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# Geometry and analysis of control-affine systems: motion planning, heat and Schrödinger evolution

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# Geometry and analysis of control-affine systems: motion planning, heat and Schrödinger evolution

## THÈSE DE DOCTORAT

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**Docteur de l'École Polytechnique**  
(spécialité mathématiques appliquées)

par

Dario Prandi

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To Meg.

*Knowledge and ability were tools,  
not things to show off.  
– Haruki Murakami, 1Q84*



Cette thèse traite de deux problèmes qui ont leur origine dans la théorie du contrôle géométrique, et qui concernent les systèmes de contrôle avec dérive, c'est-à-dire de la forme  $\dot{q} = f_0(q) + \sum_{j=1}^m u_j f_j(q)$ . Dans la première partie de la thèse, on généralise le concept de complexité de courbes non-admissibles, déjà bien compris pour les systèmes sous-riemanniens, au cas des systèmes de contrôle avec dérive, et on donne des estimations asymptotiques de ces quantités. Ensuite, dans la deuxième partie, on considère une famille de systèmes de contrôle sans dérive en dimension 2 et on s'intéresse à l'opérateur de Laplace-Beltrami associé et à l'évolution de la chaleur et des particules quantiques qu'il définit. On étudie plus particulièrement l'effet qu'a l'ensemble où les champs de vecteurs contrôlés deviennent colinéaires sur ces évolutions.

This thesis is dedicated to two problems arising from geometric control theory, regarding control-affine systems  $\dot{q} = f_0(q) + \sum_{j=1}^m u_j f_j(q)$ , where  $f_0$  is called the drift. In the first part we extend the concept of complexity of non-admissible trajectories, well understood for sub-Riemannian systems, to this more general case, and find asymptotic estimates. Then, in the second part of the thesis, we consider a family of 2-dimensional driftless control systems. For these, we study how the set where the control vector fields become collinear affects the evolution of the heat and of a quantum particle with respect to the associated Laplace-Beltrami operator.



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**Part I.**

**Preliminaries**



# 1. Introduction

Dynamics, from how a car moves up to the evolution of a quantum particle, are modelled in general by differential equations. Control theory deals with dynamics where it is possible to act on some part of the equation by means of controls, e.g., how to park a car or how to steer a quantum particle to a desired state.

More precisely, a control system on a smooth manifold  $M$  is an ordinary differential equation in the form

$$\dot{q}(t) = f(q(t), u(t)), \quad (1.1)$$

where  $u : [0, T] \rightarrow U$  is an integrable and bounded function – called control – taking values in some set  $U \subset \mathbb{R}^m$ , and  $f : M \times U \rightarrow TM$  is a continuous function such that  $f(\cdot, u)$  is a smooth vector field for each  $u \in U$ . Thus, fixing a control and an initial point  $q_0$ , system (1.1) has a unique maximal solution  $q_u(t)$ . Every curve  $\gamma : [0, T] \rightarrow M$  that can be written as solution of system (1.1) for some control  $u$  and with starting point  $\gamma(0)$ , is said to be *admissible*.

With a control system it is possible to associate an *optimal control problem*. Namely, one considers a *cost*  $J : (u, T) \mapsto [0, +\infty)$ , where  $T > 0$  and  $u \in L^1([0, T], \mathbb{R}^m)$ . Then, given two points  $q_0, q_1 \in M$  one is interested in minimizing the functional  $J$  among all admissible controls  $u \in L^1([0, T], \mathbb{R}^m)$ ,  $T > 0$ , for which the corresponding solution of (1.1) with initial condition  $q_u(0) = q_0$  is such that  $q_u(T) = q_1$ . This is written as

$$\begin{cases} \dot{q}(t) = f(q(t), u(t)), \\ q(0) = q_0, \quad q(T) = q_1, \\ J(u, T) \longrightarrow \min. \end{cases} \quad (1.2)$$

The final time  $T$  can either be fixed, or free to be selected in a certain interval of time.

From the optimal control problem associated with a cost  $J$ , one defines the value function  $V : M \times M \rightarrow [0, +\infty]$ . This is a function that associates to every pair of points  $q_0, q_1 \in M$  the infimum of the cost of controls admissible for the corresponding optimal control problem (1.2). If there are no such controls, then  $V(q_0, q_1) = +\infty$ .

The aim of this thesis is to study two different problems arising from control theory, regarding control-affine systems with unbounded controls, i.e., with  $U = \mathbb{R}^m$ . A control-affine system on a smooth manifold  $M$  is a control system in the form

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^m u_i(t) f_i(q(t)), \quad (1.3)$$

where,  $u : [0, T] \rightarrow \mathbb{R}^m$  is an integrable control function and  $\{f_0, f_1, \dots, f_m\}$  is a family of smooth vector fields. The vector fields  $f_1, \dots, f_m$  are called control vector fields,

while  $f_0$  is called the *drift*. For most of the dissertation we will consider as a cost  $\mathcal{J} : (u, T) \mapsto [0, +\infty)$  the  $L^1$ -norm of  $u$ . Namely we will be interested in the optimal control problem

$$\begin{cases} \dot{q}(t) = f_0(q(t)) + \sum_{i=1}^m u_i(t) f_i(q(t)), \\ q(0) = q_0, \quad q(T) = q_1, \\ \mathcal{J}(u, T) = \int_0^T \sqrt{\sum_{i=1}^m u_i(t)^2} dt \longrightarrow \min. \end{cases} \quad (1.4)$$

From a mathematical point of view, these systems describe the underlying geometry of hypoelliptic operators, as we will see later. In applications, they appear in the study of many mechanical systems, from the already mentioned car parking problem up to most kind of robot motion planning, and recently in research fields such as mathematical models of human behavior, quantum control or motion of self-propulsed micro-organism (see [ADL08, BDJ<sup>+</sup>08, BCG02a]). A suggestive application of these systems and of hypoelliptic diffusions, in the particular case where  $f_0 \equiv 0$ , appeared in the field of cognitive neuroscience to model the functional architecture of the area V1 of the primary visual cortex, as proposed by Petitot, Citti, and Sarti [PT99, Pet09, CS06].

We will focus on the following two general problems for these kind of systems.

1. *Complexity of non-admissible trajectories.* A common issue in control theory, used for example in robot motion planning, is to steer the system along a given curve  $\Gamma$ . Since, in general,  $\Gamma$  is not admissible, i.e., it is not a solution of system (1.3), the best one can do is to steer the system along an approximating trajectory. The first part of the thesis is dedicated to quantify the cost of this approximation – called complexity – depending on the relation between  $\Gamma$  and (1.3). As a preliminary step, it is necessary to study the value function associated with the optimal control problem (1.4), estimating its behavior along the curve  $\Gamma$ . This research appears in two papers:
  - P1. D. Prandi, *Hölder continuity of the value function for control-affine systems*, arXiv:1304.6649 [math.OC].
  - P2. F. Jean, D. Prandi, *Complexity of control-affine motion planning*, arXiv:1309.2571 [math.OC].
2. *Singular diffusions.* In the second part of the thesis we will focus on a family of two dimensional driftless control systems in the form (1.3), to which it is possible to associate intrinsically a Laplace-Beltrami operator. Due to the control vector fields becoming collinear on a curve  $\mathcal{Z}$ , this operator will present some singularities. Our interest lies on how  $\mathcal{Z}$  affects the diffusion dynamics. In particular, we will try to understand if solutions to the heat and Schrödinger equations associated with this Laplace-Beltrami operator are able to cross  $\mathcal{Z}$ , and whether some heat is absorbed in the process or not. This research appears in:

P3. U. Boscain, D. Prandi, *The heat and Schrödinger equations on conic and anticonic-type surfaces*, arXiv:1305.5271 [math.AP].

Recent results on this topic, that have been part of the research developed during the PhD, but are not presented here, are contained in the work in progress:

P4. A. Posilicano, D. Prandi, *A cornucopia of self-adjoint extensions*, in preparation.

The mathematical motivation of the problems considered in this thesis lies in sub-Riemannian geometry. Thus, we will use the next section to introduce this topic. Our contributions will then be described in Sections 1.2 and 1.3, while Section 1.4 is devoted to expose some open problems and future lines of research.

## 1.1. Sub-Riemannian geometry

Sub-Riemannian geometry can be thought of as a generalization of Riemannian geometry, where the dynamics is subject to non-holonomic constraints. Classically (see, e.g., [Mon02]), a sub-Riemannian structure on  $M$  is defined by a smooth vector distribution  $\Delta \subset TM$  – i.e., a sub-bundle of  $TM$  – of constant rank  $k$  and a Riemannian metric  $\mathbf{g}$  defined on  $\Delta$ . From this structure, one derives the so-called *Carnot-Carathéodory distance*  $d_{\text{SR}}$  on  $M$ . The length of any absolutely continuous path tangent to the distribution – called *horizontal* – is defined through the Riemannian metric, and the distance  $d_{\text{SR}}(q_0, q_1)$  is then defined as the infimum of the length of all horizontal paths joining  $q_0$  to  $q_1$ . If no such path exists,  $d_{\text{SR}}(q_0, q_1) = +\infty$ .

Locally, it is always possible to find an orthonormal frame  $\{f_1, \dots, f_m\}$  for  $\Delta$ . This allows to identify horizontal trajectories with admissible trajectories of the *non-holonomic control system*

$$\dot{q}(t) = \sum_{i=1}^m u_i(t) f_i(q(t)). \quad (1.5)$$

The problem of finding the shortest curve joining two fixed points  $q_0, q_1 \in M$  is then naturally formulated as the optimal control problem

$$\begin{cases} \dot{q}(t) = \sum_{i=1}^m u_i(t) f_i(q(t)), \\ q(0) = q_0, \quad q(T) = q_1, \\ \mathcal{J}(u, T) = \int_0^T \sqrt{\sum_{i=1}^m u_i(t)^2} dt \longrightarrow \min. \end{cases} \quad (1.6)$$

With this point of view, the Carnot-Carathéodory distance is the value function associated with (1.6).

This framework is however more general than classical sub-Riemannian geometry. Indeed, choosing  $f_1, \dots, f_m$  to be possibly non-linearly independent, this optimal control



formulation allows to define sub-Riemannian structures endowed with a *rank-varying distribution*  $\Delta(q) = \text{span}\{f_1(q), \dots, f_m(q)\}$ . Namely, it is possible to define a Riemannian norm on  $\Delta(q)$  as

$$\|v\|_q = \min \left\{ |u| \mid v = \sum_{i=1}^m u_i f_i(q) \right\}, \quad \text{for any } v \in \Delta(q),$$

from which the metric  $\mathbf{g}_q$  follows by polarization. Through this metric we obtain the Carnot-Carathéodory distance, coinciding with the value function of the optimal control problem associated with the non-holonomic control system, as in the classical case. Since it is well known that every distribution can be globally represented as the linear span of a finite family of (possibly not linearly independent) vector fields (see [Sus08, ABB12a, DLPR12]), it is always possible to represent globally a sub-Riemannian structure as a non-holonomic system.

Although it is outside the scope of the following discussion, we remark that this control theoretical setting can be stated in purely geometrical terms, as done in [ABB12a].

### 1.1.1. Metric properties

Once the Carnot-Carathéodory distance is defined, the first natural question is: is it finite? This amounts to ask if every pair of points of  $M$  is joined by an horizontal curve. This property, in the control theoretic language, is known as *controllability* or *accessibility*.

A partial answer (for analytic corank-one distributions) can be found in Carathéodory paper [Car09] on formalization of classical thermodynamics, where the role of horizontal curves is roughly taken by adiabatic processes<sup>1</sup>. However it is not until the 30's, that Rashevsky [Ras38] and Chow [Cho39] independently extended Carathéodory result to a general criterion for smooth distributions. The key assumption of this theorem is the *Hörmander condition* (or *Lie bracket-generating condition*) for  $\Delta$ , i.e., that the Lie algebra generated by the horizontal vector fields spans at any point the whole tangent space.

**Theorem 1** (Chow-Rashevsky Theorem). *Let  $M$  be a connected sub-Riemannian manifold, such that  $\Delta$  satisfies the Hörmander condition. Then, the Carnot-Carathéodory distance is finite, continuous, and induces the manifold topology.*

Heuristically, the Chow-Rashevsky theorem is a consequence of the fact that, in coordinate representation,

$$e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q) = q + t^2[f, g](q) + o(t^2). \quad (1.7)$$

Iterating this procedure shows that, if the Lie bracket-generating condition is satisfied, it is possible to move in every direction and hence to connect every couple of points on  $M$ .

---

<sup>1</sup>Indeed, the proof of this fact relies on the theory of Carnot cycles. This is why the sub-Riemannian distance is known as “Carnot-Carathéodory” distance.

Let us remark that the converse is not true without assuming  $M$  and  $\Delta$  to be analytic (see [Nag66]). From now on, we will always assume the Lie bracket-generating condition to be satisfied.

Although finite, the Carnot-Carathéodory distance presents a quite different behavior than the Riemannian one. It is a basic fact of Riemannian geometry that small balls around a fixed point are, when looked in coordinates, roughly Euclidean. This isotropic behavior is essentially due to the fact that geodesics tangent vectors are parametrized on the Euclidean sphere in the tangent space. In sub-Riemannian geometry this is no more true, and as a consequence the Carnot-Carathéodory distance is highly anisotropic. Indeed, in order to move in directions that are not contained in the distribution, it is necessary to construct curves like (1.7). This suggest that the number of brackets we have to build to attain a certain direction is directly related to the cost of moving in that direction.

In order to exploit this fact, it is necessary to choose an appropriate coordinate system. Let  $\Delta^1 = \Delta$  and define recursively  $\Delta^{s+1} = \Delta^s + [\Delta^s, \Delta]$ , for every  $s \in \mathbb{N}$ . By the Hörmander condition, the evaluations of the sets  $\Delta^s$  at  $q$  form a flag of subspaces of  $T_qM$ ,

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^r(q) = T_qM. \quad (1.8)$$

The integer  $r = r(q)$ , which is the minimum number of brackets required to recover the whole  $T_qM$  is called *degree of non-holonomy* (or *step*) of  $\Delta$  at  $q$ . Finally, let  $w_1 \leq \dots \leq w_n$  be the *weights* associated with the flag, defined by  $w_i = s$  if  $\dim \Delta^{s-1}(q) < i \leq \dim \Delta^s(q)$ , setting  $\dim \Delta^0(q) = 0$ . A system of coordinates  $z = (z_1, \dots, z_n)$  at  $q$  is *privileged* whenever the non-holonomic order of  $z_i$  is exactly  $w_i$  – i.e., if  $f_{i_1} \dots f_{i_{w_i}} z_i = 0$  for any  $\{i_1, \dots, i_{w_i}\} \subset \{1, \dots, m\}$  but  $f_{i_1} \dots f_{i_{w_i}} f_{i_{w_i+1}} z_i \neq 0$  for some  $\{i_1, \dots, i_{w_i}, i_{w_i+1}\} \subset \{1, \dots, m\}$ . In particular, any system of privileged coordinates at  $q$  induces a splitting of the tangent space as a direct sum,

$$T_qM = \Delta^1(q) \oplus \Delta^2(q) / \Delta^1(q) \oplus \dots \oplus \Delta^r(q) / \Delta^{r-1}(q),$$

where each  $\Delta^s(q) / \Delta^{s-1}(q)$  is spanned by  $\partial_{z_i}|_q$  with  $w_i = s$ .

Starting from the 80's, various authors exploited privileged coordinates to obtain the following result, showing the strong anisotropy of the Carnot-Carathéodory distance. For early versions see [NSW85, Ger90, Gro96], while a general and detailed proof can be found in [Bel96].

**Theorem 2** (Ball-box Theorem). *Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q \in M$ . Then, there exist  $C, \varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$ , it holds*

$$\text{Box} \left( \frac{1}{C} \varepsilon \right) \subset \text{B}_{\text{SR}}(q, \varepsilon) \subset \text{Box}(C\varepsilon).$$

Here, we let  $\text{B}_{\text{SR}}(q, \varepsilon)$  be both the sub-Riemannian ball of radius  $\varepsilon > 0$  centered in  $q$  and its coordinate representation  $z(\text{B}_{\text{SR}}(q, \varepsilon))$ . Moreover, we let

$$\text{Box}(\varepsilon) = \{x \in \mathbb{R}^n \mid |x_i| \leq \varepsilon^{w_i}\}. \quad (1.9)$$

An immediate consequence of this theorem is the Hölder equivalence of the Carnot-Carathéodory distance and the Euclidean one. Namely, in any coordinate system centered at  $q$  and for  $q'$  sufficiently close to  $q$ , it holds

$$|q' - q| \lesssim d_{\text{SR}}(q, q') \lesssim |q' - q|^{\frac{1}{r}}. \quad (1.10)$$

Here we used “ $\lesssim$ ” to denote an inequality up to a multiplicative constant.

As a consequence of the anisotropy of the distance, the Hausdorff dimension of a sub-Riemannian manifold is different from its topological dimension. A point  $q$  is said to be *regular* if  $\dim \Delta^s$  is constant near  $q$  for any  $1 \leq s \leq r$ . If every point is regular, then the sub-Riemannian manifold is said to be *equivregular*. This allows to prove the following celebrated theorem [Mit85].

**Theorem 3** (Mitchell’s measure theorem). *The Hausdorff dimension  $\dim_q^{\mathcal{H}} M$  of a sub-Riemannian manifold at a regular point  $q$  is given by*

$$\dim_q^{\mathcal{H}} M = \sum_{s=1}^r s(\dim \Delta^s(q) - \dim \Delta^{s-1}(q)).$$

*In particular, if  $\dim \Delta(q) < \dim M$ , then  $\dim M < \dim_q^{\mathcal{H}} M$ . Moreover, the  $(\dim_q^{\mathcal{H}} M)$ -dimensional Hausdorff measure is absolutely continuous with respect to any smooth volume, near  $q$ .*

The theorem has been proved only at regular points since near these points the Ball-Box Theorem holds with uniform constants. See [GJ13] for some more general results in this direction.

### 1.1.2. Complexity and motion planning

The concept of complexity was first developed for the non-holonomic motion planning problem in robotics. Given a non-holonomic control system on a manifold  $M$ , the motion planning problem consists in finding an admissible trajectory connecting two points, usually under further requirements, such as obstacle avoidance. Since with the control system is associated a distance, it makes sense to try to find the shortest trajectory.

Different approaches are possible to solve this problem (see [LSL98]). Here we focus on those based on the following algorithm:

1. try to find any (usually non-admissible) path  $\Gamma$  solving the problem,
2. try to approximate  $\Gamma$  with admissible paths.

The first step is independent of the sub-Riemannian structure under consideration, since it depends only on the topology of the manifold and of the obstacles. Thus, we are interested in the second step, which depends only on the local nature of the control system near the path. In particular, we try to measure the complexity of the approximation task. By complexity we mean a function of the non-admissible curve  $\Gamma \subset M$  (or path

$\gamma : [0, T] \rightarrow M$ ), and of the precision of the approximation, quantifying the difficulty of the latter by means of the cost function. Let us remark that, especially on equiregular sub-Riemannian manifolds, the asymptotic behavior of  $\sigma(\Gamma, \varepsilon)$  as  $\varepsilon \downarrow 0$  is strictly related with the Hausdorff dimension  $\dim^{\mathcal{H}} \Gamma$ .

Let  $\Gamma \subset M$  be a curve with endpoints  $x$  and  $y$ , i.e., a one dimensional submanifold with boundary of  $M$  diffeomorphic to a closed interval. For  $\varepsilon > 0$ , we let an  $\varepsilon$ -cost interpolation of  $\Gamma$  to be any horizontal curve  $\eta : [0, T] \rightarrow M$  with  $\eta(0) = x$  and  $\eta(T) = y$  such that in any segment of  $\eta$  of length  $\geq \varepsilon$ , there is a point of  $\Gamma$ . Moreover, let  $\text{Tube}(\Gamma, \varepsilon)$  to be the sub-Riemannian tubular neighborhood of radius  $\varepsilon$  around the curve  $\Gamma$ . Following Gromov [Gro96, p. 278] and Jean [Jea01a], we give two possible definitions of metric complexity of  $\Gamma$ .

- *Interpolation by cost complexity:*

$$\sigma_c(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \{ \text{length}(\eta) \mid \eta \text{ is an } \varepsilon\text{-cost interpolation of } \Gamma \}.$$

This function measures the number of pieces of cost  $\varepsilon$  necessary to interpolate  $\Gamma$ .

- *Tubular approximation complexity:*

$$\sigma_a(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ \text{length}(\eta) \left| \begin{array}{l} 0 < T \leq \mathcal{T}, \\ \eta(0) = x, \eta(T) = y, \\ \eta([0, T]) \subset \text{Tube}(\Gamma, \varepsilon) \end{array} \right. \right\}.$$

This complexity measures the number of pieces of cost  $\varepsilon$  necessary to go from  $x$  to  $y$  staying inside the sub-Riemannian tube  $\text{Tube}(\Gamma, \varepsilon)$ .

Two functions  $f(\varepsilon)$  and  $g(\varepsilon)$ , tending to  $\infty$  or to 0 when  $\varepsilon \downarrow 0$  are *weakly equivalent* (denoted by  $f(\varepsilon) \asymp g(\varepsilon)$ ) if both  $f(\varepsilon)/g(\varepsilon)$  and  $g(\varepsilon)/f(\varepsilon)$ , are bounded when  $\varepsilon \downarrow 0$ . A complete characterization of weak asymptotic equivalence of metric complexities of a path is obtained in [Jea03]. We state here this result in the special case where  $M$  is an equiregular sub-Riemannian manifold.

**Theorem 4.** *Let  $M$  be an equiregular sub-Riemannian manifold and let  $\Gamma \subset M$  be a curve. Then, if there exists  $k \in \mathbb{N}$  such that  $T_q \Gamma \subset \Delta^k(q) \setminus \Delta^{k-1}(q)$  for any  $q \in \Gamma$ , it holds*

$$\sigma_c(\Gamma, \varepsilon) \asymp \sigma_a(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

*In particular, this implies that*

$$\dim^{\mathcal{H}} \Gamma = k.$$

Here, similarly to what happened in Theorem 3, the equiregularity is needed in order to have a uniform Ball-Box theorem near  $\Gamma$ . Indeed, to get the general result of [Jea03], it is necessary to use a finer form of the Ball-Box theorem that holds uniformly around singular points, proved in [Jea01b].

Two functions  $f(\varepsilon)$  and  $g(\varepsilon)$ , tending to  $\infty$  or to 0 when  $\varepsilon \downarrow 0$  are *strongly equivalent* (denoted by  $f(\varepsilon) \simeq g(\varepsilon)$ ) if  $f(\varepsilon)/g(\varepsilon) \rightarrow 1$  when  $\varepsilon \downarrow 0$ . An asymptotic optimal synthesis

for  $\sigma_c(\Gamma, \varepsilon)$  is a control strategy, depending on  $\varepsilon$ , that realizes a strong asymptotic equivalent of  $\sigma_c(\Gamma, \varepsilon)$  for  $\varepsilon \downarrow 0$ . Namely, it is a family  $\{\eta_\varepsilon\}_{\varepsilon>0}$  of  $\varepsilon$ -cost interpolations of  $\Gamma$ , such that

$$\frac{1}{\varepsilon} \text{length}(\eta_\varepsilon) \simeq \sigma_c(\Gamma, \varepsilon).$$

The definition of asymptotic optimal synthesis for  $\sigma_a(\Gamma, \varepsilon)$  is given in a similar way.

In a series of papers by Gauthier, Zakalyukin and others (see [BG13] for a review), strong asymptotic estimates and explicit asymptotic optimal syntheses are obtained for the metric complexities on a restricted class of problems, improving the results in [Jea03]. Namely, these results holds for generic couples  $(\Gamma, (\Delta, \mathbf{g}))$  of curves and sub-Riemannian structures, where the latter are either two-step bracket-generating (i.e. such that  $\Delta + [\Delta, \Delta] = TM$ ), or with two controls and  $\dim M \leq 6$ .

We conclude this section by stating, as an example, the result regarding the strong asymptotic equivalence for a generic distribution on  $\mathbb{R}^3$  proved in the series of papers [RMGMP04, GZ05, GZ06]. Generically, a rank two distribution on  $\mathbb{R}^3$  is two-step bracket-generating except on a codimension 1 smooth surface, called the Martinet surface and denoted by  $\mathcal{M}$ . Generically the curve  $\Gamma$  crosses  $\mathcal{M}$  transversally at a finite number (possibly equal to 0) of isolated points, where  $\Delta$  is not tangent to  $\mathcal{M}$ . Letting  $\gamma : [0, T] \rightarrow M$  being a parametrization of  $\Gamma$ , let  $t_i, i = 1, \dots, \ell$ , be such that  $\gamma(t_i)$  are such points. Then, it holds the following.

**Theorem 5.** *There exists a function  $\chi : [0, T] \rightarrow [0, +\infty)$ , explicitly defined by  $\Gamma$  and  $\Delta$ , such that*

1. *If  $\ell = 0$ , then*

$$\sigma_c(\Gamma, \varepsilon) = 2\pi\sigma_a(\Gamma, \varepsilon) \simeq \frac{4\pi}{\varepsilon^2} \int_0^T \frac{dt}{\chi(t)}.$$

2. *If  $\ell \neq 0$ , letting  $\varrho(t) = |\partial_t \chi(t)|$  it holds*

$$\sigma_c(\Gamma, \varepsilon) = 2\pi\sigma_a(\Gamma, \varepsilon) \simeq -4\pi \frac{\log \varepsilon}{\varepsilon^2} \sum_{i=1}^{\ell} \frac{1}{\varrho(t_i)}.$$

### 1.1.3. The sub-Laplacian

A differential operator  $P$  is hypoelliptic if for any  $a : U \subset M \rightarrow \mathbb{R}$  it holds that  $Pa \in C^\infty(U)$  implies  $a \in C^\infty(U)$ . The deep connection between second-order hypoelliptic operators and sub-Riemannian geometry became evident after the celebrated work [Hör67]. In this paper, Hörmander proved that the Lie bracket-generating condition is sufficient for the hypoellipticity of a second order differential operators with local expression

$$\mathcal{L} = \sum_{i=1}^m f_i^2 + \text{“first-order terms”},$$

where the  $f_i$ 's are first-order differential operators. Then, interpreting  $\{f_1, \dots, f_m\}$  as a family of vector fields, it is possible to define a sub-Riemannian structure on  $M$ .

The operator  $\mathcal{L} = \sum_{i=1}^m f_i^2$  is commonly called the *sub-Laplacian* on  $M$  associated with the frame  $\{f_1, \dots, f_m\}$ . From a sub-Laplacian it is possible to recover the Carnot-Carathéodory metric  $d_{\text{SR}}$  defined by the frame. In fact, letting the sub-Riemannian gradient  $\nabla_{\text{H}} u = \sum_{i=1}^m f_i u$ , it holds that

$$d_{\text{SR}}(q_0, q_1) = \sup \{u(x) - u(y) \mid u \in C_c^\infty(M) \text{ and } |\nabla_{\text{H}} u|^2 \leq 1 \text{ a.e.}\},$$

where  $|\nabla_{\text{H}} u|^2 = \sum_{i=1}^m (f_i u)^2$ . This has allowed to find many estimates on the fundamental solution of  $\mathcal{L}$  in terms of the associated Carnot-Carathéodory distance (see, e.g., [FS74, RS76]), and was at the origin of the renewed interest in sub-Riemannian geometry in the 70's [Gav77].

However, the correspondence between hypoelliptic operators and sub-Riemannian manifolds is not one-to-one. Indeed, it is easy to check that the sub-Riemannian gradient does not depend on the family of vector fields  $\{f_1, \dots, f_m\}$ , but is intrinsically defined by the sub-Riemannian structure<sup>2</sup>. On the other hand, the sub-Laplacian  $\tilde{\mathcal{L}}$  associated with a different family of vector fields  $\{g_1, \dots, g_m\}$ , generating the same distribution, differs from  $\mathcal{L}$  by a first-order differential operator. Thus, the same sub-Riemannian structure is associated with different sub-Laplacians.

Since we are interested in having a diffusion operator intrinsically associated with the sub-Riemannian structure, we have to resolve this ambiguity. The same problem arises in Riemannian geometry, when defining the Laplace-Beltrami operator, and it is resolved through the Green identity. We will proceed in the same way. Namely, instead of defining the sub-Laplacian through a local frame of the distribution, we consider a global smooth volume form  $d\mu$ , and let the sub-Laplacian  $\mathcal{L}$  to be the only operator satisfying the Green identity:

$$-\int_M f(\mathcal{L}g) d\mu = \int_M \mathbf{g}(\nabla_{\text{H}} f, \nabla_{\text{H}} g) d\mu, \quad \text{for any } f, g \in C_c^\infty(M). \quad (1.11)$$

Hence, in order to have an intrinsically-defined sub-Laplacian, one needs the volume  $d\mu$  to be intrinsically defined by the geometric structure of the manifold.

In the Riemannian case this problem is readily settled. Indeed, on any Riemannian manifold there are three common ways to define an intrinsic volume: The Riemannian metric defines the Riemannian volume, with coordinate expression  $dV = \sqrt{\mathbf{g}} dx_1 \wedge \dots \wedge dx_n$ , while the Riemannian distance allows to define the  $n$ -dimensional Hausdorff and spherical Hausdorff volumes. Since these three volumes are proportional up to a constant (see, e.g., [Fed69]), they are equivalent for the definition of the Laplace-Beltrami operator through (1.11).

In the sub-Riemannian setting, through the sub-Riemannian structure it is possible to define an intrinsic measure – called Popp measure – that plays the role of the Riemannian volume, and which is smooth if  $M$  is equiregular. In the equiregular case, by Theorem 3, we have at our disposal also the  $(\dim^{\mathcal{H}} M)$ -dimensional Hausdorff measure and spherical

<sup>2</sup>Indeed, the sub-Riemannian gradient of  $u$  is the only vector field such that  $\mathbf{g}_q(\nabla_{\text{H}} u(q), v) = du(v)$ , for any  $q \in M$  and  $v \in \Delta(q)$ .

Hausdorff measure, which are commensurable one with respect to the other (see, e.g., [Fed69]) and are absolutely continuous with respect to the Popp measure. Recent results [ABB12b], have however proved that the density of the Hausdorff measures with respect to the Popp measure is not, in general, smooth. Thus these measures define different intrinsic sub-Laplacians. When the manifold is not equiregular, moreover, these sub-Laplacians can present terms that diverge near singular points, as we will discuss in the next section.

This said, considering sub-Riemannian manifolds endowed with additional structure can resolve this ambiguity. For example, for left-invariant sub-Riemannian structures, i.e., Lie groups equipped with a left-invariant distribution and metric, both the Popp and the Hausdorff measures are left-invariant and hence Haar measures. The uniqueness up to a constant of Haar measures, allows then to define the sub-Laplacian through (1.11), as studied in [ABGR09].

#### 1.1.4. The Laplace-Beltrami operator in almost-Riemannian geometry

We now introduce a particular class of sub-Riemannian structures, the *2-dimensional almost-Riemannian structures* (abbreviated to 2-ARS). These are rank-varying sub-Riemannian structures on a 2-dimensional manifold  $M$  that can be defined locally by a pair of smooth vector fields satisfying the Lie bracket-generating condition. The name almost-Riemannian is due to the fact that these manifolds can be regarded as equipped with a generalized Riemannian metric  $\mathbf{g}$ , whose eigenvalues are allowed to diverge approaching the singular set  $\mathcal{Z}$  where the two vector fields become collinear. Such structures were introduced in the context of hypoelliptic operators [Gru70, FL82], then appeared in problems of population transfer in quantum systems [BCG<sup>+</sup>02b, BC04, BCC05], and have applications to orbital transfer in space mechanics [BC08, BCST09].

Almost-Riemannian manifolds present very interesting phenomena. For instance, geodesics can pass through the singular set with no singularities, even if all Riemannian quantities (e.g., the metric, the Riemannian area, the curvature) diverge while approaching  $\mathcal{Z}$ . Moreover, the presence of a singular set allows the conjugate locus to be nonempty even if the Gaussian curvature, where it is defined, is always negative (see [ABS08]). See also [ABS08, ABC<sup>+</sup>10, BCG13, BCGS13] for Gauss–Bonnet-type formulas, a classification of 2-ARS from the point of view of Lipschitz equivalence and normal forms for generic 2-ARS.

Since almost-Riemannian structures are not equiregular, both the Popp measure  $d\mathcal{P}$  and the 2-dimensional Hausdorff measures diverge on  $\mathcal{Z}$ . On the other hand, on  $M \setminus \mathcal{Z}$  the Popp measure coincides with the Riemannian volume and is thus proportional to the 2-dimensional Hausdorff measures. This allows to define an intrinsic sub-Laplacian  $\mathcal{L}$  through formula (1.11) applied to smooth functions compactly supported on  $M \setminus \mathcal{Z}$ . Due to the explosion of the metric when approaching the singularity, this operator will be singular on  $\mathcal{Z}$ . Since  $\mathcal{L}$  is actually the Laplace-Beltrami operator of the Riemannian manifold  $M \setminus \mathcal{Z}$ , it is called the Laplace-Beltrami operator associated with the 2-ARS.

Let us remark that this Laplace-Beltrami operator does not coincide with the hypoelliptic operator classically associated with the 2-ARS. Indeed, on trivializable structures

over  $\mathbb{R}^2$  – i.e., structures on  $\mathbb{R}^2$  admitting a global orthonormal frame – the latter corresponds to the “sum of squares” sub-Laplacian  $L$ . This sub-Laplacian can be defined through (1.11) using the 2-dimensional Lebesgue measure, and thus it is not intrinsic. Moreover, since the Lebesgue measure is locally finite on  $\mathbb{R}^2$ , the operator  $L$  is not singular on  $\mathcal{Z}$ .

In [BL] the following has been proved for a class of 2-ARS, with strong evidence suggesting that the same is true in general.

**Theorem 6.** *Let  $M$  be a 2-dimensional compact orientable manifold endowed with a 2-ARS. Assume moreover that  $\mathcal{Z}$  is an embedded one-dimensional sub-manifold of  $M$  and that  $\Delta + [\Delta, \Delta] = TM$ . Then the Laplace-Beltrami operator  $\mathcal{L}$  is essentially self-adjoint on  $L^2(M, d\mathcal{P})$ .*

The proof proceeded through essentially two steps. First, the statement was proved for the Laplace-Beltrami operator associated with a compactified version of the Grushin plane ([Gru70, FL82]), and then this result was extended to a general compact 2-ARS, through perturbation theory [Kat95].

The Grushin plane is the 2-ARS defined globally by the couple of vector fields

$$X_1(x, y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x, y) = \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad x, y \in \mathbb{R}^2, \quad (1.12)$$

and, thanks to the normal forms obtained in [BCG13], it is a good model for general 2-ARS satisfying  $\Delta + [\Delta, \Delta] = TM$ . For this structure, the Laplace-Beltrami operator  $\mathcal{L}$  and the “sum of squares” sub-Laplacian  $L$  are, respectively,

$$\mathcal{L} = \partial_x^2 - \frac{1}{x} \partial_y + x^2 \partial_y^2 \quad \text{and} \quad L = X_1^2 + X_2^2 = \partial_x^2 + x^2 \partial_y^2.$$

Theorem 6 has a number of implications. First, it shows that the singularity splits the manifold in two connected components that a quantum particle or the heat cannot cross: The explosion of the area naturally acts as a barrier which prevents the crossing of  $\mathcal{Z}$  by the particles. Moreover, since the Carnot-Carathéodory distance between points on different sides of the singularity is finite, there is no hope to get estimates for the fundamental solution of  $\mathcal{L}$  in terms of this distance. However, such estimates have been found by [Léa87] for  $L$ , showing that this operator and  $\mathcal{L}$  have quite different properties. The first one is not intrinsic, but however keeps track of intrinsic quantities such as the Carnot-Carathéodory distance. In particular, the corresponding heat flow crosses the set  $\mathcal{Z}$ , contrarily to what happens for  $\mathcal{L}$ .

## 1.2. Control-affine systems

The first part of this thesis is devoted to generalize the concepts of sub-Riemannian complexity introduced in Section 1.1.2 to system (1.3) with a non-zero drift. Such a generalization is critical for applications. As examples we cite: mechanical systems with



controls on the acceleration (see e.g., [BL05], [BLS10]) where the drift is the velocity, or quantum control (see e.g., [D'A08], [BM06]), where the drift is the free Hamiltonian.

It has been known since the 70's that under the *strong Hörmander condition*, i.e., if  $\{f_1, \dots, f_m\}$  satisfies the Hörmander condition, and with unbounded controls, such systems are controllable. Such a result is proved for example in [BL75], considering (1.3) as a perturbation of a non-holonomic control system. From now on, we will always assume the strong Hörmander condition to be satisfied.

Although out of the scope of the present work, we have to mention that from as early as the 60's the problem of controllability of such systems under the Hörmander condition – i.e., that the Lie algebra generated by the drift *and* the control vector fields spans the whole tangent space at any point – has been subject to a lot of attention, see for example [Kal60, Her64, BL75, Sus82]. In particular, the main focus has been the so-called *small time local controllability* around an equilibrium point, i.e., if given an equilibrium point  $q \in M$  and any time  $T > 0$  the end-points of admissible trajectories defined on  $[0, T]$  and starting from  $q$  cover a neighborhood of  $q$ . This problem is important, for example, in the context of quasi-static motions for robots with controls on the acceleration. For a review on results obtained in this direction see e.g., [Kaw90].

System (1.3) can be seen, from a geometrical point of view, as a generalization of sub-Riemannian geometry, where the distribution  $\Delta(q)$  is replaced by the affine distribution  $f_0(q) + \Delta(q)$ . Thus, in addition to the  $L^1$  cost  $\mathcal{J}$  considered in (1.4), it makes sense to study also the cost

$$\mathcal{I}(u, T) = \int_0^T \sqrt{1 + \sum_{i=1}^m u_i(t)^2} dt,$$

that measures the “Riemannian” length of admissible curves. We then fix a time  $\mathcal{T} > 0$  and consider the two value functions  $V^{\mathcal{J}}(q_0, q_1)$  and  $V^{\mathcal{I}}(q_0, q_1)$  as the infima of the costs  $\mathcal{J}$  and  $\mathcal{I}$ , respectively, over all controls steering system (1.3) from  $q_0$  to  $q_1$  in time  $T \leq \mathcal{T}$ . Contrarily to what happens in sub-Riemannian geometry with the Carnot-Carathéodory distance, these value functions are not symmetric, and hence do not induce a metric space structure on  $M$ . In fact, system (1.3) is not reversible – i.e., changing orientation to an admissible trajectory does not yield an admissible trajectory.

The reason for introducing a maximal time of definition for the controls – not needed in the sub-Riemannian context – is that, by taking  $\mathcal{T}$  sufficiently small, it is possible to prevent any exploitation of the geometry of the orbits of the drift (that could be, for example, closed). Let us also remark that, since the controls can be defined on arbitrarily small times, it is possible to approximate admissible trajectories via trajectories for the sub-Riemannian associated system (i.e., the one obtained by posing  $f_0 \equiv 0$  in (1.3)) rescaled on small intervals.

### 1.2.1. Hölder continuity of the value function

Paper P1., is dedicated to generalize the Chow-Rashewsky theorem and the Ball-box theorem to system (1.3) with the cost  $\mathcal{J}$ . Indeed, the first result we obtain is a global continuity result for the value function.

**Theorem 7.** For any  $0 < \mathcal{T} \leq +\infty$ , the function  $V^{\mathcal{J}} : M \times M \rightarrow [0, +\infty)$  is continuous. Moreover, letting  $d_{\text{SR}}$  be the sub-Riemannian distance induced by  $\{f_1, \dots, f_m\}$ , it holds

$$V^{\mathcal{J}}(q, q') \leq \min_{0 < t \leq \mathcal{T}} d_{\text{SR}}(e^{tf_0}q, q'), \quad \text{for any } q, q' \in M.$$

Letting  $\mathcal{R}_{f_0}(q, \varepsilon)$  be the reachable set from  $q$  with cost  $\mathcal{J}$  less than  $\varepsilon$ , Theorem 7 shows that

$$\bigcup_{0 < t \leq \mathcal{T}} B_{\text{SR}}(e^{tf_0}q, \varepsilon) \subset \mathcal{R}_{f_0}(q, \varepsilon). \quad (1.13)$$

Thus, the cost to steer the sub-Riemannian system from one point to another is always larger or equal than the cost to steer the control-affine system between the same points. Moreover, the fact that in coordinates it holds

$$e^{tf_0 - \varepsilon f_1} \circ e^{t f_0 + \varepsilon f_1}(q) = 2t f_0(q) + t\varepsilon [f_0, f_1](q) + o(\varepsilon t),$$

suggests that exploiting the drift it is actually possible to move more easily in some directions. Indeed, we will prove that this is the case, but only on very special directions realized as brackets of the drift with the control vector fields. Although this will not suffice to improve (1.13), we will be able to obtain a ball-box-like estimation of the reachable set from the outside.

Assume that the drift is regular, in the sense that there exists  $s \in \mathbb{N}$  such that  $f_0 \in \Delta^s \setminus \Delta^{s-1}$ , where  $\Delta^s$  is defined through the vector fields  $\{f_1, \dots, f_m\}$  as in the sub-Riemannian case. In particular, this allows to build systems of privileged coordinates rectifying  $f_0$ . Let  $\{\partial_{z_i}\}_{i=1}^n$  be the canonical basis of  $\mathbb{R}^n$  and consider the following sets:

$$\begin{aligned} \Xi(\eta) &= \bigcup_{0 \leq \xi \leq \mathcal{T}} (\xi \partial_{z_\ell} + \text{Box}(\eta)) \\ \Pi(\eta) &= \bigcup_{0 \leq \xi \leq \mathcal{T}} \{z \in \mathbb{R}^n : |z_\ell - \xi| \leq \eta^s, |z_i| \leq \eta^{w_i} + \eta \xi^{\frac{w_i}{s}} \text{ for } w_i \leq s, i \neq \ell, \\ &\quad \text{and } |z_i| \leq \eta(\eta + \xi^{\frac{1}{s}})^{w_i-1} \text{ for } w_i > s\}, \end{aligned}$$

In particular, observe that  $\Pi(\eta)$  is contained in  $\text{Box}(\eta)$ , defined in (1.9), and that  $\Pi(\eta) \cap \{z_\ell < 0\} = \text{Box}(\eta) \cap \{z_\ell < 0\}$ . We then get the following generalization of the Ball-Box theorem

**Theorem 8.** Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q$  for  $\{f_1, \dots, f_m\}$ , rectifying  $f_0$  as the  $k$ -th coordinate vector field  $\partial_{z_\ell}$ , for some  $1 \leq \ell \leq n$ . Then, there exist  $C, \varepsilon_0, T_0 > 0$  such that, if the maximal time of definition of the controls satisfies  $\mathcal{T} < T_0$ , it holds

$$\Xi\left(\frac{1}{C}\varepsilon\right) \subset \mathcal{R}_{f_0}(q, \varepsilon) \subset \Pi(C\varepsilon), \quad \text{for } \varepsilon < \varepsilon_0. \quad (1.14)$$

Here, with abuse of notation, we denoted by  $\mathcal{R}_{f_0}(q, \varepsilon)$  the coordinate representation of the reachable set.

This theorem represent the key step for generalizing the estimates on the complexity of curves from sub-Riemannian control systems to control-affine systems.

Finally, as in the sub-Riemannian case, as a consequence of Theorem 8 we get the following local Hölder equivalence between the value function and the Euclidean distance.

**Theorem 9.** *Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q$  for  $\{f_1, \dots, f_m\}$ , rectifying  $f_0$  as the  $k$ -th coordinate vector field  $\partial_{z_\ell}$ , for some  $1 \leq \ell \leq n$ . Then, there exist  $T_0, \varepsilon_0 > 0$  such that, if the maximal time of definition of the controls satisfies  $\mathcal{T} < T_0$  and  $V^{\mathcal{J}}(q, q') \leq \varepsilon_0$ , it holds*

$$\text{dist} \left( z(q'), z(e^{[0, \mathcal{T}]f_0} q) \right) \lesssim V^{\mathcal{J}}(q, q') \lesssim \text{dist} \left( z(q'), z(e^{[0, \mathcal{T}]f_0} q) \right)^{\frac{1}{r}}.$$

Here for any  $x \in \mathbb{R}^n$  and  $A \subset \mathbb{R}^n$ ,  $\text{dist}(x, A) = \inf_{y \in A} |x - y|$  denotes the Euclidean distance between them and  $r$  is the degree of non-holonomy of the sub-Riemannian control system defined by  $\{f_1, \dots, f_m\}$ .

In this result, instead of the Euclidean distance from the origin that appeared in (1.10), we have the distance from the integral curve of the drift. This is due to the fact that moving in this direction has null cost.

It is worth to mention that these results regarding control-affine systems are obtained by reducing them, as in [AL10], to time-dependent control systems in the form

$$\dot{q}(t) = \sum_{i=1}^m u_i(t) f_i^t(q(t)), \quad \text{a.e. } t \in [0, T], \quad (1.15)$$

where  $f_i^t = (e^{-tf_0})_* f_i$  is the pull-back of  $f_i$  through the flow of the drift. On these systems, that are linear in the control, we are able to define a good notion of approximation of the control vector fields. Namely, we will define a generalization of the nilpotent approximation, used in the sub-Riemannian context, taking into account the fact that in system (1.15), exploiting the time, we can generate the direction of the brackets between  $f_0$  and the  $f_j$ s. This approximation and an iterated integral method yield fine estimates on the reachable set.

### 1.2.2. Complexity and motion planning

The core of the first part of the thesis is Paper P2., in collaboration with F. Jean. Here, we focus on extending the concept of complexity to the control-affine case, and to give weak estimates of these quantities in the same spirit as Theorem 4.

The lack of time-rescaling properties of system (1.3) forces us to consider separately the concepts of curves, i.e., dimension 1 connected submanifolds  $\Gamma \subset M$  diffeomorphic to a closed interval, and of paths, i.e., smooth injective functions  $\gamma : [0, T] \rightarrow M$ . In particular, when computing the complexity of paths, we will require the approximating trajectories to respect also the parametrization and not only the geometry of the path.

We consider four distinct notions of complexities, two for curves and two for paths. The two for curves are the same as the sub-Riemannian ones already introduced in Section 1.1.2. This is true also for what we call the neighboring approximation complexity of a path, since in the sub-Riemannian case it coincides with the tubular approximation complexity. On the other hand, what we call the interpolation by time complexity never appeared in the literature, to our knowledge. Here, we define them for the cost  $\mathcal{J}$ , but the same definitions holds for  $\mathcal{L}$ .

Fix a curve  $\Gamma$ . Recall the definition of  $\varepsilon$ -cost interpolation given in Section 1.1.2, and let  $\text{Tube}(\Gamma, \varepsilon)$  to be the sub-Riemannian tubular neighborhood of radius  $\varepsilon$  around the curve  $\Gamma$ . Then, denoting by  $q_u$  the trajectory associated with a control  $u$  and with starting point  $q_u(0) = x$ , we let

- *Interpolation by cost complexity:*

$$\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \{ \mathcal{J}(u, T) \mid q_u \text{ is an } \varepsilon\text{-cost interpolation of } \Gamma \}.$$

This function measures the number of pieces of cost  $\varepsilon$  necessary to interpolate  $\Gamma$ . Namely, following a trajectory given by a control admissible for  $\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon)$ , at any given moment it is possible to go back to  $\Gamma$  with a cost less than  $\varepsilon$ .

- *Tubular approximation complexity:*

$$\sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ \mathcal{J}(u, T) \left| \begin{array}{l} 0 < T \leq \mathcal{T}, \\ q_u(0) = x, q_u(T) = y, \\ q_u([0, T]) \subset \text{Tube}(\Gamma, \varepsilon) \end{array} \right. \right\}$$

This complexity measures the number of pieces of cost  $\varepsilon$  necessary to go from  $x$  to  $y$  staying inside the sub-Riemannian tube  $\text{Tube}(\Gamma, \varepsilon)$ . Such property is especially useful for motion planning with obstacle avoidance. In fact, if the sub-Riemannian distance of  $\Gamma$  from the obstacles is at least  $\varepsilon_0 > 0$ , then trajectories obtained from controls admissible for  $\sigma_a^{\mathcal{J}}(\Gamma, \varepsilon)$ ,  $\varepsilon < \varepsilon_0$ , will avoid such obstacles.

Given a path  $\gamma : [0, T] \rightarrow M$ , we let a  $\delta$ -time interpolation of  $\gamma$  to be any admissible trajectory  $q_u : [0, T] \rightarrow M$  of (1.3) such that  $q_u(0) = \gamma(0)$ ,  $q_u(T) = \gamma(T)$ , and that, for any interval  $[t_0, t_1] \subset [0, T]$  of length  $t_1 - t_0 \leq \delta$ , there exists  $t \in [t_0, t_1]$  with  $q_u(t) = \gamma(t)$ . We then let,

- *Interpolation by time complexity:*

$$\sigma_t^{\mathcal{J}}(\gamma, \varepsilon) = \delta \inf \{ \mathcal{J}(u, T) \mid u \text{ is a } \delta\text{-time interpolation of } \gamma \}.$$

Controls admissible for this complexity will define trajectories touching  $\gamma$  at intervals of time of length at most  $\delta$ . This complexity measures the minimal average cost on each of these intervals. It is thus well suited for applications where time is of great importance - e.g. motion planning in rendez-vous problem.

- *Neighboring approximation complexity:*

$$\sigma_n^{\mathcal{J}}(\gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ \mathcal{J}(u, T) \mid \begin{array}{l} q_u(0) = x, q_u(T) = y, \\ q_u(t) \in B_{\text{SR}}(\gamma(t), \varepsilon), \forall t \end{array} \right\}.$$

This complexity measures the number of pieces of cost  $\varepsilon$  necessary to go from  $x$  to  $y$  following a trajectory that at each instant  $t \in [0, T]$  remains inside the sub-Riemannian ball  $B_{\text{SR}}(\gamma(t), \varepsilon)$ . Such complexity can be applied to motion planning in rendez-vous problems where it is sufficient to attain the rendez-vous only approximately.

For these complexities, exploiting the results of the previous section, we are able to prove the following theorem, in the same spirit as Theorem 4.

**Theorem 10.** *Assume that the sub-Riemannian structure defined by  $\{f_1, \dots, f_m\}$  is equiregular, and that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$  for some  $s \geq 2$ . Then, for any curve  $\Gamma \subset M$ , whenever the maximal time of definition of the controls  $\mathcal{T}$  is sufficiently small, it holds*

$$\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_c^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^\kappa}.$$

Here  $\kappa = \max\{k: T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p), \text{ for any } p \text{ in an open subset of } \Gamma\}$ .

Moreover, for any path  $\gamma: [0, T] \rightarrow M$  such that  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \pmod{\Delta^{s-1}}$  for any  $t \in [0, T]$ , it holds

$$\sigma_t^{\mathcal{J}}(\gamma, \delta) \asymp \sigma_t^{\mathcal{I}}(\gamma, \delta) \asymp \delta^{\frac{1}{\max\{\kappa, s\}}}, \quad \sigma_n^{\mathcal{J}}(\gamma, \varepsilon) \asymp \sigma_n^{\mathcal{I}}(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{\kappa, s\}}}.$$

Here  $\kappa = \max\{k: \gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t)) \text{ for any } t \text{ in an open subset of } [0, T]\}$ .

This theorem shows that, asymptotically, the complexity of curves is uninfluenced by the drift, and only depends on the underlying sub-Riemannian system, while the one of paths depends also on how “bad” the drift is with respect to this system. We remark also that for the path complexities it is not necessary to have an a priori bound on  $\mathcal{T}$ .

We conclude this section by studying the problem of *long time local controllability* (henceforth simply *LTLC*), i.e., the problem of staying near some point for a long period of time  $T > 0$ . This is essentially a stabilization problem around a non-equilibrium point.

Since the system (1.3) satisfies the strong Hörmander condition, using unbounded controls it is always possible to satisfy some form of LTLC. Hence, we try to quantify the minimal cost needed, by posing the following. (To lighten the notation, we consider only the cost  $\mathcal{J}$ .) Let  $T > 0$ ,  $q_0 \in M$ , and  $\gamma_{q_0}: [0, T] \rightarrow M$ ,  $\gamma_{q_0}(\cdot) \equiv q_0$ .

- *LTLC complexity by time:*

$$\Theta_t(q_0, T, \delta) = \sigma_t^{\mathcal{J}}(\gamma_{q_0}, \delta).$$

Here, we require trajectories defined by admissible controls to pass through  $q_0$  at intervals of time of length at most  $\delta$ .

- *LTLC complexity by cost:*

$$\Theta_n(q_0, T, \varepsilon) = \sigma_n^{\mathcal{J}}(\gamma_{q_0}, \varepsilon).$$

Admissible controls for this complexity, will always be contained in the sub-Riemannian ball of radius  $\varepsilon$  centered at  $q_0$ .

Clearly, if  $f_0(q_0) = 0$ , then  $\Theta_t(q_0, T, \delta) = \Theta_n(q_0, T, \varepsilon) = 0$ , for any  $\varepsilon, \delta, T > 0$ . Although  $\gamma_{q_0}$  is not a path by our definition, since it is not injective and  $\dot{\gamma}_{q_0} \equiv 0$ , the arguments of Theorem 10 can be applied also to this case. Hence, we get the following asymptotic estimate for the LTLC complexities.

**Corollary 11.** *Assume that the sub-Riemannian structure defined by  $\{f_1, \dots, f_m\}$  is equiregular, and that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$  for some  $s \geq 2$ . Then, for any  $q_0 \in M$  and  $T > 0$  it holds*

$$\Theta_t(q_0, T, \delta) \asymp \delta^{\frac{1}{s}}, \quad \Theta_n(q_0, T, \varepsilon) \asymp \frac{1}{\varepsilon^s}.$$

### 1.3. The Laplace-Beltrami operator on conic and anti-conic surfaces

The second, and last, part of the thesis is devoted to generalizing the results of [BL], see Theorem 6 in Section 1.1.4, to more general singular surfaces. Namely, we will consider a family of Riemannian manifolds depending on a parameter  $\alpha \in \mathbb{R}$ , defined on the disconnected cylinder  $M = (\mathbb{R} \setminus \{0\}) \times \mathbb{S}^1$ , and whose metric has orthonormal basis

$$X_1(x, \theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x, \theta) = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}, \quad x \in \mathbb{R}, \theta \in \mathbb{S}^1. \quad (1.16)$$

In other words, we will consider the Riemannian metric  $\mathbf{g} = dx^2 + |x|^\alpha d\theta^2$ .

Through a standard procedure, it is possible to extend this metric to  $M_{\text{cylinder}} = \mathbb{R} \times \mathbb{S}^1$  when  $\alpha \geq 0$ , and to  $M_{\text{cone}} = M_{\text{cylinder}} / \sim$  when  $\alpha < 0$ . Here,  $(x_1, \theta_1) \sim (x_2, \theta_2)$  if and only if  $x_1 = x_2 = 0$ . We will let  $M_\alpha$  be this extended metric space. Notice that in the cases  $\alpha = 1, 2, 3, \dots$ ,  $M_\alpha$  is an almost Riemannian structure in the sense of 1.1.4, while in the cases  $\alpha = -1, -2, -3, \dots$  it corresponds to a singular Riemannian manifold with a semi-definite metric.

