Deux applications arithmétiques des travaux d'Arthur

Olivier Taïbi

École Doctorale 447

Thèse de doctorat
Discipline : Mathématiques

présentée par

Olivier Taïbi

Deux applications arithmétiques des travaux d'Arthur

dirigée par Gaëtan Chenevier

Soutenue le 19 septembre 2014 devant le jury composé de :

Gaëtan Chenevier	École Polytechnique	Directeur de thèse
Laurent Clozel	Université Paris XI	Examinateur
Guy Henniart	Université Paris XI	Examinateur
Colette Mœglin	Université Paris VI	Rapporteuse
David Renard	École Polytechnique	Examinateur
Jean-Loup Waldspurger	Université Paris VI	Examinateur

Rapporteurs absents lors de la soutenance :

Toby Gee	Imperial College London	Rapporteur
Benedict Gross	Harvard University	Rapporteur

Remerciements

Je tiens en premier lieu à exprimer ma reconnaissance à Gaëtan Chenevier pour avoir dirigé ma thèse. Travailler sous sa direction fut un réel plaisir. Au-delà de ses grandes connaissances et de sa disponibilité, j'ai bénéficié de son enthousiasme et de ses encouragements. Sa hauteur de vue lui permet de motiver des objets sophistiqués par des applications arithmétiques simples et élégantes. J'ai été frappé par cette qualité, précieuse pour un thésard, lorsque je l'ai vu exposer ses travaux avec Laurent Clozel sur les corps de nombres peu ramifiés.

Toby Gee, Benedict Gross et Colette Mœglin ont accepté de rapporter ma thèse. Il s'agit d'une lourde tâche et je les en remercie vivement. Laurent Clozel, Guy Henniart, David Renard et Jean-Loup Waldspurger me font l'honneur de faire partie du jury, je voudrais pour cela leur exprimer ma gratitude.

Je souhaite à nouveau remercier Colette Mœglin et Jean-Loup Waldspurger. Alors que je cherchais une démonstration trop compliquée du lemme 2.6.1.1, ils m'ont gentiment expliqué qu'il découlait simplement d'un de leurs lemmes. La seconde partie de cette thèse a nécessité d'effectuer des calculs parallèles, mobilisant presque tous les ordinateurs du DMA. Je remercie donc le SPI pour sa bienveillance.

J'ai eu la chance d'enseigner à l'ENS pendant ces trois dernières années. Cette expérience fut passionnante, et a permis un équilibre mathématique. Elle m'a également placé dans l'atmosphère conviviale et chaleureuse du DMA. Je remercie donc tous mes collègues et amis avec qui j'ai passé d'agréables moments, mathématiques ou non ; en particulier Arnaud, Henri et Quentin.

J'ai la chance de confondre travail et passion. C'est l'une des nombreuses choses que je dois à mes parents. Je profite donc de cette occasion pour leur exprimer ma gratitude. Merci enfin à Chloé, grâce à qui cette thèse fut préparée dans le bonheur.

Table des matières

Partie 1
Introduction

Dans cette introduction destinée aux mathématiciens non spécialistes, nous expliquons dans quel contexte s'inscrivent les résultats de cette thèse. Le titre annonce qu'il s'agit d'arithmétique, ce qui n'est peut-être pas évident à la lecture du texte. L'un des buts de cette branche des mathématiques est l'étude des équations *diophantiennes*, c'est-à-dire les équations "à coefficients *entiers*" dont on cherche les solutions *entières*. Un problème formulé aussi vaguement ne peut admettre de solution générale, et l'histoire de l'arithmétique est jalonnée par les découvertes d'outils permettant d'étudier seulement certaines classes d'équations diophantiennes. Citons deux outils fondamentaux pour l'étude des systèmes d'équations polynomiales, c'est-à-dire de la forme

$$\begin{cases} P_1(x_1, \dots, x_m) = 0 \\ \qquad \vdots \\ P_n(x_1, \dots, x_m) = 0 \end{cases}$$

où chaque P_i est un polynôme à coefficients entiers en les variables $X_1, \dots, X_m$. Afin de simplifier le problème, contentons-nous de travailler sur le *corps* $\mathbb{Q}$ des nombres rationnels plutôt que sur son sous-anneau $\mathbb{Z}$ des nombres entiers.

Galois nous enseigne que les solutions rationnelles d'un tel système d'équations sont les solutions $(x_1, \dots, x_m) \in \overline{\mathbb{Q}}^m$, où $\overline{\mathbb{Q}}$ désigne une clôture algébrique de $\mathbb{Q}$, qui sont fixées par $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, le groupe de Galois absolu de $\mathbb{Q}$ qui agit sur $\overline{\mathbb{Q}}$. L'intérêt de ce point de vue est que, comme souvent en mathématiques, il est plus aisé d'étudier les propriétés de solutions dont l'existence est connue a priori que de s'attaquer directement au problème d'existence. Plus généralement, il est naturel de voir un objet défini sur $\mathbb{Q}$ comme un objet défini sur $\overline{\mathbb{Q}}$ et muni d'une action de $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Notons toutefois que l'utilisation de nombres algébriques sur $\mathbb{Q}$ remonte à Gauss qui introduisit en 1832 (avant la publication des résultats de Galois) l'anneau des "entiers de Gauss" $\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}\}$ dans le but de formuler la loi de réciprocité biquadratique.

Une autre approche consiste à compléter le corps $\mathbb{Q}$ pour une de ses valuations v. D'après un théorème d'Ostrowski, à un exposant inessentiel près, v est soit la valuation archimédienne usuelle, soit la valuation p-adique pour un nombre premier p. Dans le corps complété $\mathbb{Q}_v$ correspondant, on dispose d'outils analytiques (théorème des valeurs intermédiaires, lemme de Hensel ...) simplifiant grandement la résolution du système d'équations. Afin de prendre en compte toutes ces valuations (également appelées "places") à la fois, on considère l'anneau des *adèles*

$$\mathbb{A} = \prod_v{}' \mathbb{Q}_v = \mathbb{R} \times \prod_{p \text{ premier}}{}' \mathbb{Q}_p$$

qui est le produit restreint de ces corps et qui contient $\mathbb{Q}$ comme sous-anneau, plongé diagonalement. Ici on a noté $\mathbb{Q}_p$ le complété $\mathbb{Q}_v$ de $\mathbb{Q}$ pour la valuation p-adique v. L'existence d'une solution à coefficients rationnels implique donc l'existence d'une solution à coefficients adèliques. Toute la difficulté consiste à aller dans l'autre sens, c'est-à-dire à déterminer les obstructions "globales" contrôlant l'existence d'une solution rationnelle lorsque l'on suppose l'existence d'une solution adèlique. Par exemple le théorème de Hasse-Minkowski implique que dans le cas d'une seule équation quadratique, il n'y a pas de telle obstruction. On peut en déduire le théorème de Legendre affirmant qu'un entier est somme de trois carrés si et seulement si il n'est pas de la forme $4^a(8b+7)$ pour des entiers a et b.

Outre ces outils fondamentaux, la notion récurrente dans cette thèse est celle de représentation automorphe, qui reformule et généralise celle de forme modulaire cuspidale propre pour les opérateurs de Hecke.

1.1 Formes modulaires

Soit $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ le demi-plan de Poincaré. Le groupe d'automorphismes de cette courbe complexe s'identifie à $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ via l'action définie par

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \qquad \text{pour } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ et } z \in \mathcal{H}.$$

Soit $\Gamma = \mathrm{SL}_2(\mathbb{Z})$; il s'agit d'un sous-groupe discret de $\mathrm{SL}_2(\mathbb{R})$. Le quotient $\Gamma \backslash \mathcal{H}$ a attiré l'attention des mathématiciens car il paramètre les courbes elliptiques (définies analytiquement), via l'application qui à $z \in \mathcal{H}$ associe la courbe elliptique $\mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$. Nous ne détaillerons pas davantage ce point de vue.

Definition 1.1.0.1. *Soit k un entier. On dit qu'une fonction holomorphe $f : \mathcal{H} \to \mathbb{C}$ est une* forme modulaire de poids k *si :*

- *Pour tout $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ et tout $z \in \mathcal{H}$, on a $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$. En choisissant $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on constate que f est fonction de $q = e^{2i\pi z}$, c'est-à-dire qu'il existe une unique fonction holomorphe $F : D(0,1) \smallsetminus \{0\} \to \mathbb{C}$ telle que $f(z) = F(q)$.*
- *La fonction F se prolonge en une fonction holomorphe sur $D(0,1)$. Cela revient à demander que f soit bornée sur la bande $\{z \in \mathbb{C} \mid |\mathrm{Re}(z)| \leq 1/2$ et $\mathrm{Im}(z) \geq 1\}$.*

On note $M_k(\Gamma)$ le $\mathbb{C}$-espace vectoriel des formes modulaires de poids k.

Etant donnée une fonction holomorphe F sur $D(0,1)$, la fonction $f : z \mapsto F(e^{2i\pi z})$ est une forme modulaire de poids k si et seulement si pour tout $z \in \mathcal{H}$ on a $f(-1/z) = z^k f(z)$. Cela résulte du fait que le groupe Γ est engendré par $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ et $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Dans cette définition il peut être utile de remplacer Γ par un sous-groupe convenable, mais nous ne détaillerons pas les complications que cela entraîne. En interprétant $M_k(\Gamma)$ comme (un sous-espace de) l'espace vectoriel complexe des sections globales d'un fibré en droites holomorphe sur une surface de Riemann *compacte*, on obtient que $M_k(\Gamma)$ est de dimension finie.

On peut aller plus loin et expliciter les espaces $M_k(\Gamma)$. Pour $k > 2$, la *série d'Eisenstein*

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\smallsetminus\{(0,0)\}} \frac{1}{(m+nz)^k}$$

converge uniformément sur tout compact de $\mathcal{H}$, et il est formel de vérifier qu'il s'agit d'une forme modulaire de poids k, non nulle si et seulement si k est pair. De plus il est possible d'expliciter les coefficients de G_k vue comme fonction de $q = e^{2i\pi z}$ dans son développement en série entière autour de 0. Il est remarquable qu'à un scalaire près, ces coefficients sont entiers. La théorie des séries d'Eisenstein permet donc de produire des formes modulaires tout à fait explicites.

D'autre part, la formule de Riemann-Roch ou la formule des traces permettent de calculer la dimension de chaque $M_k(\Gamma)$. Dans notre cas ($\Gamma = \mathrm{SL}_2(\mathbb{Z})$) on a formellement :

$$\sum_k \dim_{\mathbb{C}}\left(M_k(\Gamma)\right) t^k = \frac{1}{(1-t^4)(1-t^6)}$$

ce qui traduit le fait que la $\mathbb{C}$-algèbre commutative graduée $\bigoplus_k M_k(\Gamma)$ est librement engendrée par G_4 et G_6.

Cette approche concrète permet de démontrer des identités miraculeuses qui comptent le nombre de solutions de certaines équations diophantiennes. Donnons un exemple simple avec la fonction

$$r_4(m) = \mathrm{card}\left\{(x_1,\ldots,x_4)\in\mathbb{Z}^4 \mid x_1^2+\cdots+x_4^2 = m\right\}$$

qui compte le nombre de représentations de l'entier m comme somme de quatre carrés. Introduisons la série génératrice

$$\theta_4(z) = \sum_{m\geq 0} r_4(m)q^m = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^4$$

où $q = e^{2i\pi z}$, qui définit une fonction holomorphe sur $\mathcal{H}$. Il est clair que $\theta_4(z+1) = \theta_4(z)$ et grâce à la *formule sommatoire de Poisson* on a en outre

$$\theta_4(-1/4z) = -4z^2\theta_4(z)$$

ce qui implique que θ_4 est une forme modulaire de poids 2 pour le sous-groupe

$$\Gamma_1(4) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma,\ a \equiv d \equiv 1 \pmod 4 \text{ et } c \equiv 0 \pmod 4\right\}$$

de Γ. D'autre part l'espace $M_2(\Gamma_1(4))$ est de dimension 2 et la comparaison de θ_4 avec des séries d'Eisenstein permet de démontrer la formule de Jacobi :

$$r_4(m) = \begin{cases} 8 \sum\limits_{d|m} d & \text{si } m \text{ est impair,} \\ 24 \sum\limits_{\substack{d|m \\ d \text{ impair}}} d & \text{si } m \text{ est pair.} \end{cases}$$

Les formes modulaires, dont la définition a plutôt une saveur analytique, ont donc des liens avec l'arithmétique. Loin d'être anecdotique, la méthode ci-dessus admet une vaste généralisation (correspondance thêta) qui fait l'objet de recherches actuelles, mais dont il ne sera pas question dans cette thèse.

1.2 Formes automorphes et représentations galoisiennes

En 1937 Hecke définit, pour chaque nombre premier p, un opérateur $T_p : M_k(\Gamma) \to M_k(\Gamma)$. Ces opérateurs commutent entre eux et ont la propriété d'être auto-adjoints pour un produit scalaire hermitien convenable. Il est donc naturel de vouloir diagonaliser simultanément ces opérateurs. Cela suggère que les formes modulaires propres pour les opérateurs de Hecke (et s'annulant en $q = 0$, on dit d'une telle forme qu'elle est cuspidale) sont des vecteurs bien particuliers dans des *représentations irréductibles* d'un groupe adèlique, qui se trouve être $\mathrm{GL}_2(\mathbb{A})$. Une représentation irréductible convenable de $\mathrm{GL}_2(\mathbb{A})$ se décompose en un produit tensoriel restreint $\bigotimes_v' \pi_v$ où π_v est une représentation irréductible de $\mathrm{GL}_2(\mathbb{Q}_v)$, où v parcourt l'ensemble des valuations de $\mathbb{Q}$. Cette décomposition généralise le fait suivant : si G_1 et G_2 sont deux groupes finis, les représentations irréductibles de $G_1 \times G_2$ sont exactement les produits tensoriels de représentations irréductibles de G_1 et G_2. Les représentations de $\mathrm{GL}_2(\mathbb{A})$ correspondant aux formes modulaires sont celles qui interviennent dans l'espace de *formes automorphes* $L^2(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ et telles que pour la valuation archimédienne notée v, la représentation π_v du groupe $\mathrm{GL}_2(\mathbb{R})$ est "algébrique et régulière". Ce point de vue plus abstrait a au moins deux avantages : il permet d'utiliser les techniques de la théorie des représentations, et il se généralise à d'autres groupes que GL_2 pour lesquels il n'y a pas toujours d'analogues aux formes modulaires.

Les opérateurs de Hecke entrent dans la description d'un lien profond entre formes modulaires (ou plus généralement, représentations automorphes) et *représentations galoisiennes.* Afin de présenter ces dernières, revenons à un système d'équations polynomiales à coefficients rationnels. Les solutions *complexes* d'un tel système d'équations forment une variété complexe, qui possède éventuellement des singularités. On est habitué, pour étudier la topologie d'une telle variété, à considérer ses groupes de cohomologie. On obtient ainsi des *invariants* simples du système d'équations originel. Néanmoins le lien entre les solutions rationnelles du système et ces invariants n'est pas évident, de plus ces considérations "oublient" que le système de départ est à coefficients rationnels : on doit donc s'attendre à une perte d'information importante. Grâce à la géométrie algébrique, on peut affiner cette construction. Le système d'équations définit une variété algébrique X définie sur $\mathbb{Q}$, que l'on voit comme une variété algébrique $X_{\overline{\mathbb{Q}}}$ définie sur $\overline{\mathbb{Q}}$ munie d'une action de $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Pour tout nombre premier ℓ, on peut considérer les groupes de cohomologie étale ℓ-adique $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$: ce sont des $\mathbb{Q}_\ell$-espaces vectoriels de dimension finie munis d'une action continue et linéaire de $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, c'est-à-dire des représentations galoisiennes ℓ-adiques. Celles-ci "linéarisent" la variété algébrique X, et on peut espérer que ces invariants sont suffisamment fins pour permettre de retrouver des informations de nature arithmétique sur le système de départ.

En 1967, dans une lettre adressée à Weil, Langlands imagine un lien entre certaines représentations automorphes et les représentations galoisiennes se factorisant par le groupe de Galois d'une extension *finie* de $\mathbb{Q}$ (on parle de représentation d'Artin), dans l'espoir d'aboutir à des lois de réciprocité *non-abéliennes*, en termes des coefficients des formes automorphes. Plus précisément, il demande si à chaque telle représentation galoisienne en dimension n il est possible d'associer une représentation automorphe de $\mathrm{GL}_n(\mathbb{A})$, selon une "recette" explicite.

L'année suivante Deligne suit le chemin inverse en associant à tout forme modulaire de poids $k \geq 2$, cuspidale et propre pour les opérateurs de Hecke une représentation galoisienne ℓ-adique de dimension 2, caractérisée par les traces des Frobenius en p pour tout nombre premier $p \neq \ell$, données par les valeurs propres pour les opérateurs T_p. Le cadre est quelque peu différent de celui de la question posée par Langlands puisqu'aucune de ces représentations n'est d'Artin. Le cas du poids $k = 1$, correspondant aux représentations d'Artin, sera traité en 1974 par Deligne et Serre, en utilisant le résultat de Deligne.

Langlands et Tunnell démontrent un énoncé dans le sens de la question de Langlands en 1980, en utilisant le changement de base pour le groupe GL_2. Il s'agit de représentations d'Artin en dimension 2, d'images *résolubles*. Les travaux de Wiles et Taylor-Wiles en 1995 démontrent pour la première fois un cas non résoluble. Soulignons qu'il s'agit de représentations galoisiennes qui ne sont pas d'Artin. Plus précisément, ils établissent le cas semistable de la conjecture de Taniyama-Shimura, qui affirme que la représentation galoisienne de dimension 2 associée à une courbe elliptique provient d'une forme modulaire de poids 2, cuspidale et propre pour les opérateurs de Hecke. Grâce aux travaux de Weil, Hellegouarch, Frey, Serre et Ribet, cela entraîne le célèbre théorème de Fermat. Remarquons qu'ici encore, la toute dernière étape de la démonstration repose sur la connaissance concrète des formes modulaires : le fait qu'il n'y a pas de forme modulaire cuspidale non nulle en poids 2 et niveau 2 permet de conclure à l'absurdité de l'existence d'un triplet de Fermat.

À la suite de cette percée, la correspondance de Langlands a connu des avancées importantes dans les deux sens, notamment la preuve de la correspondance de Langlands locale pour les groupes linéaires (Henniart et Harris-Taylor en 2001), la construction des représentations galoisiennes ℓ-adiques dans de nombreux cas (Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin, Taylor, ...), et très récemment de nombreux cas d'automorphie potentielle généralisant les travaux de Wiles et Taylor-Wiles (Barnet-Lamb-Gee-Geraghty-Taylor, utilisant notamment des constructions de Kisin, et Patrikis-Taylor).

Il faut souligner que tous ces résultats concernent les représentations automorphes algébriques et régulières ou "quasi-régulières" aux places archimédiennes, tandis que le programme de Langlands se veut plus général. Néanmoins les représentations automorphes qui ne sont pas algébriques aux places archimédiennes ne correspondent pas conjecturalement à des objets de nature arithmétique comme les représentations galoisiennes ℓ-adiques considérées ci-dessus.

1.3 Résultats obtenus dans cette thèse

Nous proposons deux applications arithmétiques des travaux récents de James Arthur sur la classification endoscopique du spectre automorphe discret des groupes symplectiques et orthogonaux.

La première consiste à ôter une hypothèse d'irréductibilité dans un résultat de Richard Taylor décrivant l'image des conjugaisons complexes par les représentations galoisiennes p-adiques associées aux représentations automorphes cuspidales algébriques régulières essentiellement autoduales pour le groupe GL_{2n+1} sur un corps de nombres totalement réel. Cet énoncé peut être vu comme une partie de la compatibilité entre correspondances de Langlands locale et globale aux places archimédiennes, l'autre partie consistant à décrire les poids de Hodge-Tate de la représentation galoisienne en fonction des paramètres de Langlands aux places archimédiennes. Nous étendons également ce résultat au cas de GL_{2n}, sous une hypothèse de parité du caractère multiplicatif. Nous utilisons un résultat de déformation p-adique de représentations automorphes. Plus précisément, nous montrons l'abondance de points correspondant à des représentations galoisiennes (quasi-)irréductibles sur les variétés de Hecke pour les groupes symplectiques et orthogonaux pairs. La classification d'Arthur est utilisée à la fois pour définir les représentations galoisiennes et pour transférer des représentations automorphes autoduales (pas nécessairement cuspidales) de groupes linéaires aux groupes symplectiques et orthogonaux.

La deuxième application concerne le calcul explicite de dimensions d'espaces de formes automorphes ou modulaires. Notre contribution principale est un algorithme calculant les intégrales orbitales aux éléments de torsion des groupes classiques p-adiques non ramifiés, pour l'unité de l'algèbre de Hecke non ramifiée. Cela permet le calcul du côté géométrique de la formule des traces d'Arthur, et donc celui de la caractéristique d'Euler-Poincaré du spectre discret en niveau trivial. La classification d'Arthur permet l'analyse fine de cette caractéristique d'Euler, jusqu'à en déduire les dimensions des espaces de formes automorphes. De là il n'est pas difficile d'apporter une réponse à un problème plus classique : déterminer les dimensions des espaces de formes modulaires de Siegel à valeurs vectorielles.

Partie 2

Eigenvarieties for classical groups and complex conjugations in Galois representations

2.1 Introduction

Let p be a prime. Let us choose once and for all algebraic closures $\overline{\mathbb{Q}}, \overline{\mathbb{Q}}_p, \mathbb{C}$ and embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let F be a totally real number field. A regular, L-algebraic, essentially self-dual, cuspidal (RLAESDC) representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is a cuspidal automorphic representation π together with an algebraic character $\eta|\cdot|^q$ of $\mathbb{A}_F^\times/F^\times$ (η being an Artin character, and q an integer) such that

- $\pi^\vee \simeq \eta|\det|^q \otimes \pi$,
- For any real place v of F, $\mathcal{LL}(\pi_v)|_{\mathrm{W}_\mathbb{C}} \simeq \oplus_i \left(z \mapsto z^{a_{v,i}} \bar{z}^{b_{v,i}}\right)$ where $\mathcal{LL}$ is the local Langlands correspondence, $\mathrm{W}_\mathbb{C} \simeq \mathbb{C}^\times$ is the Weil group of $\mathbb{C}$, and $a_{v,i}$, $b_{v,i}$ are integers and $a_{v,i} \neq a_{v,j}$ if $i \neq j$.

By definition, π is regular, L-algebraic, essentially self-dual, cuspidal (RLAESDC) if and only if $\pi \otimes |\det|^{(n-1)/2}$ is regular, *algebraic* (in the sense of Clozel), essentially self-dual, cuspidal (RAESDC). The latter is the notion of "algebraic" usually found in the literature, and is called "C-algebraic" in [BG10]. Given a RLAESDC representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, there is (Theorem 2.4.1.2) a unique continuous, semisimple Galois representation $\rho_{\iota_p,\iota_\infty}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that $\rho_{\iota_p,\iota_\infty}(\pi)$ is unramified at any finite place v of F not lying above p for which π_v is unramified, and $\iota_\infty \iota_p^{-1} \mathrm{Tr}\left(\rho_{\iota_p,\iota_\infty}(\pi)(\mathrm{Frob}_v)\right)$ is equal to the trace of the Satake parameter of π_v (contained in this assertion is the fact that this trace is algebraic over $\mathbb{Q}$). It is conjectured that for any real place v of F, if $c_v \in G_F$ is the conjugacy class of complex conjugations associated with v, the conjugacy class of $\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$ is determined by $\mathcal{LL}(\pi_v)$ (see [BG10][Lemma 2.3.2] for the case of an arbitrary reductive group). In the present case, by Clozel's purity lemma and by regularity, $\mathcal{LL}(\pi_v)$ is completely determined by its restriction to $\mathrm{W}_\mathbb{C}$, and since $\det\left(\rho_{\iota_p,\iota_\infty}(\pi)\right)$ is known, the determination of $\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$ amounts to the following

Conjecture. *Under the above hypotheses,* $\left|\mathrm{Tr}\left(\rho_{\iota_p,\iota_\infty}(\pi)(c_v)\right)\right| \leq 1$.

There are several cases for which this is known. By [Pat] for v an infinite place of F the value of $\eta_v(-1) \in \{\pm 1\}$ does not depend on v, and we denote the common value $\eta_\infty(-1)$. When $\eta_\infty(-1)(-1)^q = -1$ (this happens only if n is even, and by [BC11] this means that $\rho_{\iota_p,\iota_\infty}(\pi)$ together with the character $\rho_{\iota_p,\iota_\infty}(\eta|\cdot|^q) = (\eta \circ \mathrm{rec})\mathrm{cyclo}^q$, is "symplectic"), $\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$ is conjugate to $-\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$, so the trace is obviously zero.

In [Tay12], Richard Taylor proves the following

Theorem (Taylor)**.** *Let F be a totally real number field, $n \geq 1$ an integer. Let π be a regular, L-algebraic, essentially self-dual, cuspidal automorphic representation of GL_{2n+1}/F. Assume that the attached Galois representation $\rho_{\iota_p,\iota_\infty}(\pi) : G_F \to \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_p)$ is irreducible. Then for any real place v of F,*

$$\mathrm{Tr}\left(\rho_{\iota_p,\iota_\infty}(\pi)(c_v)\right) = \pm 1.$$

Although one expects $\rho_{\iota_p,\iota_\infty}(\pi)$ to be always irreducible, this is not known in general. However it is known when $n \leq 2$ by [CG], and for arbitrary n but only for p in a set of positive Dirichlet density by [PT].

In this paper, the following cases are proved:

Theorem A (Theorem 2.6.3.4)**.** *Let $n \geq 2$, F a totally real number field, π a regular, L-algebraic, essentially self-dual, cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$, such that $\pi^\vee \simeq ((\eta|\cdot|^q) \circ \det) \otimes \pi$, where η is an Artin character and q an integer. Suppose that one of the following conditions holds*

1. *n is odd.*

2. *n is even, q is even, and $\eta_\infty(-1) = 1$.*

Then for any complex conjugation $c \in G_F$, $|\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c))| \leq 1$.

This is achieved thanks to the result of Taylor, Arthur's endoscopic transfer between twisted general linear groups and symplectic or orthogonal groups, and using eigenvarieties for these groups. Let us describe the natural strategy that one might consider to prove the odd-dimensional case using these tools, to explain why it fails and how a detour through the even-dimensional case allows to conclude.

Let π be a RLAESDC representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$. Up to a twist by an algebraic character π is self-dual and has trivial central character. Conjecturally, there should be an associated self-dual Langlands parameter $\phi_\pi : \mathrm{L}_F \to \mathrm{GL}_{2n+1}(\mathbb{C})$ where L_F is the conjectural Langlands group. Up to conjugation, ϕ_π takes values in $\mathrm{SO}_{2n+1}(\mathbb{C})$, and by functoriality there should be a discrete automorphic representation Π of $\mathrm{Sp}_{2n}(\mathbb{A}_F)$ such that $\mathcal{LL}(\Pi_v)$ is equal to $\mathcal{LL}(\pi_v)$ via the inclusion $\mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$ for any place of F which is either archimedean or such that π_v is unramified. Arthur's results in his book [Art13] imply that this (in fact, much more) holds. To construct p-adic families of automorphic representations (i.e. eigenvarieties) containing Π, it is preferable to work with a group which is *compact* at the real places of F, and work with representations having Iwahori-invariants at the p-adic places. A suitable solvable base change allows to assume that $[F : \mathbb{Q}]$ is even and that π_v has Iwahori-invariants for $v|p$. The last chapter of [Art13] will allow to "transfer" π to an automorphic representation Π of $\mathbf{G}$, the inner form of Sp_{2n} which is split at the finite places and compact at the real places of F. By [Loe11] (which generalizes [Che04]), the eigenvariety $\mathscr{X}$ for $\mathbf{G}$ is available. Thanks to [Art13], one can associate p-adic Galois representations $\rho_{\iota_p,\iota_\infty}(\cdot)$ to automorphic representations of $\mathbf{G}$, yielding a family of Galois representations on $\mathscr{X}$, that is to say a continuous map $T : G_F \to \mathcal{O}(\mathscr{X})$ which specializes to $\mathrm{Tr}\left(\rho_{\iota_p,\iota_\infty}(\cdot)\right)$ at the points of $\mathscr{X}$ corresponding to automorphic representations of $\mathbf{G}(\mathbb{A}_F)$. One can then hope to prove a result similar to [BC11, Lemma 3.3], i.e. show that one can "deform" Π (on $\mathscr{X}$) to reach a point corresponding to an automorphic representation Π' whose Galois representation is irreducible (even when restricted to the decomposition group of a p-adic place of F). Since $\rho_{\iota_p,\iota_\infty}(\Pi')$ comes from an automorphic representation π' of GL_{2n+1}, π' is necessarily cuspidal and satisfies the hypotheses of Taylor's theorem. Since $T(c_v)$ is locally constant on $\mathscr{X}$, we would be done.

Unfortunately, it does not appear to be possible to reach a representation Π' whose Galois representation is irreducible by using local arguments on the eigenvariety. However we will prove the following, which includes the case of some even-dimensional special orthogonal groups as it will be needed later:

Theorem B (Theorem 2.4.2.2, Theorem 2.5.0.3)**.** *Let $\mathbf{G}$ be an inner form of Sp_{2n} or SO_{4n} over a totally real number field, compact at the real places and split at the p-adic ones. Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori invariants at all the places of F above p, and having invariants under an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$. Let $\rho_{\iota_p,\iota_\infty}(\Pi)$ denote the p-adic representation of the absolute Galois group G_F of F associated with Π and embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let N be an integer. There exists an automorphic representation Π' of $\mathbf{G}(\mathbb{A}_F)$ such that:*

- *Π' is unramified at the places above p, and has invariants under U;*
- *The restriction of $\rho_{\iota_p,\iota_\infty}(\Pi')$ to the decomposition group at any place above p is either irreducible or the sum of an Artin character and an irreducible representation of dimension $2n$ (the latter occurring only in the symplectic case);*
- *For all g in G_F, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(g)) \equiv \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(g)) \mod p^N$.*

The possible presence of an Artin character (in the case of inner forms of Sp_{2n}) comes from the fact that the "standard" representation of $\mathrm{SO}_{2n+1}(\mathbb{C})$ in $\mathrm{GL}_{2n+1}(\mathbb{C})$ is not minuscule: the set of characters of a torus $T(\mathbb{C})$ of $\mathrm{SO}_{2n+1}(\mathbb{C})$ in this representation has two orbits under the Weyl group, one of which contains only the trivial character. The key fact allowing to prove the above theorem is that classical points on the eigenvariety for $\mathbf{G}$ correspond to automorphic representations Π of $\mathbf{G}(\mathbb{A}_F)$ (say, unramified at the p-adic places) *and* a refinement of each Π_v, $v|p$, that is to say a particular element in $T(\mathbb{C})$ in the conjugacy class of the Satake parameter of Π_v. The variation of the crystalline Frobenius of $\rho_{\iota_p,\iota_\infty}(\cdot)$ on the eigenvariety with respect to the weight and the freedom to change the refinement (by the action of the Weyl group) are at the heart of the proof of Theorem B.

Although the strategy outlined above fails, Theorem A can be deduced from Theorem B. Indeed the precise description of the discrete automorphic spectrum of symplectic and orthogonal groups by Arthur shows that formal sums of distinct cuspidal self-dual representations of general linear groups "contribute" to this spectrum. The even-dimensional case in Theorem A will be proved by transferring $\pi \boxplus \pi_0$, where π, π_0 are regular, L-algebraic, self-dual, cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ (resp. $\mathrm{GL}_3(\mathbb{A}_F)$) with distinct weights at any real place of F, to an automorphic representation Π of an inner form $\mathbf{G}$ of Sp_{2n+2}/F. Since $\rho_{\iota_p,\iota_\infty}(\pi) \oplus \rho_{\iota_p,\iota_\infty}(\pi_0)$ does not contain any Artin character (the zero Hodge-Tate weights come from $\rho_{\iota_p,\iota_\infty}(\pi_0)$, which is known to be irreducible), for big enough N any representation Π' as in B has an irreducible Galois representation.

To treat the original case of a regular, L-algebraic, self-dual, cuspidal representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$ having trivial central character, we appeal to Theorem B for special orthogonal groups. For example, if n is odd, $\pi \boxplus \pi_0$, where π_0 is the trivial character of $\mathbb{A}_F^\times/F^\times$, contributes to the automorphic spectrum of $\mathbf{G}$, which is now the special orthogonal group of a quadratic form on F^{2n+2} which is definite at the real places and split at the finite places of F. Note that $\pi \boxplus \pi_0$ is not regular: the zero weight appears twice at each real place of F. However the Langlands parameters of representations of the compact group $\mathrm{SO}_{2n+2}(\mathbb{R})$ are of the form

$$\bigoplus_{i=1}^{n+1} \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} \left(z \mapsto (z/\bar{z})^{k_i} \right)$$

when composed with $\mathrm{SO}_{2n+2}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+2}(\mathbb{C})$, with $k_1 > \ldots > k_{n+1} \geq 0$. Moreover $\mathcal{LL}\left((\pi \boxplus \pi_0)_v\right)$ is of the above form, with $k_{n+1} = 0$. The rest of the proof is identical to the even-dimensional case.

This fact also shows that some *non-regular*, L-algebraic, self-dual, cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ contribute to the automorphic spectrum of $\mathbf{G}$. Consequently we can also extend Taylor's result to the Galois representations associated with these slightly non-regular automorphic representations. These Galois representations were shown to exist by Wushi Goldring [Gol14].

We now fix some notations for the rest of the article. The valuation v_p of $\overline{\mathbb{Q}}_p$ is the one sending p to 1, and $|\cdot|$ will denote the norm $p^{-v_p(\cdot)}$. All the number fields in the paper will sit inside $\overline{\mathbb{Q}}$. We have chosen arbitrary embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. In fact, the constructions will only depend on the identification between the algebraic closures of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$ (informally, $\iota_p \iota_\infty^{-1}$). Observe that the choice of a p-adic place v of a number field F and of an embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_p$ is equivalent, via ι_p, to the choice of an embedding $F \hookrightarrow \overline{\mathbb{Q}}$. The same holds for the infinite places and ι_∞. Thus if F is totally real, $\iota_p \iota_\infty^{-1}$ defines a bijection between the set of infinite places of F and the set of p-adic places v of F together with an embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_p$. The eigenvarieties will be rigid analytic spaces (in the sense of Tate). If $\mathscr{X}$ is a rigid analytic space over a finite extension E of $\mathbb{Q}_p$, $|\mathscr{X}|$ will denote its points.

2.2 Assumptions on forthcoming results of Arthur

As the results of this paper rely on [Art13][Theorem 9.5.3] (the analogue of [Art13, Theorem 1.5.2] in the case of *inner forms* of quasi-split classical groups), whose proof will only be given in [Art], we have stated some properties as assumptions: Assumptions 2.4.1.1, 2.6.1.2, 2.6.2.2 and 2.6.4.1. These will all be consequences of the main global theorem of [Art], which will make more precise the statement of [Art13][Theorem 9.5.3].

The reason [Art13][Theorem 9.5.3] is not precisely stated is that at present it is not known what global data should play the role of Whittaker data in the case of inner forms of quasisplit groups. These data are needed to normalize the local Langlands correspondence, via the normalization of endoscopic transfer factors. There is a satisfactory definition in the local case: rigid inner forms as defined in [Kal]. A global analogue is necessary to formulate [Art13][Theorem 9.5.3] precisely.

A subsequent version of this paper will have the assumptions replaced by actual propositions or lemmas.

2.3 The eigenvariety for definite symplectic groups

In this section we recall the main result of [Loe11] in our particular case (existence of the eigenvariety for symplectic groups), and show that the points corresponding to unramified, "completely refinable" automorphic forms, with weight far from the walls, are "dense" in this eigenvariety.

2.3.1 The eigenvariety

2.3.1.1 Symplectic groups compact at the archimedean places

Let F be a totally real number field of even degree over $\mathbb{Q}$, and let D be a quaternion algebra over F, unramified at all the finite places of F ($F_v \otimes_F D \simeq \mathrm{M}_2(F_v)$), and definite at all the real places of F. Such a D exists thanks to the exact sequence relation the Brauer groups of F and the F_v. Let n be a positive integer, and let $\mathbf{G}$ be the algebraic group over F defined by the equation $M^*M = \mathrm{I}_n$ for $M \in \mathrm{M}_n(D)$, where $(M^*)_{i,j} = M_{j,i}^*$, and $\cdot^*$ denotes conjugation in D.

Then $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ is a compact Lie group, and for all finite places v of F, $\mathbf{G} \times_F F_v \simeq \mathbf{Sp}_{2n}/F_v$.

Fix a prime p. We will apply the results of [Loe11] to the group $\mathbf{G}' = \mathrm{Res}_{\mathbb{Q}}^F \mathbf{G}$. Let E be a finite and Galois extension of $\mathbb{Q}_p$, containing all the F_v (v over p).

2.3.1.2 The Atkin-Lehner algebra

The algebraic group $\mathbf{G}' \times_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} \mathbf{G} \times_{\mathbb{Q}} F_v$ (where v runs over the places of F) is isomorphic to $\prod_{v|p} \mathrm{Res}_{\mathbb{Q}_p}^{F_v} \mathbf{Sp}_{2n}/F_v$, which is quasi-split but not split in general. The algebraic group $\mathbf{Sp}_{2n}$ is defined over $\mathbb{Z}$ by the equation ${}^tMJM = J$ in M_{2n}, where $J = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$ and $J_n = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$. We define its algebraic subgroups $\mathbf{T}_v$, $\mathbf{B}_v$, $\bar{\mathbf{B}}_v$, $\mathbf{N}_v$, $\bar{\mathbf{N}}_v$ of diagonal, upper triangular, lower triangular, unipotent upper triangular, and unipotent lower triangular matrices of $\mathrm{Res}_{\mathbb{Q}_p}^{F_v} \mathrm{Sp}_{2n}/F_v$, and let $\mathbf{T} = \prod_{v|p} \mathbf{T}_v$, $\mathbf{B} = \prod_{v|p} \mathbf{B}_v$, and so on. In [Loe11, 2.4], only the action of the maximal split torus of $\mathbf{G}' \times_{\mathbb{Q}} \mathbb{Q}_p$ is considered. For our purpose, we will need to extend this and consider the action of a maximal (non-split in general) torus, that is $\mathbf{T}$, instead of a maximal split torus $\mathbf{S} \subset \mathbf{T}$. The results in [Loe11] are easily extended to this bigger torus, essentially because $\mathbf{T}(\mathbb{Q}_p)/\mathbf{S}(\mathbb{Q}_p)$ is compact. Moreover, we let I_v be the compact subgroup of $\mathbf{Sp}_{2n}(\mathcal{O}_v)$ consisting of matrices with invertible diagonal elements and elements of positive valuation below the diagonal. Finally, following Loeffler's notation, we let $G_0 = \prod_{v|p} I_v$. It is an Iwahori sugroup of $\mathbf{G}'(\mathbb{Q}_p)$ having an Iwahori decomposition: $G_0 \simeq \bar{N}_0 T_0 N_0$ where $*_0 = *(\mathbb{Q}_p) \cap G_0$.

For each place v of F above p, let us choose a uniformizer ϖ_v of F_v. Let Σ_v be the subgroup of $\mathbf{Sp}_{2n}(F_v)$ consisting of diagonal matrices whose diagonal elements are powers of ϖ_v, i.e. matrices of the form

$$\begin{pmatrix} \varpi_v^{r_1} & & & & & \\ & \ddots & & & & \\ & & \varpi_v^{r_n} & & & \\ & & & \varpi_v^{-r_n} & & \\ & & & & \ddots & \\ & & & & & \varpi_v^{-r_1} \end{pmatrix}$$

Let Σ_v^+ be the submonoid of Σ_v whose elements satisfy $r_1 \leq \ldots \leq r_n \leq 0$, and Σ_v^{++} the one whose elements satisfy $r_1 < \ldots < r_n < 0$. Naturally, we set $\Sigma = \prod_{v|p} \Sigma_v$, and similarly for Σ^+ and Σ^{++}.

The *Atkin-Lehner algebra* $\mathcal{H}_p^+$ is defined as the subalgebra of the Hecke-Iwahori algebra $\mathcal{H}(G_0 \backslash \mathbf{G}'(\mathbb{Q}_p)/G_0)$ (over $\mathbb{Q}$) generated by the characteristic functions $[G_0 u G_0]$, for $u \in \Sigma^+$. Let $\mathcal{H}_p$ be the subalgebra of $\mathcal{H}(G_0 \backslash \mathbf{G}'(\mathbb{Q}_p)/G_0)$ generated by the characteristic functions $[G_0 u G_0]$ and their inverses, for $u \in \Sigma^+$ (in [IM65], a presentation of the Hecke-Iwahori algebra is given, which shows that $[G_0 u G_0]$ is invertible if p is invertible in the ring of coefficients).

If S^p is a finite set of finite places of F not containing those over p, let $\mathcal{H}^S$ be the Hecke algebra (over $\mathbb{Q}$)

$$\bigotimes_{w \notin S^p \cup S_p \cup S_\infty}' \mathcal{H}(\mathbf{G}(\mathcal{O}_{F_w}) \backslash \mathbf{G}(F_w)/\mathbf{G}(\mathcal{O}_{F_w}))$$

where S_* denotes the set of places above $*$. This Hecke algebra has unit e^S. Let $\mathcal{H}_S^p$ be a commutative subalgebra of $\bigotimes_{w \in S^p} \mathcal{H}(\mathbf{G}(F_w))$, with unit e_{S^p}.

Finally, we let $\mathcal{H}^+ = \mathcal{H}_p^+ \otimes \mathcal{H}_{S^p} \otimes \mathcal{H}^S$, $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_{S^p} \otimes \mathcal{H}^S$ and $e = e_{G_0} \otimes e_{S^p} \otimes e^S$.

2.3.1.3 p-adic automorphic forms

The construction in [Loe11] depends on the choice of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}'$ and a representation V of a compact subgroup of the Levi quotient $\mathbf{M}$ of $\mathbf{P}$. The parabolic subgroup we consider here is the Borel subgroup $\mathbf{B}$, and thus, using Loeffler's notation, $\mathbf{T} = \mathbf{M}$ is a maximal (non-split in general) torus contained in $\mathbf{B}$. The representation V is taken to be trivial.

The weight space $\mathscr{W}$ is the rigid space (over E, but it is well-defined over $\mathbb{Q}_p$) parametrizing locally $\mathbb{Q}_p$-analytic (equivalently, continuous) characters of $T_0 \simeq \left(\prod_{v|p} \mathcal{O}_v^\times\right)^n$. As $1+\varpi_v\mathcal{O}_v$ is isomorphic to $(\mu_{p^\infty} \cap F_v^\times) \times \mathbb{Z}_p^{[F_v:\mathbb{Q}_p]}$, $\mathscr{W}$ is the product of an open polydisc of dimension $n[F:\mathbb{Q}]$ and a rigid space finite over E.

The construction in [Loe11] defines the k-analytic ($(G_k)_{k\geq 0}$ being a filtration of G_0) parabolic induction from T_0 to G_0 of the "universal character" $\chi : T_0 \to \mathcal{O}(\mathscr{W})^\times$, denoted by $\mathcal{C}(\mathscr{U}, k)$ (k big enough such that χ is k-analytic on the open affinoid $\mathscr{U}$), which interpolates p-adically the restriction to $\mathbf{G}'(\mathbb{Q}_p)$ of algebraic representations of $\mathbf{G}'(\overline{\mathbb{Q}}_p)$. From there one can define the spaces $M(e, \mathscr{U}, k)$ ([Loe11, Definition 3.7.1]) of p-adic automorphic forms (or overconvergent automorphic forms, by analogy with the rigid-geometric case of modular forms) above an open affinoid or a point $\mathscr{U}$ of $\mathscr{W}$ which are k-analytic and fixed by the idempotent e. This space has an action of $\mathcal{H}^+$. By [Loe11, Corollary 3.7.3], when considering p-adic automorphic forms which are eigenvectors for $[G_0uG_0]$ for some $u \in \Sigma^{++}$ and for a non-zero eigenvalue ("finite slope" p-adic eigenforms), one can forget about k, and we will do so in the sequel.

2.3.1.4 Existence and properties of the eigenvariety

We choose the element

$$\eta = \left(\begin{pmatrix} \varpi_v^{-n} & & & & & \\ & \ddots & & & & \\ & & \varpi_v^{-1} & & & \\ & & & \varpi_v & & \\ & & & & \ddots & \\ & & & & & \varpi_v^{n} \end{pmatrix} \right)_v \in \Sigma^{++}$$

Theorem 2.3.1.1. *There exists a reduced rigid space $\mathscr{X}$ over E, together with an E-algebra morphism $\Psi : \mathcal{H}^+ \to \mathcal{O}(\mathscr{X})^\times$ and a morphism of rigid spaces $w : \mathscr{X} \to \mathscr{W}$ such that:*

1. *The morphism $\left(w, \Psi([G_0\eta G_0])^{-1}\right) : \mathscr{X} \to \mathscr{W} \times \mathbb{G}_m$ is finite*
2. *For each point x of $\mathscr{X}$, $\Psi \otimes w^\natural : \mathcal{H}^+ \otimes_E \mathcal{O}_{w(x)} \to \mathcal{O}_x$ is surjective*
3. *For every finite extension E'/E, $\mathscr{X}(E')$ is in bijection with the finite slope systems of eigenvalues of $\mathcal{H}^+$ acting on the space of "overconvergent" automorphic forms, via evaluation of the image of Ψ at a given point.*

Moreover, for any point $x \in |\mathscr{X}|$, there is an arbitrarily small open affinoid $\mathscr{V}$ containing x and an open affinoid $\mathscr{U}$ of $\mathscr{W}$ such that $\mathscr{V} \subset w^{-1}(\mathscr{U})$, the morphism $w|_{\mathscr{V}} : \mathscr{V} \to \mathscr{U}$ is finite, and surjective when restricted to any irreducible component of $\mathscr{V}$.

Proof. This is [Loe11, Theorems 3.11.2 and 3.12.3], except for the last assertion. To prove it, we need to go back to the construction of the eigenvariety in [Buz07]. Buzzard begins by constructing the Fredholm hypersurface $\mathscr{Z}$ (encoding only the value of $\Psi([G_0\eta G_0])$), together with a flat morphism $\mathscr{Z} \to \mathscr{W}$, before defining the finite morphism $\mathscr{X} \to \mathscr{Z}$. By [Buz07, Theorem 4.6], $\mathscr{Z}$ can be admissibly covered by its open affinoids $\mathscr{V}_0$ such that w restricted to $\mathscr{V}_0$ induces a finite, surjective morphism to an open affinoid $\mathscr{U}$ of $\mathscr{W}$, and $\mathscr{V}_0$ is a connected component of the pullback of $\mathscr{U}$. We can assume that $\mathscr{U}$ is connected, and hence irreducible, since $\mathscr{W}$ is normal. The morphism $\mathscr{V}_0 \to \mathscr{U}$ is both open (since it is flat: [Bos09, Corollary 7.2]) and closed (since it is finite), so that any irreducible component of $\mathscr{V}_0$ is mapped onto $\mathscr{U}$. This can be seen more naturally by observing that the irreducible components of $\mathscr{V}_0$ are also Fredholm hypersurfaces, by [Con99, Theorem 4.3.2].

By [Che04, Proposition 6.4.2], if $\mathscr{V}$ denotes the pullback to $\mathscr{X}$ of $\mathscr{V}_0$, each irreducible component of $\mathscr{V}$ is mapped onto an irreducible component of $\mathscr{V}_0$ (more precisely, this is a consequence of [Che04, Lemme 6.2.10]). To conclude, we only need to show that if $x \in \mathscr{V}$, up to restricting $\mathscr{U}$, the connected component of $\mathscr{V}$ containing x can be arbitrarily small. This is a consequence of the following lemma. □

Lemma 2.3.1.2. *Let $f : \mathscr{X}_1 \to \mathscr{X}_2$ be a finite morphism of rigid analytic spaces. Then the connected components of $f^{-1}(U)$, for U admissible open of $\mathscr{X}_2$, form a basis for the canonical topology on $\mathscr{X}_1$.*

Proof. It is enough to consider the case $\mathscr{X}_1 = \mathrm{Sp}A_1$, $\mathscr{X}_2 = \mathrm{Sp}A_2$. Let x_1 be a maximal ideal of A_1. Then $f^{-1}(\{f(x_1)\}) = \{x_1, \dots, x_m\}$. We choose generators $t_1, \dots, t_n$ of $f(x_1)$, and $r_1^{(i)}, \dots, r_{k_i}^{(i)}$ of x_i. Using the maximum modulus principle, it is easily seen that $\Omega_{j,N} := \left\{y \in \mathscr{X}_2 \mid |t_j(y)| \geq p^{-N}\right\}_{j,N}$ is an admissible covering of the admissible open $\mathscr{X}_2 \setminus \{f(x)\}$ of $\mathscr{X}_2$. Let V_M be the admissible open $\left\{x \in \mathscr{X}_1 \mid \forall i, \exists k,\ |r_k^{(i)}(x)| \geq p^{-M}\right\}$, which is a finite union of open affinoids, hence quasi-compact. Consequently, the admissible open sets

$$\begin{aligned} U_{j,N} &:= V_M \cap f^{-1}(\Omega_{j,N}) \\ &= \left\{x \in \mathscr{X}_1 \mid \forall i, \exists k,\ |r_k^{(i)}(x)| \geq p^{-M} \text{ and } |f^\natural(t_j)(x)| \geq p^{-N}\right\}_{j,N} \end{aligned}$$

form an admissible covering of V_M. Therefore there is an N big enough so that

$$V_M = \bigcup_{j=1}^{r} U_{j,N}$$

which implies that

$$f^{-1}\left(\left\{y \in \mathscr{X}_2 \mid |t_j(y)| \leq p^{-N-1}\right\}\right) \subset \bigcup_i \left\{x \in \mathscr{X}_1 \mid \forall k,\ |r_k^{(i)}(x)| \leq p^{-M}\right\}$$

and when M goes to infinity, the right hand side is the disjoint union of arbitrarily small affinoid neighbourhoods of the x_i. □

We define the algebraic points of $\mathscr{W}(E)$ to be the ones of the form

$$(x_{v,i})_{v,i} \mapsto \prod_{v,\sigma} \sigma\left(\prod_{i=1}^{n} x_{v,i}^{k_{v,\sigma,i}}\right)$$

where $k_{v,\sigma,i}$ are integers, and such a point is called dominant if $k_{v,\sigma,1} \geq k_{v,\sigma,2} \geq \ldots \geq k_{v,\sigma,n} \geq 0$.

Recall that a set $S \subset |\mathscr{X}|$ is said to *accumulate* at a point $x \in |\mathscr{X}|$ if x has a basis of affinoid neighbourhoods in which S is Zariski dense.

Proposition 2.3.1.3. *Let $(\phi_r)_r$ be a finite family of linear forms on $\mathbb{R}^A$ where A is the set of triples (v, σ, i) for v a place of F above p, $\sigma : F_v \to E$ and $1 \leq i \leq n$, and let $(c_r)_r$ be a family of elements in $\mathbb{R}_{\geq 0}$. Assume that the open affine cone $C = \left\{y \in \mathbb{R}^A \mid \forall r,\ \phi_r(y) > c_r\right\}$ is nonempty. Then the set of algebraic characters in C yields a Zariski dense set in the weight space $\mathscr{W}$, which accumulates at all the algebraic points.*

Proof. [Che09, Lemma 2.7]. □

In particular the property of being dominant or "very regular" can be expressed in this way.

By finiteness of $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_{F,f})/U$ for any open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f})$, if Π is an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$, the representation Π_f is defined over $\iota_\infty(\bar{\mathbb{Q}})$. Loeffler defines ([Loe11, Definition 3.9.1]) the classical subspace of the space of p-adic automorphic forms above an algebraic and dominant point w of the weight space. This subspace is isomorphic to $\iota_p\iota_\infty^{-1}\left(e\left(\mathcal{C}^\infty(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)) \otimes W^*\right)^{\mathbf{G}(F\otimes_\mathbb{Q}\mathbb{R})}\right)$ as $\mathcal{H}^+$-module, with W the representation of $\mathbf{G}(F \otimes_\mathbb{Q} \mathbb{R})$ which is the restriction of the algebraic representation of $\mathbf{G}' \times_\mathbb{Q} \mathbb{C}$ having highest weight $\iota_\infty^{-1}\iota_p(w)$. The classical points of the eigenvariety are the ones having eigenvectors in the classical subspace.

We need to give an interpretation of classical points on the eigenvariety $\mathscr{X}$, in terms of automorphic representations of $\mathbf{G}(\mathbb{A}_F)$. Namely, there is a classical point $x \in \mathscr{X}(E')$ defining a character $\Psi_x : \mathcal{H} \to E'$ (here $E \subset E' \subset \overline{\mathbb{Q}}_p$) if and only if there is an automorphic representation $\Pi = \otimes'_v \Pi_v = \Pi_\infty \otimes \Pi_p \otimes \Pi_f^{(p)}$ of $\mathbf{G}(\mathbb{A}_F)$ such that:

- $\iota_p\iota_\infty^{-1}\left(\otimes_{v|\infty}\Pi_v\right)$ is the algebraic representation having highest weight $w(x)$;
- $\iota_p\left((e^S \otimes e_S)\Pi_f^{(p)}\right)$ contains a non-zero vector on which $\mathcal{H}^S \otimes \mathcal{H}_S$ acts according to Ψ_x;
- $\iota_p(e_{G_0}\Pi_p)$ contains a non-zero vector on which $\mathcal{H}_p$ acts according to $\mu_{w(x)}\Psi_x$, where $\mu_{w(x)}([G_0\xi G_0]) = w(x)(\xi)$ if $\xi \in \Sigma^+$.

The twist by the character $\mu_{w(x)}$ is explained by the fact that the classical overconvergent automorphic forms are constructed by induction of characters of the torus extended from T_0 (on which they are defined by w) to T trivially on Σ.

2.3.2 Unramified and "completely refinable" points

2.3.2.1 Small slope p-adic eigenforms are classical

The algebraic and dominant points of $\mathscr{W}$ are the ones of the form

$$(x_{v,i})_{v,i} \mapsto \prod_{v,\sigma} \sigma\left(\prod_{i=1}^n x_{v,i}^{k_{v,\sigma,i}}\right)$$

where $k_{v,\sigma,1} \geq k_{v,\sigma,2} \geq \ldots \geq k_{v,\sigma,n} \geq 0$ are integers. The proof of the criterion given in [Loe11, Theorem 3.9.6] contains a minor error, because it "sees" only the restriction of these characters to the maximal split torus $\mathbf{S}$ (over $\mathbb{Q}_p$), and the BGG resolution has to be applied to *split* semi-simple Lie algebras.

We correct it in the case of quasi-split reductive groups (in particular the restriction to a subfield of a quasi-split group remains quasi-split), and give a stronger criterion. This criterion could be used on an eigenvariety for which only the weights corresponding to a given p-adic place of F vary. For this purpose we use the "dual BGG resolution" given in [Jon11]. The proof will be very close to that of [Loe11, Propositions 2.6.3-2.6.4]. In the following $\mathbf{G}'$ could be any quasi-split reductive group over $\mathbb{Q}_p$, and we could replace $E/\mathbb{Q}_p$ by any extension splitting $\mathbf{G}'$.

Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}'$, $\mathbf{S}$ a maximal split torus in $\mathbf{B}$, $\mathbf{T}$ the centralizer of $\mathbf{S}$, a maximal torus. This determines an opposite Borel subgroup $\bar{\mathbf{B}}$ such that $\bar{\mathbf{B}} \cap \mathbf{B} = \mathbf{T}$. Let Φ^+ (resp. Δ) be the set of positive (resp. simple) roots of $\mathbf{G}' \times_{\mathbb{Q}_p} E$, with respect to the maximal torus $\mathbf{T}$ of the Borel subgroup $\mathbf{B}$. One can split $\Delta = \sqcup_i \Delta_i$ where α, β belong to the same Δ_i if and only if $\alpha|_{\mathbf{S}} = \beta|_{\mathbf{S}}$ (equivalently, the Δ_i are the Galois orbits of Δ). Let Σ be a subgroup of $\mathbf{T}(\mathbb{Q}_p)$ supplementary to its maximal compact subgroup, and Σ^+ the submonoid consisting of the $z \in \mathbf{T}(\mathbb{Q}_p)$ such that $|\alpha(z)| \geq 1$ for all $\alpha \in \Delta$. For each i, define η_i to be the element of $\Sigma^+ / \left(Z(\mathbf{G}')(\mathbb{Q}_p) \cap \Sigma\right)$ generating $\cap_{j \neq i} \ker |\alpha_j(\cdot)|$ (here α_j denotes any element of Δ_j, and $|\alpha_j(\cdot)|$ does not depend on this choice).

Assume that G_0 is a compact open subgroup of $\mathbf{G}'(\mathbb{Q}_p)$ having an Iwahori factorization $\bar{N}_0 T_0 N_0$. Using a lattice in the Lie algebra of N and the exponential map, it is easily seen that N_0 admits a decreasing, exhaustive filtration by open subgroups $(N_k)_{k\geq 1}$ having a canonical rigid-analytic structure. Moreover any ordering of Φ^+ endows the Banach space of $\mathbb{Q}_p$-analytic functions on N_k taking values in E with an orthonormal basis consisting of monomials on the weight spaces.

Let λ be an algebraic and dominant weight of $\mathbf{T} \times_{\mathbb{Q}_p} E$. By [Jon11], there is an exact sequence of $E[\mathbb{I}]$-modules, where $\mathbb{I} = G_0 \Sigma^+ G_0 = \bar{B}_0 \Sigma^+ N_0$ is the monoid generated by G_0 and Σ^+:

$$0 \to \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda) \otimes \text{sm-Ind}_{\bar{B}_0}^{\bar{B}_0 N_0} 1 \to \text{la-Ind}_{\bar{B}}^{\bar{B} N_0}(\lambda) \to \bigoplus_{\alpha \in \Delta} \text{la-Ind}_{\bar{B}}^{\bar{B} N_0}(s_\alpha(\lambda+\rho)-\rho) \quad (2.3.2.1)$$

where $2\rho = \sum_{\alpha \in \Phi^+} \alpha$, "sm" stands for "smooth" and "la" for "locally analytic". The relation with Loeffler's $\mathrm{Ind}(V)_k$ is $\text{la-Ind}_{\bar{B}}^{\bar{B} N_0}(\lambda) \otimes \lambda_{\text{sm}}^{-1} = \varinjlim_k \mathrm{Ind}(E_\lambda)_k$, where λ_{sm} is the character on T which is trivial on its maximal compact subgroup and agrees with λ on Σ. Naturally $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(\lambda) \otimes \text{sm-Ind}_{\bar{B}_0}^{\bar{B}_0 N_0} 1 \otimes \lambda_{\text{sm}}^{-1} = \varinjlim_k \mathrm{Ind}(E_\lambda)_k^{\text{cl}}$.

To prove a classicity criterion, we need to bound the action of η_i on the factors of the RHS of (2.3.2.1) twisted by λ_{sm}^{-1}. Let $n_\alpha = \alpha^\vee(\lambda) \in \mathbb{N}$ for $\alpha \in \Delta$, then $s_\alpha(\lambda+\rho) - \lambda - \rho = -(1+n_\alpha)\alpha$. The Banach space of k-analytic functions on N_0 is the direct sum of the spaces of analytic functions on xN_k, $x \in N_0/N_k$, and each of these spaces has an orthonormal (with respect to the supremum norm) basis $(v_{j,x})_{j \in J}$ where $J = \mathbb{N}^{\Phi^+}$ (monomials on the weights spaces). This basis depends on the choice of a representative x, but if we fix i and

$x_0 \in N_0$, we can choose $\eta_i^{-1} x_0 \eta_i$ as a representative of its class. Then if $\phi = \sum_j a_j v_{j,\eta_i^{-1} x_0 \eta_i}$ (with $a_j \to 0$) is an element of la-Ind$_{\bar{B}}^{\bar{B}N_0}(s_\alpha(\lambda+\rho)-\rho) \otimes \lambda_{\mathrm{sm}}^{-1}$, and $\xi \in N_k$,

$$\begin{aligned}(\eta_i \cdot \phi)(x_0\xi) &= \eta_i^{-(1+n_\alpha)\alpha} \sum_{j \in J} a_j v_{j,\eta_i^{-1}x_0\eta_i}(\eta_i^{-1} x_0 \xi \eta_i) \\ &= \sum_{j \in J} a_j \eta_i^{-(1+n_\alpha)\alpha - s(j)} v_{j,x_0}(x_0\xi)\end{aligned}$$

where $s(j) = \sum_{\beta \in \Phi^+} j(\beta)\beta$. This shows that $|\eta_i \cdot \phi| \leq |\alpha(\eta_i)|^{-(1+n_\alpha)}|\phi|$, and so the operator η_i has norm less than or equal to $|\alpha(\eta_i)|^{-(1+n_\alpha)}$ on la-Ind$_{\bar{B}}^{\bar{B}N_0}(s_\alpha(\lambda+\rho)-\rho) \otimes \lambda_{\mathrm{sm}}^{-1}$.

We can then apply the exact functor which to an $E[\mathbb{I}]$-module W associates the automorphic forms taking values in W, and take the invariants under the idempotent e (this functor is left exact). We obtain that $M(e, E_\lambda)/M(e, E_\lambda)_{\mathrm{cl}}$ (the space of p-adic automorphic forms modulo the classical automorphic forms) embeds in $\bigoplus_{\alpha \in \Delta} M_\alpha$ where each M_α is a Banach space on which the operator $[G_0 \eta_i G_0]$ has norm $\leq |\alpha(\eta_i)|^{-(1+n_\alpha)}$. The following criterion follows:

Lemma 2.3.2.1. *If an overconvergent eigenform $f \in M(e, E_\lambda)$ satisfies $[G_0\eta_i G_0]f = \mu_i f$ with $\mu_i \neq 0$ and*

$$v_p(\mu_i) < \inf_{\alpha \in \Delta_i} -(1+n_\alpha) v_p(\alpha(\eta_i))$$

for all i, then f is classical.

In the case of the symplectic group $\mathbf{G}'$, the family $(\eta_i)_i$ can be indexed by the couples (v, i) where v is a place of F above p and $1 \leq i \leq n$, and $\Delta_{v,i}$ is indexed by the embeddings $F_v \hookrightarrow E$. Specifically, $\eta_{v,i}$ is trivial at all the places except for v, where it equals

$$\mathrm{Diag}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1})$$

with $x_j = \begin{cases} \varpi_v^{-1} & \text{if } j \leq i \\ 1 & \text{if } j > i \end{cases}$.

The conditions in the previous lemma can be written

$$\begin{cases} v_p(\mu_{v,i}) < \frac{1}{e_v} \inf_\sigma (1 + k_{v,\sigma,i} - k_{v,\sigma,i+1}) & \text{for } i < n \\ v_p(\mu_{v,n}) < \frac{1}{e_v} \inf_\sigma (2 + 2k_{v,\sigma,n}). \end{cases}$$

2.3.2.2 Representations having Iwahori-invariants and unramified principal series

We recall results of Casselman showing that irreducible representations having Iwahori-invariants appear in unramified principal series, and giving the Atkin-Lehner eigenvalues in terms of the unramified character being induced.

In this subsection, we fix a place v of F above p. Recall I_v has an Iwahori decomposition $I_v = N_{v,0} T_{v,0} \bar{N}_{v,0}$. As in [Cas], if (Π, V) is a smooth representation of $\mathbf{G}(F_v)$, $V(\bar{N}_v)$ is the subspace of V spanned by the $\Pi(\bar{n})(x) - x$, $\bar{n} \in \bar{N}_v$, $V_{\bar{N}_v} = V/V(\bar{N}_v)$ and if $\bar{N}_{v,i}$ is a compact subgroup of $\bar{N}_v$, $V(\bar{N}_{v,i}) = \left\{ v \in V \mid \int_{\bar{N}_{v,i}} \Pi(\bar{n})(v) d\bar{n} = 0 \right\}$.

Lemma 2.3.2.2. *Let (Π, V) be an admissible representation of $\mathbf{G}(F_v)$ over $\mathbb{C}$. Then the natural (vector space) morphism from V^{I_v} to $\left(V_{\bar{N}_v}\right)^{T_{v,0}}$ is an isomorphism, inducing a Σ_v^+-equivariant isomorphism*

$$\Pi^{I_v} \xrightarrow{\sim} \left(\Pi_{\bar{N}_v}\right)^{T_{v,0}} \otimes \delta_{\bar{B}_v}^{-1}$$

where $\delta_{\bar{B}_v}$ denotes the modulus morphism of $\bar{B}_v$, and $u \in \Sigma_v^+$ acts on Π^{I_v} by $[I_v u I_v]$.

Proof. Let $\bar{N}_{v,1}$ be a compact subgroup of $\bar{N}_v$ such that $V^{I_v} \cap V(\bar{N}_v) \subset V(\bar{N}_{v,1})$. There is a $u \in \Sigma_v^+$ such that $u\bar{N}_{v,1}u^{-1} \subset \bar{N}_{v,0}$. By [Cas, Prop. 4.1.4], and using the fact that $[I_v u I_v]$ is invertible in the Hecke-Iwahori algebra, the natural morphism from V^{I_v} to $V_{\bar{N}}^{T_{v,0}}$ is an isomorphism (of vector spaces).

Lemmas 4.1.1 and 1.5.1 in [Cas] allow to compute the action of Σ_v^+. □

Corollary 2.3.2.3. *Any smooth irreducible representation of $\mathbf{G}(F_v)$ over $\mathbb{C}$ having Iwahori invariants is a subquotient of the parabolic induction (from $\bar{B}_v$) of a character of the torus T_v, which is unique up to the action of $W(T_v, \mathbf{G}(F_v))$, and unramified.*

Proof. Π is a subquotient of the parabolic induction of a character of the torus T_v if and only if $\Pi_{\bar{N}_v} \neq 0$, which is true by the previous lemma. The geometrical lemma [BZ77, 2.12] shows that if χ is a smooth character of T_v,

$$\left(\mathrm{Ind}_{\bar{B}_v}^{\mathbf{G}(F_v)} \chi\right)_{\bar{N}_v}^{\mathrm{ss}} \simeq \bigoplus_{w \in W(T_v, \mathbf{G}(F_v))} \chi^w \delta_{\bar{B}_v}^{1/2}$$

□

Since $*_{\bar{N}}$ is left adjoint to *non-normalized* induction, the first argument in the proof shows that Π is actually a subrepresentation of $\mathrm{Ind}_{\bar{B}_v}^{\mathbf{G}(F_v)}$ for at least one χ in the orbit under $W(T_v, \mathbf{G}(F_v))$. In that case we will say that (Π, χ) is a *refinement* of Π. Note that up to the action of $W(T_v, \mathbf{G}(F_v))$, there is a unique χ such that Π is a subquotient of $\mathrm{Ind}_{\bar{B}_v}^{\mathbf{G}(F_v)}$.

2.3.2.3 Most points of the eigenvariety arise from unramified, completely refinable representations

We will need a result of Tadić, characterizing the *irreducible* principal series. If $\chi_1, \dots, \chi_n$ are characters of $F_v^\times$, we denote simply by $\chi = (\chi_1, \dots, \chi_n)$ the character of T_v which maps

$$\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_n^{-1} & & \\ & & & & \ddots & \\ & & & & & x_1^{-1} \end{pmatrix}$$

to $\prod_{i=1}^n \chi_i(x_i)$. Let ν be the unramified character of $F_v^\times$ such that $\nu(\varpi_v) = |\mathbb{F}_v|^{-1}$.

Theorem 2.3.2.4. *Let $\chi = (\chi_1, \dots, \chi_n)$ be a character of T_v. Then $\mathrm{Ind}_{\bar{B}_v}^{\mathrm{Sp}_{2n}(F_v)}\chi$ is irreducible if and only if the following conditions are satisfied*

1. *For all i, χ_i is not of order 2.*
2. *For all i, $\chi_i \neq \nu^{\pm 1}$.*
3. *For all distinct i, j, $\chi_i\chi_j^{-1} \neq \nu^{\pm 1}$ and $\chi_i\chi_j \neq \nu^{\pm 1}$.*

Proof. [Tad94, Theorem 7.1] □

Definition 2.3.2.5. *An irreducible representation Π_v of $\mathbf{G}(F_v)$ is* completely refinable *if it is isomorphic to $\mathrm{Ind}_{\bar{B}_v}^{\mathrm{Sp}_{2n}(F_v)}\chi$ for some unramified character χ.*

An automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ is completely refinable if Π_v is completely refinable for any $v|p$.

Note that completely refinable representations are unramified (for any choice of hyperspecial subgroup). A representation Π_v is completely refinable if and only if $(\Pi_v)_{\bar{N}_v}^{\mathrm{ss}}$ is the sum of $|W(T_v, \mathbf{G}(F_v))|$ unramified characters.

Recall that classical points on the eigenvariety are determined by an automorphic representation Π together with a refinement of each Π_v, $v|p$. Completely refinable automorphic representations are the ones giving the greatest number of points on the eigenvariety. When one can associate Galois representations to automorphic representations, each refinement of Π comes with a "p-adic family" of Galois representations going through the same one.

Proposition 2.3.2.6. *Let $f_1, \dots, f_r \in \mathcal{O}(\mathscr{X})^\times$. The set $\mathcal{S}$ of points corresponding to completely refinable, unramified classical points at which*

$$\min_{v,\sigma} \min\{k_{v,\sigma,1} - k_{v,\sigma,2}, \dots, k_{v,\sigma,n-1} - k_{v,\sigma,n}, k_{v,\sigma,n}\} \geq \max\{v_p(f_1), \dots, v_p(f_n)\} \quad (2.3.2.2)$$

is Zariski dense and accumulates at all the algebraic points.

Compare [Che04, Proposition 6.4.7], [Loe11, Corollary 3.13.3].

