D}}(1 + (\mathsf{k}-1)\mathsf{w}(\alpha))\sqrt{1 - \frac{2n\theta p}{\mathsf{k}\bar{\varepsilon}_{\mathsf{D}}}(1-\mathsf{w}(\alpha))^{\mathsf{n}-1}},$$

$$\varepsilon_{\mathsf{DP}}(\alpha) = \frac{\mathsf{k}\bar{\varepsilon}_{\mathsf{D}}}{2\mathsf{n}\theta(1-\mathsf{w}(\alpha))^{\mathsf{n}-1}} - (1-\mathsf{w}(\alpha))^{\mathsf{n}-1}\big(\frac{\mathsf{k}}{2\mathsf{n}} - 1 - (\mathsf{k}-1)\mathsf{w}(\alpha)\big)\theta\bar{\varepsilon}_{\mathsf{D}}.$$

The conditions (14) and (16), the strain hardening condition (18), the softening condition (20) and the plastic yield softening condition (29) are automatically satisfied. Note that $\varepsilon_{\mathsf{D}}(1) = \mathsf{k}\bar{\varepsilon}_{\mathsf{D}}$ is finite. The parameter θ represents the ratio between $\bar{\varepsilon}_{\mathsf{P}}$ and $\bar{\varepsilon}_{\mathsf{D}}$ (or equivalently between $\bar{\sigma}_{\mathsf{P}}$ and $\bar{\sigma}_{\mathsf{D}}$). The limit case where $\theta = +\infty$ would correspond to a pure damage model without plasticity. It corresponds to the type of damage models which is used in the variational approach to fracture, see Amor et al. (2009); Pham et al. (2011a,b); Sicsic and Marigo (2012), and is close to those used in Lorentz et al. (2011) for quasi-brittle materials.

1. *Case* $\theta > 1$ *and* $\mathsf{n} > 1$. Since $\bar{\varepsilon}_{\mathsf{P}} > \bar{\varepsilon}_{\mathsf{D}}$ there exits no **P** stage. The **E** stage is followed by a **D** stage which ends when $\varepsilon = \varepsilon_{\mathsf{DP}}(\alpha_0)$ with α_0 such that $\sigma_{\mathsf{D}}(\alpha_0) = \sigma_{\mathsf{P}}(\alpha_0)$. That leads to

$$\alpha_0 = 1 - \theta^{\frac{1}{2(1-\mathsf{n})}}, \qquad \varepsilon_0 = \left(\mathsf{k} - (\mathsf{k}-1)\theta^{\frac{1}{1-\mathsf{n}}}\right)\bar{\varepsilon}_{\mathsf{D}}.$$

   Since $\varepsilon_{\mathsf{DP}}$ and π_{DP} are monotonically increasing, a **DP** stage starts at $\varepsilon = \varepsilon_0$. Since $\lim_{\alpha\to1}\varepsilon_{\mathsf{DP}}(\alpha) = +\infty$, the **DP** stage continues up to infinity. Thus, the evolution consists in the sequence **E**–**D**–**DP** with

$$\mathbf{E} = [0, \bar{\varepsilon}_{\mathsf{D}}), \qquad \mathbf{D} = [\bar{\varepsilon}_{\mathsf{D}}, \varepsilon_0), \qquad \mathbf{DP} = [\varepsilon_0, +\infty),$$

   see Figure 1(left).
2. *Case* $\theta > 1$ *and* $\mathsf{n} \le 1$. Since $\bar{\varepsilon}_{\mathsf{P}} > \bar{\varepsilon}_{\mathsf{D}}$ there exits no **P** stage. The **E** stage is followed by a **D** stage. Since $\sigma_{\mathsf{D}}(\alpha) < \sigma_{\mathsf{P}}(\alpha)$ for all $\alpha \in [0,1)$, the **D** stage ends only when $\alpha = 1$, *i.e.* when $\varepsilon = \varepsilon_{\mathsf{D}}(1) = \mathsf{k}\bar{\varepsilon}_{\mathsf{D}}$. Therefore, the evolution consists in the sequence **E**–**D**–**F** with

$$\mathbf{E} = [0, \bar{\varepsilon}_{\mathsf{D}}), \qquad \mathbf{D} = [\bar{\varepsilon}_{\mathsf{D}}, \mathsf{k}\bar{\varepsilon}_{\mathsf{D}}), \qquad \mathbf{F} = [\mathsf{k}\bar{\varepsilon}_{\mathsf{D}}, +\infty),$$

   see Figure 1(right). Note that the plasticity yield criterion is never reached and hence θ and n do not appear in the response.
3. *Case* $\theta < 1$ *and* $\mathsf{n} > 1$. Since $\bar{\varepsilon}_{\mathsf{P}} < \bar{\varepsilon}_{\mathsf{D}}$ there exits a **P** stage. Since both $\varepsilon_{\mathsf{DP}}$ and π_{DP} are monotonically increasing, the **P** stage is followed by a a **DP** stage and, since $\lim_{\alpha\to1}\varepsilon_{\mathsf{DP}}(\alpha) = +\infty$, the **DP** stage continues up to infinity. Thus, the evolution consists in the sequence **E**–**P**–**DP** with

$$\mathbf{E} = [0, \theta\bar{\varepsilon}_{\mathsf{D}}), \qquad \mathbf{P} = [\theta\bar{\varepsilon}_{\mathsf{D}}, \varepsilon_{\mathsf{DP}}(0)), \qquad \mathbf{DP} = [\varepsilon_{\mathsf{DP}}(0), +\infty),$$

   and

$$\varepsilon_{\mathsf{DP}}(0) = \theta\bar{\varepsilon}_{\mathsf{D}} + \frac{\mathsf{k}(1-\theta^2)}{2\mathsf{n}\theta}\bar{\varepsilon}_{\mathsf{D}},$$

   see Figure 2(left).

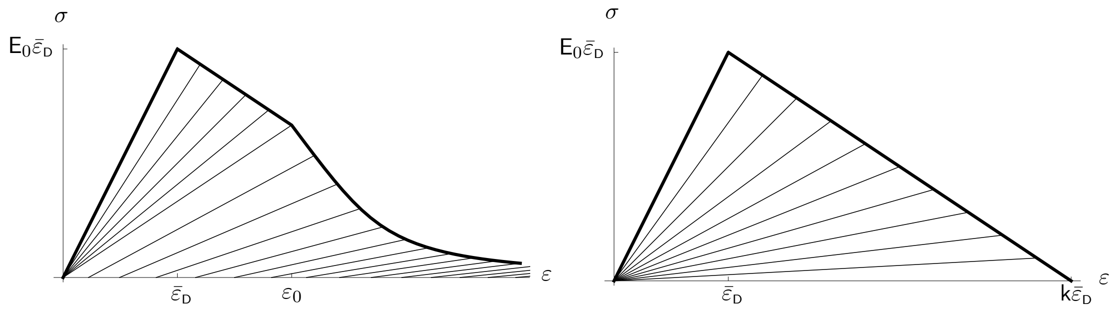

Figure 1: Response of the volume element in the case $\theta > 1$. Left: for $\mathsf{n} > 1$, the response corresponds to the sequence **E**–**D**–**DP**; Right: for $\mathsf{n} \leq 1$, the response corresponds to the sequence **E**–**D**–**F**. The thick curve corresponds to the response under a monotonic increasing stretching test. The thin lines would correspond to the responses associated with unloading and reloading tests, they are useful to represent the evolution of the Young modulus and the plastic strain.

4. *Case* $\theta < 1$ *and* $\mathsf{n} = 1$. Since $\bar{\varepsilon}_{\mathsf{P}} < \bar{\varepsilon}_{\mathsf{D}}$ there exits a **P** stage. Since $\varepsilon_{\mathsf{DP}}$ is monotonically increasing whereas π_{DP} is constant, the **P** stage is followed by a **DP** stage. However, it is a limit case of **DP** stage where $f_{\mathsf{P}} = 0$ but p does not evolve. It could be considered as a **D** stage. Since $\varepsilon_{\mathsf{DP}}(1) < \infty$, this stage ends at $\varepsilon_1 = \varepsilon_{\mathsf{DP}}(1)$ when $\alpha = 1$. Thus, the evolution consists in the sequence **E**–**P**–**DP**–**F** with:

$$\mathbf{E} = [0, \theta\bar{\varepsilon}_{\mathsf{D}}), \qquad \mathbf{P} = [\theta\bar{\varepsilon}_{\mathsf{D}}, \varepsilon_{\mathsf{DP}}(0)), \qquad \mathbf{DP} = [\varepsilon_{\mathsf{DP}}(0), \varepsilon_1), \qquad \mathbf{F} = [\varepsilon_1, +\infty)$$

and

$$\varepsilon_{\mathsf{DP}}(0) = \theta\bar{\varepsilon}_{\mathsf{D}} + \frac{\mathsf{k}(1-\theta^2)}{2\theta}\bar{\varepsilon}_{\mathsf{D}}, \qquad \varepsilon_1 = \frac{\mathsf{k}(1+\theta^2)}{2\theta}\bar{\varepsilon}_{\mathsf{D}},$$

see Figure 2(right).

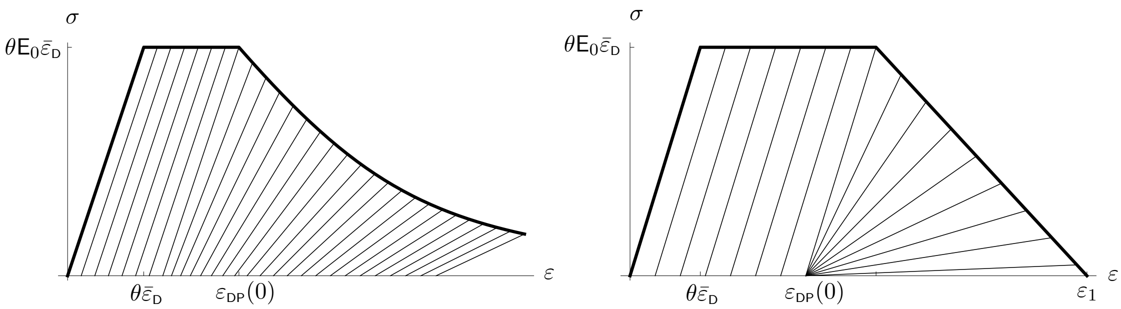

Figure 2: Response of the volume element in the case $\theta < 1$. Left: for $\mathsf{n} > 1$, the response corresponds to the sequence **E**–**P**–**DP**; Right: for $\mathsf{n} = 1$, the response corresponds to the sequence **E**–**P**–**D**–**F**.

5. *Case* $\theta < 1$ *and* $\mathsf{n} < 1$. Since $\bar{\varepsilon}_{\mathsf{P}} < \bar{\varepsilon}_{\mathsf{D}}$ there exits a **P** stage and **P**$=[\theta\bar{\varepsilon}_{\mathsf{D}}, \varepsilon_{\mathsf{DP}}(0))$. Since π_{DP} is monotonically decreasing, a **DP** stage cannot starts at $\varepsilon_{\mathsf{DP}}(0)$. Since π_{D} is monotonically decreasing and $\pi_{\mathsf{D}}(1) = 0$, $\alpha \mapsto e_{\mathsf{D}}(\pi_{\mathsf{DP}}(0), \alpha)$ is necessarily monotonically decreasing when α is greater than some value $\alpha_M \in [0, 1)$. Therefore a **D** stage starts at $\varepsilon_{\mathsf{DP}}(0)$ but will end at $\varepsilon_M = \pi_{\mathsf{DP}}(0) + e_{\mathsf{D}}(\pi_{\mathsf{DP}}(0), \alpha_M)$. Hence ε_M corresponds to a limit point after which no response exists under monotonically increasing strain. Specifically, if one considers the response given by $\varepsilon = \pi_{\mathsf{DP}}(0) + e_{\mathsf{D}}(\pi_{\mathsf{DP}}(0), \alpha)$, $p = \pi_{\mathsf{DP}}(0)$ and $\sigma = \mathsf{E}(\alpha)e_{\mathsf{D}}(\pi_{\mathsf{DP}}(0), \alpha)$ for $\alpha \geq 0$, that response corresponds to a snap-back in the $\sigma - \varepsilon$ plane as soon as $\alpha > \alpha_M$, see Figure 3. This part of the response cannot be observed under controlled increasing strain.

