

Study of dispersive phenomena in geophysical fluids mechanics

Frédéric Charve

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THÈSE DE DOCTORAT DE L'ÉCOLE

POLYTECHNIQUE

Spécialité : MATHÉMATIQUES

Présentée par **Frédéric CHARVE**

Pour obtenir le titre de

DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

Sujet de la thèse :

ÉTUDE DE PHÉNOMÈNES DISPERSIFS EN MÉCANIQUE DES FLUIDES GÉOPHYSIQUES

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Chapitre 1

Introduction

Cette introduction est constituée de deux grandes parties: la première section est consacrée à une présentation succinte des fluides géophysiques, des équations primitives (nous appellerons "équations primitives" les équations obtenues à partir de Navier-Stokes tournant tridimensionnel avec densité variable et en utilisant l'approximation de Boussinesq) ainsi que de l'équilibre géostrophique et du système quasigéostrophique tridimensionnel. Elle reprend dans une large mesure les grandes lignes de [15], [3] ainsi que [19]. Nous commençons par décrire ce qui caractérise un fluide géophysique: l'influence de la rotation de la Terre autour de son axe, matérialisée par la force de Coriolis, et la stratification du fluide qui représente l'inhomogénéité de la densité du fluide (air chaud ou froid, eau salée ou douce, etc...). Nous écrivons ensuite les équations de base (conservation de la masse, du moment cinétique...). A partir de ces équations, nous présentons les arguments formels d'analyse dimensionnelle utilisés par les physiciens pour déterminer les termes prépondérants en vue de simplifier les équations. Nous obtenons enfin les systèmes limites utilisés en météorologie et océanographie. Nous tenons à signaler que cette partie présente des approximations formelles sans justification mathématique mais qui sont cependant fidèles à la réalité. Pour des résultats mathématiquement plus précis nous renvoyons à la deuxième partie.

Dans la deuxième section nous commençons par passer en revue les travaux antérieurs d'étude de la convergence quasigéostrophique et nous présentons ensuite les résultats obtenus lors de cette thèse: nous prouvons de façon rigoureuse à partir des équations primitives la convergence en forte rotation vers le système quasigéostrophique. Le premier chapitre concerne la convergence dans le cadre des solutions faibles au sens de Leray. Le deuxième chapitre concerne des données initiales plus régulières et des solutions fortes. Toujours dans le cas des solutions fortes nous donnons deux nouveaux résultats en précisant la vitesse de convergence. Nous terminons par la convergence quasigéostrophique dans le cas des poches de tourbillons régulières.

1.1 Fluides géophysiques

1.1.1 Introduction

Généralités

Par fluide géophysique nous désignons un fluide réparti sur un volume grand par rapport à l'échelle planétaire: l'atmosphère ou les océans. Deux choses distinguent principalement

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la dynamique des fluides géophysiques des autres disciplines consacrées à l'étude de la mécanique des fluides: il s'agit de la rotation et de la stratification.

Le fait que la Terre tourne autour de son axe introduit, lorsque l'on observe le mouvement dans un repère lié à la Terre, dans les équations du mouvement deux termes d'accélération qui peuvent être interprétés comme des forces: la force de Coriolis et la force centrifuge. Même si dans la vie quotidienne la force centrifuge est la plus "palpable" des deux, ce n'est pas elle qui a le plus d'importance. On se contente de l'inclure dans le gradient de pesanteur pour constituer ce que l'on nomme le géopotentiel. Le facteur crucial est en fait la force de Coriolis dans les mouvements géophysiques.

Comme on pourra le voir, l'un des effets majeurs de la force de Coriolis est de forcer à une certaine rigidité verticale le fluide considéré, au sens où, lorsque des fluides homogènes tournent rapidement, on observe des mouvements en colonne, c'est-à-dire que toutes les particules le long de la même verticale bougent de concert en gardant leur alignement vertical.

La découverte de cette propriété est attribuée à Geoffrey I.Taylor: il serait arrivé à cette conclusion uniquement au moyen d'arguments théoriques et, croyant s'être trompé, il a effectué des expériences qui révélèrent que cette prédiction théorique était vraie: quelques gouttes de colorant dans un fluide tournant rapidement forment des traînées verticales, qui deviennent après quelques rotations des colonnes relativement à une spirale horizontale.

Dans les fluides atmosphériques ou océaniques de grande étendue, un tel état de parfaite rigidité verticale n'est pas réalisé, principalement parce que la rotation n'est pas assez rapide, et la densité, pas assez uniforme. Cependant les mouvements dans l'atmosphère, les océans, et sur d'autres planètes montrent une forte tendance vers cette répartition en colonnes: par exemple, on a pu observer certains courants situés dans l'Atlantique Nord-Ouest qui s'étendent verticalement sur 4000m sans changement significatif d'amplitude ou de direction. Les travaux exposés dans [12] vérifient cette observation, en tirant parti de propriétés dispersives.

Comme nous le verrons plus tard, en faisant une analyse d'échelle et en ne gardant que les termes d'ordre le plus bas, on obtient le modèle géostrophique, qui revient à se placer dans le cas où la rotation serait exactement équilibrée par le gradient de pression. On observe des comportements très proches de cet équilibre dans certains jets océaniques, ou dans le cas de certains vents. Mais ce modèle géostrophique présente le désavantage de ne pas permettre d'étude de l'évolution en fonction du temps. C'est dans ce but qu'a été introduit le système quasigéostrophique, qui revient à ajouter les termes d'ordre supérieur. Dans ce système, on prend en considération non seulement la rotation mais aussi la stratification.

Le second attribut de la dynamique des fluides géophysiques, la stratification, apparaît naturellement lorsque l'on considère des fluides de densité variable (par exemple air chaud ou froid, eau douce ou salée...). La pesanteur joue un rôle important puisqu'elle tend à abaisser les fluides les plus lourds et élever les fluides les plus légers.

Sous des conditions d'équilibre, le fluide est stratifié de façon stable et consiste en un empilement vertical de couches horizontales de même densité, la densité décroissant lorsque croît l'altitude. Les mouvements du fluide dérangent constamment cet équilibre que la gravité s'efforcera systématiquement de rétablir: ces petites perturbations créent des ondes internes (par exemple les vents dominants dans l'atmosphère sont dus à la différence de température pôle-équateur). Terminons par un phénomène dû à la stratification et rapporté par de nombreux océanographes: il s'agit du phénomène des "eaux mortes". Une description est donnée dans [15] et [24]. Il arrive qu'un bateau subisse une forte résistance dans son mouvement alors que l'eau en surface est apparemment très calme, presque plate. Il s'avère que ce sont ces ondes internes qui en sont la cause: le bateau navigue sur une mince couche d'eau relativement douce située sur une couche d'eau trés salée. Le bateau crée des ondes internes sur l'interface, invisibles en surface mais de grande énergie, causant la résistance au mouvement du bateau.

En résumé nous avons à considérer deux phénomènes aux effets très différents: la rotation tendant à une répartition en colonnes verticales, et la stratification, tendant à maintenir autant que possible une répartition en couches horizontales de même densité.

Nous nous placerons dans le cas de mouvements de grande échelle, situés sur des latitudes moyennes (c'est-à-dire éloignés de l'équateur). On distinguera les mouvements lents (qui seront proches du modèle quasigéostrophique, voire géostrophique) et les mouvements rapides (de l'ordre de la journée) qui eux seront trés influencés par la rotation.

Échelles de mouvement

Afin de déterminer si un processus physique est dynamiquement important, on introduit l'analyse d'échelle, qui consiste à établir de façon formelle des ordres de grandeur pour les variables physiques qui sont considérées. Ainsi par exemple pour le cyclone Hugo (1989) la longueur caractéristique vaut L = 300km (environ trois degrés de latitude), le temps caractéristique vaut $T = 2.10^5 s$ (des changements notables de direction ou de vitesse se font à des temps de l'ordre de deux journées), d'où on déduit que la vitesse caractéristique U est d'environ $U = 70m.s^{-1}$, voisine de la vitesse mesurée: environ $300km.h^{-1}$. Plus récemment le petit cyclone Isabelle, qui a balayé depuis la fin août jusqu'à la mi-septembre 2003 la côte Est de l'Amérique avait comme grandeurs caractéristiques L = 200km et $U = 250km.h^{-1}$. Cette évaluation des grandeurs caractéristiques est également faite dans le système solaire: ainsi pour la Tache Rouge de Jupiter, $L = 10^4 km$, $U = 100m.s^{-1}$.

Cette analyse d'échelle va nous permettre de nous faire une première idée de l'importance de la rotation et de la stratification par rapport aux grandeurs caractéristiques.

La rotation

Pour mesurer l'importance de la rotation grâce à l'analyse d'échelle, on compare la fréquence de rotation de la Terre et l'échelle de temps du mouvement que l'on considère:

$$\omega = \frac{2\pi}{\Omega T}$$
 où, pour la Terre, $\Omega = 7,29.10^{-5} rad.s^{-1}$ (la période est de 24h).

Ainsi la rotation est importante si ce rapport ω est (très) inférieur à l'unité, ce qui revient à dire que l'échelle de temps est de plus d'une journée. Définissons ensuite:

$$\varepsilon = \frac{\text{période d'une révolution de la Terre}}{\text{temps mis par une particule pour parcourir } L \text{ à la vitesse } U} = \frac{\frac{2\pi}{\Omega}}{\frac{L}{U}} = \frac{2\pi U}{\Omega L} \cdot$$

La rotation a un effet significatif si $\varepsilon \ll 1$ et l'effet sera d'autant plus grand que ce rapport est petit. Nous renvoyons à [15] (page 9) pour différentes valeurs caractéristiques de vitesse ou de longueur où la rotation est importante.

La stratification

Comme nous l'avons déjà vu, les mouvements à l'intérieur du fluide tendent à perturber l'empilement des masses de fluide en couches horizontales, les plus lourdes se trouvant en dessous des plus légères. Le fait que la gravité tende à rétablir cet équilibre se traduit par une augmentation d'énergie potentielle aux dépens de l'énergie cinétique, et l'on mesure l'importance dynamique de la stratification en comparant ces énergies.

Si l'on désigne par $\Delta \rho$ l'échelle de variation de densité et H l'échelle de hauteur, un modèle de perturbation de la stratification consiste à élever un élément de fluide de densité $\rho_0 + \Delta \rho$ à la hauteur H et, pour conserver le volume, d'abaisser un élément de fluide de densité ρ_0 de la même hauteur. Le changement d'énergie potentielle par unité de volume est $(\rho_0 + \Delta \rho)gH - \rho_0gH = (\Delta \rho)gH$ (où g est l'accélération de la pesanteur). Avec la vitesse caractéristique U, l'énergie cinétique est $\frac{1}{2}\rho_0 U^2$, ce qui nous fournit le rapport:

$$\sigma = \frac{\frac{1}{2}\rho_0 U^2}{(\Delta\rho)gH} \cdot$$

Si ce rapport est très supérieur à un, la modification d'énergie potentielle se fait avec un petit coût en énergie cinétique, et la stratification affecte donc très peu le fluide. Dans les autres cas, la stratification contraint de façon importante le fluide (d'autant plus que le rapport est petit devant un).

Terminons cette partie en précisant que les océans sont très stratifiés et stables, de même que la stratosphère. Par contre la troposphère est moins stratifiée et plus instable.

Différences entre l'atmosphère et les océans

Les cas qui nous intéresseront seront donc les cas où les effets de la rotation et de la stratification seront tous deux importants.

Ainsi par exemple, dans le cas où les deux rapports ε et σ sont de l'ordre de l'unité, on obtient que $U \sim \sqrt{gH\frac{\Delta\rho}{\rho_0}}$ et $L \sim \frac{U}{\Omega}$, ce qui pour les valeurs $\Omega = 7,29.10^{-5}s^{-1}$, $g = 9,81m.s^{-2}$, donne:

- Pour l'atmosphère, $\rho_0 = 1, 2kg.m^{-3}, \Delta \rho = 0, 03kg.m^{-3}, H = 5000m.$ Et donc $L_{\text{atmosphère}} \sim 500km, U_{\text{atmosphère}} \sim 30m.s^{-1}$
- Pour l'océan, $\rho_0=1028kg.m^{-3},$ $\Delta\rho=2kg.m^{-3},$ H=1000m. Et $L_{\rm océan}\sim 60Km,$ $U_{\rm océan}\sim 4m.s^{-1}$

On retrouve le fait qu'un mouvement océanique est plus lent et moins étendu qu'un mouvement atmosphérique.

D'autres différences sont à prendre en compte:

- Les reliefs des côtes (îles, continents) ont beaucoup d'impact sur les océans mais peu sur l'atmosphère.
- Les reliefs des masses montagneuses ont un impact sur l'atmosphère (montagnes sous-marines et abysses pour l'océan).
- L'humidité de l'air (nuages, précipitations, ...) n'a pas d'analogue océanique.

• Les océans sont soumis à l'attraction de la lune sous forme de marées.

1.1.2 Équations de base

Dans cette partie nous allons établir de façon succinte les diverses équations pour un fluide dans un repère tournant.

Conservation de la quantité de mouvement

Considérons le mouvement d'un fluide sur la Terre dans un repère absolu qui n'est donc pas lié à la planète. La conservation de la quantité de mouvement (ou relation fondamentale de la dynamique) dans un repère absolu s'obtient en écrivant que l'accélération est égale à la somme des forces extérieures, c'est-à-dire les forces de gravitation, les contraintes internes de pression, et celles de viscosité:

$$\rho(\frac{D}{Dt})_a U_a = -\rho \nabla \Phi_a - \nabla p + \nabla \cdot T$$

- La notation U_a désigne la vitesse dans une repère absolu fixe.
- $\rho \nabla \Phi_a$ désigne le potentiel de gravitation et s'écrit en première approximation (0, 0, -g).
- $\frac{D}{Dt} = \frac{\partial}{\partial t} + U \cdot \nabla$ est la dérivée particulaire.
- $\nabla \cdot T$ est le terme de frottement, c'est la divergence du tenseur des tensions visqueuses (ou tenseur de friction). On ne l'explicitera pas ici.

Il est plus pratique d'écrire ces équations dans un repère lié à la planète. Ce repère tournera donc avec la vitesse angulaire Ω (orientée selon l'axe des pôles et dirigée du Sud vers le Nord) et le changement de repère sera donc donné par (r = OM où O est le centre de la Terre et M le point considéré):

$$(\frac{D}{Dt})_a = \frac{D}{Dt} + \Omega \wedge, \quad \text{et} \quad U_a = U + \Omega \wedge r,$$

ce qui nous donne que:

$$(\frac{D}{Dt})_a U_a = \frac{DU}{Dt} + 2\Omega \wedge U + \Omega \wedge (\Omega \wedge r).$$

Le second terme du membre de droite représente l'accélération de Coriolis, tandis que le dernier représente l'accélération centrifuge: ainsi que nous l'avons annoncé plus haut, on va se débarasser de ce dernier terme en l'incluant dans le gradient de gravitation: en effet comme $\Omega \wedge (\Omega \wedge r) = \nabla(\frac{1}{2}|\Omega \wedge r|^2)$, on définira donc le géopotentiel par:

$$\phi = \phi_a - \frac{1}{2} |\Omega \wedge r|^2.$$

En première approximation, ce géopotentiel vérifie sensiblement $\phi \approx gr + cste \approx gz$ dans un repère cartésien localement tangent à la surface de la planète (z étant l'altitude au dessus du niveau de la mer.).

Ainsi pour un point situé à la latitude ϕ , l'axe des z sera dirigé selon le rayon terrestre, l'axe des y dans la direction Sud-Nord, et l'axe des x Ouest-Est, et $\Omega = \Omega \cos \phi \mathbf{j} + \Omega \sin \phi \mathbf{k}$ où \mathbf{i} , \mathbf{j} et \mathbf{k} désignent les vecteurs de base. Et en posant $f = 2\Omega \sin \phi$ (paramètre de Coriolis) et $f_* = 2\Omega \cos \phi$ (paramètre de Coriolis réciproque) la conservation de la quantité de mouvement s'écrit, en notant U = (u, v, w) la vitesse et τ^{ij} les composantes du terme de frottement:

$$\begin{cases} \rho(\frac{Du}{Dt} + f_*w - fv) &= -\partial_1 p + \partial_1 \tau^{11} + \partial_2 \tau^{12} + \partial_3 \tau^{13} \\ \rho(\frac{Dv}{Dt} + fu) &= -\partial_2 p + \partial_1 \tau^{12} + \partial_2 \tau^{22} + \partial_3 \tau^{23} \\ \rho(\frac{Dw}{Dt} - f_*u) &= -\partial_3 p - \rho g + \partial_1 \tau^{13} + \partial_2 \tau^{23} + \partial_3 \tau^{33} \end{cases}$$
(1.1.1)

Conservation de la masse (équation de continuité)

Si ρ désigne la densité et U la vitesse, la conservation de la masse s'écrit (de façon indépendante du repère):

$$\frac{D\rho}{Dt} + \rho \operatorname{div} U = 0.$$

On verra avec l'approximation de Boussinesq que l'on pourra faire l'hypothèse div U = 0d'incompressiblité qui correspond à la conservation du volume. On se limitera à cette approximation qui est très bien vérifiée dans l'océan (l'eau est pratiquement incompressible), et bien vérifiée dans l'atmosphère en basse altitude (troposphère, jusqu'à environ 10 km d'altitude). Par contre, en haute altitude, la densité décroit exponentiellement (raréfaction). On se limitera donc aux mouvements de la troposphère, où cela sera une bonne approximation.

Cette hypothèse d'incompressibilité du fluide simplifie les équations, sans cependant changer la nature des mouvements géophysiques. En effet la compressibilité de l'air permet l'existence d'ondes accoustiques se déplaçant à la vitesse du son et nous admettons que ces ondes n'interagissent pas avec les autres mouvements de l'atmosphère ou des océans.

Conservation de l'énergie

Nous admettons sans entrer dans les détails que d'après le premier principe de la thermodynamique, l'énergie gagnée par une masse unitaire de matière est égale à la chaleur reçue à laquelle on retranche le travail mécanique accompli, ainsi par unité de masse:

$$\frac{De}{Dt} = Q - p\frac{Dv}{Dt}$$

où $e = C_v T$ est l'énergie interne par unité de masse (C_v est la capacité thermique, T la température absolue), Q est la quantité de chaleur acquise par unité de masse, et $v = 1/\rho$ est le volume d'une unité de masse.

En général, les fluides géophysiques n'ont pas de source interne de chaleur. Cela permet d'écrire que Q provient uniquement d'une diffusion et d'après la loi de Fourier $\rho Q = k\Delta T$ où k est la conductivité thermique. En utilisant la conservation de la masse, on obtient que:

$$\rho C_v \frac{DT}{Dt} + p \operatorname{div} U = k\Delta T$$

On a introduit la variable T. Pour fermer le système on a donc besoin d'une nouvelle équation, l'équation d'état:

• dans l'air, assimilé à un gaz parfait,

$$p = \rho RT.$$

• dans l'eau, presque incompressible, on peut considérer que la densité est indépendante de la pression. D'autre part, elle est affectée non seulement par la température mais aussi par la salinité et l'on écrit en première approximation pour de faibles profondeurs:

$$\rho = \rho_0 (1 - \alpha (T - T_0) + \beta (S - S_0)),$$

où S est la salinité (exprimée en grammes par kilogramme d'eau) et α et β sont des constantes.

On boucle enfin le système en écrivant que la salinité vérifie une loi de diffusion (κ_S est le coefficient de diffusion):

$$\frac{DS}{Dt} = \kappa_S \Delta S.$$

Nous nous retrouvons donc avec six variables dans le cas de l'atmosphère $(U, p, \rho \text{ et } T)$ et sept dans le cas de l'océan $(U, p, \rho, T \text{ et } S)$. L'approximation de Boussinesq nous permettra de simplifier les équations et de traîter les deux cas simultanément.

Approximation de Boussinesq

Nous avons déjà commencé à simplifier les équations (repère tournant, coordonnées locales, approximations pour les équations d'état...) mais elles restent encore trop compliquées. L'approximation de Boussinesq, qui repose sur l'incompressibilité et le fait qu'en général, dans un système géophysique les grandeurs varient peu autour d'une valeur moyenne d'équilibre, va permettre de simplifier encore ces équations.

Par exemple, la température moyenne dans l'océan est $T = 4^{\circ}C$, la salinité $S = 34,7\%_0$ et la densité à la surface $\rho = 1028 kg.m^{-3}$. Dans la plupart des cas (océans, estuaires, rivières...), la variation de densité ne dépasse pas 2%.

Cependant dans l'atmosphère, l'air se raréfie avec l'altitude et donc la densité varie de 100%: avec les valeurs suivantes $\rho = 1kg.m^{-3}$, $g = 10m.s^{-2}$, $p_0 = 10^5Pa$ la loi hydrostatique ($\partial_z p = -\rho g$) implique que la pression de référence s'annulera à 10km d'altitude ce qui correspond bien à l'échelle de la troposphère. De plus si l'on se restreint encore à la troposphère (couche d'environ 10km), les variations de densité responsables des vents sont de moins de 5%. Il est donc raisonnable dans la plupart des cas de supposer que la densité varie peu autour d'une densité moyenne de référence:

$$\rho = \rho_0 + \rho'(x, y, z, t)$$
 avec $\rho' \ll \rho_0$.

Nous allons, grâce à cette hypothèse, simplifier les équations de (1.1.1). Commençons par la conservation de la masse :

$$\rho_0 \operatorname{div} U + \rho' \operatorname{div} U + (\partial_t \rho' + U \cdot \nabla \rho') = 0.$$

Des analyses montrent que dans la pratique les deux derniers termes et le deuxième sont du même ordre de grandeur, et négligeables devant le premier. Ainsi qu'il est présenté dans la littérature nous écrivons donc formellement que :

$$\begin{cases} \operatorname{div} \mathbf{U} = \mathbf{0} \\ \partial_t \rho' + U . \nabla \rho' = \mathbf{0} \end{cases}$$

Ainsi la conservation de la masse est devenue, grâce à l'hypothèse faite, la conservation du volume, ce qui élimine les ondes acoustiques.

Continuons par les deux premières équations de la conservation de la quantité de mouvement : nous négligeons simplement dans les membres de gauche ρ' par rapport à la densité moyenne ρ_0 . Nous n'entrerons pas dans les détails et nous admettrons qu'en utilisant l'hypothèse que le fluide est newtonien, les frottements visqueux s'expriment en fonction du gradient de vitesse et du coefficient de viscosité dynamique μ . En divisant par ρ_0 et en définissant la viscosité cinématique $\nu = \mu/\rho_0$, nous obtenons les deux premières équations :

$$\begin{cases} \frac{Du}{Dt} + f_* w - fv &= -\frac{1}{\rho_0} \partial_1 p + \nu \Delta u \\ \frac{Dv}{Dt} + fu &= -\frac{1}{\rho_0} \partial_2 p + \nu \Delta v. \end{cases}$$

En ce qui concerne l'équation sur la troisième composante w, la densité apparaît des deux côtés. Ainsi qu'il est présenté dans [15], on néglige ρ' à gauche mais pas à droite et on va absorber le terme gravitationnel dans la pression : on écrit que $p = p_0(z) + p'(x, y, z, t)$ avec $p_0(z) = P_0 - \rho_0 gz$ (c'est la pression hydrostatique) et on obtient ainsi :

$$\frac{Dw}{Dt} - f_*u = -\frac{1}{\rho_0}\partial_3 p' - \frac{\rho' g}{\rho_0} + \nu\Delta w.$$

Comme p_0 ne dépend que de z nous pouvons la soustraire aussi à p dans les équations des deux premières composantes de la vitesse ce qui ne change pas l'expression des dérivées de la pression par rapport aux première et deuxième variables.

Enfin terminons par l'équation d'énergie : en définissant la diffusivité thermique par $\kappa_T = k/(\rho_0 C_v)$ nous obtenons que (nous avons utilisé l'incompressibilité et l'hypothèse sur la densité) :

$$\frac{DT}{Dt} = \kappa_T \Delta T.$$

Nous admettons que l'on peut traîter simultanément le cas de l'atmosphère et celui des océans en notant $\kappa = \kappa_S = \kappa_T$ et combiner les équations sur la température et la salinité en :

$$\frac{D\rho'}{Dt} = \kappa \Delta \rho'.$$

En rassemblant ces équations et en notant p au lieu de p', ρ au lieu de ρ' , nous nous retrouvons avec le système :

$$\begin{cases} \partial_t u + U \cdot \nabla u + f_* w - fv &= -\frac{1}{\rho_0} \partial_1 p + \nu \Delta u \\ \partial_t v + U \cdot \nabla v + fu &= -\frac{1}{\rho_0} \partial_2 p + \nu \Delta v \\ \partial_t w + U \cdot \nabla w - f_* u &= -\frac{1}{\rho_0} \partial_3 p - \frac{\rho g}{\rho_0} + \nu \Delta w \\ \partial_t \rho + U \cdot \nabla \rho &= \kappa \Delta \rho \\ \operatorname{div} U &= 0. \end{cases}$$
(1.1.2)

1.1.3 Simplifications d'échelle

Nous continuons la simplifications des équations en utilisant les arguments d'échelle et de grandeurs caractéristiques développés dans [15]. Reprenons les mêmes notations que précédemment : la grandeur caractéristique des distances horizontales sera L (pour x et y), H pour la hauteur (z), T pour le temps t, U pour la vitesse horizontale (u et v), Wpour la vitesse verticale (w), P pour la pression (p) et $\Delta \rho$ pour la densité (ρ) . Nous faisons les approximations suivantes :

$$T \gtrsim \frac{1}{\Omega}, \quad \frac{U}{L} \lesssim \Omega, \quad H \ll L, \quad \Delta \rho \ll \rho_0.$$

En examinant les trois termes intervenant dans la dernière équation (divU = 0), nous remarquons que l'on doit comparer les échelles U/L et W/H. Il apparaît en pratique que la meilleure approximation est de convenir que $W/H \leq U/L$. Ainsi nous obtenons que $W \ll U$.

Ces simples comparaisons permettent d'éliminer, de façon purement formelle, de nombreux termes. Par exemple, f_*w (qui est de l'ordre de ΩW) sera négligé devant fu (de l'ordre de ΩU) (nous pouvons nous le permettre puisque nous nous plaçons dans les latitudes moyennes à l'écart de l'équateur). De la même façon $\nu \partial_1^2 u \ (\nu U/L^2)$ est négligeable devant $\nu \partial_3^2 u \ (\nu U/H^2)$ ou encore, dans l'équation sur w, les quatre premiers termes sont négligeables devant le cinquième, lui-même, négligeable devant les deux premiers termes du membre de droite. Nous obtenons finalement le système simplifié suivant constituant le système de base de la dynamique des fluides géophysiques (rappelons que $f = 2\Omega \sin \phi$) :

$$\begin{cases} \partial_t u + U \cdot \nabla u - fv &= -\frac{1}{\rho_0} \partial_1 p + \nu \partial_3^2 u \\ \partial_t v + U \cdot \nabla v + fu &= -\frac{1}{\rho_0} \partial_2 p + \nu \partial_3^2 v \\ 0 &= -\partial_3 p - \rho g \\ \partial_t \rho + U \cdot \nabla \rho &= \kappa \partial_3^2 \rho \\ \operatorname{div} U &= 0. \end{cases}$$
(1.1.3)

Remarquons que la troisième équation a été drastiquement simplifiée au point que nous nous retrouvons avec la relation hydrostatique.

1.1.4 Équilibre géostrophique

Effectuons maintenant une dernière analyse d'échelle afin d'évaluer les termes de premier ordre. Si l'on divise par ΩU les ordres de grandeur des deux premières équations, on obtient par ordre :

$$\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{W}{\Omega H} = \frac{WL}{UH} \cdot \frac{U}{\Omega L}, \quad 1, \quad \frac{P}{\rho_0 \Omega L U}, \quad \frac{\nu}{\Omega H^2}$$

Apparaissent ainsi trois nombres :

- Le nombre de Rossby temporel $Ro_T = \frac{1}{\Omega T}$, comparant le temps caractéristique à la période de rotation de la Terre,
- Le nombre de Rossby $Ro = \frac{U}{\Omega L}$, comparant l'advection à la force de Coriolis,
- Le nombre d'Ekman $Ek = \frac{\nu}{\Omega H^2}$ mesurant l'importance relative de la friction.

Ces nombres sont tous inférieurs à l'unité et nous allons en fait les supposer négligeables devant un. D'autre part nous admettons que l'on peut supposer que $P = \rho_0 \Omega L U$. Remarquons enfin que nulle part encore la stratification n'est encore apparue : nous n'avons encore considéré que les effets de la force de Coriolis.

Supposons maintenant que ces trois nombres sont négligeables devant un, ce qui revient à s'intéresser à des fluides tournant rapidement. Continuons et ignorons les frictions visqueuses et la variation de densité ($\rho = 0$, rappelons que nous avons choisi de noter encore $\rho' = \rho$ et p' = p), nous obtenons alors le système suivant:

$$\begin{cases}
-fv = -\frac{1}{\rho_0}\partial_1 p \\
fu = -\frac{1}{\rho_0}\partial_2 p \\
0 = -\partial_3 p \\
div U = 0.
\end{cases}$$
(1.1.4)

Il est ainsi facile d'obtenir que $\partial_3 u = \partial_3 v = 0$, résultat connu sous le nom de théorème de Taylor-Proudman, qui dit que toutes les particules à la même verticale ont la même vitesse horizontale : il s'agit de la rigidité verticale annoncée au début de cette introduction. D'autre part la vitesse est orthogonale au gradient de pression :

$$u = \frac{-1}{\rho_0 f} \partial_2 p, \quad v = \frac{1}{\rho_0 f} \partial_1 p,$$

c'est-à-dire que les particules se déplacent le long des lignes de pression constante, appelées isobares, la direction de la vitesse étant fournie par le signe de f: par exemple, dans l'hémisphère Nord, $f \ge 0$ (rotation dans le sens trigonométrique) et le courant s'écoule avec à sa droite les hautes pressions.

Terminons en précisant que le divergence de la vitesse étant nulle, la troisième composante ne dépend pas non plus de la troisième coordonnée. La composante verticale de la vitesse peut être éliminée et la vitesse est strictement bidimensionnelle. Nous renvoyons à [12] pour une étude justifiant cette limite en forte rotation pour le système des fluides tournants grâce à des méthodes utilisant les effets dispersifs de certaines ondes dites de Rossby (ou encore planétaires).

Un tel équilibre dans lequel la force de Coriolis est parfaitement contrebalancée par le gradient de pression est appelé équilibre géostrophique (ou encore du vent géostrophique), ce mot étant construit des racines grecques : $\gamma\eta$ (Terre) et $\sigma\tau\rho o\phi\eta$ (tournant).

L'équilibre géostrophique est presque réalisé dans la réalité : il suffit de lire une carte de relevés pour constater un très net parallélisme entre les vitesses des vents et les isobares.

Dans la première moitié du XX^e siècle on a voulu décrire l'atmosphère avec ce modèle du vent géostrophique. Ce système ne fournissant pas d'évolution dans le temps, les météorologues se sont vite intéressés à un système obtenu en prenant en compte les termes d'ordre supérieur en le nombre de Rossby : il s'agit du système quasigéostrophique qui fournit l'évolution dans le temps de champs lents. Ceci fait l'objet de la section suivante.

1.1.5 Dynamique quasigéostrophique tridimensionnelle

Stratification

Comme nous l'avons signalé plus haut, nous n'avons, pour l'instant, jamais parlé concrètement des effets de la stratification sur les équations. Dans cette section nous allons introduire les dernières hypothèses simplificatrices qui permettront d'obtenir non seulement le système quasigéostrophique mais aussi la forme des équations primitives sur lesquelles nous travaillerons dans le corps de cette thèse.

Rappelons que nous avons constaté formellement que la rotation impose au fluide une rigidité verticale qui donne au fluide une structure "en colonnes" au sens où les particules sur la même verticale ont la même vitesse. A l'inverse, un fluide stratifié, c'est-à-dire composé de cellules de densités différentes (par exemple air chaud ou froid, eau salée ou douce...) aura tendance sous l'effet de la pesanteur à se ré-arranger de façon à ce que des densités fortes soient situées en dessous de densités faibles. Cette répartition verticale en couches horizontales induit ainsi formellement un gradient vertical qui affectera le champ de vitesse et atténuera les effets de la rotation.

Considérons un fluide en équilibre statique composé de couches stratifiées verticalement. Il est naturel de penser que si les couches lourdes sont situées sous les couches légères alors le fluide est stable, tandis que dans le cas inverse le fluide aura tendance à subir un renversement qui le rend instable. Vérifions formellement cette intuition et considérons une petite partie de ce fluide, située à la hauteur z où la densité est $\rho(z)$. Déplaçons-la verticalement jusqu'à la hauteur z + h où la densité est $\rho(z + h)$. Si le fluide est incompressible la partie déplacée conservera sa densité même si la pression change un peu, et à sa nouvelle hauteur, cette partie de fluide de volume V, selon le principe d'Archimède, subit une force de flottaison valant :

$$g\Big(\rho(z+h)-\rho(z)\Big)V.$$

D'après le principe de Newton nous obtenons que :

$$\rho(z)V\frac{d^2h}{dt^2} = g\Big(\rho(z+h) - \rho(z)\Big)V.$$

Dans le cas d'un fluide géophysique, les variations de densité sont relativement faibles par rapport à la densité moyenne (ou de référence) du fluide. Cette remarque, qui était primoridale lors de l'approximation de Boussinesq, va nous permettre de remplacer à gauche $\rho(z)$ par cette densité moyenne ρ_0 et d'effectuer un développement limité à droite :

$$\rho(z+h) - \rho(z) \approx \frac{d\rho}{dz}h.$$

Nous obtenons ainsi l'équation :

$$\frac{d^2h}{dt^2} - \frac{g}{\rho_0}\frac{d\rho}{dz}h = 0.$$

Deux cas se présentent selon le signe du terme $-\frac{g}{\rho_0}\frac{d\rho}{dz}$:

• S'il est positif $(d\rho/dz < 0)$, la densité décroissant avec l'altitude), alors on peut définir la quantité suivante :

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz},$$

et la solution a un comportement sensiblement oscillatoire de fréquence N. Elle va d'abord redescendre vers sa hauteur d'origine, la dépasser et se retrouver dans une couche de plus forte densité, où elle sera soumise à une force de flottaison la repoussant vers le haut, et ainsi de suite jusqu'à stabilisation à cause des frottements visqueux... La quantité N s'appelle fréquence de stratification, ou fréquence de Brunt et Väisälä.

• S'il est négatif $(d\rho/dz > 0)$, la densité croissant avec l'altitude) alors la solution aura une croissance exponentielle. Le fluide est très instable et un grand mouvement de renversement se produira jusqu'à ce que le fluide se soit stabilisé avec les couches lourdes en dessous des couches légères. Si un facteur de déstabilisation est constamment appliqué au fluide, comme par exemple un chauffage par le bas ou un refroidissement par le haut, le fluide restera en constante agitation, ce phénomène s'appelant la convection.

Dans la suite, nous ne nous intéresserons qu'à des fluides stables pour les quels la fréquence N existe. On la considèrera comme constante.

Sans entrer dans les détails, précisons que dans le cas de l'atmosphère, la fréquence de stratification est définie en fait à partir d'une grandeur d'état proche de la densité, appelée densité potentielle mais nous pouvons considérer que les relations précédentes sont vraies pour la densité dans tous les cas avec une bonne approximation.

Le nombre de Froude

De la même façon que le nombre de Rossby a été défini pour mesurer l'importance de la rotation en comparant la distance parcourue par une particule de fluide durant une révolution ($\sim U/\Omega$) avec la longueur caractéristique (L) dans laquelle a lieu le mouvement (les effets de la rotation sont d'autant plus importants que le premier est grand devant le second) nous allons définir un nombre pour mesurer l'importance de la stratification. On peut déja s'attendre à ce que la fréquence de stratification N et la hauteur H interviennent de façon symétrique à Ω et L.

Considérons un fluide de fréquence de Brunt-Väisälä N s'écoulant horizontalement à la vitesse U et rencontrant un obstacle de longueur L et de hauteur H, il peut s'agir par exemple d'un vent soufflant dans la basse atmosphère sur une chaîne de montagnes. La présence de cet obstacle va forcer le fluide à être déplacé verticalement ce qui revient à lutter contre la pesanteur et à fournir de l'énergie gravitationnelle. La stratification va agir de façon à minimiser ce déplacement vertical, forçant le fluide à passer autour de l'obstacle plutôt qu'au dessus de ce dernier. Plus la stratification est importante, plus cette restriction est forte.

Le temps passé au voisinage de l'obstacle est sensiblement le temps mis pour parcourir la distance L avec la vitesse U, c'est-à-dire T = L/U. En désignant par W la composante verticale de la vitesse, les déplacements verticaux seront de l'ordre de $\Delta z = WT = WL/U$. En présence de la stratification et de la densité $\rho(z)$ ces déplacements créeront des perturbations de l'ordre de (en notant $\overline{\rho}(z)$ la répartition stratifiée de la densité, dominante devant les perturbations $\rho(x, y, z, t)$) :

$$\Delta \rho \approx |\frac{d\overline{\rho}}{dz}| \Delta z \approx \frac{\rho_0 N^2}{g} \frac{WL}{U}.$$

En retour, ces variations de densité vont créer des perturbations de la pression, qui seront grâce à l'équilibre hydrostatique de l'ordre de :

$$P \approx gH\Delta\rho \approx \frac{\rho_0 N^2 HLW}{U}.$$

Pour respecter l'équilibre des forces horizontales, ce gradient de pression doit être accompagné d'un changement de vitesse horizontale visant à l'équilibrer $(u\partial_1 u + v\partial_2 u \sim (1/\rho_0)\partial_1 p)$ et :

$$U^2 \approx \frac{P}{\rho_0} \approx \frac{N^2 H L W}{U},$$

dont nous déduisons immédiatement le rapport entre ce que nous interprêtons comme la "convergence" en vertical W/H et la "divergence" en horizontal U/L autour de l'obstacle :

$$\frac{W/H}{U/L} \approx \frac{U^2}{N^2 H^2}$$

Nous remarquons donc que si U < NH alors W/H < U/L et en utilisant la nullité de la divergence, le terme $\partial_1 u$ est plutôt compensé par $\partial_2 v$ que par $\partial_3 w$. Il y a plutôt un décalage horizontal que vertical, et plus la stratification est forte, plus U est petit devant NH et W/H devant U/L. Plus la stratification est forte, plus les déplacements verticaux et la vitesse verticale seront petits :

$$\frac{\Delta z}{H} \approx \frac{WL}{UH} \approx \frac{U^2}{N^2 H^2}.$$

Ce qui nous amène à définir le nombre de Froude par :

$$Fr = \frac{U}{NH},$$

qui mesure l'importance de la stratification : s'il est inférieur à un la stratification a des effets importants, qui le seront d'autant plus que Fr sera petit.

On peut montrer que souvent, un fluide géophysique n'étant jamais parfaitement géostrophique, le rapport (W/H)/(U/L) est de l'ordre de Ro.

Terminons en essayant de trouver un analogue aux colonnes dues aux effets d'une forte rotation ($Ro \ll 1$). Ainsi donc, dans le cas où $Fr = U/NH \ll 1$, les déplacements verticaux seront très fortement restreints, de sorte que la présence d'un obstacle force le fluide à se répandre de façon presque purement horizontale. Si l'obstacle occupe toute la longueur, le fluide est donc bloqué. Ceci peut être vu comme un analogue à la rigidité verticale en cas de forte rotation.

La courte section suivante présente l'approximation dite du β -plan qui sera utilisée dans la section suivante pour aboutir formellement au système quasigéostrophique.

Approximation du β -plan

Cette approximation consiste à développer le paramètre de Coriolis $f = 2\Omega \sin \phi$ autour d'une latitude de référence ϕ_0 . Ainsi l'on écrit $\phi = \phi_0 + y/a$, avec le rayon de la Terre a = 6371 km. Le terme y/a étant pris petit, on développe ainsi le paramètre de Coriolis :

$$f = 2\Omega \sin \phi_0 + 2\Omega \frac{y}{a} \cos \phi_0 + \dots,$$

duquel on ne gardera que les deux premiers termes : $f = f_0 + \beta_0 y$.

Pour des latitudes moyennes, $f_0 = 8.10^{-5}s^{-1}$ et $\beta_0 = 2.10^{-11}m^{-1}.s^{-1}$. Si l'on ne retient que le premier terme l'approximation est traditionnellement appelée du "*f*-plan", et du " β -plan" lorsque l'on garde les deux premiers termes.

Le système quasigéostrophique tridimensionnel

Dans cette partie les nombres de Rossby, de Rossby temporel et d'Ekman seront supposés petits. Pour rendre le nombre de Rossby $(U/\Omega L)$ petit, ou bien l'on considère des vitesses relativement faibles (U petit) ou bien des fluides très étendus (L grand). La première possibilité sera utilisée pour aboutir au système quasigéostrophique mais l'autre possibilité n'est pas à écarter car elle est vérifiée dans de nombreux mouvements atmosphériques et océaniques.

Commençons par la restriction consistant à considérer que les fluctuations autres que verticales de la densité sont faibles (rappelons que l'on avait décomposé la densité en $\rho = \rho_0 + \rho'$ ou ρ' est une petite variation, et qu'on a ensuite gardé seulement ρ' dans les équations en la notant encore ρ) :

$$\rho = \overline{\rho}(z) + \rho'(x, y, z, t), \quad |\rho'| \ll |\overline{\rho}|.$$

La densité $\overline{\rho}(z)$ est appelée la stratification basique et, lorsqu'elle n'est pas perturbée, elle crée un état d'équilibre hydrostatique. On supposera que cette stratification est établie et maintenue. Le formalisme de l'approche quasigéostrophique ne s'intéresse pas à l'origine de cet équilibre mais aux mouvements qui le perturbent faiblement. Le raisonnement suivant sera purement formel.

Sous l'approximation du β -plan, les équations s'écrivent de la façon suivante en posant $p = \overline{p}(z) + p'(x, y, z, t)$:

$$\begin{cases} \partial_t u + U \cdot \nabla u - f_0 v - \beta_0 y v &= -\frac{1}{\rho_0} \partial_1 p' + \nu \partial_3^2 u \\ \partial_t v + U \cdot \nabla v + f_0 u + \beta_0 y u &= -\frac{1}{\rho_0} \partial_2 p' + \nu \partial_3^2 v \\ 0 &= -\partial_3 p' - \rho' g \\ \partial_t \rho + u \partial_1 \rho + v \partial_2 \rho + w \partial_3 \overline{\rho} &= 0 \\ \operatorname{div} U &= 0 \end{cases}$$

L'hypothèse $|\rho'| \ll |\overline{\rho}|$ nous a permis formellement de nous débarrasser du terme $w\partial_3\rho'$, et nous avons aussi négligé la diffusion verticale de densité. Supposons de plus que l'on a $|\beta_0 y| \ll f_0$. Nous remarquons que les termes dominants correspondent exactement à l'équilibre géostrophique :

$$\begin{cases} -f_0 v = -\frac{1}{\rho_0} \partial_1 p' \\ f_0 u = -\frac{1}{\rho_0} \partial_2 p'. \end{cases}$$

Remplaçons u et v par ces valeurs géostrophiques (sauf pour les termes en f_0), nous exprimons ensuite u et v ($J(a, b) = \partial_1 a \partial_2 b - \partial_2 a \partial_1 b$) :

$$u = -\frac{1}{\rho_0 f_0} \partial_2 p' - \frac{1}{\rho_0 f_0^2} \partial_1 \partial_t p' - \frac{1}{\rho_0^2 f_0^3} J(p', \partial_1 p') + \frac{\beta_0}{\rho_0 f_0^2} y \partial_2 p' + \frac{\nu}{\rho_0 f_0^2} \partial_1 \partial_3^2 p',$$

$$v = \frac{1}{\rho_0 f_0} \partial_1 p' - \frac{1}{\rho_0 f_0^2} \partial_2 \partial_t p' - \frac{1}{\rho_0^2 f_0^3} J(p', \partial_2 p') - \frac{\beta_0}{\rho_0 f_0^2} y \partial_1 p' + \frac{\nu}{\rho_0 f_0^2} \partial_2 \partial_3^2 p'.$$

En utilisant la nullité de la divergence, nous en déduisons la valeur de $\partial_3 w$ $(\nabla_h = \partial_1^2 + \partial_2^2)$:

$$\partial_3 w = \frac{1}{\rho_0 f_0^2} \left(\partial_t \nabla_h p' + \frac{1}{\rho_0 f_0} J(p', \nabla_h p') + \beta \partial_1 p' - \nu \nabla_h \partial_3^2 p' \right).$$

Ensuite, en remplacant les vitesses par les valeurs géostrophiques, et en utilisant la fréquence de Brunt-Väisälä $(N(z)^2 = -(g/\rho_0)\partial_3\overline{\rho})$, nous obtenons que :

$$\partial_t \rho' + \frac{1}{\rho_0 f_0} J(p', \rho') - \frac{\rho_0 N^2}{g} w = 0.$$

En divisant par N^2/g et en dérivant par rapport à la troisième coordonnée, puis en utilisant l'expression de $\partial_3 w$, nous obtenons finalement, en définissant la fonction de courant $\psi = p'/(\rho_0 f_0)$:

$$\partial_t q + J(\psi, q) = \nu \partial_3^2 \nabla_h \psi,$$

où $q = \nabla_h \psi + \partial_3 (\frac{f_0^2}{N^2} \partial_3 \psi) + \beta_0 y = \partial_1 v - \partial_2 u - \partial_3 (\frac{f_0 g}{N^2 \rho_0} \rho')$ est le tourbillon potentiel. On exprime ensuite les composantes grâce aux expressions de la vitesse géostrophique, l'équilibre hydrostatique (pour ρ') et enfin l'équation simplifiée de ρ' pour obtenir w:

$$\begin{cases}
 u = -\partial_2 \psi \\
 v = \partial_1 \psi \\
 w = -\frac{f_0}{N^2} (\partial_t \partial_3 \psi + J(\psi, \partial_3 \psi)) \\
 p' = \rho_0 f_0 \psi \\
 \rho' = \frac{\rho_0 f_0}{g} \partial_3 \psi
 \end{cases}$$
(1.1.5)

Et vu ces nouvelles expressions la quantité $J(\psi, q)$ vaut donc $J(\psi, q) = u\partial_1 q + v\partial_2 q$.

Ce système consistant en l'advection de la vorticité potentielle et une inversion de celle-ci par une loi du type Biot et Savart pour obtenir la vitesse est appelé système quasigéostrophique, et a été proposé par Charney. On constate que dans la réalité, les mouvements de grande échelle sont très proches du système quasigéostrophique, voire même de l'équilibre géostrophique. L'écart entre la réalité et le système quasigéostrophique est intuitivement de l'ordre du nombre de Rossby. C'est ce que nous verrons dans la dernière partie de cette thèse.

Le système quasigéostrophique a été utilisé pour faire de la prévision météorologique (dans les années 40-50). On lui a préféré, avec la précision croissante des simulations informatiques, l'utilisation des équations primitives (à partir des années 50-60). Cependant le système quasigéostrophique est encore utilisé de nos jours pour des modèles de climat de faible variation et de basse dimension.

Nous verrons dans la deuxième partie de nombreuses preuves mathématiques de cette convergence vers le modèle quasigéostrophique. Pour cela, les équations primitives subissent encore quelques simplifications.

1.1.6 Simplification des équations primitives

Dans cette dernière section introductive, nous allons modifier les équations à l'aide d'un raisonnement de changement d'échelle pour faire apparaître les nombres de Rossby et de Froude afin obtenir le système sur lequel nous travaillerons dans le corps de cette thèse. Nous suivrons la méthode proposée par [19].

Nous prenons désormais les notations $U = (v, \rho)$, $v = (u^1, u^2, u^3)$ et ϕ pour les inconnues, et (x_1, x_2, x_3, t) pour les variables, et reprenons le système (1.1.2) en négligeant le paramètre de Coriolis réciproque :

$$\begin{cases} \partial_t u^1 + v \cdot \nabla u^1 - f u^2 &= -\frac{1}{\rho_0} \partial_1 \phi + \nu \Delta u^1 \\ \partial_t u^2 + v \cdot \nabla u^2 + f u^1 &= -\frac{1}{\rho_0} \partial_2 \phi + \nu \Delta u^2 \\ \partial_t u^3 + v \cdot \nabla u^3 &= -\frac{1}{\rho_0} \partial_3 \phi - \frac{\rho g}{\rho_0} + \nu \Delta u^3 \\ \partial_t \rho + v \cdot \nabla \rho &= \kappa \Delta \rho \\ \operatorname{div} v &= 0. \end{cases}$$

Faisons les changements $\rho = \overline{\rho}(z) + \rho'(x, y, z, t)$, $|\rho'| \ll |\overline{\rho}|$ et $\phi' = \phi + \int_0^{x_3} g\overline{\rho}(s)ds$, notons encore $\rho' = \rho$ et p' = p. En utilisant que $\partial_3\overline{\rho} = -\rho_0 N^2/g$, nous obtenons le système suivant (nous choisissons d'ignorer le terme $\kappa \partial_3^2\overline{\rho}$) :

$$\begin{cases} \partial_t u^1 + v \cdot \nabla u^1 - f u^2 &= -\frac{1}{\rho_0} \partial_1 \phi + \nu \Delta u^1 \\ \partial_t u^2 + v \cdot \nabla u^2 + f u^1 &= -\frac{1}{\rho_0} \partial_2 \phi + \nu \Delta u^2 \\ \partial_t u^3 + v \cdot \nabla u^3 + \frac{\rho g}{\rho_0} &= -\frac{1}{\rho_0} \partial_3 \phi + \nu \Delta u^3 \\ \partial_t \rho + v \cdot \nabla \rho - \frac{\rho_0 N^2}{g} u^3 &= \kappa \Delta \rho \\ \operatorname{div} v &= 0. \end{cases}$$

Introduisons les échelles suivantes :

- L longueur horizontale caractéristique
- U vitesse horizontale caractéristique
- $T = \frac{L}{U}$ temps caractéristique
- $T_R = \frac{1}{f}$ période de rotation
- $T_N = \frac{1}{N}$ période de stratification
- ρ_0 densité moyenne
- p pression moyenne

Nous effectuons ensuite les mêmes changements que dans [19] :

$$\begin{cases} x' = \frac{x}{L} \\ t' = \frac{t}{T_R} \\ v = \frac{v'}{U} \\ \rho' = \frac{\rho}{\rho_0 B} \\ \phi' = \frac{\phi}{p} \end{cases}$$

Et en définissant les nombres suivants :

$$\begin{cases} Ro = \frac{T_R}{T} = \frac{U}{Lf} & \text{nombre de Rossby} \\ Fr = \frac{T_N}{T} = \frac{U}{LN} & \text{nombre de Froude} \\ \overline{p} = \frac{p}{\rho_0 U^2} \\ \Gamma = \frac{BgL}{U^2} \end{cases},$$

nous aboutissons au système (en notant abusivement $\nu=\frac{\nu U}{L}$ et $\nu'=\frac{\kappa U}{L})$:

$$\begin{cases} \partial_t u^1 + v \cdot \nabla u^1 - \frac{1}{Ro} u^2 &= -\overline{p} \frac{1}{\rho_0} \partial_1 \phi + \nu \Delta u^1 \\ \partial_t u^2 + v \cdot \nabla u^2 + \frac{1}{Ro} u^1 &= -\overline{p} \frac{1}{\rho_0} \partial_2 \phi + \nu \Delta u^2 \\ \partial_t u^3 + v \cdot \nabla u^3 + \Gamma \rho &= -\overline{p} \frac{1}{\rho_0} \partial_3 \phi + \nu \Delta u^3 \\ \partial_t \rho + v \cdot \nabla \rho - \frac{1}{Fr^2\Gamma} u^3 &= \nu' \Delta \rho \\ \operatorname{div} : v &= 0 \end{cases}$$

Comme nous considérons des mouvements fortement influencés par la rotation et la stratification, nous avons $T_R, T_N \ll T$ et donc $Ro, Fr \ll 1$. Nous allons lier ces paramètres et poser $Ro = \varepsilon$ où ε est un petit paramètre, $\overline{p} = \varepsilon^{-1}$, $Fr = \varepsilon F$, $\Gamma = Fr^{-1}$ (ce qui équivaut à B = NU/g). Nous obtenons finalement le système des équations primitives, sur lequel nous travaillerons dans cette thèse :

$$(PE_{\varepsilon}) \qquad \begin{cases} \partial_t U_{\varepsilon} + v_{\varepsilon} \cdot \nabla U_{\varepsilon} - LU_{\varepsilon} + \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon} = \frac{1}{\varepsilon} (-\nabla \Phi_{\varepsilon}, 0) \\ div \ v_{\varepsilon} = 0 \\ U_{\varepsilon/t=0} = U_0. \end{cases}$$

L'inconnue consiste en le couple $U_{\varepsilon} = (v_{\varepsilon}, \rho_{\varepsilon})$ et le terme de pression Φ_{ε} , et on définit l'opérateur L par :

$$LU_{\varepsilon} \stackrel{\text{def}}{=} (\nu \Delta v_{\varepsilon}, \nu' \Delta \rho_{\varepsilon})$$

et \mathcal{A} par:

$$\mathcal{A} \stackrel{\text{def}}{=} \left(\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{array} \right).$$

Si l'on utilise ces changements de variable et d'échelle, l'expression du tourbillon potentiel est $q = \partial_1 u'_2 - \partial_2 u'_1 - \partial_3 (\frac{f_{0g}}{N^2 \rho_0} \rho')$, et en posant $q' = \frac{L}{U}q$ et en supposant N constante, on obtient que

$$q' = \partial_1 u_2' - \partial_2 u_1' - F \partial_3 \rho'.$$

De façon abusive, on appellera ε le nombre de Rossby et F le nombre de Froude.

1.2 Aspects mathématiques

1.2.1 Rappel des travaux antérieurs

Equations primitives

L'étude des asymptotiques des équations primitives en forte rotation (correspondant à un nombre de Rossby Ro petit) et forte stratification (nombre de Froude Fr petit) se traduit, comme nous l'avons vu dans l'introduction par les choix d'échelles et de paramètres, de façon unifiée par la convergence du seul petit paramètre ε vers zéro. Rappelons que, suivant les changements d'échelle de [19] nous avons nommé ε nombre de Rossby mais qu'il prend en compte non seulement l'évanescence du paramètre Ro mais aussi celle du paramètre Fr et il s'agit alors de vérifier mathématiquement la convergence, lorsque ε tend vers zéro, vers le système quasigéostrophique. Nous allons rappeler dans cette partie une liste de résultats concernant les asymptotiques du système des équations primitives.

Dans [31] et [32], J.-L. Lions, R. Temam et S. Wang présentent les équations primitives dans une géométrie sphérique et cartésienne, puis s'intéressent à leurs solutions et aux asymptotiques géostrophiques.

Citons ensuite le travail de T. Beale et A. Bourgeois dans [4] qui étudient les équations primitives (non visqueuses et avec une équation simplifiée pour la densité) dans le cas d'un domaine périodique en la coordonnée horizontale (et borné en la coordonnée verticale) et pour des données initiales régulières. Grâce à un changement de variable fondé sur les symétrisations, ils se ramènent à des fonctions périodiques dans les trois directions, cadre dans lequel ils démontrent leurs résultats. Il étudient ainsi le système quasigéostrophique (en temps court, puis global) ainsi que la convergence des solutions des équations primitives pour des données initiales très régulières (H^3) et bien préparées.