Proof. The hypotheses in the classicality criterion 2.3.2.1 and the ones in Theorem 2.3.2.4 are implied by inequalities of the form 2.3.2.2. First we prove the accumulation property. We can restrict to open affinoids $\mathscr{V}$ of the eigenvariety, and hence assume that the right hand side of 2.3.2.2 is replaced by a constant. By Theorem 2.3.1.1, $\mathscr{V}$ can be an arbitrarily small open affinoid containing an algebraic point x of $\mathscr{X}$, such that there is open affinoid $\mathscr{U}$ of $\mathscr{W}$ such that $\mathscr{V} \subset w^{-1}(\mathscr{U})$, the morphism $w|_{\mathscr{V}} : \mathscr{V} \to \mathscr{U}$ is finite, and surjective when restricted to any irreducible component of $\mathscr{V}$. By Proposition 2.3.1.3, the algebraic weights satisfying 2.3.2.2 are Zariski dense in the weight space $\mathscr{W}$ and accumulate at all the algebraic points of $\mathscr{W}$. [Che04, Lemme 6.2.8] shows that $\mathcal{S} \cap \mathscr{V}$ is Zariski-dense in $\mathscr{V}$.

Each irreducible component $\mathscr{X}'$ of $\mathscr{X}$ is mapped onto a Zariski-open subset of a connected component of $\mathscr{W}$, by [Che04, Corollaire 6.4.4] (which is a consequence of the decomposition of a Fredholm series into a product of prime Fredholm series, [Con99, Corollary 4.2.3]), so $\mathscr{X}'$ contains at least one algebraic point (the algebraic weights intersect all the connected components of $\mathscr{W}$), and hence the Zariski closure of $\mathcal{S} \cap \mathscr{X}'$ contains an open affinoid of $\mathscr{X}'$, which is Zariski dense in $\mathscr{X}'$. □

2.4 Galois representations associated with automorphic representations of symplectic groups

2.4.1 A consequence of Arthur's description of the discrete spectrum for classical groups

2.4.1.1 Automorphic self-dual representations of GL_{2n+1} of orthogonal type

According to Arthur's conjectural parametrization of discrete automorphic representations, each such representation of $\mathbf{G}(\mathbb{A}_F)$ should be part of an A-packet corresponding to a discrete parameter, which is a representation

$$\mathcal{L}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$$

such that (among other conditions) the commutant of the image is finite.

The standard embedding $\mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$ "transfers" this parameter to a parameter of GL_{2n+1}/F, which is not discrete in general, and thus it corresponds to an automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$. Here we define an automorphic representation π of $\mathrm{GL}_N(\mathbb{A}_F)$ as a formal sum of discrete automorphic representations π_i of GL_{n_i} such that $\sum_i n_i = N$. We will write $\pi = \boxplus_i \pi_i$. By [MW89], each π_i is the Langlands quotient of the parabolic induction of twists of a single cuspidal representation by powers of $|\det|$. We will not need this generality, as we will force the representations π_i to be cuspidal in the sequel.

Since π comes from a self-dual parameter, it is self-dual: $\pi^\vee \simeq \pi$. Even though π is not discrete in general, the discreteness of the parameter which takes values in SO_{2n+1} implies that the π_i's are self-dual.

If $\Pi = \otimes_v \Pi_v$ is an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$, then for any archimedean place v of F, the local Langlands parameter of Π_v composed with $\mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$ is of the form:

$$\mathcal{LL}(\Pi_v) \simeq \epsilon^n \oplus \bigoplus_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

where ϵ is the only non-trivial character of $W_\mathbb{C}/W_\mathbb{R}$, and the r_i are integers, with $r_n > r_{n-1} > \ldots > r_1 > 0$. We define $A_{\mathrm{Sp}_{2n}}$ to be the set of automorphic representations such that for each infinite place v of F, $r_1 \geq 2$ and $r_{i+1} \geq r_i + 2$. The equivalence above is meant as representations of $W_\mathbb{R}$ (i.e. morphisms $W_\mathbb{R} \to \mathrm{GL}_{2n+1}(\mathbb{C})$), although $\mathcal{LL}(\Pi_v)$ is a parameter taking values in $\mathrm{SO}_{2n+1}(\mathbb{C})$ (the two notions coincide).

Similarly, let $A_{\mathrm{GL}_{2n+1}}$ be the set of formal sums of self-dual cuspidal representations $\pi = \boxplus_i \pi_i = \otimes_v \pi_v$ of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$ such that for each infinite place v of F,

$$\mathcal{LL}(\pi_v) \simeq \epsilon^n \oplus \bigoplus_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

where the r_i's are integers,such that $r_1 \geq 2$, $r_{i+1} \geq r_i + 2$, and such that the product of the central characters of the π_i's is trivial.

These inequalities are imposed to ensure that the corresponding global parameters are trivial on Arthur's $\mathrm{SL}_2(\mathbb{C})$, to simplify the statements. That is why we take formal sums of *cuspidal* (not discrete) representations.

Note that there is no non-zero alternate bilinear form preserved by such a parameter (one could say that the parameter is "completely orthogonal").

Assumption 2.4.1.1. *For any $\Pi \in A_{\mathrm{Sp}_{2n}}$, there is a $\pi \in A_{\mathrm{GL}_{2n+1}}$, such that the local Langlands parameters match at the infinite places, and for any finite place v of F, π_v is unramified if Π_v is unramified, and in that case the local parameters match, by means of the inclusion $\mathrm{SO}_{2n+1}(\mathbb{C}) \subset \mathrm{GL}_{2n+1}(\mathbb{C})$.*

2.4.1.2 p-adic Galois representations associated with RLASDC representations of GL_N

An automorphic cuspidal representation π of $\mathrm{GL}_N(\mathbb{A}_F)$ is said to be *L-algebraic* if for any infinite place v of F, the restriction of $\mathcal{LL}(\pi_v)$ to $\mathbb{C}^\times$ is of the form

$$z \mapsto \mathrm{Diag}\left(\left(z^{a_{v,i}}\bar{z}^{b_{v,i}}\right)_i\right)$$

where $a_i, b_i \in \mathbb{Z}$. By the "purity lemma" [Clo88, Lemme 4.9], $a_{v,i} + b_{v,i}$ does not depend on v, i. We will say that π is L-algebraic *regular* if for any v as above, the $a_{v,i}$ are distinct. By purity, this implies that if v is real,

$$\mathcal{LL}(\pi_v)|\cdot|^{-s} = \begin{cases} \epsilon^e \oplus_i \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}}\left(z \mapsto (z/\bar{z})^{a'_{v,i}}\right) & \text{if } N \text{ is odd, with } e = 0,1 \\ \oplus_i \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}}\left(z \mapsto (z/\bar{z})^{a'_{v,i}}\right) & \text{if } N \text{ is even} \end{cases}$$

for some integer s, and integers $0 < a'_{v,1} < \ldots < a'_{v,\lfloor N/2 \rfloor}$.

As a special case of [CH13, Theorem 4.2] (which builds on previous work of Clozel, Harris, Kottwitz, Labesse, Shin, Taylor), we have the following theorem.

Theorem 2.4.1.2. *Let π be a regular L-algebraic, self-dual, cuspidal (RLASDC) representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$. Then π is L-arithmetic, and there is a continuous Galois representation*

$$\rho_{\iota_p,\iota_\infty}(\pi) : G_F \longrightarrow \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_p)$$

such that if v is a finite place of F and π_v is unramified,

1. *if v is coprime to p, then $\rho_{\iota_p,\iota_\infty}(\pi)|_{G_{F_v}}$ is unramified, and*

$$\det\left(T\mathrm{Id} - \rho_{\iota_p,\iota_\infty}(\pi)(\mathrm{Frob}_v)\right) = \iota_p\iota_\infty^{-1}\det\left(T\mathrm{Id} - A\right)$$

   where $A \in \mathrm{GL}_N(\mathbb{C})$ is associated with π_v via the Satake isomorphism.

2. *if v lies above p, $\rho_{\iota_p,\iota_\infty}(\pi)|_{G_{F_v}}$ is crystalline. The associated filtered φ-module (over $F_{v,0} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$) is such that*

$$\det_{\overline{\mathbb{Q}}_p}\left(T\mathrm{Id} - \varphi^{f_v}\right) = \iota_p\iota_\infty^{-1}\det\left(T\mathrm{Id} - A\right)^{f_v}$$

   where $A \in \mathrm{GL}_N(\mathbb{C})$ is associated with π_v via the Satake isomorphism. For any $\sigma : F_v \to \overline{\mathbb{Q}}_p$, the σ-Hodge-Tate weights are the $a_{w,i}$, where w is the real place of F defined by σ, ι_p and ι_∞.

The power f_v appearing at places above p may seem more natural to the reader (and will actually disappear) after reading subsubsection 2.4.2.1.

Combining this theorem with the transfer detailed in the last section, we obtain

Corollary 2.4.1.3. *Let Π be an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$, whose weights $k_{w,1} \geq k_{w,2} \geq \dots k_{w,n} \geq 0$ at the real places w are far from the walls ($\Pi \in A_{\mathrm{Sp}_{2n}}$ is enough), and unramified at the places above p. There exists a continuous semisimple Galois representation*

$$\rho_{\iota_p,\iota_\infty}(\Pi) : G_F \longrightarrow \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}}_p)$$

such that for any finite place v of F such that Π_v is unramified

1. *if v is coprime to p, then $\rho_{\iota_p,\iota_\infty}(\Pi)|_{G_{F_v}}$ is unramified, and*

$$\det\left(T\mathrm{Id} - \rho_{\iota_p,\iota_\infty}(\Pi)(\mathrm{Frob}_v)\right) = \iota_p\iota_\infty^{-1}\det\left(T\mathrm{Id} - A\right)$$

   where $A \in \mathrm{GL}_N(\mathbb{C})$ is associated with Π_v via the Satake isomorphism.

2. *if v lies above p, $\rho_{\iota_p,\iota_\infty}(\Pi)|_{G_{F_v}}$ is crystalline. The associated filtered φ-module is such that*

$$\det_{\overline{\mathbb{Q}}_p}\left(T\mathrm{Id} - \varphi^{f_v}\right) = \iota_p\iota_\infty^{-1}\det\left(T\mathrm{Id} - A\right)^{f_v}$$

   where $A \in \mathrm{SO}_{2n+1}(\mathbb{C}) \subset \mathrm{GL}_{2n+1}(\mathbb{C})$ is associated with Π_v via the Satake isomorphism. For any $\sigma : F_v \to \overline{\mathbb{Q}}_p$, the σ-Hodge-Tate weights are $k_{w,1}+n > k_{w,2}+n-1 > \dots > k_{w,1}+1 > 0 > -k_{w,1}-1 > \dots > -k_{w,1}-n$, where w is the real place of F defined by σ, ι_p and ι_∞.

Proof. There is an automorphic representation $\pi = \boxplus_i \pi_i$ of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$ corresponding to Π by Assumption 2.4.1.1, obtained by induction from distinct cuspidal representations π_i. Let $\rho_{\iota_p,\iota_\infty}(\Pi) = \oplus_i \rho_{\iota_p,\iota_\infty}(\pi_i)$. □

Note that in that case, since Π_∞ is C-algebraic, Π is obviously C-arithmetic (which is equivalent to L-arithmetic in the case of Sp_{2n}), and thus the coefficients of the polynomials appearing in the corollary lie in a finite extension of $\mathbb{Q}$.

2.4.1.3 The Galois pseudocharacter on the eigenvariety

To study families of representations, it is convenient to use *pseudorepresentations* (or *pseudocharacters*), which are simply the traces of semi-simple representations when the coefficient ring is an algebraically closed field of characteristic zero. We refer to [Tay91] for the definition, and [Tay91, Theorem 1] is the "converse theorem" we will need.

On $\mathcal{O}(\mathscr{X})$, we put the topology of uniform convergence on open affinoids.

The Zariski-density of the classical points at which we can define an attached Galois representation implies the following

Proposition 2.4.1.4. *There is a continuous pseudocharacter $T : G_F \to \mathcal{O}(\mathscr{X})$, such that at every classical unramified point of the eigenvariety having weight far from the walls, T specializes to the character of the Galois representation associated with the automorphic representation by Corollary 2.4.1.3.*

Proof. This is identical to the unitary case, and thus is a consequence of [Che04, Proposition 7.1.1], by Proposition 2.3.2.6. □

Thus at any (classical or not) point of the eigenvariety, there is an attached Galois representation.

2.4.2 Galois representations stemming from symplectic forms are generically almost irreducible

2.4.2.1 Crystalline representations over $\overline{\mathbb{Q}}_p$

We fix a finite extension K of $\mathbb{Q}_p$, and denote K_0 the maximal unramified subextension, $e = [K : K_0]$, $f = [K_0 : \mathbb{Q}_p]$. Let $\rho : G_K \to \mathrm{GL}(V)$ be a continuous representation of the absolute Galois group of K, where V is a finite dimensional vector space over L, a finite Galois extension of $\mathbb{Q}_p$. We will take L to be big enough so as to be able to assume in many situations that $L = \overline{\mathbb{Q}}_p$. For example, we can assume that L is an extension of K, and that ρ has a composition series $0 = V_1 \subset \ldots \subset V_r = V$ such that each quotient V_{i+1}/V_i is absolutely irreducible.

For any such ρ, we denote $D_{\mathrm{cris}}(V) = \left(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}$. From now on we assume that ρ is a crystalline representation, which means that $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$. It is well-known that $D_{\mathrm{cris}}(V)$ is a filtered φ-module over K, and since V is a vector space over L, $D_{\mathrm{cris}}(V)$ is a φ-module over $K_0 \otimes_{\mathbb{Q}_p} L$, and $D_{\mathrm{dR}}(V) = K \otimes_{K_0} D_{\mathrm{cris}}(V)$ is a module over $K \otimes_{\mathbb{Q}_p} L$ with a filtration by projective submodules.

We have a natural decomposition $K_0 \otimes_{\mathbb{Q}_p} L \simeq \prod_{\sigma_0 \in \Upsilon_0} L_{\sigma_0}$ with $\Upsilon_0 = \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{alg.}}(K_0, L)$ and $L_{\sigma_0} \simeq L$, given by the morphisms $\sigma_0 \otimes \mathrm{Id}_L$. Similarly, $K \otimes_{\mathbb{Q}_p} L \simeq \prod_{\sigma \in \Upsilon} L_\sigma$ with $\Upsilon = \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{alg.}}(K, L)$.

Hence we have decompositions

$$D_{\mathrm{cris}}(V) = \prod_{\sigma_0 \in \Upsilon_0} D_{\mathrm{cris}}(V)_{\sigma_0}, \quad D_{\mathrm{dR}}(V) = \prod_{\sigma \in \Upsilon} D_{\mathrm{dR}}(V)_\sigma.$$

The operator φ restricts as linear isomorphisms from $D_{\mathrm{cris}}(V)_{\sigma_0}$ to $D_{\mathrm{cris}}(V)_{\sigma_0 \circ \varphi^{-1}}$, and so φ^f is a L_{σ_0}-linear automorphism on each $D_{\mathrm{cris}}(V)_{\sigma_0}$, which are isomorphic as vector spaces over L equipped with the linear automorphism φ^f.

Each $D_{\mathrm{dR}}(V)_\sigma$ comes with a filtration, and hence defines $\dim_L V = N$ Hodge-Tate weights $k_{\sigma,1} \leq \ldots \leq k_{\sigma,N}$ (the jumps of the filtration).

Although we will not use it, it should be noted that by [BM02, Proposition 3.1.1.5], to verify the weak admissibility of a filtered φ-module D over K with an action of L commuting with φ and leaving the filtration stable, it is enough to check the inequality $t_N(D') \geq t_H(D')$ for sub-$K_0 \otimes L$-modules stable under φ.

If φ^f has eigenvalues $\varphi_1, \ldots, \varphi_N$, with $v_p(\varphi_1) \leq \ldots \leq v_p(\varphi_n)$, we can in particular choose $D' = \oplus_{i \leq j} \ker(\varphi^f - \varphi_i)$ (if the eigenvalues are distinct, but even if they are not, we can choose D' such that $\varphi^f|_{D'}$ has eigenvalues $\varphi_1, \ldots, \varphi_j$, counted with multiplicities).

The worst case for the filtration yields the inequalities

$$\begin{aligned} v_p(\varphi_1) &\geq \frac{1}{e}\sum_\sigma k_{\sigma,1} \\ v_p(\varphi_1\varphi_2) &\geq \frac{1}{e}\sum_\sigma k_{\sigma,1}+k_{\sigma,2} \\ &\vdots \end{aligned}$$

In the sequel, we will only use these inequalities, and we will not be concerned with the subtleties of the filtrations.

2.4.2.2 Variation of the crystalline Frobenius on the eigenvariety

In this section we explicit the formulas relating the eigenvalues of the crystalline Frobenius at classical, unramified points of the eigenvariety and the eigenvalues of the Hecke-Iwahori operators acting on p-adic automorphic forms. Let x be a classical point on the eigenvariety. There is an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ such that $\iota_p\iota_\infty^{-1}(\Pi_\infty)$ is the representation having highest weight $w(x)$. Assume that Π_p is unramified. The point x defines a refinement of Π_p, that is an unramified character $\chi_x : T_0 \to \mathbb{C}^\times$ such that $\Pi_p \hookrightarrow \mathrm{Ind}_{\bar{B}}^{\mathbf{G}'(\mathbb{Q}_p)}\chi_x$, or equivalently the character $\delta_{\bar{B}}^{1/2}\chi_x$ appearing in $(\Pi_p)_{\bar{N}}$. By 2.3.2.2, for any $u\in\Sigma^+$, $\mu_{w(x)}\Psi_x|_{\mathcal{H}_p} = (\iota_p\circ\iota_\infty^{-1}\circ\chi_x)\delta_B^{1/2}$.

The diagonal torus in $\mathrm{SO}_{2n+1}(\mathbb{C})$ and the identification of it with the dual of the diagonal torus of Sp_{2n}/F_v being fixed, the character χ_x is mapped by the unramified Langlands correspondence for tori to $y=(y_v)_{v|p}$ with $y_v = \mathrm{Diag}(y_{1,v},\dots,y_{n,v},1,y_{n,v}^{-1},\dots,y_{1,v}^{-1})$, and $y_{v,i} = \chi_x(\mathrm{Diag}(1,\dots,\varpi_v,\dots,1,1,\dots,\varpi_v^{-1},\dots,1))$ (ϖ_v being the i-th element). Thus the linearization of the crystalline Frobenius φ^{f_v} on $D_{\mathrm{cris}}(\rho_{\iota_p,\iota_\infty}(\pi)|_{G_{F_v}})_{\sigma_0}$ (for any choice of $\sigma_0 : F_v\to E$ in $\Upsilon_{0,v}$) has eigenvalues

$$\iota_p\iota_\infty^{-1}(y_{v,i}) = q_v^{n+1-i}\phi_{v,n+1-i}(x)\prod_{\sigma\in\Upsilon_v}\sigma(\varpi_v)^{k_{v,\sigma,i}}$$

and their inverses, together with the eigenvalue 1. Here $\phi_{v,n+1-i}\in\mathcal{O}(\mathscr{X})$ is defined by

$$\phi_{v,n+1-i} = \frac{\Psi\left([G_0u_{i-1}G_0]\right)}{\Psi\left([G_0u_iG_0]\right)}$$

with $u_i = \mathrm{Diag}(\varpi_{\mathrm{v}}^{-1},\dots,\varpi_{\mathrm{v}}^{-1},1,\dots,1,\varpi_{\mathrm{v}},\dots,\varpi_{\mathrm{v}})$ (the last ϖ_v^{-1} is the i-th element), and $k_{v,\sigma,i}$ the integers defining the weight $w(x)$.

Assume furthermore that Π_p admits another refinement $\chi_{x'} = \chi_x^a$ for some $a=(a_v)_{v|p}$ in the Weyl group $W(\mathbf{G}'(\mathbb{Q}_p),\mathbf{T}(\mathbb{Q}_p)) = \prod_v W(\mathbf{G}(F_v),T_v)$. Each $W(\mathbf{G}(F_v),T_v)$ can be identified with the group of permutations $a_v : \{-n,\dots,n\}\to\{-n,\dots,n\}$ such that $a_v(-i) = -a_v(i)$ for all i, acting by

$$a_v(\mathrm{Diag}(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1})) = \mathrm{Diag}(x_{a_v^{-1}(1)},\dots,x_{a_v^{-1}(n)},x_{a_v^{-1}(-n)},\dots,x_{a_v^{-1}(1)})$$

on T_v, where for commodity we set $x_{-i} = x_i^{-1}$ for $i<0$. Similarly we define $k_{v,\sigma,-i} = -k_{v,\sigma,i}$ and $\phi_{v,-i} = \phi_{v,i}^{-1}$. We also set $k_{v,\sigma,0}=0$, $\phi_{v,0}=1$. The equality $\chi_{x'}=\chi_x^a$ can also be written

$$q_v^{(n+1)\mathrm{sign}(w(i))-w(i)}\phi_{v,n+1-w(i)}(x)\prod_{\sigma\in\Upsilon_v}\sigma(\varpi_v)^{k_{v,\sigma,w(i)}} = q_v^{n+1-i}\phi_{v,n+1-i}(x')\prod_{\sigma\in\Upsilon_v}\sigma(\varpi_v)^{k_{v,\sigma,i}}$$

which is valid for any $-n\le i\le n$ if we set $\mathrm{sign}(i)=-1$ (resp. 0, 1) if i is negative (resp. zero, positive), and equivalent to

$$\phi_{v,n+1-i}(x') = \phi_{v,n+1-w(i)}(x)q_v^{i-w(i)+(n+1)(\mathrm{sign}(i)-\mathrm{sign}(w(i)))}\prod_{\sigma\in\Upsilon_v}\sigma(\varpi_v)^{k_{v,\sigma,w(i)}-k_{v,\sigma,i}}.$$

This last formula will be useful in the proof of the main result.

2.4.2.3 Main result

Lemma 2.4.2.1. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\rho : G_K \to \mathrm{GL}_N(\overline{\mathbb{Q}}_p)$ be a crystalline representation. Let $(D, \varphi, \mathrm{Fil}^i D\otimes_{K_0} K)$ be the associated filtered φ-module. Let $\kappa_{\sigma,1}\le\ldots\le\kappa_{\sigma,N}$ be the Hodge-Tate weights associated with the embedding $\sigma : K\hookrightarrow\overline{\mathbb{Q}}_p$. Let $\varphi_1,\ldots,\varphi_N$ be the eigenvalues of the linear operator φ^f (on any of the D_{σ_0}, $\sigma_0\in\Upsilon_0$), and suppose they are distinct. Finally, assume that for some $\tau\in\Upsilon$, for all i,*

$$\left|v_p(\varphi_i)-\frac{1}{e}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,i}\right|\le\frac{1}{eN}\min_{1\le j\le N-1}\kappa_{\tau,j+1}-\kappa_{\tau,j}.$$

Then if $D'\subset D$ is an admissible sub-φ-module over $K_0\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}}_p$ (corresponding to a sub-representation), there is a subset I of $\{1,\ldots,N\}$ such that D' has φ^f-eigenvalues $(\varphi_i)_{i\in I}$ and τ-Hodge-Tate weights $(\kappa_{\sigma,i})_{i\in I}$.

Proof. Since the eigenvalues of φ^f are distinct, and D' is stable under φ, there is a subset I of $\{1,\ldots,N\}$ such that $D'=\ker\prod_{i\in I}\left(\varphi^f-\varphi_i\right)$. There are unique increasing functions $\theta_{1,\sigma}: I\to\{1,\ldots,N\}$ such that the σ-weights of D' are the $\kappa_{\sigma,\theta_{1,\sigma}(i)}$, for $i\in I$. By ordering similarly the weights of D/D', we define increasing functions $\theta_{2,\sigma}:\{1,\ldots,N\}\setminus I\to\{1,\ldots,N\}$, and we can glue the $\theta_{\cdot,\sigma}$ to get bijective maps $\theta_\sigma:\{1,\ldots,N\}\to\{1,\ldots,N\}$. We will show that $\theta_\tau=\mathrm{Id}$.

We now write the admissibility condition for D' and D/D'. Let i_1 be the smallest element of I. Then $\ker\left(\varphi^f-\varphi_{i_1}\right)$ is a sub-φ-module of D'. Its induced σ-weight is one of the $\kappa_{\sigma,\theta_\sigma(i)}$ for $i\in I$, thus it is greater than or equal to $\kappa_{\sigma,\theta_\sigma(i_1)}$. This implies that $v_p(\varphi_{i_1})\ge 1/e\sum_{\sigma\in\Upsilon}\kappa_{\sigma,\theta_\sigma(i_1)}$. We can proceed similarly for the submodules

$$\ker\left(\left(\varphi^f-\varphi_{i_1}\right)\ldots\left(\varphi^f-\varphi_{i_r}\right)\right)$$

(where the $i_\cdot$ are the ordered elements of I), to get the inequality

$$\sum_{1\le x\le r}v_p(\varphi_{i_x})\ge\frac{1}{e}\sum_{1\le x\le r}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,\theta_\sigma(i_x)}$$

The same applies to D/D', and by adding both inequalities, we finally get

$$\sum_{1\le i\le s}v_p(\varphi_i)\ge\frac{1}{e}\sum_{1\le i\le s}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,\theta_\sigma(i)}$$

We now isolate τ, using the fact that $\sum_{1\le i\le s}\kappa_{\sigma,\theta_\sigma(i)} \ge \sum_{1\le i\le s}\kappa_{\sigma,i}$ for $\sigma\neq\tau$, and obtain the inequality

$$\sum_{1\le i\le s} v_p(\varphi_i) - \frac{1}{e}\sum_{1\le i\le s}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,i} \ge \frac{1}{e}\sum_{1\le i\le s}\kappa_{\tau,\theta_\tau(i)} - \kappa_{\tau,i}$$

Let r be minimal such that $\theta_\tau(s)\neq s$ (if no such s exists, we are done). In that case, we necessarily have $\theta_\tau(s)\ge s+1$, and the previous inequality yields

$$\sum_{1\le i\le s} v_p(\varphi_i) - \frac{1}{e}\sum_{1\le i\le s}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,i} \ge \frac{\kappa_{\tau,s+1}-\kappa_{\tau,s}}{e}$$

but the hypothesis implies that the left hand side is less than $\min_j\left(\kappa_{\tau,j+1}-\kappa_{\tau,j}\right)/e$, and we get a contradiction. $\square$

Theorem 2.4.2.2. *Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori invariants at all the places of F above p, and having invariants under an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$. Let N be an integer. There exists an automorphic representation Π' of $\mathbf{G}(\mathbb{A}_F)$ such that:*

- *Π' is unramified at the places above p, and has invariants under U;*
- *The restriction of $\rho_{\iota_p,\iota_\infty}(\Pi')$ to the decomposition group at any place above p is either irreducible or the sum of an Artin character and an irreducible representation of dimension $2n$;*
- *For all g in G_F, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(g)) \equiv \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(g)) \mod p^N$.*

Proof. We will write $\Pi'\equiv\Pi \mod p^N$ for the last property.

Recall that for v a place of F above p, there are elements $\phi_{v,1},\dots,\phi_{v,n}\in\mathcal{O}(\mathscr{X})^\times$ such that for any unramified classical point $x\in\mathscr{X}(\overline{\mathbb{Q}}_p)$ refining an automorphic representation Π, the filtered φ-module associated with the crystalline representation $\rho_{\iota_p,\iota_\infty}(\Pi)|_{G_{F_v}}$ has φ^{f_v}-eigenvalues

$$\Bigg(\phi_{v,-n}(x)q_v^{-n}\prod_\sigma\sigma(\varpi_v)^{k_{v,\sigma,-1}},\dots,\phi_{v,-1}(x)q_v^{-1}\prod_\sigma\sigma(\varpi_v)^{k_{v,\sigma,-n}},1,$$
$$\phi_{v,1}(x)q_v\prod_\sigma\sigma(\varpi_v)^{k_{v,\sigma,n}},\dots,\phi_{v,n}(x)q_v^{n}\prod_\sigma\sigma(\varpi_v)^{k_{v,\sigma,1}}\Bigg)$$

and σ-Hodge-Tate weights

$$k_{v,\sigma,-1}-n,\dots,k_{v,\sigma,-n}-1,0,k_{v,\sigma,n}+1,\dots,k_{v,\sigma,1}+n$$

In the following if x_b or x'_b is a classical point, $k_{v,\sigma,i}^{(b)}$ will be the weights defining $w(x_b)$. The representation Π corresponds to at least one point x of the eigenvariety $\mathscr{X}$ for $\mathbf{G}'$ and the idempotent $e_U\otimes e_{G_0}$. By Proposition 2.3.2.6, and since G_F is compact, there exists a point $x_1\in\mathscr{X}(E')$ (near x, and for some finite extension E' of E) corresponding to an unramified, completely refinable automorphic representation Π_1 and a refinement χ, such that for any v,

$$\frac{2}{e_v}\sum_{i=1}^{n}\sum_{\sigma} k^{(1)}_{v,\sigma,i} > -v_p\left(\phi_{v,1}(x_1)\dots\phi_{v,n}(x_1)\right) + 3n(n+1)f_v$$

and $\Pi_1 \equiv \Pi \mod p^N$. Since Π_1 is completely refinable, there is a point $x_1' \in \mathscr{X}(E')$ associated with the representation Π_1 and the character χ^a, where a is the element of the Weyl group acting as $-\mathrm{Id}$ on the roots. Specifically, $\Psi_{x_1}|_{\mathcal{H}^S\otimes\mathcal{H}_S\otimes e_{G_0}} = \Psi_{x_1'}|_{\mathcal{H}^S\otimes\mathcal{H}_S\otimes e_{G_0}}$, but

$$\phi_{v,n+1-i}(x_1') = \phi_{v,-n-1+i}(x_1)q_v^{2i+(2n+2)}\prod_{\sigma}\sigma(\varpi_v)^{-2k^{(1)}_{v,\sigma,i}}$$

for $i = 1,\dots,n$, and all places v. There exists a point $x_2 \in \mathscr{X}(E')$ (near x_1', and up to enlarging E') corresponding to an unramified, completely refinable automorphic representation Π_2 and a refinement, such that for any v and any $j < 0$,

$$\frac{1}{e_v}\sum_{\sigma} k^{(2)}_{v,\sigma,n+j} - k^{(2)}_{v,\sigma,n+j+1} > -v_p(\phi_{v,-j+1}(x_2)) - f_v$$

and $\Pi_2 \equiv \Pi_1 \equiv \Pi \mod p^N$. Like before, since Π_2 is completely refinable, there is a point $x_2' \in \mathscr{X}(E')$ such that $\Psi_{x_2}|_{\mathcal{H}^S\otimes\mathcal{H}_S\otimes e_{G_0}} = \Psi_{x_2'}|_{\mathcal{H}^S\otimes\mathcal{H}_S\otimes e_{G_0}}$, and

$$\begin{aligned}
\phi_{v,n}(x_2') &= \phi_{v,1}(x_2)q_v^{1-n}\prod_{\sigma}\sigma(\varpi_v)^{k^{(2)}_{v,\sigma,n}-k^{(2)}_{v,\sigma,1}}\\
\phi_{v,i}(x_2') &= \phi_{v,i+1}(x_2)q_v\prod_{\sigma}\sigma(\varpi_v)^{k^{(2)}_{v,\sigma,n-i}-k^{(2)}_{v,\sigma,n-i+1}} \text{ for } i = 1,\dots,n-1.
\end{aligned}$$

Here we used the element of the Weyl group corresponding (at each v) to the permutation

$$\begin{pmatrix} -n & -n+1 & \dots & -2 & -1 & 1 & \dots & n \\ -n+1 & -n+2 & \dots & -1 & -n & n & \dots & n-1 \end{pmatrix}.$$

Again, we can choose a point $x_3 \in \mathscr{X}(E')$ (near x_1', and up to enlarging E') corresponding to an unramified automorphic representation Π_3 and a refinement, such that for any v and any $\tau \in \Upsilon$,

$$\frac{1}{e_v(2n+1)}\min\left\{k^{(3)}_{v,\tau,1} - k^{(3)}_{v,\tau,2},\dots,k^{(3)}_{v,\tau,n-1} - k^{(3)}_{v,\tau,n}, k^{(3)}_{v,\tau,n}\right\} > \max\left\{0, |v_p(\phi_{v,\tau,1}(x_3))|,\dots,|v_p(\phi_{v,\tau,n}(x_3))|\right\}$$

and $\Pi_3 \equiv \Pi \mod p^N$. We show that Π_3 has the desired properties. First we apply the previous lemma to the local Galois representations associated with Π_3, at the places above p, which are crystalline. Since the differences $v_p(\varphi_i) - \frac{1}{e}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,i}$ in the hypotheses of the lemma are equal in our case to

$$-v_p(\phi_{v,n}(x_3)),\dots,-v_p(\phi_{v,1}(x_3)), 0, v_p(\phi_{v,1}(x_3)),\dots,v_p(\phi_{v,n}(x_3)),$$

the hypotheses of the lemma are satisfied for all $\tau \in \Upsilon$. Thus if $\rho_{\iota_p,\iota_\infty}(\pi_3)|_{G_{F_v}}$ is not irreducible, there is a subset $\emptyset \subsetneq I \subsetneq \{-n, \dots, n\}$ such that if $i_1 < \dots < i_r$ are the elements of I and $j_1 < \dots < j_{2n+1-r}$ those of $J = \{-n, \dots, n\} \setminus I$,

$$\begin{aligned}
&v_p(\phi_{v,i_1}(x_3)) \geq 0 \\
&v_p(\phi_{v,i_1}(x_3)) + v_p(\phi_{v,i_2}(x_3)) \geq 0 \\
&\vdots \\
&v_p(\phi_{v,i_1}(x_3)) + \dots + v_p(\phi_{v,i_r}(x_3)) = 0 \\
&v_p(\phi_{v,j_1}(x_3)) \geq 0 \\
&v_p(\phi_{v,j_1}(x_3)) + v_p(\phi_{v,j_2}(x_3)) \geq 0 \\
&\vdots \\
&v_p(\phi_{v,j_1}(x_3)) + \dots + v_p(\phi_{v,j_{2n+1-r}}(x_3)) = 0
\end{aligned}$$

by the admissibility of the corresponding filtered φ-modules. For all i, $v_p(\phi_{v,i}(x_2')) = v_p(\phi_{v,i}(x_3))$, so all these conditions hold also at x_2'. Up to exchanging I and J, we can assume that $i_1 = -n$. If $j_1 < 0$,

$$v_p(\phi_{v,j_1}(x_2')) = -v_p(\phi_{v,-j_1}(x_2')) = -v_p(\phi_{v,-j_1+1}(x_2)) - f_v - \frac{1}{e_v}\sum_\sigma k^{(2)}_{v,\sigma,n+j_1} - k^{(2)}_{v,\sigma,n+j_1+1}$$

and x_2 was chosen to ensure that this quantity is negative, so we are facing a contradiction. Thus J has only nonnegative elements, and $\{-n, \dots, -1\} \subset I$. If we do not assume that $i_1 = -n$, we have in general that $\{-n, \dots, -1\}$ is contained in I or J. Similarly, suppose $i_r = n$. If $j_{2n+1-r} > 0$,

$$\begin{aligned}
v_p(\phi_{v,j_{2n+1-r}}(x_2')) &= v_p(\phi_{v,j_{2n+1-r}}(x_2')) \\
&= v_p(\phi_{v,j_{2n+1-r}}(x_2)) + f_v + \frac{1}{e_v}\sum_\sigma k^{(2)}_{v,\sigma,n-j_{2n+1-r}} - k^{(2)}_{v,\sigma,n-j_{2n+1-r}+1}
\end{aligned}$$

is positive, another contradiction. Therefore $\{1, \dots, n\}$ is contained in I or J.

Assume for example that $\{-n, \dots, -1\} \subset I$ and $\{1, \dots, n\} \subset J$. In that case

$$\begin{aligned}
v_p(\phi_{v,j_1}(x_3) \dots \phi_{v,j_{2n+1-r}}(x_3)) &= v_p(\phi_{v,1}(x_2) \dots \phi_{v,n}(x_2)) \\
&= v_p(\phi_{v,1}(x_1') \dots \phi_{v,n}(x_1')) \\
&= -v_p(\phi_{v,1}(x_1) \dots \phi_{v,n}(x_1)) + 3n(n+1)f_v \\
&\quad -\frac{2}{e_v}\sum_{i=1}^n \sum_\sigma k^{(1)}_{v,\sigma,i}
\end{aligned}$$

is negative, which is yet another contradiction.

As a consequence, we can conclude that I or J is equal to $\{0\}$, and this shows that at each place v of F above p, the semisimplification of $\rho_{\iota_p,\iota_\infty}(\Pi_3)|_{G_{F_v}}$ is either irreducible or the sum of an Artin character and an irreducible representation of dimension $2n$. Consequently Π_3 has the required properties. $\square$

2.5 Similar results for even orthogonal groups

In this section we explain (very) briefly how the same method as in the previous sections applies to orthogonal groups.

Let F be a totally real number field of even degree over $\mathbb{Q}$. Then F has an even number of 2-adic places of odd degree over $\mathbb{Q}_2$, and as these are the only finite places of F at which $(-1,-1)_v = -1$ (where $(\cdot,\cdot)_v$ denotes the Hilbert symbol), we have $\prod_v (-1,-1)_v = 1$ where the product ranges over the finite places of F. Consequently, there is a unique quadratic form on F^4 which is positive definite at the real places of F, and split (isomorphic to $(x,y,z,t) \mapsto xy + zt$) at the finite places. It has Hasse invariant $(-1,-1)_v$ at each finite place v of F, and its discriminant is 1. As a consequence, for any integer $n \geq 1$, there is a connected reductive group $\mathbf{G}$ over F which is compact (and connected) at the real places (isomorphic to $\mathrm{SO}_{4n}/\mathbb{R}$) and split at all the finite places (isomorphic to the split SO_{4n}). As before, we let $\mathbf{G}' = \mathrm{Res}_{\mathbb{Q}}^{F}\mathbf{G}$. The proofs of the existence and properties of the attached eigenvariety $\mathscr{X} \to \mathscr{W}$ are identical to the symplectic case. We could not find a result as precise as Theorem 2.3.2.4 in the literature, however by [Cas80, Proposition 3.5] unramified principal series are irreducible on an explicit Zariski-open subset of the unramified characters. Specifically, if $\mathrm{SO}_{4n}(F_v) = \left\{ M \in \mathrm{M}_{4n}(F_v) \mid {}^tM J_{4n} M = J_{4n} \right\}$,

$$T = \left\{ \left. \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_n^{-1} & & \\ & & & & \ddots & \\ & & & & & x_1^{-1} \end{pmatrix} \right| x_i \in F_v^\times \right\}$$

and P is any parabolic subgroup containing T, then for an unramified character $\chi = (\chi_1, \ldots, \chi_n)$ of T (χ_i is a character of the variable x_i), $\mathrm{Ind}_P^{\mathrm{SO}_{4n}(F_v)} \chi$ is irreducible if $\chi_i(\varpi_v)^2 \neq 1$ for all i and $\chi_i(\varpi_v)\chi_j(\varpi_v)^{\pm 1} \neq 1, q_v, q_v^{-1}$ for all $i < j$. Note that this is not an equivalence.

The existence of Galois representations $\rho_{\iota_p,\iota_\infty}(\Pi)$ attached to automorphic representations Π of $\mathbf{G}(\mathbb{A}_F)$ is identical to Assumption 2.4.1.1. We now state the main result for orthogonal groups.

Theorem 2.5.0.3. *Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori invariants at all the places of F above p, and having invariants under an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$. Let N be an integer. There exists an automorphic representation Π' of $\mathbf{G}(\mathbb{A}_F)$ such that:*

- *Π' is unramified at the places above p, and has invariants under U;*
- *The restriction of $\rho_{\iota_p,\iota_\infty}(\Pi')$ to the decomposition group at any place above p is irreducible;*
- *For all g in G_F, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(g)) \equiv \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(g)) \mod p^N$.*

Proof. The proof is nearly identical to that of Theorem 2.4.2.2. In the orthogonal case the Weyl group is a bit smaller: it is the semi-direct product of S_{2n} and a hyperplane of $(\mathbb{Z}/2\mathbb{Z})^{2n}$. Alternatively, it is the group of permutations w of $\{-2n,\dots,-1,1,\dots,2n\}$ such that $w(-i) = -w(i)$ for all i and $\prod_{i=1}^{2n} w(i) > 0$. The two elements of the Weyl group used in the proof of Theorem 2.4.2.2 have natural counterparts in this Weyl group. The only difference lies in the fact that there is no Hodge-Tate weight equal to 0 in the orthogonal case, hence the simpler conclusion "$\rho_{\iota_p,\iota_\infty}(\Pi')|_{G_{F_v}}$ is irreducible for $v|p$". $\square$

2.6 The image of complex conjugation: relaxing hypotheses in Taylor's theorem

Let us apply the previous results to the determination of the image of the complex conjugations under the p-adic Galois representations associated with regular, algebraic, essentially self-dual, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, F totally real. Recall that these representations are constructed by "patching" representations of Galois groups of CM extensions of F, on Shimura varieties for unitary groups. The complex conjugations are lost when we restrict to CM fields. In [Tay12], Taylor proves that the image of any complex conjugation is given by (the "discrete" part of) the local Langlands parameter at the corresponding real place, assuming n is odd and the Galois representation is irreducible, by constructing the complex conjugation on the Shimura datum. Of course the Galois representation associated with a cuspidal representation of GL_n is conjectured to be irreducible, but unfortunately this is (at the time of writing) still out of reach in the general case (however, see [CG] for $n \leq 5$; [BLGGT14, Theorem D] for a "density one" result for arbitrary n but under the assumption that F is CM and the automorphic representation is "extremely regular" at the archimedean places; and [PT] for a "positive density" result for arbitrary n and without these assumptions).

The results of the first part of this paper allow to remove the irreducibility hypothesis in Taylor's theorem, and to extend it to some ("half") cases of even n, using Arthur's endoscopic transfer. Unfortunately some even-dimensional cases are out of reach using this method, because odd-dimensional essentially self-dual cuspidal representations are (up to a twist) self-dual, whereas some even-dimensional ones are not.

Since the proof is not direct, let us outline the strategy. First we deduce the even-dimensional self-dual case from Taylor's theorem by adding a cuspidal self-dual (with appropriate weights) representation of GL_3, we get an automorphic self-dual representation of GL_{2n+3} which (up to base change) can be "transferred" to a discrete representation of the symplectic group in dimension $2n$. Since the associated Galois representation contains no Artin character, it can be deformed irreducibly, and Taylor's theorem applies. Then the general odd-dimensional case is deduced from the even-dimensional one, by essentially the same method, using the eigenvariety for orthogonal groups.

Finally we prove a supplementary, non-regular case, thanks to the fact that discrete Langlands parameters for the group $\mathrm{SO}_{2n}/\mathbb{R}$ are not always discrete when seen as parameters for GL_{2n}, i.e. can correspond to a non-regular representation of $\mathrm{GL}_{2n}/\mathbb{R}$.

2.6.1 Regular, L-algebraic, self-dual, cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ having Iwahori-invariants

In this subsection $\mathbf{G}$ will denote the symplectic group in dimension $2n+2$ defined in section 2.3.

The following lemma is due to C. Mœglin and J.-L. Waldspurger.

Lemma 2.6.1.1. *Let K be a finite extension of $\mathbb{Q}_p$. Let $\phi : W_K \times \mathrm{SU}(2) \to \mathrm{SO}_{2n+3}(\mathbb{C})$ be a Langlands parameter (equivalently, a generic Arthur parameter). Assume that the subgroup $I \times \{1\}$ (I being the inertia subgroup of W_K) is contained in the kernel of ϕ.*

Then the A-packet associated with ϕ contains a representation having a non-zero vector fixed under the Iwahori subgroup of $\mathrm{Sp}_{2n+2}(K)$.

Proof. Let $\{\Pi_1, \ldots, \Pi_k\}$ denote the A-packet. Since Arthur's construction of the Π_i's is inductive for parameters trivial on the supplementary $\mathrm{SL}_2(\mathbb{C})$, and subquotients of parabolic inductions of representations having Iwahori-invariants have too, it is enough to prove the result when ϕ is discrete. Let τ be the irreducible smooth representation of $\mathrm{GL}_{2n+3}(K)$ having parameter ϕ, then $\tau \simeq \mathrm{Ind}_L^{\mathrm{GL}_{2n+3}} \sigma$, where σ is the tensor product of (square-integrable) Steinberg representations $\mathrm{St}(\chi_i, n_i)$ of $\mathrm{GL}_{n_i}(K)$ ($i \in \{1, \ldots, r\}$), χ_i are unramified, auto-dual characters of $K^\times$ (thus $\chi_i = 1$ or $(-1)^{v(\cdot)}$), and the couples (χ_i, n_i) are distinct. Here L denotes the standard parabolic associated with the decomposition $2n+3 = \sum_i n_i$. Since ϕ is self-dual, τ can be extended (not uniquely, but this will not matter for our purpose) to a representation of $\widetilde{\mathrm{GL}}^+_{2n+3} = \mathrm{GL}_{2n+3} \rtimes \{1, \theta\}$, where

$$\theta(g) = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} {}^t g^{-1} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

Let also $\widetilde{\mathrm{GL}}_{2n+3} = \mathrm{GL}_{2n+3} \rtimes \theta$.

Let N_0 be the number of i such that n_i is odd, and for $j \geq 1$ let N_j be the number of i such that $n_i \geq 2j$. Then $N_0 + 2\sum_{j \geq 1} N_j = 2n+3$, and if s is maximal such that $N_s > 0$, we let

$$M = \mathrm{GL}_{N_s} \times \ldots \times \mathrm{GL}_{N_1} \times \mathrm{GL}_{N_0} \times \mathrm{GL}_{N_1} \times \ldots \times \mathrm{GL}_{N_s}$$

which is a θ-stable Levi subgroup of GL_{2n+3}, allowing us to define $\widetilde{M}^+$ and $\widetilde{M}$. Since the standard (block upper triangular) parabolic containing M is also stable under θ, τ_M is naturally a representation of $\widetilde{M}^+$, denoted by $\tau_{\widetilde{M}}$. The constituents of the semi-simplification of $\tau_{\widetilde{M}}$ either stay irreducible when restricted to M, in which case they are of the form $\sigma_1 \otimes \sigma_0 \otimes \theta(\sigma_1)$ where σ_1 is a representation of $\mathrm{GL}_{N_s} \times \ldots \times \mathrm{GL}_{N_1}$ and σ_0 is a representation of $\widetilde{\mathrm{GL}}_{N_0}$; or they are induced from M to $\widetilde{M}^+$, and the restriction of their character to $\widetilde{M}$ is zero. Since we are precisely interested in that character, we can forget about the second case. By the geometrical lemma,

$$\tau_M^{\mathrm{ss}} \simeq \bigoplus_{w \in W^{L,M}} \mathrm{Ind}_{M \cap w(L)}^M w\left(\sigma_{L \cap w^{-1} M}\right)$$

where $W^{L,M}$ is the set of $w \in S_{2n+3}$ such that w is increasing on $I_1 = \{1, \dots, n_1\}$, $I_2 = \{n_1+1, \dots, n_1+n_2\}$, etc. and w^{-1} is increasing on $J_{-s} = \{1, \dots, N_s\}$, $J_{-s+1} = \{N_s+1, \dots, N_s+N_{s-1}\}$, etc. Fix the irreducible representation of $\mathrm{GL}_{N_s} \times \dots \times \mathrm{GL}_{N_1}$

$$\sigma_1 = \bigotimes_{j=1}^{s} \mathrm{Ind}_{T_j}^{\mathrm{GL}_{N_j}} \bigotimes_{i \mid n_i \geq 2j} \chi_i |\cdot|^{j-\nu_i}$$

where T_j is the standard maximal torus of GL_{N_j}, $\nu_i = \begin{cases} 0 & n_i \text{ odd} \\ 1/2 & n_i \text{ even} \end{cases}$.

There is a unique w such that $\mathrm{Ind}_{M \cap w(L)}^{M} w\left(\sigma_{L \cap w^{-1}M}\right)$ admits a subquotient of the form $\sigma_1 \otimes \sigma_0 \otimes \theta(\sigma_1)$ as above, moreover $\mathrm{Ind}_{M \cap w(L)}^{M} w\left(\sigma_{L \cap w^{-1}M}\right)$ is irreducible, and

$$\sigma_0 = \mathrm{Ind}_{T_0}^{\mathrm{GL}_{N_0}} \bigotimes_{i \mid n_i \text{ odd}} \chi_i$$

Specifically, w maps the first element of I_i in $J_{-\lfloor (n_i+1)/2 \rfloor}$, the second in $J_{-\lfloor (n_i+1)/2 \rfloor} + 1$, $\dots$, the central element (if n_i is odd) in J_0, etc.

Let M' be the parabolic subgroup of Sp_{2n+2}/K corresponding to M, i.e.

$$M' = \mathrm{GL}_{N_s} \times \dots \times \mathrm{GL}_{N_1} \times \mathrm{Sp}_{N_0-1}$$

By [Art13, 2.2.6], $\sum_i \mathrm{Tr}\Pi_i$ is a stable transfer of $\mathrm{Tr}_{\mathrm{GL}_{2n+3}^+} \tau$. By [MW06, Lemme 4.2.1] (more accurately, the proof of the lemma),

$$\sum_i \mathrm{Tr}\left((\Pi_i)_{M'}^{\mathrm{ss}}[\sigma_1]\right)$$

is a stable transfer of $\mathrm{Tr}\left(\tau_{\widetilde{M}}^{\mathrm{ss}}[\sigma_1]\right)$ (where $\cdot[\cdot]$ denotes the isotypical component on the factor $\mathrm{GL}_{N_s} \times \dots \times \mathrm{GL}_{N_1}$).

Since $\tau_{\widetilde{M}}^{\mathrm{ss}}[\sigma_1] = \sigma_1 \otimes \sigma_0 \otimes \theta(\sigma_1)$, the stable transfer of $\mathrm{Tr}\left(\tau_{\widetilde{M}}^{\mathrm{ss}}[\sigma_1]\right)$ is equal to the product of $\mathrm{Tr}(\sigma_1)$ and $\sum_l \mathrm{Tr}\Pi'_l$ where the Π'_l are the elements of the A-packet associated with the parameter

$$\bigoplus_{i \mid n_i \text{ odd}} \chi_i$$

At least one representation Π'_l is unramified for some hyperspecial compact subgroup of $\mathrm{Sp}_{N_0-1}(K)$, and so a Jacquet module of a Π_i contains a nonzero vector fixed by an Iwahori subgroup. This proves that at least one of the Π_i has Iwahori-invariants. □

Assumption 2.6.1.2. *Let F_0 be a totally real field, and let π be a regular, L-algebraic, self-dual, cuspidal (RLASDC) representation of $\mathrm{GL}_{2n}(\mathbb{A}_{F_0})$. Assume that for any place $v|p$ of F_0, π_v has vectors fixed under an Iwahori subgroup of $\mathrm{GL}_{2n}(\mathbb{A}_{F_{0,v}})$. Then there exists a RLASDC representation π_0 of $\mathrm{GL}_3(\mathbb{A}_{F_0})$, a totally real extension F/F_0 which is trivial, quadratic or quartic, and an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ such that*

1. *For any place $v|p$ of F_0, $\pi_{0,v}$ is unramified.*

2. *$\mathrm{BC}_{F/F_0}(\pi)$ and $\mathrm{BC}_{F/F_0}(\pi_0)$ remain cuspidal.*

3. *For any place v of F above p, Π_v has invariants under the action of the Iwahori subgroup G_0 of $\mathbf{G}(F_v)$.*

4. *For any finite place v of F such that $\mathrm{BC}_{F/F_0}(\pi)_v$ and $\mathrm{BC}_{F/F_0}(\pi_0)_v$ are unramified, Π_v is unramified, and via the inclusion $\mathrm{SO}_{2n+3}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+3}(\mathbb{C})$, the Satake parameter of Π_v is equal to the direct sum of those of $\mathrm{BC}_{F/F_0}(\pi)_v$ and $\mathrm{BC}_{F/F_0}(\pi_0)_v$.*

Let us comment briefly on the proof to come. First we construct π_0. Let δ be a cuspidal automorphic representation of PGL_2/F_0 which is unramified at the p-adic places, Steinberg at the ℓ-adic places for some arbitrary prime $\ell \neq p$, and whose local langlands parameters at the real places are of the form $\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}(z \mapsto (z/\bar{z})^a)$ where a is a half-integer big enough with respects to their analogues appearing in the local Langlands parameters of π. Such a representation exists thanks to [Clo86, Theorem 1B]. Let π_0 be the automorphic representation of GL_3/F_0 obtained by functoriality from δ through the adjoint representation of $\widehat{\mathrm{PGL}_2}(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ on its Lie algebra. The representation π_0 exists and is cuspidal by [GJ78, Theorem 9.3]. The condition at the ℓ-adic places ensures that no nontrivial twist of δ (seen as a representation of GL_2/F_0) is isomorphic to δ, and the cuspidality of π_0 follows. We can twist π_0 by the central character of π, to ensure that $\pi \oplus \pi_0$ has trivial central character. Clearly π_0 is a RLASDC representation of GL_3/F_0.

Note that for $\mathrm{BC}_{F/F_0}(\pi)$ and $\mathrm{BC}_{F/F_0}(\pi_0)$ to remain cuspidal, it is enough for F/F_0 to be totally ramified above a finite place of F_0 at which π and π_0 are unramified. To begin with one can choose such a quadratic extension of F_0, in order to define $\mathbf{G}$. The automorphic representation $\Psi := \mathrm{BC}_{F/F_0}(\pi) \oplus \mathrm{BC}_{F/F_0}(\pi_0)$ can be seen as a global, orthogonal parameter. This determines a global packet P_Ψ of representations of $\mathbf{G}(\mathbb{A}_F)$, and Arthur's results shall attach to each $\Pi \in P_\Psi$ a character of $S_\Psi \simeq \mathbb{Z}/2\mathbb{Z}$, and characterize the automorphic Π's as the ones whose character is trivial. We can choose the components Π_v at the finite places of F not lying above p to be associated with a trivial character of S_{Ψ_v}, and taking a quadratic extension split above the p-adic and real places of F (at which Π_v is imposed) allows to "double" the contribution of the characters, thus yielding a trivial global character.

Proposition 2.6.1.3. *Let F be a totally real field, and let π be a regular, L-algebraic, self-dual, cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$. Suppose that for any place v of F above p, π_v has invariants under an Iwahori subgroup. Then for any complex conjugation $c \in G_F$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = 0$.*

Proof. By the previous assumption, up to a (solvable) base change to a totally real extension (which only restricts the Galois representation to this totally real field, so that we get even more complex conjugations), we can take a RLASDC representation π_0 of $\mathrm{GL}_3(\mathbb{A}_F)$ and transfer $\pi \oplus \pi_0$ to an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$. The representation Π defines (at least) one point x of the eigenvariety $\mathscr{X}$ defined by $\mathbf{G}$ (and by an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$). Of course, by the Čebotarev density theorem and the compatibility of the transfer at the unramified places, the representation associated with Π is equal to $\rho_{\iota_p,\iota_\infty}(\pi) \oplus \rho_{\iota_p,\iota_\infty}(\pi_0)$. Since the Hodge-Tate weights of $\rho_{\iota_p,\iota_\infty}(\pi)|_{G_{F_v}}$ are non-zero for any place $v|p$, $\rho_{\iota_p,\iota_\infty}(\pi)$ does not contain an Artin character. By [BR92], $\rho_{\iota_p,\iota_\infty}(\pi_0)$ is irreducible and thus does not contain any character. There are only finitely many Artin characters taking values in $\{\pm 1\}$ and unramified at all the finite places at which Π is unramified. For any such character η, the pseudocharacter T on the eigenvariety is such that $T_x - \eta$ is not a pseudocharacter, hence we can find $g_{\eta,1}, \dots, g_{\eta,2n+3}$ such that

$$t_\eta := \sum_{\sigma \in S_{2n+3}} (T_x - \eta)_\sigma(g_{\eta,1}, \dots, g_{\eta,2n+3}) \neq 0$$

Let us choose N greater than all the $v_p(t_\eta)$ and such that $p^N > 2n+4$. Let Π' be an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ satisfying the requirements of Theorem 2.4.2.2 for this choice of N. Then the $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')) - \eta$ are not pseudocharacters, thus $\rho_{\iota_p,\iota_\infty}(\Pi')$ does not contain an Artin character and by Theorem 2.4.2.2 it is irreducible. This Galois representation is (by construction in the proof of Corollary 2.4.1.3) the direct sum of representations associated with cuspidal representations. Since it is irreducible, there is only one of them, and it has the property that its associated Galois representations is irreducible, so that the theorem of [Tay12] can be applied: for any complex conjugation $c \in G_F$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(c)) = \pm 1$. Since $\det \rho_{\iota_p,\iota_\infty}(\Pi') = 1$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(c)) = (-1)^{n+1}$.

As $p^N > 2n+4$ and $|\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(c)) - \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(c))| \leq 2n+4$, we can conclude that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(c)) = (-1)^{n+1}$, and hence that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) + \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi_0)(c)) = (-1)^{n+1}$. We also know that $\det \rho_{\iota_p,\iota_\infty}(\pi_0) = \det \rho_{\iota_p,\iota_\infty}(\pi)(c) = (-1)^n$, and that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi_0)(c)) = \pm 1$ by Taylor's theorem, from which we can conclude that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi_0)(c)) = (-1)^{n+1}$. Thus $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = 0$. ☐

2.6.2 Regular, L-algebraic, self-dual, cuspidal representations of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$ having Iwahori-invariants

In this subsection, $\mathbf{G}$ is the orthogonal reductive group defined in section 2.5, of dimension $2n+2$ if n is odd, $2n+4$ if n is even.

Lemma 2.6.2.1. *Let K be a finite extension of $\mathbb{Q}_p$. Let $\phi : W_K \times \mathrm{SU}(2) \to \mathrm{SO}_{2m}(\mathbb{C})$ be a Langlands parameter. Assume that the subgroup $I \times \{1\}$ (I being the inertia subgroup of W_K) is contained in the kernel of ϕ.*

Then the packet of representations of the split group $\mathrm{SO}_{2m}(K)$ associated with ϕ by Arthur contains a representation having a non-zero vector fixed under the Iwahori subgroup.

Proof. Of course this result is very similar to 2.6.1.1. However Mœglin and Waldspurger have not put their lemma in writing in this case, and the transfer factors are no longer trivial, so that one needs to modify the definition of "stable transfer". For this one needs to use the transfer factors $\Delta_{\widetilde{\mathrm{GL}}_{2m},\mathrm{SO}_{2m}}(\cdot,\cdot)$ defined in [KS99]. They depend in general on the choice of an inner class of inner twistings [KS99, 1.2] (in our case an inner class of isomorphisms between GL_{2m}/K and its quasi-split inner form defined over $\bar{K}$, which we just take to be the identity), and a Whittaker datum of the quasi-split inner form. Arthur chooses the standard splitting of GL_{2m} and an arbitrary character $K \to \mathbb{C}^\times$, but this will not matter to us since both GL_{2m} and SO_{2m} are *split*, so that the factor $\langle z_J, s_J \rangle$ of

[KS99, 4.2] (by which the transfer factors are multiplied when another splitting is chosen) is trivial. Indeed to compute this factor we can choose the split torus T_H of SO_{2m}/K, which is a norm group (see [KS99, Lemma 3.3B]) for the split torus T of GL_{2m}/K, and thus, using the notations of [KS99, 4.2], T^x is split and $H^1(K, T^x)$ is trivial, so that $z' = 1$ (z_J is the image of z' in $H^1(K, J)$, so that it is trivial). Since both groups are split the ϵ-factor of [KS99, 5.3] is also trivial, so the transfer factors are canonical.

Let $H = \mathrm{SO}_{2m}(K)$, $\widetilde{\tau}$ the representation of $\widetilde{\mathrm{GL}}_{2m}^+$ associated with ϕ, and τ^H the sum of the elements of the packet associated with ϕ by Arthur. Note that by construction, this packet is only a finite set of orbits under $\mathrm{O}_{2m}(K)/SO_{2m}(K) \simeq \mathbb{Z}/2\mathbb{Z}$ of irreducible, square-integrable representations of $\mathrm{SO}_{2m}(K)$. Each orbit has either one or two elements. In the latter case where the orbit is (say) $\{\tau_1, \tau_2\}$ one can still define a "partial" character (in the sense of Harish-Chandra):

$$\Theta_{\tau_1}(h) + \Theta_{\tau_1}(h') = \Theta_{\tau_2}(h) + \Theta_{\tau_2}(h') := \Theta_{\tau_1}(h) + \Theta_{\tau_2}(h)$$

whenever h is regular semisimple conjugacy class in $\mathrm{SO}_{2m}(K)$ and h' is the complement of h in its conjugacy class under $\mathrm{O}_{2m}(K)$. Although the individual terms on the left cannot be distinguished, their sum does not depend on the choice of a particular element (e.g. τ_1) in the orbit. In that setting, Arthur shows ([Art13, 8.3]) that the following character identity holds:

$$\sum_h |D_H(h)|^{1/2} \Theta_{\tau^H}(h) \Delta(h, g) = |D_{\widetilde{\mathrm{GL}}_{2m}}(g)|^{1/2} \Theta_{\widetilde{\tau}}(g) \tag{2.6.2.1}$$

where the sum on the left runs over the the stable conjugacy classes h in $\mathrm{SO}_{2m}(K)$ which are norms of the conjugacy class g in $\widetilde{\mathrm{GL}}_{2m}(K)$, both assumed to be strongly $\widetilde{\mathrm{GL}}_{2m}^+$-regular. There are two such stable conjugacy classes h, they are conjugate under $\mathrm{O}_{2m}(K)$ and the two transfer factors on the left are equal (this can be seen either by going back to the definition of Kottwitz and Shelstad, or by Waldspurger's formulas recalled below). This fact together with the stability of the "partial" distribution Θ_{τ^H} (which is part of Arthur's results) imply that the expression on the left is well-defined. Note that as in [MW06] and [Art13], the term Δ_{IV} is not included in the product defining the transfer factor Δ. Contrary to the case of symplectic and odd orthogonal groups treated in [MW06], the transfer factors are not trivial, and the terms $|D_H(h)|^{1/2}$ and $|D_{\widetilde{\mathrm{GL}}_{2m}}(g)|^{1/2}$ are not equal. However the latter play no particular role in the proof. This character identity 2.6.2.1 is the natural generalization of the notion of "stable transfer" of [MW06].

Let

$$M = \mathrm{GL}_{N_s} \times \ldots \times \mathrm{GL}_{N_1} \times \mathrm{GL}_{N_0} \times \mathrm{GL}_{N_1} \times \ldots \times \mathrm{GL}_{N_s}$$

be a θ-stable Levi subgroup of GL_{2m}, and $M' = \mathrm{GL}_{N_s} \times \ldots \times \mathrm{GL}_{N_1} \times \mathrm{SO}_{N_0}$ the corresponding parabolic subgroup of SO_{2m}. To mimic the proof of 2.6.1.1, we only need to show that $\mathrm{Tr}\left(\tau^H_{M'}\right)$ is a stable transfer of $\mathrm{Tr}_{\widetilde{M}}\left(\tau_{\widetilde{M}}\right)$, where "stable transfer" has the above meaning, that is the character identity 2.6.2.1 involving transfer factors. Note that $\widetilde{M}^+$ has a factor $\mathrm{GL}_{m-N_0/2} \times \mathrm{GL}_{m-N_0/2}$ together with the automorphism $\theta(a, b) = (\theta(b), \theta(a))$, for which the theory of endoscopy is trivial: θ-conjugacy classes are in bijection with conjugacy classes in $\mathrm{GL}_{m-N_0/2}$ (over K or $\bar{K}$) via $(a,b)\mapsto a\theta(b)$ and the θ-invariant irreducible representations are the ones of the form $\sigma\otimes\theta(\sigma)$.

So we need to check that if $g=(g_1,g_0)$ is a strongly regular $\mathrm{GL}_{2m}(K)$-conjugacy class in $\widetilde{\mathrm{GL}}_{2m}(K)$ determined by a conjugacy class g_1 in $\mathrm{GL}_{m-N_0/2}(K)$ and a $\mathrm{GL}_{N_0}(K)$-conjugacy class g_0 in $\widetilde{\mathrm{GL}}_{N_0}(K)$, and if h_0 is the $O_{2m}(\bar{K})$-conjugacy class in $\mathrm{SO}_{2m}(K)$ corresponding to g_0, then

$$\Delta_{\widetilde{\mathrm{GL}}_{N_0},\mathrm{SO}_{N_0}}(h_0,g_0)=\Delta_{\widetilde{\mathrm{GL}}_{2m},\mathrm{SO}_{2m}}((g_1,h_0),(g_1,g_0)).$$

Although this is most likely known by the experts (even in a general setting) we will check it. Fortunately the transfer factors have been computed by Waldspurger in [Wal10]. We recall his notations and formulas. The conjugacy class g_1, being regular enough, is parametrized by a finite set I_1, a collection of finite extensions $K_{\pm i}$ of K for $i\in I_1$, and (regular enough, i.e. generating $K_{\pm i}$ over K) elements $x_{i,1}\in K_{\pm i}$. As in [Wal10], g_0 is parametrized by a finite set I_0, finite extensions $K_{\pm i}$ of K, $K_{\pm i}$-algebras K_i, and $x_i\in K_i$. Each K_i is either a quadratic field extension of $K_{\pm i}$ or $K_{\pm i}\times K_{\pm i}$, and x_i is determined only modulo $N_{K_i/K_{\pm i}}K_i^\times$. Then g is parametrized by $I=I_1\sqcup I_0$, with $K_i=K_{\pm i}\times K_{\pm i}$ and $x_i=(x_{i,1},1)$ for $i\in I_1$, and the same data for I_0. Let τ_i be the non-trivial $K_{\pm i}$-automorphism of K_i, and $y_i=-x_i/\tau_i(x_i)$. Let I^* be the set of $i\in I$ such that K_i is a field (so $I^*\subset I_0$). For any $i\in I$, let Φ_i be the set of K-morphisms $K_i\to\bar{K}$, and let $P_I(T)=\prod_{i\in I}\prod_{\phi\in\Phi_i}(T-\phi(y_i))$. Define P_{I_0} similarly. For $i\in I^*$ (resp. I_0^*), let $C_i=x_i^{-1}P_I'(y_i)P_I(-1)y_i^{1-m}(1+y_i)$ (resp. $C_{i,0}=x_i^{-1}P_{I_0}'(y_i)P_{I_0}(-1)y_i^{1-m}(1+y_i)$). We have dropped the factor η of [Wal10, 1.10], because as remarked above, the transfer factors do not depend on the chosen splitting. Observe also that the factors computed by Waldspurger are really the factors $\Delta_0/\Delta_{\mathrm{IV}}$ of [KS99, 5.3], but the ϵ factor is trivial so they are complete.

Waldspurger shows that

$$\Delta_{\widetilde{\mathrm{GL}}_{2m},\mathrm{SO}_{2m}}((g_1,h_0),(g_1,g_0))=\prod_{i\in I^*}\mathrm{sign}_{K_i/K_{\pm i}}(C_i)$$

where $\mathrm{sign}_{K_i/K_{\pm i}}$ is the nontrivial character of $K_{\pm i}^\times/N_{K_i/K_{\pm i}}K_i^\times$. We are left to show that $\prod_{i\in I^*}\mathrm{sign}_{K_i/K_{\pm i}}(C_i/C_{i,0})=1$.

$$\begin{aligned}
C_i/C_{i,0} &= y_i^{N_0/2-m}\prod_{j\in I_1}\prod_{\phi\in\Phi_j}(y_i-\phi(y_j))(-1-\phi(y_j))\\
&= \prod_{j\in I_1}\prod_{\phi\in\Phi_{\pm j}} y_i^{-1}\left(y_i+\phi(x_{j,1})\right)\left(y_i+\phi(x_{j,1})^{-1}\right)\left(\phi(x_{j,1})-1\right)\left(\phi(x_{j,1})^{-1}-1\right)\\
&= (-1)^{m-N_0/2}N_{K_i/K_{\pm i}}\left(\prod_{j\in I_1}\prod_{\phi\in\Phi_{\pm j}}(y_i+\phi(x_{j,1}))(\phi(x_{j,1})^{-1}-1)\right)
\end{aligned}$$

where $\Phi_{\pm j}$ is the set of K-morphisms $K_{\pm j}\to\bar{K}$. Thus

$$\begin{aligned}
\prod_{i\in I^*}\mathrm{sign}_{K_i/K_{\pm i}}(C_i/C_{i,0}) &= \prod_{i\in I^*}\mathrm{sign}_{K_i/K_{\pm i}}|_{K^\times}\left((-1)^{m-N_0/2}\right)\\
&= 1
\end{aligned}$$

since $\prod_{i\in I^*} \mathrm{sign}_{K_i/K_{\pm i}}|_{K^\times}$ is easily checked to be equal to the Hilbert symbol with the discriminant of our special orthogonal group, which is 1 (this is the condition for g_0 to have a norm in the special orthogonal group). $\square$

Assumption 2.6.2.2. *Let F_0 be a totally real field, and let π be a regular, L-algebraic, self-dual, cuspidal representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_{F_0})$. Assume that for any place $v|p$ of F_0, π_v has vectors fixed under the Iwahori. Then there exists a RLASDC representation π_0 of $\mathrm{GL}_1(\mathbb{A}_{F_0})$ if n is odd (resp. $\mathrm{GL}_3(\mathbb{A}_{F_0})$ if n is even), a totally real extension F/F_0 which is trivial or quadratic, and an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ such that*

1. *For any place $v|p$ of F_0, $\pi_{0,v}$ is unramified.*
2. *$\mathrm{BC}_{F/F_0}(\pi)$ and $\mathrm{BC}_{F/F_0}(\pi_0)$ remain cuspidal.*
3. *For any place v of F above p, Π_v has invariants under the action of the Iwahori subgroup of $\mathbf{G}(F_v)$.*
4. *For any finite place v of F such that $\mathrm{BC}_{F/F_0}(\pi)_v$ and $\mathrm{BC}_{F/F_0}(\pi_0)_v$ are unramified, Π_v is unramified, and via the inclusion $\mathrm{SO}_{2n+2}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+2}(\mathbb{C})$ (resp. $\mathrm{SO}_{2n+4}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+2}(\mathbb{C})$), the Satake parameter of Π_v is equal to the direct sum of those of $\mathrm{BC}_{F/F_0}(\pi)_v$ and $\mathrm{BC}_{F/F_0}(\pi_0)_v$.*

This is very similar to Assumption 2.6.1.2. In fact in this case the group S_Ψ is trivial, which explains why it is enough to take a quadratic extension of F_0. This is only necessary to be able to define the group $\mathbf{G}$. The crucial observation is that the local Langlands parameters of $\mathrm{BC}_{F/F_0}(\pi) \oplus \mathrm{BC}_{F/F_0}(\pi_0)$ at the infinite places correspond to parameters for the compact groups $\mathrm{SO}_{2n+2}/\mathbb{R}$ (resp. SO_{2n+4}). These parameters are of the form

$$\epsilon^n \oplus \bigoplus_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

($r_1 > \ldots > r_n > 0$) for $\mathrm{BC}_{F/F_0}(\pi)$, and

$$\begin{cases} 1 & \text{if } n \text{ is odd} \\ \epsilon \oplus \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} \left(z \mapsto (z/\bar{z})^{r}\right) & \text{if } n \text{ is even} \end{cases}$$

so that the direct sum of the two is always of the form

$$1 \oplus \epsilon \oplus \bigoplus_{i=1}^{k-1} \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

for distinct, positive r_i. This is the Langlands parameter corresponding to the representation of $\mathrm{SO}_{2k}(\mathbb{R})$ having highest weight $\sum_{i=1}^{k}(r_i-(k-i))e_i$ with $r_k = 0$, where the root system consists of the $\pm e_i \pm e_j$ ($i \neq j$) and the simple roots are $e_1-e_2, \ldots, e_{k-1}-e_k, e_{k-1}+e_k$.