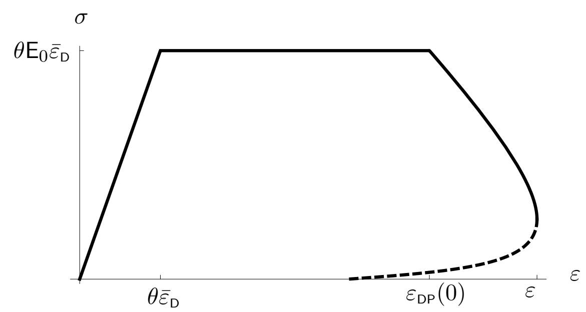

Figure 3: Response of the volume element in the case $\theta = 2/3$, $\mathsf{n} = 2/3$ and $\mathsf{k} = 4$. The response corresponds to the sequence **E**–**P**–**D** until the limit point ε_M where a snap-back occurs.

3.4. The issue of the global stability

The responses above have been obtained by only considering the first order stability conditions. It remains to see whether, at each time ε, the state $(\varepsilon, \alpha_\varepsilon, p_\varepsilon, \bar{p}_\varepsilon)$ is globally stable in the sense of (31). This verification is delicate in the general case and requires additional properties for the state functions. We merely give a partial result in the following proposition:

Proposition 6. *Provided that $\alpha \mapsto \pi_{\mathsf{D}}(\alpha)$ is non decreasing, the response given by the first order stability conditions satisfies, at each time when that response exists, the following global stability properties:*

$$\begin{cases} \mathsf{W}_{\mathsf{DP}}(\varepsilon, \alpha_\varepsilon, p_\varepsilon, \bar{p}_\varepsilon) \leq \mathsf{W}_{\mathsf{DP}}(\varepsilon, \alpha_\varepsilon, p_\varepsilon + q, \bar{p}_\varepsilon + |q|) & , \quad \forall q \in \mathbb{R} \\ \mathsf{W}_{\mathsf{DP}}(\varepsilon, \alpha_\varepsilon, p_\varepsilon, \bar{p}_\varepsilon) \leq \mathsf{W}_{\mathsf{DP}}(\varepsilon, \alpha^*, p_\varepsilon, \bar{p}_\varepsilon) & , \quad \forall \alpha^* \in [\alpha, 1]. \end{cases}$$

Thus, we are only able to prove that, at given damage state, the current state is globally stable with respect to any perturbation of the plastic strain, and, symmetrically, at given plastic strain state, the current state is globally stable with respect to any perturbation of the damage. The proof is given in the appendix. Note that these global stability properties hold true for the family of models considered in the previous subsection when $\mathsf{n} \geq 1$.

4. Introduction of the gradient of damage and the global one-dimensional problem

Throughout this section and the next one we consider a one-dimensional body called *the bar* whose reference configuration is the interval $\Omega = (0, L)$. Its end $x = 0$ is fixed and the end $x = L$ is submitted to a time dependent displacement U_t with $U_0 = 0$. The bar is made of a material whose local behavior is given by one of the plasticity-damage models described in the previous section. The purpose of these sections is first to set the problem which governs the evolution of the bar with time and then to solve this problem in some particular cases. Specifically, that consists in constructing and solving the system of equations giving $t \mapsto (u_t, \alpha_t, p_t, \bar{p}_t)$ for $t \geq 0$, where u_t, α_t, p_t and $\bar{p}_t$ denote now, respectively, the displacement field, the damage field, the plastic strain field and the cumulated plastic strain field of the bar at time t. We assume that, at time $t = 0$, the bar is sound and was never plasticized so that $\alpha_0 = p_0 = \bar{p}_0 = 0$ everywhere in Ω.

4.1. Introduction of the gradient of damage

It is well known that if one uses local damage models with softening to address the problem of the damage evolution in a whole body and no more for the volume element only, then that leads to a ill-posed mathematical problem which admits an infinite number of solutions Benallal and Marigo (2007). Moreover, in the present context of a bar under traction, if one tries to select the solutions which correspond to *stable* states at each time, it turns out that *no* solution satisfies the stability criterion formulated in terms of the total energy of the body Pham and Marigo (2013). The reason is that it is always possible to find a state more damaged and close to the tested state which decreases the energy of the bar. Roughly speaking, it is possible to break the bar without spending any energy. To overcome this pathological effect, it is necessary to regularize the model by penalizing the too localized damage fields. For that, we follow the procedure of Benallal and Marigo (2007); Pham and Marigo (2010a,b) by introducing a damage gradient term into the total energy density. Specifically, one considers that the state of the volume element is now the quintuple $(\varepsilon, \alpha, \alpha', p, \bar{p})$, where α' denotes the gradient of damage, and that the total energy density reads as

$$\mathsf{W}(\varepsilon, \alpha, \alpha', p, \bar{p}) = \frac{1}{2}\mathsf{E}(\alpha)(\varepsilon - p)^2 + \mathsf{d}(\alpha) + \sigma_{\mathsf{P}}(\alpha)\bar{p} + \mathsf{d}_1\ell^2\alpha'^2. \tag{41}$$

In (41), $\ell > 0$ represents the internal length of the material. We assume that ℓ is a given constant, independent of α, which is always possible by a change of the damage variable, see Pham and Marigo (2010b); Pham et al. (2011b). Accordingly, the total energy of the bar is the following functional of the quadruple $\boldsymbol{\xi} = (u, \alpha, p, \bar{p})$, called the global state field, made of the displacement field u, the damage field α, the plastic strain field p and the cumulated plastic strain field $\bar{p}$:

$$\mathcal{E}(u, \alpha, p, \bar{p}) = \int_\Omega \mathsf{W}(u'(x), \alpha(x), \alpha'(x), p(x), \bar{p}(x))dx$$

where the prime denotes the derivative to respect to x. The above expression of the energy makes sense provided that the global state field $\boldsymbol{\xi}$ is smooth enough. As long as the damage field is concerned, the gradient term requires that $\alpha \in H^1(\Omega)$ so that the total energy be finite and hence, in our one-dimensional setting, $\alpha \in C^0(\bar{\Omega})$. But, as long as the displacement and the plastic strain fields are concerned, it turns out that it is not always possible to find smooth evolutions because of the localization of the deformation induced by the softening character of the model. Indeed, the natural space for the displacement field is the space $BV(\Omega)$ of functions of bounded variations while the natural space of the plastic strain field is the space $\mathcal{M}_b(\Omega)$ of the measures with bounded variations, Evans and Gariepy (1992). However, to simplify the presentation, we will consider a less general framework where any global state field $\boldsymbol{\xi} = (u, \alpha, p, \bar{p})$ is piecewise smooth and its singular part is localized on a $\boldsymbol{\xi}$-dependent set $S(\boldsymbol{\xi})$ which contains a finite number of points of Ω. (For a sake of simplicity again, we assume that the ends $x = 0$ and $x = L$ are not singular points.)

Accordingly, we will use the following assumption: (i) any (extended signed) measure m is decomposed into its regular and singular parts; (ii) the regular part is denoted $m(x)dx$, where $x \mapsto m(x)$ is at least an integrable function (in practice, a piecewise continuous function) and dx is the Lebesgue measure; (iii) the singular part is a linear combination of Dirac measures concentrated on the finite number of singular points $S(m)$ with some weight $M(x_i)$ at point x_i. That leads to the following notation:

$$m = m(x)dx + \sum_{x_i \in S(m)} M(x_i)\delta_{x_i}.$$

When m is applied to a function φ continuous on $\bar{\Omega}$, one gets

$$m(\varphi) = \int_{\Omega\setminus S(m)} m(x)\varphi(x)dx + \sum_{x_i \in S(m)} M(x_i)\varphi(x_i).$$

Specifically, we assume that the displacement field u is continuously differentiable on $\Omega \setminus S(\boldsymbol{\xi})$ and admits a jump discontinuity on $S(\boldsymbol{\xi}) \subset \Omega$. Therefore, the strain field ε associated with u can be seen as the following measure

$$\varepsilon = u'(x)dx + \sum_{x_i \in S(\boldsymbol{\xi})} [\![u]\!](x_i)\delta_{x_i}.$$

In order that the elastic energy be finite, the plastic strain field p has the same singular part as the strain field and hence can be seen as the following measure

$$p = p(x)dx + \sum_{x_i \in S(\boldsymbol{\xi})} [\![u]\!](x_i)\delta_{x_i}$$

where $x \mapsto p(x)$ is at least continuous on $\Omega \setminus S(\boldsymbol{\xi})$. In the same manner, the cumulated plastic strain field is decomposed into regular and singular parts as follows

$$\bar{p} = \bar{p}(x)dx + \sum_{x_i \in S(\boldsymbol{\xi})} \bar{P}(x_i)\delta_{x_i}.$$

Finally, the total energy of the bar in the global state $\boldsymbol{\xi} = (u, \alpha, p, \bar{p})$ can read as

$$\begin{aligned} \mathcal{E}(u, \alpha, p, \bar{p}) &= \int_{\Omega \setminus S(\boldsymbol{\xi})} \left(\frac{1}{2}\mathsf{E}(\alpha(x))\Big(u'(x) - p(x)\Big)^2 + \mathsf{d}(\alpha(x)) + \sigma_{\mathsf{P}}(\alpha(x))\bar{p}(x) + \mathsf{d}_1 \ell^2 \alpha'(x)^2 \right) dx \\ &+ \sum_{x_i \in S(\boldsymbol{\xi})} \sigma_{\mathsf{P}}(\alpha(x_i))\bar{P}(x_i). \end{aligned} \tag{42}$$

In (42), the singular part of the cumulated opening only appears in the dissipated energy, because α and α' are not singular whereas ε and p have the same singular part.

4.2. The one-dimensional evolution problem

The problem giving the evolution of the state of the bar, namely $t \mapsto \boldsymbol{\xi}_t = (u_t, \alpha_t, p_t, \bar{p}_t)$, will be constructed by using, like in the case of the volume element, the three conditions of irreversibility, stability and energy balance. Assuming that the evolution $t \mapsto \boldsymbol{\xi}_t$ is smooth, $t \mapsto \bar{p}_t$ is obtained from $t \mapsto p_t$ by

$$\dot{\bar{p}}_t(x) = |\dot{p}_t(x)| \quad \forall x \in \Omega \setminus S(\boldsymbol{\xi}_t), \qquad \dot{\bar{P}}_t(x_i) = |[\![\dot{u}_t]\!](x_i)| \quad \forall x_i \in S(\boldsymbol{\xi}_t).$$

Note that $t \mapsto S(\boldsymbol{\xi}_t)$ is not decreasing, *i.e.* the number of singular points can only increase. Indeed, if a jump discontinuity of the displacement appears at a point x_i at some time t_i, then $\bar{P}_t(x_i) > 0$ for all $t \geq t_i$. Therefore those points are material points and their position does not depend on time, but their number can increase because new points can appear all along the evolution. Accordingly, using the initial conditions, we can set

$$\bar{p}_t(x) = \int_0^t |\dot{p}_s(x)|\, ds, \quad \forall x \in \Omega \setminus S(\boldsymbol{\xi}_t) \quad \text{and} \quad \bar{P}_t(x_i) = \int_0^t |[\![\dot{u}_s(x_i)]\!]|\, ds, \quad \forall x_i \in S(\boldsymbol{\xi}_t). \tag{43}$$

4.2.1. The irreversibility condition

It is essentially the same as the local one and requires that

$$\dot{\alpha}_t(x) \geq 0, \qquad 0 \leq \alpha_t(x) \leq 1, \quad \forall x \in \Omega. \tag{44}$$