One of the main features of these metrics is the fact that, except in the case  $\alpha = 0$ , the corresponding Riemannian volumes have a singularity at  $\mathcal{Z}$ ,

$$d\mu = \sqrt{\det g} dx d\theta = |x|^{-\alpha} dx d\theta.$$

Due to this fact, the corresponding Laplace-Beltrami operators contain some diverging first order terms,

$$\mathcal{L} = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^2 \partial_j \left( \sqrt{\det g} g^{jk} \partial_k \right) = \partial_x^2 + |x|^{2\alpha} \partial_\theta^2 u - \frac{\alpha}{x} \partial_x. \quad (1.17)$$

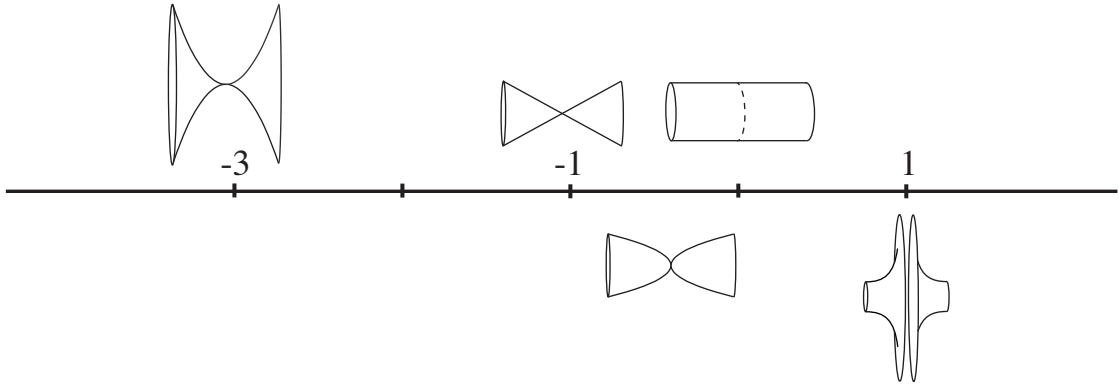


Figure 1.1.: Geometric interpretation of  $M_\alpha$ . The figures above the line are actually isometric to  $M_\alpha$ , while for the ones below the isometry is singular in  $\mathcal{Z}$ .

Here, we proceed as in Section 1.1.4 and initially define  $\mathcal{L}$  as an operator on  $C_c^\infty(M_\alpha \setminus \mathcal{Z}) = C_c^\infty(M)$ .

We have the following geometric interpretation of  $M_\alpha$  (see Figure 1.1). For  $\alpha = 0$ , this metric is that of a cylinder. For  $\alpha = -1$ , it is the metric of a flat cone in polar coordinates. For  $\alpha < -1$ , it is isometric to a surface of revolution  $\mathcal{S} = \{(t, r(t) \cos \vartheta, r(t) \sin \vartheta) \mid t > 0, \vartheta \in \mathbb{S}^1\} \subset \mathbb{R}^3$  with profile  $r(t) = |t|^{-\alpha} + O(t^{-2\alpha})$  as  $|t|$  goes to zero. For  $\alpha > -1$  ( $\alpha \neq 0$ ) it can be thought as a surface of revolution having a profile of the type  $r(t) \sim |t|^{-\alpha}$  as  $t \rightarrow 0$ , but this is only formal, since the embedding in  $\mathbb{R}^3$  is deeply singular at  $t = 0$ . The case  $\alpha = 1$  corresponds to the Grushin metric defined in (1.12), considered on the cylinder.

In Paper P3., in collaboration with U. Boscain, we considered the following problems about  $M_\alpha$ .

- (Q1) Do the heat and free quantum particles flow through the singularity? In other words, we are interested to the following: consider the heat or the Schrödinger equation

$$\partial_t \psi = \mathcal{L} \psi, \tag{1.18}$$

$$i \partial_t \psi = -\mathcal{L} \psi, \tag{1.19}$$

where  $\mathcal{L}$  is given by (1.17). Take an initial condition supported at time  $t = 0$  in  $M^- = \{x \in M \mid x < 0\}$ . Is it possible that at time  $t > 0$  the corresponding solution has some support in  $M^+ = \{x \in M \mid x > 0\}$ ?<sup>3</sup>

- (Q2) Does equation (1.18) conserve the total heat (i.e. the  $L^1$  norm of  $\psi$ )? This is known to be equivalent to the stochastic completeness of  $M_\alpha$  – i.e., the fact that the stochastic process, defined by the diffusion  $\mathcal{L}$ , almost surely has infinite lifespan.

---

<sup>3</sup>Notice that this is a necessary condition to have some positive controllability results by means of controls defined only on one side of the singularity, in the spirit of [BCG].

In particular, we are interested in understanding if the heat is absorbed by the singularity  $\mathcal{Z}$ .

The same question for the Schrödinger equation has a trivial answer, since the total probability (i.e., the  $L^2$  norm) is always conserved by Stone's theorem.

In order for this two questions to have a meaning it is necessary to interpret  $\mathcal{L}$  as a self-adjoint operator acting on  $L^2(M, d\mu)$ . However, since  $\mathcal{L}$  is defined only on  $C_c^\infty(M)$ , it cannot be self-adjoint and hence one has to apply the theory of self-adjoint extensions. As a comparison, in order to have a well defined evolution for the equation  $\partial_t \psi = \partial_x^2 \psi$  on the half-line  $[0, +\infty)$ , it is necessary to pose appropriate boundary conditions at 0: Dirichlet, Neumann, or a combination of the two. These conditions, indeed, guarantee that  $\partial_x^2$  is essentially self-adjoint on  $L^2([0, +\infty))$ .

### 1.3.1. Passage through the singularity

The rotational symmetry of  $M_\alpha$  suggests to proceed by a Fourier decomposition of  $\mathcal{L}$  in the  $\theta$  variable. Thus, we decompose the space  $L^2(M, d\mu) = \bigoplus_{k=0}^\infty H_k \cong L^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , and the corresponding operators on each  $H_k$  will be

$$\widehat{\mathcal{L}}_k = \partial_x^2 - \frac{\alpha}{x} \partial_x - |x|^{2\alpha} k^2. \quad (1.20)$$

It is a standard fact that  $\mathcal{L}$  is essentially self-adjoint on  $L^2(M, d\mu)$  if all of its Fourier components  $\widehat{\mathcal{L}}_k$  are essentially self-adjoint on  $L^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , while the contrary is not true.

As remarked at the end of Section 1.1.4, if the Laplace-Beltrami operator is essentially self-adjoint – i.e., if it admits only one self-adjoint extension that is the Friedrichs extension  $\mathcal{L}_F$  – then (Q1) has a negative answer. Indeed, by definition,  $\mathcal{L}_F$  acts separately on the two sides of the singularity hence inducing two independent dynamics. The following theorem – that extends Theorem 6 – classifies the essential self-adjointness of  $\mathcal{L}$  and of its Fourier components.

**Theorem 12.** *Consider  $M_\alpha$  for  $\alpha \in \mathbb{R}$  and the corresponding Laplace-Beltrami operator  $\mathcal{L}$  as an unbounded operator on  $L^2(M, d\mu)$ . Then the following holds.*

- *If  $\alpha \leq -3$  then  $\mathcal{L}$  is essentially self-adjoint;*
- *if  $\alpha \in (-3, -1]$ , only the first Fourier component  $\widehat{\mathcal{L}}_0$  is not essentially self-adjoint;*
- *if  $\alpha \in (-1, 1)$ , all the Fourier components of  $\mathcal{L}$  are not essentially self-adjoint;*
- *if  $\alpha \geq 1$  then  $\mathcal{L}$  is essentially self-adjoint.*

As a corollary of this theorem, we get the following answer to (Q1).



## 1. Introduction

$\alpha \leq -3$	Nothing can flow through $\mathcal{Z}$
$-3 < \alpha \leq -1$	Only the average over $\mathbb{S}^1$ of the function can flow through $\mathcal{Z}$
$-1 < \alpha < 1$	It is possible to have full communication between the two sides
$1 \leq \alpha$	Nothing can flow through $\mathcal{Z}$

More precisely, when  $-3 < \alpha \leq -1$  there exists a self-adjoint extension of  $\mathcal{L}$ , called the *bridging extension* and denoted by  $\mathcal{L}_B$ , such that the heat and Schrödinger flows allow the passage of only the first Fourier component through the singularity. On the other hand, when  $-1 < \alpha < 1$ , there exists a self-adjoint extension of  $\mathcal{L}$ , still called the bridging extension, such that (Q1) has a positive answer, i.e., all the Fourier components can flow through the singularity.

### 1.3.2. Stochastic completeness

It is a well known result that each non-positive self-adjoint operator  $A$  on a Hilbert space  $\mathcal{H}$  defines a strongly continuous contraction semigroup, denoted by  $\{e^{tA}\}_{t \geq 0}$ . If  $\mathcal{H} = L^2(M, d\mu)$  and it holds  $0 \leq e^{tA}\psi \leq 1$   $d\mu$ -a.e. whenever  $\psi \in L^2(M, d\mu)$ ,  $0 \leq \psi \leq 1$   $d\mu$ -a.e., the semigroup  $\{e^{tA}\}_{t \geq 0}$  and the operator  $A$  are called *Markovian*.

When  $\{e^{tA}\}_{t \geq 0}$  is the evolution semigroup of the heat equation, the Markov property can be seen as a physical admissibility condition. Namely, it assures that when starting from an initial datum  $\psi$  representing a temperature distribution (i.e., a positive and bounded function) the solution  $e^{tA}\psi$  remains a temperature distribution at each time, and, moreover, that the heat does not concentrate. Hence in the following we will focus only on the Markovian self-adjoint extensions of  $\mathcal{L}$ .

The interest for Markov operators lies also in the fact that, under an additional assumption which is always satisfied in the cases we consider, Markovian operators are generators of Markov processes  $\{X_t\}_{t \geq 0}$  (roughly speaking, stochastic processes which are independent of the past).

Since essentially bounded functions are approximable with functions in  $L^2(M, d\mu)$ , the Markovian property allows to extend the definition of  $e^{tA}$  from  $L^2(M, d\mu)$  to  $L^\infty(M, d\mu)$ . Let  $1$  be the constant function  $1(x, \theta) \equiv 1$ . Then (Q2) is equivalent to the following property.

**Definition 13.** A Markovian operator  $A$  is called *stochastically complete* (or conservative) if  $e^{tA}1 = 1$ , for any  $t > 0$ . It is called *explosive* if it is not stochastically complete.

It is well known that this property is equivalent to the fact that the Markov process  $\{X_t\}_{t \geq 0}$ , with generator  $A$ , has almost surely infinite lifespan.

We will consider also the following stronger property of  $\{X_t\}_{t \geq 0}$ .

**Definition 14.** A Markovian operator is called *recurrent* if the associated Markov process  $\{X_t\}_{t \geq 0}$  satisfies, for any set  $\Omega$  of positive measure and any point  $x$ ,

$$\mathbb{P}_x\{\text{there exists a sequence } t_n \rightarrow +\infty \text{ such that } X_{t_n} \in \Omega\} = 1.$$

Here  $\mathbb{P}_x$  denotes the probability measure in the space of paths emanating from a point  $x$  associated with  $\{X_t\}_{t \geq 0}$ .

We are particularly interested in distinguishing how the stochastic completeness and the recurrence are influenced by the singularity  $\mathcal{Z}$  or by the behavior at  $\infty$ . Thus we will consider the manifolds with borders  $M_0 = M \cap ([-1, 1] \times \mathbb{S}^1)$  and  $M_\infty = M \setminus [-1, 1] \times \mathbb{S}^1$ , with Neumann boundary conditions. Indeed, with these boundary conditions, when the Markov process  $\{X_t\}_{t \geq 0}$  hits the boundary it is reflected, and hence the eventual lack of recurrence or stochastic completeness on  $M_0$  (resp. on  $M_\infty$ ) is due to the singularity  $\mathcal{Z}$  (resp. to the behavior at  $\infty$ ). If a Markovian operator  $A$  on  $M$  is recurrent (resp. stochastically complete) when restricted on  $M_0$  we will call it recurrent (resp. stochastically complete) at 0. Similarly, when the same happens on  $M_\infty$ , we will call it recurrent (resp. stochastically complete) at  $\infty$ .

In this context, it makes sense to give special consideration to three specific self-adjoint extensions of  $\mathcal{L}$ , corresponding to different conditions at  $\mathcal{Z}$ . Namely, we will consider the already mentioned Friedrichs extension  $\mathcal{L}_F$ , that corresponds to an absorbing condition, the Neumann extension  $\mathcal{L}_N$ , that corresponds to a reflecting condition, and the bridging extension  $\mathcal{L}_B$ , that corresponds to a free flow through  $\mathcal{Z}$  and is Markovian only for  $\alpha \in (-1, 1)$ . Observe that  $\mathcal{L}_F$  and  $\mathcal{L}_N$  are always self-adjoint Markovian extensions, although it may happen that  $\mathcal{L}_F = \mathcal{L}_N$ . In this case  $\mathcal{L}_F$  is the only Markovian extension, and the operator  $\mathcal{L}$  is called Markov unique. This happens, for example, when  $\mathcal{L}$  is essentially self-adjoint.

The following result will answer to (Q2).

**Theorem 15.** *Consider  $M_\alpha$ , for  $\alpha \in \mathbb{R}$ , and the corresponding Laplace-Beltrami operator  $\mathcal{L}$  as an unbounded operator on  $L^2(M, d\mu)$ . Then it holds the following.*

- *If  $\alpha < -1$  then  $\mathcal{L}$  is Markov unique, and  $\mathcal{L}_F$  is stochastically complete at 0 and recurrent at  $\infty$ ;*
- *if  $\alpha = -1$  then  $\mathcal{L}$  is Markov unique, and  $\mathcal{L}_F$  is recurrent both at 0 and at  $\infty$ ;*
- *if  $\alpha \in (-1, 1)$ , then  $\mathcal{L}$  is not Markov unique and, moreover,*
  - *any Markovian extension of  $\mathcal{L}$  is recurrent at  $\infty$ ,*
  - *$\mathcal{L}_F$  is explosive at 0, while both  $\mathcal{L}_B$  and  $\mathcal{L}_N$  are recurrent at 0,*
- *if  $\alpha \geq 1$  then  $\mathcal{L}$  is Markov unique, and  $\mathcal{L}_F$  is explosive at 0 and recurrent at  $\infty$ ;*

In particular, Theorem 15 implies that for  $\alpha \in (-3, -1]$  no mixing behavior defines a Markov process. On the other hand, for  $\alpha \in (-1, 1)$  we can have a plethora of such processes. Classifying all possible Markov processes in this interval of parameters is the aim of Paper P4., in collaboration with A. Posilicano.

Since the singularity  $\mathcal{Z}$  is at finite distance from any point of  $M_\alpha$ , one can interpret a Markov process that is explosive at 0 as if  $\mathcal{Z}$  were absorbing the heat. Thus, as a corollary of Theorem 15, we get the following answer to (Q2).

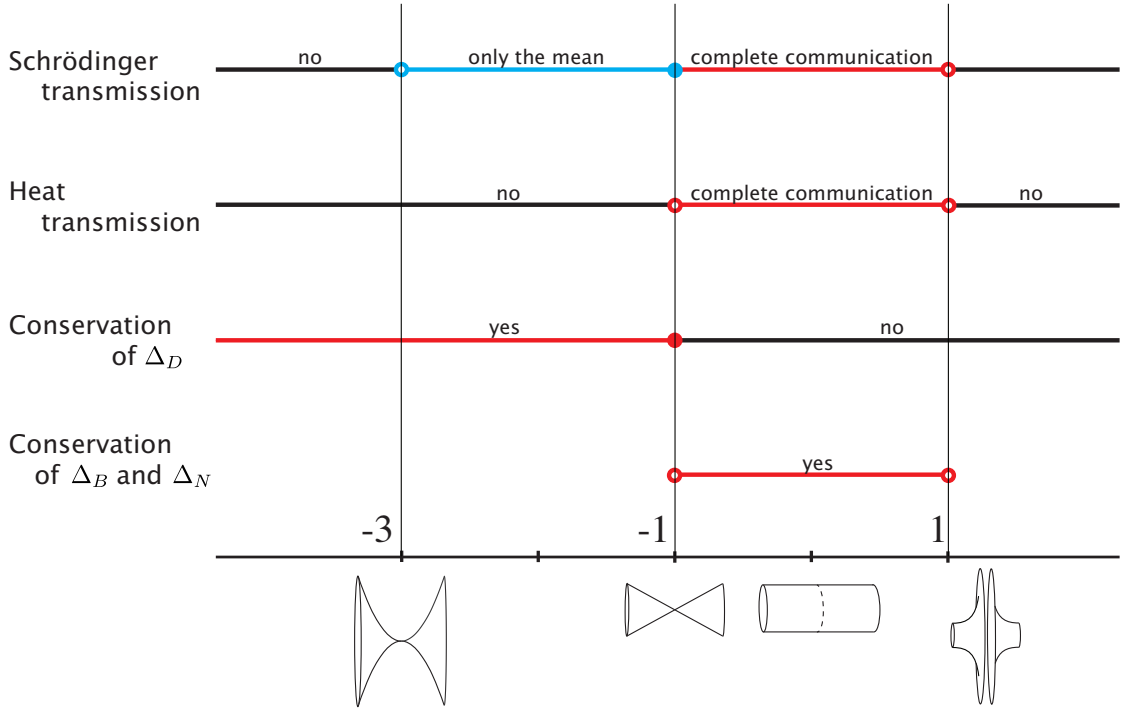


Figure 1.2.: A summary of the results obtained in Paper P3..

$\alpha \leq -1$	The heat is absorbed by $\mathcal{Z}$
$-1 < \alpha < 1$	The Friedrichs extension is absorbed by $\mathcal{Z}$ , while the Neumann and the bridging extensions are not.
$1 \leq \alpha$	The heat is absorbed by $\mathcal{Z}$

In Figure 1.2, we plotted a summary of the results we obtained.

## 1.4. Perspectives and open problems

The results exposed in this thesis are part of ongoing work. Here, we list some of the natural extensions of this work.

### 1.4.1. Complexity of non-admissible trajectories

1. Currently, we are working on improving two aspects of Theorem 10. First, we are considering the case where  $f_0(\gamma(t)) = \dot{\gamma}(t) \bmod \Delta^{s-h}(\gamma(t))$  for some  $1 \leq h \leq s$  and for any  $t \in [0, T]$ . We have strong evidence suggesting that this yields smaller complexities. Secondly, we are trying to weaken the assumption  $f_0 \in \Delta^s \setminus^{s-1}$  to  $f_0(q) \notin \Delta(q)$  for any  $q \in M$ .
2. In mechanical systems, where one controls the acceleration and the drift is the

velocity, one is usually interested in quasi-static motion planning, i.e., moving along trajectories near the zero-level set of the drift. In order to develop a complete theory of control-affine complexities for the costs  $\mathcal{J}$  and  $\mathcal{I}$ , it is then necessary to extend our results to curves or paths contained in the zero-level set of  $f_0$ .

3. We focused on costs based on the  $L^1$ -norm. While in the sub-Riemannian case, thanks to the rescaling properties of non-holonomic control systems, this is essentially equivalent to minimize the  $L^p$ -norm, such a statement is no more true for control-affine system. Thus, we intend to study what happens for this kind of costs. This problem is not just a mathematical curiosity, but is critical for a fruitful application of these results to quantum control, where the cost is usually the  $L^2$ -norm.
4. In this thesis we obtained only weak asymptotic estimates for the complexities. It is then natural to look for strong asymptotic estimates and asymptotic optimal syntheses in the spirit of Theorem 5. The techniques employed by Gauthier and Zakalyukin should indeed admit a natural generalization to the control-affine case.

#### 1.4.2. Singular diffusions

1. As already mentioned, we are currently collaborating with A. Posilicano on Paper P4. which is a direct continuation of the results exposed in Section 1.3. Our aim is to completely classify all the Markovian self-adjoint extensions of  $\mathcal{L}$  in the case  $\alpha \in (-1, 1)$ , where the deficiency indexes of the Laplace-Beltrami operator are infinite. This would allow for a better understanding of which kind of transmissions are possible in this context. The main motivation for this classification, however, is the interest these metrics have for  $\alpha \in (0, 1)$  in the control of partial differential equations, see e.g., [Mor13].
2. It would be nice to understand the scattering properties of these operators (taking the bridging extension as a reference), and to derive the associated transmission and reflection coefficients. This would give informations on how much of a wave packet would be reflected or transmitted, when hitting the singularity.
3. The study we carried out is meant as a model for more general structure, like non-quiregular sub-Riemannian manifolds or general conical manifolds. Our plan is thus to adapt the same tools to prove general results for the heat and Schrödinger equations for the Laplace-Beltrami operator corresponding to a generalized Riemannian metric including cones, horn-like manifolds and other generalizations.
4. We give results on the behavior of Markov processes by working solely on their generators. It is natural to try to understand if it is possible to obtain the same results from a purely probabilistic point of view, for example by defining such Markov processes as limits of random walks.



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**Part II.**

**Papers**



# HÖLDER CONTINUITY OF THE VALUE FUNCTION FOR CONTROL-AFFINE SYSTEMS

DARIO PRANDI<sup>†‡</sup>

**Abstract** We prove the continuity and we give a Holder estimate for the value function associated with the  $L^1$  cost of the control-affine system  $\dot{q} = f_0(q) + \sum_{j=1}^m u_j f_j(q)$ , satisfying the strong Hörmander condition. This is done by proving a result in the same spirit as the Ball-Box theorem for driftless (or sub-Riemannian) systems. The techniques used are based on a reduction of the control-affine system to a linear but time-dependent one, for which we are able to define a generalization of the nilpotent approximation. Finally, we also prove the continuity of the value function associated with the  $L^1$  cost of time-dependent systems of the form  $\dot{q} = \sum_{j=1}^m u_j f_j^t(q)$ .

**Key words:** control-affine systems, time-dependent systems, sub-Riemannian geometry, value function, Ball-Box theorem, nilpotent approximation.

**2010 AMS subject classifications:** 53C17, 93C15.

## 1. INTRODUCTION

A sub-Riemannian control system on a smooth manifold  $M$  is a control system in the form

$$(1) \quad \dot{\gamma}(t) = \sum_{i=1}^m u_i(t) f_i(\gamma(t)), \quad \text{a.e. } t \in [0, T],$$

where  $u : [0, T] \rightarrow \mathbb{R}^m$  is an integrable control function and  $\{f_1, \dots, f_m\}$  is a family of smooth vector fields satisfying the Hörmander condition, i.e., such that its iterated Lie brackets generate the whole tangent space at any point. The length of a curve  $\gamma$  solving (1), is then defined as the  $\text{length}(\gamma) = \min \int_0^T |u| dt$ , where the minimum is taken over all the possible  $u(\cdot)$  satisfying the above ODE. Due to the linearity of the system w.r.t.  $u$ , this length will be independent of the parametrization of  $\gamma$ . Finally, we define

$$d_{\text{SR}}(q, q') = \inf \{ \text{length}(\gamma) : \gamma : [0, 1] \rightarrow M \text{ is a solution of (1), } \gamma(0) = q \text{ and } \gamma(1) = q' \}.$$

By the Hörmander condition,  $d_{\text{SR}}$  is a distance, called Carnot-Carathéodory distance, endowing  $M$  with a natural metric space structure. A manifold considered together with a sub-Riemannian control system is called a sub-Riemannian manifold.

Define  $\Delta^1 = \text{span}\{f_1, \dots, f_m\}$  and  $\Delta^{s+1} = \Delta^s + [\Delta^s, \Delta^1]$ , for every  $s \in \mathbb{N}$ . Under the hypothesis that  $\{f_1, \dots, f_m\}$  is equiregular, i.e., that, for each  $s \in \mathbb{N}$ , the dimension of  $\Delta^s(q)$  is independent of  $q \in M$ , the Hörmander condition implies that there exists a (minimal)  $r \in \mathbb{N}$  such that  $\Delta^r(q) = T_q M$  for all  $q \in M$ . Such  $r$  is called the *degree of non-holonomy* of the sub-Riemannian control system.

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A fundamental result in the theory of sub-Riemannian manifolds is the celebrated Ball-Box theorem (see for example [8]). This theorem gives a rough description of the infinitesimal shape of the sub-Riemannian balls. Namely, at any point  $q$  of an equiregular sub-Riemannian manifold, the sub-Riemannian ball of small radius  $\varepsilon$  is equivalent, in privileged coordinates, to the box

$$\underbrace{[-\varepsilon, \varepsilon] \times \dots \times [-\varepsilon, \varepsilon]}_{\dim \Delta^1} \times \dots \times \underbrace{[-\varepsilon^s, \varepsilon^s] \times \dots \times [-\varepsilon^s, \varepsilon^s]}_{\dim \Delta^s - \dim \Delta^{s-1}} \times \dots \times \underbrace{[-\varepsilon^r, \varepsilon^r] \times \dots \times [-\varepsilon^r, \varepsilon^r]}_{\dim \Delta^r - \dim \Delta^{r-1}}.$$

By this we mean that, for some constant  $C > 0$ , the sub-Riemannian ball is contained in a box of side  $C\varepsilon$ , and contains a box of side  $\varepsilon/C$ .

This fact has a plethora of applications. First, it allows to prove a Hölder regularity estimate with respect to the Euclidean distance in coordinates, namely that, for  $q'$  sufficiently close to  $q$ , it holds

$$(2) \quad |q'| \lesssim d_{\text{SR}}(q, q') \lesssim |q'|^{1/r}.$$

Here we use “ $\lesssim$ ” to denote an inequality up to a multiplicative constant, independent of  $q'$ . Then, among many others, it is a fundamental step in the computation of the Hausdorff dimension of the manifold (see [21]), and it is used to obtain asymptotic estimates on the heat kernel (see e.g., [23, 20, 14, 2]). Moreover, it is the main tool in computing the asymptotic equivalents of the entropy and the complexity of curves (see e.g., [18, 19, 22, 15, 16]).

In this paper, we focus on a very important generalization of the control system (1), namely on control-affine systems. These systems are obtained by adding to (1) an uncontrolled vector field  $f_0$ , called the *drift*, and are in the form

$$(3) \quad \dot{\gamma}(t) = f_0(\gamma(t)) + \sum_{i=1}^m u_i(t) f_i(\gamma(t)), \quad \text{a.e. } t \in [0, T].$$

These kind of systems appears in plenty of applications. As an example we cite, mechanical systems with controls on the acceleration (see e.g., [10], [7]), where the drift is the velocity, or quantum control (see e.g., [12], [9]), where the drift is the free Hamiltonian. We always assume the strong Hörmander condition, i.e., that the family  $\{f_1, \dots, f_m\}$  satisfies the Hörmander condition.

The cost of a curve  $\gamma$  solving (3) is  $c_{f_0}(\gamma) = \int_0^T |u| dt$ . Unlike the sub-Riemannian length, due to the presence of the drift, the cost depends on the parametrization of the curve. Finally, the value function, between  $q, q' \in M$ , of the control system at time  $T > 0$ , is defined as

$$\rho_T^{f_0}(q, q') = \inf \{c_{f_0}(\gamma) : \gamma : [0, T'] \rightarrow M \text{ solves (3), } \gamma(0) = q, \gamma(T') = q', \text{ and } T' \leq T\}.$$

Assume now that the drift is regular, in the sense that there exists  $s \in \mathbb{N}$  such that  $f_0(q) \in \Delta^s(q) \setminus \Delta^{s-1}(q)$ , for any  $q \in M$ , where  $\Delta^s$  is defined through the vector fields  $\{f_1, \dots, f_m\}$  as before. Our main result is, then, a generalization of (2) to this context.

**Theorem 1.1.** *Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q$  for  $\{f_1, \dots, f_m\}$ , rectifying  $f_0$  as the  $k$ -th coordinate vector field  $\partial_{z_k}$ , for some  $1 \leq k \leq n$ . Then, for sufficiently small  $T$  and  $\varepsilon$ , if  $\rho_T^{f_0}(q, q') \leq \varepsilon$  then in  $z$  coordinates it holds*

$$\text{dist} \left( z(q'), z(e^{[0, T]f_0} q) \right) \lesssim \rho_T^{f_0}(q, q') \lesssim \text{dist} \left( z(q'), z(e^{[0, T]f_0} q) \right)^{1/r}.$$

Here for any  $x \in \mathbb{R}^n$  and  $A \subset \mathbb{R}^n$ ,  $\text{dist}(x, A) = \inf_{y \in A} |x - y|$  denotes the Euclidean distance between them,  $e^{tf_0}$  denotes the flow of  $f_0$ , and  $r$  is the degree of non-holonomy of the sub-Riemannian control system defined by  $\{f_1, \dots, f_m\}$ .

In this result, instead of the Euclidean distance from the origin that appeared in (2), we have the distance from the integral curve of the drift. This is due to the fact that moving in this direction has null cost. As in the sub-Riemannian case, Theorem 1.1 is a consequence of an

estimate on the shape of the reachable sets, contained in Theorem 4.27. Moreover, although Theorem 1.1 seems a natural generalization of (2), the shape of the reachable sets described in Theorem 4.27 is much more complicated than the boxes of the sub-Riemannian case, yielding a more difficult proof. Theorem 1.1 and 4.27 represent the key step for generalizing the estimates on the complexity of curves from sub-Riemannian control systems to control-affine systems.

It is worth to mention that these results regarding control-affine systems are obtained by reducing them, as in [3], to time-dependent control systems in the form

$$(4) \quad \dot{\gamma}(t) = \sum_{i=1}^m u_i(t) f_i^t(\gamma(t)), \quad \text{a.e. } t \in [0, T],$$

where  $f_i^t = (e^{-tf_0})_* f_i$  is the pull-back of  $f_i$  through the flow of the drift. On these systems, that are linear in the control, we are able to define a good notion of approximation of the control vector fields. Namely, in Section 3.3 we will define a generalization of the nilpotent approximation, used in the sub-Riemannian context, taking into account the fact that in the system (4), exploiting the time, we can generate the direction of the brackets between  $f_0$  and the  $f_j$ s. This approximation and an iterated integral method yield the correct estimates on the reachable set, contained in Theorem 3.19.

The paper is divided in three sections. In Section 2 we recall some generalities and definitions regarding sub-Riemannian control systems, used in the following sections. In Section 3 we consider control systems in the form (4), and we prove the continuity of the value function for general time-dependent vector fields. Then, in Theorem 3.19, restricting then to the case where the time dependency is explicitly given as  $f_i^t = (e^{-tf_0})_* f_i$ , we establish some estimates on the reachable sets, in the same spirit as the Ball-Box theorem. Finally, in Section 4 we consider control-affine systems. After proving the relation between control-affine systems and time-dependent systems, we prove the continuity of the value function. Then, in Lemma 4.28, exploiting the affine nature of the control system, we give an upper bound on the time needed to join two points  $q$  and  $q'$  as a function of  $\rho_T^{f_0}(q, q')$ . From this fact and the estimates of Section 3, Theorems 4.26 and 4.27 follow. Theorem 1.1 is then a particular case of Theorem 4.26, that holds under slightly milder assumptions on  $f_0$  and  $\{f_1, \dots, f_m\}$ .

## 2. SUB-RIEMANNIAN GEOMETRY

Throughout this paper,  $M$  is an  $n$ -dimensional connected smooth manifold. In this section we recall some classical notions and results of sub-Riemannian geometry.

**2.1. Sub-Riemannian control systems.** A sub-Riemannian (or non-holonomic) control system on  $M$  is a control system in the form

$$(SR) \quad \dot{q} = \sum_{i=1}^m u_i f_i(q), \quad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m,$$

where  $\{f_1, \dots, f_m\}$  is a family of smooth vector fields on  $M$ . We let  $f_u = \sum_{i=1}^m u_i f_i$ .

An absolutely continuous curve  $\gamma : [0, T] \rightarrow M$  is (SR)-*admissible* if there exists a control  $u \in L^1([0, T], \mathbb{R}^m)$  such that  $\dot{\gamma}(t) = f_{u(t)}(\gamma(t))$ , for a.e.  $t \in [0, T]$ . The curve is said to be *associated* to any such control. The length of  $\gamma$  is defined as

$$(5) \quad \text{length}(\gamma) = \min \|u\|_{L^1([0, T], \mathbb{R}^m)},$$

where the minimum is taken over all controls  $u$  such that  $\gamma$  is associated with  $u$ . It is attained, due to convexity reasons. Notice that, by definition,  $\text{length}(\gamma)$  is invariant under time



reparametrization of the curve. The *distance* induced by the sub-Riemannian system on  $M$  is then defined as

$$d_{\text{SR}}(q, q') = \inf\{\text{length}(\gamma) : \gamma \text{ (SR)-admissible and } \gamma : q \rightsquigarrow q'\},$$

where  $\gamma : q \rightsquigarrow q'$  stands for  $\gamma : [0, T] \rightarrow M$ , for some  $T > 0$ ,  $\gamma(0) = q$  and  $\gamma(T) = q'$ .

Let  $\Delta$  be the  $C^\infty$ -module generated by the vector fields  $\{f_1, \dots, f_m\}$  (in particular, it is closed under multiplication by  $C^\infty(M)$  functions and summation). Let  $\Delta^1 = \Delta$ , and define recursively  $\Delta^{s+1} = \Delta^s + [\Delta^s, \Delta]$ , for every  $s \in \mathbb{N}$ . Due to the Jacobi identity  $\Delta^s$  is the  $C^\infty$ -module of linear combinations of all commutators of  $f_1, \dots, f_m$  with length  $\leq s$ . For  $q \in M$ , let  $\Delta^s(q) = \{f(q) : f \in \Delta^s\} \subset T_q M$ . We say that  $\{f_1, \dots, f_m\}$  satisfies the *Hörmander condition* (or that it is a bracket-generating family of vector fields) if  $\bigcup_{s \geq 1} \Delta^s(q) = T_q M$  for any  $q \in M$ . In the following we will always assume this condition to be satisfied.

By the Chow–Rashevsky theorem (see for instance [1]), the hypothesis of connectedness of  $M$  and the Hörmander condition guarantee the finiteness and continuity of  $d_{\text{SR}}$  with respect to the topology of  $M$ . Hence, the function  $d_{\text{SR}}$ , called *sub-Riemannian* or *Carnot-Carathéodory distance*, induces on  $M$  a metric space structure. The open balls of radius  $\varepsilon > 0$  and centered at  $q \in M$ , with respect to  $d_{\text{SR}}$ , are denoted by  $B_{\text{SR}}(q, \varepsilon)$ .

We say that a (SR)-admissible curve  $\gamma$  is a minimizer of the sub-Riemannian distance between  $q, q' \in M$  if  $\gamma : q \rightsquigarrow q'$  and  $\text{length}(\gamma) = d_{\text{SR}}(q, q')$ . Equivalently,  $\gamma$  is a minimizer between  $q, q' \in M$  if it is a solution of the free-time optimal control problem, associated with (SR),

$$(6) \quad \|u\|_{L^1(0,T)} = \int_0^T \sqrt{\sum_{j=1}^m u_j^2(t)} dt \rightarrow \min, \quad \gamma(0) = q, \quad \gamma(T) = q', \quad T > 0.$$

Indeed, the sub-Riemannian distance is the value function associated with this problem. It is a classical result that, for any couple of points  $q, q' \in M$  sufficiently close, there exists at least one minimizer.

**Remark 2.2.** The optimal control problem (6) is equivalent to the following, with  $p \geq 1$  and  $T > 0$  fixed,

$$(7) \quad \|u\|_{L^p(0,T)} = \left( \int_0^T |u|^p dt \right)^{1/p} \rightarrow \min, \quad \gamma(0) = q, \quad \gamma(T) = q',$$

In fact, due to the invariance under time reparametrization of system (SR), in (6) we can fix either  $T > 0$  or the Euclidean norm of  $u$ . Moreover, by the Hölder inequality, for any  $p > 1$ , letting  $p'$  be the conjugated exponent to  $p$  (i.e.,  $1/p + 1/p' = 1$ ), we get  $\|u\|_{L^1(0,T)} \leq T^{1/p'} \|u\|_{L^p(0,T)}$ , with the equality holding if and only if  $|u|$  is constant. From these two facts, it is easy to check that minimizers of the optimal control problem (7) coincide with the minimizers of (6) with constant norm. Thus the two optimal control problems are equivalent.

**Remark 2.3.** This control theoretical setting can be stated in purely geometric terms. Indeed, it is equivalent to a *generalized sub-Riemannian* structure. Such a structure is defined by a rank-varying smooth distribution and a Riemannian metric on it (see [1] for a precise definition). In a sub-Riemannian control system, in fact, the map  $q \mapsto \text{span}\{f_1(q), \dots, f_m(q)\} \subset T_q M$  defines a rank-varying smooth distribution, which is naturally endowed with the Riemannian norm defined, for  $v \in \Delta(q)$ , by

$$\mathbf{g}(q, v) = \inf \left\{ |u| = \sqrt{u_1^2 + \dots + u_m^2} : f_u(q) = v \right\}.$$

The pair  $(\Delta, \mathbf{g})$  is thus a generalized sub-Riemannian structure on  $M$ . Conversely, every rank-varying distribution is finitely generated, see [6, 1, 4, 13], and thus a sub-Riemannian distance can be written, globally, as the value function of a control system of the type (SR).

**2.2. Privileged coordinates and nilpotent approximation.** We now introduce the equivalent, in the sub-Riemannian context, of the linearization of a vector field. This classical procedure, called *nilpotent approximation*, is possible only in a carefully chosen set of coordinates, called *privileged coordinates*.

Since  $\{f_1, \dots, f_m\}$  is bracket-generating, the values of the sets  $\Delta^s$  at  $q$  form a flag of subspaces of  $T_qM$ ,

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^r(q) = T_qM.$$

The integer  $r = r(q)$ , which is the minimum number of brackets required to recover the whole  $T_qM$ , is called *degree of non-holonomy* (or *step*) of the family  $\{f_1, \dots, f_m\}$  at  $q$ . Set  $n_s(q) = \dim \Delta^s(q)$ . The integer list  $(n_1(q), \dots, n_r(q))$  is called the *growth vector* at  $q$ . From now on we fix  $q \in M$ , and denote by  $r$  and  $(n_1, \dots, n_r)$  its degree of non-holonomy and its growth vector, respectively. Finally, let  $w_1 \leq \dots \leq w_n$  be the *weights* associated with the flag, defined by  $w_i = s$  if  $n_{s-1} < i \leq n_s$ , setting  $n_0 = 0$ .

For any smooth vector field  $f$ , we denote its action, as a derivation on smooth functions, by  $f : a \in C^\infty(M) \mapsto fa \in C^\infty(M)$ . For any smooth function  $a$  and every vector field  $f$  with  $f \neq 0$  near  $q$ , their (*non-holonomic*) *order* at  $q$  is

$$\begin{aligned} \text{ord}_q(a) &= \min\{s \in \mathbb{N} : \exists i_1, \dots, i_s \in \{1, \dots, m\} \text{ s.t. } (f_{i_1} \dots f_{i_s} a)(q) \neq 0\}, \\ \text{ord}_q(f) &= \max\{\sigma \in \mathbb{Z} : \text{ord}_q(fa) \geq \sigma + \text{ord}_q(a) \text{ for any } a \in C^\infty(M)\}. \end{aligned}$$

In particular it can be proved that  $\text{ord}_q(a) \geq s$  if and only if  $a(q') = \mathcal{O}(d_{\text{SR}}(q', q))^s$ .

**Definition 2.4.** A *system of privileged coordinates* at  $q$  for  $\{f_1, \dots, f_m\}$  is a system of local coordinates  $z = (z_1, \dots, z_n)$  centered at  $q$  and such that  $\text{ord}_q(z_i) = w_i$ ,  $1 \leq i \leq n$ .

For any point  $q \in M$  there always exists a system of privileged coordinates around  $q$ . Consider such a system  $z = (z_1, \dots, z_n)$ . We now show that this allows to compute the order of functions or vector fields in a purely algebraic way. Given a multiindex  $\alpha = (\alpha_1, \dots, \alpha_n)$  we define the weighted degree of the monomial  $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$  as  $w(\alpha) = w_1\alpha_1 + \dots + w_n\alpha_n$  and the weighted degree of the monomial vector field  $z^\alpha \partial_{z_j}$  as  $w(\alpha) - w_j$ . Then one can prove that, given  $a \in C^\infty(M)$  and a smooth vector field  $f$ , with Taylor expansion

$$a(z) \sim \sum_{\alpha} a_{\alpha} z^{\alpha} \quad \text{and} \quad f(z) \sim \sum_{\alpha, j} f_{\alpha, j} z^{\alpha} \partial_{z_j},$$

their orders at  $q$  can be computed as

$$\text{ord}_q(a) = \min\{w(\alpha) : a_{\alpha} \neq 0\} \quad \text{and} \quad \text{ord}_q(f) = \min\{w(\alpha) - w_j : f_{\alpha, j} \neq 0\}.$$

A function or a vector field are said to be *homogeneous* if all the nonzero terms of the expansion have the same weighted degree.

We recall that, for any  $a, b \in C^\infty(M)$  and any smooth vector fields  $f, g$ , the order satisfies the following properties

$$(8) \quad \begin{aligned} \text{ord}_q(a + b) &= \min\{\text{ord}_q(a), \text{ord}_q(b)\}, & \text{ord}_q(ab) &= \text{ord}_q(a) + \text{ord}_q(b), \\ \text{ord}_q(f + g) &= \min\{\text{ord}_q(f), \text{ord}_q(g)\}, & \text{ord}_q([f, g]) &\geq \text{ord}_q(f) + \text{ord}_q(g). \end{aligned}$$

Consider  $f_i$ ,  $1 \leq i \leq m$ . By the definition of order, it follows that  $\text{ord}_q(f_i) \geq -1$ . Then we can express  $f_i$  in coordinates as

$$z_* f_i = \sum_{j=1}^n (h_{ij} + r_{ij}) \partial_{z_j},$$

where  $z_*$  is the push-forward operator on vector fields associated with the coordinates, defined as  $z_*f = dz \circ f \circ z^{-1}$ ,  $h_{ij}$  are homogeneous polynomials of weighted degree  $w_j - 1$ , and  $r_{ij}$  are functions of order larger than or equal to  $w_j$ .

**Definition 2.5.** The nilpotent approximation at  $q$  of  $f_i$ ,  $1 \leq i \leq m$ , associated with the privileged coordinates  $z$  is the vector field with coordinate representation

$$z_*\hat{f}_i = \sum_{j=1}^n h_{ij} \partial_{z_j}.$$

The nilpotentized sub-Riemannian control system is then defined as

$$(NSR) \quad \dot{q} = \sum_{j=1}^m u_j(t) \hat{f}_j(q).$$

The family of vector fields  $\{\hat{f}_1, \dots, \hat{f}_m\}$  is bracket-generating and nilpotent of step  $r$  (i.e., every iterated bracket  $[f_{i_1}, [\dots, [f_{i_{k-1}}, f_{i_k}]]]$  of length larger than  $r$  is zero).

The main consequence of the nilpotent approximation is the following (see for example [8, Proposition 7.29]).

**Proposition 2.6.** *Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q \in M$  for  $\{f_1, \dots, f_m\}$ . For  $T > 0$  and  $u \in L^1([0, T]; \mathbb{R}^m)$ , with  $|u| \equiv 1$ , let  $\gamma(\cdot)$  and  $\hat{\gamma}(\cdot)$  be the trajectories associated with  $u$  in (SR) and (NSR), respectively, and such that  $\gamma(0) = \hat{\gamma}(0) = q$ . Then, there exist  $C, T_0 > 0$ , independent of  $u$ , such that, for any  $t < T_0$ , it holds*

$$(9) \quad |z_i(\gamma(t)) - z_i(\hat{\gamma}(t))| \leq Ct^{w_i+1}, \quad i = 1, \dots, n.$$

We recall, finally, the celebrated Ball-Box Theorem, that gives a rough description of the shape of small sub-Riemannian balls.

**Theorem 2.7** (Ball-Box Theorem). *Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q \in M$  for  $\{f_1, \dots, f_m\}$ . Then there exist  $C, \varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$ , it holds*

$$\text{Box}\left(\frac{1}{C}\varepsilon\right) \subset B_{\text{SR}}(q, \varepsilon) \subset \text{Box}(C\varepsilon),$$

where,  $B_{\text{SR}}(q, \varepsilon)$  is identified with its coordinate representation  $z(B_{\text{SR}}(q, \varepsilon))$  and, for any  $\eta > 0$ , we let

$$(10) \quad \text{Box}(\eta) = \{z \in \mathbb{R}^n : |z_i| \leq \eta^{w_i}\},$$

Observe that the first inclusion follows directly from the definition of privileged coordinates.

As a corollary of the Ball-Box Theorem, we get the following result on the regularity of the distance.

**Corollary 2.8.** *Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q \in M$  for  $\{f_1, \dots, f_m\}$ . Then there exists  $C, \varepsilon > 0$  such that*

$$\frac{1}{C}|z(q')| \leq d_{\text{SR}}(q, q') \leq C|z(q')|^{1/r}, \quad q' \in B_{\text{SR}}(q, \varepsilon).$$

### 3. TIME-DEPENDENT SYSTEMS

**3.1. Time-dependent control systems.** We now consider a more general situation. Namely, we consider on  $M$  the time-dependent non-holonomic control system

$$(TD) \quad \dot{q} = \sum_{i=1}^m u_i f_i^t(q), \quad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m, \quad t \in I,$$

where  $I = [0, b)$  for some  $b \leq +\infty$  and  $\{f_1^t, \dots, f_m^t\}$  is a family of non-autonomous smooth vector fields, with smooth dependence on the time parameter. We let  $f_u^t = \sum_{i=1}^m u_i f_i^t$ .

In analogy with the autonomous case, we define (TD)-admissible curves as absolutely continuous curves  $\gamma : [0, T] \subset I \rightarrow M$  such that  $\dot{\gamma}(t) = f_{u(t)}^t(\gamma(t))$  for a.e.  $t \in [0, T]$ , for some control  $u \in L^1([0, T], \mathbb{R}^m)$ . Observe, however, that contrary to what happens in the sub-Riemannian case, the (TD)-admissibility property is not invariant under time reparametrization, e.g., a time reversal. Thus, we define the cost (and not the length) of  $\gamma$  to be

$$c(\gamma) = \min \|u\|_{L^1([0, T], \mathbb{R}^m)},$$

where the minimum is taken over all controls  $u$  such that  $\gamma$  is associated with  $u$  and is attained due to convexity. The *value function* induced by the time-dependent system is then defined as

$$\rho(q, q') = \inf\{c(\gamma) : \gamma \text{ is (TD)-admissible and } \gamma : q \rightsquigarrow q'\}.$$

Clearly, the value function is non-negative. It is not a metric since, in general, it fails both to be symmetric and to satisfy the triangular inequality. Moreover, as the following example shows,  $\rho$  could be degenerate. Namely, it could happen that  $q \neq q'$  but  $\rho(q, q') = 0$ .

**Example 3.9.** Let  $M = \mathbb{R}$ , with coordinate  $x$  and consider the vector field  $f^t = (1-t)^{-2} \partial_x$  defined on  $[0, 1)$ . For any  $x_0 \in \mathbb{R}$ ,  $x_0 \neq 0$ , and for any sequence  $t_n \uparrow 1$ , let  $u_n \in L^1([0, t_n])$  be defined as  $u_n \equiv (1-t_n)x_0$ . By definition, each  $u_n$  steers the system from 0 to  $x_0$ . Hence,

$$\rho_1(0, x_0) \leq \inf_{n \in \mathbb{N}} \|u_n\|_{L^1([0, t_n])} = \inf_{n \in \mathbb{N}} \int_0^{t_n} (1-t_n)x_0 dt = x_0 \inf_{n \in \mathbb{N}} t_n(1-t_n) = 0.$$

This proves that, for any  $x_0 \in \mathbb{R}$ ,  $\rho_1(0, x_0) = 0$ .

For  $T > 0$ ,  $q \in M$  and  $\varepsilon > 0$ , we denote the reachable set from  $q$  with cost less than  $\varepsilon$  by

$$\mathcal{R}(q, \varepsilon) = \{q' \in M : \rho(q, q') < \varepsilon\}.$$

We will also consider the reachable set from  $q$  in time less than  $T > 0$  and cost less than  $\varepsilon$ , and denote it by  $\mathcal{R}_T(q, \varepsilon)$ . Clearly  $\mathcal{R}_T(q, \varepsilon) \subset \mathcal{R}(q, \varepsilon)$ .

In general, the existence of minimizers for the optimal control problem associated with (TD) is not guaranteed. We conclude this section with an example of this fact.

**Example 3.10.** Let  $M = \mathbb{R}$ , with coordinate  $x$ , and consider the vector field  $f^t = e^{-t} \partial_x$  for  $t \in [0, 1)$ . Fix  $x_0 \in \mathbb{R}$ ,  $x_0 \neq 0$ . Observe that, for any  $T > 0$  and any control  $u \in L^1([0, T])$  steering the system from 0 to  $x_0$ , it holds

$$(11) \quad |x_0| = \left| \int_0^T u(t) e^{-t} dt \right| \leq \int_0^T |u(t)| e^{-t} dt < \|u\|_{L^1([0, T])}.$$

This implies  $\rho(0, x_0) \geq |x_0|$ . Let now  $u_n \in L^1([0, 1/n])$  be defined as  $u_n(t) = x_0 n e^t$ . Clearly  $u_n$  steers the system from 0 to  $x_0$ . Moreover,

$$\rho(0, x_0) \leq \inf_{n \in \mathbb{N}} \|u_n\|_{L^1([0, 1/n])} = |x_0| \inf_{n \in \mathbb{N}} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = |x_0|.$$

This proves that  $\rho(0, x_0) = |x_0|$ . Hence, the non-existence of minimizers follows from (11).

**3.2. Finiteness and continuity of the value function.** In this section, we extend the Chow–Rashevsky Theorem to time-dependent non-holonomic systems, under the strong Hörmander condition, whose definition follows.

**Definition 3.11.** We say that a family of time-dependent vector fields  $\{f_1^t, \dots, f_m^t\}_{t \in I}$  satisfies the *strong Hörmander condition* if  $\{f_1^{t_0}, \dots, f_m^{t_0}\}$  satisfies the Hörmander condition for any  $t_0 \in I$ .

As we will see later on in Section 4, when considering families of time-dependent vector fields of the form  $f_i^t = (e^{-tf_0})_* f_i$  this condition is equivalent to the strong Hörmander condition for the affine control system with drift  $f_0$  and control vector fields  $\{f_1, \dots, f_m\}$ .

From now on we will assume that the following holds.

(H<sub>0</sub>) The family of smooth vector fields  $\{f_1^t, \dots, f_m^t\}_{t \in I}$ , depends smoothly on  $t$  and satisfies the strong Hörmander condition.

This section will be devoted to the proof of the following.

**Theorem 3.12.** *Assume that  $\{f_1^t, \dots, f_m^t\}_{t \in I}$  satisfies (H<sub>0</sub>). Then, the function  $\rho : M \times M \rightarrow [0, +\infty)$  is continuous. Moreover, for any  $t_0 \in I$  and any  $q, q' \in M$ , letting  $d_{\text{SR}}$  be the sub-Riemannian distance induced by  $\{f_1^{t_0}, \dots, f_m^{t_0}\}$ , it holds  $\rho(q, q') \leq d_{\text{SR}}(q, q')$ .*

Now, we need to introduce some notation. Following [5], the flows between times  $s, t \in \mathbb{R}$  of an autonomous vector field  $f$  and of a non-autonomous vector field  $\tau \mapsto f^\tau$  will be denoted by, respectively,

$$e^{(t-s)f} : M \rightarrow M \quad \text{and} \quad \overrightarrow{\text{exp}} \int_s^t f^\tau d\tau : M \rightarrow M.$$

Fix  $q \in M$  and assume, for the moment, that  $t_0 = 0$ . Let  $\ell \in \mathbb{N}$  and  $\mathcal{F} = (i_1, \dots, i_\ell) \in \{1, \dots, m\}^\ell$ . For any  $\mathcal{T} \in I$ ,  $\mathcal{T} > 0$ , we define the *switching end-point map* at time  $\mathcal{T}$  and associated with  $\mathcal{F}$  to be the function  $E_{\mathcal{T}, \mathcal{F}} : \mathbb{R}^\ell \rightarrow M$  defined as

$$\begin{aligned} E_{\mathcal{T}, \mathcal{F}}(\xi) &= \overrightarrow{\text{exp}} \int_{\frac{\ell-1}{\ell}\mathcal{T}}^{\mathcal{T}} \frac{1}{\mathcal{T}} \xi_\ell f_{i_\ell}^\tau d\tau \circ \dots \circ \overrightarrow{\text{exp}} \int_0^{\frac{\mathcal{T}}{\ell}} \frac{1}{\mathcal{T}} \xi_1 f_{i_1}^\tau d\tau (q) \\ (12) \quad &= \overrightarrow{\text{exp}} \int_{\frac{\ell-1}{\ell}}^1 \xi_\ell f_{i_\ell}^{\tau\mathcal{T}} d\tau \circ \dots \circ \overrightarrow{\text{exp}} \int_0^{\frac{1}{\ell}} \xi_1 f_{i_1}^{\tau\mathcal{T}} d\tau (q). \end{aligned}$$

Here we applied a standard change of variables formula for non-autonomous flows. Let then

$$(13) \quad g_{\mathcal{T}, \mathcal{F}}^\tau = \begin{cases} \xi_1 f_{i_1}^{\tau\mathcal{T}} & \text{if } 0 \leq \tau < 1/\ell, \\ \xi_2 f_{i_2}^{(\tau-1/\ell)\mathcal{T}} & \text{if } 1/\ell \leq \tau < 2/\ell, \\ \vdots & \\ \xi_\ell f_{i_\ell}^{(\tau-(\ell-1)/\ell)\mathcal{T}} & \text{if } (\ell-1)/\ell \leq \tau < 1, \end{cases}$$

so that we can write

$$E_{\mathcal{T}, \mathcal{F}}(\xi) = \overrightarrow{\text{exp}} \int_0^1 g_{\mathcal{T}, \mathcal{F}}^\tau(\xi) d\tau (q).$$

Clearly,  $t \mapsto \overrightarrow{\text{exp}} \int_0^t g_{\mathcal{T}, \mathcal{F}}^\tau(\xi) d\tau (q)$ ,  $t \in [0, 1]$ , is a (TD)-admissible trajectory. Thus,  $E_{\mathcal{T}, \mathcal{F}}(\xi)$ ,  $\mathcal{T} > 0$ , is the end-point of a piecewise smooth (TD)-admissible curve.

We recall that, by the series expansion of  $\overrightarrow{\text{exp}}$  (see [5]), for any non-autonomous smooth vector field  $f^\tau$ , it holds  $\overrightarrow{\text{exp}} \int_0^t f^\tau d\tau (q) = e^{t f^0} (q) + \mathcal{O}(t^2)$ . Thus, we can define

$$E_{0, \mathcal{F}}(\xi) = \lim_{\mathcal{T} \downarrow 0} E_{\mathcal{T}, \mathcal{F}}(\xi) = e^{\xi_\ell f_{i_\ell}^0} \circ \dots \circ e^{\xi_1 f_{i_1}^0} (q) = \overrightarrow{\text{exp}} \int_0^1 g_{0, \mathcal{F}}^\tau(\xi) d\tau (q),$$

where,  $g_{0, \mathcal{F}}^\tau(\xi)$  is defined in (13). Then  $t \mapsto \overrightarrow{\text{exp}} \int_0^t g_{0, \mathcal{F}}^\tau(\xi) d\tau (q)$ ,  $t \in [0, 1]$ , is an (SR)-admissible curve for the sub-Riemannian structure defined by  $\{f_1^0, \dots, f_m^0\}$  and  $E_{0, \mathcal{F}}(\xi)$  is the end-point of a piecewise smooth trajectory in (SR).