Note that, contrary to the symplectic case, there is one outer automorphism of the even orthogonal group, and so there may be two choices for the Satake parameters of Π_v, mapping to the same conjugacy class in the general linear group. Fortunately we only need the existence.

Proposition 2.6.2.3. *Let F be a totally real field, and let π be an L-algebraic, self-dual, cuspidal representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$. Suppose that for any place v of F above p, π_v has invariants under an Iwahori. Then for any complex conjugation $c \in G_F$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = \pm 1$.*

Proof. The proof is similar to that of Proposition 2.6.1.3. We use the previous assumption to be able to assume (after base change) that there is a representation π_0 (of $\mathrm{GL}_1(\mathbb{A}_F)$ if n is odd, $\mathrm{GL}_3(\mathbb{A}_F)$ if n is even) such that $\pi \oplus \pi_0$ transfers to an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$, with compatibility at the unramified places. The representation Π has Iwahori-invariants at the p-adic places of F, and thus it defines a point of the eigenvariety $\mathscr{X}$ associated with $\mathbf{G}$ (and an idempotent defined by an open subgroup of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$). By Theorem 2.5.0.3, Π is congruent (at all the complex conjugations, and modulo arbirarily big powers of p) to another automorphic representation Π' of $\mathbf{G}$, and $\rho_{\iota_p,\iota_\infty}(\Pi')$ is irreducible. Hence $\rho_{\iota_p,\iota_\infty}(\Pi') = \rho_{\iota_p,\iota_\infty}(\pi')$ for some RLASDC π' of $\mathrm{GL}_{2k}(\mathbb{A}_F)$, which is unramified at all the p-adic places of F, and we can apply Proposition 2.6.1.3 to π'. This proves that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = -\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi_0)(c)) = \pm 1$. □

2.6.3 Almost general case

We will now remove the hypothesis of being Iwahori-spherical at p, and allow more general similitude characters, using Arthur and Clozel's base change.

Lemma 2.6.3.1. *Let E be a number field, S a finite set of (possibly infinite) places of E, and for each $v \in S$, let $K^{(v)}$ be a finite abelian extension of E_v. There is an abelian extension F of E such that for any $v \in S$ and any place w of F above v, the extension F_v/E_v is isomorphic to $K^{(v)}/E_v$.*

Proof. After translation to local and global class field theory, this is a consequence of [Che51, Théorème 1]. □

Before proving the last theorem, we need to reformulate the statement, in order to make the induction argument more natural. Let π be a regular, L-algebraic, cuspidal representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$. At a real place v of F, the Langlands parameter of π_v is of the form

$$\epsilon^e \oplus \bigoplus_i \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} z \mapsto (z/\bar{z})^{n_i}$$

and according to the recipe given in [BG10, Lemma 2.3.2], $\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$ should be in the same conjugacy class as

$$\begin{pmatrix} (-1)^e & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

Since it is known that $\det \rho_{\iota_p,\iota_\infty}(\pi)(c_v) = (-1)^{e+n}$, $\rho_{\iota_p,\iota_\infty}(\pi)(c_v) \sim \mathcal{LL}(\pi_v)(j)$ if and only if $|\mathrm{Tr}\rho_{\iota_p,\iota_\infty}(\pi)(c_v)| = 1$. Similarly, in the even-dimensional case, $\rho_{\iota_p,\iota_\infty}(\pi)(c_v) \sim \mathcal{LL}(\pi_v)(j)$ if and only if $\mathrm{Tr}\rho_{\iota_p,\iota_\infty}(\pi)(c_v) = 0$.

Theorem 2.6.3.2. *Let $n \geq 2$, F a totally real number field, π a regular, L-algebraic, essentially self-dual, cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$, such that $\pi^\vee \simeq ((\eta|\cdot|^q)\circ \det) \otimes \pi$, where η is an Artin character. Suppose that one of the following conditions holds*

1. *n is odd.*

2. *n is even, q is even, and $\eta_\infty(-1) = 1$.*

Then for any complex conjugation $c \in G_F$, $|\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c))| \leq 1$.

Proof. We can twist π by an algebraic character, thus multiplying the similitude character $\eta|\cdot|^q$ by the square of an algebraic character. If n is odd, this allows to assume $\eta = 1, q = 0$ (by comparing central characters, we see that $\eta|\cdot|^q$ is a square). If n is even, we can assume that $q = 0$ (we could also assume that the order of η is a power of 2, but this is not helpful). The Artin character η defines a cyclic, totally real extension F'/F. Since local Galois groups are pro-solvable, the preceding lemma shows that there is a totally real, solvable extension F''/F' such that $\mathrm{BC}_{F''/F}(\pi)$ has Iwahori invariants at all the places of F'' above p. In general $\mathrm{BC}_{F''/F}(\pi)$ is not cuspidal, but only induced by cuspidals: $\mathrm{BC}_{F''/F}(\pi) = \pi_1 \boxplus \ldots \boxplus \pi_k$. However it is self-dual, and the particular form of the Langlands parameters at the infinite places imposes that all π_i be self-dual. We can then apply Propositions 2.6.1.3 and 2.6.2.3 to the π_i, and conclude by induction that for any complex conjugation $c \in G_F$, the conjugacy class of $\rho_{\iota_p,\iota_\infty}(\pi)(c)$ is given by the recipe found in [BG10, Lemma 2.3.2], that is to say $\left|\mathrm{Tr}\rho_{\iota_p,\iota_\infty}(\pi)(c)\right| \leq 1$. □

Remark 2.6.3.3. *The case n even, $\eta_\infty(-1) = (-1)^{q+1}$ is trivial. The case n even, q odd and $\eta_\infty(-1) = -1$ remains open.*

For the sake of clarity, we state the theorem using the more common normalization of C-algebraic representations.

Theorem 2.6.3.4. *Let $n \geq 2$, F a totally real number field, π a regular, algebraic, essentially self-dual, cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$, such that $\pi^\vee \simeq \eta|\det|^q\pi$, where η is an Artin character. Suppose that one of the following conditions holds*

1. *n is odd.*

2. *n is even, q is odd, and $\eta_\infty(-1) = 1$.*

Then for any complex conjugation $c \in G_F$, $|\mathrm{Tr}(r_{\iota_p,\iota_\infty}(\pi)(c))| \leq 1$.

Proof. Apply the previous theorem to $\pi|\det|^{(n-1)/2}$. □

2.6.4 A supplementary, non-regular case

In this subsection $\mathbf{G}$ is the orthogonal group of section 2.5.

Assumption 2.6.4.1. *Let F_0 be a totally real field, and let π be an L-algebraic, self-dual, cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A}_{F_0})$. Assume that for any place $v|p$ of F_0, π_v has vectors fixed under the Iwahori, and that for any real place v of F_0,*

$$\mathcal{LL}(\pi_v) \simeq \bigoplus_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

where $r_n > \ldots > r_1 \geq 0$ are integers (note that π is not regular if $r_1 = 0$). Then there exists a totally real extension F/F_0 which is trivial or quadratic, and an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ such that

1. *$\mathrm{BC}_{F/F_0}(\pi)$ remains cuspidal.*
2. *For any place v of F above p, Π_v has invariants under the action of the Iwahori subgroup of $\mathbf{G}(F_v)$.*
3. *For any finite place v of F such that $\mathrm{BC}_{F/F_0}(\pi)_v$ is unramified, Π_v is unramified, and via the inclusion $\mathrm{SO}_{2n+2}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$, the Satake parameter of Π_v is equal to the one of $\mathrm{BC}_{F/F_0}(\pi)_v$.*

Of course this is very similar to Assumptions 2.6.1.2 and 2.6.2.2, and as in the latter case the group S_Ψ is trivial.

For L-algebraic, self-dual, cuspidal automorphic representations of GL_{2n} having "almost regular" Langlands parameter at the archimedean places as above, the corresponding p-adic Galois representation is known to exist by [Gol14]. Exactly as in the previous subsection, we have the following:

Theorem 2.6.4.2. *Let $n \geq 2$, F a totally real number field, π an L-algebraic, essentially self-dual, cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, such that $\pi^\vee \simeq \eta\pi$, where η is an Artin character. Assume that at any real place v of F, $\eta_v(-1) = 1$ and*

$$\mathcal{LL}(\pi_v) \simeq \bigoplus_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

where $r_n > \ldots > r_1 \geq 0$ are integers. Then for any complex conjugation $c \in G_F$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = 0$.

Proof. Identical to that of Theorem 2.6.3.2. □

Proposition 2.6.4.3. *Let π be as in the previous theorem. Then for any place v of F above p, $\rho_{\iota_p,\iota_\infty}(\pi)|_{G_{F_v}}$ is Hodge-Tate. If $\sigma : F_v \to \overline{\mathbb{Q}}_p$ is a $\mathbb{Q}_p$-embedding, the σ-Hodge-Tate weights of $\rho_{\iota_p,\iota_\infty}(\pi)|_{G_{F_v}}$ are the $\pm r_{\sigma,i}$ (if $r_{\sigma,1} = 0$, it has multiplicity two), where $r_{\sigma,n} > \ldots > r_{\sigma,1} \geq 0$ are the integers appearing in $\mathcal{LL}(\pi_w)$ as in the previous theorem (where w is the real place of F determined by σ and ι_p, ι_∞).*

Proof. First observe that by totally real and solvable base change and Assumption 2.6.4.1, we can assume that π corresponds to an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori-fixed vectors at all the p-adic places, and thus corresponds to a point on the eigenvariety $\mathscr{X}/E$. Let $\rho := \rho_{\iota_p,\iota_\infty}(\pi)$, and let V denote the E-vector space underlying this representation (as usual E is a "big enough" p-adic field).

Recall that for any p-adic place v of F, $D_{\mathrm{Sen}}(V)$ is a free $E \otimes_{\mathbb{Q}_p} F_v(\mu_{p^\infty})$-module of rank $\dim_E V$, together with a linear operator Θ. As in 2.4.2.1 we can write $E \otimes_{\mathbb{Q}_p} F_v(\mu_{p^\infty}) \simeq \prod_\sigma E_\sigma \otimes_{F_v} F_v(\mu_{p^\infty})$ and thus $D_{\mathrm{Sen}}(V) = \prod_\sigma D_{\mathrm{Sen},\sigma}(V)$ (σ runs over the embeddings $F_v \hookrightarrow \overline{\mathbb{Q}}_p$ and E_σ is just a copy of E). The operator Θ is just a collection of operators Θ_σ on each $D_{\mathrm{Sen},\sigma}(V)$. Moreover Θ_σ comes from the infinitesimal action of $\mathrm{Gal}(F_v(\mu_{p^\infty})/F_v)$ on $D_{\mathrm{Sen},\sigma}(V)$, hence its characteristic polynomial has coefficients in E_σ. Therefore Θ_σ can be defined over $E_\sigma = E_\sigma \otimes_{F_v} F_v \subset E_\sigma \otimes_{F_v} F_v(\mu_{p^\infty})$, but since the result is not functorial, we will not directly use it. Note that if we write $E_\sigma \otimes_{F_v} F_v(\mu_{p^\infty})$ as a product of fields (algebraic extensions of E_σ), Θ_σ can be concretely described as a collection of matrices over these fields, all being similar to a single matrix over E_σ, so that the semisimplicity of Θ_σ is equivalent to the semisimplicity of any of these matrices. For this reason in the rest of the proof we will treat Θ_σ as an endomorphism of a vector space over $\overline{\mathbb{Q}}_p$.

The proposition is a small improvement of [BC09][Lemma 7.5.12]. By this Lemma, which states the analyticity of the Sen polynomial, we know that the characteristic polynomial of Θ_σ is

$$\prod_{i=1}^{n}(T^2 - r_{\sigma,i}^2)$$

as expected. We need to show that the Sen operator Θ_σ is semisimple. It is enough to show that $\ker \Theta_\sigma = \ker \Theta_\sigma^2$ in the case $r_{1,\sigma} = 0$. This is in turn implied by the fact that ρ is orthogonal, because then by functoriality $D_{\mathrm{Sen},\sigma}(V)$ admits a non-degenerate quadratic form for which Θ_σ is infinitesimally orthogonal, i.e. antisymmetric, and since $\ker(\Theta_\sigma^2 - r_i^2)$ is non-degenerate if $i > 1$, the orthogonal of these eigenspaces, that is $\ker \Theta_\sigma^2$, is non-degenerate too. Finally, all the elements of $\mathfrak{so}_2$ are semisimple.

Let us show that ρ is indeed orthogonal, that is that V admits a G_F-invariant non-degenerate quadratic form. Note that for automorphic RLASDC representations of GL_{2n}/F, it is known that the associated Galois representation is orthogonal by the main result of [BC11]. By the analogue of Assumption 2.4.1.1 for the special orthogonal group $\mathbf{G}$, all classical points having weight "far enough from the walls" come from such representations. We will use a deformation argument similar to [BC11][Proposition 2.4].

First we replace $\mathscr{X}$ by a curve. Of course we want this curve to contain a given classical point $z \in |\mathscr{X}|$ corresponding to Π. We also want to ensure that there are "many" classical points on $\mathscr{Y}$, that is to say we want Proposition 2.3.2.6 to hold. Let $\mathscr{Y}$ be an open affinoid of $\mathscr{X} \times_{\mathscr{W}} \mathscr{W}'$ containing z, where $\mathscr{W}'$ is the one-dimensional reduced subspace of $\mathscr{W}$ parameterizing weights of the form

$$(x_{v,i})_{v|p, i=1..n} \mapsto \gamma\left(\prod_v \prod_{i=1}^{n} N_{F_v/\mathbb{Q}_p}(x_{v,i}^{n-i})\right)$$

times $w(z)$, for γ a continuous character of $\mathbb{Z}_p^\times$. By [BC09][Lemma 7.8.11], there is a smooth connected affinoid curve $\mathscr{Y}'$ and a finite morphism $f : \mathscr{Y}' \to \mathscr{Y}$ whose image is an irreducible component of $\mathscr{Y}$ containing z, such that the $2n$-dimensional pseudocharacter $f^\sharp \circ T$ is the sum of the traces of continuous representations

$$R_j : G_F \to \mathrm{GL}_{\mathcal{O}(\mathscr{Y}')}(M_j)$$

for sheaves M_j locally free of rank n_j ($\sum_j n_j = 2n$), and such that $R_j \otimes_{\mathcal{O}(\mathscr{Y}')} k(y)$ is absolutely irreducible for y in a Zariski-open subset of $\mathscr{Y}'$.

We now work with $\mathscr{Y}'$, and still denote by z any point of $\mathscr{Y}'$ above $z \in \mathscr{Y}$. Note that $\left(R \otimes_{\mathcal{O}(\mathscr{Y}')} \overline{k(z)}\right)^{\mathrm{ss}} \simeq \rho$. The points y of $\mathscr{Y}'$ at which the semisimplification of

$$\bigoplus_j R_j \otimes_{\mathcal{O}(\mathscr{Y}')} \overline{k(y)}$$

comes from an automorphic RLASDC representation of GL_{2n} are still Zariski-dense, and by consideration of the Hodge-Tate weights, the representations R_j are pairwise non-isomorphic on a Zariski-open subset of $\mathscr{Y}'$. Since $T(g) = T(g^{-1})$ for all $g \in G_F$, each R_j is either "self-dual" (in the sense that $\mathrm{Tr}(R_j(g^{-1})) = \mathrm{Tr}(R_j(g))$ for all $g \in G_F$), or part of a pair $(R_j, R_{j'})$ ($j \neq j'$) where $\mathrm{Tr}(R_j(g^{-1})) = \mathrm{Tr}(R_{j'}(g))$ for all $g \in G_F$, and thus

$$\left(R_j \otimes_{\mathcal{O}(\mathscr{Y}')} \overline{k(y)}\right)^{\mathrm{ss},\vee} \simeq \left(R_{j'} \otimes_{\mathcal{O}(\mathscr{Y}')} \overline{k(y)}\right)^{\mathrm{ss}}$$

for any point y of $\mathscr{Y}'$.

To prove the orthogonality of ρ, it is enough to prove that for each "self-dual" R_j, $\left(R \otimes_{\mathcal{O}(\mathscr{Y}')} \overline{k(z)}\right)^{\mathrm{ss}}$ is orthogonal. We can now work locally, and simply consider R_j as a representation

$$R_j : G_F \to \mathrm{GL}_{n_j}(\mathcal{O}_z)$$

where $\mathcal{O}_z$ is the local ring of $\mathscr{Y}'$ at z, a (henselian) discrete valuation ring. We conclude using the following lemma. □

Lemma 2.6.4.4. *Let A be a discrete valuation ring, let K be its fraction field and k its residue field, and assume that* $\mathrm{char}(k) \neq 2$. *Let $R : G \to \mathrm{GL}_n(A)$ a representation such that $R \otimes_A K$ is absolutely irreducible and orthogonal. Then $(R \otimes_A k)^{\mathrm{ss}}$ is also orthogonal.*

Proof. We first remark that the semisimplification of an orthogonal representation is again orthogonal. Denote by ϖ a uniformizer of A. Let $V = K^n$ be the K-vector space underlying the representation R. By assumption V admits a G_F-stable lattice $L = A^n$. Fix a G_F-invariant, non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V^n. Replacing L by $\varpi^k L$ for some integer $k \geq 0$ if necessary, we can assume that

$$L^\vee := \{v \in V \mid \forall u \in L,\ \langle u, v \rangle \in A\}$$

contains L. We wish to find a lattice L' such that $L \subset L' \subset L^\vee$ and $(L')^\vee = L'$. This would endow $L'/\varpi L'$ with a G_F-invariant non-degenerate symmetric bilinear form, and it is well-known that $(L'/\varpi L')^{\mathrm{ss}} \simeq (R \otimes_A k)^{\mathrm{ss}}$. Even though this will not be possible in general, by attempting to do so we will show that $(R \otimes_A k)^{\mathrm{ss}}$ is orthogonal.

The A-module $L^\vee/L$ is torsion and of finite type. Let n be the smallest integer such that $\varpi^n L^\vee \subset L$. If $n > 1$, replace L by $L + \varpi^{n-1} L^\vee$, which strictly contains L and is still integral with respect to $\langle \cdot, \cdot \rangle$. After a finite number of iterations of this procedure, we are left with a lattice L such that

$$L \subset L^\vee \subset \varpi^{-1} L.$$

Therefore

$$(L/\varpi L)^{\mathrm{ss}} \simeq \left(L^\vee/L\right)^{\mathrm{ss}} \oplus \left(L/\varpi L^\vee\right)^{\mathrm{ss}}$$

and it is straightforward to check that $\langle \cdot, \cdot \rangle$ induces on both factors a G_F-invariant non-degenerate symmetric bilinear form. $\square$

Partie 3

Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula

3.1 Introduction

Let $\mathbf{G}$ be a Chevalley reductive group over $\mathbb{Z}$ admitting discrete series at the real place, i.e. one of $\mathbf{SO}_{2n+1}$, $\mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$ for $n \geq 1$. We give an algorithm to compute the geometric side in Arthur's "simple" trace formula in [Art89a] (see also [GKM97]) for $\mathbf{G}$ and the trivial Hecke operator in level one at the finite places, that is the characteristic function of $\mathbf{G}(\widehat{\mathbb{Z}})$. There are essentially three steps to compute the geometric side of the trace formula:

1. for any prime p, compute the local orbital integrals of the characteristic function of $\mathbf{G}(\mathbb{Z}_p)$ at torsion elements γ_p in $\mathbf{G}(\mathbb{Q}_p)$ (with respect to a Haar measure on the connected centraliser of γ_p),

2. for any semisimple elliptic and torsion conjugacy class $\gamma \in \mathbf{G}(\mathbb{Q})$ with connected centraliser $\mathbf{I}$, use the Smith-Minkowski-Siegel mass formula to compute $\mathrm{Vol}(\mathbf{I}(\mathbb{Q})\backslash\mathbf{I}(\mathbb{A}))$,

3. analyse the character of stable (averaged) discrete series on arbitrary maximal tori of $\mathbf{G}(\mathbb{R})$ to express the parabolic terms using elliptic terms for groups of lower semisimple rank.

We explain how to compute local orbital orbitals for special orthogonal groups (resp. symplectic groups) in sections 3.3.2.2 and 3.3.2.3, using quadratic and hermitian (resp. alternate and antihermitian) lattices. To compute the volumes appearing in local orbital integrals we rely on the local density formulae for such lattices given in [GY00], [Choa] and [Chob]. We choose a formulation similar to [Gro97] for the local and global volumes (see section 3.3.2.4). For the last step we follow [GKM97], and we only add that for the trivial Hecke operator the general formula for the archimedean factor of each parabolic term simplifies significantly (Proposition 3.3.3.2). Long but straightforward calculations lead to explicit formulae for the parabolic terms (see section 3.3.3.4).

Thus for any irreducible algebraic representation V_λ of $\mathbf{G}_\mathbb{C}$ characterised by its highest weight λ, we can compute the spectral side of the trace formula, which we now describe. Let K_∞ be a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ and let $\mathfrak{g} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{g}_0$ where $\mathfrak{g}_0 = \mathrm{Lie}(\mathbf{G}(\mathbb{R}))$. For an irreducible $(\mathfrak{g}, K_\infty)$-module π_∞, consider the Euler-Poincaré characteristic

$$\mathrm{EP}(\pi_\infty \otimes V_\lambda^*) = \sum_i (-1)^i \dim H^i\left((\mathfrak{g}, K_\infty),\ \pi_\infty \otimes V_\lambda^*\right)$$

where V_λ is seen as a representation of $\mathbf{G}(\mathbb{R})$. Let $\Pi_{\mathrm{disc}}(\mathbf{G})$ be the set of isomorphism classes of irreducible $(\mathfrak{g}, K_\infty) \times \mathbf{G}(\mathbb{A}_f)$-modules occurring in the discrete automorphic spectrum of $\mathbf{G}$. For $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G})$ denote by $m_\pi \in \mathbb{Z}_{\geq 1}$ the corresponding multiplicity. Let $\Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{G})$ be the set of $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G})$ which are unramified at all the finite places of $\mathbb{Q}$. For any dominant weight λ the set of $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{G})$ such that $H^\bullet((\mathfrak{g}, K_\infty), \pi_\infty \otimes V_\lambda^*) \neq 0$ is finite. The spectral side of Arthur's trace formula in [Art89a] for our choice of function at the finite places is

$$\sum_{\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{G})} m_\pi \mathrm{EP}(\pi_\infty \otimes V_\lambda^*). \tag{3.1.0.1}$$

This integer is interesting but it is only an alternate sum. To obtain subtler information, e.g. the sum of m_π for π_∞ isomorphic to a given $(\mathfrak{g}, K_\infty)$-module, we use Arthur's endoscopic classification of the discrete automorphic spectrum for symplectic and special orthogonal groups [Art13]. Arthur's work allows to parametrise the representations π contributing to the spectral side 3.1.0.1 using self-dual automorphic representations for general linear groups. Denote $W_\mathbb{R}$ the Weil group of $\mathbb{R}$ and $\epsilon_{\mathbb{C}/\mathbb{R}}$ the character of $W_\mathbb{R}$ having kernel $W_\mathbb{C} \simeq \mathbb{C}^\times$. For $w \in \frac{1}{2}\mathbb{Z}$ define the bounded Langlands parameter $I_w : W_\mathbb{R} \to \mathrm{GL}_2(\mathbb{C})$ as

$$\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \left(z \mapsto (z/|z|)^{2w} \right)$$

so that $I_0 \simeq 1 \oplus \epsilon_{\mathbb{C}/\mathbb{R}}$. The three families that we are led to consider are the following.

1. For $n \geq 1$ and $w_1, \dots, w_n \in \frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z}$ such that $w_1 > \dots > w_n > 0$, define $S(w_1, \dots, w_n)$ as the set of self-dual automorphic cuspidal representations of $\mathbf{GL}_{2n}/\mathbb{Q}$ which are unramified at all the finite places and with Langlands parameter at the real place

$$I_{w_1} \oplus \dots \oplus I_{w_n}.$$

   Equivalently we could replace the last condition by "with infinitesimal character having eigenvalues $\{\pm w_1, \dots, \pm w_n\}$". Here S stands for "symplectic", as the conjectural Langlands parameter of such a representation should be symplectic.

2. For $n \geq 1$ and integers $w_1 > \dots > w_n > 0$ define $O_o(w_1, \dots, w_n)$ as the set of self-dual automorphic cuspidal representations of $\mathbf{GL}_{2n+1}/\mathbb{Q}$ which are everywhere unramified and with Langlands parameter at the real place

$$I_{w_1} \oplus \dots \oplus I_{w_n} \oplus \epsilon_{\mathbb{C}/\mathbb{R}}^n.$$

   Equivalently we could replace the last condition by "with infinitesimal character having eigenvalues $\{\pm w_1, \dots, \pm w_n, 0\}$". Here O_o stands for "odd orthogonal".

3. For $n \geq 1$ and integers $w_1 > \dots > w_{2n-1} > w_{2n} \geq 0$ define $O_e(w_1, \dots, w_{2n})$ as the set of self-dual automorphic cuspidal representations of $\mathbf{GL}_{4n}/\mathbb{Q}$ which are everywhere unramified and with Langlands parameter at the real place

$$I_{w_1} \oplus \dots \oplus I_{w_{2n}}.$$

   In this case also we could replace the last condition by "with infinitesimal character having eigenvalues $\{\pm w_1, \dots, \pm w_{2n}\}$", even in the slightly singular case where $w_{2n} = 0$. Here O_e stands for "even orthogonal".

Following Arthur using these three families we can define, for any $\mathbf{G}$ and λ as above, a set $\Psi(\mathbf{G})^{\mathrm{unr},\lambda}$ of "formal Arthur-Langlands parameters" which parametrises the representations $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{G})$ contributing to 3.1.0.1. We stress that for a given $\mathbf{G}$ all three families take part in these formal parameters. Among these formal parameters, one can distinguish a subset $\Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$ of "simple" parameters, that is the tempered and non-endoscopic ones. When $\mathbf{G} = \mathbf{SO}_{2n+1}$ (resp. $\mathbf{Sp}_{2n}$, resp. $\mathbf{SO}_{4n}$), this set is exactly $S(w_1, \ldots, w_n)$ (resp. $O_o(w_1, \ldots, w_n)$, resp. $O_o(w_1, \ldots, w_{2n})$) where $(w_i)_i$ is determined by λ. The contribution of any element of $\Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$ to the spectral side 3.1.0.1 is a non-zero number depending only on $\mathbf{G}(\mathbb{R})$. Therefore it is natural to attempt to compute the cardinalities of the sets $S(\cdot)$, $O_o(\cdot)$ and $O_e(\cdot)$ inductively, the induction being on the dimension of $\mathbf{G}$. More precisely we have to compute the contribution of $\Psi(\mathbf{G})^{\mathrm{unr},\lambda} \smallsetminus \Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$ to 3.1.0.1 to deduce the cardinality of $\Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$.

When the highest weight λ is regular, any element of $\Psi(\mathbf{G})^{\mathrm{unr},\lambda}$ is tempered and consequently any $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{G})$ contributing to the spectral side is such that π_∞ is a discrete series representation having same infinitesimal character as V_λ. Thanks to the work of Shelstad on real endoscopy and using Arthur's multiplicity formula it is not difficult to compute the contribution of $\Psi(\mathbf{G})^{\mathrm{unr},\lambda} \smallsetminus \Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$ to the Euler-Poincaré characteristic on the spectral side in this case (see section 3.4.2.1). The general case is more interesting because we have to consider non-tempered representations π_∞. Since Arthur's construction of non-tempered Arthur packets at the real place in [Art13] is rather abstract, we have to make an assumption (see Assumption 3.4.2.4) in order to be able to compute explicitly the non-tempered contributions to the Euler-Poincaré characteristic. This assumption is slightly weaker than the widely believed Assumption 3.4.2.3, which states that the relevant real non-tempered Arthur packets at the real place coincide with those constructed long ago by Adams and Johnson in [AJ87].

Thus we obtain an algorithm to compute the cardinalities of the sets $S(w_1, \ldots, w_n)$, $O_o(w_1, \ldots, w_n)$ and $O_e(w_1, \ldots, w_{2n})$, under assumption 3.4.2.4 when λ is singular. For the computer the hard work consists in computing local orbital integrals. Our current implementation, using Sage [S+14], allows to compute them at least for $\mathrm{rank}(\mathbf{G}) \leq 6$. See section 3.7.2 for some values.

Once these cardinalities are known we can *count* the number of $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{G})$ such that π_∞ is isomorphic to a given $(\mathfrak{g}, K_\infty)$-module having same infinitesimal character as V_λ for some highest weight λ. A classical application is to compute dimensions of spaces of (vector-valued) Siegel cusp forms. For a genus $n \geq 1$ and $m_1 \geq \cdots \geq m_n \geq n+1$, let r be the holomorphic (equivalently, algebraic) finite-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight $(m_1, \ldots, m_n)$. Let $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$. The dimension of the space $S_r(\Gamma_n)$ of level one vector-valued cuspidal Siegel modular forms of weight r can then be computed using Arthur's endoscopic classification of the discrete spectrum for $\mathbf{Sp}_{2n}$. We emphasise that this formula depends on Assumption 3.4.2.3 when the m_k's are not pairwise distinct, in particular when considering scalar-valued Siegel cusp forms, of weight $m_1 = \cdots = m_n$. Our current implementation yields a dimension formula for $\dim S_r(\Gamma_n)$ for any $n \leq 7$ and any r as above, although for $n \geq 3$ it would be absurd to print this huge formula. See the table in section 3.5.4 for some values in the scalar case. The case $n = 1$ is well-known: $\bigoplus_{m\geq 0} M_m(\Gamma_1) = \mathbb{C}[E_4, E_6]$ where the Eisenstein series E_4, E_6 are algebraically independant over $\mathbb{C}$, and the dimension formula for $S_m(\Gamma_1)$ follows. Igusa [Igu62] determined the ring of scalar Siegel modular forms and its ideal of cusp forms when $n = 2$, which again gives a dimension formula. Tsushima [Tsu83], [Tsu84] gave a formula for the dimension of $S_r(\Gamma_2)$ for almost all representations r as above (that is for $m_1 > m_2 \geq 5$ or $m_1 = m_2 \geq 4$) using the Riemann-Roch-Hirzebruch formula along with a vanishing theorem. It follows from Arthur's classification that Tsushima's formula holds for any (m_1, m_2) such that $m_1 > m_2 \geq 3$. In genus $n = 3$ Tsuyumine [Tsu86] determined the structure of the ring of scalar Siegel modular forms and its ideal of cusp forms. Recently Bergström, Faber and van der Geer [BFvdG14] studied the cohomology of certain local systems on the moduli space $\mathcal{A}_3$ of principally polarised abelian threefolds, and conjectured a formula for the Euler-Poincaré characteristic of this cohomology (as a motive) in terms of Siegel modular forms. They are able to derive a conjectural dimension formula for spaces of Siegel modular cusp forms in genus three. Our computations corroborate their conjecture, although at the moment we have only compared values and not the formulae.

Of course the present work is not the first one to attempt to use the trace formula to obtain spectral information, and we have particularly benefited from the influence of [GP05] and [CR14]. In [GP05] Gross and Pollack use a simpler version of the trace formula, with hypotheses at a finite set S of places of $\mathbb{Q}$ containing the real place and at least one finite place. This trace formula has only elliptic terms. They use the Euler-Poincaré function defined by Kottwitz in [Kot88] at the finite places in S. These functions have the advantage that their orbital integrals were computed conceptually by Kottwitz. At the other finite places, they compute the stable orbital integrals indirectly, using computations of Lansky and Pollack [LP02] for inner forms which are compact at the real place. They do so for the groups $\mathbf{SL}_2$, $\mathbf{Sp}_4$ and $\mathbf{G}_2$. Without Arthur's endoscopic classification it was not possible to deduce the number of automorphic representations of a given type from the Euler-Poincaré characteristic on the spectral side, even for a regular highest weight λ. The condition $\mathrm{card}(S) \geq 2$ forbids the study of *level one* automorphic representations. More recently, Chenevier and Renard [CR14] computed dimensions of spaces of level one *algebraic* automorphic forms in the sense of [Gro99], for the inner forms of the groups $\mathbf{SO}_7$, $\mathbf{SO}_8$ and $\mathbf{SO}_9$ which are split at the finite places and compact at the real place. They used Arthur's classification to deduce the cardinalities of the sets $S(w_1, w_2, w_3)$ and $S(w_1, w_2, w_3, w_4)$ and, using the conjectural dimension formula of [BFvdG14], $O_e(w_1, w_2, w_3, w_4)$. Unfortunately the symplectic groups do not have such inner forms, nor do the special orthogonal groups $\mathbf{SO}_n$ when $n \mod 8 \notin \{-1, 0, 1\}$. Thus our main contribution is thus the direct computation of local orbital integrals.

3.2 Notations and definitions

Let us precise some notations. Let $\mathbb{A}_f$ denote the finite adèles $\prod'_p \mathbb{Q}_p$ and $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. We will use boldface letters to denote linear algebraic groups, for example $\mathbf{G}$. For schemes we denote base change using simply a subscript, for example $\mathbf{G}_{\mathbb{Q}_p}$ instead of $\mathbf{G} \times_{\mathbb{Q}} \mathbb{Q}_p$ where $\mathbf{G}$ is defined over $\mathbb{Q}$. For a reductive group $\mathbf{G}$ we abusively call "Levi subgroup of $\mathbf{G}$" any Levi subgroup of a parabolic subgroup of $\mathbf{G}$, i.e. the centraliser of a split torus. Rings are unital. If R is a ring and Λ a finite free R-module, $\mathrm{rk}_R(\Lambda)$ denotes its rank. If G is a finite abelian group $G^\wedge$ will denote its group of characters.

Let us define the reductive groups that we will use. For $n \geq 1$, let q_n be the quadratic form on $\mathbb{Z}^n$ defined by

$$q_n(x) = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} x_i x_{n+1-i}.$$

Let $\mathbf{O}_n$ be the algebraic group over $\mathbb{Z}$ representing the functor

$$\begin{aligned} \text{Category of commutative rings} &\to \text{Category of groups} \\ A &\mapsto \{ g \in \mathrm{GL}_n(A) \mid q_n \circ g = q_n \}. \end{aligned}$$

For n odd define $\mathbf{SO}_n$ as the kernel of $\det : \mathbf{O}_n \to \mu_2$. For n even, $\det : \mathbf{O}_n \to \mu_2$ factors through the Dickson morphism $\mathrm{Di} : \mathbf{O}_n \to \mathbb{Z}/2\mathbb{Z}$ (constant group scheme over $\mathbb{Z}$) and the morphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$ "mapping $1 \in \mathbb{Z}/2\mathbb{Z}$ to $-1 \in \mu_2$". In that case $\mathbf{SO}_n$ is defined as the kernel of Di. For any $n \geq 1$, $\mathbf{SO}_n \to \mathrm{Spec}(\mathbb{Z})$ is reductive in the sense of [SGA70][Exposé XIX, Définition 2.7]. It is semisimple if $n \geq 3$.

For $n \geq 1$ the subgroup $\mathbf{Sp}_{2n}$ of $\mathbf{GL}_{2n}/\mathbb{Z}$ defined as the stabiliser of the alternate form

$$(x, y) \mapsto \sum_{i=1}^{n} x_i y_{2n+1-i} - x_{2n+1-i} y_i$$

is also semisimple over $\mathbb{Z}$ in the sense of [SGA70][Exposé XIX, Définition 2.7].

If $\mathbf{G}$ is one of $\mathbf{SO}_{2n+1}$ ($n \geq 1$), $\mathbf{Sp}_{2n}$ ($n \geq 1$) or $\mathbf{SO}_{2n}$ ($n \geq 2$), the diagonal matrices form a split maximal torus $\mathbf{T}$, and the upper-triangular matrices form a Borel subgroup $\mathbf{B}$. We will simply denote by $\underline{t} = (t_1, \ldots, t_n)$ the element of $\mathbf{T}(A)$ (A a commutative ring) whose first n diagonal entries are $t_1, \ldots, t_n$. For $i \in \{1, \ldots, n\}$, let $e_i \in X^*(\mathbf{T})$ be the character $\underline{t} \mapsto t_i$. The simple roots corresponding to $\mathbf{B}$ are

$$\begin{cases} e_1 - e_2, \ldots, e_{n-1} - e_n, e_n & \text{if } \mathbf{G} = \mathbf{SO}_{2n+1}, \\ e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n & \text{if } \mathbf{G} = \mathbf{Sp}_{2n}, \\ e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n & \text{if } \mathbf{G} = \mathbf{SO}_{2n}. \end{cases}$$

In the first two cases (resp. third case), the dominant weights in $X^*(\mathbf{T})$ are the $\underline{k} = \sum_{i=1}^{n} k_i e_i$ with $k_1 \geq \cdots \geq k_n \geq 0$ (resp. $k_1 \geq \cdots \geq k_{n-1} \geq |k_n|$).

3.3 Computation of the geometric side of Arthur's trace formula

Arthur's invariant trace formula [Art88] for a reductive group $\mathbf{G}/\mathbb{Q}$ simplifies and becomes more explicit when $\mathbf{G}(\mathbb{R})$ has discrete series and a "nice" smooth compactly supported distribution $f_\infty(g_\infty) dg_\infty$ is used at the real place, as shown in [Art89a] (see also [GKM97] for a topological proof). In section 3.3.1 we recall the elliptic terms $T_{\mathrm{ell}}\left(f_\infty(g_\infty)dg_\infty \prod_p f_p(g_p)dg_p\right)$ on the geometric side of this trace formula, where $\prod_p f_p(g_p)dg_p$ is a smooth compactly supported distribution on $\mathbf{G}(\mathbb{A}_f)$. Then (section 3.3.2) we give an algorithm to compute these elliptic terms when $\mathbf{G}$ is a split classical group and for any prime p, $f_p(g_p)dg_p$ is the trivial element of the unramified Hecke algebra. Finally (section 3.3.3) we give explicit formulae for the parabolic terms using the elliptic terms for groups of lower semisimple rank.

3.3.1 Elliptic terms

3.3.1.1 Euler-Poincaré measures and functions

Let $\mathbf{G}$ be a reductive group over $\mathbb{R}$. Thanks to [Ser71], we have a canonical signed Haar measure on $\mathbf{G}(\mathbb{R})$, called the Euler-Poincaré measure. It is non-zero if and only if $\mathbf{G}(\mathbb{R})$ has discrete series, that is if and only if $\mathbf{G}$ has a maximal torus defined over $\mathbb{R}$ which is anisotropic.

So assume that $\mathbf{G}(\mathbb{R})$ has discrete series. Let K be a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$, $\mathfrak{g}_0 = \mathrm{Lie}(\mathbf{G}(\mathbb{R}))$ and $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$. Let V_λ be an irreducible algebraic representation of $\mathbf{G}_{\mathbb{C}}$, parametrised by its highest weight λ. We can see V_λ as an irreducible finite-dimensional representation of $\mathbf{G}(\mathbb{R})$, or as an irreducible $(\mathfrak{g}, K)$-module. If π is a $(\mathfrak{g}, K)$-module of finite length, consider

$$\mathrm{EP}(\pi, \lambda) := \sum_i (-1)^i \dim H^i\left((\mathfrak{g}, K), \pi \otimes V_\lambda^*\right).$$

Clozel and Delorme [CD90][Théorème 3] show that there is a smooth, compactly supported distribution $f_\lambda(g)dg$ on $\mathbf{G}(\mathbb{R})$ such that for any π as above,

$$\mathrm{Tr}\left(\pi\left(f_\lambda(g)dg\right)\right) = \mathrm{EP}(\pi, \lambda).$$

If π is irreducible and belongs to the L-packet $\Pi_{\mathrm{disc}}(\lambda)$ of discrete series having the same infinitesimal character as V_λ, this number is equal to $(-1)^{q(\mathbf{G}(\mathbb{R}))}$ where $2q(\mathbf{G}(\mathbb{R})) = \dim \mathbf{G}(\mathbb{R}) - \dim K$. If π is irreducible and tempered but does not belong to $\Pi_{\mathrm{disc}}(\lambda)$ it is zero.

These nice spectral properties of f_λ allow Arthur to derive nice geometric properties, similarly to the p-adic case in [Kot88]. If $\gamma \in \mathbf{G}(\mathbb{R})$, the orbital integral $O_\gamma(f_\lambda(g)dg)$ vanishes unless γ is elliptic semisimple, in which case, letting $\mathbf{I}$ denote the connected centraliser of γ in $\mathbf{G}$:

$$O_\gamma(f_\lambda(g)dg) = \mathrm{Tr}\left(\gamma | V_\lambda\right) \mu_{\mathrm{EP}, \mathbf{I}(\mathbb{R})}.$$

In fact [Art89a][Theorem 5.1] computes more generally the invariant distributions $I_{\mathbf{M}}(\gamma, f_\lambda)$ occurring in the trace formula (here $\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$), and the orbital integrals above are just the special case $\mathbf{M} = \mathbf{G}$. These more general invariant distributions will be used in the parabolic terms.

3.3.1.2 Orbital integrals for p-adic groups

We recall more precisely the definition of orbital integrals for the p-adic groups. Let p be a prime and $\mathbf{G}$ a reductive group over $\mathbb{Q}_p$. Let K be a compact open subgroup of $\mathbf{G}(\mathbb{Q}_p)$,

$\gamma \in \mathbf{G}(\mathbb{Q}_p)$ a semisimple element, and $\mathbf{I}$ its connected centraliser in $\mathbf{G}$. Lemma 19 of [HC70] implies that for any double coset KcK in $\mathbf{G}(\mathbb{Q}_p)$, the set X of $[g] \in K\backslash\mathbf{G}(\mathbb{Q}_p)/\mathbf{I}(\mathbb{Q}_p)$ such that $g\gamma g^{-1} \in KcK$ is finite. Let μ (resp. ν) be a Haar measure on $\mathbf{G}(\mathbb{Q}_p)$ (resp. $\mathbf{I}(\mathbb{Q}_p)$). Then the orbital integral at γ of the characteristic function of KcK

$$O_\gamma(\mathbf{1}_{KcK}, \mu, \nu) = \int_{\mathbf{G}(\mathbb{Q}_p)/\mathbf{I}(\mathbb{Q}_p)} \mathbf{1}_{KcK}\left(g\gamma g^{-1}\right) \frac{d\mu}{d\nu}(g)$$

is equal to

$$\sum_{[g]\in X} \frac{\mu(K)}{\nu\left(g^{-1}Kg \cap \mathbf{I}(\mathbb{Q}_p)\right)}.$$

The Haar measure $O_\gamma(\mathbf{1}_{KcK}, \mu, \nu)\nu$ is canonical, i.e. it does not depend on the choice of ν. Thus O_γ canonically maps the space of smooth compactly supported complex valued distributions on $\mathbf{G}(\mathbb{Q}_p)$ (i.e. linear combinations of distributions of the form $\mathbf{1}_{KcK}(g)d\mu(g)$) to the one-dimensional space of complex Haar measures on $\mathbf{I}(\mathbb{Q}_p)$.

Remark 3.3.1.1. *Note that any automorphism of the algebraic group* $\mathbf{I}$ *preserves* ν*, and thus if* $\mathbf{I}$ *and* ν *are fixed, for any algebraic group* $\mathbf{I}'$ *isomorphic to* $\mathbf{I}$*, there is a well-defined corresponding Haar measure on* $\mathbf{I}'$*.*

3.3.1.3 Definition of the elliptic terms

Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$ such that $\mathbf{G}(\mathbb{R})$ has discrete series. Let λ be a highest weight for the group $\mathbf{G}_{\mathbb{C}}$. Choose a Haar measure dg_∞ on $\mathbf{G}(\mathbb{R})$, and let f_∞ be a smooth compactly supported function on $\mathbf{G}(\mathbb{R})$ such that the distribution $f_{\infty,\lambda}(g_\infty)dg_\infty$ computes the Euler-Poincaré characteristic with respect to V_λ as in 3.3.1.1. Let $\prod_p f_p(g_p)dg_p$ be a smooth compactly supported distribution on $\mathbf{G}(\mathbb{A}_f)$. For almost all primes p, $\mathbf{G}_{\mathbb{Q}_p}$ is unramified, $f_p = \mathbf{1}_{K_p}$ and $\int_{K_p} dg_p = 1$ where K_p is a hyperspecial maximal compact subgroup in $\mathbf{G}(\mathbb{Q}_p)$. Let C be the set of semisimple conjugacy classes $\mathrm{cl}(\gamma)$ in $\mathbf{G}(\mathbb{Q})$ such that γ belongs to an anisotropic maximal torus in $\mathbf{G}(\mathbb{R})$. For $\mathrm{cl}(\gamma) \in C$, denote by $\mathbf{I}$ the connected centraliser of γ in $\mathbf{G}$. Given such a γ, for almost all primes p, $\mathbf{I}_{\mathbb{Q}_p}$ is unramified and $O_\gamma(f_p(g_p)dg_p)$ is the Haar measure giving measure one to a hyperspecial maximal compact subgroup of $\mathbf{I}(\mathbb{Q}_p)$ (see [Kot86, Corollary 7.3]). Thus $\prod_p O_\gamma(f_p(g_p)dg_p)$ is a well-defined complex Haar measure on $\mathbf{I}(\mathbb{A}_f)$. Let $f(g)dg = f_{\infty,\lambda}(g_\infty)dg_\infty \prod_p f_p(g_p)dg_p$. The elliptic part of the geometric side of Arthur's trace formula is

$$T_{\mathrm{ell}}(f(g)dg) = \sum_{\mathrm{cl}(\gamma)\in C} \frac{\mathrm{Vol}(\mathbf{I}(\mathbb{Q})\backslash\mathbf{I}(\mathbb{A}))}{\mathrm{card}\left(\mathrm{Cent}(\gamma, \mathbf{G}(\mathbb{Q}))/\mathbf{I}(\mathbb{Q})\right)} \mathrm{Tr}(\gamma \,|\, V_\lambda) \tag{3.3.1.1}$$

where $\mathbf{I}(\mathbb{R})$ is endowed with the Euler-Poincaré measure, $\mathbf{I}(\mathbb{A}_f)$ the complex Haar measure $\prod_p O_\gamma(f_p(g_p)dg_p)$ and $\mathbf{I}(\mathbb{Q})$ the counting measure. The set of $\mathrm{cl}(\gamma) \in C$ such that for any prime p, γ is conjugate in $\mathbf{G}(\mathbb{Q}_p)$ to an element belonging to the support of f_p is finite, so that the sum has only a finite number of nonzero terms.

3.3.2 Computation of the elliptic terms in the trace formula

Our first task is to explicitly compute $T_{\mathrm{ell}}(f(g)dg)$ when $\mathbf{G}$ is one of $\mathbf{SO}_{2n+1}$, $\mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$ and moreover for any prime p, $f_p = \mathbf{1}_{\mathbf{G}(\mathbb{Z}_p)}$ and $\int_{\mathbf{G}(\mathbb{Z}_p)} dg_p = 1$. In this case any $\gamma \in \mathbf{G}(\mathbb{Q})$ whose contribution to $T_{\mathrm{ell}}(f(g)dg)$ is nonzero is torsion ($\gamma^r = 1$ for some integer $r > 0$), since γ is compact in $\mathbf{G}(\mathbb{Q}_v)$ for any place v. Here "compact" means that the smallest closed subgroup of $\mathbf{G}(\mathbb{Q}_v)$ containing γ is compact, and it is equivalent to the fact that the eigenvalues of γ in any faithful algebraic representation of $\mathbf{G}_{\overline{\mathbb{Q}_v}}$ have norm one.

First we describe the semisimple conjugacy classes in $\mathbf{G}(\mathbb{Q})$ and their centralisers, a necessary first step to compute the set C and the groups $\mathbf{I}$. Then we explain how to enumerate the conjugacy classes of torsion elements in the group $\mathbf{G}(\mathbb{Z}_p)$. To be precise we can compute a collection of subsets $(Y_s)_s$ of $\mathbf{G}(\mathbb{Z}_p)$ such that

$$\{g \in \mathbf{G}(\mathbb{Z}_p) \,|\, \exists r > 0,\, g^r = 1\} = \bigsqcup_s \{xyx^{-1} \,|\, y \in Y_s,\, x \in \mathbf{G}(\mathbb{Z}_p)\}.$$

Note that this leaves the possibility that for a fixed s, there exist distinct $y, y' \in Y_s$ which are conjugated under $\mathbf{G}(\mathbb{Z}_p)$. Thus it seems that to compute local orbital integrals we should check for such cases and throw away redundant elements in each Y_s, and then compute the measures of the centralisers of y in $\mathbf{G}(\mathbb{Z}_p)$. This would be a computational nightmare. Instead we will show in section 3.3.2.3 that the fact that such orbital integrals are masses (as in "mass formula") implies that we only need to compute the cardinality of each Y_c. Finally the Smith-Minkowski-Siegel mass formulae of [GY00] provide a means to compute the global volumes.

3.3.2.1 Semisimple conjugacy classes in classical groups

Let us describe the absolutely semisimple conjugacy classes in classical groups over a field, along with their centralisers. It is certainly well-known, but we could not find a reference. We explain in detail the case of quadratic forms (orthogonal groups). The case of alternate forms (symplectic groups) is similar but simpler since characteristic 2 is not "special" and symplectic automorphisms have determinant 1. The case of (anti-)hermitian forms (unitary groups) is even simpler but it will not be used hereafter.

Let V be a vector space of finite dimension over a (commutative) field K, equipped with a regular ("ordinaire" in the sense of [SGA73, Exposé XII]) quadratic form q. Let $\gamma \in \mathrm{O}(q)$ be absolutely semisimple, i.e. $\gamma \in \mathrm{End}_K(V)$ preserves q and the finite commutative K-algebra $K[\gamma]$ is étale. Since γ preserves q, the K-automorphism τ of $K[\gamma]$ sending γ to γ^{-1} is well-defined: if $\dim_K V$ is even or $2 \neq 0$ in K, τ is the restriction to $K[\gamma]$ of the antiautomorphism of $\mathrm{End}_K(V)$ mapping an endomorphism to its adjoint with respect to the bilinear form B_q corresponding to q, defined by the formula $B_q(x, y) := q(x+y)-q(x)-q(y)$.

In characteristic 2 and odd dimension, (V, q) is the direct orthogonal sum of its γ-stable subspaces $V' = \ker(\gamma-1)$ and $V'' = \ker P(\gamma)$ where $(X-1)P(X) \in K[X] \setminus \{0\}$ is separable and annihilates γ. If V'' were odd-dimensional, the kernel of $B_q|_{V''\times V''}$ would be a γ-stable line Kx with $q(x) \neq 0$, which imposes $\gamma(x) = x$, in contradiction with $P(1) \neq 0$. Thus $K[\gamma] = K\left[\gamma|_{V''}\right] \times K$ if $V'' \neq 0$, and τ is again well-defined.

Thanks to τ we have a natural decomposition as a finite product:

$$(K[\gamma], \gamma) = \prod_i (A_i, \gamma_i)$$

where for any i, A_i is a finite étale K-algebra generated by γ_i such that $\gamma_i \mapsto \gamma_i^{-1}$ is a well-defined K-involution τ_i of A_i and $F_i = \{x \in A_i \,|\, \tau_i(x) = x\}$ is a field. Moreover the minimal polynomials P_i of γ_i are pairwise coprime. For any i, either:

- $\gamma_i^2 = 1$ and $A_i = K$,
- $\gamma_i^2 \neq 1$ and A_i is a separable quadratic extension of F_i, $\mathrm{Gal}(A_i/F_i) = \{1, \tau_i\}$,
- $\gamma_i^2 \neq 1$, $A_i \simeq F_i \times F_i$ and τ_i swaps the two factors.

Let I_{triv}, I_{field} and I_{split} be the corresponding sets of indices. There is a corresponding orthogonal decomposition of V:

$$V = \bigoplus_i V_i$$

where V_i is a projective A_i-module of constant finite rank.

Lemma 3.3.2.1. *For any i, there is a unique τ_i-hermitian (if τ_i is trivial, this simply means quadratic) form $h_i : V_i \to F_i$ such that for any $v \in V_i$, $q(v) = \mathrm{Tr}_{F_i/K}(h_i(v))$.*

Proof. If $i \in I_{\mathrm{triv}}$ this is obvious, so we can assume that $\dim_{F_i} A_i = 2$. Let us show that the K-linear map

$$\begin{array}{rcl} T : \{\tau_i\text{-hermitian forms on } V_i\} & \longrightarrow & \{K\text{-quadratic forms on } V_i \text{ preserved by } \gamma_i\} \\ h_i & \mapsto & \big(v \mapsto \mathrm{Tr}_{F_i/K} h_i(v)\big) \end{array}$$

is injective. If h_i is a τ_i-hermitian form on V_i, denote by B_{h_i} the unique τ_i-sesquilinear map $V_i \times V_i \to A_i$ such that for any $v, w \in V_i$, $h_i(v + w) - h_i(v) - h_i(w) = \mathrm{Tr}_{A_i/F_i} B_{h_i}(v, w)$, so that in particular $h_i(v) = B_{h_i}(v, v)$. Moreover for any $v, w \in V_i$, $B_{T(h_i)}(v, w) = \mathrm{Tr}_{A_i/K} B_{h_i}(v, w)$. If $h_i \in \ker T$, then $B_{T(h_i)} = 0$ and by non-degeneracy of $\mathrm{Tr}_{A_i/K}$ we have $B_{h_i} = 0$ and thus $h_i = 0$.

To conclude we have to show that the two K-vector spaces above have the same dimension. Let $d = \dim_K F_i$ and $n = \dim_{A_i} V_i$, then $\dim_K\{\tau_i\text{-hermitian forms on } V_i\} = dn^2$. To compute the dimension of the vector space on the right hand side, we can tensor over K with a finite separable extension K'/K such that γ_i is diagonalizable over K'. Since $\gamma_i^2 \neq 1$ the eigenvalues of $1 \otimes \gamma_i$ on $K' \otimes_K V_i$ are $t_1, t_1^{-1}, \dots, t_d, t_d^{-1}$ where the $t_k^{\pm 1}$ are distinct and $\neq 1$. Furthermore each eigenspace $U_k^+ := \ker(1 \otimes \gamma_i - t_k \otimes 1), U_k^- := \ker(1 \otimes \gamma_i - t_k^{-1} \otimes 1)$ has dimension n over K'. If q' is a K'-quadratic form on $K' \otimes_K V_i$ preserved by $1 \otimes \gamma_i$, then:

- for any k, $q'|_{U_k^\pm} = 0$ since $t_k^2 \neq 1$,
- for any $k \neq l$, $B_{q'}|_{U_k^\pm \times U_l^\pm} = 0$ since $t_k/t_l, t_k t_l \neq 1$.

Hence q' is determined by the restrictions of $B_{q'}$ to $U_k^+ \times U_k^-$, and conversely any family of K'-bilinear forms $U_k^+ \times U_k^- \to K'$ ($k \in \{1, \dots, d\}$) give rise to a K'-quadratic form on $K' \otimes_K V_i$ preserved by $1 \otimes \gamma_i$, and we conclude that the dimension is again dn^2. $\square$

The regularity of q implies that of h_i (when $\gamma_i^2 \neq 1$, regularity means non-degeneracy of B_{h_i}). In the split case, V_i can be more concretely described as a pair (W_i, W_i') of vector spaces over F_i having the same dimension, h_i identifies W_i' with the dual W_i^* of W_i over F_i, and thus the pair (V_i, h_i) is isomorphic to $((W_i, W_i^*), (w, f) \mapsto f(w))$.

If instead of q we consider a non-degenerate alternate form $\langle \cdot, \cdot \rangle$, we have the same kind of decomposition for $(K[\gamma], \gamma)$. Moreover the above lemma still holds if instead of considering hermitian forms h_i we consider τ_i-sesquilinear forms $B_i : V_i \times V_i \to A_i$ such that for any $v \in V_i$, $\mathrm{Tr}_{A_i/F_i}(B_i(v, v)) = 0$.

Proposition 3.3.2.2. *Two absolutely semisimple elements γ, γ' of $\mathrm{O}(V, q)$ are conjugate if and only if there is a bijection σ between their respective sets of indices I and I' and compatible isomorphisms $(A_i, \gamma_i) \simeq \left(A'_{\sigma(i)}, \gamma'_{\sigma(i)}\right)$ and $(V_i, h_i) \simeq \left(V'_{\sigma(i)}, h'_{\sigma(i)}\right)$. Moreover the algebraic group $\mathrm{Cent}(\gamma, \mathbf{O}(V, q))$ is naturally isomorphic to*

$$\prod_{i \in I_{\mathrm{triv}}} \mathbf{O}(V_i, h_i) \times \prod_{i \in I_{\mathrm{field}}} \mathrm{Res}_{F_i/K} \mathbf{U}(V_i, h_i) \times \prod_{i \in I_{\mathrm{split}}} \mathrm{Res}_{F_i/K} \mathbf{GL}(W_i).$$

If $\dim_K V$ is odd $\mathbf{O}(V, q) = \mathbf{SO}(V, q) \times \mu_2$, so this proposition easily yields a description of absolutely semisimple conjugacy classes in $\mathrm{SO}(V, q) = \mathbf{SO}(V, q)(K)$ and their centralisers. If $\dim_K V$ is even the proposition still holds if we replace $\mathbf{O}(V, q)$ by $\mathbf{SO}(V, q)$ and $\prod_{i \in I_{\mathrm{triv}}} \mathbf{O}(V_i, h_i)$ by $\mathbf{S}\left(\prod_{i \in I_{\mathrm{triv}}} \mathbf{O}(V_i, h_i)\right)$ and add the assumption $I_{\mathrm{triv}} \neq \emptyset$. If $\dim_K V$ is even and $I_{\mathrm{triv}} = \emptyset$, the datum $(A_i, \gamma_i, V_i, h_i)_{i \in I}$ determines two conjugacy classes in $\mathrm{SO}(V, q)$.

In the symplectic case there is a similar proposition, but now the indices $i \in I_{\mathrm{triv}}$ yield symplectic groups.

Note that if K is a local or global field in which $2 \neq 0$, the simple and explicit invariants in the local case and the theorem of Hasse-Minkowski (and its simpler analogue for hermitian forms, see [Jac40]) in the global case allow to classify the semisimple conjugacy classes explicitly. For example if $K = \mathbb{Q}$, given $M > 0$ one can enumerate the semisimple conjugacy classes in $\mathrm{SO}(V, q)$ annihilated by a non-zero polynomial having integer coefficients bounded by M.

3.3.2.2 Semisimple conjugacy classes in hyperspecial maximal compact subgroups

To compute orbital integrals in the simplest case of the unit in the unramified Hecke algebra of a split classical group over a p-adic field, it would be ideal to have a similar description of conjugacy classes and centralisers valid over $\mathbb{Z}_p$. It is straightforward to adapt the above description over any ring (or any base scheme). However, it is not very useful as the conjugacy classes for which we would like to compute orbital integrals are not all "semisimple over $\mathbb{Z}_p$", i.e. $\mathbb{Z}_p[\gamma]$ is not always an étale $\mathbb{Z}_p$-algebra. Note that the "semisimple over $\mathbb{Z}_p$" case is covered by [Kot86, Corollary 7.3] (with the natural choice of Haar measures, the orbital integral is equal to 1). Nevertheless using the tools of the previous section, we give in this section a method to exhaust the isomorphism classes of triples (Λ, q, γ) where Λ is a finite free $\mathbb{Z}_p$-module, q is a regular quadratic form on Λ and $\gamma \in \mathrm{SO}(\Lambda, q)$. The symplectic case is similar. This means that we will be able to enumerate them, but a priori we will obtain some isomorphism classes several times. In the next section we will nonetheless see that the results of this section can be used to compute the orbital integrals, without checking for isomorphisms.

Let Λ be a free $\mathbb{Z}_p$-module of finite rank endowed with a regular quadratic form q, and let $\gamma \in \mathrm{Aut}_{\mathbb{Z}_p}(\Lambda)$ preserving q and semisimple over $\mathbb{Q}_p$. We apply the notations and considerations of section 3.3.2.1 to the isometry γ of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda$, to obtain quadratic or hermitian spaces $\left(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda\right)_i$. Consider the lattices

$$\Lambda_i := \Lambda \cap \left(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda\right)_i = \ker\left(P_i(\gamma) \mid \Lambda\right).$$

Let $N \geq 0$ be such that p^N belongs to the ideal of $\mathbb{Z}_p[X]$ generated by the $\prod_{j \neq i} P_j$ for all i. Then $\Lambda / \left(\oplus_i \Lambda_i\right)$ is annihilated by p^N, so this group is finite. Since Λ_i is saturated in Λ and q is regular, for any $v \in \Lambda_i \smallsetminus p\Lambda_i$,

$$\begin{cases} p^N \in B(v, \Lambda_i) & \text{if } p \geq 3 \text{ or } \mathrm{rk}_{\mathbb{Z}_p}\Lambda_i \text{ is even,} \\ p^N \in B(v, \Lambda_i) \text{ or } q(v) \in \mathbb{Z}_2^\times & \text{if } p = 2 \text{ and } \mathrm{rk}_{\mathbb{Z}_p}\Lambda_i \text{ is odd.} \end{cases} \tag{3.3.2.1}$$

The $\mathbb{Z}_p[\gamma_i]$-module Λ_i is endowed with a hermitian (quadratic if $\gamma_i^2 = 1$) form h_i taking values in F_i. The sesquilinear (bilinear if $\gamma_i^2 = 1$) form $B_i : \Lambda_i \times \Lambda_i \to A_i$ associated with h_i has the property that for all $v, w \in \Lambda_i$,

$$B(v, w) = \mathrm{Tr}_{A_i/\mathbb{Q}_p}\left(B_i(v, w)\right).$$

From now on we assume for simplicity that $\mathbb{Z}_p[\gamma_i]$ is normal (i.e. either it is the integer ring of an extension of $\mathbb{Q}_p$, or the product of two copies of such an integer ring), as it will be the case in our global situation which imposes that the γ_i's be roots of unity. The structure of quadratic or hermitian modules over such rings is known: see [O'M00] for the quadratic case, [Jac62] for the hermitian case. The "split" case amounts to the comparison of two lattices in a common vector space (isomorphism classes of such pairs are parametrised by "invariant factors"). Choose a uniformiser ϖ_i of $\mathbb{Z}_p[\gamma_i]$ (by definition, in the split case ϖ_i is a uniformiser of $\mathcal{O}_{F_i}$). In all cases, there is a (non-canonical) orthogonal decomposition $\Lambda_i = \bigoplus_{r \in \mathbb{Z}} \Lambda_i^{(r)}$ such that $\varpi_i^{-r} B_i|_{\Lambda_i^{(r)} \times \Lambda_i^{(r)}}$ is integral and non-degenerate. If $(\varpi_i^{d_i})$ is the different of $\mathbb{Z}_p[\gamma_i]/\mathbb{Z}_p$ and $(p) = (\varpi_i^{e_i})$, condition 3.3.2.1 imposes (but in general stays stronger than) the following:

$$\begin{cases} \Lambda_i^{(r)} = 0 \text{ unless } -d_i \leq r \leq -d_i + Ne_i & \text{if } p \geq 3 \text{ or } \mathrm{rk}_{\mathbb{Z}_p}\Lambda_i \text{ is even,} \\ \Lambda_i^{(r)} = 0 \text{ unless } 0 \leq r \leq \max(1, N) & \text{if } p = 2 \text{ and } \mathrm{rk}_{\mathbb{Z}_p}\Lambda_i \text{ is odd.} \end{cases} \tag{3.3.2.2}$$

Note that in the second case $\gamma_i^2 = 1$ and h_i is a quadratic form over $\mathbb{Z}_2$. These conditions provide an explicit version of the finiteness result in section 3.3.1.2, since for any i and r

there is a finite number of possible isomorphism classes for $\Lambda_i^{(r)}$, and when the Λ_i's are fixed, there is only a finite number of possible γ-stable q-regular Λ's since

$$\bigoplus_i \Lambda_i \subset \Lambda \subset p^{-\max(1,N)} \bigoplus_i \Lambda_i.$$

For efficiency it is useful to sharpen these conditions. Denote by o an orbit of $\mathbb{Z}/2\mathbb{Z} \times \mathrm{Gal}\left(\overline{\mathbb{F}_p}/\mathbb{F}_p\right)$ acting on $\overline{\mathbb{F}_p}^{\times}$, where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts by $x \mapsto x^{-1}$. Concretely, o is an orbit in the set of primitive m-th roots of unity (m coprime to p) under the subgroup $\langle p, -1\rangle$ of $(\mathbb{Z}/m\mathbb{Z})^{\times}$. Let I_o be the set of indices i such that γ_i modulo some (at most two possibilities) maximal ideal of $\mathbb{Z}_p[\gamma_i]$ belongs to o. Then for $o \neq o'$, $\prod_{i\in I_o} P_i$ and $\prod_{i\in I_{o'}} P_i$ generate the unit ideal in $\mathbb{Z}_p[X]$, thus $\Lambda = \bigoplus_o \Lambda_{I_o}$ where

$$\Lambda_{I_o} = \mathrm{Sat}_\Lambda\left(\bigoplus_{i\in I_o} \Lambda_i\right) = \ker\left(\prod_{i\in I_o} P_i(\gamma) \mid \Lambda\right).$$

Here $\mathrm{Sat}_\Lambda(\Lambda')$, the *saturation* of Λ' in Λ, is defined as $\Lambda \cap (\mathbb{Q}_p\Lambda')$. Our task is now to enumerate the γ-stable q-regular lattices containing $\bigoplus_{i\in I_o}\Lambda_i$ in which each Λ_i is saturated. For $i \in I_o$, there is a canonical ("Jordan-Chevalley over $\mathbb{Z}_p$") decomposition $\gamma_i = \alpha_i\beta_i$ where $\Phi_m(\alpha_i) = 0$ (m associated with o as above) and

$$\beta_i^{p^n} \xrightarrow[n\to+\infty]{} 1.$$

Since we assumed that $\mathbb{Z}_p[\gamma_i] = \mathbb{Z}_p[\alpha_i][\beta_i]$ is normal, either $\beta_i \in \mathbb{Z}_p[\alpha_i]$ or over each factor of $\mathbb{Q}_p[\alpha_i]$, $\mathbb{Q}_p[\gamma_i]$ is a non-trivial totally ramified field extension and $\beta_i - 1$ is a uniformiser. In any case, define $h_i' := \mathrm{Tr}_{F_i/\mathbb{Q}_p[\alpha_i+\alpha_i^{-1}]}(h_i)$, a quadratic or hermitian (with respect to $\tau_i : \alpha_i \mapsto \alpha_i^{-1}$) form on the $\mathbb{Z}_p[\alpha_i]$-module Λ_i. On Λ_{I_o}, $\gamma = \alpha_{I_o}\beta_{I_o}$ as above, the restriction of α_{I_o} to Λ_i ($i \in I_o$) is α_i, and the minimal polynomial of α_i over $\mathbb{Q}_p$ does not depend on $i \in I_o$. Thus we can see the Λ_i, $i \in I_o$ as finite free quadratic or hermitian modules over the same ring $\mathbb{Z}_p\left[\alpha_{I_o}\right]$, each of these modules being endowed with an automorphism β_i satisfying $\beta_i^{p^n} \to 1$. Moreover since $\mathbb{Z}_p\left[\alpha_{I_o}\right]$ is an étale $\mathbb{Z}_p$-algebra, the regularity of q (restricted to Λ_{I_o}) is equivalent to the regularity of $h' = \oplus_i h_i'$ on Λ_{I_o}. Knowing the Λ_i's, finding the possible Λ_{I_o}'s amounts to finding the β-stable h'-regular lattices containing $\bigoplus_{i\in I_o}\Lambda_i$ in which each Λ_i is saturated, where $\beta = \oplus_i\beta_i$.

Let us now specialise to the case where each γ_i is a root of unity, i.e. $\beta_i^{p^n} = 1$ for some $n \geq 0$. Denote by Φ_r the r-th cyclotomic polynomial.