4.2.2. Stability condition

To simplify the presentation, we will only consider the evolution before the rupture of the bar, *i.e.* we consider the times t such that $\max_{x \in \Omega} \alpha_t(x) < 1$. Let $\boldsymbol{\xi}_t = (u_t, \alpha_t, p_t, \bar{p}_t)$ be the state of the bar at such a time and let $\boldsymbol{\xi}^* = (u^*, \alpha^*, p^*, \bar{p}^*)$ be the following virtual state $\boldsymbol{\xi}^* = \boldsymbol{\xi}_t + h(v, \beta, q, |q|)$ where h is a positive constant. In order that u^* be kinematically admissible, the field v must be such that $v(0) = v(L) = 0$. Moreover, the field v is assumed, like u_t, piecewise smooth and we denote by $S(v)$ the set of points where v is discontinuous. Thus the strain field ε^* associated with u^* can be seen as the following measure

$$\varepsilon^* = \varepsilon_t + hv'(x)dx + h \sum_{x_i \in S(v)} [\![v]\!](x_i)\delta_{x_i}.$$

In order that the elastic energy associated with $\boldsymbol{\xi}^*$ be finite, the virtual plastic strain field p^* must have the same singular part and hence is the following measure

$$p^* = p_t + hq(x)dx + h \sum_{x_i \in S(v)} [\![v]\!](x_i)\delta_{x_i}.$$

Therefore the singular set of $\boldsymbol{\xi}^*$ is $S(\boldsymbol{\xi}^*) = S(\boldsymbol{\xi}_t) \cup S(v)$. Finally in order that $\alpha_t \leq \alpha^* < 1$ and that $\alpha^* \in H^1(\Omega)$, it is necessary and sufficient that $\beta \geq 0$, $\beta \in H^1(\Omega)$ and h be small enough. A triple of fields (v, β, q) which satisfies the above conditions will be called an admissible direction of perturbation. We are now in a position to define the condition of stability.

Definition 2. *The state $(u_t, \alpha_t, p_t, \bar{p}_t)$ of the bar at a time t before rupture is said locally stable if, for every admissible direction of perturbation (v, β, q), there exists $\bar{h} > 0$ such that for all $h \in [0, \bar{h}]$*

$$\mathcal{E}(u_t + hv, \alpha_t + h\beta, p_t + hq, \bar{p}_t + h\,|q|) \geq \mathcal{E}(u_t, \alpha_t, p_t, \bar{p}_t). \tag{45}$$

Dividing (45) by h and passing to the limit when $h \to 0$ yields *the first order stability conditions*:

$$\frac{d}{dh}\mathcal{E}(u_t + hv, \alpha_t + h\beta, p_t + hq, \bar{p}_t + h\,|q|)\bigg|_{h=0} \geq 0, \quad \forall (v, \beta, q) \text{ admissible.}$$

Using the definition (42) of the energy and the assumed forms of the field leads to

$$\begin{aligned}
0 \quad \leq \quad & \int_{\Omega\setminus S(v)} \Big(\sigma_t(x)(v'(x) - q(x)) + \sigma_{\mathsf{P}}(\alpha_t(x))\,|q(x)|\Big)dx + \sum_{x_i \in S(v)} \sigma_{\mathsf{P}}(\alpha_t(x_i))\,|[\![v]\!](x_i)| \\
+ \quad & \int_{\Omega\setminus S(\boldsymbol{\xi}_t)} \Big(\Big(-\frac{1}{2}\mathsf{S}'(\alpha_t(x))\sigma_t(x)^2 + \mathsf{d}'(\alpha_t(x)) + \sigma_{\mathsf{P}}'(\alpha_t(x))\bar{p}_t(x)\Big)\beta(x) + 2\mathsf{d}_1\ell^2\alpha_t'(x)\beta'(x)\Big)\,dx \\
+ \quad & \sum_{x_i \in S(\boldsymbol{\xi}_t)} \sigma_{\mathsf{P}}'(\alpha_t(x_i))\bar{P}_t(x_i)\beta(x_i)
\end{aligned} \tag{46}$$

where $\sigma_t(x) = \mathsf{E}(\alpha_t(x))(u_t'(x) - p_t(x))$ denotes the stress field at time t. The inequality (46) must hold for all v such that $v(0) = v(L) = 0$, all $\beta \geq 0$ and all q. Let us derive the different local conditions which are given by (46).

1. *Equilibrium equation.* Taking first $\beta = q = 0$ and $S(v) = \emptyset$, one easily obtains that the stress field is constant, *i.e.*
$$\sigma_t(x) = \sigma_t, \quad \forall x \in \Omega. \tag{47}$$

2. *Plasticity yield criterion.* Taking $v = \beta = 0$ and using the equilibrium equation, (46) gives
$$\int_\Omega \Big(\sigma_{\mathsf{P}}(\alpha_t(x))\,|q(x)| - \sigma_t q(x)\Big)dx \geq 0, \quad \forall q \text{ smooth,}$$
from which one immediately deduces that the stress must satisfy the plasticity yield criterion at every point of the bar, *i.e.*
$$|\sigma_t| \leq \sigma_{\mathsf{P}}(\alpha_t(x)), \quad \forall x \in \Omega. \tag{48}$$

3. *The damage yield criteria.* Taking $v = q = 0$, (46) becomes
$$0 \leq \int_{\Omega\setminus S(\boldsymbol{\xi}_t)} \Big(\Big(-\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t^2 + \mathsf{d}'(\alpha_t) + \sigma_{\mathsf{P}}'(\alpha_t)\bar{p}_t\Big)\beta + 2\mathsf{d}_1\ell^2\alpha_t'\beta'\Big)\,dx + \sum_{S(\boldsymbol{\xi}_t)} \sigma_{\mathsf{P}}'(\alpha_t)\bar{P}_t\beta, \quad \forall \beta \geq 0,$$
where the dependence on x of the fields is omitted. Integrating by parts the term in $\alpha_t'\beta'$, we obtain by classical arguments of Calculus of Variations the following damage yield criteria in the regular part, the singular part and the ends of the bar:
$$\begin{aligned}
\text{In } \Omega \setminus S(\boldsymbol{\xi}_t) \quad &: \quad -\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t^2 + \mathsf{d}'(\alpha_t) + \sigma_{\mathsf{P}}'(\alpha_t)\bar{p}_t - 2\mathsf{d}_1\ell^2\alpha_t'' \geq 0; & (49)\\
\text{On } S(\boldsymbol{\xi}_t) \quad &: \quad \sigma_{\mathsf{P}}'(\alpha_t)\bar{P}_t - 2\mathsf{d}_1\ell^2[\![\alpha_t']\!] \geq 0; & (50)\\
\text{On } \partial\Omega \quad &: \quad \alpha_t'(0) \leq 0, \qquad \alpha'(L) \geq 0. & (51)
\end{aligned}$$

In the case where σ_{P} does not depend on α, we recover the damage yield criteria obtained in Pham and Marigo (2010b); Pham et al. (2011a,b) by the same variational approach and in Comi (1999). But note that here, because of the coupling term between damage and plasticity, the localization of the plastic strain will in general induce a discontinuity of the gradient of damage and *vice versa*. More precisely, if α' is continuous at some point and some time, then (50) implies that $\bar{P}=0$ at that point and that time and hence, by virtue of (43), this point was never a singular point before that time.

4. Noting that $v(0)=v(L)=0$ implies $\int_{\Omega\setminus S(v)} v'(x)dx+\sum_{x_i\in S(v)}[\![v]\!](x_i)=0$, one easily verifies that the equilibrium equation (47), the plasticity yield criterion (48) and the damage yield criteria (49)-(50) are sufficient so that (46) be satisfied.

4.2.3. Energy balance

Following the presentation of Pham and Marigo (2010b); Pham et al. (2011b), the energy balance principle in our one-dimensional setting reads as

$$\frac{d}{dt}\mathcal{E}(u_t,\alpha_t,p_t,\bar{p}_t)=\sigma_t(L)\dot{U}_t$$

and hence requires that the variation of the total energy of the bar be equal to the power of the external force at L. Using the equilibrium equation and expanding the time derivative of the energy, the energy balance becomes

$$\begin{aligned} 0 &= \int_{\Omega\setminus S(\boldsymbol{\xi}_t)}\Big(\sigma_t(\dot{u}'_t-\dot{p}_t)+\sigma_{\mathsf{P}}(\alpha_t)\,|\dot{p}_t|+\Big(\mathsf{d}'(\alpha_t)+\sigma'_{\mathsf{P}}(\alpha_t)\bar{p}_t-\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t^2\Big)\dot{\alpha}_t+2\mathsf{d}_1\ell^2\alpha'_t\dot{\alpha}'_t\Big)\,dx \\ &+ \sum_{S(\boldsymbol{\xi}_t)}\Big(\sigma'_{\mathsf{P}}(\alpha_t)\bar{P}_t\dot{\alpha}_t+\sigma_{\mathsf{P}}(\alpha_t)\,|[\![\dot{u}_t]\!]|\Big)-\sigma_t\dot{U}_t. \end{aligned}$$

Integrating by parts the term in $\alpha'_t\dot{\alpha}'_t$ and using the identity $\int_{\Omega\setminus S(\boldsymbol{\xi}_t)}\dot{u}'_t dx+\sum_{S(\boldsymbol{\xi}_t)}[\![\dot{u}_t]\!]=\dot{U}_t$ given by the boundary conditions lead to

$$\begin{aligned} 0 &= \int_{\Omega\setminus S(\boldsymbol{\xi}_t)}\Big(\sigma_{\mathsf{P}}(\alpha_t)\,|\dot{p}_t|-\sigma_t\dot{p}_t\Big)dx+\int_{\Omega\setminus S(\boldsymbol{\xi}_t)}\Big(\mathsf{d}'(\alpha_t)+\sigma'_{\mathsf{P}}(\alpha_t)\bar{p}_t-\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t^2-2\mathsf{d}_1\ell^2\alpha''_t\Big)\dot{\alpha}_t dx \\ &+ \sum_{S(\boldsymbol{\xi}_t)}\Big(\sigma'_{\mathsf{P}}(\alpha_t)\bar{P}_t-2\mathsf{d}_1\ell^2[\![\alpha'_t]\!]\Big)\dot{\alpha}_t+\sum_{S(\boldsymbol{\xi}_t)}\Big(\sigma_{\mathsf{P}}(\alpha_t)\,|[\![\dot{u}_t]\!]|-\sigma_t[\![\dot{u}_t]\!]\Big)+2\mathsf{d}_1\ell^2\Big(\alpha'_t(L)\dot{\alpha}_t(L)-\alpha'_t(0)\dot{\alpha}_t(0)\Big). \end{aligned}$$

Using the plasticity yield criterion (48), the damage yield criteria (49)–(51) and the irreversibility condition (44), one finally obtains the consistency equations and the plasticity flow rules:

$$\text{In } \Omega\setminus S(\boldsymbol{\xi}_t) \;:\; \Big(\mathsf{d}'(\alpha_t)+\sigma'_{\mathsf{P}}(\alpha_t)\bar{p}_t-\frac{1}{2}\mathsf{S}'(\alpha_t)\sigma_t^2-2\mathsf{d}_1\ell^2\alpha''_t\Big)\dot{\alpha}_t=0; \tag{52}$$

$$\text{In } \Omega\setminus S(\boldsymbol{\xi}_t) \;:\; \dot{p}_t\geq 0 \text{ if } \sigma_t=\sigma_{\mathsf{P}}(\alpha_t),\quad \dot{p}_t\leq 0 \text{ if } \sigma_t=-\sigma_{\mathsf{P}}(\alpha_t),\quad \dot{p}_t=0 \text{ if } |\sigma_t|<\sigma_{\mathsf{P}}(\alpha_t); \tag{53}$$

$$\text{On } S(\boldsymbol{\xi}_t) \;:\; \Big(\sigma'_{\mathsf{P}}(\alpha_t)\bar{P}_t-2\mathsf{d}_1\ell^2[\![\alpha'_t]\!]\Big)\dot{\alpha}_t=0; \tag{54}$$

$$\text{On } S(\boldsymbol{\xi}_t) \;:\; [\![\dot{u}_t]\!]\geq 0 \text{ if } \sigma_t=\sigma_{\mathsf{P}}(\alpha_t),\quad [\![\dot{u}_t]\!]\leq 0 \text{ if } \sigma_t=-\sigma_{\mathsf{P}}(\alpha_t),\quad [\![\dot{u}_t]\!]=0 \text{ if } |\sigma_t|<\sigma_{\mathsf{P}}(\alpha_t); \tag{55}$$

$$\text{On } \partial\Omega \;:\; \alpha'_t\dot{\alpha}_t=0. \tag{56}$$

Note that the consistency equation and the plasticity flow rule hold on the singular set too.