La notion de données "bien préparées" signifie que l'on prend des données initiales déjà proches de l'état quasigéostrophique au sens où la suite des données initiales converge lorsque ε tend vers zéro, vers une fonction quasigéostrophique. Historiquement, tant pour le système des équations primitives que pour celui des fluides tournants, les méthodes ne permettaient pas de considérer des données initiales générales, aussi devait-on choisir des données proches de la structure des solutions du système limite.

La raison se trouve sous forme de conjecture dans ces trois articles: on prend conscience que ce qui force à se placer dans un cas bien préparé est la présence de solutions oscillant très rapidement (dont la fréquence est inversement proportionnelle au nombre de Rossby) ce sont les ondes de Poincaré (ou ondes planétaires). Et si l'on ne se plaçait pas dans le cas bien préparé (ce qui veut dire qu'on essaie autant que possible de neutraliser ces ondes dans la donnée initiale) les méthodes ne permettaient pas de prouver la convergence vers le système quasigéostrophique. Comme nous le verrons plus tard, les méthodes de dispersion et les estimations de Strichartz nous permettront dans cette thèse d'étudier précisément ces ondes et de connaître leur comportement : ces ondes tendent vers zéro dans des espaces du type $L^p([0, T], L^{\infty}/\text{Lip}/C^s)$.

Dans [19] et [20], P. Embid et A. Majda présentent une formulation générale pour les mouvements de fluides géophysiques, suivie de l'explication de l'approximation de Boussinesq et de changements d'échelle. Le choix d'un unique petit paramètre rendant compte des phénomènes dus à la rotation et la stratification permet ensuite une simplification aboutissant au système des équations primitives que nous allons utiliser tout le long de cette thèse. Suivent une étude dans le cas de fluides peu profonds en rotation rapide et dans le cas périodique en les trois variables (\mathbb{T}^3).

Citons ensuite le travail de J.-Y. Chemin dans [11] qui propose une étude du système primitif dans le cadre d'espaces de Sobolev $\dot{H}^{s}(\mathbb{R}^{3})$ (on se place dans tout l'espace) et sous l'hypothèse où F = 1, cas, nous le verrons plus tard, où l'on n'a aucun effet dispersif. La structure du système permet dans ce contexte d'utiliser les théorèmes de Leray et Fujita-Kato et d'obtenir ainsi des solutions faibles et des solutions fortes. J.-Y. Chemin prouve que la solution U_{ε} converge dans l'espace $L^{\infty}(\mathbb{R}_{+}, \dot{H}^{1-\eta}) \cap L^{2}(\mathbb{R}_{+}, \dot{H}^{2-\eta})$ vers la solution unique du système quasigéostrophique ayant pour donnée initiale $U_{0,QG}$. Ceci est fait sous des hypothèses de proximité des deux viscosités ν et ν' , et pour des données initiales régulières et bien préparées: la suite des parties quasigéostrophiques des données initiales converge dans \dot{H}^{1} vers $U_{0,QG}$, et la suite des parties oscillantes initiales converge vers zéro dans \dot{H}^{-1} . De façon plus précise:

Théorème 1.2.1 [11] Il existe une constante 0 < c < 1 telle que si $|\nu - \nu'| \leq c\nu$ et si la donnée initiale $U_0 \in \dot{H}^1 \cap \dot{H}^{-1}$ vérifie:

• $||U_{0,osc}||_{\dot{H}^{-1}} \le \frac{c\nu^4}{||U_0||_{\dot{H}^1}^3} e^{-\frac{||U_0||_{\dot{H}^1}||U_0||_{L^2}}{c\nu^2}},$

•
$$\varepsilon \leq \frac{c\nu^4}{\|U_0\|_{\dot{H}^1}^4\|U_0\|_{\dot{H}^{\frac{1}{2}}}}e^{-\frac{\|U_0\|_{\dot{H}^1}\|U_0\|_{L^2}}{c\nu^2}}$$

Alors le système primitif est bien posé: il admet une unique solution globale appartenant à l'espace $\mathcal{C}(\mathbb{R}_+, \dot{H}^1) \cap L^2(\mathbb{R}_+, \dot{H}^2)$ et vérifiant pour tout temps t:

$$\|U(t)\|_{\dot{H}^1}^2 + \nu \int_0^\infty \|U(t')\|_{\dot{H}^2}^2 dt' \le \|U_0\|_{\dot{H}^1}^2$$

Le théorème suivant donne la convergence dans le cadre des données initiales bien préparées:

Théorème 1.2.2 [11] Soit $(U_{0,\varepsilon})_{\varepsilon\in]0,\varepsilon_0}$ une famille de données initiales bornées dans \dot{H}^1 telles qu'il existe $U_{0,QG}$ tel que l'on ait:

$$\lim_{\varepsilon \to 0} \|U_{0,\varepsilon,QG} - U_{0,QG}\|_{\dot{H}^1} = 0 \quad et \quad \lim_{\varepsilon \to 0} \|U_{0,\varepsilon,osc}\|_{\dot{H}^{-1}} = 0.$$

Alors la famille $(U_{\varepsilon})_{\varepsilon \in [0,\varepsilon_0]}$ du système des équations primitives avec donnée initiale $U_{0,\varepsilon}$ converge vers la solution U_{QG} du système quasigéostrophique avec donnée initiale $U_{0,QG}$ et ce dans l'espace $L^{\infty}(\mathbb{R}_+, \dot{H}^{1-\eta}) \cap L^2(\mathbb{R}_+, \dot{H}^{2-\eta})$ pour tout $\eta > 0$. Plus précisément, on a:

$$\overline{\lim_{\varepsilon \to 0}} (U_{\varepsilon} - U_{\varepsilon, osc}) = U_{QG} \quad dans \quad L^{\infty}(\mathbb{R}_+, \dot{H}^1) \cap L^2(\mathbb{R}_+, \dot{H}^2).$$

Le cas F = 1 ne donne aucun effet dispersif (pour la simple raison que des simplifications rendent dans ce cas la phase indépendante de la fréquence). Dans la présente thèse, où nous travaillons dans la continuité de cet article, nous nous placerons dans le cas $F \neq 1$ et utiliserons systématiquement la dispersion pour prouver la convergence vers zéro de la partie oscillante. Nous insistons sur le fait que nous nous plaçons d'emblée dans le cas mal préparé et n'aurons nulle part à faire l'hypothèse de données bien préparées. D'autre part nous n'aurons pas besoin de l'hypothèse de proximité des viscosités, ni de la petitesse des données initiales. Notons enfin que nous n'aurons besoin au départ que d'une régularité minimale: $U_0 \in L^2$.

Dans la continuité de cet article, citons l'étude de D. Iftimie ([30]) qui s'est intéressé au cas du système primitif non visqueux avec $F \neq 1$, et prouve la convergence vers le système quasigéostrophique dans l'espace $L^{\infty}_{loc}(\mathbb{R}_+, H^{\sigma}), \sigma < s$ où $s > \frac{5}{2}$ est la régularité initiale. Ici aussi on fait l'hypothèse de données initiales bien préparées en demandant que la suite des parties oscillantes des données initiales tende vers zéro dans l'espace L^2 , et que les parties quasigéostrophiques tendent dans L^2 vers une fonction quasigéostrophique.

Toujours dans la continuité de [11] nous mentionnons aussi le travail d'I. Gallagher sur les équations primitives dans [21], obtenant la convergence quasigéostrophique dans le cas périodique (cas du tore \mathbb{T}^3) avec F, ν , ν' quelconques et pour des données initiales mal préparées. Ce résultat fait partie de résultats plus généraux appliquant des méthodes développées par S. Schochet dans [35]. Citons enfin les travaux de B. Desjardins et E. Grenier dans [17] qui se placent dans un domaine de type bande, borné en haut et en bas par une paroi solide et pour une viscosité evanescente et en particulier une viscosité verticale petite devant ε . Ils prouvent l'existence d'une solution faible globale pour le système quasigéostrophique et étudient pour une solution régulière non seulement la convergence quasigéostrophique mais aussi les couches limites d'Ekman sur les parois verticales lorsque ε tend vers zéro.

Nous insistons sur le fait que dans la hiérarchie des systèmes obtenus par simplifications successives des équations ou hypothèses sur les grandeurs caractéristiques et domaines, de nombreuses équations font l'objet d'études très poussées : par exemple le système des fluides tournants, le système géostrophique, le système des lacs peu profonds... Citons ainsi les travaux de D. Bresch, J. Lemoine et J. Simon dans [5] et [6].

Fluides tournants

Le système dit des fluides tournants prend en compte la force de Coriolis mais pas la stratification. On l'obtient à partir des équations de base comme dans la première partie de cette introduction, et si l'on considère les approximations que l'on a faites pour obtenir notre système des équations primitives, on retrouve le système des fluides tournants en y éliminant la densité.

$$\begin{cases} \partial_t u_{\varepsilon} + u_{\varepsilon} \cdot \nabla u_{\varepsilon} - \nu \Delta u_{\varepsilon} + \frac{u_{\varepsilon} \times e_3}{\varepsilon} &= -\nabla p_{\varepsilon} \\ \operatorname{div} u_{\varepsilon} = 0 \\ u_{\varepsilon/t=0} = u_0 \end{cases}$$

Commençons par mentionner le travail d'E. Grenier qui s'est intéressé dans [27] aux ondes induites par le terme de rotation et les filtre grâce à un groupe d'isométries, à la manière de [35].

Citons ensuite les travaux de A. Babin, A. Mahalov et B. Nicolaenko qui se sont intéressés au cas mal préparé : dans [1] ils s'intéressent au système des fluides tournants avec ou sans viscosité et dans un domaine périodique non résonant et dans [2] ils prouvent l'existence en temps infini de solutions régulières pour le système des fluides tournants dans la limite d'une forte rotation et dans des domaines périodiques quelconques.

Continuons avec les travaux de T. Colin et P. Fabrie : dans [13] sont prouvés des résultats d'existence en temps long et la convergence en forte rotation et avec une viscosité verticale evanescente (de l'ordre du nombre de Rossby) vers le système de Navier-Stokes bidimensionnel pour des conditions au bord périodiques et des données initiales particulières. Dans [14], pour le même système, sont prouvés des résultats d'existence globale et de convergence vers Navier-Stokes bidimensionnel pour des conditions au bord périodiques et des données initiales partie de convergence vers Navier-Stokes bidimensionnel pour des conditions au bord périodiques et des données bien préparées.

Citons ensuite travail de J.-Y. Chemin, B. Desjardins, I. Gallagher et E. Grenier dans [12] qui ont obtenu une vérification du théorème de Taylor-Proudman dans tout l'espace en utilisant les phénomènes dispersifs engendrés par la force de Coriolis. Ils ont prouvé des estimations de type Strichartz et les ont utilisées pour obtenir des convergences vers la solution (unique) du système de Navier-Stokes bidimensionnel (on retrouve ainsi la vitesse bidimensionnelle et le fait que sur une même verticale, la vitesse est la même : c'est la répartition en colonnes évoquée dans la première section) dans différents cas pour des données initiales mal préparées : ils ont examiné les solutions faibles avec $u_0 \in L^2(\mathbb{R}^3)$. Dans ce cas, la suite u_{ε} converge vers zéro dans $L^2_{loc}(\mathbb{R}_+, L^q(\mathbb{R}^3))$ $(q \in]2, 6]$) : grâce à une étude spectrale et des estimations de Strichartz, ils prouvent que l'énergie est dispersée par les ondes de Rossby.

Toujours pour des solutions faibles avec données initiales mal préparées : si on part d'une donnée initiale se décomposant en $u_0 = \overline{u}_0 + w_0$ avec $\overline{u}_0 \in L^2(\mathbb{R}^2)$ et $w_0 \in L^2(\mathbb{R}^3)$ alors la solution u_{ε} converge dans $L^2_{loc}(\mathbb{R}_+, L^q(\mathbb{R}^3))$ $(q \in]2, 6[)$ vers la solution unique \overline{u} du système de Navier-Stokes bidimensionnel avec donnée initiale \overline{u}_0 .

Terminons par le cas où pour la deuxième partie on suppose $w_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$. Dans ce cas on dispose de solutions fortes et d'une bien meilleure convergence : en effet, si l'on note $w_{\varepsilon} = u_{\varepsilon} - \overline{u}$ et v_{ε}^F la solution de l'équation linéarisée homogène alors, si ε est suffisamment petit, la solution est globale, unique et $w_{\varepsilon} - v_{\varepsilon}^F$ converge vers zéro dans $L^{\infty}(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}_+, \dot{H}^{\frac{3}{2}}).$

Anticipons sur la partie suivante en disant que dans cette thèse, nous utiliserons abondamment ces méthodes pour des données initiales systématiquement mal préparées, avec des hypothèses minimales de régularité et sans supposer que $\nu = \nu'$ ou que F = 1 (le cas $F \neq 1$ est celui dans lequel la dispersion a lieu). Nous améliorerons ensuite les résultats en accordant plus de régularité aux données initiales et tout au long de cette thèse nous affinerons successivement les estimations dispersives et les estimations de Strichartz selon nos exigences de régularité ou de convergence. Précisons que dans le dernier chapitre, nous obtenons une estimation de la vitesse de convergence des solutions U_{ε} vers la solution du système quasigéostrophique en fonction du nombre de Rossby ε . Nous nous placerons enfin dans le cas $\nu = \nu'$ pour obtenir une vitesse de convergence bien meilleure.

Poches de tourbillon

Terminons par des résultats de type poche de tourbillon pour le système primitif non visqueux. Rappelons tout d'abord qu'un résultat de type poche de tourbillon consiste en la preuve de la persistance au cours du temps de la régularité tangentielle du bord de la poche. Il s'agit en fait de prendre une donnée initiale dont le tourbillon est l'indicatrice d'un domaine borné et régulier, et de vérifier que le tourbillon reste la fonction indicatrice d'un domaine, qui varie au cours du temps mais conserve tout de même une certaine régularité. Historiquement on a commencé à s'intéresser à ce problème pour le système d'Euler bidimensionnel suite à des observations expérimentales. Nous renvoyons à [10] pour le cas du système d'Euler bidimensionnel, à [16] et [29] pour le cas de Navier-Stokes bidimensionnel et enfin à [23] pour le cas de Navier-Stokes tridimensionnel. Citons aussi le travail de H. Chaocheng dans [9].

Concernant le système primitif, A. Dutrifoy a étudié dans [18] la convergence quasigéostrophique dans le cadre des poches de tourbillon, obtenant la persistance de la régularité tangentielle lorque ε tend vers zéro et $F \neq 1$. Le tourbillon potentiel, caractéristique des équations primitives, a été utilisé à la place du tourbillon. Les données initiales sont mal préparées et les résultats de dispersion de [12] sont appliqués au cas $\nu = \nu' = 0$ où les éléments propres sont plus simples à déterminer.

Le dernier résultat de cette thèse reprendra les principes de cet article en les adaptant au cas visqueux, et pour résoudre la difficulté technique imposée par la viscosité, nous utiliserons les travaux récents de T. Hmidi ([29]), qui complètent ceux de R. Danchin ([16]) concernant les poches de tourbillons visqueuses pour le système de Navier-Stokes bidimensionnel. Lors de cette étude, nous nous placerons dans le cas $\nu = \nu'$.

1.2.2 Présentation des résultats obtenus

Le but de cette thèse est d'utiliser dans un cadre plus général les méthodes développées dans [12], notamment de dispersion, afin obtenir dans la continuité des résultats de [11], la convergence en forte rotation et forte stratification du modèle des équations primitives (appelé aussi système primitif) vers le modèle quasigéostrophique tridimensionnel, ceci dans le cas de tout l'espace \mathbb{R}^3 pour $F \neq 1$. Nous insistons sur le fait que nous travaillerons systématiquement avec des données initiales mal préparées ce qui ne posera pas de problème étant donné que notre approche a consisté notamment en une étude très précise du comportement de la partie oscillante, et tout particulièrement des ondes de Poincaré, étude suffisamment robuste pour que nous puissions même permettre aux parties oscillantes des données initiales d'exploser.

Dans cette étude des oscillations, le fait de prendre $\nu \neq \nu'$ complique considérablement les calculs sur le système linéarisé et les estimations: non seulement pour l'étude des valeurs propres où, par rapport au cas des fluides tournants, la simple détermination des expressions des valeurs propres nécessite beaucoup plus de travail (ceci est présenté en détail dans le chapitre suivant), mais aussi pour celle des vecteurs propres qui nécessite des estimations précises pour les projecteurs spectraux du fait de la non orthogonalité des vecteurs propres (dans le cas des fluides tournants les vecteurs propres sont orthogonaux). Ainsi parfois nous présenterons des résultats dans le cas $\nu = \nu'$ seulement (cas où les valeurs propres seront plus simples et les vecteurs propres orthogonaux), notamment dans le cas des poches de tourbillon où la présence de la viscosité, qui était un atout dans le cadre des solutions faibles (chapitre 2) et des solutions fortes (chapitres 3 et 4), dans le sens où elle permettait d'utiliser le concept de solution faible et les théorèmes de Leray et Fujita-Kato, ainsi que des estimations d'energie, pose ici de sérieux problèmes techniques qui seront surmontés en adaptant les résultats récents de T. Hmidi dans [29] concernant les poches de tourbillons pour le système de Navier-Stokes bidimensionnel.

Précisons qu'il est important de pouvoir traîter le cas $\nu \neq \nu'$ puisque dans la réalité ces deux grandeurs sont effectivement distinctes (bien que selon les cas pouvant être proches). À titre d'exemple, nous renvoyons à [28] où nous trouvons diverses valeurs de la diffusivité thermique ν' et de la viscosité cinématique ν : pour l'eau $\nu' = 10^{-7}m^2.s^{-1}$ et $\nu = 10^{-6}m^2.s^{-1}$, et pour l'air $\nu' = 2,24.10^{-5}m^2.s^{-1}$ et $\nu = 1,43.10^{-5}m^2.s^{-1}$. Si l'on définit $R = \frac{|\nu - \nu'|}{n}$, alors dans le premier cas on a environ R = 0.9 et dans le deuxième, R = 0.56.

De façon générale, même si nos données initiales sont mal préparées au sens où nous ne leur imposons aucune structure particulière, nous tirerons constamment parti de la séparation de la solution en sa partie oscillante et sa partie quasigéostrophique, représentation qui n'est en fait, comme nous le verrons, qu'une décomposition orthogonale associée à des opérateurs pseudo-différentiels homogènes d'ordre zéro.

Le deuxième chapitre traîte le cas de solutions faibles avec des données initiales mal préparées ayant une régularité minimale pour utiliser le concept de solution de Leray : nous les prendrons dans L^2 sans hypothèse supplémentaire sur leurs parties oscillantes.

Le troisième chapitre étudiera les solutions fortes avec une meilleure régularité pour les données initiales : les conclusions sont nettement plus précises.

Le quatrième chapitre présente, pour les solutions fortes et lorsque l'on demande encore un peu plus de régularité, des estimations beaucoup plus précises de la vitesse de convergence, avec un résultat plus précis encore sous l'hypothèse $\nu = \nu'$. Signalons que dans ce quatrième chapitre, on autorise la partie oscillante des données initiales à être non bornée en ε . La deuxième partie de ce chapitre prouve la convergence quasigéostrophique dans le cadre des poches de tourbillons et adapte des résultats de [29] et [18] pour prouver la persistence de la régularité tangentielle.

Nous insistons sur le fait que le fil conducteur de cette thèse est l'obtention d'estimations dispersives de plus en plus fines, sachant que l'uniformité en la viscosité se fait au détriment du caractère global en temps de ces estimations (nous verrons que la viscosité apparaît uniquement à travers ses puissances négatives, à part dans les dernières estimations de Strichartz où nous obtenons des estimations qui restent bornées lorsque la viscosité tend vers zéro, rendant ces résultats utilisables en viscosité evanescente, mais alors ces estimations ne sont que locales en temps).

Solutions faibles

Nous nous plaçons dans le cadre des solutions de Leray. La matrice \mathcal{A} étant antisymétrique, les méthodes d'énergie, utilisées pour la démonstration du théorème de Leray dans le cas du système de Navier-Stokes, sont inchangées et fournissent le même résultat d'existence de solutions faibles globales que l'on énonce dans le théorème suivant :

Théorème 1.2.3 Supposons que la donnée initiale $U_0 \in L^2(\mathbb{R}^3)$, alors il existe pour tout $\varepsilon > 0$ une solution de Leray du système (PE_{ε}) , U_{ε} , globale en temps, appartenant à l'espace $L^{\infty}(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ et satisfaisant l'estimation d'énergie suivante (avec $\nu_0 = \min(\nu, \nu') > 0$) :

$$\forall t \in \mathbb{R}_+, \quad \|U_{\varepsilon}(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\nu_0 \int_0^t \|\nabla U_{\varepsilon}(t)\|_{L^2(\mathbb{R}^3)}^2 dt \le \|U_0\|_{L^2(\mathbb{R}^3)}^2.$$

Dans le premier chapitre de cette thèse nous allons travailler sur ces solutions faibles et obtenir le formalisme qui nous servira dans toute la suite, ainsi que le premier résultat de convergence. Commençant par un calcul formel nous introduirons de façon naturelle le système quasigéostrophique ainsi que la quantité suivante que l'on appelle tourbillon potentiel :

$$\Omega_{\varepsilon} = \partial_1 v_{\varepsilon}^2 - \partial_2 v_{\varepsilon}^1 - F \partial_3 \theta_{\varepsilon},$$

ce qui nous permet dans la suite de scinder la solution U_{ε} en sa partie quasigéostrophique (l'objectif est d'éliminer les termes comportant ε en dénominateur en définissant un élément du noyau de l'opérateur $\mathbb{P}\mathcal{A}$) :

$$U_{\varepsilon,QG} = \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \Omega_{\varepsilon}$$

où l'on a posé $\Delta_F = \partial_1^2 + \partial_2^2 + F^2 \partial_3^2$, et enfin sa partie oscillante

$$U_{\varepsilon,osc} = U_{\varepsilon} - U_{\varepsilon,QG}.$$

Nous verrons que cette décomposition se résume en fait en une décomposition orthogonale du même type que celle obtenue lorsque l'on définit le projecteur de Leray \mathbb{P} sur les champs à divergence nulle : il existe deux opérateurs pseudo-différentiels homogènes de degré zéro, notés \mathcal{P} et \mathcal{Q} , tels que $U_{\varepsilon,QG} = \mathcal{Q}U_{\varepsilon}$, et $U_{\varepsilon,osc} = \mathcal{P}U_{\varepsilon}$.

Citons maintenant les deux résultats obtenus dans ce chapitre :

Théorème 1.2.4 Si la donnée initiale U_0 appartient à $L^2(\mathbb{R}^3)$ et si l'on considère une suite de solutions de Leray $(U_{\varepsilon})_{\varepsilon>0}$ alors, la partie oscillante $U_{\varepsilon,osc}$ tend vers zéro quand ε tend vers zéro, dans l'espace $L^2_{loc}(\mathbb{R}_+, L^q(\mathbb{R}^3))$ pour tout $q \in]2, 6[$.

Théorème 1.2.5 Sous les mêmes hypothèses, il existe une suite extraite de la partie quasigéostrophique $U_{\varepsilon,QG}$ qui converge pour tout $q \in]2, 6[$ dans l'espace $L^2_{loc}(\mathbb{R}_+, L^q_{loc}(\mathbb{R}^3))$ vers une fonction U_{QG} de la forme $(v^1, v^2, 0, \theta)$ et solution du système:

$$(QG) \qquad \qquad \partial_t U - \Gamma U = \begin{pmatrix} \partial_2 \\ -\partial_1 \\ 0 \\ F\partial_3 \end{pmatrix} \Delta_F^{-1}(v.\nabla\Omega),$$

avec Γ l'opérateur elliptique d'ordre deux défini par $\Gamma = \Delta \Delta_F^{-1} (\nu \partial_1^2 + \nu \partial_2^2 + \nu' F^2 \partial_3^2).$

Il faut voir que malgré cette forme particulière, le système (QG) est très proche de celui de Navier-Stokes: pour le voir, nous pouvons l'écrire sous forme de système sur le tourbillon accompagné d'une loi de Biot et Savart:

$$\partial_t \Omega + U \cdot \nabla \Omega - \Gamma \Omega = 0 \quad \text{et} \quad U = \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F \partial_3 \end{pmatrix} \Delta_F^{-1} \Omega,$$

mais la proximité avec Navier-Stokes devient flagrante lorsque nous utilisons les propriétés des projecteurs (et le fait que pour un champ quasigéostrophique U, $\Gamma U = QLU$, que nous utiliserons dans le troisième chapitre):

$$\partial_t U + U \cdot \nabla U - LU = \mathcal{P}P,$$

où le deuxième membre est l'équivalent du gradient de pression pour Navier-Stokes. Précisons cependant que ce n'est pas sous cette forme que le système quasigéostrophique est le plus facile à manipuler : nous préfèrerons systématiquement utiliser la première forme (QG).

Signalons que dans ce chapitre nous étudions les propriétés spectrales de la matrice (en variable de Fourier) $-\widehat{L} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A}$. Elles seront capitales dans les estimations dispersives et celles de Strichartz.

Solutions fortes

Toujours en suivant les méthodes développées dans [12], nous allons dans ce chapitre obtenir des résultats beaucoup plus précis lorsque les données initiales sont plus régulières.

Alors que dans le premier chapitre, nous n'avions aucun résultat d'unicité pour les solutions du système quasigéostrophique, ici nous avons une structure beaucoup plus forte : nous verrons que l'utilisation du tourbillon potentiel permet d'éviter le terme de stretching qui apparaît lorsque l'on utilise le tourbillon classique, ce qui rend ainsi le système quasigéostrophique plus proche du système de Navier-Stokes bidimensionnel que du système tridimensionnel. Cette structure proche du système de Navier-Stokes bidimensionnel que d'obtenir, sans hypothèse de petitesse des données initiales, l'existence et l'unicité d'une solution globale pour le système limite lorsque les données initiales sont plus régulières :

Théorème 1.2.6 Supposons que $U_{0,QG} \in H^1(\mathbb{R}^3)$, alors le système (QG) a une unique solution, globale et appartenant à l'espace $L^{\infty}(\mathbb{R}_+, H^1(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, H^2(\mathbb{R}^3))$.

Nous obtenons ensuite un théorème qui donne de façon plus précise le comportement en fortes rotation et stratification des solutions fortes. Choisissons la donnée initiale $U_0 \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ en demandant seulement que $U_{0,QG} \in L^2(\mathbb{R}^3)$. Précisons que même si, à cause du terme antisymétrique linéaire $\frac{1}{\varepsilon}\mathcal{A}U_{\varepsilon}$, le système n'a plus de scaling, il est tout de même naturel de choisir le même scaling que pour Navier-Stokes tridimensionnel, c'est-à-dire l'espace $\dot{H}^{\frac{1}{2}}$ (espace invariant par la transformation $u \mapsto \lambda u(\lambda^2 t, \lambda x)$). Encore une fois, grâce à cette antisymétrie, les méthodes d'énergie utilisées dans la démonstration du théorème de Fujita et Kato (qui consistent à étudier les variations du carré de la norme $\dot{H}^{\frac{1}{2}}$) fournissent le résultat correspondant pour le système primitif.

Définissons W_{ε} comme étant la solution du système linéaire suivant :

$$\begin{cases} \partial_t W_{\varepsilon} - L W_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_{\varepsilon} = -G \\ W_{\varepsilon/t=0} = U_{0,osc} = \mathcal{P}(U_0) \end{cases}$$
(1.2.6)

dans lequel G est un terme de force extérieure construit à partir de la solution limite U_{QG} : ce choix de l'auxiliaire W_{ε} s'explique par le fait qu'il s'agit d'éliminer le terme G, qui apparaît lorsque l'on écrit le système vérifié par $U_{\varepsilon} - U_{QG}$, et qui en tant que terme constant ruine toute méthode de type Gronwall pour obtenir des résultats de convergence vers zéro. Nous faisons donc osciller ce terme en tirant parti des propriétés dispersives de l'opérateur $-L + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A}$.

Alors nous obtenons facilement que W_{ε} existe globalement et est unique dans l'espace $E^s \stackrel{\text{def}}{=} L^{\infty}(\mathbb{R}_+, \dot{H}^s) \cap L^2(\mathbb{R}_+, \dot{H}^{s+1})$ pour tout $s \in [\frac{1}{2}, 1].$

Nous démontrons ensuite grâce aux estimations de Strichartz qu'elle tend vers zéro dans $L^2(\mathbb{R}_+, L^{\infty})$ lorsque ε tend vers zéro.

Et l'essentiel de l'article consiste en la démonstration du fait que lorsque ε est suffisamment petit (le voisinage dépend des données initiales et des paramètres ν , ν' et F), la différence entre la solution du système primitif et celle du système limite, modulée par les oscillations de W_{ε} visant à stabiliser le système, que nous noterons $\gamma_{\varepsilon} = U_{\varepsilon} - U_{QG} - W_{\varepsilon}$, est définie globalement et converge vers zéro dans E^s pour tout $s \in [\frac{1}{2}, 1]$.

Précisons que, contrairement au système de Navier-Stokes tridimensionnel qui est pourtant très proche du système des équations primitives, les solutions de (PE_{ε}) sont uniques et globales lorsque ε est suffisamment petit, ceci sans aucune hypothèse de petitesse des données initiales. La rotation et la stratification induisent ici des effets dispersifs qui stabilisent le système de la même façon que la forte rotation stabilise le système des fluides tournants.

Nous obtenons enfin que $U_{\varepsilon} - U_{QG}$ converge vers zéro dans l'espace $L^2(\mathbb{R}_+, L^{\infty})$, lorsque le paramètre ε tend vers zéro.

Asymptotiques précisées

Dans ce dernier chapitre, nous allons obtenir des résultats beaucoup plus précis sur la vitesse de convergence vers la limite quasigéostrophique sous des hypothèses de plus grande régularité (en permettant même une explosion des normes des données initiales lorsque le nombre de Rossby tend vers zéro).

Dans le cas où nous partons de données initiales dépendant de ε décomposées de la façon suivante : $U_{0,\varepsilon} = U_{0,\varepsilon,QG} + U_{0,\varepsilon,osc}$, avec

- $||U_{0,\varepsilon,QG} U_{0,QG}||_{H^1} \leq C\varepsilon$, avec $U_{0,QG} \in H^{1+\beta}$ (β est fixée strictement positive)
- $U_{0,\varepsilon,osc} \in L^1 \cap \dot{H}^{\frac{1}{2}} \cap \dot{H}^{1+\beta}$, régulière mais avec des normes pouvant être explosives :

 $\|U_{0,\varepsilon,osc}\|_{L^1} + \|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}}} + \|U_{0,\varepsilon,osc}\|_{\dot{H}^{1+\beta}} \le \alpha \log|\log \varepsilon|.$

Si W_{ε} est définie comme précédemment, alors nous prouvons que l'on peut majorer la vitesse de convergence par toute puissance négative de $|\log \varepsilon|$ pour peu que ε soit suffisamment petit : pour tout $\omega > 0$, il existe une constante C et ε_0 (dépendant des données initiales et de ω) tels que si $\varepsilon \leq \varepsilon_0$,

- $||W_{\varepsilon}||_{L^2(\mathbb{R}_+,L^{\infty})} \leq C|\log \varepsilon|^{-\omega}.$
- $\|\gamma_{\varepsilon}\|_{E^s} \le C |\log \varepsilon|^{-\omega}$
- $||U_{\varepsilon} U_{QG}||_{L^2(\mathbb{R}_+, L^{\infty})} \le C |\log \varepsilon|^{-\omega}.$

Dans le cas où la viscosité ν est égale à la diffusivité thermique ν' nous obtenons de très nombreuses simplifications (la première étant que $L = \nu \Delta = \Gamma$ et que les valeurs propres du système linéarisé sont explicites et plus simples) ce qui nous permet d'obtenir des résultats beaucoup plus fins : sans donner de détails précisons qu'avec les mêmes hypothèses pour les parties quasigéostrophiques et lorsque $||U_{0,\varepsilon,osc}||_{H^{1+\beta}} \leq \alpha |\log \varepsilon|$, alors les estimations deviennent lorsque α est suffisamment petit, et pour un certain ω dépendant des données initiales, de α et de β :

- $||W_{\varepsilon}||_{L^2(\mathbb{R}_+,L^{\infty})} \leq C\varepsilon^{\omega}$
- $||U_{\varepsilon} U_{QG}||_{L^2(\mathbb{R}_+, L^{\infty})} \le C\varepsilon^{\omega}.$

Nous insistons sur le fait que ces résultats ont été obtenus grâce à des raffinements des estimations dispersives qui, dans l'optique d'une estimation de la vitesse de convergence, nécessitent des estimations beaucoup plus précises sur les éléments spectraux du système linéarisé (les preuves détaillées sont placées dans l'appendice du dernier chapitre).

On comprend ainsi mieux le sens de la remarque en première section qui disait que l'écart entre le système quasigéostrophique et la réalité est de l'ordre de ε .

La dernière partie de ce chapitre concerne des méthodes de type poches de tourbillon. Il est naturel de s'intéresser aux poches de tourbillon lorsque l'on pense à diverses manifestations tourbillonnaires que l'on peut rencontrer dans les océans ou l'atmosphère : tourbillons, tornades, cyclones, typhons ou maëlströms... Nous utiliserons ici le formalisme de la régularité tangentielle développé dans [10], [16], [18], [23], et [29]. Nous nous placerons dans cette partie dans le cadre des espaces C^s de Hölder.

Définition 1.2.1 On dit que Ω_0 est une poche de tourbillon de classe C^s si pour un $s \in]0,1[,$

$$\Omega_0 = \Omega_{0,i} \mathbf{1}_D + \Omega_{0,e} \mathbf{1}_{\mathbb{R}^3 - D},$$

où $\Omega_{0,i} \in C^s(\overline{D}), \ \Omega_{0,e} \in C^s(\mathbb{R}^3 - D)$ et D est un domaine ouvert de classe C^{s+1} .

INTRODUCTION

Définissons maintenant le concept de régularité tangentielle par rapport à une famille de champs de vecteurs X :

Définition 1.2.2 Si $X = (X_{\lambda})_{\lambda=1,\dots,N}$ est une famille finie de champs de vecteurs on dit que cette famille est admissible si et seulement si (\wedge désigne le produit vectoriel usuel de \mathbb{R}^3) :

$$[X]^{-1} \stackrel{\text{def}}{=} \left(\frac{2}{N(N-1)} \sum_{\lambda < \lambda'} |X_{\lambda} \wedge X_{\lambda'}|^2\right)^{-\frac{1}{4}} < \infty.$$

Si $s \in [0,1]$ et X est une famille de champs de vecteurs admissibles C^s on définit l'espace :

$$C^{s}(X) = \{ w \in L^{\infty} \quad tel \ que \quad X_{\lambda}(x, D)w \stackrel{\text{def}}{=} \operatorname{div}(w \otimes X_{\lambda}) \in C^{s-1} \}$$

en désignant par div $(u \otimes v)$ la matrice de composantes $\partial_i(uv^i)$.

Enonçons maintenant le résultat. Prenons une famille de données initiales telles que :

- Ω_0 est une poche de tourbillon C^s avec $s \in]0, 1[,$
- $U_{0,QG} \in L^2$ est un champ de vecteurs quasigéostrophique tel que $\Omega(U_{0,QG}) = \Omega_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$,
- $U_{0,\varepsilon,osc}$ est une famille de champs de vecteurs oscillants,
- et décomposons $U_{0,\varepsilon} = U_{0,\varepsilon,QG} + U_{0,\varepsilon,osc}$ ($U_{0,\varepsilon,QG}$ est une régularisation, par exemple $\chi(\varepsilon|D|)U_{0,QG}$, et $V_{0,\varepsilon}$ la famille des parties oscillantes initiales).

Supposons que $X_0 = \{X_{0,\lambda}, \lambda = 1, ..., N\}$ est un système admissible de champs de vecteurs C^s , et qu'il existe une constante $C_0 > 0$ telle que :

$$\|\Omega_{0,\varepsilon}\|_{L^2} \le C_0, \quad \|\Omega_{0,\varepsilon}\|_{C^s(X_0)} \le C_0, \quad \|U_{0,\varepsilon}\|_{H^5} \le C_0 \varepsilon^{-\alpha},$$

où $\alpha > 0$ est une constante que nous ne préciserons pas ici. Remarquons qu'on demande beaucoup de régularité initiale mais que l'on permet aux normes des parties oscillantes d'exploser.

Alors si ε est suffisamment petit, on peut minorer le temps d'existence T_{ε}^* par un certain temps $T_{\varepsilon}^{\gamma} = \gamma \log |\log \varepsilon|$ (où $0 < \gamma \leq \gamma_0$).

Nous avons convergence vers zéro de la partie oscillante :

$$\|U_{\varepsilon,osc}\|_{L^8_{T^{\gamma}_{\varepsilon}}(Lip)} \le C\varepsilon^{\alpha}$$

ainsi que la convergence locale en temps des parties quasigéostrophiques : pour tout $\tilde{T} > 0$, $U_{\varepsilon,QG}$ converge dans $L^{\infty}([0,\tilde{T}],L^2)$ vers une solution Lipschitzienne du système quasigéostrophique (QG) avec $U_{0,QG}$ comme donnée initiale.

Par rapport au travail de [18], la présence des viscosités a été une complication. Dans les autres chapitres, même si elles nous ont compliqué les calculs, les viscosités nous ont été toujours d'une très grande utilité, que ce soit en nous permettant d'utiliser quasiment automatiquement de nombreux résultats sur le système de Navier-Stokes dans tout l'espace (solutions faibles, fortes, énergie...), ou bien en nous permettant d'obtenir des estimations globales en temps. Or ici, le formalisme des poches de tourbillon ayant été à l'origine développé pour des systèmes non visqueux, nous avons dû utiliser des résultats récents développés par T. Hmidi dans [29] concernant les poches de tourbillon pour le système de Navier-Stokes bidimensionnel. Nous présentons ici le résultat dans le cas $\nu = \nu'$. Le cas $\nu \neq \nu'$, pose de façon non surprenante des problèmes techniques qui feront l'objet d'un développement ultérieur.

Perspectives

Nous allons donner ici quelques directions envisagées pour des travaux ultérieurs:

- Tout d'abord, nous prévoyons de continuer l'étude de la convergence quasigéostrophique dans le cadre des poches de tourbillon et de traîter le cas où $\nu \neq \nu'$. Il apparaît que dans un premier temps nous aurons à faire une hypothèse de proximité des viscosités $|\nu - \nu'| \leq \frac{\nu}{2}$. L'objectif serait ensuite de s'affranchir de cette contrainte.
- Ensuite, un travail en collaboration avec V. Roussier a débuté. Il s'agit d'étudier le comportement en temps long et à nombre de Rossby fixé des solutions des équations primitives. Une étude de ce problème pour les solutions du système des fluides tournants dans une bande tridimensionnelle a été menée par V. Roussier dans [34] en adaptant des résultats développés par T. Gallay et C.-E. Wayne, notamment dans [22] où sont mis en évidence les tourbillons d'Oseen comme asymptotiques en temps long.
- Enfin, il semble intéressant de chercher à tirer parti des phénomènes dispersifs et d'adapter ces méthodes à d'autres systèmes de la hiérarchie des différents modèles utilisés pour décrire des fluides géophysiques notamment au système des équations planétaires géostrophiques obtenu à partir d'un système d'équations primitives un peu différent de celui que nous avons utilisé dans cette thèse : en particulier l'analogue de la matrice A n'est plus antisymétrique. Citons à ce sujet les travaux de D. Bresch, T. Huck et M. Sy dans [7] ainsi que ceux de D. Bresch, D. Gérard-Varet et E. Grenier dans [8].

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Solutions faibles

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Chapitre 2

Solutions faibles

Résumé: On s'intéresse dans ce chapitre¹ au comportement des solutions faibles (au sens de Leray) du système primitif lorsque le nombre de Rossby tend vers zéro. On sépare notamment la solution en sa partie oscillante et sa partie quasigéostrophique, chacune solution d'un système particulier. La première tend vers zéro lorsque la rotation devient forte et de la seconde on peut extraire une sous-suite tendant vers une solution du système quasigéostrophique.

¹Les résultats présentés dans ce chapitre sont acceptés pour publication dans la revue "Asymptotic Analysis".

2.1 Introduction

The primitive system is the following:

$$(PE_{\varepsilon}) \qquad \begin{cases} \partial_t U_{\varepsilon} + v_{\varepsilon} \cdot \nabla U_{\varepsilon} - LU_{\varepsilon} + \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon} = \frac{1}{\varepsilon} (-\nabla \Phi_{\varepsilon}, 0) \\ div \ v_{\varepsilon} = 0 \\ U_{\varepsilon/t=0} = U_0 \in L^2(\mathbb{R}^3). \end{cases}$$

The unknowns are U_{ε} and Φ_{ε} . We denote by U_{ε} a pair $(v_{\varepsilon}, \theta_{\varepsilon})$ where v_{ε} is a vector field on \mathbb{R}^3 (three dimensional velocity), θ_{ε} a scalar function (the density fluctuation : in the case of the atmosphere it depends on the scalar (potential) temperature and in the case of the ocean it depends on the temperature and the salinity), and Φ_{ε} the pressure, all of them depending on (t, x).

The operator L is defined by

$$LU_{\varepsilon} = (\nu \Delta v_{\varepsilon}, \nu' \Delta \theta_{\varepsilon})$$

and \mathcal{A} by:

$$\mathcal{A} = \left(\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{array} \right).$$

This system is obtained by combining the effects of the Coriolis force and those of the vertical stratification induced by the Boussinesq approximation (one can see [7] as well as [1], about rotating fluids). We refer to [3], [8], [9], [12], [14], [17], and [19] for a discussion on this model, and its derivation.

The coefficient $\varepsilon > 0$ denotes the Rossby number, $\nu > 0$ is the viscosity and $\nu' > 0$ the heat diffusivity (which we will also call a viscosity in the following). As the characteristic displacement of a particle in the ocean within a day is very small compared to the displacement caused by the rotation of the earth, the Rossby number is supposed to be small, about 10^{-1} to 10^{-3} , and we focus on the limit of a strong rotation (ε goes to zero).

The coefficient F is called the Froude number. We refer to [6] for a study of the case F = 1, to [11] for the periodic case, and the aim is here to study when $F \neq 1$ the convergence, as ε goes to zero, of the solutions in the whole space. Let us also refer to [2] for the case $\nu = \nu'$ and $F \neq 1$.

We will show in this paper that, in the case $\nu \neq \nu'$, although the computations are more complicated (non-orthogonal eigenvectors, asymptotic expansions in the phase...), everything behaves up to an ε like when $\nu = \nu'$.

The fact that the parameter ε goes to zero gives a high importance to the term $\mathcal{A}U$, which is said to be penalized. But, as we will see, the term $\frac{1}{\varepsilon}$ will not play any role in the L^2 energy estimate thanks to the skew-symmetry of \mathcal{A} . This skew-symmetry allows also, if the initial data U_0 belongs to $L^2(\mathbb{R}^3)$ to build a sequence $(U_{\varepsilon})_{0<\varepsilon\leq\varepsilon_0}$ of weak solutions given by the Leray method: they are uniformly bounded with respect to ε in the space $L^{\infty}(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ where \dot{H}^s is the homogenous Sobolev space of order s. Precisely: **Definition 2.1.1** If s is a real number, the homogenous Sobolev space of order s, denoted by \dot{H}^s , is defined as the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^3)$ whose Fourier transform \hat{u} is locally integrable and has the following property:

$$||u||_{\dot{H}^s}^2 = \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty.$$

Remark 2.1.1 For more generality one can add an external force belonging to the space $L^2(\mathbb{R}_+, \dot{H}^{-1})$, but with a wish of simplification we will abstrain from it.

Theorem 2.1.1 Let the initial data $U_0 \in L^2(\mathbb{R}^3)$, then there exists for all $\varepsilon > 0$ a Leray solution of the system $(PE_{\varepsilon}), U_{\varepsilon}$, globally defined in time, belonging to $L^{\infty}(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and satisfying the following energy inequality (let $\nu_0 = \min(\nu, \nu') > 0$):

$$\forall t \in \mathbb{R}_+, \quad \|U_{\varepsilon}(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\nu_0 \int_0^t \|\nabla U_{\varepsilon}(t)\|_{L^2(\mathbb{R}^3)}^2 dt \le \|U_0\|_{L^2(\mathbb{R}^3)}^2.$$

We will not prove this theorem which is very close to the Leray theorem concerning the incompressible Navier-Stokes system, not only by its formulation but also by its proof.

The aim is to know the behaviour of these solutions in the limit of a strong rotation (i-e when the Rossby number ε goes to zero). Formally, one expects these solutions to converge (with finally an extraction) to a solution of the system obtained when making ε go to zero in the equations of the system. We will show that it is indeed the way it is.

Like in [6] and [11] let us introduce the potential vorticity (the aim is to get rid of the terms with ε in their denominator) $\Omega_{\varepsilon} = \partial_1 v_{\varepsilon}^2 - \partial_2 v_{\varepsilon}^1 - F \partial_3 \theta_{\varepsilon}$ and the quasigeostrophic part

$$U_{\varepsilon,QG} = \begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} \Omega_{\varepsilon}$$

where we have noted $\Delta_F = \partial_1^2 + \partial_2^2 + F^2 \partial_3^2$, and in order to define the inverse operator Δ_F^{-1} (we refer to [5] p.37) we build for all $i, j \in \{1, ..., d\}$ the operator $T_{i,j}$ satisfying $\Delta_F T_{i,j} w = \partial_i \partial_j w$.

This operator is, in fact, the pseudo-differential operator associated to

$$\frac{\xi_i \xi_j}{\xi_1^2 + \xi_2^2 + F^2 \xi_3^2}.$$
(2.1.1)

Let us finally introduce the oscillating part $U_{\varepsilon,osc} = U_{\varepsilon} - U_{\varepsilon,QG}$ and state the results:

Theorem 2.1.2 If the initial data U_0 belongs to $L^2(\mathbb{R}^3)$ and if we consider a sequence of Leray solutions (U_{ε}) (in the way of Theorem 2.1.1) then, the oscillating part $U_{\varepsilon,osc}$ goes to zero (when ε goes to zero) in the space $L^2_{loc}(\mathbb{R}_+, L^q(\mathbb{R}^3))$ for all $q \in]2, 6[$.

Theorem 2.1.3 Under the same assumptions, there exists an extracted sequence from the quasigeostrophic part $U_{\varepsilon,QG}$ that converges for all $q \in]2, 6[$ in the space $L^2_{loc}(\mathbb{R}_+, L^q_{loc}(\mathbb{R}^3))$ to a fonction \tilde{U}_{QG} which has the form $(\tilde{v}^1, \tilde{v}^2, 0, \tilde{\theta})$ and solution of the system:

$$(\widetilde{QG}) \qquad \qquad \partial_t \tilde{U} - \Gamma \tilde{U} = \begin{pmatrix} \partial_2 \\ -\partial_1 \\ 0 \\ F \partial_3 \end{pmatrix} \Delta_F^{-1}(\tilde{v}.\nabla \tilde{\Omega}),$$

with Γ the elliptic operator of order two defined by

$$\Gamma = \Delta \Delta_F^{-1} (\nu \partial_1^2 + \nu \partial_2^2 + \nu' F^2 \partial_3^2).$$
(2.1.2)

The study of the case when the initial data is more regular (global existence and convergence) is achieved and will be published later.

This paper is structured in the following way: first of all, inspired by [6] we will formally determine a limit and establish the differential system it solves in Section 2.2. We will then go back, in Section 2.3, to the primitive system whose solution we will cut in two parts, with very different asymptotic behaviour (the oscillating part, and the quasigeostrophic part). With a view to prove the convergence of the oscillating part (Theorem 2.1.2), most of the work will consist in the study of the eigenelements of the linearized system projected on the divergence-free vector fields, and its application to look for dispersive inequalities and Strichartz estimates (relatively to only two of the four eigenvalues). We will finally try to get rid of the last eigenvalue (the one that doesn't give oscillations). The method is very similar to those in [7] except for the fact that we have to cope with the additionnal eigenvalue and with the nonorthogonality of the eigenvectors. This will be dealt in Sections 2.4.2 and 2.4.3.

When it is done, the extraction of a convergent subsequence from the quasigeostrophic part (Theorem 2.1.3) follows a classical method (Section 2.5).

2.2 Formal approach of the limit system

Let us write in extension the system (PE_{ε}) :

$$(PE\varepsilon) \begin{cases} \partial_t v_{\varepsilon}^1 + v_{\varepsilon} . \nabla v_{\varepsilon}^1 - \nu \Delta v_{\varepsilon}^1 - \frac{1}{\varepsilon} v_{\varepsilon}^2 = -\frac{1}{\varepsilon} \partial_1 \Phi_{\varepsilon} \\ \partial_t v_{\varepsilon}^2 + v_{\varepsilon} . \nabla v_{\varepsilon}^2 - \nu \Delta v_{\varepsilon}^2 + \frac{1}{\varepsilon} v_{\varepsilon}^1 = -\frac{1}{\varepsilon} \partial_2 \Phi_{\varepsilon} \\ \partial_t v_{\varepsilon}^3 + v_{\varepsilon} . \nabla v_{\varepsilon}^3 - \nu \Delta v_{\varepsilon}^3 + \frac{1}{\varepsilon F} \theta_{\varepsilon} = -\frac{1}{\varepsilon} \partial_3 \Phi_{\varepsilon} \\ \partial_t \theta_{\varepsilon} + v_{\varepsilon} . \nabla \theta_{\varepsilon} - \nu' \Delta \theta_{\varepsilon} - \frac{1}{\varepsilon F} v_{\varepsilon}^3 = 0 \\ div \ v_{\varepsilon} = 0 \\ (v_{\varepsilon}, \theta_{\varepsilon})_{/t=0} = (v_0, \theta_0) \end{cases}$$

The aim of this short section is to give an idea of what the limit system should look like (if it exists); that is why we will make a formal argument, exactly like in [6].

Let us consider a family of solutions $(v_{\varepsilon}, \theta_{\varepsilon}, \Phi_{\varepsilon})_{0 < \varepsilon < \varepsilon_0}$ (Leray solutions) of the system (PE_{ε}) , and assume that this family converges towards a triplet $(\tilde{v}, \tilde{\theta}, \tilde{\Phi})$ in a sufficiently strong way so that we can go to the limit in the non-linear term of the system.

Necessarily, when ε goes to zero, we must have:

$$\begin{cases} v_{\varepsilon}^2 - \partial_1 \Phi_{\varepsilon} \longrightarrow 0 \\ v_{\varepsilon}^1 + \partial_2 \Phi_{\varepsilon} \longrightarrow 0 \\ \frac{\theta_{\varepsilon}}{F} + \partial_3 \Phi_{\varepsilon} \longrightarrow 0 \\ v_{\varepsilon}^3 \longrightarrow 0. \end{cases}$$

and thus

$$\tilde{v} = \begin{pmatrix} -\partial_2 \tilde{\Phi} \\ \partial_1 \tilde{\Phi} \\ 0 \end{pmatrix} = \nabla_2^{\perp} \tilde{\Phi}$$

and
$$\tilde{\theta} = -F\partial_3\tilde{\Phi}$$
.

If we denote by $\tilde{\Omega} = \partial_1 \tilde{v}^2 - \partial_2 \tilde{v}^1 - F \partial_3 \tilde{\theta}$, then $\tilde{\Omega} = (\partial_1^2 + \partial_2^2 + F^2 \partial_3^2) \tilde{\Phi} = \Delta_F \tilde{\Phi}$ and thus:

$$\tilde{v} = \nabla_2^{\perp} \Delta_F^{-1} \tilde{\Omega}$$

and $\tilde{\theta} = -F \partial_3 \Delta_F^{-1} \tilde{\Omega}$.

Assume we have, when ε goes to zero, the following convergences:

$$\begin{cases} \frac{v_{\varepsilon}^2 - \partial_1 \Phi_{\varepsilon}}{\varepsilon} \longrightarrow Fw^2 \\ \frac{v_{\varepsilon}^1 + \partial_2 \Phi_{\varepsilon}}{\varepsilon} \longrightarrow Fw^1 \\ \frac{F^{-1}\theta_{\varepsilon} + \partial_3 \Phi_{\varepsilon}}{\varepsilon} \longrightarrow w^4 \\ \frac{F^{-1}v_{\varepsilon}^3}{\varepsilon} \longrightarrow w^3 \end{cases}$$

Then, $\operatorname{div} w = 0$ and we get the following system:

$$\begin{cases} \partial_t \tilde{v}^1 + \tilde{v} \cdot \nabla \tilde{v}^1 - \nu \Delta \tilde{v}^1 - F w^2 &= 0\\ \partial_t \tilde{v}^2 + \tilde{v} \cdot \nabla \tilde{v}^2 - \nu \Delta \tilde{v}^1 + F w^1 &= 0\\ \partial_t \tilde{v}^3 + \tilde{v} \cdot \nabla \tilde{v}^3 - \nu \Delta \tilde{v}^3 + w^4 &= 0\\ \partial_t \tilde{\theta} + \tilde{v} \cdot \nabla \tilde{\theta} - \nu' \Delta \tilde{\theta} - w^3 &= 0 \end{cases}$$

The fact that \tilde{v}^3 is zero implies that w^4 is also zero and we finally get:

$$\begin{cases} \partial_t \tilde{v}^1 + \tilde{v} \cdot \nabla \tilde{v}^1 - \nu \Delta \tilde{v}^1 - F w^2 &= 0\\ \partial_t \tilde{v}^2 + \tilde{v} \cdot \nabla \tilde{v}^2 - \nu \Delta \tilde{v}^1 + F w^1 &= 0\\ \partial_t \tilde{\theta} + \tilde{v} \cdot \nabla \tilde{\theta} - \nu' \Delta \tilde{\theta} - w^3 &= 0 \end{cases}$$

Derivating the lines of this system and adding them following the formula giving $\tilde{\Omega}$ in terms of the coordinates of \tilde{U} give the equation satisfied by $\tilde{\Omega}$:

$$\partial_t \Omega + \tilde{v} \cdot \nabla \Omega - \Gamma \Omega = 0 \tag{2.2.3}$$

with
$$\Gamma = \Delta \Delta_F^{-1} (\nu \partial_1^2 + \nu \partial_2^2 + \nu' F^2 \partial_3^2).$$

Thanks to this equation, let us now reformulate the limit system satisfied by \tilde{U} using the expressions of \tilde{v}^1 , \tilde{v}^2 , and $\tilde{\theta}$ in terms of Ω :

$$(\widetilde{QG}) \qquad \qquad \partial_t \begin{pmatrix} \tilde{v}^1 \\ \tilde{v}^2 \\ 0 \\ \tilde{\theta} \end{pmatrix} - \Gamma \begin{pmatrix} \tilde{v}^1 \\ \tilde{v}^2 \\ 0 \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} \partial_2 \\ -\partial_1 \\ 0 \\ F\partial_3 \end{pmatrix} \Delta_F^{-1}(\tilde{v}.\nabla\tilde{\Omega}).$$

Remark 2.2.1 This allows us, comparing both systems, to get the expressions of the w^j in terms of the coordinates of \tilde{U} .

2.3 Reformulation of the primitive system

Let us go back to the original primitive system. Due to the previous computations, we will separate the solution of the system (PE_{ε}) in two parts, one of them will satisfy a system very close to (\widetilde{QG}) .