Lemma 3.3.2.3. *Let $m \geq 1$ be coprime to p. In $\mathbb{Z}_p[X]$, for any $k \geq 1$, p belongs to the ideal generated by $\Phi_{p^km}(X)$ and $\Phi_m\left(X^{p^{k-1}}\right)$.*

Proof. For $k = 1$, since $\Phi_m(X^p) = \Phi_{pm}(X)\Phi_m(X)$, by derivating we obtain the following equality in the finite étale $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[X]/\Phi_m(X)$:

$$\Phi_{pm}(X) = pX^{p-1}\Phi_m'(X^p)/\Phi_m'(X) = p \times \text{unit}.$$

Hence there exists $U, V \in \mathbb{Z}_p[X]$ such that $\Phi_{pm}(X)U(X) + \Phi_m(X)V(X) = p$. For any $k \geq 1$ we have $\Phi_{p^km}(X) = \Phi_{pm}\left(X^{p^{k-1}}\right)$, and the general case follows. $\square$

Having chosen quadratic or hermitian lattices $(\Lambda_i)_{i\in I_o}$, there is a natural order in which to proceed to enumerate the possible Λ_{I_o}. Let us focus on one orbit o. To lighten notation name the indices $I_o = \{1, \dots, s\}$ in such a way that for $1 \le t \le s$, $P_t | \Phi_{mp^{k_t}}$ where $0 \le k_1 < \dots < k_s$. Having fixed o we also drop the indices I_o from our notations. The lemma tells us that for any $1 \le t < s$, p annihilates

$$\mathrm{Sat}_\Lambda(\Lambda_1 \oplus \dots \oplus \Lambda_{t+1}) / (\mathrm{Sat}_\Lambda(\Lambda_1 \oplus \dots \oplus \Lambda_t) \oplus \Lambda_{t+1})$$

and thus we also have that p^{s-t} annihilates

$$\Lambda / (\mathrm{Sat}_\Lambda(\Lambda_1 \oplus \dots \oplus \Lambda_t) \oplus \Lambda_{t+1} \oplus \dots \Lambda_s).$$

This will provide a sharper version of condition 3.3.2.1. Let B' be the sesquilinear (bilinear if $\alpha^2 = 1$) form on Λ associated with h'. For any $i \in I_o$ there is an orthogonal decomposition with respect to B': $\Lambda_i = \bigoplus_r L_i^{(r)}$ where each $L_i^{(r)}$ is p^r-modular for B', i.e. $p^{-r} B'|_{L_i^{(r)} \times L_i^{(r)}}$ takes values in $\mathbb{Z}_p[\alpha]$ and is non-degenerate. For $1 \le t \le s$ denote $M_t = \mathrm{Sat}_\Lambda(\Lambda_1 \oplus \dots \oplus \Lambda_t)$, which can similarly be decomposed orthogonally with respect to B': $M_t = \bigoplus_r M_t^{(r)}$. Note that $M_1 = \Lambda_1$. Analogously to condition 3.3.2.1, for $1 \le t < s$ we have

$$L_{t+1}^{(r)} = M_t^{(r)} = 0 \quad \text{unless } 0 \le r \le s - t. \tag{3.3.2.3}$$

and if $s = 1$ we simply have that the hermitian (or quadratic) module (Λ_1, h') over $\mathbb{Z}_p[\alpha]$ is regular. We can deduce a sharper version of condition 3.3.2.2. If $s > 1$ then

$$\Lambda_1^{(r)} = 0 \text{ unless } -d_1 \le r \le -d_1 + (s-1)e_1 \tag{3.3.2.4}$$

$$\text{for } 1 < t \le s,\ \Lambda_t^{(r)} = 0 \text{ unless } -d_t \le r \le -d_t + (s-t+1)e_t. \tag{3.3.2.5}$$

while for $s = 1$:

$$\begin{cases} \Lambda_1^{(r)} = 0 \text{ if } r \ne -d_1 & \text{if } p \ge 3 \text{ or } m > 1, \\ \Lambda_1 \text{ is a regular quadratic } \mathbb{Z}_2\text{-module} & \text{if } p = 2 \text{ and } m = 1. \end{cases} \tag{3.3.2.6}$$

Let us recapitulate the algorithm thus obtained to enumerate *non-uniquely* the isomorphism classes of triples (Λ, q, γ) such that (Λ, q) is regular and γ is torsion. Begin with a datum $(A_i, \gamma_i)_{i\in I}$, i.e. fix the characteristic polynomial of γ. For any orbit o for which $s = \mathrm{card}(I_o) > 1$:

1. For any $i \in I_o$, enumerate the isomorphism classes of quadratic or hermitian $\mathbb{Z}_p[\alpha_i]$-modules Λ_i subject to conditions 3.3.2.4 and 3.3.2.5, compute B' on $\Lambda_i \times \Lambda_i$ and throw away those which do not satisfy condition 3.3.2.3.

2. For any such family $(\Lambda_i)_{i\in I_o}$, enumerate inductively the possible $\mathrm{Sat}_\Lambda(\Lambda_1 \oplus \dots \oplus \Lambda_t)$. At each step $t = 1, \dots, s$, given a candidate M_t for $\mathrm{Sat}_\Lambda(\Lambda_1 \oplus \dots \oplus \Lambda_t)$, we have to enumerate the candidates M_{t+1} for $\mathrm{Sat}_\Lambda(\Lambda_1 \oplus \dots \oplus \Lambda_t)$, i.e. the β-stable lattices containing $M_t \oplus \Lambda_{t+1}$ such that

   (a) h' is integral on M_{t+1},

(b) both M_t and Λ_{t+1} are saturated in M_{t+1},

(c) if $t < s-1$, M_{t+1} satisfies condition 3.3.2.3,

(d) if $t = s-1$, M_{t+1} (a candidate for Λ) is regular for h'.

Remark 3.3.2.4. *The first step can be refined, since already over $\mathbb{Q}_p$ there are obstructions to the existence of a regular lattice. These obstructions exist only when $h' = q$ is a quadratic form, i.e. $\alpha_{I_o}^2 = 1$, so let us make this assumption for a moment. Consider its discriminant $D = \mathrm{disc}(q) \in \mathbb{Q}_p^\times/\mathrm{squares}(\mathbb{Q}_p^\times)$. If $\mathrm{rk}_{\mathbb{Z}_p}\Lambda = 2n$ is even, then $\mathbb{Q}_p[X]/(X^2 - (-1)^n D)$ is unramified over $\mathbb{Q}_p$. If $\mathrm{rk}_{\mathbb{Z}_p}\Lambda$ is odd, the valuation of $\mathrm{disc}(q)/2$ is even. Moreover in any case, once we fix the discriminant, the Hasse-Witt invariant of q is determined. We do not go into more detail. A subtler obstruction is given by the spinor norm of γ. Assume that $N = \mathrm{rk}_{\mathbb{Z}_p}\Lambda$ is at least 3, and for simplicity assume also that $\det(\gamma) = 1$. The regular lattice (Λ, q) defines a reductive group $\mathbf{SO}(q)$ over $\mathbb{Z}_p$. The fppf exact sequence of groups over $\mathbb{Z}_p$*

$$1 \to \mu_2 \to \mathbf{Spin}(q) \to \mathbf{SO}(q) \to 1$$

yields for any $\mathbb{Z}_p$-algebra R the spinor norm $\mathbf{SO}(q)(R) \to H^1_{\mathrm{fppf}}(R, \mu_2)$ whose kernel is the image of $\mathbf{Spin}(q)(R)$. Moreover if $\mathrm{Pic}(R) = 1$ (which is the case if $R = \mathbb{Q}_p$ or $\mathbb{Z}_p$) we have $H^1_{\mathrm{fppf}}(R, \mu_2) = R^\times/\mathrm{squares}(R^\times)$. Thus another obstruction is that the spinor norm of γ must have even valuation. We can compute the spinor norm of each γ_i easily. If $\gamma_i = -1$ its spinor norm is simply the discriminant of the quadratic form h_i. If $i \notin I_{\mathrm{triv}}$ a straightforward computation shows that the spinor norm of γ_i is $N_{A_i/\mathbb{Q}_p}(1+\gamma_i)^{\dim_{A_i} V_i}$. Note that it does not depend on the isomorphism class of the hermitian form h_i.

Let us elaborate on the second step of the algorithm. For an orbit o for which $s = 1$, we simply have to enumerate the modules Λ_1 satisfying 3.3.2.6 and such that the resulting quadratic form q (equivalently, h') is regular.

We have not given an optimal method for the case $s > 1$. A very crude one consists in enumerating all the free $\mathbb{F}_p[\alpha]$-submodules in $p^{-1}\mathbb{Z}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} (M_t \oplus \Lambda_{t+1})$ and keeping only the relevant ones. The following example illustrates that one can do much better in many cases.

Example 3.3.2.5. *Consider the "second simplest" case $s = 2$. Assume for simplicity that $p > 2$ or $m > 1$. Then condition 3.3.2.3 shows that for any pair $((\Lambda_1, h_1), (\Lambda_2, h_2))$ found at the first step of the algorithm, we have*

$$\Lambda_1 = L_1^{(0)} \oplus L_1^{(1)} \qquad \textit{and} \qquad \Lambda_2 = L_2^{(0)} \oplus L_2^{(1)}$$

where each $L_i^{(r)}$ is p^r-modular. Moreover for any $i \in \{1,2\}$ the topologically unipotent automorphism β_i stabilises

$$pL_i^{(0)} \oplus L_i^{(1)} = \{v \in \Lambda_i \,|\, \forall w \in \Lambda_i,\ B_i'(v,w) \in p\mathbb{Z}_p[\alpha]\}$$

and thus β_i induces a unipotent automorphism $\overline{\beta_i}$ of (V_i, η_i) where $V_i = L_i^{(1)}/pL_i^{(1)}$ and η_i is a the non-degenerate quadratic or hermitian form $p^{-1}h_i' \bmod p$ on V_i. It is easy to *check that any relevant $\Lambda \supset \Lambda_1 \oplus \Lambda_2$ is such that*

$$p\Lambda/(p\Lambda_1 \oplus p\Lambda_2) = \{v_1 \oplus f(v_1) \,|\, v_1 \in V_1\}$$

for a unique isomorphism $f : (V_1, \eta_1, \beta_1) \to (V_2, -\eta_2, \beta_2)$. Conversely such an isomorphism yields a relevant Λ.

For $p = 2$ and $m = 1$ there is a similar but a bit more complicated description of the relevant lattices $\Lambda \supset \Lambda_1 \oplus \Lambda_2$. In that case each form η_i is a "quadratic form modulo 4", i.e. $x \mapsto \langle x, x \rangle \mod 4$ where $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form on a free $\mathbb{Z}_2$-module N. Note that $\langle x, x \rangle \mod 4$ only depends on the class of x in $\mathbb{F}_2 \otimes N$. A further complication comes into play when $\mathrm{rk}_{\mathbb{Z}_2}(\Lambda_1) + \mathrm{rk}_{\mathbb{Z}_2}(\Lambda_2)$ is odd, but we do not go into more detail.

In the case of an arbitrary $s > 1$, the observation made in example 3.3.2.5 still applies at the last step $t = s-1$, replacing (Λ_1, Λ_2) with (M_{s-1}, Λ_s). We do not go into the details of our implementation of the previous steps ($t < s-1$). We merely indicate that in general $pM_{t+1}/(M_t \oplus \Lambda_{t+1})$ is still described using an isomorphism f between a β-stable subspace of $\bigoplus_{r \geq 1} M_t^{(r)} \mod p$ and a β-stable subspace of $\bigoplus_{r \geq 1} L_t^{(r)} \mod p$.

Remark 3.3.2.6. *Regarding all the results of this section, the symplectic case is similar, replacing "quadratic" by "symplectic" and "hermitian" by "antihermitian", and even simpler because the prime 2 is "less exceptional". More precisely, the classification of hermitian modules for e.g. the quadratic extension $\mathbb{Z}_p[\zeta_{p^k}]/\mathbb{Z}_p[\zeta_{p^k} + \zeta_{p^k}^{-1}]$ is more involved for $p = 2$ than for the other primes (see [Jac62]), but once we have enumerated the possible isomorphism classes of Λ_i's, the enumeration of the relevant $\Lambda \supset \oplus_i \Lambda_i$ can be done uniformly in p.*

3.3.2.3 Orbital integrals for the unit in the unramified Hecke algebra of a p-adic classical group

In this section we show that thanks to the fact that orbital integrals are formally sums of masses (where "mass" takes the same meaning as in "mass formula", or in overly fancy terms, the "measure of a groupoid"), they can be computed by counting instead of enumerating and checking isomorphisms. As before we focus on the case of special orthogonal groups, the case of symplectic groups being easier.

Let Λ_0 be a free $\mathbb{Z}_p$-module of finite rank endowed with a regular quadratic form q_0 and consider the algebraic group $\mathbf{G} = \mathbf{SO}(\Lambda_0, q_0)$ which is reductive over $\mathbb{Z}_p$. Let $f = \mathbf{1}_{\mathbf{G}(\mathbb{Z}_p)}$ be the characteristic function of $\mathbf{G}(\mathbb{Z}_p)$ and fix the Haar measure on $\mathbf{G}(\mathbb{Q}_p)$ such that $\int_{\mathbf{G}(\mathbb{Z}_p)} dg = 1$. Let $\gamma_0 \in \mathbf{G}(\mathbb{Q}_p)$ be semisimple (for now we do not assume that it is torsion), and let $\mathbf{I}_0$ be its connected centraliser in $\mathbf{G}_{\mathbb{Q}_p}$. Fix a Haar measure ν on $\mathbf{I}_0(\mathbb{Q}_p)$. Consider the isomorphism classes of triples (Λ, q, γ) such that

- Λ is a free $\mathbb{Z}_p$-module of finite rank endowed with a regular quadratic form q,
- $\gamma \in \mathrm{SO}(\Lambda, q)$,
- there exists an isomorphism between $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda, q, \gamma)$ and $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_0, q_0, \gamma_0)$.

We apply the previous section's notations and results to such (Λ, q, γ). The last condition can be expressed simply using the classical invariants of quadratic (over $\mathbb{Q}_p$) or hermitian (over $\mathbb{Q}_p[\gamma_i]$) forms, as in Proposition 3.3.2.2. It implies that $\mathbf{I}_0$ and the connected centraliser $\mathbf{I}$ of γ in $\mathbf{SO}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda, q)$ are isomorphic, and by Remark 3.3.1.1 we can see ν as a Haar measure on $\mathbf{I}(\mathbb{Q}_p)$. Then

$$O_{\gamma_0}(f(g)dg) = \left(\sum_{(\Lambda,q,\gamma)} \nu\left(\mathbf{I}(\mathbb{Q}_p) \cap \mathrm{SO}(\Lambda, q)\right)^{-1} \right) \nu$$

where the sum ranges over isomorphism classes as above. Note that $\mathbf{I}(\mathbb{Q}_p) \cap \mathrm{SO}(\Lambda, q)$ stabilises each Λ_i, so that it is a subgroup of $\prod_i \Gamma_i \subset \mathbf{I}(\mathbb{Q}_p)$ where

$$\Gamma_i = \begin{cases} \mathrm{SO}(\Lambda_i, h_i) & \text{if } i \in I_{\text{triv}} \\ \mathrm{U}(\Lambda_i, h_i) & \text{if } i \in I_{\text{field}} \cup I_{\text{split}}. \end{cases}$$

In fact $\mathbf{I}(\mathbb{Q}_p) \cap \mathrm{SO}(\Lambda, q)$ is the stabiliser of $\Lambda / \bigoplus_i \Lambda_i$ for the action of $\prod_i \Gamma_i$ on $(\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} (\bigoplus_i \Lambda_i)$. Grouping the terms in the above sum according to the isomorphism classes of the quadratic or hermitian modules Λ_i, we obtain

$$O_{\gamma_0}(f(g)dg) = \left(\sum_{(\Lambda_i,h_i)_{i\in I}} \frac{\mathrm{ext}\left((\Lambda_i, h_i)_i\right)}{\nu\left(\prod_i \Gamma_i\right)} \right) \nu. \tag{3.3.2.7}$$

Now the sum ranges over the isomorphism classes of quadratic or hermitian lattices (Λ_i, h_i) over $\mathbb{Z}_p[\gamma_i]$, which become isomorphic to the corresponding datum for $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_0, q_0, \gamma_0)$ when p is inverted, and

$$\mathrm{ext}\left((\Lambda_i, h_i)_i\right) := \mathrm{card} \left\{ q\text{-regular } (\oplus_i \gamma_i)\text{-stable } \Lambda \supset \bigoplus_i \Lambda_i \mid \forall i,\ \Lambda_i \text{ saturated in } \Lambda \right\}.$$

We will study the volumes appearing at the denominator below, but for the moment we consider these numerators. Motivated by the global case, assume from now on that γ_0 is torsion as in the end of the previous section. It is harmless to restrict our attention to a single orbit o, and assume $I = I_o$. For the computation of orbital integrals, the benefit resulting from the transformation above is that instead of enumerating the possible M_{t+1} knowing M_t at the last step $t = s - 1$, we only have to count them. Let us discuss the various cases that can occur, beginning with the simplest ones.

The unramified case corresponds to $s = 1$ and $A_1 = \mathbb{Q}_p[\gamma_1] = \mathbb{Q}_p[\alpha]$, and in that case there is a unique relevant isomorphism class (Λ_1, h_1). It is easy to check that we recover Kottwitz's result [Kot86][Corollary 7.3] that the orbital integral equals 1 for the natural choice of Haar measures.

The case where $s = 1$ but $\mathbb{Q}_p[\gamma_1]/\mathbb{Q}_p[\alpha]$ can be non-trivial (i.e. ramified) is not much harder: the algorithm given in the previous section identifies the relevant isomorphism classes (Λ_1, h_1) appearing below the sum, and $\mathrm{ext}(\Lambda_1, h_1) = 1$. In this case we have reduced the problem of computing the orbital integral by that of computing the volume of the stabilisers of some lattices. When $\mathbf{G} = \mathbf{Sp}_2 = \mathbf{SL}_2$ it is the worst that can happen.

The first interesting case is $s = 2$. Assume for simplicity that $p > 2$ or $m > 1$, and let us look back at example 3.3.2.5, using the same notations. Then $\mathrm{ext}((\Lambda_1, h_1), (\Lambda_2, h_2)) = 0$ unless $(V_1, \eta_1, \beta_1) \simeq (V_2, -\eta_2, \beta_2)$, in which case $\mathrm{ext}((\Lambda_i, h_i)_i) = \mathrm{card}\,(\mathrm{Aut}(V_1, \eta_1, \beta_1))$. This group is the centraliser of a unipotent element in a classical group over a finite field. Results of Wall [Wal63] give the invariants of such conjugacy classes as well as formulae for their centralisers. In many cases (e.g. if $\mathrm{rk}_{\mathbb{Z}_p}(\Lambda) < p^2 - 1$) the automorphism β_1 of V_1 is trivial, and thus we do not need the general results of Wall, but merely the simple cardinality formulae of finite classical groups. For $\mathbf{G} = \mathbf{Sp}_4$ or $\mathbf{SO}_4$ we have $s \leq 2$ and $\beta_1|_{V_1} = 1$ at worst.

When $s > 2$ the situation is of course more complicated, and it seems that we cannot avoid the enumeration of successive lattices $M_{t+1} \supset M_t \oplus \Lambda_{t+1}$ for $t < s - 1$, although the last step $t = s - 1$ is identical to the above case. Note however that these "very ramified" cases are rare in low rank. More precisely $\mathrm{rk}_{\mathbb{Z}_p}\Lambda \geq p^{s-1}$, e.g. in rank less than 25 it can happen that $s > 2$ only for $p = 2, 3$. Thus the "worst cases" have $p = 2$. This is fortunate because for fixed k and n the number of k-dimensional subspaces in an n-dimensional vector space over a finite field with q elements increases dramatically with q.

Remark 3.3.2.7. *In the case where* $\mathbf{G}$ *is an* even *special orthogonal group, some of the semisimple conjugacy classes in* $\mathbf{G}(\mathbb{Q}_p)$ *were parametrised only up to outer conjugation. Since* $\mathbf{G}(\mathbb{Z}_p)$ *is invariant by an outer automorphism of* $\mathbf{G}$*, for any* $\gamma_0, \gamma_0' \in \mathbf{G}(\mathbb{Q}_p)$ *which are conjugate by an outer automorphism of* $\mathbf{G}_{\mathbb{Q}_p}$*, the orbital integrals for* $f(g_p)dg_p$ *at* γ_0 *and* γ_0' *are equal. Of course the above formula for the orbital integral is valid for both.*

3.3.2.4 Local densities and global volumes

To complete the computation of adèlic orbital integrals we still have to evaluate the denominators in formula 3.3.2.7 and the global volumes. Formulae for local densities and Smith-Minkowski-Siegel mass formulae are just what we need. But we will use the point of view suggested by [Gro97] and used in [GP05], i.e. fix canonical Haar measures to see local orbital integrals as numbers. For this we need to work in a slightly more general setting than cyclotomic fields.

If k is a number field or a p-adic field, denote by $\mathcal{O}_k$ its ring of integers. If k is a number field $\mathbb{A}_k = k \otimes_{\mathbb{Q}} \mathbb{A}$ will denote the adèles of k.

Let k be a number field or a local field of characteristic zero, and let K be a finite commutative étale k-algebra such that $\dim_k K \leq 2$, i.e. $K = k$ or $k \times k$ or K is a quadratic field extension of k. Let τ be such that $\mathrm{Aut}_k(K) = \{\mathrm{Id}_K, \tau\}$. This determines τ. Let V be a vector space over K of dimension $r \geq 0$. Let $\alpha \in \{1, -1\}$, and assume that $\alpha = 1$ if $\dim_k K = 2$. Assume that V is endowed with a non-degenerate τ-sesquilinear form $\langle \cdot, \cdot \rangle$ such that for any $v_1, v_2 \in V$ we have $\langle v_2, v_1 \rangle = \alpha\tau\left(\langle v_1, v_2 \rangle\right)$. Let $\mathbf{G} = \mathbf{Aut}(V, \langle \cdot, \cdot \rangle)^0$ be the connected reductive group over k associated with this datum. Then $\mathbf{G}$ is a special orthogonal ($K = k$ and $\alpha = 1$), symplectic ($K = k$ and $\alpha = -1$), unitary (K/k is a quadratic field extension and $\alpha = 1$) or general linear ($K = k \times k$ and $\alpha = 1$) group.

If k is a number field, by Weil [Wei82] the Tamagawa number $\tau(\mathbf{G})$ equals 2 (resp. 1) in the orthogonal case if $r \geq 2$ and V is not a hyperbolic plane (resp. if $r = 1$ or V is a hyperbolic plane), 1 in the symplectic case, 2 in the unitary case if $r > 0$ and 1 in the general linear case.

If k is a p-adic field, consider a lattice N in V, i.e. a finite free $\mathcal{O}_K$-module $N \subset V$ such that $V = KN$. Denote $N^\vee = \{v \in V \,|\, \forall w \in N,\, \langle v, w\rangle \in \mathcal{O}_K\}$. If $\langle\cdot,\cdot\rangle|_{N\times N}$ takes values in $\mathcal{O}_K$ then $N^\vee \supset N$ and we can consider $[N^\vee : N]$, i.e. the cardinality of the finite abelian group $N^\vee/N$. In general define $[N^\vee : N]$ as $[N^\vee : N^\vee \cap N]/[N : N^\vee \cap N]$. Recall also [GY00][Definition 3.5] the *density* β_N associated with $(N, \langle\cdot,\cdot\rangle)$.

In [Gro97] Gross associates a motive M of Artin-Tate type to any reductive group over a field. For the groups $\mathbf{G}$ defined above, letting n be the rank of $\mathbf{G}$, we have

$$M = \begin{cases} \bigoplus_{x=1}^n \mathbb{Q}(1-2x) & \text{orthogonal case with } r \text{ odd and symplectic case,} \\ \chi\mathbb{Q}(1-n) \oplus \bigoplus_{x=1}^{n-1} \mathbb{Q}(1-2x) & \text{orthogonal case with } r > 0 \text{ even,} \\ \bigoplus_{x=1}^n \chi^x \mathbb{Q}(1-x) & \text{unitary and general linear cases.} \end{cases}$$

In the orthogonal case with $r > 0$ even let $(-1)^n D$ be the discriminant of $(V, \langle\cdot,\cdot\rangle)$ (i.e. the determinant of the Gram matrix), then χ is defined as the character $\mathrm{Gal}(k(\sqrt{D})/k) \to \{\pm 1\}$ which is non-trivial if D is not a square in k. In the general linear case χ is trivial, and in the unitary case χ is the non-trivial character of $\mathrm{Gal}(K/k)$. For L-functions and ϵ-factors we will use the same notations as [Gro97].

If k is a number field D_k will denote the absolute value of its discriminant. For $K = k$ or $K = k \times k$ denote $D_{K/k} = 1$, whereas for a quadratic field extension K of k we denote $D_{K/k} = |N_{K/\mathbb{Q}}(\mathfrak{D}_{K/k})|$ where $\mathfrak{D}_{K/k}$ is the different ideal of K/k and the absolute value of the ideal $m\mathbb{Z}$ of $\mathbb{Z}$ is m if $m \geq 1$. There are obvious analogues over any p-adic field, and D_k (resp. $D_{K/k}$) is the product of D_{k_v} (resp. D_{K_v/k_v} where $K_v = k_v \otimes_k K$) over the finite places v of k.

For $(k, K, \alpha, V\langle\cdot,\cdot\rangle)$ (local or global) as above define as in [GY00]

$$n(V) = \begin{cases} r + \alpha & \text{if } K = k, \\ r & \text{if } \dim_k K = 2 \end{cases}$$

and

$$\mu = \begin{cases} 2^r & \text{in the orthogonal case with } r \text{ even,} \\ 2^{(r+1)/2} & \text{in the orthogonal case with } r \text{ odd,} \\ 1 & \text{in the symplectic, unitary and general linear cases.} \end{cases}$$

Finally, consider the case where $k = \mathbb{R}$ and $\mathbf{G}(\mathbb{R})$ has discrete series, i.e. the Euler-Poincaré measure on $\mathbf{G}(\mathbb{R})$ is non-zero, i.e. $\mathbf{G}$ has a maximal torus $\mathbf{T}$ which is anisotropic. Recall Kottwitz's sign $e(\mathbf{G}) = (-1)^{q(\mathbf{G})}$ and the positive rational number $c(\mathbf{G})$ defined in [Gro97][§8]. Explicitly,

$$c(\mathbf{G}) = \begin{cases} 1 & \text{in the symplectic case,} \\ 2^n/\binom{n}{\lfloor a/2 \rfloor} & \text{in the orthogonal case with signature } (a,b), b \text{ even,} \\ 2^n/\binom{n}{a} & \text{in the unitary case with signature } (a,b). \end{cases}$$

The following theorem is a reformulation of the mass formula [GY00][Theorem 10.20] in our special cases.

Theorem 3.3.2.8. *Let k be a totally real number field and let K, α, $(V, \langle\cdot,\cdot\rangle)$ and $\mathbf{G}$ be as above. Let M denote the Gross motive of $\mathbf{G}$. Assume that for any real place v of k, $\mathbf{G}(k_v)$ has discrete series. Define a signed Haar measure $\nu = \prod_v \nu_v$ on $\mathbf{G}(\mathbb{A}_k)$ as follows. For any real place v of k, ν_v is the Euler-Poincaré measure on $\mathbf{G}(k_v)$. For any finite place v of k, ν_v is the canonical measure $L_v(M^\vee(1))|\omega_{\mathbf{G}_{k_v}}|$ on $\mathbf{G}(k_v)$ (see [Gro97][§4]). In particular, for any finite place v such that $\mathbf{G}_{k_v}$ is unramified, the measure of a hyperspecial compact subgroup of $\mathbf{G}(k_v)$ is one. Then for any $\mathcal{O}_K$-lattice N in V,*

$$\int_{\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}_k)} \nu = \tau(\mathbf{G}) \times L(M) \times \frac{D_k^{\dim \mathbf{G}/2} D_{K/k}^{r(r+1)/4}}{\epsilon(M)} \times \prod_{v|\infty} \frac{(-1)^{q(\mathbf{G}_{k_v})}}{c(\mathbf{G}_{k_v})}$$
$$\times\, \mu^{\dim_{\mathbb{Q}} k} \prod_{v \text{ finite}} \frac{[N_v^\vee : N_v]^{n(V)/2} \times \nu_v\left(\mathbf{G}(k_v) \cap \mathrm{GL}(N_v)\right)}{L_v(M^\vee(1))\beta_{N_v}}$$

Proof. To get this formula from [GY00][Theorem 10.20], use the comparison of measure at real places [Gro97][Proposition 7.6], the fact that $L_v(M^\vee(1))\beta_{N_v} = 1$ for almost all finite places of k, and the functional equation $\Lambda(M) = \epsilon(M)\Lambda(M^\vee(1))$ (see [Gro97][9.7]). $\square$

Note that the choice of ν at the finite places does not play any role. This choice was made to compare with the very simple formula [Gro97][Theorem 9.9]:

$$\int_{\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}_k)} \nu \;=\; \tau(\mathbf{G}) \times L(M) \times \prod_{v|\infty} \frac{(-1)^{q(\mathbf{G}_{k_v})}}{c(\mathbf{G}_{k_v})}. \tag{3.3.2.8}$$

We obtain that under the hypotheses of the theorem,

$$\prod_{v \text{ finite}} \nu_v\left(\mathbf{G}(k_v) \cap \mathrm{GL}(N_v)\right) = \frac{\epsilon(M)\mu^{-\dim_{\mathbb{Q}} k}}{D_k^{\dim \mathbf{G}/2} D_{K/k}^{r(r+1)/4}} \prod_{v \text{ finite}} \frac{L_v(M^\vee(1))\beta_{N_v}}{[N_v^\vee : N_v]^{n(V)/2}}. \tag{3.3.2.9}$$

We can compute explicitely

$$\frac{\epsilon(M)}{D_k^{\dim \mathbf{G}/2} D_{K/k}^{r(r+1)/4}} = \begin{cases} D_{K/k}^{-n/2} & \text{in the unitary case if } r = n \text{ is even,} \\ \left|N_{k/\mathbb{Q}}(\delta)\right|^{n-1/2} & \text{in the orthogonal case if } r \text{ is even,} \\ 1 & \text{otherwise,} \end{cases}$$

where in the second case $(-1)^n D$ is the discriminant of $\langle\cdot,\cdot\rangle$ and δ is the discriminant of $k(\sqrt{D})/k$. As the proof of the following proposition shows, the factor $\mu^{-\dim_{\mathbb{Q}} k}$, which is nontrivial only in the orthogonal cases, is local at the dyadic places.

Proposition 3.3.2.9. *Let p be a prime. Let k_0 be a p-adic field and let $(K_0, \alpha, V_0, \langle\cdot,\cdot\rangle_0)$ and $\mathbf{G}_0$ be as above. Let ν_0 be the canonical Haar measure $L(M^\vee(1))|\omega_{\mathbf{G}_0}|$ on $\mathbf{G}_0(k_0)$. If $p = 2$, $K_0 = k_0$ and $\alpha = 1$, let $x_0 = \mu^{-\dim_{\mathbb{Q}_2} k_0}$, otherwise let $x_0 = 1$. Then for any $\mathcal{O}_{K_0}$-lattice N_0 in V_0,*

$$\nu_0\left(\mathbf{G}_0(k_0) \cap \mathrm{GL}(N_0)\right) = L(M^\vee(1)) \times x_0 \times \beta_{N_0} \times [N_0^\vee : N_0]^{-n(V_0)/2}$$
$$\times \begin{cases} D_{K_0/k_0}^{-n/2} & \textit{in the unitary case if } r = n \textit{ is even,} \\ \left|N_{k_0/\mathbb{Q}_p}(\delta_0)\right|^{n-1/2} & \textit{in the orthogonal case if } r \textit{ is even,} \\ 1 & \textit{otherwise,} \end{cases}$$

where in the second case $(-1)^n D_0$ is the discriminant of $\langle\cdot,\cdot\rangle_0$ and δ_0 is the discriminant of $k_0(\sqrt{D_0})/k_0$.

Proof. We apologise for giving a global proof of this local statement. We only give details for the hardest case of orthogonal groups.

When $p > 2$ and the symmetric bilinear form $\langle\cdot,\cdot\rangle_0|_{N_0\times N_0}$ is integer-valued and non-degenerate, $\mathbf{G}_0$ is the generic fiber of a reductive group over $\mathcal{O}_{k_0}$ and the equality is obvious. Note that this does not apply for $p = 2$, even assuming further that the quadratic form $v \mapsto \langle v, v\rangle_0/2$ is integer-valued on N_0, because the local density β_{N_0} is defined using the bilinear form $\langle\cdot,\cdot\rangle_0$, not the quadratic form $v \mapsto \langle v, v\rangle_0/2$.

Next consider the case $p = 2$ and N_0 arbitrary. By Krasner's lemma there exists a totally real number field k and a quadratic vector space $(V, \langle\cdot,\cdot\rangle)$ which is positive definite at the real places of k and such that k has a unique dyadic place v_0 and $(k_0, V_0, \langle\cdot,\cdot\rangle_0) \simeq (k_{v_0}, k_{v_0} \otimes_k V, \langle\cdot,\cdot\rangle)$. Let S be the finite set of finite places $v \neq v_0$ of k such that $(k_v \otimes_k V, \langle\cdot,\cdot\rangle)$ is ramified, i.e. does not admit an integer-valued non-degenerate $\mathcal{O}_{k_v}$-lattice. For any $v \in S$ there is a finite extension $E^{(v)}$ of k_v over which $(k_v \otimes_k V, \langle\cdot,\cdot\rangle)$ becomes unramified. By Krasner's lemma again there exists a finite extension k' of k which is totally split over the real places of k and over v_0 and such that for any $v \in S$, the k_v-algebra $k_v \otimes_k k'$ is isomorphic to a product of copies of $E^{(v)}$. Let S_0 be the set of dyadic places of k', i.e. the set of places of k' above v_0. There exists a lattice N' in $k' \otimes_k V$ such that for any finite $v \notin S_0$ the symmetric bilinear form $\langle\cdot,\cdot\rangle|_{N'_v\times N'_v}$ is integer-valued and non-degenerate, and for any $v \in S_0$ we have $\langle\cdot,\cdot\rangle_{N'_v\times N'_v} \simeq \langle\cdot,\cdot\rangle_0|_{N_0\times N_0}$. Applying formula 3.3.2.9 we obtain the desired equality to the power $\mathrm{card}(S_0)$, which is enough because all the terms are positive real numbers. Having established the dyadic case, the general case can be established similarly.

The unitary case is similar but simpler, because the dyadic places are no longer exceptional and it is sufficient to take a quadratic extension k'/k in the global argument. The symplectic and general linear cases are even simpler. □

Remark 3.3.2.10.

1. *In this formula, one can check case by case that the product of $[N_0^\vee : N_0]^{-n(V_0)/2}$ and the last term is always rational, as expected since all other terms are rational by definition.*

2. *We did not consider the case where $\alpha = -1$ and K/k is a quadratic field extension, i.e. the case of antihermitian forms, although this case is needed to compute orbital integrals for symplectic groups. If $y \in K^\times$ is such that $\tau(y) = -y$, multiplication by y induces a bijection between hermitian and antihermitian forms, and of course the automorphism groups are equal.*

3. *There are other types of classical groups considered in [GY00] and which we left aside. For a central simple algebra K over k with $\dim_k K = 4$ (i.e. $K = M_2(k)$ or K is a quaternion algebra over k) they also consider hermitian (resp. antihermitian) forms over a K-vector space. The resulting automorphism groups are inner forms of symplectic (resp. even orthogonal) groups. Using the same method as in the proof*

of the proposition leads to a formula relating the local density β_{N_0} to the canonical measure of $\mathrm{Aut}(N_0)$ *in these cases as well.*

We use the canonical measure defined by Gross (called ν_v above) when computing local orbital integrals. In the previous section we explained how to compute the numerators in formula 3.3.2.7 for the local orbital integrals. Proposition 3.3.2.9 reduces the computation of the denominators to that of local densities. Using an elegant method of explicitly constructing smooth models, Gan and Yu [GY00] give a formula for β_{N_0} for $p > 2$ in general and for $p = 2$ only in the case of symplectic and general linear groups and in the case of unitary groups if K_0/k_0 is unramified. Using a similar method Cho [Choa] gives a formula in the case of special orthogonal groups when $p = 2$ and $k_0/\mathbb{Q}_2$ is unramified. This is enough for our computations since we only need the case $k_0 = \mathbb{Q}_2$. For $m \geq 1$ and $\zeta = \zeta_m$ the quadratic extension $\mathbb{Q}(\zeta)/\mathbb{Q}(\zeta + \zeta^{-1})$ is ramified over a dyadic place if and only if m is a power of 2. In this case the different $\mathfrak{D}_{\mathbb{Q}_2(\zeta)/\mathbb{Q}_2(\zeta+\zeta^{-1})}$ is generated by a uniformiser of $\mathbb{Q}_2(\zeta + \zeta^{-1})$, which is the minimal ramification that one can expect from a ramified quadratic extension in residue characteristic 2. Cho [Chob][Case 1] also proved an explicit formula for the local density in this case. To be honest [Chob] only asserts it in the case where k_0 is unramified over $\mathbb{Q}_2$. Nevertheless the proof in "Case 1" does not use this assumption. This completes the algorithm to compute the local orbital integrals in all cyclotomic cases over $\mathbb{Q}$. Note that the result is rational and the computations are exact (i.e. no floating point numbers are used).

Finally, the global volume is evaluated using Gross' formula 3.3.2.8. The value of $L(M)$ is known to be rational and computable by [Sie69]. However, we only need the values of $L(M)$ for M which is a direct sum of Tate twists of *cyclotomic* Artin motives (concretely, representations of $\mathrm{Gal}(E/F)$ where E is contained in a cyclotomic extension of $\mathbb{Q}$). Thus we only need the values of Dirichlet L-functions at non-negative integers, i.e. the values of generalised Bernoulli numbers (see e.g. [Was97]).

Remark 3.3.2.11. *Formally it is not necessary to use the results of [Gro97] to compute the factors* $\mathrm{Vol}(\mathbf{I}(\mathbb{Q})\backslash\mathbf{I}(\mathbb{A}))$ *in formula 3.3.1.1, the mass formula in [GY00] along with the formulae for the local densities β_{N_0} would suffice. Apart from the fact that it is less confusing and more elegant to clearly separate local and global measures, using Gross' canonical measure, which is compatible between inner forms by definition, allows to compute κ-orbital integrals once we have computed orbital integrals. The fundamental lemma gives a meaningful way to check the results of computations of orbital integrals. More precisely we need the formulation of the fundamental lemma for semisimple singular elements [Kot86][Conjecture 5.5] which has been reduced to the semisimple regular case by [Kot88][§3] and [LS90][Lemma 2.4.A]. For an unramified endoscopic group the fundamental lemma for the unit of the unramified Hecke algebra at regular semisimple elements is a consequence of the work of Hales, Waldspurger and Ngô. The case of a ramified endoscopic group is [Kot86][Proposition 7.5]: the κ-orbital integral simply vanishes.*

3.3.2.5 Short description of the global algorithm

Let $\mathbf{G}$ be one of $\mathbf{SO}_{2n+1}$ or $\mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$ over $\mathbb{Z}$, let $\prod_p f_p$ be the characteristic function of $\mathbf{G}(\widehat{\mathbb{Z}})$ and $\prod_p dg_p$ the Haar measure on $\mathbf{G}(\mathbb{A}_f)$ such that $\mathbf{G}(\widehat{\mathbb{Z}})$ has measure one. Let λ be a dominant weight for $\mathbf{G}_{\mathbb{C}}$ and let $f_{\infty,\lambda}(g_\infty)dg_\infty$ be the distribution on $\mathbf{G}(\mathbb{R})$ defined in section 3.3.1.1. Denote $f(g)dg = f_{\infty,\lambda}(g_\infty)dg_\infty \prod_p f_p(g_p)dg_p$. We give a short summary of the algorithm computing $T_{\mathrm{ell}}(f(g)dg)$ for a family of dominant weights λ, by outlining the main steps. Realise $\mathbf{G}$ as $\mathbf{SO}(\Lambda, q)$ (resp. $\mathbf{Sp}(\Lambda, a)$) where Λ is a finite free $\mathbb{Z}$-module endowed with a regular quadratic form q (resp. nondegenerate alternate form a). Denote $N = \mathrm{rank}_{\mathbb{Z}}(\Lambda)$.

1. Enumerate the possible characteristic polynomials in the standard representation of $\mathbf{G}$ for $\gamma \in C(\mathbf{G}(\mathbb{Q}))$. That is, enumerate the polynomials $P \in \mathbb{Q}[X]$ unitary of degree d such that all the roots of P are roots of unity, and the multiplicity of -1 as root of P is even.

2. For each such P, and for any prime number p, in $\mathbb{Q}_p[X]$ write $P = \prod_i P_i$ as in section 3.3.2.1. For any i, enumerate the finite set of isomorphism classes of quadratic or hermitian (resp. alternate or antihermitian) lattices (Λ_i, h_i) as in section 3.3.2.2. For almost all primes p, the minimal polynomial $\mathrm{rad}(P) = P/\mathrm{gcd}(P, P')$ is separable modulo p, there is a unique isomorphism class (Λ_i, h_i) to consider and h_i is non-degenerate. Thus we only need to consider a finite set of primes.

3. The combinations of these potential local data determine a finite set of conjugacy classes in $\mathbf{G}(\mathbb{Q})$.

4. For any such conjugacy class over $\mathbb{Q}$, compute the local orbital integrals using section 3.3.2.3 and Proposition 3.3.2.9. Compute the global volumes using Gross' formula 3.3.2.8.

5. Let C' be the set of $\mathbf{G}(\overline{\mathbb{Q}})$-conjugacy classes in $C(\mathbf{G}(\mathbb{Q}))$. For $c \in C'$ define the "mass" of c
$$m_c = \sum_{\mathrm{cl}(\gamma)\in c} \frac{\mathrm{Vol}(\mathbf{I}(\mathbb{Q})\backslash \mathbf{I}(\mathbb{A}))}{\mathrm{card}(\mathrm{Cent}(\gamma, \mathbf{G}(\mathbb{Q}))/\mathbf{I}(\mathbb{Q}))}$$
so that
$$T_{\mathrm{ell}}(f(g)dg) = \sum_{c\in C'} m_c \mathrm{Tr}(c \,|\, V_\lambda).$$
Using Weyl's character formula, we can finally compute $T_{\mathrm{ell}}(f(g)dg)$ for the dominant weights λ we are interested in. Some conjugacy classes $c \in C'$ are singular, so that a refinement of Weyl's formula is needed: see [CC09][Proposition 1.9] and [CR14][Proposition 2.3].

We give tables of the masses m_c in section 3.7.1, for the groups of rank ≤ 4. Our current implementation allows to compute these masses at least up to rank 6 (and for $\mathbf{Sp}_{14}$ also), but starting with rank 5 they no longer fit on a single page.

Remark 3.3.2.12. *In the orthogonal case the group* $\mathbf{G}$ *is not simply connected and thus in* $\mathbf{G}(\mathbb{Q})$ *there is a distinction between stable conjugacy and conjugacy in* $\mathbf{G}(\overline{\mathbb{Q}})$. *However, if* $\gamma, \gamma' \in C(\mathbf{G}(\mathbb{Q}))$ *both contribute non-trivially to* $T_{\mathrm{ell}}(f(g)dg)$ *and are conjugated in* $\mathbf{G}(\overline{\mathbb{Q}})$, *then they are stably conjugate. Indeed their spinor norms have even valuation at every finite prime, and are trivial at the archimedean place since they each belong to a compact connected torus, therefore their spinor norms are both trivial. This implies that they lift to elements* $\tilde{\gamma}, \tilde{\gamma}'$ *in the spin group* $\mathbf{G}_{\mathrm{sc}}(\mathbb{Q})$, *and moreover we can assume that* $\tilde{\gamma}$ *and* $\tilde{\gamma}'$ *are conjugated in* $\mathbf{G}_{\mathrm{sc}}(\overline{\mathbb{Q}})$, *which means that they are stably conjugate.*

This observation allows to avoid unnecessary computations: if the spinor norm of γ *is not equal to* 1, *the global orbital integral* $O_\gamma(f(g)dg)$ *vanishes.*

3.3.3 Computation of the parabolic terms using elliptic terms for groups of lower semisimple rank

In the previous sections we gave an algorithm to compute the elliptic terms in Arthur's trace formula in [Art89a]. After recalling the complete geometric side of the trace formula, i.e. the parabolic terms, we explain how the archimedean contributions to these terms simplify in our situation where the functions f_p at the finite places have support contained in a compact subgroup. The result is that we can express the parabolic terms very explicitely (perhaps too explicitely) using elliptic terms for groups of lower semisimple rank in section 3.3.3.4.

3.3.3.1 Parabolic terms

Let us recall the geometric side of the trace formula given in [Art89a][§6]. We will slightly change the formulation by using Euler-Poincaré measures on real groups instead of transferring Haar measures to compact inner forms. The translation is straightforward using [Kot88][Theorem 1]. Let $\mathbf{G}$ be one of $\mathbf{SO}_{2n+1}$, $\mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$. Of course the following notions and Arthur's trace formula apply to more general groups.

First we recall the definition of the constant term at the finite places. Let p be a finite prime, and denote $K = \mathbf{G}(\mathbb{Z}_p)$. Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$ having unipotent radical $\mathbf{N}$ admitting $\mathbf{M}$ as a Levi subgroup. Since K is a hyperspecial maximal compact subgroup of $\mathbf{G}(\mathbb{Q}_p)$ it is "good": there is an Iwasawa decomposition $\mathbf{G}(\mathbb{Q}_p) = K\mathbf{P}(\mathbb{Q}_p)$. When p is not ambiguous write $\delta_{\mathbf{P}}(m) = |\det(m\,|\,\mathrm{Lie}(\mathbf{N}))|_p$. In formulae we require the Haar measures on the unimodular groups $\mathbf{G}(\mathbb{Q}_p)$, $\mathbf{M}(\mathbb{Q}_p)$ and $\mathbf{N}(\mathbb{Q}_p)$ to be compatible in the sense that for any continuous $h : \mathbf{G}(\mathbb{Q}_p) \to \mathbb{C}$ having compact support,

$$\int_{\mathbf{G}(\mathbb{Q}_p)} h(g)dg = \int_{K\times\mathbf{N}(\mathbb{Q}_p)\times\mathbf{M}(\mathbb{Q}_p)} h(knm)\,dk\,dn\,dm = \int_{K\times\mathbf{N}(\mathbb{Q}_p)\times\mathbf{M}(\mathbb{Q}_p)} h(kmn)\delta_{\mathbf{P}}(m)\,dk\,dn\,dm.$$

If $f_p(g)dg$ is a smooth compactly supported distribution on $\mathbf{G}(\mathbb{Q}_p)$, the formula

$$f_{p,\mathbf{M}}(m) = \delta_{\mathbf{P}}(m)^{1/2}\int_K\int_{\mathbf{N}(\mathbb{Q}_p)} f_p(kmnk^{-1})dndk$$

defines a smooth compactly supported distribution $f_{p,\mathbf{M}}(m)dm$ on $\mathbf{M}(\mathbb{Q}_p)$. Although it seems to depend on the choice of $\mathbf{N}$ and the good compact subgroup K, the orbital integrals of $f_{p,\mathbf{M}}(m)dm$ at semisimple $\mathbf{G}$-regular elements of $\mathbf{M}(\mathbb{Q}_p)$ only depend on f_p (see [vD72][Lemma 9]). The case of arbitrary semisimple elements follows using [Kaz86][Theorem 0]. When f_p is the characteristic function $\mathbf{1}_{\mathbf{G}(\mathbb{Z}_p)}$ of $\mathbf{G}(\mathbb{Z}_p)$ (and $\mathrm{vol}(\mathbf{G}(\mathbb{Z}_p)) = 1$), the fact that $\mathbf{T}_0$ is defined over $\mathbb{Z}_p$ and the choice $K = \mathbf{G}(\mathbb{Z}_p)$ imply that for any choice of $\mathbf{N}$, $f_{p,\mathbf{M}} = \mathbf{1}_{\mathbf{M}(\mathbb{Z}_p)}$ (if $\mathrm{vol}(\mathbf{M}(\mathbb{Z}_p)) = 1$).

We can now define the factors appearing on the geometric side of the trace formula. As for elliptic terms, consider a smooth compactly supported distribution $\prod_p f_p(g_p)dg_p$ on $\mathbf{G}(\mathbb{A}_f)$. Fix a split maximal torus $\mathbf{T}_0$ of $\mathbf{G}$ (over $\mathbb{Z}$). The geometric side is a sum over Levi subgroups $\mathbf{M}$ containing $\mathbf{T}_0$, they are also defined over $\mathbb{Z}$. For such $\mathbf{M}$, denote by $\mathbf{A}_\mathbf{M}$ the connected center of $\mathbf{M}$ and let $C(\mathbf{M}(\mathbb{Q}))$ be the set of semisimple conjugacy classes of elements $\gamma \in \mathbf{M}(\mathbb{Q})$ which belong to a maximal torus of $\mathbf{M}_\mathbb{R}$ which is anisotropic modulo $(\mathbf{A}_\mathbf{M})_\mathbb{R} = \mathbf{A}_{\mathbf{M}_\mathbb{R}}$. If γ is (a representative of) an element of $C(\mathbf{M}(\mathbb{Q}))$, let $\mathbf{I}$ denote the connected centraliser of γ in $\mathbf{M}$. Define $\iota^{\mathbf{M}}(\gamma) = |\mathrm{Cent}(\gamma, \mathbf{M}(\mathbb{Q}))/\mathbf{I}(\mathbb{Q})|$. For any finite prime p, to $f_p(g_p)dg_p$ we associate the complex Haar measure $O_\gamma(f_{p,\mathbf{M}})$ on $\mathbf{I}(\mathbb{Q}_p)$. For p outside a finite set (containing the primes at which $\mathbf{I}$ is ramified), the measure of a hyperspecial maximal compact subgroup of $\mathbf{I}(\mathbb{Q}_p)$ is 1. Define a complex Haar measure on $\mathbf{I}(\mathbb{A})/\mathbf{A}_\mathbf{M}(\mathbb{A})$ as follows:

- Give $\mathbf{I}(\mathbb{R})/\mathbf{A}_\mathbf{M}(\mathbb{R})$ its Euler-Poincaré measure. It is nonzero by our assumption on γ.
- Give $\mathbf{A}_\mathbf{M}(\mathbb{Q}_p)$ its Haar measure such that its maximal compact subgroup (in the case at hand $\mathbf{A}_\mathbf{M}(\mathbb{Z}_p)$) has measure 1, and endow $\mathbf{I}(\mathbb{Q}_p)/\mathbf{A}_\mathbf{M}(\mathbb{Q}_p)$ with the quotient measure.

Now fix a dominant weight λ for $\mathbf{G}$ and denote $\tau = \lambda + \rho$ (where 2ρ is the sum of the positive roots) the associated infinitesimal character. For $f(g)dg = f_{\infty,\lambda}(g_\infty)dg_\infty \prod_p f_p(g_p)dg_p$, the last ingredient occurring in $T_{\mathrm{geom}}(f(g)dg)$ is the continuous function $\gamma \mapsto \Phi_\mathrm{M}(\gamma, \tau)$ defined for semisimple $\gamma \in \mathbf{M}(\mathbb{R})$ which belong to a maximal torus of $\mathbf{M}_\mathbb{R}$ which is anisotropic modulo $(\mathbf{A}_\mathbf{M})_\mathbb{R}$. This function will be defined in terms of characters of discrete series and studied at compact elements γ in section 3.3.3.3. If γ does not satisfy these properties define $\Phi_\mathrm{M}(\gamma, \tau) = 0$.

The geometric side $T_{\mathrm{geom}}(f(g)dg)$ of the trace formula is

$$\sum_{\mathbf{M} \supset \mathbf{T}_0} \left(\frac{-1}{2}\right)^{\dim \mathbf{A}_\mathbf{M}} \frac{|W(\mathbf{T}_0, \mathbf{M})|}{|W(\mathbf{T}_0, \mathbf{G})|} \sum_{\gamma \in C(\mathbf{M}(\mathbb{Q}))} \frac{\mathrm{vol}\left(\mathbf{I}(\mathbb{Q}) \backslash \mathbf{I}(\mathbb{A}) / \mathbf{A}_\mathbf{M}(\mathbb{A})\right)}{\mathrm{card}\left(\mathrm{Cent}(\gamma, \mathbf{M}(\mathbb{Q}))/\mathbf{I}(\mathbb{Q})\right)} \Phi_\mathbf{M}(\gamma, \tau). \tag{3.3.3.1}$$

After the definition of the function $\Phi_\mathbf{M}$ it will be clear that the term corresponding to $\mathbf{M} = \mathbf{G}$ is $T_{\mathrm{ell}}(f(g)dg)$.

3.3.3.2 Sums of averaged discrete series constants

Harish-Chandra gave a formula for the character of discrete series representations of a real reductive group at regular elements of any maximal torus. This formula is similar to Weyl's character formula but it also includes certain integers which can be computed inductively. In the case of averaged discrete series this induction is particularly simple. We recall the characterisation of these integers given in [GKM97][§3] and compute their sum and alternate sum. When the support of $\prod_p f_p(g_p)dg_p$ is contained in a compact subgroup of $\mathbf{G}(\mathbb{A}_f)$, in the trace formula only these alternate sums need to be computed, not the individual constants.

Let X be a real finite-dimensional vector space and R a reduced root system in X^*. Assume that $-\mathrm{Id} \in W(R)$, i.e. any irreducible component of R is of type A_1, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_{2n} $(n \geq 2)$, E_7, E_8, F_4 or G_2. If R_1 is a subsystem of R having the same property, letting R_2 be the subsystem of R consisting of roots orthogonal to all the roots in R_1, $-\mathrm{Id}_{\mathbb{R}R_2} \in W(R_2)$ by [Bou68][ch. V, §3, Proposition 2], and $\mathrm{rank}(R) = \mathrm{rank}(R_1) + \mathrm{rank}(R_2)$. In particular for $\alpha \in R$, $R_\alpha := \{\beta \in R \mid \alpha(\beta^\vee) = 0\}$ is a root system in Y^* where $Y = \ker \alpha$.

Recall that $X_{\mathrm{reg}} := \{x \in X \mid \forall \alpha \in R,\ \alpha(x) \neq 0\}$, and define X^*_{reg} similarly with respect to $R^\vee$. For $x \in X_{\mathrm{reg}}$ we denote by Δ_x the basis of simple roots of R associated with the chamber containing x. There is a unique collection of functions $\bar{c}_R : X_{\mathrm{reg}} \times X^*_{\mathrm{reg}} \to \mathbb{Z}$ for root systems R as above such that:

1. $\bar{c}_\emptyset(0,0) = 1$,
2. for all $(x, \lambda) \in X_{\mathrm{reg}} \times X^*_{\mathrm{reg}}$ such that $\lambda(x) > 0$, $\bar{c}_R(x, \lambda) = 0$,
3. for all $(x, \lambda) \in X_{\mathrm{reg}} \times X^*_{\mathrm{reg}}$ and $\alpha \in \Delta_x$, $\bar{c}_R(x, \lambda) + \bar{c}_R(s_\alpha(x), \lambda) = 2\bar{c}_{R_\alpha}(y, \lambda|_Y)$ where $Y = \ker \alpha$ and $y = (x + s_\alpha(x))/2$.

In the third property note that for any $\beta \in R \smallsetminus \{\pm\alpha\}$ such that $\beta(x) > 0$, $\beta(y) > 0$: writing $\beta = \sum_{\gamma \in \Delta_x} n_\gamma \gamma$ with $n_\gamma \geq 0$, we have

$$\beta(y) = \beta(x) - \frac{\alpha(x)\beta(\alpha^\vee)}{2} = \sum_{\gamma \in \Delta_x \smallsetminus \{\alpha\}} n_\gamma \left(\gamma(x) - \frac{\gamma(\alpha^\vee)\alpha(x)}{2}\right) > 0. \qquad (3.3.3.2)$$

In the second property we could replace "$\lambda(x) > 0$" by the stronger condition that $R \neq \emptyset$ and x and λ define the same order: $\{\alpha \in R \mid \alpha(x) > 0\} = \{\alpha \in R \mid \lambda(\alpha^\vee) > 0\}$. By induction $\bar{c}_R$ is locally constant, and $W(R)$-invariant for the diagonal action of $W(R)$ on $X_{\mathrm{reg}} \times X^*_{\mathrm{reg}}$.

The existence of these functions follows from Harish-Chandra's formulae and the existence of discrete series for the split semisimple groups over $\mathbb{R}$ having a root system as above. However, [GKM97] give a direct construction.

Let $x_0 \in X_{\mathrm{reg}}$ and $\lambda_0 \in X^*_{\mathrm{reg}}$ define the same order. For $w \in W(R)$ define $d(w) = \bar{c}_R(x_0, w(\lambda_0)) = \bar{c}_R(w^{-1}(x_0), \lambda_0)$.

Proposition 3.3.3.1. *Let R be a root system as above, and denote by $q(R)$ the integer $(|R|/2 + \mathrm{rank}(R))/2$. Then*

$$\sum_{w \in W(R)} d(w) = |W(R)| \text{ and } \sum_{w \in W(R)} \epsilon(w) d(w) = (-1)^{q(R)} |W(R)|.$$

Proof. The two formulae are equivalent by [GKM97][Theorem 3.2] so let us prove the first one by induction on the rank of R. The case of $R = \emptyset$ is trivial. Assume that R is not empty and that the formula holds in lower rank. Denote $W = W(R)$. For $\alpha \in R$ let $\mathcal{C}_\alpha = \{x \in W x_0 \mid \alpha \in \Delta_x\}$ and $\mathcal{D}_\alpha$ the orthogonal projection of $\mathcal{C}_\alpha$ on $Y = \ker \alpha$. Geometrically, $\mathcal{C}_\alpha$ represents the chambers adjacent to the wall Y on the side determined by α. For $x \in \mathcal{C}_\alpha$, by a computation similar to 3.3.3.2, orthogonal projection on Y maps the chamber containing x onto a connected component of $Y \smallsetminus \bigcup_{\beta \in R \setminus \{\pm\alpha\}} \ker \beta$, i.e. a chamber in Y *relative to* R. Thus the projection $\mathcal{C}_\alpha \to \mathcal{D}_\alpha$ is bijective and in any R_α-chamber of Y there is the same number $|\mathcal{D}_\alpha|/|W(R_\alpha)|$ of elements in $\mathcal{D}_\alpha$.

$$\begin{aligned}
\mathrm{rank}(R) \sum_{w \in W} d(w) &= \sum_{x \in W x_0} \sum_{\alpha \in \Delta_x} \bar{c}_R(x, \lambda_0) \\
&= \frac{1}{2} \sum_{\alpha \in R} \sum_{x \in \mathcal{C}_\alpha} \bar{c}_R(x, \lambda_0) + \bar{c}_R(s_\alpha(x), \lambda_0) \\
&= \sum_{\alpha \in R} \sum_{y \in \mathcal{D}_\alpha} \bar{c}_{R_\alpha}(y, \lambda_0|_Y) \\
&= \sum_{\alpha \in R} |\mathcal{D}_\alpha| = \sum_{x \in W x_0} |\Delta_x| = \mathrm{rank}(R)|W|.
\end{aligned}$$

At the second line we used the permutation $\alpha \mapsto -\alpha$ of R and the fact that $x \in \mathcal{C}_\alpha \Leftrightarrow s_\alpha(x) \in \mathcal{C}_{-\alpha}$. □

3.3.3.3 Character of averaged discrete series on non-compact tori

In this section we consider a reductive group $\mathbf{G}$ over $\mathbb{R}$ which has discrete series. To simplify notations we assume that $\mathbf{G}$ is semisimple, as it is the case for the symplectic and special orthogonal groups. Fix a dominant weight λ for $\mathbf{G}_\mathbb{C}$, and let $\tau = \lambda + \rho$ where 2ρ is the sum of the positive roots. Let $\mathbf{M}$ be a Levi subgroup of $\mathbf{G}$ and denote by $\mathbf{A_M}$ the biggest split central torus in $\mathbf{M}$. If $\gamma \in \mathbf{M}(\mathbb{R})$ is semisimple, $\mathbf{G}$-regular and belongs to a maximal torus anisotropic modulo $\mathbf{A_M}$, define

$$\Phi_\mathbf{M}(\gamma, \tau) := (-1)^{q(\mathbf{G}(\mathbb{R}))} \left| D_\mathbf{M}^\mathbf{G}(\gamma) \right|^{1/2} \sum_{\pi_\infty \in \Pi_{\mathrm{disc}}(\tau)} \Theta_{\pi_\infty}(\gamma)$$

where $D_\mathbf{M}^\mathbf{G}(\gamma) = \det(\mathrm{Id} - \mathrm{Ad}(\gamma) \,|\, \mathfrak{g}/\mathfrak{m})$. Note that for $\gamma \in \mathbf{G}(\mathbb{R})$ semisimple elliptic regular, $\Phi_\mathbf{G}(\gamma, \tau) \mu_{\mathrm{EP}, \mathbf{I}(\mathbb{R})} = \mathrm{Tr}(\gamma | V_\lambda) \mu_{\mathrm{EP}, \mathbf{I}(\mathbb{R})} = O_\gamma(f_\lambda(g) dg)$ where $f_\lambda(g) dg$ is the smooth compactly supported distribution of section 3.3.1.1.

When $\mathbf{M} \times_\mathbb{Q} \mathbb{R}$ admits a maximal torus $\mathbf{T}$ anisotropic modulo $\mathbf{A_M} \times_\mathbb{Q} \mathbb{R}$, Arthur shows that $\Phi_\mathbf{M}(\cdot, \tau)$ extends continuously to $\mathbf{T}(\mathbb{R})$ (beware that the statement [Art89a][(4.7)] is erroneous: in general $\Phi_\mathbf{M}(\gamma, \tau)$ is not identically zero outside the connected components that intersect the center of $\mathbf{G}$). Following [GKM97][§4], to which we refer for details, let us write the restriction of $\Phi_{\mathbf{M}}(\cdot,\tau)$ to any connected component of $\mathbf{T}(\mathbb{R})_{\mathbf{G}-\mathrm{reg}}$ as a linear combination of traces in algebraic representations of $\mathbf{M}$.

Let R be the set of roots of $\mathbf{T}$ on $\mathbf{G}$ (over $\mathbb{C}$). Let $R_{\mathbf{M}}$ be the set of roots of $\mathbf{T}$ on $\mathbf{M}$. Let $\gamma \in \mathbf{T}(\mathbb{R})$ be $\mathbf{G}$-regular, and let Γ be the connected component of γ in $\mathbf{T}(\mathbb{R})$. Let R_Γ be the set of real roots $\alpha \in R$ such that $\alpha(\gamma) > 0$. As the notation suggests, it only depends on Γ. Moreover R_Γ and $R_{\mathbf{M}}$ are orthogonal sub-root systems of R: the coroots of $R_{\mathbf{M}}$ factor through $\mathbf{T}\cap\mathbf{M}_{\mathrm{der}}$ which is anisotropic, while the roots in R_Γ factor through the biggest split quotient of $\mathbf{T}$. Finally $\Phi_{\mathbf{M}}(\gamma,\tau)=0$ unless γ belongs to the image of $\mathbf{G}_{\mathrm{sc}}(\mathbb{R})$, and in that case the Weyl group $W(R_\Gamma)$ of R_Γ contains $-\mathrm{Id}$ and $\mathrm{rk}(R_\Gamma)=\dim \mathbf{A}_{\mathbf{M}}$. In the following we assume that $\gamma \in \mathrm{Im}(\mathbf{G}_{\mathrm{sc}}(\mathbb{R}) \to \mathbf{G}(\mathbb{R}))$.

Since γ is $\mathbf{G}$-regular, it defines a set of positive roots $R_\gamma^+ = \{\alpha \in R_\gamma \mid \alpha(\gamma) > 1\}$ in R_Γ. Choose a parabolic subgroup $\mathbf{P}=\mathbf{M}\mathbf{N}$ with unipotent radical $\mathbf{N}$ such that R_γ^+ is included in the set of roots of $\mathbf{T}$ on $\mathbf{N}$. In general this choice is not unique. Choose any set of positive roots $R_{\mathbf{M}}^+$ for $R_{\mathbf{M}}$. There is a unique Borel subgroup $\mathbf{B}\subset\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{T}$ such that the set of roots of $\mathbf{T}$ on $\mathbf{B}\cap\mathbf{M}$ is $R_{\mathbf{M}}^+$. Let R^+ be the set of positive roots in R corresponding to $\mathbf{B}$.

There is a unique $x_\gamma \in (\mathbb{R}R_\Gamma)^* = \mathbb{R}\otimes_{\mathbb{Z}} X_*(\mathbf{A}_{\mathbf{M}})$ such that for any $\alpha\in R_\Gamma$, $\alpha(x_\gamma)=\alpha(\gamma)$. Then x_γ is R_Γ-regular and the chamber in which x_γ lies only depends on the connected component of γ in $\mathbf{T}(\mathbb{R})_{\mathbf{G}-\mathrm{reg}}$. Denote by pr the orthogonal projection $\mathbb{R}\otimes_{\mathbb{Z}} X^*(\mathbf{T}) \to \mathbb{R}R_\Gamma$. When we identify $\mathbb{R}R_\Gamma$ with $\mathbb{R}\otimes_{\mathbb{Z}} X^*(\mathbf{A}_{\mathbf{M}})$, pr is simply "restriction to $\mathbf{A}_{\mathbf{M}}$". By [GKM97][proof of Lemma 4.1 and end of §4] we have

$$\Phi_{\mathbf{M}}(\gamma,\tau) = \frac{\delta_{\mathbf{P}}(\gamma)^{1/2}}{\prod_{\alpha\in R_{\mathbf{M}}^+}(1-\alpha(\gamma)^{-1})} \sum_{w\in W(R)} \epsilon(w)\bar{c}_{R_\Gamma}(x_\gamma, \mathrm{pr}(w(\tau_{\mathbf{B}}))) \left[w(\tau_{\mathbf{B}})-\rho_{\mathbf{B}}\right](\gamma)$$

where

$$\delta_{\mathbf{P}}(\gamma) = \left|\det\left(\gamma\,|\,\mathrm{Lie}(\mathbf{N})\right)\right| = \prod_{\alpha\in R^+-R_{\mathbf{M}}^+} |\alpha(\gamma)|\,.$$

Since $\rho_{\mathbf{B}}-\rho_{\mathbf{B}\cap\mathbf{M}}$ is invariant under $W(R_{\mathbf{M}})$, in the above sum we can combine terms in the same orbit under $W(R_{\mathbf{M}})$ to identify Weyl's character formula for algebraic representations of $\mathbf{M}$. Let $E = \left\{w\in W(R) \mid \forall \alpha\in R_\gamma^+\cup R_{\mathbf{M}}^+,\ w^{-1}(\alpha)\in R^+\right\}$, a set of representatives for the action of $W(R_\Gamma)\times W(R_{\mathbf{M}})$ on the left of $W(R)$. Denoting $V_{\mathbf{M},\lambda'}$ the algebraic representation of $\mathbf{M}$ with highest weight λ', we obtain

$$\Phi_{\mathbf{M}}(\gamma,\tau) = \delta_{\mathbf{P}}(\gamma)^{1/2} \sum_{w_0\in E}\sum_{w_1\in W(R_\Gamma)} \epsilon(w_1w_0)d(w_1)\mathrm{Tr}\left(\gamma|V_{\mathbf{M},w_1w_0(\tau_{\mathbf{B}})-\rho_{\mathbf{B}}}\right)$$

Furthermore $w_1w_0(\tau_{\mathbf{B}})-w_0(\tau_{\mathbf{B}})\in\mathbb{Z}R_\Gamma$ is invariant under $W(R_{\mathbf{M}})$, hence in the above sum

$$\mathrm{Tr}\left(\gamma|V_{\mathbf{M},w_1w_0(\tau_{\mathbf{B}})-\rho_{\mathbf{B}}}\right) = \left[w_1w_0(\tau_{\mathbf{B}})-w_0(\tau_{\mathbf{B}})\right](\gamma)\times\mathrm{Tr}\left(\gamma|V_{\mathbf{M},w_0(\tau_{\mathbf{B}})-\rho_{\mathbf{B}}}\right)$$

and $\left[w_1w_0(\tau_{\mathbf{B}})-w_0(\tau_{\mathbf{B}})\right](\gamma)$ is a positive real number, which does not really depend on γ

but only on the coset $(\mathbf{T} \cap \mathbf{M}_{\mathrm{der}})(\mathbb{R})\gamma$ (equivalently, on x_γ). Finally we obtain

$$\Phi_{\mathbf{M}}(\gamma, \tau) = \delta_{\mathbf{P}}(\gamma)^{1/2} \sum_{w_0 \in E} \epsilon(w_0) \left[\sum_{w_1 \in W(R_\Gamma)} \epsilon(w_1) d(w_1) \left[w_1 w_0(\tau_{\mathbf{B}}) - w_0(\tau_{\mathbf{B}}) \right](\gamma) \right.$$
$$\left. \times \operatorname{Tr}\left(\gamma | V_{\mathbf{M}, w_0(\tau_{\mathbf{B}}) - \rho_{\mathbf{B}}} \right) \right].$$

This formula is valid for γ in the closure (in $\mathbf{T}(\mathbb{R})$) of a connected component of $\mathbf{T}(\mathbb{R})_{\mathbf{G}-\mathrm{reg}}$.

Proposition 3.3.3.2. *If γ is compact, i.e. the smallest closed subgroup of $\mathbf{G}(\mathbb{R})$ containing γ is compact, then we have*

$$\Phi_{\mathbf{M}}(\gamma, \tau) = (-1)^{q(R_\Gamma)} |W(R_\Gamma)| \sum_{w_0 \in E} \epsilon(w_0) \operatorname{Tr}\left(\gamma | V_{\mathbf{M}, w_0(\tau_{\mathbf{B}}) - \rho_{\mathbf{B}}} \right).$$

Proof. This formula follows from $\left[w_1 w_0(\tau_{\mathbf{B}}) - w_0(\tau_{\mathbf{B}}) \right](\gamma) = 1$ and Proposition 3.3.3.1. □

3.3.3.4 Explicit formulae for the parabolic terms

Let $\mathbf{G}$ be one of $\mathbf{SO}_{2n+1}$ or $\mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$ over $\mathbb{Z}$, let $\prod_p f_p$ be the characteristic function of $\mathbf{G}(\widehat{\mathbb{Z}})$ and $\prod_p dg_p$ the Haar measure on $\mathbf{G}(\mathbb{A}_f)$ such that $\mathbf{G}(\widehat{\mathbb{Z}})$ has measure one. Let λ be a dominant weight for $\mathbf{G}_{\mathbb{C}}$ and let $f_{\infty,\lambda}(g_\infty) dg_\infty$ be the distribution on $\mathbf{G}(\mathbb{R})$ defined in section 3.3.1.1. Denote $f(g)dg = f_{\infty,\lambda}(g_\infty) dg_\infty \prod_p f_p(g_p) dg_p$. Using Proposition 3.3.3.2 and tedious computations, we obtain explicit formulae for the geometric side $T_{\mathrm{geom}}(f(g)dg)$ of Arthur's trace formula defined in section 3.3.3.1. For a dominant weight $\lambda = k_1 e_1 + \cdots + k_n e_n$ it will be convenient to write $T_{\mathrm{geom}}(\mathbf{G}, \underline{k})$ for $T_{\mathrm{geom}}(f(g)dg)$ to precise the group $\mathbf{G}$, and similarly for T_{ell}. If $\mathbf{G}$ is trivial ($\mathbf{SO_0}$ or $\mathbf{SO}_1$ or $\mathbf{Sp}_0$) then T_{ell} is of course simply equal to 1.