We can summarize our construction of the global evolution problem by the following

Proposition 7. *A smooth evolution $t\mapsto\boldsymbol{\xi}_t=(u_t,\alpha_t,p_t,\bar{p}_t)$ satisfies the irreversibility condition, the first order stability condition and the energy balance if and only if the cumulated plastic strain relations* (43)*, the local irreversibility condition* (44)*, the equilibrium equation* (47)*, the plasticity yield criterion* (48)*, the damage yield criteria* (49)*–*(51)*, the consistency conditions and the plasticity flow rules* (52)*–*(56) *are satisfied at every instant t.*

5. One-dimensional non homogeneous responses

It is easy to check that the homogeneous response, *i.e.* the response such that $u_t(x) = U_t x/L$, $\alpha_t(x) = \alpha_t$ and $p_t(x) = p_t$ for all $x \in \Omega$, is still possible. In the case where $U_t = tL$, *i.e.* for a monotonically increasing traction test, the homogeneous response is precisely that obtained in the previous section for the volume element. This evolution satisfies the irreversibility condition, the first order stability conditions and the energy balance because, in particular, the gradient of damage vanishes and $S(\boldsymbol{\xi}_t) = \emptyset$. However, we are no more ensured that it is the unique solution. Moreover, we are not ensured that the local stability condition (45) is satisfied by the homogeneous response. If we refer to what happens in the case of gradient damage models with softening (without plasticity), we know that the homogeneous response is unique and stable if and only if the length L of the bar is sufficiently small by comparison with the internal length ℓ of the material Pham et al. (2011b); Pham and Marigo (2013). When the length of the bar is large enough, the homogeneous response is not stable and it is possible to construct non homogeneous responses. (It is even possible in general to construct an infinite number of responses Benallal and Marigo (2007).) Accordingly, we propose here to follow the same procedure and, assuming that L is sufficiently large by comparison with ℓ, to construct a response where the damage, when it appears, remains localized on a time-dependent part of the bar. To construct such an evolution, we follow the method proposed in Pham et al. (2011b); Pham and Marigo (2012) and the reader must refer to these papers to have some details or proofs which are omitted here.

5.1. General assumptions

To simplify the presentation and to prevent from considering too many cases, we construct such non homogeneous responses for the family of models considered at Section 3.3 only. Therefore, E, d and σ_{P} are given by (39)-(40). The analysis starts at a time when the damage yield criterion is reached somewhere in the bar. This time t_c corresponds to the end of the $\mathbf{E}$ stage or of a $\mathbf{P}$ stage according to whether $\theta > 1$ or $\theta < 1$. In the former case, when $\theta > 1$, there is no plasticity before t_c, the state of the bar at t_c is $\boldsymbol{\xi}_{t_c} = (\bar{\varepsilon}_{\mathsf{D}} x, 0, 0, 0)$, the stress is $\sigma_{t_c} = \bar{\sigma}_{\mathsf{D}}$ and hence the damage yield criterion is reached at every point of the bar. In the latter case, when $\theta < 1$, a $\mathbf{P}$ stage occurs before t_c and we will assume to simplify the presentation that the plastic strain field and the cumulated plastic strain field are uniform at t_c. (This property is not ensured because we are in a perfect plasticity setting and hence the uniqueness of the plastic strain field is not guaranteed.) Hence, the state of the bar at t_c is $\boldsymbol{\xi}_{t_c} = (\varepsilon_{\mathsf{DP}}(0)x, 0, \pi_{\mathsf{DP}}(0), \pi_{\mathsf{DP}}(0))$, the stress is $\sigma_{t_c} = \theta\bar{\sigma}_{\mathsf{D}}$ and the damage yield criterion is also reached at every point of the bar. When $t > t_c$, we assume that σ_t is monotonically decreasing from σ_{t_c} to 0. We seek for non homogeneous evolutions such that the damage zone is the interval $(x_1 - \Delta_t, x_1 + \Delta_t)$ where x_1 is an arbitrary point of the bar sufficiently far from its ends so that $0 < x_1 - \Delta_t \leq x_1 + \Delta_t < L$. Thus, we exclude the case where the damage zone is at the boundary. The half width Δ_t of the damage zone, which can depend on time, has to be determined. We will assume that the center x_1 of the damage zone is the unique possible singular point, *i.e.* $S(\boldsymbol{\xi}_t) = \emptyset$ or $S(\boldsymbol{\xi}_t) = \{x_1\}$. (This property could be deduced from some natural assumptions on the form of the response.) The analysis is made by discriminating different cases and, when there is no risk of confusion, we do not explicit the dependence on time t of the fields.

5.2. Cases where $\theta > 1$

At the beginning of the localization process, damage grows without plasticity. We are in the situation studied in Pham et al. (2011b); Pham and Marigo (2012), there is no singular point and, by hypothesis,

$$-\mathsf{S}'(\alpha)\sigma^2 + 2\mathsf{d}'(\alpha) - 4\mathsf{d}_1\ell^2\alpha'' = 0 \quad \text{in} \quad I = (x_1 - \Delta, x_1 + \Delta)$$

with $\alpha(x_1 \pm \Delta) = \alpha'(x_1 \pm \Delta) = 0$. The autonomous second order differential equation for α above is an Euler-Lagrange equation which admits a first integral. Indeed, multiplying the equation by α' gives

$$-\mathsf{S}(\alpha(x))\sigma^2 + 2\mathsf{d}(\alpha(x)) - 2\mathsf{d}_1\ell^2\alpha'(x)^2 = C, \quad \forall x \in I,$$

and the constant C is obtained from the boundary conditions at $x_1 \pm \Delta$. Hence, $C = -\mathsf{S}(0)\sigma^2$. Using (39)-(40) and making the change of variable $\alpha \to \omega = \mathsf{w}(\alpha)$, the first integral becomes

$$\ell^2\omega'^2 = 4\omega(\bar{\omega}_\sigma - \omega) \quad \text{in} \quad I,$$

where

$$\bar{\omega}_\sigma = 1 - \frac{\sigma^2}{\bar{\sigma}_{\mathsf{D}}^2}. \tag{57}$$

Hence, in the normalized phase plane $(\omega, \ell\omega'/2)$, the first integral is a circle of center $\bar{\omega}_\sigma/2$ and radius $\bar{\omega}_\sigma/2$. Setting $\omega(x) = \bar{\omega}_\sigma \cos^2 \frac{\varphi(x)}{2}$ gives $\ell^2\varphi'^2 = 4$ and hence $\ell\varphi(x) = 2(x_1 - x)$. Therefore, the half width Δ of the damage zone is

$$\Delta = \frac{\pi}{2}\ell.$$

So, it is independent of σ and proportional to the internal length. For a given σ, the damage profile in the damage zone is given by

$$\alpha(x) = 1 - \sqrt{1-\omega(x)} = 1 - \sqrt{1 - \bar{\omega}_\sigma \cos^2 \frac{x - x_1}{\ell}}.$$

Thus, the profile is symmetric, α is maximal at the center x_1 of the zone where it is equal to $\bar{\alpha}_\sigma = 1 - \sigma/\bar{\sigma}_{\mathsf{D}}$.

This evolution with localization of damage and without plasticity remains admissible as long as the plasticity yield criterion is satisfied everywhere in the bar, *i.e.* as long as $\sigma \le \sigma_{\mathsf{P}}(\alpha(x))$ for all $x \in \Omega$. Hence, since $\alpha \mapsto \sigma_{\mathsf{P}}(\alpha)$ is decreasing and since the damage field is maximal at x_1, this evolution is admissible as long as $\sigma \le \sigma_{\mathsf{P}}(\bar{\alpha}_\sigma)$. By virtue of (39), this inequality reads as

$$\theta\left(\frac{\sigma}{\bar{\sigma}_{\mathsf{D}}}\right)^{2\mathsf{n}-1} \ge 1 \tag{58}$$

and one has to discriminate two cases according to $\mathsf{n} \le 1/2$ or $\mathsf{n} > 1/2$.

5.2.1. Case I: $\theta > 1$ and $\mathsf{n} \le 1/2$, damage localization until rupture without plasticity

In that case, (58) is automatically satisfied for all $\sigma \le \bar{\sigma}_{\mathsf{D}}$. There is no plasticity and the previous analysis remains true until $\sigma = 0$. The damage profile at a given σ and its evolution with σ are given in Figure 4.

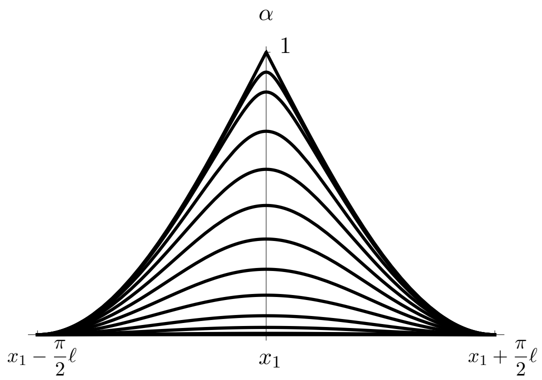

Figure 4: Case I: $\theta > 1$ and $\mathsf{n} \le 1/2$, evolution of the damage profile from the nucleation up to the rupture

When $\sigma = 0$, then $\alpha(x_1) = 1$, a crack appears and the displacement is discontinuous at that point. The damage profile itself becomes singular in the sense that α' is discontinuous at x_1. This final damage profile is given by

$$\alpha(x) = 1 - \sin\frac{|x - x_1|}{\ell} \quad \text{for} \quad x \in I, \quad \text{when } \sigma = 0.$$

As long as the dissipated energy in the bar during the damage process is concerned, we have

$$\mathcal{D}_\sigma := \int_{x_1-\Delta}^{x_1+\Delta} \left(\mathsf{d}(\alpha) + \mathsf{d}_1\ell^2\alpha'^2\right) dx = \ell\sqrt{2\mathsf{d}_1}\int_0^{\bar\alpha_\sigma} \frac{4\mathsf{d}(\alpha) - (\mathsf{S}(\alpha)-\mathsf{S}(0))\sigma^2}{\sqrt{2\mathsf{d}(\alpha) - (\mathsf{S}(\alpha)-\mathsf{S}(0))\sigma^2}} d\alpha.$$

Therefore, the energy $\mathsf{G_c}$ which is dissipated during all the damage process until rupture is given by

$$\mathsf{G_c} := \mathcal{D}_0 = \frac{\pi\mathsf{k}}{2}\frac{\bar\sigma_{\mathsf{D}}^2\ell}{\mathsf{E}_0}. \tag{59}$$

Thus $\mathsf{G_c}$ is proportional to ℓ and involves all the material constants but those concerning the plastic behavior, *i.e.* θ and n.