Guided by the expression of $\tilde{\Omega}$ let us introduce the following definition:

Definition 2.3.1 We call potential vorticity of the quadruplet $U_{\varepsilon} = (v_{\varepsilon}, \theta_{\varepsilon})$ the quantity

$$\Omega_{\varepsilon} = \partial_1 v_{\varepsilon}^2 - \partial_2 v_{\varepsilon}^1 - F \partial_3 \theta_{\varepsilon}$$

Then, copying the expression of \tilde{v} we define $U_{\varepsilon,QG}$ and $U_{\varepsilon,osc}$ by (we address to (2.1.1) for the inverse of the Laplacian):

$$U_{\varepsilon,QG} = \begin{pmatrix} -\partial_2 \Delta_F^{-1} \Omega_{\varepsilon} \\ \partial_1 \Delta_F^{-1} \Omega_{\varepsilon} \\ 0 \\ -F \partial_3 \Delta_F^{-1} \Omega_{\varepsilon} \end{pmatrix}$$
$$U_{\varepsilon,osc} = U_{\varepsilon} - U_{\varepsilon,QG} = \begin{pmatrix} v_{\varepsilon}^1 + \partial_2 \Delta_F^{-1} \Omega_{\varepsilon} \\ v_{\varepsilon}^2 - \partial_1 \Delta_F^{-1} \Omega_{\varepsilon} \\ v_{\varepsilon}^3 \\ \theta_{\varepsilon} + F \partial_3 \Delta_F^{-1} \Omega_{\varepsilon} \end{pmatrix}$$

Let us observe that, as U_{ε} and $U_{\varepsilon,QG}$ are divergence-free (with respect to the three first coordinates), their difference $U_{\varepsilon,osc}$ has the same property.

Remark 2.3.1 U_{ε} and $U_{\varepsilon,QG}$ have the same potential vorticity, and the potential vorticity of $U_{\varepsilon,osc}$ is zero.

Actually the previous remark can be reformulated in a far more interesting way if we compute the Fourier transform of $U_{\varepsilon,QG}$ and $U_{\varepsilon,osc}$:

$$\widehat{U_{\varepsilon,QG}} = \begin{pmatrix} \xi_2 \\ -\xi_1 \\ 0 \\ F\xi_3 \end{pmatrix} i \frac{\widehat{\Omega_{\varepsilon}}}{|\xi|_F^2},$$

$$\widehat{U_{\varepsilon,osc}} = \begin{pmatrix} \widehat{v_{\varepsilon}^1} \\ \widehat{v_{\varepsilon}^2} \\ \widehat{v_{\varepsilon}^3} \\ \widehat{\theta_{\varepsilon}} \end{pmatrix} - \begin{pmatrix} \xi_2 \\ -\xi_1 \\ 0 \\ F\xi_3 \end{pmatrix} i \frac{\widehat{\Omega_{\varepsilon}}}{|\xi|_F^2}$$

If we calculate their (vectorial) scalar product in \mathbb{C}^4 we find zero. The reason is obviously that computing the scalar product of $\widehat{U_{\varepsilon,osc}}$ with the vector $(\xi_2, -\xi_1, 0, F\xi_3)$ is exactly computing the Fourier transform of the potential vorticity of $U_{\varepsilon,osc}$, which is zero.

Thus we have cut the solution U_{ε} in two quantities whose Fourier transforms are, in the case of $U_{\varepsilon,QG}$ collinear to the vector $(\xi_2, -\xi_1, 0, F\xi_3)$, and in the case of $U_{\varepsilon,osc}$ orthogonal to the same vector.

That is, we exactly obtained the decomposition corresponding to the orthogonal projection in the subspace of \mathbb{C}^4 generated by this vector.

The same way as we defined the Leray projector \mathbb{P} onto the divergence free vector fields, we will define here two orthogonal projectors.

Definition 2.3.2 Let us denote by \mathcal{P} the orthogonal projector in the potential vorticity free vector fields, and $\mathcal{Q} = Id - \mathcal{P}$.

Proposition 2.3.1 In terms of the Fourier transform, these two operators are written in the following way:

$$\widehat{\mathcal{Q}U} = \frac{1}{|\xi|_F^2} \begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 & 0 & F\xi_2\xi_3 \\ -\xi_1\xi_2 & \xi_1^2 & 0 & -F\xi_1\xi_3 \\ 0 & 0 & 0 & 0 \\ F\xi_2\xi_3 & -F\xi_1\xi_3 & 0 & F^2\xi_3^2 \end{pmatrix} \widehat{U}$$
(2.3.4)
$$\widehat{\mathcal{P}U} = \frac{1}{|\xi|_F^2} \begin{pmatrix} \xi_1^2 + F^2\xi_3^2 & \xi_1\xi_2 & 0 & -F\xi_2\xi_3 \\ \xi_1\xi_2 & \xi_2^2 + F^2\xi_3^2 & 0 & F\xi_1\xi_3 \\ 0 & 0 & |\xi|_F^2 & 0 \\ -F\xi_2\xi_3 & F\xi_1\xi_3 & 0 & \xi_1^2 + \xi_2^2 \end{pmatrix} \widehat{U}.$$

Thus this decomposition of $U_{\varepsilon} = U_{\varepsilon,QG} + U_{\varepsilon,osc}$ where $U_{\varepsilon,osc} = \mathcal{P}U_{\varepsilon}$ and $U_{\varepsilon,QG} = \mathcal{Q}U_{\varepsilon}$ involves two homogenous pseudo-differential operators of order zero, and this will be very useful in the following for many estimates.

Proposition 2.3.2 Like the Leray projector, these projectors have a few immediate properties:

1. $\mathcal{P}U = U \Leftrightarrow \Omega(U) = 0$ 2. $\mathcal{Q}U = U \Leftrightarrow \mathcal{P}U = 0 \Leftrightarrow U$ has the form $(\partial_2 \Phi, -\partial_1 \Phi, 0, F \partial_3 \Phi)$.

Further in this paper we will go back in more details on these projectors. Let us now determine the systems satisfied by Ω_{ε} , $U_{\varepsilon,QG}$, and $U_{\varepsilon,osc}$.

The same computations as in the previous section allow us to get, following [6]:

$$\partial_t \Omega_{\varepsilon} + v_{\varepsilon} \cdot \nabla \Omega_{\varepsilon} - \Gamma \Omega_{\varepsilon} = (\nu - \nu') F \Delta \partial_3 \theta_{\varepsilon, osc} + q(U_{\varepsilon, osc}, U_{\varepsilon})$$
(2.3.5)

with

$$q(U_{\varepsilon,osc}, U_{\varepsilon}) = \partial_3 v_{\varepsilon,osc}^3 (\partial_1 v_{\varepsilon}^2 - \partial_2 v_{\varepsilon}^1) - \partial_1 v_{\varepsilon,osc}^3 \partial_3 v_{\varepsilon}^2 + \partial_2 v_{\varepsilon,osc}^3 \partial_3 v_{\varepsilon}^1$$

$$+ F \partial_3 (v_{\varepsilon} - v_{\varepsilon,osc}) \nabla \theta_{\varepsilon,osc} + F \partial_3 v_{\varepsilon,osc} \nabla \theta_{\varepsilon}.$$

$$(2.3.6)$$

Then, thanks to this equation and the definition of $U_{\varepsilon,QG}$ (which only uses Ω_{ε}):

$$\partial_t U_{\varepsilon,QG} - \Gamma U_{\varepsilon,QG} = \begin{pmatrix} -\partial_2 \Delta_F^{-1} \\ \partial_1 \Delta_F^{-1} \\ 0 \\ -F \partial_3 \Delta_F^{-1} \end{pmatrix} \Big(-v_{\varepsilon} \cdot \nabla \Omega_{\varepsilon} + q(U_{\varepsilon,osc}, U_{\varepsilon}) + (\nu - \nu') F \Delta \partial_3 \theta_{\varepsilon,osc} \Big).$$
(2.3.7)

In order to express the system satisfied by $U_{\varepsilon,osc}$, we use the previous equation: let us begin by writing:

$$\partial_t U_{\varepsilon,osc} + v_{\varepsilon} \cdot \nabla U_{\varepsilon} - LU_{\varepsilon,osc} + \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon,osc} = (\partial_t U_{\varepsilon} + v_{\varepsilon} \cdot \nabla U_{\varepsilon} - LU_{\varepsilon} + \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon}) \\ - (\partial_t U_{\varepsilon,QG} - LU_{\varepsilon,QG} + \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon,QG}).$$

Using the systems (PE_{ε}) and (2.3.7):

$$\partial_t U_{\varepsilon,osc} + v_{\varepsilon} \cdot \nabla U_{\varepsilon} - L U_{\varepsilon,osc} + \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon,osc} = \frac{1}{\varepsilon} (-\nabla \Phi_{\varepsilon}, 0) - (\partial_t U_{\varepsilon,QG} - \Gamma U_{\varepsilon,QG}) \\ - (\Gamma - L) U_{\varepsilon,QG} - \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon,QG}$$

Thus, as $\mathcal{A}U_{\varepsilon,QG} = -(\nabla \Delta_F^{-1}\Omega_{\varepsilon}, 0)$, and

$$(\widehat{\Gamma - L})\widehat{U_{\varepsilon,QG}} = iF(\nu - \nu')\frac{|\xi|^2}{|\xi|_F^4} \begin{pmatrix} -F\xi_2\xi_3^2\\F\xi_1\xi_3^2\\0\\(\xi_1^2 + \xi_2^2)\xi_3 \end{pmatrix}\widehat{\Omega_{\varepsilon}},$$

the following expression can be simplified into:

$$(\Gamma - L)U_{\varepsilon,QG} + \begin{pmatrix} -\partial_2 \Delta_F^{-1} \\ \partial_1 \Delta_F^{-1} \\ 0 \\ -F\partial_3 \Delta_F^{-1} \end{pmatrix} (\nu - \nu')F\Delta \partial_3 \theta_{\varepsilon,osc}$$
$$= -F(\nu - \nu')\Delta \Delta_F^{-1} \partial_3 \begin{pmatrix} \partial_2 \theta_{\varepsilon} \\ -\partial_1 \theta_{\varepsilon} \\ 0 \\ \partial_1 v_{\varepsilon}^2 - \partial_2 v_{\varepsilon}^1 \end{pmatrix}.$$

Projecting onto the divergence free vector fields (the Leray projector commutes with derivations), we finally obtain the following system:

$$\partial_{t}U_{\varepsilon,osc} - LU_{\varepsilon,osc} + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}U_{\varepsilon,osc} = \begin{pmatrix} -\partial_{2}\Delta_{F}^{-1} \\ \partial_{1}\Delta_{F}^{-1} \\ 0 \\ -F\partial_{3}\Delta_{F}^{-1} \end{pmatrix} \begin{pmatrix} -v_{\varepsilon}.\nabla\Omega_{\varepsilon} + q(U_{\varepsilon,osc},U_{\varepsilon}) \end{pmatrix} \quad (2.3.8)$$
$$-\mathbb{P}(v_{\varepsilon}.\nabla U_{\varepsilon}) + F(\nu - \nu')\Delta\Delta_{F}^{-1}\partial_{3} \begin{pmatrix} -\partial_{2}\theta_{\varepsilon} \\ \partial_{1}\theta_{\varepsilon} \\ 0 \\ \partial_{2}v_{\varepsilon}^{1} - \partial_{1}v_{\varepsilon}^{2} \end{pmatrix}.$$

2.4 Study of the oscillating part and proof of Theorem 2.1.2

2.4.1 Recall and definitions

Energy inequalities

The energy inequality seen in Theorem 2.1.1 is provided by that satisfied by each of the approached solutions given by the Friedrichs scheme (and from this sequence we weakly extract a Leray solution U_{ε}). As the energy is convex and strongly from-below semicontinuous, we get the following inequality (independent of ε):

$$\|U_{\varepsilon}(t)\|_{L^{2}}^{2} + 2\nu \int_{0}^{t} \|\nabla v_{\varepsilon}(s)\|_{L^{2}}^{2} ds + 2\nu' \int_{0}^{t} \|\nabla \theta_{\varepsilon}(s)\|_{L^{2}}^{2} ds \le \|U_{0}\|_{L^{2}}^{2},$$

and if we put $\nu_0 = \min(\nu, \nu')$ $(\nu, \nu' > 0)$, we get the energy inequality:

$$\|U_{\varepsilon}(t)\|_{L^{2}}^{2} + 2\nu_{0} \int_{0}^{t} \|\nabla U_{\varepsilon}(s)\|_{L^{2}}^{2} ds \leq \|U_{0}\|_{L^{2}}^{2}.$$
(2.4.9)

Remark 2.4.1 As \mathcal{P} and \mathcal{Q} are homogenous pseudo-differential operators of order zero, $U_{\varepsilon,QG}$ and $U_{\varepsilon,osc}$ also satisfy this energy inequality up to the same multiplicative constant.

Remark 2.4.2 Let us consider the associate homogenous linear equation

$$\partial_t U_{\varepsilon} + (-L + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A}) U_{\varepsilon} = 0$$

Then, denoting by S(t) the corresponding semi-group, the solution satisfies again the previous energy inequality, together with, a fortiori:

$$\|S(t)U_0\|_{L^2} \le \|U_0\|_{L^2}^2.$$
(2.4.10)

Truncations

Definition 2.4.1 Let us choose a function $\chi \in C_0^{\infty}(\mathbb{R})$ and equal to 1 near zero: for example let us choose $supp \chi \subset B(0,1)$, and $\chi \equiv 1$ near $B(0,\frac{1}{2})$.

Let us fix an arbitrary $\eta > 0$.

In the following we will cut the oscillating part into three parts:

$$U_{\varepsilon,osc} = (1 - \chi(\frac{|D|}{R}))U_{\varepsilon,osc} + \chi(\frac{|D_3|}{r})\chi(\frac{|D|}{R})U_{\varepsilon,osc} + (1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})U_{\varepsilon,osc}$$
$$= (a) + (b) + (c), \qquad (2.4.11)$$

where we note $(\mathcal{F}^{-1}$ is the inverse Fourier transform):

• $\chi(|D|)f = \mathcal{F}^{-1}(\chi(|\xi|)\hat{f}(\xi))$ • $\chi(|D_3|)f = \mathcal{F}^{-1}(\chi(|\xi_3|)\hat{f}(\xi)).$

Next we will successively fix R and r so that each of the first two parts has a norm less or equal to $\frac{\eta}{3}$: this will be the purpose of Section 2.4.2.

Most of the work will consist in the study of the third part (which requires the use of dispersion inequalities and Strichartz estimates): Section 2.4.3 will be entirely devoted to this task.

2.4.2 Study of the first two truncations

Let us begin by recalling that the following Sobolev embedding holds: If $s \ge 0$ and $s < \frac{d}{2}$, then $\dot{H}^s(\mathbb{R}^d)$ is continuously embedded in the space $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$.

Thus according to this embedding, and denoting $\alpha = \frac{3}{q} - \frac{1}{2}$, we can write, like in [7]:

$$\|(1-\chi(\frac{|D|}{R}))U_{\varepsilon,osc}\|_{L^{2}(\mathbb{R}_{+},L^{q}(\mathbb{R}^{3}))} \leq C\|(1-\chi(\frac{|D|}{R}))U_{\varepsilon,osc}\|_{L^{2}(\mathbb{R}_{+},\dot{H}^{1-\alpha}(\mathbb{R}^{3}))},$$

Then, according the energy inequality (2.4.9), we get:

$$\|(1-\chi(\frac{|D|}{R}))U_{\varepsilon,osc}\|_{L^{2}(\mathbb{R}_{+},L^{q}(\mathbb{R}^{3}))} \leq CR^{-\alpha}\|U_{\varepsilon,osc}\|_{L^{2}(\mathbb{R}_{+},\dot{H}^{1}(\mathbb{R}^{3}))} \leq \frac{C}{\sqrt{\nu_{0}}}R^{-\alpha}\|U_{0}\|_{L^{2}(\mathbb{R}^{3})}.$$

If $q \in [2, 6[$, then $\alpha \in]0, 1]$ and we will fix, until the end of the paper R large enough so that this norm is less than $\frac{\eta}{3}$, thus it follows that (independently of ε)

$$\|(1-\chi(\frac{|D|}{R}))U_{\varepsilon,osc}\|_{L^{2}(\mathbb{R}_{+},L^{q}(\mathbb{R}^{3}))} \leq \frac{\eta}{3}.$$
(2.4.12)

A fortiori, for all $T < \infty$, we get the same estimate in $L^2_T(L^q(\mathbb{R}^3)) = L^2([0,T], L^q(\mathbb{R}^3))$. Hence we have bounded the first truncature (a) from (2.4.11), and R is then fixed.

In the following, let us begin with the statement of two useful lemmas. Let us note that we will denote by C every universal constant (i.e which does not depend on the parameters T, R, r...).

The following lemma is an anisotropic majoration, and we address to [15] for more results about this subject.

Lemma 2.4.1 There exist a constant C > 0 (independent of R or r) so that for all $p \in [1, \infty]$ and all $u \in L^p$ we have:

$$\|\chi(\frac{|D_3|}{r})\chi(\frac{|D|}{R})u\|_{L^p} \le C\|u\|_{L^p}.$$

Proof: Let us remark that, by definition of the function χ , we can write that:

$$\chi(\frac{|D_3|}{r})\chi(\frac{|D|}{R})u = \chi(\frac{|\xi_3|}{r})\chi(\frac{|\xi|}{R})\hat{u}(\xi) = \chi(\frac{|\xi|}{R})\hat{u}(\xi)\chi(\frac{|\xi_1|}{2R})\chi(\frac{|\xi_2|}{2R})\chi(\frac{|\xi_3|}{2R}\cdot\frac{2R}{r})\hat{u}(\xi)$$

Thus, if we call

$$g = \mathcal{F}^{-1}(\chi(|\xi|)\chi(\frac{|\xi_1|}{2})\chi(\frac{|\xi_2|}{2})\chi(\frac{|\xi_3|}{2}\cdot\frac{2R}{r})),$$

we get $\chi(\frac{|D_3|}{r})\chi(\frac{|D|}{R})u = R^3g(R.) * u$ and then

$$\|\chi(\frac{|D_3|}{r})\chi(\frac{|D|}{R})u\|_{L^p} \le \|g\|_{L^1} \|u\|_{L^p}.$$

It is then about to bound $||g||_{L^1}$. Let us note $h(x) = \mathcal{F}^{-1}(\chi(|\xi|))$ and $k(x_1) = \mathcal{F}^{-1}(\chi(|\xi_1|))$, then: $||g||_{L^1(\mathbb{R}^3)} \leq ||h||_{L^1(\mathbb{R}^3)} ||k||_{L^1(\mathbb{R})}^3$ which concludes the proof of this lemma.

The following lemma is an anisotropic version of the Bernstein Lemma (one can find it in [5] p 16) and can be easily proved thanks to this latter lemma (we refer to [7]):

Lemma 2.4.2 There exists a constant C > 0 (independent of R or r) so that for all $u \in L^2(\mathbb{R}^3)$ (actually, it is sufficient for \hat{u} to be locally integrable) we have:

$$\begin{aligned} &\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})u\|_{L^{\infty}(\mathbb{R}^3)} \le C(R^2r)^{\frac{1}{2}}\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})u\|_{L^{2}(\mathbb{R}^3)} \\ &\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})u\|_{L^{2}(\mathbb{R}^3)} \le C(R^2r)^{\frac{1}{2}}\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})u\|_{L^{1}(\mathbb{R}^3)}. \end{aligned}$$

We will now bound the truncature (b) from the equality (2.4.11), R is now fixed and this time we will play with r: using the previous lemma and the energy estimate given by the Leray theorem together with the fact that \mathcal{P} is a pseudo-differential operator of order zero we get:

• $\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})U_{\varepsilon,osc}\|_{L^2_TL^2} \le T^{\frac{1}{2}}\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})U_{\varepsilon,osc}\|_{L^{\infty}_TL^2} \le CT^{\frac{1}{2}}\|U_0\|_{L^2}$

•
$$\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})U_{\varepsilon,osc}\|_{L^2_T L^{\infty}} \le C(R^2r)\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})U_{\varepsilon,osc}\|_{L^2_T L^2} \le C(R^2r)T^{\frac{1}{2}}\|U_0\|_{L^2}$$

An interpolation argument finally gives that $\|\chi(\frac{|D|}{R})\chi(\frac{|D_3|}{r})U_{\varepsilon,osc}\|_{L^2_T L^q} \leq C(R^2r)^{1-\frac{2}{q}}$. Let us then fix r until the end of the paper so that this expression is less than $\frac{\eta}{3}$.

Before going any further, let us summarize what we got for an arbitrary T. We have cut the oscillating part into three parts (see (2.4.11)). First, R was fixed so that the following inequality holds:

$$\|(1-\chi(\frac{|D|}{R}))U_{\varepsilon,osc}\|_{L^2_T(L^q(\mathbb{R}^3))} \le \frac{\eta}{3}$$

Then, r was fixed so that:

$$\|\chi(\frac{|D_3|}{r})\chi(\frac{|D|}{R})U_{\varepsilon,osc}\|_{L^2_T(L^q(\mathbb{R}^3))} \leq \frac{\eta}{3}$$

Therefore, in the following, we will bound the third truncature with r and R fixed.

2.4.3 Study of the third truncation

Study of the linearization of system (2.3.8)

In this section, we will study the eigenvectors of system (2.3.8) (in terms of the Fourier variable). We will see that the matrix is diagonalizable, that one of the eigenvalues does not play any role, and that among the rest of them, only two are oscillating. We will meet two main problems: first, the eigenvectors are no longer orthogonal (unlike the case of rotating fluids in [7]), then we will have to study the behaviour of the part of $U_{\varepsilon,osc}$ that corresponds to the non-oscillating eigenvalue.

Recall that $U_{\varepsilon,osc}$ satisfies system (2.3.8). Let us now begin with the homogeneous equation associated with this system and apply the Fourier transform to it:

$$\partial_t \widehat{W}_{\varepsilon,osc} = \widehat{LW_{\varepsilon,osc}} - \frac{1}{\varepsilon} \mathbb{P} \widehat{\mathcal{A}W_{\varepsilon,osc}}$$
(2.4.13)

$$= \left(\begin{pmatrix} -\nu & 0 & 0 & 0 \\ 0 & -\nu & 0 & 0 \\ 0 & 0 & -\nu & 0 \\ 0 & 0 & 0 & -\nu' \end{pmatrix} |\xi|^2 - \frac{1}{\varepsilon} \widehat{\mathbb{P}A} \right) \widehat{W}_{\varepsilon,osc}.$$

In terms of Fourier variables, the Leray projector can be written:

$$\widehat{\mathbb{P}} = \begin{pmatrix} \frac{\xi_2^2 + \xi_3^2}{|\xi|^2} & -\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{\xi_1\xi_3}{|\xi|^2} & 0\\ -\frac{\xi_1\xi_2}{|\xi|^2} & \frac{\xi_1^2 + \xi_3^2}{|\xi|^2} & -\frac{\xi_2\xi_3}{|\xi|^2} & 0\\ -\frac{\xi_1\xi_3}{|\xi|^2} & -\frac{\xi_2\xi_3}{|\xi|^2} & \frac{\xi_1^2 + \xi_2^2}{|\xi|^2} & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, the linearized equation becomes:

$$\partial_t \widehat{W}_{\varepsilon,osc} = \mathbb{B}(\xi,\varepsilon) \widehat{W}_{\varepsilon,osc} , \qquad (2.4.14)$$

where

$$\mathbb{B}(\xi,\varepsilon) = \begin{pmatrix} -\nu|\xi|^2 + \frac{\xi_1\xi_2}{\varepsilon|\xi|^2} & \frac{\xi_2^2 + \xi_3^2}{\varepsilon|\xi|^2} & 0 & \frac{\xi_1\xi_3}{\varepsilon F|\xi|^2} \\ -\frac{\xi_1^2 + \xi_3^2}{\varepsilon|\xi|^2} & -\nu|\xi|^2 - \frac{\xi_1\xi_2}{\varepsilon|\xi|^2} & 0 & \frac{\xi_2\xi_3}{\varepsilon F|\xi|^2} \\ \frac{\xi_2\xi_3}{\varepsilon|\xi|^2} & -\frac{\xi_1\xi_3}{\varepsilon|\xi|^2} & -\nu|\xi|^2 & -\frac{\xi_1^2 + \xi_2^2}{\varepsilon F|\xi|^2} \\ 0 & 0 & \frac{1}{\varepsilon F} & -\nu'|\xi|^2 \end{pmatrix}.$$

Eigenelements of the matrix \mathbb{B} and asymptotic expansions

Let us begin with the statement of the main results from this section:

Definition 2.4.2 If 0 < r < R, let us note $C_{r,R} = \{\xi \in \mathbb{R}^3 \text{ so that } |\xi| \le R, \text{ and } |\xi_3| \ge r\}.$

In the following computations, when we talk about a frequency ξ , we will always take it in $C_{r,R}$, it is this particular case that will be useful in the following.

Recall that we note $|\xi|_F$ the quantity $(\xi_1^2 + \xi_2^2 + F^2 \xi_3^2)^{\frac{1}{2}}$.

Lemma 2.4.3 Let us note:

$$p = \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2} - \frac{(\nu - \nu')^2}{3} |\xi|^4,$$

$$q = \frac{\nu - \nu'}{\varepsilon^2} \left(\frac{|\xi|_F^2}{3F^2} - \xi_3^2\right) - \frac{2}{27}(\nu - \nu')^3 |\xi|^6,$$

$$\alpha = \left(-\frac{q}{2} + \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}, \quad \text{and} \quad \beta = \left(-\frac{q}{2} - \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}$$

Then the eigenvalues of $\mathbb{B}(\xi, \varepsilon)$ are:

$$\begin{cases} \mu_0 &= -\nu |\xi|^2 \\ \mu &= -\frac{\nu' + 2\nu}{3} |\xi|^2 + \alpha + \beta \\ \lambda &= -\frac{\nu' + 2\nu}{3} |\xi|^2 + \alpha j + \beta j^2 \\ \overline{\lambda} &= -\frac{\nu' + 2\nu}{3} |\xi|^2 + \alpha j^2 + \beta j. \end{cases}$$

In particular, these eigenvalues are all different, and the matrix is diagonalizable.

Lemma 2.4.4 When $\xi \in C_{r,R}$ and ε is close to zero,

$$\mu = -(\nu\xi_1^2 + \nu\xi_2^2 + \nu'F^2\xi_3^2)\frac{|\xi|^2}{|\xi|_F^2} + \varepsilon^2 I(\varepsilon)$$
$$\lambda = -\tau(\xi)|\xi|^2 + i\frac{|\xi|_F}{\varepsilon F|\xi|} + i\varepsilon S(\xi,\varepsilon) + \varepsilon^2 S'(\xi,\varepsilon)$$

with the function τ defined by:

$$\tau(\xi) = \frac{\nu}{2} \left(1 + \frac{F^2 \xi_3^2}{|\xi|_F^2} \right) + \frac{\nu'}{2} \left(1 - \frac{F^2 \xi_3^2}{|\xi|_F^2} \right),$$

and I, S, and S' (that we do not precise more here) are functions of ε and ξ uniformly bounded on $\mathcal{C}_{r,R}$ relatively to ε .

Remark 2.4.3 The leading part of the asymptotic expansion of μ is exactly the expression in term of the Fourier variable of operator Γ .

The following lemma which can be easily proved by a simple computation on the above expression of τ , will be essential when we want to get dispersive estimates with respect to the eigenvalues λ and $\overline{\lambda}$:

Lemma 2.4.5 The function τ is bounded from below by a strictly positive constant on $C_{r,R}$:

$$\forall \xi \in \mathcal{C}_{r,R}, \ \tau(\xi) \ge \nu_0 > 0$$

We will now prove lemmas 2.4.3 and 2.4.4.

Proof: determination of the eigenvalues The characteristic polynomial of \mathbb{B} is:

$$\chi_{\mathbb{B}}(X) = det(XI_4 - \mathbb{B}) =$$

$$\frac{\xi_1^2 + \xi_2^2}{\varepsilon^2 F^2 |\xi|^2} (X + \nu |\xi|^2)^2 + (X + \nu |\xi|^2)^3 (X + \nu' |\xi|^2) + \frac{\xi_3^2}{\varepsilon^2 |\xi|^2} (X + \nu |\xi|^2) (X + \nu' |\xi|^2),$$

In order to simplify, let us write this polynomial in terms of the variable $(X + \nu |\xi|^2)$, which gives:

$$\chi_{\mathbb{B}}(X) = (X + \nu |\xi|^2) P(X),$$

with

$$P(X) = (X + \nu|\xi|^2)^3 - (\nu - \nu')|\xi|^2 (X + \nu|\xi|^2)^2 + \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2} (X + \nu|\xi|^2) - (\nu - \nu') \frac{\xi_3^2}{\varepsilon^2}$$

To obtain the roots of this polynomial the key point is the use of the Cardan formulas (see for example [16] page 172) which require the polynomial to be re-written into the particular form $x^3 + px + q$.

Let us change the unknown according to $X = Y - \frac{\nu'+2\nu}{3}|\xi|^2$ (considering not only the first change of variable (writing in $(X + \nu|\xi|^2)$) but also the one done to turn the polynomial into the particular formulation $x^3 + px + q$): we get the following polynomial:

 $Q(Y) = Y^3 + pY + q$

where
$$p = \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2} - \frac{(\nu - \nu')^2}{3} |\xi|^4$$
,
and $q = \frac{\nu - \nu'}{\varepsilon^2} \left(\frac{|\xi|_F^2}{3F^2} - \xi_3^2\right) - \frac{2}{27} (\nu - \nu')^3 |\xi|^6$.

Recall that we have taken $|\xi|_F^2 = \xi_1^2 + \xi_2^2 + F^2 \xi_3^2$.

All that remains is to apply the Cardan formulas: first of all, let us define the discriminent of the equation $D = \frac{q^2}{4} + \frac{p^3}{27}$.

Remark 2.4.4 It is important to keep in mind that the Cardan formulas give different results depending on the sign of the discriminent. In our case, because of the expression of p and q, it is not at all obvious to know the sign of D in function of ξ (even if ε is fixed). But, as R and r are fixed in the set $C_{r,R}$, if ε is small enough, we have the following equivalents:

$$p \sim \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2}$$
 and $q \sim \frac{\nu - \nu'}{\varepsilon^2} \left(\frac{|\xi|_F^2}{3F^2} - \xi_3^2\right).$

So the equivalent of D is:

$$D\sim \frac{|\xi|_F^6}{\varepsilon^6 F^6 |\xi|^6}>0$$

Thus, in the following computations, the discriminent will always be considered as (strictly) positive (as is the case after taking ε small enough, when R and r are fixed).

Let us note:

$$\alpha = \left(-\frac{q}{2} + D^{\frac{1}{2}}\right)^{\frac{1}{3}}$$
 and $\beta = \left(-\frac{q}{2} - D^{\frac{1}{2}}\right)^{\frac{1}{3}}$

Then the Cardan formulas (see [16]) give us the roots of polynomial Q:

$$\begin{cases} x_1 &= \alpha + \beta \\ x_2 &= \alpha j + \beta j^2 \\ x_3 &= \alpha j^2 + \beta j \end{cases}$$

From what we can immediately deduce the expression of the eigenvalues given in Lemma 2.4.3.

In order to determine the asymptotic expansions of Lemma 2.4.4, let us go back to the expressions of p and q.

As previously seen

$$p = \frac{p_1}{\varepsilon^2} + p_2, \quad \text{and} \quad q = \frac{q_1}{\varepsilon^2} + q_2,$$

with $p_1 = \frac{|\xi|_F^2}{F^2 |\xi|^2}$ and $p_2 = -\frac{(\nu - \nu')^2}{3} |\xi|^4,$
 $q_1 = (\nu - \nu') \left(\frac{|\xi|_F^2}{3F^2} - \xi_3^2\right)$ and $q_2 = -\frac{2}{27} (\nu - \nu')^3 |\xi|^6.$

The following lemma can be easily proved.

Lemma 2.4.6 When ε is close to zero, we have the following expansions:

$$\begin{aligned} \alpha &= \frac{p_1^{\frac{1}{2}}}{\varepsilon\sqrt{3}} - \frac{q_1}{2p_1} + \frac{p_1^{\frac{1}{2}}}{\sqrt{3}} \Big(\frac{\delta_2}{6} - \frac{3q_1^2}{4p_1^3}\Big)\varepsilon + \varepsilon^2 E(\xi,\varepsilon), \\ \beta &= -\frac{p_1^{\frac{1}{2}}}{\varepsilon\sqrt{3}} - \frac{q_1}{2p_1} - \frac{p_1^{\frac{1}{2}}}{\sqrt{3}} \Big(\frac{\delta_2}{6} - \frac{3q_1^2}{4p_1^3}\Big)\varepsilon + \varepsilon^2 E(\xi,\varepsilon), \end{aligned}$$

where δ_2 is a coefficient (depending on p_1 , p_2 , q_1 , q_2 as a rational fraction of these coefficients, and uniformly bounded with respect to ε and ξ) that we will not precise, and E a function of ξ and ε uniformly bounded in $C_{r,R}$ with respect to ε .

Then to obtain the asymptotic expansions of the eigenvalues, we have to use this lemma on the expressions of the roots of Q, then to do the inverse change of unknown in order to return to polynomial P which finally achieves the proof of lemma 2.4.4.

Let us conclude with an important remark.

Remark 2.4.5 When we compute the coefficients of the various asymptotic expansions seen before, we realize that they are rational fractions of p_1 , p_2 , q_1 , q_2 , and that in their denominators only appear powers of p_1 , which is function bounded from below and from above on $C_{r,R}$ by strictly positive constants.

That is all these coefficients are bounded by constants depending on r, R, F, and the viscosities, on $C_{r,R}$.

Computation of the eigenvectors

The following lemma can be proved by classical computation:

Lemma 2.4.7 The matrix $\mathbb{B}(\xi, \varepsilon)$ accepts the following vectors as a basis of eigenvectors:

1. Corresponding to the eigenvalue $\mu_0 = -\nu |\xi|^2$:

$$W_1(\xi) = \begin{pmatrix} \xi_2 \xi_3 \\ -\xi_1 \xi_3 \\ -\varepsilon F^2(\nu - \nu') |\xi|^2 \xi_3^2 \\ F\xi_3^2 \end{pmatrix}.$$

2. Corresponding to the eigenvalue $\mu = -\frac{\nu'+2\nu}{3}|\xi|^2 + \alpha + \beta$, which is real:

$$W_{2}(\xi) = \begin{pmatrix} \xi_{3}(\varepsilon\xi_{1}A + \xi_{2}) \\ \xi_{3}(\varepsilon\xi_{2}A - \xi_{1}) \\ -\varepsilon A(\xi_{1}^{2} + \xi_{2}^{2}) \\ F(\varepsilon^{2}|\xi|^{2}A^{2} + \xi_{3}^{2}) \end{pmatrix},$$

with $A = \mu + \nu |\xi|^2 = \frac{\nu - \nu'}{3} |\xi|^2 + \alpha + \beta$.

3. Corresponding to the eigenvalue $\lambda = -\frac{\nu'+2\nu}{3}|\xi|^2 + \alpha j + \beta j^2$,

$$W_{3}(\xi) = \begin{pmatrix} \xi_{3}(\varepsilon\xi_{1}B + \xi_{2}) \\ \xi_{3}(\varepsilon\xi_{2}B - \xi_{1}) \\ -\varepsilon B(\xi_{1}^{2} + \xi_{2}^{2}) \\ F(\varepsilon^{2}|\xi|^{2}B^{2} + \xi_{3}^{2}) \end{pmatrix}.$$

with $B = \lambda + \nu |\xi|^2 = \frac{\nu - \nu'}{3} |\xi|^2 + \alpha j + \beta j^2$.

4. W_4 corresponding to $\overline{\lambda}$, is the conjugate of W_3 .

Remark 2.4.6 It is interesting to know the behaviour of these vectors relatively to the divergence and to the potential vorticity, i.e when we look at their scalar product with (repectively) $(\xi_1, \xi_2, \xi_3, 0)$ and $(-\xi_2, \xi_1, 0, -F\xi_3)$.

- W_1 is orthogonal to none of them.
- The three others are orthogonal to the former but not to the latter.

As a consequence, it is immediate that the family (W_2, W_3, W_4) is a basis of the hyperplane formed by the vectors orthogonal to $(\xi_1, \xi_2, \xi_3, 0)$. Thus, every divergence free function has its Fourier transform that can be written as a linear combination (with coefficients depending on ξ) of these vectors. As the solution is also divergence free, we can say that its Fourier transform has no component on W_1 .

Let us prove a relation between the coordinates of these eigenvectors, this relation will be essential in Section 2.4.3 when it is about getting majorations for the coefficients (in the basis composed by the eigenvectors) of the solution in $C_{r,R}$:

Lemma 2.4.8 The last two coordinates of the eigenvectors W_2 , W_3 , and W_4 are linked by the relations

$$\begin{split} W_2^3 &= \varepsilon F(A - (\nu - \nu')|\xi|^2)W_2^4, \\ W_3^3 &= \varepsilon F(B - (\nu - \nu')|\xi|^2)W_3^4, \\ W_4^3 &= \varepsilon F(\overline{B} - (\nu - \nu')|\xi|^2)W_4^4, \end{split}$$

Proof:

When computing the coordinates of W_2 (it is the same computation for the two other vectors), we get a 4×4 system composed by a 3×3 system and one simple equation:

$$\begin{cases} \left(\mu+\nu|\xi|^2 - \frac{\xi_1\xi_2}{\varepsilon|\xi|^2}\right)x - \frac{\xi_2^2 + \xi_3^2}{\varepsilon|\xi|^2}y - \frac{\xi_1\xi_3}{\varepsilon F|\xi|^2}t &= 0\\ \frac{\xi_1^2 + \xi_3^2}{\varepsilon|\xi|^2}x + \left(\mu+\nu|\xi|^2 + \frac{\xi_1\xi_2}{\varepsilon|\xi|^2}\right)y - \frac{\xi_2\xi_3}{\varepsilon F|\xi|^2}t &= 0\\ -\frac{\xi_2\xi_3}{\varepsilon|\xi|^2}x + \frac{\xi_1\xi_3}{\varepsilon|\xi|^2}y + \left(\mu+\nu|\xi|^2\right)z + \frac{\xi_1^2 + \xi_2^2}{\varepsilon F|\xi|^2}t &= 0\\ -\frac{1}{\varepsilon F}z + \left(\mu+\nu'|\xi|^2\right)t &= 0 \end{cases}$$

Taking t as a parameter, we solve the 3×3 system. In order to simplify the expressions we take $t = F(\varepsilon^2 |\xi|^2 A^2 + \xi_3^2)$ and all that remains is to see if the last relation is satisfied, which is equivalent to proving that μ is a root of the polynomial P (it is true!).

Remark 2.4.7 If we want to know the limit of these vectors when ε goes to zero, we observe that the first two go to colinear vectors to $(\xi_2\xi_3, -\xi_1\xi_3, 0, F\xi_3^2)$, and the last two go to vectors orthogonal to it.

Now that we have all the eigenelements of the linearized system matrix, we meet an obvious difference with the case of the rotating fluids (where eigenelements are more simple, see [7]): our eigenvectors are not mutually orthogonal (in \mathbb{C}^4) which makes the work more difficult when it is about to get Strichartz estimates. This is the subject of the following section.

The problem of non orthogonal eigenvectors

In the case of the rotating fluids (see [7]), we obtain three eigenvalues, one of them is real and its associated eigenvector is not divergence-free (then it has no role to play in the solution), and the others are conjugated and give dispersion. After getting separate Strichartz inequalities for the projection of the solution in these two eigen directions, it is immediate, thanks to the orthogonality of the eigenvectors, to deduce the same inequalities for the whole solution.

In our case, the eigenvectors are not orthogonal (one convinces oneself after tedious computations). We still can get separate Strichartz inequalities for each projection of the solution corresponding to λ and $\overline{\lambda}$, but it seems that we need a majoration of these projections (with frequencies in $C_{r,R}$) in terms of the norm of U_{osc} .

Let us be more precise, the solution $U_{\varepsilon,osc,R,r} = (1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})U_{\varepsilon,osc}$ satisfies:

$$\partial_{t} U_{\varepsilon,osc,R,r} - LU_{\varepsilon,osc,R,r} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} U_{\varepsilon,osc,R,r} = -\mathbb{P}(1 - \chi(\frac{|D_{3}|}{r}))\chi(\frac{|D|}{R})(v_{\varepsilon}.\nabla U_{\varepsilon}) + (1 - \chi(\frac{|D_{3}|}{r}))\chi(\frac{|D|}{R})\left(\begin{array}{c} -\partial_{2}\Delta_{F}^{-1} \\ \partial_{1}\Delta_{F}^{-1} \\ 0 \\ -F\partial_{3}\Delta_{F}^{-1} \end{array}\right) \left(-v_{\varepsilon}.\nabla\Omega_{\varepsilon} + q(U_{\varepsilon,osc},U_{\varepsilon})\right)$$

$$+F(\nu-\nu')(1-\chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})\Delta\Delta_F^{-1}\partial_3\begin{pmatrix}-\partial_2\theta_{\varepsilon}\\\partial_1\theta_{\varepsilon}\\0\\\partial_2v_{\varepsilon}^1-\partial_1v_{\varepsilon}^2\end{pmatrix}$$

Let us note $F_{\varepsilon,R,r}$ the right-hand side of this equation.

Each term from this equation is divergence-free (everything is projected in the subspace of divergence-free vectors). Thus in the equation $\partial_t U_{\varepsilon,osc,R,r} - \mathbb{B}U_{\varepsilon,osc,R,r} = \widehat{F_{\varepsilon,R,r}}$, if we write the various terms in the basis of \mathbb{C}^4 composed by the eigenvectors $W_1(\xi)$, $W_2(\xi)$, $W_3(\xi)$, and $W_4(\xi)$ (in which $\mathbb{B}(\xi)$ is a diagonal matrix), none of them has any coordinate on the first vector. Then, we have:

$$\widehat{U_{\varepsilon,osc,R,r}}(\xi) = K_2(\xi)W_2(\xi) + K_3(\xi)W_3(\xi) + K_4(\xi)W_4(\xi)$$

$$\widehat{F_{\varepsilon,R,r}}(\xi) = F_2(\xi)W_2(\xi) + F_3(\xi)W_3(\xi) + F_4(\xi)W_4(\xi)$$

And, noting $Q(\xi) = (W_1(\xi), W_2(\xi), W_3(\xi), W_4(\xi))$, we have

$$\begin{pmatrix} 0\\K_2\\K_3\\K_4 \end{pmatrix} = Q^{-1}\widehat{U_{\varepsilon,osc}} \quad \text{and} \quad \begin{pmatrix} 0\\F_2\\F_3\\F_4 \end{pmatrix} = Q^{-1}\widehat{F_{\varepsilon}}.$$

As

$$Q^{-1}\mathbb{B}Q = \begin{pmatrix} \mu_0 & 0 & 0 & 0\\ 0 & \mu & 0 & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix},$$

what we have just done is only a diagonalization of the system into:

$$\partial_t \begin{pmatrix} 0 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} - \begin{pmatrix} \mu_0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix} \begin{pmatrix} 0 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} = \begin{pmatrix} 0 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}.$$

If we note $\mathcal{P}_2(\xi, \varepsilon)$, $\mathcal{P}_3(\xi, \varepsilon)$, and $\mathcal{P}_4(\xi, \varepsilon)$, the projectors in the last three eigen spaces of matrix $B(\xi)$ (that depend also on ε), and define the following pseudo-differential operators:

$$\mathbb{P}_{i}(u) = \mathcal{F}^{-1}(\mathcal{P}_{i}(\xi,\varepsilon)(\widehat{u}(\xi))), \qquad (2.4.15)$$

the diagonalization implies that each component K_iW_i satisfies the same equation as $\widehat{U_{osc}}$ but projecting initial data and right-hand side in the corresponding eigen space. Thus, for all $i \in \{2, 3, 4\}$ we have:

$$\widehat{\mathbb{P}_i(U_{osc})}(\xi) = K_i(\xi)W_i(\xi)$$

and

$$\begin{cases} \partial_t \mathbb{P}_i(\widehat{U_{\varepsilon,osc,R,r}}) = & \mathbb{B}(\xi,\varepsilon)\mathbb{P}_i(\widehat{U_{\varepsilon,osc,R,r}}) + \mathbb{P}_i(\widehat{F_{\varepsilon,R,r}}) \\ \mathbb{P}_i(\widehat{U_{osc,R,r}})_{/t=0} = & \mathbb{P}_i(\widehat{U_{osc,0,R,r}}). \end{cases}$$

The Duhamel formula can be written:

$$K_i W_i(t) = e^{t\lambda_i} K_i W_i(0) + \int_0^t e^{(t-s)\lambda_i} F_i W_i(s) ds$$

which is also

$$\mathbb{P}_i(U_{osc})(t) = e^{t\lambda_i} \mathbb{P}_i(U_{osc})(0) + \int_0^t e^{(t-s)\lambda_i} \mathbb{P}_i(F)(s) ds.$$

Then we will have to study an operator of the following form:

$$f \mapsto \int_{\mathbb{R}^3} e^{ix.\xi} e^{\lambda(\xi,\varepsilon)t} \widehat{f}(\xi) d\xi$$

with $\lambda(\xi,\varepsilon) = -\tau(\xi) |\xi|^2 + i \frac{|\xi|_F}{\varepsilon F |\xi|} + i\varepsilon S(\xi,\varepsilon) + \varepsilon^2 S'(\xi,\varepsilon)$

$$et \quad \tau(\xi) = \frac{\nu}{2} \left(1 + \frac{F^2 \xi_3^2}{|\xi|_F^2} \right) + \frac{\nu'}{2} \left(1 - \frac{F^2 \xi_3^2}{|\xi|_F^2} \right).$$

Moreover, according to Lemma 2.4.5, τ is bounded from below on $C_{r,R}$ by $\nu_0 > 0$.

We will then study the operator:

$$K(\varepsilon,t,z) = \int_{\mathbb{R}^3} \psi(\xi) e^{-t\tau(\xi)|\xi|^2 + i\frac{t|\xi|_F}{\varepsilon F|\xi|} + i\varepsilon tS(\xi,\varepsilon) + \varepsilon^2 tS'(\xi,\varepsilon) + iz.\xi} d\xi,$$

where

- the function $\psi \in \mathcal{C}_0^{\infty}(\mathbb{R}^3)$ is radial, supported in $\mathcal{C}_{\frac{r}{2},2R}$, and equals 1 near $\mathcal{C}_{r,R}$.
- $\forall \xi \in \mathcal{C}_{r,R}, \ \tau(\xi) \ge \nu_0 > 0.$
- S, S' and all of their derivatives with respect to ξ are bounded on $C_{r,R}$ by a constant $A_{r,R}$ (see Remark 2.4.5).

Let us precise now the Strichartz inequalities we have been talking about.

Lemma 2.4.9 Dispersion estimates

There exists a constant $C = C_{r,R,F,\nu,\nu'}$ such that for all $z \in \mathbb{R}^3$ and all ε small enough, we have:

$$|K(\varepsilon,t,z)| \le Ce^{\frac{-\nu_0 r^2 t}{16}} (\frac{\varepsilon}{t})^{\frac{1}{2}}.$$

The proof of these dipersion estimates is very close to those in [7]. One will find it in Section 2.4.3. When it is proved, we will deduce the following fact, that will be useful when achieving the proof of Theorem 1.

Corollary 2.4.1 Strichartz estimates

Assume that U solves on [0, T] the system:

$$\begin{cases} \partial_t U - LU + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} U = F \\ div \ v = 0 \\ U_{/t=0} = U_0 \in L^2(\mathbb{R}^3) \end{cases}$$

Assume also that

$$supp\widehat{U_0} \cup supp\widehat{F}(t) \subset \mathcal{C}_{r,R}$$

Then there exists a constant $C = C_{r,R,F}$ so that we have, for i = 3 or 4:

$$\|\mathbb{P}_{i}(U)\|_{L^{4}([0,T],L^{\infty}(\mathbb{R}^{3}))} \leq C\varepsilon^{\frac{1}{4}} \left(\|\mathbb{P}_{i}(U_{0})\|_{L^{2}} + \|\mathbb{P}_{i}(F)\|_{L^{1}([0,T],L^{2}(\mathbb{R}^{3}))}\right)$$

As the eigenvectors are not orthogonal, and as the solution has a non zero projection in the second eigen space, it would be interesting to bound the operator norm of these projectors in eigen spaces, which would allow not only to provide Strichartz estimates for the projection of U_{osc} in the last two eigen spaces (corresponding to λ and $\overline{\lambda}$, i.e $(\mathbb{P}_3 + \mathbb{P}_4)((1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})U_{osc}))$ and then bound it with the norms of $U_{osc,0}$ and F(that can be bounded using the Leray energy estimate), but also to estimate the projection $\mathbb{P}_2((1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})U_{osc})$ (we will see that it also goes to zero). That's why the following section is devoted to the matrix Q.

We establish the following estimate:

Lemma 2.4.10 There exists a constant $C = C_{r,R,F}$ so that for all vector f orthogonal to $(\xi_1, \xi_2, \xi_3, 0)$, for all $\xi \in C_{r,R}$, and for all $\varepsilon \in]0, 1[$, the projections of f in the three eigen spaces (see 2.4.15) satisfy, with respect to the norm in \mathbb{C}^4 ,

$$|\mathcal{P}_i(\xi,\varepsilon)(f)(\xi)| \le C|f(\xi)| \tag{2.4.16}$$

This lemma finally leads to the following bound:

Lemma 2.4.11 There exists a constant $C = C_{r,R,F}$ so that for all $i \in \{2,3,4\}$ and $s \in \mathbb{R}$:

$$\|\mathbb{P}_{i}((1-\chi(\frac{|D_{3}|}{r}))\chi(\frac{|D|}{R})U_{\varepsilon,osc})\|_{\dot{H}^{s}(\mathbb{R}^{3})} \leq C\|(1-\chi(\frac{|D_{3}|}{r}))\chi(\frac{|D|}{R})U_{\varepsilon,osc}\|_{\dot{H}^{s}(\mathbb{R}^{3})}$$

Combining this result with what precedes, we will prove the corollary:

Lemma 2.4.12 Strichartz inequalities

Under the same assumptions, there exists a constant $C = C_{r,R,F}$ so that:

$$\| (\mathbb{P}_3 + \mathbb{P}_4) ((1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})U) \|_{L^4(\mathbb{R}_+, L^\infty(\mathbb{R}^3))}$$

$$\leq C\varepsilon^{\frac{1}{4}} \Big(\| U_0 \|_{L^2} + \| F \|_{L^1(\mathbb{R}_+, L^2(\mathbb{R}^3))} \Big)$$

Study of the change of basis, proof of Lemma 2.4.10

Given the eigenvectors, the matrix of the change of basis can be written the following way:

$$Q = \begin{pmatrix} \xi_2\xi_3 & \xi_3(\varepsilon\xi_1A + \xi_2) & \xi_3(\varepsilon\xi_1B + \xi_2) & \xi_3(\varepsilon\xi_1\overline{B} + \xi_2) \\ -\xi_1\xi_3 & \xi_3(\varepsilon\xi_2A - \xi_1) & \xi_3(\varepsilon\xi_2B - \xi_1) & \xi_3(\varepsilon\xi_2\overline{B} - \xi_1) \\ -\varepsilon F^2(\nu - \nu')|\xi|^2\xi_3^2 & -\varepsilon A(\xi_1^2 + \xi_2^2) & -\varepsilon B(\xi_1^2 + \xi_2^2) & -\varepsilon \overline{B}(\xi_1^2 + \xi_2^2) \\ F\xi_3^2 & F(\varepsilon^2|\xi|^2A^2 + \xi_3^2) & F(\varepsilon^2|\xi|^2B^2 + \xi_3^2) & F(\varepsilon^2|\xi|^2\overline{B}^2 + \xi_3^2) \end{pmatrix}$$

Assume that a vector orthogonal to $(\xi_1, \xi_2, \xi_3, 0)$ (recall that this implies it has no coordinate on W_1) is written $f = K_2 W_2 + K_3 W_3 + K_4 W_4$, then

$$\left(\begin{array}{c} 0\\ K_2\\ K_3\\ K_4 \end{array}\right) = Q^{-1}f$$

It is then a matter of solving the linear system:

$$Q\begin{pmatrix}0\\K_2\\K_3\\K_4\end{pmatrix} = \begin{pmatrix}X\\Y\\Z\\T\end{pmatrix}$$

If we put

$$\begin{cases} U &= K_2 + K_3 + K_4 \\ V &= AK_2 + BK_3 + \overline{B}K_4 \\ W &= A^2K_2 + B^2K_3 + \overline{B}^2K_4 \end{cases}$$

then, the system is equivalent to

$$\begin{cases} \xi_3(\xi_2 U + \varepsilon \xi_1 V) = X\\ \xi_3(-\xi_1 U + \varepsilon \xi_2 V) = Y\\ -\varepsilon(\xi_1^2 + \xi_2^2) V = Z\\ F\xi_3^2 U + F\varepsilon^2 |\xi|^2 W = T \end{cases}$$

The first two lines allow to obtain:

$$U = \frac{\xi_2 X - \xi_1 Y}{\xi_3(\xi_1^2 + \xi_2^2)}$$
$$V = \frac{\xi_1 X + \xi_2 Y}{\varepsilon \xi_3(\xi_1^2 + \xi_2^2)}$$

Then thanks to the last line we can write:

$$W = \frac{T}{F\varepsilon^2 |\xi|^2} - \frac{\xi_3(\xi_2 X - \xi_1 Y)}{\varepsilon^2 |\xi|^2 (\xi_1^2 + \xi_2^2)}$$

Finally, we have to compute U, V, and W in function of the components of f, that is a linear system whose matrix is a Van der Monde matrix of parameters A, B, et \overline{B} . These coefficients are one to one distinct, so the matrix is invertible, and:

$$VdM(A,B,\overline{B})^{-1} = \left(\begin{array}{ccc} 1 & 1 & 1 \\ A & B & \overline{B} \\ A^2 & B^2 & \overline{B}^2 \end{array}\right)^{-1}$$

$$= M = \begin{pmatrix} \frac{|B|^2}{|B|^2 - AB + A^2 - A\overline{B}} & \frac{-(B + \overline{B})}{|B|^2 - AB + A^2 - A\overline{B}} & \frac{1}{|B|^2 - AB + A^2 - A\overline{B}} \\ \frac{-A\overline{B}}{-A\overline{B} + AB - B^2 + |B|^2} & \frac{A + \overline{B}}{-A\overline{B} + AB - B^2 + |B|^2} & \frac{-1}{-A\overline{B} + AB - B^2 + |B|^2} \\ \frac{AB}{-A\overline{B} + AB - B^2 + |B|^2} & \frac{-(A + B)}{-A\overline{B} + AB - B^2 + |B|^2} & \frac{1}{-A\overline{B} + AB - B^2 + |B|^2} \end{pmatrix}$$

Thus, we get that

$$\begin{pmatrix} K_2 \\ K_3 \\ K_4 \end{pmatrix} = M \begin{pmatrix} \frac{\xi_2 X - \xi_1 Y}{\xi_3(\xi_1^2 + \xi_2^2)} \\ \frac{\xi_1 X + \xi_2 Y}{\varepsilon \xi_3(\xi_1^2 + \xi_2^2)} \\ \frac{T}{F\varepsilon^2 |\xi|^2} - \frac{\xi_3(\xi_2 X - \xi_1 Y)}{\varepsilon^2 |\xi|^2(\xi_1^2 + \xi_2^2)} \end{pmatrix}$$

Remark 2.4.8 At this stage there are two annoying points to be cleared: first the terms with negative powers of ε , second the denominators with $(\xi_1^2 + \xi_2^2)$ in factor (we want uniform majorations with respect to the Fourier variable).