Any Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ is isomorphic to $\prod_i \mathbf{GL}_{n_i} \times \mathbf{G}'$ where $\mathbf{G}'$ is of the same type as $\mathbf{G}$. Note that $\mathbf{M}(\mathbb{R})$ has essentially discrete series (i.e. $\Phi_{\mathbf{M}}(\cdot,\cdot)$ is not identically zero) if and only if for all $i, n_i \leq 2$ and in case $\mathbf{G}$ is even orthogonal, $\mathbf{G}'$ has even rank. Thus the Levi subgroups M whose contribution to T_{geom} (that is formula 3.3.3.1) is nonzero are isomorphic to $\mathbf{GL}_1^a \times \mathbf{GL}_2^c \times \mathbf{G}'$ for some integers a, c.

Since $\mathbf{PGL}_2 \simeq \mathbf{SO}_3$, for $k \in \mathbb{Z}_{\geq 0}$ we denote $T_{\mathrm{ell}}(\mathbf{PGL}_2, k) = T_{\mathrm{ell}}(\mathbf{SO}_3, k)$. For non-negative $k \in 1/2\mathbb{Z} \smallsetminus \mathbb{Z}$ it is convenient to define $T_{\mathrm{ell}}(\mathbf{PGL}_2, k) = 0$, so that for any $k \in \mathbb{Z}_{\geq 0}$ we have $T_{\mathrm{ell}}(\mathbf{PGL}_2, k/2) = T_{\mathrm{ell}}(\mathbf{Sp}_2, k)/2$.

For $a, c, d \in \mathbb{Z}_{\geq 0}$, let $\Xi_{a,c,d}$ be the set of σ in the symmetric group S_{a+2c+d} such that

- $\sigma(1) < \cdots < \sigma(a)$,
- $\sigma(a+1) < \sigma(a+3) < \cdots < \sigma(a+2c-1)$,
- for any $1 \leq i \leq c$, $\sigma(a+2i-1) < \sigma(a+2i)$,
- $\sigma(a+2c+1) < \cdots < \sigma(n)$.

For $a \geq 0$ and $x \in \{0, \dots, a\}$, define

$$\eta^{(B)}(a,x) = \frac{(-1)^{a(a-1)/2}}{2^a} \sum_{b=0}^{\lfloor a/2 \rfloor} (-1)^b \sum_{r=0}^{2b} \binom{x}{r} \binom{a-x}{2b-r} (-1)^r.$$

It is easy to check that

$$\eta^{(B)}(a,x) = \frac{(-1)^{a(a-1)/2}}{2^{a+1}} \mathrm{Tr}_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}} \left((1+\sqrt{-1})^{a-x}(1-\sqrt{-1})^x\right) \in \frac{1}{2^{\lfloor (a+1)/2 \rfloor}}\mathbb{Z}.$$

For $n \geq a$, $\sigma \in S_n$ and $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, let

$$\eta^{(B)}(a, \underline{k}, \sigma) = \eta^{(B)}\left(a, \mathrm{card}\{i \in \{1,\dots,a\} \mid k_{\sigma(i)} + \sigma(i) + i = 1 \pmod 2\}\right).$$

Theorem 3.3.3.3 (Parabolic terms for $\mathbf{G} = \mathbf{SO}_{2n+1}$). *Let $a, c, d \in \mathbb{Z}_{\geq 0}$ not all zero and $n = a + 2c + d$. The sum of the contributions to $T_{\mathrm{geom}}(\mathbf{SO}_{2n+1}, \underline{k})$ in formula 3.3.3.1 of the Levi subgroups $\mathbf{M}$ in the orbit of $\mathbf{GL}_1^a \times \mathbf{GL}_2^c \times \mathbf{SO}_{2d+1}$ under the Weyl group $W(\mathbf{T}_0, \mathbf{G})$ is*

$$\begin{aligned}
&\sum_{\sigma \in \Xi_{a,c,d}} \eta^{(B)}(a, \underline{k}, \sigma) \\
&\quad \times \prod_{i=1}^{c} \Big[T_{\mathrm{ell}}\left(\mathbf{PGL}_2, (k_{\sigma(a+2i-1)} - k_{\sigma(a+2i)} + \sigma(a+2i) - \sigma(a+2i-1) - 1)/2\right) \\
&\qquad - T_{\mathrm{ell}}(\mathbf{PGL}_2, (k_{\sigma(a+2i-1)} + k_{\sigma(a+2i)} - \sigma(a+2i) - \sigma(a+2i-1) + 2n)/2) \Big] \\
&\quad \times T_{\mathrm{ell}}(\mathbf{SO}_{2d+1}, (k_{\sigma(n-d+1)} + n - d + 1 - \sigma(n-d+1), \dots, k_{\sigma(n)} + n - \sigma(n))).
\end{aligned}$$

We have a similar formula for the symplectic group. For $a \geq 0$ and $x \in \{0, \dots, a\}$, define

$$\eta^{(C)}(a,x) = \frac{(-1)^{a(a-1)/2}}{2^a} \sum_{b=0}^{a} (-1)^{b(a-b)} \sum_{r=0}^{b} \binom{x}{r} \binom{a-x}{b-r} (-1)^r.$$

Then we have

$$\eta^{(C)}(a,x) = \begin{cases} (-1)^{a/2} & \text{if } a \text{ is even and } x = a, \\ (-1)^{(a-1)/2} & \text{if } a \text{ is odd and } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $n \geq a$, $\sigma \in S_n$ and $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, let

$$\eta^{(C)}(a, \underline{k}, \sigma) = \eta^{(C)}\left(a, \mathrm{card}\{i \in \{1,\dots,a\} \mid k_{\sigma(i)} + \sigma(i) + i = 1 \pmod 2\}\right).$$

Theorem 3.3.3.4 (Parabolic terms for $\mathbf{G} = \mathbf{Sp}_{2n}$). *Let $a, c, d \in \mathbb{Z}_{\geq 0}$ not all zero and $n = a + 2c + d$. The sum of the contributions to $T_{\mathrm{geom}}(\mathbf{Sp}_{2n}, \underline{k})$ in formula 3.3.3.1 of the Levi subgroups $\mathbf{M}$ in the orbit of $\mathbf{GL}_1^a \times \mathbf{GL}_2^c \times \mathbf{Sp}_{2d}$ under the Weyl group $W(\mathbf{T}_0, \mathbf{G})$ is*

$$\begin{aligned}
&\sum_{\sigma \in \Xi_{a,c,d}} \eta^{(C)}(a, \underline{k}, \sigma) \\
&\quad \times \prod_{i=1}^{c} \Big[T_{\mathrm{ell}}\left(\mathbf{PGL}_2, (k_{\sigma(a+2i-1)} - k_{\sigma(a+2i)} + \sigma(a+2i) - \sigma(a+2i-1) - 1)/2\right) \\
&\qquad - T_{\mathrm{ell}}(\mathbf{PGL}_2, (k_{\sigma(a+2i-1)} + k_{\sigma(a+2i)} - \sigma(a+2i) - \sigma(a+2i-1) + 2n + 1)/2) \Big] \\
&\quad \times T_{\mathrm{ell}}(\mathbf{Sp}_{2d}, (k_{\sigma(n-d+1)} + n - d + 1 - \sigma(n-d+1), \dots, k_{\sigma(n)} + n - \sigma(n)))
\end{aligned}$$

For $a \geq 0$ and $x \in \{0, \dots, 2a\}$, define

$$\eta^{(D)}(a,x) = \frac{1}{2^{2a}} \sum_{b=0}^{a} \sum_{r=0}^{2b} \binom{x}{r} \binom{2a-x}{2b-r} (-1)^r.$$

We have

$$\eta^{(D)}(a,x) = \begin{cases} 1 & \text{if } a = 0, \\ 1/2 & \text{if } a > 0 \text{ and } x(2a-x) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $n \geq a$, $\sigma \in S_{2n}$ and $\underline{k} = (k_1, \dots, k_{2n}) \in \mathbb{Z}^{2n}$, let

$$\eta^{(D)}(a, \underline{k}, \sigma) = \eta^{(D)} \left(a, \operatorname{card}\{i \in \{1, \dots, 2a\} \mid k_{\sigma(i)} + \sigma(i) + i = 1 \pmod 2\}\right).$$

For the group $\mathbf{SO}_{4n}$, we need only consider dominant weights $\underline{k}$ with $k_{2n} \geq 0$ (i.e. the same inequalities as for the other two infinite families) since the end result is invariant under the outer automorphism of $\mathbf{SO}_{4n}$, that is $T_{\text{geom}}(\mathbf{SO}_{4n}, (k_1, \dots, k_{2n-1}, -k_{2n})) = T_{\text{geom}}(\mathbf{SO}_{4n}, (k_1, \dots, k_{2n-1}, k_{2n}))$.

Theorem 3.3.3.5 (Parabolic terms for $\mathbf{G} = \mathbf{SO}_{4n}$)**.** *Let $a, c, d \in \mathbb{Z}_{\geq 0}$ not all zero and $n = a + c + d$. The sum of the contributions to $T_{\text{geom}}(\mathbf{SO}_{4n}, \underline{k})$ in formula 3.3.3.1 of the Levi subgroups $\mathbf{M}$ in the orbit of $\mathbf{GL}_1^{2a} \times \mathbf{GL}_2^c \times \mathbf{SO}_{4d}$ under the Weyl group $W(\mathbf{T}_0, \mathbf{G})$ is*

$$\begin{aligned} &\sum_{\sigma \in \Xi_{2a,c,2d}} \eta^{(D)}(a, \underline{k}, \sigma) \\ &\times \prod_{i=1}^{c} \Big[T_{\text{ell}} \left(\mathbf{PGL}_2, (k_{\sigma(2a+2i-1)} - k_{\sigma(2a+2i)} + \sigma(2a+2i) - \sigma(2a+2i-1) - 1)/2 \right) \\ &\quad + T_{\text{ell}} (\mathbf{PGL}_2, (k_{\sigma(2a+2i-1)} + k_{\sigma(2a+2i)} - \sigma(2a+2i) - \sigma(2a+2i-1) + 4n - 1)/2) \Big] \\ &\times T_{\text{ell}}(\mathbf{SO}_{4d}, (k_{\sigma(2n-2d+1)} + 2n - 2d + 1 - \sigma(2n-2d+1), \dots, k_{\sigma(2n)} + 2n - \sigma(2n))). \end{aligned}$$

3.4 Endoscopic decomposition of the spectral side

3.4.1 The spectral side of the trace formula

The previous sections give an algorithm to compute the geometric side of Arthur's trace formula in [Art89a]. Let us recall the spectral side of this version of the trace formula. As before $\mathbf{G}$ denotes one of the reductive groups $\mathbf{SO}_{2n+1}$, $\mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$ over $\mathbb{Z}$. Let K_∞ be a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ and denote $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \operatorname{Lie}(\mathbf{G}(\mathbb{R}))$. Let $\mathcal{A}_{\text{disc}}(\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}))$ be the space of $K_\infty \times \mathbf{G}(\widehat{\mathbb{Z}})$-finite and $Z(U(\mathfrak{g}))$-finite functions in the discrete spectrum $L^2_{\text{disc}}(\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}))$. It is also the space of automorphic forms in the sense of [BJ79] which are square-integrable. There is an orthogonal decomposition

$$\mathcal{A}_{\text{disc}}(\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})) = \bigoplus_{\pi \in \Pi_{\text{disc}}(\mathbf{G})} m_\pi \pi$$

where $\Pi_{\text{disc}}(\mathbf{G})$ is a countable set of distinct isomorphism classes of unitary $(\mathfrak{g}, K_\infty) \times \mathbf{G}(\mathbb{A}_f)$-modules and $m_\pi \in \mathbb{Z}_{\geq 1}$. Denote $\Pi_{\text{disc}}^{\text{unr}}(\mathbf{G}) \subset \Pi_{\text{disc}}(\mathbf{G})$ the set of π such that for any prime number p the representation π_p is unramified, i.e. $\pi_p^{\mathbf{G}(\mathbb{Z}_p)} \neq 0$.

Let λ be a dominant weight for $\mathbf{G}_{\mathbb{C}}$, and denote V_λ the corresponding algebraic representation of $\mathbf{G}(\mathbb{C})$, which by restriction to $\mathbf{G}(\mathbb{R})$ we see as a $(\mathfrak{g}, K_\infty)$-module. If X is an admissible $(\mathfrak{g}, K_\infty)$-module, define its Euler-Poincaré characteristic with respect to V_λ

$$\mathrm{EP}(X \otimes V_\lambda^*) = \sum_{i \geq 0} (-1)^i \dim H^i((\mathfrak{g}, K_\infty), X \otimes V_\lambda^*).$$

We refer to [BW00] for definitions and essential properties of $(\mathfrak{g}, K_\infty)$-cohomology. By [BW00][Chapter I, Corollary 4.2] for any irreducible $(\mathfrak{g}, K_\infty)$-module X, we have that $H^\bullet((\mathfrak{g}, K_\infty), X \otimes V_\lambda^*) = 0$ unless X has the same infinitesimal character as V_λ.

For our particular choice of function on $\mathbf{G}(\mathbb{A}_f)$ the spectral side of Arthur's trace formula in [Art89a] is

$$\sum_{\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{G})} m_\pi \mathrm{EP}(\pi_\infty \otimes V_\lambda^*). \tag{3.4.1.1}$$

By [HC68][Theorem 1] there is only a finite number of nonzero terms. Vogan and Zuckerman [VZ84] (see also [BW00][Chapter VI, §5]) have classified the irreducible unitary $(\mathfrak{g}, K_\infty)$-modules having cohomology with respect to V_λ, and computed this cohomology. However, the integer 3.4.1.1 alone is not enough to recover the number $m(X)$ of $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{G})$ such that π_∞ is isomorphic to a given irreducible unitary $(\mathfrak{g}, K_\infty)$-module X having the same infinitesimal character as V_λ.

Arthur's endoscopic classification of the discrete automorphic spectrum of $\mathbf{G}$ [Art13] allows to express $m(X)$ using numbers of certain *self-dual* cuspidal automorphic representations of general linear groups. Conversely these numbers can be obtained from the Euler-Poincaré characteristic 3.4.1.1 for various groups $\mathbf{G}$ and weights λ. For explicit computations we will have to make Assumption 3.4.2.4 that relates the rather abstract Arthur packets at the real place with the ones previously defined by Adams and Johnson in [AJ87].

Note that it will not be necessary to use [VZ84] since the Euler-Poincaré characteristic is a much simpler invariant than the whole cohomology.

3.4.1.1 Arthur's endoscopic classification

Let us review how Arthur's very general results in [Art13] specialise in our particular situation: level one and regular infinitesimal character. We are brief since this was done in [CR14][§3], though with a slightly different formulation. We refer to [Bor79] for the definition of L-groups. For $\mathbf{G}$ a reductive group over F we will denote $\widehat{\mathbf{G}}$ the connected component of the neutral element in ${}^L\mathbf{G}$ (which Borel denotes ${}^L\mathbf{G}^0$).

Let F be a local field of characteristic zero. The Weil-Deligne group of F is denoted by W_F': if F is archimedean $W_F' = W_F$, whereas in the p-adic case $W_F' = W_F \times \mathrm{SU}(2)$. Consider a quasisplit special orthogonal or symplectic group $\mathbf{G}$ over F. Let $\psi : W_F' \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$ be a local Arthur parameter, i.e. $\psi|_{W_F'}$ is a continuous semisimple splitting of ${}^L\mathbf{G} \to W_F'$ with bounded image and $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic. If $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is trivial then ψ is a tempered Langlands parameter. The general case is considered for global purposes, which we will discuss later. Consider the group $C_\psi = \mathrm{Cent}(\psi, \widehat{\mathbf{G}})$ and the finite group

$$S_\psi = C_\psi / C_\psi^0 Z(\widehat{\mathbf{G}})^{\mathrm{Gal}(\overline{F}/F)}.$$

For the groups $\mathbf{G}$ considered here the group S_ψ is isomorphic to a product of copies of $\{\pm 1\}$. Arthur [Art13][Theorem 1.5.1] associates with ψ a finite multiset Π_ψ of irreducible unitary representations of $\mathbf{G}(F)$, along with a character $\langle \cdot, \pi \rangle$ of S_ψ for any $\pi \in \Pi_\psi$. In the even orthogonal case this is not exactly true: instead of actual representations, Π_ψ is comprised of orbits of the group $\mathrm{Out}(\mathbf{G}) \simeq \mathbb{Z}/2\mathbb{Z}$ of outer automorphisms of $\mathbf{G}$ on the set of isomorphism classes of irreducible representations of $\mathbf{G}(F)$. These orbits can be described as modules over the $\mathrm{Out}(\mathbf{G})$-invariants of the Hecke algebra $\mathcal{H}(\mathbf{G}(F))$ of $\mathbf{G}(F)$, which we denote $\mathcal{H}'(\mathbf{G}(F))$. Here we have fixed a splitting $\mathrm{Out}(\mathbf{G}) \to \mathrm{Aut}(\mathbf{G})$ defined over F. Note that if F is p-adic, $\mathbf{G}$ is unramified and K is a hyperspecial subgroup of $\mathbf{G}(F)$ we can choose a splitting $\mathrm{Out}(\mathbf{G}) \to \mathrm{Aut}(\mathbf{G})$ that preserves K. If F is archimedean and K is a maximal compact subgroup of $\mathbf{G}(F)$, we can also choose a splitting that preserves K, and $\mathcal{H}'(\mathbf{G}(F))$ is the algebra of left and right K-finite $\mathrm{Out}(\mathbf{G})$-invariant distributions on $\mathbf{G}(F)$ with support in K. Note that the choice of splitting does not matter when one considers invariant objects, such as orbital integrals or traces in representations.

Denote $\mathrm{Std} : {}^L\mathbf{G} \to \mathrm{GL}_N(\mathbb{C})$ the standard representation, where

$$N = \begin{cases} 2n & \text{if } \mathbf{G}_{\bar{F}} \simeq (\mathbf{SO}_{2n+1})_{\bar{F}}\text{, i.e. } \widehat{\mathbf{G}} \simeq \mathrm{Sp}_{2n}(\mathbb{C}), \\ 2n+1 & \text{if } \mathbf{G}_{\bar{F}} \simeq (\mathbf{Sp}_{2n})_{\bar{F}}\text{, i.e. } \widehat{\mathbf{G}} \simeq \mathrm{SO}_{2n+1}(\mathbb{C}), \\ 2n & \text{if } \mathbf{G}_{\bar{F}} \simeq (\mathbf{SO}_{2n})_{\bar{F}}\text{, i.e. } \widehat{\mathbf{G}} \simeq \mathrm{SO}_{2n}(\mathbb{C}). \end{cases}$$

In the first two cases $\det \circ \mathrm{Std}$ is trivial, whereas in the third case it takes values in $\{\pm 1\}$ and factors through a character $\mathrm{Gal}(\overline{F}/F) \to \{\pm 1\}$, which by local class field theory we can also see as a character $\eta_{\mathbf{G}} : F^\times \to \{\pm 1\}$. If $\widehat{\mathbf{G}} = \mathrm{Sp}_{2n}(\mathbb{C})$ (resp. $\widehat{\mathbf{G}} = \mathrm{SO}_{2n+1}(\mathbb{C})$), the standard representation Std induces a bijection from the set of conjugacy classes of Arthur parameters $\psi : W'_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}$ to the set of conjugacy classes of Arthur parameters $\psi' : W'_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ such that $\det \circ \psi'$ is trivial and there exists a non-degenerate alternate (resp. symmetric) bilinear form on $\mathbb{C}^N$ preserved by $\mathrm{Im}(\psi')$. The third case, where $\mathbf{G}$ is an even special orthogonal group, induces a small complication. Composing with Std still induces a surjective map from the set of conjugacy classes of Arthur parameters $\psi : W'_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$ to the set of conjugacy classes of Arthur parameters $\psi' : W'_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ having determinant $\eta_{\mathbf{G}}$ and such that there exists a non-degenerate bilinear form on $\mathbb{C}^N$ preserved by $\mathrm{Im}(\psi')$. However, the fibers of this map can have cardinality one or two, the latter case occurring if and only if all the self-dual irreducible constituents of ψ' have even dimension. The Arthur packet Π_ψ along with the characters $\langle \cdot, \pi \rangle$ of S_ψ are characterised [Art13][Theorem 2.2.1] using the representation of $\mathbf{GL}_N(F)$ associated with $\mathrm{Std} \circ \psi$ by the local Langlands correspondence, and twisted and ordinary endoscopic character identities. The characters $(\langle \cdot, \pi \rangle)_{\pi \in \Pi_\psi}$ of S_ψ are well-defined only once we have fixed an equivalence class of Whittaker datum for $\mathbf{G}$, since this choice has to be made to normalise the transfer factors involved in the ordinary endoscopic character identities.

In the p-adic case, we will mainly be interested in *unramified* Arthur parameters ψ, i.e. such that $\psi|_{W'_F}$ is trivial on the inertia subgroup and on $\mathrm{SU}(2)$. Of course these exist only if $\mathbf{G}$ is unramified, so let us make this assumption. We refer to [CS80] for the definition of unramified Whittaker data with respect to a choice of hyperspecial maximal compact subgroup. Note that several conjugacy classes of Whittaker data can correspond to the same conjugacy class of hyperspecial subgroups, and that $\mathbf{G}_{\mathrm{ad}}(F)$ acts transitively on both sets of conjugacy classes.

The following lemma is implicit in [Art13]. Note that a weak version of it is needed to make sense of the main global theorem [Art13][Theorem 1.5.2].

Lemma 3.4.1.1. *Let $\psi : W_F' \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$ be an Arthur parameter for the p-adic field F. Then Π_ψ contains an unramified representation if and only if ψ is unramified. In that case, Π_ψ contains a unique unramified representation π, which satisfies $\langle \cdot, \pi \rangle = 1$.*

Proof. This is a consequence of the proof of [Art13][Lemma 7.3.4]. We borrow Arthur's notations for this (sketch of) proof. Let $\widetilde{f}$ be the characteristic function of $\mathbf{GL}_N(\mathcal{O}_F) \rtimes \theta \subset \widetilde{\mathbf{GL}}_N(F)$. Arthur shows that $\widetilde{f}_N(\psi) = 1$ if ψ is unramified. If ψ is ramified, the representation of $\mathbf{GL}_N(F)$ associated with $\mathrm{Std} \circ \psi$ is ramified, thus $\widetilde{f}_N(\psi) = 0$. The statement of the lemma follows easily from these two identities, the characterization [Art13][Theorem 2.2.1] of the local Arthur packets by endoscopic character relations, and the twisted fundamental lemma (which applies even when the residual characteristic of F is small!) proved in [Art13][Lemma 7.3.4]. $\square$

To state Arthur's global theorem we only consider the split groups $\mathbf{SO}_{2n+1}$, $\mathbf{Sp}_{2n}$ and $\mathbf{SO}_{2n}$ over $\mathbb{Q}$. From now on $\mathbf{G}$ denotes one of these groups. By [Art13][Theorem 1.4.1], any self-dual cuspidal automorphic representation π of $\mathbf{GL}_M$ over a number field has a sign $s(\pi) \in \{\pm 1\}$, which intuitively is the type of the conjectural Langlands parameter of π: $s(\pi) = 1$ (resp. -1) if this parameter is orthogonal (resp. symplectic). Unsurprisingly if M is odd then $s(\pi) = 1$, and if M is even and $s(\pi) = -1$ then the central character χ_π of π is trivial. Moreover Arthur characterises $s(\pi)$ using Sym^2 and $\bigwedge^2$ L-functions [Art13][Theorem 1.5.3]. This partition of the set of self-dual cuspidal automorphic representations of general linear groups allows to define substitutes for discrete Arthur-Langlands parameters for the group $\mathbf{G}$. Define $s(\mathbf{G}) = -1$ in the first case ($\widehat{\mathbf{G}} = \mathrm{Sp}_{2n}(\mathbb{C})$) and $s(\mathbf{G}) = 1$ in the last two cases ($\widehat{\mathbf{G}} = \mathrm{SO}_{2n+1}(\mathbb{C})$ or $\mathrm{SO}_{2n}(\mathbb{C})$). Define $\Psi(\mathbf{G})$ as the set of formal sums $\psi = \boxplus_{i \in I} \pi_i[d_i]$ where

1. for all $i \in I$, π_i is a self-dual cuspidal automorphic representation of $\mathbf{GL}_{n_i}/\mathbb{Q}$,
2. for all $i \in I$, $d_i \in \mathbb{Z}_{\geq 1}$ is such that $s(\pi_i)(-1)^{d_i - 1} = s(\mathbf{G})$,
3. $N = \sum_{i \in I} n_i d_i$,
4. the pairs (π_i, d_i) are distinct,
5. $\prod_{i \in I} \chi_{\pi_i}^{d_i} = 1$, where χ_{π_i} is the central character of π_i.

The last condition is automatically satisfied if $\widehat{\mathbf{G}} = \mathrm{Sp}_{2n}(\mathbb{C})$. The notation $\pi_i[d_i]$ suggests taking the tensor product of the putative Langlands parameter of π_i with the d_i-dimensional algebraic representation of $\mathrm{SL}_2(\mathbb{C})$. Each factor $\pi_i[d_i]$ corresponds to a discrete automorphic representation of $\mathbf{GL}_{n_i d_i}$ over $\mathbb{Q}$ by [MW89].

Let v denote a place of $\mathbb{Q}$. Thanks to the local Langlands correspondence for general linear groups applied to the $(\pi_i)_v$'s, for $\psi \in \Psi(\mathbf{G})$, ψ specialises into a local Arthur parameter $\psi_v : W'_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$. By [Art13][Theorem 1.4.2] we can see ψ_v as a genuine local Arthur parameter $W'_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$, but in the even orthogonal case ψ_v is well-defined only up to outer automorphism. To be honest it is not known in general that $\psi_v(W'_{\mathbb{Q}_v})$ is bounded (this would be the Ramanujan-Petersson conjecture), but we will not comment more on this technicality and refer to the discussion preceding [Art13][Theorem 1.5.2] for details. Thus we have a finite multiset Π_{ψ_v} of irreducible unitary representations of $\mathbf{G}(\mathbb{Q}_v)$, each of these representations being well-defined only up to outer conjugacy in the even orthogonal case.

As in the local case we want to define $C_\psi = \mathrm{Cent}(\psi, \widehat{\mathbf{G}})$ and

$$S_\psi = C_\psi / C_\psi^0 Z(\widehat{\mathbf{G}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = C_\psi / Z(\widehat{\mathbf{G}}).$$

Observe that this can be done formally for $\psi = \boxplus_{i \in I} \pi_i[d_i]$. An element s of C_ψ is described by $J \subset I$ such that $\sum_{i \in J} n_i d_i$ is even, and s corresponds formally to $-\mathrm{Id}$ on the space of $\boxplus_{i \in J} \pi_i[d_i]$ and Id on the space of $\boxplus_{i \in I \smallsetminus J} \pi_i[d_i]$. Thus one can define a finite 2-group S_ψ along with a natural morphism $S_\psi \to S_{\psi_v}$ for any place v of $\mathbb{Q}$. The last ingredient in Arthur's global theorem is the character ϵ_ψ of S_ψ. It is defined in terms of the root numbers $\epsilon(\pi_i \times \pi_j, 1/2)$ just after [Art13][Theorem 1.5.2]. If all the d_i's are equal to 1, in which case we say that ψ is formally tempered, then $\epsilon_\psi = 1$.

Fix a global Whittaker datum for $\mathbf{G}$, inducing a family of Whittaker data for $\mathbf{G}_{\mathbb{Q}_v}$ where v ranges over the places of $\mathbb{Q}$. Our reductive group is defined over $\mathbb{Z}$, and the global Whittaker datum can be chosen so that for any prime number p it induces an unramified Whittaker datum on $\mathbf{G}(\mathbb{Q}_p)$ with respect to the hyperspecial subgroup $\mathbf{G}(\mathbb{Z}_p)$. Let K_∞ be a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$, and denote $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}(\mathbf{G}(\mathbb{R}))$. The following is a specialization of the general theorem [Art13][Theorem 1.5.2] to the "everywhere unramified" case, using Lemma 3.4.1.1.

Theorem 3.4.1.2. *Recall that $\mathcal{A}_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}))$ is the space of $K_\infty \times \mathbf{G}(\widehat{\mathbb{Z}})$-finite and $Z(U(\mathfrak{g}))$-finite functions in the discrete spectrum $L^2_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}))$. Let $\Psi(\mathbf{G})^{\mathrm{unr}}$ be the set of $\psi = \boxplus_i \pi_i[d_i] \in \Psi(\mathbf{G})$ such that for any i, π_i is unramified at every prime. There is a $\mathcal{H}'(\mathbf{G}(\mathbb{R}))$-equivariant isomorphism*

$$\mathcal{A}_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}))^{\mathbf{G}(\widehat{\mathbb{Z}})} \simeq \bigoplus_{\psi \in \Psi(\mathbf{G})^{\mathrm{unr}}} \bigoplus_{\substack{\pi_\infty \in \Pi_{\psi_\infty} \\ \langle \cdot, \pi_\infty \rangle = \epsilon_\psi}} m_\psi \pi_\infty$$

where $m_\psi = 1$ except if $\mathbf{G}$ is even orthogonal and for all i $n_i d_i$ is even, in which case $m_\psi = 2$.

For $\pi_\infty \in \Pi_{\psi_\infty}$ the character $\langle \cdot, \pi_\infty \rangle$ of S_{ψ_∞} induces a character of S_ψ using the morphism $S_\psi \to S_{\psi_\infty}$, and the inner direct sum ranges over the π_∞'s such that this character of S_ψ is equal to ϵ_ψ.

In the even orthogonal case, π_∞ is only an $\mathrm{Out}(\mathbf{G}_{\mathbb{R}})$-orbit of irreducible representations, and it does not seem possible to resolve this ambiguity at the moment. Nevertheless it disappears in the global setting. There is a splitting $\mathrm{Out}(\mathbf{G}) \to \mathrm{Aut}(\mathbf{G})$ such that $\mathrm{Out}(\mathbf{G})$ preserves $\mathbf{G}(\widehat{\mathbb{Z}})$, and thus if $\{X_1, X_2\}$ is an $\mathrm{Out}(\mathbf{G}_{\mathbb{R}})$-orbit of isomorphism classes of irreducible unitary $(\mathfrak{g}, K_\infty)$-modules, then X_1 and X_2 have the same multiplicity in $\mathcal{A}_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))^{\mathbf{G}(\widehat{\mathbb{Z}})}$.

3.4.1.2 The spectral side from an endoscopic perspective

We keep the notations from the previous section. Suppose now that $\mathbf{G}(\mathbb{R})$ has discrete series, i.e. $\mathbf{G}$ is not $\mathbf{SO}_{2n}$ with n odd. Let λ be a dominant weight for $\mathbf{G}_{\mathbb{C}}$. Using Theorem 3.4.1.2 we can write the spectral side of the trace formula 3.4.1.1 as

$$\sum_{\psi \in \Psi(\mathbf{G})^{\mathrm{unr}}} \sum_{\substack{\pi_\infty \in \Pi_{\psi_\infty} \\ \langle \cdot, \pi_\infty \rangle = \epsilon_\psi}} m_\psi \mathrm{EP}(\pi_\infty \otimes V_\lambda^*). \tag{3.4.1.2}$$

We need to be cautious here since $\mathrm{EP}(\pi_\infty \otimes V_\lambda^*)$ is not well-defined in the even orthogonal case. If π_∞ is the restriction to $\mathcal{H}'(\mathbf{G}(\mathbb{R}))$ of two non-isomorphic $(\mathfrak{g}, K_\infty)$-modules $\pi_\infty^{(1)}$ and $\pi_\infty^{(2)}$, we *define*

$$\mathrm{EP}(\pi_\infty \otimes V_\lambda^*) = \frac{1}{2}\mathrm{EP}\left((\pi_\infty^{(1)} \oplus \pi_\infty^{(2)}) \otimes V_\lambda^*\right).$$

In 3.4.1.2 we can restrict the sum to π_∞'s whose infinitesimal character equals that of V_λ (up to outer automorphism in the even orthogonal case), which is $\lambda + \rho$ via Harish-Chandra's isomorphism, where 2ρ is the sum of the positive roots. Thanks to the work of Mezo, we can identify the infinitesimal character of the elements of Π_{ψ_∞}. To lighten notation, we drop the subscript ∞ temporarily and consider an archimedean Arthur parameter $\psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$. Recall that $W_{\mathbb{C}} = \mathbb{C}^\times$, $W_{\mathbb{R}} = W_{\mathbb{C}} \sqcup jW_{\mathbb{C}}$ where $j^2 = -1 \in W_{\mathbb{C}}$ and for any $z \in W_{\mathbb{C}}$, $jzj^{-1} = \bar{z}$. Define a Langlands parameter φ_ψ by composing ψ with $W_{\mathbb{R}} \to W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ mapping $w \in W_{\mathbb{R}}$ to

$$\left(w, \begin{pmatrix} ||w||^{1/2} & 0 \\ 0 & ||w||^{-1/2} \end{pmatrix}\right)$$

where $||\cdot|| : W_{\mathbb{R}} \to \mathbb{R}_{>0}$ is the unique morphism mapping $z \in W_{\mathbb{C}}$ to $z\bar{z}$. Let $\mathcal{T}$ be a maximal torus in $\widehat{\mathbf{G}}$. Conjugating by an element of $\widehat{\mathbf{G}}$ if necessary, we can assume that $\varphi_\psi(W_{\mathbb{C}}) \subset \mathcal{T}$ and write $\varphi_\psi(z) = \mu_1(z)\mu_2(\bar{z})$ for $z \in W_{\mathbb{C}}$, where $\mu_1, \mu_2 \in \mathbb{C} \otimes_{\mathbb{Z}} X_*(\mathcal{T})$ are such that $\mu_1 - \mu_2 \in X_*(\mathcal{T})$. The conjugacy class of (μ_1, μ_2) under the Weyl group $W(\mathcal{T}, \widehat{\mathbf{G}})$ is well-defined. Note that for any maximal torus $\mathbf{T}$ of $\mathbf{G}_{\mathbb{C}}$ we can see μ_1, μ_2 as elements of $\mathbb{C} \otimes_{\mathbb{Z}} X^*(\mathbf{T})$, again canonically up to the action of the Weyl group.

Lemma 3.4.1.3. *The Weyl group orbit of μ_1 is the infinitesimal character of any element of Π_ψ.*

Proof. Recall [Art13][Theorem 2.2.1] that the packet Π_ψ is characterised by twisted and standard endoscopic character identities involving the representation of $\mathbf{GL}_N(\mathbb{R})$ having Langlands parameter $\mathrm{Std} \circ \varphi_\psi$. The lemma follows from [Mez13][Lemma 24] (see also [Wal][Corollaire 2.8]), which establishes the equivariance of twisted endoscopic transfer for the actions of the centers of the enveloping algebras. □

Attached to λ is a unique (up to $\widehat{\mathbf{G}}$-conjugacy) *discrete* parameter $\varphi_\lambda : W_{\mathbb{R}} \to {}^L\mathbf{G}$ having infinitesimal character $\lambda+\rho$. We explicit the $\mathrm{GL}_N(\mathbb{C})$-conjugacy class of $\mathrm{Std}\circ\varphi_\lambda$ in each case. For $w \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ it is convenient to denote the Langlands parameter $W_{\mathbb{R}} \to \mathrm{GL}_2(\mathbb{C})$

$$I_w = \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}\left(z \mapsto (z/|z|)^{2w}\right) : \; z \in W_{\mathbb{C}} \mapsto \begin{pmatrix} (z/|z|)^{2w} & 0 \\ 0 & (z/|z|)^{-2w} \end{pmatrix}, \; j \mapsto \begin{pmatrix} 0 & (-1)^{2w} \\ 1 & 0 \end{pmatrix}.$$

Note that this was denoted I_{2w} in [CR14] to emphasise motivic weight in a global setting. We choose to emphasise Hodge weights, i.e. eigenvalues of the infinitesimal character: our I_w has Hodge weights w and $-w$. Let $\epsilon_{\mathbb{C}/\mathbb{R}}$ be the non-trivial continuous character $W_{\mathbb{R}} \to \{\pm 1\}$, so that $I_0 = 1 \oplus \epsilon_{\mathbb{C}/\mathbb{R}}$. If $\mathbf{G} = \mathbf{SO}_{2n+1}$, we can write $\lambda = k_1 e_1 + \cdots + k_n e_n$ where $k_1 \geq \cdots \geq k_n \geq 0$ are integers, and $\rho = (n-\frac{1}{2})e_1 + (n-\frac{3}{2})e_2 + \cdots + \frac{1}{2}e_n$. In this case $\mathrm{Std}\circ\varphi_\lambda$ is

$$\bigoplus_{r=1}^{n} I_{k_r+n+1/2-r}.$$

If $\mathbf{G} = \mathbf{Sp}_{2n}$, we can write $\lambda = k_1 e_1 + \cdots + k_n e_n$ where $k_1 \geq \cdots \geq k_n \geq 0$ are integers, and $\rho = ne_1 + (n-1)e_2 + \cdots + e_n$. Then $\mathrm{Std}\circ\varphi_\lambda$ is

$$\epsilon_{\mathbb{C}/\mathbb{R}}^n \oplus \bigoplus_{r=1}^{n} I_{k_r+n+1-r}.$$

Finally, if $\mathbf{G} = \mathbf{SO}_{4n}$, we can write $\lambda = k_1 e_1 + \cdots + k_{2n} e_{2n}$ where $k_1 \geq \cdots \geq k_{2n-1} \geq |k_{2n}|$ are integers, and $\rho = (2n-1)e_1 + (2n-2)e_2 + \cdots + e_{2n-1}$. Then $\mathrm{Std}\circ\varphi_\lambda$ is

$$\bigoplus_{r=1}^{2n} I_{k_r+2n-r}.$$

Replacing $(k_1, \dots, k_{2n-1}, k_{2n})$ with $(k_1, \dots, k_{2n-1}, -k_{2n})$ yields the same conjugacy class under $\mathrm{GL}_N(\mathbb{C})$.

From this explicit description one can deduce several restrictions on the global parameters $\psi \in \Psi(\mathbf{G})^{\mathrm{unr}}$ contributing non-trivially to the spectral side 3.4.1.2. These observations were already made in [CR14], using a different formulation. We define $\Psi(\mathbf{G})^{\lambda}$ as the subset of $\Psi(\mathbf{G})$ consisting of ψ such that the infinitesimal character of ψ_∞ is equal to $\lambda+\rho$. Define also $\Psi(\mathbf{G})^{\mathrm{unr},\lambda} = \Psi(\mathbf{G})^{\mathrm{unr}} \cap \Psi(\mathbf{G})^{\lambda}$.

1. In the first two cases ($\mathbf{G} = \mathbf{SO}_{2n+1}$ of $\mathbf{Sp}_{2n}$) the infinitesimal character of $\mathrm{Std}\circ\varphi_\lambda$ is algebraic and regular in the sense of Clozel [Clo88]. Clozel's definition of "algebraic" is "C-algebraic" in the sense of [BG10], and we will also use the term "C-algebraic" to avoid confusion. In the third case ($\mathbf{G} = \mathbf{SO}_{4n}$) we have that $||\cdot||^{1/2} \otimes (\mathrm{Std}\circ\varphi_\lambda)$ is C-algebraic, but not always regular. It is regular if and only if $k_{2n} \neq 0$. In all cases, Clozel's purity lemma [Clo88][Lemme 4.9] implies that if $\psi = \boxplus_i \pi_i[d_i] \in \Psi(\mathbf{G})^{\lambda}$, then for all i the self-dual cuspidal automorphic representation π_i of $\mathbf{GL}_{n_i}/\mathbb{Q}$ is *tempered* at the real place. Equivalently, $\psi_\infty(W_{\mathbb{R}})$ is bounded.

2. Let $\Psi(\mathbf{G})_{\mathrm{sim}}$ be the set of *simple* formal Arthur parameters in $\Psi(\mathbf{G})$, i.e. those $\psi = \boxplus_{i\in I}\pi_i[d_i]$ such that $I = \{i_0\}$ and $d_{i_0} = 1$. Denote $\Psi(\mathbf{G})_{\mathrm{sim}}^{\lambda} = \Psi(\mathbf{G})_{\mathrm{sim}} \cap \Psi(\mathbf{G})^{\lambda}$.

Then $\Psi(\mathbf{G})_{\mathrm{sim}}^{\lambda}$ is the set of self-dual cuspidal automorphic representations of $\mathbf{GL}_N/\mathbb{Q}$ such that the central character of π is trivial and the local Langlands parameter of π_∞ is $\mathrm{Std}\circ\varphi_\lambda$. Indeed in all three cases $\mathrm{Std}\circ\varphi_\lambda$ is either orthogonal or symplectic, and thus π_∞ *determines* $s(\pi)$.

3. Let $m \geq 1$ and consider a self-dual cuspidal automorphic representation π of $\mathbf{GL}_{2m}/\mathbb{Q}$ such that $|\det|^{1/2}\otimes\pi$ is C-algebraic regular. Self-duality implies that the central character χ_π of π is quadratic, i.e. $\chi_\pi : \mathbb{A}^\times/\mathbb{Q}^\times \to \{\pm 1\}$. Since $|\det|^{1/2}\otimes\pi$ is C-algebraic and regular, there are unique integers $w_1 > \cdots > w_m > 0$ such that the local Langlands parameter of π_∞ is

$$\bigoplus_{r=1}^{m} I_{w_r},$$

which implies that $\chi_\pi|_{\mathbb{R}^\times}(-1) = (-1)^m$. If moreover we assume that π is everywhere unramified, then χ_π is trivial on $\prod_p \mathbb{Z}_p^\times$. Since $\mathbb{A}^\times = \mathbb{Q}^\times\mathbb{R}_{>0}\prod_p \mathbb{Z}_p^\times$, this implies that χ_π is trivial, and thus m must be even.

4. The previous point has the following important consequence for our inductive computations. Let $\mathbf{G}$ be a split symplectic or special orthogonal group admitting discrete series at the real place, and λ a dominant weight for $\mathbf{G}$. Let $\psi = \boxplus_i \pi_i[d_i] \in \Psi(\mathbf{G})^{\mathrm{unr},\lambda}$. Then for any i, there is a split symplectic or special orthogonal group $\mathbf{G}'$ admitting discrete series at the real place and a dominant weight λ' for $\mathbf{G}'$ such that $\pi_i \in \Psi(\mathbf{G}')_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$. We emphasise that this holds even if $\mathbf{G} = \mathbf{SO}_{4n}$ and $\lambda = k_1e_1 + \cdots + k_{2n}e_{2n}$ with $k_{2n} = 0$. To be precise, we have the following classification:

   (a) $\mathbf{G} = \mathbf{SO}_{2n+1}$ and thus $\widehat{\mathbf{G}} = \mathrm{Sp}_{2n}(\mathbb{C})$. For a dominant weight λ and $\psi = \boxplus_{i\in I}\pi_i[d_i] \in \Psi(\mathbf{G})^{\mathrm{unr},\lambda}$, there is a canonical decomposition $I = I_1 \sqcup I_2 \sqcup I_3$ where

      i. for all $i \in I_1$, d_i is odd, n_i is even and $\pi_i \in \Psi(\mathbf{SO}_{n_i+1})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$,

      ii. for all $i \in I_2$, d_i is even, n_i is divisible by 4 and $\pi_i \in \Psi(\mathbf{SO}_{n_i})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$,

      iii. $\mathrm{card}(I_3) \in \{0,1\}$ and if $I_3 = \{i\}$, d_i is even, n_i is odd and $\pi_i \in \Psi(\mathbf{Sp}_{n_i-1})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$.

   (b) $\mathbf{G} = \mathbf{Sp}_{2n}$ and thus $\widehat{\mathbf{G}} = \mathrm{SO}_{2n+1}(\mathbb{C})$. For a dominant weight λ and $\psi = \boxplus_{i\in I}\pi_i[d_i] \in \Psi(\mathbf{G})^{\mathrm{unr},\lambda}$, there is a canonical decomposition $I = I_1 \sqcup I_2 \sqcup I_3$ where

      i. $I_1 = \{j\}$, d_j is odd, n_j is odd and $\pi_j \in \Psi(\mathbf{Sp}_{n_j-1})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$,

      ii. for all $i \in I_2$, d_i is odd, n_i is divisible by 4 and $\pi_i \in \Psi(\mathbf{SO}_{n_i})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$,

      iii. for all $i \in I_3$, d_i is even, n_i is even and $\pi_i \in \Psi(\mathbf{SO}_{n_i+1})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$.

      Note that $n_jd_j = 2n+1 \mod 4$.

   (c) $\mathbf{G} = \mathbf{SO}_{4n}$ and thus $\widehat{\mathbf{G}} = \mathrm{SO}_{4n}(\mathbb{C})$. For a dominant weight λ and $\psi = \boxplus_{i\in I}\pi_i[d_i] \in \Psi(\mathbf{G})^{\mathrm{unr},\lambda}$, there is a canonical decomposition $I = I_1 \sqcup I_2 \sqcup I_3$ where

i. for all $i \in I_1$, d_i is odd, n_i is divisible by 4 and $\pi_i \in \Psi(\mathbf{SO}_{n_i})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$,

ii. for all $i \in I_2$, d_i is even, n_i is even and $\pi_i \in \Psi(\mathbf{SO}_{n_i+1})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$,

iii. $\mathrm{card}(I_3) \in \{0,2\}$. If $I_3 = \{i,j\}$ and up to exchanging i and j, $d_i = 1$ and d_j is odd, n_i and n_j are odd, and $\pi_i \in \Psi(\mathbf{Sp}_{n_i-1})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$ and $\pi_j \in \Psi(\mathbf{Sp}_{n_j-1})_{\mathrm{sim}}^{\mathrm{unr},\lambda'}$.

Note that in all three cases, if λ is *regular* then for any $\psi = \boxplus_{i\in I}\pi_i[d_i] \in \Psi(\mathbf{G})^{\mathrm{unr},\lambda}$ we have that $\psi_\infty = \varphi_\lambda$ and thus all d_i's are equal to 1 (i.e. ψ is formally tempered) and moreover in the third case $I_3 = \emptyset$.

As in the introduction, it will be convenient to have a more concrete notation for the sets $\Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$.

1. For $n \geq 1$, the dominant weights for $\mathbf{G} = \mathbf{SO}_{2n+1}$ are the characters $\lambda = k_1e_1 + \cdots + k_ne_n$ such that $k_1 \geq \cdots \geq k_n \geq 0$. Then $\lambda + \rho = w_1e_1 + \cdots + w_ne_n$ where $w_r = k_r + n + \frac{1}{2} - r$, so that $w_1 > \cdots > w_n > 0$ belong to $\frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z}$. Define $S(w_1,\dots,w_n) = \Psi(\mathbf{SO}_{2n+1})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$, that is the set of self-dual automorphic cuspidal representations of $\mathbf{GL}_{2n}/\mathbb{Q}$ which are everywhere unramified and with Langlands parameter at the real place
$$I_{w_1} \oplus \cdots \oplus I_{w_n}.$$
Equivalently we could replace the last condition by "with infinitesimal character having eigenvalues $\{\pm w_1,\dots,\pm w_n\}$". Here S stands for "symplectic", as $\widehat{\mathbf{G}} = \mathrm{Sp}_{2n}(\mathbb{C})$.

2. For $n \geq 1$, the dominant weights for $\mathbf{G} = \mathbf{Sp}_{2n}$ are the characters $\lambda = k_1e_1+\cdots+k_ne_n$ such that $k_1 \geq \cdots \geq k_n \geq 0$. Then $\lambda+\rho = w_1e_1+\cdots+w_ne_n$ where $w_r = k_r+n+1-r$, so that $w_1 > \cdots > w_n > 0$ are integers. Define $O_o(w_1,\dots,w_n) = \Psi(\mathbf{Sp}_{2n})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$, that is the set of self-dual automorphic cuspidal representations of $\mathbf{GL}_{2n+1}/\mathbb{Q}$ which are everywhere unramified and with Langlands parameter at the real place
$$I_{w_1} \oplus \cdots \oplus I_{w_n} \oplus \epsilon_{\mathbb{C}/\mathbb{R}}^n.$$
Equivalently we could replace the last condition by "with infinitesimal character having eigenvalues $\{\pm w_1,\dots,\pm w_n,0\}$". Here O_o stands for "odd orthogonal", as $\widehat{\mathbf{G}} = \mathrm{SO}_{2n+1}(\mathbb{C})$.

3. For $n \geq 1$, the dominant weights for $\mathbf{G} = \mathbf{SO}_{4n}$ are the characters $\lambda = k_1e_1 + \cdots + k_{2n}e_{2n}$ such that $k_1 \geq \cdots \geq k_{2n-1} \geq |k_{2n}|$. Since we only consider quantities invariant under outer conjugation we assume $k_{2n} \geq 0$. Then $\lambda+\rho = w_1e_1+\cdots+w_{2n}e_{2n}$ where $w_r = k_r + n - r$, so that $w_1 > \cdots > w_{2n-1} > w_{2n} \geq 0$ are integers. Define $O_e(w_1,\dots,w_{2n}) = \Psi(\mathbf{SO}_{4n})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$, that is the set of self-dual automorphic cuspidal representations of $\mathbf{GL}_{4n}/\mathbb{Q}$ which are everywhere unramified and with Langlands parameter at the real place
$$I_{w_1} \oplus \cdots \oplus I_{w_{2n}}.$$
In this case also we could replace the last condition by "with infinitesimal character having eigenvalues $\{\pm w_1,\dots,\pm w_{2n}\}$", even when $k_{2n} = 0$. Here O_e stands for "even orthogonal", as $\widehat{\mathbf{G}} = \mathrm{SO}_{4n}(\mathbb{C})$.

It is now natural to try to compute the cardinality of $\Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$, inductively on the dimension of $\mathbf{G}$. Observe that for $\psi \in \Psi(\mathbf{G})_{\mathrm{sim}}$, the group S_ψ is trivial. Thus the contribution of any $\psi \in \Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$ to the spectral side 3.4.1.2 is simply

$$\sum_{\pi_\infty \in \Pi_{\psi_\infty}} \mathrm{EP}\left(\pi_\infty \otimes V_\lambda^*\right).$$

Recall that for such a ψ, the local Arthur parameter ψ_∞ is φ_λ. In that case Arthur defines Π_{φ_λ} as the L-packet that Langlands [Lan89] associates with φ_λ. In the next section we will review these packets in more detail, in particular Shelstad's definition of $\langle \cdot, \pi_\infty \rangle$ for $\pi_\infty \in \Pi_{\varphi_\lambda}$, but since S_ψ is trivial all that matters for now is that $\mathrm{card}(\Pi_{\varphi_\lambda})$ is positive (and easily computed) and that all the representations in Π_{φ_λ} are discrete series. By [BW00][ch. III, Thm. 5.1] for any $\pi_\infty \in \Pi_{\varphi_\lambda}$,

$$\mathrm{EP}\left(\pi_\infty \otimes V_\lambda^*\right) = (-1)^{q(\mathbf{G}(\mathbb{R}))}$$

and thus to compute the cardinality of $\Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$ we want to compute the contribution of $\Psi(\mathbf{G})^{\mathrm{unr},\lambda} \smallsetminus \Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$ to the spectral side 3.4.1.2.

This is particularly easy if λ is regular, since as we observed above in that case any $\psi \in \Psi(\mathbf{G})^{\mathrm{unr},\lambda}$ is "formally tempered" or "formally of Ramanujan type", i.e. $\psi_\infty = \varphi_\lambda$. Moreover ϵ_ψ is trivial. Shelstad's results reviewed in the next section allow the explicit determination of the number of $\pi_\infty \in \Pi_{\varphi_\lambda}$ such that $\langle \cdot, \pi_\infty \rangle$ is equal to a given character of S_{ψ_∞}.

The general case is more interesting. The determination of ϵ_ψ in the "conductor one" case was done in [CR14], and the result is simple since it involves only epsilon factors at the real place of $\mathbb{Q}$. In all three cases, for any $\psi = \boxplus_{i \in I} \pi_i[d_i] \in \Psi(\mathbf{G})^{\mathrm{unr},\lambda}$ the abelian 2-group S_ψ is generated by $(s_i)_{i \in J}$ where $J = \{i \in I \,|\, n_i d_i \text{ is even}\}$ and $s_i \in C_\psi$ is formally $-\mathrm{Id}$ on the space of $\pi_i[d_i]$ and Id on the space of $\pi_j[d_j]$ for $j \neq i$. By [CR14][(3.10)]

$$\epsilon_\psi(s_i) = \prod_{j \in I \smallsetminus \{i\}} \epsilon(\pi_i \times \pi_j)^{\min(d_i, d_j)}$$

and since π_i and π_j are everywhere unramified $\epsilon(\pi_i \times \pi_j)$ can be computed easily from the tensor product of the local Langlands parameters of $(\pi_i)_\infty$ and $(\pi_j)_\infty$. Note that by [Art13][Theorem 1.5.3] $\epsilon(\pi_i \times \pi_j) = 1$ if $s(\pi_i)s(\pi_j) = 1$. The explicit computation of Π_{ψ_∞}, along with the map $\Pi_{\psi_\infty} \to S_{\psi_\infty}^\wedge$, does not follow directly from Arthur's work, even in our special case where the infinitesimal character of ψ_∞ is that of an algebraic representation V_λ. We will need to make an assumption (Assumption 3.4.2.4) relating Arthur's packet Π_{ψ_∞} to the packets constructed by Adams and Johnson in [AJ87]. The latter predate Arthur's recent work, in fact [AJ87] has corroborated Arthur's general conjectures: see [Art89b][§5]. Under this assumption, we will also be able to compute the Euler-Poincaré characteristic of any element of Π_{ψ_∞} in section 3.4.2.2.

Remark 3.4.1.4. *Our original goal was to compute, for a given group $\mathbf{G}/\mathbb{Q}$ as above, dominant weight λ and simple $(\mathfrak{g}, K_\infty)$-module module X with infinitesimal character $\lambda+\rho$, the multiplicity of X in $\mathcal{A}_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))^{\mathbf{G}(\widehat{\mathbb{Z}})}$. This is possible once the cardinalities of*

$\Psi(\mathbf{G}')^{\mathrm{unr},\lambda'}_{\mathrm{sim}}$ *are computed, under Assumption 3.4.2.3 if we do not assume that* λ *is regular. However, Arthur's endoscopic classification shows that computing* $\mathrm{card}\left(\Psi(\mathbf{G}')^{\mathrm{unr},\lambda'}_{\mathrm{sim}}\right)$ *is a more interesting problem from an arithmetic perspective, since conjecturally we are counting the number of self-dual motives over* $\mathbb{Q}$ *with conductor* 1 *and given Hodge weights.*

Remark 3.4.1.5. *Except in the even orthogonal case with* $\lambda = k_1 e_1 + \cdots + k_{2n} e_{2n}$ *and* $k_{2n} = 0$*, it is known that any* $\psi \in \Psi(\mathbf{G})^{\mathrm{unr},\lambda}_{\mathrm{sim}}$ *is tempered also at the finite places by [Clo13].*

Remark 3.4.1.6. *If* $\mathbf{G}$ *is symplectic or even orthogonal, it has non-trivial center* $\mathbf{Z}$ *isomorphic to* μ_2*. Thus* $\mathbf{Z}(\mathbb{R}) \subset \mathbf{Z}(\mathbb{Q})\mathbf{Z}(\widehat{\mathbb{Z}})$*, and* $\mathbf{Z}(\mathbb{R})$ *acts trivially on* $\mathcal{A}_{\mathrm{disc}}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))^{\mathbf{G}(\widehat{\mathbb{Z}})}$*. This implies that* $\Psi(\mathbf{G})^{\mathrm{unr},\lambda}_{\mathrm{sim}}$ *is empty if* $\lambda|_{\mathbf{Z}(\mathbb{R})}$ *is not trivial, since* $\mathbf{Z}(\mathbb{R})$ *acts by* λ *on any discrete series representation with infinitesimal character* $\lambda + \rho$*. Using the concrete description above, it is elementary to deduce that in fact* $\Psi(\mathbf{G})^{\mathrm{unr},\lambda}$ *is empty if* $\lambda|_{\mathbf{Z}(\mathbb{R})}$ *is not trivial.*

3.4.2 Euler-Poincaré characteristic of cohomological archimedean Arthur packets

3.4.2.1 Tempered case: Shelstad's parametrization of L-packets

For archimedean local fields in the tempered case the A-packets Π_ψ in [Art13] are not defined abstractly using the global twisted trace formula. Rather, Arthur *defines* Π_{φ_λ} as the L-packet that Langlands [Lan89] associates with φ_λ, and the map $\Pi_{\varphi_\lambda} \to S^\wedge_{\varphi_\lambda}, \pi \mapsto \langle \cdot, \pi \rangle$ is defined by Shelstad's work, which we review below. Mezo [Mez] has shown that these Langlands-Shelstad L-packets satisfy the twisted endoscopic character relation [Art13][Theorem 2.2.1 (a)], and Shelstad's work contains the "standard" endoscopic character relations [Art13][Theorem 2.2.1 (b)].

In this section we will only be concerned with the local field $\mathbb{R}$ and thus we drop the subscripts ∞, and we denote $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$. Let $\mathbf{G}$ be a reductive group over $\mathbb{R}$, and denote by $\mathbf{A}_\mathbf{G}$ the biggest split torus in the connected center $\mathbf{Z}_\mathbf{G}$ of $\mathbf{G}$. Let us assume that $\mathbf{G}$ has a maximal torus (defined over $\mathbb{R}$) which is anisotropic modulo $\mathbf{A}_\mathbf{G}$, i.e. $\mathbf{G}(\mathbb{R})$ has essentially discrete series. Consider a dominant weight λ_0 for $(\mathbf{G}_{\mathrm{der}})_\mathbb{C}$ defining an algebraic representation V_{λ_0} of $\mathbf{G}_{\mathrm{der}}(\mathbb{C})$ and a continuous character $\chi_0 : \mathbf{Z}_\mathbf{G}(\mathbb{R}) \to \mathbb{C}^\times$ such that χ_0 and λ_0 coincide on $\mathbf{Z}_\mathbf{G}(\mathbb{R}) \cap \mathbf{G}_{\mathrm{der}}(\mathbb{C})$. Let $\Pi_{\mathrm{disc}}(\lambda_0, \chi_0)$ be the finite set of essentially discrete series representations π of $\mathbf{G}(\mathbb{R})$ such that

- $\pi|_{\mathbf{G}_{\mathrm{der}}(\mathbb{R})}$ has the same infinitesimal character as $V_{\lambda_0}|_{\mathbf{G}_{\mathrm{der}}(\mathbb{R})}$,
- $\pi|_{\mathbf{Z}_\mathbf{G}(\mathbb{R})} = \chi_0$.

Harish-Chandra has shown that inside this L-packet of essentially discrete series, the representations are parameterised by the conjugacy classes (under $\mathbf{G}(\mathbb{R})$) of pairs $(\mathbf{B}, \mathbf{T})$ where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$ anisotropic modulo $\mathbf{A}_\mathbf{G}$ and $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}_\mathbb{C}$ containing $\mathbf{T}_\mathbb{C}$. For such a pair $(\mathbf{B}, \mathbf{T})$, χ_0 and the character λ_0 of $\mathbf{T}_{\mathrm{der}}(\mathbb{R})$ which is dominant for $\mathbf{B}$ extend uniquely to a character $\lambda_\mathbf{B}$ of $\mathbf{T}(\mathbb{R})$. If we fix such a pair $(\mathbf{B}, \mathbf{T})$, the pairs $(\mathbf{B}', \mathbf{T})$ which are in the same conjugacy class form an orbit under the subgroup

$W_c := W(\mathbf{G}(\mathbb{R}), \mathbf{T}(\mathbb{R}))$ of $W := W(\mathbf{G}(\mathbb{C}), \mathbf{T}(\mathbb{C}))$. Concretely, if $\pi \in \Pi_{\mathrm{disc}}(\lambda_0, \chi_0)$ is the representation associated with this conjugacy class, then for any $\gamma \in \mathbf{T}(\mathbb{R})_{\mathbf{G}-\mathrm{reg}}$,

$$\Theta_\pi(\gamma) = (-1)^{q(\mathbf{G})} \sum_{w \in W_c} \frac{\lambda_{w\mathbf{B}w^{-1}}(\gamma)}{\Delta_{w\mathbf{B}w^{-1}}(\gamma)}$$

where Θ_π is Harish-Chandra's character for π, and $\Delta_{\mathbf{B}}(\gamma) = \prod_{\alpha \in R(\mathbf{T},\mathbf{B})}(1 - \alpha(\gamma)^{-1})$. Therefore the choice of $(\mathbf{B}, \mathbf{T})$ as a base point identifies the set of conjugacy classes with $W_c \backslash W$, by $g \in N(\mathbf{G}(\mathbb{C}), \mathbf{T}(\mathbb{C})) \mapsto (g\mathbf{B}g^{-1}, \mathbf{T})$.

Langlands [Lan89] and Shelstad [She08a], [She10], [She08b] gave another formulation for the parameterisation inside an L-packet, more suitable for writing endoscopic character relations. By definition of the L-group we have a splitting $(\mathcal{B}, \mathcal{T}, (\mathcal{X}_\alpha)_{\alpha \in \Delta})$ of $\widehat{\mathbf{G}}$ which defines a section of $\mathrm{Aut}(\widehat{\mathbf{G}}) \to \mathrm{Out}(\widehat{\mathbf{G}})$ and ${}^L\mathbf{G} = \widehat{\mathbf{G}} \rtimes W_{\mathbb{R}}$. Let $(\mathbf{B}, \mathbf{T})$ be as above. Thanks to $\mathbf{B}$ we have a canonical isomorphism $\widehat{\mathbf{T}} \to \mathcal{T}$, which can be extended into an embedding of L-groups $\iota : {}^L\mathbf{T} \to {}^L\mathbf{G}$ as follows. For $z \in W_{\mathbb{C}}$, define $\iota(z) = \prod_{\alpha \in R_{\mathcal{B}}} \alpha^\vee(z/|z|) \rtimes z$ where $R_{\mathcal{B}}$ is the set of roots of $\mathcal{T}$ in $\mathcal{B}$. Define $\iota(j) = n_0 \rtimes j$ where $n_0 \in N(\widehat{\mathbf{G}}, \mathcal{T}) \cap \widehat{\mathbf{G}}_{\mathrm{der}}$ represents the longest element of the Weyl group $W(\widehat{\mathbf{G}}, \mathcal{T})$ for the order defined by $\mathcal{B}$. Then ι is well-defined thanks to [Lan89][Lemma 3.2]. Since conjugation by $n_0 \rtimes j$ acts by $t \mapsto t^{-1}$ on $\mathcal{T} \cap \widehat{\mathbf{G}}_{\mathrm{der}}$, the conjugacy class of ι does not depend on the choice of n_0. The character $\lambda_{\mathbf{B}}$ of $\mathbf{T}(\mathbb{R})$ corresponds to a Langlands parameter $\varphi_{\lambda_{\mathbf{B}}} : W_{\mathbb{R}} \to {}^L\mathbf{T}$. If $\mathbf{G}$ is semisimple, $\lambda_{\mathbf{B}}$ is the restriction to $\mathbf{T}(\mathbb{R})$ of an element of $X^*(\mathbf{T}) = X_*(\mathcal{T})$ and for any $z \in W_{\mathbb{C}}$, $\varphi_{\lambda_{\mathbf{B}}}(z) = \lambda_{\mathbf{B}}(z/|z|)$. Composing $\varphi_{\lambda_{\mathbf{B}}}$ with ι we get a Langlands parameter $\varphi : W_{\mathbb{R}} \to {}^L\mathbf{G}$, whose conjugacy class under $\widehat{\mathbf{G}}$ does not depend on the choice of $(\mathbf{B}, \mathbf{T})$. Langlands has shown that the map $(\lambda_0, \chi_0) \mapsto \varphi$ is a bijection onto the set of conjugacy classes of discrete Langlands parameters, i.e. Langlands parameters φ such that $S_\varphi := \mathrm{Cent}(\varphi, \widehat{\mathbf{G}})/Z(\widehat{\mathbf{G}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$ is finite.

Consider a discrete Langlands parameter φ, and denote by $\Pi_\varphi = \Pi(\lambda_0, \chi_0)$ the corresponding L-packet. Assume that $\mathbf{G}$ is quasisplit and fix a Whittaker datum (see [Kal] for the general case). Then Shelstad defines an injective map $\Pi_\varphi \to S_\varphi^\wedge$, $\pi \mapsto \langle \cdot, \pi \rangle$. It has the property that $\langle \cdot, \pi \rangle$ is trivial if π is the unique generic (for the given Whittaker datum) representation in the L-packet.

Recall the relation between these two parametrizations of the discrete L-packets. Let $(\mathbf{B}, \mathbf{T})$ be as above, defining an embedding $\iota : {}^L\mathbf{T} \to {}^L\mathbf{G}$ and recall that W and W_c denote the complex and real Weyl groups. Let $C_\varphi = \mathrm{Cent}(\varphi, \widehat{\mathbf{G}})$, so that $S_\varphi = C_\varphi / Z(\widehat{\mathbf{G}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$. Using ι we have an isomorphism between $H^1(\mathbb{R}, \mathbf{T})$ and $\pi_0(C_\varphi)^\wedge$. We have a bijection

$$W_c \backslash W \to \ker\left(H^1(\mathbb{R}, \mathbf{T}) \to H^1(\mathbb{R}, \mathbf{G})\right)$$

mapping $g \in N_{\mathbf{G}(\mathbb{C})}(\mathbf{T}(\mathbb{C}))$ to $(\sigma \mapsto g^{-1}\sigma(g))$. Kottwitz [Kot86] has defined a natural morphism $H^1(\mathbb{R}, \mathbf{G}) \to \pi_0\left(Z(\widehat{\mathbf{G}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}\right)^\wedge$ and thus the above bijection yields an injection $\eta : W_c \backslash W \to S_\varphi^\wedge$. If $\pi \in \Pi_\varphi$ corresponds to (the conjugacy class of) $(\mathbf{B}, \mathbf{T})$ and $\pi' \in \Pi_\varphi$ corresponds to $(g\mathbf{B}g^{-1}, \mathbf{T})$, then for any $s \in S_\varphi$,

$$\frac{\langle s, \pi \rangle}{\langle s, \pi' \rangle} = \eta(g)(s).$$

Finally, the generic representation in Π_φ corresponds to a pair $(\mathbf{B}, \mathbf{T})$ as above such that all the simple roots for $\mathbf{B}$ are noncompact. This is a consequence of [Kos78][Theorem 3.9] and [Vog78][Theorem 6.2]. In particular there *exists* such a pair $(\mathbf{B}, \mathbf{T})$. We will make use of the converse in the non-tempered case.

Lemma 3.4.2.1. *Let $\mathbf{H}$ be a reductive group over $\mathbb{R}$. Assume that $\mathbf{T}$ is a maximal torus of $\mathbf{H}$ which is anisotropic modulo $\mathbf{A}_\mathbf{H}$, and assume that there exists a Borel subgroup $\mathbf{B} \supset \mathbf{T}_\mathbb{C}$ of $\mathbf{H}_\mathbb{C}$ such that all the simple roots of $\mathbf{T}$ in $\mathbf{B}$ are non-compact. Then $\mathbf{H}$ is quasisplit.*

Proof. We can assume that $\mathbf{H}$ is semisimple. We use the "$\mathbb{R}$-opp splittings" of [She08b][§12]. Let Δ be the set of simple roots of $\mathbf{T}$ in $\mathbf{B}$. For any $\alpha \in \Delta$ we can choose an $\mathfrak{sl}_2$-triple $(H_\alpha, X_\alpha, Y_\alpha)$ in $\mathfrak{h} = \mathbb{C} \otimes_\mathbb{R} \mathrm{Lie}(\mathbf{H}(\mathbb{R}))$. The pair (X_α, Y_α) is not unique: it could be replaced by $(xX_\alpha, x^{-1}Y_\alpha)$ for any $x \in \mathbb{C}^\times$. Since $\sigma(\alpha) = -\alpha$, $\sigma(X_\alpha) = yY_\alpha$ for some $y \in \mathbb{C}^\times$, and $y \in \mathbb{R}^\times$ because σ is an involution. The sign of y does not depend on the choice of (X_α, Y_α), and making some other choice if necessary, we can assume that $y = \pm 1$. It is easy to check that α is non-compact if and only if $y > 0$. Thus the hypotheses imply the existence of an $\mathbb{R}$-opp splitting, that is a splitting $(X_\alpha)_{\alpha \in \Delta}$ such that $\sigma(X_\alpha) = Y_\alpha$ for any α. Note that this splitting is unique up to the action of $\mathbf{T}(\mathbb{R})$.

Let $\mathbf{H}'$ be the quasisplit reductive group over $\mathbb{R}$ such that $\mathbf{H}'$ admits an anisotropic maximal torus and $\mathbf{H}_\mathbb{C} \simeq \mathbf{H}'_\mathbb{C}$. We know that $\mathbf{H}'$ admits a pair $(\mathbf{B}', \mathbf{T}')$ where $\mathbf{T}'$ is an anisotropic maximal torus and all the simple roots of $\mathbf{B}'$ are non-compact. Therefore there exists an $\mathbb{R}$-opp splitting $(X'_\alpha)_{\alpha' \in \Delta'}$ for $(\mathbf{B}', \mathbf{T}')$.