5.2.2. Case II: $\theta > 1$ and $\mathsf{n} > 1/2$, damage localization with nucleation of a cohesive crack before rupture

In that case, the condition (58) is satisfied as long as $\sigma \geq \sigma_0$ with

$$\sigma_0 = \theta^{-1/(2\mathsf{n}-1)}\bar\sigma_{\mathsf{D}}.$$

At $\sigma = \sigma_0$, the plasticity yield criterion is reached at the center x_1 of the damage zone. At that time, the damage profile reads as:

$$\alpha_{\sigma_0}(x) = 1 - \sqrt{1-\omega_{\sigma_0}(x)}, \quad \omega_{\sigma_0}(x) = \left(1-\theta^{-\frac{2}{2\mathsf{n}-1}}\right)\cos^2\left(\frac{x_1-x}{\ell}\right).$$

Then, when $\sigma < \sigma_0$, the equality $\sigma = \sigma_{\mathsf{P}}(\alpha(x))$ is satisfied only at the points where α is maximal and all these points have the same damage state α^*_σ,

$$\alpha^*_\sigma = 1-\sqrt{1-\omega^*_\sigma}, \qquad \omega^*_\sigma = 1-\left(\frac{\sigma}{\theta\bar\sigma_{\mathsf{D}}}\right)^{1/\mathsf{n}}.$$

Since $\alpha^*_\sigma < \bar\alpha_\sigma$, a part of the damage zone I is necessarily unloaded and the damage does not grow in that unloaded zone. Accordingly, we search a solution of the following form:

(i) the damage is maximal at x_1 only, *i.e.* at the center of the damage zone;

(ii) the still damaging zone is the interval $I_\sigma = (x_1 - \varphi_\sigma\ell/2, x_1 + \varphi_\sigma\ell/2)$ where φ_σ (to be determined) is increasing when σ decreases;

(iii) the remaining part of the damage zone $I \setminus I_\sigma$ is unloaded and α remains equal to α_{σ_0}.

The above conditions could be deduced from weaker assumptions, but we start here from them to simplify the presentation. By virtue of (i), plasticity can only occur at x_1 and we will see that it is the case. So, x_1 is a singular point at $\sigma < \sigma_0$ and, since the damage grows at x_1, (43), (54) and (55) give

$$\sigma'_{\mathsf{P}}(\alpha^*_\sigma)[\![u]\!](x_1) = 2\mathsf{d}_1\ell^2[\![\alpha']\!](x_1). \tag{60}$$

In other words, *a cohesive crack has nucleated* and the cohesive law relating $[\![u]\!]$ and σ will be obtained once the jump of α' will be found. In $I \setminus I_\sigma$, the plasticity criterion (48) and the damage criterion (50) are automatically satisfied because $\sigma < \sigma_0$, σ_{P} is decreasing and $\alpha_{\sigma_0} < \alpha^*_\sigma$. At the ends of I_σ, α and α' must be continuous. The continuity of α' follows from (50) and the irreversibility condition. Indeed, let us consider the point $x' = x_1 - \varphi_\sigma\ell/2$. Since there is no plasticity at this point, (50) gives $[\![\alpha']\!](x') \leq 0$. On the other hand, since α'_{σ_0} is continuous and since α can only grows on the right hand side of x', one has $[\![\alpha']\!](x') \geq 0$. Hence $[\![\alpha']\!](x') = 0$. The same argument holds at the other end $x_1 + \varphi_\sigma\ell/2$. Therefore, the set of equations giving the damage field in I_σ is the following one:

$$-\mathsf{S}'(\alpha)\sigma^2 + 2\mathsf{d}'(\alpha) - 4\mathsf{d}_1\ell^2\alpha'' = 0 \quad \text{in} \quad I_\sigma \setminus \{x_1\} \tag{61}$$

$$\alpha = \alpha_{\sigma_0}, \qquad \alpha' = \alpha'_{\sigma_0} \qquad \text{at} \quad \partial I_\sigma \tag{62}$$

$$\alpha = \alpha^*_\sigma \qquad \text{at} \quad x_1 \tag{63}$$

Introducing the variable ω, one still deduces from (61) that there exists a first integral and that the curve in the $(\omega, \ell\omega'/2)$ plane is still a circle. But the constant has changed and the first integral reads now

$$\frac{\ell^2}{4}\omega'^2 = r_\sigma^2 - (\omega - c_\sigma)^2 \quad \text{in} \quad I_\sigma \setminus \{x_1\} \tag{64}$$

where the center c_σ and the radius r_σ of the circle are related to the stress by

$$\frac{\sigma^2}{\bar{\sigma}_{\mathsf{D}}^2} = (1 - c_\sigma)^2 - r_\sigma^2. \tag{65}$$

(Note that the constant is the same on both parts of I_σ by symmetry.) The first integral gives

$$\omega = c_\sigma + r_\sigma \cos\psi, \qquad \ell^2\psi'^2 = 4 \quad \text{in} \quad I_\sigma \setminus \{x_1\}. \tag{66}$$

The continuity conditions (62) give

$$r_\sigma \sin\psi_\sigma = \frac{1}{2}\Big(1 - \frac{\sigma_0^2}{\bar{\sigma}_{\mathsf{D}}^2}\Big)\sin\varphi_\sigma, \qquad c_\sigma + r_\sigma\cos\psi_\sigma = \frac{1}{2}\Big(1 - \frac{\sigma_0^2}{\bar{\sigma}_{\mathsf{D}}^2}\Big)(1 + \cos\varphi_\sigma), \tag{67}$$

where $\psi_\sigma = \psi(x_1 - \varphi_\sigma\ell/2) = -\psi(x_1 + \varphi_\sigma\ell/2)$, see Figure 5. Finally, the plasticity condition (63) at x_1 reads as

$$1 - \Big(\frac{\sigma}{\theta\bar{\sigma}_{\mathsf{D}}}\Big)^{1/\mathsf{n}} = c_\sigma + r_\sigma\cos(\psi_\sigma - \varphi_\sigma). \tag{68}$$

Thus, (65), (67) and (68) constitute a set of four (non linear) equations for the four unknowns c_σ, r_σ, φ_σ

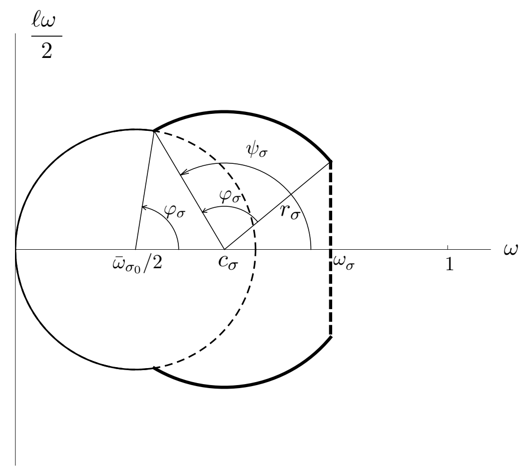

Figure 5: Construction of the solution in the phase plane at a given stress $\sigma < \sigma_0$. The thin circle represents the damage field at σ_0 (its dashed part corresponds to the interval I_σ where the damage has already evolved), the thick circle gives the new damage field at σ in I_σ, the dashed line corresponds to the jump of ω' at the center of the damage zone.

and ψ_φ. It turns out that it can be solved in a closed form and after tedious calculations one eventually obtains that the size of the still damaging zone is given by

$$\tan^2\frac{\varphi_\sigma}{2} = \frac{1 - \Big(\dfrac{\sigma}{\sigma_0}\Big)^{1/\mathsf{n}}}{\Big(\dfrac{\sigma}{\sigma_0}\Big)^{1/\mathsf{n}} - \Big(\dfrac{\sigma}{\sigma_0}\Big)^2}\frac{\sigma_0^2}{\bar{\sigma}_{\mathsf{D}}^2} \tag{69}$$

Passing to the limit when $\sigma \uparrow \sigma_0$ and using l'Hopital's rule, one sees that φ_σ tends to a finite limit, say φ_{σ_0}, given by

$$\tan^2 \frac{\varphi_{\sigma_0}}{2} = \frac{\theta^{-2/(2\mathsf{n}-1)}}{2\mathsf{n}-1},$$

which means that a finite part of but not the whole damage zone is unloaded when the plasticity occurs. Then φ_σ is increasing when σ decreases. When σ goes to 0, φ_σ tends to π and the whole initial damage zone is damaging again.

The evolution of the center c_σ of the circle is given by

$$2c_\sigma = 1 - \Big(\frac{\sigma}{\sigma_0}\Big)^2 + \Big(1 - \frac{\sigma_0^2}{\bar{\sigma}_{\mathsf{D}}^2}\Big)\Big(\frac{\sigma}{\sigma_0}\Big)^{1/\mathsf{n}}. \tag{70}$$

Therefore, using (64) and (65), one obtains

$$\frac{\ell}{2}\omega'(x_1\pm) = \mp\frac{\sigma_0}{\bar{\sigma}_{\mathsf{D}}}\left(1 - \Big(\frac{\sigma}{\sigma_0}\Big)^{1/\mathsf{n}}\right)^{1/2}\left(\Big(\frac{\sigma}{\sigma_0}\Big)^{1/\mathsf{n}} - \Big(\frac{\sigma}{\sigma_0}\Big)^2\right)^{1/2}$$

which proves that ω' is discontinuous at the center x_1 of the damage zone and hence that a cohesive crack has really be created. Inserting into (60) gives the cohesive law

$$[\![u]\!] = \frac{\mathsf{k}}{\mathsf{n}}\bar{\varepsilon}_{\mathsf{D}}\ell\left(1 - \Big(\frac{\sigma}{\sigma_0}\Big)^{1/\mathsf{n}}\right)^{1/2}\left(\Big(\frac{\sigma_0}{\sigma}\Big)^{(2\mathsf{n}-1)/\mathsf{n}} - 1\right)^{1/2}. \tag{71}$$

This law involves all the parameters of the model (the parameter θ entering in the definition of σ_0) and $\mathsf{k}\bar{\varepsilon}_{\mathsf{D}}\ell$ plays the role of a characteristic length which gives the order of magnitude of the crack opening. Note that $[\![u]\!]$ increases from 0 to infinity when σ decreases from σ_0 to 0, and hence $\sigma = 0$ is only reached asymptotically, see Figure 6.

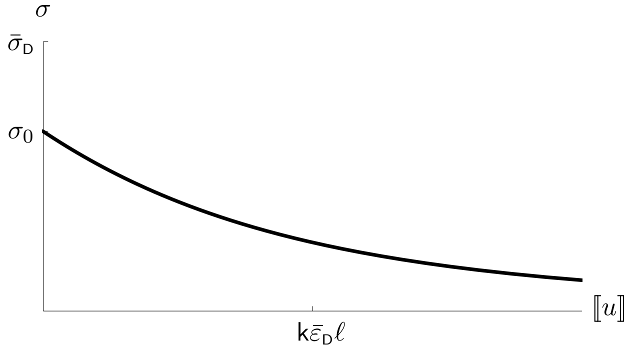

Figure 6: Case *II* with $\theta = 3/2$, $\mathsf{n} = 1$ and $\mathsf{k} = 4$: the cohesive law giving the relation between the stress and the jump of the displacement at the center of the damage zone once the plasticity occurs and remains concentrated at that point.

Finally the evolution of the damage field follows from (66). Indeed, ψ can read as $\psi(x) = \psi_\sigma - \varphi_\sigma + 2|x - x_1|/\ell$ and hence

$$\alpha(x) = 1 - \sqrt{1 - c_\sigma - r_\sigma \cos\left(\psi_\sigma - \varphi_\sigma + \frac{2|x-x_1|}{\ell}\right)}, \quad \forall x \in I_\sigma,$$

where r_σ and ψ_σ can be obtained from (65) and (67). The final expression is too long to be reproduced here, but one can see the evolution of the damage field in Figure 7 where the irreversibility condition $\dot{\alpha} \geq 0$ can be checked.