After [24], we say that a point  $q' \in M$  is (TD)-*reachable* from  $q$  at time  $t_0 = 0$ , if there exist  $\ell \in \mathbb{N}$ ,  $\mathcal{F} \in \{1, \dots, m\}^\ell$ ,  $\mathcal{T} > 0$  and  $\xi \in \mathbb{R}^\ell$ , such that  $E_{\mathcal{T}, \mathcal{F}}(\xi) = q'$ . In this case it is clear that  $\rho(q, q') \leq \sum_i |\xi_i|$ . Moreover, if  $\xi' \mapsto E_{\mathcal{T}, \mathcal{F}}(\xi')$  has rank  $n$  at  $\xi$ , the point  $q'$  is said to be

(TD)-normally reachable at time  $t_0 = 0$ . Finally, the point  $q'$  is said to be (SR)-reachable or (SR)-normally reachable for the vector fields  $\{f_1^0, \dots, f_m^0\}$ , if these properties holds for  $\mathcal{T} = 0$ .

In the case  $t_0 > 0$ , taking  $\mathcal{T} > 0$  such that  $\mathcal{T} + t_0 \in I$  and changing the interval of integration in (12) from  $[0, \mathcal{T}]$  to  $[t_0, t_0 + \mathcal{T}]$ , it is clear how to define (TD)-reachable and (TD)-normally reachable points from  $q$  at time  $t_0$ , and (SR)-reachable and (SR)-normally reachable points for the vector fields  $\{f_1^{t_0}, \dots, f_m^{t_0}\}$ .

The proof of the following lemma is an adaptation of [24, Lemma 3.1].

**Lemma 3.13.** *Let  $q' \in M$  be (SR)-normally reachable for the vector fields  $\{f_1^{t_0}, \dots, f_m^{t_0}\}$  from  $q$ , by some  $\ell \in \mathbb{N}$ ,  $\xi \in \mathbb{R}^\ell$  and  $\mathcal{F} \in \{1, \dots, m\}^\ell$ . Then, there exist  $\varepsilon_0, \mathcal{T}_0 > 0$  such that, for any  $\varepsilon < \varepsilon_0$ , the point  $q'$  is (TD)-normally reachable at time  $t_0$ , by the same  $\ell$  and  $\mathcal{F}$ , and some  $\xi' \in \mathbb{R}^\ell$ , with  $\sum_j |\xi_j - \xi'_j| \leq \varepsilon$ , and any  $\mathcal{T} < \mathcal{T}_0$ .*

*Proof.* Without loss of generality, we assume  $t_0 = 0$ .

Let  $U \subset \mathbb{R}^\ell$  be a neighborhood of  $\xi$  such that  $E_{0, \mathcal{F}}$  has still rank  $n$  when restricted to it. Then, there exists  $B = \{x: \sum_j |x_j - \xi_j| \leq \varepsilon\} \subset U$  such that  $E_{0, \mathcal{F}}$  maps diffeomorphically a neighborhood of  $B$  in  $U$  onto a neighborhood of  $q$ . It follows, from standard properties of differential equations, that, for  $\mathcal{T} > 0$  sufficiently small, the map  $E_{\mathcal{T}, \mathcal{F}}$  is well defined on  $B$  and that  $E_{\mathcal{T}, \mathcal{F}} \rightarrow E_{0, \mathcal{F}}$  as  $\mathcal{T} \downarrow 0$  in the  $C^1$ -topology over  $B$ . Thus, there exists  $\mathcal{T}_1 > 0$  such that, for  $\mathcal{T} < \mathcal{T}_1$ ,  $E_{\mathcal{T}, \mathcal{F}}$  has rank  $n$  at every point of  $B$ .

Since the map  $E_{0, \mathcal{F}}$  is an homeomorphism from  $B$  onto a neighborhood of  $q$ , and  $E_{\mathcal{T}, \mathcal{F}} \rightarrow E_{0, \mathcal{F}}$  uniformly as  $\mathcal{T} \downarrow 0$ , it follows that there exists a fixed neighborhood  $V$  of  $q$  and  $\mathcal{T}_2 > 0$  such that  $V \subset E_{\mathcal{T}, \mathcal{F}}(B)$ , for any  $\mathcal{T} < \mathcal{T}_2$ . Then, for any  $\mathcal{T} < \min\{\mathcal{T}_1, \mathcal{T}_2\}$ , there exists  $\xi' \in B$  such that the point  $q' = E_{\mathcal{T}, \mathcal{F}}(\xi')$  is (TD)-normally reachable.  $\square$

We will use the following consequence of Lemma 3.13. We remark that the result holds even if  $\{f_1^t, \dots, f_m^t\}_{t \in I}$  satisfies the Hörmander condition only at time  $t_0 \in I$ .

**Lemma 3.14.** *Let  $d_{\text{SR}}$  be the sub-Riemannian distance induced by  $\{f_1^{t_0}, \dots, f_m^{t_0}\}$ , then for any  $t_1 \in I$ , such that  $t_1 - t_0 > 0$  is sufficiently small, and for any  $q, q' \in M$  it holds*

$$\inf\{c(\gamma): \gamma: [t_0, t_1] \rightarrow M \text{ is (TD)-admissible, } \gamma(t_0) = q \text{ and } \gamma(t_1) = q'\} \leq d_{\text{SR}}(q, q').$$

*In particular,  $\rho(q, q') \leq d_{\text{SR}}(q, q')$ .*

*Proof.* Fix  $\varepsilon > 0$ . By Chow's theorem it is clear that  $q'$  is (SR)-reacheable from  $q$ . Moreover, since there exist (SR)-normally reachable points from  $q'$  arbitrarily close to  $q'$  (see e.g., [1, Lemma 3.21]), follows that  $q'$  is always (SR)-normally reacheable from  $q$  by  $\xi$  such that  $\sum_j |\xi_j| \leq d_{\text{SR}}(q, q') + \varepsilon/2$ . Hence, by Lemma 3.13, if  $\varepsilon$  and  $\eta > 0$  are sufficiently small, we have that  $q'$  is (TD)-normally reachable from  $q$  at time  $t_0$  by  $\xi'$  such that  $\sum_j |\xi'_j| \leq d_{\text{SR}}(q, q') + \varepsilon$  and  $T < t_1$ . This clearly implies that

$$\inf\{c(\gamma): \gamma \text{ is (TD)-admissible, } \gamma(t_0) = q \text{ and } \gamma(t_1) = q'\} \leq d_{\text{SR}}(q, q') + \varepsilon.$$

Finally, the lemma follows letting  $\varepsilon \downarrow 0$ .  $\square$

We now prove the main theorem of the section.

*Proof of Theorem 3.12.* By Lemma 3.14, we only need to prove the continuity of  $\rho$ . We will prove only the lower semicontinuity, sinche the upper semicontinuity follows by similar arguments.

We start by proving the lower semicontinuity of  $\rho(q, \cdot)$  at  $q'$ . Consider a sequence  $q_k \rightarrow q'$  and let  $u_k \in L^1([0, T_k], \mathbb{R}^m)$  be controls such that each one steers system (TD) from  $q$  to  $q_k$  and  $\liminf_n \rho^{f_0}(q, q_k) = \liminf_n \|u_k\|_{L^1}$ . Then, by Lemma 3.14, for any  $\varepsilon > 0$  there exists a sequence

of  $\tilde{T}_k > 0$  and a sequence of controls  $v_k \in L^1([T_k, \tilde{T}_k], \mathbb{R}^m)$  all steering system (TD) from  $q_k$  to  $q'$  and such that  $\|v_k\|_{L^1([T_k, \tilde{T}_k], \mathbb{R}^m)} \leq d_{\text{SR}}(q_k, q') + \varepsilon$ . Since  $d_{\text{SR}}(q_k, q') \rightarrow 0$ , this implies that

$$\rho(q, q') \leq \lim_{n \rightarrow \infty} \left( \|u_k\|_{L^1([0, T_k], \mathbb{R}^m)} + \|v_k\|_{L^1([T_k, \tilde{T}_k], \mathbb{R}^m)} \right) = \liminf_n \rho(q, q_k) + \varepsilon.$$

Letting  $\varepsilon \downarrow 0$  proves that  $\rho(q, \cdot)$  is lower semicontinuous at  $q'$ .

In order to prove the lower semicontinuity of  $\rho(\cdot, q')$  at  $q$ , let us define

$$\varphi_\varepsilon(p) = \inf\{c(\gamma) : \gamma : [\varepsilon, T] \subset I \rightarrow M \text{ is (TD)-admissible and } \gamma : p \rightsquigarrow q'\}.$$

We claim that for any  $p \in M$  it holds that  $\varphi_\varepsilon(p) \rightarrow \rho(p, q')$  as  $\varepsilon \downarrow 0$ . Since it is clear that  $\varphi_\varepsilon(\cdot) \geq \rho(\cdot, q')$ , it suffices to prove that

$$(14) \quad \lim_{\varepsilon \downarrow 0} \varphi_\varepsilon(p) \leq \rho(p, q') \quad \text{for any } p \in M.$$

To this aim, fix  $p \in M$  and  $\eta > 0$  and let  $\gamma : [0, T] \rightarrow M$  be such that  $c(\gamma) \leq \rho(p, q') + \eta$ . It is clear that  $\gamma(2\varepsilon) \rightarrow p$  as  $\varepsilon \downarrow 0$ , and hence that  $\rho(p, \gamma(2\varepsilon)) \rightarrow 0$  as  $\varepsilon \downarrow 0$ , by the first part of the proof. Thus, for any  $\varepsilon > 0$  sufficiently small, there exists a (TD)-admissible curve  $\gamma_\varepsilon : [\varepsilon, 2\varepsilon] \rightarrow M$  such that  $\gamma_\varepsilon : p \rightsquigarrow \gamma(2\varepsilon)$  and  $c(\gamma_\varepsilon) \leq \rho(p, \gamma(2\varepsilon)) + \eta$ . By concatenating  $\gamma_\varepsilon$  with  $\gamma|_{[2\varepsilon, T]}$ , we get that

$$\varphi_\varepsilon(p) \leq c(\gamma_\varepsilon) + c(\gamma) \leq \rho(p, \gamma(2\varepsilon)) + \rho(p, q') + 2\eta.$$

Letting  $\varepsilon \downarrow 0$  and then  $\eta \downarrow 0$ , this proves (14) and thus the claim.

Let now  $q_k \rightarrow q$  and fix  $\eta > 0$ . By Lemma 3.14 this implies that  $\rho(q_k, q) \rightarrow 0$  and that for any  $\varepsilon > 0$  sufficiently small, there exists a (TD)-admissible curve  $\gamma_\varepsilon : [0, \varepsilon] \rightarrow M$  such that  $\gamma_\varepsilon : q_k \rightsquigarrow q$  and  $c(\gamma_\varepsilon) \leq \rho(q_k, q) + \eta$ . Hence

$$\rho(q_k, q') \leq c(\gamma_\varepsilon) + \varphi_\varepsilon(q) \leq \rho(q_k, q) + \varphi_\varepsilon(q) + \eta.$$

By the previous claim, letting  $\varepsilon, \eta \downarrow 0$ , this implies that  $\rho(q_k, q') \leq \rho(q_k, q) + \rho(q, q')$ . Since  $\rho(q_k, q) \rightarrow 0$ , taking the liminf as  $k \rightarrow +\infty$ , this proves the lower semicontinuity of  $\rho(\cdot, q')$  at  $q$ , completing the proof.  $\square$

**Remark 3.15.** From the proof of Theorem 3.12, it follows that hypothesis (H<sub>0</sub>) is not sharp. Indeed, the following is sufficient to prove the theorem.

- The family of smooth vector fields  $\{f_1^t, \dots, f_m^t\}_{t \in I}$ , depends smoothly on  $t$ ,  
(H<sub>1</sub>) and satisfies the strong Hörmander condition at  $t = 0$  and in an open neighborhood of  $\sup I$ .

We will conclude this section by showing that, in our framework, it is necessary to assume the Hörmander condition on both ends of  $I$ . Although outside the scope of the present work, we remark that stronger assumptions on the regularity of the vector fields, i.e., that they are uniformly Lipschitz, would allow to prove Theorem 3.12 assuming only that  $\{f_1^t, \dots, f_m^t\}_{t \in I}$  satisfies the Hörmander condition at one time  $t_0 \in I$ .

The following example proves that if the family  $\{f_1^t, \dots, f_m^t\}_{t \in I}$  satisfies the Hörmander condition only near  $t = 0$ , then the value function is in general not continuous. Through a slight modification, the same argument can also be used to prove that the same holds if the Hörmander condition is satisfied only at a neighborhood of  $\sup I$  or of any  $t_0 \in I$ .

**Example 3.16.** Let  $M = (-2, 2) \times (-1, +\infty)$ , with coordinates  $(x, y)$ , and consider the vector fields

$$f(x, y) = \frac{((y+1)(1-x^2), -x)}{\sqrt{(y+1)^2(1-x^2)^2 + x^2}}, \quad g(x, y) = \frac{(x, h(x)(y+2))}{\sqrt{x^2 + h(x)^2(y+2)^2}},$$

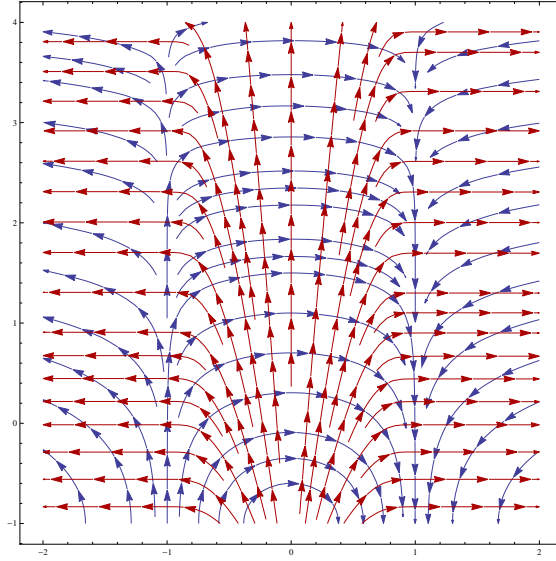


FIGURE 1. The two vector fields of Example 3.16 with  $h(x) = ce^{-\frac{1}{1-x^2}}$  for  $x \in [-1, 1]$ .

where  $h : [-2, 2] \rightarrow \mathbb{R}$  is a smooth cutoff function such that  $\text{supp } h \subset [-1, 1]$ ,  $h \geq 0$  and  $h(0) = 1$  (see Figure 1). Fix  $0 < \varepsilon < 1$ ,  $C \geq 16$  and let  $\phi, \psi : [0, 1] \rightarrow \mathbb{R}$  be two smooth functions such that

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \varepsilon, \\ 0 & \text{if } 2\varepsilon \leq t \leq 1, \end{cases} \quad \psi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 2\varepsilon, \\ C & \text{if } 3\varepsilon \leq t \leq 1, \end{cases}$$

and such that  $\phi$  is nonincreasing while  $\psi$  is nondecreasing. Finally, consider the time-dependent system on  $M$  specified by the vector fields  $f^t(x, y) = \phi(t)f(x, y)$  and  $g^t(x, y) = \psi(t)g(x, y)$ ,  $t \in [0, 1]$ . We will show that  $\{f^t, g^t\}$  satisfies the Hörmander condition for  $t \in [0, \varepsilon]$ , but that the value function associated with the family  $\{f^t, g^t\}_{t \in [0, 1]}$  is not lower semicontinuous.

We start by showing that  $f(p)$  and  $g(p)$  are transversal for any  $p = (x, y) \in M$ , implying the Hörmander condition for  $\{f^t, g^t\}$ ,  $t \in [0, \varepsilon]$ . If  $x \in (-2, -1] \cup [1, 2)$ , then, by definition of  $h$ ,  $g(p) = (1, 0)$  is clearly transversal to  $f(p)$ . On the other hand, if  $x \in (-1, 1) \setminus \{0\}$  and  $g(p)$  is parallel to  $f(p)$ , a simple computation shows that  $h(x) < 0$ , which is a contradiction. Finally, for  $x = 0$ , it is clear that  $g(p) = (0, y + 2)$  and  $f(p) = (y + 1, 0)$  are never parallel. We remark that this implies also that the value function  $\rho_\varepsilon$ , induced by controls defined on  $[0, \varepsilon]$ , is a distance equivalent to the Euclidean one. In particular,  $|p_1 - p_2| \leq 2\rho_\varepsilon(p_1, p_2)$  for any  $p_1, p_2 \in M$ .

Fix now  $q' = (1, 0)$ . The set of points from which  $q'$  is reachable using only  $f$  is exactly  $\mathcal{O}_{q'} = \{(1, y) : y > -1\}$ . Let then  $q_0 \in (-1, 0) \times \{0\}$  be such that  $\rho_\varepsilon(q_0, (-1, 0)) \leq \frac{1}{4} \min_{p \in \mathcal{O}_{q'}} \rho_\varepsilon(q_0, p)$ . In order to show that  $\rho_1(q_0, \cdot)$  is not lower semicontinuous at  $q'$ , consider any sequence  $\{q_n\}_{n \in \mathbb{N}} \subset (1/2, 1) \times \{0\}$  such that  $q_n \rightarrow q'$ . By continuity of  $\rho_\varepsilon$  and the fact that  $-q_n \rightarrow (-1, 0)$ , we can always assume that, up to subsequences,  $\rho_\varepsilon(q_0, -q_n) \leq \frac{1}{2} \min_{p \in \mathcal{O}_{q'}} \rho_\varepsilon(q_0, p)$ .

Since  $g^t \equiv 0$  for  $t \geq 2\varepsilon$ , if  $u \in L^1([0, 1], \mathbb{R}^2)$  is a control steering the system from  $q_0$  to  $q'$ , the control  $u|_{[0, 2\varepsilon]}$  steers the system from  $q_0$  to some  $p \in \mathcal{O}_{q'}$ . Exploiting the fact that  $\rho_{2\varepsilon} \geq \rho_\varepsilon$  by monotonicity of  $\psi$ , this implies that

$$(15) \quad \rho_1(q_0, q') \geq \min_{p \in \mathcal{O}_{q'}} \rho_\varepsilon(q_0, p) \geq 2\rho_\varepsilon(q_0, -q_n).$$



Let now  $u \in L^1([0, 1], \mathbb{R}^m)$  be the control constructed as follows. From time 0 to  $\varepsilon$ ,  $u|_{[0, \varepsilon]}$  is the minimizer of  $\rho_\varepsilon$  steering the system from  $q_0$  to  $-q_n$ . Then,  $u|_{(\varepsilon, 3\varepsilon)} \equiv 0$  and, after this, the control acts only on  $f^t$  for time  $t \in [3\varepsilon, 1]$ , steering the system from  $-q_n$  to  $q_n$ . Hence,

$$(16) \quad |q_n - (-q_n)| = \left| \int_{3\varepsilon}^1 u(t) f^t(x(t), y(t)) dt \right| = C \int_{3\varepsilon}^1 |u(t)| dt.$$

Since  $|q_n - (-q_n)| < 2$ ,  $C \geq 16/|q_0 - q'| \geq 8/\rho_\varepsilon(q_0, q')$ , and by (16), it holds that

$$\rho_1(q_0, q_n) \leq \int_0^1 |u(t)| dt = \rho_\varepsilon(q_0, -q_n) + \frac{1}{C} |q_n - (-q_n)| \leq \frac{3}{4} \rho_1(q_0, q').$$

Taking the  $\liminf$  as  $n \rightarrow \infty$  shows that  $\rho_1(q_0, \cdot)$  is not l.s.c. at  $q'$ .

**3.3. Estimates on reachable sets.** In this section, we concentrate on a particular class of time-dependent systems. Namely, let  $\{f_1, \dots, f_m\}$  be a bracket-generating family of smooth vector fields,  $f_0$  be a smooth vector field, and consider the time-dependent system

$$(17) \quad \dot{q} = \sum_{i=1}^m u_i f_i^t, \quad f_i^t = (e^{-tf_0})_* f_i(q), \quad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m.$$

Here,  $(e^{-tf_0})_*$  is the push-forward operator associated with the flow of  $f_0$ .

As we will see in the next section, this class of systems arises naturally when dealing with control systems that are affine with respect to the control. Observe, in particular, that from the bracket-generating property of  $\{f_1, \dots, f_m\}$  it follows immediately that the time-dependent family  $\{(e^{-tf_0})_* f_1, \dots, (e^{-tf_0})_* f_m\}$  satisfies the strong Hörmander condition, as per Definition 3.11.

Before proceeding with the estimates of the reachable sets, we need to define a suitable approximation of system (17). Namely, fix a system of privileged coordinates (in the sub-Riemannian sense) at  $q$  for  $\{f_1, \dots, f_m\}$ . Assume that  $f_0(q) \neq 0$ , and let  $s \in \{1, \dots, r\}$  be such that  $\text{ord}_q(f_0) = -s$ . In this case, there exist, in coordinates, an homogeneous vector field  $f_0^{-s}$ , of weighted degree  $-s$ , and a vector field  $f_0^{>-s}$ , of weighted degree  $\geq -s + 1$ , such that

$$(18) \quad z_* f_0 = f_0^{-s} + f_0^{>-s}.$$

In particular, it holds that  $f_0^{-s} \neq 0$  near  $z(q) = 0$ .

**Remark 3.17.** The fact that  $\text{ord}_q(f_0) = -s$  is not equivalent, in general, to  $f_0 \in \Delta^s$  near  $q$ . In particular, if the growth vector is non-constant around  $q$ , from  $\text{ord}_q(f_0) \geq -s$  it does not follow that  $f_0 \in \Delta^s$ . For example, consider the sub-Riemannian control system on  $\mathbb{R}^2$  with (privileged) coordinates  $(x, y)$ , defined by the vector fields  $\partial_x$  and  $x\partial_y$ , called the Grushin plane. Outside  $\{x = 0\}$ , the non-holonomic degree of these vector fields is 1, while, on  $\{x = 0\}$ , we need one bracket to generate the  $y$  direction, and thus it is 2. Hence, if  $\bar{y} \neq 0$  the vector field  $y\partial_y$  is never in  $\Delta$  near  $(0, \bar{y})$ , but  $\text{ord}_{(0, \bar{y})}(y\partial_y) = \text{ord}_{(0, \bar{y})}(y) + \text{ord}_{(0, \bar{y})}(\partial_y) = 0$ .

However, due to the properties (8) and the fact that  $\Delta^s$  is a module, the converse is always true. Namely, if  $f_0 \in \Delta^s$ , then  $\text{ord}_q(f_0) \geq -s$ .

For any smooth vector field  $f$ , let  $(\text{ad}^1 f_0)f = [f_0, f]$  and  $(\text{ad}^\ell f_0)f = [f_0, (\text{ad}^{\ell-1} f_0)f]$ , for any  $\ell \in \mathbb{N}$ . We recall (see for example [17]) that the following Taylor expansion holds

$$(19) \quad (e^{-tf_0})_* f \sim \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} (\text{ad}^\ell f_0)f.$$

Since  $\text{ord}_q(f_j) \geq -1$ , by (8) we have that  $\text{ord}_q((\text{ad}^\ell f_0)f_j) \geq -\ell s - 1$ . Then, using the decomposition (18), for any  $\ell \geq 0$ , there exists, in coordinates, an homogeneous vector field  $F_j^\ell$  of weighted

degree  $-\ell s$ , and a remainder  $r^\ell$  of order  $\geq -\ell s - 1$ , such that

$$(20) \quad z_*[(\text{ad}^\ell f_0)f_j] = F_j^\ell + r^\ell.$$

**Definition 3.18.** The *homogeneous series approximation* at  $q$  of  $f_j^t$ ,  $1 \leq j \leq m$ , associated with the privileged coordinates  $z$ , is the vector field with coordinate representation

$$(21) \quad \hat{f}_j^t = \sum_{\ell=0}^{\varrho} \frac{t^\ell}{\ell!} F_j^\ell,$$

where  $\varrho = \lfloor r^{-1}/s \rfloor$  and  $r$  is the non-holonomic degree of  $\{f_1, \dots, f_m\}$  at  $q$ . The approximated time-dependent control system is then defined as

$$(ATD) \quad \dot{q} = \sum_{j=1}^m u_j(t) \hat{f}_j^t(q).$$

If a system, in some system of privileged coordinates, coincides with its homogeneous series approximation, we will say that it is series homogeneous.

The homogeneous series approximation encodes the idea that the time  $t$  is of weight  $s = -\text{ord}_q(f_0)$ . This is a consequence of the fact that, due to the expansion (19),  $t$  allows to build brackets of  $f_0$  with the  $f_j$ s. In this sense, the homogeneous series approximation is a generalization of the nilpotent approximation.

We are now ready to state the main theorem of this section.

**Theorem 3.19.** *Let  $\{f_1, \dots, f_m\}$  satisfy the Hörmander condition, let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q \in M$  for  $\{f_1, \dots, f_m\}$ . Then there exist  $C, T, \varepsilon_0 > 0$  such that, for any  $\varepsilon < \varepsilon_0$  and any  $q' \in \mathcal{R}_T(q, \varepsilon)$ , setting  $s = -\text{ord}_q(f_0)$  it holds*

$$(22) \quad |z_i(q')| \leq C \left( \varepsilon^{w_i} + \varepsilon T^{\frac{w_i}{s}} \right) \quad \text{if } w_i \leq s,$$

$$(23) \quad |z_i(q')| \leq C \varepsilon \left( \varepsilon + T^{\frac{1}{s}} \right)^{w_i - 1} \quad \text{if } w_i > s.$$

Moreover, if the system is series homogeneous, then it holds the stronger estimate

$$(24) \quad |z_i(q')| \leq C \varepsilon^{w_i} \quad \text{if } w_i \leq s.$$

To prove this theorem we need the following proposition, estimating the difference between (17) and (ATD).

**Proposition 3.20.** *Let  $\{f_1, \dots, f_m\}$  satisfy the Hörmander condition, and let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q \in M$  for  $\{f_1, \dots, f_m\}$ . For  $T > 0$  and  $u \in L^1([0, T]; \mathbb{R}^m)$ , let  $\gamma(\cdot)$  and  $\hat{\gamma}(\cdot)$  be the trajectories associated with  $u$  in (17) and (ATD), respectively, and such that  $\gamma(0) = \hat{\gamma}(0) = q$ . Then there exist  $C, \varepsilon_0, T_0 > 0$ , independent of  $u$ , such that, if  $t < T_0$  and  $\int_0^t |u| ds = \varepsilon < \varepsilon_0$ , and setting  $s = -\text{ord}_q(f_0)$  it holds*

$$(25) \quad |z_i(\gamma(t)) - z_i(\hat{\gamma}(t))| \leq C \varepsilon \left( \varepsilon + t^{\frac{1}{s}} \right)^{w_i}, \quad i = 1, \dots, n.$$

**Remark 3.21.** This proposition generalizes Proposition 2.6. In fact, in the sub-Riemannian case, since  $f_0 \equiv 0$ , if, for  $t > 0$ , the curve  $\gamma$  is associated to  $u \in L^1([0, t], \mathbb{R}^m)$ , it is associated also to  $u_\tau(\cdot) = \frac{\tau}{t} u(\frac{\cdot}{t})$ , for any  $\tau > 0$ . Thus, since  $\int_0^\tau |u_\tau| ds = \int_0^t |u| ds = \varepsilon$ , (25) reduces to

$$|z_i(\gamma(t)) - z_i(\hat{\gamma}(t))| \leq \lim_{\tau \downarrow 0} C (\varepsilon^{w_i+1} + \tau \varepsilon^\tau) = C \varepsilon^{w_i+1}.$$

Finally, assuming that  $u$  satisfies the hypotheses of Proposition 2.6, i.e., that  $|u| = 1$ , we get  $t = \varepsilon$ .

*Proof.* Let  $z(\gamma(\cdot)) = x(\cdot)$ ,  $z(\hat{\gamma}(\cdot)) = y(\cdot)$ , and  $\|z\| = \sum_{\ell=1}^n |z_\ell|^{1/w_i}$ . We mimic the proof of Proposition 7.29 in [8]. The first step is to prove that there exists a constant  $C > 0$  such that  $\|x(t)\|, \|y(t)\| \leq C\varepsilon$  for  $t$  and  $\varepsilon = \int_0^t |u| ds$  small enough. We prove this for  $\|x(t)\|$ , the same argument works also for  $\|y(t)\|$ .

In  $z$  coordinates, the equation of the control system (17) is,

$$\dot{x}_i(t) = \sum_{j=1}^m u_j(t) (z_i)_* f_j^t(\gamma(t)), \quad i = 1, \dots, n.$$

Due to the fact that  $z_* f_j^t = z_* f_j + \mathcal{O}(t)$ , uniformly in a neighborhood of  $q$  that  $\text{ord}_q(z_i) = w_i$  and that  $\text{ord}_q(f_j) \geq -1$ , we have that there exist  $T_0$  and  $C > 0$  such that  $|(z_i)_* f_j^t(q)| \leq \frac{C}{2} |(z_i)_* f_j(q)| \leq C \|x(t)\|^{w_i-1}$ , for any  $t < T_0$ . Thus we get

$$(26) \quad |\dot{x}_i(t)| \leq C \sum_{j=1}^m |u_j(t)| \|x_i(t)\|^{w_j-1}.$$

As in the proof for the sub-Riemannian case, choosing  $N$  sufficiently large, so that all  $N/w_i$  are even integers, and setting  $\|z\| = (\sum_{\ell=1}^n |z_\ell|^{N/w_i})^{1/N}$  we get a norm equivalent to  $\|z\|$ . Deriving with respect to time and using (26) we get  $\frac{d}{dt} \|x(t)\| \leq C \sum_{j=1}^n |u_j(t)|$ . Finally, by integration, equivalence of the norms, and the fact that  $x(0) = z(q) = 0$ , we conclude that  $\|x(t)\| \leq C\varepsilon$ .

Now we move to proving (25). By construction of (ATD) and the Taylor expansion of  $f_j^t$ , for any  $\ell \leq \varrho = \lfloor r-1/s \rfloor$ , there exist homogeneous polynomials  $h_{ji}^\ell$  of order  $w_i - \ell s - 1$  and remainders  $r_{ji}^\ell$  of order larger than or equal to  $w_i - \ell s$ , such that we can write

$$(z_i)_* f_j^t = \sum_{\ell=0}^{\varrho} \frac{t^\ell}{\ell!} (h_{ji}^\ell + r_{ji}^\ell) + \mathcal{O}(t^{\varrho+1}),$$

$$(z_i)_* \hat{f}_j^t = \sum_{\ell=0}^{\varrho} \frac{t^\ell}{\ell!} h_{ji}^\ell.$$

Here, the  $\mathcal{O}$  is intended as  $t \downarrow 0$  and is uniform in a compact neighborhood of the origin. Then,

$$\begin{aligned} \dot{x}_i(t) - \dot{y}_i(t) &= \sum_{j=1}^m u_j(t) \left( \sum_{\ell=0}^{\varrho} \frac{t^\ell}{\ell!} (h_{ji}^\ell(x) - h_{ji}^\ell(y) + r_{ji}^\ell(x)) + \mathcal{O}(t^{\varrho+1}) \right) \\ &= \sum_{j=1}^m u_j(t) \left( \sum_{\ell=0}^{\varrho} \frac{t^\ell}{\ell!} \left( \sum_{w_k < w_i - \ell s} (x_k(t) - y_k(t)) Q_{jik}^\ell(x, y) + r_{ji}^\ell(x) \right) + \mathcal{O}(t^{\varrho+1}) \right), \end{aligned}$$

where  $Q_{jik}^\ell$  are homogeneous polynomial in  $x$  and  $y$ , of order  $w_i - w_k - \ell s - 1$ . We observe that, if  $w_i - w_k - \ell s - 1 < 0$ , then  $Q_{jik}^\ell \equiv 0$ . Thus, for sufficiently small  $\|x\|$  and  $\|y\|$ , we have

$$|Q_{jik}^\ell(x, y)| \leq C (\|x\|^{(w_i - w_k - \ell s - 1)^+} + \|y\|^{(w_i - w_k - \ell s - 1)^+}), \quad |r_{ji}^\ell(x)| \leq C \|x\|^{(w_i - \ell s)^+}.$$

Here we let  $(\xi)^+ = \max\{\xi, 0\}$ , for any  $\xi \in \mathbb{R}$ . Using the inequalities of the first step, taking  $t < T$  sufficiently small, and enlarging the constant  $C$ , we get

$$\begin{aligned} &|\dot{x}_i(t) - \dot{y}_i(t)| \\ &\leq C |u(t)| \left( \sum_{\ell=0}^{\varrho} \frac{t^\ell}{\ell!} \left( \sum_{w_k < w_i - \ell s} |x_k(t) - y_k(t)| \varepsilon^{w_i - w_k - \ell s - 1} + \varepsilon^{(w_i - \ell s)^+} \right) + t^{\varrho+1} \right) \\ &\leq C |u(t)| \left( \sum_{\ell=0}^{\varrho} t^\ell \left( \sum_{w_h < w_i} |x_h(t) - y_h(t)| \varepsilon^{w_i - w_h - 1} + \varepsilon^{(w_i - \ell s)^+} \right) + t^{\varrho+1} \right). \end{aligned}$$

In the last inequality we applied the change of variable  $\bar{w}_k \mapsto w_h - \ell s$  in each of the sums.

We can integrate the system by induction, since it is in triangular form. For  $w_i = 1$ , since  $(w_i - \ell s)^+ = 0$  for any  $\ell \geq 1$ , the inequality reduces to

$$|\dot{x}_i(t) - \dot{y}_i(t)| \leq C|u(t)| \left( \sum_{\ell=0}^{\varrho} t^\ell \varepsilon^{(w_i - \ell s)^+} + t^{\varrho+1} \right) \leq C|u(t)|(\varepsilon^{w_i} + t).$$

Here we enlarged the constant  $C$ . Thus, integrating the previous inequality, we get  $|x_i(t) - y_i(t)| \leq C\varepsilon(\varepsilon^{w_i} + t) \leq C\varepsilon(\varepsilon + t^{\frac{1}{s}})^{w_i}$ .

Let, then,  $w_i > 1$  and assume that  $|x_h(t) - y_h(t)| \leq C\varepsilon(\varepsilon + t^{\frac{1}{s}})^{w_h}$  for any  $w_h < w_i$ . To complete the proof it suffices to show that  $|\dot{x}_i(t) - \dot{y}_i(t)| \leq C|u(t)|(\varepsilon + t^{\frac{1}{s}})^{w_i}$ , since (25) will follow, as above, by integration. Thus, we have, enlarging again the constant  $C$  and taking  $t$  sufficiently small,

$$\begin{aligned} & |\dot{x}_i(t) - \dot{y}_i(t)| \\ (27) \quad & \leq C|u(t)| \left( \sum_{\ell=0}^{\varrho} t^\ell \left( \sum_{w_h < w_i} \left( \varepsilon + t^{\frac{1}{s}} \right)^{w_h} \varepsilon^{w_i - w_h} + \varepsilon^{(w_i - \ell s)^+} \right) + t^{\varrho+1} \right) \\ & \leq C|u(t)| \left( \sum_{\ell=0}^{\varrho} t^\ell \left( \sum_{w_h < w_i} t^{\frac{w_h}{s}} \varepsilon^{w_i - w_h} + \varepsilon^{(w_i - \ell s)^+} \right) + t^{\varrho+1} \right). \end{aligned}$$

If  $t \leq \varepsilon^s$ , from (27) it is clear that  $|\dot{x}_i(t) - \dot{y}_i(t)| \leq C|u(t)|\varepsilon^{w_i}$ . Here we used the fact that  $\varrho + 1 \geq w_i/s$ . On the other hand, if  $\varepsilon < t^{1/s}$ , it holds

$$|\dot{x}_i(t) - \dot{y}_i(t)| \leq C|u(t)| \left( \sum_{\ell=0}^{\varrho} \left( \sum_{w_h < w_i} t^{\frac{w_h}{s} + \ell + \frac{w_i - w_h}{s}} + t^{\ell + \frac{w_i - \ell s}{s}} \right) + t^{\varrho+1} \right) \leq C|u(t)|t^{\frac{w_i}{s}}.$$

Putting together these two estimates, we get that  $|\dot{x}_i(t) - \dot{y}_i(t)| \leq C|u(t)|(\varepsilon^{w_i} + t^{\frac{w_i}{s}}) \leq C|u(t)|(\varepsilon + t^{\frac{1}{s}})^{w_i}$ , completing the proof of the proposition.  $\square$

*Proof of Theorem 3.19.* We start by claiming that (24) implies (22). In fact, if  $\gamma : q \rightsquigarrow q'$  is the trajectory associated in (17) to a control  $u \in L^1([0, T], \mathbb{R}^m)$ , and  $\hat{\gamma}$  is the trajectory associated with the same control in the homogeneous series approximation (ATD), with  $\hat{\gamma}(0) = q$ , it holds

$$|z_i(q')| \leq |z_i(\hat{\gamma}(T))| + |z_i(\hat{\gamma}(T)) - z_i(\gamma(T))|.$$

Thus, by Proposition 3.20, the claim is proved.

Hence, from now on we assume our system to be in the form (ATD). Let us define, for  $1 \leq j \leq n$  and  $0 \leq \alpha \leq r$ , the vector fields  $\varphi_j^\alpha$  as

$$\varphi_j^\alpha = \sum_{\ell=0}^{\alpha} \frac{t^\ell}{\ell!} F_j^\ell,$$

where  $F_j^\ell$  are defined in (20). We do not explicitly denote the dependence on time, to lighten the notation. Observe that, if  $\alpha = \varrho$ , then, by (21),  $\varphi_j^\alpha = \hat{f}_j^t$ .

We claim that, letting  $x^{(\alpha)}(\cdot)$  be the trajectory associated with a control  $u \in L^1([0, T], \mathbb{R}^m)$  in system (TD) with  $\{\varphi_1^\alpha, \dots, \varphi_m^\alpha\}$  as vector fields, then, for some constant  $C > 0$  and any  $i \in \{1, \dots, n\}$  and  $\alpha \geq 1$ , it holds

$$(28) \quad |x_i^{(\alpha)}(T) - x_i^{(\alpha-1)}(T)| \leq \begin{cases} 0 & \text{if } w_i \leq \alpha s, \\ C\varepsilon(\varepsilon + T^{\frac{1}{s}})^{w_i-1} & \text{if } w_i > \alpha s. \end{cases}$$

In fact, due to the homogeneity of the  $F_j^\ell$ , proceeding as in the proof of Proposition 3.20, we get that for  $w_i \leq \alpha s$  it holds

$$|\dot{x}_i^{(\alpha)}(t) - \dot{x}_i^{(\alpha-1)}(t)| \leq C|u(t)| \sum_{\ell=0}^{\alpha-1} t^\ell \sum_{w_h < w_i} |x_h^{(\alpha)}(t) - x_h^{(\alpha-1)}(t)| \varepsilon^{w_i - w_h - 1}.$$

By induction on  $1 \leq w_i \leq \alpha s$ , this proves the first part of the claim. On the other hand, if  $w_i > \alpha s$ , it holds

$$|\dot{x}_i^{(\alpha)}(t) - \dot{x}_i^{(\alpha-1)}(t)| \leq C|u(t)| \left( \sum_{\ell=0}^{\alpha-1} t^\ell \sum_{w_h < w_i} |x_h^{(\alpha)}(t) - x_h^{(\alpha-1)}(t)| \varepsilon^{w_i - w_h - 1} + t^\alpha \varepsilon^{w_i - \alpha s - 1} \right).$$

Then, again by induction over  $w_i$ , we get that  $|x_i^{(\alpha)}(T) - x_i^{(\alpha-1)}(T)| \leq CT^\alpha \varepsilon^{w_i - \alpha s}$ . Finally, the claim follows considering the cases  $T \leq \varepsilon^s$  and  $T > \varepsilon^s$ .

Due to the fact that  $\varphi_j^0 = \widehat{f}_j$ , by Theorem 2.7 it holds  $|x_i^{(0)}(T)| \leq C\varepsilon^{w_i}$ . Thus, applying (28) and enlarging the constant  $C$ , we get

$$|z_i(q')| = |x_i^{(r)}(T)| \leq \sum_{\ell=1}^r |x_i^{(\ell)}(T) - x_i^{(\ell-1)}(T)| + |x_i^{(0)}(T)| \leq \begin{cases} C\varepsilon^{w_i} & \text{if } w_i \leq s, \\ C\varepsilon(\varepsilon + T^{\frac{1}{s}})^{w_i-1} & \text{if } w_i > s. \end{cases}$$

This proves (23) and (24), completing the proof of the theorem.  $\square$

We end this section by showing that the estimate (23) is sharp, at least in some directions. Indeed, for a system which is series homogeneous at  $q$  in some privileged coordinates  $z$ , and satisfies the hypotheses of Theorem 3.19, it holds that  $z_*((\text{ad}^k f_0) f_j)$  is an homogeneous vector field of weighted degree  $-sk - 1$ . Thus, since  $\varepsilon t^k \leq \varepsilon(\varepsilon + t^{\frac{1}{s}})^{sk}$ , the following proposition shows that (23) is sharp in this direction. The proof is an adaption of an argument from [11].

**Proposition 3.22.** *Let  $\{f_1, \dots, f_m\}$  satisfy the Hörmander condition. Let, moreover  $q \in M$ ,  $i \in \{1, \dots, m\}$  and  $k \geq 0$ . Then, for any coordinate system  $y$  at  $q$ , there exist  $T, \varepsilon_0 > 0$  such that, for any  $\varepsilon < \varepsilon_0$  and  $t < T$  there exists a (TD)-admissible curve  $\gamma : [0, t] \rightarrow M$ , with  $c(\gamma) \leq \varepsilon$ , and such that*

$$y(\gamma(t)) = \varepsilon t^k dy((\text{ad}^k f_0) f_j(q)) + \mathcal{O}(\varepsilon t^{k+1}) \quad \text{as } \varepsilon t \rightarrow 0.$$

*Proof.* Let  $t, \eta > 0$  be fixed, and define  $u \in L^1([0, T], \mathbb{R}^m)$  as  $u_i(\tau) \equiv \eta$ ,  $u_j(\tau) \equiv 0$  for  $j \neq i$ ,  $\tau \in [0, t]$ . Then, fix any  $\Phi \in C^k([0, 1])$  such that  $\Phi^{(i)}(0) = \Phi^{(i)}(1) = 0$ , for  $0 \leq i < k$ . Thus, by integrating by parts and the fact that  $\frac{d}{dt}(e^{-\tau f_0})_* g = (e^{-\tau f_0})_*(\text{ad}(f_0)g)$ , we get

$$\int_0^t \Phi^{(k)}(\tau/t)(e^{-\tau f_0})_* f_i(q) d\tau = t^k \int_0^t \Phi(\tau/t)(e^{-\tau f_0})_* \left( (\text{ad}^k f_0) f_i \right) (q) d\tau,$$

for any  $t$  and  $q$ . This implies that the flows generated by  $\Phi^{(k)}(\tau/t)(e^{-\tau f_0})_* f_i$  and  $t^k \Phi(\tau/t)(e^{-\tau f_0})_* \left( (\text{ad}^k f_0) f_i \right)$  coincide. Using the series expansions of the chronological exponential and  $(e^{-\tau f_0})_*$ , see [5, Section

2.4], there holds, then,

$$\begin{aligned}
\overrightarrow{\text{exp}} \int_0^t \sum_{j=1}^m \Phi^{(k)}(\tau/t) u_j(\tau) (e^{-\tau f_0})_* f_j d\tau &= \overrightarrow{\text{exp}} \int_0^t \eta \Phi^{(k)}(\tau/t) (e^{-\tau f_0})_* f_i d\tau \\
&= \overrightarrow{\text{exp}} \int_0^t \eta t^k \Phi(\tau/t) (e^{-\tau f_0})_* \left( (\text{ad}^k f_0) f_i \right) d\tau \\
&= \overrightarrow{\text{exp}} \int_0^1 \eta t^{k+1} \Phi(s) (e^{-ts f_0})_* \left( (\text{ad}^k f_0) f_i \right) ds \\
&= \overrightarrow{\text{exp}} \int_0^1 \eta t^{k+1} \Phi(s) \left( (\text{ad}^k f_0) f_i + \mathcal{O}(t) \right) ds \\
&= \text{Id} + \eta t^{k+1} (\text{ad}^k f_0) f_i + \mathcal{O}(\eta t^{k+2})
\end{aligned}$$

Finally, considering any coordinate system and letting  $\varepsilon = \eta t$ , this completes the proof.  $\square$

#### 4. CONTROL-AFFINE SYSTEMS

In this section we apply the results of Section 3 to control-affine systems. Let  $\{f_1, \dots, f_m\}$  be a bracket-generating family of vector fields,  $f_0$  be a smooth vector field, called the *drift*, and consider the control-affine system

$$(D) \quad \dot{q} = f_0(q) + \sum_{i=1}^m u_i f_i(q), \quad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m.$$

The assumption on  $\{f_1, \dots, f_m\}$  to be bracket-generating, is called *strong Hörmander condition* for (D).

An absolutely continuous curve  $\gamma : [0, T] \rightarrow M$  is (D)-admissible if  $\dot{\gamma}(t) = f_0(\gamma(t)) + f_{u(t)}(\gamma(t))$  for some control  $u \in L^1([0, T], \mathbb{R}^m)$ . Its cost is defined as

$$c_{f_0}(\gamma) = \min \|u\|_{L^1([0, T], \mathbb{R}^m)},$$

where the minimum is taken over all controls  $u$  such that  $\gamma$  is associated with  $u$ . Then, we define the two value functions we are interested in as

$$\begin{aligned}
\rho_T^{f_0}(q, q') &= \inf \{c(\gamma) : \gamma : [0, T'] \rightarrow M \text{ is (D)-admissible, } \gamma : q \rightsquigarrow q', T' \leq T\}, \\
\rho^{f_0}(q, q') &= \inf \{c(\gamma) : \gamma \text{ (D)-admissible and } \gamma : q \rightsquigarrow q'\}.
\end{aligned}$$

It is clear that  $\rho_T^{f_0}(q, q') \searrow \rho^{f_0}(q, q')$  as  $T \rightarrow +\infty$ , for any  $q, q' \in M$ . Moreover, we observe that,  $\rho_T^{f_0}(q, e^{t f_0} q) = 0$  for any  $0 \leq t \leq T$ . Finally, the reachable sets with respect to these value functions, from any  $q \in M$  and for  $\varepsilon, T > 0$ , are

$$\mathcal{R}_T^{f_0}(q, \varepsilon) = \{q' \in M : \rho_T^{f_0}(q, q') < \varepsilon\}, \quad \mathcal{R}^{f_0}(q, \varepsilon) = \{q' \in M : \rho^{f_0}(q, q') < \varepsilon\}.$$

**4.1. Connection with time-dependent systems.** Applying the variations formula (see [5]), system (D) can be written as a composition of a time-dependent system in the form (17) and of a translation along the drift. Namely, for any  $u \in L^1([0, T], \mathbb{R}^m)$ , it holds

$$(29) \quad \overrightarrow{\text{exp}} \int_0^T \left( f_0 + \sum_{i=1}^m u_i(t) f_i \right) dt = e^{T f_0} \circ \overrightarrow{\text{exp}} \int_0^T \sum_{i=1}^m u_i(t) (e^{-t f_0})_* f_i dt.$$

We call *time-dependent system associated with* (D) the following control system,

$$(30) \quad \dot{q} = \sum_{i=1}^m u_i (e^{-t f_0})_* f_i(q), \quad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m.$$

Observe that, since diffeomorphisms preserve linear independence, the strong Hörmander condition for (D), implies that  $\{(e^{-tf_0})_*f_1, \dots, (e^{-tf_0})_*f_m\}_{t \in [0, +\infty)}$  satisfies the strong Hörmander condition, defined in Definition 3.11.

Exploiting these facts, we can prove the following.

**Proposition 4.23.** *Assume that (D) satisfies the strong Hörmander condition. Then, for any  $T > 0$ , the functions  $\rho_T^{f_0}, \rho^{f_0} : M \times M \rightarrow [0, +\infty)$  are continuous. Moreover, letting  $d_{\text{SR}}$  be the sub-Riemannian distance induced by  $\{f_1, \dots, f_m\}$ , for any  $q, q' \in M$  it holds*

$$\rho_T^{f_0}(q, q') \leq \min_{0 \leq t \leq T} d_{\text{SR}}(e^{tf_0}q, q'), \quad \rho^{f_0}(q, q') \leq \min_{t \geq 0} d_{\text{SR}}(e^{tf_0}q, q').$$

*Proof.* The continuity of the two functions, and the fact that  $\rho_T^{f_0}(q, q'), \rho^{f_0}(q, q') \leq d_{\text{SR}}(q, q')$ , for any  $q, q' \in M$ , follows from the same arguments used in Theorem 3.12, adapting Lemmata 3.13 and 3.14 to the system (D). In particular, one has to consider  $(\mathcal{T}, \mathcal{F}, \xi) \mapsto e^{\mathcal{T}f_0} \circ E_{\mathcal{T}, \mathcal{F}}(\xi)$  instead of  $(\mathcal{T}, \mathcal{F}, \xi) \mapsto E_{\mathcal{T}, \mathcal{F}}(\xi)$ .

To prove the second part of the statement, we let, for any  $t \in [0, T)$ ,

$$\varphi_t(p) = \inf\{c(\gamma) : \gamma : [t, T'] \rightarrow M \text{ is (D)-admissible, } \gamma : p \rightsquigarrow q', T' \leq T\}.$$

It is clear that, as above, it holds  $\varphi_t(p) \leq d_{\text{SR}}(p, q')$ . Moreover, we observe that  $\rho_T^{f_0}(q, e^{tf_0}q) = 0$  for any  $0 \leq t < T$ , and hence that for any such  $t$  it holds

$$\rho_T^{f_0}(q, q') \leq \varphi_t(e^{tf_0}q) \leq d_{\text{SR}}(e^{tf_0}q, q').$$

Taking the minimum for  $0 \leq t < T$ , proves the inequality regarding  $\rho_T^{f_0}$ . To complete the proof it suffices to observe that  $\rho^{f_0}(q, q') \leq \rho_T^{f_0}(q, q')$  for any  $T > 0$ .  $\square$

We point out that in system (D), as in time-dependent systems, the existence of minimizers is not assured. Moreover, this lack of minimizers is possible even if they exist for the associated time-dependent system, as the following example points out.

**Example 4.24.** Consider the following vector fields on  $\mathbb{R}^3$ , with coordinates  $(x, y, z)$ ,

$$f_1(x, y, z) = \partial_x, \quad f_2(x, y, z) = \partial_y + x\partial_z.$$

Since  $[f_1, f_2] = \partial_z$ ,  $\{f_1, f_2\}$  is a bracket-generating family of vector fields. The sub-Riemannian control system associated to  $\{f_1, f_2\}$  on  $\mathbb{R}^3$  corresponds to the Heisenberg group.

Let now  $f_0 = \partial_z$  be the drift. Since  $[f_1, \partial_z] = [f_2, \partial_z] = 0$  it holds that  $(e^{-tf_0})_*f_1 = f_1$  and  $(e^{-tf_0})_*f_2 = f_2$ . Hence, the associated time-dependent system is actually not time-dependent. Thus, by (29), a curve  $\gamma : [0, T] \rightarrow \mathbb{R}^3$  is (SR)-admissible for  $\{f_1, f_2\}$  if and only if  $\tilde{\gamma}(\cdot) = e^{f_0} \circ \gamma(\cdot)$  is (D)-admissible. As a consequence of this, for any  $q \in \mathbb{R}^3$  and any  $\varepsilon > 0$ ,

$$\mathcal{R}^{f_0}(q, \varepsilon) = \bigcup_{t \geq 0} e^{tf_0} \circ B_{\text{SR}}(q, \varepsilon).$$

As pointed out in Section 2, minimizers for the sub-Riemannian system exist between any pair of points in  $B_{\text{SR}}(q, \varepsilon)$ , if  $\varepsilon$  is sufficiently small. Let us show that, for any point in  $\mathcal{R}^{f_0}(q, \varepsilon)$  with positive  $z$  coordinate, we have an explicit minimizer, while for the others there exists no minimizer. Without loss of generality we can consider  $q = 0$ . Then, since  $e^{t'f_0}(x', y', z') = (x', y', z' + t')$ , every point  $(x, y, z) \in \mathcal{R}^{f_0}(0, \varepsilon)$  with  $z > 0$ , can be reached optimally considering the sub-Riemannian minimizing curve between the origin and  $(x, y, 0)$  rescaled on time  $z$ .

If, instead,  $z \leq 0$ , we cannot construct any sub-Riemannian trajectory from 0 to  $(x, y, z - t)$ ,  $t > 0$ , with cost  $\leq d_{\text{SR}}(0, (x, y, z))$ . This is due to the fact that the minimizing trajectories in Heisenberg group are the lifts of arcs on the plane  $(x, y)$ , spanning area equal to the  $z$  coordinates, and that  $|z - t| = -z + t > |z|$ . Since, by Proposition 4.23,  $\rho^{f_0}(0, (x, y, z)) \leq d_{\text{SR}}(0, (x, y, z))$ , this implies that there exists no minimizer for  $\rho^{f_0}(q, (x, y, z))$ .

**4.2. Estimates on reachable sets.** In this section we apply Theorem 3.19, in order to obtain results in the spirit of Theorem 2.7 and Corollary 2.8. First, we need the following definition.

**Definition 4.25.** The point  $q$  is said to be *regular with respect to the drift  $f_0$* , if  $q' \mapsto \text{ord}_{q'}(f_0)$  is locally constant at  $q$ .

The main result of this section are the following local regularity estimates for  $\rho^{f_0}$ . We cannot expect anything global, since in general the sets  $\mathcal{R}^{f_0}(q, \varepsilon)$  are noncompact.

**Theorem 4.26.** *Assume that (D) satisfies the strong Hörmander condition, and let  $q$  be regular with respect to the drift  $f_0$ . Assume, moreover, that  $z = (z_1, \dots, z_n)$  is a system of privileged coordinates at  $q$  for  $\{f_1, \dots, f_m\}$ , such that  $z_* f_0 = \partial_{z_k}$ , for some  $1 \leq k \leq n$ . Then, there exist  $T_0, \varepsilon_0, C > 0$  such that*

$$\frac{1}{C} \text{dist} \left( z(q'), z(e^{[0, T] f_0} q) \right) \leq \rho_T^{f_0}(q, q') \leq C \text{dist} \left( z(q'), z(e^{[0, T] f_0} q) \right)^{1/r}, \quad q' \in \mathcal{R}_T^{f_0}(q, \varepsilon),$$

where, for any  $x \in \mathbb{R}^n$  and  $A \subset \mathbb{R}^n$ ,  $\text{dist}(x, A) = \inf_{y \in A} |x - y|$  is the Euclidean distance between them and  $r$  is the degree of non-holonomy of  $\{f_1, \dots, f_m\}$  at  $q$ .

Let us define the following sets, for parameters  $\eta > 0$  and  $T > 0$ . We remark that  $\text{Box}(\eta)$  is defined as in (10) and that  $\{\partial_{z_i}\}_{i=1}^n$  is the canonical basis in  $\mathbb{R}^n$ .

$$\begin{aligned} \Xi_T(\eta) &= \bigcup_{0 \leq \xi \leq T} \left( \xi \partial_{z_k} + \text{Box}(\eta) \right), \\ \Pi_T(\eta) &= \text{Box}(\eta) \cup \bigcup_{0 < \xi \leq T} \{z \in \mathbb{R}^n : z_k = \xi, |z_i| \leq \eta^{w_i} + \eta \xi^{\frac{w_i}{s}} \text{ for } w_i \leq s, i \neq k, \\ &\quad \text{and } |z_i| \leq \eta(\eta + \xi^{\frac{1}{s}})^{w_i-1} \text{ for } w_i > s\}, \\ \widehat{\Pi}_T(\eta) &= \text{Box}(\eta) \cup \bigcup_{0 < \xi \leq T} \{z \in \mathbb{R}^n : z_k = \xi, |z_i| \leq \eta^{w_i} \text{ for } w_i \leq s, i \neq k, \\ &\quad \text{and } |z_i| \leq \eta(\eta + \xi^{\frac{1}{s}})^{w_i-1} \text{ for } w_i > s\}. \end{aligned}$$

As in the sub-Riemannian case, Theorem 4.26 is a direct consequence of some estimates on the shape of the accessible sets, contained in the following.