There is a unique isomorphism $f : \mathbf{H}_\mathbb{C} \to \mathbf{H}'_\mathbb{C}$ identifying $(\mathbf{B}, \mathbf{T}_\mathbb{C}, (X_\alpha)_{\alpha \in \Delta})$ with $(\mathbf{B}', \mathbf{T}'_\mathbb{C}, (X'_\alpha)_{\alpha \in \Delta'})$ and to conclude we only have to show that it is defined over $\mathbb{R}$, i.e. that it is Galois-equivariant. It is obviously the case on $\mathbf{T}$, since any automorphism of $\mathbf{T}_\mathbb{C}$ is defined over $\mathbb{R}$. Moreover by construction $f(\sigma(X_\alpha)) = \sigma(X'_{f(\alpha)})$ for any $\alpha \in \Delta$. Since $\mathbf{T}_\mathbb{C}$ and the one-dimensional unipotent groups corresponding to $\pm\alpha$ for $\alpha \in \Delta$ generate $\mathbf{H}_\mathbb{C}$, f is σ-equivariant. □

There are as many conjugacy classes of such pairs $(\mathbf{B}, \mathbf{T})$ such that all the simple roots are non-compact as there are conjugacy classes of Whittaker datum. For the adjoint group $\mathbf{SO}_{2n+1}$ there is a single conjugacy class, whereas for $\mathbf{G} = \mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$ there are two. However, for our purposes it will fortunately not be necessary to precise which pair $(\mathbf{B}, \mathbf{T})$ corresponds to each conjugacy class of Whittaker datum.

For the quasi-split group $\mathbf{G} = \mathbf{SO}(V, q)$ where $\dim V \geq 3$ and $\mathrm{disc}(q) > 0$, $\mathbf{T}$ is the stabiliser of a direct orthogonal sum

$$P_1 \oplus \cdots \oplus P_n$$

where each P_i is a definite plane and $n = \lfloor \dim V/2 \rfloor$. Let I_+ (resp. I_-) be the set of $i \in \{1, \ldots, n\}$ such that P_i is positive (resp. negative), $V_- = \bigoplus_{i \in I_-} P_i$ and $V_+ = V_-^\perp$. The group K of real points of

$$\mathbf{S}\left(\mathbf{O}(V_+, q) \times \mathbf{O}(V_-, q)\right)$$

is the maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ containing $\mathbf{T}(\mathbb{R})$. For each i, choose an isomorphism $e_i : \mathbf{SO}(P_i, q)_{\mathbb{C}} \to \mathbf{G}_m$ arbitrarily. For $\dim V$ even, the roots $e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n$ are all noncompact if and only if

$$\{I_+, I_-\} = \{\{1, 3, 5, \ldots\}, \{2, 4, \ldots\}\}$$

and modulo conjugation by $W_c = N(K, \mathbf{T}(\mathbb{R}))/\mathbf{T}(\mathbb{R})$ there are two Borel subgroups $\mathbf{B} \supset \mathbf{T}_{\mathbb{C}}$ whose simple roots are all noncompact. For $\dim V$ odd the roots $e_1 - e_2, \ldots, e_{n-1} - e_n, e_n$ are all noncompact if and only if

$$I_- = \{n, n-2, n-4, \ldots\} \text{ and } I_+ = \{n-1, n-3, \ldots\}$$

and there is just one conjugacy class of such Borel subgroups. In both cases

$$\ker\left(H^1(\mathbb{R}, \mathbf{T}) \to H^1(\mathbb{R}, \mathbf{G})\right)$$

is isomorphic to the set of $(\epsilon_i)_{1 \leq i \leq n}$ where $\epsilon_i \in \{\pm 1\}$ and

$$\mathrm{card}\{i \in I_+ \mid \epsilon_i = -1\} = \mathrm{card}\{i \in I_- \mid \epsilon_i = -1\}.$$

For the symplectic group $\mathbf{G} = \mathbf{Sp}(V, a)$ (where a is a non-degenerate alternate form) $H^1(\mathbb{R}, \mathbf{G})$ is trivial, so that the set of $\langle \pi, \cdot \rangle$ ($\pi \in \Pi_\varphi$) is simply the whole group $S_\varphi^\wedge$. However, for the non-tempered case and for the application to Siegel modular forms it will be necessary to have an explicit description of the pairs $(\mathbf{B}, \mathbf{T})$ as for the special orthogonal groups. There exists $J \in \mathbf{G}(\mathbb{R})$ such that $J^2 = -\mathrm{Id}$ and for any $v \in V \smallsetminus \{0\}$, $a(Jv, v) > 0$. Then J is a complex structure on V and

$$h(v_1, v_2) := a(Jv_1, v_2) + i a(v_1, v_2)$$

defines a positive definite hermitian form h on V. Choose an orthogonal (for h) decomposition $V = \bigoplus_{i=1}^n P_i$ where each P_i is a complex line, then we can define $\mathbf{T}$ as the stabiliser of this decomposition. The maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ containing $\mathbf{T}(\mathbb{R})$ is $K = \mathbf{U}(V, h)(\mathbb{R})$, and $W_c \simeq S_n$. Thanks to the complex structure there are canonical isomorphisms $e_i : \mathbf{U}(P_i, h) \to \mathbf{U}_1$ (for $i \in \{1, \ldots, n\}$). Modulo conjugation by W_c, the two Borel subgroups containing $\mathbf{T}_{\mathbb{C}}$ and having non-compact simple roots correspond to the sets of simple roots

$$\{e_1 + e_2, -e_2 - e_3, \ldots, (-1)^n(e_{n-1} + e_n), (-1)^{n+1} 2e_n\},$$

$$\{-e_1 - e_2, e_2 + e_3, \ldots, (-1)^{n-1}(e_{n-1} + e_n), (-1)^n 2e_n\}.$$

3.4.2.2 Adams-Johnson packets and Euler-Poincaré characteristics

Let us now consider the general case, which as we observed above is necessary only when the dominant weight λ is not regular. For a quasisplit special orthogonal or symplectic group $\mathbf{G}$ and an Arthur parameter $\psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$ having infinitesimal character $\lambda + \rho$, we would like to describe explicitly the multiset Π_ψ along with the map $\Pi_\psi \to$

$S_\psi^\wedge$. We would also like to compute the Euler-Poincaré characteristic $\mathrm{EP}(\pi \otimes V_\lambda^*)$ for any $\pi \in \Pi_\psi$. Unfortunately it does not seem possible to achieve these tasks directly from Arthur's characterisation [Art13][Theorem 2.2.1]. We will review Adams and Johnson's construction of packets Π_ψ^{AJ} using Arthur's formulation, which will lead us naturally to Assumption 3.4.2.4 relating Arthur's Π_ψ with Π_ψ^{AJ}. This review was done in [Art89b], [Kot90] and [CR14] but we need to recall Adams and Johnson's results precisely in order to compute Euler-Poincaré characteristics. Moreover we will uncover a minor problem in [Art89b][§5]. Finally, [AJ87] was written before Shahidi's conjecture [Sha90][Conjecture 9.4] was formulated, and thus we need to adress the issue of normalization of transfer factors by Whittaker datum. This is necessary to get a precise and explicit formulation of [AJ87] in our setting, which is a prerequisite for writing an algorithm.

As in the previous section $\mathbf{G}$ could be any reductive algebraic group over $\mathbb{R}$ such that $\mathbf{G}(\mathbb{R})$ has essentially discrete series. To simplify notations we assume that $\mathbf{G}$ is semisimple. To begin with, we consider general Arthur parameters $\psi : W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$, i.e. continuous morphisms such that

- composing with ${}^L\mathbf{G} \to W_\mathbb{R}$, we get $\mathrm{Id}_{W_\mathbb{R}}$,
- $\psi|_{W_\mathbb{C}}$ is semisimple and bounded,
- $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic.

As before we fix a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant splitting $(\mathcal{B}, \mathcal{T}, (\mathcal{X}_\alpha)_{\alpha\in\Delta})$ in $\widehat{\mathbf{G}}$. Assume that ψ is *pure*, i.e. the restriction of ψ to $\mathbb{R}_{>0} \subset W_\mathbb{C}$ is trivial. Otherwise ψ would factor through a Levi subgroup of ${}^L\mathbf{G}$. After conjugating by an element of $\widehat{\mathbf{G}}$ we have a $\mathcal{B}$-dominant $\tau_0 \in \frac{1}{2}X_*(\mathcal{T})$ such that for any $z \in W_\mathbb{C}$, $\psi(z) = (2\tau_0)(z/|z|)$. The set of roots $\alpha \in R(\mathcal{T}, \widehat{\mathbf{G}})$ such that $\langle \tau_0, \alpha\rangle \geq 0$ defines a parabolic subgroup $\mathcal{Q} = \mathcal{L}\mathcal{U}$ of $\widehat{\mathbf{G}}$ with Levi $\mathcal{L} = \mathrm{Cent}(\psi(W_\mathbb{C}), \widehat{\mathbf{G}})$ and $\psi(\mathrm{SL}_2(\mathbb{C})) \subset \mathcal{L}_{\mathrm{der}}$. After conjugating we can assume that

$$z \in \mathbb{C}^\times \mapsto \psi\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})\right)$$

takes values in $\mathcal{T} \cap \mathcal{L}_{\mathrm{der}}$ and is dominant with respect to $\mathcal{B} \cap \mathcal{L}_{\mathrm{der}}$. Let us restrict our attention to parameters ψ such that $\psi|_{\mathrm{SL}_2(\mathbb{C})} : \mathrm{SL}_2(\mathbb{C}) \to \mathcal{L}_{\mathrm{der}}$ is the principal morphism. After conjugating we can assume that

$$\mathrm{d}\left(\psi|_{\mathrm{SL}_2(\mathbb{C})}\right)\left(\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} \in \mathfrak{sl}_2\right) = \sum_{\alpha\in\Delta_\mathcal{L}} \mathcal{X}_\alpha.$$

We claim that $\psi(j) \in \widehat{\mathbf{G}} \rtimes \{j\}$ is now determined modulo left multiplication by $Z(\mathcal{L})$. Let $n : W(\widehat{\mathbf{G}}, \mathcal{T}) \rtimes W_\mathbb{R} \to N({}^L\mathbf{G}, \mathcal{T}) = N(\widehat{\mathbf{G}}, \mathcal{T}) \rtimes W_\mathbb{R}$ be the set-theoretic section defined in [LS87][§2.1]. Let $w_0 \in W(\widehat{\mathbf{G}}, \mathcal{T})$ be the longest element in the Weyl group (with respect to $\mathcal{B}$). Since $\mathbf{G}$ has an anisotropic maximal torus, conjugation by (any representative of) $w_0 \rtimes j$ acts by $t \mapsto t^{-1}$ on $\mathcal{T}$. Let w_1 be the longest element of the Weyl group $W(\mathcal{L}, \mathcal{T})$. Then $w_1 w_0 \rtimes j$ preserves $\Delta_\mathcal{L}$ and acts by $t \mapsto t^{-1}$ on $Z(\mathcal{L})$. By [Spr98][Proposition 9.3.5] $n(w_1 w_0 \rtimes j) = n(w_1 w_0) \rtimes j$ preserves the splitting $(\mathcal{X}_\alpha)_{\alpha\in\Delta_\mathcal{L}}$, and thus commutes with $\psi(\mathrm{SL}_2(\mathbb{C}))$. The following lemma relates $\psi(j)$ and $n(w_1 w_0 \rtimes j)$.

Lemma 3.4.2.2. *There is a unique element $a \in Z(\mathcal{L})\backslash\left(\widehat{\mathbf{G}} \rtimes \{j\}\right)$ commuting with $\psi(\mathrm{SL}_2(\mathbb{C}))$ and such that for any $z \in W_{\mathbb{C}}$, $a\psi(z)a^{-1} = \psi(z^{-1})$.*

Proof. If a and b are two such elements, $ab^{-1} \in \widehat{\mathbf{G}}$ commutes with $\psi(W_{\mathbb{C}})$, thus $ab^{-1} \in \mathcal{L}$. Furthermore ab commutes with $\psi(\mathrm{SL}_2(\mathbb{C}))$, hence $ab^{-1} \in Z(\mathcal{L})$. □

Since $n(w_1 w_0 \rtimes j)$ and $\psi(j)$ satisfy these two conditions, they coincide modulo $Z(\mathcal{L})$. In particular conjugation by $\psi(j)$ acts by $t \mapsto t^{-1}$ on $Z(\mathcal{L})$, and thus the group

$$C_\psi := \mathrm{Cent}(\psi, \widehat{\mathbf{G}}) = \{t \in Z(\mathcal{L}) \mid t^2 = 1\}$$

is finite, and so is $S_\psi := C_\psi / Z(\widehat{\mathbf{G}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$. In addition, $(2\tau_0)(-1) = \psi(j)^2 = n(w_1 w_0 \rtimes j)^2$ only depends on $\mathcal{L}$. By [LS87][Lemma 2.1.A], $n(w_1 w_0 \rtimes j)^2 = \prod_{\alpha \in R_{\mathcal{Q}}} \alpha^\vee(-1)$ where $R_{\mathcal{Q}}$ is the set of roots of $\mathcal{T}$ occurring in the unipotent radical $\mathcal{U}$ of $\mathcal{Q}$. Thus

$$\tau_0 \in X_*(Z(\mathcal{L})^0) + \frac{1}{2} \sum_{\alpha \in R_{\mathcal{Q}}} \alpha^\vee.$$

Conversely, using the element $n(w_1 w_0 \rtimes j)$ we see that for any standard parabolic subgroup $\mathcal{Q} = \mathcal{L}\mathcal{U} \supset \mathcal{B}$ of $\widehat{\mathbf{G}}$ and any strictly dominant (for $R_{\mathcal{Q}}$) $\tau_0 \in X_*(Z(\mathcal{L})^0) + \frac{1}{2}\sum_{\alpha \in R_{\mathcal{Q}}} \alpha^\vee$, there is at least one Arthur parameter mapping $z \in W_{\mathbb{C}}$ to $(2\tau_0)(z/|z|)$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2$ to $\sum_{\alpha \in \Delta_{\mathcal{L}}} \mathcal{X}_\alpha$. Finally, for any $u \in Z(\mathcal{L})$, we can form another Arthur parameter ψ' by imposing $\psi'|_{W_{\mathbb{C}} \times \mathrm{SL}_2(\mathbb{C})} = \psi|_{W_{\mathbb{C}} \times \mathrm{SL}_2(\mathbb{C})}$ and $\psi'(j) = u\psi(j)$. It follows that the set of conjugacy classes of Arthur parameters ψ' such that $\psi'|_{W_{\mathbb{C}} \times \mathrm{SL}_2(\mathbb{C})}$ is conjugated to $\psi|_{W_{\mathbb{C}} \times \mathrm{SL}_2(\mathbb{C})}$ is a torsor under

$$Z(\mathcal{L})/\{t^2 \mid t \in Z(\mathcal{L})\} = H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), Z(\mathcal{L})) \text{ where } \sigma \text{ acts by } w_1 w_0 \rtimes j \text{ on } Z(\mathcal{L}).$$

Recall the norm $||\cdot|| : W_{\mathbb{R}} \to \mathbb{R}_{>0}$ which maps j to 1 and $z \in W_{\mathbb{C}}$ to $z\bar{z}$, which is used to define the morphism $W_{\mathbb{R}} \to W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ mapping w to

$$\left(w, \begin{pmatrix} ||w||^{1/2} & 0 \\ 0 & ||w||^{-1/2} \end{pmatrix}\right).$$

Composing ψ with this morphism we get a Langlands parameter $\varphi_\psi : W_{\mathbb{R}} \to {}^L\mathbf{G}$ which is not tempered in general. For $z \in W_{\mathbb{C}}$, $\varphi_\psi(z) = (\tau - \tau')(z/|z|)(\tau + \tau')(|z|)$ (formally $\tau(z)\tau'(\bar{z})$) where

$$\tau = \tau_0 + \frac{1}{2} \sum_{\alpha \in R_{\mathcal{B} \cap \mathcal{L}}} \alpha^\vee \quad \text{and} \quad \tau' = -\tau_0 + \frac{1}{2} \sum_{\alpha \in R_{\mathcal{B} \cap \mathcal{L}}} \alpha^\vee.$$

Then $\tau \in \frac{1}{2}\sum_{\alpha \in R_{\mathcal{B}}} \alpha^\vee + X_*(\mathcal{T})$ and the following are equivalent:

1. τ is regular,
2. $\tau - \frac{1}{2}\sum_{\alpha \in R_{\mathcal{B}}} \alpha^\vee$ is dominant with respect to $R_{\mathcal{B}}$,
3. $\tau_0 - \frac{1}{2}\sum_{\alpha \in R_{\mathcal{Q}}} \alpha^\vee$ is dominant with respect to $R_{\mathcal{Q}}$.

In fact for any pure Arthur parameter ψ, without assuming a priori that $\psi|_{\mathrm{SL}_2(\mathbb{C})} \to \mathcal{L}$ is principal, if the holomorphic part τ of $\varphi_\psi|_{W_\mathbb{C}}$ is regular, then $\psi|_{\mathrm{SL}_2(\mathbb{C})} \to \mathcal{L}$ is principal. The orbit of τ under the Weyl group is the infinitesimal character associated with ψ, and we have seen that it is the infinitesimal character of any representation in the packet Π_ψ associated with ψ (Lemma 3.4.1.3). For quasisplit special orthogonal or symplectic groups we checked this (up to outer conjugacy in the even orthogonal case) in Lemma 3.4.1.3.

From now on we also assume that the infinitesimal character τ of ψ is regular. Note that τ is then the infinitesimal character of the restriction to $\mathbf{G}(\mathbb{R})$ of the irreducible algebraic representation V_λ of $\mathbf{G}_\mathbb{C}$, where $\tau = \lambda + \rho$. Let us describe the set of representations Π_ψ^{AJ} that Adams and Johnson associate with ψ as well as the pairing $\Pi_\psi \to S_\psi^\wedge$. To be honest Adams and Johnson do not consider parameters ψ, they only work with representations, but [Art89b][§5] interpreted their construction in terms of parameters. We will only add details concerning Whittaker normalisation. As in the tempered case we begin by considering pairs $(\mathbf{B}, \mathbf{T})$ where $\mathbf{T}$ is an anisotropic maximal torus of $\mathbf{G}$ and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}_\mathbb{C}$ containing $\mathbf{T}_\mathbb{C}$. We have a canonical isomorphism between the based root data

$$(X^*(\mathbf{T}_\mathbb{C}), \Delta_\mathbf{B}, X_*(\mathbf{T}_\mathbb{C}), \Delta_\mathbf{B}^\vee) \quad \text{and} \quad (X_*(\mathcal{T}), \Delta_\mathcal{B}^\vee, X^*(\mathcal{T}), \Delta_\mathcal{B})$$

and we can associate with $(\mathcal{Q}, \mathcal{L})$ a parabolic subgroup $\mathbf{Q} \supset \mathbf{B}$ of $\mathbf{G}_\mathbb{C}$ and a Levi subgroup $\mathbf{L}_\mathbb{C} \supset \mathbf{T}_\mathbb{C}$ of $\mathbf{G}_\mathbb{C}$. As the notation suggests $\mathbf{L}_\mathbb{C}$ is defined over $\mathbb{R}$ (for any root α of $\mathbf{T}_\mathbb{C}$ in $\mathbf{G}_\mathbb{C}$, $\sigma(\alpha) = -\alpha$), and we denote this real subgroup of $\mathbf{G}$ by $\mathbf{L}$. Consider the set $\Sigma_\mathcal{Q}$ of conjugacy classes of pairs $(\mathbf{Q}, \mathbf{L})$ ($\mathbf{Q}$ a parabolic subgroup of $\mathbf{G}_\mathbb{C}$ and $\mathbf{L}$ a real subgroup of $\mathbf{G}$ such that $\mathbf{L}_\mathbb{C}$ is a Levi subgroup of $\mathbf{Q}$) obtained this way. The finite set $\Sigma_\mathcal{B}$ of conjugacy classes of pairs $(\mathbf{B}, \mathbf{T})$ surjects to $\Sigma_\mathcal{Q}$. If we fix a base point $(\mathbf{B}, \mathbf{T})$, we have seen that $\Sigma_\mathcal{B}$ is identified with $W_c \backslash W$. This base point allows to identify $\Sigma_\mathcal{Q}$ with $W_c \backslash W / W_\mathbf{L}$ where $W_\mathbf{L} = W(\mathbf{L}(\mathbb{C}), \mathbf{T}(\mathbb{C}))$, and

$$W_c \backslash W / W_\mathbf{L} \simeq \ker\left(H^1(\mathbb{R}, \mathbf{L}) \to H^1(\mathbb{R}, \mathbf{G})\right).$$

For any $\mathrm{cl}(\mathbf{Q}, \mathbf{L}) \in \Sigma_\mathcal{Q}$ there is a canonical isomorphism $\widehat{\mathbf{L}} \simeq \mathcal{L}$ identifying the splittings. Given another $\mathrm{cl}(\mathbf{Q}', \mathbf{L}') \in \Sigma_\mathcal{Q}$, there is a unique $g \in \mathbf{G}(\mathbb{C})/\mathbf{L}(\mathbb{C})$ conjugating $(\mathbf{Q}, \mathbf{L})$ into $(\mathbf{Q}', \mathbf{L}')$, yielding a canonical isomorphism of L-groups ${}^L\mathbf{L} \simeq {}^L\mathbf{L}'$. As in the tempered case we want to extend $\widehat{\mathbf{L}} \simeq \mathcal{L}$ into an embedding $\iota : {}^L\mathbf{L} \to {}^L\mathbf{G}$ as follows. For $z \in W_\mathbb{C}$, define $\iota(z) = \prod_{\alpha \in R_\mathcal{Q}} \alpha^\vee(z/|z|) \rtimes z$. Define $\iota(j) = n(w_1 w_0 \rtimes j)$. We have computed $n(w_1 w_0 \rtimes j)^2 = \prod_{\alpha \in R_\mathcal{Q}} \alpha^\vee(-1)$ above and thus ι is well-defined. Note that contrary to the tempered case, there are other choices for $\iota(j)$ even up to conjugation by $Z(\mathcal{L})$: we could replace $\iota(j)$ by $u\iota(j)$ where $u \in Z(\mathcal{L})$, and it can happen that u is not a square in $Z(\mathcal{L})$. This issue seems to have been overlooked in [Art89b][§5]. We will not try to determine whether $n(w_1 w_0 \rtimes j)$ is the correct choice here and we will consider this problem in a separate note, since for our present purpose this choice does not matter.

For any class $\mathrm{cl}(\mathbf{Q}, \mathbf{L}) \in \Sigma_\mathcal{Q}$ there is a unique Arthur parameter

$$\psi_{\mathbf{Q},\mathbf{L}} : W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{L}$$

such that up to conjugation by $\widehat{\mathbf{G}}$, $\psi = \iota \circ \psi_{\mathbf{Q},\mathbf{L}}$. Now $\psi_{\mathbf{Q},\mathbf{L}}|_{\mathrm{SL}_2(\mathbb{C})} : \mathrm{SL}_2(\mathbb{C}) \to \widehat{\mathbf{L}}$ is the principal morphism. Thus $\psi_{\mathbf{Q},\mathbf{L}}|_{W_\mathbb{R}}$ takes values in $Z(\widehat{\mathbf{L}}) \rtimes W_\mathbb{R}$, and the conjugacy class of $\psi_{\mathbf{Q},\mathbf{L}}$ is determined by the resulting element of $H^1(W_\mathbb{R}, Z(\widehat{\mathbf{L}}))$, which has compact image.

Recall that for any real reductive group $\mathbf{H}$ there is a natural morphism

$$\nu_\mathbf{H} : H^1(W_\mathbb{R}, Z(\widehat{\mathbf{H}})) \to \mathrm{Hom}_{\mathrm{cont}}(\mathbf{H}(\mathbb{R}), \mathbb{C}^\times)$$

which is surjective and maps cocyles with compact image to unitary characters of $\mathbf{H}(\mathbb{R})$. To define this morphism we can use the same arguments as [Kot86][§1]. If $\mathbf{H}$ is simply connected, then $\widehat{\mathbf{H}}$ is adjoint and $\mathbf{H}(\mathbb{R})$ is connected. More generally, if $\mathbf{H}_{\mathrm{der}}$ is simply connected then the torus $\mathbf{C} = \mathbf{H}/\mathbf{H}_{\mathrm{der}}$ is such that $Z(\widehat{\mathbf{H}}) = \widehat{\mathbf{C}}$ and

$$\mathbf{H}(\mathbb{R})^{\mathrm{ab}} = \ker\left(\mathbf{C}(\mathbb{R}) \to H^1(\mathbb{R}, \mathbf{H}_{\mathrm{der}})\right).$$

Finally if $\mathbf{H}$ is arbitrary there exists a z-extension $\mathbf{C} \hookrightarrow \widetilde{\mathbf{H}} \twoheadrightarrow \mathbf{H}$ where $\mathbf{C}$ is an induced torus and $\widetilde{\mathbf{H}}_{\mathrm{der}}$ is simply connected. Then $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbf{C}(\mathbb{C}))$ is trivial, thus $\widetilde{\mathbf{H}}(\mathbb{R}) \twoheadrightarrow \mathbf{H}(\mathbb{R})$ and

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbf{H}(\mathbb{R}), \mathbb{C}^\times) = \ker\left(\mathrm{Hom}_{\mathrm{cont}}(\widetilde{\mathbf{H}}(\mathbb{R}), \mathbb{C}^\times) \to \mathrm{Hom}_{\mathrm{cont}}(\mathbf{C}(\mathbb{R}), \mathbb{C}^\times)\right).$$

Parallelly, $\widehat{\mathbf{C}}^{W_\mathbb{R}}$ is connected so that $\widehat{\mathbf{C}}^{W_\mathbb{R}} \to H^1(W_\mathbb{R}, Z(\widehat{\mathbf{H}}))$ is trivial and thus

$$H^1(W_\mathbb{R}, Z(\widehat{\mathbf{H}})) = \ker\left(H^1(W_\mathbb{R}, Z(\widehat{\widetilde{\mathbf{H}}})) \to H^1(W_\mathbb{R}, \widehat{\mathbf{C}})\right).$$

As in [Kot86][§1] the morphism $\nu_\mathbf{H}$ obtained this way does not depend on the choice of a z-extension. Note that when $\mathbf{H}$ is quasi-split, $\nu_\mathbf{H}$ is an isomorphism, by reduction to the case where $\mathbf{H}_{\mathrm{der}}$ is simply connected and using the fact that a maximally split maximal torus in a simply connected quasi-split group is an induced torus. It is not injective in general, e.g. when $\mathbf{H}$ is the group of invertible quaternions.

Hence $\psi_{\mathbf{Q},\mathbf{L}}$ defines a one-dimensional unitary representation $\pi^0_{\psi,\mathbf{Q},\mathbf{L}}$ of $\mathbf{L}(\mathbb{R})$, and applying cohomological induction as defined by Zuckerman, Adams and Johnson define the representation $\pi_{\psi,\mathbf{Q},\mathbf{L}} = R^i_\mathfrak{q}(\pi^0_{\psi,\mathbf{Q},\mathbf{L}})$ of $\mathbf{G}(\mathbb{R})$, where $\mathfrak{q} = \mathrm{Lie}(\mathbf{Q})$ and $i = q(\mathbf{G}) - q(\mathbf{L})$. Vogan has shown that this representation is unitary. They define the set Π^{AJ}_ψ in bijection with $\Sigma_\mathcal{Q}$:

$$\Pi^{\mathrm{AJ}}_\psi = \{\pi_{\psi,\mathbf{Q},\mathbf{L}} \mid \mathrm{cl}(\mathbf{Q},\mathbf{L}) \in \Sigma_\mathcal{Q}\}.$$

The endoscopic character relations that they prove [AJ87][Theorem 2.21] allow to identify the map $\Pi_\psi \to S_\psi^\wedge$, as Arthur did in [Art89b][§5]. Assume that $\mathbf{G}$ is quasisplit (this is probably unnecessary as in the tempered case using the constructions of [Kal]), and fix a Whittaker datum for $\mathbf{G}$. Then any $\mathrm{cl}(\mathbf{B},\mathbf{T}) \in \Sigma_\mathcal{B}$ determines an element of $S_\varphi^\wedge$ (here φ could be any discrete parameter, the group S_φ is described in terms of $\mathcal{B},\mathcal{T}$ independently). It is easy to check that if $(\mathbf{B},\mathbf{T})$ and $(\mathbf{B}',\mathbf{T}')$ give rise to pairs $(\mathbf{Q},\mathbf{L})$ and $(\mathbf{Q}',\mathbf{L}')$ which are conjugated under $\mathbf{G}(\mathbb{R})$, then the restrictions to S_ψ of the characters of S_φ associated with $(\mathbf{B},\mathbf{T})$ and $(\mathbf{B}',\mathbf{T}')$ coincide. We get a map $\Pi^{\mathrm{AJ}}_\psi \to S_\psi^\wedge$ which is not injective in general.

Adams and Johnson ([AJ87][Theorem 8.2], reformulating the main result of [Joh84]) give a resolution of $\pi_{\psi,\mathbf{Q},\mathbf{L}}$ by direct sums of standard modules

$$0 \to \pi_{\psi,\mathbf{Q},\mathbf{L}} \to X^{q(\mathbf{L})} \to \cdots \to X^0 \to 0. \tag{3.4.2.1}$$

Recall that a standard module is a parabolic induction of an essentially tempered representation of a Levi subgroup of $\mathbf{G}$, with a certain positivity condition on its central character. Johnson's convention is opposite to that of Langlands, so that $\pi_{\psi,\mathbf{Q},\mathbf{L}}$ *embeds* in a standard module. Apart from its length, the only two properties of this resolution that we need are

1. X^0 is the direct sum of the discrete series representations of $\mathbf{G}(\mathbb{R})$ having infinitesimal character τ and corresponding to the $\mathrm{cl}(\mathbf{B},\mathbf{T}) \in \Sigma_{\mathcal{B}}$ mapping to $\mathrm{cl}(\mathbf{Q},\mathbf{L}) \in \Sigma_{\mathcal{Q}}$,
2. for any $i > 0$, X^i is a direct sum of standard modules induced from *proper* parabolic subgroups of $\mathbf{G}$, therefore $\mathrm{EP}(X^i \otimes V_\lambda^*) = 0$.

Thus we have the simple formula

$$\mathrm{EP}(\pi_{\psi,\mathbf{Q},\mathbf{L}} \otimes V_\lambda^*) = (-1)^{q(\mathbf{G})-q(\mathbf{L})}\mathrm{card}\left(\text{fiber of } \mathrm{cl}(\mathbf{Q},\mathbf{L}) \text{ by } \Sigma_{\mathcal{B}} \to \Sigma_{\mathcal{Q}}\right).$$

Note that $\pi_{\psi,\mathbf{Q},\mathbf{L}}$ is a discrete series representation if and only if $\mathbf{L}$ is anisotropic.

Let us be more precise about the endoscopic character relations afforded by Adams-Johnson representations, since Shahidi's conjecture was only formulated after both [AJ87] and [Art89b]. Let s_ψ be the image by ψ of $-1 \in \mathrm{SL}_2(\mathbb{C})$, which we will see as an element of S_ψ. Arthur and Kottwitz have shown that for $\mathrm{cl}(\mathbf{Q},\mathbf{L}), \mathrm{cl}(\mathbf{Q}',\mathbf{L}') \in \Sigma_{\mathcal{Q}}$, we have $\langle s_\psi, \pi_{\psi,\mathbf{Q},\mathbf{L}}\rangle = (-1)^{q(\mathbf{L})-q(\mathbf{L}')}\langle s_\psi, \pi_{\psi,\mathbf{Q}',\mathbf{L}'}\rangle$. Let $(\mathbf{B}_0,\mathbf{T}_0)$ be a pair in $\mathbf{G}$ corresponding to the base point (i.e. the generic representation for our fixed Whittaker datum) for any discrete L-packet. It determines a pair $(\mathbf{Q}_0,\mathbf{L}_0)$ such that $\mathrm{cl}(\mathbf{Q}_0,\mathbf{L}_0) \in \Sigma_{\mathcal{Q}}$. The simple roots of $\mathbf{B}_0$ are all non-compact and thus the same holds for the Borel subgroup $\mathbf{B}_0 \cap (\mathbf{L}_0)_{\mathbb{C}}$ of $(\mathbf{L}_0)_{\mathbb{C}}$. By Lemma 3.4.2.1 the group $\mathbf{L}_0$ is quasisplit. Thus for any $\mathrm{cl}(\mathbf{Q},\mathbf{L}) \in \Sigma_{\mathcal{Q}}$ we have $\langle s_\psi, \pi_{\psi,\mathbf{Q},\mathbf{L}}\rangle = (-1)^{q(\mathbf{L}_0)-q(\mathbf{L})}$. Note that if $(\mathbf{B}_1,\mathbf{T}_1)$ corresponds to the generic element in tempered L-packets for *another* Whittaker datum, the pair $(\mathbf{L}_1,\mathbf{Q}_1)$ that it determines also has the property that $\mathbf{L}_1$ is quasisplit. Since $\mathbf{L}_0$ and $\mathbf{L}_1$ are inner forms of each other, they are isomorphic and $q(\mathbf{L}_0) = q(\mathbf{L}_1)$. This shows that the map

$$f(g)dg \mapsto \sum_{\pi \in \Pi_\psi^{\mathrm{AJ}}} \langle s_\psi, \pi\rangle \mathrm{Tr}\left(\pi(f(g)dg)\right),$$

defined on smooth compactly supported distributions on $\mathbf{G}(\mathbb{R})$, is canonical: it does not depend on the choice of a Whittaker datum for the quasisplit group $\mathbf{G}$. By [AJ87][Theorem 2.13] it is *stable*, i.e. it vanishes if all the stable orbital integrals of $f(g)dg$ vanish. Consider an arbitrary element $x \in S_\psi$. It determines an endoscopic group $\mathbf{H}$ of $\mathbf{G}$ and an Arthur parameter $\psi_{\mathbf{H}} : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{H}$ whose infinitesimal character is regular. Thanks to the choice of a Whittaker datum we have a well-defined *transfer map* $f(g)dg \mapsto f^{\mathbf{H}}(h)dh$ from smooth compactly supported distributions on $\mathbf{G}(\mathbb{R})$ to smooth compactly supported distributions on $\mathbf{H}(\mathbb{R})$. Adams and Johnson have proved [AJ87][Theorem 2.21] that there is some $t \in \mathbb{C}^\times$ such that

$$\sum_{\pi \in \Pi_\psi^{\mathrm{AJ}}} \langle s_\psi x, \pi \rangle \mathrm{Tr}\left(\pi(f(g)dg)\right) = t \sum_{\pi \in \Pi_{\psi_{\mathbf{H}}}^{\mathrm{AJ}}} \langle s_\psi, \pi \rangle \mathrm{Tr}\left(\pi(f^{\mathbf{H}}(h)dh)\right) \tag{3.4.2.2}$$

for any smooth compactly supported distribution $f(g)dg$ on $\mathbf{G}(\mathbb{R})$. We check that $t = 1$. Let $\varphi : W_{\mathbb{R}} \to {}^L\mathbf{G}$ be the discrete Langlands parameter having infinitesimal character τ. Conjugating if necessary, we can assume that the holomorphic parts of $\varphi|_{W_{\mathbb{C}}}$ and $\varphi_\psi|_{W_{\mathbb{C}}}$ are equal and not just conjugated. In this way we see S_ψ as a subgroup of S_φ. We restrict to distributions $f(g)dg$ whose support is contained in the set of semisimple regular elliptic elements of $\mathbf{G}(\mathbb{R})$. In that case by Johnson's resolution 3.4.2.1

$$\begin{aligned}
\sum_{\pi \in \Pi_\psi^{\mathrm{AJ}}} \langle s_\psi x, \pi \rangle \mathrm{Tr}\left(\pi(f(g)dg)\right) &= (-1)^{q(\mathbf{L}_0)} \sum_{\pi \in \Pi_\varphi} \langle x, \pi \rangle \mathrm{Tr}\left(\pi(f(g)dg)\right) \\
&= (-1)^{q(\mathbf{L}_0)} \sum_{\pi \in \Pi_{\varphi_{\mathbf{H}}}} \mathrm{Tr}\left(\pi(f^{\mathbf{H}}(h)dh)\right)
\end{aligned}$$

where the second equality is the endoscopic character relation for (φ, x). Let $(\mathbf{B}_0^{\mathbf{H}}, \mathbf{T}_0^{\mathbf{H}})$ be a pair for $\mathbf{H}$ such that the simple roots of $\mathbf{B}_0^{\mathbf{H}}$ are all non-compact. Then the pair $(\mathbf{Q}_0^{\mathbf{H}}, \mathbf{L}_0^{\mathbf{H}})$ that it determines is such that $\mathbf{L}_0^{\mathbf{H}}$ is quasisplit and has same Langlands dual group as $\mathbf{L}_0$, thus $\mathbf{L}_0^{\mathbf{H}} \simeq \mathbf{L}_0$. In particular $q(\mathbf{L}_0^{\mathbf{H}}) = q(\mathbf{L}_0)$ and

$$(-1)^{q(\mathbf{L}_0)} \sum_{\pi \in \Pi_{\varphi_{\mathbf{H}}}} \mathrm{Tr}\left(\pi(f^{\mathbf{H}}(h)dh)\right) = \sum_{\pi \in \Pi_{\psi_{\mathbf{H}}}^{\mathrm{AJ}}} \langle s_\psi, \pi \rangle \mathrm{Tr}\left(\pi(f^{\mathbf{H}}(h)dh)\right).$$

Therefore the endoscopic character relation 3.4.2.2 holds with $t = 1$ for such distributions $f(g)dg$. By choosing $f(g)dg$ positive with small support around a well-chosen semisimple regular elliptic element we can ensure that both sides do not vanish, so that $t = 1$.

This concludes the precise determination of the map $\pi \mapsto \langle \cdot, \pi \rangle$, normalised using Whittaker datum as in the tempered case. Note that this normalised version of [AJ87][Theorem 2.21] is completely analogous to [Art13][Theorem 2.2.1(b)]. We are led to make the following assumption.

Assumption 3.4.2.3. *Let $\mathbf{G}$ be a quasisplit special orthogonal or symplectic group over $\mathbb{R}$ having discrete series. Fix a Whittaker datum for $\mathbf{G}$. Let ψ be an Arthur parameter for $\mathbf{G}$ with regular infinitesimal character $\tau = \lambda + \rho$. Then for any $\chi \in S_\psi^\wedge$,*

$$\bigoplus_{\substack{\pi \in \Pi_\psi^{\mathrm{AJ}} \\ \langle \cdot, \pi \rangle = \chi}} \pi \simeq \bigoplus_{\substack{\pi \in \Pi_\psi \\ \langle \cdot, \pi \rangle = \chi}} \pi. \tag{3.4.2.3}$$

Note that in the even orthogonal case, this only assumes an isomorphism of $\mathcal{H}'(\mathbf{G}(\mathbb{R}))$-modules.

To compute Euler-Poincaré characteristics we only need the character of the direct sum appearing in Assumption 3.4.2.3 on an anisotropic maximal torus. This follows from the fact that the standard modules form a basis of the Grothendieck group of finite length $(\mathfrak{g}, K)$-modules. Using also the fact that Arthur and Adams-Johnson packets satisfy the same endoscopic relations, we can formulate a weaker assumption which is enough to compute the Euler-Poincaré characteristic of the right hand side of 3.4.2.3 for any $\chi \in S_\psi^\wedge$.

Assumption 3.4.2.4. *Let* $\mathbf{G}$ *be a quasisplit special orthogonal or symplectic group over* $\mathbb{R}$ *having discrete series. Let* ψ *be an Arthur parameter for* $\mathbf{G}$ *with regular infinitesimal character* $\tau = \lambda + \rho$*, and let* $\mathbf{T}$ *be a maximal torus of* $\mathbf{G}$ *which is anisotropic. Let* $\mathbf{L}_0$ *denote the quasisplit reductive group defined in the discussion above. If* $\mathbf{G}$ *is symplectic or odd orthogonal, the assumption is that for any* $\gamma \in \mathbf{T}_{\mathrm{reg}}(\mathbb{R})$*,*

$$\sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi \rangle \Theta_\pi(\gamma) = (-1)^{q(\mathbf{G})-q(\mathbf{L}_0)} \mathrm{Tr}(\gamma | V_\lambda).$$

In the even orthogonal case, this identity takes the following meaning. Let $\gamma \in \mathbf{T}_{\mathrm{reg}}(\mathbb{R})$ *and consider a* $\gamma' \in \mathbf{G}(\mathbb{R})$ *outer conjugated to* γ*. For* π *in* Π_ψ*, which is only an* $\mathrm{Out}(\mathbf{G})$*-orbit of representations, we still denote by* π *any element of this orbit. The assumption is*

$$\sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi \rangle \left(\Theta_\pi(\gamma) + \Theta_\pi(\gamma')\right) = (-1)^{q(\mathbf{G})-q(\mathbf{L}_0)} \left(\mathrm{Tr}(\gamma | V_\lambda) + \mathrm{Tr}(\gamma' | V_\lambda)\right).$$

Of course it does not depend on the choice made in each orbit.

Thus under this assumption we have an algorithm to compute inductively the cardinality of each $\Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}$.

Remark 3.4.2.5. *For this algorithm it is not necessary to enumerate the sets*

$$W_c \backslash W / W_{\mathbf{L}} \simeq \ker\left(H^1(\mathbb{R}, \mathbf{L}) \to H^1(\mathbb{R}, \mathbf{G})\right)$$

parametrizing the elements of each Π_ψ*. It is enough to compute, for each discrete series* π *represented by a collection of signs as in the previous section, the restriction of* $\langle \cdot, \pi \rangle$ *to* S_ψ *and the sign* $(-1)^{q(\mathbf{L})}$*.*

See the tables in section 3.7.2 for some values for card $\left(\Psi(\mathbf{G})_{\mathrm{sim}}^{\mathrm{unr},\lambda}\right)$ in low weight λ ordered lexicographically.

3.5 Application to vector-valued Siegel modular forms

Let us give a classical application of the previous results, to the computation of dimensions of spaces $S_r(\Gamma_n)$ of vector-valued Siegel cusp forms in genus $n \geq 1$, weight r and level one. It is certainly well-known that, under a natural assumption on the weight r, this dimension is equal to the multiplicity in $L^2_{\mathrm{disc}}(\mathbf{PGSp}_{2n}(\mathbb{Q}) \backslash \mathbf{PGSp}_{2n}(\mathbb{A}) / \mathbf{PGSp}_{2n}(\widehat{\mathbb{Z}}))$ of the holomorphic discrete series representation corresponding to r. Although [AS01] contains "half" of the argument, we could not find a complete reference for the full statement. To set our mind at rest we give details for the other half. We begin with a review of holomorphic discrete series. We do so even though it is redundant with [Kna86] and [AS01], in order to give precise references, to set up notation and to identify the holomorphic discrete series in Shelstad's parametrisation.

Note that it is rather artificial to restrict our attention to symplectic groups. For any $n \geq 3$ such that $n \neq 2 \mod 4$, the split group $\mathbf{G} = \mathbf{SO}_n$ has an inner form $\mathbf{H}$ which is split at all the finite places of $\mathbb{Q}$ and such that

- if $n = -1, 0, 1 \mod 8$, $\mathbf{H}(\mathbb{R})$ is compact,
- if $n = 3, 4, 5 \mod 8$, $\mathbf{H}(\mathbb{R}) \simeq \mathrm{SO}(n-2, 2)$.

In the second case $\mathbf{H}(\mathbb{R})$ has holomorphic discrete series which can be realised on a hermitian symmetric space of complex dimension $n-2$. In the first case $\mathbf{H}(\mathbb{R})$ also has holomorphic discrete series which can be realised on a zero-dimensional hermitian symmetric space.

3.5.1 Bounded symmetric domains of symplectic type and holomorphic discrete series

Let us recall Harish-Chandra's point of view on bounded symmetric domains and his construction of holomorphic discrete series (see [Bru59], [HC55], [HC56a], [HC56b]) in the case of symplectic groups. Let $n \geq 1$ and $\mathbf{G} = \mathbf{Sp}_{2n}$, over $\mathbb{R}$ in this section, and denote $G = \mathbf{G}(\mathbb{R})$, $\mathfrak{g}_0 = \mathrm{Lie}(G)$ and $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$. Then $\mathbf{G}$ is the stabiliser of a non-degenerate alternate form a on a $2n$-dimensional real vector space V. As before choose $J \in G$ such that $J^2 = -1$ and for any $v \in V \smallsetminus \{0\}$, $a(Jv, v) > 0$, which endows V with a complex structure and realises a as the imaginary part of the positive definite hermitian form h defined by

$$h(v_1, v_2) = a(Jv_1, v_2) + ia(v_1, v_2).$$

Then $\mathbf{K} = \mathbf{U}(V, h)$ is a reductive subgroup of $\mathbf{G}$, and $K = \mathbf{K}(\mathbb{R})$ is a maximal compact subgroup of G. Note that both G and K are connected. The center $\mathbf{Z_K}$ of $\mathbf{K}$ is one-dimensional and anisotropic, and the complex structure J yields a canonical isomorphism $\mathbf{Z_K} \simeq \mathbf{U}_1$. Let $\mathfrak{u}_+$ (resp. $\mathfrak{u}_-$) be the subspace of $\mathfrak{g}$ such that the adjoint action of $z \in \mathbf{Z_K}(\mathbb{R})$ on $\mathfrak{u}_+$ (resp. $\mathfrak{u}_-$) is by multiplication by z^2 (resp. z^{-2}). Then $\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{k} \oplus \mathfrak{u}_-$ and $[\mathfrak{u}_+, \mathfrak{u}_+] = [\mathfrak{u}_-, \mathfrak{u}_-] = 0$. Moreover $\mathfrak{u}_+ \oplus \mathfrak{u}_- = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{p}_0$ where $\mathfrak{p}_0$ is the subspace of $\mathfrak{g}_0 = \mathrm{Lie}(G)$ on which J acts by -1, i.e. $\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0$ is the Cartan decomposition of $\mathfrak{g}_0$ for the Cartan involution $\theta = \mathrm{Ad}(J)$. There are unipotent abelian subgroups $\mathbf{U}_+, \mathbf{U}_-$ of $\mathbf{G}_{\mathbb{C}}$ associated with $\mathfrak{u}_+, \mathfrak{u}_-$, and the subgroups $\mathbf{K}_{\mathbb{C}}\mathbf{U}_+$ and $\mathbf{K}_{\mathbb{C}}\mathbf{U}_-$ are opposite parabolic subgroups of $\mathbf{G}_{\mathbb{C}}$ with common Levi subgroup $\mathbf{K}_{\mathbb{C}}$. It follows that the multiplication map $\mathbf{U}_+ \times \mathbf{K}_{\mathbb{C}} \times \mathbf{U}_- \to \mathbf{G}_{\mathbb{C}}$ is an open immersion. Furthermore $G \subset \mathbf{U}_+(\mathbb{C})\mathbf{K}(\mathbb{C})\mathbf{U}_-(\mathbb{C})$. For $g \in G$, we can thus write $g = g_+ g_0 g_-$ where $(g_+, g_0, g_-) \in \mathbf{U}_+(\mathbb{C}) \times \mathbf{K}(\mathbb{C}) \times \mathbf{U}_-(\mathbb{C})$, and Harish-Chandra showed that $g \mapsto \log(g_+)$ identifies G/K with a bounded domain $D \subset \mathfrak{u}_+$. This endows G/K with a structure of complex manifold, and for any $g \in G$, left multiplication by g yields a holomorphic map $G/K \to G/K$.

Remark 3.5.1.1. *Let us compare this point of view with the classical one. Let $V = \mathbb{R}^{2n}$ and choose the alternate form $a(\cdot, \cdot)$ having matrix $A = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$, that is $a(v_1, v_2) =$*

${}^t v_1 A v_2$. *The complex structure J whose matrix is also A satisfies the above conditions, and the resulting maximal compact subgroup K is the stabiliser of $i1_n$ for the usual action of G on the Siegel upper half plane $\mathcal{H}_g = \{\tau \in M_n(\mathbb{C}) \mid {}^t\tau = \tau$ and $\mathrm{Im}(\tau) > 0\}$: for $a,b,c,d \in M_n(\mathbb{R})$ such that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $\tau \in \mathcal{H}_g$, $g(\tau) = (a\tau + b)(c\tau + d)^{-1}$. We now have two identifications of G/K with domains, D and $\mathcal{H}_n$, and they differ by the Cayley transform $\mathcal{H}_n \to D$, $\tau \mapsto (\tau - i1_n)(\tau + i1_n)^{-1}$.*

Observe that $G\mathbf{K}(\mathbb{C})\mathbf{U}_-(\mathbb{C}) = \exp(D)\mathbf{K}(\mathbb{C})\mathbf{U}_-(\mathbb{C})$ is open in $\mathbf{G}(\mathbb{C})$. Consider an irreducible unitary representation $r : K \to \mathrm{GL}(W)$, i.e. an irreducible algebraic representation of $\mathbf{K}_\mathbb{C}$ endowed with a K-invariant positive definite hermitian form. Harish-Chandra considered the space of holomorphic functions $f : G\mathbf{K}(\mathbb{C})\mathbf{U}_-(\mathbb{C}) \to W$ such that

1. for any $(s,k,n) \in G\mathbf{K}(\mathbb{C})\mathbf{U}_-(\mathbb{C}) \times \mathbf{K}(\mathbb{C}) \times \mathbf{U}_-(\mathbb{C})$, $f(skn) = r(k)^{-1} f(s)$,

2. $\int_G ||f(g)||^2 dg < \infty$.

It has an action of G defined by $(g \cdot f)(s) = f(g^{-1}s)$, and we get a unitary representation of G on a Hilbert space $\mathscr{H}_r$. Since $G/K \simeq G\mathbf{K}(\mathbb{C})\mathbf{U}_-(\mathbb{C})/\mathbf{K}(\mathbb{C})\mathbf{U}_-(\mathbb{C})$, $\mathscr{H}_r$ is isomorphic to the space of $f \in L^2(G,W)$ such that

1. for any $(g,k) \in G \times K$, $f(gk) = r(k)^{-1} f(g)$,

2. the function $G/K \to W$, $g \mapsto r(g_0)f(g)$ is holomorphic.

Harish-Chandra proved that $\mathscr{H}_r$ is zero or irreducible, by observing that in any closed invariant subspace, there is an f such that $G/K \to W$, $g \mapsto r(g_0)f(g)$ is constant and nonzero. Actually this a special case of [HC56a][Lemma 12, p. 20]). Hence when $\mathscr{H}_r \neq 0$, there is a K-equivariant embedding $\phi : W \to \mathscr{H}_r$, and any vector in its image is $\mathfrak{u}_+$-invariant. More generally, using the simple action of $\mathbf{Z}_\mathbf{K}(\mathbb{R})$ on $\mathbf{U}_+$ we see that when $\mathscr{H}_r \neq 0$ the K-finite vectors of $\mathscr{H}_r$ are exactly the polynomial functions on D. Note that when $\mathscr{H}_r \neq 0$ it is square-integrable by definition, i.e. it belongs to the discrete series of G.

Harish-Chandra determined necessary and sufficient conditions for $\mathscr{H}_r \neq 0$. Let $\mathbf{T}$ be a maximal torus of $\mathbf{K}$, and choose an order on the roots of $\mathbf{T}$ in $\mathbf{K}$. This determines a unique order on the roots of $\mathbf{T}$ in $\mathbf{G}$ such that the parabolic subgroup $\mathbf{K}_\mathbb{C}\mathbf{U}_+$ is standard, i.e. contains the Borel subgroup B of $\mathbf{G}_\mathbb{C}$ such that the positive roots are the ones occurring in $\mathbf{B}$. To be explicit in the symplectic case, $\mathbf{T}$ is determined by a decomposition of V as an orthogonal (for the hermitian form h) direct sum $V = V_1 \oplus \cdots \oplus V_n$ where each V_k is a line over $\mathbb{C}$. For any k we have a canonical isomorphism $e_k : \mathbf{U}(V_k, h) \simeq \mathbf{U}_1$. We can choose the order on the roots so that the simple roots are $e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n$. Note that among these simple roots, only $2e_n$ is noncompact. Let $\lambda = m_1 e_1 + \cdots + m_n e_n$ be the highest weight of r, so that $m_1 \geq \cdots \geq m_n$. This means that up to multiplication by a scalar there is a unique highest weight vector $v \in W \smallsetminus \{0\}$, that is such that for any $b \in \mathbf{K}(\mathbb{C}) \cap \mathbf{B}(\mathbb{C})$, $r(b)v = \lambda(b)v$. Let $\rho = ne_1 + \cdots + e_n$ be half the sum of the positive roots of $\mathbf{T}$ in $\mathbf{G}$. Then $\mathscr{H}_r \neq 0$ if and only if for any root α of $\mathbf{T}$ in $\mathbf{U}_+$, $\langle \alpha^\vee, \lambda + \rho \rangle < 0$ (see [HC56b][Lemma 29, p. 608]). In our case this condition is equivalent to $m_1 + n < 0$.

Assume that $\mathscr{H}_r \neq 0$. Note that $\phi(v)$ is a highest weight in the $\mathfrak{g}$-module $(\mathscr{H}_r)_{K-\mathrm{fin}}$, i.e. the Lie algebra of the unipotent radical of $\mathbf{B}$ cancels $\phi(v)$. Since $\mathscr{H}_r$ is irreducible and unitary, $(\mathscr{H}_r)_{K-\mathrm{fin}}$ is a simple $\mathfrak{g}$-module whose isomorphism class determines that of $\mathscr{H}_r$ (see [Kna86][chapter VIII]), and thus it is the unique simple quotient of the Verma module defined by $\mathbf{B}$ and λ. In particular, $\lambda+\rho$ is a representative for the infinitesimal character of $\mathscr{H}_r$. One can show that $(\mathscr{H}_r)_{K-\mathrm{fin}} = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}\oplus\mathfrak{u}_+)} W$, where W is seen as a $\mathfrak{k}\oplus\mathfrak{u}_+$-module by letting $\mathfrak{u}_+$ act trivially.

Remark 3.5.1.2. *Before Harish-Chandra realised these holomorphic discrete series concretely, in [HC55] he considered the simple quotient of the Verma module defined by λ and $\mathbf{B}$, for λ an arbitrary dominant weight for $\mathbf{K}_{\mathbb{C}} \cap \mathbf{B}$. He determined a necessary condition for this $\mathfrak{g}$-module to be unitarisable [HC55][Corollary 1 p.768]: for any root α of $\mathbf{T}$ in $\mathbf{U}_+$, $\langle \alpha^\vee, \lambda\rangle \leq 0$ (in our case this is equivalent to $m_1 \leq 0$). He also determined a sufficient condition [HC55][Theorem 3 p.770]: for any root α of $\mathbf{T}$ in $\mathbf{U}_+$, $\langle \alpha^\vee, \lambda+\rho\rangle \leq 0$ (in our case this is equivalent to $m_1+n \leq 0$). For classical groups Enright and Parthasarathy [EP81] gave a necessary and sufficient condition for unitarisability. In our symplectic case, this condition is*

$$-m_1 \geq \min_{1\leq j\leq n} \left(n-i+\sum_{2\leq j\leq i} \frac{m_1-m_j}{2} \right).$$

It would be interesting to determine whether all these unitary representations are globally relevant, i.e. belong to some Arthur packet.

The character of $\mathscr{H}_r$ was computed explicitely in [Sch75], [Mar75] and [Hec76]. There exists a unique Borel subgroup $\mathbf{B}' \supset \mathbf{T}_{\mathbb{C}}\mathbf{U}_-$ of $\mathbf{G}_{\mathbb{C}}$ such that $\mathbf{B}' \cap \mathbf{K}_{\mathbb{C}} = \mathbf{B} \cap \mathbf{K}_{\mathbb{C}}$. The order on the roots defined by $\mathbf{B}'$ is such that $\lambda+\rho$ is strictly dominant, i.e. for any root α occurring in $\mathbf{B}'$, $\langle \alpha^\vee, \lambda+\rho\rangle > 0$. Let $W_c = W(\mathbf{T}(\mathbb{R}), G) = W(\mathbf{T}(\mathbb{R}), K)$. Then among the discrete series of G with infinitesimal character $\lambda+\rho$, $\mathscr{H}_r$ is determined by the G-conjugacy class of the pair $(\mathbf{B}', \mathbf{T})$ (see section 3.4.2.1). In our case the simple roots for $\mathbf{B}'$ are $e_1-e_2, \dots, e_{n-1}-e_n$ and $-2e_1$.

Remark 3.5.1.3. *This characterisation of the holomorphic discrete series in their L-packet is enough to determine which Adams-Johnson representations are holomorphic discrete series. Using the notations of section 3.4.2.2, the representation $\pi_{\psi,\mathbf{Q},\mathbf{L}}$ is a holomorphic discrete series if and only if $\mathbf{Q} \supset \mathbf{B}'$ and $\mathbf{L}$ is anisotropic. By [CR14][Lemma 9.4] the packet Π_ψ^{AJ} contains a holomorphic discrete series representation if and only if $\mathrm{Std}\circ\psi$ does not contain $[d]$ or $\epsilon_{\mathbb{C}/\mathbb{R}}[d]$ as a factor for some $d>1$ (necessarily odd).*

We have made an arbitrary choice between $\mathbf{U}_+$ and $\mathbf{U}_-$. We could have also identified G/K with a bounded domain $D' \subset \mathfrak{u}_-$:

$$G/K \subset \mathbf{U}_-(\mathbb{C})\mathbf{K}(\mathbb{C})\mathbf{U}_+(\mathbb{C})/\mathbf{K}(\mathbb{C})\mathbf{U}_+(\mathbb{C}) \simeq \mathbf{U}_-(\mathbb{C}).$$

The resulting isomorphism of manifolds $D \simeq D'$ is antiholomorphic. Given an infinitesimal character τ which occurs in a finite-dimensional representation of G, we have a discrete series representations of G in the L-packet associated with τ, $\pi_{\tau,+}^{\mathrm{hol}} := (\mathscr{H}_r)_{K-\mathrm{fin}}$ (resp. $\pi_{\tau,-}^{\mathrm{hol}}$). It is characterised among irreducible unitary representations having infinitesimal character τ by the fact that it has a nonzero K-finite vector cancelled by $\mathfrak{u}_+$ (resp. $\mathfrak{u}_-$). Since K stabilises $\mathfrak{u}_+$ and $\mathfrak{u}_-$, $\pi_{\tau,+}^{\mathrm{hol}} \not\simeq \pi_{\tau,-}^{\mathrm{hol}}$.

Let us now define holomorphic discrete series for the group $G' = \mathrm{PGSp}(V, a)$. Assume that $\sum_{k=1}^n m_k$ is even, i.e. the center of G acts trivially in $\pi_{\tau,+}^{\mathrm{hol}}$ (and $\pi_{\tau,-}^{\mathrm{hol}}$). The image of G in G' has index two, and there is an element of G' normalizing K and exchanging $\mathbf{U}_+$ and $\mathbf{U}_-$. Thus if τ is such that the kernel of $\pi_{\tau,\pm}^{\mathrm{hol}}$ contains the center of G, $\pi_{\tau}^{\mathrm{hol}} := \mathrm{Ind}_G^{G'}\left(\pi_{\tau,+}^{\mathrm{hol}}\right)$ is irreducible and isomorphic to $\mathrm{Ind}_G^{G'}\left(\pi_{\tau,-}^{\mathrm{hol}}\right)$. Among irreducible unitary representations having infinitesimal character τ, $\pi_{\tau}^{\mathrm{hol}}$ is characterised by the fact that it has a nonzero K-finite vector cancelled by $\mathfrak{u}_+$. Of course we could replace $\mathfrak{u}_+$ by $\mathfrak{u}_-$.

3.5.2 Siegel modular forms and automorphic forms

Let us recall the link between Siegel modular forms and automorphic cuspidal representations for the group $\mathbf{PGSp}$. Almost all that we will need is contained in [AS01], in which the authors construct an isometric Hecke-equivariant map from the space of cuspidal Siegel modular forms to a certain space of cuspidal automorphic forms. We will simply add a characterisation of the image of this map.

For the definitions and first properties of Siegel modular forms, see [BvdGHZ08] or [Fre83]. We will use the classical conventions and consider the alternate form a on $\mathbb{Z}^{2n}$ whose matrix is $A = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{Z})$ for some integer $n \geq 1$. Let $\mu : \mathbf{GSp}(A) \to \mathbf{GL}_1$ be the multiplier, defined by the relation $a(g(v_1), g(v_2)) = \mu(g)a(v_1, v_2)$. Let $\mathbf{G} = \mathbf{Sp}(A) = \ker(\mu)$ and $\mathbf{G}' = \mathbf{PGSp}(A) = \mathbf{G}_{\mathrm{ad}}$, both reductive over $\mathbb{Z}$.

Recall the automorphy factor $j(g, \tau) = c\tau + d \in \mathrm{GL}_n(\mathbb{C})$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GSp}(A, \mathbb{R})$ and $\tau \in \mathcal{H}_n$. As in the previous section denote by K the stabiliser of $i1_n \in \mathcal{H}_n$ under the action of $\mathbf{G}(\mathbb{R})$. Let K' be the maximal compact subgroup of $\mathbf{G}'(\mathbb{R})$ containing the image of K by the natural morphism $\mathbf{G}(\mathbb{R}) \to \mathbf{G}'(\mathbb{R})$. Observe that the map $k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K \mapsto j(k, i1_n) = a - ib$ is an isomorphism between K and the unitary group $\mathrm{U}(1_n)$. In the previous section, using the complex structure J whose matrix is equal to A, we have identified K with the unitary group $\mathrm{U}(h)$ for a positive definite hermitian form h on $\mathbb{R}^{2n}$ with the complex structure J. We emphasise that the the resulting isomorphism $\mathrm{U}(1_n) \simeq \mathrm{U}(h)$ is *not* induced by an isomorphism between the hermitian spaces: one has to compose with the outer automorphism $x \mapsto {}^t x^{-1}$ on one side.

Let (V, r) be an algebraic representation of $\mathbf{GL}_n$. We can see the highest weight of r as $(m_1, \dots, m_g)$ where $m_1 \geq \dots m_g$ are integers. The representation $k \in K \mapsto r(j(k, i1_n))$ is the restriction to K of an algebraic representation r' of $\mathbf{K}_{\mathbb{C}}$. As in the previous section we choose a Borel pair $(\mathbf{B}_c, \mathbf{T})$ in $\mathbf{K}$ and denote by $e_1 - e_2, \dots, e_{n-1} - e_n$ the corresponding simple roots. Then the highest weight of r' is $-m_n e_1 - \dots - m_1 e_n$.

Let $\Gamma_n = \mathrm{Sp}(A, \mathbb{Z})$, and denote by $S_r(\Gamma_n)$ the space of vector-valued Siegel modular forms of weight r. When $m_1 = \dots = m_g$, that is when r is one-dimensional, this is the space of scalar Siegel modular forms of weight m_1. Asgari and Schmidt associate with any $f \in S_r(\Gamma_n)$ a function $\widetilde{\Phi}_f \in L^2(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}), V)$ such that

1. $\widetilde{\Phi}_f$ is right $\mathbf{G}'(\widehat{\mathbb{Z}})$-invariant,
2. for any $g \in \mathbf{G}'(\mathbb{A})$, the function $\mathbf{G}'(\mathbb{R}) \to W, h \mapsto \widetilde{\Phi}_f(gh)$ is smooth,
3. for any $X \in \mathfrak{u}_-$ and any $g \in \mathbf{G}'(\mathbb{A})$, $(X \cdot \widetilde{\Phi}_f)(g) = 0$,
4. for any $g \in \mathbf{G}'(\mathbb{A})$ and any $k \in K$, $\widetilde{\Phi}_f(gk) = r(j(k, i1_n))\widetilde{\Phi}_f(g)$,
5. $\widetilde{\Phi}_f$ is cuspidal.

The third condition translates the Cauchy-Riemann equation for the holomorphy of f into a condition on $\widetilde{\Phi}_f$. If the measures are suitably normalised, $f \mapsto \widetilde{\Phi}_f$ is isometric for the Petersson hermitian product on $S_r(\Gamma_n)$. Finally, $f \mapsto \widetilde{\Phi}_f$ is equivariant for the action of the unramified Hecke algebra at each finite place.

Let $\mathbf{N}_c$ be the unipotent radical of $\mathbf{B}_c$, let $\mathfrak{n}_c$ be its Lie algebra and let $\mathfrak{h}_0$ be the Lie algebra of $\mathbf{T}$. The representation r' allows to see V as a simple $\mathfrak{k}$-module, and $\mathfrak{n}_c V$ has codimension one in V. Let L be a linear form on V such that $\ker(L) = \mathfrak{n}_c V$. We can see $X^*(\mathbf{T})$ as a lattice in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_0, i\mathbb{R}) \subset \mathfrak{h}^*$. Let $\lambda = m_1 e_1 + \cdots + m_n e_n$ which we can see as an element of $(\mathfrak{h} \oplus \mathfrak{n}_c \oplus \mathfrak{u}_-)^*$ trivial on $\mathfrak{n}_c \oplus \mathfrak{u}_-$. For any $v \in V$ and any $X \in \mathfrak{h} \oplus \mathfrak{n}_c \oplus \mathfrak{u}_-$, $L(-r(X)v) = \lambda(X)$. For $g \in \mathbf{G}'(\mathbb{A})$, define $\Phi_f(g) = L(\widetilde{\Phi}_f(g))$. Then $\Phi_f \in L^2(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}))$ satisfies the following properties

1. Φ_f is right $\mathbf{G}'(\widehat{\mathbb{Z}})$-invariant and right K'-finite,
2. for any $g \in \mathbf{G}'(\mathbb{A})$, the function $\mathbf{G}'(\mathbb{R}) \to W, h \mapsto \Phi_f(gh)$ is smooth,
3. for any $X \in \mathfrak{h} \oplus \mathfrak{n}_c \oplus \mathfrak{u}_-$ and any $g \in \mathbf{G}'(\mathbb{A})$, $(X \cdot \Phi_f)(g) = \lambda(X)\Phi_f(g)$,
4. Φ_f is cuspidal.

Again $f \mapsto \Phi_f$ is equivariant for the action of the unramified Hecke algebras at the finite places, and is isometric (up to a scalar). The third condition implies that Φ_f is an eigenvector for $Z(U(\mathfrak{g}))$ and the infinitesimal character $\lambda + \rho_{\mathfrak{n}_c \oplus \mathfrak{u}_-} = (m_1 - 1)e_1 + \cdots + (m_n - n)e_n$. In particular Φ_f is a cuspidal automorphic form in the sense of [BJ79], which we denote by $\Phi_f \in \mathcal{A}_{\mathrm{cusp}}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}))$.