The first problem will be solved thanks to asymptotic expansions of the coefficients of the matrix M, and the second problem will be solved trying not to only bound coefficients K_i , but the whole component K_iW_i .

As A, B, and \overline{B} only depend on coefficients p_1 , p_2 , q_1 , q_2 (see Remark 2.4.5), and as the denominators are powers of p_1 , bounded from below as well as from above by strictly positive constants on $C_{r,R}$, then all the coefficients of the matrix M are bounded and we will have to look for their equivalents when ε goes to zero.

Using lemma 2.4.4 we immediately obtain:

Lemma 2.4.13 We have the following asymptotic expansions:

$$A = (\nu - \nu')F^2 \xi_3^2 \frac{|\xi|^2}{|\xi|_F^2} + \mathcal{O}(\varepsilon^2)$$
$$B = \frac{i|\xi|_F}{\varepsilon F|\xi|} + \frac{\nu - \nu'}{2} \frac{|\xi|^2}{|\xi|_F^2} (\xi_1^2 + \xi_2^2) + \mathcal{O}(\varepsilon),$$

where the $\mathcal{O}(\varepsilon)$ are uniform in $\xi \in \mathcal{C}_{r,R}$

Up to a multiplicative constant, when ε is small enough, the matrix M has the following form:

$$M \sim \left(\begin{array}{ccc} 1 & \varepsilon^2 & \varepsilon^2 \\ \varepsilon & \varepsilon & \varepsilon^2 \\ \varepsilon & \varepsilon & \varepsilon^2 \end{array}\right),$$

The right-hand member has the following form:

$$\left(\begin{array}{c} M_1\\ \frac{M_2}{\varepsilon}\\ \frac{M_3}{\varepsilon^2} \end{array}\right).$$

One can easily deduce from it that the coordinates only have positive powers of ε which solves the first problem.

Roughly speaking we can write that coefficients K_2 , K_3 , and K_4 are polynomial expressions, of degree one (linear) in X, Y, Z, and T (components of f), whose coefficients are of the form:

$$G(\varepsilon, \frac{1}{|\xi|}, \frac{1}{\xi_3}, \frac{\xi_1}{\xi_1^2 + \xi_2^2}, \frac{\xi_2}{\xi_1^2 + \xi_2^2}),$$

which are rational fractions whose degrees are positive with respect to the first variable, and 0 or 1 with respect to the last two ones, recall that $\frac{1}{|\xi|}$, and $\frac{1}{\xi_3}$ are bounded from below and from above by strictly positive constants on $C_{r,R}$.

Then, when we look at K_2W_2 , K_3W_3 , and K_4W_4 , the annoying coefficients in the first two coordinates of the vector (see lemma 2.4.7) become:

$$\frac{\xi_1^2}{\xi_1^2 + \xi_2^2} \quad \frac{\xi_2^2}{\xi_1^2 + \xi_2^2} \quad \text{or} \quad \frac{\xi_1 \xi_2}{\xi_1^2 + \xi_2^2}$$

which are bounded by 1.

Concerning the third coordinate, as W_i^3 contains $(\xi_1^2 + \xi_2^2)$ as a factor, the annoying coefficients simply become ξ_1 or ξ_2 , bounded on $C_{r,R}$.

At first sight, it seems that this method no longer works for the fourth coordinate. But if we recall lemma 2.4.8, and the fact that A or εB are uniformly bounded in terms of ε and ξ on $C_{r,R}$ then this case is dealt the same way as the previous one.

We obtain (2.4.16):

There exists a constant $C = C_{r,R,F}$ so that for all vector f orthogonal to $(\xi_1, \xi_2, \xi_3, 0)$, for all $\xi \in C_{r,R}$, and all $\varepsilon \in]0, 1[$, the projections of f onto the three eigenspaces satisfy, relatively to the norm in \mathbb{C}^4 :

$$|(K_i W_i)(\xi)| \le C_{r,R} |f(\xi)|$$

Lemma 2.4.10 is proved.

Proof of the dispersion estimate from Lemma 2.4.9

We use here the same method as in [7]: a simplified stationnary phase (thanks to the set $C_{r,R}$ we can get rid of the singularity, which implies that everything works as if there were only a non-stationary phase). Let us begin with noting $b(\xi) = \frac{|\xi|_F}{F|\xi|}$.

As we chose ψ to be a radial function, a change of variable (rotation around the third axis of coordinates) allows us to assume $z_2 = 0$ (the invariance by such a rotation is justified by remark 2.4.5 and the invariance under the same rotation of coefficients p_1 , p_2 , q_1 and q_2).

Then, note $\beta = -\partial_{\xi_2} b(\xi)$ and define the following operator:

$$\mathcal{L} = \frac{1}{1 + \frac{t}{\varepsilon}\beta(\xi)^2} (1 + i\beta(\xi)\partial_{\xi_2})$$

which works on variable ξ_2 and satisfies $\mathcal{L}(e^{i\frac{t}{\varepsilon}b}) = e^{i\frac{t}{\varepsilon}b}$.

The fact that we could assume $z_2 = 0$ implies that $e^{iz.\xi}$ doesn't depend on variable ξ_2 so:

$$\mathcal{L}(e^{iz.\xi}e^{i\frac{t}{\varepsilon}b}) = e^{iz.\xi}e^{i\frac{t}{\varepsilon}b}.$$

By integration by parts, we obtain:

$$K(\varepsilon,t,z) = \int_{\mathcal{C}_{\frac{r}{2},2R}} e^{i\frac{\varepsilon}{\varepsilon}b(\xi) + iz.\xi} \mathcal{L}^{T}(\psi(\xi)e^{-t\tau(\xi)|\xi|^{2} + i\varepsilon tS(\xi,\varepsilon) + \varepsilon^{2}tS'(\xi,\varepsilon) + iz.\xi}))d\xi_{2}d\xi_{1}d\xi_{3}.$$

As in [7], we express the transposed of operator \mathcal{L} :

$$\mathcal{L}^{T}(g) = (\frac{1}{1 + \frac{t}{\varepsilon}\beta^{2}} - i\partial_{\xi_{2}}\beta \frac{1 - \frac{t}{\varepsilon}\beta^{2}}{(1 + \frac{t}{\varepsilon}\beta^{2})^{2}})g - \frac{i\beta}{1 + \frac{t}{\varepsilon}\beta^{2}}\partial_{\xi_{2}}g$$

Replacing \mathcal{L}^T in the integral, computations lead to:

$$\begin{split} |K(\varepsilon,t,z)| &\leq \int_{\mathcal{C}_{\frac{r}{2},2R}} \Big[\Big(\frac{1}{1+\frac{t}{\varepsilon}\beta^2} + A_0(\xi) \frac{1+\frac{t}{\varepsilon}\beta^2}{(1+\frac{t}{\varepsilon}\beta^2)^2} \Big) \|\psi\|_{L^{\infty}} e^{-t\nu_0(\frac{r}{2})^2} e^{\varepsilon t A_1(\varepsilon,\xi)}, \\ &+ \frac{A_2(\xi)}{1+\frac{t}{\varepsilon}\beta^2} \Big(\|\partial_{\xi_2}\psi\|_{L^{\infty}} + t \|\psi\|_{L^{\infty}} A_3(\xi) + \varepsilon t \|\psi\|_{L^{\infty}} A_4(\xi) \Big) e^{-t\nu_0(\frac{r}{2})^2} e^{\varepsilon t A_1(\varepsilon,\xi)} \Big] d\xi \end{split}$$

where coefficients $A_0, ..., A_4$ are rationnal fractions in terms of ξ but in fact only depending on $|\xi|$, $|\xi|_F$, and the derivatives of S and S' which depend on p_1, p_2, q_1, q_2 .

So, all these coefficients are easily bounded from above and from below by (strictly) positive constants on $C_{\frac{r}{2},2R}$ (see remark 2.4.5) it follows that:

$$|K(\varepsilon,t,z)| \le \int_{\mathcal{C}_{\frac{r}{2},2R}} \frac{1}{1+\frac{t}{\varepsilon}\beta^2} C_{r,R,F,\psi} (1+t+\varepsilon t) e^{-t\nu_0 \frac{r^2}{4}} e^{\varepsilon t C'_{r,R}} d\xi$$

there exists an ε_{r,R,ν_0} small enough so that for all $\varepsilon \leq \varepsilon_{r,R,\nu_0}$:

$$-\nu_0 \frac{r^2}{4} + \varepsilon C'_{r,R} \le -\nu_0 \frac{r^2}{16}$$

and

$$(1+t+\varepsilon t)e^{-t\nu_0\frac{r^2}{4}+\varepsilon tC'_{r,R}} \le C_{r,R,F,\psi}e^{-t\nu_0\frac{r^2}{16}}$$

Then:

$$\begin{aligned} |K(\varepsilon,t,z)| &\leq C_{r,R,F,\psi} e^{-t\nu_0 \frac{r^2}{16}} \int_{\mathcal{C}_{\frac{r}{2},2R}} \frac{1}{1 + \frac{t}{\varepsilon} \frac{(F^2 - 1)^2 \xi_3^4 \xi_2^2}{|\xi|_F^2 |\xi|^6}} d\xi \\ &\leq C_{r,R,F,\psi} e^{-t\nu_0 \frac{r^2}{16}} \int_{\mathcal{C}_{\frac{r}{2},2R}} \frac{1}{1 + \frac{t}{\varepsilon} \xi_2^2 C_{r,R,F}} d\xi_2 \end{aligned}$$

Then, a change of variable gives the expected dispersion estimate:

$$|K(\varepsilon,t,z)| \le C_{r,R,F,\nu,\nu'} e^{\frac{-\nu_0 r^2 t}{16}} (\frac{\varepsilon}{t})^{\frac{1}{2}}.$$

Strichartz estimates, proof of Corollary 2.4.1

This result can be proved using a classical TT^* type argument, exactly like in [7], we use it on both projections \mathbb{P}_3 and \mathbb{P}_4 of $U_{\varepsilon,osc}$, which satisfy the same equation as $U_{\varepsilon,osc}$, but with the projections of initial data and right-hand side. We will not give more details here.

2.4.4 Convergence of the oscillating part, end of the proof of Theorem 2.1.2

Let us go back to (2.4.11): the solution $U_{\varepsilon,osc}$ was cut into three parts, each of them given by frequency truncature.

We bounded two of them, the proof of theorem 2.1.2 will be achieved when we study the last part, and that's where the Strichartz estimates and projectors \mathcal{P} and \mathcal{Q} will be essential.

Note
$$U_{\varepsilon,osc,R,r} = (1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})U_{\varepsilon,osc}$$
, it is a solution of the following system:

$$\partial_t U_{\varepsilon,osc,R,r} - LU_{\varepsilon,osc,R,r} + \frac{1}{\varepsilon} \mathbb{P} A U_{\varepsilon,osc,R,r} = F'_{R,r}$$
$$F'_{R,r} = (F'1) + (F'2) + (F'3) + (F'4)$$

where:

$$(F'1) = -(1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})\mathbb{P}(v_{\varepsilon}.\nabla U_{\varepsilon}),$$

$$(F'2) = (1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R}) \begin{pmatrix} -\partial_2\\ \partial_1\\ 0\\ -F\partial_3 \end{pmatrix} \Delta_F^{-1}(-v_{\varepsilon}.\nabla\Omega_{\varepsilon}),$$
$$(F'3) = (1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R}) \begin{pmatrix} -\partial_2\\ \partial_1\\ 0\\ -F\partial_3 \end{pmatrix} \Delta_F^{-1}(q(U_{\varepsilon,osc}, U_{\varepsilon})),$$

and
$$(F'4) = F(\nu - \nu')(1 - \chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})\Delta\Delta_F^{-1}\partial_3\begin{pmatrix} -\partial_2\theta_{\varepsilon}\\ \partial_1\theta_{\varepsilon}\\ 0\\ \partial_2v_{\varepsilon}^1 - \partial_1v_{\varepsilon}^2 \end{pmatrix}$$
.

The fact that the eigenvectors of the matrix \mathbb{B} are not orthogonal imposes to study each of the three projections of $U_{\varepsilon,osc,R,r}$.

For $\mathbb{P}_3 U_{\varepsilon,osc,R,r}$ and $\mathbb{P}_4 U_{\varepsilon,osc,R,r}$, which both satisfy the assumptions of corollary 2.4.1, we obtain the existence of a constant $C = C_{r,R,F}$ so that:

$$\begin{split} \|\mathbb{P}_{i}(1-\chi(\frac{|D_{3}|}{r}))\chi(\frac{|D|}{R})U_{\varepsilon,osc}\|_{L^{4}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} &\leq C\varepsilon^{\frac{1}{4}}\Big(\|\mathbb{P}_{i}(1-\chi(\frac{|D_{3}|}{r}))\chi(\frac{|D|}{R})U_{osc}(0)\|_{L^{2}} \\ &+\|\mathbb{P}_{i}(1-\chi(\frac{|D_{3}|}{r}))\chi(\frac{|D|}{R})F_{R,r}'\|_{L^{1}(\mathbb{R}_{+},L^{2}(\mathbb{R}^{3}))}\Big) \end{split}$$

Which we bound thanks to lemmas 2.4.11 and 2.4.12 by:

$$C\varepsilon^{\frac{1}{4}}\Big(\|(1-\chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})U_{osc}(0)\|_{L^2}+\|(1-\chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})F'_{R,r}\|_{L^1(\mathbb{R}_+,L^2(\mathbb{R}^3))}\Big)$$

Then we use the following estimate which is easily proved from the Plancherel formula and the fact that \mathcal{P} is an homogenous pseudo-differential operator of order zero. With the usual notations:

$$\|(1-\chi(\frac{|D_3|}{r}))\chi(\frac{|D|}{R})U_{osc}(0)\|_{L^2(\mathbb{R}^3)} \le C\|U_0\|_{L^2(\mathbb{R}^3)}$$

Then we can state the estimates:

Lemma 2.4.14 With the notations recalled in the beginning of this section, the following bounds hold:

$$\left\| (F'4) \right\|_{L^1([0,T], L^2(\mathbb{R}^3))} \le CF |\nu - \nu'| R^2 T \| U_0 \|_{L^2(\mathbb{R}^3)}$$
(2.4.17)

$$\left\| (F'1) \right\|_{L^1([0,T], L^2(\mathbb{R}^3))} \le CR^{\frac{5}{2}} T \| U_0 \|_{L^2(\mathbb{R}^3)}^2$$
(2.4.18)

$$\left\| (F'2) \right\|_{L^1([0,T], L^2(\mathbb{R}^3))} \le \frac{C}{\sqrt{\nu_0}} R^{\frac{3}{2}} T^{\frac{1}{2}} \| U_0 \|_{L^2(\mathbb{R}^3)}$$
(2.4.19)

$$\left\| (F'3) \right\|_{L^1([0,T], L^2(\mathbb{R}^3))} \le \frac{C}{\sqrt{\nu_0}} R^{\frac{3}{2}} T^{\frac{1}{2}} \| U_0 \|_{L^2(\mathbb{R}^3)}$$
(2.4.20)

Proof: We won't give details but the proofs of these inequalities consist in the use of classical tools (Plancherel, majoration given by Leray's theorem, Hölder inequality...), and the rewriting of q_{ε} (2.3.6) into a more suitable form (let us recall that v, $v_{\varepsilon,osc}$ and $v_{\varepsilon,QG}$ are divergence-free):

$$q(U_{\varepsilon,osc}, U_{\varepsilon}) = (\partial_3(v_{\varepsilon,osc}^3 \partial_1 v_{\varepsilon}^2) - v_{\varepsilon,osc}^3 \partial_3 \partial_1 v_{\varepsilon}^2) - (\partial_3(v_{\varepsilon,osc}^3 \partial_2 v_{\varepsilon}^1) - v_{\varepsilon,osc}^3 \partial_3 \partial_2 v_{\varepsilon}^1))$$

$$-(\partial_1(v_{\varepsilon,osc}^3\partial_3 v_{\varepsilon}^2) - v_{\varepsilon,osc}^3\partial_1\partial_3 v_{\varepsilon}^2) + (\partial_2(v_{\varepsilon,osc}^3\partial_3 v_{\varepsilon}^1) - v_{\varepsilon,osc}^3\partial_2\partial_3 v_{\varepsilon}^1) + Fdiv(\partial_3(v_{\varepsilon} - v_{\varepsilon,osc})\theta_{\varepsilon,osc}) + Fdiv(\partial_3 v_{\varepsilon,osc}\theta_{\varepsilon})$$

and, simplifying, we get:

$$\begin{split} &= \partial_3(v_{\varepsilon,osc}^3\partial_1v_{\varepsilon}^2) - \partial_3(v_{\varepsilon,osc}^3\partial_2v_{\varepsilon}^1) - \partial_1(v_{\varepsilon,osc}^3\partial_3v_{\varepsilon}^2) \\ &\quad + \partial_2(v_{\varepsilon,osc}^3\partial_3v_{\varepsilon}^1) + Fdiv(\partial_3(v_{\varepsilon} - v_{\varepsilon,osc})\theta_{\varepsilon,osc}) + Fdiv(\partial_3v_{\varepsilon,osc}\theta_{\varepsilon}). \end{split}$$

Thus, $q(U_{\varepsilon,osc}, U_{\varepsilon})$ has the following form:

$$q(U_{\varepsilon,osc}, U_{\varepsilon}) = \nabla(v_{\varepsilon,osc}^3 \nabla v_{\varepsilon}) + F \nabla(\nabla(v_{\varepsilon} - v_{\varepsilon,osc})\theta_{\varepsilon,osc}) + F \nabla(\nabla v_{\varepsilon,osc}\theta_{\varepsilon}).\blacksquare$$

It follows that:

Corollary 2.4.2 For i = 3 or 4, we have:

$$\begin{split} \|\mathbb{P}_{i}(1-\chi(\frac{|D_{3}|}{r}))\chi(\frac{|D|}{R})U_{\varepsilon,osc}\|_{L^{4}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} &\leq C\varepsilon^{\frac{1}{4}}\Big(\|U_{0}\|_{L^{2}(\mathbb{R}^{3})} \\ &+R^{\frac{5}{2}}T\|U_{0}\|_{L^{2}(\mathbb{R}^{3})}^{2} + \frac{R^{\frac{3}{2}}}{\sqrt{\nu_{0}}}T^{\frac{1}{2}}\|U_{0}\|_{L^{2}(\mathbb{R}^{3})} + F|\nu-\nu'|R^{2}T\|U_{0}\|_{L^{2}(\mathbb{R}^{3})}\Big) \end{split}$$

To complete the bound of truncation (c) (see (2.4.11)), we will focus on the second projection (\mathbb{P}_2) .

As $U_{\varepsilon,osc}$ is the orthogonal projection onto the potential vorticity free vector fields, we have:

$$U_{\varepsilon,osc,R,r} = \mathcal{P}U_{\varepsilon,osc,R,r} = \mathcal{P}\mathbb{P}_2 U_{\varepsilon,osc,R,r} + \mathcal{P}\mathbb{P}_3 U_{\varepsilon,osc,R,r} + \mathcal{P}\mathbb{P}_4 U_{\varepsilon,osc,R,r}.$$

As \mathcal{P} is an homogenous pseudo-differential operator of order zero, the last two terms satisfy the majoration given by the previous corollary, thus let us consider $\mathcal{PP}_2U_{\varepsilon,osc,R,r}$.

In terms of the Fourier transform, we have: $\mathcal{PP}_{2}\widehat{U_{\varepsilon,osc,R,r}} =$

$$\begin{split} K_{2}(\xi)(1-\chi(\frac{|\xi_{3}|}{r}))\chi(\frac{|\xi|}{R})\frac{1}{|\xi|_{F}^{2}} \begin{pmatrix} \xi_{1}^{2}+F^{2}\xi_{3}^{2} & \xi_{1}\xi_{2} & 0 & -F\xi_{2}\xi_{3} \\ \xi_{1}\xi_{2} & \xi_{2}^{2}+F^{2}\xi_{3}^{2} & 0 & F\xi_{1}\xi_{3} \\ 0 & 0 & |\xi|_{F}^{2} & 0 \\ -F\xi_{2}\xi_{3} & F\xi_{1}\xi_{3} & 0 & \xi_{1}^{2}+\xi_{2}^{2} \end{pmatrix} \cdot \begin{pmatrix} \xi_{3}(\varepsilon\xi_{1}A+\xi_{2}) \\ \xi_{3}(\varepsilon\xi_{2}A-\xi_{1}) \\ -\varepsilon A(\xi_{1}^{2}+\xi_{2}^{2}) \\ F(\varepsilon^{2}|\xi|^{2}A^{2}+\xi_{3}^{2}) \end{pmatrix} \\ = K_{2}(\xi)(1-\chi(\frac{|\xi_{3}|}{r}))\chi(\frac{|\xi|}{R}) \begin{pmatrix} \varepsilon\xi_{3}A(\xi_{1}-\varepsilon F^{2}\frac{AF^{2}|\xi|^{2}}{|\xi|_{F}^{2}}\xi_{2}) \\ \varepsilon\xi_{3}A(\xi_{2}+\varepsilon F^{2}\frac{AF^{2}|\xi|^{2}}{|\xi|_{F}^{2}}\xi_{1}) \\ -\varepsilon A(\xi_{1}^{2}+\xi_{2}^{2}) \\ F\varepsilon^{2}|\xi|^{2}A^{2}\frac{(\xi_{1}^{2}+\xi_{2}^{2})}{|\xi|_{F}^{2}} \end{pmatrix} \end{split}$$

And we use the arguments given in the end of section 2.4.3 to point out that in this term there are no negative powers of ε occurrence and that the denominators having $(\xi_1^2 + \xi_2^2)$ in their factors are compensated. So we have the following vectorial majoration

$$|\mathcal{P}\mathbb{P}_2 U_{\varepsilon,osc,R,r}(\xi)| \le \varepsilon c_{r,R,F} |U_{\varepsilon,osc}(\xi)|$$

Then, the Bernstein, Plancherel lemmas and the Leray majoration give:

 $\|\mathcal{P}\mathbb{P}_{2}U_{\varepsilon,osc,R,r}\|_{L^{4}([0,T],L^{\infty}(\mathbb{R}^{3}))} \leq \varepsilon c_{r,R,F}R^{\frac{3}{2}}\|U_{0}\|_{L^{2}(\mathbb{R}^{3})}T^{\frac{1}{4}}.$

Finally, as r and R are fixed, we can make ε small enough to obtain that the norm of the third truncature is less than $\frac{\eta}{3}$.

Then, an interpolation argument achieves the proof of Theorem 2.1.2.

Remark 2.4.9 The use of projector \mathcal{P} is essential to make this ε appear allowing us to conclude.

Let us end this part by a remark about the case when viscosities are equal:

Remark 2.4.10 If $\nu = \nu'$ the previous computations are far more simple and we obtain as eigenvalues $-\nu|\xi|^2$, $-\nu|\xi|^2 \pm \frac{i|\xi|_F}{\varepsilon F|\xi|}$, but the matrix is no longer diagonalizable. It is not a problem because the three eigenvectors corresponding to these eigen values are still a basis of the orthogonal space of $(\xi_1, \xi_2, \xi_3, 0)$ which implies that the system is still diagonalizable in the subspace of divergence-free vectors. Then, the majorations are done the same way, and so are the Strichartz estimates, but computations are easier because now the eigenvectors are orthogonal, so we have $\mathcal{PP} = \mathbb{P}_{3+4}$ and $\mathcal{QP} = \mathbb{P}_2$.

I.e we can see that when the viscosities are different everything goes as in the case of equal viscosities, with an error of the order of ε , provided that the frequencies are localized in $C_{r,R}$.

In [2] this particular case is studied with more regularity on the initial data.

2.5 The quasigeostrophic part, proof of Theorem 2.1.3

2.5.1 Extraction and limit system

In this small section, the argument is classical, so we will not give too many details.

After having proved the convergence to zero of the oscillating part we will now prove Theorem 2.1.3 concerning the quasigeostrophic part, which satisfies system (2.3.7).

In Section 2.2 we obtained in a formal way that the "formal limit" should satisfy system (2.1.3) (very close to (2.3.7)).

As the energy estimates provided by the Leray theorem give that the family $(U_{\varepsilon})_{\varepsilon}$ is bounded in $L^{\infty}(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ by $||U_0||_{L^2}$, and as we know that $U_{\varepsilon,QG} = \mathcal{Q}U_{\varepsilon}$ with \mathcal{Q} homogenous pseudo-differential operator of order zero, the family of the quasigeostrophic parts $(U_{\varepsilon,QG})_{\varepsilon}$ is also bounded. Thus we can extract, by weak compacity arguments, a subsequence $(U_{\varepsilon',QG})_{\varepsilon'}$ that weakly converges towards a limit that we will note $\widetilde{U_{QG}}$, and this convergence occurs in the space $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$, which implies that it also occurs in $\mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{R}^3)$.

Moreover, as the oscillating part $U_{osc,\varepsilon}$ goes to zero in the space $L^2_{loc}(\mathbb{R}_+, L^q(\mathbb{R}^3))$, we also have that $U_{\varepsilon',osc}$ goes strongly to zero, and that $U_{\varepsilon'}$ converges to U_{QG} in the sense of distributions.

So, in the equation:

$$\begin{array}{l} (QG'_{\varepsilon}) \\ \partial_t U_{\varepsilon',QG} - \Gamma U_{\varepsilon',QG} \end{array} = \begin{pmatrix} -\partial_2 \Delta_F^{-1} \\ \partial_1 \Delta_F^{-1} \\ 0 \\ -F \partial_3 \Delta_F^{-1} \end{pmatrix} \left(-v_{\varepsilon'} \cdot \nabla \Omega_{\varepsilon'} + q(U_{\varepsilon',osc}, U_{\varepsilon'}) + (\nu - \nu') F \Delta \partial_3 \theta_{\varepsilon',osc} \right)$$

The following terms have a limit in the sense of distributions (and so do all their derivatives), in \mathcal{D}' :

$$\begin{cases} \partial_t U_{\varepsilon',QG} \longrightarrow \partial_t \widetilde{U_{QG}}, \\ \Gamma U_{\varepsilon',QG} \longrightarrow \Gamma \widetilde{U_{QG}}, \\ q(U_{\varepsilon',osc}, U_{\varepsilon'}) \longrightarrow q(0, \widetilde{U_{QG}}) = 0, \\ (\nu - \nu') F \Delta \partial_3 \theta_{\varepsilon',osc} \longrightarrow 0. \end{cases}$$

Only $v_{\varepsilon'} \cdot \nabla \Omega_{\varepsilon'}$ is annoying. The aim of this section will be to check that this term effectively converges to $\widetilde{v_{QG}} \cdot \nabla \widetilde{\Omega_{QG}}$ in the sense of distributions: $\forall \phi \in \mathcal{D}(\mathbb{R}^*_+ \times \mathbb{R}^3)$, when ε' goes to zero:

$$\int \left((v_{\varepsilon'} \cdot \nabla \Omega_{\varepsilon'}) \phi \right)(t, x) dt dx \longrightarrow \int \left((\widetilde{v_{QG}} \cdot \nabla \widetilde{\Omega_{QG}}) \phi \right)(t, x) dt dx$$

The argument used in the proof of the Leray theorem, which uses weak compacity, the Ascoli theorem and a diagonal extraction allows us to show that we really can go to the limit in the sense of distributions in this term, and finally, in every term of system (2.3.7), so we have:

$$\partial_t \widetilde{U_{QG}} - \Gamma \widetilde{U_{QG}} = \begin{pmatrix} -\partial_2 \Delta_F^{-1} \\ \partial_1 \Delta_F^{-1} \\ 0 \\ -F \partial_3 \Delta_F^{-1} \end{pmatrix} \left(-\widetilde{v_{QG}} \cdot \nabla \widetilde{\Omega_{QG}} \right)$$

We have shown that there exists a subsequence of U_{ε} that weakly converges towards a solution (in the sense of distributions) of the system obtained in the first section.

2.5.2 Strong convergence

Once that we have obtained that the limit of the subsequence $U_{\varepsilon'',QG}$ is a solution of system (\widetilde{QG}) , we can go back to the results given by the Ascoli theorem: for all $T_N, N > 0$, and for all function χ_N such that:

$$\begin{cases} \chi_N \in \mathcal{D}(\mathbb{R}^3) \\ supp\chi_N \subset B(0, N+1) \\ \chi_N \equiv 1 \text{ near } B(0, N), \end{cases}$$

then $(\chi_N U_{\varepsilon'',QG})_{\varepsilon'}$ converges in $L^{\infty}([0,T_N], H^{-1}(B(0,N+1)))$ towards $\chi_N \widetilde{U_{QG}}$.

As χ_N equals 1 near B(0, N), $(U_{\varepsilon'',QG})_{\varepsilon'}$ converges in $L^{\infty}([0, T_N], H^{-1}(B(0, N)))$ towards $\widetilde{U_{QG}}$. Using Leray's energy estimates and the fact that T_N is finite, we obtain, with an interpolation that:

$$\lim_{\varepsilon''\to 0} U_{\varepsilon'',QG} = \widetilde{U_{QG}}.$$

This convergence occurs in the space $L^{\frac{4}{2-\eta}}([0,T_N], H^{1-\eta}(B(0,N)))$. Thus, using the Sobolev embedding, it also occurs in $L^{\frac{4}{2-\eta}}([0,T_N], L^{\frac{6}{1+2\eta}}(B(0,N)))$. Taking η in the segment $]0, 1[, \frac{4}{2-\eta} \in]2, 4[$ and $\frac{6}{1+2\eta} \in]2, 6[$. Hölder's inequality leads to L^2 in time, so the convergence occurs in the space $L^2([0,T_N], L^q(B(0,N)))$ for all $q \in]2, 6[$. and this holds for all $T_N, N > 0$.

Thus, for all $q \in]2, 6[$,

$$\lim_{\varepsilon''\to 0} U_{\varepsilon'',QG} = \widetilde{U_{QG}} \text{ in the space } L^2_{loc}(\mathbb{R}_+, L^q_{loc}(\mathbb{R}^3)).$$

Finally, the behaviour of the oscillating part allows to write:

$$\lim_{\varepsilon''\to 0} U_{\varepsilon''} = U_{QG} \text{ in the space } L^2_{loc}(\mathbb{R}_+, L^q_{loc}(\mathbb{R}^3)).$$

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Chapitre 3

Solutions fortes

Résumé: On s'intéresse dans ce chapitre¹ au comportement des solutions fortes du système primitif lorsque le nombre de Rossby tend vers zéro. On montre que, sous des hypothèses de régularité suffisantes, lorsque le nombre de Rossby est assez petit, la solution du système primitif est unique et globale, et qu'elle converge fortement vers la solution du système quasigéostrophique, elle aussi unique et globale, ceci sans aucune hypothèse de petitesse des données initiales.

 $^{^{1}}$ Les résultats présentés dans ce chapitre sont publiés dans la revue "Communications in Partial Differential Equations", volume 29 (11 & 12), 2004.
3.1 Introduction

In this paper we will consider the primitive system (or primitive equations):

$$(PE_{\varepsilon}) \qquad \begin{cases} \partial_t U_{\varepsilon} + v_{\varepsilon} \cdot \nabla U_{\varepsilon} - LU_{\varepsilon} + \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon} = \frac{1}{\varepsilon} (-\nabla \Phi_{\varepsilon}, 0) \\ \operatorname{div} v_{\varepsilon} = 0 \\ U_{\varepsilon/t=0} = U_0. \end{cases}$$

The unknowns are U_{ε} and Φ_{ε} . We denote by U_{ε} a pair $(v_{\varepsilon}, \theta_{\varepsilon})$ where v_{ε} is a vector field on \mathbb{R}^3 (three dimensional velocity), θ_{ε} a scalar function (the density fluctuation : in the case of the atmosphere it depends on the scalar (potential) temperature and in the case of the ocean it depends on the temperature and the salinity), and Φ_{ε} the pressure, all of them depending on (t, x).

The operator L is defined by

$$LU_{\varepsilon} \stackrel{\text{def}}{=} (\nu \Delta v_{\varepsilon}, \nu' \Delta \theta_{\varepsilon})$$

and \mathcal{A} by:

$$\mathcal{A} \stackrel{\text{def}}{=} \left(\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{array} \right).$$

This system is obtained by combining the effects of the Coriolis force and those of the vertical stratification induced by the Boussinesq approximation (one can see [8] as well as [1], about rotating fluids). The coefficient $\varepsilon > 0$ denotes the Rossby number, $\nu > 0$ is the viscosity and $\nu' > 0$ the heat diffusivity (which we will also call a viscosity in the following).

As the characterisic displacement of a particle in the ocean within a day is very small compared to the displacement caused by the rotation of the earth, the Rossby number is supposed to be small, about 10^{-1} to 10^{-3} , and we focus on the limit of a strong rotation (ε goes to zero).

The coefficient F is called the Froude number. We refer to [7] for a study of the case F = 1, to [12] for the periodic case, and to [4] for the case $\nu \neq \nu'$, $F \neq 1$ (convergence of Leray solution). Let us also refer to [2] for the case $\nu = \nu'$ and $F \neq 1$.

We refer to [3], [9], [10], [13], [15], [17], and [18] for a discussion on this model, and its derivation.

The fact that the parameter ε goes to zero gives high importance to the term $\mathcal{A}U$, which is said to be penalized. But the term $\frac{1}{\varepsilon}$ doesn't play any role in the energy estimates thanks to the skew-symmetry of \mathcal{A} .

Definition 3.1.1 If s is a real number, the homogenous (resp. inhomogenous) Sobolev space of order s, which we will denote by \dot{H}^s (resp. H^s), is defined as the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^3)$ whose Fourier transform \hat{u} is locally integrable and has the following property:

$$\|u\|_{\dot{H}^{s}}^{2} \stackrel{\text{def}}{=} \int_{\mathbb{R}^{3}} |\xi|^{2s} |\hat{u}(\xi)|^{2} d\xi < \infty \quad (\text{resp.} \quad \|u\|_{H^{s}}^{2} \stackrel{\text{def}}{=} \int_{\mathbb{R}^{3}} (1+|\xi|^{2})^{s} |\hat{u}(\xi)|^{2} d\xi < \infty).$$

Remark 3.1.1 For more generality one can add an external force to (PE_{ε}) belonging to the space $L^2(\mathbb{R}_+, L^2)$, but with a wish of simplification we will abstrain from it.

The Leray theorem (see [16]) is true for the primitive system: if the initial data $U_0 \in L^2(\mathbb{R}^3)$, then there exists for all $\varepsilon > 0$ a Leray solution of the system $(PE_{\varepsilon}), U_{\varepsilon}$, globally defined in time, belonging to $L^{\infty}(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and satisfying the following energy inequality (let $\nu_0 = \min(\nu, \nu') > 0$):

$$\forall t \in \mathbb{R}_+, \quad \|U_{\varepsilon}(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\nu_0 \int_0^t \|\nabla U_{\varepsilon}(t)\|_{L^2(\mathbb{R}^3)}^2 dt \le \|U_0\|_{L^2(\mathbb{R}^3)}^2.$$

We refer to [4] where we studied the limit of Leray solutions when ε , the Rossby number, goes to zero and introduced the following notations and results in the case of weak solutions: the potential vorticity is defined by

$$\Omega_{\varepsilon} \stackrel{\text{def}}{=} \partial_1 v_{\varepsilon}^2 - \partial_2 v_{\varepsilon}^1 - F \partial_3 \theta_{\varepsilon}.$$

Then from this, we define the quasigeostrophic part of U_{ε} , which is very close to the solution of the limit system (obtained by a formal method in the beginning of [4] and given in (\widetilde{QG}) below, called system of the quasigeostrophic equations):

$$U_{\varepsilon,QG} \stackrel{\text{def}}{=} \begin{pmatrix} -\partial_2 \Delta_F^{-1} \Omega_{\varepsilon} \\ \partial_1 \Delta_F^{-1} \Omega_{\varepsilon} \\ 0 \\ -F \partial_3 \Delta_F^{-1} \Omega_{\varepsilon} \end{pmatrix}, \quad \text{with} \quad \Delta_F \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2 + F^2 \partial_3^2.$$

The other part is called the oscillating part and has a very different behaviour:

$$U_{\varepsilon,osc} \stackrel{\text{def}}{=} U_{\varepsilon} - U_{\varepsilon,QG} = \begin{pmatrix} v_{\varepsilon}^{1} + \partial_{2}\Delta_{F}^{-1}\Omega_{\varepsilon} \\ v_{\varepsilon}^{2} - \partial_{1}\Delta_{F}^{-1}\Omega_{\varepsilon} \\ v_{\varepsilon}^{3} \\ \theta_{\varepsilon} + F\partial_{3}\Delta_{F}^{-1}\Omega_{\varepsilon} \end{pmatrix}$$

It is important to look at this decomposition in terms of orthogonal projections: if we denote by \mathcal{P} the orthogonal projector onto the potential vorticity free vector fields (which is built the same way as the orthogonal projector \mathbb{P} on divergence free vector fields, also called the Leray projector) and $\mathcal{Q} = Id - \mathcal{P}$, then in the Fourier variable, these two operators are written:

$$\widehat{\mathcal{Q}U} = \frac{1}{|\xi|_F^2} \begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 & 0 & F\xi_2\xi_3 \\ -\xi_1\xi_2 & \xi_1^2 & 0 & -F\xi_1\xi_3 \\ 0 & 0 & 0 & 0 \\ F\xi_2\xi_3 & -F\xi_1\xi_3 & 0 & F^2\xi_3^2 \end{pmatrix} \widehat{U}$$
(3.1.1)
$$\widehat{\mathcal{P}U} = \frac{1}{|\xi|_F^2} \begin{pmatrix} \xi_1^2 + F^2\xi_3^2 & \xi_1\xi_2 & 0 & -F\xi_2\xi_3 \\ \xi_1\xi_2 & \xi_2^2 + F^2\xi_3^2 & 0 & F\xi_1\xi_3 \\ 0 & 0 & |\xi|_F^2 & 0 \\ -F\xi_2\xi_3 & F\xi_1\xi_3 & 0 & \xi_1^2 + \xi_2^2 \end{pmatrix} \widehat{U},$$

where $|\xi|_F^2 \stackrel{\text{def}}{=} \xi_1^2 + \xi_2^2 + F^2 \xi_3^2$.

So this decomposition $U_{\varepsilon} = U_{\varepsilon,QG} + U_{\varepsilon,osc}$ where $U_{\varepsilon,osc} = \mathcal{P}U_{\varepsilon}$ and $U_{\varepsilon,QG} = \mathcal{Q}U_{\varepsilon}$ involves two homogeneous pseudo differential operators of degree zero. We also have the following obvious properties:

$$\mathcal{P}U = U \Leftrightarrow \Omega(U) = 0, \quad \text{and}$$
(3.1.2)
$$\mathcal{Q}U = U \Leftrightarrow \mathcal{P}U = 0 \Leftrightarrow U \text{ is of the form } (\partial_2 \Phi, -\partial_1 \Phi, 0, F \partial_3 \Phi).$$

The main result in [4] is the determination of the asymptotic behaviour of the Leray solution when ε goes to zero:

Theorem 3.1.1 [4] If the initial data U_0 belongs to $L^2(\mathbb{R}^3)$ and if we consider a sequence of Leray solutions (U_{ε}) then, there exists a subsequence that converges for all $q \in]2, 6[$ in the space $L^2_{loc}(\mathbb{R}_+, L^q_{loc}(\mathbb{R}^3))$ to a function $\widetilde{U_{QG}}$ which satisfies $\mathcal{Q}\widetilde{U_{QG}} = \widetilde{U_{QG}}$ and solution of the system:

$$(\widetilde{QG}) \qquad \begin{cases} \partial_t \widetilde{U_{QG}} - \Gamma \widetilde{U_{QG}} = -\begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F\partial_3 \end{pmatrix} \Delta_F^{-1} (\widetilde{U_{QG}} \cdot \nabla \widetilde{\Omega_{QG}}) \\ \widetilde{U_{QG}}_{/t=0} = U_{0,QG} = \mathcal{Q}(U_0) \in L^2(\mathbb{R}^3), \end{cases}$$

where $U.\nabla U$ is a simplified notation for $v.\nabla U$ if $U = (v, \theta)$, and where Γ is the operator of order two defined by

$$\Gamma \stackrel{\text{def}}{=} \Delta \Delta_F^{-1} (\nu \partial_1^2 + \nu \partial_2^2 + \nu' F^2 \partial_3^2). \tag{3.1.3}$$

We can easily show that if U_{QG} is a vector field such that $\mathcal{Q}U_{QG} = U_{QG}$, U_{QG} . $\nabla \Omega_{QG} = \Omega(U_{QG}, \nabla U_{QG})$ which easily allows us to rewrite system (\widetilde{QG}) into:

$$\begin{cases} \partial_t \widetilde{U_{QG}} - \Gamma \widetilde{U_{QG}} + \mathcal{Q}(\widetilde{U_{QG}}, \nabla \widetilde{U_{QG}}) = 0\\ \widetilde{U_{QG}}_{/t=0} = U_{0,QG}. \end{cases}$$
(3.1.4)

Remark 3.1.2 We can notice that $\Gamma \widetilde{U_{QG}} = \mathcal{Q} L \widetilde{U_{QG}}$, and that the projection of the advection term $\mathbb{P}\mathcal{P}(\widetilde{U_{QG}}, \nabla \widetilde{\Omega_{QG}})$ has a zero potential vorticity.

In this paper we will focus on the case of a more regular initial data: $U_0 \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $U_{0,QG} \in L^2(\mathbb{R}^3)$, with the aim of studying solutions which are unique.

Even if there is no scale invariance for this system, it is very natural to choose initial data in $\dot{H}^{\frac{1}{2}}$. Indeed such a space is scale invariant for the three dimensional Navier-Stokes equations and as \mathcal{A} is skewsymmetric and vanishes from any energy estimates we can, as for the Leray theorem, adapt the Fujita-Kato theorem to the primitive system (see for instance [11] or [5]): there exist a unique maximal time $T_{\varepsilon}^* > 0$, and a unique solution $U_{\varepsilon} \in \mathcal{C}([0, T_{\varepsilon}^*[, \dot{H}^{\frac{1}{2}}) \cap L^2_{loc}([0, T_{\varepsilon}^*[, \dot{H}^{\frac{3}{2}}))$. Moreover, if T_{ε}^* is finite, then we have $\int_0^{T_{\varepsilon}^*} \|U_{\varepsilon}(t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)}^2 dt = +\infty$; finally there exists a constant c such that if $\|U_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq c\nu_0$

then $T_{\varepsilon}^* = +\infty$. Contrary to the Leray solutions, the solutions are unique but we do not know whether they are global in general. The Fujita-Kato theorem also works on system (\widetilde{QG}) , and again, it does not say whether the unique solution is global if we do not have a small initial data.

In the following, we show like in [12] or [10] (the only difference is that we are in the whole space and not in the periodic case, but that does not change anything in the arguments) that the quasigeostrophic system has a unique and global solution without any assumption on the smallness of the initial data:

Theorem 3.1.2 Assume $U_{0,QG} \in H^1(\mathbb{R}^3)$, then the limit system (\widetilde{QG}) has a unique solution, global in time in the space $L^{\infty}(\mathbb{R}_+, H^1) \cap L^2(\mathbb{R}_+, H^2)$.

Note that $U_0 \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $U_{0,QG} \in L^2(\mathbb{R}^3)$ imply $U_{0,QG} \in H^1(\mathbb{R}^3)$. The global lifespan holds thanks to the special form of the system satisfied by the potential vorticity $\widetilde{\Omega_{QG}}$, where like in 2-D Navier-Stokes, and unlike in 3-D Navier-Stokes, there is no stretching term such as $\Omega.\nabla U$.

The main point of this paper is to show, using dispersion phenomena induced by the operator $-L + \frac{1}{\varepsilon} \mathcal{A}$, that if ε is small enough, then system (PE_{ε}) has a unique solution, global in time, and which converges to $\widetilde{U_{QG}}$ in a certain sense, as ε goes to zero. This is the purpose of:

Theorem 3.1.3 Assume that $U_0 \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $U_{0,QG} \in L^2(\mathbb{R}^3)$. Let us define for $s \in \mathbb{R}$, $E^s \stackrel{\text{def}}{=} L^{\infty}(\mathbb{R}_+, \dot{H}^s) \cap L^2(\mathbb{R}_+, \dot{H}^{s+1})$ and let W_{ε} be a solution of the following linear system:

$$\begin{cases} \partial_t W_{\varepsilon} - LW_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_{\varepsilon} = -G \\ W_{\varepsilon/t=0} = U_{0,osc} = \mathcal{P}(U_0) \end{cases}$$
(3.1.5)

with
$$G \stackrel{\text{def}}{=} \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}, \nabla \widetilde{U_{QG}}) - F(\nu - \nu')\Delta\Delta_F^{-2} \begin{pmatrix} -F\partial_2\partial_3^2 \\ F\partial_1\partial_3^2 \\ 0 \\ (\partial_1^2 + \partial_2^2)\partial_3 \end{pmatrix} \widetilde{\Omega_{QG}}.$$
 (3.1.6)

Then we have the following results:

- W_{ε} exists globally and is unique in the space E^s for every $s \in [\frac{1}{2}, 1]$.
- Moreover $||W_{\varepsilon}||_{L^2(\mathbb{R}_+,L^{\infty})} \to 0$ as $\varepsilon \to 0$.
- If we denote by $\gamma_{\varepsilon} \stackrel{\text{def}}{=} U_{\varepsilon} \widetilde{U_{QG}} W_{\varepsilon}$, then if ε is small enough, $\gamma_{\varepsilon} \in E^s$ and converges to zero in this space E^s for every $s \in [\frac{1}{2}, 1]$.
- Finally if ε is small enough U_{ε} is defined for all time in E^s and $U_{\varepsilon} \widetilde{U_{QG}} = \gamma_{\varepsilon} + W_{\varepsilon}$ goes to zero as ε goes to zero, in the space $L^2(\mathbb{R}_+, L^{\infty})$.

This paper is structured in the following way: Section 3.2 will be devoted to the quasigeostrophic system (\widetilde{QG}) (Theorem 3.1.2), and in Section 3.3 we will prove Theorem 3.1.3. We will use many results or computations developped in [4], and some technical results are given in an appendix.

3.2 The limit system (\widetilde{QG}) , proof of Theorem 3.1.2

The following method is the same as for the periodic case ([12] or [10]) so we will explain it without giving many details. As we have seen we can study equivalently (\widetilde{QG}) or (3.1.4).

As the initial data of (3.1.4) is in particular in L^2 , the Leray theorem gives a (global) Leray solution. As the initial data also belongs to $\dot{H}^{\frac{1}{2}}$ the Fujita-Kato theorem is easily adapted to this system and gives the existence and uniqueness of a maximal time $T^* > 0$ and of a unique solution $\widetilde{U}_{QG} \in \mathcal{C}([0, T^*[, \dot{H}^{\frac{1}{2}}) \cap L^2_{loc}([0, T^*[, \dot{H}^{\frac{3}{2}}).$

Classical weak-strong unicity theorem (one can see for instance [19]) tells us that on this interval of time this unique solution coincides with every Leray solution (which are therefore unique on that time interval). Finally, as the initial data is in \dot{H}^1 classical regularity propagation arguments give that the solution \widetilde{U}_{QG} is also in $\mathcal{C}([0, T^*[, \dot{H}^1) \cap L^2_{loc}([0, T^*[, \dot{H}^2)$. So our solution is in $\mathcal{C}([0, T^*[, H^1) \cap L^2_{loc}([0, T^*[, H^2)$.

In our case, the \dot{H}^1 regularity of the initial data will give another result: combined with the special form of the equation satisfied by the potential vorticity, it will allow us to show that the solution is global and then prove Theorem 3.1.2.

After a quick computation the potential vorticity satisfies the equation (see (3.1.3) for the definition of Γ):

$$\partial_t \widetilde{\Omega_{QG}} - \Gamma \widetilde{\Omega_{QG}} + \widetilde{U_{QG}} \cdot \nabla \widetilde{\Omega_{QG}} = 0.$$
(3.2.7)

The regularity of $\widetilde{U_{QG}}$ gives us that $\widetilde{\Omega_{QG}} \in \mathcal{C}([0, T^*[, L^2) \cap L^2_{loc}([0, T^*[, \dot{H}^1), \text{ and this regularity is enough for us to compute a scalar product in <math>L^2$ ($\nu_0 = \min(\nu, \nu') > 0$):

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\Omega_{QG}}\|_{L^2}^2 + c\nu_0\|\nabla\widetilde{\Omega_{QG}}\|_{L^2}^2 + (\widetilde{U_{QG}}\cdot\nabla\widetilde{\Omega_{QG}}|\widetilde{\Omega_{QG}})_{L^2} = 0.$$

The regularity is such that we can effectively define the last term, and according to the fact that the velocity is divergence free, this term is zero. Thus, integrating from zero to t we obtain that for all $t < T^*$,

$$\|\widetilde{\Omega_{QG}}(t)\|_{L^2}^2 + 2c\nu_0 \int_0^t \|\nabla\widetilde{\Omega_{QG}}(s)\|_{L^2}^2 ds = \|\widetilde{\Omega_{QG}}(0)\|_{L^2}^2 \le C' \|U_0\|_{\dot{H}^1}^2.$$

This latter inequality is obtained by using the expression of the vorticity in terms of the first derivatives of the coordinates of the velocity.

On the other hand, using that

$$\widetilde{U_{QG}} = \begin{pmatrix} -\partial_2 \Delta_F^{-1} \\ \partial_1 \Delta_F^{-1} \\ 0 \\ -F \partial_3 \Delta_F^{-1} \end{pmatrix} \widetilde{\Omega_{QG}},$$

we get for every $t < T^*$,

$$\|\widetilde{U_{QG}}\|_{\dot{H}^1}^2 \le C \|\widetilde{\Omega_{QG}}\|_{L^2}^2$$
 and $\|\widetilde{U_{QG}}\|_{\dot{H}^2}^2 \le C \|\widetilde{\Omega_{QG}}\|_{\dot{H}^1}^2$,

which gives that $\forall t < T^*$,

$$\int_{0}^{t} \|\widetilde{U_{QG}}(s)\|_{\dot{H}^{2}}^{2} ds \leq C \int_{0}^{t} \|\nabla \widetilde{\Omega_{QG}}(s)\|_{L^{2}}^{2} ds < \infty.$$

Then supposing that $T^* < +\infty$, the blow up condition in the Fujita-Kato theorem (see [5] or [11]) writes:

$$\int_0^{T^*} \|\nabla \widetilde{U_{QG}}(t')\|_{\dot{H}^{\frac{1}{2}}}^2 dt' = +\infty.$$

Thanks to the interpolation between Sobolev spaces we can write that for every $T < T^*$:

$$\int_0^T \|\nabla \widetilde{U_{QG}}(t')\|_{\dot{H}^{\frac{1}{2}}}^2 dt' \le C \int_0^T \|\widetilde{U_{QG}}(t')\|_{\dot{H}^1} \|\widetilde{U_{QG}}(t')\|_{\dot{H}^2} dt'.$$

Using the Hölder inequality we find

$$\int_0^T \|\nabla \widetilde{U_{QG}}(t')\|_{\dot{H}^{\frac{1}{2}}}^2 dt' \le C(\int_0^T \|\widetilde{U_{QG}}(t')\|_{\dot{H}^1}^2 dt')^{\frac{1}{2}} (\int_0^T \|\widetilde{U_{QG}}(t')\|_{\dot{H}^2}^2 dt')^{\frac{1}{2}}.$$

Then the energy inequalities given by the Leray theorem and the previous computation about the potential vorticity imply that:

$$\int_0^T \|\nabla \widetilde{U_{QG}}(t')\|_{\dot{H}^{\frac{1}{2}}}^2 dt' \le C \frac{\|U_{0,QG}\|_{L^2}}{(2\nu_0)^{\frac{1}{2}}} \frac{\|U_0\|_{\dot{H}^1}}{(2c\nu_0)^{\frac{1}{2}}} < \infty,$$

which by the usual blow-up criterion implies that (\widetilde{QG}) has a unique, global solution when $U_{0,QG} \in H^1$, moreover:

$$\forall t \in \mathbb{R}_+, \|\widetilde{U_{QG}}(t)\|_{\dot{H}^1}^2 + 2c\nu_0 \int_0^t \|\widetilde{U_{QG}}(s)\|_{\dot{H}^2}^2 ds \le C \|U_0\|_{\dot{H}^1}^2.$$
(3.2.8)

So using the following Leray estimate

$$\forall t \in \mathbb{R}_+, \|\widetilde{U_{QG}}(t)\|_{L^2}^2 + c\nu_0 \int_0^t \|\widetilde{U_{QG}}(s)\|_{\dot{H}^1}^2 ds \le C \|U_{0,QG}\|_{L^2}^2, \tag{3.2.9}$$

thanks to an interpolation argument, we obtain that there exists a constant C such that for all $s \in [0, 1]$,

$$\forall t \in \mathbb{R}_{+}, \|\widetilde{U_{QG}}(t)\|_{\dot{H}^{s}}^{2} + 2c\nu_{0}\int_{0}^{t}\|\widetilde{U_{QG}}(t')\|_{\dot{H}^{s+1}}^{2}dt' \leq C\|U_{0,QG}\|_{\dot{H}^{s}}^{2}.$$
(3.2.10)

And thanks to interpolation arguments and convexity we get

$$\|U_{0,QG}\|_{\dot{H}^{s}} \le C \|U_{0,QG}\|_{L^{2}}^{1-s} \|U_{0,QG}\|_{\dot{H}^{1}}^{s} \le C (\|U_{0,QG}\|_{L^{2}} + \|U_{0,QG}\|_{\dot{H}^{1}}).$$

So we have that

$$\forall t \in \mathbb{R}_+, \|\widetilde{U_{QG}}(t)\|_{\dot{H}^s}^2 + 2c\nu_0 \int_0^t \|\widetilde{U_{QG}}(t')\|_{\dot{H}^{s+1}}^2 dt' \le C(U_0), \tag{3.2.11}$$

and Theorem 3.1.2 is proved \blacksquare .

Remark 3.2.1 What allowed us to prove that the solution is global in time is exactly the same phenomenon that occurs when one computes the equation satisfied by the vorticity for 2-D Navier-Stokes: contrary to the case of the 3-D Navier Stokes vorticity, there is no stretching term of the form $\Omega.\nabla U$ (which complicates the energy estimate). This special form also appears in [12].

3.3 Convergence of the strong solutions

In this section we will prove the main theorem of this paper : Theorem 3.1.3. We will first reformulate the limit system and define the systems on which we will work. Then we will provide energy estimates for these auxiliary systems, and we will finally be able to conclude.