**Theorem 4.27.** *Assume that (D) satisfies the strong Hörmander condition, and let  $q \in M$  be regular with respect to the drift  $f_0$ . Assume, moreover, that  $z = (z_1, \dots, z_n)$  is a system of privileged coordinates at  $q$  for  $\{f_1, \dots, f_m\}$ , such that  $z_* f_0 = \partial_{z_k}$ , for some  $1 \leq k \leq n$ . Then, there exist  $C, \varepsilon_0, T_0 > 0$  such that*

$$(31) \quad \Xi_T \left( \frac{1}{C} \varepsilon \right) \subset \mathcal{R}_T^{f_0}(q, \varepsilon) \subset \Pi_T(C\varepsilon), \quad \text{for } \varepsilon < \varepsilon_0 \text{ and } T < T_0.$$

Here, with abuse of notation, we denoted by  $\mathcal{R}_T^{f_0}(q, \varepsilon)$  the coordinate representation of the reachable set. In particular,

$$\text{Box} \left( \frac{1}{C} \varepsilon \right) \cap \{z_k \leq 0\} \subset \mathcal{R}_T^{f_0}(q, \varepsilon) \cap \{z_k \leq 0\} \subset \text{Box}(C\varepsilon) \cap \{z_k \leq 0\}.$$

Moreover, if the system is nilpotent, it holds

$$(32) \quad \Xi_T \left( \frac{1}{C} \varepsilon \right) \subset \mathcal{R}_T^{f_0}(q, \varepsilon) \subset \widehat{\Pi}_T(C\varepsilon), \quad \text{for } \varepsilon < \varepsilon_0 \text{ and } T < T_0.$$

In order to prove Theorem 4.27, we need the following lemma.



**Lemma 4.28.** *Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q \in M$ . Then there exist  $C, \varepsilon_0, T_0 > 0$  such that, for any  $q' \in \mathcal{R}_T^{f_0}(q, \varepsilon_0)$  for  $\varepsilon < \varepsilon_0$  and  $T < T_0$ , and such that*

- (i) *for any  $t < \varepsilon_0$ ,  $\text{ord}_{q'(t)} f_0 = -s$ , where  $q'(t) = e^{-tf_0}(q')$ ,*
- (ii)  *$dz_k(f_0(z(q'))) \neq 0$ , for some  $k$  with  $w_k = -s$ ,*

*it holds that, if  $u \in L^1([0, T], \mathbb{R}^m)$  is a control steering the system (D) from  $q$  to  $q'$ , with  $\|u\|_1 = \varepsilon$ , then*

$$T \leq C(\varepsilon^s + \max\{z_k(q'), 0\}).$$

*Proof.* For any  $\eta > 0$ , let  $\gamma$  be the trajectory associated with  $u \in L^1([0, T], \mathbb{R}^m)$  in the system (D), and satisfying  $\gamma : q \rightsquigarrow q'$ . Let  $\tilde{\gamma}$  be the trajectory associated with  $u$  and starting from  $q$ , in the time-dependent system (30). Thus  $\gamma(t) = e^{tf_0} \circ \tilde{\gamma}(t)$  and  $\rho(q, \tilde{\gamma}(T)) \leq \varepsilon$ .

Recall that, for any vector field  $g$  and point  $p \in M$ , it holds that  $z_k(e^{Tg}(p)) - z_k(p) = \int_0^T dz_k(g(e^{tg}(p)))$ . Thus, by the mean value theorem, there exists  $\tau \in [0, T]$  such that

$$(33) \quad z_k(q') = z_k(\gamma(T)) = T dz_k(f_0(e^{\tau f_0}(\tilde{\gamma}(T)))) + z_k(\tilde{\gamma}(T)).$$

Since  $e^{\tau f_0}(\tilde{\gamma}(T)) = e^{-(T-\tau)f_0}(q')$ , by hypothesis (ii) and the smoothness of  $f_0$ , there exist  $T_0, C_1 > 0$ , independent of  $\gamma$ , such that  $dz_k(f_0(e^{\tau f_0}(\tilde{\gamma}(T)))) \geq C_1$  for  $T < T_0$ . Hence, by Theorem 3.19 (since  $w_k = s$ ), there exist  $C_2, \bar{\varepsilon} > 0$  such that, if  $\varepsilon < \bar{\varepsilon}$  and  $T < T_0$ ,

$$T \leq \frac{|z_k(\tilde{\gamma}(T))| + \max\{z_k(q'), 0\}}{C_1} \leq \frac{C_2(\varepsilon^s + T\varepsilon) + \max\{z_k(q'), 0\}}{C_1}.$$

Since the constants are independent of  $\gamma$ , taking  $C = C_2/C_1$ ,  $\varepsilon_0 \leq \min\{T_0, \bar{\varepsilon}, (C-1)/C^2\}$  completes the proof.  $\square$

*Proof of Theorem 4.27.* The first inclusion in (31) follows from Theorem 2.7 and Theorem 4.23. In fact, combining them, we have that, for any  $\varepsilon < \varepsilon_0$  and any  $T > 0$ ,

$$\Xi_T \left( \frac{1}{C} \varepsilon \right) \subset \bigcup_{0 \leq t \leq T} B_{\text{SR}}(e^{tf_0} q, \varepsilon) \subset \mathcal{R}_T^{f_0}(q, \varepsilon).$$

To prove the second inclusion, we let  $q' \in \mathcal{R}_T^{f_0}(q, \varepsilon)$ . Fix any  $\eta > 0$  and consider a control  $u \in L^1([0, T], \mathbb{R}^m)$  such that its associated trajectory  $\gamma$ , in the system (D), satisfies  $\gamma : q \rightsquigarrow q'$  and  $c_{f_0}(\gamma) \leq \varepsilon + \eta$ . We distinguish two cases. First we assume that  $z_k(q') \leq 0$ . In this case, by Lemma 4.28 it follows there exists  $C, \varepsilon_0, T_0 > 0$  such that if  $T < T_0$  and  $\varepsilon < \varepsilon_0$ , then  $T \leq C\varepsilon^s$ . Moreover (29) implies that  $e^{-Tf_0}(q') \in \mathcal{R}_T(q, \varepsilon)$ . Then, enlarging the constant  $C$ , Theorem 3.19 yields

$$\begin{aligned} |z_i(q')| &= |z_i(e^{-Tf_0}(q'))| \leq C(\varepsilon^{w_i} + \varepsilon T^{\frac{w_i}{s}}) \leq C\varepsilon^{w_i}, \quad \text{if } w_i \leq s \text{ and } i \neq k, \\ |z_k(q')| &\leq T + |z_k(e^{-Tf_0}(q'))| \leq T + C(\varepsilon + T^{\frac{1}{s}})^s \leq C\varepsilon^s, \\ |z_i(q')| &= |z_i(e^{-Tf_0}(q'))| \leq C\varepsilon(\varepsilon + T^{\frac{1}{s}})^{w_i-1} \leq C\varepsilon^{w_i}, \quad \text{if } w_i > s. \end{aligned}$$

Here, we used the fact that, for any  $p \in M$ , from  $z_* f_0 = \partial_{z_k}$ , it holds  $z_i(p) = z_i(e^{-Tf_0}(p))$  and  $|dz_k(f_0(p))| \equiv 1$ . Thus,  $q' \subset \text{Box}(C\varepsilon) \subset \Pi(C\varepsilon)$ .

On the other hand, if  $z_k(q') > 0$ , Lemma 4.28 yields that  $T \leq C(\varepsilon^s + z_k(q'))$ . Then, applying again Theorem 3.19, we get

$$\begin{aligned} |z_i(q')| &\leq C(\varepsilon^{w_i} + \varepsilon z_k(q')^{\frac{w_i}{s}}), \quad \text{if } w_i \leq s \text{ and } i \neq k, \\ |z_k(q')| &\leq T + C\varepsilon^s, \\ |z_i(q')| &\leq C\varepsilon(\varepsilon + z_k(q')^{\frac{1}{s}})^{w_i-1}, \quad \text{if } w_i > s. \end{aligned}$$

This proves that  $q' \subset \Pi(C\varepsilon)$ , completing the proof of (31).

To prove (32) it suffices to use the same argument as above, applying the result on nilpotent systems in Theorem 3.19.  $\square$

**Remark 4.29.** Theorem 4.27 suggests that the behavior of the system (D), when moving in the direction  $-f_0$ , is essentially sub-Riemannian. However, although this is true locally in time, it is false in general. For example, consider the Euclidean plane endowed with a rotational drift, i.e., such that  $\{e^{tf_0}(q)\}_{t \in (0, +\infty)}$  is diffeomorphic to  $\mathcal{S}^1$  for any  $q \neq 0$ . Then,  $\rho^{f_0}(q, e^{-tf_0}(q)) = 0$  for any  $t > 0$  and thus we can move in the direction  $-f_0$  for free.

*Proof of Theorem 4.26.* Since every norm on  $\mathbb{R}^n$  is equivalent,  $\text{dist}(z(q'), [0, T] \partial_{z_k})$  is equivalent to

$$a(q') = \sum_{\substack{1 \leq i \leq n \\ i \neq k}} |z_i(q')| + \min_{t \in [0, T]} |z_k(q') - t|.$$

Thus, to complete the proof it suffices to prove that it holds  $C^{-1}a(q') \leq \rho_T^{f_0}(q, q') \leq Ca(q')^{1/r}$ .

By Theorem 3.19,  $\Xi_T(C^{-1}\varepsilon) \subset \mathcal{R}_T^{f_0}(q, \varepsilon) \subset \Pi_T(C\varepsilon)$  for any  $\varepsilon < \varepsilon_0$ . The first inclusion is equivalent to the fact that, for every  $\varepsilon < \varepsilon_0$  such that  $Ca(q') \leq \varepsilon^r$ , one has  $\rho_T^{f_0}(q, q') \leq \varepsilon$ . From this follows that  $\rho_T^{f_0}(q, q') \leq C^{1/r}a(q')^{1/r}$ . The same reasoning applied to the other inclusion proves that

$$\begin{aligned} |z_i(q')| &\leq C(\rho_T^{f_0}(q, q')^{w_i} + \rho_T^{f_0}(q, q')T^{\frac{w_i}{s}}) \text{ if } w_i \leq s, i \neq k, \\ \min_{t \in [0, T]} |z_k(q') - t| &\leq C\rho_T^{f_0}(q, q')^s, \\ |z_i(q')| &\leq C(\rho_T^{f_0}(q, q')^{w_i} + \rho_T^{f_0}(q, q')T^{\frac{w_i-1}{s}}) \text{ if } w_i > s. \end{aligned}$$

Clearly, this implies that  $a(q') \leq C\rho_T^{f_0}(q, q')$ , for some larger constant, completing the proof of the theorem.  $\square$

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# COMPLEXITY OF CONTROL-AFFINE MOTION PLANNING

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**Abstract** In this paper we study the complexity of the motion planning problem for control-affine systems. Such complexities are already defined and rather well-understood in the particular case of nonholonomic (or driftless) systems. Our aim is to generalize these notions and results to systems with a drift. Accordingly, we present various definitions of complexity, as functions of the curve that is approximated, and of the precision of the approximation. Due to the lack of time-rescaling invariance of these systems, we consider geometric and parametrized curves separately. Then, we give some asymptotic estimates for these quantities.

**Key words:** control-affine systems, sub-Riemannian geometry, motion planning, complexity.

**2010 AMS subject classifications:** 53C17, 93C15.

## 1. INTRODUCTION

The concept of complexity was first developed for the non-holonomic motion planning problem in robotics. Given a control system on a manifold  $M$ , the motion planning problem consists in finding an admissible trajectory connecting two points, usually under further requirements, such as obstacle avoidance. If a cost function is given, it makes sense to try to find the trajectory costing the least.

Different approaches are possible to solve this problem (see [21]). Here we focus on those based on the following scheme:

- (1) try to find any (usually non-admissible) path solving the problem,
- (2) try to approximate it with admissible paths.

The first step is independent of the control system, since it depends only on the topology of the manifold and of the obstacles. Thus, we are interested in the second step, which depends only on the local nature of the control system near the path. The goal of the paper is to understand how to measure the complexity of the approximation task. By complexity we mean a function of the non-admissible curve  $\Gamma \subset M$  (or path  $\gamma : [0, T] \rightarrow M$ ), and of the precision of the approximation, quantifying the difficulty of the latter by means of the cost function.

**1.1. Control theoretical setting.** A sub-Riemannian (or non-holonomic) control system on a smooth manifold  $M$  is a control system in the form

$$(1) \quad \dot{q}(t) = \sum_{i=1}^m u_i(t) f_i(q(t)), \quad \text{a.e. } t \in [0, T],$$

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where  $u : [0, T] \rightarrow \mathbb{R}^m$  is an integrable and bounded control function and  $\{f_1, \dots, f_m\}$  is a family of smooth vector fields on  $M$  satisfying the Hörmander condition, i.e. such that its iterated Lie brackets generate the whole tangent space at every point. Moreover, we will always assume the sub-Riemannian structure to be equiregular (see Section 2.1). Given a sub-Riemannian control system, a natural choice for the cost is the  $L^1$ -norm of the controls. Due to the linearity and the reversibility in time of such a system, the associated value function is in fact a distance, called Carnot-Carathéodory distance, that endows  $M$  with a metric space structure.

In this paper, we focus on a very important generalization of control system (1), namely on control-affine systems. These systems are obtained by adding to (1) an uncontrolled vector field  $f_0$ , called the *drift*, and are in the form

$$(2) \quad \dot{q}(t) = f_0(q(t)) + \sum_{i=1}^m u_i(t) f_i(q(t)), \quad \text{a.e. } t \in [0, T].$$

These kind of systems appears in plenty of applications. As an example we cite, mechanical systems with controls on the acceleration (see e.g., [7], [5]), where the drift is the velocity, or quantum control (see e.g., [8], [6]), where the drift is the free Hamiltonian. We always assume the strong Hörmander condition, i.e., that the family  $\{f_1, \dots, f_m\}$  satisfies the Hörmander condition.

Our work will focus on the following cost functions,

$$(3) \quad \mathcal{J}(u, T) = \int_0^T \sqrt{\sum_{j=1}^m u_j(t)^2} dt \quad \text{and} \quad \mathcal{I}(u, T) = \int_0^T \sqrt{1 + \sum_{j=1}^m u_j(t)^2} dt.$$

Let  $q \in M$  and define  $q_u : [0, T] \rightarrow M$  as the trajectory associated with a control  $u \in L^1([0, T], \mathbb{R}^m)$  such that  $q_u(0) = q$ . The cost  $\mathcal{J}$ , measuring the  $L^1$ -norm of the control, quantifies the cost spent by the controller to steer the system (2) along  $q_u$ . On the other hand,  $\mathcal{I}$  measures the Riemannian length of  $q_u$  with respect to a Riemannian metric<sup>1</sup> such that  $f_0, f_1, \dots, f_m$  are orthonormal.

Fix a time  $\mathcal{T} > 0$  and consider the two value functions  $V^{\mathcal{J}}(q, q')$  and  $V^{\mathcal{I}}(q, q')$  as the infima of the costs  $\mathcal{J}$  and  $\mathcal{I}$ , respectively, over all controls  $u \in \mathcal{U}_{\mathcal{T}} = \bigcup_{0 < T \leq \mathcal{T}} L^1([0, T], \mathbb{R}^m)$  steering the system from  $q$  to  $q'$ . Contrarily to what happens in sub-Riemannian geometry with the Carnot-Carathéodory distance, these value functions are not symmetric, and hence do not induce a metric space structure on  $M$ . In fact, system (2) is not reversible – i.e., changing orientation to an admissible trajectory does not yield an admissible trajectory.

We consider controls defined on  $T \leq \mathcal{T}$  since we are interested in the local behavior of system (2). Indeed, without an upper-bound for the time of definition of the controls, the reachable sets  $\mathcal{R}^{f_0}(q, \varepsilon) = \{q' \in M \mid V^{\mathcal{J}}(q, q') \leq \varepsilon\}$  are in general non-compact for any  $q \in M$  and  $\varepsilon > 0$ . As a byproduct of this choice, by taking  $\mathcal{T}$  sufficiently small, it is then possible to prevent any exploitation of the geometry of the orbits of the drift (that could be, for example, closed). Let us also remark that, since the controls can be defined on arbitrarily small times, it is possible to approximate admissible trajectories for system (2) via trajectories for the sub-Riemannian associated system (i.e., the one obtained by posing  $f_0 \equiv 0$ ) rescaled on small intervals.

**1.2. Complexities.** Depending on whether we want to approximate curves, i.e., dimension 1 connected submanifolds of  $\Gamma \subset M$  diffeomorphic to a closed interval, or paths, i.e., smooth injective functions  $\gamma : [0, T] \rightarrow M$ , we will consider different notions of complexities. In particular, when computing the complexity of paths, we will require the approximating trajectories to respect also the parametrization and not only the geometry of the path. While in the sub-Riemannian case,

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<sup>1</sup>Whenever it exists, which is not always the case since the vector fields  $f_0, f_1, \dots, f_m$  could be not linearly independent

due to the time rescaling properties of the control system, these concepts coincide, this is not the case for control-affine systems.

We are interested in four distinct notions of complexity, two for curves and two for paths. The two for curves are the same as the sub-Riemannian ones already introduced in [17, 18]. This is true also for what we call the neighboring approximation complexity of a path, since in the sub-Riemannian case it coincides with the tubular approximation complexity. On the other hand, what we call the interpolation by time complexity never appeared in the literature, to our knowledge. Here we give the definitions for a generic cost  $J : \mathcal{U}_{\mathcal{T}} \rightarrow [0, +\infty)$ .

Fix a curve  $\Gamma$  – i.e., a one dimensional oriented submanifold with boundary of  $M$  diffeomorphic to a closed interval. Then, for any  $\varepsilon > 0$  we define the following complexities for  $\Gamma$ .

- *Interpolation by cost complexity:* For  $\varepsilon > 0$ , let an  $\varepsilon$ -cost interpolation of  $\Gamma$  to be any control  $u \in \mathcal{U}_{\mathcal{T}}$  such that there exist  $0 = t_0 < t_1 < \dots < t_N = T \leq \mathcal{T}$  for which the trajectory  $q_u$  with initial condition  $q_u(0) = x$  satisfies  $q_u(T) = y$ ,  $q_u(t_i) \in \Gamma$  and  $J(u|_{[t_{i-1}, t_i]}, t_i - t_{i-1}) \leq \varepsilon$ , for any  $i = 1, \dots, N$ . Then, let

$$\sigma_c^J(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \{ J(u, T) \mid u \text{ is an } \varepsilon\text{-cost interpolation of } \Gamma \}.$$

This function measures the number of pieces of cost  $\varepsilon$  necessary to interpolate  $\Gamma$ . Namely, following a trajectory given by a control admissible for  $\sigma_c^J(\Gamma, \varepsilon)$ , at any given moment it is possible to go back to  $\Gamma$  with a cost less than  $\varepsilon$ .

- *Tubular approximation complexity:* Let  $\text{Tube}(\Gamma, \varepsilon)$  to be the tubular neighborhood of radius  $\varepsilon$  around the curve  $\Gamma$  w.r.t. the small sub-Riemannian system associated with (2) (obtained by putting  $f_0 \equiv 0$ , see Section 2.2), and define

$$\sigma_a^J(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ J(u, T) \left| \begin{array}{l} 0 < T \leq \mathcal{T}, \\ q_u(0) = x, q_u(T) = y, \\ q_u([0, T]) \subset \text{Tube}(\Gamma, \varepsilon) \end{array} \right. \right\}$$

This complexity measures the number of pieces of cost  $\varepsilon$  necessary to go from  $x$  to  $y$  staying inside the sub-Riemannian tube  $\text{Tube}(\Gamma, \varepsilon)$ . Such property is especially useful for motion planning with obstacle avoidance. In fact, if the sub-Riemannian distance of  $\Gamma$  from the obstacles is at least  $\varepsilon_0 > 0$ , then trajectories obtained from controls admissible for  $\sigma_a^J(\Gamma, \varepsilon)$ ,  $\varepsilon < \varepsilon_0$ , will avoid such obstacles.

Fix now a path  $\gamma : [0, T] \rightarrow M$  – i.e., a smooth injective function with  $\dot{\gamma}(t) \neq 0$  for any  $t \in [0, T]$ . For any  $\delta > 0$  and  $\varepsilon > 0$  we then define the following complexities for the path  $\gamma$ .

- *Interpolation by time complexity:* Let a  $\delta$ -time interpolation of  $\gamma$  to be any control  $u \in L^1([0, T], \mathbb{R}^m)$  such that its trajectory  $q_u : [0, T] \rightarrow M$  in (2) with  $q_u(0) = \gamma(0)$  is such that  $q_u(T) = \gamma(T)$  and that, for any interval  $[t_0, t_1] \subset [0, T]$  of length  $t_1 - t_0 \leq \delta$ , there exists  $t \in [t_0, t_1]$  with  $q_u(t) = \gamma(t)$ . Then, let

$$\sigma_t^J(\gamma, \varepsilon) = \delta \inf \{ J(u, T) \mid u \text{ is a } \delta\text{-time interpolation of } \gamma \}.$$

Controls admissible for this complexity will define trajectories touching  $\gamma$  at intervals of time of length at most  $\delta$ . This complexity measures the minimal average cost on each of these intervals. It is thus well suited for applications where time is of great importance – e.g. motion planning in rendez-vous problem.

- *Neighboring approximation complexity:* Let  $B_{\text{SR}}(p, \varepsilon)$  denote the ball of radius  $\varepsilon$  centered at  $p \in M$  w.r.t. the small sub-Riemannian system associated with (2) (obtained by putting  $f_0 \equiv 0$ , see Section 2.2), and define

$$\sigma_n^J(\gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ J(u, T) \left| \begin{array}{l} q_u(0) = x, q_u(T) = y, \\ q_u(t) \in B_{\text{SR}}(\gamma(t), \varepsilon), \forall t \in [0, T] \end{array} \right. \right\}.$$

This complexity measures the number of pieces of cost  $\varepsilon$  necessary to go from  $x$  to  $y$  following a trajectory that at each instant  $t \in [0, T]$  remains inside the sub-Riemannian ball  $B_{\text{SR}}(\gamma(t), \varepsilon)$ . Such complexity can be applied to motion planning in rendez-vous problems where it is sufficient to attain the rendez-vous only approximately.

Two functions  $f(\varepsilon)$  and  $g(\varepsilon)$ , tending to  $\infty$  or to 0 when  $\varepsilon \downarrow 0$  are *weakly equivalent* (denoted by  $f(\varepsilon) \asymp g(\varepsilon)$ ) if both  $f(\varepsilon)/g(\varepsilon)$  and  $g(\varepsilon)/f(\varepsilon)$ , are bounded when  $\varepsilon \downarrow 0$ . When  $f(\varepsilon)/g(\varepsilon)$  (resp.  $g(\varepsilon)/f(\varepsilon)$ ) is bounded, we will write  $f(\varepsilon) \preccurlyeq g(\varepsilon)$  (resp.  $f(\varepsilon) \succcurlyeq g(\varepsilon)$ ). In the sub-Riemannian context, for any curve  $\Gamma \subset M$  and path  $\gamma : [0, T] \rightarrow M$  such that  $\gamma([0, T]) = \Gamma$  it holds  $\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_n^{\mathcal{J}}(\gamma, \varepsilon)$ . A complete characterization of weak asymptotic equivalence of sub-Riemannian complexities is obtained in [20]. We state here such characterization in the special case where  $\{f_1, \dots, f_m\}$  defines an equiregular structure.

**Theorem 1.1.** *Assume that  $\{f_1, \dots, f_m\}$  defines an equiregular sub-Riemannian structure. Let  $\Gamma \subset M$  be a curve and  $\gamma : [0, T] \rightarrow M$  be a path such that  $\gamma([0, T]) = \Gamma$ . Then, if there exists  $k \in \mathbb{N}$  such that  $T_q \Gamma \subset \Delta^k(q) \setminus \Delta^{k-1}(q)$  for any  $q \in \Gamma$ , it holds*

$$\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_n^{\mathcal{J}}(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

We mention also that for a restricted set of sub-Riemannian systems, i.e., one-step bracket generating or with two controls and dimension not larger than 6, strong asymptotic estimates and explicit asymptotic optimal syntheses are obtained in the series of papers [23, 12, 13, 14, 15, 16, 11] (see [10] for a review).

**1.3. Main results.** As a first result, in Section 5 we prove the following weak asymptotic equivalence for the interpolation by time complexity, in the same spirit as Theorem 1.1.

**Theorem 1.2.** *Assume that  $\{f_1, \dots, f_m\}$  defines an equiregular sub-Riemannian structure and let  $\gamma : [0, T] \rightarrow M$  be a path. Then, if there exists  $k \in \mathbb{N}$  such that  $\dot{\gamma}(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$  for any  $t \in [0, T]$ , it holds*

$$\sigma_t^{\mathcal{J}}(\gamma, \delta) \asymp \delta^{\frac{1}{k}}.$$

Since in the sub-Riemannian context one is only interested in the cost  $\mathcal{J}$ , Theorems 1.1 and 1.2 completely characterize the weak asymptotic equivalences of complexities of equiregular sub-Riemannian manifolds.

The main result of the paper is then a weak asymptotic equivalence of the above defined complexities in control-affine systems, generalizing Theorems 1.1 and 1.2.

**Theorem 1.3.** *Assume that  $\{f_1, \dots, f_m\}$  defines an equiregular sub-Riemannian structure and that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$  for some  $s \geq 2$ . Let  $\Gamma \subset M$  be a curve and define  $\kappa = \max\{k : T_p \Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p), \text{ for any } p \text{ in an open subset of } \Gamma\}$ . Then, whenever the maximal time of definition of the controls  $\mathcal{T}$  is sufficiently small, it holds*

$$\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_c^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^\kappa}.$$

*On the other hand, let  $\gamma : [0, T] \rightarrow M$  be a path such that  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \bmod \Delta^{s-1}(\gamma(t))$  for any  $t \in [0, T]$  and define  $\kappa = \max\{k : \gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t)) \text{ for any } t \text{ in an open subset of } [0, T]\}$ . Then, it holds*

$$\sigma_t^{\mathcal{J}}(\gamma, \delta) \asymp \sigma_t^{\mathcal{I}}(\gamma, \delta) \asymp \delta^{\frac{1}{\max\{\kappa, s\}}}, \quad \sigma_n^{\mathcal{J}}(\gamma, \varepsilon) \asymp \sigma_n^{\mathcal{I}}(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{\kappa, s\}}}.$$

This theorem shows that, asymptotically, the complexity of curves is uninfluenced by the drift, and only depends on the underlying sub-Riemannian system, while the one of paths depends also on how “bad” the drift is with respect to this system. We remark also that for the path complexities it is not necessary to have an a priori bound on  $\mathcal{T}$ .

**1.4. Long time local controllability.** As an application of the above theorem, let us briefly mention the problem of *long time local controllability* (henceforth simply *LTLC*), i.e., the problem of staying near some point for a long period of time  $T > 0$ . This is essentially a stabilization problem around a non-equilibrium point.

Since the system (2) satisfies the strong Hörmander condition, it is always possible to satisfy some form of LTLC. Hence, it makes sense to quantify the minimal cost needed, by posing the following. (To lighten the notation, we consider only the cost  $\mathcal{J}$ .) Let  $T > 0$ ,  $q_0 \in M$ , and  $\gamma_{q_0} : [0, T] \rightarrow M$ ,  $\gamma_{q_0}(\cdot) \equiv q_0$ .

- *LTLC complexity by time:*

$$\Xi_t(q_0, T, \delta) = \sigma_t^{\mathcal{J}}(\gamma_{q_0}, \delta).$$

Here, we require trajectories defined by admissible controls to pass through  $q_0$  at intervals of time of length at most  $\delta$ .

- *LTLC complexity by cost:*

$$\Xi_c(q_0, T, \varepsilon) = \sigma_n^{\mathcal{J}}(\gamma_{q_0}, \varepsilon).$$

Admissible controls for this complexity, will always be contained in the sub-Riemannian ball of radius  $\varepsilon$  centered at  $q_0$ .

Clearly, if  $f_0(q_0) = 0$ , then  $\Xi_t(q_0, T, \delta) = \Xi_c(q_0, T, \varepsilon) = 0$ , for any  $\varepsilon, \delta, T > 0$ . Although  $\gamma_{q_0}$  is not a path by our definition, since it is not injective and  $\dot{\gamma}_{q_0} \equiv 0$ , the arguments of Theorem 1.3 can be applied also to this case. Hence, we get the following asymptotic estimate for the LTLC complexities.

**Corollary 1.4.** *Assume that  $\{f_1, \dots, f_m\}$  defines an equiregular sub-Riemannian structure and that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$  for some  $s \geq 2$ . Then, for any  $q_0 \in M$  and  $T > 0$  it holds*

$$\Xi_t(q_0, T, \delta) \asymp \delta^{\frac{1}{s}}, \quad \Xi_c(q_0, T, \varepsilon) \asymp \frac{1}{\varepsilon^s}.$$

**1.5. Structure of the paper.** In Section 2 we introduce more in detail the setting of the problem. In Section 3 we present families of coordinates adapted to the drift and to curves or paths, that will be useful in the sequel. Section 4 collects some useful properties of the costs  $\mathcal{J}$  and  $\mathcal{I}$ , proved mainly in [22]. Then, Section 5 is devoted to relate the complexities of the control-affine system with those of the associated sub-Riemannian systems and to prove Theorem 1.2. In this section we also prove Proposition 5.3, that gives a first result in the direction of Theorem 1.3 showing when the sub-Riemannian and control-affine complexities coincide. Finally, the proof of the main result is contained in Sections 6 and 7, for curves and paths respectively.

## 2. PRELIMINARIES

Throughout this paper,  $M$  is an  $n$ -dimensional connected smooth manifold.

**2.1. Sub-Riemannian control systems.** As already stated, a sub-Riemannian (or non-holonomic) control system on a connected smooth manifold  $M$  is a control system in the form

$$(SR) \quad \dot{q}(t) = \sum_{i=1}^m u_i(t) f_i(q(t)), \quad a.e. t \in [0, T]$$

where  $u : [0, T] \rightarrow \mathbb{R}^m$  is an integrable and bounded control function and  $\{f_1, \dots, f_m\}$  is a family of smooth vector fields on  $M$ . We let  $f_u = \sum_{i=1}^m u_i f_i$ . The value function  $d_{SR}$  associated with the



$L^1$  cost is in fact a distance, called *Carnot-Carathéodory* (or *sub-Riemannian*) *distance*. Namely, for any  $q, q' \in M$ ,

$$d_{\text{SR}}(q, q') = \inf \int_0^T \sqrt{\sum_{j=1}^m u_j(t)^2} dt,$$

where the infimum is taken between any control  $u \in L^1([0, T], \mathbb{R}^m)$  for some  $T > 0$  such that its trajectory in (SR) is such that  $q_u(0) = q$  and  $q_u(T) = q'$ . An absolutely continuous curve  $\gamma : [0, T] \rightarrow M$  is admissible for (SR) if there exists  $u \in L^1([0, T], \mathbb{R}^m)$  such that  $\dot{\gamma}(t) = f_u(t)$ .

Let  $\Delta$  be the  $C^\infty$ -module generated by the vector fields  $\{f_1, \dots, f_m\}$  (in particular, it is closed under multiplication by  $C^\infty(M)$  functions and summation). Let  $\Delta^1 = \Delta$ , and define recursively  $\Delta^{s+1} = \Delta^s + [\Delta^s, \Delta]$ , for every  $s \in \mathbb{N}$ . Due to the Jacobi identity  $\Delta^s$  is the  $C^\infty$ -module of linear combinations of all commutators of  $f_1, \dots, f_m$  with length  $\leq s$ . For  $q \in M$ , let  $\Delta^s(q) = \{f(q) : f \in \Delta^s\} \subset T_q M$ . We say that  $\{f_1, \dots, f_m\}$  satisfies the *Hörmander condition* (or that it is a bracket-generating family of vector fields) if  $\bigcup_{s \geq 1} \Delta^s(q) = T_q M$  for any  $q \in M$ . Moreover,  $\{f_1, \dots, f_m\}$  defines an *equiregular* sub-Riemannian structure if  $\dim \Delta^i(q)$  does not depend on the point for any  $i \in \mathbb{N}$ . In the following we will always assume these two conditions to be satisfied.

By the Chow–Rashevsky theorem (see for instance [1]), the hypothesis of connectedness of  $M$  and the Hörmander condition guarantee the finiteness and continuity of  $d_{\text{SR}}$  with respect to the topology of  $M$ . Hence, the sub-Riemannian distance, induces on  $M$  a metric space structure. The open balls of radius  $\varepsilon > 0$  and centered at  $q \in M$ , with respect to  $d_{\text{SR}}$ , are denoted by  $B_{\text{SR}}(q, \varepsilon)$ .

We say that a control  $u \in L^1([0, T], \mathbb{R}^m)$ ,  $T > 0$ , is a minimizer of the sub-Riemannian distance between  $q, q' \in M$  if the associated trajectory  $q_u$  with  $q_u(0) = q$  is such that  $q_u(T) = q'$  and  $\|u\|_{L^1([0, T], \mathbb{R}^m)} = d_{\text{SR}}(q, q')$ . Equivalently,  $u$  is a minimizer between  $q, q' \in M$  if it is a solution of the free-time optimal control problem, associated with (SR),

$$(4) \quad \|u\|_{L^1(0, T)} = \int_0^T \sqrt{\sum_{j=1}^m u_j^2(t)} dt \rightarrow \min, \quad q_u(0) = q, \quad q_u(T) = q', \quad T > 0.$$

It is a classical result that, for any couple of points  $q, q' \in M$  sufficiently close, there exists at least one minimizer.

**Remark 2.1.** This control theoretical setting can be stated in purely geometric terms even if we drop the equiregularity assumption. Indeed, it is equivalent to a *generalized sub-Riemannian* structure. Such a structure is defined by a rank-varying smooth distribution and a Riemannian metric on it (see [1] for a precise definition).

In a sub-Riemannian control system, in fact, the map  $q \mapsto \text{span}\{f_1(q), \dots, f_m(q)\} \subset T_q M$  defines a rank-varying smooth distribution, which is naturally endowed with the Riemannian norm defined, for  $v \in \Delta(q)$ , by

$$(5) \quad \mathbf{g}(q, v) = \inf \left\{ |u| = \sqrt{u_1^2 + \dots + u_m^2} : f_u(q) = v \right\}.$$

The pair  $(\Delta, \mathbf{g})$  is thus a generalized sub-Riemannian structure on  $M$ . Conversely, every rank-varying distribution is finitely generated, see [2, 1, 3, 9], and thus a sub-Riemannian distance can be written, globally, as the value function of a control system of the type (SR).

Since  $\{f_1, \dots, f_m\}$  is bracket-generating, the values of the sets  $\Delta^s$  at  $q$  form a flag of subspaces of  $T_q M$ ,

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^r(q) = T_q M.$$

The integer  $r$ , which is the minimum number of brackets required to recover the whole  $T_q M$ , is called *degree of non-holonomy* (or *step*) of the family  $\{f_1, \dots, f_m\}$  at  $q$ . The degree of non-holonomy is

independent of  $q$  since we assumed the family  $\{f_1, \dots, f_m\}$  to define an equiregular sub-Riemannian structure. Let  $n_s = \dim \Delta^s(q)$  for any  $q \in M$ . The integer list  $(n_1, \dots, n_r)$  is called the *growth vector* associated with (SR). Finally, let  $w_1 \leq \dots \leq w_n$  be the *weights* associated with the flag, defined by  $w_i = s$  if  $n_{s-1} < i \leq n_s$ , setting  $n_0 = 0$ .

For any smooth vector field  $f$ , we denote its action, as a derivation on smooth functions, by  $f : a \in C^\infty(M) \mapsto fa \in C^\infty(M)$ . For any smooth function  $a$  and every vector field  $f$  with  $f \neq 0$  near  $q$ , their (*non-holonomic*) *order* at  $q$  is

$$\begin{aligned} \text{ord}_q(a) &= \min\{s \in \mathbb{N} : \exists i_1, \dots, i_s \in \{1, \dots, m\} \text{ s.t. } (f_{i_1} \dots f_{i_s} a)(q) \neq 0\}, \\ \text{ord}_q(f) &= \max\{\sigma \in \mathbb{Z} : \text{ord}_q(fa) \geq \sigma + \text{ord}_q(a) \text{ for any } a \in C^\infty(M)\}. \end{aligned}$$

In particular it can be proved that  $\text{ord}_q(a) \geq s$  if and only if  $a(q') = \mathcal{O}(d_{\text{SR}}(q', q))^s$ .

**Definition 2.2.** A *system of privileged coordinates* at  $q$  for  $\{f_1, \dots, f_m\}$  is a system of local coordinates  $z = (z_1, \dots, z_n)$  centered at  $q$  and such that  $\text{ord}_q(z_i) = w_i$ ,  $1 \leq i \leq n$ .

Let  $q \in M$ . A set of vector fields  $\{f_1, \dots, f_n\}$  such that

$$(6) \quad \{f_1(q), \dots, f_n(q)\} \text{ is a basis of } T_q M, \quad \text{and} \quad f_i \in \Delta^{w_i} \text{ for } i = 1, \dots, n,$$

is called an *adapted frame* at  $q$ . We remark that to any system of privileged coordinates  $z$  at  $q$  is associated a (non-unique) adapted frame at  $q$  such that  $\partial_{z_i} = z_* f_i(q)$  (i.e., privileged coordinates are always linearly adapted to the flag).

For any ordering  $\{i_1, \dots, i_n\}$ , the inverse of the local diffeomorphisms

$$(z_1, \dots, z_n) \mapsto e^{z_{i_1} f_{i_1} + \dots + z_{i_n} f_{i_n}}(q), \quad (z_1, \dots, z_n) \mapsto e^{z_{i_n} f_{i_n}} \circ \dots \circ e^{z_{i_1} f_{i_1}}(q),$$

define privileged coordinates at  $q$ , called *canonical coordinates of the first kind* and of the *second kind*, respectively. We remark that, for the canonical coordinates of the second kind, it holds  $z_* f_{i_n}(z) \equiv \partial_{z_{i_n}}$ .

We recall the celebrated Ball-Box Theorem, that gives a rough description of the shape of small sub-Riemannian balls.

**Theorem 2.3** (Ball-Box Theorem). *Let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinates at  $q \in M$  for  $\{f_1, \dots, f_m\}$ . Then there exist  $C, \varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$ , it holds*

$$\text{Box}\left(\frac{1}{C}\varepsilon\right) \subset B_{\text{SR}}(q, \varepsilon) \subset \text{Box}(C\varepsilon),$$

where,  $B_{\text{SR}}(q, \varepsilon)$  is identified with its coordinate representation  $z(B_{\text{SR}}(q, \varepsilon))$  and, for any  $\eta > 0$ , we let

$$(7) \quad \text{Box}(\eta) = \{z \in \mathbb{R}^n : |z_i| \leq \eta^{w_i}\},$$

**Remark 2.4.** Let  $N \subset M$  be compact and let  $\{z^q\}_{q \in N}$  be a family of systems of privileged coordinates at  $q$  depending continuously on  $q$ . Then there exist uniform constants  $C, \varepsilon_0 > 0$  such that the Ball-Box Theorem holds for any  $q \in N$  in the system  $z^q$ .

**2.2. Control-affine systems.** Let  $f_0$  and  $\{f_1, \dots, f_m\}$  be smooth vector fields on  $M$  and, for some  $\mathcal{T} > 0$ , define  $\mathcal{U}_{\mathcal{T}} = \bigcup_{0 < T < \mathcal{T}} L^1([0, T], \mathbb{R}^m)$ . Consider the control-affine control system

$$(D) \quad \dot{q}(t) = f_0(q(t)) + \sum_{j=1}^m u_j f_j(q(t)), \quad u \in \mathcal{U}_{\mathcal{T}}.$$

An absolutely continuous curve  $\gamma : [0, T] \rightarrow M$  is admissible for (D) if  $\dot{\gamma}(t) = f_0(\gamma(t)) + f_{u(t)}(\gamma(t))$  for some control  $u \in L^1([0, T], \mathbb{R}^m)$ . Observe, however, that contrary to what happens in the sub-Riemannian case, the admissibility for (D) is not invariant under time reparametrization, e.g., a

time reversal. Thus there is no canonical choice for the cost, and we will focus on the two costs given in (3).

In the rest of the paper we will always assume the following hypotheses to be satisfied.

(H1) *Equiregularity*:  $\dim \Delta^k(q)$  does not depend on  $q \in M$ ;

(H2) *Strong Hörmander condition*: there exists  $r \in \mathbb{N}$  such that  $\Delta^r(q) = T_q M$  for any  $q \in M$ ;

Hypotheses (H1) is made for technical reasons and to lighten the notation. It would be possible to avoid it through a desingularization procedure similar to the one in [19]. On the other hand, (H2) is essential to apply our methods.

We will also often assume that, for some  $s \in \mathbb{N}$ , the following ‘‘equiregularity’’ for (D) holds.

$$(H_s) \quad f_0 \subset \Delta^s \setminus \Delta^{s-1}.$$

Due to hypothesis (H1), this is equivalent to the fact that  $\text{ord}_q(f_0) = -s$  for any  $q \in M$ .

For any  $u \in L^1([0, T], \mathbb{R}^m)$ , by the variation formula (see [4]), it holds

$$(8) \quad \overrightarrow{\text{exp}} \int_0^T \left( f_0 + \sum_{i=1}^m u_i(t) f_i \right) dt = e^{Tf_0} \circ \overrightarrow{\text{exp}} \int_0^T \sum_{i=1}^m u_i(t) (e^{-tf_0})_* f_i dt.$$

This shows that a control steering system (D) from  $p \in M$  to  $q \in M$  in time  $T > 0$ , steers from  $p$  to  $e^{-Tf_0}q$  the time-dependent control system

$$(TD) \quad \dot{q}(t) = \sum_{j=1}^m u_j(t) (e^{-tf_0})_* f_j(q(t)).$$

Sometimes proofs will be eased by considering (TD) instead of (D), due to the linearity w.r.t. the control of the former.

In the following we will often consider also the two sub-Riemannian control systems associated with (D), called respectively *small* and *big*, and defined as

$$(SR-s) \quad \dot{q}(t) = \sum_{j=1}^m u_j(t) f_j(q(t)),$$

$$(SR-b) \quad \dot{q}(t) = u_0(t) f_0(q(t)) + \sum_{j=1}^m u_j(t) f_j(q(t)).$$

We will denote by  $d_{\text{SR}}$  and  $B_{\text{SR}}$  the Carnot-Carathéodory metric and metric balls, respectively, associated with (SR-s). This distance is well-defined due to Hypothesis (H2)

### 3. CONTINUOUS FAMILIES OF COORDINATES

In this section we consider properties of families of coordinates depending continuously on the points of the curve or path, in order to be able to exploit Remark 2.4.

From the definition of privileged coordinates, we immediately get the following.

**Proposition 3.1.** *Let  $\gamma : [0, T] \rightarrow M$  be a path. Let  $t > 0$  and let  $z$  be a system of privileged coordinates at  $\gamma(t)$  for  $\{f_1, \dots, f_m\}$ . Then, there exists  $C > 0$  such that*

$$(9) \quad |z_j(\gamma(t + \xi))| \leq C|\xi| \quad \text{for any } j = 1, \dots, n \text{ and any } t + \xi \in [0, T].$$

*Moreover, if for  $k \in \mathbb{N}$  it holds that  $\dot{\gamma}(t) \notin \Delta^{k-1}(\gamma(t))$ , then there exist  $C_1, C_2, \xi_0 > 0$  and a coordinate  $z_\alpha$ , of weight  $\geq k$ , such that for any  $t \in [0, T]$  and any  $|\xi| \leq \xi_0$  with  $t + \xi \in [0, T]$  it holds*

$$(10) \quad C_1 \xi \leq z_\alpha(\gamma(t + \xi)) \leq C_2 \xi.$$

*Finally, if  $\dot{\gamma}(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ , the coordinate  $z_\alpha$  can be chosen to be of weight  $k$ .*

*Proof.* By the smoothness of  $\gamma$ , there exists a constant  $C > 0$  such that  $|(z_j)_*\dot{\gamma}(t + \xi)| \leq C$  for any  $j = 1, \dots, n$  and any  $t + \xi \in [0, T]$ . Thus, we obtain

$$|z_j(\gamma(t + \xi))| \leq \left| \int_t^{t+\xi} |(z_j)_*\dot{\gamma}(t + \eta)| d\eta \right| \leq C |\xi|.$$

Let us prove (10). Let  $\{f_1, \dots, f_n\}$  be an adapted basis associated with the system of coordinates  $z$ . In particular it holds that  $z_*f_i(\gamma(t)) = \partial_{z_i}$ . Moreover, let  $k' \geq k$  be such that  $\dot{\gamma}(t) \in \Delta^{k'}(\gamma(t)) \setminus \Delta^{k'-1}(\gamma(t))$  and write  $\dot{\gamma}(t) = \sum_{w_i \leq k'} a_i(t) f_i(\gamma(t))$  for some  $a_i \in C^\infty([0, T])$ . Hence

$$z_*\dot{\gamma}(t) = \sum_{w_i \leq k'} a_i(t) z_*f_i(\gamma(t)) = \sum_{w_i \leq k'} a_i(t) \partial_{z_i}.$$

Since there exists  $i$  with  $w_i = k'$  such that  $a_i(t) \neq 0$ , this implies that  $(z_i)_*\dot{\gamma}(t) \neq 0$ . Since  $k' \geq k$ , we have then proved (9).  $\square$

As already observed in Remark 2.4, in order to be able to apply the estimates of Theorem 2.3 uniformly on  $\gamma$  we need to consider a continuous family of coordinates  $\{z^t\}_{t \in [0, T]}$  such that each  $z^t$  is privileged at  $\gamma(t)$  for  $\{f_1, \dots, f_m\}$ . We will call such a family a *continuous coordinate family* for  $\gamma$ .

Let us remark that, fixed any basis  $\{f_1, \dots, f_n\}$  adapted to the flag in a neighborhood of  $\gamma([0, T])$ , letting  $z^t$  be the inverse of the diffeomorphism

$$(11) \quad (z_1, \dots, z_n) \mapsto e^{z_1 f_1} \circ \dots \circ e^{z_n f_n}(\gamma(t)),$$

defines a continuous coordinate family for  $\gamma$ .

The following proposition precises Proposition 3.1.

**Proposition 3.2.** *Let  $\gamma : [0, T] \rightarrow M$  be a path and let  $k \in \mathbb{N}$  such that  $\dot{\gamma}(s) \in \Delta^k(\gamma(s))$  for any  $t \in [0, T]$ . Then, for any continuous coordinate family  $\{z^t\}_{t \in [0, T]}$  for  $\gamma$  there exists constants  $C, \xi_0 > 0$  such that for any  $t \in [0, T]$  and  $0 \leq \xi \leq \xi_0$  it holds*

$$(12) \quad |z_j^t(\gamma(t + \xi))| \leq C\xi \quad \text{if } w_j \leq k \quad \text{and} \quad |z_j^t(\gamma(t + \xi))| \leq C\xi^{\frac{w_j}{k}} \quad \text{if } w_j > k.$$

*Proof.* Fix  $t \in [0, T]$  and let  $\{f_1, \dots, f_n\}$  be an adapted basis associated with the privileged coordinate system  $z^t$ . To lighten we do not explicitly write the dependence on time of such basis. Writing  $z_*^t f_i(z) = \sum_{j=1}^n f_i^j(z) \partial_{z_j^t}$ , it holds that  $f_i^j$  is of weighted order  $\geq w_j - w_i$ , and hence there exists a constant  $C > 0$  such that

$$(13) \quad |f_i^j(z)| \leq C \|z\|^{(w_j - w_i)^+}.$$

Here  $\|z\|$  is the pseudo-norm  $|z_1|^{\frac{1}{w_1}} + \dots + |z_n|^{\frac{1}{w_n}}$  and  $h^+ = \max\{0, h\}$  for any  $h \in \mathbb{R}$ . Due to the compactness of  $[0, T]$ , the constant  $C$  can be chosen to be uniform w.r.t. the time.

Since  $\dot{\gamma}(\xi) \in \Delta^k(\gamma(\xi))$  for  $\xi > 0$ , there exist functions  $a_i \in C^\infty([0, T])$  such that

$$(14) \quad \dot{\gamma}(\xi) = \sum_{w_i \leq k} a_i(\xi) f_i(\gamma(\xi)) \quad \text{for any } \xi \in [0, T].$$

Observe that, for any  $t \in [0, T]$ , it holds that

$$(15) \quad \frac{1}{\xi} \int_t^{t+\xi} |a_i(\eta)| d\eta = |a_i(t)| + \mathcal{O}(\xi) \quad \text{as } \xi \downarrow 0,$$

where  $\mathcal{O}(\xi)$  is uniform w.r.t.  $t$ . In particular, for any  $\xi$  sufficiently small, this integral is bounded.

By (14), for any  $t \in [0, T]$  we get

$$(16) \quad z_j^t(\gamma(t + \xi)) = \sum_{w_i \leq k} \int_t^{t+\xi} a_i(\eta) f_i^j(z^t(\gamma(\eta))) d\eta, \quad \text{for any } t + \xi \in [0, T]$$

Then, applying (13) we obtain

$$\begin{aligned}
(17) \quad \max_{\rho \in [0, \xi]} |z_j^t(\gamma(t + \rho))| &\leq \sum_{w_i \leq k} \int_t^{t+\xi} |a_i(\eta)| |f_i^j(z^t(\gamma(\eta)))| d\eta \\
&\leq C \left( \max_{\rho \in [0, \xi]} \|z^t(\gamma(t + \rho))\| \right)^{(w_j - k)^+} \sum_{w_i \leq k} \int_t^{t+\xi} |a_i(\eta)| d\eta.
\end{aligned}$$

Up to enlarging the constant  $C$ , this and (15) yield

$$\begin{aligned}
(18) \quad \frac{\max_{\rho \in [0, \xi]} |z_j^t(\gamma(t + \rho))|}{\xi^{w_j}} &\leq C \left( \frac{\max_{\rho \in [0, \xi]} \|z^t(\gamma(t + \rho^k))\|}{\xi} \right)^{(w_j - k)^+} \sum_{w_i \leq k} \frac{1}{\xi^k} \int_t^{t+\xi^k} |a_i(\eta)| d\eta \\
&\leq C \left( \frac{\max_{\rho \in [0, \xi]} \|z^t(\gamma(t + \rho^k))\|}{\xi} \right)^{(w_j - k)^+}.
\end{aligned}$$

Clearly, if  $\max_{\rho \in [0, \xi]} \|z^t(\gamma(t + \rho^k))\|/\xi \leq C$  uniformly in  $t$ , inequality (18) proves (12). Then, let us assume by contradiction that  $\max_{\rho \in [0, \xi]} \|z^t(\gamma(t + \rho^k))\|/\xi$  is unbounded as  $\xi \downarrow 0$ . For any  $\xi$  let  $\bar{\xi} \in [0, \xi]$  to be such that  $\|z^t(\gamma(t + \bar{\xi}^k))\| = \max_{\rho \in [0, \xi]} \|z^t(\gamma(t + \rho^k))\|$ . Then, there exists a sequence  $\xi_\nu \rightarrow +\infty$  such that

$$b_\nu = \frac{|z_j^t(\gamma(t + \bar{\xi}_\nu^k))|}{\xi_\nu^{w_j}} \rightarrow +\infty \quad \text{and} \quad \frac{1}{n} \frac{\|z^t(\gamma(t + \bar{\xi}_\nu^k))\|}{\xi_\nu} \leq b_\nu^{\frac{1}{w_j}} \leq \frac{\|z^t(\gamma(t + \bar{\xi}_\nu^k))\|}{\xi_\nu}.$$

Moreover, by (18), it has to hold that  $w_j > k$ . Then, again by (18), follows that

$$b_\nu \leq C n b_\nu^{1 - \frac{k}{w_j}} \rightarrow 0 \quad \text{as } \nu \rightarrow +\infty.$$

This contradicts the fact that  $b_\nu \rightarrow +\infty$ , and proves that there exists  $\xi_0 > 0$ , a priori depending on  $t$ , such that  $\|z^t(\gamma(t + \bar{\xi}^k))\|/\xi \leq C$  for any  $\xi < \xi_0$ . Since  $[0, T]$  is compact, both constants  $\xi_0, C$  are uniform for  $t \in [0, T]$ , thus completing the proof of (12) and of the proposition.  $\square$

We now focus on coordinate systems adapted to the drift. In particular, if for some  $s \in \mathbb{N}$  it holds that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ , it makes sense to consider the following definition.

**Definition 3.3.** A *privileged coordinate system adapted to  $f_0$  at  $q$*  is a system of privileged coordinates  $z$  at  $q$  for  $\{f_1, \dots, f_m\}$  such that there exists a coordinate  $z_\ell$  such that  $z_* f_0 \equiv \partial_{z_\ell}$ .

Observe that completing  $f_0$  to an adapted basis  $\{f_1, \dots, f_0, \dots, f_n\}$  allows us to consider the coordinate system adapted to  $f_0$  at  $q$ , given by the inverse of the diffeomorphism

$$(19) \quad (z_1, \dots, z_n) \mapsto e^{z_\ell f_0} \circ \dots \circ e^{z_n f_n}(q).$$

The following definition combines continuous coordinate families for a path  $\gamma : [0, T] \rightarrow M$  with coordinate systems adapted to a drift.

**Definition 3.4.** A *continuous coordinate family for  $\gamma$  adapted to  $f_0$*  is a continuous coordinate family  $\{z^t\}_{t \in [0, T]}$  for  $\gamma$ , such that each  $z^t$  is a privileged coordinate system adapted to  $f_0$  at  $\gamma(t)$ .

Such coordinates systems can be built as per (19), letting the point  $q$  vary on the curve.

Recall that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$  for some  $s$ , and consider a path  $\gamma : [0, T] \rightarrow M$  such that  $\dot{\gamma}(t) \in \Delta^s(\gamma(t))$  and that  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \pmod{\Delta^{s-1}(\gamma(t))}$  for any  $t \in [0, T]$ . In this case, there exists  $f_\alpha \subset \Delta^s \setminus \Delta^{s-1}$  and two functions  $\varphi_\ell, \varphi_\alpha \in C^\infty([0, T])$ ,  $\varphi_\alpha \geq 0$ , such that

$$\dot{\gamma}(t) \pmod{\Delta^{s-1}(\gamma(t))} = \varphi_\ell(t) f_0(\gamma(t)) + \varphi_\alpha(t) f_\alpha(\gamma(t)).$$

Moreover, by the assumption  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \pmod{\Delta^{s-1}(\gamma(t))}$ , if  $\varphi_\ell(t) = 1$  then  $\varphi_\alpha(t) > 0$ . Then, using  $f_\alpha$  as an element of the adapted basis used to define a continuous coordinate family for  $\gamma$  adapted to  $f_0$ , it holds  $(z_i^t)_* \dot{\gamma}(t) = \varphi_i(t)$  for  $i = \alpha, \ell$  and any  $t \in [0, T]$ . The following lemma will be essential to study this case.