At $\sigma = 0$, *i.e.* when the rupture occurs, $\varphi_0 = \psi_0 = \pi$, $c_0 = r_0 = 1/2$ and we recover the final damage field

$$\alpha(x) = 1 - \sin\frac{|x - x_1|}{\ell} \quad \text{for} \quad x \in I$$

as in the case I. Accordingly, the total dissipated energy is also the same and equal to $\mathsf{G_c}$ given by (59). Indeed, the contribution to the damage field is the same since α is the same and the contribution of the cohesive crack goes to 0 when σ tends to 0. This latter point comes from

$$\sigma_{\mathsf{P}}(\alpha(x_1)) = \sigma_{\mathsf{P}}(\alpha^*_\sigma) = \sigma, \qquad \bar{P}(x_1) = [\![u]\!] \approx \frac{\mathsf{k}}{\mathsf{n}}\bar{\varepsilon}_{\mathsf{D}}\ell\left(\frac{\sigma_0}{\sigma}\right)^{1-1/2\mathsf{n}}$$

and hence $\lim_{\sigma\to 0}\sigma_{\mathsf{P}}(\alpha(x_1))\bar{P}(x_1) = 0$.

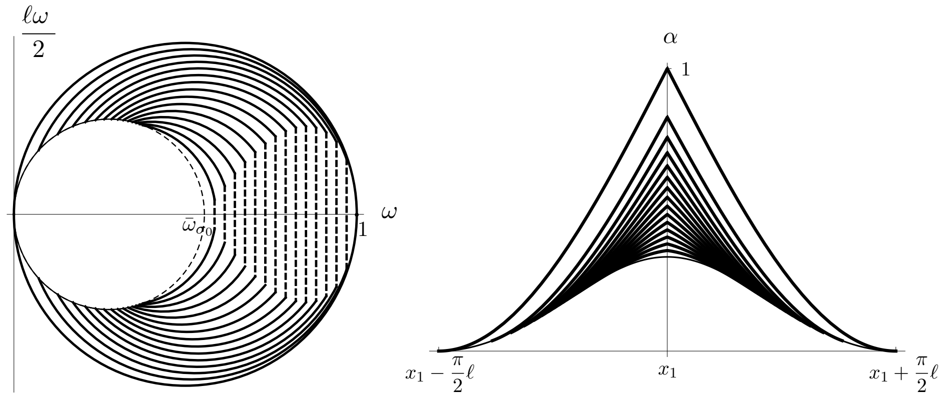

Figure 7: Case II with $\theta = 3/2$, $\mathsf{n} = 1$ and $\mathsf{k} = 4$: evolution of the damage after the plasticity has occured at the center. On the left, evolution in the phase plane $(\omega, \ell\omega'/2)$; on the right, evolution in the physical space. The presence of a cohesive crack can be seen through the jump of ω' in the phase space or the jump of α' in the physical space.

5.3. Cases where $\theta < 1$

We treat this case in a complete way only when $\mathsf{n} = 1$ because the solution can be obtained in a closed form and all the steps for constructing the solution are easier. The other cases will be briefly discussed at the end of the section.

5.3.1. Case III: $\theta < 1$ and $\mathsf{n} = 1$, plasticity stage followed by damage localization with nucleation and growth of a cohesive crack

Accordingly, the state function σ_{P} reads here as

$$\sigma_{\mathsf{P}}(\alpha) = \theta(1-\alpha)^2\bar{\sigma}_{\mathsf{D}}.$$

The damage localization process is preceded by a plasticity stage. Assuming that the plastic strain field is uniform during this stage, one has at the end of the plastic stage

$$\alpha(x) = 0, \quad p(x) = \bar{p}(x) = \pi_{\mathsf{DP}}(0) = \frac{(1-\theta^2)\mathsf{k}\bar{\varepsilon}_{\mathsf{D}}}{2\theta}, \quad \sigma = \theta\bar{\sigma}_{\mathsf{D}}.$$

When σ decreases from $\theta\bar{\sigma}_{\mathsf{D}}$ to 0, we search an evolution such that the damage grows in an interval centered at x_1, with the damage field maximal at x_1 and with x_1 as the unique possible singular point. Therefore, the plasticity can only evolve at x_1 by virtue of (48) and remains equal to $\pi_{\mathsf{DP}}(0)$ otherwise. Accordingly, by virtue of (52), the damage field must satisfy when $0 < \sigma < \theta\bar{\sigma}_{\mathsf{D}}$:

$$-\mathsf{S}'(\alpha)\sigma^2 + 2\mathsf{d}'(\alpha) + 2\sigma'_{\mathsf{P}}(\alpha)\pi_{\mathsf{DP}}(0) - 4\mathsf{d}_1\ell^2\alpha'' = 0 \quad \text{in} \quad I_\sigma \setminus \{x_1\}, \tag{72}$$

and

$$\alpha(x_1 \pm \Delta_\sigma) = \alpha'(x_1 \pm \Delta_\sigma) = 0, \tag{73}$$

where $I_\sigma = (x_1 - \Delta_\sigma, x_1 + \Delta_\sigma)$ denotes the damage zone and Δ_σ is its half-width which has to be determined. The conditions at x_1 depends on whether x_1 is singular or not, but in any case the plasticity criterion requires that

$$\sigma \leq \sigma_{\mathsf{P}}(\alpha(x_1)) = \theta(1-\alpha(x_1))^2\bar{\sigma}_{\mathsf{D}}. \tag{74}$$

Multiplying (72) by $2\alpha'$, one obtains a first integral with the constant given by (73). Specifically, one gets

$$2\mathsf{d}_1\ell^2\alpha'^2 = 2\mathsf{d}(\alpha) - 2(\sigma_{\mathsf{P}}(0) - \sigma_{\mathsf{P}}(\alpha))\pi_{\mathsf{DP}}(0) - (\mathsf{S}(\alpha) - \mathsf{S}(0))\sigma^2 \quad \text{in} \quad I_\sigma \setminus \{x_1\}.$$

This property holds for any plasticity-damage model. In the case of the models given by (39)-(40) with $\mathsf{n} = 1$, after introducing the variable ω, the first integral reads as

$$\ell^2\omega'^2 = 4\theta^2\omega(\bar{\omega}_\sigma - \omega) \quad \text{where} \quad \bar{\omega}_\sigma = 1 - \frac{\sigma^2}{\theta^2\bar{\sigma}_{\mathsf{D}}^2}.$$

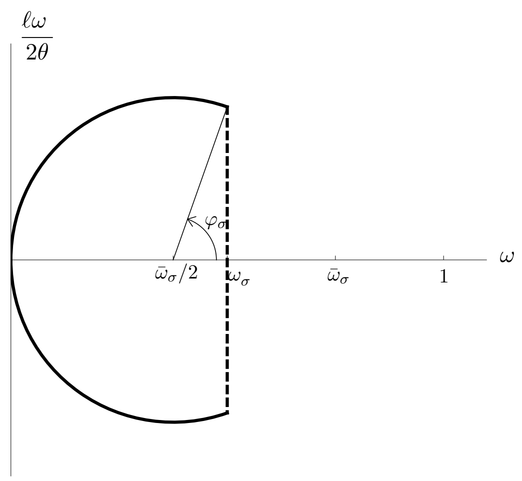

Figure 8: Case *III*: $\theta < 1$ and $\mathsf{n} = 1$, construction of the solution in the phase plane at a given stress $\sigma < \theta\bar{\sigma}_{\mathsf{D}}$. The circle represents the damage field and the dashed line corresponds to the jump of ω' at the center of the damage zone.

Hence, in the normalized phase plane $(\omega, \ell\omega'/2\theta)$, the first integral is a circle of center $\bar{\omega}_\sigma/2$ and radius $\bar{\omega}_\sigma/2$. The plasticity criterion (74) requires that

$$\omega(x_1) \leq \omega_\sigma^* := 1 - \frac{\sigma}{\theta\bar{\sigma}_{\mathsf{D}}}.$$

But since $\omega_\sigma^* < \bar{\omega}_\sigma$ as soon as $\sigma < \theta\bar{\sigma}_{\mathsf{D}}$, ω' is necessarily discontinuous at x_1 and hence x_1 is a singular point, see Figure 8. Therefore, a cohesive crack appears as soon as $\sigma < \theta\bar{\sigma}_{\mathsf{D}}$ and, by virtue of (54)-(55), one must have

$$\omega(x_1) = \omega_\sigma^*, \qquad \sigma_{\mathsf{P}}'(\alpha_\sigma^*)[\![u]\!](x_1) = 2\mathsf{d}_1\ell^2[\![\alpha']\!](x_1).$$

Setting $\omega(x) = \bar{\omega}_\sigma \cos^2\frac{\varphi(x)}{2}$ gives $\ell^2\varphi'^2 = 4\theta^2$. Therefore, the half width Δ_σ of the damage zone is given by

$$\Delta_\sigma = (\pi - \varphi_\sigma)\frac{\ell}{2\theta} \qquad \text{with} \qquad \varphi_\sigma = \arccos\frac{\theta\bar{\sigma}_{\mathsf{D}} - \sigma}{\theta\bar{\sigma}_{\mathsf{D}} + \sigma}.$$

So Δ_σ increases from $\pi\ell/4\theta$ to $\pi\ell/2\theta$ when σ goes from $\theta\bar{\sigma}_{\mathsf{D}}$ to 0. For a given σ, the damage profile in the damage zone is given by

$$\alpha(x) = 1 - \sqrt{1-\omega(x)} = 1 - \sqrt{1 - \bar{\omega}_\sigma \cos^2\left(\frac{\varphi_\sigma}{2} + \frac{\theta\,|x-x_1|}{\ell}\right)} \quad \text{in} \quad I_\sigma.$$

Since $\bar{\omega}_\sigma$ is increasing and φ_σ is decreasing (when σ decreases), the damage grows at given x and hence the irreversibility condition is satisfied. The damage evolution is represented on Figure 9 for $\theta = 2/3$.

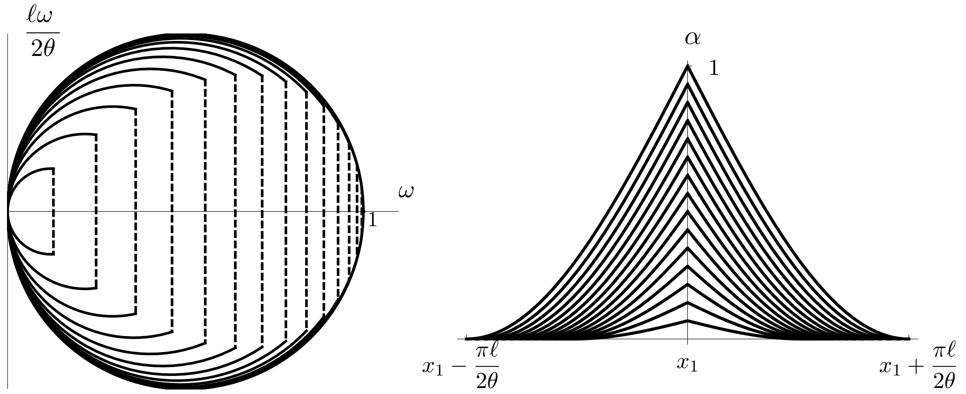

Figure 9: Case *III* with $\theta = 2/3$, $\mathsf{n} = 1$ and $\mathsf{k} = 4$: evolution of the damage after the plasticity stage. On the left, evolution in the phase plane $(\omega, \ell\omega'/2\theta)$; on the right, evolution in the physical space. The presence of a cohesive crack from the beginning of the damage process is visible on both spaces.

As long as the cohesive law is concerned, one gets

$$[\![u]\!] = \mathsf{k}\bar{\varepsilon}_{\mathsf{D}}\ell\left(\sqrt{\frac{\theta\bar{\sigma}_{\mathsf{D}}}{\sigma}} - \sqrt{\frac{\sigma}{\theta\bar{\sigma}_{\mathsf{D}}}}\right) \tag{75}$$

which is similar to that obtained in case *II*, see (71). However, the difference is that here the cohesive crack appears as soon as the damage starts.