3.3.1 The different systems : strategy of proof

Let us rewrite the primitive system, using the Leray projector on the divergence-free vector fields :

$$\begin{cases} \partial_t U_{\varepsilon} + \mathbb{P}(U_{\varepsilon} \cdot \nabla U_{\varepsilon}) - LU_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} U_{\varepsilon} = 0\\ U_{\varepsilon/t=0} = U_0 \end{cases}$$
(3.3.12)

Let us begin by noticing that (\widetilde{QG}) is equivalent to the following system:

$$\begin{cases} \partial_t \widetilde{U_{QG}} - L\widetilde{U_{QG}} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \widetilde{U_{QG}} = -\mathbb{P}(\widetilde{U_{QG}} \cdot \nabla \widetilde{U_{QG}}) + G \\ \widetilde{U_{QG}}_{/t=0} = U_{0,QG} \in \dot{H}^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3), \end{cases}$$
(3.3.13)

where
$$G = \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}, \nabla \widetilde{U_{QG}}) - F(\nu - \nu')\Delta\Delta_F^{-2} \begin{pmatrix} -F\partial_2\partial_3^2 \\ F\partial_1\partial_3^2 \\ 0 \\ (\partial_1^2 + \partial_2^2)\partial_3 \end{pmatrix} \widetilde{\Omega_{QG}}.$$
 (3.3.14)

Indeed, let us write that:

$$\partial_t \widetilde{U_{QG}} - L\widetilde{U_{QG}} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \widetilde{U_{QG}} = (\partial_t \widetilde{U_{QG}} - \Gamma \widetilde{U_{QG}}) + (\Gamma \widetilde{U_{QG}} - L\widetilde{U_{QG}}) + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \widetilde{U_{QG}}$$
$$= -\mathcal{Q} (\widetilde{U_{QG}} \cdot \nabla \widetilde{U_{QG}}) + (\Gamma - L) \widetilde{U_{QG}} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \widetilde{U_{QG}}.$$

As we have (see chapter 2, Section 2.3):

•
$$\mathcal{A}U_{QG} = -(\nabla \Delta_F^{-1} \Omega_{QG}, 0)$$
 which is in the kernel of \mathbb{P} ,

•
$$(\Gamma - L)\widetilde{U_{QG}} = -iF(\nu - \nu')\frac{|\xi|^2}{|\xi|_F^4} \begin{pmatrix} -F\xi_2\xi_3^2 \\ F\xi_1\xi_3^2 \\ 0 \\ (\xi_1^2 + \xi_2^2)\xi_3 \end{pmatrix} \widehat{\Omega_{QG}},$$

• and $-\mathcal{Q}(\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}}) = -\mathbb{P}\mathcal{Q}(\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}}) = -\mathbb{P}(Id - \mathcal{P})(\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}})$
 $= -\mathbb{P}(\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}}) + \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}})$

It is important to notice that G is the sum of two terms, both divergence-free and whose potential vorticity is zero (it is an application of remark 3.1.2 for the first term, and a simple computation for the second one), which will be important in the following.

As in [8] for rotating fluids, we realize that the study of the simple difference $V_{\varepsilon} \stackrel{\text{def}}{=} U_{\varepsilon} - \widetilde{U_{QG}}$ will not give any clue for the convergence. Indeed, on the one hand if we compute

the difference of the two systems we obtain a new system that has a fixed external force which prevents us from getting any convergence to zero by Gronwall methods, and on the other hand, we need truncations in frequency space to use dispersion results.

So in the spirit of [8] we will use the solution of the linear oscillating system in order to make the term G oscillate, in the sense that we consider the linearized system, with the term G as an external force. And, in order to use the Strichartz estimates proved in the appendix we are led to introducing the following systems :

$$\begin{cases} \partial_t W_{\varepsilon} - L W_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_{\varepsilon} = -G \\ W_{\varepsilon/t=0} = U_{0,osc} \end{cases}$$
(3.3.15)

$$\begin{cases} \partial_t W_{\varepsilon}^{r,R} - L W_{\varepsilon}^{r,R} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_{\varepsilon}^{r,R} = -\mathcal{P}_{r,R} \mathbb{P}_{3+4} G\\ W_{\varepsilon}^{r,R}_{\varepsilon/t=0} = \mathcal{P}_{r,R} \mathbb{P}_{3+4} U_{0,osc}, \end{cases}$$
(3.3.16)

where \mathbb{P}_{3+4} is the projection onto the last two eigenvectors of the matrix $L - \frac{1}{\varepsilon} \mathbb{P}A$ (see in the appendix), and where $\mathcal{P}_{r,R}$ is a frequency cut-off :

$$\mathcal{P}_{r,R} = \chi(\frac{|D|}{R})(1 - \chi(\frac{|D_3|}{r})), \qquad (3.3.17)$$

with χ a fixed \mathcal{C}^{∞} function whose support is included in [-1,1] and equal to 1 in $[-\frac{1}{2},\frac{1}{2}]$, and $(\mathcal{F}^{-1}$ is the inverse Fourier transform): $\chi(|D|)f = \mathcal{F}^{-1}(\chi(|\xi|)\hat{f}(\xi))$ and $\chi(|D_3|)f = \mathcal{F}^{-1}(\chi(|\xi_3|)\hat{f}(\xi))$.

As we will see in the following, with this external force, when we compute the equation satisfied by $U_{\varepsilon} - \widetilde{U_{QG}} - W_{\varepsilon}^{r,R}$, the new external force will be $G - \mathcal{P}_{r,R}\mathbb{P}_{3+4}G$ which we will be able to make arbitrary small. Then we define the two following quantities :

$$\begin{cases} \delta_{\varepsilon} \stackrel{\text{def}}{=} U_{\varepsilon} - \widetilde{U_{QG}} - W_{\varepsilon}^{r,R} \\ \delta_{\varepsilon}' \stackrel{\text{def}}{=} W_{\varepsilon} - W_{\varepsilon}^{r,R}. \end{cases}$$
(3.3.18)

In the rest of this paper we will state and prove energy estimates for (3.3.15) and (3.3.16), then we will use them to bound the energies of δ_{ε}' and δ_{ε} , and show they are small. It is here that we see the reason to use system (3.3.16): truncation of the frequencies in $C_{r,R}$ allows us not only to use the estimates from [4] and the Strichartz estimates, but also to control the initial data and the external force in the energy estimates of δ_{ε} and δ_{ε}' : once rand R are fixed to make these two terms small enough (smaller than an arbitrary η), we can show thanks to the ε given by the Strichartz estimates that these two energy estimates are also smaller than 2η .

So γ_{ε} , which is the difference between δ_{ε} and δ'_{ε} , is finally smaller than 4η , for ε small enough (depending on r and R).

We will next deduce that W_{ε} goes to zero in $L^2(\mathbb{R}_+, L^{\infty})$, and using the Sobolev injection $\dot{H}^2(\mathbb{R}^3) \hookrightarrow L^{\infty}(\mathbb{R}^3)$ we will finally get that $U_{\varepsilon} - \widetilde{U_{QG}}$ goes to zero in the space $L^2(\mathbb{R}_+, L^{\infty})$.

3.3.2 Energy estimates for the linear systems

Until the end of the paper we will use the following notation for constants: a constant which only depends on U_0 , ν_0 or $\nu - \nu'$ shall be written K^0 , and if it also depends on r

and R it shall be written $K^{r,R}$.

Using the fact that $\widetilde{U_{QG}}$ is globally defined and belongs to $L^{\infty}(\mathbb{R}_+, H^1) \cap L^2(\mathbb{R}_+, H^2)$ (recall Theorem 3.1.2), we shall show that systems (3.3.15) and (3.3.16) have a unique global solution in $E^s = L^{\infty}(\mathbb{R}_+, \dot{H}^s) \cap L^2(\mathbb{R}_+, \dot{H}^{s+1})$ for all $s \in [\frac{1}{2}, 1]$.

From now on we take $U_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ and require that its quasigeostrophic part $U_{0,QG}$ belongs to $L^2(\mathbb{R}^3)$.

Lemma 3.3.1 There exists a constant K^0 such that, for all $s \in [\frac{1}{2}, 1]$ and $t \in \mathbb{R}_+$, the solutions of (3.3.15) and (3.3.16) satisfy respectively (with $\nu_0 = \min(\nu, \nu') > 0$):

$$\begin{split} \|W_{\varepsilon}^{r,R}(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla W_{\varepsilon}^{r,R}(t')\|_{\dot{H}^{s}}^{2} dt' &\leq K^{0}(1 + |\nu - \nu'|\varepsilon K^{r,R})^{2} \\ and \quad \|W_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla W_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} dt' &\leq K^{0}. \end{split}$$

Proof: We will prove the first estimate (the other one is proved using the very same computation and is in fact simpler). Taking the inner product (in \dot{H}^s) of (3.3.16) with $W_{\varepsilon}^{r,R}$ and using Lemma 3.4.6, we get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{2} + \nu_{0} \|\nabla W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{2} &\leq (1 + |\nu - \nu'|\varepsilon K^{r,R}) \|G^{b}\|_{\dot{H}^{s}} \|W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}} \\ &+ (1 + |\nu - \nu'|\varepsilon K^{r,R}) \|G^{l}\|_{\dot{H}^{s-1}} \|W_{\varepsilon}^{r,R}\|_{\dot{H}^{s+1}}, \end{aligned}$$

where we have defined

$$G^{b} = \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}, \nabla\widetilde{U_{QG}}) \quad \text{and} \quad G^{l} = -F(\nu - \nu')\Delta\Delta_{F}^{-2} \begin{pmatrix} -F\partial_{2}\partial_{3}^{2} \\ F\partial_{1}\partial_{3}^{2} \\ 0 \\ (\partial_{1}^{2} + \partial_{2}^{2})\partial_{3} \end{pmatrix} \widetilde{\Omega_{QG}}.$$
 (3.3.19)

Using the classical inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{2} + \nu_{0} \|\nabla W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{2} &\leq \frac{1}{2} (1 + |\nu - \nu'|\varepsilon K^{r,R})^{2} \|G^{b}\|_{\dot{H}^{s}} + \frac{1}{2} \|G^{b}\|_{\dot{H}^{s}} \|W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{2} \\ &+ \frac{1}{2\nu_{0}} (1 + |\nu - \nu'|\varepsilon K^{r,R})^{2} \|G^{l}\|_{\dot{H}^{s-1}}^{2} + \frac{\nu_{0}}{2} \|W_{\varepsilon}^{r,R}\|_{\dot{H}^{s+1}}^{2}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dt} \|W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{2} &- \|G^{b}\|_{\dot{H}^{s}} \|W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{2} + \nu_{0} \|\nabla W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{2} \\ &\leq (1 + |\nu - \nu'|\varepsilon K^{r,R})^{2} (\|G^{b}\|_{\dot{H}^{s}} + \frac{1}{\nu_{0}} \|G^{l}\|_{\dot{H}^{s-1}}^{2}). \end{aligned}$$

then, multiplying by $e^{-\int_0^t \|G^b(\tau)\|_{\dot{H}^s} d\tau}$, integrating from 0 to t, and after that multiplying by $e^{\int_0^t \|G^b(\tau)\|_{\dot{H}^s} d\tau}$, we obtain that $\forall t \ge 0$:

$$\begin{split} \|W_{\varepsilon}^{r,R}(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla W_{\varepsilon}^{r,R}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} \|G^{b}(\tau)\|_{\dot{H}^{s}} d\tau} dt' &\leq \|W_{\varepsilon}^{r,R}(0)\|_{\dot{H}^{s}}^{2} e^{\int_{0}^{t} \|G^{b}(\tau)\|_{\dot{H}^{s}} d\tau} \\ + (1 + |\nu - \nu'|\varepsilon K^{r,R})^{2} \int_{0}^{t} (\|G^{b}\|_{\dot{H}^{s}} + \frac{1}{\nu_{0}} \|G^{l}\|_{\dot{H}^{s-1}}^{2}) e^{\int_{t'}^{t} \|G^{b}(\tau)\|_{\dot{H}^{s}} d\tau} dt'. \end{split}$$

Let us state the following lemma which will conclude the proof of these energy estimates: Lemma 3.3.2 There exists a constant K^0 such that for all $s \in [\frac{1}{2}, 1]$ we have:

$$\int_0^\infty \|G^b(t)\|_{\dot{H}^s} dt + \int_0^\infty \|G^l(t)\|_{\dot{H}^{s-1}}^2 dt \le K^0$$

Proof of Lemma 3.3.2: Because of the conditions about the orders when we compute a product in Sobolev spaces, we have to distinguish the case when $s = \frac{1}{2}$ from the case when $s \in]\frac{1}{2}, 1]$ and obtain:

$$\int_0^\infty \|G^b\|_{\dot{H}^s} dt \le C \begin{cases} \|\widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+, \dot{H}^1)} \|\widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+, \dot{H}^2)} & \text{if } s = \frac{1}{2} \\ \|\widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+, \dot{H}^{s+1})}^2 & \text{if } s \in]\frac{1}{2}, 1]. \end{cases}$$

Then, thanks to the energy estimate (3.2.11), and classical interpolation arguments :

$$\|G^b\|_{L^1(\mathbb{R}_+,\dot{H}^s)} \le \frac{C}{\nu_0} (\|U_{0,QG}\|_{L^2} + \|U_{0,QG}\|_{\dot{H}^1})^2.$$

Similarly, using (3.1.6) we also obtain the estimate for $\|G^l\|_{L^2(\mathbb{R}_+,\dot{H}^{s-1})}^2$ and this concludes the proof of Lemma 3.3.2.

Let us go back to the proof of Lemma 3.3.1: thanks to Lemma 3.4.6 and the interpolation results in Sobolev spaces, we can estimate $||W_{\varepsilon}^{r,R}(0)||_{\dot{H}^s}^2$ and using Lemma 3.3.2, for all $t \in \mathbb{R}_+$,

$$\|W_{\varepsilon}^{r,R}(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla W_{\varepsilon}^{r,R}(t')\|_{\dot{H}^{s}}^{2} dt' \leq K^{0} (1 + |\nu - \nu'|\varepsilon K^{r,R})^{2},$$

which proves Lemma 3.3.1 (the computation for W_{ε} being identical).

3.3.3 Energy estimates for δ_{ε} and δ'_{ε}

The aim of this subsection is the proof of the following energy estimates:

Proposition 3.3.1 There exist constants K^0 , $K^{r,R}$, a function $V \in L^1(\mathbb{R})$ and a function I(r, R) (also depending on $|\nu - \nu'|$, ν_0 and U_0) such that if ε is small enough (in a neighbourhood of zero which depends on r and R) for all s in $[\frac{1}{2}, 1]$ and for all $t \ge 0$,

$$\|\delta_{\varepsilon}'(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla\delta_{\varepsilon}'(t')\|_{\dot{H}^{s}}^{2} dt' \leq \left(I(r,R) + \varepsilon K^{r,R}\right) e^{K^{0} + \varepsilon K^{r,R}}$$

and

$$\begin{aligned} \|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu_{0} - 2C \|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V(\tau)d\tau} dt' \\ &\leq \left(I(r,R) + \varepsilon^{\frac{1}{16}} K^{r,R}\right) e^{K^{0} + \varepsilon K^{r,R}}, \end{aligned}$$

where we have the estimate $\|V\|_{L^1(\mathbb{R}_+)} \leq K^0 + \varepsilon K^{r,R}$ and I(r,R) goes to zero as r goes to zero and R goes to ∞ .

Notation and estimates

From systems (3.3.15) and (3.3.16), and using that for any function g (see (3.3.17) for the definition of $\mathcal{P}_{r,R}$ and definition 3.4.2 in the appendix for \mathbb{P}_2),

$$g - \mathcal{P}_{r,R}\mathbb{P}_{3+4}g = (1 - \mathcal{P}_{r,R})g + \mathcal{P}_{r,R}\mathbb{P}_2g,$$

we can write that δ_{ε}' satisfies the following system:

$$\begin{cases} \partial_t \delta'_{\varepsilon} - L \delta'_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \delta'_{\varepsilon} = f'_1 + f'_2 \\ \delta'_{\varepsilon/t=0} = \delta_0, \end{cases}$$
(3.3.20)

with

$$\begin{cases} f'_{1} = -(1 - \mathcal{P}_{r,R})G^{b} - \mathcal{P}_{r,R}\mathbb{P}_{2}G^{b}, \\ f'_{2} = -(1 - \mathcal{P}_{r,R})G^{l} - \mathcal{P}_{r,R}\mathbb{P}_{2}G^{l}, \\ \delta_{0} = (1 - \mathcal{P}_{r,R})U_{0,osc} + \mathcal{P}_{r,R}\mathbb{P}_{2}U_{0,osc}. \end{cases}$$

Remark 3.3.1 Remember that G^l , G^b and $U_{0,osc}$ are divergence free, and have a zero potential vorticity (see the expression of G in (3.3.13) and of G^l and G^b in (3.3.19)).

From systems (3.3.12), (3.3.13), and (3.3.16), we get the system satisfied by δ_{ε} :

$$\begin{cases} \partial_t \delta_{\varepsilon} - L \delta_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \delta_{\varepsilon} = & -\mathbb{P}(\delta_{\varepsilon} . \nabla \delta_{\varepsilon}) \\ & -\mathbb{P}(\delta_{\varepsilon} . \nabla (\widetilde{U_{QG}} + W_{\varepsilon}^{r,R})) - \mathbb{P}((\widetilde{U_{QG}} + W_{\varepsilon}^{r,R}) . \nabla \delta_{\varepsilon}) \\ & + f_1 + f_2 \end{cases}$$
(3.3.21)
$$\delta_{\varepsilon/t=0} = \delta_0,$$

with

$$\begin{cases} f_1 = f'_1 \\ f_2 = f'_2 - \mathbb{P}(W_{\varepsilon}^{r,R}.\nabla \widetilde{U_{QG}}) - \mathbb{P}(\widetilde{U_{QG}}.\nabla W_{\varepsilon}^{r,R}) - \mathbb{P}(W_{\varepsilon}^{r,R}.\nabla W_{\varepsilon}^{r,R}) \end{cases}$$
(3.3.22)

The aim of the following subsections is the proof of Proposition 3.3.1: we will successively establish estimates on δ_{ε} and δ'_{ε} , estimates on the external forces and then we will end the proof.

Energy for δ'_{ε}

Let us take the inner product in \dot{H}^s of (3.3.20) with δ'_{ε} (where $s \in [\frac{1}{2}, 1]$):

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta_{\varepsilon}'\|_{\dot{H}^{s}}^{2} + \nu_{0} \|\nabla\delta_{\varepsilon}'\|_{\dot{H}^{s}}^{2} + 0 &\leq (f_{1}'|\delta_{\varepsilon}')_{\dot{H}^{s}} + (f_{2}'|\delta_{\varepsilon}')_{\dot{H}^{s}} \\ &\leq \|f_{1}'\|_{\dot{H}^{s}} \|\delta_{\varepsilon}'\|_{\dot{H}^{s}} + \|f_{2}'\|_{\dot{H}^{s-1}} \|\delta_{\varepsilon}'\|_{\dot{H}^{s+1}}, \end{aligned}$$

using the usual argument $(ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2)$, we get:

$$\begin{split} \frac{1}{2} \frac{d}{dt} \|\delta_{\varepsilon}'\|_{\dot{H}^{s}}^{2} + \nu_{0} \|\nabla\delta_{\varepsilon}'\|_{\dot{H}^{s}}^{2} &\leq \frac{1}{2} \|f_{1}'\|_{\dot{H}^{s}} + \frac{1}{2} \|f_{1}'\|_{\dot{H}^{s}} \|\delta_{\varepsilon}'\|_{\dot{H}^{s}}^{2} \\ &+ \frac{\nu_{0}}{2} \|\delta_{\varepsilon}'\|_{\dot{H}^{s+1}}^{2} + \frac{1}{2\nu_{0}} \|f_{2}'\|_{\dot{H}^{s-1}} \|\delta_{\varepsilon}'\|_{\dot{H}^{s+1}}^{2}, \end{split}$$

then, thanks to the same Gronwall argument as in the proof of Lemma 3.3.1, we obtain that $\forall t \geq 0$:

$$\begin{split} \|\delta_{\varepsilon}'(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla\delta_{\varepsilon}'(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} \|f_{1}'(\tau)\|_{\dot{H}^{s}} d\tau} dt' &\leq \|\delta_{\varepsilon}'(0)\|_{\dot{H}^{s}}^{2} e^{\int_{0}^{t} \|f_{1}'(\tau)\|_{\dot{H}^{s}} d\tau} \\ &+ \int_{0}^{t} (\|f_{1}'(t')\|_{\dot{H}^{s}} + \frac{1}{\nu_{0}} \|f_{2}'(t')\|_{\dot{H}^{s-1}}^{2}) e^{\int_{t'}^{t} \|f_{1}'(\tau)\|_{\dot{H}^{s}} d\tau} dt' \end{split}$$

And noting $||u||_{L^p\dot{H}^s} \stackrel{\text{def}}{=} ||u||_{L^p(\mathbb{R}_+,\dot{H}^s(\mathbb{R}^3))}, \forall t \ge 0$:

$$\|\delta_{\varepsilon}'(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla\delta_{\varepsilon}'(t')\|_{\dot{H}^{s}}^{2} dt' \leq \left(\|\delta_{0}\|_{\dot{H}^{s}}^{2} + \|f_{1}'\|_{L^{1}\dot{H}^{s}} + \frac{1}{\nu_{0}}\|f_{2}'(t')\|_{L^{2}\dot{H}^{s-1}}^{2}\right) e^{\|f_{1}'\|_{L^{1}\dot{H}^{s}}}.$$
(3.3.23)

Energy for δ_{ε}

The inner product in \dot{H}^s , yields:

$$\frac{1}{2}\frac{d}{dt}\|\delta_{\varepsilon}\|^{2}_{\dot{H}^{s}} + \nu_{0}\|\nabla\delta_{\varepsilon}\|^{2}_{\dot{H}^{s}} \leq |(\mathbb{P}(\delta_{\varepsilon}.\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| + |(\mathbb{P}(\delta_{\varepsilon}.\nabla(\widetilde{U_{QG}} + W^{r,R}_{\varepsilon}))|\delta_{\varepsilon})_{\dot{H}^{s}}| + |(\mathbb{P}((\widetilde{U_{QG}} + W^{r,R}_{\varepsilon})).\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| + |(f_{1}|\delta_{\varepsilon})_{\dot{H}^{s}}| + |(f_{2}|\delta_{\varepsilon})_{\dot{H}^{s}}|$$

Lemma 3.3.3 There exists a constant C such that the following estimates are true for all $s \in [\frac{1}{2}, 1]$:

$$|(\mathbb{P}(\delta_{\varepsilon}.\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| \le C \|\delta_{\varepsilon}\|_{\dot{H}^{s}} \|\nabla\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2}$$
(3.3.24)

$$|(\mathbb{P}(\delta_{\varepsilon}.\nabla(\widetilde{U_{QG}} + W_{\varepsilon}^{r,R}))|\delta_{\varepsilon})_{\dot{H}^{s}}| \leq \frac{\nu_{0}}{6} \|\delta_{\varepsilon}\|_{\dot{H}^{s+1}}^{2} + \frac{C}{\nu_{0}} \|\nabla(\widetilde{U_{QG}} + W_{\varepsilon}^{r,R})\|_{\dot{H}^{\frac{1}{2}}}^{2} \|\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2} \quad (3.3.25)$$

$$\begin{split} \| (\mathbb{P}((\widetilde{U_{QG}} + W_{\varepsilon}^{r,R}) \cdot \nabla \delta_{\varepsilon}) | \delta_{\varepsilon})_{\dot{H}^{s}} \| &\leq \frac{\nu_{0}}{6} \| \delta_{\varepsilon} \|_{\dot{H}^{s+1}}^{2} + e_{s}(t) \| \delta_{\varepsilon} \|_{\dot{H}^{s}}^{2} \tag{3.3.26} \\ \text{where} \quad e_{s}(t) &= \begin{cases} \frac{C}{\nu_{0}^{3}} \| \widetilde{U_{QG}} + W_{\varepsilon}^{r,R} \|_{\dot{H}^{\frac{1}{2}}}^{2} \| \widetilde{U_{QG}} + W_{\varepsilon}^{r,R} \|_{\dot{H}^{\frac{1}{2}}}^{2} & \text{if} \quad s = \frac{1}{2} \\ \frac{C}{\nu_{0}} \| \widetilde{U_{QG}} + W_{\varepsilon}^{r,R} \|_{\dot{H}^{s+1}}^{2} & \text{if} \quad s \in]\frac{1}{2}, 1]. \end{cases} \end{split}$$

Proof: we use the fact that δ_{ε} is divergence-free together with Sobolev product laws as well as the fact that if $s > \frac{1}{2}$, \dot{H}^{s+1} is a Banach algebra. As δ_{ε} is divergence-free, we can write:

$$|(\mathbb{P}(\delta_{\varepsilon}.\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| \leq \begin{cases} \|\delta_{\varepsilon}.\nabla\delta_{\varepsilon}\|_{L^{2}}\|\delta_{\varepsilon}\|_{\dot{H}^{1}} & \text{if } s = \frac{1}{2} \\ \|\operatorname{div}\left(\delta_{\varepsilon}\otimes\delta_{\varepsilon}\right)\|_{\dot{H}^{s}}\|\delta_{\varepsilon}\|_{\dot{H}^{s}} & \text{if } s \in]\frac{1}{2}, 1]. \end{cases}$$

Using an interpolation argument, the Sobolev product laws $(\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \dot{H}^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3))$, and the fact that if $s > \frac{1}{2}$, \dot{H}^{s+1} is a Banach algebra, we get the estimate:

$$|(\mathbb{P}(\delta_{\varepsilon}.\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| \leq \begin{cases} C\|\delta_{\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\delta_{\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}^{2} & \text{if } s = \frac{1}{2}\\ C\|\delta_{\varepsilon}\|_{\dot{H}^{s}}\|\nabla\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2} & \text{if } s \in]\frac{1}{2}, 1]. \end{cases}$$

So we have

$$|(\mathbb{P}(\delta_{\varepsilon}.\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| \le C \|\delta_{\varepsilon}\|_{\dot{H}^{s}} \|\nabla\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2}$$

which is exactly (3.3.24). Let us now deal with (3.3.25): we have

$$|\mathbb{P}(\delta_{\varepsilon}.\nabla(\widetilde{U_{QG}}+W^{r,R}_{\varepsilon}))|\delta_{\varepsilon})_{\dot{H}^{s}}| \leq C \|\delta_{\varepsilon}.\nabla(\widetilde{U_{QG}}+W^{r,R}_{\varepsilon})\|_{\dot{H}^{s-1}}\|\delta_{\varepsilon}\|_{\dot{H}^{s+1}}$$

so, using the product of \dot{H}^s by $\dot{H}^{\frac{1}{2}}$ we get

$$|\mathbb{P}(\delta_{\varepsilon}.\nabla(\widetilde{U_{QG}}+W_{\varepsilon}^{r,R}))|\delta_{\varepsilon})_{\dot{H}^{s}}| \leq C \|\delta_{\varepsilon}\|_{\dot{H}^{s}} \|\nabla(\widetilde{U_{QG}}+W_{\varepsilon}^{r,R})\|_{\dot{H}^{\frac{1}{2}}} \|\delta_{\varepsilon}\|_{\dot{H}^{s+1}}.$$

And finally, a classical argument gives

$$\left|\mathbb{P}(\delta_{\varepsilon}.\nabla(\widetilde{U_{QG}}+W_{\varepsilon}^{r,R}))|\delta_{\varepsilon})_{\dot{H}^{s}}\right| \leq \frac{\nu_{0}}{6} \|\delta_{\varepsilon}\|_{\dot{H}^{s+1}}^{2} + \frac{C}{\nu_{0}} \|\nabla(\widetilde{U_{QG}}+W_{\varepsilon}^{r,R})\|_{\dot{H}^{\frac{1}{2}}}^{2} \|\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2}.$$

The last estimate (3.3.26) is dealt the same way:

$$|(\mathbb{P}((\widetilde{U_{QG}} + W_{\varepsilon}^{r,R}).\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| \leq ||(\widetilde{U_{QG}} + W_{\varepsilon}^{r,R}).\nabla\delta_{\varepsilon}||_{\dot{H}^{s}}||\delta_{\varepsilon}||_{\dot{H}^{s}}.$$

It is necessary to separate cases relatively to the Sobolev order (and using Sobolev product laws):

$$|(\mathbb{P}((\widetilde{U_{QG}} + W_{\varepsilon}^{r,R}).\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| \leq \begin{cases} C\|\widetilde{U_{QG}} + W_{\varepsilon}^{r,R}\|_{\dot{H}^{1}}\|\nabla\delta_{\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}\|\delta_{\varepsilon}\|_{\dot{H}^{1}} & \text{if } s = \frac{1}{2}\\ C\|\text{div}\left((\widetilde{U_{QG}} + W_{\varepsilon}^{r,R})\otimes\delta_{\varepsilon}\right)\|_{\dot{H}^{s}}\|\delta_{\varepsilon}\|_{\dot{H}^{s}} & \text{if } s \in]\frac{1}{2}, 1] \end{cases}$$

Using Sobolev inclusions and interpolation, together with the fact that \dot{H}^{s+1} is a Banach algebra when $s > \frac{1}{2}$ and the classical inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ if $\frac{1}{p} + \frac{1}{q} = 1$ with p = 2 and p = 4 leads to:

$$|(\mathbb{P}((\widetilde{U_{QG}} + W_{\varepsilon}^{r,R}).\nabla\delta_{\varepsilon})|\delta_{\varepsilon})_{\dot{H}^{s}}| \leq \frac{\nu_{0}}{6} \|\delta_{\varepsilon}\|_{\dot{H}^{s+1}}^{2} + e_{s}(t)\|\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2}$$

where

$$e_s(t) = \begin{cases} \frac{C}{\nu_0^3} \| \widetilde{U_{QG}} + W_{\varepsilon}^{r,R} \|_{\dot{H}^{\frac{1}{2}}}^2 \| \widetilde{U_{QG}} + W_{\varepsilon}^{r,R} \|_{\dot{H}^{\frac{3}{2}}}^2 & \text{if } s = \frac{1}{2} \\ \frac{C}{\nu_0} \| \widetilde{U_{QG}} + W_{\varepsilon}^{r,R} \|_{\dot{H}^{s+1}}^2 & \text{if } s \in]\frac{1}{2}, 1]. \end{cases}$$

And that proves Lemma $3.3.3 \blacksquare$.

Let us go back to the energy estimate: for all $t \ge 0$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2} + \nu_{0} \|\nabla\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2} &\leq C \|\delta_{\varepsilon}\|_{\dot{H}^{s}} \|\nabla\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2} + \frac{\nu_{0}}{6} \|\delta_{\varepsilon}\|_{\dot{H}^{s+1}}^{2} \\ &+ \frac{C}{2\nu_{0}} \|\nabla(\widetilde{U_{QG}} + W_{\varepsilon}^{r,R})\|_{\dot{H}^{\frac{1}{2}}}^{2} \|\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2} + \frac{\nu_{0}}{6} \|\delta_{\varepsilon}\|_{\dot{H}^{s+1}}^{2} + e_{s}(t) \|\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2} \\ &+ \frac{1}{2} \|f_{1}\|_{\dot{H}^{s}} + \frac{1}{2} \|f_{1}\|_{\dot{H}^{s}} \|\delta_{\varepsilon}\|_{\dot{H}^{s}}^{2} + \frac{1}{2\nu_{0}} \|f_{2}\|_{\dot{H}^{s-1}}^{2} + \frac{\nu_{0}}{6} \|\delta_{\varepsilon}\|_{\dot{H}^{s+1}}^{2}. \end{aligned}$$

Posing $V(t) = \frac{C}{\nu_0} \|\nabla(\widetilde{U_{QG}} + W_{\varepsilon}^{r,R})\|_{\dot{H}^{\frac{1}{2}}}^2 + e_s(t) + \|f_1\|_{\dot{H}^s}$, the usual Gronwall argument implies that for all $t \ge 0$:

$$\|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu_{0} - 2C \|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V(\tau)d\tau} dt' \leq (3.3.27)$$
$$\left(\|\delta_{\varepsilon}(0)\|_{\dot{H}^{s}}^{2} + \|f_{1}\|_{L^{1}\dot{H}^{s}} + \frac{1}{\nu_{0}} \|f_{2}\|_{L^{2}\dot{H}^{s-1}}^{2}\right) e^{\|V\|_{L^{1}(\mathbb{R}_{+})}},$$

In the following we will obtain that if ε is in a small neighbourhood of zero (depending on the radiuses r and R) we can bound all these quantities (including the L^1 norm of V).

Estimates on the external forces

The estimates concerning the external forces are the following:

Lemma 3.3.4 There exist a constant K^0 , a function $\alpha(r, R)$ bounded by K^0 , and going to zero as r goes to zero, and R goes to infinity, and a constant $K^{r,R}$ such that if $\varepsilon < 1$ we have the following estimates

$$\|f_1\|_{L^1(\mathbb{R}_+,\dot{H}^s)} + \|f_2'\|_{L^2(\mathbb{R}_+,\dot{H}^{s-1})}^2 + \|\delta_0\|_{\dot{H}^s}^2 \le \alpha(r,R) + \varepsilon K^{r,R},$$

and if ε is in a small neighbourhood of zero (depending on r and R),

$$\|f_2\|_{L^2(\mathbb{R}_+,\dot{H}^{s-1})} \le \alpha(r,R) + \varepsilon^{\frac{1}{16}} K^{r,R}.$$

Proof: The function α is easily given by the Lebesgue Theorem together with Lemma 3.3.2, precisely:

$$\alpha(r,R) = \| (1 - \mathcal{P}_{r,R}) G^b \|_{L^1 \dot{H}^s} + \| (1 - \mathcal{P}_{r,R}) G^l \|_{L^2 \dot{H}^{s-1}}^2 + \| (1 - \mathcal{P}_{r,R}) U_{0,osc} \|_{\dot{H}^s}^2.$$

Lemmas 3.4.3 and 3.3.2 give the first estimate. In order to get the last estimate, all we have to do is to estimate the last three terms in the expression of f_2 (because all the other terms have just been estimated). By interpolation, we have:

$$\|\widetilde{U_{QG}}.\nabla W^{r,R}_{\varepsilon}\|_{\dot{H}^{s-1}} \leq \|\widetilde{U_{QG}}.\nabla W^{r,R}_{\varepsilon}\|_{\dot{H}^{2s-2}}^{\frac{1}{2}} \|\widetilde{U_{QG}}.\nabla W^{r,R}_{\varepsilon}\|_{L^{2}}^{\frac{1}{2}}.$$

Thanks to the product laws in Sobolev spaces $(s - \frac{1}{2} + s - \frac{3}{2} = 2s - 2$ and $s \in [\frac{1}{2}, 1])$ we can write that:

$$\|\widetilde{U_{QG}}.\nabla W_{\varepsilon}^{r,R}\|_{\dot{H}^{s-1}} \le C \|\widetilde{U_{QG}}\|_{\dot{H}^{s-\frac{1}{2}}}^{\frac{1}{2}} \|\nabla W_{\varepsilon}^{r,R}\|_{\dot{H}^{s}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{2}}^{\frac{1}{2}} \|\nabla W_{\varepsilon}^{r,R}\|_{L^{\infty}}^{\frac{1}{2}}$$

Using the energy estimates for $\widetilde{U_{QG}}$ and $W_{\varepsilon}^{r,R}$ as well as the fact that $s - \frac{1}{2} \in [0, \frac{1}{2}]$ and the Bernstein Lemma $(\|\nabla W_{\varepsilon}^{r,R}\|_{L^{\infty}} \leq CR \|W_{\varepsilon}^{r,R}\|_{L^{\infty}})$, we obtain (where the notation $L^{p}L^{q}$ means $L^{p}(\mathbb{R}_{+}, L^{q}(\mathbb{R}^{3}))$ and similarly $L^{p}\dot{H}^{s}$ means $L^{p}(\mathbb{R}_{+}, \dot{H}^{s}(\mathbb{R}^{3}))$:

$$\|\widetilde{U_{QG}}.\nabla W_{\varepsilon}^{r,R}\|_{L^{2}(\mathbb{R}_{+},\dot{H}^{s-1})} \leq C_{R} \|\widetilde{U_{QG}}\|_{L^{\infty}\dot{H}^{s-\frac{1}{2}}}^{\frac{1}{2}} \|W_{\varepsilon}^{r,R}\|_{L^{2}\dot{H}^{s+1}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{\infty}L^{2}}^{\frac{1}{2}} \|W_{\varepsilon}^{r,R}\|_{L^{2}L^{\infty}}^{\frac{1}{2}}$$

Let us now state the Strichartz estimates whose proof is given in the appendix:

Lemma 3.3.5 There exists a constant $K^{r,R}$ such that if ε is small enough (the neighbourhood of zero depends on r, and R), we have the following estimate:

$$\|W_{\varepsilon}^{r,R}\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq \varepsilon^{\frac{1}{8}}K^{r,R}$$

Using these Strichartz estimates together with Lemma 3.3.1, and (3.2.11) we obtain :

$$\|\widetilde{U_{QG}} \cdot \nabla W_{\varepsilon}^{r,R}\|_{L^2(\mathbb{R}_+,\dot{H}^{s-1})} \le \varepsilon^{\frac{1}{16}} K^{r,R}.$$
(3.3.28)

Similarly :

$$\|W_{\varepsilon}^{r,R}.\nabla \widetilde{U_{QG}}\|_{L^{2}(\mathbb{R}_{+},\dot{H}^{s-1})} \leq \varepsilon^{\frac{1}{16}}K^{r,R} \quad \text{and} \quad \|W_{\varepsilon}^{r,R}.\nabla W_{\varepsilon}^{r,R}\|_{L^{2}(\mathbb{R}_{+},\dot{H}^{s-1})} \leq \varepsilon^{\frac{1}{16}}K^{r,R}.$$
(3.3.29)

This concludes the proof of Lemma 3.3.4. \blacksquare

Conclusion: end of the proof of Proposition 3.3.1

We are now able to achieve the proof of Proposition 3.3.1. First let us bound the norm of function V introduced with (3.3.27): there exists a constant K^0 such that:

$$\|V\|_{L^1(\mathbb{R}_+)} \le K^0(1 + |\nu - \nu'|\varepsilon C_{r,R})^4 + \|G^b_{r,R}\|_{L^1\dot{H}^s} + \varepsilon K^{r,R}$$

which we can rewrite into (using Lemma 3.3.4):

$$\|V\|_{L^1(\mathbb{R}_+)} \le K^0 + \varepsilon K^{r,R}.$$

Using the estimates given by (3.3.27), (3.3.23), and the ones on the external forces in Lemma 3.3.4 we can write that for all $t \ge 0$:

$$\|\delta_{\varepsilon}'(t)\|_{\dot{H}^s}^2 + \nu_0 \int_0^t \|\nabla \delta_{\varepsilon}'(t')\|_{\dot{H}^s}^2 ds \le \Big(I(r,R) + \big(\varepsilon^2 K^{r,R} + \varepsilon K^{r,R} + \frac{1}{\nu_0} \varepsilon K^{r,R}\big)\Big)e^{K^0 + \varepsilon K^{r,R}},$$

and if ε is close to zero (in a neighbourhood of zero depending on r and R):

$$\|\delta_{\varepsilon}(t)\|_{\dot{H}^s}^2 + \int_0^t (\nu_0 - 2C \|\delta_{\varepsilon}(t')\|_{\dot{H}^s}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^s}^2 e^{\int_{t'}^t V(\tau)d\tau} dt' \le$$

$$\Big(I(r,R) + \big(\varepsilon^2 K^{r,R} + \varepsilon K^{r,R} + \frac{1}{\nu_0}\varepsilon^{\frac{1}{16}}K^{r,R}\big)\Big)e^{K^0 + \varepsilon K^{r,R}}$$

where we define (see Lemma 3.3.4 for the definition of α):

$$I(r,R) \stackrel{\text{def}}{=} (2 + \frac{1}{\nu_0})\alpha(r,R),$$

since Lemma 3.3.4 implies that I(r, R) goes to zero when r goes to zero and R goes to $+\infty$, and that $\varepsilon \leq 1$, we finally obtain all the results stated in Proposition 3.3.1.

3.3.4 Convergence of solutions : proof of Theorem 3.1.3

Let $0 < \eta \leq \frac{\nu_0}{2C}$ and fix r small and R large such that we have (see Lemma 3.3.4):

$$I(r,R)e^{2K^0} \le \frac{\eta}{4}.$$
(3.3.30)

From now on and until the end of the paper, r and R are fixed like this. Moreover, in order to simplify notation in the following computations, we will write directly the estimate on δ_{ε} instead of an approximation (such as for example the one given by the Friedrichs scheme) as should be done. Let us fix $0 < \varepsilon_0(\nu, \nu', U_0, r, R) \leq 1$ such that $\forall \varepsilon \leq \varepsilon_0, \varepsilon K^{r,R} \leq K^0$ and let us define the time

$$T_{\varepsilon} = \sup \left\{ t \ge 0 \quad \text{such that} \quad \delta_{\varepsilon} \in \mathcal{C}([0,t], \dot{H}^s) \quad and \quad \forall t' \le t, \quad \|\delta_{\varepsilon}(t')\|_{\dot{H}^s}^2 \le \eta \right\}.$$

Let T_{ε}^1 be a lifespan of a solution δ_{ε} in $E^{\frac{1}{2}} = L^{\infty}([0, T_{\varepsilon}^1], \dot{H}^{\frac{1}{2}}) \cap L^2([0, T_{\varepsilon}^1], \dot{H}^{\frac{3}{2}})$: with our initial data an adaptation of the Fujita and Kato Theorem with external force proves that such a $T_{\varepsilon}^1 > 0$ exists and on this interval the solution is in E^s for all $s \in [\frac{1}{2}, 1]$. Let us fix $0 < \varepsilon_1(\nu, \nu', U_0, r, R) \le \varepsilon_0$ such that $\forall \varepsilon \le \varepsilon_1$,

$$\|\delta_{\varepsilon}(0)\|_{\dot{H}^{s}}^{2} \leq I(r,R) + \varepsilon K^{r,R} \leq \frac{\eta}{4} + \varepsilon K^{r,R} \leq \frac{\eta}{2}.$$

So consequently, T_{ε} is well defined and strictly positive: $\forall 0 < t < T_{\varepsilon}$ $\nu_0 - C \|\delta_{\varepsilon}\|_{\dot{H}^s} \geq \frac{\nu_0}{2}$. By (3.3.30) $\forall t < T_{\varepsilon}$, the energy estimate on δ_{ε} becomes:

$$\|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \frac{\nu_{0}}{2} \int_{0}^{t} \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} dt' \leq \frac{\eta}{4} + \varepsilon^{\frac{1}{16}} K^{r,R} e^{2K^{0}}.$$

The definition of ε_1 also implies that the estimate for δ'_{ε} becomes:

$$\|\delta_{\varepsilon}'(t)\|_{\dot{H}^s}^2 + \nu_0 \int_0^t \|\nabla \delta_{\varepsilon}'(t')\|_{\dot{H}^s}^2 dt' \le \frac{\eta}{4} + \varepsilon K^{r,R} e^{2K^0}$$

So there exists $\varepsilon_2 = \varepsilon_2(\nu, \nu', \nu_0, U_0, r, R) \le \varepsilon_1$ such that $\forall \varepsilon \le \varepsilon_2$ and $\forall t < T_{\varepsilon}$,

$$\|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \frac{\nu_{0}}{2} \int_{0}^{t} \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} dt' \leq \frac{\eta}{2} \leq \frac{\nu_{0}}{4C} \quad \text{and} \quad \|\delta_{\varepsilon}'(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla\delta_{\varepsilon}'(t')\|_{\dot{H}^{s}}^{2} dt' \leq \frac{\eta}{2}.$$

This implies that $\forall \varepsilon \leq \varepsilon_2, T_{\varepsilon} = +\infty$ (bootstrap) and also that $\delta_{\varepsilon} - \delta'_{\varepsilon} = U_{\varepsilon} - \widetilde{U_{QG}} - W_{\varepsilon}$ satisfies for all $\varepsilon \leq \varepsilon_2$ and $t \geq 0$:

$$\|(\delta_{\varepsilon} - \delta_{\varepsilon}')(t)\|_{\dot{H}^s}^2 + \frac{\nu_0}{2} \int_0^t \|\nabla(\delta_{\varepsilon} - \delta_{\varepsilon}')(s)\|_{\dot{H}^s}^2 ds \le \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

and that achieves the proof of the convergence of γ_{ε} . As $W_{\varepsilon} = \delta'_{\varepsilon} + W^{r,R}_{\varepsilon}$, and knowing that, thanks to the Strichartz estimate given in Lemma 3.3.5, if ε is close to zero: $\|W^{r,R}_{\varepsilon}\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \leq \varepsilon^{\frac{1}{8}}K^{r,R}$, we can define $\varepsilon_3 = \varepsilon_3(\nu,\nu',\nu_0,U_0,r,R) \leq \varepsilon_2$ such that $\forall \varepsilon \leq \varepsilon_3$ and $\forall t$, $\|W^{r,R}_{\varepsilon}\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \leq \frac{\eta}{2}$ but with the case s = 1, we also have the estimate $\|\delta'_{\varepsilon}(t)\|^2_{\dot{H}^1} + \nu_0 \int_0^t \|\nabla\delta'_{\varepsilon}(t')\|^2_{\dot{H}^1} dt' \leq \frac{\eta}{2}$, so as the dimension is 3, $\dot{H}^2 \hookrightarrow L^{\infty}$, we have $\|\delta'_{\varepsilon}\|^2_{L^2(\mathbb{R}_+,L^{\infty})} \leq \frac{\eta}{\nu_0}$ and finally:

$$\|W_{\varepsilon}\|_{L^2(\mathbb{R}_+,L^\infty(\mathbb{R}^3))} \leq (\frac{1}{2} + \frac{1}{\nu_0})\eta$$

Which gives (remember that $U_{\varepsilon} - \widetilde{U_{QG}} = \gamma_{\varepsilon} + W_{\varepsilon}$ and use another time the injection $\dot{H}^2(\mathbb{R}^3) \hookrightarrow L^{\infty}(\mathbb{R}^3)$) that $\|U_{\varepsilon} - \widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+, L^{\infty}(\mathbb{R}^3))}$ goes to zero and achieves the proof of Theorem 3.1.3.

3.4 Appendix

3.4.1 Technical results

We won't give details for the proofs of the lemmas from this section.

Definition 3.4.1 If 0 < r < R, let $C_{r,R} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3 \text{ such that } |\xi| \leq R, \text{ and } |\xi_3| \geq r\}.$

Let us denote by $\mathbb{B}(\xi, \varepsilon)$ the following matrix (in Fourier variables):

$$\mathbb{B}(\xi,\varepsilon) \stackrel{\text{def}}{=} \widehat{L - \frac{1}{\varepsilon}} \mathbb{P} \mathcal{A} = \begin{pmatrix} -\nu|\xi|^2 + \frac{\xi_1\xi_2}{\varepsilon|\xi|^2} & \frac{\xi_2^2 + \xi_3^2}{\varepsilon|\xi|^2} & 0 & \frac{\xi_1\xi_3}{\varepsilon F|\xi|^2} \\ -\frac{\xi_1^2 + \xi_3^2}{\varepsilon|\xi|^2} & -\nu|\xi|^2 - \frac{\xi_1\xi_2}{\varepsilon|\xi|^2} & 0 & \frac{\xi_2\xi_3}{\varepsilon F|\xi|^2} \\ \frac{\xi_2\xi_3}{\varepsilon|\xi|^2} & -\frac{\xi_1\xi_3}{\varepsilon|\xi|^2} & -\nu|\xi|^2 & -\frac{\xi_1^2 + \xi_2^2}{\varepsilon F|\xi|^2} \\ 0 & 0 & \frac{1}{\varepsilon F} & -\nu'|\xi|^2 \end{pmatrix}.$$

Then we have the following lemmas (we refer to [4] for the proofs):

Lemma 3.4.1 The matrix \mathbb{B} is diagonalizable with four distinct eigenvalues we will denote the following way:

$$\begin{cases} \mu_0 = -\nu |\xi|^2 \\ \mu \quad \text{which is real} \\ \lambda \quad \text{and} \quad \overline{\lambda}. \end{cases}$$

Lemma 3.4.2 When $\xi \in C_{r,R}$, and ε is close to zero (the neighbourhood of zero depends on r and R),

$$\mu = -(\nu\xi_1^2 + \nu\xi_2^2 + \nu'F^2\xi_3^2)\frac{|\xi|^2}{|\xi|_F^2} + \varepsilon^2 I(\varepsilon)$$
$$\lambda = -\tau(\xi)|\xi|^2 + i\frac{|\xi|_F}{\varepsilon F|\xi|} + i\varepsilon S(\xi,\varepsilon) + \varepsilon^2 S'(\xi,\varepsilon)$$

where the function τ is defined by:

$$\tau(\xi) \stackrel{\text{def}}{=} \frac{\nu}{2} \left(1 + \frac{F^2 \xi_3^2}{|\xi|_F^2} \right) + \frac{\nu'}{2} \left(1 - \frac{F^2 \xi_3^2}{|\xi|_F^2} \right) \ge \nu_0 > 0 \quad \forall \xi \in \mathcal{C}_{r,R}$$
(3.4.31)

and I, S, and S' are functions of ε and ξ uniformly bounded in $\mathcal{C}_{r,R}$ with respect to ε .

Remark 3.4.1 The leading part of the asymptotic expansion of μ is exactly the expression in terms of the Fourier variable of operator Γ .

A basis of eigenvectors $W_1(\xi, \varepsilon), ..., W_4(\xi, \varepsilon)$ corresponding respectively to the previous eigenvalues is given by the following matrix of change of basis (noting $A \stackrel{\text{def}}{=} \mu + \nu |\xi|^2$ and $B \stackrel{\text{def}}{=} \lambda + \nu |\xi|^2$):

$$\begin{pmatrix} \xi_{2}\xi_{3} & \xi_{3}(\varepsilon\xi_{1}A + \xi_{2}) & \xi_{3}(\varepsilon\xi_{1}B + \xi_{2}) & \xi_{3}(\varepsilon\xi_{1}\overline{B} + \xi_{2}) \\ -\xi_{1}\xi_{3} & \xi_{3}(\varepsilon\xi_{2}A - \xi_{1}) & \xi_{3}(\varepsilon\xi_{2}B - \xi_{1}) & \xi_{3}(\varepsilon\xi_{2}\overline{B} - \xi_{1}) \\ -\varepsilon F^{2}(\nu - \nu')|\xi|^{2}\xi_{3}^{2} & -\varepsilon A(\xi_{1}^{2} + \xi_{2}^{2}) & -\varepsilon B(\xi_{1}^{2} + \xi_{2}^{2}) & -\varepsilon \overline{B}(\xi_{1}^{2} + \xi_{2}^{2}) \\ F\xi_{3}^{2} & F(\varepsilon^{2}|\xi|^{2}A^{2} + \xi_{3}^{2}) & F(\varepsilon^{2}|\xi|^{2}B^{2} + \xi_{3}^{2}) & F(\varepsilon^{2}|\xi|^{2}\overline{B}^{2} + \xi_{3}^{2}) \end{pmatrix}$$

As W_1 is not "divergence free", a divergence free vector field has no coordinate along this vector, so $\widehat{U_{\varepsilon,osc}}$ depends only on the last three eigenvectors : $\widehat{U_{\varepsilon,osc}} = K_2W_2 + K_3W_3 + K_4W_4$. Let us give the following definition

Definition 3.4.2 If g is a divergence free vector field (with four components but depending on $x \in \mathbb{R}^3$), we can write: $g = \sum_{i=2}^4 \mathbb{P}_i g$, where for $i \in \{2, 3, 4\}$, $\mathbb{P}_i g = \mathcal{F}^{-1}(K_i W_i)$ is the inverse Fourier of the projection onto the span of W_i .

Let us now state the following lemma showing that \mathbb{P}_2 has a small norm:

Lemma 3.4.3 Let 0 < r < R, then there exists a constant $c_{r,R} > 0$ such that for all functions g satisfying $\mathbb{P}g = g$ and $\mathcal{P}g = g$, (i.e with zero divergence and zero potential vorticity), and $\operatorname{supp}(\hat{g}) \subset C_{r,R}$, we have: $\forall \xi \in C_{r,R}, |\widehat{\mathbb{P}_{2}g}(\xi)| \leq c_{r,R}\varepsilon|\nu - \nu'||\hat{g}(\xi)|$ and consequently:

$$\forall s \in \mathbb{R}, \|\mathbb{P}_2 g\|_{\dot{H}^s} \le c_{r,R} \varepsilon |\nu - \nu'| \|g\|_{\dot{H}^s}.$$

Proof: It comes directly from the following two lemmas

Lemma 3.4.4 Under the same assumptions as Lemma 3.4.3 and if we denote by (X, Y, Z, T) the coordinates of \hat{g} , we have (with the notation of definition 3.4.2):

$$K_{2} = \frac{A}{(B-A)(\overline{B}-A)} \frac{1}{F^{2}(\varepsilon^{2}A^{2}|\xi|^{2} + \xi_{3}^{2})} (\frac{Z}{\varepsilon} + AT)$$

Proof: This lemma can be proved using the relations between the coefficients and the roots of the polynomial $(X - A)(X - B)(X - \overline{B})$ (which is given by a translation of variable of the characteristic polynomial, whose coefficients are well known) and other algebraic relations between the coordinates of the eigenvectors (see Lemma 4.9 in [4]).

Lemma 3.4.5 We have the following asymptotic expansions when $\xi \in C_{r,R}$ and ε is close to zero:

$$A = (\nu - \nu')F^{2}\xi_{3}^{2}\frac{|\xi|^{2}}{|\xi|_{F}^{2}} + \mathcal{O}(\varepsilon^{2})$$
$$B = \frac{i|\xi|_{F}}{\varepsilon F|\xi|} + \frac{\nu - \nu'}{2}\frac{|\xi|^{2}}{|\xi|_{F}^{2}}(\xi_{1}^{2} + \xi_{2}^{2}) + \mathcal{O}(\varepsilon),$$

where the $\mathcal{O}(\varepsilon)$ are uniform with respect to $\xi \in \mathcal{C}_{r,R}$.

Proof: This lemma is obtained from the asymptotic expansions of the eigenvalues of $\mathbb{B}(\xi, \varepsilon)$ (see [4]).

Let us state a useful lemma which is a consequence of Lemma 3.4.3: we define $\chi \ a \ C^{\infty}$ function whose support is included in [-1, 1] and equal to 1 in $[-\frac{1}{2}, \frac{1}{2}]$.

Lemma 3.4.6 There exists a constant $C_{r,R}$ such that for all $s \in \mathbb{R}$ and for all g divergence free and with a zero potential vorticity, denoting by \mathbb{P}_{3+4} the projector $\mathbb{P}_3 + \mathbb{P}_4$ (see 3.3.17 for the definition of operator $\mathcal{P}_{r,R}$): $\|\mathcal{P}_{r,R}\mathbb{P}_{3+4}g\|_{\dot{H}^s} \leq (1 + |\nu - \nu'|\varepsilon C_{r,R})\|g\|_{\dot{H}^s}$.

Proof: As
$$\mathcal{P}_{r,R}\mathbb{P}_{3+4}g = \chi(\frac{|D|}{R})(1-\chi(\frac{|D_3|}{r}))\mathbb{P}_{3+4}g$$

= $\chi(\frac{|D|}{R})(1-\chi(\frac{|D_3|}{r}))g - \chi(\frac{|D|}{R})(1-\chi(\frac{|D_3|}{r}))\mathbb{P}_2g$

we can use the triangle inequality:

$$\|\chi(\frac{|D|}{R})(1-\chi(\frac{|D_3|}{r}))\mathbb{P}_{3+4}g\|_{\dot{H}^s} \le \|\chi(\frac{|D|}{R})(1-\chi(\frac{|D_3|}{r}))g\|_{\dot{H}^s} + \|\chi(\frac{|D|}{R})(1-\chi(\frac{|D_3|}{r}))\mathbb{P}_2g\|_{\dot{H}^s},$$

and then the Plancherel theorem for the first term and Lemma 3.4.3 for the other term:

$$\|\chi(\frac{|D|}{R})(1-\chi(\frac{|D_3|}{r}))\mathbb{P}_{3+4}g\|_{\dot{H}^s} \le \|g\|_{\dot{H}^s} + |\nu-\nu'|\varepsilon C_{r,R}\|g\|_{\dot{H}^s}$$

And Lemma 3.4.6 is proved. \blacksquare

3.4.2 Proof of the Strichartz estimates

We refer to [14] for Strichartz estimates in a more general case for the wave equation, to [8] for Strichartz estimates suited for the rotating fluid system, and to [4] for the case of the primitive system.