**Lemma 3.5.** *Assume that there exists  $s \in \mathbb{N}$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $\gamma : [0, T] \rightarrow M$  be a path such that  $\dot{\gamma}(t) \in \Delta^s(\gamma(t))$  and such that  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \pmod{\Delta^{s-1}(\gamma(t))}$  for any  $t \in [0, T]$ . Consider the continuous coordinate family  $\{z^t\}_{t \in [0, T]}$  for  $\gamma$  adapted to  $f_0$  defined above. Then, there exist constants  $\xi_0, \rho, m > 0$  and a coordinate  $\alpha \neq \ell$  of weight  $s$  such that for any  $t \in [0, T]$  and  $0 \leq \xi \leq \xi_0$ , it holds*

$$(20) \quad (z_\ell^t)_* \dot{\gamma}(t + \xi) \leq 1 - \rho \quad \text{if } t \in E_1 = \{\varphi_\ell < 1 - 2\rho\},$$

$$(21) \quad (z_\alpha^t)_* \dot{\gamma}(t + \xi) \geq m \quad \text{if } t \in E_2 = \{1 - 2\rho \leq \varphi_\ell \leq 1 + 2\rho\},$$

$$(22) \quad (z_\ell^t)_* \dot{\gamma}(t + \xi) \geq 1 + \rho \quad \text{if } t \in E_3 = \{\varphi_\ell > 1 + 2\rho\}.$$

In particular, it holds that  $E_1 \cup E_2 \cup E_3 = [0, T]$ .

*Proof.* Since  $\varphi_\alpha > 0$  on  $\varphi_\ell^{-1}(1)$ , by continuity of  $\varphi_\ell$  and  $\varphi_\alpha$  there exists  $\rho > 0$  such that  $\varphi_\alpha > 0$  on  $\varphi_\ell^{-1}([1 - 2\rho, 1 + 2\rho])$ . Since  $E_2 = \varphi_\ell^{-1}([1 - 2\rho, 1 + 2\rho])$  is closed, letting  $2m = \min_{E_2} \varphi_\alpha > 0$  property (21) follows by the uniform continuity of  $(t, \xi) \mapsto (z_\alpha^t)_* \dot{\gamma}(t + \xi)$  on  $E_2 \times [0, \xi_0]$ , for sufficiently small  $\xi_0$ . Finally, the uniform continuity of  $(t, \xi) \mapsto (z_\ell^t)_* \dot{\gamma}(t + \xi)$  over  $E_1 \times [0, \xi_0]$  and  $E_3 \times [0, \xi_0]$  yields (20) and (22).  $\square$

We end this section by observing that when the path is well-behaved with respect to the sub-Riemannian structure, it is possible to construct a very special continuous coordinate family, rectifying both  $\gamma$  and  $f_0$  at the same time.

**Proposition 3.6.** *Let  $\gamma : [0, T] \rightarrow M$  be a path and  $k \in \mathbb{N}$  be such that  $\dot{\gamma}(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$  for any  $t \in [0, T]$ , there exists a continuous coordinate family  $\{z^t\}_{[0, T]}$  for  $\gamma$  adapted such that*

- (1) *there exists a coordinate  $z_\alpha$  of weight  $k$  such that  $z_\alpha^t \dot{\gamma} \equiv \partial_{z_\alpha}$ ;*
- (2) *for any  $\xi, t \in [0, T]$  it holds that  $z_\alpha^t = z_\alpha^{t-\xi} + \xi$  and  $z_i^t = z_i^\xi$  if  $i \neq \alpha$ .*

*Moreover, if there exists  $s \in \mathbb{N}$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$  and such that  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \pmod{\Delta^{s-1}(\gamma(t))}$  for any  $t \in [0, T]$  whenever  $s = k$ , such family can be chosen adapted to  $f_0$ .*

*Proof.* By the assumptions on  $\dot{\gamma}$ , it is possible to choose  $f_\alpha \subset \Delta^k \setminus \Delta^{k-1}$  such that  $\dot{\gamma}(t) = f_\alpha(\gamma(t))$ . Let then  $\{f_1, \dots, f_n\}$  be the adapted basis obtained by completing  $f_\alpha$  and  $f_0$ . Finally, to complete the proof it is enough to consider the family of coordinates given by the inverse of the diffeomorphisms

$$(z_1, \dots, z_n) \mapsto e^{z_\ell f_0} \circ \dots \circ e^{z_\alpha f_\alpha}(\gamma(t)). \quad \square$$

#### 4. COST FUNCTIONS

In this section we focus on properties of the cost functions defined in (3) and of the associated value functions, respectively denoted by  $V^{\mathcal{J}}(\cdot, \cdot)$  and  $V^{\mathcal{I}}(\cdot, \cdot)$ . For  $\mathcal{J}$  such function is defined by

$$(23) \quad V^{\mathcal{J}}(q, q') = \inf \{ \mathcal{J}(u, T) \mid T > 0, q_u(0) = q, q_u(T) = q' \}.$$

The definition of  $V^{\mathcal{I}}$  is analogous.

**4.1. Regularity of the value function.** The following result, in the case of  $\mathcal{J}$  is contained in [22, Proposition 4.1], The proof can easily be extended to  $\mathcal{I}$ .

**Theorem 4.1.** *For any  $\mathcal{T} > 0$ , the functions  $V^{\mathcal{J}}$  and  $V^{\mathcal{I}}$  are continuous from  $M \times M \rightarrow [0, +\infty)$  (in particular they are finite). Moreover, for any  $q, q' \in M$  it holds*

$$\begin{aligned} V^{\mathcal{J}}(q, q') &\leq \min_{0 \leq t \leq \mathcal{T}} d_{SR}(e^{tf_0}q, q'), \\ V^{\mathcal{I}}(q, q') &\leq \min_{0 \leq t \leq \mathcal{T}} (t + d_{SR}(e^{tf_0}q, q')). \end{aligned}$$

Here  $e^{tf_0}$  denotes the flow of  $f_0$  at time  $t$  and we recall that  $d_{SR}$  denotes the Carnot-Carathéodory distance w.r.t. the system (SR-s).

We remark that this fact follows from the following proposition (obtained adapting [22, Lemma 3.6] to control-affine systems).

**Proposition 4.2.** *For any  $\eta > 0$  sufficiently small and for any  $q_0, q_1 \in M$ , it holds*

$$\inf\{\mathcal{J}(u, \eta) \mid \text{if } q_u(0) = q_0 \text{ then } q_u(\eta) = q_1\} \leq d_{SR}(q_0, q_1).$$

In the same paper, it is proved a more general version of the following result, in the same spirit of Theorem 2.3. We denote the reachable set from the point  $q \in M$  with cost  $\mathcal{J}$  less than  $\varepsilon > 0$  as

$$(24) \quad \mathcal{R}_T^{f_0}(q, \varepsilon) = \{p \in M \mid V^{\mathcal{J}}(q, p) \leq \varepsilon\}.$$

Recall de definition of  $\text{Box}(\eta)$  in (7) and that  $\{\partial_{z_i}\}_{i=1}^n$  is the canonical basis in  $\mathbb{R}^n$ . Then, we define the following sets, for parameters  $\eta > 0$  and  $T > 0$ :

$$\begin{aligned} \Xi_T(\eta) &= \bigcup_{0 \leq \xi \leq T} \left( \xi \partial_{z_\ell} + \text{Box}(\eta) \right), \\ \Pi_T(\eta) &= \text{Box}(\eta) \cup \bigcup_{0 < \xi \leq T} \{z \in \mathbb{R}^n : |z_\ell - \xi| \leq \eta^s, |z_i| \leq \eta^{w_i} + \eta \xi^{\frac{w_i}{s}} \text{ for } w_i \leq s, i \neq \ell, \\ &\quad \text{and } |z_i| \leq \eta(\eta + \xi^{\frac{1}{s}})^{w_i-1} \text{ for } w_i > s\}. \end{aligned}$$

**Theorem 4.3.** *Assume that there exists  $s \in \mathbb{N}$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Assume, moreover, that  $z = (z_1, \dots, z_n)$  is a privileged coordinate system adapted to  $f_0$ , i.e., such that  $z_* f_0 = \partial_{z_\ell}$ . Then, there exist  $C, \varepsilon_0, T_0 > 0$  such that*

$$(25) \quad \Xi_T \left( \frac{1}{C} \varepsilon \right) \subset \mathcal{R}_T^{f_0}(q, \varepsilon) \subset \Pi_T(C\varepsilon), \quad \text{for } \varepsilon < \varepsilon_0 \text{ and } T < T_0.$$

Here, with abuse of notation, we denoted by  $\mathcal{R}_T^{f_0}(q, \varepsilon)$  the coordinate representation of the reachable set. In particular,

$$(26) \quad \text{Box} \left( \frac{1}{C} \varepsilon \right) \cap \{z_\ell \leq 0\} \subset \mathcal{R}_T^{f_0}(q, \varepsilon) \cap \{z_\ell \leq 0\} \subset \text{Box}(C\varepsilon) \cap \{z_\ell \leq 0\}.$$

**Remark 4.4.** Let  $N \subset M$  be compact and let  $\{z^q\}_{q \in N}$  be a family of systems of privileged coordinates at  $q$  depending continuously on  $q$ . Then, as for Theorem 2.3 (see Remark 2.4), there exist uniform constants  $C, \varepsilon_0, T_0 > 0$  such that Theorem 4.3 holds for any  $q \in N$  in the system  $z^q$ .

We notice also that, since [22, Example 21] is easily extendable to  $\mathcal{I}$ , it follows that, for neither  $\mathcal{J}$  nor  $\mathcal{I}$ , the existence of minimizers is assured. Recall that a control  $u \in \mathcal{U}_{\mathcal{T}}$  is a minimizer between  $q_1, q_2 \in M$  for the cost  $J$  if its associated trajectory with initial condition  $q_u(0) = q_1$  is such that  $q_u(T) = q_2$  and  $V^{\mathcal{J}}(q_1, q_2) = J(u, T)$ .

**4.2. Behavior along the drift.** The following proposition assures that a minimizer for  $\mathcal{J}$  and  $\mathcal{I}$  always exists when moving in the drift direction.

**Proposition 4.5.** *Assume that there exists  $s \in \mathbb{N}$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . For any  $0 < t < \mathcal{T}$ , the unique minimizer between any  $q_0 \in M$  and  $e^{tf_0}q_0$  for the cost  $\mathcal{J}$  is the null control on  $[0, t]$ . Moreover, if  $f_0 \notin \Delta(q_0)$ , i.e.  $s \geq 2$ , and the maximal time of definition of the controls  $\mathcal{T}$  is sufficiently small, the same is true for  $\mathcal{I}$ .*

*Proof.* Since, for  $t \in [0, \mathcal{T}]$ , we have that  $V^{\mathcal{J}}(q, e^{tf_0}q) = 0$ , the first statement is trivial.

To prove the second part of the statement we proceed by contradiction. Namely, we assume that there exists a sequence  $\mathcal{T}_n \rightarrow 0$  such that for any  $n \in \mathbb{N}$  there exists a control  $v_n \in L^1([0, t_n], \mathbb{R}^m) \subset \mathcal{U}_{\mathcal{T}_n}$ ,  $v_n \neq 0$ , steering the system from  $q_0$  to  $e^{\mathcal{T}_n f_0}(q_0)$  and such that

$$(27) \quad t_n + \|v_n\|_{L^1([0, t_n], \mathbb{R}^m)} = \mathcal{I}(v_n, t_n) \leq \mathcal{I}(0, \mathcal{T}_n) = \mathcal{T}_n.$$

Let  $z = (z_1, \dots, z_n)$  be a privileged coordinate system adapted to  $f_0$  at  $q$ , as per Definition 3.3. Thus, by Theorem 4.3, it holds

$$(28) \quad |z_\ell(e^{\mathcal{T}_n f_0}(q_0))| \leq t_n + C\|v_n\|_{L^1([0, t_n], \mathbb{R}^m)}^2.$$

Since  $z_\ell(e^{\mathcal{T}_n f_0}(q_0)) = \mathcal{T}_n$ , putting together (27) and (28) yields  $\|v_n\|_{L^1([0, t_n], \mathbb{R}^m)} \leq C\|v_n\|_{L^1([0, t_n], \mathbb{R}^m)}^2$  for any  $n \in \mathbb{N}$ . Since by the continuity of  $V^{\mathcal{I}}$  we have that  $\|v_n\|_{L^1([0, t_n], \mathbb{R}^m)} \rightarrow 0$ , this is a contradiction.  $\square$

We remark that, in the case of  $\mathcal{I}$ , the assumption  $f_0 \notin \Delta(q_0)$  of Proposition 4.5 is essential. In particular, in the following example we show that when  $f_0 \subset \Delta$  even if a minimizer between  $q_0$  and  $e^{tf_0}(q_0)$  exists, it could not coincide with an integral curve of the drift.

**Example 4.6.** Consider the control-affine system on  $\mathbb{R}^2$ ,

$$(29) \quad \frac{d}{dt}x = f_0(x) + u_1 f_1(x) + u_2 f_2(x),$$

where  $f_0 = (1, 0)$  and  $f = (\phi_1, \phi_2)$  for some  $\phi_1, \phi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ , with  $\phi_2 \neq 0$  and  $\partial_x(\phi_1/\phi_2)|_{(0,0)} \neq 0$ . Since  $f_0$  and  $f$  are always linearly independent, the underlying small sub-Riemannian system is indeed Riemannian with metric

$$g = \begin{pmatrix} 1 & -\phi_1/\phi_2 \\ -\phi_1/\phi_2 & \frac{1-\phi_1^2}{\phi_2^2} \end{pmatrix}.$$

Let us now prove that the curve  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ ,  $\gamma(t) = (tT, 0)$  is not a minimizer of the Riemannian distance between  $(0, 0)$  and  $(T, 0)$ . In particular, it is enough to prove that  $\gamma$  is not a geodesic for small  $T > 0$ . For  $\gamma$  the geodesic equation writes

$$\begin{cases} t^2 \Gamma_{11}^1(\gamma(t)) = 0, \\ t^2 \Gamma_{11}^2(\gamma(t)) = 0, \end{cases} \quad \text{for any } t \in [0, 1] \quad \iff \quad \Gamma_{11}^1(\cdot, 0) = \Gamma_{11}^2(\cdot, 0) = 0 \text{ near } 0.$$

Here,  $\Gamma_{k\ell}^i$  are the Christoffel numbers of the second kind associated with  $g$ . A simple computation shows that

$$\Gamma_{11}^1 = \frac{\phi_1}{\phi_2} \partial_{x_1} \left( \frac{\phi_1}{\phi_2} \right), \quad \Gamma_{11}^2 = \partial_{x_1} \left( \frac{\phi_1}{\phi_2} \right).$$

Thus, if  $\partial_{x_1}(\phi_1/\phi_2)|_{(0,0)} \neq 0$ , then  $\Gamma_{11}^2(0, 0) \neq 0$ , showing that  $\gamma$  is not a geodesic.

We now show that this fact implies that for any minimizing sequence  $u_n = (u_n^1, u_n^2) \in L^1([0, t_n], \mathbb{R}^2)$  for  $V^{\mathcal{I}}$  between  $(0, 0)$  and  $e^{Tf_0}((0, 0)) = (T, 0)$ , such that  $J(u_{n+1}, t_{n+1}) \leq J(u_n, t_n)$ , then  $u_n^2 \neq 0$  for sufficiently big  $n$ . To this aim, fix any  $t_n \rightarrow 0$ , let  $u_n(s) = u(s/t_n)$  and  $q_n(\cdot)$  be the trajectory associated with  $u_n$  in system (29). Moreover, let  $v = (v_1, 0) \in L^1([0, S], \mathbb{R}^2)$  be the minimizer



of  $\mathcal{I}$  between  $(0,0)$  and  $(T,0)$  in the system  $\dot{x}_1 = 1 + v_1$ . Since the trajectory of  $v$  is exactly  $\gamma$ , by rescaling it holds  $\text{length}(\gamma) = \mathcal{I}(v, S)$ . Then, by standard results in the theory of ordinary differential equations, it follows that  $q_n(t_n) \rightarrow (T,0)$  and the fact that  $\gamma$  is not a Riemannian minimizing curve implies that

$$\|u_n\|_{L^1} = \|u\|_{L^1} < \text{length}(\gamma) = \mathcal{I}(v, S).$$

Hence, for sufficiently big  $n$  it holds that  $\mathcal{I}(u_n, t_n) < \mathcal{I}(v, S)$ , proving the claim.

As a consequence of Proposition 4.5, we get the following property for the complexities defined in the previous section with respect to the costs  $\mathcal{J}$  and  $\mathcal{I}$ . It generalizes to the control-affine setting the trivial minimality of the sub-Riemannian complexity on the path  $\Gamma = \{q\}$ .

**Corollary 4.7.** *Assume that there exists  $s \geq 2$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $x \in M$  and  $y = e^{Tf_0}x$ , for some  $0 < T < \mathcal{T}$ . Then, for any  $\varepsilon > 0$ , the minimum over all curves  $\Gamma \subset M$  (resp. paths  $\gamma : [0, T] \rightarrow M$ ) connecting  $x$  and  $y$  of  $\sigma_c^{\mathcal{J}}(\cdot, \varepsilon)$  and  $\sigma_a^{\mathcal{J}}(\cdot, \varepsilon)$  (resp.  $\sigma_t^{\mathcal{J}}(\cdot, \delta)$  and  $\sigma_n^{\mathcal{J}}(\cdot, \varepsilon)$ ) is attained at  $\Gamma = \{e^{tf_0}\}_{t \in [0, T]}$  (resp. at  $\gamma(t) = e^{tf_0}x$ ). Moreover, the same is true for the cost  $\mathcal{I}$ , whenever  $\mathcal{T}$  is sufficiently small.*

**4.3. Behavior transversally to the drift.** When we consider two points on different integral curves of the drift, it turns out that the two costs  $\mathcal{J}$  and  $\mathcal{I}$  are indeed equivalent, as proved in the following.

**Proposition 4.8.** *Assume that there exists  $s \in \mathbb{N}$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $q, q' \in M$  be such that there exists a set of privileged coordinates adapted to  $f_0$  at  $q$ . Then, there exists  $C, \varepsilon_0, \mathcal{T} > 0$  such that, for any  $u \in \mathcal{U}_{\mathcal{T}}$  such that, for some  $T < \mathcal{T}$ ,  $q_u(T) = q'$  and  $\mathcal{J}(u, T) < \varepsilon_0$ , it holds*

$$\mathcal{J}(u, T) \leq \mathcal{I}(u, T) \leq C\mathcal{J}(u, T).$$

The proof of this fact relies on the following particular case of [22, Lemma 25].

**Lemma 4.9.** *Assume that there exists  $s \in \mathbb{N}$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $q \in M$  and let  $z = (z_1, \dots, z_n)$  be a system of privileged coordinate system adapted to  $f_0$  at  $q$ . Then, there exist  $C, \varepsilon_0, \mathcal{T} > 0$  such that, for any  $u \in \mathcal{U}_{\mathcal{T}}$ , with  $\mathcal{J}(u, T) < \varepsilon_0$  for some  $T < \mathcal{T}$ , it holds*

$$T \leq C(\mathcal{J}(u, T)^s + z_\ell(q_u(T))^+).$$

Here, we let  $\xi^+ = \max\{\xi, 0\}$ .

This Lemma is crucial, since it allows to bound the time of definition of any control through its cost. We now prove Proposition 4.8.

*Proof of Proposition 4.8.* The first inequality is trivial. The second one follows by applying Lemma 4.9, and computing

$$\mathcal{I}(u, T) \leq T + \mathcal{J}(u, T) \leq (C\varepsilon_0^{s-1} + 1)\mathcal{J}(u, T).$$

□

## 5. FIRST RESULTS ON COMPLEXITIES

In this section we collect some first results regarding the various complexities we defined.

For the complexity of a generic cost function  $J : \mathcal{U}_{\mathcal{T}} \rightarrow [0, +\infty)$ , satisfying some weak assumptions, we have the following.

**Proposition 5.1.** *Assume that, for any  $q_1 \in M$  and any  $q_2 \notin \{e^{tf_0}q_1\}_{t \geq 0}$ , it holds  $V^J(q_1, q_2) > 0$ . Then, for any curve  $\Gamma \subset M$  it holds the following.*

- (1) *if  $\Gamma$  is not an integral curve of the drift, then  $\lim_{\varepsilon \downarrow 0} \sigma_c^J(\Gamma, \varepsilon) = \lim_{\varepsilon \downarrow 0} \sigma_a^J(\Gamma, \varepsilon) = +\infty$ .*

(2) if  $\Gamma$  is an admissible curve for (D), then  $\varepsilon\sigma_c^J(\Gamma, \varepsilon)$  and  $\varepsilon\sigma_a^J(\Gamma, \varepsilon)$  are bounded from above, for any  $\varepsilon > 0$ .

Moreover, for any path  $\gamma : [0, T] \rightarrow M$  it holds the following.

- (1)  $\lim_{\delta \downarrow 0} \sigma_t^J(\gamma, \delta) = 0$  and, if  $\gamma$  is not a solution of (D),  $\lim_{\varepsilon \downarrow 0} \sigma_n^J(\gamma, \varepsilon) = +\infty$ .
- (2) if  $\gamma$  is an admissible curve for (D), then  $\sigma_t^J(\gamma, \varepsilon)/\delta$  and  $\varepsilon\sigma_n^J(\gamma, \varepsilon)$  are bounded by above, for any  $\delta, \varepsilon > 0$ .

*Proof.* We only prove the statements regarding the cost interpolation complexity of a curve  $\Gamma$ . The same reasonings will hold for the others.

Since  $V^J(x, y) > 0$ , the first statement follows from

$$\lim_{\varepsilon \downarrow 0} \sigma_c^J(\Gamma, \varepsilon) \geq V(x, y) \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} = +\infty.$$

On the other hand, let  $u$  be a control such that  $q_u([0, T]) = \Gamma$ . Then, it is clear that  $\sigma_c^J(\Gamma, \varepsilon) \leq J(u, T)/\varepsilon$ , completing the proof of the proposition.  $\square$

Before starting to work with complexities in control-affine systems, we prove Theorem 1.2.

*Proof of Theorem 1.2.* Let  $\{z^t\}_{t \in [0, T]}$  to be the continuous family of coordinates for  $\gamma$  given by Proposition 3.6. We start by proving that  $\sigma_t^{\text{SR-s}}(\gamma, \delta) \preceq \delta^{\frac{1}{k}}$ . Fix any partition  $0 = t_0 < t_1 < \dots < t_N = T$  such that  $\delta/2 \leq t_i - t_{i-1} \leq \delta$ . If  $\delta$  is sufficiently small, from Theorem 2.3 follows that there exists a constant  $C > 0$  such that for any  $i = 0, \dots, N$  in the coordinate system  $z^{t_i}$  it holds that  $\text{Box}(\gamma(t_i), C\delta^{\frac{1}{k}}) \subset B_{\text{SR}}(\gamma(t_i), \delta^{\frac{1}{k}})$ . Hence, since  $z_\alpha^{t_{i-1}}(\gamma(t_i)) = t_i - t_{i-1}$ , that  $z_j^{t_{i-1}}(\gamma(t_i)) = 0$  for any  $j \neq \alpha$ , and that  $N \leq \lceil 2T/\delta \rceil \leq CT/\delta$ , we get

$$\sigma_t^{\text{SR-s}}(\gamma, \delta) \leq \delta \sum_{i=1}^N d_{\text{SR}}(\gamma(t_{i-1}), \gamma(t_i)) \leq C\delta \sum_{i=1}^N \sum_{j=1}^n |z_j^{t_{i-1}}(\gamma(t_i))|^{w_j} = C\delta \sum_{i=1}^N (t_i - t_{i-1})^{\frac{1}{k}} \leq CT\delta^{\frac{1}{k}}.$$

This proves that  $\sigma_t^{\text{SR-s}}(\gamma, \delta) \preceq \delta^{\frac{1}{k}}$ .

On the other hand, to prove that  $\sigma_t^{\text{SR-s}}(\gamma, \delta) \succcurlyeq \delta^{\frac{1}{k}}$ , let  $\eta > 0$  and  $u \in L^1([0, T])$  be a control admissible for  $\sigma_t^{\text{SR-s}}(\gamma, \delta)$  such that

$$\|u\|_{L^1([t_{i-1}, t_i])} \leq \frac{\sigma_t^{\text{SR-s}}(\gamma, \delta)}{\delta} + \eta.$$

Let  $0 = t_0 < t_1 < \dots < t_N = T$  be times such that  $q_u(t_i) = \gamma(t_i)$ ,  $i = 0, \dots, N$ ,  $0 < t_i - t_{i-1} \leq \delta$ . Moreover, let  $u_i \in L^1([t_{i-1}, t_i])$  be the restriction of  $u$  between  $t_{i-1}$  and  $t_i$ . Observe that, up to removing some  $t_i$ 's, we can assume that  $t_i - t_{i-1} \in (\frac{\delta}{2}, \frac{3}{2}\delta]$ . This implies that  $\lceil 2T/(3\delta) \rceil \leq N \leq \lceil 2T/\delta \rceil$ .

To complete the proof it suffices to show that  $\|u_i\|_{L^1([t_{i-1}, t_i])} \geq C\delta^{\frac{1}{k}}$ . In fact, for any  $\eta > 0$ , this yields

$$\frac{\sigma_t^{\text{SR-s}}(\gamma, \delta)}{\delta} \geq \|u\|_{L^1([0, T], \mathbb{R}^m)} - \eta = \sum_{i=1}^N \|u_i\|_{L^1([t_{i-1}, t_i])} - \eta \geq C \sum_{i=1}^N \delta^{\frac{1}{k}} - \eta \geq C \frac{2T}{3\delta} \delta^{\frac{1}{k}} - \eta.$$

Letting  $\eta \downarrow 0$ , this will prove that  $\sigma_t^{\text{SR-s}}(\gamma, \delta) \succcurlyeq \delta^{\frac{1}{k}}$ , completing the proof.

Observe that, by Theorem 2.3, for any  $i = 1, \dots, N$  in the coordinate system  $z^{t_{i-1}}$  it holds  $B_{\text{SR}}(\gamma(t_i), \|u_i\|_{L^1([t_{i-1}, t_i])}) \subset \text{Box}(\gamma(t_i), C\|u_i\|_{L^1([t_{i-1}, t_i])})$ . Since  $z_\alpha^{t_{i-1}}(t_i) = t_i - t_{i-1}$ , this implies that

$$\frac{\delta}{2} \leq t_i - t_{i-1} = |z_\alpha^{t_{i-1}}(\gamma(t_i))| \leq C \|u_i\|_{L^1([t_{i-1}, t_i])}^k,$$

proving the claim and the theorem.  $\square$

In the following, we will denote with an apex “SR-s” – e.g.  $\sigma_c^{\text{SR-s}}$  – the complexities associated with the small sub-Riemannian system (SR-s) defined at p. 8, and with an apex “SR-b”, e.g.  $\sigma_c^{\text{SR-b}}$ , the ones associated with the big sub-Riemannian system (SR-b).

We immediately get the following.

**Proposition 5.2.** *Let  $\Gamma \subset M$  be a curve and  $\gamma : [0, T] \rightarrow M$  be a path.*

- (i) *Any complexity relative to the cost  $\mathcal{J}$  is smaller than the same complexity relative to  $\mathcal{I}$ . Namely, for any  $\varepsilon, \delta > 0$ , it holds*

$$\begin{aligned} \sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) &\leq \sigma_c^{\mathcal{I}}(\Gamma, \varepsilon), & \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) &\leq \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon), \\ \sigma_t^{\mathcal{J}}(\gamma, \delta) &\leq \sigma_t^{\mathcal{I}}(\gamma, \delta), & \sigma_n^{\mathcal{J}}(\gamma, \varepsilon) &\leq \sigma_n^{\mathcal{I}}(\gamma, \varepsilon). \end{aligned}$$

- (ii) *For any cost, the neighboring approximation complexity of some path is always bigger than the tubular approximation complexity of its support. Namely, for any  $\gamma : [0, T] \rightarrow M$  and any  $\varepsilon > 0$ , it holds*

$$\sigma_a^{\mathcal{J}}(\gamma([0, T]), \varepsilon) \leq \sigma_n^{\mathcal{J}}(\gamma, \varepsilon), \quad \sigma_a^{\mathcal{I}}(\gamma([0, T]), \varepsilon) \leq \sigma_n^{\mathcal{I}}(\gamma, \varepsilon)$$

- (iii) *Any complexity relative to the cost  $\mathcal{I}$  is bigger than the same complexity computed for the system (SR-b). Namely, for any  $\varepsilon, \delta > 0$ , it holds*

$$\begin{aligned} \sigma_c^{\text{SR-b}}(\Gamma, \varepsilon) &\leq \sigma_c^{\mathcal{I}}(\Gamma, \varepsilon), & \sigma_a^{\text{SR-b}}(\Gamma, \varepsilon) &\leq \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon), \\ \sigma_t^{\text{SR-b}}(\gamma, \delta) &\leq \sigma_t^{\mathcal{I}}(\gamma, \delta), & \sigma_n^{\text{SR-b}}(\gamma, \varepsilon) &\leq \sigma_n^{\mathcal{I}}(\gamma, \varepsilon). \end{aligned}$$

- (iv) *In the case of curves, the complexities relative to the cost  $\mathcal{I}$  are always smaller than the same complexities computed for the system (SR-s). Namely, for any  $\varepsilon > 0$  it holds*

$$\sigma_c^{\mathcal{I}}(\Gamma, \varepsilon) \leq \sigma_c^{\text{SR-s}}(\Gamma, \varepsilon), \quad \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \leq \sigma_a^{\text{SR-s}}(\Gamma, \varepsilon).$$

*Proof.* The inequality in (ii) is immediate, since any control admissible for the  $\sigma_n(\gamma, \varepsilon)$  is also admissible for  $\sigma_a(\gamma([0, T]), \varepsilon)$ .

On the other hand, the inequalities in (iii) between the complexities in (SR-b) and the ones in (D), with cost  $\mathcal{I}$ , is a consequence of the fact that, for every control  $u \in \mathcal{U}_{\mathcal{T}}$ , the trajectory  $q_u$  is admissible for (SR-b) and associated with the control  $u_0 = (1, u) : [0, T] \rightarrow \mathbb{R}^{m+1}$  with  $\|u_0\|_{L^1([0, T], \mathbb{R}^{m+1})} = \mathcal{I}(u, T)$ . The inequalities in (i) between the complexities in (D) with respect to the different costs follows from the fact that  $\mathcal{J} \leq \mathcal{I}$ .

Finally, to complete the proof of the proposition, observe that, by Theorem 4.1, it holds that

$$V^{\mathcal{I}}(q, q') \leq d_{\text{SR}}(q, q'), \quad \text{for any } q, q' \in M.$$

This shows, in particular, that every  $\varepsilon$ -cost interpolation for (SR-s), is an  $\varepsilon$ -cost interpolation for (D), proving the statement regarding the cost interpolation complexity in (iv). The part concerning the tubular approximation follows in the same way.  $\square$

We conclude this section by proving an asymptotic equivalence for the complexities of a control-affine system in a very special case. In particular, we will prove that if we cannot generate the direction of  $\Gamma$  with an iterated bracket of  $f_0$  and some  $f_1, \dots, f_m$ , then the curve complexities for the systems (D), (SR-s) and (SR-b) behaves in the same way.

Let  $\mathcal{L}_{f_0}$  be the ideal of the Lie algebra  $\text{Lie}(f_0, f_1, \dots, f_m)$  generated by the adjoint endomorphism  $\text{ad}(f_0) : f \mapsto \text{ad}(f_0)f = [f_0, f]$ ,  $f \in \text{Vec}(M)$ . Then the following holds.

**Proposition 5.3.** *Assume that there exists  $s \in \mathbb{N}$  such that  $f_0 \in \Delta^s \setminus \Delta^{s-1}$ , and let  $\Gamma \subset M$  be a curve such that there exists  $k \in \mathbb{N}$  for which  $T\Gamma \subset \Delta^k \setminus \Delta^{k-1}$ . Assume, moreover, that for any  $q \in \Gamma$  it holds that  $T_q\Gamma \not\subset \mathcal{L}_{f_0}(q)$ . Then, for sufficiently small  $\mathcal{T}$ ,*

$$(30) \quad \sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_c^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

*Proof.* By the fact that  $T_q\Gamma \not\subset \mathcal{L}_{f_0}(q)$ , follows that  $T_q\Gamma \subset \text{Lie}_q^k(f_0, f_1, \dots, f_m) \setminus \text{Lie}_q^{k-1}(f_0, f_1, \dots, f_m)$ . Thus, approximating  $\Gamma$  in the big or in the small sub-Riemannian system is equivalent, and by Theorem 1.1 follows

$$\sigma_c^{\text{SR-s}}(\Gamma, \varepsilon) \asymp \sigma_c^{\text{SR-b}}(\Gamma, \varepsilon) \asymp \sigma_a^{\text{SR-s}}(\Gamma, \varepsilon) \asymp \sigma_a^{\text{SR-b}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

The statement then follows by applying Proposition 5.2.  $\square$

**Remark 5.4.** Observe that if  $f_0 \in \Delta$  in a neighborhood  $U$  of  $\Gamma$ , it holds that  $\text{Lie}_q^k(f_0, f_1, \dots, f_m) = \Delta^k(q)$  for any  $q \in U$ . Then, by the same argument as above, we get that (30) holds. This shows that, where  $f_0 \subset \Delta$ , the asymptotic behavior of complexities of curves is the same as in the sub-Riemannian case.

## 6. COMPLEXITY OF CURVES

This section is devoted to prove the statement on curves of Theorem 1.3. Namely, we will prove the following.

**Theorem 6.1.** *Assume that there exists  $s \geq 2$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $\Gamma \subset M$  be a curve and define  $\kappa = \max\{k: T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p) \text{ for some } p \in \Gamma\}$ . Then, if the maximal time of definition of the controls  $\mathcal{T}$  is small enough,*

$$\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_c^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^\kappa},$$

Due to the fact that the value functions associated with the costs  $\mathcal{J}$  and  $\mathcal{I}$  are always smaller than the sub-Riemannian distance associated with system (SR-s), the  $\asymp$  immediately follows from the results in [20].

**Proposition 6.2.** *Let  $\Gamma \subset M$  be a curve such that there exists  $k \in \mathbb{N}$  for which  $T\Gamma \subset \Delta^k$ . Then,*

$$\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_c^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}, \quad \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

*Proof.* By (i) in Proposition 5.2, follows that we only have to prove the upper bound for the complexities relative to the cost  $\mathcal{I}$ . Moreover, by the same proposition and [20, Theorem 3.14], follows immediately that  $\sigma_c^{\mathcal{I}}(\Gamma, \varepsilon)$  and  $\sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \varepsilon^{-k}$ , completing the proof of the proposition.  $\square$

In order to prove  $\asymp$ , we will need to exploit a sub-additivity property of the complexities. In order to have this property, it is necessary to exclude certain bad behaving points, called cusps. Near these points, the value function behaves like the Euclidean distance does near algebraic cusps (e.g.,  $(0, 0)$  for the curve  $y = \sqrt{|x|}$  in  $\mathbb{R}^2$ ). In the sub-Riemannian context, they have been introduced in [20].

**Definition 6.3.** The point  $q \in \Gamma$  is a *cusp* for the cost  $J$  if it is not an endpoint of  $\Gamma$  and if, for every  $c, \eta > 0$ , there exist two points  $q_1, q_2 \in \Gamma$  such that  $q$  lies between  $q_1$  and  $q_2$ , with  $q_1$  before  $q$  and  $q_2$  after  $q$  w.r.t. the orientation of  $\Gamma$  (in particular  $q \neq q_1, q_2$ ),  $V^J(q_1, q_2) \leq \eta$  and  $V^J(q, q_2) \geq cV(q_1, q_2)$ .

In [20] is proved that no curve has cusps in an equiregular sub-Riemannian structure. As the following example shows, the equiregularity alone is not enough for control-affine systems.

**Example 6.4.** Consider the following vector fields on  $\mathbb{R}^3$ , with coordinates  $(x, y, z)$ ,

$$f_1(x, y, z) = \partial_x, \quad f_2(x, y, z) = \partial_y + x\partial_z.$$

Since  $[f_1, f_2] = \partial_z$ ,  $\{f_1, f_2\}$  is a bracket-generating family of vector fields. The sub-Riemannian control system associated with  $\{f_1, f_2\}$  on  $\mathbb{R}^3$  corresponds to the Heisenberg group.

Let now  $f_0 = \partial_z \subset \Delta^2 \setminus \Delta$  be the drift, and let us consider the curve  $\Gamma = \{(t^2, 0, t) \mid t \in (-\eta, \eta)\}$ . Let  $q = (0, 0, 0)$ . Since  $T_q\Gamma \notin \Delta(q)$ , by smoothness of  $\Gamma$  and  $\Delta$ , for  $\eta$  sufficiently small  $T\Gamma \subset \Delta^2 \setminus \Delta$ . We now show that the point  $q$  is indeed a cusp for the cost  $\mathcal{J}$ . In fact, for any  $\xi > 0$  such that  $2\xi < \mathcal{T}$ , it holds that the null control defined over time  $[0, 2\xi]$  steers the control affine system from  $q_1 = (\xi^2, 0, -\xi) \in \Gamma$  to  $q_2 = (\xi^2, 0, \xi) \in \Gamma$ . Hence, by Proposition 4.5,  $V^{\mathcal{J}}(q_1, q_2) = 0$ . Moreover, since  $q$  and  $q_2$  are not on the same integral curve of the drift,  $V^{\mathcal{J}}(q, q_2) > 0 = V^{\mathcal{J}}(q_1, q_2)$ . This proves that  $q$  is a cusp for  $\mathcal{J}$ .

The following proposition shows that cusps appear only where the drift becomes tangent to the curve at isolated points, as in the above example.

**Proposition 6.5.** *Assume that there exists  $s \geq 2$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $\Gamma \subset M$  be a curve such that  $T\Gamma \subset \Delta^k \setminus \Delta^{k-1}$ . Moreover, if  $s = k$ , let  $\Gamma$  be such that either  $f_0(p) \notin T_p\Gamma \oplus \Delta^{s-1}(p)$  for any  $p \in \Gamma$  or  $f_0|_{\Gamma} \subset T\Gamma \oplus \Delta^{s-1}$ . Then  $\Gamma$  has no cusps for the cost  $V^{\mathcal{J}}$ .*

*Proof.* If  $f_0|_{\Gamma} \subset T_p\Gamma \oplus \Delta^{s-1}(p)$ , the statement is a consequence of Proposition 4.5. Hence, we assume that  $f_0(p) \notin T_p\Gamma \oplus \Delta^{s-1}(p)$  for any  $p \in \Gamma$ . Let  $\gamma : [0, \mathfrak{T}] \rightarrow M$  be a path parametrizing  $\Gamma$  and consider the continuous coordinate family  $\{z^t\}_{t \in [0, \mathfrak{T}]}$  adapted to  $f_0$  given by Proposition 3.6. In particular, it holds that  $z_*^t \dot{\gamma}(\cdot) \equiv \partial_{z_\alpha}$  for some coordinate  $z_\alpha$  of weight  $k$  and for any  $t \in [0, \mathfrak{T}]$ . We now fix any  $t_0 \in (0, \mathfrak{T})$  and prove that  $\gamma(t_0)$  is not a cusp. In fact, letting  $\eta > 0$  be sufficiently small, by Theorem 4.3 and the fact that  $z_\alpha^t(\gamma(\cdot)) \equiv 0$  we get

$$\begin{aligned} V^{\mathcal{J}}(\gamma(t_0), \gamma(t_0 + \eta)) &\leq C \sum_{j=1}^n |z_j^{t_0}(\gamma(t_0 + \eta))|^{\frac{1}{w_j}} = C |z_\alpha^{t_0} \gamma(t_0 + \eta)|^{\frac{1}{k}} \\ &= 2C |z_\alpha^{t_0 - \eta}(\gamma(t_0 + \eta))|^{\frac{1}{k}} \leq CV(\gamma(t_0 - \eta), \gamma(t_0 + \eta)). \end{aligned}$$

Letting  $t_1 = t_0 - \eta$  and  $t_2 = t_0 + \eta$ , this proves that  $V^{\mathcal{J}}(\gamma(t_0), \gamma(t_2)) \leq V^{\mathcal{J}}(\gamma(t_1), \gamma(t_2))$ . By definition, this implies that  $\gamma(t_0)$  is not a cusp, completing the proof of the proposition.  $\square$

Finally, we can prove the sub-additivity of the curve complexities.

**Proposition 6.6.** *Let  $\Gamma' \subset \Gamma \subset M$  be two curves. Then, if the endpoints of  $\Gamma'$  are not cusps for the cost  $\mathcal{J}$ , there exists a constant  $C > 0$  such that for sufficiently small  $\mathcal{T}$  it holds*

$$\sigma_c^{\mathcal{J}}(\Gamma', \varepsilon) \preceq \sigma_c^{\mathcal{J}}(\Gamma, \varepsilon), \quad \sigma_a^{\mathcal{J}}(\Gamma', \varepsilon) \preceq \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon).$$

*Proof. Cost interpolation complexity.* Let  $u \in L^1([0, T], \mathbb{R}^m)$  be a control admissible for  $\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon)$ , and let  $0 = t_1 < \dots < t_N = T$  be such that  $\|u\|_{L^1([t_{i-1}, t_i])} \leq \varepsilon$ . Recall that by Theorem 4.1,  $V^{\mathcal{J}}$  is a continuous function. Since for small  $\mathcal{T} > 0$ , for any  $\varepsilon > 0$  and for any  $q_0 \in M$  the reachable set  $\mathcal{R}_{\mathcal{T}}(q, \varepsilon)$  is bounded, it holds that  $\mathcal{R}_{\mathcal{T}}(q, \varepsilon) \searrow \{e^{t f_0}(q_0) \mid t \in [0, \mathcal{T}]\}$  as  $\varepsilon \downarrow 0$ , in the sense of pointwise convergence of characteristic functions. From this follows that, for  $\varepsilon$  and  $\mathcal{T}$  sufficiently small, there exist  $i_1 \neq i_2$  such that  $q_u(t_i) \in \Gamma'$  for any  $i \in \{i_1, \dots, i_2\}$  and  $q_u(t_i) \notin \Gamma'$  for any  $i \notin \{i_1, \dots, i_2\}$ . Since  $x'$  and  $y'$  are not cusps, there exists  $c > 0$  such that, letting  $x'$  and  $y'$  be the endpoints of  $\Gamma'$ , it holds  $V^{\mathcal{J}}(x', q_u(t_{i_1})) \leq cV^{\mathcal{I}}(q_u(t_{i_1-1}), q_u(t_{i_1})) \leq \varepsilon$  and  $V^{\mathcal{J}}(q_u(t_{i_2}), y') \leq V^{\mathcal{J}}(q_u(t_{i_2}), q_u(t_{i_2+1})) \leq c\varepsilon$ . Thus, there exists a constant  $C > 0$  such that

$$\sigma_c^{\mathcal{J}}(\Gamma', \varepsilon) \leq \frac{\mathcal{J}(u|_{[t_{i_1}, t_{i_2}]})}{\varepsilon} + 2c \leq C \frac{\mathcal{J}(u|_{[t_{i_1-1}, t_{i_2+1}]})}{\varepsilon} \leq C \frac{\mathcal{J}(u)}{\varepsilon}.$$

Taking the infimum over all controls  $u$ , admissible for  $\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon)$  completes the proof.

*Tubular approximation complexity.* Let  $u \in L^1([0, T], \mathbb{R}^m)$  be a control admissible for  $\sigma_a^{\mathcal{J}}(\Gamma, \varepsilon)$ . Then, letting  $q_u$  be its trajectory such that  $q_u(0) = x$ , there exists two times  $t_1$  and  $t_2$  such that  $q_u(t_1) \in B_{\text{SR}}(x', C\varepsilon)$  and  $q_u(t_2) \in B_{\text{SR}}(y', C\varepsilon)$ . Then, since  $V^{\mathcal{J}} \leq d_{\text{SR}}$  by Theorem 4.1, the same argument as above applies.  $\square$

Thanks to the sub-additivity, we can prove the  $\succcurlyeq$  part of Theorem 6.1 in the case where the curve is always tangent to the same stratum  $\Delta^k \setminus \Delta^{k-1}$ .

**Proposition 6.7.** *Assume, that there exists  $s \in \mathbb{N}$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $\Gamma \subset M$  be a curve such that there exists  $k \in \mathbb{N}$  for which  $T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p)$  for any  $p \in \Gamma$ . Then, for sufficiently small time  $\mathcal{T}$ , it holds*

$$\sigma_c^{\mathcal{I}}(\Gamma, \varepsilon) \succcurlyeq \sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \succcurlyeq \frac{1}{\varepsilon^k}, \quad \sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \succcurlyeq \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \succcurlyeq \frac{1}{\varepsilon^k}.$$

*Proof.* By Proposition 5.2,  $\sigma_c^{\mathcal{I}}(\Gamma, \varepsilon) \succcurlyeq \sigma_c^{\mathcal{J}}(\Gamma, \varepsilon)$  and  $\sigma_a^{\mathcal{I}}(\Gamma, \varepsilon) \succcurlyeq \sigma_a^{\mathcal{J}}(\Gamma, \varepsilon)$ . We will only prove that  $\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \succcurlyeq \varepsilon^{-k}$ , since the same arguments apply to  $\sigma_a^{\mathcal{J}}(\Gamma, \varepsilon)$ .

Let  $\gamma : [0, \mathfrak{T}] \rightarrow M$  be a path parametrizing  $\Gamma$ . We will distinguish three cases.

**Case 1**  $f_0(p) \notin \Delta^{s-1}(p) \oplus T_p\Gamma$  for any  $p \in \Gamma$ : Fix  $\eta > 0$  and consider a control  $u \in L^1([0, T])$ , admissible for  $\sigma_c(\Gamma, \varepsilon)$  such that

$$(31) \quad \frac{\|u\|_{L^1}}{\varepsilon} \leq \sigma_c(\Gamma, \varepsilon) + \eta.$$

Let  $u_i = u|_{[t_{i-1}, t_i]}$ ,  $i = 1, \dots, N = \left\lceil \frac{\|u\|_{L^1}}{\varepsilon} \right\rceil$  to be such that  $\|u_i\|_{L^1} = \varepsilon$  for any  $1 \leq i < N$ ,  $\|u_N\|_{L^1} \leq \varepsilon$ . Moreover, let  $s_i$  be the times such that  $\gamma(s_i) = q_u(t_i)$ .

By (31), it holds  $N \leq \lceil \sigma_c(\Gamma, \varepsilon) + \eta + 1 \rceil$ . However, we can assume w.l.o.g. that  $N \leq \lceil \sigma_c(\Gamma, \varepsilon) + \eta \rceil$ . In fact,  $N > \lceil \sigma_c(\Gamma, \varepsilon) + \eta \rceil$  only if  $\|u_N\| < \varepsilon$ . In this case we can simply restrict ourselves to compute  $\sigma_c(\tilde{\Gamma}, \varepsilon)$  where  $\tilde{\Gamma}$  is the segment of  $\Gamma$  comprised between  $x$  and  $q_u(t_{N-1})$ . Indeed, by Propositions 6.5 and 6.6, it follows that  $\sigma_c(\tilde{\Gamma}, \varepsilon) \preccurlyeq \sigma_c(\Gamma, \varepsilon)$ .

We now assume that  $\varepsilon$  and  $\mathcal{T}$  are sufficiently small, in order to satisfy the hypotheses of Theorem 4.3 at any point of  $\Gamma$ . Moreover, let  $\{z^t\}_{t \in [0, \mathfrak{T}]}$  be the continuous coordinate family for  $\Gamma$  adapted to  $f_0$  given by Proposition 3.6. Then, it holds

$$(32) \quad \mathfrak{T} = \sum_{i=1}^N (s_i - s_{i-1}) = \sum_{i=1}^N |z_\alpha^{s_i-1}(\gamma(s_i))| = \sum_{i=1}^N |z_\alpha^{s_i-1}(q_u(t_i))| \leq C(\sigma_c(\Gamma, \varepsilon) + \eta)\varepsilon^k.$$

Here, in the last inequality we applied Theorem 4.3 and the fact that  $z_\ell^{s_i-1}(q_u(t_i)) = 0$  by Proposition 3.6. Finally, letting  $\eta \downarrow 0$  in (32), we get that for any  $\varepsilon$  sufficiently small it holds  $\sigma_c(\Gamma, \varepsilon) \geq C\mathfrak{T}\varepsilon^{-k}$ . This completes the proof in this case.

**Case 2**  $s = k$  and  $f_0(p) \in \Delta^{s-1}(p) \oplus T_p\Gamma$  for any  $p \in \Gamma$ : Let  $\{z^t\}_{t \in [0, \mathfrak{T}]}$  be a continuous coordinate family for  $\gamma$  adapted to  $f_0$ . In this case, since  $(z_\ell^t)_* f_0 = 1$ , it holds that  $(z_\ell^t)_* \dot{\gamma}(\cdot) \neq 0$ . Hence, there exist  $C_1, C_2 > 0$  such that for any  $t, \xi \in [0, T]$

$$(33) \quad C_1(t - \xi) \leq z_\ell^t(\gamma(\xi)) \leq C_2(t - \xi), \quad \text{if } (z_\ell^t)_* \dot{\gamma}(\cdot) > 0;$$

$$(34) \quad C_1(t - \xi) \leq -z_\ell^t(\gamma(\xi)) \leq C_2(t - \xi), \quad \text{if } (z_\ell^t)_* \dot{\gamma}(\cdot) < 0.$$

If (34) holds, then we can proceed as in Case 1 with  $\alpha = \ell$ . In fact,  $|z_\ell^{s_i-1}(q_u(t_i))| \leq C\varepsilon^s$  by Theorem 4.3. On the other hand, if (33) holds, by applying Theorem 4.3 we get

$$\begin{aligned} \mathfrak{T} &= \sum_{i=1}^N (s_i - s_{i-1}) \leq \frac{1}{C_1} \sum_{i=1}^N |z_\ell^{s_i-1}(\gamma(s_i))| = \frac{1}{C_1} \sum_{i=1}^N |z_\ell^{s_i-1}(q_u(t_i))| \\ &\leq \frac{1}{C_1} \sum_{i=1}^N (C\varepsilon^s + t_i - t_{i-1}) \leq C(\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) + \eta)\varepsilon^s + T. \end{aligned}$$

By taking  $\mathcal{T}$  sufficiently small, it holds  $T \leq \mathcal{T} < \mathfrak{T}$ . Then, letting  $\eta \downarrow 0$  this proves that  $\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon) \geq ((T - \mathcal{T})/C)\varepsilon^{-s} \succcurlyeq \varepsilon^{-s}$ . This completes the proof of this case.

**Case 3**  $s = k$  and  $f_0(p) \in \Delta^{s-1}(p) \oplus T_p\Gamma$  for some  $p \in \Gamma$ : In this case, there exists an open interval  $(t_1, t_2) \subset [0, \mathfrak{T}]$  such that  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \pmod{\Delta^{s-1}(\gamma(t))}$  for any  $t \in (t_1, t_2)$ . Thus,  $\Gamma' = \gamma((t_1, t_2))$ , satisfies the assumption of Case 1 and hence  $\sigma_c^{\mathcal{J}}(\Gamma', \varepsilon) \asymp \varepsilon^{-k}$ . Moreover, by Proposition 6.5, we can assume that  $\gamma(t_1)$  and  $\gamma(t_2)$  are not cusps. Then, by Proposition 6.6 we get

$$\frac{1}{\varepsilon^k} \asymp \sigma_c^{\mathcal{J}}(\Gamma', \varepsilon) \asymp \sigma_c^{\mathcal{J}}(\Gamma, \varepsilon),$$

completing the proof of the proposition.  $\square$

Finally, we are in a condition to prove the main theorem of this section.

*Proof of Theorem 6.1.* Since it is clear that  $T\Gamma \subset \Delta^\kappa$ , the upper bound follows by Proposition 6.2. Moreover, by Proposition 5.2 it suffices to prove that  $\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon)$  and  $\sigma_a^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \varepsilon^{-\kappa}$ . Since the arguments are analogous, we only prove this for  $\sigma_c^{\mathcal{J}}$ .

By smoothness of  $\Gamma$ , the set  $A = \{p \in \Gamma \mid T_p\Gamma \in \Delta^\kappa(p) \setminus \Delta^{\kappa-1}(p)\}$  has non-empty interior. Let then  $\Gamma' \subset A$  be a non-trivial curve such that either  $f_0(p) \notin T_p\Gamma' \oplus \Delta^{s-1}(p)$  for any  $p \in \Gamma'$  or that  $f_0|_{\Gamma'} \subset T\Gamma' \oplus \Delta^{s-1}$ . Then, since by Proposition 6.5 we can choose  $\Gamma'$  such that it does not contain any cusps, applying Proposition 6.6 yields that  $\sigma_c^{\mathcal{J}}(\Gamma', \varepsilon) \asymp \sigma_c^{\mathcal{J}}(\Gamma, \varepsilon)$ . Finally, the result follows from the fact that, by Proposition 6.7, it holds  $\sigma_c^{\mathcal{J}}(\Gamma', \varepsilon) \asymp \varepsilon^{-\kappa}$ .  $\square$

## 7. COMPLEXITY OF PATHS

In this section we will prove the statement on paths of Theorem 1.3. Namely, we will prove the following.

**Theorem 7.1.** *Assume that there exists  $s \geq 2$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $\gamma : [0, T] \rightarrow M$  be a path such that  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \pmod{\Delta^{s-1}(\gamma(t))}$  for any  $t \in [0, T]$  and define  $\kappa = \max\{k : \gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t)) \text{ for any } t \text{ in an open subset of } [0, T]\}$ . Then, it holds*

$$\sigma_t(\gamma, \delta) \asymp \delta^{\frac{1}{\max\{\kappa, s\}}}, \quad \sigma_n(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{\kappa, s\}}}.$$

Differently to what happened for curves, the  $\asymp$  part does not immediately follow from the estimates of sub-Riemannian complexities, but requires additional care. It is contained in the following proposition.

**Proposition 7.2.** *Assume that there exists  $s \in \mathbb{N}$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $\gamma : [0, T] \rightarrow M$  be a path such that  $\dot{\gamma}(t) \in \Delta^k(\gamma(t))$ . Then, it holds*

$$(35) \quad \sigma_t^{\mathcal{J}}(\gamma, \delta) \asymp \sigma_t^{\mathcal{I}}(\gamma, \delta) \asymp \delta^{\frac{1}{\max\{s, k\}}}, \quad \sigma_n^{\mathcal{J}}(\Gamma, \varepsilon) \asymp \sigma_n^{\mathcal{I}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{s, k\}}}.$$

*Proof.* By (i) in Proposition 5.2, follows that we only have to prove the upper bound for the complexities relative to the cost  $\mathcal{I}$ . We will start by proving (35) for  $\sigma_t^{\mathcal{I}}$ .

Let  $\{z^t\}_{t \in [0, T]}$  be a continuous coordinate family for  $\gamma$  adapted to  $f_0$ . Let  $\tilde{\gamma}_t(\xi) = e^{-(\xi-t)f_0}(\gamma(\xi))$ . Then, since  $z_*^t f_0 = \partial_{z_t}$ , it holds

$$(36) \quad z_\ell^t(\tilde{\gamma}_t(\xi)) = z_\ell^t(\gamma(\xi)) - (\xi - t), \quad z_i^t(\tilde{\gamma}_t(\xi)) = z_i^t(\gamma(\xi)) \quad \text{for any } i \neq \ell.$$

Fix  $\xi > 0$  sufficiently small for Proposition 3.2 to hold and choose a partition  $0 < t_1 < \dots < t_N = T$  such that  $\delta/2 \leq t_i - t_{i-1} \leq \delta$ . In particular,  $N \leq \lceil 2T/\delta \rceil$ . We then select a control  $u \in L^1([0, T], \mathbb{R}^m)$  such that its trajectory  $q_u$  in (D), with  $q_u(0) = x$ , satisfies  $q_u(t_i) = \gamma(t_i)$  for any  $i = 1, \dots, N$  as follows. For each  $i$ , we choose  $u_i \in L^1([t_{i-1}, t_i], \mathbb{R}^m)$  steering system (TD) from  $\gamma(t_{i-1}) = \tilde{\gamma}_{t_{i-1}}(t_{i-1})$  to  $\tilde{\gamma}_{t_{i-1}}(t_i)$ . Then, by (8) and the definition of  $\tilde{\gamma}_{t_{i-1}}$ , the control  $u_i$  steers system (D) from  $\gamma(t_{i-1})$  to  $\gamma(t_i)$ .