Lemma 3.5.2.1. *Any $\Phi \in \mathcal{A}_{\mathrm{cusp}}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}))$ satisfying the four conditions above is equal to Φ_f for a unique $f \in S_r(\Gamma_n)$.*

Proof. Since Φ is K'-finite and transforms under $\mathfrak{h} \oplus \mathfrak{n}_c$ according to λ, $\Phi = L(\widetilde{\Phi})$ for a unique function $\widetilde{\Phi} : \mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}) \to V$ such that for $k \in K$, $\widetilde{\Phi}(gk) = r(j(k, i1_n))^{-1}\widetilde{\Phi}(g)$. It is completely formal to check that there is a unique $f \in M_r(\Gamma_n)$ such that $\widetilde{\Phi} = \widetilde{\Phi}_f$, and thanks to the Koecher principle we only need to use that Φ has moderate growth when $n = 1$. We are left to show that f is cuspidal. Write $f(\tau) = \sum_{s \in \mathrm{Sym}_n} c(s) e^{2i\pi \mathrm{Tr}(s\tau)}$ where $c_s \in V$ and the sum ranges over the set Sym_n of symmetric half-integral semi-positive definite $n\times n$. We need to show that for any $s'\in \mathrm{Sym}_{n-1}$, $c\left(\begin{pmatrix}0&0\\0&s'\end{pmatrix}\right)=0$. We use the cuspidality condition on Φ for the parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ defined over $\mathbb{Z}$ by

$$\mathbf{P}=\left\{\begin{array}{c|c} & \begin{array}{cccc}1 & n-1 & 1 & n-1\end{array}\\ \begin{array}{c}1\\ n-1\\ 1\\ n-1\end{array} & \begin{pmatrix}* & * & * & *\\ 0 & * & * & *\\ 0 & 0 & * & 0\\ 0 & * & * & *\end{pmatrix}\in \mathbf{G}\end{array}\right\}.$$

Denote $\mathbf{N}$ the unipotent radical of $\mathbf{P}$, and observe that $\mathbf{N}=\mathbf{N}_0\rtimes\mathbf{N}_1$ where

$$\mathbf{N}_0=\left\{\begin{pmatrix}1&0&t_1&t_2\\0&1_{n-1}&{}^tt_2&0\\0&0&1&0\\0&0&0&1_{n-1}\end{pmatrix}\right\}\qquad\text{and}\qquad \mathbf{N}_1=\left\{\begin{pmatrix}1&t_3&0&0\\0&1_{n-1}&0&0\\0&0&1&0\\0&0&-{}^tt_3&1_{n-1}\end{pmatrix}\right\}$$

are vector groups. Moreover $\mathbf{N}_0(\mathbb{Q})\backslash\mathbf{N}_0(\mathbb{A})\simeq\mathbf{N}_0(\mathbb{Z})\backslash\mathbf{N}_0(\mathbb{R})$ and similarly for $\mathbf{N}_1$. Therefore for any $g\in\mathbf{G}(\mathbb{R})$,

$$\int_{\mathbf{N}_1(\mathbb{Z})\backslash\mathbf{N}_1(\mathbb{R})}\int_{\mathbf{N}_0(\mathbb{Z})\backslash\mathbf{N}_0(\mathbb{R})}\widetilde{\Phi}(n_0n_1g)dn_0dn_1=0.$$

By definition of $\widetilde{\Phi}$., for some $m\in\mathbb{R}$ depending only on r,

$$\widetilde{\Phi}(n_0n_1g)=\mu(g)^m r(j(n_0n_1g,i1_n))^{-1}f(n_0n_1g(i1_n)).$$

Fix $\tau\in\mathcal{H}_n$ of the form $\begin{pmatrix}iT&0\\0&\tau'\end{pmatrix}$ where $T\in\mathbb{R}_{>0}$ and $\tau'\in\mathcal{H}_{n-1}$, and let $g\in\mathbf{G}(\mathbb{R})$ be such that $\tau=g(i1_n)$. We will evaluate the inner integral first. Fix $n_1\in\mathbf{N}_1(\mathbb{R})$ determined by $t_3\in\mathbb{R}^{n-1}$ as above. For any $n_0\in\mathbf{N}(\mathbb{R})$ determined by $(t_1,t_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ as above, $j(n_0n_1g,i1_n)=j(n_1g,i1_n)$ and we have the Fourier expansion

$$\begin{aligned}\widetilde{\Phi}(n_0n_1g) &= \mu(g)^m r(j(n_1g,i1_n))^{-1}\sum_{s_1\in\mathbb{Z},s_2\in 1/2\mathbb{Z}^{n-1}}\left(\sum_{s'\in\mathrm{Sym}_{n-1}}c\left(\begin{pmatrix}s_1&s_2\\{}^ts_2&s'\end{pmatrix}\right)e^{2i\pi\mathrm{Tr}(s'\tau')}\right)\\ &\qquad\times\exp\left(2i\pi(s_1(t_3\tau'{}^tt_3+iT+t_1)+2s_2(\tau'{}^tt_3{}^tt_2))\right)\end{aligned}$$

and thus

$$\begin{aligned}\int_{\mathbf{N}_0(\mathbb{Z})\backslash\mathbf{N}_0(\mathbb{R})}\widetilde{\Phi}(n_0n_1g)dn_0 &= \mu(g)^m r(j(n_1g,i1_n))^{-1}\sum_{s'\in\mathrm{Sym}_{n-1}}c\left(\begin{pmatrix}0&0\\0&s'\end{pmatrix}\right)e^{2i\pi\mathrm{Tr}(s'\tau')}\\ &= \mu(g)^m r(j(g,i1_n))^{-1}\sum_{s'\in\mathrm{Sym}_{n-1}}c\left(\begin{pmatrix}0&0\\0&s'\end{pmatrix}\right)e^{2i\pi\mathrm{Tr}(s'\tau')}\end{aligned}$$

does not depend on n_1. Note that to get the last expression we used

$$r(j(n_1,\tau))^{-1}c\left(\begin{pmatrix}0&0\\0&s'\end{pmatrix}\right)=c\left(\begin{pmatrix}1&0\\{}^tt_3&1\end{pmatrix}\begin{pmatrix}0&0\\0&s'\end{pmatrix}\begin{pmatrix}1&t_3\\0&1\end{pmatrix}\right)=c\left(\begin{pmatrix}0&0\\0&s'\end{pmatrix}\right).$$

Hence we can conclude that for any $s'\in\mathrm{Sym}_{n-1}$, $c\left(\begin{pmatrix}0&0\\0&s'\end{pmatrix}\right)=0$. $\square$

Assume that $m_n \geq n+1$, i.e. that $\lambda + \rho_{\mathfrak{n}_c \oplus \mathfrak{u}_-}$ is the infinitesimal character of an L-packet of discrete series for $\mathbf{G}'(\mathbb{R})$. Assume also that $\sum_{k=1}^n m_k$ is even, since otherwise $S_r(\Gamma_n) = 0$. By the theorem of Gelfand, Graev and Piatetski-Shapiro

$$\mathcal{A}_{\mathrm{cusp}}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A})) \simeq \bigoplus_{\pi \in \Pi_{\mathrm{cusp}}(\mathbf{G}')} m_\pi \pi$$

where $\Pi_{\mathrm{cusp}}(\mathbf{G}')$ is the set of isomorphism classes of irreducible admissible $(\mathfrak{g}, K') \times \mathbf{G}'(\mathbb{A}_f)$-modules occurring in $\mathcal{A}_{\mathrm{cusp}}(\mathbf{G}'(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{A}))$ and $m_\pi \in \mathbb{Z}_{\geq 1}$. Consider a $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{G}')$. For any prime p, $\pi_p^{\mathbf{G}'(\mathbb{Z}_p)} \neq 0$ if and only if π_p is unramified, and in that case $\dim_{\mathbb{C}} \pi_p^{\mathbf{G}'(\mathbb{Z}_p)} = 1$. Since π_∞ is unitary, it has a highest weight vector for $(\lambda, \mathfrak{n}_c \oplus \mathfrak{u}_-)$ if and only if π_∞ is the holomorphic discrete series with infinitesimal character $(m_1 - 1)e_1 + \cdots + (m_n - n)e_n$, and in that case the space of highest weight vectors has dimension one. Thus $\dim S_r(\Gamma_n)$ is equal the sum of the m_π for $\pi = \otimes'_v \pi_v \in \Pi_{\mathrm{cusp}}(\mathbf{G}')$ such that π_∞ is a holomorphic discrete series with infinitesimal character $(m_1 - 1)e_1 + \cdots + (m_n - n)e_n$ and for any prime number p, π_p is unramified. By [Wal84] any $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G}') \smallsetminus \Pi_{\mathrm{cusp}}(\mathbf{G}')$ is such that π_∞ is not tempered. Therefore $\dim S_r(\Gamma_n)$ is equal to the sum of the multiplicities m_π for $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G}')$ such that

- for any prime number p, π_p is unramified,
- π_∞ is the holomorphic discrete series representation π_τ^{hol} with infinitesimal character $\tau = (m_1 - 1)e_1 + \cdots + (m_n - n)e_n$.

Recall that $\mathbf{G} = \mathbf{Sp}_{2n}$. Thanks to [CR14][Proposition 4.7] we have that $\dim S_r(\Gamma_n)$ is also equal to the sum of the multiplicities m_π for $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G})$ such that π is unramified everywhere and $\pi_\infty \simeq \pi_{\tau,+}^{\mathrm{hol}}$.

Remark 3.5.2.2. *For any central isogeny $\mathbf{G} \to \mathbf{G}'$ between semisimple Chevalley groups over $\mathbb{Z}$, the integer denoted $[\pi_\infty, \pi'_\infty]$ in [CR14][Proposition 4.7] is always equal to 1. This follows from the fact that $\mathbf{G}'(\mathbb{R})/\mathbf{G}(\mathbb{R})$ is a finite abelian group.*

Thus we have an algorithm to compute $\dim S_r(\Gamma_n)$ from the cardinalities of $S(\cdot)$, $O_o(\cdot)$ and $O_e(\cdot)$, under Assumption 3.4.2.3 if $m_1, \ldots, m_n$ are not distinct. Note that since the Adams-Johnson packets Π_ψ^{AJ} have multiplicity one, under Assumption 3.4.2.3 the multiplicites m_π for π as above are all equal to 1, and thus Siegel eigenforms in level one and weight r satisfying $m_n \geq n+1$ have multiplicity one: up to a scalar they are determined by their Hecke eigenvalues at primes in a set of density one. This was already observed in [CR14][Corollary 4.10].

Remark 3.5.2.3. *Without assuming that $m_n \geq n+1$, the construction in [AS01] shows that $f \mapsto \Phi_f$ is an isometry from the space of square-integrable modular forms (for the Petersson scalar product) to the space of square-integrable automorphic forms which are λ-equivariant under $\mathfrak{n}_c \oplus \mathfrak{u}_-$ and $\mathbf{G}'(\mathbb{Z})$-invariant.*

In fact for $m_n \geq n+1$ (even $m_n \geq n$) we could avoid using [Wal84] and Lemma 3.5.2.1 and use the fact [Wei83][Satz 3] that for $m_n \geq n$ square-integrable Siegel modular forms are cusp forms.

3.5.3 Example: genus 4

Let us give more details in case $n = 4$, which is interesting because there an endoscopic contribution from the group $\mathbf{SO}_8$ (the formal parameter $O_e(w_1, w_2, w_3, w_4) \boxplus 1$ below) which cannot be explained using lower genus Siegel eigenforms. First we list the possible Arthur parameters for the group $\mathbf{Sp}_8$ in terms of the sets $S(w_1, \dots)$, $O_o(w_1, \dots)$ and $O_e(w_1, \dots)$. The non-tempered ones only occur when $\lambda' = (m_1 - n - 1)e_1 + \dots + (m_n - n - 1)e_n$ is orthogonal to a non-empty subset of the simple coroots $\{e_1^* - e_2^*, \dots, e_{n-1}^* - e_n^*, e_n^*\}$. The convention in the following table is that the weights $w_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ are decreasing with i. For example $S(w_3)[2] \boxplus O_o(w_1, w_2)$ occurs only if $m_3 = m_4$, and if this is the case then

$$(m_1, m_2, m_3, m_4) = \left(w_1 + 1, w_2 + 2, w_3 + \frac{7}{2}, w_3 + \frac{7}{2}\right).$$

Table 3.1: Unramified cohomological Arthur parameters for $\mathbf{Sp}_8$

$O_o(w_1, w_2, w_3, w_4)$	$O_e(w_1, w_2, w_3, w_4) \boxplus 1$	$O_e(w_1, w_4) \boxplus O_e(w_2, w_3) \boxplus 1$
$O_e(w_2, w_3) \boxplus O_o(w_1, w_4)$	$O_e(w_1, w_4) \boxplus O_o(w_2, w_3)$	$O_e(w_1, w_3) \boxplus O_e(w_2, w_4) \boxplus 1$
$O_e(w_2, w_4) \boxplus O_o(w_1, w_3)$	$O_e(w_1, w_3) \boxplus O_o(w_2, w_4)$	$O_e(w_1, w_2) \boxplus O_e(w_3, w_4) \boxplus 1$
$O_e(w_3, w_4) \boxplus O_o(w_1, w_2)$	$O_e(w_1, w_2) \boxplus O_o(w_3, w_4)$	$O_e(w_1, w_2) \boxplus S(w_3)[2] \boxplus 1$
$S(w_3)[2] \boxplus O_o(w_1, w_2)$	$O_e(w_1, w_4) \boxplus S(w_2)[2] \boxplus 1$	$S(w_2)[2] \boxplus O_o(w_1, w_4)$
$O_e(w_3, w_4) \boxplus S(w_1)[2] \boxplus 1$	$S(w_1)[2] \boxplus O_o(w_3, w_4)$	$S(w_1, w_3)[2] \boxplus 1$
$S(w_1)[2] \boxplus S(w_3)[2] \boxplus 1$	$S(w_1)[4] \boxplus 1$	$S(w_1)[2] \boxplus [5]$
$O_e(w_1, w_2) \boxplus [5]$	$O_o(w_1)[3]$	$[9]$

Among these 24 types for $\psi \in \Psi(\mathbf{Sp}_8)^{\mathrm{unr}, \lambda'}$, some never yield Siegel modular forms. In the last four cases ($S(w_1)[2] \boxplus [5]$, $O_e(w_1, w_2) \boxplus [5]$, $O_o(w_1)[3]$ and $[9]$), Π_{ψ_∞} does not contain the holomorphic discrete series. In the other 20 cases, Π_{ψ_∞} contains the holomorphic discrete series representation $\pi^{\mathrm{hol}}_{\tau,+}$ but it can happen that $\langle \cdot, \pi^{\mathrm{hol}}_{\tau,+} \rangle|_{S_\psi}$ never equals ϵ_ψ. For example if ψ is tempered (the first 11 cases) ϵ_ψ is always trivial, whereas $\langle \cdot, \pi^{\mathrm{hol}}_{\tau,+} \rangle|_{S_\psi}$ is trivial if and only if ψ does not contain $O_e(w_1, w_2)$ or $O_e(w_1, w_4)$ or $O_e(w_2, w_3)$ as a factor.

In the following table we list the 11 types that yield Siegel modular forms for some dominant weight λ' for $\mathbf{Sp}_8$. In the last column we give a necessary and sufficient condition on the weights for having $\langle \cdot, \pi^{\mathrm{hol}}_{\tau,+} \rangle|_{S_\psi} = \epsilon_\psi$.

Table 3.2: The 11 possible Arthur parameters of Siegel eigenforms for Γ_4

Type	(m_1, m_2, m_3, m_4)	Occurs iff
$O_o(w_1, w_2, w_3, w_4)$	$(w_1+1, w_2+2, w_3+3, w_4+4)$	always
$O_e(w_1, w_2, w_3, w_4) \boxplus 1$	$(w_1+1, w_2+2, w_3+3, w_4+4)$	always
$O_e(w_1, w_3) \boxplus O_e(w_2, w_4) \boxplus 1$	$(w_1+1, w_2+2, w_3+3, w_4+4)$	always
$O_e(w_2, w_4) \boxplus O_o(w_1, w_3)$	$(w_1+1, w_2+2, w_3+3, w_4+4)$	always
$O_e(w_1, w_3) \boxplus O_o(w_2, w_4)$	$(w_1+1, w_2+2, w_3+3, w_4+4)$	always
$S(w_3)[2] \boxplus O_o(w_1, w_2)$	$(w_1+1, w_2+2, w_3+\frac{7}{2}, w_3+\frac{7}{2})$	$w_3+\frac{1}{2}$ is odd
$S(w_2)[2] \boxplus O_e(w_1, w_4) \boxplus 1$	$(w_1+1, w_2+\frac{5}{2}, w_2+\frac{5}{2}, w_4+4)$	$w_2+\frac{1}{2}$ is even
$S(w_2)[2] \boxplus O_o(w_1, w_4)$	$(w_1+1, w_2+\frac{5}{2}, w_2+\frac{5}{2}, w_4+4)$	$w_2+\frac{1}{2}$ is even
$S(w_1)[2] \boxplus O_o(w_3, w_4)$	$(w_1+\frac{3}{2}, w_1+\frac{3}{2}, w_3+3, w_4+4)$	$w_1+\frac{1}{2}$ is odd
$S(w_1, w_3)[2] \boxplus 1$	$(w_1+\frac{3}{2}, w_1+\frac{3}{2}, w_3+\frac{7}{2}, w_3+\frac{7}{2})$	w_1+w_3 is odd
$S(w_1)[4] \boxplus 1$	$(w_1+\frac{3}{2}, w_1+\frac{3}{2}, w_1+\frac{3}{2}, w_1+\frac{3}{2})$	$w_1+\frac{1}{2}$ is even

3.5.4 Some dimensions in the scalar case

In genus n greater than 4 the enumeration of the possible Arthur parameters of Siegel eigenforms is best left to a computer. Our implementation currently allows to compute $\dim S_r(\Gamma_n)$ for $n \leq 7$ and any algebraic representation r of $\mathbf{GL}_n$ such that its highest weight $m_1 \geq \cdots \geq m_n$ satisfies $m_n \geq n+1$.

Table 3.3 displays the dimensions of some spaces of *scalar* Siegel cusp forms. Note that our method does *not* allow to compute $\dim S_k(\Gamma_n)$ when $k \leq n$ (question marks in the bottom left corner), and that for scalar weights is is necessary to make Assumption 3.4.2.3. We do not include the values $\dim S_k(\Gamma_n)$ when $n+1 \leq k \leq 7$ because they all vanish. The question marks on the right side could be obtained simply by computing more traces in algebraic representations ($\mathrm{Tr}(\gamma \,|\, V_\lambda)$ in the geometric side of the trace formula). For more data see `http://www.math.ens.fr/~taibi/dimtrace/`. For $n \geq 8$ we have not (yet) managed to compute the masses for $\mathbf{Sp}_{2n}$. Nevertheless we can enumerate some endoscopic parameters, and thus give lower bounds for $\dim S_k(\Gamma_n)$: these are the starred numbers.

Table 3.3: Dimensions of spaces of scalar Siegel cusp forms

k	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$\dim S_k(\Gamma_1)$	0	0	0	0	1	0	0	0	1	0	1	0	1	0	1
$\dim S_k(\Gamma_2)$	0	0	1	0	1	0	1	0	2	0	2	0	3	0	4
$\dim S_k(\Gamma_3)$	0	0	0	0	1	0	1	0	3	0	4	0	6	0	9
$\dim S_k(\Gamma_4)$	1	0	1	0	2	0	3	0	7	0	12	1	22	1	38
$\dim S_k(\Gamma_5)$	0	0	0	0	2	0	3	0	13	0	28	0	76	0	186
$\dim S_k(\Gamma_6)$	0	0	1	0	3	0	9	0	33	0	117	1	486	?	?
$\dim S_k(\Gamma_7)$	0	0	0	0	3	0	9	0	83	0	?	0	?	0	?
$\dim S_k(\Gamma_8)$	?	0^*	1^*	0^*	4^*	1^*	23^*	2^*	234^*						
$\dim S_k(\Gamma_9)$	?	?	0^*	0^*	2^*	0^*	25^*	0^*	843^*						
$\dim S_k(\Gamma_{10})$	?	?	?	0^*	2^*	0^*	43^*	1^*	1591^*						
$\dim S_k(\Gamma_{11})$	?	?	?	?	1^*	0^*	32^*	0^*	6478^*						

In principle for $n \leq 7$ one can compute the generating series $\sum_{k \geq n+1} (\dim S_k(\Gamma_n))\, T^k$. We have not attempted to do so for $n \geq 4$.

3.6 Reliability

The complete algorithm computing the three families of numbers

- $\operatorname{card}(S(w_1, \dots, w_n))$ for $n \geq 1$, $w_i \in \frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z}$ and $w_1 > \cdots > w_n > 0$,
- $\operatorname{card}(O_o(w_1, \dots, w_n))$ for $n \geq 1$, $w_i \in \mathbb{Z}$ and $w_1 > \cdots > w_n > 0$,
- $\operatorname{card}(O_e(w_1, \dots, w_{2n}))$ for $n \geq 1$, $w_i \in \mathbb{Z}$ and $w_1 > \cdots > w_{2n} \geq 0$,

is long and complicated. Our implementation consists of more than 5000 lines of source code (mainly in Python, using Sage [S+14]), therefore it certainly contains errors. There are several mathematically meaningful checks suggesting that the tables produced by our program are valid:

1. When computing the geometric side of the trace formula we obviously always find a rational number. The trace formula asserts that it is equal to the spectral side, which is an integer, being an Euler-Poincaré characteristic. The first check that our tables pass is thus that the geometric sides are indeed integral.

2. With a one-line modification, our algorithm can be used to compute global orbital integrals for special orthogonal groups $\mathbf{G}/\mathbb{Q}$ which are split at every finite place and such that $\mathbf{G}(\mathbb{R})$ is compact. On a space of dimension d such a group exists if and only $d = -1, 0, 1 \mod 8$. Recall that for $d \in \{7, 8, 9\}$, up to isomorphism there is a unique regular and definite positive quadratic form $q : \mathbb{Z}^d \to \mathbb{Z}$. These are the lattices E_7, E_8 and $E_8 \oplus A_1$. Each one of these three lattices defines a reductive group $\mathbf{G}$ over $\mathbb{Z}$ such that $\mathbf{G}_{\mathbb{Q}}$ is as above, and their uniqueness is equivalent to the fact that the arithmetic genus $\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(\widehat{\mathbb{Z}})$ has one element. Chenevier and Renard [CR14] computed the geometric side of the trace formula, which is elementary and does not depend on Arthur's work in the anisotropic case, to *count* level one automorphic

representations for these groups. This is possible because $\mathbf{G}(\mathbb{Z})$ is closely related to the Weyl groups of the root systems E_7 and E_8, for which Carter [Car72] described the conjugacy classes and their orders. We checked that we obtain the same "masses" (see section 3.3.2.5).

3. The numbers $\operatorname{card}(S(w_1,\dots,w_n))$, $\operatorname{card}(O_o(w_1,\dots,w_n))$ and $\operatorname{card}(O_e(w_1,\dots,w_{2n}))$ belong to $\mathbb{Z}_{\geq 0}$. Our tables pass this check.

4. In low rank there are exceptional isogenies between the groups that we consider: $\mathbf{PGSp}_2 \simeq \mathbf{SO}_3$, $\mathbf{PGSp}_4 \simeq \mathbf{SO}_5$, $(\mathbf{SO}_4)_{\mathrm{sc}} \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$, which by [CR14][Proposition 4.7] imply:

   (a) For any odd $w_1 \in \mathbb{Z}_{>0}$, $\operatorname{card}(S(w_1/2)) = \operatorname{card}(O_o(w_1))$. Note that $\operatorname{card}(O_o(w_1)) = 0$ if w_1 is even.

   (b) For any integers $w_1 > w_2 > 0$ such that $w_1 + w_2$ is odd,
   $$\operatorname{card}\left(S\left(\frac{w_1+w_2}{2}, \frac{w_1-w_2}{2}\right)\right) = \operatorname{card}(O_o(w_1,w_2)).$$
   Note that $\operatorname{card}(O_o(w_1,w_2)) = 0$ if $w_1 + w_2$ is even.

   (c) For any integers $w_1 > w_2 > 0$ such that $w_1 + w_2$ is odd,
   $$\operatorname{card}\left(S\left(\frac{w_1+w_2}{2}\right)\right) \times \operatorname{card}\left(S\left(\frac{w_1-w_2}{2}\right)\right) = \operatorname{card}(O_e(w_1,w_2)),$$
   and for any odd integer $w > 0$,
   $$\binom{\operatorname{card}\left(S(\frac{w}{2})\right)}{2} = O_e(w,0).$$
   Note that $\operatorname{card}(O_e(w_1,w_2)) = 0$ if $w_1 + w_2$ is even.

5. By results of Mestre [Mes86], Fermigier [Fer96] and Miller [Mil02], in low motivic weight (that is $2w_1$) some of the cardinalities of $S(w_1,\dots)$, $O_o(w_1,\dots)$ and $O_e(w_1,\dots)$ are known to vanish. In forthcoming work, Chenevier and Lannes improve their method to show that if $n \geq 1$ and π is a self-dual cuspidal automorphic representation of $\mathbf{GL}_n/\mathbb{Q}$ such that

   - for any prime number p, π_p is unramified,
   - the local Langlands parameter φ of π_∞ is either
     - a direct sum of copies of 1, $\epsilon_{\mathbb{C}/\mathbb{R}}$ and I_r for integers $1 \leq r \leq 10$, or
     - a direct sum of copies of I_r for $r \in \frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z}$ and $\frac{1}{2} \leq r \leq \frac{19}{2}$.

   then φ belongs to the following list:

   - 1,
   - $I_{11/2}$, $I_{15/2}$, $I_{17/2}$, $I_{19/2}$,
   - $\epsilon_{\mathbb{C}/\mathbb{R}} \oplus I_{10}$, $\epsilon_{\mathbb{C}/\mathbb{R}} \oplus I_9$,

- $I_{r/2} \oplus I_{19/2}$ with $r \in \{5, 7, 9, 11, 13\}$,
- $I_4 \oplus I_9$, $I_r \oplus I_{10}$ with $r \in \{2, 3, 4, 5, 6, 7\}$,
- $1 \oplus I_6 \oplus I_{10}$, $1 \oplus I_7 \oplus I_{10}$.

Note that they make no regularity assumption. This implies the vanishing of 2521 values in our tables for groups of rank ≤ 6. In our tables, the only non-vanishing $\operatorname{card}(S(w_1, \dots))$, $\operatorname{card}(O_o(w_1, \dots))$ or $\operatorname{card}(O_e(w_1, \dots))$ with $w_1 \leq 10$ are the following.

- For $w_1 \in \left\{\frac{11}{2}, \frac{15}{2}, \frac{17}{2}, \frac{19}{2}\right\}$, $\operatorname{card}(S(w_1)) = 1$. These are the well-known modular forms.
- $\operatorname{card}\left(S\left(\frac{19}{2}, \frac{7}{2}\right)\right) = 1$.

6. Finally, we can compare the values that we obtain for the dimensions of spaces of Siegel modular forms with known ones. Our formulae coincide with those given in [Igu62] (genus two, scalar) and [Tsu83] and [Tsu84] (genus two, vector-valued). Tsuyumine [Tsu86] gave a dimension formula in the scalar case in genus 3. There seems to be a typographical error in the formula on page 832 of [Tsu86], the denominator should be

$$(1-T^4)(1-T^{12})^2(1-T^{14})(1-T^{18})(1-T^{20})(1-T^{30})$$

instead of

$$(1-T^4)(1-T^{12})^3(1-T^{14})(1-T^{18})(1-T^{20})(1-T^{30}).$$

With this correction we find the same formula as Tsuyumine. In [BFvdG14] Bergström, Faber and van der Geer conjecture a formula for the cohomology of local systems on the moduli space $\mathcal{A}_3$ in terms of motives conjecturally associated with Siegel cusp forms. As a corollary they obtain a conjectural formula for $\dim S_r(\Gamma_3)$ where r is an algebraic representation of $\mathbf{GL}_3$ of highest weight $m_1 \geq m_2 \geq m_3 \geq 4$. For $m_1 \leq 24$ (1771 values) we have checked that our values coincide. We have also checked that our tables agree with Nebe and Venkov's theorem and conjecture in weight 12 [NV01] and Poor and Yuen's results in low weight [PY07].

3.7 Tables

3.7.1 Masses

Table 3.4: Masses for the group $\mathbf{SO}_3$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^3	$-1/12$	$\Phi_1\Phi_2^2$	$1/4$	$\Phi_1\Phi_3$	$1/3$

Table 3.5: Masses for the group $\mathbf{SO}_5$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^5	$-1/1440$	$\Phi_1^3\Phi_2^2$	$-1/48$	$\Phi_1\Phi_2^4$	$7/288$	$\Phi_1\Phi_2^2\Phi_4$	$1/4$
$\Phi_1\Phi_4^2$	$-1/24$	$\Phi_1^3\Phi_3$	$-1/36$	$\Phi_1\Phi_2^2\Phi_3$	$1/12$	$\Phi_1\Phi_3^2$	$-1/36$
$\Phi_1\Phi_2^2\Phi_6$	$2/9$	$\Phi_1\Phi_6^2$	$-1/36$	$\Phi_1\Phi_{12}$	$1/6$	$\Phi_1\Phi_5$	$2/5$

Table 3.6: Masses for the group $\mathbf{SO}_7$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^7	$1/483840$	$\Phi_1^5\Phi_2^2$	$-19/23040$	$\Phi_1^3\Phi_2^4$	$-331/13824$
$\Phi_1\Phi_2^6$	$1/7680$	$\Phi_1^3\Phi_2^2\Phi_4$	$-11/192$	$\Phi_1\Phi_2^4\Phi_4$	$1/64$
$\Phi_1^3\Phi_4^2$	$25/1152$	$\Phi_1\Phi_2^2\Phi_4^2$	$-7/384$	$\Phi_1\Phi_2^2\Phi_8$	$3/16$
$\Phi_1\Phi_4\Phi_8$	$3/16$	$\Phi_1^5\Phi_3$	$-1/1440$	$\Phi_1^3\Phi_2^2\Phi_3$	$-1/36$
$\Phi_1\Phi_2^4\Phi_3$	$7/864$	$\Phi_1\Phi_2^2\Phi_3\Phi_4$	$1/24$	$\Phi_1\Phi_3\Phi_4^2$	$-1/72$
$\Phi_1^3\Phi_3^2$	$7/144$	$\Phi_1\Phi_2^2\Phi_3^2$	$-1/144$	$\Phi_1\Phi_3^3$	$1/216$
$\Phi_1^3\Phi_2^2\Phi_6$	$-23/432$	$\Phi_1\Phi_2^4\Phi_6$	$1/48$	$\Phi_1\Phi_2^2\Phi_4\Phi_6$	$1/8$
$\Phi_1\Phi_2^2\Phi_3\Phi_6$	$5/27$	$\Phi_1^3\Phi_6^2$	$1/432$	$\Phi_1\Phi_2^2\Phi_6^2$	$1/48$
$\Phi_1\Phi_3\Phi_6^2$	$1/216$	$\Phi_1^3\Phi_{12}$	$-1/72$	$\Phi_1\Phi_2^2\Phi_{12}$	$1/24$
$\Phi_1\Phi_3\Phi_{12}$	$5/36$	$\Phi_1\Phi_9$	$1/3$	$\Phi_1^3\Phi_5$	$-1/15$
$\Phi_1\Phi_2^2\Phi_5$	$1/10$	$\Phi_1\Phi_3\Phi_5$	$1/15$	$\Phi_1\Phi_2^2\Phi_{10}$	$3/10$
$\Phi_1\Phi_7$	$3/7$				

Table 3.7: Masses for the group $\mathbf{SO}_9$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^9	$1/116121600$	$\Phi_1^7\Phi_2^2$	$1/1935360$	$\Phi_1^5\Phi_2^4$	$-4963/1658880$
$\Phi_1^3\Phi_2^6$	$-31/92160$	$\Phi_1\Phi_2^8$	$121/116121600$	$\Phi_1^5\Phi_2^2\Phi_4$	$-67/23040$
$\Phi_1^3\Phi_2^4\Phi_4$	$-7/768$	$\Phi_1\Phi_2^6\Phi_4$	$1/2560$	$\Phi_1^5\Phi_4^2$	$109/138240$
$\Phi_1^3\Phi_2^2\Phi_4^2$	$37/4608$	$\Phi_1\Phi_2^4\Phi_4^2$	$-331/27648$	$\Phi_1\Phi_2^2\Phi_4^3$	$1/128$
$\Phi_1\Phi_4^4$	$1/7680$	$\Phi_1^3\Phi_2^2\Phi_8$	$-1/64$	$\Phi_1\Phi_2^4\Phi_8$	$1/64$
$\Phi_1^3\Phi_4\Phi_8$	$-1/64$	$\Phi_1\Phi_2^2\Phi_4\Phi_8$	$21/64$	$\Phi_1\Phi_8^2$	$1/32$
$\Phi_1^7\Phi_3$	$1/1451520$	$\Phi_1^5\Phi_2^2\Phi_3$	$-49/23040$	$\Phi_1^3\Phi_2^4\Phi_3$	$-331/41472$
$\Phi_1\Phi_2^6\Phi_3$	$1/23040$	$\Phi_1^3\Phi_2^2\Phi_3\Phi_4$	$5/576$	$\Phi_1\Phi_2^4\Phi_3\Phi_4$	$1/192$
$\Phi_1^3\Phi_3\Phi_4^2$	$25/3456$	$\Phi_1\Phi_2^2\Phi_3\Phi_4^2$	$-7/1152$	$\Phi_1\Phi_2^2\Phi_3\Phi_8$	$1/16$
$\Phi_1\Phi_3\Phi_4\Phi_8$	$1/16$	$\Phi_1^5\Phi_3^2$	$67/17280$	$\Phi_1^3\Phi_2^2\Phi_3^2$	$7/576$
$\Phi_1\Phi_2^4\Phi_3^2$	$-7/10368$	$\Phi_1\Phi_2^2\Phi_3^2\Phi_4$	$-1/144$	$\Phi_1\Phi_3^2\Phi_4^2$	$1/864$
$\Phi_1^3\Phi_3^3$	$-25/2592$	$\Phi_1\Phi_2^2\Phi_3^3$	$1/864$	$\Phi_1\Phi_3^4$	$1/25920$
$\Phi_1^5\Phi_2^2\Phi_6$	$-83/51840$	$\Phi_1^3\Phi_2^4\Phi_6$	$-7/576$	$\Phi_1\Phi_2^6\Phi_6$	$37/51840$
$\Phi_1^3\Phi_2^2\Phi_4\Phi_6$	$-1/96$	$\Phi_1\Phi_2^4\Phi_4\Phi_6$	$1/32$	$\Phi_1\Phi_2^2\Phi_4^2\Phi_6$	$-23/864$
$\Phi_1\Phi_2^2\Phi_6\Phi_8$	$1/8$	$\Phi_1^3\Phi_2^2\Phi_3\Phi_6$	$-11/324$	$\Phi_1\Phi_2^4\Phi_3\Phi_6$	$1/36$
$\Phi_1\Phi_2^2\Phi_3\Phi_4\Phi_6$	$1/6$	$\Phi_1\Phi_2^2\Phi_3^2\Phi_6$	$1/324$	$\Phi_1^5\Phi_6^2$	$1/51840$
$\Phi_1^3\Phi_2^2\Phi_6^2$	$-1/576$	$\Phi_1\Phi_2^4\Phi_6^2$	$-133/3456$	$\Phi_1\Phi_2^2\Phi_4\Phi_6^2$	$-1/16$
$\Phi_1\Phi_4^2\Phi_6^2$	$1/864$	$\Phi_1^3\Phi_3\Phi_6^2$	$-1/2592$	$\Phi_1\Phi_2^2\Phi_3\Phi_6^2$	$-13/288$
$\Phi_1\Phi_3^2\Phi_6^2$	$41/2592$	$\Phi_1\Phi_2^2\Phi_6^3$	$1/324$	$\Phi_1\Phi_6^4$	$1/25920$
$\Phi_1^5\Phi_{12}$	$-1/8640$	$\Phi_1^3\Phi_2^2\Phi_{12}$	$-1/288$	$\Phi_1\Phi_2^4\Phi_{12}$	$7/1728$
$\Phi_1\Phi_2^2\Phi_4\Phi_{12}$	$1/8$	$\Phi_1\Phi_4^2\Phi_{12}$	$1/48$	$\Phi_1^3\Phi_3\Phi_{12}$	$-5/432$
$\Phi_1\Phi_2^2\Phi_3\Phi_{12}$	$5/144$	$\Phi_1\Phi_3^2\Phi_{12}$	$1/432$	$\Phi_1\Phi_2^2\Phi_6\Phi_{12}$	$5/54$
$\Phi_1\Phi_6^2\Phi_{12}$	$1/432$	$\Phi_1\Phi_{12}^2$	$1/48$	$\Phi_1\Phi_{24}$	$1/4$
$\Phi_1^3\Phi_9$	$-1/36$	$\Phi_1\Phi_2^2\Phi_9$	$1/12$	$\Phi_1\Phi_3\Phi_9$	$4/9$
$\Phi_1\Phi_2^2\Phi_{18}$	$2/9$	$\Phi_1\Phi_6\Phi_{18}$	$1/9$	$\Phi_1^5\Phi_5$	$-7/3600$
$\Phi_1^3\Phi_2^2\Phi_5$	$-1/60$	$\Phi_1\Phi_2^4\Phi_5$	$7/720$	$\Phi_1\Phi_2^2\Phi_4\Phi_5$	$1/20$
$\Phi_1\Phi_4^2\Phi_5$	$-1/60$	$\Phi_1^3\Phi_3\Phi_5$	$1/180$	$\Phi_1\Phi_2^2\Phi_3\Phi_5$	$1/60$
$\Phi_1\Phi_3^2\Phi_5$	$-1/90$	$\Phi_1\Phi_2^2\Phi_5\Phi_6$	$4/45$	$\Phi_1\Phi_5\Phi_6^2$	$-1/90$
$\Phi_1\Phi_5\Phi_{12}$	$1/15$	$\Phi_1\Phi_5^2$	$1/100$	$\Phi_1^3\Phi_2^2\Phi_{10}$	$-1/40$
$\Phi_1\Phi_2^4\Phi_{10}$	$11/200$	$\Phi_1\Phi_2^2\Phi_4\Phi_{10}$	$3/20$	$\Phi_1\Phi_2^2\Phi_3\Phi_{10}$	$1/10$
$\Phi_1\Phi_2^2\Phi_6\Phi_{10}$	$1/5$	$\Phi_1\Phi_{10}^2$	$1/100$	$\Phi_1\Phi_{20}$	$3/10$
$\Phi_1\Phi_{15}$	$1/5$	$\Phi_1\Phi_{30}$	$1/5$	$\Phi_1^3\Phi_7$	$-1/28$
$\Phi_1\Phi_2^2\Phi_7$	$3/28$	$\Phi_1\Phi_3\Phi_7$	$1/7$	$\Phi_1\Phi_2^2\Phi_{14}$	$3/7$

Table 3.8: Masses for the group $\mathbf{Sp}_2$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^2	$-1/12$	Φ_2^2	$-1/12$	Φ_4	$1/2$
Φ_3	$1/3$	Φ_6	$1/3$		

Table 3.9: Masses for the group $\mathbf{Sp}_4$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^4	$-1/1440$	$\Phi_1^2\Phi_2^2$	$7/144$	Φ_2^4	$-1/1440$	$\Phi_1^2\Phi_4$	$-1/24$
$\Phi_2^2\Phi_4$	$-1/24$	Φ_4^2	$-1/24$	Φ_8	$1/2$	$\Phi_1^2\Phi_3$	$-1/36$
$\Phi_2^2\Phi_3$	$-1/36$	$\Phi_3\Phi_4$	$1/6$	Φ_3^2	$-1/36$	$\Phi_1^2\Phi_6$	$-1/36$
$\Phi_2^2\Phi_6$	$-1/36$	$\Phi_4\Phi_6$	$1/6$	$\Phi_3\Phi_6$	$4/9$	Φ_6^2	$-1/36$
Φ_{12}	$1/6$	Φ_5	$2/5$	Φ_{10}	$2/5$		

Table 3.10: Masses for the group $\mathbf{Sp}_6$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^6	$1/362880$	$\Phi_1^4\Phi_2^2$	$31/17280$	$\Phi_1^2\Phi_2^4$	$31/17280$
Φ_2^6	$1/362880$	$\Phi_1^4\Phi_4$	$-1/2880$	$\Phi_1^2\Phi_2^2\Phi_4$	$7/288$
$\Phi_2^4\Phi_4$	$-1/2880$	$\Phi_1^2\Phi_4^2$	$7/288$	$\Phi_2^2\Phi_4^2$	$7/288$
Φ_4^3	$1/48$	$\Phi_1^2\Phi_8$	$-1/24$	$\Phi_2^2\Phi_8$	$-1/24$
$\Phi_4\Phi_8$	$3/4$	$\Phi_1^4\Phi_3$	$-1/4320$	$\Phi_1^2\Phi_2^2\Phi_3$	$7/432$
$\Phi_2^4\Phi_3$	$-1/4320$	$\Phi_1^2\Phi_3\Phi_4$	$-1/72$	$\Phi_2^2\Phi_3\Phi_4$	$-1/72$
$\Phi_3\Phi_4^2$	$-1/72$	$\Phi_3\Phi_8$	$1/6$	$\Phi_1^2\Phi_3^2$	$25/432$
$\Phi_2^2\Phi_3^2$	$1/432$	$\Phi_3^2\Phi_4$	$-1/72$	Φ_3^3	$1/162$
$\Phi_1^4\Phi_6$	$-1/4320$	$\Phi_1^2\Phi_2^2\Phi_6$	$7/432$	$\Phi_2^4\Phi_6$	$-1/4320$
$\Phi_1^2\Phi_4\Phi_6$	$-1/72$	$\Phi_2^2\Phi_4\Phi_6$	$-1/72$	$\Phi_4^2\Phi_6$	$-1/72$
$\Phi_6\Phi_8$	$1/6$	$\Phi_1^2\Phi_3\Phi_6$	$-1/27$	$\Phi_2^2\Phi_3\Phi_6$	$-1/27$
$\Phi_3\Phi_4\Phi_6$	$2/9$	$\Phi_3^2\Phi_6$	$1/54$	$\Phi_1^2\Phi_6^2$	$1/432$
$\Phi_2^2\Phi_6^2$	$25/432$	$\Phi_4\Phi_6^2$	$-1/72$	$\Phi_3\Phi_6^2$	$1/54$
Φ_6^3	$1/162$	$\Phi_1^2\Phi_{12}$	$-1/72$	$\Phi_2^2\Phi_{12}$	$-1/72$
$\Phi_4\Phi_{12}$	$5/12$	$\Phi_3\Phi_{12}$	$2/9$	$\Phi_6\Phi_{12}$	$2/9$
Φ_9	$4/9$	Φ_{18}	$4/9$	$\Phi_1^2\Phi_5$	$-1/30$
$\Phi_2^2\Phi_5$	$-1/30$	$\Phi_4\Phi_5$	$1/5$	$\Phi_3\Phi_5$	$2/15$
$\Phi_5\Phi_6$	$2/15$	$\Phi_1^2\Phi_{10}$	$-1/30$	$\Phi_2^2\Phi_{10}$	$-1/30$
$\Phi_4\Phi_{10}$	$1/5$	$\Phi_3\Phi_{10}$	$2/15$	$\Phi_6\Phi_{10}$	$2/15$
Φ_7	$4/7$	Φ_{14}	$4/7$		

Table 3.11: Masses for the group $\mathbf{Sp}_8$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^8	1/87091200	$\Phi_1^6\Phi_2^2$	−127/4354560	$\Phi_1^4\Phi_2^4$	871/2073600	$\Phi_1^2\Phi_2^6$	−127/4354560
Φ_2^8	1/87091200	$\Phi_1^6\Phi_4$	1/725760	$\Phi_1^4\Phi_2^2\Phi_4$	31/34560	$\Phi_1^2\Phi_2^4\Phi_4$	31/34560
$\Phi_2^6\Phi_4$	1/725760	$\Phi_1^4\Phi_4^2$	31/34560	$\Phi_1^2\Phi_2^2\Phi_4^2$	−361/3456	$\Phi_2^4\Phi_4^2$	31/34560
$\Phi_1^2\Phi_4^3$	−7/576	$\Phi_2^2\Phi_4^3$	−7/576	Φ_4^4	1/5760	$\Phi_1^4\Phi_8$	−1/2880
$\Phi_1^2\Phi_2^2\Phi_8$	7/288	$\Phi_2^4\Phi_8$	−1/2880	$\Phi_1^2\Phi_4\Phi_8$	−3/16	$\Phi_2^2\Phi_4\Phi_8$	−3/16
$\Phi_4^2\Phi_8$	−1/48	Φ_8^2	1/24	Φ_{16}	1	$\Phi_1^6\Phi_3$	1/1088640
$\Phi_1^4\Phi_2^2\Phi_3$	31/51840	$\Phi_1^2\Phi_2^4\Phi_3$	31/51840	$\Phi_2^6\Phi_3$	1/1088640	$\Phi_1^4\Phi_3\Phi_4$	−1/8640
$\Phi_1^2\Phi_2^2\Phi_3\Phi_4$	7/864	$\Phi_2^4\Phi_3\Phi_4$	−1/8640	$\Phi_1^2\Phi_3\Phi_4^2$	7/864	$\Phi_2^2\Phi_3\Phi_4^2$	7/864
$\Phi_3\Phi_4^3$	1/144	$\Phi_1^2\Phi_3\Phi_8$	−1/72	$\Phi_2^2\Phi_3\Phi_8$	−1/72	$\Phi_3\Phi_4\Phi_8$	1/4
$\Phi_1^4\Phi_3^2$	241/51840	$\Phi_1^2\Phi_2^2\Phi_3^2$	−175/5184	$\Phi_2^4\Phi_3^2$	1/51840	$\Phi_1^2\Phi_3^2\Phi_4$	25/864
$\Phi_2^2\Phi_3^2\Phi_4$	1/864	$\Phi_3^2\Phi_4^2$	1/864	$\Phi_3^2\Phi_8$	−1/72	$\Phi_1^2\Phi_3^3$	−25/1944
$\Phi_2^2\Phi_3^3$	−1/1944	$\Phi_3^3\Phi_4$	1/324	Φ_3^4	1/19440	$\Phi_1^6\Phi_6$	1/1088640
$\Phi_1^4\Phi_2^2\Phi_6$	31/51840	$\Phi_1^2\Phi_2^4\Phi_6$	31/51840	$\Phi_2^6\Phi_6$	1/1088640	$\Phi_1^4\Phi_4\Phi_6$	−1/8640
$\Phi_1^2\Phi_2^2\Phi_4\Phi_6$	7/864	$\Phi_2^4\Phi_4\Phi_6$	−1/8640	$\Phi_1^2\Phi_4^2\Phi_6$	7/864	$\Phi_2^2\Phi_4^2\Phi_6$	7/864
$\Phi_4^3\Phi_6$	1/144	$\Phi_1^2\Phi_6\Phi_8$	−1/72	$\Phi_2^2\Phi_6\Phi_8$	−1/72	$\Phi_4\Phi_6\Phi_8$	1/4
$\Phi_1^4\Phi_3\Phi_6$	−1/3240	$\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	7/324	$\Phi_2^4\Phi_3\Phi_6$	−1/3240	$\Phi_1^2\Phi_3\Phi_4\Phi_6$	−1/54
$\Phi_2^2\Phi_3\Phi_4\Phi_6$	−1/54	$\Phi_3\Phi_4^2\Phi_6$	−1/54	$\Phi_3\Phi_6\Phi_8$	2/9	$\Phi_1^2\Phi_3^2\Phi_6$	−25/648
$\Phi_2^2\Phi_3^2\Phi_6$	−1/648	$\Phi_3^2\Phi_4\Phi_6$	1/108	$\Phi_3^3\Phi_6$	5/243	$\Phi_1^4\Phi_6^2$	1/51840
$\Phi_1^2\Phi_2^2\Phi_6^2$	−175/5184	$\Phi_2^4\Phi_6^2$	241/51840	$\Phi_1^2\Phi_4\Phi_6^2$	1/864	$\Phi_2^2\Phi_4\Phi_6^2$	25/864
$\Phi_4^2\Phi_6^2$	1/864	$\Phi_6^2\Phi_8$	−1/72	$\Phi_1^2\Phi_3\Phi_6^2$	−1/648	$\Phi_2^2\Phi_3\Phi_6^2$	−25/648
$\Phi_3\Phi_4\Phi_6^2$	1/108	$\Phi_3^2\Phi_6^2$	11/648	$\Phi_1^2\Phi_6^3$	−1/1944	$\Phi_2^2\Phi_6^3$	−25/1944
$\Phi_4\Phi_6^3$	1/324	$\Phi_3\Phi_6^3$	5/243	Φ_6^4	1/19440	$\Phi_1^4\Phi_{12}$	−1/8640
$\Phi_1^2\Phi_2^2\Phi_{12}$	7/864	$\Phi_2^4\Phi_{12}$	−1/8640	$\Phi_1^2\Phi_4\Phi_{12}$	−5/144	$\Phi_2^2\Phi_4\Phi_{12}$	−5/144
$\Phi_4^2\Phi_{12}$	7/144	$\Phi_8\Phi_{12}$	1/12	$\Phi_1^2\Phi_3\Phi_{12}$	−1/54	$\Phi_2^2\Phi_3\Phi_{12}$	−1/54
$\Phi_3\Phi_4\Phi_{12}$	5/9	$\Phi_3^2\Phi_{12}$	1/108	$\Phi_1^2\Phi_6\Phi_{12}$	−1/54	$\Phi_2^2\Phi_6\Phi_{12}$	−1/54
$\Phi_4\Phi_6\Phi_{12}$	5/9	$\Phi_3\Phi_6\Phi_{12}$	14/27	$\Phi_6^2\Phi_{12}$	1/108	Φ_{12}^2	1/36
Φ_{24}	1/3	$\Phi_1^2\Phi_9$	−1/27	$\Phi_2^2\Phi_9$	−1/27	$\Phi_4\Phi_9$	2/9
$\Phi_3\Phi_9$	16/27	$\Phi_6\Phi_9$	4/27	$\Phi_1^2\Phi_{18}$	−1/27	$\Phi_2^2\Phi_{18}$	−1/27
$\Phi_4\Phi_{18}$	2/9	$\Phi_3\Phi_{18}$	4/27	$\Phi_6\Phi_{18}$	16/27	$\Phi_1^4\Phi_5$	−1/3600
$\Phi_1^2\Phi_2^2\Phi_5$	7/360	$\Phi_2^4\Phi_5$	−1/3600	$\Phi_1^2\Phi_4\Phi_5$	−1/60	$\Phi_2^2\Phi_4\Phi_5$	−1/60
$\Phi_4^2\Phi_5$	−1/60	$\Phi_5\Phi_8$	1/5	$\Phi_1^2\Phi_3\Phi_5$	−1/90	$\Phi_2^2\Phi_3\Phi_5$	−1/90
$\Phi_3\Phi_4\Phi_5$	1/15	$\Phi_3^2\Phi_5$	−1/90	$\Phi_1^2\Phi_5\Phi_6$	−1/90	$\Phi_2^2\Phi_5\Phi_6$	−1/90
$\Phi_4\Phi_5\Phi_6$	1/15	$\Phi_3\Phi_5\Phi_6$	8/45	$\Phi_5\Phi_6^2$	−1/90	$\Phi_5\Phi_{12}$	1/15
Φ_5^2	1/75	$\Phi_1^4\Phi_{10}$	−1/3600	$\Phi_1^2\Phi_2^2\Phi_{10}$	7/360	$\Phi_2^4\Phi_{10}$	−1/3600
$\Phi_1^2\Phi_4\Phi_{10}$	−1/60	$\Phi_2^2\Phi_4\Phi_{10}$	−1/60	$\Phi_4^2\Phi_{10}$	−1/60	$\Phi_8\Phi_{10}$	1/5
$\Phi_1^2\Phi_3\Phi_{10}$	−1/90	$\Phi_2^2\Phi_3\Phi_{10}$	−1/90	$\Phi_3\Phi_4\Phi_{10}$	1/15	$\Phi_3^2\Phi_{10}$	−1/90
$\Phi_1^2\Phi_6\Phi_{10}$	−1/90	$\Phi_2^2\Phi_6\Phi_{10}$	−1/90	$\Phi_4\Phi_6\Phi_{10}$	1/15	$\Phi_3\Phi_6\Phi_{10}$	8/45
$\Phi_6^2\Phi_{10}$	−1/90	$\Phi_{10}\Phi_{12}$	1/15	$\Phi_5\Phi_{10}$	24/25	Φ_{10}^2	1/75
Φ_{20}	2/5	Φ_{15}	4/15	Φ_{30}	4/15	$\Phi_1^2\Phi_7$	−1/21
$\Phi_2^2\Phi_7$	−1/21	$\Phi_4\Phi_7$	2/7	$\Phi_3\Phi_7$	4/21	$\Phi_6\Phi_7$	4/21
$\Phi_1^2\Phi_{14}$	−1/21	$\Phi_2^2\Phi_{14}$	−1/21	$\Phi_4\Phi_{14}$	2/7	$\Phi_3\Phi_{14}$	4/21
$\Phi_6\Phi_{14}$	4/21						

For even orthogonal groups and when the characteristic polynomial is coprime to $\Phi_1\Phi_2$, the characteristic polynomial defines two conjugacy classes over $\overline{\mathbb{Q}}$. They have the same mass.

Table 3.12: Masses for the group $\mathbf{SO}_4$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^4	$1/144$	$\Phi_1^2\Phi_2^2$	$1/8$	Φ_2^4	$1/144$
Φ_4^2	$-1/24$	$\Phi_1^2\Phi_3$	$1/9$	Φ_3^2	$-1/36$
$\Phi_2^2\Phi_6$	$1/9$	Φ_6^2	$-1/36$	Φ_{12}	$1/6$

Table 3.13: Masses for the group $\mathbf{SO}_8$

Char. pol.	mass	Char. pol.	mass	Char. pol.	mass
Φ_1^8	$1/58060800$	$\Phi_1^6\Phi_2^2$	$1/15360$	$\Phi_1^4\Phi_2^4$	$1357/165888$
$\Phi_1^2\Phi_2^6$	$1/15360$	Φ_2^8	$1/58060800$	$\Phi_1^4\Phi_2^2\Phi_4$	$1/64$
$\Phi_1^2\Phi_2^4\Phi_4$	$1/64$	$\Phi_1^4\Phi_4^2$	$-55/13824$	$\Phi_1^2\Phi_2^2\Phi_4^2$	$17/768$
$\Phi_2^4\Phi_4^2$	$-55/13824$	Φ_4^4	$1/7680$	$\Phi_1^2\Phi_2^2\Phi_8$	$3/16$
$\Phi_1^2\Phi_4\Phi_8$	$3/32$	$\Phi_2^2\Phi_4\Phi_8$	$3/32$	Φ_8^2	$1/32$
$\Phi_1^6\Phi_3$	$1/25920$	$\Phi_1^4\Phi_2^2\Phi_3$	$1/96$	$\Phi_1^2\Phi_2^4\Phi_3$	$41/5184$
$\Phi_1^2\Phi_2^2\Phi_3\Phi_4$	$1/8$	$\Phi_1^2\Phi_3\Phi_4^2$	$1/432$	$\Phi_1^4\Phi_3^2$	$-19/1728$
$\Phi_1^2\Phi_2^2\Phi_3^2$	$1/96$	$\Phi_2^4\Phi_3^2$	$-1/5184$	$\Phi_3^2\Phi_4^2$	$1/864$
$\Phi_1^2\Phi_3^3$	$1/648$	Φ_3^4	$1/25920$	$\Phi_1^4\Phi_2^2\Phi_6$	$41/5184$
$\Phi_1^2\Phi_2^4\Phi_6$	$1/96$	$\Phi_2^6\Phi_6$	$1/25920$	$\Phi_1^2\Phi_2^2\Phi_4\Phi_6$	$1/8$
$\Phi_2^2\Phi_4^2\Phi_6$	$1/432$	$\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$23/81$	$\Phi_2^2\Phi_3^2\Phi_6$	$1/648$
$\Phi_1^4\Phi_6^2$	$-1/5184$	$\Phi_1^2\Phi_2^2\Phi_6^2$	$1/96$	$\Phi_2^4\Phi_6^2$	$-19/1728$
$\Phi_4^2\Phi_6^2$	$1/864$	$\Phi_1^2\Phi_3\Phi_6^2$	$1/648$	$\Phi_3^2\Phi_6^2$	$41/2592$
$\Phi_2^2\Phi_6^3$	$1/648$	Φ_6^4	$1/25920$	$\Phi_1^4\Phi_{12}$	$1/864$
$\Phi_1^2\Phi_2^2\Phi_{12}$	$1/48$	$\Phi_2^4\Phi_{12}$	$1/864$	$\Phi_4^2\Phi_{12}$	$1/48$
$\Phi_1^2\Phi_3\Phi_{12}$	$5/108$	$\Phi_3^2\Phi_{12}$	$1/432$	$\Phi_2^2\Phi_6\Phi_{12}$	$5/108$
$\Phi_6^2\Phi_{12}$	$1/432$	Φ_{12}^2	$1/48$	Φ_{24}	$1/4$
$\Phi_1^2\Phi_9$	$1/9$	$\Phi_3\Phi_9$	$1/9$	$\Phi_2^2\Phi_{18}$	$1/9$
$\Phi_6\Phi_{18}$	$1/9$	$\Phi_1^4\Phi_5$	$1/100$	$\Phi_1^2\Phi_2^2\Phi_5$	$3/20$
$\Phi_1^2\Phi_3\Phi_5$	$1/5$	Φ_5^2	$1/100$	$\Phi_1^2\Phi_2^2\Phi_{10}$	$3/20$
$\Phi_2^4\Phi_{10}$	$1/100$	$\Phi_2^2\Phi_6\Phi_{10}$	$1/5$	Φ_{10}^2	$1/100$
Φ_{20}	$3/10$	Φ_{15}	$1/5$	Φ_{30}	$1/5$
$\Phi_1^2\Phi_7$	$3/7$	$\Phi_2^2\Phi_{14}$	$3/7$		

3.7.2 Some essentially self-dual, algebraic, level one, automorphic cuspidal representations of GL_n for $n \leq 13$

The following tables list the *non-zero*

$$\mathrm{card}(S(w_1,\dots,w_n)),\ \mathrm{card}(O_o(w_1,\dots,w_n)) \text{ and } \mathrm{card}(O_e(w_1,\dots,w_{2n}))$$

as defined in the introduction. These values depend on Assumption 3.4.2.4 when $w_i = w_{i+1}+1$ for some i or

- $w_n = \frac{1}{2}$ for $\mathrm{card}(S(w_1,\dots,w_n))$,

- $w_n = 1$ for $\mathrm{card}(O_o(w_1, \ldots, w_n))$,
- $w_n = 0$ for $\mathrm{card}(O_e(w_1, \ldots, w_{2n}))$.

Much more data is available at http://www.math.ens.fr/~taibi/dimtrace/.

Table 3.14: $\mathrm{card}\,(S(w))$

$2w$	card.	$2w$	card.	$2w$	card.	$2w$	card.
11	1	23	2	33	2	43	3
15	1	25	1	35	3	45	3
17	1	27	2	37	2	47	4
19	1	29	2	39	3	49	3
21	1	31	2	41	3	51	4

Table 3.15: card $(S(w_1, w_2))$

$(2w_1, 2w_2)$	card.	$(2w_1, 2w_2)$	card.	$(2w_1, 2w_2)$	card.	$(2w_1, 2w_2)$	card.
(19, 7)	1	(31, 5)	1	(35, 25)	5	(39, 33)	4
(21, 5)	1	(31, 7)	3	(35, 27)	3	(39, 35)	1
(21, 9)	1	(31, 9)	2	(35, 29)	2	(39, 37)	1
(21, 13)	1	(31, 11)	3	(35, 31)	1	(41, 1)	1
(23, 7)	1	(31, 13)	4	(37, 1)	1	(41, 3)	1
(23, 9)	1	(31, 15)	4	(37, 5)	4	(41, 5)	6
(23, 13)	1	(31, 17)	3	(37, 7)	3	(41, 7)	4
(25, 5)	1	(31, 19)	4	(37, 9)	7	(41, 9)	9
(25, 7)	1	(31, 21)	3	(37, 11)	5	(41, 11)	6
(25, 9)	2	(31, 23)	2	(37, 13)	9	(41, 13)	13
(25, 11)	1	(31, 25)	2	(37, 15)	6	(41, 15)	10
(25, 13)	2	(33, 5)	3	(37, 17)	9	(41, 17)	13
(25, 15)	1	(33, 7)	2	(37, 19)	8	(41, 19)	11
(25, 17)	1	(33, 9)	5	(37, 21)	10	(41, 21)	14
(25, 19)	1	(33, 11)	2	(37, 23)	7	(41, 23)	11
(27, 3)	1	(33, 13)	6	(37, 25)	9	(41, 25)	15
(27, 7)	2	(33, 15)	4	(37, 27)	6	(41, 27)	11
(27, 9)	1	(33, 17)	6	(37, 29)	5	(41, 29)	11
(27, 11)	2	(33, 19)	5	(37, 31)	4	(41, 31)	9
(27, 13)	2	(33, 21)	5	(37, 33)	2	(41, 33)	8
(27, 15)	2	(33, 23)	3	(39, 3)	3	(41, 35)	4
(27, 17)	1	(33, 25)	4	(39, 5)	2	(41, 37)	3
(27, 19)	1	(33, 27)	2	(39, 7)	7	(43, 3)	5
(27, 21)	1	(33, 29)	1	(39, 9)	5	(43, 5)	3
(29, 5)	2	(35, 3)	2	(39, 11)	8	(43, 7)	9
(29, 7)	1	(35, 5)	1	(39, 13)	8	(43, 9)	7
(29, 9)	3	(35, 7)	5	(39, 15)	10	(43, 11)	11
(29, 11)	1	(35, 9)	4	(39, 17)	8	(43, 13)	11
(29, 13)	4	(35, 11)	5	(39, 19)	11	(43, 15)	15
(29, 15)	2	(35, 13)	5	(39, 21)	10	(43, 17)	13
(29, 17)	3	(35, 15)	6	(39, 23)	10	(43, 19)	17
(29, 19)	2	(35, 17)	5	(39, 25)	10	(43, 21)	14
(29, 21)	2	(35, 19)	7	(39, 27)	9	(43, 23)	16
(29, 25)	1	(35, 21)	6	(39, 29)	7	(43, 25)	16
(31, 3)	2	(35, 23)	5	(39, 31)	6	(43, 27)	16

Table 3.16: card $(S(w_1, w_2, w_3))$

$(2w_i)_i$	card.	$(2w_i)_i$	card.	$(2w_i)_i$	card.	$(2w_i)_i$	card.
(23, 13, 5)	1	(27, 21, 7)	2	(29, 19, 13)	5	(31, 13, 5)	3
(23, 15, 3)	1	(27, 21, 9)	4	(29, 19, 15)	1	(31, 13, 7)	2
(23, 15, 7)	1	(27, 21, 11)	2	(29, 19, 17)	1	(31, 13, 9)	4
(23, 17, 5)	1	(27, 21, 13)	3	(29, 21, 3)	5	(31, 15, 3)	3
(23, 17, 9)	1	(27, 21, 15)	1	(29, 21, 5)	1	(31, 15, 5)	2
(23, 19, 3)	1	(27, 21, 17)	1	(29, 21, 7)	10	(31, 15, 7)	5
(23, 19, 11)	1	(27, 23, 3)	1	(29, 21, 9)	4	(31, 15, 9)	3
(25, 13, 3)	1	(27, 23, 5)	3	(29, 21, 11)	8	(31, 15, 11)	2
(25, 13, 7)	1	(27, 23, 7)	1	(29, 21, 13)	4	(31, 17, 1)	2
(25, 15, 5)	1	(27, 23, 9)	2	(29, 21, 15)	5	(31, 17, 5)	7
(25, 15, 9)	1	(27, 23, 11)	2	(29, 21, 17)	1	(31, 17, 7)	4
(25, 17, 3)	2	(27, 23, 13)	1	(29, 21, 19)	1	(31, 17, 9)	9
(25, 17, 7)	2	(27, 23, 15)	1	(29, 23, 1)	1	(31, 17, 11)	3
(25, 17, 11)	1	(27, 23, 17)	1	(29, 23, 3)	2	(31, 17, 13)	5
(25, 19, 1)	1	(27, 25, 5)	2	(29, 23, 5)	5	(31, 19, 3)	6
(25, 19, 5)	2	(27, 25, 7)	1	(29, 23, 7)	5	(31, 19, 5)	4
(25, 19, 9)	2	(27, 25, 9)	1	(29, 23, 9)	6	(31, 19, 7)	10
(25, 19, 13)	1	(27, 25, 11)	1	(29, 23, 11)	7	(31, 19, 9)	8
(25, 21, 3)	2	(27, 25, 13)	1	(29, 23, 13)	5	(31, 19, 11)	9
(25, 21, 7)	2	(27, 25, 15)	1	(29, 23, 15)	5	(31, 19, 13)	6
(25, 21, 11)	2	(27, 25, 17)	1	(29, 23, 17)	3	(31, 19, 15)	4
(25, 21, 15)	1	(29, 9, 7)	1	(29, 23, 19)	1	(31, 21, 1)	3
(27, 9, 5)	1	(29, 11, 5)	1	(29, 25, 3)	3	(31, 21, 3)	1
(27, 13, 5)	2	(29, 13, 3)	1	(29, 25, 5)	3	(31, 21, 5)	11
(27, 13, 7)	1	(29, 13, 5)	1	(29, 25, 7)	7	(31, 21, 7)	7
(27, 13, 9)	1	(29, 13, 7)	3	(29, 25, 9)	4	(31, 21, 9)	15
(27, 15, 3)	1	(29, 13, 9)	1	(29, 25, 11)	7	(31, 21, 11)	9
(27, 15, 5)	1	(29, 15, 1)	1	(29, 25, 13)	4	(31, 21, 13)	12
(27, 15, 7)	2	(29, 15, 5)	3	(29, 25, 15)	5	(31, 21, 15)	6
(27, 15, 9)	1	(29, 15, 7)	2	(29, 25, 17)	3	(31, 21, 17)	6
(27, 17, 5)	4	(29, 15, 9)	3	(29, 25, 19)	2	(31, 23, 1)	1
(27, 17, 7)	1	(29, 15, 13)	1	(29, 25, 21)	1	(31, 23, 3)	6
(27, 17, 9)	3	(29, 17, 3)	3	(29, 27, 1)	1	(31, 23, 5)	6
(27, 17, 11)	1	(29, 17, 5)	1	(29, 27, 5)	1	(31, 23, 7)	12
(27, 17, 13)	1	(29, 17, 7)	6	(29, 27, 7)	2	(31, 23, 9)	11
(27, 19, 3)	2	(29, 17, 9)	3	(29, 27, 9)	3	(31, 23, 11)	13
(27, 19, 5)	2	(29, 17, 11)	3	(29, 27, 11)	1	(31, 23, 13)	10
(27, 19, 7)	3	(29, 17, 13)	1	(29, 27, 13)	2	(31, 23, 15)	10
(27, 19, 9)	3	(29, 19, 1)	1	(29, 27, 15)	1	(31, 23, 17)	6
(27, 19, 11)	3	(29, 19, 3)	1	(29, 27, 17)	1	(31, 23, 19)	3
(27, 19, 13)	2	(29, 19, 5)	6	(29, 27, 19)	1	(31, 25, 1)	3
(27, 19, 15)	1	(29, 19, 7)	3	(31, 9, 5)	1	(31, 25, 3)	2
(27, 21, 1)	1	(29, 19, 9)	7	(31, 11, 3)	1	(31, 25, 5)	11
(27, 21, 5)	4	(29, 19, 11)	4	(31, 11, 7)	1	(31, 25, 7)	9

Table 3.17: card $(S(w_1, w_2, w_3, w_4))$

$(2w_i)_i$	card.	$(2w_i)_i$	card.	$(2w_i)_i$	card.	$(2w_i)_i$	card.
(25, 17, 9, 5)	1	(27, 19, 13, 9)	3	(27, 23, 15, 7)	7	(27, 25, 19, 7)	3
(25, 17, 13, 5)	1	(27, 19, 15, 3)	2	(27, 23, 15, 9)	4	(27, 25, 19, 9)	6
(25, 19, 9, 3)	1	(27, 19, 15, 5)	1	(27, 23, 15, 11)	5	(27, 25, 19, 11)	3
(25, 19, 11, 5)	1	(27, 19, 15, 7)	1	(27, 23, 15, 13)	1	(27, 25, 19, 13)	3
(25, 19, 13, 3)	1	(27, 19, 15, 9)	1	(27, 23, 17, 1)	5	(27, 25, 21, 3)	4
(25, 19, 13, 5)	1	(27, 19, 17, 5)	1	(27, 23, 17, 3)	2	(27, 25, 21, 7)	4
(25, 19, 13, 7)	1	(27, 19, 17, 9)	1	(27, 23, 17, 5)	6	(27, 25, 21, 9)	2
(25, 19, 13, 9)	1	(27, 21, 9, 3)	2	(27, 23, 17, 7)	5	(27, 25, 21, 11)	3
(25, 19, 15, 5)	1	(27, 21, 9, 7)	1	(27, 23, 17, 9)	7	(27, 25, 21, 13)	1
(25, 21, 11, 7)	1	(27, 21, 11, 3)	1	(27, 23, 17, 11)	3	(27, 25, 21, 15)	1
(25, 21, 13, 5)	1	(27, 21, 11, 5)	2	(27, 23, 17, 13)	4	(27, 25, 23, 3)	1
(25, 21, 13, 7)	1	(27, 21, 11, 7)	2	(27, 23, 19, 3)	5	(27, 25, 23, 9)	1
(25, 21, 15, 3)	1	(27, 21, 13, 3)	5	(27, 23, 19, 5)	1	(27, 25, 23, 11)	1
(25, 21, 15, 5)	1	(27, 21, 13, 5)	2	(27, 23, 19, 7)	6	(29, 15, 7, 5)	1
(25, 21, 15, 7)	2	(27, 21, 13, 7)	6	(27, 23, 19, 9)	2	(29, 15, 9, 3)	1
(25, 21, 15, 9)	1	(27, 21, 13, 9)	2	(27, 23, 19, 11)	3	(29, 15, 13, 3)	1
(25, 21, 17, 5)	1	(27, 21, 15, 1)	1	(27, 23, 19, 13)	1	(29, 17, 7, 3)	1
(25, 21, 17, 7)	1	(27, 21, 15, 3)	2	(27, 23, 19, 15)	1	(29, 17, 9, 5)	3
(25, 21, 17, 9)	1	(27, 21, 15, 5)	4	(27, 23, 21, 1)	1	(29, 17, 11, 3)	2
(25, 23, 9, 3)	1	(27, 21, 15, 7)	4	(27, 23, 21, 5)	1	(29, 17, 11, 7)	1
(25, 23, 11, 1)	1	(27, 21, 15, 9)	4	(27, 23, 21, 9)	1	(29, 17, 13, 1)	1
(25, 23, 11, 5)	2	(27, 21, 15, 11)	2	(27, 25, 9, 3)	2	(29, 17, 13, 5)	4
(25, 23, 13, 3)	1	(27, 21, 17, 3)	5	(27, 25, 11, 1)	1	(29, 17, 13, 7)	1
(25, 23, 13, 7)	1	(27, 21, 17, 7)	6	(27, 25, 11, 3)	1	(29, 17, 13, 9)	2
(25, 23, 15, 1)	1	(27, 21, 17, 9)	2	(27, 25, 11, 5)	2	(29, 17, 15, 3)	1
(25, 23, 15, 5)	3	(27, 21, 17, 11)	3	(27, 25, 13, 3)	5	(29, 17, 15, 7)	1
(25, 23, 15, 9)	1	(27, 21, 19, 3)	1	(27, 25, 13, 5)	1	(29, 19, 7, 5)	1
(25, 23, 15, 11)	1	(27, 21, 19, 5)	1	(27, 25, 13, 7)	4	(29, 19, 9, 3)	4
(25, 23, 17, 3)	1	(27, 21, 19, 7)	1	(27, 25, 13, 9)	1	(29, 19, 9, 5)	1
(25, 23, 17, 5)	1	(27, 21, 19, 9)	1	(27, 25, 15, 1)	3	(29, 19, 9, 7)	1
(25, 23, 17, 7)	1	(27, 21, 19, 11)	1	(27, 25, 15, 3)	2	(29, 19, 11, 1)	1
(25, 23, 17, 11)	1	(27, 23, 7, 3)	2	(27, 25, 15, 5)	5	(29, 19, 11, 3)	1
(25, 23, 19, 5)	1	(27, 23, 9, 1)	1	(27, 25, 15, 7)	3	(29, 19, 11, 5)	4
(27, 17, 9, 3)	1	(27, 23, 9, 5)	2	(27, 25, 15, 9)	5	(29, 19, 11, 7)	1
(27, 17, 9, 7)	1	(27, 23, 11, 3)	5	(27, 25, 15, 11)	1	(29, 19, 11, 9)	1
(27, 17, 13, 3)	2	(27, 23, 11, 5)	1	(27, 25, 17, 3)	7	(29, 19, 13, 3)	8
(27, 17, 13, 7)	2	(27, 23, 11, 7)	4	(27, 25, 17, 5)	2	(29, 19, 13, 5)	4
(27, 19, 9, 5)	1	(27, 23, 13, 1)	4	(27, 25, 17, 7)	7	(29, 19, 13, 7)	6
(27, 19, 11, 3)	2	(27, 23, 13, 3)	1	(27, 25, 17, 9)	4	(29, 19, 13, 9)	4
(27, 19, 11, 5)	1	(27, 23, 13, 5)	6	(27, 25, 17, 11)	5	(29, 19, 13, 11)	1
(27, 19, 13, 1)	1	(27, 23, 13, 7)	3	(27, 25, 17, 13)	1	(29, 19, 15, 1)	2
(27, 19, 13, 3)	1	(27, 23, 13, 9)	6	(27, 25, 19, 1)	3	(29, 19, 15, 3)	2
(27, 19, 13, 5)	4	(27, 23, 15, 3)	7	(27, 25, 19, 3)	2	(29, 19, 15, 5)	5
(27, 19, 13, 7)	1	(27, 23, 15, 5)	3	(27, 25, 19, 5)	5	(29, 19, 15, 7)	3