Let us recall that r and R were fixed in (3.3.30).

In this section we will prove the Strichartz estimates given in Lemma 3.3.5 about $W_{\varepsilon}^{r,R}$ which satisfies system (3.3.16).

Let us begin by stating the corresponding dispersion inequality (see [4] for the proof). Let us denote by K the following operator (see definition 3.4.2 for the notations):

$$K(\varepsilon,t,z) = \int_{\mathbb{R}^3} \psi(\xi) e^{-t\tau(\xi)|\xi|^2 + i\frac{t|\xi|_F}{\varepsilon F|\xi|} + i\varepsilon tS(\xi,\varepsilon) + \varepsilon^2 tS'(\xi,\varepsilon) + iz.\xi} d\xi,$$

where

- the function $\psi \in \mathcal{C}_0^{\infty}(\mathbb{R}^3)$ is radial, supported in $\mathcal{C}_{\frac{r}{2},2R}$, and equals 1 near $\mathcal{C}_{r,R}$.
- $\forall \xi \in \mathcal{C}_{r,R}, \tau(\xi) \ge \nu_0 > 0$, according to (3.4.31).
- S, S' and their derivatives with respect to ξ are bounded on $C_{r,R}$ by a constant $A_{r,R}$. Then, according to [4] there exists a constant $C_{r,R}$ such that if ε is small enough:

$$\sup_{z\in\mathbb{R}^3}|K(\varepsilon,t,z)|\leq C_{r,R}e^{-\frac{\nu_0}{16}r^2t}(\frac{\varepsilon}{t})^{\frac{1}{2}}.$$

If we note S_{λ} the semigroup associated with (3.3.16) and:

$$\begin{cases} w_0 = \mathcal{P}_{r,R} \mathbb{P}_{3+4} U_{0,osc} \\ g^b = -\mathcal{P}_{r,R} \mathbb{P}_{3+4} G^b \\ g^l = -\mathcal{P}_{r,R} \mathbb{P}_{3+4} G^l \end{cases}$$
(3.4.32)

then the Duhamel formula writes:

$$W_{\varepsilon}^{r,R}(t) = S_{\lambda}(t)W_{\varepsilon}^{r,R}(0) + \int_0^t S_{\lambda}(t-t')g^b(t')dt' + \int_0^t S_{\lambda}(t-t')g^l(t')dt'.$$

So it depends only on the last two "Fourier eigenvalues", and like in [4] (section 4.3.4) we will deal with the computation for the projection \mathbb{P}_3 (the computation is the same for \mathbb{P}_4) and use Lemma 3.4.6 to conclude.

In the following we will state the estimates separately for each of the three terms in the previous expression. We will only prove the estimates separately for each of the three terms. The last one differs a little from the classical proof.

Homogeneous case

Lemma 3.4.7 There exists a constant $C_{r,R}$ (r and R are fixed) such that if ε is small enough,

$$\nu_0^{\vec{8}} \| S_{\lambda}(t) w_0 \|_{L^2(\mathbb{R}_+, L^{\infty}(\mathbb{R}^3))} \le C_{r, R} \varepsilon^{\frac{1}{8}} \| w_0 \|_{L^2}.$$

Proof: Let us follow here the very same method used in [4] (same method as in [8] except we have to deal with an asymptotic expansion): we first use a duality argument to do the case $L^1(\mathbb{R}_+, L^{\infty}(\mathbb{R}^3))$, then we estimate the $L^{\infty}(\mathbb{R}_+, L^{\infty}(\mathbb{R}^3))$ norm, and conclude with an interpolation.

If
$$\mathcal{B} = \{ \phi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3) \text{ such that } \|\phi\|_{L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3))} \le 1 \}$$
, we can write:

$$\|S_{\lambda}(t)w_0\|_{L^1(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} = \sup_{\phi \in \mathcal{B}} \int_{\mathbb{R}_+ \times \mathbb{R}^3} S_{\lambda}(t)w_0(x)\phi(t,x)dxdt$$

Using the Plancherel inequality, multiplying by the well known function χ (see the beginning of section 3.3):

$$\|S_{\lambda}(t)w_{0}\|_{L^{1}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C \sup_{\phi \in \mathcal{B}} \int_{\mathbb{R}^{3}} \widehat{w_{0}}(\xi) \Big(\int_{\mathbb{R}_{+}} e^{t\lambda(\xi,\varepsilon)} \chi(\xi) \widehat{\phi}(t,\xi) dt \Big) d\xi.$$

The Cauchy-Schwarz inequality gives that:

 $||S_{\lambda}(t)w_0||_{L^1(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))}$

$$\leq C \|\widehat{w_0}\|_{L^2} \sup_{\phi \in \mathcal{B}} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} e^{t\lambda(\xi,\varepsilon)} e^{s\overline{\lambda(\xi,\varepsilon)}} |\chi(\xi)|^2 \widehat{\phi}(t,\xi) \overline{\widehat{\phi}(s,\xi)} dt ds d\xi \right)^{\frac{1}{2}}.$$

Then, we use the asymptotic expansion of λ given by Lemma 3.4.2 and denote by \mathcal{O} the term $(t-s)i\varepsilon S(\xi,\varepsilon) + \varepsilon^2(tS'(\xi,\varepsilon) + s\overline{S'(\xi,\varepsilon)})$ (this term is easily dealt within the proof of the dispersion estimate) and again the Plancherel theorem:

$$\|S_{\lambda}(t)w_{0}\|_{L^{1}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C\|w_{0}\|_{L^{2}}\sup_{\phi\in\mathcal{B}}$$

$$\left(\int_{\mathbb{R}_{+}}\int_{\mathbb{R}^{3}}\int_{\mathbb{R}^{3}}\left(\int_{\mathbb{R}^{3}}e^{-(t+s)\tau(\xi)|\xi|^{2}+\frac{i(t-s)|\xi|_{F}}{\varepsilon F|\xi|}+\mathcal{O}}|\chi(\xi)|^{2}e^{ix.\xi}d\xi\right)(\phi(t)\ast\overline{\phi(s)})(x)dxdtds\right)^{\frac{1}{2}}.$$

Using the dispersion estimate and the fact that $L^1 * L^1 \hookrightarrow L^1$ we obtain that:

$$\begin{split} |S_{\lambda}(t)w_{0}\|_{L^{1}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} &\leq C \|w_{0}\|_{L^{2}} \sup_{\phi \in \mathcal{B}} \\ & \left(\int_{\mathbb{R}_{+}} \int_{\mathbb{R}_{+}} C_{r,R} e^{-\frac{\nu_{0}}{16}r^{2}(t+s)} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}} \|\phi(t)\|_{L^{1}} \|\phi(s)\|_{L^{1}} dt ds \right)^{\frac{1}{2}}. \end{split}$$

The definition of \mathcal{B} allows us to write:

$$\|S_{\lambda}(t)w_{0}\|_{L^{1}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{r,R}\|w_{0}\|_{L^{2}} \sup_{\phi \in \mathcal{B}} \min\left(\int_{\mathbb{R}_{+}} \int_{\mathbb{R}_{+}} e^{-\frac{\nu_{0}}{16}r^{2}(t+s)} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}} dt ds\right)^{\frac{1}{2}}$$

A computation gives that:

$$\int_{\mathbb{R}_{+}} \int_{\mathbb{R}_{+}} e^{-\frac{\nu_{0}}{16}r^{2}(t+s)} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}} dt ds = C' \frac{\varepsilon^{\frac{1}{2}}}{\nu_{0}^{\frac{3}{2}}r^{3}}$$

So if ε is small enough,

$$\|S_{\lambda}(t)w_{0}\|_{L^{1}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{r,R} \frac{\varepsilon^{\frac{1}{4}}}{\nu_{0}^{\frac{3}{4}}} \|w_{0}\|_{L^{2}}.$$

On the other hand, according to the Bernstein Lemma, there exists a constant C_R such that if ε is small enough (this condition is imposed by the last terms of the asymptotic expansion):

$$||S_{\lambda}(t)w_{0}||_{L^{\infty}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{R}||w_{0}||_{L^{2}}$$

then, an interpolation argument gives that for all $1 \le p < \infty$,

$$\|S_{\lambda}(t)w_{0}\|_{L^{p}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{r,R,p} \frac{\varepsilon^{\frac{1}{4p}}}{\nu_{0}^{\frac{3}{4p}}} \|w_{0}\|_{L^{2}}$$

In particular,

$$\|S_{\lambda}(t)w_{0}\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{r,R} \frac{\varepsilon^{\frac{1}{8}}}{\nu_{0}^{\frac{3}{8}}} \|w_{0}\|_{L^{2}}.$$
(3.4.33)

Bilinear inhomogeneous case

Lemma 3.4.8 There exists a constant $C_{r,R}$ (r and R are fixed) such that if ε is small enough,

$$\nu_0^{\frac{3}{8}} \left\| \int_0^t S_{\lambda}(t-t') g^b(t') dt' \right\|_{L^2(\mathbb{R}_+, L^{\infty}(\mathbb{R}^3))} \le C_{r,R} \varepsilon^{\frac{1}{8}} \|g^b\|_{L^1 L^2}$$

Proof: We will use the homogeneous case:

$$\begin{aligned} \left\| \int_{0}^{t} S_{\lambda}(t-t')g^{b}(t')dt' \right\|_{L^{1}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} &= \int_{0}^{\infty} \| \int_{0}^{t} S_{\lambda}(t-t')g^{b}(t')dt' \|_{L^{\infty}(\mathbb{R}^{3})}dt \\ &\leq \int_{0}^{\infty} \int_{0}^{t} \| S_{\lambda}(t-t')g^{b}(t') \|_{L^{\infty}(\mathbb{R}^{3})}dt'dt \end{aligned}$$

Then, using the Fubini theorem, we get:

$$\left\|\int_0^t S_{\lambda}(t-t')g^b(t')dt'\right\|_{L^1(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \le \int_0^{\infty} \left(\int_{t'}^{\infty} \|S_{\lambda}(t-t')g^b(t')\|_{L^{\infty}(\mathbb{R}^3)}dt\right)dt'$$

A change of variable $(t \mapsto t - t')$ followed by the use of the estimate obtained in the previous section give:

$$\begin{split} \left\| \int_{0}^{t} S_{\lambda}(t-t') g^{b}(t') dt' \right\|_{L^{1}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} &\leq \int_{0}^{\infty} \| S_{\lambda}(t) g^{b}(t') \|_{L^{1}_{t}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} dt dt' \\ &\leq \int_{0}^{\infty} C_{r,R} \frac{\varepsilon^{\frac{1}{4}}}{\nu_{0}^{\frac{3}{4}}} \| g^{b}(t') \|_{L^{2}(\mathbb{R}^{3})} dt', \end{split}$$

which gives

$$\left\|\int_0^t S_{\lambda}(t-t')g^b(t')dt'\right\|_{L^1(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \le C_{r,R}\frac{\varepsilon^{\frac{1}{4}}}{\nu_0^{\frac{3}{4}}}\|g^b(t')\|_{L^1(\mathbb{R}_+,L^2(\mathbb{R}^3))}.$$

Like previously we can easily show that there exists a constant C_R such that, if ε is small enough,

$$\left\| \int_0^t S_{\lambda}(t-t')g^b(t')dt' \right\|_{L^{\infty}(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \le C_R \|g^b(t')\|_{L^1(\mathbb{R}_+,L^2(\mathbb{R}^3))}.$$

So, with interpolation we have that for all $1 \leq p < \infty$,

$$\left\| \int_0^t S_{\lambda}(t-t')g^b(t')dt' \right\|_{L^p(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \le C_{r,R,p} \frac{\varepsilon^{\frac{1}{4p}}}{\nu_0^{\frac{3}{4p}}} \|g^b\|_{L^1L^2}.$$

In particular,

$$\left\| \int_{0}^{t} S_{\lambda}(t-t')g^{b}(t')dt' \right\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{r,R} \frac{\varepsilon^{\frac{1}{8}}}{\nu_{0}^{\frac{3}{8}}} \|g^{b}\|_{L^{1}L^{2}}.$$
 (3.4.34)

Linear inhomogeneous case

Lemma 3.4.9 There exists a constant (r and R are fixed) such that if ε is small enough,

$$\nu_0^{\frac{3}{4}} \left\| \int_0^t S_{\lambda}(t-t')g^l(t')dt' \right\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \le C_{r,R}\varepsilon^{\frac{1}{4}} \|g^l\|_{L^2L^2}$$

Proof: This case is a little more complicated because we want a different bound on the right-hand side. We will directly prove the $L^2 L^{\infty}$ estimate without using interpolation arguments like previously.

If we note $\mathcal{B} = \{ \phi \in \mathcal{C}_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^3) \text{ such that } \|\phi\|_{L^2(\mathbb{R}_+, L^1(\mathbb{R}^3))} \leq 1 \}$, then we have:

$$\left\|\int_0^t S_{\lambda}(t-t')g^l(t')dt'\right\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} = \sup_{\phi\in\mathcal{B}}\int_0^{\infty}\int_{\mathbb{R}^3} \left(\int_0^t S_{\lambda}(t-t')g^l(t')dt'\right)\phi(t,x)dxdt.$$

The Fubini theorem allows us to write that:

$$\left\|\int_0^t S_{\lambda}(t-t')g^l(t')dt'\right\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} = \sup_{\phi\in\mathcal{B}}\int_0^{\infty}\int_{t'}^{\infty}\int_{\mathbb{R}^3} S_{\lambda}(t-t')g^l(t')\phi(t,x)dtdxdt'.$$

The Plancherel theorem gives that (with the same notation \mathcal{O} as in the homogeneous case):

$$\begin{split} \left\| \int_0^t S_{\lambda}(t-t')g^l(t')dt' \right\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \\ &\leq C \sup_{\phi\in\mathcal{B}} \int_0^{\infty} \int_{t'}^{\infty} \int_{\mathbb{R}^3} e^{-(t-t')\tau(\xi)|\xi|^2 + i(t-t')\frac{|\xi|_F}{\varepsilon F|\xi|} + \mathcal{O}} \widehat{g^l(t')}\widehat{\phi}(t,\xi)\chi(\xi)d\xi dt dt' \end{split}$$

Another use of the Fubini interversion theorem implies:

$$\begin{split} \left\| \int_0^t S_{\lambda}(t-t')g^l(t')dt' \right\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \\ &\leq C \sup_{\phi\in\mathcal{B}} \int_0^{\infty} \Big(\int_{\mathbb{R}^3} \widehat{g^l(t')} \int_{t'}^{\infty} e^{-(t-t')\tau(\xi)|\xi|^2 + i(t-t')\frac{|\xi|_F}{\varepsilon F|\xi|} + \mathcal{O}} \widehat{\phi}(t,\xi)\chi(\xi)d\xi dt \Big) dt' \end{split}$$

The Cauchy-Schwarz inequality (in space) and the writing of the square of the L^2 norm bring us to:

$$\left\| \int_0^t S_{\lambda}(t-t')g^l(t')dt' \right\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} \leq C \sup_{\phi \in \mathcal{B}} \int_0^{\infty} \|g^l(t')\|_{L^2(\mathbb{R}^3)} \\ \left(\int_{\mathbb{R}^3} \int_{t'}^{\infty} \int_{t'}^{\infty} e^{-(t+s-2t')\tau(\xi)|\xi|^2 + i(t-s)\frac{|\xi|_F}{\varepsilon F|\xi|} + \mathcal{O}} |\chi(\xi)|^2 \widehat{\phi}(t,\xi)\overline{\widehat{\phi}(s,\xi)} dt ds d\xi \right)^{\frac{1}{2}} dt'.$$

The Cauchy-Schwarz inequality (in time) allows us to write:

$$\left\|\int_{0}^{t} S_{\lambda}(t-t')g^{l}(t')dt'\right\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C\left(\int_{0}^{\infty}\|g^{l}(t')\|_{L^{2}(\mathbb{R}^{3})}^{2}dt'\right)^{\frac{1}{2}}$$

$$\sup_{\phi\in\mathcal{B}}\left(\int_0^\infty\int_{\mathbb{R}^3}\int_{t'}^\infty\int_{t'}^\infty e^{-(t+s-2t')\tau(\xi)|\xi|^2+i(t-s)\frac{|\xi|_F}{\varepsilon F|\xi|}+\mathcal{O}}|\chi(\xi)|^2\widehat{\phi}(t,\xi)\overline{\widehat{\phi}(s,\xi)}dtdsd\xi dt'\right)^{\frac{1}{2}}.$$

Thanks to the Fubini and the Plancherel theorems:

$$\begin{split} \left\| \int_0^t S_{\lambda}(t-t')g^l(t')dt' \right\|_{L^2(\mathbb{R}_+,L^{\infty}(\mathbb{R}^3))} &\leq C \|g^l(t')\|_{L^2(\mathbb{R}_+,L^2(\mathbb{R}^3))} \\ \sup_{\phi\in\mathcal{B}} \left(\int_0^{\infty} \int_{t'}^{\infty} \int_{t'}^{\infty} \left[\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} e^{ix.\xi} e^{-(t+s-2t')\tau(\xi)|\xi|^2 + i(t-s)\frac{|\xi|_F}{\varepsilon F|\xi|} + \mathcal{O}} |\chi(\xi)|^2 d\xi \right) \right. \\ \left. \left. \left(\phi(t) * \overline{\phi}(s) \right)(x) dx \right] dt ds dt' \right)^{\frac{1}{2}}. \end{split}$$

Where $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$ is the usual convolution. Then, the dispersive estimate, and the fact that $L^1 * L^1 \hookrightarrow L^1$ imply:

$$\begin{aligned} \left\| \int_{0}^{t} S_{\lambda}(t-t') g^{l}(t') dt' \right\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} &\leq C \|g^{l}(t')\|_{L^{2}(\mathbb{R}_{+},L^{2}(\mathbb{R}^{3}))} \\ \sup_{\phi \in \mathcal{B}} \left(\int_{0}^{\infty} \int_{t'}^{\infty} \int_{t'}^{\infty} C_{r,R} e^{-\frac{\nu_{0}}{16}r^{2}(t+s-2t')} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}} \|\phi(t)\|_{L^{1}(\mathbb{R}^{3})} \|\phi(s)\|_{L^{1}(\mathbb{R}^{3})} dt ds dt' \right)^{\frac{1}{2}} \end{aligned}$$

And

$$\left\| \int_{0}^{t} S_{\lambda}(t-t')g^{l}(t')dt' \right\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{r,R} \|g^{l}(t')\|_{L^{2}(\mathbb{R}_{+},L^{2}(\mathbb{R}^{3}))} \sup_{\phi \in \mathcal{B}} \left(\varepsilon^{\frac{1}{2}}I\right)^{\frac{1}{2}}$$

Where

$$I \stackrel{\text{def}}{=} \int_0^\infty \int_{t'}^\infty \int_{t'}^\infty \frac{e^{-\frac{\nu_0}{16}r^2(t+s-2t')}}{|t-s|^{\frac{1}{2}}} \|\phi(t)\|_{L^1(\mathbb{R}^3)} \|\phi(s)\|_{L^1(\mathbb{R}^3)} dt ds dt'$$

All we need is to compute this integral:

$$I = \int_0^\infty \int_0^\infty \int_0^\infty \mathbf{1}_{\{t' \le t\}} \mathbf{1}_{\{t' \le s\}} \frac{e^{-\frac{\nu_0}{16}r^2(t+s-2t')}}{|t-s|^{\frac{1}{2}}} \|\phi(t)\|_{L^1(\mathbb{R}^3)} \|\phi(s)\|_{L^1(\mathbb{R}^3)} dt ds dt'$$

The Fubini theorem implies:

$$I = \int_0^\infty \int_0^\infty \frac{e^{-\frac{\nu_0}{16}r^2(t+s)}}{|t-s|^{\frac{1}{2}}} \|\phi(t)\|_{L^1(\mathbb{R}^3)} \|\phi(s)\|_{L^1(\mathbb{R}^3)} \Big(\int_0^\infty \mathbf{1}_{\{t' \le t\}} \mathbf{1}_{\{t' \le s\}} e^{-\frac{\nu_0}{16}r^2(-2t')} dt'\Big) dt ds$$

Calculating the integral in t', which is exactly $\int_0^{\min(s,t)} e^{\frac{\nu_0}{8}r^2t'} dt'$ gives

$$I \leq \int_0^\infty \int_0^\infty \frac{e^{-\frac{\nu_0}{16}r^2(t+s)}}{|t-s|^{\frac{1}{2}}} \|\phi(t)\|_{L^1(\mathbb{R}^3)} \|\phi(s)\|_{L^1(\mathbb{R}^3)} \Big(\frac{e^{2\frac{\nu_0}{16}r^2\min(s,t)}}{\frac{\nu_0}{8}r^2}\Big) dt ds$$

As s + t - 2min(s, t) = |t - s|, we have:

$$\begin{split} I &\leq \frac{8}{r^2 \nu_0} \int_0^\infty \int_0^\infty \frac{e^{-\frac{\nu_0}{16}r^2 |t-s|}}{|t-s|^{\frac{1}{2}}} \|\phi(t)\|_{L^1(\mathbb{R}^3)} \|\phi(s)\|_{L^1(\mathbb{R}^3)} dt ds \\ &\leq \frac{8}{r^2 \nu_0} \int_0^\infty \|\phi(t)\|_{L^1(\mathbb{R}^3)} \Big(\int_0^\infty \frac{e^{-\frac{\nu_0}{16}r^2 |t-s|}}{|t-s|^{\frac{1}{2}}} \|\phi(s)\|_{L^1(\mathbb{R}^3)} ds \Big) dt \end{split}$$

Let us define the two following functions:

$$\begin{cases} a: y \longmapsto \frac{e^{-\frac{\nu_0}{16}r^2|y|}}{|y|^{\frac{1}{2}}} \in L^1(\mathbb{R}) \\ b: y \longmapsto \|\phi(y)\|_{L^1} \mathbf{1}_{\{y \ge 0\}} \in L^2(\mathbb{R}) \end{cases}$$

Then $a * b \in L^2$ and the Cauchy-Schwarz inequality gives:

$$I \le \frac{8}{r^2 \nu_0} \|b\|_{L^2}^2 \|a\|_{L^1} \le \frac{8}{r^2 \nu_0} \|\phi\|_{L^2 L^1}^2 \frac{8\sqrt{\pi}}{\sqrt{\nu_0} r} \le \frac{C}{r^3 \nu_0^{\frac{3}{2}}}$$

So, if ε is small enough,

$$\left\| \int_{0}^{t} S_{\lambda}(t-t')g^{l}(t')dt' \right\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{r,R} \frac{\varepsilon^{\frac{1}{4}}}{\nu_{0}^{\frac{3}{4}}} \|g^{l}\|_{L^{2}L^{2}}$$
(3.4.35)

Conclusion

According to the previous results and the Duhamel formula, we can write that:

$$\|W_{\varepsilon}^{r,R}\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq C_{r,R} \frac{\varepsilon^{\frac{1}{8}}}{\nu_{0}^{\frac{3}{8}}} \left(\|w_{0}\|_{L^{2}} + \|g^{b}\|_{L^{1}L^{2}} \right) + C_{r,R} \frac{\varepsilon^{\frac{1}{4}}}{\nu_{0}^{\frac{3}{4}}} \|g^{l}\|_{L^{2}L^{2}}$$
(3.4.36)

In order to achieve the proof of the Strichartz estimates we still have to estimate the norms of w_0 , g^g , and g^l (see (3.4.32) for their definition): as $supp(\widehat{w}_0) \subset \mathcal{C}_{r,R}$, there exists a constant C such that: $||w_0||_{L^2} \leq \frac{C}{r} ||w_0||_{\dot{H}^1}$ and using Lemma 3.4.6 and (3.4.32), we have:

$$\|w_0\|_{L^2} \le \frac{C}{r} (1 + |\nu - \nu'|\varepsilon C_{r,R}) \|U_0\|_{\dot{H}^1}.$$

The same argument, and the use of estimates close to those from Lemma 3.3.4 imply :

$$\|g^b\|_{L^1L^2} \le \frac{C}{r\nu_0} (1 + |\nu - \nu'|\varepsilon C_{r,R}) \|U_0\|_{\dot{H}^1}^2$$

And similarly we also obtain that

$$\|g^l\|_{L^2L^2} \le \frac{C}{\sqrt{\nu_0}} (1 + |\nu - \nu'|\varepsilon C_{r,R}) \|U_0\|_{\dot{H}^1}.$$

Using these estimates in (3.4.36) achieves the proof of the Strichartz estimates and we obtain ($\varepsilon \leq 1$):

$$\|W_{\varepsilon}^{r,R}\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq \varepsilon^{\frac{1}{8}}K^{r,R},$$

which concludes the proof of Lemma 3.3.5. \blacksquare

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Chapitre 4

Poches de tourbillon

Résumé: On s'intéresse dans la première partie de cet article à l'estimation, explicite en fonction du nombre de Rossby ε , de la vitesse de convergence de la solution du système des équations primitives vers la solution unique du système quasigéostrophique, avec de meilleurs résultats dans le cas $\nu = \nu'$. La deuxième partie est consacrée à la preuve de la persistence, lorsque le nombre de Rossby tend vers zéro, de la structure de régularité tangentielle pour les équations primitives visqueuses et diffusives.

4.1 Introduction

4.1.1 The primitive equations

The primitive system writes:

$$(PE_{\varepsilon}) \qquad \begin{cases} \partial_t U_{\varepsilon} + v_{\varepsilon} \cdot \nabla U_{\varepsilon} - LU_{\varepsilon} + \frac{1}{\varepsilon} \mathcal{A} U_{\varepsilon} = \frac{1}{\varepsilon} (-\nabla \Phi_{\varepsilon}, 0) \\ div \ v_{\varepsilon} = 0 \\ U_{\varepsilon/t=0} = U_{0,\varepsilon}. \end{cases}$$

The unknowns are U_{ε} and Φ_{ε} . We denote by U_{ε} a pair $(v_{\varepsilon}, \theta_{\varepsilon})$ where v_{ε} is a vector field on \mathbb{R}^3 (three dimensional velocity), θ_{ε} a scalar function (the density fluctuation : in the case of the atmosphere it depends on the scalar (potential) temperature and in the case of the ocean it depends on the temperature and the salinity), and Φ_{ε} the pressure, all of them depending on (t, x). The operator L is defined by

$$LU_{\varepsilon} \stackrel{\text{def}}{=} (\nu \Delta v_{\varepsilon}, \nu' \Delta \theta_{\varepsilon}),$$

and the matrix \mathcal{A} by:

$$\mathcal{A} \stackrel{\text{def}}{=} \left(\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{array} \right).$$

In the first two parts of this paper we will follow the methods of [5] and get more precise results on the convergence speed (with additionnal assumptions). In the case $\nu = \nu'$ the speed of convergence will be much better. Let us decompose the initial data into:

$$U_{0,\varepsilon} = U_{0,\varepsilon,QG} + U_{0,\varepsilon,osc},$$

where $U_{0,QG} \in H^{1+\beta}$ $(\beta > 0)$, $||U_{0,\varepsilon,QG} - U_{0,QG}||_{H^1} \leq K^0 \varepsilon$, and $U_{0,\varepsilon,osc}$ is regular and we allow its norm to blow up when ε goes to zero (see the statement of the theorems for more details). We will show that, as ε goes to zero, U_{ε} goes to the solution of the quasigeostrophic system:

$$(\widetilde{QG}) \qquad \qquad \begin{cases} \partial_t \widetilde{U_{QG}} - \Gamma \widetilde{U_{QG}} = -\begin{pmatrix} -\partial_2 \\ \partial_1 \\ 0 \\ -F \partial_3 \end{pmatrix} \Delta_F^{-1} (\widetilde{U_{QG}} \cdot \nabla \widetilde{\Omega_{QG}}) \\ \widetilde{U_{QG}}_{/t=0} = U_{0,QG} \end{cases}$$

where Γ is the pseudo-differential operator of order two defined by :

$$\Gamma \stackrel{\text{def}}{=} \Delta \Delta_F^{-1} (\nu \partial_1^2 + \nu \partial_2^2 + \nu' \partial_3^2).$$

The third part is devoted to the persistence of the tangential regularity. We will consider vortex patches.

4.1.2 Statement of the results

In the following, a constant depending only on the initial data, F, ν or ν' will be noted K^0 , and the notation $\|u\|_{L^p\dot{H}^s}$ means $\|u\|_{L^p(\mathbb{R}_+,\dot{H}^s(\mathbb{R}^3)}$. The following theorem is an easy adaptation of Theorem 3.1.2 from the previous chap-

The following theorem is an easy adaptation of Theorem 3.1.2 from the previous chapter.

Theorem 4.1.1 Let $\beta > 0$ be given and assume $U_{0,QG} \in H^{1+\beta}(\mathbb{R}^3)$, then the limit system has a unique solution, global in time and belonging to the space $L^{\infty}(\mathbb{R}_+, H^{1+\beta}) \cap L^2(\mathbb{R}_+, H^{2+\beta})$ and satisfying for all $s \in [0, 1+\beta]$ ($\nu_0 = \min(\nu, \nu')$):

$$\forall t \in \mathbb{R}_+, \|\widetilde{U_{QG}}(t)\|_{\dot{H}^s}^2 + 2c\nu_0 \int_0^t \|\widetilde{U_{QG}}(t')\|_{\dot{H}^{s+1}}^2 dt' \le C(U_{0,QG})$$
(4.1.1)

where $\nu_0 = \min(\nu, \nu')$

Let us state our first result :

Theorem 4.1.2 Let $\beta > 0$, $\alpha > 0$ and assume that $U_{0,\varepsilon} = U_{0,\varepsilon,QG} + U_{0,\varepsilon,osc}$, where

- $||U_{0,\varepsilon,QG} U_{0,QG}||_{H^1} \le C\varepsilon$, with $U_{0,QG} \in H^{1+\beta}$
- $U_{0,\varepsilon,osc} \in L^1 \cap \dot{H}^{\frac{1}{2}} \cap \dot{H}^{1+\beta}$, regular but with blowing up norms :

$$\|U_{0,\varepsilon,osc}\|_{L^1} + \|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}}} + \|U_{0,\varepsilon,osc}\|_{\dot{H}^{1+\beta}} \le \alpha \log |\log \varepsilon|.$$

Let us define for $s \in \mathbb{R}$, $E^s \stackrel{\text{def}}{=} L^{\infty}(\mathbb{R}_+, \dot{H}^s) \cap L^2(\mathbb{R}_+, \dot{H}^{s+1})$ and let W_{ε} be a solution of the following linear system:

$$\begin{cases} \partial_t W_{\varepsilon} - L W_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_{\varepsilon} = -G \\ W_{\varepsilon/t=0} = U_{0,\varepsilon,osc} \end{cases}$$
(4.1.2)

with
$$G \stackrel{\text{def}}{=} \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}, \nabla \widetilde{U_{QG}}) - F(\nu - \nu')\Delta \Delta_F^{-2} \begin{pmatrix} -F\partial_2 \partial_3^2 \\ F\partial_1 \partial_3^2 \\ 0 \\ (\partial_1^2 + \partial_2^2)\partial_3 \end{pmatrix} \widetilde{\Omega_{QG}}.$$
 (4.1.3)

Then for all M > 0, there exist a constant $K^0 > 0$ and $\varepsilon_0 > 0$ (depending on β and M) such that, if we define $\omega = \beta M - \alpha K^0$ and if $\alpha < \frac{\beta M}{K^0}$, we have the following results:

- W_{ε} exists globally, is unique in the space E^s for every $s \in [\frac{1}{2}, 1]$ and if $\varepsilon \leq \varepsilon_0$, $\|W_{\varepsilon}\|_{L^2(\mathbb{R}_+, L^{\infty})} \leq K^0 |\log \varepsilon|^{-\omega}$.
- If we denote by $\gamma_{\varepsilon} \stackrel{\text{def}}{=} U_{\varepsilon} \widetilde{U_{QG}} W_{\varepsilon}$, then if ε is small enough, $\gamma_{\varepsilon} \in E^s$ for every $s \in [\frac{1}{2}, 1]$ and goes to zero in E^s , for every $s \in [\frac{1}{2}, 1]$: $\|\gamma_{\varepsilon}\|_{E^s} \leq K^0 |\log \varepsilon|^{-\omega}$.
- Finally if ε is small enough, U_{ε} is defined for all time and $U_{\varepsilon} \widetilde{U_{QG}} = \gamma_{\varepsilon} + W_{\varepsilon}$ satisfies $\|U_{\varepsilon} \widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+, L^{\infty})} \leq K^0 |\log \varepsilon|^{-\omega}$.

The structure of the proof of this theorem will be very close to the one from the previous chapter so we will mainly focus on the differences induced by the fact that the initial data depend on the Rossby number and we will concentrate on the Strichartz estimates.

When the viscosity ν is equal to the diffusivity ν' many simplifications arise (first of all $L = \nu \Delta = \Gamma$ and the eigenvalues of the linear system are explicit) allowing us to get better results :

Theorem 4.1.3 Assume $\nu = \nu'$ and that $U_{0,\varepsilon} = U_{0,\varepsilon,QG} + U_{0,\varepsilon,osc}$, where $||U_{0,\varepsilon,QG} - U_{0,QG}||_{H^1} \leq C\varepsilon$, $U_{0,QG} \in H^{1+\beta}$ and $||U_{0,\varepsilon,osc}||_{H^{1+\beta}} \leq \alpha |\log \varepsilon|$. Let E^s and W_{ε} be the same as in the previous theorem, G being simplified into :

$$G = \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}, \nabla \widetilde{U_{QG}}) \tag{4.1.4}$$

Then there exist a constant $K^0 > 0$ and $\varepsilon_0 > 0$ such that if we define $\gamma = \frac{1}{16(1+\beta)}$ and $\omega = \beta \gamma - \alpha K^0$ and if $\alpha < \frac{\beta \gamma}{K^0}$ and $\varepsilon \leq \varepsilon_0$ we have the following results:

- W_{ε} exists globally, is unique in the space E^s for every $s \in [\frac{1}{2}, 1]$ and we have the estimate $||W_{\varepsilon}||_{L^2(\mathbb{R}_+, L^{\infty})} \leq K^0 \varepsilon^{\omega}$.
- If we denote by $\delta_{\varepsilon} \stackrel{\text{def}}{=} U_{\varepsilon} \widetilde{U_{QG}} \chi(\frac{|D|}{R_{\varepsilon}})W_{\varepsilon}$, where $R_{\varepsilon} = \varepsilon^{-\gamma}$, then $\delta_{\varepsilon} \in E^s$ for every $s \in [\frac{1}{2}, 1]$ and $\|\delta_{\varepsilon}\|_{E^s} \leq K^0 \varepsilon^{\omega} \quad \forall s \in [\frac{1}{2}, 1]$.
- Finally if $\varepsilon \leq \varepsilon_0$, $\|U_{\varepsilon} \widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+, L^{\infty})} \leq K^0 \varepsilon^{\omega}$.

4.1.3 Vortex patches $(\nu = \nu')$

In this section, C^s is the usual Hölder space. We will only use the definition involving the Littlewood-Paley theory (we refer to [7] for a complete presentation of the theory) :

$$C^{s} = \{ u \in \mathcal{S}'(\mathbb{R}^{3}), \quad \|u\|_{C^{s}} \stackrel{\text{def}}{=} \sup_{q \ge -1} 2^{qs} \|\Delta_{q} u\|_{L^{\infty}} < \infty \},$$

where Δ_q is the usual frequency localization operator defined as follows : consider two regular functions χ and ϕ whose supports are respectively a ball and an annulus, such that for all $\xi \in \mathbb{R}^3$,

$$\chi(\xi) + \sum_{q \ge 0} \phi(2^{-q}\xi) = 1$$
 and $\frac{1}{3} \le \chi(\xi)^2 + \sum_{q \ge 0} \phi(2^{-q}\xi)^2 \le 1.$

Then for all tempered ditribution we define :

•
$$\Delta_{-1} = \mathcal{F}^{-1}(\chi(\xi)\hat{u}(\xi))$$

• $\forall q \leq -2, \quad \Delta_q = 0$

•
$$\forall q \ge 0$$
, $\Delta_q = \mathcal{F}^{-1}(\phi(2^{-q}\xi)\hat{u}(\xi))$ and $S_q u = \sum_{p < q-1} \Delta_p u = \chi(2^{-q}D)u$.

•
$$\forall q \in \mathbb{Z}$$
, $\dot{\Delta}_q = \mathcal{F}^{-1}(\phi(2^{-q}\xi)\hat{u}(\xi))$ and $\dot{S}_q u = \sum_{p < q-1} \dot{\Delta}_p u = \chi(2^{-q}D)u$.

A Hölder space is a particular case of a Besov space $C^s = B^s_{\infty,\infty}$, where :

$$B_{p,r}^{s} = \{ u \in \mathcal{S}'(\mathbb{R}^{3}), \quad \|u\|_{B_{p,r}^{s}} \stackrel{\text{def}}{=} \| \left(2^{qs} \|\Delta_{q} u\|_{L^{p}} \right)_{q \geq -1} \|_{\ell^{r}} < \infty \},$$

Concerning the vortex patches we also refer to [7] and [15] for a description of the persistence of the vortex patches structure in the case of the Euler system, to [10] and [17] for the case of the Navier-Stokes system, and to [11] for the case of the primitive equations when $\nu = \nu' = 0$. We choose to take here the definitions of vortex patches and tangential regularity of [11] : a vortex patch will be defined with respect to the scalar potential vorticity instead of the vorticity (rotational of the velocity). Basically the potential vorticity is a vortex patch if it is the characteristic function of a regular open set :

Definition 4.1.1 We say that Ω_0 is a vortex patch of class C^s if, for some $s \in [0, 1[$,

$$\Omega_0 = \Omega_{0,i} \mathbf{1}_D + \Omega_{0,e} \mathbf{1}_{\mathbb{R}^3 - D},$$

where $\Omega_{0,i} \in C^{s}(\overline{D}), \Omega_{0,e} \in C^{s}(\mathbb{R}^{3} - D)$ and D is an open bounded domain of class C^{s+1} .

Following those papers we will state a general theorem of persistence of the geometrical structures for the primitive equations in the case $\nu = \nu'$, and the concept of tangential regularity with respect to a set X of vector fields will be important :

Definition 4.1.2 If $X = (X_{\lambda})_{\lambda=1,\dots,N}$ is a finite family of vector fields we will say that this family is admissible if and only if (\wedge is the usual vectorial product in \mathbb{R}^3):

$$[X]^{-1} \stackrel{\text{def}}{=} \left(\frac{2}{N(N-1)} \sum_{\lambda < \lambda'} |X_{\lambda} \wedge X_{\lambda'}|^2\right)^{-\frac{1}{4}} < \infty.$$

If $s \in [0, 1]$ and X is an admissible family of vector fields C^s we define the space :

$$C^{s}(X) = \{ w \in L^{\infty} \text{ such that } X_{\lambda}(x, D) w \stackrel{\text{def}}{=} \operatorname{div}(w \otimes X_{\lambda}) \in C^{s-1} \}$$

and as corresponding norm we take :

$$\|w\|_{C^{s}(X)} \stackrel{\text{def}}{=} \|w\|_{L^{\infty}} + \|[X]^{-1}\|_{L^{\infty}} + \sum_{\lambda=1}^{N} \left(\|X_{\lambda}\|_{C^{s}} + \|X_{\lambda}(x,D)w\|_{C^{s-1}}\right).$$
(4.1.5)

Remark 4.1.1 We took here, with a view of simplicity the same definition as [11] for $X_{\lambda}(x, D)w$.

Let us now state the result :

Theorem 4.1.4 Suppose that:

- Ω_0 is a C^s vortex patch with $s \in]0, 1[$,
- $U_{0,QG} \in L^2$ is a quasigeostrophic vector field such that $\Omega(U_{0,QG}) = \Omega_0 \in L^2(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3)$,
- $U_{0,\varepsilon,osc}$ is a family of regular oscillating vector fields,

• and define $U_{0,\varepsilon,QG}$ as a regularization of $U_{0,QG}$: $||U_{0,\varepsilon,QG} - U_{0,QG}||_{L^2} \leq \varepsilon$ (for example $U_{0,\varepsilon,QG} = \chi(\varepsilon|D|)U_{0,QG}$). So $U_{0,\varepsilon} = U_{0,\varepsilon,QG} + U_{0,\varepsilon,osc}$.

Assume that $X_0 = \{X_{0,\lambda}, \lambda = 1, ..., N\}$ is an admissible system of C^s vector fields, and that there exists a constant $C_0 > 0$ such that :

 $\|\Omega_{0,\varepsilon}\|_{L^2} \le C_0, \quad \|\Omega_{0,\varepsilon}\|_{C^s(X_0)} \le C_0, \quad \|U_{0,\varepsilon}\|_{H^5} \le C_0 \varepsilon^{-\alpha}, \quad \text{where} \quad 0 < \alpha < \frac{1}{64}.$

Then there exist two constants $\gamma_0 = \gamma_0(F, s)$ and $C_{F,s} > 0$, a time T_{ε}^{γ} depending on the Rossby number, $T_{\varepsilon}^{\gamma} = \gamma \log |\log \varepsilon|$ where $0 < \gamma \leq \gamma_0$, and $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the lifespan T_{ε}^{*} of the solution U_{ε} satisfies $T_{\varepsilon}^{*} > T_{\varepsilon}^{\gamma}$ and :

- $\|U_{\varepsilon,osc}\|_{L^8_{T^{\gamma}_{\varepsilon}}(Lip)} \leq C_{F,s}\varepsilon^{\frac{1}{32}-2\alpha},$
- For all $t \leq T_{\varepsilon}^{\gamma}$, $V_{\varepsilon}(t) = \int_{0}^{t} \|v_{\varepsilon}(\tau)\|_{Lip} d\tau \leq 2|\log \varepsilon|^{\gamma C_{F,s}}$,
- $||U_{\varepsilon,QG}||_{L^1_{\tau^\gamma}(Lip)} \le 2|\log \varepsilon|^{6\gamma C_{F,s}},$
- locally in time, if T > 0, there exists a constant C_T such that if ε is small enough (the neighbourhood of zero depending also on T), we have : $\int_0^T \|v_{\varepsilon}(\tau)\|_{Lip} d\tau + \|U_{\varepsilon,QG}\|_{L_T^1(Lip)} \leq C_T$, and $\|U_{\varepsilon,osc}\|_{L_T^8(Lip)} \leq C_T \varepsilon^{\frac{1}{32}-\alpha}$.
- we have local convergence : for all T > 0, $U_{\varepsilon,QG}$ converges in $L^{\infty}([0,T],L^2)$ to a Lipschitzian solution of the quasigeostrophic system with $U_{0,QG}$ as initial data.

Remark 4.1.2 As $\nu = \nu'$, a vertor field which has a null potential vorticity is automatically divergence-free.

This paper consists in three sections: in the first section we will separate the two cases $(\nu \neq \nu' \text{ and } \nu = \nu')$ and for each we will adapt the methods developped in [4] and [5] to prove Theorems 3 and 4. The third one is devoted to the vortex patches, and we put all the technical results (spectral majorations, a priori estimates, Strichartz estimates) in the appendix.

4.2 Proof of Theorem 4.1.2

4.2.1 Preliminaries

We will use here the same arguments as in [5], except that we will localize the frequencies in $C_{r,R}$ where r and R are not fixed radiuses but depend on the Rossby number $r_{\varepsilon} = |\log \varepsilon|^{-m}$ and $R_{\varepsilon} = |\log \varepsilon|^{M}$. We will take M = 1 and m = 10 but will leave these variables to see the dependency.

We define $\mathcal{P}_{r,R}$ as the following frequency cut-off :

$$\mathcal{P}_{r,R} = \chi(\frac{|D|}{R})(1-\chi(\frac{|D_3|}{r})),$$

with χ a fixed C^{∞} function whose support is included in [-1, 1] and equal to 1 in $[-\frac{1}{2}, \frac{1}{2}]$, and $(\mathcal{F}^{-1}$ is the inverse Fourier transform) :

$$\chi(|D|)f = \mathcal{F}^{-1}(\chi(|\xi|)\hat{f}(\xi)), \text{ and } \chi(|D_3|)f = \mathcal{F}^{-1}(\chi(|\xi_3|)\hat{f}(\xi)).$$

Let us begin with the statement of the following proposition whose proof is given in the appendix (section 4.5) together with the notation \mathbb{P}_i , for i = 1, ..., 4 which consists in the spectral projectors onto the matrix $\widehat{L - \frac{1}{\varepsilon}\mathbb{P}A}$, and $\mathbb{P}_{3+4} = \mathbb{P}_3 + \mathbb{P}_4$.

Proposition 4.2.1 There exist a constant C_F and $\varepsilon_0 = \varepsilon_0(F, \nu - \nu', m, M) > 0$ such that for all $\varepsilon \leq \varepsilon_0$:

• for all divergence-free vector field $f \in \dot{H}^s$, for i = 3, 4,

$$\|\mathbb{P}_i \mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}} f\|_{\dot{H}^s} \le C_F |\log \varepsilon|^{M+m} \|f\|_{\dot{H}^s},$$

• for all divergence-free and potential vorticity-free vector field $f \in \dot{H}^s$,

$$\begin{split} \|\mathbb{P}_{2}\mathcal{P}_{r_{\varepsilon},R_{\varepsilon}}f\|_{\dot{H}^{s}} &\leq C_{F}|\nu-\nu'|\varepsilon^{\frac{1}{2}}\|f\|_{\dot{H}^{s}},\\ \text{and} \quad \|\mathbb{P}_{3+4}\mathcal{P}_{r_{\varepsilon},R_{\varepsilon}}f\|_{\dot{H}^{s}} &\leq (1+C_{F}|\nu-\nu'|\varepsilon^{\frac{1}{2}})\|f\|_{\dot{H}^{s}}. \end{split}$$

4.2.2 The different systems

Let us write all the systems we will consider in the following (we refer to (4.1.3) for the expression of G.):

$$\begin{cases} \partial_t \widetilde{U_{QG}} - L\widetilde{U_{QG}} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \widetilde{U_{QG}} = -\mathbb{P}(\widetilde{U_{QG}} \cdot \nabla \widetilde{U_{QG}}) + G \\ \widetilde{U_{QG}}_{/t=0} = U_{0,QG} \in \dot{H}^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3), \end{cases}$$
(4.2.6)

$$\begin{cases} \partial_t W_{\varepsilon} - L W_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_{\varepsilon} = -G \\ W_{\varepsilon/t=0} = U_{0,\varepsilon,osc} \end{cases}$$
(4.2.7)

$$\begin{cases} \partial_t W_{\varepsilon}^T - L W_{\varepsilon}^T + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_{\varepsilon}^T = -\mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}} \mathbb{P}_{3+4} G \\ W_{\varepsilon}^T |_{t=0} = \mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}} \mathbb{P}_{3+4} U_{0, \varepsilon, osc}. \end{cases}$$
(4.2.8)

Then we define the two following quantities :

$$\begin{cases} \delta_{\varepsilon} \stackrel{\text{def}}{=} U_{\varepsilon} - \widetilde{U_{QG}} - W_{\varepsilon}^{T} \\ \delta_{\varepsilon}' \stackrel{\text{def}}{=} W_{\varepsilon} - W_{\varepsilon}^{T}. \end{cases}$$
(4.2.9)

From (PE_{ε}) , (4.2.6), (4.2.7) and (4.2.8) they satisfy the following system :

$$\begin{cases} \partial_t \delta'_{\varepsilon} - L \delta'_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \delta'_{\varepsilon} = f'_1 + f'_2 \\ \delta'_{\varepsilon/t=0} = \delta'_{0,\varepsilon}, \end{cases}$$
(4.2.10)

$$\begin{cases} \partial_t \delta_{\varepsilon} - L \delta_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \delta_{\varepsilon} = & -\mathbb{P}(\delta_{\varepsilon} . \nabla \delta_{\varepsilon}) \\ & -\mathbb{P}(\delta_{\varepsilon} . \nabla (\widetilde{U_{QG}} + W_{\varepsilon}^T)) - \mathbb{P}((\widetilde{U_{QG}} + W_{\varepsilon}^T)) . \nabla \delta_{\varepsilon}) \\ & + f_1 + f_2 \end{cases}$$

$$(4.2.11)$$

$$\delta_{\varepsilon/t=0} = \delta_{0,\varepsilon},$$

with
$$\begin{cases} f_1 = f'_1 = -(1 - \mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}})G^b - \mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}}\mathbb{P}_2G^b, \\ f'_2 = -(1 - \mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}})G^l - \mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}}\mathbb{P}_2G^l, \\ f_2 = f'_2 - \mathbb{P}(\widetilde{U_{QG}}.\nabla W_{\varepsilon}^T) - \mathbb{P}(W_{\varepsilon}^T.\nabla\widetilde{U_{QG}}) - \mathbb{P}(W_{\varepsilon}^T.\nabla W_{\varepsilon}^T), \\ \delta'_{0,\varepsilon} = (1 - \mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}})U_{0,\varepsilon, osc} + \mathcal{P}_{r_{\varepsilon}, R_{\varepsilon}}\mathbb{P}_2U_{0,\varepsilon, osc}, \\ \delta_{0,\varepsilon} = \delta'_{0,\varepsilon} + (U_{0,\varepsilon, QG} - U_{0, QG}), \end{cases}$$
(4.2.12)

where we have defined

$$G^{b} = \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}, \nabla \widetilde{U_{QG}}) \quad \text{and} \quad G^{l} = -F(\nu - \nu')\Delta\Delta_{F}^{-2} \begin{pmatrix} -F\partial_{2}\partial_{3}^{2} \\ F\partial_{1}\partial_{3}^{2} \\ 0 \\ (\partial_{1}^{2} + \partial_{2}^{2})\partial_{3} \end{pmatrix} \widetilde{\Omega_{QG}}.$$

Remark 4.2.1 Remember that G^l , G^b and $U_{0,\varepsilon,osc}$ are divergence free, and have a zero potential vorticity (see (4.1.3) for the expression of G, G^l , G^b).

4.2.3 Energy estimates

The object of this section is the proof of the following result :

Proposition 4.2.2 If we take $m = (6\beta + 8)M$ there exist a constant $K^0 > 0$, a function $V_{\varepsilon} \in L^1(\mathbb{R}_+)$ and $\varepsilon_0 > 0$ (also depending on β and ε) such that, if $\alpha < \frac{\beta M}{K^0}$ and $\varepsilon \leq \varepsilon_0$, then for all $s \in [\frac{1}{2}, 1]$ and $t \geq 0$,

$$\|\delta_{\varepsilon}'(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla\delta_{\varepsilon}'(t')\|_{\dot{H}^{s}}^{2} dt' \leq K^{0} |\log\varepsilon|^{-(\beta M - \alpha K^{0})},$$
$$\|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu_{0} - 2C \|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V_{\varepsilon}(\tau) d\tau} dt' \leq K^{0} |\log\varepsilon|^{-(\beta M - \alpha K^{0})},$$
where $\|V\|_{\dot{H}^{s}} \leq K^{0} |\log\varepsilon|^{-(\beta M - \alpha K^{0})}$

where $\|V_{\varepsilon}\|_{L^1(\mathbb{R}_+)} \leq K^0 \alpha \log |\log \varepsilon|$.

Linear estimates

We use here the same convention to denote by K^0 a constant only depending on the initial data, F, ν or ν' . We won't prove the following two lemmas because they are easy adaptations of lemmas 3.1 and 3.2 from [5]. The only change is that we have to care about the initial data, now depending on the Rossby number.