Since by [22, Theorem 8] it holds  $V_{\text{TD}}^{\mathcal{I}} \leq d_{\text{SR}}$ , by (36), Proposition 3.2 and Theorem 2.3, if  $\delta$  is sufficiently small we can choose  $u_i$  such that there exists  $C > 0$  for which

$$(37) \quad \begin{aligned} \mathcal{I}(u_i, t_i - t_{i-1}) &\leq C \sum_{j=1}^n |z_j^{t_i-1}(\tilde{\gamma}_{t_{i-1}}(t_i))|^{\frac{1}{w_j}} \leq C \sum_{j=1}^n |z_j^{t_i-1}(\gamma(t_i))|^{\frac{1}{w_j}} + \delta^{\frac{1}{s}} \\ &\leq C \left( \sum_{w_j \leq k} \delta^{\frac{1}{w_j}} + \delta^{\frac{1}{s}} + \sum_{w_j > k} \delta^{\frac{1}{k}} \right) \leq C \delta^{\frac{1}{\max\{k, s\}}}. \end{aligned}$$

Hence, we obtain that

$$(38) \quad \mathcal{I}(u, T) \leq N \mathcal{I}(u_i, t_i - t_{i-1}) \leq 3C \frac{T}{\delta} \delta^{\frac{1}{\max\{k, s\}}}.$$

Since the control  $u$  is admissible for  $\sigma_t^{\mathcal{I}}(\gamma, \delta)$ , this proves that  $\sigma_t^{\mathcal{I}}(\gamma, \delta) \asymp \delta^{\frac{1}{\max\{k, s\}}}$ .

To complete the proof for  $\sigma_n(\gamma, \varepsilon)$ , let  $\delta = \varepsilon^{\max\{k, s\}}$ . Then, by Theorems 2.3 and 4.3, there exists a constant  $C > 0$  such that  $\mathcal{R}_\delta^{f_0}(\gamma(t), \varepsilon) \subset B_{\text{SR}}(\gamma(t), C\varepsilon)$  for any  $t \in [0, T]$ . In particular,  $d_{\text{SR}}(\gamma(t_i), q_u(t)) \leq C\varepsilon$  for any  $t \in [t_{i-1}, t_i]$ . Moreover, again by Theorem 2.3, Proposition 3.2, and the fact that  $\dot{\gamma}(\cdot) \in \Delta^k(\gamma(\cdot))$ , this choice of  $\delta$  implies also that  $d_{\text{SR}}(\gamma(t_{i-1}), \gamma(t)) \leq C\varepsilon$  for any  $t \in [t_{i-1}, t_i]$ . Hence, for any  $t \in [t_{i-1}, t_i]$ , we get

$$d_{\text{SR}}(\gamma(t), q_u(t)) \leq d_{\text{SR}}(\gamma(t_{i-1}), q_u(t)) + d_{\text{SR}}(\gamma(t_{i-1}), \gamma(t)) \leq 2C\varepsilon.$$

Thus,  $u$  is admissible for  $\sigma_n^{\mathcal{I}}(\gamma, C\varepsilon)$ . Finally, from (38) we get that  $\sigma_n^{\mathcal{I}}(\gamma, C\varepsilon) \leq \varepsilon^{-1} \mathcal{I}(u, T) \leq 3CT\varepsilon^{-\max\{k, s\}}$ , proving that  $\sigma_n(\gamma, \varepsilon) \asymp \varepsilon^{-\max\{k, s\}}$ . This completes the proof.  $\square$

Now, we prove the  $\succcurlyeq$  part of the statement, in the case where  $\dot{\gamma}$  is always contained in the same stratum  $\Delta^k \setminus \Delta^{k-1}$ .

**Proposition 7.3.** *Assume that there exists  $s \geq 2$  such that  $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ . Let  $\gamma : [0, T] \rightarrow M$  be a path, such that  $\dot{\gamma}(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$  for any  $t \in [0, T]$ . Moreover, if  $s = k$ , assume that  $f_0(\gamma(t)) \neq \dot{\gamma}(t) \pmod{\Delta^{s-1}}$  for any  $t \in [0, T]$ . Then, it holds*

$$\sigma_t^{\mathcal{I}}(\gamma, \delta) \succcurlyeq \sigma_t^{\mathcal{J}}(\gamma, \delta) \succcurlyeq \delta^{\frac{1}{\max\{s, k\}}}, \quad \sigma_n^{\mathcal{I}}(\gamma, \varepsilon) \succcurlyeq \sigma_n^{\mathcal{J}}(\gamma, \varepsilon) \succcurlyeq \frac{1}{\varepsilon^{\max\{s, k\}}}.$$

*Proof.* By Proposition 5.2,  $\sigma_t^{\mathcal{I}}(\gamma, \delta) \asymp \sigma_t^{\mathcal{J}}(\gamma, \delta)$  and  $\sigma_n^{\mathcal{I}}(\gamma, \varepsilon) \asymp \sigma_n^{\mathcal{J}}(\gamma, \varepsilon)$ . Hence, to complete the proof it suffices to prove the asymptotic lower bound for  $\sigma_t^{\mathcal{J}}(\gamma, \delta)$  and  $\sigma_n^{\mathcal{J}}(\gamma, \varepsilon)$ . In the following, to lighten the notation, we write  $\sigma_t$  and  $\sigma_n$  instead of  $\sigma_t^{\mathcal{J}}$  and  $\sigma_n^{\mathcal{J}}$ .

**Interpolation by time complexity.** Let  $\eta > 0$  and  $u \in L^1([0, T])$  be a control admissible for  $\sigma_t(\gamma, \delta)$  such that

$$(39) \quad \mathcal{J}(u, T) = \|u\|_{L^1([0, T], \mathbb{R}^m)} \leq \frac{\sigma_t(\gamma, \delta)}{\delta} + \eta.$$

Let  $N = \lceil T/\delta \rceil$  and  $0 = t_0 < t_1 < \dots < t_N = T$  be times such that  $q_u(t_i) = \gamma(t_i)$ ,  $i = 0, \dots, N$ , and  $0 < t_i - t_{i-1} \leq \delta$ . Observe that, up to removing some  $t_i$ 's, we can always assume  $\delta/2 \leq t_i - t_{i-1} \leq (3/2)\delta$  and  $N \geq \lceil (2T)/(3\delta) \rceil$ . Moreover, let  $u_i = u|_{[t_{i-1}, t_i]}$ . Proceeding as in the proof of Theorem 1.2, p. 15, we get that in order to show that  $\sigma_t(\gamma, \delta) \succcurlyeq \delta^{\frac{1}{\max\{k, s\}}}$  it suffices to prove

$$(40) \quad \|u_i\|_{L^1([t_{i-1}, t_i])} \geq C \delta^{\frac{1}{\max\{s, k\}}}, \quad i = 1, \dots, N.$$

We distinguish three cases.



**Case 1**  $k > s$ : Let  $\{z^t\}$  be the continuous coordinate family for  $\gamma$  adapted to  $f_0$  given by Proposition 3.6. Then, since  $z_\ell^t(\gamma(\cdot)) = 0$  and  $z_\alpha^t(\gamma(\xi)) = \xi - t$ , by Theorem 4.3 it holds

$$(41) \quad \frac{\delta}{2} \leq (t_i - t_{i-1}) = |z_\alpha^{t_i-1}(\gamma(t_i))| \leq C \|u_i\|_{L^1([t_{i-1}, t_i])}^k.$$

This proves (40).

**Case 2**  $k < s$ : Also in this case, let  $\{z^t\}$  be the continuous coordinate family for  $\gamma$  adapted to  $f_0$  given by Proposition 3.6. Then, by Lemma 4.9 we get

$$\frac{\delta}{2} \leq t_i - t_{i-1} \leq C \|u_i\|_{L^1([t_{i-1}, t_i])}^s,$$

which immediately proves (40).

**Case 3**  $k = s$ : Let  $\{z^t\}_{t \in [0, T]}$  be a continuous coordinate family for  $\gamma$  adapted to  $f_0$ . By the mean value theorem there exists  $\xi \in [t_{i-1}, t_i]$  such that

$$(42) \quad z_\ell^{t_i-1}(\gamma(t_i)) = \int_{t_{i-1}}^{t_i} (z_\ell^t)_* \dot{\gamma}(t) dt = ((z_\ell^{t_i-1})_* \dot{\gamma}(\xi))(t_i - t_{i-1}).$$

Consider the partition  $\{E_1, E_2, E_3\}$  of  $[0, T]$  given by Lemma 3.5 and let  $\delta \leq \delta_0$ . Then, depending to which  $E_j$  belongs  $t_{i-1}$ , we proceed differently.

(a)  $t_{i-1} \in E_1$ : By Lemma 4.9 and (42) we get

$$t_i - t_{i-1} \leq C \|u_i\|_{L^1([t_{i-1}, t_i])}^s + z_\ell^{t_i-1}(\gamma(t_i))^+ = C \|u_i\|_{L^1([t_{i-1}, t_i])}^s + ((z_\ell^{t_i-1})_* \dot{\gamma}(\xi))(t_i - t_{i-1}).$$

Then, by (20) of Lemma 3.5, we get

$$\|u_i\|_{L^1([t_{i-1}, t_i])} \geq \left( \frac{1 - (z_\ell^{t_i-1})_* \dot{\gamma}(\xi)}{C} \right)^{\frac{1}{s}} (t_i - t_{i-1})^{\frac{1}{s}} \geq \left( \frac{\rho}{C} \right)^{\frac{1}{s}} \delta^{\frac{1}{s}}.$$

This proves (40).

(b)  $t_{i-1} \in E_2$ : By (21) of Lemma 3.5, (42) and Theorem 4.3 we get

$$m(t_i - t_{i-1}) \leq |z_\alpha^{t_i-1}(\gamma(t_i))| \leq C \left( \|u_i\|_{L^1([t_{i-1}, t_i])}^s + \|u_i\|_{L^1([t_{i-1}, t_i])} |z_\ell^{t_i-1}(\gamma(t_i))| \right).$$

Reasoning as in (37) yields that we can assume  $\|u_i\|_{L^1([t_{i-1}, t_i])} \leq C \delta^{\frac{1}{s}}$ . Then, by (42) and letting  $\delta \leq (m/(2 + 4\rho))^s$ , we get

$$\|u_i\|_{L^1([t_{i-1}, t_i])} \geq (m - \delta^{\frac{1}{s}}(1 + 2\rho))^{\frac{1}{s}} (t_i - t_{i-1})^{\frac{1}{s}} \geq \left( \frac{m}{2} \right)^{\frac{1}{s}} \delta^{\frac{1}{s}},$$

proving (40).

(c)  $t_{i-1} \in E_3$ : By Theorem 4.3 it follows that

$$(43) \quad |z_\ell^{t_i-1}(\gamma(t_i))| \leq C \|u_i\|_{L^1([t_{i-1}, t_i])}^s + (t_i - t_{i-1}).$$

Then, by (42) and (43) we obtain

$$\|u_i\|_{L^1([t_{i-1}, t_i])} \geq \left( \frac{(z_\ell^{t_i-1})_* \dot{\gamma}(\xi) - 1}{C} \right)^{\frac{1}{s}} (t_i - t_{i-1})^{\frac{1}{s}} \geq \left( \frac{\rho}{C} \right)^{\frac{1}{s}} \delta^{\frac{1}{s}}.$$

The last inequality follows from (22) of Lemma 3.5. This proves (40).

**Neighboring approximation complexity.** Fix  $\eta > 0$  and let  $u \in L^1([0, T], \mathbb{R}^m)$  be admissible for  $\sigma_n(\gamma, \varepsilon)$  and such that  $\|u\|_{L^1([0, T], \mathbb{R}^m)} \leq \sigma_n(\gamma, \varepsilon) + \eta$ . Let  $q_u : [0, T] \rightarrow M$  be the trajectory of  $u$  with  $q_u(0) = \gamma(0)$ . Let then  $N = \lceil \sigma_n(\gamma, \varepsilon) + \eta \rceil$  and  $0 = t_0 < t_1 < \dots < t_N = T$  be such that  $\|u\|_{L^1([t_{i-1}, t_i])} \leq \varepsilon$  for any  $i = 1, \dots, N$ . By Proposition 4.2 and the fact that  $q_u(t) \in B_{\text{SR}}(\gamma(t), \varepsilon)$  for

any  $t \in [0, T]$ , we can build a new control, still denoted by  $u$ , such that  $q_u(t_i) = \gamma(t_i)$ ,  $i = 1, \dots, N$ , and  $\|u\|_{L^1([t_{i-1}, t_i])} \leq 3\varepsilon$ .

Fixed a  $\delta_0 > 0$ , w.l.o.g. we can assume that  $t_i - t_{i-1} \leq \delta_0$ . In fact, we can split each interval  $[t_{i-1}, t_i]$  not satisfying this property as  $t_{i-1} = \xi_1 < \dots < \xi_M = t_i$ , with  $\xi_\nu - \xi_{\nu-1} \leq \delta_0$ . Then, as above, it is possible to modify the control  $u$  so that  $q_u(\xi_\nu) = \gamma(\xi_\nu)$  for any  $\nu = 1, \dots, M$ . Since  $M \leq \lceil T/\delta_0 \rceil$  and  $q_u(\cdot) \in B_{\text{SR}}(\gamma(\cdot), \varepsilon)$ , we have  $\|u\|_{L^1([\xi_i, \xi_{i-1}])} \leq 5\varepsilon$  and the new total number of intervals is  $\leq (1 + \lceil T/\delta_0 \rceil)[\sigma_n(\gamma, \varepsilon) + \eta] \leq C(\sigma_n(\gamma, \varepsilon) + \eta)$ .

We claim that to prove  $\sigma_n(\gamma, \varepsilon) \succcurlyeq \varepsilon^{-\max\{s, k\}}$ , it suffices to show that there exists a constant  $C > 0$ , independent of  $u$ , such that

$$(44) \quad t_i - t_{i-1} \leq C\varepsilon^{\max\{s, k\}}, \quad \text{for any } i = 1, \dots, N.$$

In fact, since  $N \leq C(\sigma_n(\gamma, \varepsilon) + \eta)$ , this will imply that

$$T = \sum_{i=1}^N t_i - t_{i-1} \leq C(\sigma_n(\gamma, \varepsilon) + \eta)\varepsilon^{\max\{s, k\}}.$$

Letting  $\eta \downarrow 0$ , we get that  $\sigma_n(\gamma, \varepsilon) \succcurlyeq \varepsilon^{-\max\{s, k\}}$ , proving the claim.

We now let  $\delta_0$  sufficiently small in order to apply Lemma 3.5, Theorem 4.3, and Lemma 4.9. As before, we distinguish three cases.

**Case 1**  $k > s$ : Let  $\{z^t\}$  be the continuous coordinate family for  $\gamma$  adapted to  $f_0$  given by Proposition 3.6. By Theorem 4.3, using the fact that  $\gamma(t_i) = q_u(t_i)$  for  $i = 1, \dots, N$ , we get

$$(45) \quad (t_i - t_{i-1}) = |z_\alpha^{t_i-1}(\gamma(t_i))| \leq C\varepsilon^k.$$

This proves (44).

**Case 2**  $k < s$ : Again, let  $\{z^t\}$  be the continuous coordinate family for  $\gamma$  adapted to  $f_0$  given by Proposition 3.6. As for the interpolation by time complexity, by Lemma 4.9 and the fact that  $q_u(t_i) = \gamma(t_i)$ , we get

$$(t_i - t_{i-1}) \leq C\varepsilon^s,$$

thus proving (44).

**Case 3**  $k = s$ : Let  $\{z^t\}_{t \in [0, T]}$  to be a continuous coordinate family for  $\gamma$  adapted to  $f_0$ . Consider the partition  $\{E_1, E_2, E_3\}$  of  $[0, T]$  given by Lemma 3.5 and recall (42). We distinguish three cases.

(a)  $t_i - 1 \in E_1$ : By Lemma 4.9 and (42) we get

$$t_i - t_{i-1} \leq C\varepsilon^s + z_\ell^{t_i-1}(\gamma(t_i)) = 2C\varepsilon^s + ((z_\ell^{t_i-1})_* \dot{\gamma}(\xi))(t_i - t_{i-1}).$$

By (20) of Lemma 3.5, this implies

$$t_i - t_{i-1} \leq \left( \frac{2C}{1 - (z_\ell^{t_i-1})_* \dot{\gamma}(\xi)} \right) \varepsilon^s \leq \frac{2C}{\rho} \varepsilon^s.$$

Hence, (44) is proved.

(b)  $t_i - 1 \in E_2$ : By (21) of Lemma 3.5, (42) and Theorem 4.3 we get

$$m(t_i - t_{i-1}) \leq |z_\alpha^{t_i-1}(\gamma(t_i))| \leq C \left( \varepsilon^s + \varepsilon |z_\ell^{t_i-1}(\gamma(t_i))| \right) \leq C \left( \varepsilon^s + \varepsilon^{s+1} + \varepsilon(t_i - t_{i-1}) \right).$$

This, by taking  $\varepsilon$  sufficiently small and enlarging  $C$ , implies (44).

(c)  $t_i - 1 \in E_3$ : By Theorem 4.3 it follows that

$$(46) \quad |z_\ell^{t_i-1}(\gamma(t_i))| \leq C\varepsilon^s + (t_i - t_{i-1}).$$

Then, by (42) and (46) we obtain

$$t_i - t_{i-1} \leq \frac{C}{(z_\ell^{t_{i-1}})_* \dot{\gamma}(\xi) - 1} \varepsilon^s \leq \frac{C}{\rho} \varepsilon^s.$$

The last inequality follows from (22) of Lemma 3.5, and proves (44).  $\square$

As for the case of curves, in order to extend Proposition 7.3 to paths not always tangent to the same strata, we will need the following sub-additivity property. Let us remark that due to the definition of the path complexities, we do not need to make any assumption regarding cusps.

**Proposition 7.4.** *Let  $\gamma : [0, T] \rightarrow M$  be a path and let  $t_1, t_2 \subset [0, T]$ . Then,*

$$\sigma_t^{\mathcal{J}}(\gamma|_{[t_1, t_2]}, \delta) \preceq \sigma_t^{\mathcal{J}}(\gamma, \delta), \quad \sigma_n^{\mathcal{J}}(\gamma|_{[t_1, t_2]}, \varepsilon) \preceq \sigma_n^{\mathcal{J}}(\gamma, \varepsilon).$$

*Proof. Time interpolation complexity.* Let  $u \in L^1([0, T], \mathbb{R}^m)$  be a control admissible for  $\sigma_c^{\mathcal{J}}(\Gamma, \varepsilon)$ , and let  $0 = \xi_1 < \dots < \xi_N = T$  be the times where  $q_u(\xi_i) = \gamma(\xi_i)$ . Let  $i_1 \neq i_2$  such that  $t_1 \leq \xi_{i_1} \leq t_2$  for any  $i \in \{i_1, \dots, i_2\}$ . Observe that, by Theorems 2.3 and 4.1, we have  $V^{\mathcal{J}}(\gamma(t_1), \gamma(\xi_{i_1})) \leq d_{\text{SR}}(\gamma(t_1), \gamma(\xi_{i_1})) \leq C\delta^{\frac{1}{r}}$  and  $V^{\mathcal{J}}(\gamma(\xi_{i_2}), \gamma(t_2)) \leq d_{\text{SR}}(\gamma(\xi_{i_2}), \gamma(t_2)) \leq C\delta^{\frac{1}{r}}$ , where  $\delta$  is sufficiently small,  $C$  is independent of  $\delta$ , and  $r$  is the nonholonomic degree of the distribution. Thus, assuming w.l.o.g.  $C \geq 1$ ,

$$\sigma_t^{\mathcal{J}}(\gamma|_{[t_1, t_2]}, \delta) \leq \delta \mathcal{J}(u|_{[t_1, t_2]}) + 2C\delta^{1+\frac{1}{r}} \leq C\delta \mathcal{J}(u) + C\delta^{1+\frac{1}{r}}.$$

Taking the infimum over all controls  $u$  admissible for  $\sigma_t^{\mathcal{J}}(\gamma, \delta)$ , and recalling that, by Proposition 7.2, it holds  $\sigma_t^{\mathcal{J}}(\gamma, \delta) \preceq \delta^{\frac{1}{r}}$ , completes the proof.

*Neighboring approximation complexity.* In this case, the proof is identical to the one of Proposition 6.6 for the tubular approximation complexity. The sole difference is that here, by definition of  $\sigma_n^{\mathcal{J}}$ , we do not need to assume the absence of cusps.  $\square$

We can now complete the proof of Theorem 1.3, by proving Theorem 7.1.

*Proof.* The proof is analogous to the one of Theorem 6.1, using Propositions 7.2, 7.3 and 7.4.  $\square$

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# THE HEAT AND SCHRÖDINGER EQUATIONS ON CONIC AND ANTICONIC SURFACES

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**Abstract** We study the evolution of the heat and of a free quantum particle (described by the Schrödinger equation) on two-dimensional manifolds endowed with the degenerate Riemannian metric  $ds^2 = dx^2 + |x|^{-2\alpha}d\theta^2$ , where  $x \in \mathbf{R}$ ,  $\theta \in \mathbb{T}$  and the parameter  $\alpha \in \mathbf{R}$ . For  $\alpha \leq -1$  this metric describes cone-like manifolds (for  $\alpha = -1$  it is a flat cone). For  $\alpha = 0$  it is a cylinder. For  $\alpha \geq 1$  it is a Grushin-like metric. We show that the Laplace-Beltrami operator  $\Delta$  is essentially self-adjoint if and only if  $\alpha \notin (-3, 1)$ . In this case the only self-adjoint extension is the Friedrichs extension  $\Delta_F$ , that does not allow communication through the singular set  $\{x = 0\}$  both for the heat and for a quantum particle. For  $\alpha \in (-3, -1]$  we show that for the Schrödinger equation only the average on  $\theta$  of the wave function can cross the singular set, while the solutions of the only Markovian extension of the heat equation (which indeed is  $\Delta_F$ ) cannot. For  $\alpha \in (-1, 1)$  we prove that there exists a canonical self-adjoint extension  $\Delta_B$ , called bridging extension, which is Markovian and allows the complete communication through the singularity (both of the heat and of a quantum particle). Also, we study the stochastic completeness (i.e., conservation of the  $L^1$  norm for the heat equation) of the Markovian extensions  $\Delta_F$  and  $\Delta_B$ , proving that  $\Delta_F$  is stochastically complete at the singularity if and only if  $\alpha \leq -1$ , while  $\Delta_B$  is always stochastically complete at the singularity.

**Key words:** heat and Schrödinger equation, degenerate Riemannian manifold, Grushin plane, stochastic completeness.

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## 1. INTRODUCTION

In this paper we consider the Riemannian metric on  $M = (\mathbf{R} \setminus \{0\}) \times \mathbb{T}$  whose orthonormal basis has the form:

$$(1) \quad X_1(x, \theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x, \theta) = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}, \quad \alpha \in \mathbf{R}.$$

Here  $x \in \mathbf{R}$ ,  $\theta \in \mathbb{T}$  and  $\alpha \in \mathbf{R}$  is a parameter. In other words we are interested in the Riemannian manifold  $(M, g)$ , where

$$(2) \quad g = dx^2 + |x|^{-2\alpha}d\theta^2, \text{ i.e., in matrix notation } g = \begin{pmatrix} 1 & 0 \\ 0 & |x|^{-2\alpha} \end{pmatrix}.$$

Define

$$M_{\text{cylinder}} = \mathbf{R} \times \mathbb{T}, \quad M_{\text{cone}} = M_{\text{cylinder}} / \sim,$$

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where  $(x_1, \theta_1) \sim (x_2, \theta_2)$  if and only if  $x_1 = x_2 = 0$ . In the following we are going to suitably extend the metric structure to  $M_{\text{cylinder}}$  through (1) when  $\alpha \geq 0$ , and to  $M_{\text{cone}}$  through (2) when  $\alpha < 0$ .

Recall that, on a general two dimensional Riemannian manifold for which there exists a global orthonormal frame, the distance between two points can be defined equivalently as

$$(3) \quad d(q_1, q_2) = \inf \left\{ \int_0^1 \sqrt{u_1(t)^2 + u_2(t)^2} dt \mid \gamma : [0, 1] \rightarrow M \text{ Lipschitz}, \gamma(0) = q_1, \gamma(1) = q_2 \right. \\ \left. \text{and } u_1, u_2 \text{ are defined by } \dot{\gamma}(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t)) \right\},$$

$$(4) \quad d(q_1, q_2) = \inf \left\{ \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt \mid \gamma : [0, 1] \rightarrow M \text{ Lipschitz}, \gamma(0) = q_1, \gamma(1) = q_2 \right\},$$

where  $\{X_1, X_2\}$  is the global orthonormal frame for  $(M, g)$ .

**Case  $\alpha \geq 0$ .** Similarly to what is usually done in sub-Riemannian geometry (see e.g., [1]), when  $\alpha \geq 0$ , formula (3) can be used to define a distance on  $M_{\text{cylinder}}$  where  $X_1$  and  $X_2$  are given by formula (1). We have the following (for the proof see Appendix A.1).

**Lemma 1.1.** *For any  $\alpha \geq 0$ , formula (3) endows  $M_{\text{cylinder}}$  with a metric space structure, which is compatible with its original topology.*

**Case  $\alpha < 0$ .** In this case  $X_1$  and  $X_2$  are not well defined in  $x = 0$ . However, to extend the metric structure, one can use formula (4), where  $g$  is given by (2). Notice that this metric identifies points on  $\{x = 0\}$ , in the sense that they are at zero distance. Hence, formula (4) gives a structure of well-defined metric space not to  $M_{\text{cylinder}}$  but to  $M_{\text{cone}}$ . Indeed, we have the following (for the proof see Appendix A.1).

**Lemma 1.2.** *For  $\alpha < 0$ , formula (4) endows  $M_{\text{cone}}$  with a metric space structure, which is compatible with its original topology.*

**Remark 1.3** (Notation). In the following we call  $M_\alpha$  the generalized Riemannian manifold given as follows

- $\alpha \geq 0$ :  $M_\alpha = M_{\text{cylinder}}$  and metric structure induced by (1);
- $\alpha < 0$ :  $M_\alpha = M_{\text{cone}}$  and metric structure induced by (2).

The corresponding metric space is called  $(M_\alpha, d)$ . Moreover, we call  $\mathcal{Z}$  the singular set, i.e.,

$$\mathcal{Z} = \begin{cases} \{0\} \times \mathbb{T}, & \alpha \geq 0, \\ \{0\} \times \mathbb{T} / \sim & \alpha < 0. \end{cases}$$

The singularity splits the manifold  $M_\alpha$  in two sides  $M^+ = (0, +\infty) \times \mathbb{T}$  and  $M^- = (-\infty, 0) \times \mathbb{T}$ .

Notice that in the cases  $\alpha = 1, 2, 3, \dots$ ,  $M_\alpha$  is an almost Riemannian structure in the sense of [3, 2, 6, 7, 8], while in the cases  $\alpha = -1, -2, -3, \dots$  it corresponds to a singular Riemannian manifold with a semi-definite metric.

One of the main features of these metrics is the fact that, except in the case  $\alpha = 0$ , the corresponding Riemannian volumes have a singularity at  $\mathcal{Z}$ ,

$$d\omega = \sqrt{\det g} dx d\theta = |x|^{-\alpha} dx d\theta.$$

Due to this fact, the corresponding Laplace-Beltrami operators contain some diverging first order terms,

$$(5) \quad \Delta = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^2 \partial_j \left( \sqrt{\det g} g^{jk} \partial_k \right) = \partial_x^2 + |x|^{2\alpha} \partial_\theta^2 u - \frac{\alpha}{x} \partial_x$$

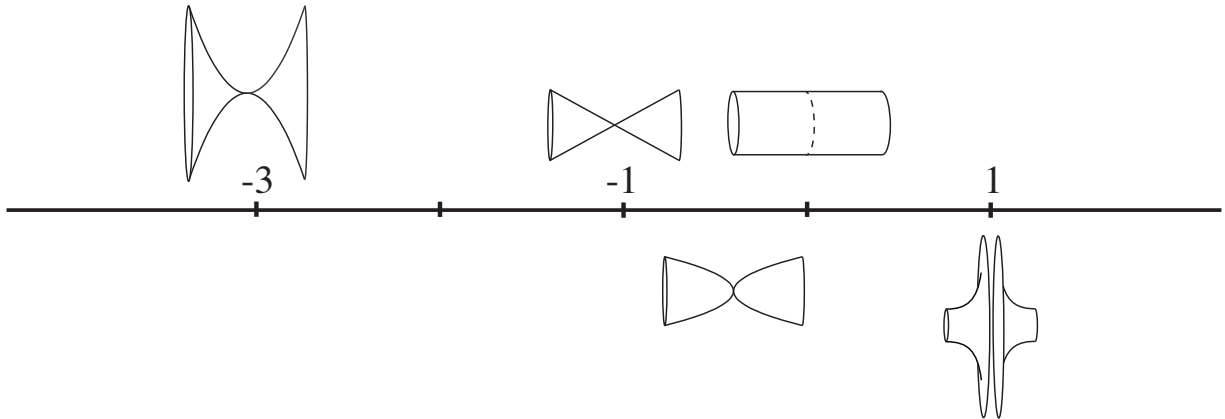


FIGURE 1. Geometric interpretation of  $M_\alpha$ . The figures above the line are actually isometric to  $M_\alpha$ , while for the ones below the isometry is singular in  $\mathcal{Z}$ .

We have the following geometric interpretation of  $M_\alpha$  (see Figure 1). For  $\alpha = 0$ , this metric is that of a cylinder. For  $\alpha = -1$ , it is the metric of a flat cone in polar coordinates. For  $\alpha < -1$ , it is isometric to a surface of revolution  $\mathcal{S} = \{(t, r(t) \cos \vartheta, r(t) \sin \vartheta) \mid t > 0, \vartheta \in \mathbb{T}\} \subset \mathbf{R}^3$  with profile  $r(t) = |t|^{-\alpha} + \mathcal{O}(t^{-2\alpha})$  as  $|t|$  goes to zero. For  $\alpha > -1$  ( $\alpha \neq 0$ ) it can be thought as a surface of revolution having a profile of the type  $r(t) \sim |t|^{-\alpha}$  as  $t \rightarrow 0$ , but this is only formal, since the embedding in  $\mathbf{R}^3$  is deeply singular at  $t = 0$ . The case  $\alpha = 1$  corresponds to the Grushin metric on the cylinder. This geometric interpretation is explained in Appendix A.2.

**Remark 1.4.** The curvature of  $M_\alpha$  is given by  $K_\alpha(x) = -\alpha(1 + \alpha)x^{-2}$ . Notice that  $M_\alpha$  and  $M_\beta$  with  $\beta = -(\alpha + 1)$  have the same curvature for any  $\alpha \in \mathbf{R}$ . For instance, the cylinder with Grushin metric has the same curvature as the cone corresponding to  $\alpha = -2$ , but they are not isometric even locally (see [7]).

1.1. **The problem.** About  $M_\alpha$ , we are interested to the following problems.

(Q1) Do the heat and free quantum particles flow through the singularity? In other words, we are interested to the following: consider the heat or the Schrödinger equation

$$(6) \quad \partial_t \psi = \Delta \psi,$$

$$(7) \quad i\partial_t \psi = -\Delta \psi,$$

where  $\Delta$  is given by (5). Take an initial condition supported at time  $t = 0$  in  $M^-$ . Is it possible that at time  $t > 0$  the corresponding solution has some support in  $M^+$ ? <sup>1</sup>

(Q2) Does the equation (6) conserve the total heat (i.e. the  $L^1$  norm of  $\psi$ )? This is known to be equivalent to the fact that the stochastic process, defined by the diffusion  $\Delta$ , almost surely has infinite lifespan. This is known as the problem of the stochastic completeness of  $M_\alpha$ . In particular, we are interested in understanding if the heat is absorbed by the singularity  $\mathcal{Z}$ .

The same question for the Schrödinger equation has a trivial answer, since the total probability (i.e., the  $L^2$  norm) is always conserved by Stone's theorem.

Of course, the first thing to do in attacking this problem is to give a meaning to  $\Delta$  at  $\mathcal{Z}$ , and to define in which functional spaces we are working. In particular, it is classical that to have a well defined dynamic associated to  $\Delta$ , it is necessary for  $\Delta$  to be a self-adjoint operator on  $L^2(M, d\omega)$  (see Theorem 2.1). Thus, we will consider the operator  $\Delta|_{C_c^\infty(M)}$ , and characterize all its self-adjoint

<sup>1</sup>Notice that this is a necessary condition to have some positive controllability results by means of controls defined only on one side of the singularity, in the spirit of [5].



extensions. This will be achieved by prescribing opportune boundary conditions at the singularity  $\mathcal{Z}$ .

**Remark 1.5.** By making the unitary change of coordinate in the Hilbert space  $U : L^2(M, d\omega) \rightarrow L^2(M, dx d\theta)$ , defined by  $Uv(x) = |x|^{-\alpha/2}v(x)$ , the Laplace-Beltrami operator is transformed in

$$\mathbf{\Delta} = U\Delta U^{-1} = \partial_x^2 - \frac{\alpha}{2} \left(1 + \frac{\alpha}{2}\right) \frac{1}{x^2} + |x|^{2\alpha} \partial_\theta^2.$$

This transformation was used to study the essential self-adjointness of  $\Delta|_{C_c^\infty(M)}$  for  $\alpha = 1$  in [9]. Let us remark that, when acting on functions independent of  $\theta$ , the operator  $\mathbf{\Delta}$  reduces to the Laplace operator on  $\mathbf{R} \setminus \{0\}$  in presence of an inverse square potential, usually called Calogero potential (see, e.g., [18]).

**1.2. Self-adjoint extensions.** The problem of determining the self-adjoint extensions of  $\Delta|_{C_c^\infty(M)}$  on  $L^2(M, d\omega)$  has been widely studied in different fields. A lot of work has been done in the case  $\alpha = -1$ , in the setting of Riemannian manifolds with conical singularities (see e.g., [11, 24]), and the same methods have been applied in the more general context of metric cusps or horns (see e.g., [12, 10]) that covers the case  $\alpha < -1$ . See also [22]. Concerning  $\alpha > -1$ , on the other hand, the literature regarding  $\Delta$  is more thin (see e.g., [25]).

In the following we will consider only the real self-adjoint extensions, i.e., all the function spaces taken into consideration are composed of real-valued functions. We refer to Appendix B for a discussion of the complex case.

Any closed symmetric extension  $A$  of  $\Delta|_{C_c^\infty(M)}$  is such that  $D_{\min}(\Delta|_{C_c^\infty(M)}) \subset D(A) \subset D_{\max}(\Delta|_{C_c^\infty(M)})$ , where the minimal and maximal domains are defined as

$$D_{\min}(\Delta|_{C_c^\infty(M)}) = D(\overline{\Delta}) = \text{closure of } C_c^\infty(M) \text{ with respect to the norm } \|\Delta \cdot\|_{L^2(M_\alpha, d\omega)} + \|\cdot\|_{L^2(M_\alpha, d\omega)},$$

$$D_{\max}(\Delta|_{C_c^\infty(M)}) = D(\Delta^*) = \{u \in L^2(M_\alpha, d\omega) : \Delta u \in L^2(M_\alpha, d\omega) \text{ in the sense of distributions}\}.$$

Thus, it has to hold that  $Au = \Delta^*u$  for any  $u \in D(A)$ , and hence determining the self-adjoint extensions of  $\Delta|_{C_c^\infty(M)}$  amounts to classify the so-called domains of self-adjointness. Recall that the Riemannian gradient is given by  $\nabla u(x, \theta) = (\partial_x u(x, \theta), |x|^{2\alpha} \partial_\theta u(x, \theta))$ . Following [19], we let the Sobolev spaces on the Riemannian manifold  $M$  endowed with measure  $d\omega$  to be

$$H^1(M, d\omega) = \{u \in L^2(M, d\omega) : |\nabla u| \in L^2(M, d\omega)\}, \quad H_0^1(M, d\omega) = \text{closure of } C_c^\infty(M) \text{ in } H^1(M, d\omega),$$

$$H^2(M, d\omega) = \{u \in H^1(M, d\omega) : \Delta u \in L^2(M, d\omega)\}, \quad H_0^2(M, d\omega) = \{u \in H_0^1(M, d\omega) : \Delta u \in L^2(M, d\omega)\}.$$

We define the Sobolev spaces  $H^1(M_\alpha, d\omega)$  and  $H^2(M_\alpha, d\omega)$  in the same way. We remark that, in general, it may happen that  $H^1(M, d\omega) = H_0^1(M, d\omega)$ . Indeed this property will play an important role in the next section. In Proposition 2.10, is contained a description of  $D_{\max}(\Delta|_{C_c^\infty(M)})$  in terms of these Sobolev spaces.

Although in general the structure of the self-adjoint extensions of  $\Delta|_{C_c^\infty(M)}$  can be very complicated, the Friedrichs (or Dirichlet) extension  $\Delta_F$ , is always well defined and self-adjoint. Namely,

$$D(\Delta_F) = H_0^2(M, d\omega).$$

Observe that, since  $L^2(M, d\omega) = L^2(M^+, d\omega) \oplus L^2(M^-, d\omega)$  and  $H_0^1(M, d\omega) = H_0^1(M^+, d\omega) \oplus H_0^1(M^-, d\omega)$ , it follows that

$$D(\Delta_F) = \{u \in H_0^1(M^+, d\omega) \mid \Delta u \in L^2(M^+, d\omega)\} \oplus \{u \in H_0^1(M^-, d\omega) \mid \Delta u \in L^2(M^-, d\omega)\}.$$

This implies that  $\Delta_F$  actually defines two separate dynamics on  $M^+$  and on  $M^-$  and, hence, there is no hope for an initial datum concentrated in  $M^+$  to pass to  $M^-$ , and vice versa. Thus, if  $\Delta|_{C_c^\infty(M)}$  is essentially self-adjoint (i.e., the only self-adjoint extension is  $\Delta_F$ ) the question (Q1) has negative answer.

1.2.1. *Essential self-adjointness of  $\Delta|_{C_c^\infty(M)}$ .* The rotational symmetry of the cones, suggests to proceed by a Fourier decomposition in the  $\theta$  variable, through the orthonormal basis  $\{e_k\}_{k \in \mathbf{Z}} \subset L^2(\mathbb{T})$ . Thus, we decompose the space  $L^2(M, d\omega) = \bigoplus_{k=0}^\infty H_k \cong L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , and the corresponding operators on each  $H_k$  will be

$$(8) \quad \widehat{\Delta}_k = \partial_x^2 - \frac{\alpha}{x} \partial_x - |x|^{2\alpha} k^2.$$

Observe (see Proposition 2.3) that if all the  $\widehat{\Delta}_k$  are essentially self-adjoint on  $C_c^\infty(\mathbf{R} \setminus \{0\})$ , then the same holds for  $\Delta|_{C_c^\infty(M)}$ .

The following theorem (that extends a result in [9] and is proved in Section 2.3), classifies the essential self-adjointness of  $\Delta|_{C_c^\infty(M)}$  and its Fourier components. We remark that the same result holds if  $\Delta|_{C_c^\infty(M)}$  acts on complex-valued functions (see Theorem B.2).

**Theorem 1.6.** *Consider  $M_\alpha$  for  $\alpha \in \mathbf{R}$  and the corresponding Laplace-Beltrami operator  $\Delta|_{C_c^\infty(M)}$  as an unbounded operator on  $L^2(M, d\omega)$ . Then it holds the following.*

- (i) *If  $\alpha \leq -3$  then  $\Delta|_{C_c^\infty(M)}$  is essentially self-adjoint;*
- (ii) *if  $\alpha \in (-3, -1]$ , only the first Fourier component  $\widehat{\Delta}_0$  is not essentially self-adjoint;*
- (iii) *if  $\alpha \in (-1, 1)$ , all the Fourier components of  $\Delta|_{C_c^\infty(M)}$  are not essentially self-adjoint;*
- (iv) *if  $\alpha \geq 1$  then  $\Delta|_{C_c^\infty(M)}$  is essentially self-adjoint.*

As a corollary of this theorem, we get the following preliminary answer to (Q1).

$\alpha \leq -3$	Nothing can flow through $\mathcal{Z}$
$-3 < \alpha \leq -1$	Only the average over $\mathbb{T}$ of the function can flow through $\mathcal{Z}$
$-1 < \alpha < 1$	It is possible to have full communication between the two sides
$1 \leq \alpha$	Nothing can flow through $\mathcal{Z}$

In particular, to understand the possible evolutions in the case  $\alpha \in (-3, -1]$ , it suffices to study the equation on the first Fourier component. Indeed, in this case any self-adjoint extension  $A$  of  $\Delta|_{C_c^\infty(M)}$  can be decomposed as

$$(9) \quad A = \widehat{A}_0 \oplus \left( \bigoplus_{k \in \mathbf{Z} \setminus \{0\}} \widehat{\Delta}_k \right),$$

where  $\widehat{A}_0$  is a self-adjoint extension of  $\widehat{\Delta}_0$  and, with abuse of notation, we denoted the only self-adjoint extension of  $\widehat{\Delta}_k$  by  $\widehat{\Delta}_k$  as well.

**Remark 1.7.** Notice that in the case  $\alpha \in (-3, 0)$ , since the singularity reduces to a single point, one would expect to be able to “transmit” through  $\mathcal{Z}$  only a function independent of  $\theta$  (i.e. only the average over  $\mathbb{T}$ ). Theorem 1.6 shows that this is the case for  $\alpha \in (-3, -1]$ , but not for  $\alpha \in (-1, 0)$ . Looking at  $M_\alpha$ ,  $\alpha \in (-1, 0)$ , as a surface embedded in  $\mathbf{R}^3$  the possibility of transmitting Fourier components other than  $k = 0$ , is due to the deep singularity of the embedding. In this case we say that the contact between  $M^+$  and  $M^-$  is *non-apophantic*.

1.2.2. *The first Fourier component  $\widehat{\Delta}_0$ .* We now focus on the first Fourier component  $\widehat{\Delta}_0|_{C_c^\infty(\mathbf{R} \setminus \{0\})}$  on  $L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , when  $\alpha \in (-3, 1)$ , and we describe its real self-adjoint extensions. For a description of the complex self-adjoint extensions of  $\widehat{\Delta}_0|_{C_c^\infty(\mathbf{R} \setminus \{0\})}$ , we refer to Theorem B.3. We remark that this operator is regular at the origin, in the sense of Sturm-Liouville problems (see Definition 2.5), if and only if  $\alpha > -1$ . Hence, for  $\alpha \leq -1$ , the boundary conditions will be asymptotic, and not punctual.

Let  $\phi_D^+$  and  $\phi_N^+$  be two smooth functions on  $\mathbf{R} \setminus \{0\}$ , supported in  $[0, 2)$ , and such that, for any  $x \in [0, 1]$  it holds

$$(10) \quad \phi_D^+(x) = 1, \quad \phi_N^+(x) = \begin{cases} (1 + \alpha)^{-1} x^{1+\alpha} & \text{if } \alpha \neq -1, \\ \log(x) & \text{if } \alpha = -1. \end{cases}$$

Let also  $\phi_D^-(x) = \phi_D^+(-x)$  and  $\phi_N^-(x) = \phi_N^+(-x)$ . It holds the following.

**Theorem 1.8.** *Let  $D_{\min}(\widehat{\Delta}_0)$  and  $D_{\max}(\widehat{\Delta}_0)$  be the minimal and maximal domains of  $\widehat{\Delta}_0|_{C_c^\infty(\mathbf{R} \setminus \{0\})}$  on  $L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , for  $\alpha \in (-3, 1)$ . Then,*

$$D_{\min}(\widehat{\Delta}_0) = \text{closure of } C_c^\infty(\mathbf{R} \setminus \{0\}) \text{ in } H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$$

$$D_{\max}(\widehat{\Delta}_0) = \{u = u_0 + u_D^+ \phi_D^+ + u_N^+ \phi_N^+ + u_D^- \phi_D^- + u_N^- \phi_N^- : u_0 \in D_{\min}(\widehat{\Delta}_0) \text{ and } u_D^\pm, u_N^\pm \in \mathbf{R}\},$$

Moreover,  $A$  is a self-adjoint extension of  $\widehat{\Delta}_0$  if and only if  $Au = (\widehat{\Delta}_0)^*u$ , for any  $u \in D(A)$ , and one of the following holds

(i) Disjoint dynamics: there exist  $c_+, c_- \in (-\infty, +\infty]$  such that

$$D(A) = \{u \in D_{\max}(\widehat{\Delta}_0) : u_N^+ = c_+ u_D^+ \text{ and } u_N^- = c_- u_D^+\}.$$

(ii) Mixed dynamics: there exist  $K \in SL_2(\mathbf{R})$  such that

$$D(A) = \{u \in D_{\max}(\widehat{\Delta}_0) : (u_D^-, u_N^-) = K (u_D^+, u_N^+)^T\}.$$

Finally, the Friedrichs extension  $(\widehat{\Delta}_0)_F$  is the one corresponding to the disjoint dynamics with  $c_+ = c_- = 0$  if  $\alpha \leq -1$  and with  $c_+ = c_- = +\infty$  if  $\alpha > -1$ .

From the above theorem (see Remark 2.9) it follows that  $u_N^\pm = \lim_{x \rightarrow 0^\pm} |x|^{-\alpha} \partial_x u(x)$  and, if  $-1 < \alpha < 1$ , that  $u_D^\pm = u(0^\pm)$ . Moreover, the last statement implies that

$$D((\widehat{\Delta}_0)_F) = \begin{cases} \{u \in D_{\max}(\widehat{\Delta}_0) : u_N^+ = u_N^- = 0\} & \text{if } \alpha \leq -1, \\ \{u \in D_{\max}(\widehat{\Delta}_0) : u(0^+) = u(0^-) = 0\} & \text{if } \alpha > -1. \end{cases}$$

In particular, if  $\alpha \leq -1$  the Friedrichs extension does not impose zero boundary conditions.

Clearly, the disjoint dynamics extensions will give an evolution for which (Q1) has negative answer. On the other hand, the mixed dynamics extensions, will permit information transfer between the two halves of the space. Since by Theorem 1.6, to classify the self-adjoint extensions for  $\alpha \in (-3, -1]$  it is enough to study  $\widehat{\Delta}_0$ , this analysis completely classifies the self-adjoint extensions in this case. On the other hand, since for  $\alpha \in (-1, 1)$  all the Fourier components are not essentially self-adjoint, a complete classification requires more sophisticated techniques. We will, in turn, study some selected extensions.

**Remark 1.9.** We call the mixed dynamics extension with  $K = \text{Id}$  the *bridging extension* of the first Fourier component, and denote it by  $(\widehat{\Delta}_0)_B$ . Then, if  $\alpha \in (-3, -1]$ , we let the bridging extension  $\Delta_B$  of  $\Delta|_{C_c^\infty(M)}$  to be defined by (9) with  $A_0 = (\widehat{\Delta}_0)_B$ . This extension allows for a maximal communication between the two sides. The bridging extension for  $\alpha \in (-1, 1)$  is described in the following section.

**1.3. Markovian extensions.** It is a well known result, that each non-positive self-adjoint operator  $A$  on an Hilbert space  $\mathcal{H}$  defines a strongly continuous contraction semigroup, denoted by  $\{e^{tA}\}_{t \geq 0}$ . If  $\mathcal{H} = L^2(M, d\omega)$  and it holds  $0 \leq e^{tA}u \leq 1$   $d\omega$ -a.e. whenever  $u \in L^2(M, d\omega)$ ,  $0 \leq u \leq 1$   $d\omega$ -a.e., the semigroup  $\{e^{tA}\}_{t \geq 0}$  and the operator  $A$  are called Markovian. The interest for Markov operators lies in the fact that, under an additional assumption which is always satisfied in the cases we consider

(see Section 3), Markovian operators are generators of Markov processes  $\{X_t\}_{t \geq 0}$  (roughly speaking, stochastic processes which are independent of the past).

Since essentially bounded functions are approximable from  $L^2(M, d\omega)$ , the Markovian property allows to extend the definition of  $e^{tA}$  from  $L^2(M, d\omega)$  to  $L^\infty(M, d\omega)$ . Let  $1$  be the constant function  $1(x, \theta) \equiv 1$ . Then (Q2) is equivalent to the following property.

**Definition 1.10.** A Markovian operator  $A$  is called *stochastically complete* (or conservative) if  $e^{tA}1 = 1$ , for any  $t > 0$ . It is called *explosive* if it is not stochastically complete.

It is well known that this property is equivalent to the fact that the Markov process  $\{X_t\}_{t \geq 0}$ , with generator  $A$ , has almost surely infinite lifespan.

We will be interested also in the following property of  $\{X_t\}_{t \geq 0}$ .

**Definition 1.11.** A Markovian operator is called *recurrent* if the associated Markov process  $\{X_t\}_{t \geq 0}$  satisfies, for any set  $\Omega$  of positive measure and any point  $x$ ,

$$\mathbb{P}_x\{\text{there exists a sequence } t_n \rightarrow +\infty \text{ such that } X_{t_n} \in \Omega\} = 1.$$

Here  $\mathbb{P}_x$  denotes the measure in the space of paths emanating from a point  $x$  associated to  $\{X_t\}_{t \geq 0}$ .

**Remark 1.12.** Notice that recurrence of an operator implies its stochastic completeness. Equivalently, any explosive operator is not recurrent.

We are particularly interested in distinguish how the stochastically completeness and the recurrence are influenced by the singularity  $\mathcal{Z}$  or by the behavior at  $\infty$ . Thus we will consider the manifolds with borders  $M_0 = M \cap ([-1, 1] \times \mathbb{T})$  and  $M_\infty = M \setminus [-1, 1] \times \mathbb{T}$ , with Neumann boundary conditions. Indeed, with these boundary conditions, when the Markov process  $\{X_t\}_{t \geq 0}$  hits the boundary it is reflected, and hence the eventual lack of recurrence or stochastic completeness on  $M_0$  (resp. on  $M_\infty$ ) is due to the singularity  $\mathcal{Z}$  (resp. to the behavior at  $\infty$ ). If a Markovian operator  $A$  on  $M$  is recurrent (resp. stochastically complete) when restricted on  $M_0$  we will call it recurrent (resp. stochastically complete) at 0. Similarly, when the same happens on  $M_\infty$ , we will call it recurrent (resp. stochastically complete) at  $\infty$ . As proven in Proposition 3.14, a Markovian extension of  $\Delta|_{C_c^\infty(M)}$  is recurrent (resp. stochastically complete) if and only if it is recurrent (resp. stochastically complete) both at 0 and at  $\infty$ .

In this context, it makes sense to give special consideration to three specific self-adjoint extensions, corresponding to different conditions at  $\mathcal{Z}$ . Namely, we will consider the already mentioned Friedrichs extension  $\Delta_F$ , that corresponds to an absorbing condition, the Neumann extension  $\Delta_N$ , that corresponds to a reflecting condition, and the bridging extension  $\Delta_B$ , that corresponds to a free flow through  $\mathcal{Z}$  and is defined only for  $\alpha \in (-1, 1)$ . In particular, the latter two have the following domains (see Proposition 3.12),

$$D(\Delta_N) = \{u \in H^1(M, d\omega) \mid (\Delta u, v) = (\nabla u, \nabla v) \text{ for any } v \in H^1(M, d\omega)\},$$

$$D(\Delta_B) = \{H^2(M_\alpha, d\omega) \mid u(0^+, \cdot) = u(0^-, \cdot), \lim_{x \rightarrow 0^+} |x|^{-\alpha} u(x, \cdot) = \lim_{x \rightarrow 0^-} |x|^{-\alpha} u(x, \cdot) \text{ for a.e. } \theta \in \mathbb{T}\}.$$

Each one of  $\Delta_F$ ,  $\Delta_N$  and  $\Delta_B$  is a self-adjoint Markovian extensions. However, it may happen that  $\Delta_F = \Delta_N$ . In this case  $\Delta_F$  is the only Markovian extension, and the operator  $\Delta|_{C_c^\infty(M)}$  is called Markov unique. This is the case, for example, when  $\Delta|_{C_c^\infty(M)}$  is essentially self-adjoint.

The following result, proved in Section 3.3, will answer to (Q2).

**Theorem 1.13.** Consider  $M_\alpha$ , for  $\alpha \in \mathbf{R}$ , and the corresponding Laplace-Beltrami operator  $\Delta|_{C_c^\infty(M)}$  as an unbounded operator on  $L^2(M, d\omega)$ . Then it holds the following.

- (i) If  $\alpha < -1$  then  $\Delta|_{C_c^\infty(M)}$  is Markov unique, and  $\Delta_F$  is stochastically complete at 0 and recurrent at  $\infty$ ;
- (ii) if  $\alpha = -1$  then  $\Delta|_{C_c^\infty(M)}$  is Markov unique, and  $\Delta_F$  is recurrent both at 0 and at  $\infty$ ;

- (iii) if  $\alpha \in (-1, 1)$ , then  $\Delta|_{C_c^\infty(M)}$  is not Markov unique and, moreover,
  - (a) any Markovian extension of  $\Delta|_{C_c^\infty(M)}$  is recurrent at  $\infty$ ,
  - (b)  $\Delta_F$  is explosive at 0, while both  $\Delta_B$  and  $\Delta_N$  are recurrent at 0,
- (iv) if  $\alpha \geq 1$  then  $\Delta|_{C_c^\infty(M)}$  is Markov unique, and  $\Delta_F$  is explosive at 0 and recurrent at  $\infty$ ;

In particular, Theorem 1.13 implies that for  $\alpha \in (-3, -1]$  no mixing behavior defines a Markov process. On the other hand, for  $\alpha \in (-1, 1)$  we can have a plethora of such processes.

**Remark 1.14.** Notice that, since the singularity  $\mathcal{Z}$  is at finite distance from any point of  $M_\alpha$ , one can interpret a Markov process that is explosive at 0 as if  $\mathcal{Z}$  were absorbing the heat.

As a corollary of 1.13, we get the following answer to (Q2).

$\alpha \leq -1$	The heat is absorbed by $\mathcal{Z}$
$-1 < \alpha < 1$	The Friedrichs extension is absorbed by $\mathcal{Z}$ , while the Neumann and the bridging extensions are not.
$1 \leq \alpha$	The heat is absorbed by $\mathcal{Z}$

**1.4. Structure of the paper.** The structure of the paper is the following. In Section 2, after some preliminaries regarding self-adjointness, we analyze in detail the Fourier components of the Laplace-Beltrami operator on  $M_\alpha$ , proving Theorems 1.6 and 1.8. We conclude this section with a description of the maximal domain of the Laplace-Beltrami operator in terms of the Sobolev spaces on  $M_\alpha$ , contained in Proposition 2.10.

Then, in Section 3, we introduce and discuss the concepts of Markovianity, stochastic completeness and recurrence through the potential theory of Dirichlet forms. After this, we study the Markov uniqueness of  $\Delta|_{C_c^\infty(M)}$  and characterize the domains of the Friedrichs, Neumann and bridging extensions (Propositions 3.11 and 3.12). Then, we define stochastic completeness and recurrence at 0 and at  $\infty$ , and, in Proposition 3.15, we discuss how these concepts behave if the  $k = 0$  Fourier component of the self-adjoint extension is itself self-adjoint. In particular, we show that the Markovianity of such an operator  $A$  implies the Markovianity of its first Fourier component  $\hat{A}_0$ , and that the stochastic completeness (resp. recurrence) at 0 (resp. at  $\infty$ ) of  $A$  and  $\hat{A}_0$  are equivalent. Then, in Proposition 3.14 we prove that stochastic completeness or recurrence are equivalent to stochastically completeness or recurrence both at 0 and at  $\infty$ . Finally, we prove Theorem 1.13.