Table 3.18: card $(S(w_1, w_2, w_3, w_4))$

$(2w_i)_i$	card.	$(2w_i)_i$	card.	$(2w_i)_i$	card.
(23, 21, 17, 11, 3)	1	(27, 19, 15, 13, 5)	1	(27, 23, 13, 7, 5)	1
(25, 19, 15, 9, 3)	1	(27, 19, 17, 9, 3)	2	(27, 23, 13, 9, 3)	4
(25, 21, 13, 9, 3)	1	(27, 19, 17, 11, 5)	2	(27, 23, 13, 9, 7)	1
(25, 21, 15, 7, 3)	1	(27, 19, 17, 13, 3)	1	(27, 23, 13, 11, 5)	1
(25, 21, 15, 9, 1)	1	(27, 21, 13, 7, 3)	2	(27, 23, 15, 7, 3)	2
(25, 21, 15, 9, 5)	1	(27, 21, 13, 9, 1)	1	(27, 23, 15, 7, 5)	1
(25, 21, 15, 11, 3)	1	(27, 21, 13, 9, 5)	1	(27, 23, 15, 9, 1)	2
(25, 21, 15, 13, 1)	1	(27, 21, 13, 11, 3)	1	(27, 23, 15, 9, 3)	1
(25, 21, 17, 9, 3)	2	(27, 21, 15, 7, 1)	1	(27, 23, 15, 9, 5)	8
(25, 21, 17, 11, 1)	1	(27, 21, 15, 7, 5)	2	(27, 23, 15, 9, 7)	1
(25, 21, 17, 11, 5)	2	(27, 21, 15, 9, 3)	4	(27, 23, 15, 11, 3)	3
(25, 21, 17, 13, 3)	1	(27, 21, 15, 9, 5)	1	(27, 23, 15, 11, 5)	4
(25, 21, 17, 13, 7)	1	(27, 21, 15, 9, 7)	2	(27, 23, 15, 11, 7)	2
(25, 21, 19, 11, 3)	2	(27, 21, 15, 11, 1)	2	(27, 23, 15, 13, 1)	1
(25, 21, 19, 13, 1)	1	(27, 21, 15, 11, 5)	4	(27, 23, 15, 13, 3)	1
(25, 21, 19, 15, 3)	1	(27, 21, 15, 13, 3)	2	(27, 23, 15, 13, 5)	4
(25, 23, 15, 9, 3)	1	(27, 21, 15, 13, 5)	1	(27, 23, 15, 13, 7)	2
(25, 23, 17, 7, 3)	1	(27, 21, 15, 13, 7)	1	(27, 23, 15, 13, 9)	1
(25, 23, 17, 11, 3)	2	(27, 21, 17, 7, 3)	5	(27, 23, 17, 5, 3)	1
(25, 23, 17, 11, 5)	1	(27, 21, 17, 9, 1)	3	(27, 23, 17, 7, 1)	2
(25, 23, 17, 11, 7)	1	(27, 21, 17, 9, 5)	5	(27, 23, 17, 7, 5)	5
(25, 23, 17, 13, 1)	1	(27, 21, 17, 9, 7)	1	(27, 23, 17, 9, 3)	11
(25, 23, 17, 13, 5)	1	(27, 21, 17, 11, 3)	7	(27, 23, 17, 9, 5)	4
(25, 23, 17, 13, 7)	1	(27, 21, 17, 11, 5)	2	(27, 23, 17, 9, 7)	4
(25, 23, 17, 13, 9)	1	(27, 21, 17, 11, 7)	5	(27, 23, 17, 11, 1)	3
(25, 23, 19, 9, 5)	1	(27, 21, 17, 13, 1)	3	(27, 23, 17, 11, 3)	3
(25, 23, 19, 11, 3)	2	(27, 21, 17, 13, 5)	7	(27, 23, 17, 11, 5)	14
(25, 23, 19, 11, 5)	1	(27, 21, 17, 13, 7)	2	(27, 23, 17, 11, 7)	6
(25, 23, 19, 11, 7)	1	(27, 21, 17, 13, 9)	2	(27, 23, 17, 11, 9)	2
(25, 23, 19, 13, 3)	1	(27, 21, 17, 15, 3)	2	(27, 23, 17, 13, 3)	12
(25, 23, 19, 13, 5)	1	(27, 21, 17, 15, 7)	1	(27, 23, 17, 13, 5)	7
(25, 23, 19, 13, 7)	1	(27, 21, 19, 7, 1)	1	(27, 23, 17, 13, 7)	12
(25, 23, 19, 15, 3)	1	(27, 21, 19, 9, 3)	5	(27, 23, 17, 13, 9)	4
(25, 23, 19, 15, 7)	2	(27, 21, 19, 11, 1)	4	(27, 23, 17, 13, 11)	1
(25, 23, 19, 15, 9)	1	(27, 21, 19, 11, 5)	6	(27, 23, 17, 15, 1)	1
(25, 23, 21, 11, 3)	1	(27, 21, 19, 11, 7)	1	(27, 23, 17, 15, 5)	5
(25, 23, 21, 11, 5)	1	(27, 21, 19, 11, 9)	1	(27, 23, 17, 15, 7)	2
(25, 23, 21, 13, 3)	1	(27, 21, 19, 13, 3)	5	(27, 23, 17, 15, 9)	2
(25, 23, 21, 13, 5)	1	(27, 21, 19, 13, 5)	1	(27, 23, 19, 7, 3)	4
(27, 19, 13, 9, 3)	1	(27, 21, 19, 13, 7)	3	(27, 23, 19, 7, 5)	1
(27, 19, 15, 7, 3)	1	(27, 21, 19, 15, 1)	3	(27, 23, 19, 9, 1)	3
(27, 19, 15, 9, 1)	1	(27, 21, 19, 15, 5)	4	(27, 23, 19, 9, 3)	4
(27, 19, 15, 9, 5)	2	(27, 21, 19, 15, 9)	2	(27, 23, 19, 9, 5)	13
(27, 19, 15, 11, 3)	1	(27, 21, 19, 17, 3)	1	(27, 23, 19, 9, 7)	2

Table 3.19: card $(S(w_1, w_2, w_3, w_4, w_5, w_6))$

$(2w_i)_i$	card.	$(2w_i)_i$	card.	$(2w_i)_i$	card.
(25, 21, 17, 13, 7, 3)	1	(25, 23, 21, 19, 11, 7)	1	(27, 23, 17, 15, 7, 1)	4
(25, 23, 17, 11, 7, 3)	1	(25, 23, 21, 19, 13, 5)	1	(27, 23, 17, 15, 7, 5)	3
(25, 23, 17, 13, 7, 1)	1	(27, 21, 15, 13, 7, 3)	1	(27, 23, 17, 15, 9, 3)	8
(25, 23, 17, 13, 9, 3)	2	(27, 21, 17, 11, 7, 3)	1	(27, 23, 17, 15, 9, 5)	1
(25, 23, 19, 13, 7, 3)	2	(27, 21, 17, 13, 5, 3)	1	(27, 23, 17, 15, 9, 7)	1
(25, 23, 19, 13, 9, 1)	2	(27, 21, 17, 13, 7, 1)	1	(27, 23, 17, 15, 11, 1)	2
(25, 23, 19, 13, 9, 5)	3	(27, 21, 17, 13, 7, 5)	2	(27, 23, 17, 15, 11, 5)	3
(25, 23, 19, 13, 11, 3)	2	(27, 21, 17, 13, 9, 3)	4	(27, 23, 17, 15, 13, 3)	1
(25, 23, 19, 15, 7, 1)	2	(27, 21, 17, 13, 9, 5)	1	(27, 23, 19, 9, 5, 3)	1
(25, 23, 19, 15, 9, 3)	2	(27, 21, 17, 15, 7, 3)	2	(27, 23, 19, 9, 7, 1)	1
(25, 23, 19, 15, 9, 5)	2	(27, 21, 17, 15, 9, 5)	2	(27, 23, 19, 11, 5, 1)	1
(25, 23, 19, 15, 11, 1)	3	(27, 21, 19, 13, 7, 3)	3	(27, 23, 19, 11, 7, 3)	7
(25, 23, 19, 15, 11, 5)	3	(27, 21, 19, 13, 9, 1)	1	(27, 23, 19, 11, 9, 1)	2
(25, 23, 19, 15, 13, 3)	1	(27, 21, 19, 13, 9, 3)	2	(27, 23, 19, 11, 9, 5)	4
(25, 23, 19, 17, 9, 1)	2	(27, 21, 19, 13, 9, 5)	2	(27, 23, 19, 13, 5, 3)	7
(25, 23, 19, 17, 9, 5)	2	(27, 21, 19, 13, 11, 3)	1	(27, 23, 19, 13, 7, 1)	7
(25, 23, 19, 17, 11, 3)	2	(27, 21, 19, 13, 11, 5)	1	(27, 23, 19, 13, 7, 3)	3
(25, 23, 19, 17, 13, 1)	1	(27, 21, 19, 15, 5, 3)	1	(27, 23, 19, 13, 7, 5)	8
(25, 23, 19, 17, 13, 5)	1	(27, 21, 19, 15, 7, 1)	1	(27, 23, 19, 13, 9, 3)	25
(25, 23, 21, 11, 7, 3)	1	(27, 21, 19, 15, 9, 3)	4	(27, 23, 19, 13, 9, 5)	9
(25, 23, 21, 13, 7, 1)	2	(27, 21, 19, 15, 9, 5)	2	(27, 23, 19, 13, 9, 7)	6
(25, 23, 21, 13, 9, 3)	2	(27, 21, 19, 15, 11, 3)	2	(27, 23, 19, 13, 11, 1)	6
(25, 23, 21, 13, 11, 1)	1	(27, 21, 19, 15, 11, 5)	1	(27, 23, 19, 13, 11, 3)	3
(25, 23, 21, 15, 7, 3)	2	(27, 21, 19, 15, 11, 7)	1	(27, 23, 19, 13, 11, 5)	7
(25, 23, 21, 15, 9, 1)	2	(27, 21, 19, 17, 9, 5)	1	(27, 23, 19, 13, 11, 7)	2
(25, 23, 21, 15, 9, 5)	2	(27, 21, 19, 17, 11, 3)	2	(27, 23, 19, 15, 5, 1)	6
(25, 23, 21, 15, 11, 3)	5	(27, 23, 15, 11, 7, 3)	1	(27, 23, 19, 15, 7, 3)	13
(25, 23, 21, 15, 11, 5)	1	(27, 23, 15, 13, 7, 1)	1	(27, 23, 19, 15, 7, 5)	3
(25, 23, 21, 15, 11, 7)	2	(27, 23, 15, 13, 9, 3)	2	(27, 23, 19, 15, 9, 1)	15
(25, 23, 21, 15, 13, 5)	1	(27, 23, 17, 9, 7, 3)	2	(27, 23, 19, 15, 9, 3)	8
(25, 23, 21, 17, 7, 1)	2	(27, 23, 17, 11, 5, 3)	3	(27, 23, 19, 15, 9, 5)	24
(25, 23, 21, 17, 7, 5)	1	(27, 23, 17, 11, 7, 1)	2	(27, 23, 19, 15, 9, 7)	5
(25, 23, 21, 17, 9, 3)	3	(27, 23, 17, 11, 7, 5)	2	(27, 23, 19, 15, 11, 1)	1
(25, 23, 21, 17, 9, 7)	1	(27, 23, 17, 11, 9, 3)	3	(27, 23, 19, 15, 11, 3)	18
(25, 23, 21, 17, 11, 1)	3	(27, 23, 17, 13, 5, 1)	1	(27, 23, 19, 15, 11, 5)	8
(25, 23, 21, 17, 11, 5)	3	(27, 23, 17, 13, 7, 3)	12	(27, 23, 19, 15, 11, 7)	9
(25, 23, 21, 17, 11, 7)	1	(27, 23, 17, 13, 7, 5)	1	(27, 23, 19, 15, 11, 9)	1
(25, 23, 21, 17, 13, 3)	3	(27, 23, 17, 13, 9, 1)	6	(27, 23, 19, 15, 13, 1)	8
(25, 23, 21, 17, 13, 7)	1	(27, 23, 17, 13, 9, 3)	1	(27, 23, 19, 15, 13, 3)	2
(25, 23, 21, 17, 15, 1)	1	(27, 23, 17, 13, 9, 5)	9	(27, 23, 19, 15, 13, 5)	8
(25, 23, 21, 17, 15, 5)	1	(27, 23, 17, 13, 9, 7)	1	(27, 23, 19, 15, 13, 7)	3
(25, 23, 21, 19, 7, 3)	1	(27, 23, 17, 13, 11, 3)	4	(27, 23, 19, 15, 13, 9)	3
(25, 23, 21, 19, 9, 1)	1	(27, 23, 17, 13, 11, 7)	1	(27, 23, 19, 17, 5, 3)	4
(25, 23, 21, 19, 9, 5)	1	(27, 23, 17, 15, 5, 3)	2	(27, 23, 19, 17, 7, 1)	6

Table 3.20: card $(O_o(w))$

w	card.	w	card.	w	card.	w	card.
11	1	21	1	29	2	37	2
15	1	23	2	31	2	39	3
17	1	25	1	33	2	41	3
19	1	27	2	35	3	43	3

Table 3.21: card $(O_o(w_1, w_2))$

(w_1, w_2)	card.	(w_1, w_2)	card.	(w_1, w_2)	card.	(w_1, w_2)	card.
(13, 6)	1	(21, 12)	5	(25, 6)	4	(28, 7)	6
(13, 8)	1	(21, 14)	5	(25, 8)	6	(28, 9)	8
(15, 6)	1	(21, 16)	4	(25, 10)	6	(28, 11)	8
(15, 8)	1	(21, 18)	3	(25, 12)	9	(28, 13)	10
(15, 10)	1	(21, 20)	1	(25, 14)	8	(28, 15)	11
(15, 12)	1	(22, 3)	1	(25, 16)	9	(28, 17)	9
(16, 7)	1	(22, 5)	1	(25, 18)	9	(28, 19)	10
(16, 9)	1	(22, 7)	2	(25, 20)	8	(28, 21)	9
(17, 4)	1	(22, 9)	4	(25, 22)	5	(28, 23)	5
(17, 8)	2	(22, 11)	2	(25, 24)	2	(28, 25)	3
(17, 10)	2	(22, 13)	4	(26, 5)	3	(29, 4)	4
(17, 12)	2	(22, 15)	3	(26, 7)	5	(29, 6)	5
(17, 14)	2	(22, 17)	2	(26, 9)	5	(29, 8)	10
(18, 5)	1	(22, 19)	1	(26, 11)	6	(29, 10)	11
(18, 7)	1	(23, 4)	1	(26, 13)	8	(29, 12)	13
(18, 9)	1	(23, 6)	3	(26, 15)	6	(29, 14)	15
(18, 11)	1	(23, 8)	4	(26, 17)	7	(29, 16)	17
(18, 13)	1	(23, 10)	6	(26, 19)	6	(29, 18)	15
(19, 6)	2	(23, 12)	5	(26, 21)	4	(29, 20)	17
(19, 8)	2	(23, 14)	7	(26, 23)	1	(29, 22)	15
(19, 10)	3	(23, 16)	7	(27, 2)	1	(29, 24)	13
(19, 12)	3	(23, 18)	6	(27, 4)	2	(29, 26)	10
(19, 14)	3	(23, 20)	5	(27, 6)	5	(29, 28)	4
(19, 16)	2	(23, 22)	2	(27, 8)	7	(30, 3)	2
(19, 18)	1	(24, 3)	1	(27, 10)	9	(30, 5)	5
(20, 5)	1	(24, 5)	2	(27, 12)	10	(30, 7)	7
(20, 7)	2	(24, 7)	3	(27, 14)	13	(30, 9)	10
(20, 9)	1	(24, 9)	4	(27, 16)	11	(30, 11)	11
(20, 11)	2	(24, 11)	5	(27, 18)	13	(30, 13)	13
(20, 13)	2	(24, 13)	5	(27, 20)	12	(30, 15)	13
(20, 15)	1	(24, 15)	5	(27, 22)	10	(30, 17)	15
(21, 4)	1	(24, 17)	4	(27, 24)	8	(30, 19)	13
(21, 6)	2	(24, 19)	3	(27, 26)	3	(30, 21)	13
(21, 8)	4	(24, 21)	1	(28, 3)	2	(30, 23)	10
(21, 10)	3	(25, 4)	2	(28, 5)	3	(30, 25)	8

Table 3.22: $\mathrm{card}\,(O_o(w_1, w_2, w_3))$

$(w_i)_i$	card.	$(w_i)_i$	card.	$(w_i)_i$	card.
(12, 8, 4)	1	(16, 9, 5)	1	(17, 12, 7)	5
(13, 8, 5)	1	(16, 9, 7)	1	(17, 12, 9)	6
(13, 10, 3)	1	(16, 10, 2)	1	(17, 12, 11)	2
(13, 10, 5)	1	(16, 10, 4)	2	(17, 13, 2)	1
(13, 10, 7)	1	(16, 10, 6)	2	(17, 13, 4)	4
(13, 12, 5)	1	(16, 10, 8)	1	(17, 13, 6)	5
(13, 12, 7)	1	(16, 11, 3)	1	(17, 13, 8)	3
(13, 12, 9)	1	(16, 11, 5)	2	(17, 13, 10)	2
(14, 7, 3)	1	(16, 11, 7)	1	(17, 14, 3)	4
(14, 8, 4)	1	(16, 11, 9)	1	(17, 14, 5)	6
(14, 9, 5)	1	(16, 12, 2)	1	(17, 14, 7)	8
(14, 10, 4)	1	(16, 12, 4)	3	(17, 14, 9)	7
(14, 10, 6)	1	(16, 12, 6)	3	(17, 14, 11)	6
(14, 12, 2)	1	(16, 12, 8)	2	(17, 14, 13)	2
(14, 12, 6)	1	(16, 12, 10)	2	(17, 15, 2)	2
(14, 12, 8)	1	(16, 13, 3)	2	(17, 15, 4)	2
(15, 8, 3)	1	(16, 13, 5)	2	(17, 15, 6)	3
(15, 8, 5)	1	(16, 13, 7)	2	(17, 15, 8)	4
(15, 8, 7)	1	(16, 13, 9)	1	(17, 15, 10)	2
(15, 9, 4)	1	(16, 13, 11)	1	(17, 15, 12)	1
(15, 10, 3)	1	(16, 14, 2)	2	(17, 16, 1)	1
(15, 10, 5)	2	(16, 14, 4)	2	(17, 16, 3)	2
(15, 10, 7)	1	(16, 14, 6)	3	(17, 16, 5)	4
(15, 10, 9)	1	(16, 14, 8)	3	(17, 16, 7)	6
(15, 11, 4)	1	(16, 14, 10)	2	(17, 16, 9)	7
(15, 11, 6)	1	(16, 14, 12)	1	(17, 16, 11)	3
(15, 12, 3)	2	(17, 6, 3)	1	(17, 16, 13)	4
(15, 12, 5)	2	(17, 7, 4)	1	(18, 6, 4)	1
(15, 12, 7)	3	(17, 8, 3)	1	(18, 7, 3)	1
(15, 12, 9)	2	(17, 8, 5)	3	(18, 7, 5)	1
(15, 13, 4)	1	(17, 8, 7)	1	(18, 8, 2)	1
(15, 13, 6)	1	(17, 9, 2)	1	(18, 8, 4)	3
(15, 13, 8)	1	(17, 9, 4)	1	(18, 8, 6)	2
(15, 14, 1)	1	(17, 9, 6)	1	(18, 9, 3)	2
(15, 14, 5)	2	(17, 10, 3)	3	(18, 9, 5)	3
(15, 14, 7)	3	(17, 10, 5)	3	(18, 9, 7)	2
(15, 14, 9)	3	(17, 10, 7)	4	(18, 10, 2)	2
(15, 14, 13)	1	(17, 10, 9)	2	(18, 10, 4)	4
(16, 6, 4)	1	(17, 11, 2)	1	(18, 10, 6)	4
(16, 7, 5)	1	(17, 11, 4)	3	(18, 10, 8)	2
(16, 8, 2)	1	(17, 11, 6)	1	(18, 11, 3)	3
(16, 8, 4)	1	(17, 11, 8)	1	(18, 11, 5)	4
(16, 8, 6)	1	(17, 12, 3)	3	(18, 11, 7)	4
(16, 9, 3)	1	(17, 12, 5)	7	(18, 11, 9)	2

Table 3.23: card $(O_o(w_1, w_2, w_3, w_4))$

$(w_i)_i$	card.	$(w_i)_i$	card.	$(w_i)_i$	card.
(13, 10, 9, 4)	1	(15, 12, 11, 6)	4	(15, 14, 13, 10)	3
(13, 12, 7, 4)	1	(15, 12, 11, 8)	2	(16, 8, 7, 3)	1
(13, 12, 9, 4)	1	(15, 12, 11, 10)	1	(16, 10, 7, 3)	1
(13, 12, 9, 6)	1	(15, 13, 7, 3)	2	(16, 10, 7, 5)	1
(13, 12, 11, 4)	1	(15, 13, 7, 5)	1	(16, 10, 8, 4)	1
(14, 10, 9, 5)	1	(15, 13, 8, 2)	1	(16, 10, 9, 3)	2
(14, 12, 7, 3)	1	(15, 13, 8, 4)	1	(16, 10, 9, 5)	2
(14, 12, 8, 4)	1	(15, 13, 8, 6)	1	(16, 10, 9, 7)	1
(14, 12, 9, 3)	1	(15, 13, 9, 3)	3	(16, 11, 6, 3)	1
(14, 12, 9, 5)	2	(15, 13, 9, 5)	4	(16, 11, 7, 4)	1
(14, 12, 9, 7)	1	(15, 13, 9, 7)	2	(16, 11, 8, 5)	1
(14, 12, 10, 4)	1	(15, 13, 10, 4)	2	(16, 11, 9, 2)	1
(14, 12, 10, 6)	1	(15, 13, 10, 6)	2	(16, 11, 9, 4)	2
(14, 12, 11, 3)	1	(15, 13, 10, 8)	1	(16, 11, 9, 6)	2
(14, 12, 11, 5)	1	(15, 13, 11, 1)	1	(16, 11, 9, 8)	1
(14, 12, 11, 7)	1	(15, 13, 11, 3)	2	(16, 11, 10, 3)	1
(14, 13, 8, 5)	1	(15, 13, 11, 5)	3	(16, 11, 10, 7)	1
(14, 13, 10, 5)	1	(15, 13, 11, 7)	3	(16, 12, 5, 3)	1
(14, 13, 10, 7)	1	(15, 13, 11, 9)	1	(16, 12, 6, 4)	2
(15, 10, 5, 4)	1	(15, 14, 5, 2)	1	(16, 12, 7, 3)	2
(15, 10, 7, 4)	1	(15, 14, 7, 2)	1	(16, 12, 7, 5)	3
(15, 10, 7, 6)	1	(15, 14, 7, 4)	4	(16, 12, 8, 2)	1
(15, 10, 9, 2)	1	(15, 14, 7, 6)	2	(16, 12, 8, 4)	3
(15, 10, 9, 4)	1	(15, 14, 8, 3)	1	(16, 12, 8, 6)	3
(15, 10, 9, 6)	1	(15, 14, 8, 5)	1	(16, 12, 9, 1)	1
(15, 10, 9, 8)	1	(15, 14, 9, 2)	3	(16, 12, 9, 3)	5
(15, 11, 7, 5)	1	(15, 14, 9, 4)	6	(16, 12, 9, 5)	6
(15, 11, 9, 3)	1	(15, 14, 9, 6)	7	(16, 12, 9, 7)	5
(15, 11, 9, 5)	1	(15, 14, 9, 8)	3	(16, 12, 10, 2)	2
(15, 11, 9, 7)	1	(15, 14, 10, 3)	2	(16, 12, 10, 4)	5
(15, 12, 5, 4)	2	(15, 14, 10, 5)	3	(16, 12, 10, 6)	4
(15, 12, 7, 2)	1	(15, 14, 10, 7)	1	(16, 12, 10, 8)	3
(15, 12, 7, 4)	2	(15, 14, 11, 2)	2	(16, 12, 11, 3)	5
(15, 12, 7, 6)	3	(15, 14, 11, 4)	7	(16, 12, 11, 5)	6
(15, 12, 8, 3)	1	(15, 14, 11, 6)	8	(16, 12, 11, 7)	3
(15, 12, 9, 2)	1	(15, 14, 11, 8)	7	(16, 12, 11, 9)	1
(15, 12, 9, 4)	5	(15, 14, 11, 10)	2	(16, 13, 4, 3)	1
(15, 12, 9, 6)	4	(15, 14, 12, 3)	3	(16, 13, 5, 4)	1
(15, 12, 9, 8)	3	(15, 14, 12, 5)	3	(16, 13, 6, 3)	2
(15, 12, 10, 1)	1	(15, 14, 12, 7)	2	(16, 13, 6, 5)	2
(15, 12, 10, 3)	1	(15, 14, 13, 2)	1	(16, 13, 7, 2)	1
(15, 12, 10, 5)	2	(15, 14, 13, 4)	4	(16, 13, 7, 4)	2
(15, 12, 11, 2)	2	(15, 14, 13, 6)	3	(16, 13, 7, 6)	2
(15, 12, 11, 4)	4	(15, 14, 13, 8)	3	(16, 13, 8, 3)	4

Table 3.24: card $(O_o(w_1, w_2, w_3, w_4, w_5))$

$(w_i)_i$	card.	$(w_i)_i$	card.	$(w_i)_i$	card.
(13, 10, 9, 6, 3)	1	(14, 12, 11, 6, 4)	1	(14, 13, 11, 9, 6)	2
(13, 11, 9, 8, 4)	1	(14, 12, 11, 7, 1)	1	(14, 13, 12, 7, 1)	1
(13, 12, 7, 6, 3)	1	(14, 12, 11, 7, 3)	1	(14, 13, 12, 7, 3)	1
(13, 12, 9, 4, 3)	1	(14, 12, 11, 7, 5)	1	(14, 13, 12, 8, 4)	1
(13, 12, 9, 6, 1)	1	(14, 12, 11, 8, 2)	2	(14, 13, 12, 9, 5)	1
(13, 12, 9, 6, 3)	1	(14, 12, 11, 8, 4)	2	(15, 10, 7, 6, 3)	1
(13, 12, 9, 6, 5)	1	(14, 12, 11, 8, 6)	1	(15, 10, 9, 4, 3)	1
(13, 12, 9, 8, 3)	2	(14, 12, 11, 9, 3)	1	(15, 10, 9, 6, 1)	1
(13, 12, 9, 8, 5)	1	(14, 12, 11, 9, 5)	1	(15, 10, 9, 6, 3)	2
(13, 12, 10, 6, 2)	1	(14, 12, 11, 10, 4)	1	(15, 10, 9, 6, 5)	1
(13, 12, 10, 8, 4)	1	(14, 13, 7, 5, 4)	1	(15, 10, 9, 7, 2)	1
(13, 12, 11, 6, 3)	2	(14, 13, 8, 6, 4)	1	(15, 10, 9, 8, 3)	2
(13, 12, 11, 8, 3)	1	(14, 13, 8, 7, 3)	1	(15, 11, 7, 6, 2)	1
(13, 12, 11, 8, 5)	1	(14, 13, 8, 7, 5)	1	(15, 11, 8, 6, 3)	1
(14, 10, 7, 6, 2)	1	(14, 13, 9, 3, 2)	1	(15, 11, 9, 5, 3)	1
(14, 10, 9, 6, 2)	1	(14, 13, 9, 5, 2)	1	(15, 11, 9, 6, 2)	2
(14, 10, 9, 7, 1)	1	(14, 13, 9, 5, 4)	2	(15, 11, 9, 6, 4)	2
(14, 10, 9, 8, 2)	1	(14, 13, 9, 6, 1)	1	(15, 11, 9, 7, 1)	1
(14, 11, 9, 5, 2)	1	(14, 13, 9, 7, 2)	1	(15, 11, 9, 7, 3)	2
(14, 11, 9, 6, 3)	1	(14, 13, 9, 7, 4)	3	(15, 11, 9, 7, 5)	1
(14, 11, 9, 7, 2)	1	(14, 13, 9, 7, 6)	2	(15, 11, 9, 8, 2)	3
(14, 11, 9, 7, 4)	1	(14, 13, 9, 8, 3)	1	(15, 11, 9, 8, 4)	2
(14, 11, 9, 8, 1)	1	(14, 13, 10, 5, 3)	1	(15, 11, 9, 8, 6)	1
(14, 11, 9, 8, 5)	1	(14, 13, 10, 6, 2)	1	(15, 11, 10, 5, 4)	1
(14, 12, 7, 6, 4)	1	(14, 13, 10, 6, 4)	1	(15, 11, 10, 7, 4)	1
(14, 12, 8, 6, 3)	1	(14, 13, 10, 7, 1)	1	(15, 12, 7, 4, 3)	1
(14, 12, 9, 4, 2)	1	(14, 13, 10, 7, 3)	2	(15, 12, 7, 6, 1)	1
(14, 12, 9, 5, 3)	1	(14, 13, 10, 7, 5)	2	(15, 12, 7, 6, 3)	2
(14, 12, 9, 6, 2)	2	(14, 13, 10, 8, 2)	1	(15, 12, 7, 6, 5)	1
(14, 12, 9, 6, 4)	2	(14, 13, 10, 8, 4)	2	(15, 12, 8, 4, 2)	1
(14, 12, 9, 7, 3)	2	(14, 13, 10, 8, 6)	1	(15, 12, 8, 5, 3)	1
(14, 12, 9, 7, 5)	1	(14, 13, 10, 9, 3)	2	(15, 12, 8, 6, 2)	2
(14, 12, 9, 8, 2)	1	(14, 13, 10, 9, 5)	1	(15, 12, 8, 6, 4)	2
(14, 12, 9, 8, 4)	3	(14, 13, 10, 9, 7)	1	(15, 12, 9, 4, 1)	1
(14, 12, 9, 8, 6)	1	(14, 13, 11, 5, 2)	2	(15, 12, 9, 4, 3)	2
(14, 12, 10, 5, 2)	1	(14, 13, 11, 5, 4)	1	(15, 12, 9, 5, 2)	2
(14, 12, 10, 6, 1)	1	(14, 13, 11, 6, 3)	1	(15, 12, 9, 6, 1)	2
(14, 12, 10, 6, 3)	2	(14, 13, 11, 7, 2)	3	(15, 12, 9, 6, 3)	8
(14, 12, 10, 6, 5)	1	(14, 13, 11, 7, 4)	3	(15, 12, 9, 6, 5)	4
(14, 12, 10, 7, 2)	2	(14, 13, 11, 7, 6)	1	(15, 12, 9, 7, 2)	4
(14, 12, 10, 7, 4)	1	(14, 13, 11, 8, 3)	1	(15, 12, 9, 7, 4)	4
(14, 12, 10, 8, 3)	2	(14, 13, 11, 8, 5)	1	(15, 12, 9, 8, 1)	4
(14, 12, 10, 8, 5)	2	(14, 13, 11, 9, 2)	1	(15, 12, 9, 8, 3)	6
(14, 12, 11, 6, 2)	2	(14, 13, 11, 9, 4)	2	(15, 12, 9, 8, 5)	8

Table 3.25: card $(O_o(w_1, w_2, w_3, w_4, w_5, w_6))$

$(w_i)_i$	card.	$(w_i)_i$	card.	$(w_i)_i$	card.
(13, 11, 10, 8, 4, 3)	1	(14, 12, 11, 8, 4, 2)	1	(14, 13, 10, 8, 7, 1)	1
(13, 12, 9, 8, 5, 4)	1	(14, 12, 11, 8, 5, 1)	2	(14, 13, 10, 8, 7, 3)	2
(13, 12, 9, 8, 7, 2)	1	(14, 12, 11, 8, 5, 3)	4	(14, 13, 10, 8, 7, 5)	1
(13, 12, 10, 8, 5, 3)	1	(14, 12, 11, 8, 6, 2)	3	(14, 13, 10, 9, 5, 2)	1
(13, 12, 11, 6, 5, 4)	1	(14, 12, 11, 8, 6, 4)	3	(14, 13, 10, 9, 5, 4)	1
(13, 12, 11, 8, 3, 2)	1	(14, 12, 11, 8, 7, 1)	2	(14, 13, 10, 9, 7, 2)	2
(13, 12, 11, 8, 5, 2)	2	(14, 12, 11, 8, 7, 3)	3	(14, 13, 10, 9, 7, 4)	1
(13, 12, 11, 8, 5, 4)	2	(14, 12, 11, 8, 7, 5)	2	(14, 13, 11, 6, 4, 3)	2
(13, 12, 11, 8, 7, 2)	1	(14, 12, 11, 9, 4, 3)	1	(14, 13, 11, 7, 4, 2)	2
(13, 12, 11, 8, 7, 4)	2	(14, 12, 11, 9, 5, 2)	3	(14, 13, 11, 7, 5, 1)	1
(13, 12, 11, 8, 7, 6)	1	(14, 12, 11, 9, 5, 4)	3	(14, 13, 11, 7, 5, 3)	2
(13, 12, 11, 9, 5, 3)	1	(14, 12, 11, 9, 6, 3)	3	(14, 13, 11, 7, 6, 2)	2
(13, 12, 11, 9, 7, 1)	1	(14, 12, 11, 9, 7, 2)	2	(14, 13, 11, 7, 6, 4)	2
(13, 12, 11, 10, 5, 2)	1	(14, 12, 11, 9, 7, 4)	3	(14, 13, 11, 8, 4, 1)	1
(13, 12, 11, 10, 5, 4)	2	(14, 12, 11, 9, 7, 6)	1	(14, 13, 11, 8, 4, 3)	5
(13, 12, 11, 10, 7, 2)	2	(14, 12, 11, 10, 4, 2)	1	(14, 13, 11, 8, 5, 2)	3
(13, 12, 11, 10, 7, 4)	2	(14, 12, 11, 10, 5, 1)	1	(14, 13, 11, 8, 6, 1)	2
(13, 12, 11, 10, 7, 6)	2	(14, 12, 11, 10, 5, 3)	3	(14, 13, 11, 8, 6, 3)	6
(13, 12, 11, 10, 9, 2)	1	(14, 12, 11, 10, 6, 2)	3	(14, 13, 11, 8, 6, 5)	4
(13, 12, 11, 10, 9, 6)	1	(14, 12, 11, 10, 6, 4)	3	(14, 13, 11, 8, 7, 2)	1
(14, 11, 9, 8, 4, 3)	1	(14, 12, 11, 10, 7, 1)	2	(14, 13, 11, 8, 7, 4)	1
(14, 11, 9, 8, 6, 3)	1	(14, 12, 11, 10, 7, 3)	5	(14, 13, 11, 9, 4, 2)	3
(14, 11, 10, 8, 5, 3)	1	(14, 12, 11, 10, 7, 5)	3	(14, 13, 11, 9, 5, 1)	1
(14, 12, 9, 7, 5, 2)	1	(14, 12, 11, 10, 8, 2)	2	(14, 13, 11, 9, 5, 3)	6
(14, 12, 9, 7, 5, 4)	1	(14, 12, 11, 10, 8, 4)	3	(14, 13, 11, 9, 6, 2)	4
(14, 12, 9, 8, 5, 3)	2	(14, 12, 11, 10, 8, 6)	1	(14, 13, 11, 9, 6, 4)	6
(14, 12, 9, 8, 6, 2)	1	(14, 12, 11, 10, 9, 1)	1	(14, 13, 11, 9, 7, 1)	2
(14, 12, 9, 8, 6, 4)	1	(14, 12, 11, 10, 9, 3)	1	(14, 13, 11, 9, 7, 3)	4
(14, 12, 9, 8, 7, 3)	1	(14, 12, 11, 10, 9, 5)	1	(14, 13, 11, 9, 7, 5)	3
(14, 12, 10, 6, 5, 2)	1	(14, 13, 9, 6, 4, 3)	1	(14, 13, 11, 9, 8, 2)	1
(14, 12, 10, 7, 5, 1)	1	(14, 13, 9, 7, 4, 2)	1	(14, 13, 11, 9, 8, 4)	3
(14, 12, 10, 7, 5, 3)	1	(14, 13, 9, 7, 5, 3)	1	(14, 13, 11, 9, 8, 6)	2
(14, 12, 10, 8, 4, 3)	1	(14, 13, 9, 7, 6, 4)	1	(14, 13, 11, 10, 4, 1)	1
(14, 12, 10, 8, 5, 2)	2	(14, 13, 9, 8, 4, 3)	2	(14, 13, 11, 10, 4, 3)	4
(14, 12, 10, 8, 5, 4)	2	(14, 13, 9, 8, 6, 3)	3	(14, 13, 11, 10, 5, 2)	1
(14, 12, 10, 8, 6, 1)	1	(14, 13, 9, 8, 6, 5)	1	(14, 13, 11, 10, 5, 4)	1
(14, 12, 10, 8, 6, 3)	2	(14, 13, 10, 6, 4, 2)	1	(14, 13, 11, 10, 6, 1)	2
(14, 12, 10, 8, 7, 2)	2	(14, 13, 10, 6, 5, 3)	1	(14, 13, 11, 10, 6, 3)	8
(14, 12, 10, 8, 7, 4)	1	(14, 13, 10, 7, 5, 2)	2	(14, 13, 11, 10, 6, 5)	4
(14, 12, 10, 9, 5, 3)	1	(14, 13, 10, 7, 5, 4)	1	(14, 13, 11, 10, 7, 2)	3
(14, 12, 11, 6, 5, 1)	1	(14, 13, 10, 8, 4, 2)	2	(14, 13, 11, 10, 7, 4)	1
(14, 12, 11, 7, 5, 2)	1	(14, 13, 10, 8, 5, 3)	4	(14, 13, 11, 10, 8, 1)	1
(14, 12, 11, 7, 5, 4)	2	(14, 13, 10, 8, 6, 2)	1	(14, 13, 11, 10, 8, 3)	5
(14, 12, 11, 8, 3, 1)	1	(14, 13, 10, 8, 6, 4)	2	(14, 13, 11, 10, 8, 5)	4

Table 3.26: card $(O_e(w_1, w_2))$

(w_1, w_2)	card.	(w_1, w_2)	card.	(w_1, w_2)	card.	(w_1, w_2)	card.
(13, 2)	1	(24, 7)	2	(29, 6)	6	(33, 0)	1
(14, 3)	1	(24, 9)	2	(29, 8)	2	(33, 2)	6
(15, 4)	1	(24, 13)	2	(29, 10)	3	(33, 4)	4
(16, 1)	1	(25, 2)	4	(29, 12)	3	(33, 6)	6
(16, 5)	1	(25, 4)	2	(29, 14)	3	(33, 8)	3
(17, 2)	1	(25, 6)	2	(29, 18)	4	(33, 10)	6
(17, 6)	2	(25, 8)	2	(30, 1)	4	(33, 12)	3
(18, 1)	1	(25, 10)	3	(30, 3)	4	(33, 14)	4
(18, 3)	1	(25, 14)	3	(30, 5)	3	(33, 16)	3
(18, 7)	1	(26, 1)	2	(30, 7)	4	(33, 18)	4
(19, 2)	1	(26, 3)	4	(30, 9)	3	(33, 22)	4
(19, 4)	2	(26, 5)	2	(30, 11)	3	(34, 1)	6
(19, 8)	2	(26, 7)	2	(30, 13)	3	(34, 3)	4
(20, 1)	1	(26, 9)	3	(30, 15)	3	(34, 5)	6
(20, 3)	2	(26, 11)	2	(30, 19)	3	(34, 7)	6
(20, 5)	1	(26, 15)	3	(31, 0)	1	(34, 9)	3
(20, 9)	2	(27, 0)	1	(31, 2)	4	(34, 11)	6
(21, 2)	2	(27, 2)	2	(31, 4)	6	(34, 13)	4
(21, 4)	1	(27, 4)	4	(31, 6)	2	(34, 15)	3
(21, 6)	2	(27, 6)	2	(31, 8)	6	(34, 17)	4
(21, 10)	2	(27, 8)	3	(31, 10)	3	(34, 19)	4
(22, 1)	2	(27, 10)	2	(31, 12)	3	(34, 23)	4
(22, 3)	1	(27, 12)	3	(31, 14)	3	(35, 0)	3
(22, 5)	2	(27, 16)	3	(31, 16)	4	(35, 2)	4
(22, 7)	2	(28, 1)	4	(31, 20)	4	(35, 4)	6
(22, 11)	2	(28, 3)	2	(32, 1)	4	(35, 6)	6
(23, 0)	1	(28, 5)	4	(32, 3)	6	(35, 8)	6
(23, 2)	1	(28, 7)	3	(32, 5)	4	(35, 10)	3
(23, 4)	2	(28, 9)	2	(32, 7)	3	(35, 12)	8
(23, 6)	2	(28, 11)	3	(32, 9)	6	(35, 14)	3
(23, 8)	2	(28, 13)	3	(32, 11)	3	(35, 16)	4
(23, 12)	3	(28, 17)	3	(32, 13)	3	(35, 18)	4
(24, 1)	2	(29, 0)	1	(32, 15)	4	(35, 20)	4
(24, 3)	2	(29, 2)	4	(32, 17)	3	(35, 24)	5
(24, 5)	2	(29, 4)	2	(32, 21)	4	(36, 1)	6

Table 3.27: card $(O_e(w_1, w_2, w_3, w_4))$

$(w_i)_i$	card.	$(w_i)_i$	card.	$(w_i)_i$	card.
(12, 9, 5, 2)	1	(14, 12, 10, 2)	1	(15, 12, 7, 2)	2
(12, 10, 7, 1)	1	(14, 12, 10, 4)	1	(15, 12, 7, 4)	3
(13, 9, 5, 1)	1	(14, 13, 6, 1)	1	(15, 12, 8, 1)	3
(13, 9, 7, 3)	1	(14, 13, 7, 2)	1	(15, 12, 8, 3)	2
(13, 10, 5, 2)	1	(14, 13, 8, 1)	2	(15, 12, 8, 5)	2
(13, 10, 7, 4)	1	(14, 13, 9, 0)	1	(15, 12, 9, 2)	4
(13, 11, 5, 3)	1	(14, 13, 9, 4)	1	(15, 12, 9, 4)	1
(13, 11, 7, 1)	1	(14, 13, 10, 3)	1	(15, 12, 9, 6)	2
(13, 12, 7, 2)	1	(14, 13, 11, 2)	1	(15, 12, 10, 1)	2
(13, 12, 8, 1)	1	(15, 7, 4, 2)	1	(15, 12, 10, 3)	2
(13, 12, 9, 4)	1	(15, 8, 5, 2)	1	(15, 12, 10, 5)	1
(13, 12, 10, 3)	1	(15, 9, 4, 2)	1	(15, 12, 10, 7)	1
(14, 8, 5, 3)	1	(15, 9, 5, 1)	1	(15, 12, 11, 4)	1
(14, 9, 4, 1)	1	(15, 9, 5, 3)	1	(15, 12, 11, 8)	1
(14, 9, 6, 1)	1	(15, 9, 6, 2)	1	(15, 13, 3, 1)	1
(14, 9, 7, 2)	1	(15, 9, 7, 1)	1	(15, 13, 4, 2)	1
(14, 10, 5, 1)	1	(15, 9, 7, 3)	1	(15, 13, 5, 1)	1
(14, 10, 6, 2)	1	(15, 10, 3, 2)	1	(15, 13, 5, 3)	2
(14, 10, 7, 1)	1	(15, 10, 5, 0)	1	(15, 13, 6, 2)	2
(14, 10, 7, 3)	1	(15, 10, 5, 2)	1	(15, 13, 6, 4)	1
(14, 10, 8, 2)	1	(15, 10, 5, 4)	1	(15, 13, 7, 1)	3
(14, 10, 8, 4)	1	(15, 10, 6, 1)	1	(15, 13, 7, 3)	1
(14, 11, 4, 1)	1	(15, 10, 7, 2)	3	(15, 13, 7, 5)	2
(14, 11, 5, 2)	1	(15, 10, 7, 4)	1	(15, 13, 8, 0)	1
(14, 11, 6, 1)	1	(15, 10, 7, 6)	1	(15, 13, 8, 2)	2
(14, 11, 6, 3)	1	(15, 10, 8, 1)	1	(15, 13, 8, 4)	1
(14, 11, 7, 0)	1	(15, 10, 8, 3)	1	(15, 13, 8, 6)	1
(14, 11, 7, 4)	1	(15, 10, 9, 4)	1	(15, 13, 9, 1)	3
(14, 11, 8, 1)	1	(15, 11, 4, 2)	1	(15, 13, 9, 3)	2
(14, 11, 8, 3)	1	(15, 11, 5, 1)	2	(15, 13, 9, 5)	1
(14, 11, 8, 5)	1	(15, 11, 5, 3)	1	(15, 13, 9, 7)	1
(14, 11, 9, 2)	1	(15, 11, 6, 2)	1	(15, 13, 10, 2)	3
(14, 12, 4, 2)	1	(15, 11, 7, 1)	2	(15, 13, 10, 4)	1
(14, 12, 5, 1)	1	(15, 11, 7, 3)	3	(15, 13, 11, 1)	1
(14, 12, 5, 3)	1	(15, 11, 8, 2)	2	(15, 13, 11, 3)	2
(14, 12, 6, 2)	1	(15, 11, 8, 4)	1	(15, 13, 11, 5)	1
(14, 12, 6, 4)	1	(15, 11, 9, 1)	1	(15, 14, 5, 2)	1
(14, 12, 7, 1)	2	(15, 11, 9, 3)	1	(15, 14, 5, 4)	1
(14, 12, 7, 5)	1	(15, 11, 9, 5)	1	(15, 14, 6, 1)	1
(14, 12, 8, 0)	1	(15, 12, 4, 1)	1	(15, 14, 7, 2)	3
(14, 12, 8, 2)	1	(15, 12, 5, 2)	3	(15, 14, 7, 6)	1
(14, 12, 8, 6)	1	(15, 12, 6, 1)	2	(15, 14, 8, 1)	2
(14, 12, 9, 1)	1	(15, 12, 6, 3)	2	(15, 14, 8, 3)	1
(14, 12, 9, 3)	1	(15, 12, 7, 0)	1	(15, 14, 9, 0)	1

Table 3.28: card $(O_e(w_1, w_2, w_3, w_4, w_5, w_6))$

$(w_i)_i$	card.	$(w_i)_i$	card.	$(w_i)_i$	card.
(13, 11, 10, 6, 4, 3)	1	(14, 12, 10, 6, 3, 2)	1	(14, 12, 11, 9, 5, 0)	1
(13, 12, 8, 7, 4, 1)	1	(14, 12, 10, 6, 4, 1)	2	(14, 12, 11, 9, 5, 2)	2
(13, 12, 9, 6, 3, 2)	1	(14, 12, 10, 6, 4, 3)	2	(14, 12, 11, 9, 6, 1)	1
(13, 12, 9, 7, 5, 1)	1	(14, 12, 10, 6, 5, 2)	1	(14, 12, 11, 9, 6, 3)	1
(13, 12, 9, 8, 5, 2)	1	(14, 12, 10, 7, 3, 1)	1	(14, 12, 11, 9, 7, 2)	2
(13, 12, 10, 6, 4, 2)	1	(14, 12, 10, 7, 4, 0)	1	(14, 12, 11, 9, 7, 4)	1
(13, 12, 10, 7, 4, 1)	1	(14, 12, 10, 7, 4, 2)	2	(14, 12, 11, 9, 8, 3)	1
(13, 12, 10, 7, 4, 3)	1	(14, 12, 10, 7, 5, 1)	2	(14, 12, 11, 10, 5, 1)	2
(13, 12, 10, 8, 5, 1)	1	(14, 12, 10, 7, 5, 3)	2	(14, 12, 11, 10, 7, 3)	1
(13, 12, 10, 8, 6, 2)	1	(14, 12, 10, 7, 6, 2)	1	(14, 13, 7, 6, 4, 1)	1
(13, 12, 10, 9, 4, 1)	2	(14, 12, 10, 8, 4, 1)	3	(14, 13, 8, 6, 3, 1)	1
(13, 12, 10, 9, 6, 1)	1	(14, 12, 10, 8, 5, 0)	1	(14, 13, 8, 7, 4, 1)	1
(13, 12, 10, 9, 6, 3)	1	(14, 12, 10, 8, 5, 2)	2	(14, 13, 8, 7, 5, 2)	1
(13, 12, 11, 6, 5, 2)	1	(14, 12, 10, 8, 6, 1)	2	(14, 13, 9, 6, 2, 1)	1
(13, 12, 11, 7, 5, 3)	1	(14, 12, 10, 8, 6, 3)	2	(14, 13, 9, 6, 3, 2)	1
(13, 12, 11, 8, 3, 2)	1	(14, 12, 10, 8, 7, 2)	1	(14, 13, 9, 6, 4, 1)	2
(13, 12, 11, 8, 5, 4)	1	(14, 12, 10, 9, 4, 0)	2	(14, 13, 9, 6, 4, 3)	1
(13, 12, 11, 8, 7, 2)	1	(14, 12, 10, 9, 4, 2)	1	(14, 13, 9, 7, 3, 1)	1
(14, 10, 8, 6, 3, 2)	1	(14, 12, 10, 9, 5, 1)	3	(14, 13, 9, 7, 4, 2)	1
(14, 11, 8, 7, 3, 2)	1	(14, 12, 10, 9, 6, 0)	1	(14, 13, 9, 7, 5, 1)	2
(14, 11, 9, 6, 4, 1)	1	(14, 12, 10, 9, 6, 2)	2	(14, 13, 9, 8, 3, 0)	1
(14, 11, 9, 8, 3, 2)	1	(14, 12, 10, 9, 7, 1)	1	(14, 13, 9, 8, 4, 1)	3
(14, 11, 10, 6, 4, 2)	1	(14, 12, 10, 9, 7, 3)	2	(14, 13, 9, 8, 5, 2)	2
(14, 11, 10, 6, 5, 3)	1	(14, 12, 11, 5, 3, 2)	1	(14, 13, 9, 8, 6, 1)	1
(14, 11, 10, 7, 4, 1)	1	(14, 12, 11, 6, 3, 1)	2	(14, 13, 9, 8, 6, 3)	1
(14, 11, 10, 7, 5, 2)	1	(14, 12, 11, 6, 4, 2)	2	(14, 13, 10, 5, 4, 1)	1
(14, 11, 10, 8, 4, 2)	1	(14, 12, 11, 6, 5, 1)	1	(14, 13, 10, 6, 2, 0)	1
(14, 11, 10, 8, 5, 1)	1	(14, 12, 11, 6, 5, 3)	1	(14, 13, 10, 6, 3, 1)	2
(14, 11, 10, 9, 4, 3)	1	(14, 12, 11, 7, 2, 1)	1	(14, 13, 10, 6, 4, 2)	3
(14, 12, 8, 5, 3, 1)	1	(14, 12, 11, 7, 3, 2)	1	(14, 13, 10, 6, 5, 1)	2
(14, 12, 8, 6, 4, 1)	1	(14, 12, 11, 7, 4, 1)	2	(14, 13, 10, 6, 5, 3)	1
(14, 12, 8, 7, 4, 0)	1	(14, 12, 11, 7, 4, 3)	1	(14, 13, 10, 7, 2, 1)	1
(14, 12, 8, 7, 5, 1)	1	(14, 12, 11, 7, 5, 2)	3	(14, 13, 10, 7, 3, 2)	2
(14, 12, 9, 6, 3, 1)	1	(14, 12, 11, 7, 5, 4)	1	(14, 13, 10, 7, 4, 1)	3
(14, 12, 9, 6, 4, 2)	1	(14, 12, 11, 7, 6, 3)	1	(14, 13, 10, 7, 4, 3)	2
(14, 12, 9, 6, 5, 1)	1	(14, 12, 11, 8, 3, 1)	2	(14, 13, 10, 7, 5, 0)	1
(14, 12, 9, 7, 4, 1)	1	(14, 12, 11, 8, 4, 2)	2	(14, 13, 10, 7, 5, 2)	2
(14, 12, 9, 7, 5, 0)	1	(14, 12, 11, 8, 5, 1)	3	(14, 13, 10, 7, 5, 4)	2
(14, 12, 9, 7, 5, 2)	1	(14, 12, 11, 8, 6, 2)	2	(14, 13, 10, 7, 6, 1)	1
(14, 12, 9, 8, 5, 1)	2	(14, 12, 11, 8, 6, 4)	1	(14, 13, 10, 8, 3, 1)	2
(14, 12, 9, 8, 6, 2)	1	(14, 12, 11, 8, 7, 1)	1	(14, 13, 10, 8, 4, 0)	2
(14, 12, 10, 5, 3, 1)	1	(14, 12, 11, 8, 7, 3)	1	(14, 13, 10, 8, 4, 2)	1
(14, 12, 10, 5, 4, 2)	1	(14, 12, 11, 9, 3, 2)	1	(14, 13, 10, 8, 5, 1)	5
(14, 12, 10, 6, 2, 1)	1	(14, 12, 11, 9, 4, 1)	1	(14, 13, 10, 8, 5, 3)	1

Bibliography

[AJ87] Jeffrey Adams and Joseph F. Johnson, *Endoscopic groups and packets of nontempered representations*, Compositio Math. **64** (1987), no. 3, 271–309.

[Art] James Arthur, *Automorphic representations of inner twists*, in preparation.

[Art88] ______, *The invariant trace formula. II. Global theory*, J. Amer. Math. Soc. **1** (1988), no. 3, 501–554.

[Art89a] ______, *The L^2-Lefschetz numbers of Hecke operators*, Invent. Math. **97** (1989), no. 2, 257–290.

[Art89b] ______, *Unipotent automorphic representations: conjectures*, Astérisque (1989), no. 171-172, 13–71, Orbites unipotentes et représentations, II.

[Art13] James Arthur, *The Endoscopic Classification of Representations: Orthogonal and Symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, 2013.

[AS01] Mahdi Asgari and Ralf Schmidt, *Siegel modular forms and representations*, Manuscripta Math. **104** (2001), no. 2, 173–200.

[BC09] Joël Bellaïche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups*, Astérisque (2009), no. 324, xii+314.

[BC11] Joël Bellaïche and Gaëtan Chenevier, *The sign of galois representations attached to automorphic forms for unitary groups*, Compositio Math. **147** (2011), 1337–1352.

[BFvdG14] Jonas Bergström, Carel Faber, and Gerard van der Geer, *Siegel modular forms of degree three and the cohomology of local systems*, Selecta Math. (N.S.) **20** (2014), no. 1, 83–124.

[BG10] Kevin Buzzard and Toby Gee, *The conjectural connections between automorphic representations and Galois representations*, http://arxiv.org/abs/1009.0785v1, 2010.

[BJ79] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, Automorphic forms, representations and L-functions (Proc. Sympos.

Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 189–207.

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609.

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbb{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *en* $\ell = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart.

[Bor79] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[Bos09] Siegfried Bosch, *Half a century of rigid analytic spaces*, Pure Appl. Math. Q. **5** (2009), no. 4, Special Issue: In honor of John Tate. Part 1, 1435–1467.

[Bou68] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[BR92] Don Blasius and Jonathan D. Rogawski, *Tate classes and arithmetic quotients of the two-ball.*, The zeta functions of Picard modular surfaces. CRM Workshop, Montreal / Can. 1988, 421-444 (1992)., 1992.

[Bru59] François Bruhat, *Travaux de Harish-Chandra*, Séminaire Bourbaki, Vol. 4, Soc. Math. France, Paris, 1959, pp. Exp. No. 143, 85–93.

[Buz07] Kevin Buzzard, *Eigenvarieties*, L-functions and Galois representations, London Math. Soc. Lecture Notes, vol. 320, Cambridge Univ. Press, 2007, Durham, 2004, pp. 59–120.

[BvdGHZ08] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder, and Don Zagier, *The 1-2-3 of modular forms*, Universitext, Springer-Verlag, Berlin, 2008, Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[BW00] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000.

[BZ77] Joseph Bernstein and Andrei Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. I*, Ann. Sci. École Norm. Sup. **10** (1977), no. 4, 441–472.

[Car72] R. W. Carter, *Conjugacy classes in the Weyl group*, Compositio Math. **25** (1972), 1–59.

[Cas] William Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, available at http://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf.

[Cas80] ______, *The unramified principal series of p-adic groups. I. the spherical function*, Compositio Mathematica **40** (1980), no. 3, 387–406.

[CC09] G. Chenevier and L. Clozel, *Corps de nombres peu ramifiés et formes automorphes autoduales*, J. Amer. Math. Soc. **22** (2009), no. 2, 467–519.

[CD90] Laurent Clozel and Patrick Delorme, *Le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs. II*, Ann. Sci. École Norm. Sup. (4) **23** (1990), no. 2, 193–228.

[CG] Frank Calegari and Toby Gee, *Irreducibility of automorphic Galois representations of* $\mathrm{GL}(n)$*, n at most* 5, http://arxiv.org/abs/1104.4827.

[CH13] Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations. II.*, Camb. J. Math. **1** (2013), no. 1, 53–73.

[Che51] Claude Chevalley, *Deux Théorèmes d'Arithmétique*, J. Math. Soc. Japan **3** (1951), no. 1, 36–44.

[Che04] Gaëtan Chenevier, *Familles p-adiques de formes automorphes pour* GL_n, J. Reine Angew. Math. **570** (2004), 143–217.

[Che09] ______, *Une application des variétés de Hecke des groupes unitaires*, Part of the Book Project under the direction of Michael Harris at Paris 7: http://fa.institut.math.jussieu.fr/node/29, 2009.

[Choa] Sungmun Cho, *Group schemes and local densities of quadratic lattices in residue characteristic 2*, http://arxiv.org/abs/1210.7625.

[Chob] ______, *Group schemes and local densities of ramified hermitian lattices in residue characteristic 2*, http://arxiv.org/abs/1210.7894.

[Clo86] Laurent Clozel, *On limit multiplicities of discrete series representations in spaces of automorphic forms*, Invent. Math. **83** (1986), no. 2, 265–284.

[Clo88] Laurent Clozel, *Motifs et Formes Automorphes: Applications du Principe de Fonctorialité*, Automorphic Forms, Shimura Varieties, and L-functions (Laurent Clozel and James S. Milne, eds.), vol. 1, 1988, Proceedings of a Conference held at the University of Michigan, Ann Arbor.

[Clo13] Laurent Clozel, *Purity reigns supreme*, Int. Math. Res. Not. IMRN (2013), 328–346.

[Con99] Brian Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier **49** (1999), no. 2, 473–541.

[CR14] Gaëtan Chenevier and David Renard, *Level one algebraic cuspforms of classical groups of small ranks*, Mem. Amer. Math. Soc. **9999** (2014), no. 9999, 120.

[CS80] William Casselman and J. Shalika, *The unramified principal series of p-adic groups. II: The Whittaker function.*, Compos. Math. **41** (1980), 207–231.

[EP81] T. J. Enright and R. Parthasarathy, *A proof of a conjecture of Kashiwara and Vergne*, Noncommutative harmonic analysis and Lie groups (Marseille, 1980), Lecture Notes in Math., vol. 880, Springer, Berlin-New York, 1981, pp. 74–90.

[Fer96] Stéfane Fermigier, *Annulation de la cohomologie cuspidale de sous-groupes de congruence de* $\mathrm{GL}_n(\mathbf{Z})$, Math. Ann. **306** (1996), no. 2, 247–256.

[Fre83] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der Mathematischen Wissenschaften, vol. 254, Springer-Verlag, Berlin, 1983.

[GJ78] Stephen Gelbart and Hervé Jacquet, *A relation between automorphic representations of* $\mathrm{GL}(2)$ *and* $\mathrm{GL}(3)$, Annales scientifiques de l'ENS **11** (1978), no. 4, 471–542.

[GKM97] M. Goresky, R. Kottwitz, and R. MacPherson, *Discrete series characters and the Lefschetz formula for Hecke operators*, Duke Math. J. **89** (1997), no. 3, 477–554.

[Gol14] Wushi Goldring, *Galois representations associated to holomorphic limits of discrete series*, Compos. Math. **150** (2014), no. 2, 191–228.

[GP05] Benedict H. Gross and David Pollack, *On the Euler characteristic of the discrete spectrum*, J. Number Theory **110** (2005), no. 1, 136–163.

[Gro97] Benedict H. Gross, *On the motive of a reductive group*, Invent. Math. **130** (1997), no. 2, 287–313.

[Gro99] ______, *Algebraic modular forms*, Israel J. Math. **113** (1999), 61–93.

[GY00] Wee Teck Gan and Jiu-Kang Yu, *Group schemes and local densities*, Duke Math. J. **105** (2000), no. 3, 497–524.

[HC55] Harish-Chandra, *Representations of semisimple Lie groups. IV*, Amer. J. Math. **77** (1955), 743–777.

[HC56a] ______, *Representations of semisimple Lie groups. V*, Amer. J. Math. **78** (1956), 1–41.

[HC56b] ______, *Representations of semisimple Lie groups. VI. Integrable and square-integrable representations*, Amer. J. Math. **78** (1956), 564–628.

[HC68] ______, *Automorphic forms on semisimple Lie groups*, Notes by J. G. M. Mars. Lecture Notes in Mathematics, No. 62, Springer-Verlag, Berlin-New York, 1968.

[HC70] ______, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Mathematics, Vol. 162, Springer-Verlag, Berlin, 1970, Notes by G. van Dijk.

[Hec76] Henryk Hecht, *The characters of some representations of Harish-Chandra*, Math. Ann. **219** (1976), no. 3, 213–226.

[Igu62] Jun-ichi Igusa, *On Siegel modular forms of genus two*, Amer. J. Math. **84** (1962), 175–200.

[IM65] Nagayoshi Iwahori and Hideya Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Publications Mathématiques de l'IHÉS **25** (1965), 5–48.

[Jac40] N. Jacobson, *A note on hermitian forms*, Bull. Amer. Math. Soc. **46** (1940), 264–268.

[Jac62] Ronald Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. **84** (1962), 441–465.

[Joh84] Joseph F. Johnson, *Lie algebra cohomology and the resolution of certain Harish-Chandra modules*, Math. Ann. **267** (1984), no. 3, 377–393.

[Jon11] Owen T. R. Jones, *An analogue of the BGG resolution for locally analytic principal series*, J. Number Theory **131** (2011), no. 9, 1616–1640.

[Kal] Tasho Kaletha, *Rigid inner forms of real and p-adic groups*, available at https://web.math.princeton.edu/~tkaletha/.

[Kaz86] David Kazhdan, *Cuspidal geometry of p-adic groups*, J. Analyse Math. **47** (1986), 1–36.

[Kna86] Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986, An overview based on examples.

[Kos78] Bertram Kostant, *On Whittaker vectors and representation theory*, Invent. Math. **48** (1978), no. 2, 101–184.

[Kot86] Robert Kottwitz, *Stable trace formula: elliptic singular terms*, Mathematische Annalen **275** (1986), no. 3, 365–399.

[Kot88] Robert E. Kottwitz, *Tamagawa numbers*, Ann. of Math. (2) **127** (1988), no. 3, 629–646.

[Kot90] ———, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 161–209.

[KS99] Robert E. Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190.

[Lan89] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., vol. 31, Amer. Math. Soc., Providence, RI, 1989, pp. 101–170.

[Loe11] David Loeffler, *Overconvergent algebraic automorphic forms*, Proc. London Math. Soc. **102** (2011), no. 2, 193–228.

[LP02] Joshua Lansky and David Pollack, *Hecke algebras and automorphic forms*, Compositio Math. **130** (2002), no. 1, 21–48.

[LS87] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1-4, 219–271.

[LS90] R. Langlands and D. Shelstad, *Descent for transfer factors*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 485–563.

[Mar75] Susan Martens, *The characters of the holomorphic discrete series*, Proc. Nat. Acad. Sci. U.S.A. **72** (1975), no. 9, 3275–3276.

[Mes86] Jean-François Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques*, Compositio Math. **58** (1986), no. 2, 209–232.

[Mez] Paul Mezo, *Spectral transfer in the twisted endoscopy of real groups*, available at http://people.math.carleton.ca/~mezo/research.html.

[Mez13] Paul Mezo, *Character identities in the twisted endoscopy of real reductive groups*, Mem. Amer. Math. Soc. **222** (2013), no. 1042, vi+94.

[Mil02] Stephen D. Miller, *The highest lowest zero and other applications of positivity*, Duke Math. J. **112** (2002), no. 1, 83–116.

[MW89] Colette Mœglin and Jean-Loup Waldspurger, *Le spectre résiduel de* $\mathrm{GL}(n)$, Annales scientifiques de l'ENS **22** (1989), no. 4, 605–674.

[MW06] ———, *Sur le transfert des traces d'un groupe classique p-adique à un groupe linéaire tordu*, Selecta Math. **12** (2006), no. 3-4, 433–515.

[NV01] Gabriele Nebe and Boris Venkov, *On Siegel modular forms of weight 12*, J. Reine Angew. Math. **531** (2001), 49–60.

[O'M00] O. Timothy O'Meara, *Introduction to quadratic forms*, Classics in Mathematics, Springer-Verlag, Berlin, 2000, Reprint of the 1973 edition.

[Pat] Stefan Patrikis, *On the sign of regular algebraic polarizable automorphic representations*, http://arxiv.org/abs/1306.1242.

[PT] Stefan Patrikis and Richard Taylor, *http://arxiv.org/abs/1307.1640*, http://arxiv.org/abs/1307.1640.

[PY07] Cris Poor and David S. Yuen, *Computations of spaces of Siegel modular cusp forms*, J. Math. Soc. Japan **59** (2007), no. 1, 185–222.

[S+14] W. A. Stein et al., *Sage Mathematics Software (Version 6.1.1)*, The Sage Development Team, 2014, http://www.sagemath.org.

[Sch75] Wilfried Schmid, *On the characters of the discrete series. The Hermitian symmetric case*, Invent. Math. **30** (1975), no. 1, 47–144.

[Ser71] Jean-Pierre Serre, *Cohomologie des groupes discrets*, Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970), Princeton Univ. Press, 1971, pp. 77–169. Ann. of Math. Studies, No. 70.

[SGA70] *Schémas en groupes. III: Structure des schémas en groupes réductifs*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin-New York, 1970.

[SGA73] *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, Vol. 340, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz.

[Sha90] Freydoon Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) **132** (1990), no. 2, 273–330.

[She08a] D. Shelstad, *Tempered endoscopy for real groups. I. Geometric transfer with canonical factors*, Representation theory of real reductive Lie groups, Contemp. Math., vol. 472, Amer. Math. Soc., Providence, RI, 2008, pp. 215–246.

[She08b] ———, *Tempered endoscopy for real groups. III. Inversion of transfer and L-packet structure*, Represent. Theory **12** (2008), 369–402.

[She10] ———, *Tempered endoscopy for real groups. II. Spectral transfer factors*, Automorphic forms and the Langlands program, Adv. Lect. Math. (ALM), vol. 9, Int. Press, Somerville, MA, 2010, pp. 236–276.

[Sie69] Carl Ludwig Siegel, *Berechnung von Zetafunktionen an ganzzahligen Stellen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1969** (1969), 87–102.

[Spr98] T. A. Springer, *Linear algebraic groups*, second ed., Progress in Mathematics, vol. 9, Birkhäuser Boston, Inc., Boston, MA, 1998.

[Tad94] Marko Tadić, *Representations of p-adic symplectic groups*, Compositio Mathematica **90** (1994), no. 2, 123–181.

[Tay91] Richard Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. **63** (1991), no. 2, 281–332.

[Tay12] Richard Taylor, *The image of complex conjugation in l-adic representations associated to automorphic forms*, Algebra Number Theory **6** (2012), no. 3, 405–435.

[Tsu83] Ryuji Tsushima, *An explicit dimension formula for the spaces of generalized automorphic forms with respect to* $\mathrm{Sp}(2, \mathbf{Z})$, Proc. Japan Acad. Ser. A Math. Sci. **59** (1983), no. 4, 139–142.

[Tsu84] ———, *An explicit dimension formula for the spaces of generalized automorphic forms with respect to* $\mathrm{Sp}(2, \mathbf{Z})$, Automorphic forms of several variables (Katata, 1983), Progr. Math., vol. 46, Birkhäuser Boston, Boston, MA, 1984, pp. 378–383.

[Tsu86] Shigeaki Tsuyumine, *On Siegel modular forms of degree three*, Amer. J. Math. **108** (1986), no. 4, 755–862.

[vD72] G. van Dijk, *Computation of certain induced characters of p-adic groups*, Math. Ann. **199** (1972), 229–240.

[Vog78] David A. Vogan, Jr., *Gel'fand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48** (1978), no. 1, 75–98.

[VZ84] David A. Vogan, Jr. and Gregg J. Zuckerman, *Unitary representations with nonzero cohomology*, Compositio Math. **53** (1984), no. 1, 51–90.

[Wal] Jean-Loup Waldspurger, *Stabilisation de la formule des traces tordue I: endoscopie tordue sur un corps local*, http://arxiv.org/abs/1401.4569.

[Wal63] G. E. Wall, *On the conjugacy classes in the unitary, symplectic and orthogonal groups*, J. Austral. Math. Soc. **3** (1963), 1–62.

[Wal84] N. R. Wallach, *On the constant term of a square integrable automorphic form*, Operator algebras and group representations, Vol. II (Neptun, 1980), Monogr. Stud. Math., vol. 18, Pitman, Boston, MA, 1984, pp. 227–237.

[Wal10] Jean-Loup Waldspurger, *Les facteurs de transfert pour les groupes classiques: un formulaire*, Manuscripta Math. **133** (2010), no. 1-2, 41–82.

[Was97] Lawrence C. Washington, *Introduction to cyclotomic fields*, second ed., Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1997.

[Wei82] André Weil, *Adeles and algebraic groups*, Progress in Mathematics, vol. 23, Birkhäuser, Boston, Mass., 1982, With appendices by M. Demazure and Takashi Ono.

[Wei83] Rainer Weissauer, *Vektorwertige Siegelsche Modulformen kleinen Gewichtes*, J. Reine Angew. Math. **343** (1983), 184–202.