When $\sigma = 0$, then $\varphi_0 = 0$, $\bar{\omega}_0 = 1$ and $\alpha(x_1) = 1$. A true crack has nucleated at x_1 and the damage profile is

$$\alpha(x) = 1 - \sin\left(\frac{\theta\,|x-x_1|}{\ell}\right) \quad \text{in} \quad \left(x_1 - \frac{\pi\ell}{2\theta}, x_1 + \frac{\pi\ell}{2\theta}\right), \qquad \text{when} \quad \sigma = 0. \tag{76}$$

That final profile differs from those obtained in the previous cases only by the size of the damage zone. The dissipated energy inside the damage zone at the end of the damage localization process, *i.e.* when $\sigma = 0$ is given by

$$\mathcal{D}_0 = \int_{x_1-\Delta_0}^{x_1+\Delta_0} \left(\mathsf{d}(\alpha(x)) + \sigma_{\mathsf{P}}(\alpha(x))\pi_{\mathsf{DP}}(0) + \mathsf{d}_1\ell^2\alpha'(x)^2\right) dx$$

with α given by (76). A part, namely $2\sigma_{\mathsf{P}}(0)\pi_{\mathsf{DP}}(0)\Delta_0$, was dissipated during the **P** stage. So, if we define $\mathsf{G_c}$ as the dissipated energy due to the damage process alone, then we obtain

$$\mathsf{G_c} := \int_{x_1-\Delta_0}^{x_1+\Delta_0} \left(\mathsf{d}(\alpha(x)) + (\sigma_{\mathsf{P}}(\alpha(x)) - \sigma_{\mathsf{P}}(0))\pi_{\mathsf{DP}}(0) + \mathsf{d}_1\ell^2\alpha'(x)^2\right) dx.$$

After some calculations, one gets

$$\mathsf{G_c} = \frac{\pi\mathsf{k}}{2}\frac{\theta\bar{\sigma}_{\mathsf{D}}^2}{\mathsf{E}_0}\ell$$

and hence this value differs from that of cases *I* and *II* by the factor θ only, see (59). (However, the fact that $\mathsf{n} = 1$ plays a role.) Note that this value of $\mathsf{G_c}$ involves all the parameters of the model.

5.3.2. Other cases with $\theta < 1$

For $\mathsf{n} > 0$, at $\sigma = \theta\bar{\sigma}_{\mathsf{D}}$, the initial plastic strain $\pi_{\mathsf{DP}}(0)$ reads as

$$\pi_{\mathsf{DP}}(0) = \frac{\mathsf{k}\bar{\varepsilon}_{\mathsf{D}}}{2\mathsf{n}}\frac{1-\theta^2}{\theta}.$$

For $\sigma < \theta\bar{\sigma}_{\mathsf{D}}$, the damage field is still governed by the first integral

$$2\mathsf{d}_1\ell^2\alpha'^2 = 2\mathsf{d}(\alpha) - 2(\sigma_{\mathsf{P}}(0) - \sigma_{\mathsf{P}}(\alpha))\pi_{\mathsf{DP}}(0) - (\mathsf{S}(\alpha) - \mathsf{S}(0))\sigma^2 \quad \text{in} \quad I_\sigma \setminus \{x_1\}$$

which reads now in terms of the variable ω as

$$\ell^2\omega'^2 = 4\mathsf{F}_\sigma(\omega)$$

with

$$\mathsf{F}_\sigma(\omega) := \omega\Big(1 - \frac{\sigma^2}{\bar{\sigma}_{\mathsf{D}}^2} - \omega\Big) - \frac{1-\theta^2}{\mathsf{n}}(1-\omega)(1-(1-\omega)^{\mathsf{n}}). \tag{77}$$

One easily checks that $\mathsf{F}_\sigma(0) = 0$ and F_σ is positive in an interval $(0, \bar{\omega}_\sigma)$ where $\bar{\omega}_\sigma < 1$ is the (first) positive zero of F_σ. This root depends both on n, θ and σ and must be obtained in general by a numerical procedure.

Moreover the plasticity criterion requires that

$$\omega(x_1)) \le \omega_\sigma^* := 1 - \Big(\frac{\sigma}{\theta\bar{\sigma}_{\mathsf{D}}}\Big)^{1/\mathsf{n}}. \tag{78}$$

Accordingly, x_1 is a singular point (and hence a cohesive crack nucleates) if $\omega_\sigma^* < \bar{\omega}_\sigma$ for some $\sigma < \theta\bar{\sigma}_{\mathsf{D}}$. On the other hand, if $\omega_\sigma^* \ge \bar{\omega}_\sigma$ for all $\sigma \le \theta\bar{\sigma}_{\mathsf{D}}$, then damage will localize around x_1 up to the rupture without concentration of the plastic strain at x_1. When $\mathsf{n} = 1$ we are in the first situation and even $\omega_\sigma^* < \bar{\omega}_\sigma$ for all $\sigma \in (0, \theta\bar{\sigma}_{\mathsf{D}})$. In the other cases, for discriminating between the two possibilities, one must carefully study the dependence of $\bar{\omega}_\sigma$ on n, θ and σ. It is outside the scope of this paper.

6. Review and Perspectives

Let us first recall the main ingredients of our models and the main properties which result. By virtue of our variational approach, all comes from the form of the total energy

$$\mathsf{W}(\varepsilon, \alpha, \alpha', p, \bar{p}) = \frac{1}{2}\mathsf{E}(\alpha)(\varepsilon - p)^2 + \mathsf{d}(\alpha) + \sigma_{\mathsf{P}}(\alpha)\bar{p} + \mathsf{d}_1\ell^2\alpha'^2$$

and the assumed properties of the state functions $\mathsf{E}(\alpha)$, $\mathsf{d}(\alpha)$ and $\sigma_{\mathsf{P}}(\alpha)$. The first important parameter of those models is the ratio between the critical damage stress $\bar{\sigma}_{\mathsf{D}}$ (which is given in terms of $\mathsf{E}(0)$, $\mathsf{E}'(0)$ and $\mathsf{d}'(0)$) and the critical plastic stress $\bar{\sigma}_{\mathsf{P}} = \sigma_{\mathsf{P}}(0)$. When $\bar{\sigma}_{\mathsf{P}} < \bar{\sigma}_{\mathsf{D}}$ plasticity precedes damage whereas when $\bar{\sigma}_{\mathsf{D}} < \bar{\sigma}_{\mathsf{P}}$ damage occurs first. However, the situation is not symmetric because damage necessarily happens in any case whereas it is possible that plasticity never occurs. Specifically, the stress softening assumption and the hypothesis that $\sigma_{\mathsf{P}}(\alpha)$ goes to 0 when α goes to 1 play an essential role in the qualitative properties of the model. Since both assumptions force the stress to decrease once the first damage occurs, homogeneous solutions are unstable and the damage will localize as for quasi-brittle materials. But here, because of the coupling with plasticity, there exists a competition between the plasticity criterion and the damage criterion. If the decrease of $\sigma_{\mathsf{P}}(\alpha)$ to 0 is not sufficiently fast, then damage localizes without plasticity (case *I*) and we recover the same results as for quasi-brittle materials. On the other hand, if the decrease of $\sigma_{\mathsf{P}}(\alpha)$ to 0 is sufficiently fast, then the plasticity criterion plays a role during the localization process (cases *II* and *III*). In this latter situation, the localization process of damage up to the rupture is quite different from that of quasi-brittle materials because a cohesive crack is necessarily created. The first cause is the stress softening condition which forces the damage to be non homogeneous and maximal at the center of the damage zone. In turn, plasticity can only evolve at that point and, since our model allows the plasticity to be concentrated (it is the concept of measure or equivalently of shear band like in perfect plasticity), then plasticity really localizes at the center of the damage zone. Consequently, a jump of the displacement occurs whereas the damage as not reached its ultimate value and the stress is not zero. It is important to note that the value of the jump of the displacement (and hence the cohesive law) is obtained in terms of the jump of the gradient of damage by virtue of the coupling term. This situation is really new by comparison with quasi-brittle materials and specific to our model which couples damage with plasticity. Moreover, it suffices to read the different steps of the construction of the solution to see the importance of the variational approach.

Let us finish now by some perspectives.

1. We have chosen here a form of the total energy which is the simplest one to couple damage with plasticity. Moreover we have considered almost all the cases that one can encounter with this type of energy. On one hand many other choices are possible and on the other hand some cases have no practical interest. (For instance, is the case $\bar{\sigma}_{\mathsf{D}} < \bar{\sigma}_{\mathsf{P}}$ really interesting?) Consequently, the natural task should be to propose a method of experimental identification of the different parameters of the model or more generally of the different state functions.
2. Even if the model has been presented here in a one-dimensional framework only, there is no difficulty to extend it in 3D. Indeed, it suffices to propose a form of the total energy which works in a vectorial setting. A natural extension could be to define W by

$$\mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha, \mathbf{p}, \bar{p}) = \frac{1}{2}\mathsf{E}(\alpha)(\boldsymbol{\varepsilon} - \mathbf{p}) \cdot (\boldsymbol{\varepsilon} - \mathbf{p}) + \mathsf{d}(\alpha) + \sigma_{\mathsf{P}}(\alpha)\bar{p} + \mathsf{d}_1\ell^2\nabla\alpha \cdot \nabla\alpha$$

where $\mathsf{E}(\alpha)$, $\boldsymbol{\varepsilon}$ and $\mathbf{p}$ stand now for the stiffness tensor, the strain tensor and the plastic strain tensors, respectively. The state functions d and σ_{P} remain unchanged, $\bar{p}$ still denotes the cumulated plastic strain but is now defined by $\dot{\bar{p}} = \sqrt{2\dot{\mathbf{p}} \cdot \dot{\mathbf{p}}/3}$ whereas $\nabla\alpha$ represents the gradient of damage. If one adopts the plastic incompressibility hypothesis, *i.e.* tr $\mathbf{p} = \mathbf{0}$, then the stability condition will give at the first order the Von Mises plastic yield criterion

$$\sqrt{\frac{3}{2}\boldsymbol{\sigma}^D \cdot \boldsymbol{\sigma}^D} \le \sigma_{\mathsf{P}}(\alpha),$$

where $\boldsymbol{\sigma}^D$ stands for the deviator of the stress tensor $\boldsymbol{\sigma} = \mathsf{E}(\alpha)(\boldsymbol{\varepsilon} - \mathbf{p})$. We will also obtain the equilibrium equations and the damage yield criterion will read as

$$\frac{1}{2}\mathsf{E}'(\alpha)(\boldsymbol{\varepsilon} - \mathbf{p}) \cdot (\boldsymbol{\varepsilon} - \mathbf{p}) + \mathsf{d}'(\alpha) + \sigma_{\mathsf{P}}'(\alpha)\bar{p} - 2\mathsf{d}_1\ell^2\Delta\alpha \ge 0,$$

where Δ stands for the Laplace operator. So, the three-dimensional model is a simple extension of the one-dimensional one. Of course, the properties of this model in 3D will be much more difficult to establish.

7. Conclusions

An important challenge in the variational approach to fracture is to propose a regularization of Dugdale-Barenblatt models (Dugdale (1960); Barenblatt (1962)) in the same manner as gradient damage models were proposed for regularize Griffith's model. It is really interesting from a numerical point of view to find such regularizations. Indeed, once this is done, it is no more necessary to consider discontinuous displacement fields across free surface of discontinuities. The jumps are replaced by high gradients and one can use the classical finite element method. Unfortunately, gradient damage models alone are not able to give rise to true cohesive cracks. Symmetrically, perfect plasticity models give rise to shear bands where the displacement is discontinuous, but the shear stress remains constant and hence these shear bands cannot be considered as true cohesive cracks. Our study shows that, by coupling damage with plasticity, the nucleation of a "Griffith's crack" is in general preceded by the nucleation of a true cohesive crack at the center of the damage zone since both the displacement is discontinuous and the stress progressively decreases to 0. Accordingly, gradient damage models coupled with plasticity could be good candidates. This favorable impression must be confirmed by further theoretical works and by numerical tests.

Appendix A. Proof of Propositions 1, 2, 3, 6

Proof of Proposition 1. Noting that the stress-strain relation (1) is equivalent to (7), let us first prove that (7)–(9) imply (2)–(3).