The proofs of Lemmata 1.1 and 1.2 are contained in Appendix A.1, while in Appendix A.2 we justify the geometric interpretation of Figure 1. Appendix B contains the description of the complex self-adjoint extension of  $\hat{\Delta}_0$ .

## 2. SELF-ADJOINT EXTENSIONS

**2.1. Preliminaries.** Let  $\mathcal{H}$  be a Hilbert space with scalar product  $(\cdot, \cdot)_\mathcal{H}$  and norm  $\|\cdot\|_\mathcal{H} = \sqrt{(\cdot, \cdot)_\mathcal{H}}$ . Given an operator  $A$  on  $\mathcal{H}$  we will denote its domain by  $D(A)$  and its adjoint by  $A^*$ . Namely, if  $A$  is densely defined,  $D(A^*)$  is the set of  $\varphi \in \mathcal{H}$  such that there exists  $\eta \in \mathcal{H}$  with  $(A\psi, \varphi)_\mathcal{H} = (\psi, \eta)_\mathcal{H}$ , for all  $\psi \in D(A)$ . For each such  $\varphi$ , we define  $A^*\varphi = \eta$ .

An operator  $A$  is symmetric if

$$(A\psi, \varphi)_\mathcal{H} = (\psi, A\varphi)_\mathcal{H}, \quad \text{for all } \psi \in D(A).$$

A densely defined operator  $A$  is *self-adjoint* if and only if it is symmetric and  $D(A) = D(A^*)$ , and is *non-positive* if and only if  $(A\psi, \psi) \leq 0$  for any  $\psi \in D(A)$ .

Given a strongly continuous group  $\{T_t\}_{t \in \mathbf{R}}$  (resp. semigroup  $\{T_t\}_{t \geq 0}$ ), its generator  $A$  is defined as

$$Au = \lim_{t \rightarrow 0} \frac{T_t u - u}{t}, \quad D(A) = \{u \in \mathcal{H} \mid Au \text{ exists as a strong limit}\}.$$

When a group (resp. semigroup) has generator  $A$ , we will write it as  $\{e^{tA}\}_{t \in \mathbf{R}}$  (resp.  $\{e^{tA}\}_{t \geq 0}$ ). Then, by definition,  $u(t) = e^{tA}u_0$  is the solution of the functional equation

$$\begin{cases} \partial_t u(t) = Au(t) \\ u(0) = u_0 \in \mathcal{H}. \end{cases}$$

Recall the following classical result.

**Theorem 2.1.** *Let  $\mathcal{H}$  be an Hilbert space, then*

(1) (Stone's theorem) *The map  $A \mapsto \{e^{itA}\}_{t \in \mathbf{R}}$  induces a one-to-one correspondence*

$$A \text{ self-adjoint operator} \iff \{e^{itA}\}_{t \in \mathbf{R}} \text{ strongly continuous unitary group};$$

(2) *The map  $A \mapsto \{e^{tA}\}_{t \geq 0}$  induces a one-to-one correspondence*

$$A \text{ non-positive self-adjoint operator} \iff \{e^{tA}\}_{t \geq 0} \text{ strongly continuous semigroup};$$

For any Riemannian manifold  $\mathcal{M}$  with measure  $dV$ , via the Green identity follows that  $\Delta|_{C_c^\infty(\mathcal{M})}$  is symmetric. However, from the same formula, follows that

$$D(\Delta|_{C_c^\infty(\mathcal{M})}^*) = \{u \in L^2(\mathcal{M}, dV) \mid \Delta u \in L^2(\mathcal{M}, dV)\} \not\subseteq C_c^\infty(\mathcal{M}),$$

where  $\Delta u$  is intended in the sense of distributions. Hence,  $\Delta$  is not self-adjoint on  $C_c^\infty(\mathcal{M})$ .

Since, by Theorem 2.1, in order to have a well defined solution of the Schrödinger equation the Laplace-Beltrami operator has to be self-adjoint, we have to extend its domain in order to satisfy this property. For the heat equation, on the other hand, we will need also to worry about the fact that it stays non-positive while doing so. We will tackle this problem in the next section, where we will require the stronger property of being Markovian (i.e., that the evolution preserves both the non-negativity and the boundedness).

Mathematically speaking, given two operators  $A, B$ , we say that  $B$  is an extension of  $A$  (and we will write  $A \subset B$ ) if  $D(A) \subset D(B)$  and  $A\psi = B\psi$  for any  $\psi \in D(A)$ . The simplest extension one can build starting from  $A$  is the closure  $\bar{A}$ . Namely,  $D(\bar{A})$  is the closure of  $D(A)$  with respect to the graph norm  $\|\cdot\|_A = \|A \cdot\|_{\mathcal{H}} + \|\cdot\|_{\mathcal{H}}$ , and  $\bar{A}\psi = \lim_{n \rightarrow +\infty} A\psi_n$  where  $\{\psi_n\}_{n \in \mathbf{N}} \subset D(A)$  is such that  $\psi_n \rightarrow \psi$  in  $\mathcal{H}$ . Since if  $A$  is symmetric  $A \subset \bar{A} \subset A^*$ , any self-adjoint extension  $B$  of  $A$  will be such that  $\bar{A} \subset B \subset A^*$ . For this reason, we let  $D_{\min}(A) = D(\bar{A})$  and  $D_{\max}(A) = D(A^*)$ . Moreover, from this fact follows that any self-adjoint extension  $B$  will be defined as  $B\psi = A^*\psi$  for  $\psi \in D(B)$ , so we are only concerned in specifying the domain of  $B$ . The simplest case is the following.

**Definition 2.2.** The densely defined operator  $A$  is *essentially self-adjoint* if its closure  $\bar{A}$  is self-adjoint.

It is a well known fact, dating as far back as the series of papers [16, 17], that the Laplace-Beltrami operator is essentially self-adjoint on any complete Riemannian manifold. On the other hand, it is clear that if the manifold is incomplete this is no more the case, in general (see [23, 20]). It suffices, for example, to consider the case of an open set  $\Omega \subset \mathbf{R}^n$ , where to have the self-adjointness of the Laplacian, we have to pose boundary conditions (Dirichlet, Neumann or a mixture of the two). In our case, Theorem 1.6 will give an answer to the problem of whether  $\Delta|_{C_c^\infty(M)}$  is essentially self-adjoint or not.

## 2.2. Fourier decomposition and self-adjoint extensions of Sturm-Liouville operators.

There exist various theories allowing to classify the self-adjoint extensions of symmetric operators. We will use some tools from the Neumann theory (see [26]) and, when dealing with one-dimensional problems, from the Sturm-Liouville theory. Let  $\mathcal{H}$  be a complex Hilbert space and  $i$  be the imaginary unit. The *deficiency indexes* of  $A$  are then defined as

$$n_+(A) = \dim \ker(A + i), \quad n_-(A) = \dim \ker(A - i).$$

Then  $A$  admits self-adjoint extensions if and only if  $n_+(A) = n_-(A)$ , and they are in one to one correspondence with the set of partial isometries between  $\ker(A - i)$  and  $\ker(A + i)$ . Obviously,  $A$  is essentially self-adjoint if and only if  $n_+(A) = n_-(A) = 0$ .

Following [27], we say that a self-adjoint extension  $B$  of  $A$  in  $\mathcal{H}$  is a *real self-adjoint extension* if  $u \in D(B)$  implies that  $\bar{u} \in D(B)$  and  $B(\bar{u}) = \overline{Bu}$ . When  $\mathcal{H} = L^2(M, d\omega)$ , i.e. the real Hilbert space of square-integrable real-valued function on  $M$ , the self-adjoint extensions of  $A$  in  $L^2(M, d\omega)$  are the restrictions to this space of the real self-adjoint extensions of  $A$  in  $L^2_{\mathbb{C}}(M, d\omega)$ , i.e. the complex Hilbert space of square-integrable complex-valued functions. This proves that  $A$  is essentially self-adjoint in  $L^2(M, d\omega)$  if and only if it is essentially self-adjoint in  $L^2_{\mathbb{C}}(M, d\omega)$ . Hence, when speaking of the deficiency indexes of an operator acting on  $L^2(M, d\omega)$ , we will implicitly compute them on  $L^2_{\mathbb{C}}(M, d\omega)$ .

We start by proving the following general proposition that will allow us to study only the Fourier components of  $\Delta|_{C^\infty(M)}$ , in order to understand its essential self-adjointness.

**Proposition 2.3.** *Let  $A_k$  be symmetric on  $D(A_k) \subset H_k$ , for any  $k \in \mathbf{Z}$  and let  $D(A)$  be the set of vectors in  $\mathcal{H} = \bigoplus_{k \in \mathbf{Z}} H_k$  of the form  $\psi = (\psi_1, \psi_2, \dots)$ , where  $\psi_k \in D(A_k)$  and all but finitely many of them are zero. Then  $A = \sum_{k \in \mathbf{Z}} A_k$  is symmetric on  $D(A)$ ,  $n_+(A) = \sum_{k \in \mathbf{Z}} n_+(A_k)$  and  $n_-(A) = \sum_{k \in \mathbf{Z}} n_-(A_k)$ .*

*Proof.* Let  $\psi = (\psi_1, \psi_2, \dots) \in D(A)$ . Then, by symmetry of the  $A_k$ 's and the fact that only finitely many  $\psi_k$  are nonzero, it holds

$$(Au, v)_{\mathcal{H}} = \sum_{k \in \mathbf{Z}} (A_k u_k, v_k)_{H_k} = \sum_{k \in \mathbf{Z}} (u_k, A_k v_k)_{H_k} = (u, Av)_{\mathcal{H}}.$$

This proves the symmetry of  $A$ .

Observe now that  $\psi = (\psi_1, \psi_2, \dots) \in \ker(A \pm i)$  if and only if  $0 = A\psi \pm i\psi = (A_1\psi_1 \pm i\psi_1, A_2\psi_2 \pm i\psi_2, \dots)$ . This clearly implies that  $\dim \ker(A \pm i) = \sum_{k \in \mathbf{Z}} \dim \ker(A_k \pm i)$ , completing the proof.  $\square$

Observe that, for any  $k \in \mathbf{Z}$ , the Fourier component  $\widehat{\Delta}_k$ , defined in (8), is a second order differential operator of one variable. Thus, it can be studied through the Sturm-Liouville theory (see [27, 14]). Let  $J = (a_1, b_1) \cup (a_2, b_2)$ ,  $-\infty \leq a_1 < b_1 \leq a_2 < b_2 \leq +\infty$ , and for  $1/p, q, w \in L^1_{\text{loc}}(J)$  consider the Sturm-Liouville operator on  $L^2(J, w(x)dx)$  defined by

$$(11) \quad Au = \frac{1}{w} \left( -\partial_x(p \partial_x u) + qu \right).$$

Letting  $J = \mathbf{R} \setminus \{0\}$ ,  $w(x) = p(x) = |x|^{-\alpha}$ ,  $q(x) = k^2|x|^\alpha$ , we recover  $\widehat{\Delta}_k$ .

**Definition 2.4.** The endpoint (finite or infinite)  $a_1$ , is limit-circle if all solutions of the equation  $Au = 0$  are in  $L^2((a_1, d), w(x)dx)$  for some (and hence any)  $d \in (a_1, b_1)$ . Otherwise  $a_1$  is limit-point.

Analogous definitions can be given for  $b_1, a_2$  and  $b_2$ .

Let us define the Lagrange parenthesis of  $u, v : J \rightarrow \mathbf{R}$  associated to (11) as the bilinear anti-symmetric form

$$[u, v] = up \partial_x v - vp \partial_x u.$$

By [27, (10.4.41)] or [14, Lemma 3.2], we have that  $[u, v](d)$  exists and is finite for any  $u, v \in D_{\max}(\widehat{\Delta}_k)$  and any endpoint  $d$  of  $J$ . In particular, if  $d$  is limit-point, it holds  $[u, v](d) = 0$ . By Lemma 2.8, the Patching Lemma (see [27, Lemma 10.4.1]) and [27, Lemma 13.3.1], we then have the following characterization of the minimal domain of  $\widehat{\Delta}_k$ ,

$$(12) \quad D_{\min}(\widehat{\Delta}_k) = \left\{ u \in D_{\max}(\widehat{\Delta}_k) \mid [u, v](0^+) = [u, v](0^-) = 0 \text{ for all } v \in D_{\max}(\widehat{\Delta}_k) \right\}.$$

We recall also that the maximal domain can be written as

(13)

$$D_{\max}(A) = \{u : J \rightarrow \mathbf{R} \mid u, p \partial_x u \text{ are absolutely continuous on } J, \text{ and } u, Au \in L^2(J, w(x)dx)\}.$$

**Definition 2.5.** The Sturm-Liouville operator (11) is *regular* at the endpoint  $a_1$  if for some (and hence any)  $d \in (a_1, b_1)$ , it holds

$$\frac{1}{p}, q, w \in L^1((a_1, d)).$$

A similar definition holds for  $b_1, a_2, b_2$ .

In particular, for any  $k \in \mathbf{Z}$ , the operator  $\widehat{\Delta}_k$  is never regular at the endpoints  $+\infty$  and  $-\infty$ , and is regular at  $0^+$  and  $0^-$  if and only if  $\alpha \in (-1, 1)$ .

We will need the following theorem, that we state only for real extensions and in the cases we will use.

**Theorem 2.6** (Theorem 13.3.1 in [27]). *Let  $A$  be the Sturm-Liouville operator on  $L^2(J, w(x)dx)$  defined in (11). Then*

$$n_+(A) = n_-(A) = \#\{\text{limit-circle endpoints of } J\}.$$

Assume now that  $n_+(A) = n_-(A) = 2$ , and let  $a$  and  $b$  be the two limit-circle endpoints of  $J$ . Moreover, let  $\phi_1, \phi_2 \in D_{\max}(A)$  be linearly independent modulo  $D_{\min}(A)$  and normalized by  $[\phi_1, \phi_2](a) = [\phi_1, \phi_2](b) = 1$ . Then,  $B$  is a self-adjoint extension of  $A$  over  $L^2(J, w(x)dx)$  if and only if  $Bu = A^*u$ , for any  $u \in D(B)$ , and one of the following holds

- (1) Disjoint dynamics: there exists  $c_+, c_- \in (-\infty, +\infty]$  such that  $u \in D(B)$  if and only if

$$[u, \phi_1](0^+) = c_+[u, \phi_2](0^+) \quad \text{and} \quad [u, \phi_1](0^-) = d_+[u, \phi_2](0^-).$$

- (2) Mixed dynamics: there exist  $K \in SL_2(\mathbf{R})$  such that  $u \in D(B)$  if and only if

$$U(0^-) = K U(0^+), \quad \text{for } U(x) = \begin{pmatrix} [u, \phi_1](x) \\ [u, \phi_2](x) \end{pmatrix}.$$

**Remark 2.7.** Let  $\phi_1^a$  and  $\phi_2^a$  be, respectively, the functions  $\phi_1$  and  $\phi_2$  of the above theorem, multiplied by a cutoff function  $\eta : \bar{J} \rightarrow [0, 1]$  supported in a (right or left) neighborhood of  $a$  in  $J$  and such that  $\eta(a) = 1$  and  $\eta'(a) = 0$ . Let  $\phi_1^b$  and  $\phi_2^b$  be defined analogously. Then, from (12), follows that we can write

$$(14) \quad D_{\max}(A) = D_{\min}(A) + \text{span}\{\phi_1^a, \phi_1^b, \phi_2^a, \phi_2^b\}.$$

The following lemma classifies the end-points of  $\mathbf{R} \setminus \{0\}$  with respect to the Fourier components of  $\Delta|_{C_c^\infty(M)}$ .

**Lemma 2.8.** *Consider the Sturm-Liouville operator  $\widehat{\Delta}_k$  on  $\mathbf{R} \setminus \{0\}$ . Then, for any  $k \in \mathbf{Z}$  the endpoints  $+\infty$  and  $-\infty$  are limit-point. On the other hand, regarding  $0^+$  and  $0^-$  the following holds.*

- (1) If  $\alpha \leq -3$  or if  $\alpha \geq 1$ , then they are limit-point for any  $k \in \mathbf{Z}$ ;
- (2) if  $-3 < \alpha \leq -1$ , then they are limit-circle if  $k = 0$  and limit-point otherwise;
- (3) if  $-1 < \alpha < 1$ , then they are limit-circle for any  $k \in \mathbf{Z}$ .

*Proof.* By symmetry with respect to the origin of  $\widehat{\Delta}_k$ , it suffices to check only  $0^+$  and  $+\infty$ .

Let  $k = 0$ , then for  $\alpha \neq -1$  the equation  $\widehat{\Delta}_0 u = u'' - (\alpha/x)u' = 0$  has solutions  $u_1(x) = 1$  and  $u_2(x) = x^{1+\alpha}$ . Clearly,  $u_1$  and  $u_2$  are both in  $L^2((0, 1), |x|^{-\alpha}dx)$ , i.e.,  $0^+$  is limit-circle, if and only if  $\alpha \in (-3, 1)$ . On the other hand,  $u_1$  and  $u_2$  are never in  $L^2((1, +\infty), |x|^{-\alpha}dx)$  simultaneously, and hence  $+\infty$  is always limit-point. If  $\alpha = -1$ , the statement follows by the same argument applied to the solutions  $u_1(x) = 1$  and  $u_2(x) = \log(x)$ .



Let now  $k \neq 0$  and  $\alpha \neq -1$ . Then  $\widehat{\Delta}_k u = u'' - (\alpha/x)u' - x^{2\alpha}k^2 = 0$ ,  $x > 0$ , has solutions  $u_1(x) = \exp\left(\frac{kx^{1+\alpha}}{1+\alpha}\right)$  and  $u_2(x) = \exp\left(-\frac{kx^{1+\alpha}}{1+\alpha}\right)$ . If  $\alpha > -1$ , both  $u_1$  and  $u_2$  are bounded and nonzero near  $x = 0$ , and either  $u_1$  or  $u_2$  has exponential growth as  $x \rightarrow +\infty$ . Hence, in this case,  $u_1, u_2 \in L^2((0, 1), |x|^{-\alpha})$  if and only if  $\alpha < 1$ , while  $+\infty$  is always limit-point. On the other hand, if  $\alpha < -1$ ,  $u_1$  and  $u_2$  are bounded as  $x \rightarrow +\infty$  and one of them has exponential growth at  $x = 0$ . Since the measure  $|x|^{-\alpha}dx$  blows up at infinity, this implies that both  $0^+$  and  $+\infty$  are limit-point. Finally, the same holds for  $\alpha = -1$ , considering the solutions  $u_1(x) = x^k$  and  $u_2(x) = x^{-k}$ .  $\square$

**2.3. Proofs of Theorem 1.6 and 1.8.** We are now able to classify the essential self-adjointness of the operator  $\Delta|_{C_c^\infty(M)}$ .

*Proof of Theorem 1.6.* Let  $D \subset C_c^\infty(M)$  be the set of  $C_c^\infty(M)$  functions which are finite linear combinations of products  $u(x)v(\theta)$ . Since  $L^2(M, d\omega) = L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx) \otimes L^2(\mathbb{T}, d\theta)$ , the set  $D$  is dense in  $L^2(M, d\omega)$  and hence, by Proposition 2.3 the operator  $\Delta|_D$  is essentially self adjoint if and only if so are all  $\widehat{\Delta}_k|_{D \cap H_k}$ . Since  $n_\pm(\Delta|_D) = n_\pm(\Delta|_{C_c^\infty(M)})$ , this is equivalent to  $\Delta|_{C_c^\infty(M)}$  being essentially self-adjoint.

To conclude, recall that by Theorem 2.6 the operator  $\widehat{\Delta}_k$  is not essentially self-adjoint on  $L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)$  if and only if it is in the limit-circle case at at least one of the four end-points  $-\infty, 0^-, 0^+$  and  $+\infty$ . Hence applying Lemma 2.8 is enough to complete the proof.  $\square$

Now we proceed to study the self-adjoint extensions of the first Fourier component, proving Theorem 1.8 through Theorem 2.6 and Remark 2.7.

*Proof of Theorem 1.8.* We start by proving the statement on  $D_{\min}(\widehat{\Delta}_0)$ . The operator  $\widehat{\Delta}_0$  is transformed by the unitary map  $U_0 : L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx) \rightarrow L^2(\mathbf{R} \setminus \{0\})$ ,  $U_0 v(x) = |x|^{-\alpha/2}v(x)$ , in

$$\mathbf{\Delta}_0 = \partial_x^2 - \frac{\alpha}{2} \left( \frac{\alpha}{2} + 1 \right) \frac{1}{x^2}.$$

By [4] and [27, Lemma 13.3.1], it holds that  $D_{\min}(\mathbf{\Delta}_0)$  is the closure of  $C_c^\infty(\mathbf{R} \setminus \{0\})$  in the norm of  $H^2(\mathbf{R} \setminus \{0\}, dx)$ , i.e.,

$$\|u\|_{H^2(\mathbf{R} \setminus \{0\}, dx)} = \|u\|_{L^2(\mathbf{R} \setminus \{0\}, dx)} + \|\partial_x u\|_{L^2(\mathbf{R} \setminus \{0\}, dx)} + \|\partial_x^2 u\|_{L^2(\mathbf{R} \setminus \{0\}, dx)}.$$

From this follows that  $D_{\min}(\widehat{\Delta}_0) = U_0^{-1}D_{\min}(\mathbf{\Delta}_0)$  is given by the closure of  $C_c^\infty(\mathbf{R} \setminus \{0\})$  in  $W = U_0^{-1}H^2(\mathbf{R} \setminus \{0\}, dx)$ , w.r.t. the induced norm

$$\|v\|_W = \|v\|_{L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)} + \||x|^{\alpha/2}\partial_x(|x|^{-\alpha/2}v)\|_{L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)} + \||x|^{\alpha/2}\partial_x^2(|x|^{-\alpha/2}v)\|_{L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)}.$$

Thus, to prove the claim, we need to show that the convergences in  $W$  and in  $H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)^2$  are equivalent on  $C_c^\infty(\mathbf{R} \setminus \{0\})$ .

To this aim, fix a cutoff function  $\psi \in C_c^\infty(\mathbf{R})$  such that  $\psi(0) = 0$ ,  $\partial_x \psi(0) = 1$ , and  $\text{supp } \psi \subset (-1, 1)$ . Moreover, let  $\{v_n\}_{n \in \mathbf{N}} \subset C_c^\infty(\mathbf{R} \setminus \{0\})$  be a sequence such that  $v_n \rightarrow v$  w.r.t.  $\|\cdot\|_W$ . In particular,  $\psi v_n \rightarrow \psi v$  and  $(1 - \psi)v_n \rightarrow (1 - \psi)v$  w.r.t.  $\|\cdot\|_W$ . Since  $x^{-1} \leq 1$  if  $|x| \geq 1$ , by

$$(15) \quad \partial_x v(x) = |x|^{\alpha/2}\partial_x(|x|^{-\alpha/2}v) + \frac{\alpha}{2} \frac{v}{x}, \quad \widehat{\Delta}_0 v = |x|^{\alpha/2}\partial_x^2(|x|^{-\alpha/2}v) + \frac{\alpha}{2} \left( \frac{\alpha}{2} + 1 \right) \frac{v}{x^2}.$$

<sup>2</sup>recall that the norm on  $H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)$  is

$$\|v\|_{H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)} = \|v\|_{L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)} + \|\partial_x v\|_{L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)} + \|\widehat{\Delta}_0 v\|_{L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha}dx)}$$

follows immediately that  $(1 - \psi)v_n \rightarrow (1 - \psi)v$  in  $H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ . Recall now the Hardy inequality (see [13])

$$(16) \quad \int_0^1 \frac{u^2}{x^2} dx \leq 4 \int_0^1 (\partial_x u)^2 dx, \quad \text{for any } u \in H_0^1((0, 1), dx).$$

Let  $u_n = U_0(\psi(v_n - v)) = \psi|x|^{-\alpha/2}(v_n - v)$ . Since  $\psi|x|^{-\alpha/2}v_n \in C_c^\infty((0, 1))$  and  $\psi|x|^{-\alpha/2}v_n \rightarrow \psi|x|^{-\alpha/2}v$  in  $H^2((0, 1), dx)$ , it holds that  $u_n \in H_0^1((0, 1), dx)$ . Thus, by (16),

$$\begin{aligned} \int_0^1 \frac{(\psi v_n - \psi v)^2}{x^2} x^{-\alpha} dx &= \int_0^1 \frac{u_n^2}{x^2} dx \\ &\leq 4 \int_0^1 (\partial_x u_n)^2 dx = 4 \int_0^1 \left( |x|^{-\alpha/2} \partial_x (|x|^{-\alpha/2} (\psi v_n - \psi v)) \right)^2 x^{-\alpha} dx \rightarrow 0. \end{aligned}$$

By (15), the same argument applied on  $(-1, 0)$  proves that  $\partial_x \psi v_n \rightarrow \partial_x \psi v$  in  $L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , and hence that  $\partial_x v_n \rightarrow \partial_x v$ . Observe now that by [4, (3.5)] there exists  $C > 0$  such that for any  $u \in D_{\min}(\hat{\Delta}_0)$  it holds

$$(17) \quad \left\| \frac{\alpha}{2} \left( \frac{\alpha}{2} + 1 \right) \frac{u}{x^2} \right\| \leq C \|u\|_{H^2(\mathbf{R} \setminus \{0\}, dx)}.$$

Hence, for any  $\varphi \in D_{\min}(\hat{\Delta}_0)$  it holds

$$\|\hat{\Delta}_0 \varphi\|_{L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)} = \|\hat{\Delta}_0 (|x|^{\alpha/2} \varphi)\|_{L^2(\mathbf{R} \setminus \{0\}, dx)} \leq C \| |x|^{\alpha/2} \varphi \|_{H^2(\mathbf{R} \setminus \{0\}, dx)} = C \|\varphi\|_W.$$

Hence, choosing  $\varphi = \psi v_n - \psi v$ , this proves that  $\hat{\Delta}_0(\psi v_n) \rightarrow \hat{\Delta}_0(\psi v)$  in  $L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ . This completes the proof that  $v_n \rightarrow v$  in  $H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$  and hence that  $D_{\min}(\hat{\Delta}_0) \subset H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ .

In order to complete the first part of the proof, we have to show that if  $\{v_n\}_{n \in \mathbf{N}} \subset C_c^\infty(\mathbf{R} \setminus \{0\})$  is such that  $v_n \rightarrow v$  in  $H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , then  $v_n \rightarrow v$  also in  $W$ . This can be proved as above, by (15), (16), and (17).

We now proceed to the classification of the self-adjoint extensions of  $\hat{\Delta}_0$ . For this purpose, recall the definition of  $\phi_D^\pm$  and  $\phi_N^\pm$  given in (10) and let

$$\phi_N(x) = \phi_N^+(x) + \phi_N^-(x), \quad \phi_D(x) = \phi_D^+(x) + \phi_D^-(x).$$

Observe that  $\phi_D \in L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$  and that  $\hat{\Delta}_0 \phi_D(x) = 0$  for any  $x \notin (-2, -1) \cup (1, 2)$ . Since the function is smooth, this implies that  $\phi_D \in D_{\max}(\hat{\Delta}_0)$ . The same holds for  $\phi_N$ . Moreover, a simple computation shows that  $[\phi_D^+, \phi_N^+](0^+) = [\phi_D^+, \phi_N^+](0^-) = 1$ , and hence  $\phi_N$  and  $\phi_D$  satisfy the hypotheses of Theorem 2.6. In particular, by Remark 2.7, this implies that

$$D_{\max}(\hat{\Delta}_0) = D_{\min}(\hat{\Delta}_0) + \text{span}\{\phi_D^+, \phi_N^+, \phi_D^-, \phi_N^-\}.$$

We claim that for any  $u = u_0 + u_D^+ \phi_D^+ + u_N^+ \phi_N^+ + u_D^- \phi_D^- + u_N^- \phi_N^- \in D_{\max}$  it holds

$$(18) \quad [u, \phi_N](0^+) = u_D^+, \quad [u, \phi_D](0^+) = u_N^+, \quad [u, \phi_N](0^-) = u_D^-, \quad [u, \phi_N](0^-) = u_N^-.$$

This, by Theorem 2.6 will complete the classification of the self-adjoint extensions. Observe that, (12) and the bilinearity of the Lagrange parentheses imply that  $[u_0, \phi_N](0^\pm) = [u_0, \phi_D](0^\pm) = 0$ . The claim then follows from the fact that

$$\begin{aligned} [\phi_D^+, \phi_N](0^+) &= [\phi_N^+, \phi_D](0^+) = [\phi_D^-, \phi_N](0^-) = [\phi_N^-, \phi_D](0^-) = 1, \\ [\phi_D^-, \phi_N](0^+) &= [\phi_N^-, \phi_D](0^+) = [\phi_D^+, \phi_N](0^-) = [\phi_N^+, \phi_D](0^-) = 0. \end{aligned}$$

To complete the proof, it remains only to identify the Friedrichs extension  $(\widehat{\Delta}_0)_F$ . Recall that such extension is always defined, and has domain

$$D((\widehat{\Delta}_0)_F) = \{u \in H_0^1(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx) \mid \widehat{\Delta}_0 u \in L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)\}.$$

Since if  $\alpha \leq -1$ ,  $\phi_N \notin H^1(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , it is clear that the Friedrichs extension corresponds to the case where  $u_N^+ = u_N^- = 0$ , i.e., to  $c_+ = c_- = 0$ . On the other hand, if  $\alpha > -1$ , since all the end-points are regular, by [14, Corollary 10.20] holds that the Friedrichs extension corresponds to the case where  $u(0^\pm) = u_D^\pm = 0$ , i.e., to  $c_+ = c_- = +\infty$ .  $\square$

**Remark 2.9.** If  $u \in D_{\max}(\widehat{\Delta}_0)$ , it holds

$$u_D^+ = [u, \phi_N](0^+) = \lim_{x \downarrow 0} (u(x) - x \partial_x u(x)) \quad \text{and} \quad u_N^+ = [u, \phi_D](0^+) = \lim_{x \downarrow 0} x^{-\alpha} \partial_x u(x).$$

This implies, in particular, that if  $\alpha > -1$  then  $u_D^+ = u(0^+)$ . Indeed this holds if and only if the end-point  $0^+$  is regular in the sense of Sturm-Liouville operators, see Definition 2.5. Clearly the same computations hold at  $0^-$ .

We conclude this section with a description of the maximal domain, in the case  $\alpha \in (-1, 1)$ .

**Proposition 2.10.** For any  $\alpha \in \mathbf{R}$ , it holds that

$$D_{\max}(\Delta|_{C_c^\infty(M)}) = \begin{cases} H^2(M, d\omega) = H_0^2(M, d\omega) & \text{if } \alpha \leq -3 \text{ or } \alpha \geq 1, \\ H^2(M, d\omega) \oplus \text{span}\{\phi_N^+, \phi_N^-\} & \text{if } -3 < \alpha \leq -1, \\ H^2(M, d\omega) \supsetneq H_0^2(M, d\omega) & \text{if } -1 < \alpha < 1. \end{cases}$$

Here we let, with abuse of notation,  $\phi_N^\pm(x, y) = \phi_N^\pm(x)$ .

*Proof.* Recall that, by definition,  $H^2(M, d\omega) \subset D_{\max}(\Delta|_{C_c^\infty(M)})$ . Moreover, if  $\alpha \leq -3$  or if  $\alpha \geq 1$ , by Theorem 1.6 it holds  $D_{\max}(\Delta|_{C_c^\infty(M)}) = D(\Delta_F) = H_0^2(M, d\omega) \subset H^2(M, d\omega)$ . This proves the first statement.

On the other hand, by Remark 2.7, if  $\alpha \in (-3, -1]$ , since  $\widehat{\Delta}_k$  is essentially self-adjoint for any  $k \neq 0$  we can decompose the maximal domain as

$$D_{\max}(\Delta|_{C_c^\infty(M)}) = D_{\max}(\widehat{\Delta}_0) \oplus \left( \bigoplus_{k \in \mathbf{Z} \setminus \{0\}} D(\widehat{\Delta}_k) \right)$$

Moreover, letting  $\pi_0$  be the projection on the  $k = 0$  Fourier component and defining  $(\pi_0^{-1} u_0)(x, \theta) = u_0(x)$  for any  $u_0 \in L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , the previous decomposition and the fact that  $D_{\min}(\Delta|_{C_c^\infty(M)}) \subset H^2(M, d\omega) \subset D_{\max}(\Delta|_{C_c^\infty(M)})$  implies that

$$\begin{aligned} D_{\max}(\Delta|_{C_c^\infty(M)}) &= \{u = u_0 + \pi_0^{-1} \tilde{u} \mid u_0 \in D_{\min}(\Delta|_{C_c^\infty(M)}), \tilde{u} \in \text{span}\{\phi_D^+, \phi_N^+, \phi_D^-, \phi_N^-\}\} \\ &= H^2(M, d\omega) + \text{span}\{\phi_D^+, \phi_N^+, \phi_D^-, \phi_N^-\}. \end{aligned}$$

Here, in the last equality, we let  $\phi_D(x, y) = \phi_D(x)$  and  $\phi_N(x, y) = \phi_N(x)$ . A simple computation shows that  $\phi_D \in H^1(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$  and  $\phi_N \notin H^1(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ . Since  $\widehat{\Delta}_0 \phi_D = 0$ , it follows that  $\phi_D \in H^2(M, d\omega)$ , while  $\phi_N \notin H^2(M, d\omega)$ . This implies the statement.

To complete the proof it suffices to prove that if  $\alpha \in (-1, 1)$  it holds  $D_{\max}(\Delta|_{C_c^\infty(M)}) \subset H^2(M, d\omega)$ . In fact, the inequality  $H^2(M, d\omega) \neq H_0^2(M, d\omega)$  will then follow from the fact that  $\Delta_F$  is not the only self-adjoint extension of  $\Delta|_{C_c^\infty(M)}$ . By Parseval identity,  $\phi, \Delta\phi \in L^2(M, d\omega)$  if and only if  $\phi_k, \widehat{\Delta}_k \phi_k \in L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$  for any  $k \in \mathbf{Z}$  and thus the statement is equivalent to  $D_{\max}(\widehat{\Delta}_k) \subset H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$  for any  $k \in \mathbf{Z}$ . Let  $u \in D_{\max}(\widehat{\Delta}_k)$ . Since  $\lim_{x \rightarrow 0^\pm} x^{-\alpha} \partial_x u(x) = [u, \phi_D](0^\pm)$ , this limit exists and is finite. Moreover, since  $\pm\infty$  are limit-point, it holds  $\lim_{x \rightarrow \pm\infty} x^{-\alpha} \partial_x u(x) =$

$[u, \phi_D](\pm\infty) = 0$ . Hence,  $x^{-\alpha}\partial_x u$  is square integrable near 0 and at infinity, and from the characterization (13) follows that  $\widehat{\Delta}_k u \in L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ . This proves that  $u \in H^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$  and thus the proposition.  $\square$

### 3. BILINEAR FORMS

**3.1. Preliminaries.** This introductory section is based on [15]. Let  $\mathcal{H}$  be an Hilbert space with scalar product  $(\cdot, \cdot)_{\mathcal{H}}$ . A non-negative symmetric bilinear form densely defined on  $\mathcal{H}$ , henceforth called only a *symmetric form on  $\mathcal{H}$* , is a map  $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \rightarrow \mathbf{R}$  such that  $D(\mathcal{E})$  is dense in  $\mathcal{H}$  and  $\mathcal{E}$  is bilinear, symmetric, and non-negative (i.e.,  $\mathcal{E}(u, u) \geq 0$  for any  $u \in D(\mathcal{E})$ ). A symmetric form is *closed* if  $D(\mathcal{E})$  is a complete Hilbert space with respect to the scalar product

$$(19) \quad (u, v)_{\mathcal{E}} = (u, v)_{\mathcal{H}} + \mathcal{E}(u, v), \quad u, v \in D(\mathcal{E}).$$

To any densely defined non-positive definite self-adjoint operator  $A$  it is possible to associate a symmetric form  $\mathcal{E}_A$  such that

$$\begin{aligned} \mathcal{E}_A(u, v) &= (-Au, v) \\ D(A) &= \{u \in D(\mathcal{E}_A) : \exists v \in \mathcal{H} \text{ s.t. } \mathcal{E}(u, \phi) = (v, \phi) \text{ for all } \phi \in D(\mathcal{E}_A)\}. \end{aligned}$$

Indeed, we have the following.

**Theorem 3.1** ([21, 15]). *Let  $\mathcal{H}$  be an Hilbert space, then the map  $A \mapsto \mathcal{E}_A$  induces a one to one correspondence*

$$A \text{ non-positive definite self-adjoint operator} \iff \mathcal{E}_A \text{ closed symmetric form.}$$

*In particular, this correspondence can be characterized by  $D(A) \subset D(\mathcal{E}_A)$  and  $\mathcal{E}_A(u, v) = (-Au, v)$  for all  $u \in D(A)$ ,  $v \in D(\mathcal{E}_A)$ .*

Consider now a  $\sigma$ -finite measure space  $(X, \mathcal{F}, m)$ .

**Definition 3.2.** A symmetric form  $\mathcal{E}$  on  $L^2(X, m)$  is *Markovian* if for any  $\varepsilon > 0$  there exists  $\psi_\varepsilon : \mathbf{R} \rightarrow \mathbf{R}$  such that  $-\varepsilon \leq \psi_\varepsilon \leq 1 + \varepsilon$ ,  $\psi_\varepsilon(t) = t$  if  $t \in [0, 1]$ ,  $0 \leq \psi'_\varepsilon(t) - \psi'_\varepsilon(s) \leq t - s$  whenever  $s < t$  and

$$u \in D(\mathcal{E}) \implies \psi_\varepsilon(u) \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(\psi_\varepsilon(u), \psi_\varepsilon(u)) \leq \mathcal{E}(u, u).$$

A closed Markovian symmetric form is a *Dirichlet form*.

A semigroup  $\{T_t\}_{t \geq 0}$  on  $L^2(X, m)$  is *Markovian* if

$$u \in L^2(X, m) \text{ s.t. } 0 \leq u \leq 1 \quad m - \text{a.e.} \implies 0 \leq T_t u \leq 1 \quad m - \text{a.e. for any } t > 0.$$

A non-positive self-adjoint operator is *Markovian* if it generates a Markovian semigroup.

When the form is closed, the Markov property can be simplified, as per the following Theorem. For any  $u : X \rightarrow \mathbf{R}$  let  $u_{\#} = \min\{1, \max\{u, 0\}\}$ .

**Theorem 3.3** (Theorem 1.4.1 of [15]). *The closed symmetric form  $\mathcal{E}$  is Markovian if and only if*

$$u \in D(\mathcal{E}) \implies u_{\#} \in D(\mathcal{E}) \text{ and } \mathcal{E}(u_{\#}, u_{\#}) \leq \mathcal{E}(u, u).$$

Since any function of  $L^\infty(X, m)$  is approximable by functions in  $L^2(X, m)$ , the Markov property allows to extend the definition of  $\{T_t\}_{t \geq 0}$  to  $L^\infty(X, m)$ , and moreover implies that it is a contraction semigroup on this space. When  $\{T_t\}_{t \geq 0}$  is the evolution semigroup of the heat equation, the Markov property can be seen as a physical admissibility condition. Namely, it assures that when starting from an initial datum  $u$  representing a temperature distribution (i.e., a positive and bounded function) the solution  $T_t u$  remains a temperature distribution at each time, and, moreover, that the heat does not concentrate.

The following theorem extends the one-to-one correspondence given in Theorems 2.1 and 3.1 to the Markovian setting.

**Theorem 3.4** ([15]). *Let  $A$  be a non-positive self-adjoint operator on  $L^2(X, m)$ . The following are equivalents*

- (1)  $A$  is a Markovian operator;
- (2)  $\mathcal{E}_A$  is a Dirichlet form;
- (3)  $\{e^{tA}\}_{t \geq 0}$  is a Markovian semigroup.

Given a non-positive symmetric operator  $A$  we can always define the (non-closed) symmetric form

$$\mathcal{E}(u, v) = (-Au, v), \quad D(\mathcal{E}) = D(A).$$

The Friedrichs extension  $A_F$  of  $A$  is then defined as the self-adjoint operator associated via Theorem 3.1 to the closure  $\mathcal{E}_0$  of this form. Namely,  $D(\mathcal{E}_0)$  is the closure of  $D(A)$  with respect to the scalar product (19), and  $\mathcal{E}_0(u, v) = \lim_{n \rightarrow +\infty} \mathcal{E}(u_n, v_n)$  for  $u_n \rightarrow u$  and  $v_n \rightarrow v$  w.r.t.  $(\cdot, \cdot)_{\mathcal{E}}$ . It is a well-known fact that the Friedrichs extension of a Markovian operator is always a Dirichlet form (see, e.g., [15, Theorem 3.1.1]).

A Dirichlet form  $\mathcal{E}$  is said to be *regular* on  $X$  if  $D(\mathcal{E}) \cap C_c(X)$  is both dense in  $D(\mathcal{E})$  w.r.t. the scalar product (19) and dense in  $C_c(X)$  w.r.t. the  $L^\infty(X)$  norm. To any regular Dirichlet form  $\mathcal{E}_A$  it is possible to associate a Markov process  $\{X_t\}_{t \geq 0}$  which is generated by  $A$  (indeed they are in one-to-one correspondence to a particular class of Markov processes, the so-called Hunt processes, see [15] for the details).

If its associated Dirichlet form is regular, by Definitions 1.10 and 1.11, a Markovian operator is said *stochastically complete* if its associated Markov process has almost surely infinite lifespan, and *recurrent* if it intersects any subset of  $X$  with positive measure an infinite number of times. If it is not stochastically complete, an operator is called *explosive*. Observe that recurrence is a stronger property than stochastic completeness. Since we will only consider regular Dirichlet forms, we refer to [15] for a definition of recurrence valid for general Dirichlet forms.

We will need the following characterizations.

**Theorem 3.5** (Theorem 1.6.6 in [15]). *A Dirichlet form  $\mathcal{E}$  is stochastically complete if and only if there exists a sequence  $\{u_n\} \subset D(\mathcal{E})$  satisfying*

$$0 \leq u_n \leq 1, \quad \lim_{n \rightarrow +\infty} u_n = 1 \quad m - a.e.,$$

such that

$$\mathcal{E}(u_n, v) \rightarrow 0 \quad \text{for any } v \in D(\mathcal{E}) \cap L^1(X, m).$$

We let the *extended domain*  $D(\mathcal{E})_e$  of a Dirichlet form  $\mathcal{E}$  to be the family of functions  $u \in L^\infty(X, m)$  such that there exists  $\{u_n\}_{n \in \mathbb{N}} \subset D(\mathcal{E})$ , Cauchy sequence w.r.t. the scalar product (19), such that  $u_n \rightarrow u$   $m$ -a.e. . The Dirichlet form  $\mathcal{E}$  can be extended to  $D(\mathcal{E})_e$  as a non-negative definite symmetric bilinear form, by  $\mathcal{E}(u, u) = \lim_{n \rightarrow +\infty} \mathcal{E}(u_n, u_n)$ .

**Theorem 3.6** (Theorems 1.6.3 and 1.6.5 in [15]). *Let  $\mathcal{E}$  be a Dirichlet form. The following are equivalent.*

- (1)  $\mathcal{E}$  is recurrent;
- (2) there exists a sequence  $\{u_n\} \subset D(\mathcal{E})$  satisfying

$$0 \leq u_n \leq 1, \quad \lim_{n \rightarrow +\infty} u_n = 1 \quad m - a.e.,$$

such that

$$\mathcal{E}(u_n, v) \rightarrow 0 \quad \text{for any } v \in D(\mathcal{E})_e.$$

- (3)  $1 \in D(\mathcal{E})_e$ , i.e., there exists a sequence  $\{u_n\} \subset D(\mathcal{E})$  such that  $\lim_{n \rightarrow +\infty} u_n = 1$   $m - a.e.$  and  $\mathcal{E}(u_n, u_n) \rightarrow 0$ .

**Remark 3.7.** As a consequence of this two theorems we have that if  $m(X) < +\infty$ , stochastic completeness and recurrence are equivalent.

We conclude this preliminary part, by introducing a notion of restriction of closed forms associated to self-adjoint extensions of  $\Delta|_{C_c^\infty(M)}$ .

**Definition 3.8.** Given a self-adjoint extension  $A$  of  $\Delta|_{C_c^\infty(M)}$  and an open set  $U \subset M$ , we let the *Neumann restriction*  $\mathcal{E}_A|_U$  of  $\mathcal{E}_A$  to be the form associated with the self-adjoint operator  $A|_U$  on  $L^2(U, d\omega)$ , obtained by putting Neumann boundary conditions on  $\partial U$ .

In particular, by Theorem 3.1 and an integration by parts, it follows that  $D(\mathcal{E}_A|_U) = \{u|_U \mid u \in D(\mathcal{E}_A)\}$ .

**3.2. Markovian extensions of  $\Delta|_{C_c^\infty(M)}$ .** The bilinear form associated with  $\Delta|_{C_c^\infty(M)}$  is

$$\mathcal{E}(u, v) = \int_{M_\alpha} g(\nabla u, \nabla v) d\omega = \int_{M_\alpha} (\partial_x u \partial_x v + |x|^{2\alpha} \partial_\theta u \partial_\theta v) d\omega, \quad D(\mathcal{E}) = C_c^\infty(M).$$

By [15, Example 1.2.1],  $\mathcal{E}$  is a Markovian form. The Friederichs extension is then associated with the form

$$\mathcal{E}_F(u, v) = \int_M (\partial_x u \partial_x v + |x|^{2\alpha} \partial_\theta u \partial_\theta v) d\omega, \quad D(\mathcal{E}_F) = H_0^1(M, d\omega),$$

where the derivatives are taken in the sense of Schwartz distributions. By its very definition, and the fact that  $D(\mathcal{E}_F) \cap C_c^\infty(M) = C_c^\infty(M)$ , follows that  $\mathcal{E}_F$  is always a regular Dirichlet form on  $M$  (equivalently, on  $M^+$  or on  $M^-$ ). Its associated Markov process is absorbed by the singularity.

The following Lemma will be crucial to study the properties of the Friederichs extension. Let  $M_0 = (-1, 1) \times \mathbb{T}$ ,  $M_\infty = (1, +\infty) \times \mathbb{T}$  and recall the notion of Neumann restriction given in Definition 3.8.

**Lemma 3.9.** *If  $\alpha \leq -1$ , it holds that  $1 \in D(\mathcal{E}_F|_{M_0})$ . Moreover,  $1 \notin D(\mathcal{E}_F|_{M_0})_e$  if  $\alpha > -1$  and  $1 \in D(\mathcal{E}_F|_{M_\infty})_e$  if and only if  $\alpha \geq -1$ .*

*Proof.* To ease the notation, we let  $\widehat{\mathcal{E}}_k$  to be the Dirichlet form associated to the Friederichs extension of  $\widehat{\Delta}_k$ . In particular, for  $k = 0$ ,

$$\widehat{\mathcal{E}}_0(u, v) = \int_{\mathbf{R} \setminus \{0\}} \partial_x u \partial_x v |x|^{-\alpha} dx, \quad D(\widehat{\mathcal{E}}_0) = H_0^1(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx).$$

Let  $\pi_k : L^2(M, d\omega) \rightarrow H_k = L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$  be the projection on the  $k$ -th Fourier component. Then, from the rotational invariance of  $D(\mathcal{E}_F)$  follows that

$$D(\mathcal{E}_F) = \bigoplus_{k \in \mathbf{Z}} D(\widehat{\mathcal{E}}_k), \quad \mathcal{E}_F(u, v) = \sum_{k \in \mathbf{Z}} \widehat{\mathcal{E}}_k(\pi_k u, \pi_k v).$$

In particular, since  $\pi_0 1 = 1$  and  $\pi_k 1 = 0$  for  $k \neq 0$ , follows that  $1 \in D(\mathcal{E}_F|_{M_0})$  (resp.  $1 \in D(\mathcal{E}_F|_{M_\infty})_e$ ) if and only if  $1 \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$  (resp.  $1 \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$ ). Here, with abuse of notation, we denoted as 1 both the functions  $1 : M \rightarrow \{1\}$  and  $1 : \mathbf{R} \rightarrow \{1\}$ . Thus, to complete the proof of the lemma, it suffices to prove that  $1 \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$  if  $\alpha \leq -1$ , that  $1 \notin D(\widehat{\mathcal{E}}_0|_{(0,1)})_e$  if  $\alpha \geq -1$  and that  $1 \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$  if and only if  $\alpha \geq -1$ .

For any  $0 < r < R < +\infty$ , let  $f_{r,R}^\alpha$  be the only solution to the Cauchy problem

$$\begin{cases} \widehat{\Delta}_0 f = 0, \\ f(r) = 1, \quad f(R) = 0. \end{cases}$$

Namely,

$$f_{r,R}^\alpha(x) = \begin{cases} \frac{R^{1+\alpha} - x^{1+\alpha}}{R^{1+\alpha} - r^{1+\alpha}} & \text{if } \alpha \neq -1, \\ \frac{\log(\frac{R}{x})}{\log(\frac{R}{r})} & \text{if } \alpha = -1. \end{cases}$$

Then, the 0-equilibrium potential (see [15] and Remark 3.10) of  $[0, r]$  in  $[0, R]$ , is given by

$$(20) \quad u_{r,R}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq r, \\ f_{r,R}^\alpha(x) & \text{if } r < x \leq R, \\ 0 & \text{if } x > R. \end{cases}$$

It is a well-known fact that  $u_{r,R}$  is the minimizer for the capacity of  $[0, r]$  in  $[0, R)$ . Namely, for any locally Lipschitz function  $v$  with compact support contained in  $[0, R]$  and such that  $v(x) = 1$  for any  $0 < x < r$ , it holds

$$(21) \quad \int_0^{+\infty} |\partial_x u_{r,R}|^2 x^{-\alpha} dx \leq \int_0^{+\infty} |\partial_x v|^2 x^{-\alpha} dx$$

Since it is compactly supported on  $[0, +\infty)$  and locally Lipschitz, it follows that  $u_{r,R} \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})$  and  $1 - u_{r,R} \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$  for any  $0 < r < R < +\infty$ .

Consider now  $\alpha \geq -1$ , and let us prove that  $1 \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$ . To this aim, it suffices to show that there exists a sequence  $\{u_n\}_{n \in \mathbf{N}} \subset D(\widehat{\mathcal{E}}_0|_{(1,+\infty)}) = \{u|_{(1,+\infty)} \mid u \in H^1((0, +\infty), x^{-\alpha} dx)\}$  such that  $u_n \rightarrow 1$  a.e. and  $\widehat{\mathcal{E}}_0|_{(1,+\infty)}$ . Let

$$u_n = \begin{cases} u_{n,2n} & \text{if } \alpha \neq -1, \\ u_{n,n^2} & \text{if } \alpha = -1. \end{cases}$$

It is clear that  $u_n \rightarrow 1$  a.e., moreover, a simple computation shows that

$$\widehat{\mathcal{E}}_0|_{(1,+\infty)}(u_n, u_n) = \int_1^{+\infty} |\partial_x u_n|^2 x^{-\alpha} dx = \begin{cases} \frac{1+\alpha}{2^{1+\alpha}-1} n^{-(1+\alpha)} & \text{if } \alpha \neq -1, \\ \frac{1}{\log(n)} & \text{if } \alpha = -1. \end{cases}$$

Hence  $\widehat{\mathcal{E}}_0|_{(1,+\infty)} \rightarrow 0$  if  $\alpha \geq -1$ , proving that  $1 \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$ .

We now prove that  $1 \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$  if  $\alpha \leq -1$ . Consider the following sequence in  $H^1((0, 1), x^{-\alpha} dx)$ ,

$$u_n = \begin{cases} u_{1/2n, 1/n} & \text{if } \alpha \neq -1, \\ u_{1/n^2, 1/n} & \text{if } \alpha = -1. \end{cases}$$

A direct computation of  $\int_0^1 |\partial_x u_n|^2 x^{-\alpha} dx$ , the fact that  $\text{supp } u_n \subset [0, 1/n]$  and  $0 \leq u_n \leq 1$ , prove that  $u_n \rightarrow 0$  in  $H^1((0, 1), x^{-\alpha} dx)$ . Since  $1 - u_n \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$ , which is closed, this proves that  $1 - u_n \rightarrow 1$  in  $D(\widehat{\mathcal{E}}_0|_{(0,1)})$ , and hence the claim.

To complete the proof, it remains to show that  $1 \notin D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$  if  $\alpha < -1$ . The same argument can be then used to prove that  $1 \notin D(\widehat{\mathcal{E}}_0|_{(0,1)})_e$  if  $\alpha > -1$ . We proceed by contradiction, assuming that there exists a sequence  $\{v_n\}_{n \in \mathbf{N}} \subset D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})$  such that  $v_n \rightarrow 1$  a.e. and  $\widehat{\mathcal{E}}_0|_{(1,+\infty)}(v_n, v_n) \rightarrow 0$ . Since the form  $\widehat{\mathcal{E}}_0|_{(1,+\infty)}$  is regular on  $[1, +\infty)$ , we can take  $v_n \in C_c^\infty([1, +\infty))$ . Moreover, we can assume that  $v_n(1) = 1$  for any  $n \in \mathbf{N}$ . In fact, if this is not the case, it suffices to consider the sequence  $\tilde{v}_n(x) = v_n(x)/v_n(1)$ . Let  $R_n > 0$  be such that  $\bigcup_{m \leq n} \text{supp } v_m \subset [1, R_n]$ . Moreover, extend  $v_n$  to 1 on  $(0, 1)$ , so that  $\widehat{\mathcal{E}}_0|_{(1,+\infty)}(v_n, v_n) =$

$\int_0^{+\infty} |\partial_x v_n|^2 x^{-\alpha} dx$ . Since the same holds for  $u_{1,R_n}$ , by (21), the fact that  $R_n \rightarrow +\infty$  and  $\alpha < -1$ , we get

$$\lim_{n \rightarrow +\infty} \widehat{\mathcal{E}}_0|_{(1,+\infty)}(v_n, v_n) \geq \lim_{n \rightarrow +\infty} \widehat{\mathcal{E}}_0|_{(1,+\infty)}(u_{1,R_n}, u_{1,R_n}) = \lim_{n \rightarrow +\infty} \frac{1+\alpha}{R_n^{1+\alpha} - 1} = -(1+\alpha) > 0.$$

This contradicts the fact that  $\widehat{\mathcal{E}}_0|_{(1,+\infty)}(v_n, v_n) \rightarrow 0$ , completing the proof.  $\square$

**Remark 3.10.** The 0-equilibrium potential defined in (20) admits a probabilistic interpretation. Namely, it is the probability that the Markov process associated with  $\widehat{\Delta}_0$  and starting from  $x$ , exits the first time from the interval  $\{r < x < R\}$  through the inner boundary  $\{x = r\}$ .

It is possible to define a semi-order on the set of the Markovian extensions of  $\Delta|_{C_c^\infty(M)}$  as follows. Given two Markovian extensions  $A$  and  $B$ , we say that  $A \subset B$  if  $D(\mathcal{E}_A) \subset D(\mathcal{E}_B)$  and  $\mathcal{E}_A(u, u) \geq \mathcal{E}_B(u, u)$  for any  $u \in D(\mathcal{E}_A)$ . With respect to this semi-order, the Friedrichs extension is the minimal Markovian extension. Let  $\Delta_N$  be the maximal Markovian extension (see [15]). This extension is associated with the Dirichlet form  $\mathcal{E}^+$  defined by

$$\mathcal{E}^+(u, v) = \int_M (\partial_x u \partial_x v + |x|^{2\alpha} \partial_\theta u \partial_\theta v) d\omega,$$

$$D(\mathcal{E}^+) = \{u \in L^2(M, d\omega) \mid \mathcal{E}^+(u, u) < +\infty\} = H^1(M, d\omega),$$

where the derivatives are taken in the sense of Schwartz distributions. We remark that  $\mathcal{E}^+$  is a regular Dirichlet form on  $M^+ = M_\alpha \setminus M^-$  and  $M^- = M_\alpha \setminus M^+$  (see, e.g., [15, Lemma 3.3.3]). Its associated Markov process is reflected by the singularity.