Lemma 4.2.1 There exists a constant K^0 such that we have the following estimates for all $s \in [0, 1 + \beta]$,

$$\int_0^t \|G^b(t')\|_{\dot{H}^s} dt' + \int_0^t \|G^l(t')\|_{\dot{H}^{s-1}}^2 dt' \le K^0.$$

Lemma 4.2.2 There exists a constant K^0 such that, for all $s \in [\frac{1}{2}, 1]$ and $t \in \mathbb{R}_+$, the solutions of (4.2.7) and (4.2.8) satisfy respectively:

$$\|W_{\varepsilon}^{T}(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla W_{\varepsilon}^{T}(t')\|_{\dot{H}^{s}}^{2} dt' \leq K^{0} (1 + \alpha \log|\log\varepsilon|) (1 + |\nu - \nu'|C_{F}\varepsilon^{\frac{1}{2}})^{2},$$

and

$$\|W_{\varepsilon}(t)\|_{\dot{H}^s}^2 + \nu_0 \int_0^t \|\nabla W_{\varepsilon}(t')\|_{\dot{H}^s}^2 dt' \le K^0 (1 + \alpha \log|\log \varepsilon|).$$

Energy for δ_{ε}' and δ_{ε}

For more simplicity with the formulas we will only write the cases $s \in]\frac{1}{2}, 1]$, the case $s = \frac{1}{2}$ is dealt the same way except that the product laws change a little. The same formulas as estimate (27) from [5] hold: $\forall t \in \mathbb{R}_+$,

$$\|\delta_{\varepsilon}'(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla\delta_{\varepsilon}'(t')\|_{\dot{H}^{s}}^{2} dt' \leq \left(\|\delta_{0,\varepsilon}'\|_{\dot{H}^{s}}^{2} + \|f_{1}'\|_{L^{1}\dot{H}^{s}} + \frac{1}{\nu_{0}}\|f_{2}'\|_{L^{2}\dot{H}^{s-1}}^{2}\right) e^{\|f_{1}'\|_{L^{1}\dot{H}^{s}}},$$

$$(4.2.13)$$

and

$$\begin{split} \|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu_{0} - 2C \|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V_{\varepsilon}(\tau) d\tau} dt' \leq \\ & \left(\|\delta_{0,\varepsilon}\|_{\dot{H}^{s}}^{2} + \|f_{1}\|_{L^{1}\dot{H}^{s}} + \frac{1}{\nu_{0}}\|f_{2}\|_{L^{2}\dot{H}^{s-1}}^{2}\right) e^{\|V_{\varepsilon}\|_{L^{1}(\mathbb{R}_{+})}}, \end{split}$$
with $V_{\varepsilon}(t) = \frac{C}{\nu_{0}} \left(\|\nabla(\widetilde{U_{QG}} + W_{\varepsilon}^{T})\|_{\dot{H}^{\frac{1}{2}}}^{2} + \|\nabla(\widetilde{U_{QG}} + W_{\varepsilon}^{T})\|_{\dot{H}^{s}}^{2}\right) + \|f_{1}\|_{\dot{H}^{s}}.$

$$(4.2.14)$$

Estimates on the forcing terms and initial data

Lemma 4.2.3 If $m = (6\beta + 8)M$ there exist a constant K^0 and ε_0 (also depending on M) such that for all $\varepsilon \leq \varepsilon_0$,

$$\|\delta_{0,\varepsilon}\|_{\dot{H}^{s}} + \|\delta_{0,\varepsilon}'\|_{\dot{H}^{s}} + \|f_{1}\|_{L^{1}\dot{H}^{s}} + \|f_{2}'\|_{L^{2}\dot{H}^{s-1}} + \|f_{2}\|_{L^{2}\dot{H}^{s-1}} \le K^{0} |\log \varepsilon|^{-\beta M}$$

Proof: we will estimate separately each term recalling that we have the following decomposition:

$$\begin{cases} f_1 = f'_1 = -\left(1 - \chi(\frac{|D|}{R_{\varepsilon}})\right)G^b - \chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}})G^b - \chi(\frac{|D|}{R_{\varepsilon}})\left(1 - \chi(\frac{|D_3|}{r_{\varepsilon}})\right)\mathbb{P}_2G^b, \\ f'_2 = -\left(1 - \chi(\frac{|D|}{R_{\varepsilon}})\right)G^l - \chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}})G^l - \chi(\frac{|D|}{R_{\varepsilon}})\left(1 - \chi(\frac{|D_3|}{r_{\varepsilon}})\right)\mathbb{P}_2G^l, \\ f_2 = f'_2 - \mathbb{P}(\widetilde{U_{QG}}.\nabla W_{\varepsilon}^T) - \mathbb{P}(W_{\varepsilon}^T.\nabla\widetilde{U_{QG}}) - \mathbb{P}(W_{\varepsilon}^T.\nabla W_{\varepsilon}^T), \\ \delta'_{0,\varepsilon} = \left(1 - \chi(\frac{|D|}{R_{\varepsilon}})\right)U_{0,\varepsilon,osc} + \chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}})U_{0,\varepsilon,osc} + \chi(\frac{|D|}{R_{\varepsilon}})\left(1 - \chi(\frac{|D_3|}{r_{\varepsilon}})\right)\mathbb{P}_2U_{0,\varepsilon,osc}, \\ \delta_{0,\varepsilon} = \delta'_{0,\varepsilon} + \left(U_{0,\varepsilon,QG} - U_{0,QG}\right). \end{cases}$$

• Using the Plancherel lemma,

$$\|\left(1-\chi(\frac{|D|}{R_{\varepsilon}})\right)G^b\|_{L^1\dot{H}^s} \le CR_{\varepsilon}^{-\beta}\|G^b\|_{L^1\dot{H}^{s+\beta}} \le K^0|\log\varepsilon|^{-\beta M}.$$
• Similarly, we obtain that

$$\|\left(1-\chi(\frac{|D|}{R_{\varepsilon}})\right)G^l\|_{L^2\dot{H}^{s-1}} \le K^0|\log\varepsilon|^{-\beta M}.$$

• Using the definition of G^b and the fact that \mathbb{P} and \mathcal{P} are pseudo-differential operators of degree zero,

$$\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}})G^b\|_{L^1\dot{H}^s} \le R^s_{\varepsilon}\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}})\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}}\|_{L^1L^2}.$$

Using an anisotropic Bernstein Lemma (see lemma 4.2 from [4]) followed by lemma 4.1 from [4] to get rid of the truncation operators,

$$\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}})G^b\|_{L^1\dot{H}^s} \le R^s_{\varepsilon}(R^2_{\varepsilon}r_{\varepsilon})^{\frac{2}{3}-\frac{1}{2}}\|\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}}\|_{L^1L^{\frac{3}{2}}}.$$

and thanks to a Hölder inequality $(L^2 \cdot L^6 \hookrightarrow L^{\frac{3}{2}})$ and the fact that $s \leq 1$, we finally obtain:

$$\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_{3}|}{r_{\varepsilon}})G^{b}\|_{L^{1}\dot{H}^{s}} \leq R_{\varepsilon}^{\frac{4}{3}}r_{\varepsilon}^{\frac{1}{6}}\|\widetilde{U_{QG}}\|_{L^{2}L^{6}}\|\widetilde{U_{QG}}\|_{L^{2}\dot{H}^{1}} \leq \frac{K^{0}}{\nu_{0}}|\log\varepsilon|^{-(\frac{m}{6}-\frac{4M}{3})}.$$

• Using the same arguments we obtain :

$$\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}}))U_{0,\varepsilon,osc}\|_{\dot{H}^s} \leq R_{\varepsilon}^s \|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}}))U_{0,\varepsilon,osc}\|_{L^2},$$

thanks to the anisotropic Bernstein lemma (we refer to [4] or [9]):

$$\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}}))U_{0,\varepsilon,osc}\|_{\dot{H}^s} \le R_{\varepsilon}^s (R_{\varepsilon}^2 r_{\varepsilon})^{\frac{1}{2}} \|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}}))U_{0,\varepsilon,osc}\|_{L^1},$$

and then $\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}}))U_{0,\varepsilon,osc}\|_{\dot{H}^s} \leq K^0 |\log \varepsilon|^{-(\frac{m}{2}-2M)} \log |\log \varepsilon|.$

• since $G^l = -F|\nu - \nu'|\Delta \Delta_F^{-2} \partial^3 \partial_3 \widetilde{U_{QG}}$,

$$\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}})G^l\|_{L^2\dot{H}^{s-1}} \le C_F|\nu-\nu'|\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_3|}{r_{\varepsilon}})\partial\partial_3\widetilde{U_{QG}}\|_{L^2\dot{H}^{s-1}},$$

which we easily estimate by

$$\|\chi(\frac{|D|}{R_{\varepsilon}})\chi(\frac{|D_{3}|}{r_{\varepsilon}})G^{l}\|_{L^{2}\dot{H}^{s-1}} \leq C_{F}|\nu-\nu'|r_{\varepsilon}^{\frac{1}{2}}\|\widetilde{U_{QG}}\|_{L^{2}\dot{H}^{s+\frac{1}{2}}} \leq \frac{K^{0}}{\nu_{0}}|\log\varepsilon|^{-\frac{m}{2}}.$$

• Using proposition 4.2.1, we get:

$$\begin{aligned} \|\chi(\frac{|D|}{R_{\varepsilon}})(1-\chi(\frac{|D_{3}|}{r_{\varepsilon}}))\mathbb{P}_{2}G^{b}\|_{L^{1}\dot{H}^{s}} + \|\chi(\frac{|D|}{R_{\varepsilon}})(1-\chi(\frac{|D_{3}|}{r_{\varepsilon}}))\mathbb{P}_{2}G^{l}\|_{L^{2}\dot{H}^{s-1}} &\leq K^{0}\varepsilon^{\frac{1}{2}} \\ \\ \text{and} \quad \|\chi(\frac{|D|}{R_{\varepsilon}})(1-\chi(\frac{|D_{3}|}{r_{\varepsilon}}))\mathbb{P}_{2}U_{0,\varepsilon,osc}\|_{\dot{H}^{1}} &\leq K^{0}\varepsilon^{\frac{1}{2}}. \end{aligned}$$

• We easily obtain that :

$$\begin{split} \| \left(1 - \chi(\frac{|D|}{R_{\varepsilon}}) \right) U_{0,\varepsilon,osc} \|_{\dot{H}^{s}} &\leq R_{\varepsilon}^{-\beta} \| U_{0,\varepsilon} \|_{\dot{H}^{s+\beta}}, \end{split}$$

then $\| \left(1 - \chi(\frac{|D|}{R_{\varepsilon}}) \right) U_{0,\varepsilon,osc} \|_{\dot{H}^{s}} &\leq K^{0} |\log \varepsilon|^{-\beta M} \log |\log \varepsilon|$

and recall that $\|U_{0,\varepsilon,QG} - U_{0,QG}\|_{\dot{H}^1} \le K^0 \varepsilon.$

• All that remains is to estimate the last three terms appearing in the expression of f_2 . We use here the same estimates used in [5]. By interpolation we have :

$$\|\widetilde{U_{QG}} \cdot \nabla W_{\varepsilon}^{T}\|_{\dot{H}^{s-1}} \leq \|\widetilde{U_{QG}} \cdot \nabla W_{\varepsilon}^{T}\|_{\dot{H}^{2s-2}}^{\frac{1}{2}} \|\widetilde{U_{QG}} \cdot \nabla W_{\varepsilon}^{T}\|_{L^{2}}^{\frac{1}{2}}$$

Thanks to the product laws in Sobolev spaces $(s - \frac{1}{2} + s - \frac{3}{2} = 2s - 2$ and $s \in [\frac{1}{2}, 1])$ we can write that:

$$\begin{split} \|\widetilde{U_{QG}} \cdot \nabla W_{\varepsilon}^{T}\|_{L^{2}(\mathbb{R}_{+},\dot{H}^{s-1})} &\leq C \|\widetilde{U_{QG}}\|_{L^{\infty}\dot{H}^{s-\frac{1}{2}}}^{\frac{1}{2}} \|\nabla W_{\varepsilon}^{T}\|_{L^{2}\dot{H}^{s}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{\infty}L^{2}}^{\frac{1}{2}} \|W_{\varepsilon}^{T}\|_{L^{2}Lip}^{\frac{1}{2}} \\ &\leq C(R_{\varepsilon})^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{\infty}\dot{H}^{s-\frac{1}{2}}}^{\frac{1}{2}} \|W_{\varepsilon}^{T}\|_{L^{2}\dot{H}^{s+1}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{\infty}L^{2}}^{\frac{1}{2}} \|W_{\varepsilon}^{T}\|_{L^{2}L\infty}^{\frac{1}{2}} \end{split}$$

Similarly,

$$\|W_{\varepsilon}^{T}.\nabla\widetilde{U_{QG}}\|_{L^{2}(\mathbb{R}_{+},\dot{H}^{s-1})} \leq C\|W_{\varepsilon}^{T}\|_{L^{\infty}\dot{H}^{s}}^{\frac{1}{2}}\|\nabla\widetilde{U_{QG}}\|_{L^{2}\dot{H}^{s-\frac{1}{2}}}^{\frac{1}{2}}\|W_{\varepsilon}^{T}\|_{L^{2}L^{\infty}}^{\frac{1}{2}}\|\widetilde{U_{QG}}\|_{L^{\infty}\dot{H}^{1}}^{\frac{1}{2}}.$$

and

$$\|W_{\varepsilon}^{T}.\nabla W_{\varepsilon}^{T}\|_{L^{2}(\mathbb{R}_{+},\dot{H}^{s-1})} \leq C\|W_{\varepsilon}^{T}\|_{L^{\infty}\dot{H}^{s}}^{\frac{3}{4}}\|W_{\varepsilon}^{T}\|_{L^{2}\dot{H}^{s+1}}^{\frac{1}{4}}\|W_{\varepsilon}^{T}\|_{L^{2}L^{\infty}}^{\frac{1}{2}}\|W_{\varepsilon}^{T}\|_{L^{\infty}\dot{H}^{\frac{1}{2}}}^{\frac{1}{4}}\|W_{\varepsilon}^{T}\|_{L^{2}\dot{H}^{\frac{3}{2}}}^{\frac{1}{4}}$$

Thus, we need to estimate $||W_{\varepsilon}^{T}||_{L^{2}L^{\infty}}$ and this is the object of the Strichartz estimates (we refer to section 4.5 for the proof) :

Lemma 4.2.4 There exist $\varepsilon_0 = \varepsilon_0(F, \nu_0, m, M)$ and a constant K^0 (depending on a negative power of ν_0) such that for all $\varepsilon \leq \varepsilon_0$,

$$\|W_{\varepsilon}^{T}\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq K^{0}\varepsilon^{\frac{1}{16}}.$$

We finally obtain that :

$$\begin{cases} \|f_1\|_{L^1\dot{H}^s} \leq K^0 |\log \varepsilon|^{-\beta M} + K^0 |\log \varepsilon|^{-(\frac{m}{6} - \frac{4M}{3})} + K^0 \varepsilon^{\frac{1}{2}}, \\ \|f'_2\|_{L^2\dot{H}^{s-1}} \leq K^0 |\log \varepsilon|^{-\beta M} + K^0 |\log \varepsilon|^{-\frac{m}{2}} + K^0 \varepsilon^{\frac{1}{2}}, \\ \|f_2\|_{L^2\dot{H}^{s-1}} \leq K^0 |\log \varepsilon|^{-\beta M} + K^0 |\log \varepsilon|^{-\frac{m}{2}} + K^0 \varepsilon^{\frac{1}{2}} + K^0 \varepsilon^{\frac{1}{16}} \alpha \log |\log \varepsilon|, \\ \|\delta'_{0,\varepsilon}\|_{\dot{H}^s} \leq K^0 |\log \varepsilon|^{-\beta M} \log |\log \varepsilon| + K^0 |\log \varepsilon|^{-(\frac{m}{2} - 2M)} \log |\log \varepsilon| + K^0 \varepsilon^{\frac{1}{2}}, \\ \|\delta_{0,\varepsilon}\|_{\dot{H}^s} \leq K^0 |\log \varepsilon|^{-\beta M} \log |\log \varepsilon| + K^0 |\log \varepsilon|^{-(\frac{m}{2} - 2M)} \log |\log \varepsilon| + K^0 \varepsilon^{\frac{1}{2}}. \end{cases}$$

$$(4.2.15)$$

Fixing m > 8M, we try to boud from below the exponent of $|\log \varepsilon|^{-1}$ so we define :

$$\eta = \min(\beta M, \frac{m}{2} - 2M, \frac{1}{2}(\beta M + \frac{m}{2} - 2M), \frac{m}{6} - \frac{4M}{3}, 2\beta M, m, \beta M + \frac{m}{2})$$
$$= \min(\beta M, \frac{m}{6} - \frac{4M}{3}) > 0.$$

and, as ε goes to zero, we don't have any problem to estimate the powers of ε occuring in the energy estimates. For example, if we take *m* such that $\beta M = \frac{m}{6} - \frac{4M}{3}$, that is $m = (6\beta + 8)M > 8M$, we naturally obtain that $\eta = \beta M$ and then if ε is small enough :

$$\|\delta_{0,\varepsilon}\|_{\dot{H}^s} + \|\delta'_{0,\varepsilon}\|_{\dot{H}^s} + \|f_1\|_{L^1\dot{H}^s} + \|f'_2\|_{L^2\dot{H}^{s-1}} + \|f_2\|_{L^2\dot{H}^{s-1}} \le K^0 |\log\varepsilon|^{-\beta M}$$

which concludes the proof of Lemma 4.2.3. \blacksquare

Using Lemmas 4.2.3 and 4.2.2 we easily obtain that there exist a constant K^0 and ε_0 such that for all $\varepsilon \leq \varepsilon_0$,

$$\|f_1\|_{L^1\dot{H}^s} \le K^0 \quad \text{and} \quad \|V\|_{L^1(\mathbb{R}_+)} \le K^0 \alpha \log|\log \varepsilon|,$$

Then, we obtain from (4.2.13) and (4.2.14) that if $\varepsilon \leq \varepsilon_0$:

$$\begin{aligned} \|\delta_{\varepsilon}'(t)\|_{\dot{H}^{s}}^{2} + \nu_{0} \int_{0}^{t} \|\nabla\delta_{\varepsilon}'(t')\|_{\dot{H}^{s}}^{2} dt' &\leq K^{0} |\log\varepsilon|^{-\beta M} e^{K^{0} |\log\varepsilon|^{-\beta M}} \leq K^{0} |\log\varepsilon|^{-\beta M} \\ &\leq K^{0} |\log\varepsilon|^{-(\beta M - \alpha K^{0})}, \end{aligned}$$

and

$$\begin{split} \|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu_{0} - 2C \|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V(\tau)d\tau} dt' \\ & \leq K^{0} |\log\varepsilon|^{-(\beta M - \alpha K^{0})} \quad \text{if} \quad \alpha < \frac{\beta M}{K^{0}}, \end{split}$$

which ends the proof of proposition 4.2.2. \blacksquare

4.2.4 Conclusion

We are now able to conclude the proof of the theorem : using Proposition 4.2.2, we obtain that if $\varepsilon \leq \varepsilon_0$, for all t:

$$\|\delta_{\varepsilon}'(t)\|_{\dot{H}^s}^2 + \nu_0 \int_0^t \|\nabla \delta_{\varepsilon}'(t')\|_{\dot{H}^s}^2 dt' \le K^0 |\log \varepsilon|^{-(\beta M - \alpha K^0)},$$

and

$$\|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu_{0} - 2C\|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V(\tau)d\tau} dt' \le K^{0} |\log\varepsilon|^{-(\beta M - \alpha K^{0})}.$$

Let ε be small enough so that $K^0 |\log \varepsilon|^{-(\beta M - \alpha K^0)} \le \frac{\nu_0}{4C}$ and let us define the time :

$$T_{\varepsilon} = \sup \left\{ t \ge 0 \quad \text{such that} \quad \delta_{\varepsilon} \in \mathcal{C}([0,t], \dot{H}^1) \quad \text{and} \quad \forall t' \le t, \quad \|\delta_{\varepsilon}(t')\|_{\dot{H}^1}^2 \le \frac{\nu_0}{2C} \right\}.$$

Thanks to the estimates on the initial data, $T_{\varepsilon} > 0$ and like in [5] the previous estimates imply that $T_{\varepsilon} = \infty$, and as $\delta_{\varepsilon} - \delta'_{\varepsilon} = U_{\varepsilon} - \widetilde{U_{QG}} - W_{\varepsilon} = \gamma_{\varepsilon}$, we obtain that :

$$\|\gamma_{\varepsilon}\|_{E^s} \le K^0 |\log \varepsilon|^{-(\beta M - \alpha K^0)}.$$

Moreover, using the injection $\dot{H}^2 \hookrightarrow L^\infty$ we get :

$$\forall \varepsilon \leq \varepsilon_0, \quad \|\delta_{\varepsilon}\|_{L^2 L^{\infty}} \leq K^0 |\log \varepsilon|^{-(\beta M - \alpha K^0)}.$$

So, as $\delta_{\varepsilon} + W_{\varepsilon}^T = U_{\varepsilon} - \widetilde{U_{QG}}$, thanks to the Strichartz estimates, if ε is small enough,

$$|U_{\varepsilon} - \widetilde{U_{QG}}||_{L^2(\mathbb{R}_+, L^{\infty})} \le K^0 |\log \varepsilon|^{-(\beta M - \alpha K^0)}$$

and this ends the proof of Theorem 4.1.2. \blacksquare

4.3 Proof of Theorem 4.1.3

4.3.1 Preliminaries

Remark 4.3.1 Theorem 4.1.1 and the energy estimates are unchanged (except that ν_0 is replaced by ν) for all $s \in [0, 1 + \beta]$:

$$\forall t \in \mathbb{R}_+, \|\widetilde{U_{QG}}(t)\|_{\dot{H}^s}^2 + 2c\nu \int_0^t \|\widetilde{U_{QG}}(t')\|_{\dot{H}^{s+1}}^2 dt' \le C(U_{0,QG})$$
(4.3.16)

We refer to the appendix for precise statement of the simplifications involved by the fact that $\nu = \nu'$: for now let us just say that concerning the following matrix (in Fourier variables) :

$$\mathbb{B}(\xi,\varepsilon) \stackrel{\text{def}}{=} -\nu\widehat{\Delta - \frac{1}{\varepsilon}}\mathbb{P}\mathcal{A} = \mathcal{F}\left(-\nu\Delta - \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}\right)$$

we have explicit, simple expressions for its eigenelements. Moreover, the eigenvectors are simpler and orthogonal and we can show that the oscillating part is here exactly the part which will go to zero due to dispersion effects contrary to the case when $\nu \neq \nu'$ (see [5] and sections 4.5 and 4.6 for explicit computation).

4.3.2 The different systems

There are many simplifications from the systems used in the previous section due to the fact that $\nu = \nu'$ (no projection with \mathbb{P}_{3+4} , no truncation in frequency, disappearance of the term G). Precisely, let us write the different systems involved in the definition of δ_{ε} :

$$\begin{cases} \partial_{t}\widetilde{U_{QG}} - \nu\Delta\widetilde{U_{QG}} + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}\widetilde{U_{QG}} = -\mathbb{P}(\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}}) + G \\ \widetilde{U_{QG}}_{/t=0} = U_{0,QG}, \end{cases}$$

$$\begin{cases} \partial_{t}U_{\varepsilon} - \nu\Delta U_{\varepsilon} + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}U_{\varepsilon} = -\mathbb{P}(U_{\varepsilon}.\nabla U_{\varepsilon}) \\ U_{\varepsilon/t=0} = U_{0,\varepsilon} \end{cases}$$

$$\partial_{t}W_{\varepsilon} - \nu\Delta W_{\varepsilon} + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}W_{\varepsilon} = -G \\ W_{\varepsilon/t=0} = U_{0,\varepsilon,osc} \end{cases}$$
where $G = \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}})$

Remark 4.3.2 Recall that the term G is appearing as a constant term when one writes the system satisfied by the difference $U_{\varepsilon} - U_{QG}$ so in order to compensate this annoying constant term, we introduce it in the linear system and take advantage of dispersive effects of the operator $-\nu\Delta - \frac{1}{\varepsilon}A$.

Remark 4.3.3 Notice that G is divergence-free and with a zero potential vorticity. And in the case $\nu = \nu'$ it implies that \hat{G} depends only on the last two eigenvectors and we don't need to use a projector such as \mathbb{P}_{3+4} .

Let us begin with some notations : from the systems recalled in this section we can write the system satisfied by $\delta_{\varepsilon} = U_{\varepsilon} - \widetilde{U_{QG}} - W_{\varepsilon}^{T}$, where we have defined $W_{\varepsilon}^{T} = \chi(\frac{|D|}{R_{\varepsilon}})W_{\varepsilon}$ with $R_{\varepsilon} = \varepsilon^{-\gamma}$:

$$\begin{cases} \partial_t \delta_{\varepsilon} - L \delta_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \delta_{\varepsilon} &= -\mathbb{P}(\delta_{\varepsilon} \cdot \nabla \delta_{\varepsilon}) \\ & -\mathbb{P}(\delta_{\varepsilon} \cdot \nabla (\widetilde{U_{QG}} + W_{\varepsilon}^T)) - \mathbb{P}((\widetilde{U_{QG}} + W_{\varepsilon}^T)) \cdot \nabla \delta_{\varepsilon}) \\ & + f_1 + f_2 \\ \delta_{\varepsilon/t=0} &= (1 - \chi(\frac{|D|}{R_{\varepsilon}})) U_{0,\varepsilon,osc} + U_{0,\varepsilon,QG} - U_{0,QG} \end{cases}$$
(4.3.18)

with

$$\begin{cases} f_1 = -(1 - \chi(\frac{|D|}{R_{\varepsilon}}))G\\ f_2 = -\mathbb{P}(\widetilde{U_{QG}}.\nabla W_{\varepsilon}^T) - \mathbb{P}(W_{\varepsilon}^T.\nabla \widetilde{U_{QG}}) - \mathbb{P}(W_{\varepsilon}^T.\nabla W_{\varepsilon}^T) \end{cases}$$
(4.3.19)

4.3.3 Energy estimates

The object of this section is the proof of the following result :

Lemma 4.3.1 There exists a constant K^0 , a function $V_{\varepsilon} \in L^1(\mathbb{R}_+)$, and ε_0 such that if $\alpha < \frac{\beta}{16K^0(1+\beta)}$, for all $\varepsilon \leq \varepsilon_0$ and if we choose $\gamma = \frac{1}{16(1+\beta)}$:

$$\|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu_{0} - 2C\|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V(\tau)d\tau} dt' \le K^{0} \varepsilon^{\frac{\beta}{16(1+\beta)} - \alpha K^{0}}$$

where $||V_{\varepsilon}||_{L^1(\mathbb{R}_+)} \leq K^0 \alpha |\log \varepsilon|$

Energy for the linear system

We begin with linear energy estimates : like for Lemma 4.2.2 we will only write the result :

Lemma 4.3.2 For all $s \in [\frac{1}{2}, 1]$ and $t \in \mathbb{R}_+$, the solution of (4.1.2) satisfies:

$$\|W_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \nu \int_{0}^{t} \|\nabla W_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} dt' \leq C\alpha |\log \varepsilon|.$$

Energy for δ_{ε}

with V

We easily adapt the proof of the estimate (28) from [5] to obtain that for all $t \in \mathbb{R}_+$ and $s > \frac{1}{2}$ (like previously for more simplicity with the formulas we will only write the cases $s \in]\frac{1}{2}, 1$], the case $s = \frac{1}{2}$ is dealt the same way except for the product laws.

$$\begin{aligned} \|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu - 2C \|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V_{\varepsilon}(\tau)d\tau} dt' \leq \\ & \left(\|\delta_{\varepsilon}(0)\|_{\dot{H}^{s}}^{2} + \|f_{1}\|_{L^{1}\dot{H}^{s}} + \frac{1}{\nu}\|f_{2}\|_{L^{2}\dot{H}^{s-1}}^{2}\right) e^{\|V_{\varepsilon}\|_{L^{1}(\mathbb{R}_{+})}}, \\ f_{\varepsilon}(t) &= \frac{C}{\nu} \left(\|\nabla(\widetilde{U_{QG}} + W_{\varepsilon}^{T})\|_{\dot{H}^{\frac{1}{2}}}^{2} + \|\nabla(\widetilde{U_{QG}} + W_{\varepsilon}^{T})\|_{\dot{H}^{s}}^{2}\right) + \|f_{1}\|_{L^{1}\dot{H}^{s}}. \end{aligned}$$
(4.3.20)

Estimate on the external force and initial data

Let us begin with the initial data : recall that

$$\delta_{\varepsilon/t=0} = (1 - \chi(\frac{|D|}{R_{\varepsilon}}))U_{0,\varepsilon,osc} + U_{0,\varepsilon,QG} - U_{0,QG}$$

With $R_{\varepsilon} = \varepsilon^{-\gamma}$, and using the assumption of theorem 4.1.3 we have :

$$\|\delta_{\varepsilon}(0)\|_{\dot{H}^{s}} \leq C(R_{\varepsilon}^{-\beta} \|U_{0,\varepsilon,osc}\|_{\dot{H}^{s+\beta}} + \|U_{0,\varepsilon,QG} - U_{0,QG}\|_{\dot{H}^{s}}) \leq K^{0}\alpha |\log \varepsilon|\varepsilon^{\beta\gamma} + \varepsilon \leq K^{0}\varepsilon^{\frac{\beta\gamma}{2}} + \varepsilon.$$

$$(4.3.21)$$

The following lemma gives estimates on the external forces. In the case $\nu = \nu'$ the external force f_2 consists only in terms which we will estimate thanks to the Strichartz estimates :

Lemma 4.3.3 There exist a constant K^0 and ε_0 such that if $\varepsilon \leq \varepsilon_0$:

$$\|f_1\|_{L^1(\mathbb{R}_+,\dot{H}^s)} \le K^0 \varepsilon^{\beta\gamma}$$
 and $\|f_2\|_{L^2(\mathbb{R}_+,\dot{H}^{s-1})} \le K^0 \varepsilon^{\frac{1}{2}(\frac{1}{16}-\gamma)}$

Proof: We could use all the following arguments with W_{ε} instead of W_{ε}^{T} , but as it is not localized in frequencies, we couldn't use the Bernstein lemma to estimate $\|\nabla W_{\varepsilon}\|_{L^{\infty}}$ in terms of $\|W_{\varepsilon}\|_{L^{\infty}}$ and it would require a little more regularity to the initial data to deal with $\|W_{\varepsilon}\|_{L^{2}Lip}$ thanks to the Strichartz estimates. That is why we choose to localize W_{ε} . Using the computations from [5] we get first by interpolation that :

$$\|\widetilde{U_{QG}}.\nabla W_{\varepsilon}^{T}\|_{\dot{H}^{s-1}} \leq \|\widetilde{U_{QG}}.\nabla W_{\varepsilon}^{T}\|_{\dot{H}^{2s-2}}^{\frac{1}{2}} \|\widetilde{U_{QG}}.\nabla W_{\varepsilon}^{T}\|_{L^{2}}^{\frac{1}{2}}$$

And then using the product laws in Sobolev spaces $(s - \frac{1}{2} + s - \frac{3}{2} = 2s - 2$ and $s \in [\frac{1}{2}, 1])$ we obtain :

$$\|\widetilde{U_{QG}} \cdot \nabla W_{\varepsilon}^{T}\|_{\dot{H}^{s-1}} \leq C \|\widetilde{U_{QG}}\|_{\dot{H}^{s-\frac{1}{2}}}^{\frac{1}{2}} \|\nabla W_{\varepsilon}^{T}\|_{\dot{H}^{s}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{2}}^{\frac{1}{2}} \|\nabla W_{\varepsilon}^{T}\|_{L^{\infty}}^{\frac{1}{2}}.$$

Then, using the Bernstein Lemma and the fact that $W_{\varepsilon}^T = \chi(\frac{|D|}{R_{\varepsilon}})W_{\varepsilon}$ in order to estimate the truncation W_{ε}^T in terms of W_{ε} we obtain (where the notation L^pL^q means $L^p(\mathbb{R}_+, L^q(\mathbb{R}^3))$ etc...):

$$\|\widetilde{U_{QG}} \cdot \nabla W_{\varepsilon}^{T}\|_{L^{2}(\mathbb{R}_{+}, \dot{H}^{s-1})} \leq C \|\widetilde{U_{QG}}\|_{L^{\infty}\dot{H}^{s-\frac{1}{2}}}^{\frac{1}{2}} \|W_{\varepsilon}^{T}\|_{L^{2}\dot{H}^{s+1}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{\infty}L^{2}}^{\frac{1}{2}} \|W_{\varepsilon}^{T}\|_{L^{2}Lip}^{\frac{1}{2}}$$

$$\leq CR_{\varepsilon} \|\widetilde{U_{QG}}\|_{L^{\infty}\dot{H}^{s-\frac{1}{2}}}^{\frac{1}{2}} \|W_{\varepsilon}\|_{L^{2}\dot{H}^{s+1}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{\infty}L^{2}}^{\frac{1}{2}} \|W_{\varepsilon}\|_{L^{2}L^{\infty}}^{\frac{1}{2}}$$

There are no changes for the other terms of f_2 :

$$\|W_{\varepsilon}^{T}.\nabla \widetilde{U_{QG}}\|_{L^{2}\dot{H}^{s-1}} \leq C \|W_{\varepsilon}\|_{L^{\infty}\dot{H}^{s}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{2}\dot{H}^{s+\frac{1}{2}}}^{\frac{1}{2}} \|W_{\varepsilon}\|_{L^{2}L^{\infty}}^{\frac{1}{2}} \|\widetilde{U_{QG}}\|_{L^{\infty}\dot{H}^{1}}^{\frac{1}{2}},$$

and

$$\|W_{\varepsilon}^{T}.\nabla W_{\varepsilon}\|_{L^{2}\dot{H}^{s-1}} \leq C\|W_{\varepsilon}\|_{L^{\infty}\dot{H}^{s}}^{\frac{3}{4}}\|W_{\varepsilon}\|_{L^{2}\dot{H}^{s+1}}^{\frac{1}{4}}\|W_{\varepsilon}\|_{L^{2}L^{\infty}}^{\frac{1}{2}}\|W_{\varepsilon}\|_{L^{\infty}\dot{H}^{\frac{1}{2}}}^{\frac{1}{4}}\|W_{\varepsilon}\|_{L^{2}\dot{H}^{\frac{3}{2}}}^{\frac{1}{4}}.$$

All that remains is then to estimate the norms, for this purpose, we will use Strichartz estimates whose proof is given in section 4.6 :

Lemma 4.3.4 There exist $\varepsilon_0 = \varepsilon_0(F, \nu)$ and a constant K^0 a constant depending on F, $\nu_0, U_{0,\varepsilon,osc}$, such that for all $\varepsilon \leq \varepsilon_0$,

$$\|W_{\varepsilon}\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq \|W_{\varepsilon}\|_{L^{2}(\mathbb{R}_{+},B^{0}_{\infty^{-1}}(\mathbb{R}^{3}))} \leq K^{0}\varepsilon^{\frac{1}{8}}|\log\varepsilon|$$

Using these Strichartz estimates, the energy estimate (4.3.16) for the limit system and the one for the linear system (Lemma 4.3.2), and the fact that if ε is small enough $(\varepsilon^{\frac{1}{8}}|\log \varepsilon|)^{\frac{1}{2}}|\log \varepsilon|^{K}\varepsilon^{-\gamma} \leq K^{0}\varepsilon^{\frac{1}{2}(\frac{1}{16}-\gamma)}$ we obtain the estimate of lemma 4.3.3.

Let us end this section by stating an estimate on the function V_{ε} which is, considering the expression of V_{ε} , just a matter of using lemmas 4.3.3 and 4.3.2 :

Lemma 4.3.5 There exist a constant K^0 and ε_0 such that for all $\varepsilon \leq \varepsilon_0$,

$$\|V_{\varepsilon}\|_{L^1(\mathbb{R}_+)} \le K^0 \alpha |\log \varepsilon$$

4.3.4 Conclusion

We can conclude the proof of the theorem like in section 4.2.4 : using the estimate (4.3.21) and lemma 4.3.3 in the estimates from (4.3.20) we obtain that if $\varepsilon \leq \varepsilon_0$ and if we choose γ such that $\gamma\beta = \frac{1}{16} - \gamma$, that is $\gamma = \frac{1}{16(1+\beta)}$:

$$\|\delta_{\varepsilon}(t)\|_{\dot{H}^{s}}^{2} + \int_{0}^{t} (\nu_{0} - 2C\|\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}) \|\nabla\delta_{\varepsilon}(t')\|_{\dot{H}^{s}}^{2} e^{\int_{t'}^{t} V_{\varepsilon}(\tau)d\tau} dt' \leq K^{0} \varepsilon^{\frac{\beta}{16(1+\beta)} - \alpha K^{0}}$$

The argument is then exactly the same as in section 4.2.4 and we obtain that if $\alpha < \frac{\beta}{16K^0(1+\beta)}$ then $\|\delta_{\varepsilon}\|_{E^s} \leq K^0 \varepsilon^{\frac{\beta}{16(1+\beta)}-\alpha K^0}$. And using the injection $\dot{H}^2 \hookrightarrow L^{\infty}$ we get :

 $\forall \varepsilon \leq \varepsilon_0, \quad \|\delta_\varepsilon\|_{L^2 L^\infty} \leq K^0 \varepsilon^{\frac{\beta}{16(1+\beta)} - \alpha K^0}$

and
$$||U_{\varepsilon} - \widetilde{U_{QG}}||_{L^2(\mathbb{R}_+, L^{\infty})} \le K^0 \varepsilon^{\frac{\beta}{16(1+\beta)} - \alpha K^0}$$

which ends the proof of Theorem 4.1.3. \blacksquare

4.4 **Proof of Theorem 4.1.4 : vortex patches**

In this section we will work in the case $\nu = \nu'$ only. We will use here the definitions introduced in Definition 4.1.2.

This section, is devoted to the proof of Theorem 4.1.4 : in the first subsection we recall some important results and focus on the two systems on which relies the persistence of the vortex patch structure ; in the second subsection we will establish a first estimate on the potential vorticity, which will then be refined so that we can conclude the proof.

The a priori and Strichartz estimates are put in the appendix.

4.4.1 Preliminaries

Let us begin with these important lemmas (we refer to [7], [11] or [15] for the proofs) :

Lemma 4.4.1 There exists a constant C > 0 only depending on $s \in]0,1[$ such that, for any divergence-free vector field $U \in L^2(\mathbb{R}^3)$ which is quasigeostrophic and whose potential vorticity $\Omega = \Omega(U) \in L^2(\mathbb{R}^3) \cap C^s(X)$ for a fixed admissible family X of C^s vector fields, U is Lipschitzian and we have :

$$\|U\|_{Lip} = \|\nabla U\|_{L^{\infty}} \le C \Big(\|\Omega\|_{L^{2}} + \|\Omega\|_{L^{\infty}} \log \Big(e + \frac{\|\Omega\|_{C^{s}(X)}}{\|\Omega\|_{L^{\infty}}} \Big) \Big)$$

Lemma 4.4.2 If $s \in [0, 1[$ then there exists a constant $C_s > 0$ such that for all U quasigeostrophic vector field, and X family of C^s vector fields, we have :

$$\|X.\nabla U\|_{C^s} \le C_s (\|U\|_{Lip} \|X\|_{C^s} + \|\operatorname{div} (\Omega(U)X)\|_{C^{s-1}})$$

In order to prove the persistence of the tangential regularity, we will not use a fixed family of vector fields : like in [7], [11], [15], and [17], we will use a fixed initial family, and we will measure the tangential regularity with respect to a special family, advected by the velocity : precisely if $(X_{0,\lambda})$ is a family of C^s vector fields, we define the transported family $X_t^{\varepsilon} = (X_{t,\lambda}^{\varepsilon})$ by :

$$\begin{cases} \partial_t X_{t,\lambda}^{\varepsilon} + v_{\varepsilon} \cdot \nabla X_{t,\lambda}^{\varepsilon} = X_{t,\lambda}^{\varepsilon} \cdot \nabla v_{\varepsilon} \\ X_{t,\lambda/t=0}^{\varepsilon} = X_{0,\lambda}. \end{cases}$$
(4.4.22)

Remark 4.4.1 This formulation is equivalent to the fact that, if $\psi_{\varepsilon}(t)$ is the flow associated to v_{ε} , $X_{t,\lambda}^{\varepsilon} = (X_{0,\lambda} \cdot \nabla \psi_{\varepsilon}(t)) o \psi_{\varepsilon}(t)^{-1}$.

The regularity is preserved by this transformation : if the initial family is regular, the advected family has the same regularity : we refer to [15] for the proof of the fact that $X_{0,\lambda} \in C^s \Rightarrow X_{t,\lambda}^{\varepsilon} \in C^s$.

We want to establish the transport-diffusion system satisfied by $X_{t,\lambda}^{\varepsilon}(x,D)\Omega_{\varepsilon}$. For this purpose we refer to [4] for the equation of the potential vorticity (with initial data $\Omega(U_{0,\varepsilon}) = \Omega_{0,\varepsilon}$):

$$\partial_t \Omega_\varepsilon + v_\varepsilon \cdot \nabla \Omega_\varepsilon - \nu \Delta \Omega_\varepsilon = q_\varepsilon, \tag{4.4.23}$$

with

$$q_{\varepsilon} = q(U_{\varepsilon,osc}, U_{\varepsilon}) \stackrel{\text{def}}{=} \partial_3 v^3_{\varepsilon,osc} (\partial_1 v^2_{\varepsilon} - \partial_2 v^1_{\varepsilon}) - \partial_1 v^3_{\varepsilon,osc} \partial_3 v^2_{\varepsilon} + \partial_2 v^3_{\varepsilon,osc} \partial_3 v^1_{\varepsilon} + F \partial_3 (v_{\varepsilon} - v_{\varepsilon,osc}) \nabla \theta_{\varepsilon,osc} + F \partial_3 v_{\varepsilon,osc} \nabla \theta_{\varepsilon}.$$

In order to simplify we will take the following notation :

$$q_{\varepsilon} = \partial U_{\varepsilon,osc} \partial U_{\varepsilon} + \partial U_{\varepsilon,osc} \partial U_{\varepsilon,QG}$$

$$(4.4.24)$$

Then we obtain :

$$\begin{cases} (\partial_t + v_{\varepsilon} \cdot \nabla - \nu \Delta) X_{t,\lambda}^{\varepsilon}(x,D) \Omega_{\varepsilon} = X_{t,\lambda}^{\varepsilon}(x,D) q_{\varepsilon} + \nu [X_{t,\lambda}^{\varepsilon}(x,D),\Delta] \Omega_{\varepsilon} \\ X_{t,\lambda}^{\varepsilon}(x,D) \Omega_{\varepsilon_{/t=0}} = X_{0,\lambda}(x,D) \Omega_{0,\varepsilon}. \end{cases}$$
(4.4.25)

Our aim is to study the two systems (4.4.22) and (4.4.25) with a view to use some coupled Gronwall estimates and then estimate the quantity $\|\Omega\|_{C^s(X)}$ that appears in Lemma 4.4.1. When it is done, we can control the lifespan of the solution U_{ε} thanks to the following remark :

Remark 4.4.2 As the initial data $U_{0,\varepsilon} \in H^5$, the Leray and Fujita-Kato theorems apply and give us, thanks to a weak-strong unicity theorem existence and uniqueness of a regular solution on a maximal intervall $[0, T_{\varepsilon}^*[$. Thanks to a regularity-propagation theorem, the solution U_{ε} is bounded in the space $L^{\infty}([0,T], H^5) \cap L^2([0,T], H^6)$ for all $T < T_{\varepsilon}^*$.

The blowup condition remains :

$$T_{\varepsilon}^* < \infty \Rightarrow \int_0^{T_{\varepsilon}^*} \|U_{\varepsilon}(t)\|_{\dot{H}^{\frac{3}{2}}}^2 dt = +\infty$$

So if $T < T_{\varepsilon}^*$, using the a priori estimates (see section 4.7 for details), we get :

$$\int_0^T \|U_{\varepsilon}(t)\|_{\dot{H}^{\frac{3}{2}}}^2 dt \le C_0 \varepsilon^{-2\alpha} T e^{2C_s \int_0^T \|U_{\varepsilon}(t)\|_{Lip} dt}$$

So if $T_{\varepsilon}^* < \infty$, and if we define a time T_{ε}^{γ} and manage to prove that for all $T < \min(T_{\varepsilon}^{\gamma}, T_{\varepsilon}^*)$,

$$\int_0^T \|U_{\varepsilon}(t)\|_{Lip} dt \le K_{\varepsilon},$$

then for all $T < \min(T_{\varepsilon}^{\gamma}, T_{\varepsilon}^{*})$ we have the estimate

$$\int_0^T \|U_{\varepsilon}(t)\|_{\dot{H}^{\frac{3}{2}}}^2 dt \le C_0 \varepsilon^{-2\alpha} T_{\varepsilon}^* e^{2K_{\varepsilon}} < \infty.$$

This proves by contradiction that $\min(T_{\varepsilon}^{\gamma}, T_{\varepsilon}^{*}) < T_{\varepsilon}^{*}$ and then $T_{\varepsilon}^{\gamma} < T_{\varepsilon}^{*}$.

But as we want to deal with Hölder norms C^{s-1} (s-1<0) and given the regularity of $X_{t,\lambda}^{\varepsilon}$, the term $\nu[X_{t,\lambda}^{\varepsilon}(x,D),\Delta]\Omega_{\varepsilon}$ is not necessarily defined because of the Laplacian (with a wish of simplicity we won't write the index λ). We therefore decompose the term in the following way :

$$\nu[X_{t,\lambda}^{\varepsilon}(x,D),\Delta]\Omega_{\varepsilon} = -\nu\sum_{i}\partial_{i}(\Delta X^{\varepsilon,i}\Omega_{\varepsilon}) - 2\nu\sum_{i}\partial_{i}(\nabla X^{\varepsilon,i}\nabla\Omega_{\varepsilon}) = \nu F + \nu G,$$

with $F = -\sum_{i}\partial_{i}R(\Delta X^{\varepsilon,i},\Omega_{\varepsilon}) - 2\sum_{i}\partial_{i}R(\nabla X^{\varepsilon,i},\nabla\Omega_{\varepsilon})$ (4.4.26)
 $G = -\sum_{i}\partial_{i}T_{\Delta X^{\varepsilon,i}}\Omega_{\varepsilon} - \sum_{i}\partial_{i}T_{\Omega_{\varepsilon}}\Delta X^{\varepsilon,i} - 2\sum_{i}\partial_{i}T_{\nabla X^{\varepsilon,i}}\nabla\Omega_{\varepsilon} - 2\sum_{i}\partial_{i}T_{\nabla \Omega_{\varepsilon}}\nabla X^{\varepsilon,i},$

$$G = -\sum_{i} \partial_{i} T_{\Delta X^{\varepsilon,i}} \Omega_{\varepsilon} - \sum_{i} \partial_{i} T_{\Omega_{\varepsilon}} \Delta X^{\varepsilon,i} - 2\sum_{i} \partial_{i} T_{\nabla X^{\varepsilon,i}} \nabla \Omega_{\varepsilon} - 2\sum_{i} \partial_{i} T_{\nabla \Omega_{\varepsilon}} \nabla X^{\varepsilon,i},$$

where R and T are the operators of the Bony decomposition (we refer to [3] and [7] for precise studies of these operators) :

- $uv = T_uv + T_vu + R(u, v),$
- T is the paraproduct : $T_u v = \sum_{p \le q-2} \Delta_p u \Delta_q v = \sum_q S_{q-1} u \Delta_q v$,
- R is the remainder : $R(u, v) = \sum_{|p-q| \le 1} \Delta_p u \Delta_q v.$

As, X^{ε} is in C^s and $\Omega_{\varepsilon} \in L^{\infty}$ there is no problem to define the paraproducts, but the remainder is not necessarily defined (for s-1 should be positive).

So G is well defined but F is only formally defined (we have not determined to which space it belongs). Thanks to the smoothing effect developped in [17] we will be able to show that in fact $\nu F \in \widetilde{L^1}C^{s-1}$ ($||u||_{\widetilde{L}^p_T C^{s-1}} = \sup_{q \ge -1} ||2^{q(s-1)}||\Delta_q u||_{L^\infty} ||_{L^p_T}$) and estimate it uniformly with respect to ν contrary to the previous chapters, where the power in ν_0 was negative. We will also estimate uniformly $G \in L^\infty C^{s-3}$.

Remark 4.4.3 The viscosity ν in $\nu F \in \widetilde{L^1}C^{s-1}$ is essential to have uniform estimates with respect to ν .

4.4.2 A first estimate on the potential vorticity

In this section we wish to estimate the quasigeostrophic part of U_{ε} . A simple use of Lemma 4.4.1 gives that $U_{\varepsilon,QG}$ is lipschitzian and for all $t \in [0, T_{\varepsilon}^*]$:

$$\|U_{\varepsilon,QG}(t)\|_{Lip} \le C\Big(\|\Omega_{\varepsilon}(t)\|_{L^2} + \|\Omega_{\varepsilon}(t)\|_{L^{\infty}}\log\Big(e + \frac{\|\Omega_{\varepsilon}(t)\|_{C^s(X^{\varepsilon})}}{\|\Omega_{\varepsilon}(t)\|_{L^{\infty}}}\Big)\Big).$$

Within this estimate appear five quantities to estimate (see Definition 4.1.2):

- 1. $\|\Omega_{\varepsilon}(t)\|_{L^2}$,
- 2. $\|\Omega_{\varepsilon}(t)\|_{L^{\infty}}$,
- 3. $[X_{t,\lambda}^{\varepsilon}]^{-1}$,
- 4. $\|X_{t,\lambda}^{\varepsilon}\|_{C^s}$,
- 5. $\|X_{t,\lambda}^{\varepsilon}(x,D)\Omega_{\varepsilon}\|_{C^{s-1}}$.

This section is devoted to the estimates of these terms. We will need a priori estimates that are given in the appendix. Estimates of 1 and 2 are given directly by Lemma 2.2.5 from [17] which reads as follows :

Lemma 4.4.3 [17] Let $p \in [1, +\infty]$, $v \in L^1_{loc}(\mathbb{R}_+, Lip(\mathbb{R}^d))$ a divergence-free vector field, $Q \in L^1_{loc}(\mathbb{R}_+, L^p(\mathbb{R}^d))$, and $a_0 \in L^p(\mathbb{R}^d)$. Then, if $a \in C(\mathbb{R}_+, L^p(\mathbb{R}^d))$ is a solution of the transport-diffusion system :

$$\begin{cases} \partial_t a + v \cdot \nabla a - \nu \Delta a = Q\\ a_{/t=0} = a_0. \end{cases}$$

Then for all $t \in \mathbb{R}_+$,

$$||a(t)||_{L^p} \le ||a_0||_{L^p} + \int_0^t ||Q(\tau)||_{L^p} d\tau.$$

Then, as Ω_{ε} satisfies system (4.4.23) and using the notation of (4.4.24) for q_{ε} , we get that for all $t \in [0, T_{\varepsilon}^*[$:

$$\begin{split} \|\Omega_{\varepsilon}(t)\|_{L^{2}} &\leq \|\Omega_{0,\varepsilon}\|_{L^{2}} + \int_{0}^{t} \left(\|\partial U_{\varepsilon,osc}(\tau).\partial U_{\varepsilon}(\tau)\|_{L^{2}} + \|\partial U_{\varepsilon,osc}(\tau).\partial U_{\varepsilon,QG}(\tau)\|_{L^{2}} d\tau \right) \\ &\leq \|\Omega_{0,\varepsilon}\|_{L^{2}} + \int_{0}^{t} \left(\|\partial U_{\varepsilon,osc}(\tau)\|_{L^{\infty}} \|\partial U_{\varepsilon}(\tau)\|_{L^{2}} + \|\partial U_{\varepsilon,osc}(\tau)\|_{L^{\infty}} \|\partial U_{\varepsilon,QG}(\tau)\|_{L^{2}} d\tau \right), \end{split}$$

which, using the injection $H^5(\mathbb{R}^3) \hookrightarrow \dot{H}^1(\mathbb{R}^3)$, the fact that \mathcal{Q} is a homogeneous pseudodifferential operator of order zero, and the a priori estimates (lemma 4.7.1), turns into :

$$\|\Omega_{\varepsilon}(t)\|_{L^{2}} \leq C_{0} + C \int_{0}^{t} \|U_{\varepsilon,osc}(\tau)\|_{Lip} \|U_{\varepsilon,0}\|_{H^{5}} e^{C \int_{0}^{\tau} \|U_{\varepsilon}(t')\|_{Lip} dt'} d\tau.$$

Then, using the initial data estimates, and a Hölder inequality,

$$\|\Omega_{\varepsilon}(t)\|_{L^2} \leq C_0 + C_0 \varepsilon^{-\alpha} e^{C \int_0^t \|U_{\varepsilon}(t')\|_{Lip} dt'} \|U_{\varepsilon,osc}\|_{L^8_t Lip} t^{\frac{7}{8}}.$$

Finally, the Strichartz estimates (Lemma 4.8.1) give :

$$\|\Omega_{\varepsilon}(t)\|_{L^{2}} \leq C_{0} + C_{0}t^{\frac{7}{8}}\varepsilon^{\frac{1}{8}-2\alpha}e^{CV_{\varepsilon}(t)} \quad \text{where} \quad V_{\varepsilon}(t) = \int_{0}^{t} \|U_{\varepsilon}(t')\|_{Lip}dt'.$$
(4.4.27)

Similarly, using the injection $H^s(\mathbb{R}^3) \hookrightarrow Lip(\mathbb{R}^3)$ if $s > \frac{5}{2}$, we get :

$$\|\Omega_{\varepsilon}(t)\|_{L^{\infty}} \le C_0 + C_0 t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}.$$
(4.4.28)

Now, let us return to the estimate on the quasigeostrophic part : for all $t \in [0, T_{\varepsilon}^*]$,

$$\begin{split} \|U_{\varepsilon,QG}(t)\|_{Lip} &\leq C\Big(\|\Omega_{\varepsilon}(t)\|_{L^{2}} + \|\Omega_{\varepsilon}(t)\|_{L^{\infty}}\log\Big(e + \|\Omega_{\varepsilon}(t)\|_{L^{\infty}}^{-1} \times \\ \Big(\|\Omega_{\varepsilon}(t)\|_{L^{\infty}} + \|[X_{t,\lambda}^{\varepsilon}]^{-1}\|_{L^{\infty}} + \sum_{\lambda=1}^{N} (\|X_{t,\lambda}^{\varepsilon}\|_{C^{s}} + \|X_{t,\lambda}^{\varepsilon}(x,D)\Omega_{\varepsilon}\|_{C^{s-1}})\Big)\Big)\Big). \end{split}$$

So, using the estimates (4.4.27) and (4.4.28), and the fact that the function $x \mapsto x \log(e + 1 + \frac{a}{x})$ is increasing if a > 0:

$$\begin{split} \|U_{\varepsilon,QG}(t)\|_{Lip} &\leq C(C_0 + C_0 t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}) \Big(1 + \log\left(e + 1 + \frac{\|[X_{t,\lambda}^{\varepsilon}]^{-1}\|_{L^{\infty}} + \sum_{\lambda=1}^{N} \left(\|X_{t,\lambda}^{\varepsilon}\|_{C^s} + \|X_{t,\lambda}^{\varepsilon}(x,D)\Omega_{\varepsilon}\|_{C^{s-1}} \right)}{C_0 + C_0 t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}} \Big) \Big). \end{split}$$

Using the fact that function $x \mapsto \log(e+1+\frac{a}{x})$ is decreasing if a > 0:

$$\begin{split} \|U_{\varepsilon,QG}(t)\|_{Lip} &\leq C_0 (1 + t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}) \Big(1 + \log \big(e + 1 + \frac{1}{C_0} \| [X_{t,\lambda}^{\varepsilon}]^{-1} \|_{L^{\infty}} + \frac{1}{C_0} \sum_{\lambda=1}^N (\|X_{t,\lambda}^{\varepsilon}\|_{C^s} + \|X_{t,\lambda}^{\varepsilon}(x,D)\Omega_{\varepsilon}\|_{C^{s-1}}) \Big) \Big). \end{split}$$

We refer to [7] or [15] (Corollary 4.3) for the proof of the following estimate :

$$\forall t \in [0, T_{\varepsilon}^*[, \quad \| [X_{t,\lambda}^{\varepsilon}]^{-1} \|_{L^{\infty}} \le C_0 e^{CV_{\varepsilon}(t)}.$$

With a wish of simplicity we will not write the coordinates λ anymore, so we finally obtain :

$$\|U_{\varepsilon,QG}(t)\|_{Lip} \le C_0 (1 + t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}) \Big(1 + \log \big(e + 1 + C_0 e^{CV_{\varepsilon}(t)} + \frac{1}{C_0} (\|X^{\varepsilon}(t)\|_{C^s} + \|X^{\varepsilon}_t(x, D)\Omega_{\varepsilon}\|_{C^{s-1}}) \Big) \Big).$$
(4.4.29)

In the following section where we will use the results on transport-diffusion equations from [17] and [7] to estimate the quantity :

$$\|X^{\varepsilon}(t)\|_{C^{s}} + \|X^{\varepsilon}_{t}(x,D)\Omega_{\varepsilon}\|_{C^{s-1}}$$

4.4.3 Transport-diffusion estimates

This entire section is an adaptation of the results developped in [17] about the hölderian regularity of viscous Vortex patches for 2-D Navier-Stokes equations. Let us begin by recalling systems (4.4.22) and (4.4.25) where F and G are defined in (4.4.26) :

$$\begin{cases} \partial_t X^{\varepsilon}(t) + v_{\varepsilon} . \nabla X^{\varepsilon}(t) = X^{\varepsilon}(t) . \nabla v_{\varepsilon} \\ X^{\varepsilon}_{/t=0} = X_0 \end{cases}$$
$$\begin{cases} (\partial_t + v_{\varepsilon} . \nabla - \nu \Delta) X^{\varepsilon}(x, D) \Omega_{\varepsilon} = X^{\varepsilon}(x, D) q_{\varepsilon} + \nu F + \nu G \\ X^{\varepsilon}(x, D) \Omega_{\varepsilon/t=0} = X_0(x, D) \Omega_{0,\varepsilon} \end{cases}$$

For reasons exposed in the following section, the estimates will be given in some bounded interval $[T_1, T_2]$.