1. Take $p^* = p \pm h$ with $h > 0$ and insert it in (8). Dividing by h and passing to the limit when h goes to 0 give $\pm\sigma \le \sigma_{\mathsf{P}}$ which is precisely (2).
2. By virtue of (7) and (4), developing (9) by using the chain rule of differentiation of composite functions leads to $-\sigma\dot{p} + \sigma_{\mathsf{P}}\,|\dot{p}| = 0$. Hence $\dot{p} = 0$ if $|\sigma| < \sigma_{\mathsf{P}}$ and $\operatorname{sign}\dot{p} = \operatorname{sign}\sigma$ if $|\sigma| = \sigma_{\mathsf{P}}$ which are precisely (3).

Let us now prove that (1)–(3) imply (8)–(9).

1. Using (1), (4), (6) and the chain rule give $\dot{W} - \sigma\dot{\varepsilon} = -\sigma\dot{p} + \sigma_{\mathsf{P}}\,|\dot{p}|$. Then (3) leads to (9).
2. Let $p^* \in \mathbb{R}$. After easy calculations, one deduces from (6) that

$$\mathsf{W}_{\mathsf{P}}(\varepsilon, p^*, \bar{p} + |p^* - p|) - \mathsf{W}_{\mathsf{P}}(\varepsilon, p, \bar{p}) = \frac{1}{2}\mathsf{E}(p^* - p)^2 - \sigma(p^* - p) + \sigma_{\mathsf{P}}\,|p^* - p|$$

and (8) follows from (2) and the positivity of E. □

Proof of Proposition 2. Noting that the stress-strain relation (10) is equivalent to (23), let us first prove that (23)–(26) imply (12)–(13). The case when $\alpha = 1$ being trivial, we can assume that $0 \le \alpha < 1$.

1. Take $\alpha^* = \alpha + h$ with $h > 0$ small enough and insert it in (25). Dividing by h and passing to the limit when h goes to 0 give (12).
2. By virtue of (21), (22) and (23), developing (26) with the chain rule of differentiation of composite functions gives (13).

Let us now prove that (10)–(13) imply (25)–(26).

1. Using (10), (22) and the chain rule gives $\dot{W} - \sigma\dot{\varepsilon} = \left(\frac{1}{2}\mathsf{E}'(\alpha)\varepsilon^2 + \mathsf{d}'(\alpha)\right)\dot{\alpha}$. Then (13) leads to (26).
2. One deduces from (22) that

$$\mathsf{W}_{\mathsf{P}}(\varepsilon, \alpha^*) - \mathsf{W}_{\mathsf{P}}(\varepsilon, \alpha) = \mathsf{d}(\alpha^*) - \mathsf{d}(\alpha) - \frac{\varepsilon^2}{2}\big(\mathsf{E}(\alpha) - \mathsf{E}(\alpha^*)\big). \tag{A.1}$$

By virtue of (14) and (18), $2\mathsf{d}'(\beta) \ge -\varepsilon_{\mathsf{D}}(\alpha)^2\mathsf{E}'(\beta)$ for $\alpha \le \beta \le \alpha^*$. Integrating this inequality over $[\alpha, \alpha^*]$ gives $2(\mathsf{d}(\alpha^*) - \mathsf{d}(\alpha)) > \varepsilon_{\mathsf{D}}(\alpha)^2(\mathsf{E}(\alpha) - \mathsf{E}(\alpha^*))$. Inserting into (A.1) and using both (14) and (17) give (26). □

Proof of Proposition 3 We only detail the proofs of the most delicate parts.

1. By virtue of the initial condition and by continuity, neither the plastic strain nor the damage evolve at the beginning of the test. When $\bar{\varepsilon}_{\mathsf{D}} > \bar{\varepsilon}_{\mathsf{P}}$ the plastic yield criterion is reached the first, when $\bar{\varepsilon}_{\mathsf{D}} < \bar{\varepsilon}_{\mathsf{P}}$ the damage yield criterion is reached the first and when $\bar{\varepsilon}_{\mathsf{D}} = \bar{\varepsilon}_{\mathsf{P}}$ both criteria are reached at the same time.
2. It suffices to prove that there does not exist $\varepsilon_0 > \bar{\varepsilon}_{\mathsf{I}}$ such that $\alpha_{\varepsilon_0} < 1$ and $\sigma_{\varepsilon_0} = 0$. Assume that there exist $\varepsilon_i > \bar{\varepsilon}_{\mathsf{I}}$ such that $\sigma_{\varepsilon_i} = 0$. By continuity of $\varepsilon \mapsto \sigma_\varepsilon$ the set of such ε_i is closed and hence admits an infimum, say ε_0. If $\alpha_{\varepsilon_0} = 1$, then there is nothing to prove by virtue of the irreversibility condition. Consider the case $\alpha_{\varepsilon_0} < 1$. Then $\varepsilon_0 = p_{\varepsilon_0}$. By continuity, there exists an interval $(\varepsilon_0 - h, \varepsilon_0)$ with $h > 0$ such that $\sigma_{\mathsf{P}}(\alpha_\varepsilon) > \sigma_\varepsilon > 0$ for all ε in that interval. By virtue of the plasticity rule, p_ε remains constant in this interval and hence equal to $p_{\varepsilon_0} = \varepsilon_0$. But, in the interval $(\varepsilon_0 - h, \varepsilon_0)$, we should have both $\sigma_\varepsilon > 0$ and $\sigma_\varepsilon = \mathsf{E}(\alpha_\varepsilon)(\varepsilon - \varepsilon_0) < 0$ which is impossible. Hence either $\alpha_{\varepsilon_0} = 1$ or there does not exist a time where σ vanishes.

3. Let us prove the property by contradiction. Assume there exists $\varepsilon > \bar{\varepsilon}_{\mathsf{I}}$ such that $\alpha_\varepsilon < 1$ with both $f_{\mathsf{D}}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) < 0$ and $f_{\mathsf{P}}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) < 0$. Then, by continuity, that remains true in an interval $(\varepsilon_0, \varepsilon_1)$. Consequently, $(\alpha_\varepsilon, p_\varepsilon) = (\alpha_{\varepsilon_0}, p_{\varepsilon_0})$ during this interval. Taking for ε_0 the greatest lower bound, then necessarily $f_{\mathsf{D}}(\varepsilon_0, \alpha_{\varepsilon_0}, p_{\varepsilon_0}) = 0$ or $f_{\mathsf{P}}(\varepsilon_0, \alpha_{\varepsilon_0}, p_{\varepsilon_0}) = 0$. But since
$$\forall \varepsilon > \varepsilon_0, \quad f_{\mathsf{D}}(\varepsilon, \alpha_{\varepsilon_0}, p_{\varepsilon_0}) > f_{\mathsf{D}}(\varepsilon_0, \alpha_{\varepsilon_0}, p_{\varepsilon_0}), \quad f_{\mathsf{P}}(\varepsilon, \alpha_{\varepsilon_0}, p_{\varepsilon_0}) > f_{\mathsf{P}}(\varepsilon_0, \alpha_{\varepsilon_0}, p_{\varepsilon_0}),$$
at least one yield criterion is violated in the interval $(\varepsilon_0, \varepsilon_1)$, which is a contradiction.
4. Also by contradiction. If the damage yield criterion was never reached, we should have $\alpha_\varepsilon = 0$ for all $\varepsilon \geq 0$. Then $f_{\mathsf{P}}(\varepsilon, 0, p_\varepsilon) \leq 0$ gives $p_\varepsilon \geq \varepsilon - \bar{\varepsilon}_{\mathsf{P}}$. Consequently, $f_{\mathsf{D}}(\varepsilon, 0, p_\varepsilon) \geq -\mathsf{d}'(0) + |\sigma_{\mathsf{P}}'(0)|\,(\varepsilon - \bar{\varepsilon}_{\mathsf{P}})$ and hence $f_{\mathsf{D}}(\varepsilon, 0, p_\varepsilon) > 0$ for ε large enough, which is a contradiction.
5. Still by contradiction. Assume there exists $\varepsilon > \varepsilon_d$ such that $\alpha_\varepsilon < 1$ and $f_{\mathsf{D}}(\varepsilon, \alpha_\varepsilon, p_\varepsilon) < 0$. Then, by continuity, that remains true in an interval $(\varepsilon_0, \varepsilon_1)$. Taking for ε_0 the greatest lower bound, then necessarily $f_{\mathsf{D}}(\varepsilon_0, \alpha_{\varepsilon_0}, p_{\varepsilon_0}) = 0$. Moreover $\alpha_\varepsilon = \alpha_{\varepsilon_0}$ and, by virtue of Property 3, $f_{\mathsf{P}}(\varepsilon, \alpha_{\varepsilon_0}, p_\varepsilon) = 0$ in the interval $(\varepsilon_0, \varepsilon_1)$. Hence $p_\varepsilon = \varepsilon - \sigma_{\mathsf{P}}(\alpha_{\varepsilon_0})/\mathsf{E}(\alpha_{\varepsilon_0})$ for $\varepsilon \in (\varepsilon_0, \varepsilon_1)$. Consequently,
$$\forall \varepsilon \in (\varepsilon_0, \varepsilon_1), \quad f_{\mathsf{D}}(\varepsilon, \alpha_{\varepsilon_0}, p_\varepsilon) = |\sigma_{\mathsf{P}}'(\alpha_{\varepsilon_0})|\,(\varepsilon - \varepsilon_0) > 0,$$
which is a contradiction.
6. Let $\alpha_m = \lim_{\varepsilon \to \infty} \alpha_\varepsilon$, by the irreversibility condition we have $0 \leq \alpha_\varepsilon \leq \alpha_m$ for all $\varepsilon \geq 0$. If $\alpha_m < 1$, then we should deduce from the damage and the plasticity yield criteria that p_ε and $\varepsilon - p_\varepsilon$ are uniformly bounded:
$$\forall \varepsilon \geq 0, \quad p_\varepsilon \leq \max_{\alpha \in [0, \alpha_m]} \frac{\mathsf{d}'(\alpha)}{|\sigma_{\mathsf{P}}'(\alpha)|}, \quad \varepsilon - p_\varepsilon \leq \max_{\alpha \in [0, \alpha_m]} \frac{\sigma_{\mathsf{P}}(\alpha)}{\mathsf{E}(\alpha)}.$$
Hence ε should be bounded, which is a contradiction. So $\alpha_m = 1$. Note that, according to some particular properties of the state functions E, d and σ_{P}, $\alpha = 1$ could be reached at a finite strain ε_r. In such a case, $\alpha_\varepsilon = 1$, $\sigma_\varepsilon = 0$ and p_ε remains undetermined for $\varepsilon \geq \varepsilon_r$.

Proof of Proposition 6. The proof is a simple adaptation of the proofs of Propositions 1 and 2. At given α_ε, it suffices to use Proposition 1 to obtain the desired inequality. At given p_ε, one can repeat the proof of Proposition 2 with $\mathsf{d}(\alpha^*) + \sigma_{\mathsf{P}}(\alpha^*) p_\varepsilon$ instead of $\mathsf{d}(\alpha^*)$. The key point in the proof is the strain hardening condition which reads now
$$\alpha \mapsto \frac{\mathsf{d}'(\alpha) + \sigma_{\mathsf{P}}'(\alpha) p_\varepsilon}{|\mathsf{E}'(\alpha)|} \text{ must be monotonically increasing.}$$
By definition of π_{D}, one gets
$$\frac{\mathsf{d}'(\alpha^*) + \sigma_{\mathsf{P}}'(\alpha^*) p_\varepsilon}{|\mathsf{E}'(\alpha^*)|} = \frac{\mathsf{d}'(\alpha^*)}{|\mathsf{E}'(\alpha^*)|} \left(1 - \frac{p_\varepsilon}{\pi_{\mathsf{D}}(\alpha^*)}\right)$$
and hence the monotonicity property is ensured by (18) provided that π_{D} is non decreasing.

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