When  $\Delta|_{C_c^\infty(M)}$  has only one Markovian extension, i.e., whenever  $\Delta_F = \Delta_N$ , we say that it is *Markov unique*. Clearly, if  $\Delta|_{C_c^\infty(M)}$  is essentially self-adjoint, it is also Markov unique. The next proposition shows that Markov uniqueness is a strictly stronger property than essential self-adjointness.

**Proposition 3.11.** *The operator  $\Delta|_{C_c^\infty(M)}$  is Markov unique if and only if  $\alpha \notin (-1, 1)$ .*

*Proof.* As observed above, the statement is an immediate consequence of Theorem 1.6 for  $\alpha \leq -3$  and  $\alpha \geq 1$ . If  $\alpha \in (-3, -1]$ , since by Theorem 1.6 all  $\widehat{\Delta}_k$  for  $k \neq 0$  are essentially self-adjoint, it holds that  $\Delta_N = \widehat{A}_0 \oplus (\bigoplus_{k \in \mathbb{N}} \widehat{\Delta}_k)$  for some self-adjoint extension  $\widehat{A}_0$  of  $\widehat{\Delta}_0$ . Recall the definition of  $\phi_D^\pm$  and  $\phi_N^\pm$  given in (10) and with abuse of notation let  $\phi_D^\pm(x, \theta) = \phi_D^\pm(x)$  and  $\phi_N^\pm(x, \theta) = \phi_N^\pm(x)$ . Since  $\mathcal{E}^+(\phi_N^\pm, \phi_N^\pm) = +\infty$  if and only if  $\alpha \leq -1$ , we get that  $\phi_N^+, \phi_N^- \notin D(\mathcal{E}^+) \supset D(\Delta_N)$  if  $\alpha \leq -1$ . Hence, by Theorem 1.8, it holds that  $\widehat{A}_0 = (\widehat{\Delta}_0)_F$  and hence that  $\Delta_N = \Delta_F$ .

On the other hand, if  $\alpha \in (-1, 1)$ , the result follows from Lemma 3.9. In fact, it implies that  $\phi_D \notin H_0^1(M, d\omega) = D(\mathcal{E}_F)$  but, since  $\mathcal{E}^+(\phi_D, \phi_D) < +\infty$ , we have that  $\phi_D \in D(\mathcal{E}^+)$ . This proves that  $\Delta_F \subsetneq \Delta_N$ .  $\square$

By the previous result, when  $\alpha \in (-1, 1)$  it makes sense to consider the bridging extension, associated to the operator  $\Delta_B$  and the form  $\mathcal{E}_B$ , defined by

$$\mathcal{E}_B(u, v) = \int_{M_\alpha} (\partial_x u \partial_x v + |x|^{2\alpha} \partial_\theta u \partial_\theta v) d\omega,$$

$$D(\mathcal{E}_B) = \{u \in H^1(M, d\omega) \mid u(0^+, \theta) = u(0^-, \theta) \text{ for a.e. } \theta \in \mathbb{T}\}.$$

From Theorem 3.3 and the fact that  $\mathcal{E}_B = \mathcal{E}^+|_{D(\mathcal{E}_B)}$  follows immediately that  $\mathcal{E}_B$  is a Dirichlet form, and hence  $\Delta_F \subset \Delta_B \subset \Delta_N$ . Moreover, due to the regularity of  $\mathcal{E}^+$  and the symmetry of the boundary conditions appearing in  $D(\mathcal{E}_B)$ , follows that  $\mathcal{E}_B$  is regular on the whole  $M_\alpha$ . Its associated Markov process can cross, with continuous trajectories, the singularity.



We conclude this section by specifying the domains of the Markovian self-adjoint extensions associated with  $\mathcal{E}_F$ ,  $\mathcal{E}^+$  and, when it is defined,  $\mathcal{E}_B$ .

**Proposition 3.12.** *It holds that  $D(\Delta_F) = H_0^2(M, d\omega)$ , while*

$$D(\Delta_N) = \{u \in H^1(M, d\omega) \mid (\Delta u, v) = (\nabla u, \nabla v) \text{ for any } v \in H^1(M, d\omega)\}.$$

Moreover, if  $\alpha \in (-1, 1)$ , the domain of  $\Delta_B$  is

$$D(\Delta_B) = \{H^2(M_\alpha, d\omega) \mid u(0^+, \cdot) = u(0^-, \cdot), \lim_{x \rightarrow 0^+} |x|^{-\alpha} \partial_x u(x, \cdot) = \lim_{x \rightarrow 0^-} |x|^{-\alpha} \partial_x u(x, \cdot) \text{ for a.e. } \theta \in \mathbb{T}\}.$$

*Proof.* In view of Theorem 3.1, to prove that  $A$  is the operator associated with  $\mathcal{E}_A$  it suffices to prove that  $D(A) \subset D(\mathcal{E}_A)$  and that  $\mathcal{E}_A(u, v) = (-Au, v)$  for any  $u \in D(A)$  and  $v \in D(\mathcal{E}_A)$ . The requirement on the domain is satisfied by definition in all three cases. We proceed to prove the second fact.

*Friedrichs extension.* By integration by parts it follows that  $\mathcal{E}_F(u, v) = (-\Delta_F u, v)$  for any  $u, v \in C_c^\infty(M)$ , and this equality can be extended to  $u \in H_0^2(M, d\omega) = D(\Delta_F)$  and  $v \in H_0^1(M, d\omega) = D(\mathcal{E}_F)$ .

*Neumann extension.* The property that  $\mathcal{E}^+(u, v) = (-\Delta_N u, v)$  for any  $u \in D(\Delta_N)$  and  $v \in D(\mathcal{E}^+)$  is contained in the definition.

*Bridging extension.* By an integration by parts, it follows that

$$\int_{M_\alpha} (\partial_x u \partial_x v + x^{2\alpha} \partial_\theta u \partial_\theta v) d\omega = (-\Delta_B u, v) - \int_{\mathbb{T}} v |x|^{-\alpha} \partial_x u \Big|_{x=0^-}^{0^+} d\theta = (-\Delta_B u, v).$$

□

**3.3. Stochastic completeness and recurrence on  $M_\alpha$ .** We are interested in localizing the properties of stochastic completeness and recurrence of a Markovian self-adjoint extension  $A$  of  $\Delta|_{C_c^\infty(M)}$ . Due to the already mentioned repulsing properties of Neumann boundary conditions, the natural way to operate is to consider the Neumann restriction introduced in Definition 3.8.

Observe that, if  $U \subset M$  is an open set such that  $\bar{U} \cap (\{-\infty, 0, +\infty\} \times \mathbb{T}) = \emptyset$ , then the Neumann restriction  $\mathcal{E}_A|_U$  is always recurrent on  $U$ . In fact, in this case, there exist two constants  $0 < C_1 < C_2$  such that  $C_1 dx d\theta \leq d\omega \leq C_2 dx d\theta$  on  $U$  and clearly  $1 \in D(\mathcal{E}_A|_U) = H^1(U, dx d\theta)$ , that by Theorem 3.6 implies the recurrence. For this reason, we will concentrate only on the properties “at 0” or “at  $\infty$ ”.

**Definition 3.13.** Given a Markovian extension  $A$  of  $\Delta|_{C_c^\infty(M)}$ , we say that it is *stochastically complete at 0* (resp. *recurrent at 0*) if its Neumann restriction to  $M_0 = (-1, 1) \times \mathbb{T}$ , is stochastically complete (resp. recurrent). We say that  $A$  is *exploding at 0* if it is not stochastically complete at 0. Considering  $M_\infty = (1, \infty) \times \mathbb{T}$ , we define stochastic completeness, recurrence and explosiveness at  $\infty$  in the same way.

In order to justify this approach, we will need the following.

**Proposition 3.14.** *A Markovian extension  $A$  of  $\Delta|_{C_c^\infty(M)}$  is stochastically complete (resp. recurrent) if and only if it is stochastically complete (resp. recurrent) both at 0 and at  $\infty$ .*

*Proof.* Let  $\{u_n\}_{n \in \mathbb{N}} \subset D(\mathcal{E}_A)$  such that  $u_n \rightarrow 1$  a.e. and  $\mathcal{E}_A(u_n, u_n) \rightarrow 0$ . Since  $D(\mathcal{E}_A|_{M_0}) = \{u|_{M_0} \mid u \in D(\mathcal{E}_A)\}$  and  $D(\mathcal{E}_A|_{M_\infty}) = \{u|_{M_\infty} \mid u \in D(\mathcal{E}_A)\}$  follows that  $\{u_n|_{M_0}\}_{n \in \mathbb{N}} \subset D(\mathcal{E}_A|_{M_0})$  and  $\{u_n|_{M_\infty}\}_{n \in \mathbb{N}} \subset D(\mathcal{E}_A|_{M_\infty})$ . Moreover, it is clear that  $u_n|_{M_0}, u_n|_{M_\infty} \rightarrow 1$  a.e. and  $\mathcal{E}_A|_{M_0}(u_n|_{M_0}, u_n|_{M_0}), \mathcal{E}_A|_{M_\infty}(u_n|_{M_\infty}, u_n|_{M_\infty}) \rightarrow 0$ . By Theorem 3.6, this proves that if  $\mathcal{E}_A$  is recurrent it is recurrent also at 0 and  $\infty$ .

On the other hand, if  $A|_{M_0}$  and  $A|_{M_\infty}$  are recurrent, we can always choose the sequences  $\{u_n\}_{n \in \mathbb{N}} \subset D(\mathcal{E}_A|_{M_0})$  and  $\{v_n\}_{n \in \mathbb{N}} \subset D(\mathcal{E}_A|_{M_\infty})$  approximating 1 such that they equal 1 in a neighborhood  $N$  of  $\partial_{M_0} = \partial_{M_\infty} = (\{1\} \times \mathbb{T}) \cup (\{-1\} \times \mathbb{T})$ . In fact the constant function satisfies the

Neumann boundary conditions we posed on  $\partial M_0 = \partial M_\infty$  for the operators associated with  $\mathcal{E}_A|_{M_0}$  and  $\mathcal{E}_A|_{M_\infty}$ . Hence, by gluing  $u_n$  and  $v_n$  we get a sequence of functions in  $D(\mathcal{E}_A)$  approximating 1. The same argument gives also the equivalence of the stochastic completeness, exploiting the characterization given in Theorem 3.5.  $\square$

Before proceeding with the classification of the stochastic completeness and recurrence of  $\Delta_F$ ,  $\Delta_N$  and  $\Delta_B$ , we need the following result. For an operator acting on  $L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$ , the definition of stochastic completeness and recurrence at 0 or at  $\infty$  is given substituting  $M_0$  and  $M_\infty$  in Definition 3.13 with  $(-1, 1)$  and  $(1, +\infty)$ .

**Proposition 3.15.** *Let  $A$  be a Markovian self-adjoint extension of  $\Delta|_{C_c^\infty(M)}$  and assume it decomposes as  $A = \widehat{A}_0 \oplus \tilde{A}$ , where  $\widehat{A}_0$  is a self-adjoint operator on  $H_0$  and  $\tilde{A}$  is a self-adjoint operator on  $\bigoplus_{k \neq 0} H_k$ . Then,  $\widehat{A}_0$  is a Markovian self-adjoint extension of  $\widehat{\Delta}_0$ . Moreover,  $A$  is stochastically complete (resp. recurrent) at 0 or at  $\infty$  if and only if so is  $\widehat{A}_0$ .*

*Proof.* Let  $\pi_k : L^2(M, d\omega) \rightarrow H_k = L^2(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$  be the projection on the  $k$ -th Fourier component. In particular, recall that  $\pi_0 u = (2\pi)^{-1} \int_0^{2\pi} u(x, \theta) d\theta$ . Let  $u \in D(\widehat{A}_0) \subset L^2(\mathbf{R}, |x|^{-\alpha} dx)$  be such that  $0 \leq u \leq 1$ . Hence, posing  $\tilde{u}(x, \theta) = u(x)$ , due to the splitting of  $A$  follows that  $\tilde{u} \in D(A)$  and by the markovianity follows that  $0 \leq A\tilde{u} \leq 1$ . The first part of the statement is then proved by observing that, since  $\pi_0 \tilde{u} = u$  and  $\pi_k \tilde{u} = 0$  for  $k \neq 0$ , we have  $A\tilde{u}(x, \theta) = \widehat{A}_0 u(x)$  for any  $(x, \theta) \in M$ .

We prove the second part of the statement only at 0, since the arguments to treat the at  $\infty$  case are analogous. First of all, we show that the stochastic completeness of  $A$  and  $\widehat{A}_0$  at 0 are equivalent. If  $1 : M_0 \rightarrow \mathbf{R}$  is the constant function, it holds that  $\pi_0 1 = 1 : (-1, 1) \rightarrow \mathbf{R}$ . Moreover, due to the splitting of  $A$ , we have that  $e^{tA} = e^{t\widehat{A}_0} \oplus e^{t\tilde{A}}$ . Hence, it follows that  $e^{tA} 1 = e^{t\widehat{A}_0} 1$ . This, by Definition 1.10, proves the claim.

To prove the equivalence of the recurrences at 0, we start by observing that  $D(\mathcal{E}_A) = D(\mathcal{E}_{\widehat{A}_0}) \oplus D(\mathcal{E}_{\tilde{A}})$  and that

$$(22) \quad \mathcal{E}_A(u, v) = \mathcal{E}_{\widehat{A}_0}(\pi_0 u, \pi_0 v) + \mathcal{E}_{\tilde{A}}(\bigoplus_{k \neq 0} \pi_k u, \bigoplus_{k \neq 0} \pi_k v), \quad \text{for any } u, v \in D(\mathcal{E}_A)$$

In particular, since  $\pi_0 1 = 1$  this implies that  $\mathcal{E}_A|_{M_0}(1, 1) = \mathcal{E}_{\widehat{A}_0}|_{(-1, 1)}(1, 1)$ . By Theorem 3.6, this proves that if  $\widehat{A}_0$  is recurrent at 0, so is  $A$ . Assume now that  $A|_{M_0}$  is recurrent. By Theorem 3.6 there exists  $\{u_n\}_{n \in \mathbf{N}} \subset D(\mathcal{E}_A|_{M_0})$  such that  $0 \leq u_n \leq 1$  a.e.,  $u_n \rightarrow 1$  a.e. and  $\mathcal{E}_A|_{M_0}(u_n, v) \rightarrow 0$  for any  $v$  in the extended domain  $D(\mathcal{E}_A|_{M_0})_e$ . By dominated convergence, it follows that  $\pi_0 u_n = (2\pi)^{-1} \int_0^{2\pi} u_n(\cdot, \theta) d\theta \rightarrow 1$  for a.e.  $x \in (-1, 1)$ . For any  $v \in D(\mathcal{E}_{\widehat{A}_0}|_{(-1, 1)})_e$ , let  $\tilde{v}(x, \theta) = v(x)$ . It is easy to see that  $\tilde{v} \in D(\mathcal{E}_A|_{M_0})_e$ . Then, by applying (22) we get

$$\mathcal{E}_{\widehat{A}_0}|_{(-1, 1)}(\pi_0 u_n, v) = \mathcal{E}_A|_{M_0}(u_n, \tilde{v}) \rightarrow 0, \quad \text{for any } v \in D(\mathcal{E}_{\widehat{A}_0}|_{(-1, 1)})_e.$$

Since  $0 \leq \pi_0 u_n \leq 1$ , this proves that  $\widehat{A}_0|_{(-1, 1)}$  is recurrent  $\square$

The following proposition answers the problem of stochastic completeness or recurrence of the Friedrichs extension.

**Proposition 3.16.** *Let  $\Delta_F$  be the Friedrichs extension of  $\Delta|_{C_c^\infty(M)}$ . Then, the following holds*

	at 0	at $\infty$
$\alpha < -1$	recurrent	stochastically complete
$\alpha = -1$	recurrent	recurrent
$\alpha > -1$	explosive	recurrent

*In particular,  $\Delta_F$  is stochastically complete for  $\alpha < -1$ , recurrent for  $\alpha = -1$  and explosive for  $\alpha > -1$ .*

*Proof.* The part regarding the recurrence is a consequence of Lemma 3.9 and Theorem 3.6, while the last statement is a consequence of Proposition 3.14. Thus, to complete the proof it suffices to prove that  $\Delta_F$  is stochastically complete at  $+\infty$  if  $\alpha < -1$  and not stochastically complete at 0 if  $\alpha > -1$ .

By Proposition 3.15 and the fact that  $\Delta_F = \oplus_{k \in \mathbf{Z}} (\widehat{\Delta}_k)_F$ , we actually need to prove this fact only for  $(\widehat{\Delta}_0)_F$ . Moreover, since the Friedrichs extension decouples the dynamics on the two sides of the singularity, we can work only on  $(0, +\infty)$  instead that on  $\mathbf{R} \setminus \{0\}$ . As in Lemma 3.9, we let  $\widehat{\mathcal{E}}_0$  to be the Dirichlet form associated to the Friedrichs extension of  $\widehat{\Delta}_0$ .

We start by proving the explosion for  $\alpha > -1$  on  $(0, 1)$ . Let us proceed by contradiction and assume that  $(\widehat{\Delta}_0)_F$  is stochastically complete on  $(0, 1)$ . By Theorem 3.5, there exists  $u_n \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$ ,  $0 \leq u_n \leq 1$ ,  $u_n \rightarrow 1$  a.e. and such that  $\widehat{\mathcal{E}}_0|_{(0,1)}(u_n, v) \rightarrow 0$  for any  $v \in D(\widehat{\mathcal{E}}_0|_{(0,1)}) \cap L^1((0, 1), x^{-\alpha} dx)$ . Since  $\widehat{\mathcal{E}}_0|_{(0,1)}$  is regular on  $(0, 1]$ , we can choose the sequence such that  $u_n \in C_c^\infty((0, 1])$ . In particular  $u_n(0) = \lim_{x \downarrow 0} u_n(x) = 0$  for any  $n$ . Let us define, for any  $0 < R \leq 1$ ,

$$v_R(x) = \lim_{r \downarrow 0} (1 - u_{r,R}(x)) = \begin{cases} x^{1+\alpha}/R^{1+\alpha} & \text{if } 0 \leq x < R, \\ 1 & \text{if } 0 \leq x \geq R, \end{cases}$$

where  $u_{r,R}$  is defined in (20). Observe that, by the probabilistic interpretation of  $u_{r,R}$  given in Remark 3.10, follows that  $v_R(x)$  is the probability that the Markov process associated with  $(\widehat{\Delta}_0)_F$  and starting from  $x$  exits the interval  $(0, R)$  before being absorbed by the singularity at 0. A simple computation shows that  $v_R \in D(\widehat{\mathcal{E}}_0|_{(0,1)}) \cap L^1((0, 1), x^{-\alpha} dx)$ . Thus, by definition of  $\{u_n\}_{n \in \mathbf{N}}$  and a direct computation we get

$$0 = \lim_{n \rightarrow +\infty} \widehat{\mathcal{E}}_0|_{(0,1)}(u_n, v_R) = \frac{1+\alpha}{R^{1+\alpha}} \lim_{n \rightarrow +\infty} \int_0^R \partial_x u_n dx = \frac{1+\alpha}{R^{1+\alpha}} \lim_{n \rightarrow +\infty} u_n(R).$$

Hence,  $u_n(R) \rightarrow 0$  for any  $0 < R < 1$ , contradicting the fact that  $u_n \rightarrow 1$  a.e..

To complete the proof, we show that if  $\alpha < -1$ ,  $(\widehat{\Delta}_0)_F$  is stochastically complete on  $(1, +\infty)$ . Let  $v \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)}) \cap L^1((1, +\infty), x^{-\alpha} dx) \subset H^1((1, +\infty), dx)$ . Thus, by Morrey's inequality  $v$  is  $1/2$ -Hölder continuous with constant  $C_H$ . Since, for any  $1 < r < R$ , by (20) it holds that  $u_{r,R} \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})$ , letting  $u_n = u_{n,2n}$  a direct computation yields

$$(23) \quad \widehat{\mathcal{E}}_0|_{(1,+\infty)}(u_n, v) = (1+\alpha) \frac{v(2n) - v(n)}{n^{1+\alpha}(2^{1+\alpha} - 1)}.$$

Since  $u_n \rightarrow 1$  pointwise, by Theorem 3.5, to complete the proof it suffices to show that

$$(24) \quad \frac{v(2n) - v(n)}{n^{1+\alpha}(2^{1+\alpha} - 1)} \rightarrow 0, \quad \text{for any } v \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)}) \cap L^1((1, +\infty), x^{-\alpha} dx).$$

Fix  $C > 0$  and let  $\{n_i\}_{n \in \mathbf{N}} = \{n \in \mathbf{N} \mid v(n) > Cn^{\alpha-1}\}$  and  $\varepsilon_i = \inf\{\varepsilon > 0 \mid v(n_i + \varepsilon) \leq C(n_i + \varepsilon)^{\alpha-1}\}$ . By the continuity of  $v$  it holds  $v(n_i + \varepsilon_i) = C(n_i + \varepsilon_i)^{\alpha-1}$ . Moreover  $\varepsilon_i < \infty$  since  $x^{\alpha-1} \notin L^1((1, +\infty), x^{-\alpha} dx)$ . Notice that

$$\int_1^{+\infty} |v| x^{-\alpha} dx \geq C \sum_{i \in \mathbf{N}} \int_{n_i}^{n_i + \varepsilon_i} \frac{1}{x} dx = C \sum_{i \in \mathbf{N}} \log \left( \frac{n_i + \varepsilon_i}{n_i} \right).$$

Thus, since  $v \in L^1((1, +\infty), x^{-\alpha} dx)$ , the sum on the r.h.s. has to be finite. In particular we have that, for  $i$  sufficiently big,  $\log((n_i + \varepsilon_i)/n_i) \leq 1/n_i$ . Hence, there exists  $C' > 0$  such that  $\varepsilon_i \leq C' e^{1/n_i}$ , for  $i$  sufficiently big. Due to the  $1/2$ -Hölder continuity of  $v$  and the fact that  $x \mapsto x^{\alpha-1}$  is decreasing,

we get

$$e^{1/2n_i} \geq \frac{\varepsilon_i^{1/2}}{C'} \geq \frac{|v(n_i) - v(n_i + \varepsilon_i)|}{C_H C'} = \frac{|v(n_i) - C(n_i + \varepsilon_i)^{\alpha-1}|}{C_H C'} \geq \frac{|v(n_i) - C n_i^{\alpha-1}|}{C_H C'}.$$

Finally, this implies that there exists  $C''$  such that  $|v(n)| \leq C''(n^{\alpha-1} + e^{1/2n})$  for  $n$  sufficiently big, and hence that

$$\left| \frac{v(2n) - v(n)}{n^{1+\alpha}(2^{1+\alpha} - 1)} \right| \leq C'' \frac{2^{\alpha-1} + 1}{2^{\alpha+1} - 1} \frac{1}{n^2} + \frac{C''}{2^{\alpha+1} - 1} \frac{e^{1/4n} + e^{1/2n}}{n^{\alpha+1}} \longrightarrow 0,$$

completing the proof of (24), and hence of the theorem.  $\square$

We are now in a position to prove Theorem 1.13.

*Proof of Theorem 1.13.* By Propositions 3.11 and 3.16, we are left only to prove statement (iii)-(a) and the second part of (iii)-(b), i.e., the stochastic completeness of  $\Delta_N$  and  $\Delta_B$  at 0 when  $\alpha \in (-1, 1)$ .

Statement (iii)-(a) follows from [15, Theorem 1.6.4], since for  $\alpha \in (-1, 1)$  the Friedrichs extension (which is the minimal extension of  $\Delta|_{C_c^\infty(M)}$ ) is recurrent at  $\infty$ . To complete the proof it suffices to observe that, for these values of  $\alpha$ , it holds that  $1 \in H^1(M_0, d\omega) = D(\mathcal{E}^+|_{M_0})$  and clearly  $\mathcal{E}^+|_{M_0}(1, 1) = 0$ . By Theorem 3.6, this implies the recurrence of  $\mathcal{E}^+$  at 0. The recurrence of  $\mathcal{E}_B$  at 0 follows analogously, observing that 1 is also continuous on  $\mathcal{Z}$  and hence it belongs to  $D(\mathcal{E}_B|_{M_0})$ .  $\square$

## APPENDIX A. GEOMETRIC INTERPRETATION

In this appendix we prove Lemmata 1.1 and 1.2, and justify the geometric interpretation of Figure 1.

### A.1. Topology of $M_\alpha$ .

*Proof of Lemma 1.1.* By (3), it is clear that  $d : M_{\text{cylinder}} \times M_{\text{cylinder}} \rightarrow [0, +\infty)$  is symmetric, satisfies the triangular inequality and  $d(q, q) = 0$  for any  $q \in M_{\text{cylinder}}$ . Observe that the topology on  $M_{\text{cylinder}}$  is induced by the distance  $d_{\text{cylinder}}((x_1, \theta_1), (x_2, \theta_2)) = |x_1 - x_2| + |\theta_1 - \theta_2|$ . Here and henceforth, for any  $\theta_1, \theta_2 \in \mathbb{T}$  when writing  $\theta_1 - \theta_2$  we mean  $\theta_1 - \theta_2 \pmod{2\pi}$ . In order to complete the proof it suffices to show that for any  $\{q_n\}_{n \in \mathbf{N}} \subset M_{\text{cylinder}}$  and  $\bar{q} \in M_{\text{cylinder}}$  it holds

$$(25) \quad d(q_n, \bar{q}) \longrightarrow 0 \text{ if and only if } d_{\text{cylinder}}(q_n, \bar{q}) \longrightarrow 0.$$

In fact, this clearly implies that if  $d(q_1, q_2) = 0$  then  $q_1 = q_2$ , proving that  $d$  is a distance, and moreover proves that  $d$  and  $d_{\text{cylinder}}$  induce the same topology.

Assume that  $d(q_n, \bar{q}) \rightarrow 0$  for some  $\{q_n\}_{n \in \mathbf{N}} \subset M_{\text{cylinder}}$  and  $\bar{q} = (\bar{x}, \bar{\theta}) \in M_{\text{cylinder}}$ . In this case, for any  $n \in \mathbf{N}$  there exists a control  $u_n : [0, 1] \rightarrow \mathbf{R}^2$  such that  $\|u_n\|_{L^1([0,1], \mathbf{R}^2)} \rightarrow 0$  and that the associated trajectory  $\gamma_n(\cdot) = (x_n(\cdot), \theta_n(\cdot))$  satisfies  $\gamma_n(0) = q_n$  and  $\gamma_n(1) = \bar{q}$ . This implies that, for any  $t \in [0, 1]$

$$|x_n(t) - \bar{x}| \leq \int_0^t |u_1(t)| dt \leq \|u_n\|_{L^1([0,1], \mathbf{R}^2)} \longrightarrow 0.$$

Hence,  $x_n(t) \rightarrow \bar{x}$ . In particular, this implies that  $|x_n(t)| \leq \|u_n\|_{L^1([0,1], \mathbf{R}^2)} + |\bar{x}|$  for any  $t \in [0, 1]$ , and hence

$$\begin{aligned} |\theta_n(0) - \bar{\theta}| &\leq \int_0^1 |u_2(t)| |x_n(t)|^\alpha dt \leq (\|u_n\|_{L^1([0,1], \mathbf{R}^2)} + |\bar{x}|)^\alpha \int_0^1 |u_2(t)| dt \\ &\leq \|u_n\|_{L^1([0,1], \mathbf{R}^2)} (\|u_n\|_{L^1([0,1], \mathbf{R}^2)} + |\bar{x}|)^\alpha \longrightarrow 0. \end{aligned}$$

Here, when taking the limit, we exploited the fact that  $\alpha \geq 0$ . Thus also  $\theta_n(0) \rightarrow \bar{\theta}$ , and hence  $q_n = (x_n(0), \theta_n(0)) \rightarrow (\bar{x}, \bar{\theta}) = \bar{q}$  w.r.t.  $d_{\text{cylinder}}$ .

In order to complete the proof of (25), we now assume that for some  $q_n = (x_n, \theta_n)$  and  $\bar{q} = (\bar{x}, \bar{\theta})$  it holds  $d_{\text{cylinder}}(q_n, \bar{q}) \rightarrow 0$  and claim that  $d(q_n, \bar{q}) \rightarrow 0$ . We start by considering the case  $\bar{q} \notin \mathcal{Z}$ , and w.l.o.g. we assume  $\bar{q} \in M^+$ . Since  $M^+$  is open with respect to  $d_{\text{cylinder}}$ , up to subsequences it holds  $q_n \in M^+$ . Consider now the controls

$$u_n(t) = \begin{cases} 2(\bar{x} - x_n)(1, 0) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2(\bar{\theta} - \theta_n)|\bar{x}|^{-\alpha}(0, 1) & \text{if } \frac{1}{2} < t \leq 1, \end{cases}$$

A simple computation shows that each  $u_n$  steers the system from  $q_n$  to  $\bar{q}$ . The claim then follows from

$$d(q_n, \bar{q}) \leq \|u_n\|_{L^1([0,1], \mathbf{R}^2)} \leq |\bar{x} - x_n| + |\bar{\theta} - \theta_n||\bar{x}|^\alpha \leq (1 + |\bar{x}|^\alpha) d_{\text{cylinder}}(q_n, \bar{q}) \rightarrow 0.$$

Let now  $\bar{q} \in \mathcal{Z}$  and observe that w.l.o.g. we can assume  $q_n \notin \mathcal{Z}$  for any  $n \in \mathbf{N}$ . In fact, if this is not the case it suffices to consider  $\tilde{q}_n = (x_n + 1/n, \theta_n) \notin \mathcal{Z}$ , observe that  $d(q_n, \tilde{q}_n) \rightarrow 0$  and apply the triangular inequality. Then, we consider the following controls, steering the system from  $q_n$  to  $\bar{q}$ ,

$$v_n(t) = \begin{cases} 3((\bar{\theta} - \theta_n)^{1/2\alpha} - x_n)(1, 0) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ 3(\bar{\theta} - \theta_n)^{1/2}(0, 1) & \text{if } \frac{1}{3} < t \leq \frac{2}{3}, \\ 3(\theta_n - \bar{\theta})^{1/2\alpha}(1, 0) & \text{if } \frac{2}{3} < t \leq 1. \end{cases}$$

Since  $\bar{x} = 0$  and  $\alpha \geq 0$ , we have

$$d(q_n, \bar{q}) \leq \|v_n\|_{L^1([0,1], \mathbf{R}^2)} \leq |(\theta_n - \bar{\theta})^{1/2\alpha} - x_n| + |\bar{\theta} - \theta_n|^{1/2} + |\theta_n - \bar{\theta}|^{1/2\alpha} \rightarrow 0.$$

This proves (25) and hence the lemma.  $\square$

*Proof of Lemma 1.2.* By (4), it is clear that  $d : M_{\text{cone}} \times M_{\text{cone}} \rightarrow [0, +\infty)$  is symmetric, satisfies the triangular inequality and  $d(q, q) = 0$  for any  $q \in M_{\text{cone}}$ .

Observe that the topology on  $M_{\text{cone}}$  is induced by the following metric

$$d_{\text{cone}}((x_1, \theta_1), (x_2, \theta_2)) = \begin{cases} |x_1 - x_2| + |\theta_1 - \theta_2| & \text{if } x_1 x_2 > 0, \\ |x_1 - x_2| & \text{if } x_1 = 0 \text{ or } x_2 = 0, \\ |x_1 - x_2| + |\theta_1| + |\theta_2| & \text{if } x_1 x_2 < 0. \end{cases}$$

By symmetry, to show the equivalence of the topologies induced by  $d$  and by  $d_{\text{cone}}$ , it is enough to show that the two distances are equivalent on  $[0, +\infty) \times \mathbb{T}$ . Moreover, since by definition of  $g$  it is clear that  $d(q_1, q_2) = 0$  for any  $q_1, q_2 \in \mathcal{Z}$  and that  $d$  is equivalent to the Euclidean metric on  $(0, +\infty) \times \mathbb{T}$ , we only have to show that for any  $\{q_n\} \subset (0, +\infty) \times \mathbb{T}$ ,  $q_n = (x_n, \theta_n)$ , and  $\bar{q} = (0, \bar{\theta}) \in \mathcal{Z}$ , it holds that

$$(26) \quad d(q_n, \bar{q}) \rightarrow 0 \text{ if and only if } d_{\text{cone}}(q_n, \bar{q}) \rightarrow 0.$$

We start by assuming that  $d(q_n, \bar{q}) \rightarrow 0$ . Then, there exists  $\gamma_n : [0, 1] \rightarrow M$  such that  $\gamma_n(0) = q_n$  and  $\gamma_n(1) = \bar{q}$  and  $\int_0^1 \sqrt{g(\gamma_n(t), \gamma_n(t))} dt \rightarrow 0$ . This implies that

$$|x_n| \leq \int_0^1 \sqrt{g(\gamma_n(t), \gamma_n(t))} dt \rightarrow 0,$$

and thus that  $x_n \rightarrow 0$ . This suffices to prove that  $d_{\text{cone}}(q_n, \bar{q}) \rightarrow 0$ .

On the other hand, if  $d_{\text{cone}}(q_n, \bar{q}) \rightarrow 0$ , it suffices to consider the curves

$$\gamma_n(t) = \begin{cases} ((1 - 2t)x_n, \theta_n) & \text{if } 0 \leq t < \frac{1}{2}, \\ (0, \theta_n + (2t - 1)(\bar{\theta} - \theta_n)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

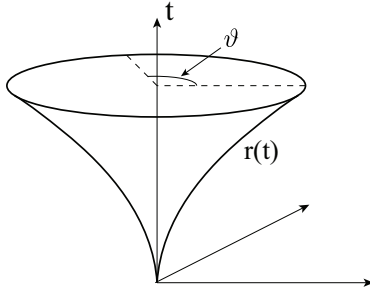


FIGURE 2. The surface of revolution of Proposition A.1 with  $\alpha = -2$ , i.e.  $r(t) = t^2$ .

Clearly  $\gamma_n$  is Lipschitz and  $\gamma_n(0) = q_n$  and  $\gamma_n(1) = \bar{q}$ . Finally, since  $g|_{\mathcal{Z}} = 0$ , the proof is completed by

$$d(q_n, \bar{q}) \leq \int_0^1 \sqrt{g_{\gamma_n(t)}(\dot{\gamma}_n(t), \dot{\gamma}_n(t))} dt = \int_0^{\frac{1}{2}} \sqrt{g_{\gamma_n(t)}((-2x_n, 0), (-2x_n, 0))} dt = x_n \longrightarrow 0.$$

□

**A.2. Surfaces of revolution.** Given two manifolds  $M$  and  $N$ , endowed with two (possibly semi-definite) metrics  $g^M$  and  $g^N$ , we say that  $M$  is  $C^1$ -isometric to  $N$  if and only if there exists a  $C^1$ -diffeomorphism  $\Phi : M \rightarrow N$  such that  $\Phi^*g_N = g_M$ . Here  $\Phi^*$  is the pullback of  $\Phi$ . Recall that, in matrix notation, for any  $q \in M$  it holds

$$(27) \quad (\Phi^*g^N)_q(\xi, \eta) = (J_\Phi)^T g_{\Phi(q)}^M J_\Phi(\xi, \eta).$$

Here  $J_\Phi$  is the Jacobian matrix of  $\Phi$ .

We have the following.

**Proposition A.1.** *If  $\alpha < -1$  the manifold  $M_\alpha$  is  $C^1$ -isometric to a surface of revolution  $\mathcal{S} = \{(t, r(t) \cos \vartheta, r(t) \sin \vartheta) \mid t \in \mathbf{R}, \vartheta \in \mathbb{T}\} \subset \mathbf{R}^3$  with profile  $r(t) = |t|^{-\alpha} + \mathcal{O}(t^{-2\alpha})$  as  $|t| \downarrow 0$  (see figure 2), endowed with the metric induced by the embedding in  $\mathbf{R}^3$ .*

*If  $\alpha = -1$ ,  $M_\alpha$  is globally  $C^1$ -isometric to the surface of revolution with profile  $r(t) = t$ , endowed with the metric induced by the embedding in  $\mathbf{R}^3$ .*

*Proof.* For any  $r \in C^1(\mathbf{R})$ , consider the surface of revolution  $\mathcal{S} = \{(t, r(t) \cos \vartheta, r(t) \sin \vartheta) \mid t > 0, \vartheta \in \mathbb{T}\} \subset \mathbf{R}^3$ . By standard formulae of calculus, we can calculate the corresponding (continuous) semi-definite Riemannian metric on  $\mathcal{S}$  in coordinates  $(t, \vartheta) \in \mathbf{R} \times \mathbb{T}$  to be

$$g_{\mathcal{S}}(t, \vartheta) = \begin{pmatrix} 1 + r'(t)^2 & 0 \\ 0 & r^2(t) \end{pmatrix}.$$

Let now  $\alpha < -1$  and consider the  $C^1$  diffeomorphism  $\Phi : (x, \theta) \in \mathbf{R} \times \mathbb{T} \mapsto (t(x), \vartheta(\theta)) \in \mathcal{S}$  defined as the inverse of

$$(28) \quad \Phi^{-1}(t, \vartheta) = \begin{pmatrix} x(t) \\ \theta(\vartheta) \end{pmatrix} = \begin{pmatrix} \int_0^t \sqrt{1 + r'(s)^2} ds \\ \vartheta \end{pmatrix}.$$

Observe that  $\Phi$  is well defined due to the fact that  $r'$  is bounded near 0. Since  $\partial_t(\Phi^{-1}) = \partial_t x(t) = \sqrt{1 + r'(t)^2}$ , by (27) the metric is transformed in

$$\Phi^*g_{\mathcal{S}}(x, \theta) = (J_\Phi^{-1})^T g_{\mathcal{S}}(\Phi(x, \theta)) J_\Phi^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r(\Phi(x, \theta))^2 \end{pmatrix}.$$

We now claim that, if  $\alpha < -1$ , there exists  $r(\cdot) \in C^1(\mathbf{R})$  such that  $r(t(x)) = |x|^{-\alpha}$  near  $\{x = 0\}$ , given by the expression

$$r(t) = \begin{cases} t^{-\alpha} + \mathcal{O}(t^{-2\alpha}), & \text{if } t \geq 0, \\ -(-t)^{-\alpha} + \mathcal{O}(t^{-2\alpha}) & \text{if } t < 0. \end{cases}$$

Notice that, this function generates the same surface of revolution as  $r(t) = |t|^{-\alpha} + \mathcal{O}(t^{-2\alpha})$ , but is  $C^1$  in 0 while the latter is not.

Take  $r(t(x)) = |x|^{-\alpha}$ , and assume w.l.o.g. that  $t$ , and hence  $x(t)$ , is positive. Thus, we get

$$(29) \quad r'(t) = \partial_t r(t(x(t))) = \partial_t (x(t))^{-\alpha} = -\alpha (x(t))^{-(\alpha+1)} \partial_t x(t) = -\alpha (x(t))^{-(\alpha+1)} \sqrt{1 + r'(t)^2}.$$

Since  $x(0) = 0$  and  $\alpha > -1$ , this implies that  $r'(0) = 0$ . Finally, a Taylor expansion around  $t = 0$  yields

$$r(t) = (x(t))^{-\alpha} = (t \partial_t x(0) + \mathcal{O}(t^2))^{-\alpha} = t^{-\alpha} (1 + r'(0)^2)^{-\alpha/2} + \mathcal{O}(t^{-2\alpha}) = t^{-\alpha} + \mathcal{O}(t^{-2\alpha}),$$

completing the proof of the claim.

To complete the proof of the proposition, let  $\alpha = -1$ . In this case, by letting  $r(t) = t$ , the metric on the surface of revolution is

$$g_{\mathcal{S}}(t, \vartheta) = \begin{pmatrix} 2 & 0 \\ 0 & t^2 \end{pmatrix}.$$

Consider the diffeomorphism  $\Psi : (x, \theta) \in \mathbf{R} \times \mathbb{T} \mapsto (t, \vartheta) \in \mathcal{S}$  defined as

$$(30) \quad \Psi(x, \theta) = \sqrt{2} \begin{pmatrix} x \\ \theta \end{pmatrix}.$$

Then the statement follows from the following computation,

$$\Phi^* g_{\mathcal{S}}(x, \theta) = (J_{\Psi}^{-1})^T g_{\mathcal{S}}(\Psi(x, \theta)) J_{\Psi}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r(\Psi(x, \theta))^2/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}.$$

□

**Remark A.2.** If  $\alpha > -1$  we cannot have a result like the above, since the change of variables (28) is no more regular. In fact, the function  $r(t) = t^{-\alpha}$  has an unbounded first derivative near 0. On the other hand, if  $\alpha$  is a negative integer, by iterating (29) follows that the change of variables (28) (or (30) for  $\alpha = -1$ ) is indeed  $C^\infty$ . A similar argument can be used to prove that, if  $\alpha < -k$  for some  $k \in \mathbf{N}$ , the change of variable is of class  $C^k$ .

## APPENDIX B. COMPLEX SELF-ADJOINT EXTENSIONS

The natural functional setting for the Schrödinger equation on  $M_\alpha$  is the space of square integrable complex-valued function  $L^2_{\mathbb{C}}(M, d\omega)$ . Recall that a self-adjoint extension  $B$  of an operator  $A$  over  $L^2_{\mathbb{C}}(M, d\omega)$  is a *real self-adjoint extensions* if and only if  $u \in D(B)$  implies  $\bar{u} \in D(B)$  and  $B(\bar{u}) = \overline{Bu}$ . The self-adjoint extension of  $A$  over  $L^2(M, d\omega)$  are exactly the restrictions to this space of the real self-adjoint extension of  $A$  over  $L^2_{\mathbb{C}}(M, d\omega)$ .

All the theory of Section 2 extends to the complex case, in particular, we have the following generalization of Theorem 2.6.

**Theorem B.1** (Theorem 13.3.1 in [27]). *Let  $A$  be the Sturm-Liouville operator on  $L^2_{\mathbb{C}}(J, w(x)dx)$  defined in (11). Then*

$$n_+(A) = n_-(A) = \#\{\text{limit-circle endpoints of } J\}.$$

*Assume now that  $n_+(A) = n_-(A) = 2$ , and let  $a$  and  $b$  be the two limit-circle endpoints of  $J$ . Moreover, let  $\phi_1, \phi_2 \in D_{\max}(A)$  be linearly independent modulo  $D_{\min}(A)$  and normalized by*

$[\phi_1, \phi_2](a) = [\phi_1, \phi_2](b) = 1$ . Then,  $B$  is a self-adjoint extension of  $A$  over  $L^2_{\mathbb{C}}(J, w(x)dx)$  if and only if  $Bu = A^*u$ , for any  $u \in D(B)$ , and one of the following holds

(1) Disjoint dynamics: there exists  $c_+, c_- \in (-\infty, +\infty]$  such that  $u \in D(B)$  if and only if

$$[u, \phi_1](0^+) = c_+[u, \phi_2](0^+) \quad \text{and} \quad [u, \phi_1](0^-) = c_-[u, \phi_2](0^-).$$

(2) Mixed dynamics: there exist  $K \in SL_2(\mathbf{R})$  and  $\gamma \in (-\pi, \pi]$  such that  $u \in D(B)$  if and only if

$$U(0^-) = e^{i\gamma} K U(0^+), \quad \text{for } U(x) = \begin{pmatrix} [u, \phi_1](x) \\ [u, \phi_2](x) \end{pmatrix}.$$

Finally,  $B$  is a real self-adjoint extension if and only if it satisfies (1) the disjoint dynamic or (2) the mixed dynamic with  $\gamma = 0$ .

As a consequence of Theorem B.1, we get a complete description of the essential self-adjointness of  $\Delta|_{C^\infty(M)}$  over  $L^2_{\mathbb{C}}(M, d\omega)$ , extending Theorem 1.6, and of the complex self-adjoint extensions of  $\widehat{\Delta}_0$ , extending Theorem 1.8.

**Theorem B.2.** Consider  $M_\alpha$  for  $\alpha \in \mathbf{R}$  and the corresponding Laplace-Beltrami operator  $\Delta|_{C^\infty(M)}$  as an unbounded operator on  $L^2_{\mathbb{C}}(M, d\omega)$ . Then it holds the following.

- (i) If  $\alpha \leq -3$  then  $\Delta|_{C^\infty(M)}$  is essentially self-adjoint;
- (ii) if  $\alpha \in (-3, -1]$ , only the first Fourier component  $\widehat{\Delta}_0$  is not essentially self-adjoint;
- (iii) if  $\alpha \in (-1, 1)$ , all the Fourier components of  $\Delta|_{C^\infty(M)}$  are not essentially self-adjoint;
- (iv) if  $\alpha \geq 1$  then  $\Delta|_{C^\infty(M)}$  is essentially self-adjoint.

**Theorem B.3.** Let  $D_{\min}(\widehat{\Delta}_0)$  and  $D_{\max}(\widehat{\Delta}_0)$  be the minimal and maximal domains of  $\widehat{\Delta}_0|_{C^\infty(\mathbf{R} \setminus \{0\})}$  on  $L^2_{\mathbb{C}}(\mathbf{R} \setminus \{0\}, |x|^{-\alpha})$ , for  $\alpha \in (-3, 1)$ . Then,

$$D_{\min}(\widehat{\Delta}_0) = \text{closure of } C_c^\infty(\mathbf{R} \setminus \{0\}) \text{ in } H^2_{\mathbb{C}}(\mathbf{R} \setminus \{0\}, |x|^{-\alpha} dx)$$

$$D_{\max}(\widehat{\Delta}_0) = \{u = u_0 + u_D^+ \phi_D^+ + u_N^+ \phi_N^+ + u_D^- \phi_D^- + u_N^- \phi_N^- : u_0 \in D_{\min}(\widehat{\Delta}_0) \text{ and } u_D^\pm, u_N^\pm \in \mathbb{C}\},$$

Moreover,  $A$  is a self-adjoint extension of  $\widehat{\Delta}_0$  if and only if  $Au = (\widehat{\Delta}_0)^*u$ , for any  $u \in D(A)$ , and one of the following holds

(i) Disjoint dynamics: there exist  $c_+, c_- \in (-\infty, +\infty]$  such that

$$D(A) = \{u \in D_{\max}(\widehat{\Delta}_0) : u_N^+ = c_+ u_D^+ \text{ and } u_N^- = c_- u_D^+\}.$$

(ii) Mixed dynamics: there exist  $K \in SL_2(\mathbf{R})$  and  $\gamma \in (-\pi, \pi]$  such that

$$D(A) = \{u \in D_{\max}(\widehat{\Delta}_0) : (u_D^-, u_N^-) = e^{i\gamma} K (u_D^+, u_N^+)^T\}.$$

Finally, the Friedrichs extension  $(\widehat{\Delta}_0)_F$  is the one corresponding to the disjoint dynamics with  $c_+ = c_- = 0$  if  $\alpha \leq -1$  and with  $c_+ = c_- = +\infty$  if  $\alpha > -1$ .

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**Part III.**  
**Abstracts**



## 5. Abstract

We study two problems arising from geometric control theory, regarding control-affine systems  $\dot{q} = f_0(q) + \sum_{j=1}^m u_j f_j(q)$ . The main contributions are those given in the three articles attached to the thesis.

The first part of this dissertation is devoted to the problem of complexity of non-admissible trajectories in the case of control-affine systems satisfying the strong Hörmander condition. This problem amounts to quantify the cost of approximating a non-admissible curve up to a certain precision. In the particular case of sub-Riemannian (or driftless) systems – i.e., with  $f_0 \equiv 0$  – various notions of complexities are already defined and rather well-understood. Our contributions appear in two papers:

- P1. D. Prandi, *Hölder continuity of the value function for control-affine systems*, arXiv:1304.6649 [math.OC].
- P2. F. Jean, D. Prandi, *Complexity of control-affine motion planning*, arXiv:1309.2571 [math.OC].

In the first, as a preliminary step, we prove a result in the same spirit as the Ball-Box theorem for sub-Riemannian systems, in the context of control-affine systems equipped with the  $L^1$  cost. The techniques used are based on a reduction of the control-affine system to a linear but time-dependent one. As a byproduct, besides consequences on the Hölder regularity of the associated value function, we get the continuity of the value function of time-dependent systems of the form  $\dot{q} = \sum_{j=1}^m u_j f_j^t(q)$ .

Paper P2. is the core of this part of the thesis. Here, we present various definitions of complexity, as functions of the curve that is approximated, and of the precision of the approximation. Due to the lack of time-rescaling invariance of these systems, we consider geometric and parametrized curves separately. Then, using the instruments developed in (P1.), we give some asymptotic estimates for these quantities.

In the second part of the thesis we consider the family of two-dimensional driftless control systems in the form  $\dot{q} = u_1 f_1(q) + u_2 f_2(q)$  with  $q = (x, \theta) \in \mathbb{R} \times \mathbb{S}^1$ , where  $f_1(x, \theta) = (1, 0)$  and  $f_2(x, \theta) = (0, |x|^\alpha)$  for a parameter  $\alpha \in \mathbb{R}$ . This family of control system describes dynamics on cone-like manifolds, on the cylinder and on Grushin-type manifolds. Due to the control vector fields becoming collinear on  $\{x = 0\}$ , the intrinsic Laplace-Beltrami operator  $\mathcal{L}$  presents some singularities. We study how the singular set affects the evolution of the heat and of a quantum particle (described by the Schrödinger equation) associated with  $\mathcal{L}$  are affected by this singularity. This research appears in:

- P3. U. Boscain, D. Prandi, *The heat and Schrödinger equations on conic and anticonic-type surfaces*, arXiv:1305.5271 [math.AP].

Here, we show that  $\mathcal{L}$  is essentially self-adjoint if and only if  $\alpha \notin (-3, 1)$ . In this case the only self-adjoint extension is the Friedrichs extension  $\mathcal{L}_F$ , that does not allow communication through the singular set  $\{x = 0\}$  both for the heat and for a quantum particle. For  $\alpha \in (-3, -1]$  we show that for the Schrödinger equation only the average on  $\theta$  of the wave function can cross the singular set, while the solutions of the only Markovian extension of the heat equation (which indeed is  $\mathcal{L}_F$ ) cannot. For  $\alpha \in (-1, 1)$  we prove that there exists a canonical self-adjoint extension  $\mathcal{L}_B$ , called bridging extension, which is Markovian and allows the complete communication through the singularity (both of the heat and of a quantum particle). Also, we study the stochastic completeness (i.e., conservation of the  $L^1$  norm for the heat equation) of the Markovian extensions  $\mathcal{L}_F$  and  $\mathcal{L}_B$ , proving that  $\mathcal{L}_F$  is stochastically complete at the singularity if and only if  $\alpha \leq -1$ , while  $\mathcal{L}_B$  is always stochastically complete at the singularity.

## 6. Résumé

On étudie deux problèmes qui ont leur origine dans la théorie du contrôle géométrique et qui concernent les systèmes de contrôle avec dérive  $\dot{q} = f_0(q) + \sum_{j=1}^m u_j f_j(q)$ . Les contributions originales sont celles contenues dans les trois articles insérés dans la thèse.

La première partie de ce manuscrit traite du problème de la complexité des courbes non-admissibles pour les systèmes de contrôle avec dérive qui satisfont l'hypothèse d'Hörmander forte. Il s'agit de quantifier pour ces systèmes le coût nécessaire à l'approximation d'un chemin non-admissible avec une précision donnée. Dans le cas particulier des systèmes sous-riemanniens, c'est-à-dire sans dérive, plusieurs notions de complexité ont déjà été définies et sont maintenant bien comprises. Nos contributions sont contenues dans deux papiers:

P1. D. Prandi, *Hölder continuity of the value function for control-affine systems*, arXiv:1304.6649 [math.OC].

P2. F. Jean, D. Prandi, *Complexity of control-affine motion planning*, arXiv:1309.2571 [math.OC].

Dans le premier, on généralise le théorème du Ball-Box valable pour les systèmes sous-riemanniens aux systèmes avec dérive, le coût étant la norme  $L^1$ . Ce résultat permet d'obtenir des estimations de type Hölder pour la fonction valeur associée et constitue un outil indispensable pour P2.. Les techniques que l'on utilise sont basées sur une réduction du système de contrôle avec dérive à un système sans dérive mais qui dépend explicitement du temps.

Ensuite, dans le papier P2., on présente plusieurs définitions de complexité, comme fonctions de la courbe à approximer et de la précision requise. En particulier, le système n'étant pas invariant par changement de paramétrage temporel, on est forcé de considérer séparément les courbes géométriques et les courbes paramétrées. En utilisant les résultats obtenus dans P1., on arrive à donner des estimations asymptotiques de ces quantités quand la précision se rapproche de zéro.

Dans la deuxième partie de la thèse on considère une famille de systèmes de contrôle sans dérive en dimension 2 de la forme  $\dot{q} = u_1 f_1(q) + u_2 f_2(q)$  où  $q = (x, \theta) \in \mathbb{R} \times \mathbb{S}^1$ ,  $f_1(x, \theta) = (1, 0)$  et  $f_2(x, \theta) = (0, |x|^\alpha)$ , avec un paramètre  $\alpha \in \mathbb{R}$ . Selon la valeur de  $\alpha$ , cette famille décrit une dynamique sur des variétés coniques, sur le cylindre ou sur une variété de type Grushin. On s'intéresse à l'opérateur de Laplace-Beltrami associé ainsi qu'à l'évolution de la chaleur et des particules quantiques qu'il définit. On étudie,

plus particulièrement, l'effet qu'a l'ensemble singulier  $\{x = 0\}$ , où les champs de vecteurs contrôlés deviennent colinéaires, sur ces évolutions. Cette recherche fait l'objet du papier,

P3. U. Boscain, D. Prandi, *The heat and Schrödinger equations on conic and anticonic-type surfaces*, arXiv:1305.5271 [math.AP].

On montre que  $\mathcal{L}$  est essentiellement auto-adjoint si et seulement si  $\alpha \notin (-3, 1)$ . Dans ce cas-là, la seule extension auto-adjointe est l'extension de Friedrichs  $\mathcal{L}_F$ , qui ne permet pas la communication à travers l'ensemble singulier  $\{x = 0\}$  aussi bien pour la chaleur que pour les particules quantiques. Pour  $\alpha \in (-3, -1]$  on montre que pour l'équation de Schrödinger seulement la moyenne sur  $\theta$  de la fonction d'onde peut traverser l'ensemble singulier, tandis que les solutions de la seule extension markovienne de l'équation de la chaleur, qui est  $\mathcal{L}_F$ , ne le peuvent pas. Finalement, pour  $\alpha \in (-1, 1)$  on montre qu'il existe une extension auto-adjointe canonique  $\mathcal{L}_B$ , qu'on appelle extension "bridging", qui est markovienne et permet une communication complète à travers la singularité (aussi bien pour la chaleur que pour les particules quantiques). En outre, on étudie la complexité stochastique des extensions markoviennes  $\mathcal{L}_F$  et  $\mathcal{L}_B$ , c'est-à-dire la conservation de la norme  $L^1$  pour l'équation de la chaleur, et on prouve que  $\mathcal{L}_F$  est stochastiquement complète sur la singularité si et seulement si  $\alpha < -1$  et que  $\mathcal{L}_B$  est toujours stochastiquement complète sur la singularité.