Estimate for the vector field $X^{\varepsilon}(t)$

System (4.4.22) has been studied in [6], [15], and [11] so the following estimate is well known (for example we refer to [7] (Lemma 4.1.1)) : there exists a constant C > 0 such that for all $t \in [T_1, T_2]$,

$$\|X^{\varepsilon}(t)\|_{C^{s}} \leq \|X^{\varepsilon}(T_{1})\|_{C^{s}} + C \int_{T_{1}}^{t} \|X^{\varepsilon} \cdot \nabla v_{\varepsilon}\|_{C^{s}} dt' + C \int_{T_{1}}^{t} \|X^{\varepsilon}(t')\|_{C^{s}} \cdot \|v_{\varepsilon}(t')\|_{Lip} dt'$$
(4.4.30)

Now let us decompose the solution U_{ε} into its oscillating and quasigeostrophic parts :

$$\|X^{\varepsilon} \cdot \nabla v_{\varepsilon}\|_{C^{s}} \le \|X^{\varepsilon} \cdot \nabla v_{\varepsilon,osc}\|_{C^{s}} + \|X^{\varepsilon} \cdot \nabla v_{\varepsilon,QG}\|_{C^{s}}$$

Then use product laws for the former and Lemma 4.4.2 for the latter :

$$\begin{aligned} \|X^{\varepsilon} \cdot \nabla U\|_{C^{s}} &\leq C_{s} \left(\|X^{\varepsilon}\|_{L^{\infty}} \|\nabla U_{\varepsilon,osc}\|_{C^{s}} + \|X^{\varepsilon}\|_{C^{s}} \|\nabla U_{\varepsilon,osc}\|_{L^{\infty}} \right) \\ &+ C_{s} \left(\|U_{\varepsilon,QG}\|_{Lip} \|X^{\varepsilon}\|_{C^{s}} + \|\operatorname{div} (X^{\varepsilon} \Omega_{\varepsilon})\|_{C^{s-1}} \right) \end{aligned}$$

So, plugging this into (4.4.30) we finally obtain (also decomposing the speed) for all $t \in [T_1, T_2]$:

$$\begin{split} \|X^{\varepsilon}(t)\|_{C^{s}} &\leq \|X^{\varepsilon}(T_{1})\|_{C^{s}} + C_{s} \int_{T_{1}}^{t} \|X^{\varepsilon}(t')\|_{C^{s}} (\|U_{\varepsilon,osc}\|_{Lip} + \|U_{\varepsilon,QG}\|_{Lip})dt' \\ &+ C_{s} \int_{T_{1}}^{t} \|X^{\varepsilon}(x,D)\Omega_{\varepsilon}\|_{C^{s-1}}dt' + C_{s} \int_{T_{1}}^{t} \|X^{\varepsilon}(t')\|_{L^{\infty}} \|U_{\varepsilon,osc}\|_{C^{s+1}}dt'. \end{split}$$

A weaker version of (4.4.30) (see [7] or [15]) gives :

$$\forall t \in [0, T_{\varepsilon}^*], \quad \|X^{\varepsilon}(t)\|_{L^{\infty}} \le \|X_0\|_{L^{\infty}} e^{C \int_0^t \|U_{\varepsilon}(t')\|_{Lip} dt'}$$

$$(4.4.31)$$

Then we use this estimation in the last integral, together with the strichartz estimates and an Hölder inequality, and finally obtain $\forall t \in [T_1, T_2]$:

$$\|X^{\varepsilon}(t)\|_{C^{s}} \leq \|X^{\varepsilon}(T_{1})\|_{C^{s}} + C_{s} \int_{T_{1}}^{t} \|X^{\varepsilon}(t')\|_{C^{s}} (\|U_{\varepsilon,osc}\|_{Lip} + \|U_{\varepsilon,QG}\|_{Lip})dt' \qquad (4.4.32)$$
$$+ C_{s} \int_{T_{1}}^{t} \|X^{\varepsilon}(x,D)\Omega_{\varepsilon}\|_{C^{s-1}}dt' + C_{F,s}\varepsilon^{\frac{1}{8}-\alpha}e^{CV_{\varepsilon}(t)}(t-T_{1})^{\frac{7}{8}}.$$

Estimate for $X^{\varepsilon}(x,D)\Omega_{\varepsilon}$: general lemmas

Let us define the following transport-diffusion system :

$$\begin{cases} \partial_t a + v \cdot \nabla a - \nu \Delta a = \nu F + \nu G + Q\\ a_{/t=0} = a_0. \end{cases}$$

$$(4.4.33)$$

Recall Lemma 4.4.3 which gives estimates in L^p when F = G = 0.

In this section we will use and adapt the results from [17] concerning the hölderian regularity of viscous Vortex patches for 2-D Navier-Stokes equations on vanishing viscosity (sections 2.3 and 2.4).

We will not give any proof here because the adaptation basically consists in adding another source term Q in the transport-diffusion equation, so we refer to [17] for the details. In the case of the equation of the potential vorticity, $Q = q_{\varepsilon}$ and in the case of system (4.4.25), $Q = X^{\varepsilon}(x, D)q_{\varepsilon}$.

Before stating these estimates, let us point out that we will only use the local in time versions of the results, we will globalize only once in the end. Another important point is that these results provide estimates that are uniformly bounded with respect to the viscosity. Although the viscosity complicates everything, we will use lemmas from [17] in order to get estimates close to those from [11] (where $\nu = \nu' = 0$). Let us begin with the smoothing effect (we refer to [17] Section 2.3 for the proof) :

Lemma 4.4.4 There exist two constants C > 0 and $0 < C_m \le 1$ such that if a is a solution of (4.4.33), with div v = 0, F = G = 0 and $T_1 < T_2$ satisfy the condition

$$\int_{T_1}^{T_2} \|\nabla v_{\varepsilon}(\tau)\|_{L^{\infty}} d\tau \le C_m$$

Then for all $t \in [T_1, T_2]$, we have :

$$\nu \|a\|_{\widetilde{L^{1}}([T_{1},t],C_{*}^{2})} \leq C(1+\nu(t-T_{1}))\big(\|a(T_{1})\|_{L^{\infty}}+\|Q\|_{L^{1}([T_{1},t],L^{\infty})}\big),$$

where C_*^2 is the Zygmund class of order 2 (see the definition of Hölder spaces in the introduction) and $\|u\|_{\tilde{L}^p_T C^2_*} = \sup_{q \ge -1} \|2^{2q}\|\Delta_q u\|_{L^\infty}\|_{L^p_T}$.

Now, let us state the lemma of propagation in Hölder spaces (See section 2.4 in [17] for the proof) :

Lemma 4.4.5 If $s \in [0, 1[$ there exist two constants C > 0 and $c_m > 0$ such that if a is a solution of (4.4.33), and $T_1 < T_2$ satisfy the condition

$$\int_{T_1}^{T_2} \|\nabla v_{\varepsilon}(\tau)\|_{L^{\infty}} d\tau \le c_m \min(s, 2-s) \le C_m$$

Then for all $t \in [T_1, T_2]$, we have :

$$\begin{aligned} \|a(t)\|_{C^{s-1}} &\leq C \left(\|a(T_1)\|_{C^{s-1}} + \nu \|F\|_{\widetilde{L^1}([T_1,t],C^{s-1})} + \|Q\|_{\widetilde{L^1}([T_1,t],C^{s-1})} \right) \\ &+ C (1 + \nu (t-T_1)) \|G\|_{\widetilde{L^\infty}([T_1,t],C^{s-3})}. \end{aligned}$$

Recall that we will apply these estimates to systems (4.4.23) and (4.4.25) and then we will have to estimate F and G (see (4.4.26)). This is the object of the following lemma, which is the analog of Lemma 2.5.1 from [17]:

Lemma 4.4.6 Let $s \in]0,1[$ and c_m the same as in the previous lemma. There exists a constant C such that for all $0 \le T_1 < T_2$ satisfying :

$$T_2 - T_1 + \int_{T_1}^{T_2} \|v_{\varepsilon}(\tau)\|_{Lip} d\tau \le c_m \min(s, 2-s) \le C_m,$$

and for all $t \in [T_1, T_2]$, we have :

$$\nu \|F\|_{\widetilde{L}^{1}([T_{1},t],C^{s-1})} + \|G\|_{C^{s-3}} \le C_{F,s}(1+t^{\frac{7}{8}}\varepsilon^{\frac{1}{8}-2\alpha}e^{CV_{\varepsilon}(t)}) \sup_{\tau\in[T_{1},t]} \|X^{\varepsilon}(\tau)\|_{C^{s}}$$

Proof: This is here that we will use the smoothing effect on the potential vorticity.

$$\nu \|F\|_{\widetilde{L^1}([T_1,t],C^{s-1})} = \sup_q 2^{q(s-1)} \int_{T_1}^t \nu \|\Delta_q F(\tau)\|_{L^\infty} d\tau$$

Using the expression of F we get :

$$\nu \|F\|_{\widetilde{L^{1}}([T_{1},t],C^{s-1})} \leq \sup_{q \geq -1} 2^{q(s-1)} \sum_{i} \int_{T_{1}}^{t} (\nu \|\Delta_{q}\partial_{i}R(\Delta X^{\varepsilon,i},\Omega_{\varepsilon})\|_{L^{\infty}} + 2\nu \|\partial_{i}R(\nabla X^{\varepsilon,i},\nabla\Omega_{\varepsilon})\|_{L^{\infty}}) d\tau$$

Developping the remainder in terms of the Littlewood-Paley operators, and using the Bernstein Lemma to get rid of the derivative we get :

$$\nu \|F\|_{\widetilde{L^{1}}([T_{1},t],C^{s-1})} \leq \sup_{q \geq -1} 2^{qs} \sum_{i} \int_{T_{1}}^{t} \left(\nu \sum_{q' \geq q-N_{0}} \sum_{\alpha=-1}^{1} \|\Delta_{q'}\Omega_{\varepsilon}\|_{L^{\infty}} \|\Delta_{q'+\alpha}\Delta X^{\varepsilon,i}\|_{L^{\infty}} \right)$$
$$+ 2\nu \sum_{q' \geq q-N_{0}} \sum_{\alpha=-1}^{1} \|\Delta_{q'}\nabla\Omega_{\varepsilon}\|_{L^{\infty}} \|\Delta_{q'+\alpha}\nabla X^{\varepsilon,i}\|_{L^{\infty}}) d\tau.$$

Using another time the Bernstein Lemma to get rid of the derivatives, and the fact that $X^{\varepsilon,i}\in C^s$ we obtain :

$$\nu \|F\|_{\widetilde{L^{1}}([T_{1},t],C^{s-1})} \leq \sup_{q \geq -1} 2^{qs} \sum_{i} \int_{T_{1}}^{t} \left(\nu \sum_{q' \geq q-N_{0}} \sum_{\alpha=-1}^{1} 2^{2q'} \|\Delta_{q'}\Omega_{\varepsilon}\|_{L^{\infty}} 2^{-q's} \|X^{\varepsilon,i}\|_{C^{s}}\right) d\tau.$$

The smoothing effect (Lemma 4.4.4) applied to system (4.4.23) gives us that for all $t \in [T_1, T_2]$:

$$\nu \|\Omega_{\varepsilon}\|_{\widetilde{L^{1}}([T_{1},t],C_{*}^{2})} \leq C(1+\nu(t-T_{1}))\big(\|\Omega_{\varepsilon}(T_{1})\|_{L^{\infty}}+\|q_{\varepsilon}\|_{L^{1}([T_{1},t],L^{\infty})}\big).$$

And, thanks to Lemma 4.4.3 we can write that :

$$\|\Omega_{\varepsilon}(T_1)\|_{L^{\infty}} \le \|\Omega_{\varepsilon}(0)\|_{L^{\infty}} + \|q_{\varepsilon}\|_{L^1([0,T_1],L^{\infty})}$$

which implies :

$$\sup_{q} \int_{T^{1}}^{t} \nu 2^{2q} \|\Delta_{q} \Omega_{\varepsilon}(\tau)\|_{L^{\infty}} \leq C(1 + \nu(t - T_{1})) \big(\|\Omega_{\varepsilon}(0)\|_{L^{\infty}} + \|q_{\varepsilon}\|_{L^{1}([0,t],L^{\infty})} \big)$$

So that we obtain :

$$\nu \|F\|_{\widetilde{L^{1}}([T_{1},t],C^{s-1})} \leq \sup_{q \geq -1} 2^{qs} \sum_{i} \int_{T_{1}}^{t} \left(\sum_{q' \geq q-N_{0}} \sum_{\alpha=-1}^{1} 2^{-q's}\right) C(1+\nu(\tau-T_{1})) \\ \left(\|\Omega_{\varepsilon}(0)\|_{L^{\infty}} + \|q_{\varepsilon}\|_{L^{1}([0,\tau],L^{\infty})}\right) \|X^{\varepsilon,i}\|_{C^{s}} d\tau.$$

Using the fact that $T_2 - T_1 \leq c_m \min(s, 2 - s) \leq C_m$, we finally obtain that (estimating the integral on X^{ε} with the L^{∞} norm) :

$$\nu \|F\|_{\widetilde{L^{1}}([T_{1},t],C^{s-1})} \leq \frac{C}{s} (1+\nu C_{m}) \big(\|\Omega_{\varepsilon}(0)\|_{L^{\infty}} + \|q_{\varepsilon}\|_{L^{1}([0,t],L^{\infty})} \big) \Big(C_{m} \sup_{\tau \in [T_{1},t]} \|X^{\varepsilon,i}(\tau)\|_{C^{s}} \Big).$$

The assumptions give that $\|\Omega_{\varepsilon}(0)\|_{L^{\infty}} \leq C_0$ and all that remains is to estimate q_{ε} .

Lemma 4.4.7 There exists a constant $C_{F,s}$, such that for all t,

$$\|q_{\varepsilon}\|_{L^{1}([0,t],L^{\infty})} \leq C_{F,s} t^{\frac{7}{8}} \varepsilon^{\frac{1}{8}-2\alpha} e^{CV_{\varepsilon}(t)}.$$

Proof: : Using the expression of q_{ε} :

$$\|q_{\varepsilon}\|_{L^{1}([0,t],L^{\infty})} \leq \int_{0}^{t} \left(\|\partial U_{\varepsilon,osc}\|_{L^{\infty}} \|\partial U_{\varepsilon}\|_{L^{\infty}} + \|\partial U_{\varepsilon,osc}\|_{L^{\infty}} \|\partial U_{\varepsilon,QG}\|_{L^{\infty}} \right) d\tau,$$

and thanks to the Sobolev embedding $H^{s+\frac{5}{2}} \hookrightarrow Lip$ and the apriori estimates (Lemma 4.7.1) we obtain :

$$\|q_{\varepsilon}\|_{L^{1}([0,t],L^{\infty})} \leq C \int_{0}^{t} \left(\|U_{\varepsilon,osc}\|_{Lip} \|U_{\varepsilon,0}\|_{H^{s+\frac{5}{2}}} e^{CV_{\varepsilon}(\tau)} d\tau \right)$$

which gives, using a Hölder inequality and the Strichartz estimates (Lemma 4.8.1) :

$$\|q_{\varepsilon}\|_{L^{1}([0,t],L^{\infty})} \leq C_{F,s} t^{\frac{7}{8}} \varepsilon^{\frac{1}{8}-2\alpha} e^{CV_{\varepsilon}(t)}.$$

And this completes the proof of Lemma 4.4.7. \blacksquare

Finally we get :

$$\nu \|F\|_{\widetilde{L^{1}}([T_{1},t],C^{s-1})} \leq C_{F,s}(1+t^{\frac{7}{8}}\varepsilon^{\frac{1}{8}-2\alpha}e^{CV_{\varepsilon}(t)})\sup_{\tau\in[T_{1},t]}\|X^{\varepsilon}(\tau)\|_{C^{s}}.$$

The estimate on G is the same as in [17] only for the fact that we have to take care of the term q_{ε} , so we won't give many details :

 $\|G(t)\|_{C^{s-3}} \le C_s \|X^{\varepsilon}(t)\|_{C^s} \|\Omega_{\varepsilon(t)}\|_{L^{\infty}}$

and using lemma (4.4.3), and the estimate on q_{ε} we obtain :

$$||G||_{C^{s-3}} \le C_{F,s} (1 + t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}) \sup_{\tau \in [T_1, t]} ||X^{\varepsilon}(\tau)||_{C^s},$$

which concludes the proof of Lemma 4.4.6. \blacksquare

Conclusion

We are now able to estimate $X^{\varepsilon}(x, D)\Omega_{\varepsilon}$, let us make the same assumption :

$$T_2 - T_1 + \int_{T_1}^{T_2} \|v_{\varepsilon}(\tau)\|_{Lip} d\tau \le c_m \min(s, 2-s) \le C_m.$$

Then thanks to Lemma 4.4.5, for all $t \in [T_1, T_2]$, we have :

$$\begin{aligned} \|X^{\varepsilon}(x,D)\Omega_{\varepsilon}(t)\|_{C^{s-1}} &\leq C\big(\|X^{\varepsilon}(x,D)\Omega_{\varepsilon}(T_{1})\|_{C^{s-1}} + \nu\|F\|_{\widetilde{L}^{1}([T_{1},t],C^{s-1})} \\ &+ \|X^{\varepsilon}(x,D)q_{\varepsilon}\|_{\widetilde{L}^{1}([T_{1},t],C^{s-1})}\big) + C(1+\nu(t-T_{1}))\big(\|G\|_{\widetilde{L^{\infty}}([T_{1},t],C^{s-3})}\big). \end{aligned}$$

Let us begin with the estimate on $X^{\varepsilon}(x, D)q_{\varepsilon}$:

Lemma 4.4.8 Under the same assumptions on T_1 and T_2 there exists a constant $C_{F,s}$ such that for $t \in [T_1, T_2]$:

$$\|X^{\varepsilon}(x,D)q_{\varepsilon}\|_{L^{1}([T_{1},t],C^{s-1})} \leq C_{F,s}(1+t^{\frac{7}{8}}\sup_{[T_{1},t]}\|X^{\varepsilon}(\tau)\|_{C^{s}})\varepsilon^{\frac{1}{8}-2\alpha}e^{CV_{\varepsilon}(t)}$$

Proof: according to the definition :

$$\|X^{\varepsilon}(x,D)q_{\varepsilon}\|_{L^{1}([T_{1},t],C^{s-1})} \leq \int_{T_{1}}^{t} \|X^{\varepsilon}.q_{\varepsilon}\|_{C^{s}} d\tau.$$

Then, thanks to the product laws in the Hölder spaces, to (4.4.31) and to the Hölder inequality, we have :

$$\|X^{\varepsilon}(x,D)q_{\varepsilon}\|_{L^{1}([T_{1},t],C^{s-1})} \leq C_{F}\|X^{\varepsilon}\|_{L^{\infty}([T_{1},t],C^{s})}\|q_{\varepsilon}\|_{L^{1}L^{\infty}} + \|X_{0}\|_{L^{\infty}}e^{V_{\varepsilon(t)}}\|q_{\varepsilon}\|_{L^{1}C^{s}}$$

then using methods close to those in the proof of Lemma 4.4.7, and the fact that $T_2 - T_1 \leq C_m$, we can finally write :

$$\|X^{\varepsilon}(x,D)q_{\varepsilon}\|_{L^{1}([T_{1},t],C^{s-1})} \leq C_{F,s}(1+t^{\frac{7}{8}}\sup_{[T_{1},t]}\|X^{\varepsilon}(\tau)\|_{C^{s}})\varepsilon^{\frac{1}{8}-2\alpha}e^{CV_{\varepsilon}(t)}.\blacksquare$$

And then

$$\|X^{\varepsilon}(x,D)\Omega_{\varepsilon}(t)\|_{C^{s-1}} \leq C_{F,s} \Big(\|X^{\varepsilon}(x,D)\Omega_{\varepsilon}(T_{1})\|_{C^{s-1}} + \varepsilon^{\frac{1}{8}-2\alpha} e^{CV_{\varepsilon}(t)} + (1+t^{\frac{7}{8}}\varepsilon^{\frac{1}{8}-2\alpha} e^{CV_{\varepsilon}(t)}) \sup_{\tau \in [T_{1},t]} \|X^{\varepsilon}(\tau)\|_{C^{s}}\Big)$$

$$(4.4.34)$$

In order to get rid of the term $\sup_{\tau \in [T_1,t]} \|X^{\varepsilon}(\tau)\|_{C^s}$ we estimate it thanks to (4.4.32). As we want to use a Gronwall argument, instead of reasoning separately on (4.4.32), or (4.4.34) we will work on $\Gamma(t) \stackrel{\text{def}}{=} \|X^{\varepsilon}(x,D)\Omega_{\varepsilon}(t)\|_{C^{s-1}} + \|X^{\varepsilon}(t)\|_{C^s}$. From the previous estimates we have that for all $t \in [T_1,T_2]$:

$$\Gamma(t) \leq C_{F,s} \left(1 + t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}\right) \left(\Gamma(T_1) + \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}\right) + C_{F,s} \left(1 + t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}\right) \int_{T_1}^t \Gamma(\tau) \left(1 + \|U_{\varepsilon,osc}\|_{Lip} + \|U_{\varepsilon,QG}\|_{Lip}\right) d\tau$$

4.4.4 Proof of the theorem

End of the proof

As we have already said, for every $\varepsilon > 0$, the solution is regular and defined on a maximal time intervall $[0, T_{\varepsilon}^*]$.

Let us define the following times : $T_{\varepsilon}^{\gamma} \stackrel{\text{def}}{=} \gamma \log |\log \varepsilon|$, where γ is a small constant, whose value will be fixed later, and :

$$T_{\varepsilon} = \sup\left\{t \in [0, T_{\varepsilon}^{\gamma}] \cap [0, T_{\varepsilon}^{*}[\text{ such that } V_{\varepsilon}(t) = \int_{0}^{t} \|v_{\varepsilon}(\tau)\|_{Lip} d\tau \le K_{\varepsilon}\right\}, \quad (4.4.35)$$

where $K_{\varepsilon} = 2e^{6C_{F,s}T_{\varepsilon}^{\gamma}} = 2|\log \varepsilon|^{6\gamma C_{F,s}}$, and $C_{F,s}$ is basically the same as in the previous lemmas.

We will prove the theorem by contradiction : assume that $T_{\varepsilon} < \min(T_{\varepsilon}^{\gamma}, T_{\varepsilon}^{*})$. Assume that we have proved that for all $t \leq T_{\varepsilon}$, if we define $g_{\varepsilon}(t) = t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}$.

$$V_{\varepsilon}(t) \le C_{F,s} (1 + g_{\varepsilon}(t)) e^{2C_{F,s} \int_0^t (1 + g_{\varepsilon}(\tau)) d\tau}.$$
(4.4.36)

As g_{ε} is a increasing function, and $T_{\varepsilon} \leq T_{\varepsilon}^{\gamma}$ we have:

$$V_{\varepsilon}(t) \le C_{F,s} (1 + g_{\varepsilon}(t)) e^{2C_{F,s} T_{\varepsilon}^{\gamma}(1 + g_{\varepsilon}(t))}.$$

As we know that for all $x \ge 0$, $xe^{2T_{\varepsilon}^{\gamma}x} \le \frac{1}{T_{\varepsilon}^{\gamma}e}e^{3T_{\varepsilon}^{\gamma}x}$, and that $eT_{\varepsilon}^{\gamma} \ge 1$, we get that:

$$V_{\varepsilon}(t) \le e^{3C_{F,s}T_{\varepsilon}^{\gamma}(1+g_{\varepsilon}(t))}$$

Thanks to the choice of γ , for all $\varepsilon \leq \varepsilon_0$, $g_{\varepsilon(t)} \leq 1$, which implies that for all $t \leq T_{\varepsilon}$, $V_{\varepsilon}(t) \leq \frac{K_{\varepsilon}}{2}$, which contradicts the definition of the maximality of T_{ε} .

So $T_{\varepsilon} = min(T_{\varepsilon}^{\gamma}, T_{\varepsilon}^*)$ and then, as T_{ε}^* is the maximal time of existence of U_{ε} , we can write that:

$$T_{\varepsilon}^* \geq T_{\varepsilon}^{\gamma} = \gamma \log |\log \varepsilon|$$

This concludes the proof of the first part of theorem 4.1.4. \blacksquare

Proof of (4.4.36)

Now we will prove (4.4.36): let $t \leq T_{\varepsilon}$ and subdivide [0, t] into $0 = T_0 < T_1 < ... < T_N = t$ such that for all i = 0, ..., N - 1,

$$T_{i+1} - T_i + \int_{T_i}^{T_{i+1}} \|\nabla v_{\varepsilon}(\tau)\|_{L^{\infty}} d\tau \le c_m \min(s, 2-s) \le C_m,$$
(4.4.37)

and from the previous estimates on Γ , for all $t \in [T_i, T_{i+1}]$:

$$\Gamma(t) \le C_{F,s} (1 + t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}) \left(\Gamma(T_i) + \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)} \right)$$

$$+C_{F,s}\left(1+t^{\frac{7}{8}}\varepsilon^{\frac{1}{8}-2\alpha}e^{CV_{\varepsilon}(t)}\right)\int_{T_{i}}^{t}\Gamma(\tau)\left(1+\|U_{\varepsilon,osc}\|_{Lip}+\|U_{\varepsilon,QG}\|_{Lip}\right)d\tau$$
(4.4.38)

Taking advantage of the fact that ε goes to zero, let us state the following lemma :

Lemma 4.4.9 If $0 < \gamma \leq \frac{1}{6C_{F,s}} - \frac{\log 32C}{6C_{F,s} \log 2}$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$, for all $t \leq T_{\varepsilon}$, $(1 + t^{\frac{7}{8}})\varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)} \leq 1$

Proof: for all $t \leq T_{\varepsilon} \leq T_{\varepsilon}^{\gamma}$, we have :

$$(1+t^{\frac{7}{8}})\varepsilon^{\frac{1}{8}-2\alpha}e^{CV_{\varepsilon}(t)} \leq (1+(\gamma\log|\log\varepsilon|)^{\frac{7}{8}})\varepsilon^{\frac{1}{8}-2\alpha}e^{C2|\log\varepsilon|^{6\gamma C_{F,s}}}.$$

We want this quantity to be less than 1, and begin with taking γ such that for all $x \geq 2$:

$$2Cx^{6\gamma C_{F,s}} \le \frac{x}{16}.$$

It is easy to show that this is equivalent to $0 < \gamma \leq \frac{1}{6C_{F,s}} - \frac{\log 32C}{6C_{F,s} \log 2}$. So if γ is chosen this way and if $\varepsilon \leq e^{-2}$, for all $t \leq T_{\varepsilon}$,

$$(1+t^{\frac{7}{8}})\varepsilon^{\frac{1}{8}-2\alpha}e^{CV_{\varepsilon}(t)} \le (1+(\gamma\log|\log\varepsilon|)^{\frac{7}{8}})\varepsilon^{\frac{1}{8}-2\alpha}e^{\frac{|\log\varepsilon|}{16}} \le (1+(\gamma\log|\log\varepsilon|)^{\frac{7}{8}})\varepsilon^{\frac{1}{16}-2\alpha}.$$

So there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ this is less than 1.

It simplifies (4.4.38) into : $\forall t \in [T_i, T_{i+1}],$

$$\Gamma(t) \le C_{F,s} \big(\Gamma(T_i) + 1 \big) + C_{F,s} \int_{T_i}^t \Gamma(\tau) \Big(1 + \| U_{\varepsilon,osc} \|_{Lip} + \| U_{\varepsilon,QG} \|_{Lip} \Big) d\tau.$$

Then, the Gronwall lemma implies that for all i and $t \in [T_i, T_{i+1}]$,

$$\Gamma(t) \le C_{F,s} \big(\Gamma(T_i) + 1 \big) e^{C_{F,s} \int_{T_i}^t f_{\varepsilon}(\tau) d\tau}, \quad \text{with} \quad f_{\varepsilon}(\tau) = 1 + \| U_{\varepsilon,osc} \|_{Lip} + \| U_{\varepsilon,QG} \|_{Lip},$$

which we transform into the recursive relation : $a_{i+1} \leq C_{F,s}a_i + C_{F,s}b_i$ with :

$$a_i \stackrel{\text{def}}{=} \Gamma(T_i) e^{-C_{F,s} \int_0^{T_i} f_{\varepsilon}}$$
 and $b_i = e^{-C_{F,s} \int_0^{T_i} f_{\varepsilon}}$.

An easy recurrence gives then :

$$a_N \le (C_{F,s})^N a_0 + \sum_{k=0}^{N-1} b_k (C_{F,s})^{N-k}.$$

We can suppose that $C_{F,s} > 1$ (if not, just take $C_{F,s} = \max(2, C_{F,s})$), so after a computation we find :

$$a_N \le (C_{F,s})^N (a_0 + \frac{C_{F,s}}{C_{F,s} - 1}),$$

which is nothing else than (returning to the original quantities) :

$$\Gamma(T_N) \le e^{C_{F,s} \int_0^{T_N} f_{\varepsilon}} (C_{F,s})^N \Big(\Gamma(0) + \frac{C_{F,s}}{C_{F,s} - 1} \Big).$$

Thanks to (4.4.37) we have

$$N \sim \frac{1}{C_m} (T_N + \int_0^{T_N} \|\nabla v_{\varepsilon}(\tau)\|_{L^{\infty}} d\tau)$$

and then (as $T_N = t$) :

$$\Gamma(t) \le e^{C_{F,s} \int_0^t f_{\varepsilon}} e^{\frac{\log C_{F,s}}{C_m} (t + \int_0^t \|v_{\varepsilon}(\tau)\|_{Lip} d\tau)} \Big(\Gamma(0) + \frac{C_{F,s}}{C_{F,s} - 1} \Big).$$

If we pose again : $C_{F,s} = \max(C_{F,s}, \frac{\log C_{F,s}}{C_m}, \frac{\Gamma(0)+C_{F,s}}{C_{F,s}-1})$, and as :

$$t + \int_0^t \|v_{\varepsilon}(\tau)\|_{Lip} d\tau \le \int_0^t \left(1 + \|v_{\varepsilon,osc}(\tau)\|_{Lip} + \|v_{\varepsilon,QG}(\tau)\|_{Lip}\right) d\tau,$$

we can write that :

$$\forall t \leq T_{\varepsilon}, \quad \Gamma(t) \leq C_{F,s} e^{C_{F,s} \int_0^t \left(1 + \|v_{\varepsilon,osc}\|_{Lip}(\tau) + \|v_{\varepsilon,QG}(\tau)\|_{Lip}\right) d\tau}.$$

And, using the Strichartz estimates (see Section 4.8) and posing $C_{F,s} = \max(C_{F,s}, C_{F,s}^2)$, we finally obtain that for all $\varepsilon \leq \varepsilon_0$, and all $t \leq T_{\varepsilon}$:

$$\Gamma(t) \le C_{F,s} e^{C_{F,s} \left(t + t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - \alpha} e^{CV_{\varepsilon}(t)} + \int_0^t \|U_{\varepsilon,QG}(\tau)\|_{Lip} d\tau \right)}.$$
(4.4.39)

Now we are able to deal with (4.4.29):

$$\|U_{\varepsilon,QG}(t)\|_{Lip} \le C_0 (1 + t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}) \Big(1 + \log\left(e + 1 + C_0 e^{CV_{\varepsilon}(t)} + \frac{1}{C_0} \Gamma(t)\right) \Big).$$

As $\varepsilon \leq \varepsilon_0$ and $t \leq T_{\varepsilon}$, and posing :

$$g_{\varepsilon}(t) = t^{\frac{7}{8}} \varepsilon^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon}(t)}, \text{ and } C = \max(1, e+1, C_0, C, \frac{1}{C_0}),$$

$$\frac{\|U_{\varepsilon,QG}(t)\|_{Lip}}{(1+g_{\varepsilon}(t))} \leq C\Big(1+\log\left(C(1+e^{CV_{\varepsilon}(t)}+C_{F,s}e^{C_{F,s}\left(t+g_{\varepsilon}(t)+\int_{0}^{t}\|U_{\varepsilon,QG}(\tau)\|_{Lip}d\tau\right)}\right)\Big).$$

As $1 + e^{CV_{\varepsilon}(t)} \leq e^{C_{F,s}(t+g_{\varepsilon}(t)+\int_{0}^{t} \|U_{\varepsilon,QG}(\tau)\|_{Lip}d\tau)})$, and taking $C_{F,s} = \max(3C_{F,s}, C)$, we obtain :

$$\frac{\|U_{\varepsilon,QG}(t)\|_{Lip}}{(1+g_{\varepsilon}(t))} \le C_{F,s} \Big(1+\log C_{F,s} + C_{F,s} \big(t+g_{\varepsilon}(t) + \int_0^t \|U_{\varepsilon,QG}(\tau)\|_{Lip} d\tau\big)\Big)$$

As $0 < \log C_{F,s} \le C_{F,s} \le C_{F,s}^2$, we obtain :

$$\frac{\|U_{\varepsilon,QG}(t)\|_{Lip}}{(1+g_{\varepsilon}(t))} \le C_{F,s} \Big(1+t+g_{\varepsilon}(t)+\int_0^t \|U_{\varepsilon,QG}(\tau)\|_{Lip} d\tau\Big),$$

which we rewrite into, defining $h_{\varepsilon}(t) = \frac{\|U_{\varepsilon,QG}(t)\|_{Lip}}{1+g_{\varepsilon}(t)}$,

$$h_{\varepsilon}(t) \leq C_{F,s} (1+t+g_{\varepsilon}(t)) + C_{F,s} \int_{0}^{t} (1+g_{\varepsilon}(\tau))h_{\varepsilon}(\tau)d\tau),$$

Using the Gronwall lemma gives : $h_{\varepsilon}(t) \leq C_{F,s} (1 + t + g_{\varepsilon}(t)) e^{C_{F,s} \int_0^t (1 + g_{\varepsilon}(\tau)) d\tau}$, so :

$$\|U_{\varepsilon,QG}(t)\|_{Lip} \le C_{F,s}(1+g_{\varepsilon}(t))\big(1+t+g_{\varepsilon}(t)\big)e^{C_{F,s}\int_0^t (1+g_{\varepsilon}(\tau))d\tau}\big),$$

and then, integrating on [0, t], and using that $t \mapsto t + g_{\varepsilon}(t)$ is an increasing function, we obtain :

$$\int_0^t \|U_{\varepsilon,QG}(t')\|_{Lip} dt' \le (1+t+g_{\varepsilon}(t))(e^{C_{F,s}\int_0^t (1+g_{\varepsilon}(\tau)d\tau)} - 1).$$

$$\le C_{F,s}(1+t+g_{\varepsilon}(t))(e^{C_{F,s}\int_0^t (1+g_{\varepsilon}(\tau))d\tau} - 1).$$

$$(4.4.40)$$

And we can finally go back to V_{ε} (we use the Strichartz estimates for the oscillating part):

$$\begin{aligned} V_{\varepsilon}(t) &= \int_{0}^{t} \|U_{\varepsilon}(\tau)\|_{Lip} d\tau \leq \int_{0}^{t} \left(\|U_{\varepsilon,osc}(\tau)\|_{Lip} + \|v_{\varepsilon,QG}(\tau)\|_{Lip} \right) d\tau \\ &\leq C_{F,s} g_{\varepsilon}(t) + \int_{0}^{t} \|v_{\varepsilon,QG}(\tau)\|_{Lip} d\tau. \end{aligned}$$

Using (4.4.40) and the fact that $xe^x \leq \frac{1}{e}e^{2x}$, we get:

$$V_{\varepsilon}(t) \leq C_{F,s} (1 + g_{\varepsilon}(t)) e^{2C_{F,s} \int_0^t (1 + g_{\varepsilon}(\tau)) d\tau}$$
.

4.4.5 Quasigeostrophic limit

This section follows the lines of the last part of [11] : we will show that $(U_{\varepsilon,QG})_{\varepsilon}$ is a Cauchy sequence in $L^{\infty}_{loc}(\mathbb{R}_+, L^2)$ when ε goes to zero.

Let U_{ε_1} and U_{ε_2} solutions of (respectively) PE_{ε_1} and PE_{ε_2} : using the diagonalization explained in (4.8.53):

$$\begin{cases} \partial_t U_{\varepsilon_1,QG} - \nu \Delta U_{\varepsilon_1,QG} = -\mathcal{Q}\mathbb{P}(v_{\varepsilon_1}.\nabla U_{\varepsilon_1}) \\ \partial_t U_{\varepsilon_2,QG} - \nu \Delta U_{\varepsilon_2,QG} = -\mathcal{Q}\mathbb{P}(v_{\varepsilon_2}.\nabla U_{\varepsilon_2}) \end{cases}$$

Let us define $U = U_{\varepsilon_1} - U_{\varepsilon_2}$, we obtain the system :

$$\begin{cases} \partial_t U_{QG} - \nu \Delta U_{QG} = -\mathcal{QP}(v_{\varepsilon_1} \cdot \nabla U) - \mathcal{QP}(v \cdot \nabla U_{\varepsilon_2}) \\ U_{QG/t=0} = U_{\varepsilon_1,0,QG} - U_{\varepsilon_2,0,QG} \end{cases}$$

A usual scalar product in L^2 gives :

$$\frac{1}{2}\frac{d}{dt}\|U_{QG}\|_{L^2}^2 + \nu\|\nabla U_{QG}\|_{L^2}^2 = -(v_{\varepsilon_1} \cdot \nabla U_{osc} + v \cdot \nabla U_{\varepsilon_2}|U_{QG}).$$

We try to separate as much as possible oscillating parts (which goes to zero) :

$$v_{\varepsilon_1}.\nabla U_{osc} + v.\nabla U_{\varepsilon_2} = v_{\varepsilon_1}.\nabla U_{\varepsilon_1,osc} + v.\nabla U_{\varepsilon_2,QG} - v_{\varepsilon_2}.\nabla U_{\varepsilon_2,osc}$$

So, our energy estimate becomes :

$$\frac{1}{2} \frac{d}{dt} \|U_{QG}\|_{L^{2}}^{2} + \nu \|\nabla U_{QG}\|_{L^{2}}^{2} \leq \left(\|v_{\varepsilon_{1}}\|_{L^{2}} \|U_{\varepsilon_{1},osc}\|_{Lip} + \|v_{QG}\|_{L^{2}} \|U_{\varepsilon_{2},QG}\|_{Lip} + \left(\|v_{\varepsilon_{1},osc}\|_{C^{s+1}} + \|v_{\varepsilon_{2},osc}\|_{C^{s+1}} \right) \|U_{\varepsilon_{2},QG}\|_{H^{1}} + \|v_{\varepsilon_{2}}\|_{L^{2}} \|U_{\varepsilon_{2},osc}\|_{Lip} \right) \|U_{QG}\|_{L^{2}}.$$

We take advantage of the fact that Q is an homogeneous pseudo-differential operator of order zero, and group terms:

$$\frac{1}{2}\frac{d}{dt}\|U_{QG}\|_{L^{2}}^{2} \leq \left(\|v_{\varepsilon_{1}}\|_{L^{2}} \|U_{\varepsilon_{1},osc}\|_{Lip} + \left(\|v_{\varepsilon_{1},osc}\|_{Lip} + \|v_{\varepsilon_{2},osc}\|_{Lip}\right)\|U_{\varepsilon_{2}}\|_{H^{1}} + \|v_{\varepsilon_{2}}\|_{L^{2}}\|U_{\varepsilon_{2},osc}\|_{Lip}\right)\|U_{QG}\|_{L^{2}} + \|U_{\varepsilon_{2},QG}\|_{Lip}\|U_{QG}\|_{L^{2}}^{2}.$$

Using a Gronwall estimate we obtain that :

$$\begin{aligned} \|U_{QG}\|_{L^{2}} &\leq \|U_{QG}(0)\|_{L^{2}} e^{\int_{0}^{t} \|U_{\varepsilon_{2},QG}(\tau)\|_{Lip}d\tau} + \int_{0}^{t} \left(\|v_{\varepsilon_{1}}\|_{L^{2}} \|U_{\varepsilon_{1},osc}\|_{Lip} + \|v_{\varepsilon_{2}}\|_{L^{2}} \|U_{\varepsilon_{2},osc}\|_{Lip} + \|U_{\varepsilon_{2},osc}\|_{Lip}\right) \|U_{\varepsilon_{2}}\|_{H^{1}} \right) e^{\int_{t'}^{t} \|U_{\varepsilon_{2},QG}(\tau)\|_{Lip}d\tau} dt' \end{aligned}$$

If $\varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_0$, we can fix a time $\widetilde{T} \leq T_{\varepsilon_i}^{\gamma} \leq T_{\varepsilon_i}^*$ for i = 1, 2 and according to lemma 4.4.9 for $i = 1, 2, g_{\varepsilon_i}(t) \leq \widetilde{T}^{\frac{7}{8}} \varepsilon_i^{\frac{1}{8} - 2\alpha} e^{CV_{\varepsilon_i}(t)} \leq 1$, and (4.4.40) implies that :

$$\int_0^t \|U_{\varepsilon_i,QG}(t')\|_{Lip}dt' \le (1+\widetilde{T}+1)(e^{2C_{F,s}\widetilde{T}}-1) \le 2(1+\widetilde{T})e^{2C_{F,s}\widetilde{T}},$$

which implies that :

$$\begin{aligned} \|U_{QG}\|_{L^{2}} &\leq e^{2\left(1+\widetilde{T}\right)}e^{2C_{F,s}\widetilde{T}} \Big(\|U_{QG}(0)\|_{L^{2}} + \int_{0}^{t} \big(\|v_{\varepsilon_{1}}\|_{L^{2}} \cdot \|U_{\varepsilon_{1},osc}\|_{Lip} \\ &+ \|v_{\varepsilon_{2}}\|_{L^{2}} \|U_{\varepsilon_{2},osc}\|_{Lip} + \big(\|U_{\varepsilon_{1},osc}\|_{Lip} + \|U_{\varepsilon_{2},osc}\|_{Lip}\big)\|U_{\varepsilon_{2}}\|_{H^{1}}\big)dt'\Big). \end{aligned}$$

All that remains is then to use the Strichartz estimates and we obtain (returning to the notaion $U_{\varepsilon_1} - U_{\varepsilon_2}$):

$$\begin{split} \|U_{\varepsilon_{1},QG} - U_{\varepsilon_{2},QG}\|_{L^{\infty}([0,\widetilde{T}],L^{2})} &\leq C_{F,s,\widetilde{T}} \Big(\|U_{\varepsilon_{1},0,QG} - U_{\varepsilon_{2},0,QG}\|_{L^{2}} \\ &+ \varepsilon_{1}^{\frac{1}{8}-2\alpha} + \varepsilon_{2}^{\frac{1}{8}-2\alpha} + \varepsilon_{2}^{-\alpha} \big(\varepsilon_{1}^{\frac{1}{8}-\alpha} + \varepsilon_{2}^{\frac{1}{8}-\alpha}\big) \Big). \end{split}$$

As $\varepsilon_1 \leq \varepsilon_2$, $\varepsilon_2^{-\alpha} \varepsilon_1^{\frac{1}{8}-\alpha} \leq \varepsilon_1^{\frac{1}{8}-2\alpha}$, and then $U_{\varepsilon,QG}$ is a Cauchy sequence in $L^{\infty}([0,\widetilde{T}], L^2)$ if ε is small enough.

From [4] we already know the existence of a extracted sequence that converges to a solution of the quasigeostrophic system with $U_{0,QG}$ as an initial data (even if the initial data depends on ε we can easily adapt the method), which is Lipschitzian like every $U_{\varepsilon,QG}$ and this concludes the proof of Theorem 4.1.4.

4.5 Appendix 1 : Strichartz estimates when $\nu \neq \nu'$

The object of this section is to prove the following result which is used in the proof of Theorem 4.1.2 when we want to estimate the external force (Lemma 4.2.3):

Lemma 4.5.1 Assume f solves on \mathbb{R}_+ the system

$$\begin{cases} \partial_t f - Lf + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} f = g^l + g^b \\ f_{/t=0} = f_0, \end{cases}$$

where $f_0 \in L^2(\mathbb{R}^3)$, $g^b \in L^1(\mathbb{R}_+, L^2)$ and $g^b \in L^2(\mathbb{R}_+, L^2)$

Assume that f_0 and $g^{b(l)}(t)$ for all $t \ge 0$ have their frequencies localized in $\mathcal{C}_{r_{\varepsilon},R_{\varepsilon}}$, where $r_{\varepsilon} = |log\varepsilon|^{-m}$, $R_{\varepsilon} = |log\varepsilon|^M$ and $\mathcal{C}_{r,R} = \{\xi \in \mathbb{R}^3 / |\xi| \le R \text{ and } |\xi_3| \ge r\}$. Assume also that f_0 and $g^{b(l)}(t)$ for all $t \in [0,T]$ only depend on the last two eigenvalues

Assume also that f_0 and $g^{b(l)}(t)$ for all $t \in [0, T]$ only depend on the last two eigenvalues of matrix \mathbb{B} . Then there exists $\varepsilon_0 = \varepsilon_0(F, \nu_0, \nu - \nu', m, M)$ a constant K^0 such that we have the following estimate $\forall \varepsilon \leq \varepsilon_0$:

$$\|f\|_{L^{2}(\mathbb{R}_{+},L^{\infty})} \leq K^{0} \varepsilon^{\frac{1}{8}} |log\varepsilon|^{6(m+M)} \Big(\|f_{0}\|_{L^{2}} + \|g^{b}\|_{L^{1}(\mathbb{R}_{+},L^{2})} + \|g^{l}\|_{L^{2}(\mathbb{R}_{+},L^{2})} \Big)$$

4.5.1 Estimates on the projectors

The object of this section is to prove proposition 4.2.1 which gives estimates on the projectors \mathbb{P}_i when the radiuses r_{ε} and R_{ε} depend on the Rossby number (more precise than those from [4]). This proposition is used in the proof of the Strichartz estimates.

Preliminary remarks

Remember that in [4] we provided asymptotic expansions of quantities depending on $\xi \in C_{r,R}$ with respect to ε . Here, r and R will also depend on the Rossby number. So the method will be the same as in [4] but we will have to be far more precise because the important points here, are the precise tracking in terms of ε of all the estimations, norms, and the asymptotic expansions in the stationnary phase.

Recall that matrix $\mathbb{B}(\xi,\varepsilon) = L - \frac{1}{\varepsilon} \mathbb{P} \mathcal{A}$ writes:

$$\mathbb{B}(\xi,\varepsilon) = \begin{pmatrix} -\nu|\xi|^2 + \frac{\xi_1\xi_2}{\varepsilon|\xi|^2} & \frac{\xi_2^2 + \xi_3^2}{\varepsilon|\xi|^2} & 0 & \frac{\xi_1\xi_3}{\varepsilon F|\xi|^2} \\ -\frac{\xi_1^2 + \xi_3^2}{\varepsilon|\xi|^2} & -\nu|\xi|^2 - \frac{\xi_1\xi_2}{\varepsilon|\xi|^2} & 0 & \frac{\xi_2\xi_3}{\varepsilon F|\xi|^2} \\ \frac{\xi_2\xi_3}{\varepsilon|\xi|^2} & -\frac{\xi_1\xi_3}{\varepsilon|\xi|^2} & -\nu|\xi|^2 & -\frac{\xi_1^2 + \xi_2^2}{\varepsilon F|\xi|^2} \\ 0 & 0 & \frac{1}{\varepsilon F} & -\nu'|\xi|^2 \end{pmatrix}.$$

Its characteristic polynomial is:

$$\chi_{\mathbb{B}}(X) = det(XI_4 - \mathbb{B}) =$$

$$\frac{\xi_1^2 + \xi_2^2}{\varepsilon^2 F^2 |\xi|^2} (X + \nu |\xi|^2)^2 + (X + \nu |\xi|^2)^3 (X + \nu' |\xi|^2) + \frac{\xi_3^2}{\varepsilon^2 |\xi|^2} (X + \nu |\xi|^2) (X + \nu' |\xi|^2),$$

and this polynomial writes, in terms of the variable $(X + \nu |\xi|^2)$:

$$\chi_{\mathbb{B}}(X) = (X + \nu |\xi|^2) P(X),$$

with

$$P(X) = (X + \nu|\xi|^2)^3 - (\nu - \nu')|\xi|^2 (X + \nu|\xi|^2)^2 + \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2} (X + \nu|\xi|^2) - (\nu - \nu')\frac{\xi_3^2}{\varepsilon^2}$$

In [4], as we wanted to use the Cardan formulas, we had to make the change the unknown $X = Y - \frac{\nu'+2\nu}{3}|\xi|^2$ (which takes account of the first change of variable (writing in $(X + \nu|\xi|^2)$) of the one done to turn the polynomial into the particular formulation $x^3 + px + q$): we get the following polynomial:

$$Q(Y) = Y^3 + pY + q, (4.5.41)$$

where

$$p = \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2} - \frac{(\nu - \nu')^2}{3} |\xi|^4, \quad \text{and} \quad q = \frac{\nu - \nu'}{\varepsilon^2} \left(\frac{|\xi|_F^2}{3F^2} - \xi_3^2\right) - \frac{2}{27} (\nu - \nu')^3 |\xi|^6.$$

Before applying the Cardan formulas we have to define the discriminent of the equation $D = \frac{q^2}{4} + \frac{p^3}{27}.$

In [4], the radiuses r and R of $C_{r,R}$ were fixed, and all we had to do was taking ε small enough to make D > 0 so that we can use the formula. Here, even though the radiuses depend on epsilon we can do the same: as $r_{\varepsilon} = |log\varepsilon|^{-m}$ and $R_{\varepsilon} = |log\varepsilon|^{M}$, in the expression of D every term is negligible in front of $\frac{|\xi|_{F}^{2}}{\varepsilon^{2}F^{2}|\xi|^{2}}$, for example:

$$\forall \xi \in \mathcal{C}_{r_{\varepsilon},R_{\varepsilon}} \quad \frac{\left(\frac{(\nu-\nu')^2}{3}|\xi|^4\right)^3}{\left(\frac{|\xi|_F^2}{\varepsilon^2F^2|\xi|^2}\right)^3} \le C_F|\nu-\nu'|^6|log\varepsilon|^{12M}\varepsilon^6.$$

So when ε is close to zero, say $\varepsilon \leq \varepsilon_0(F, m, M)$, and for every $\xi \in C_{r_\varepsilon, R_\varepsilon}$ the discriminant is equivalent to $\frac{|\xi|_F^6}{\varepsilon^6 F^6 |\xi|^6}$ and then is strictly positive so we can freely use the Cardan formulas, that is define:

$$\alpha = \left(-\frac{q}{2} + D^{\frac{1}{2}}\right)^{\frac{1}{3}} \quad \text{and} \quad \beta = \left(-\frac{q}{2} - D^{\frac{1}{2}}\right)^{\frac{1}{3}}, \tag{4.5.42}$$

and, then returning to the original variable the eigenvalues of $\mathbb{B}(\xi,\varepsilon)$ are:

$$\begin{cases} \mu_{0} = -\nu|\xi|^{2} \\ \mu = -\frac{\nu'+2\nu}{3}|\xi|^{2} + \alpha + \beta \\ \lambda = -\frac{\nu'+2\nu}{3}|\xi|^{2} + \alpha j + \beta j^{2} \\ \overline{\lambda} = -\frac{\nu'+2\nu}{3}|\xi|^{2} + \alpha j^{2} + \beta j. \end{cases}$$

$$(4.5.43)$$

If we note W_i the eigenvectors corresponding to these eigenvalues we obtain the matrix:

$$Q = \begin{pmatrix} \xi_2\xi_3 & \xi_3(\varepsilon\xi_1A + \xi_2) & \xi_3(\varepsilon\xi_1B + \xi_2) & \xi_3(\varepsilon\xi_1\overline{B} + \xi_2) \\ -\xi_1\xi_3 & \xi_3(\varepsilon\xi_2A - \xi_1) & \xi_3(\varepsilon\xi_2B - \xi_1) & \xi_3(\varepsilon\xi_2\overline{B} - \xi_1) \\ -\varepsilon F^2(\nu - \nu')|\xi|^2\xi_3^2 & -\varepsilon A(\xi_1^2 + \xi_2^2) & -\varepsilon B(\xi_1^2 + \xi_2^2) & -\varepsilon \overline{B}(\xi_1^2 + \xi_2^2) \\ F\xi_3^2 & F(\varepsilon^2|\xi|^2A^2 + \xi_3^2) & F(\varepsilon^2|\xi|^2B^2 + \xi_3^2) & F(\varepsilon^2|\xi|^2\overline{B}^2 + \xi_3^2) \end{pmatrix}$$

If we note $\mathcal{P}_i(\xi, \varepsilon)$, the projectors in the last three eigenspaces of matrix $B(\xi)$ (that depend also on ε), we define the following pseudo-differential operators:

$$\mathbb{P}_{i}(u) = \mathcal{F}^{-1}(\mathcal{P}_{i}(\xi,\varepsilon)(\widehat{u}(\xi))), \qquad (4.5.44)$$

The aim of this section will be to estimate the norms of these operators (defined on Sobolev spaces). We refer to [4] for the fact that (W_2, W_3, W_4) is a basis of the hyperplane of vectors orthogonal to $(\xi_1, \xi_2, \xi_3, 0)$ and that a vector h = (X, Y, Z, T) orthogonal to $(\xi_1, \xi_2, \xi_3, 0)$ writes $h = K_2W_2 + K_3W_3 + K_4W_4$ and that the solution of the system:

$$\begin{pmatrix} 0\\K_2\\K_3\\K_4 \end{pmatrix} = Q^{-1}h$$

is given by:

$$\begin{pmatrix} K_2 \\ K_3 \\ K_4 \end{pmatrix} = M \begin{pmatrix} \frac{\xi_2 X - \xi_1 Y}{\xi_3(\xi_1^2 + \xi_2^2)} \\ \frac{\xi_1 X + \xi_2 Y}{\varepsilon \xi_3(\xi_1^2 + \xi_2^2)} \\ \frac{T}{F\varepsilon^2 |\xi|^2} - \frac{\xi_3(\xi_2 X - \xi_1 Y)}{\varepsilon^2 |\xi|^2(\xi_1^2 + \xi_2^2)} \end{pmatrix}$$

with

$$M = \begin{pmatrix} \frac{|B|^2}{|B|^2 - AB + A^2 - A\overline{B}} & \frac{-(B + \overline{B})}{|B|^2 - AB + A^2 - A\overline{B}} & \frac{1}{|B|^2 - AB + A^2 - A\overline{B}} \\ \frac{-A\overline{B}}{-A\overline{B} + AB - B^2 + |B|^2} & \frac{A + \overline{B}}{-A\overline{B} + AB - B^2 + |B|^2} & \frac{-1}{-A\overline{B} + AB - B^2 + |B|^2} \\ \frac{AB}{-A\overline{B} + AB - B^2 + |B|^2} & \frac{-(A + B)}{-A\overline{B} + AB - B^2 + |B|^2} & \frac{1}{-A\overline{B} + AB - B^2 + |B|^2} \end{pmatrix}$$

where $A = \mu + \nu |\xi|^2$ and $B = \lambda + \nu |\xi|^2$

Projectors

In this section we will compute exactly the coefficients K_i in order to make precise estimates.

Lemma 4.5.2 If h is a function orthogonal to $(\xi_1, \xi_2, \xi_3, 0)$ and $\varepsilon \leq \varepsilon_0$ (so that the discriminent is positive see (4.5.42) in the previous section), then for all $\xi \in C_{r_{\varepsilon},R_{\varepsilon}}$ we have the estimate:

$$\begin{aligned} \|\mathcal{P}_{3}(\xi,\varepsilon)(h(\xi))\| \\ \leq \frac{1}{|(A-B)(B-\overline{B})|\varepsilon^{2}F^{2}} \Big(\frac{\varepsilon^{2}F^{2}|\xi|^{2}|B|^{2}+|\xi|_{F}^{2}}{\varepsilon^{2}|\xi|^{2}|B|^{2}+\xi_{3}^{2}}\Big)^{\frac{1}{2}} \Big(\frac{|\xi_{3}|}{|\xi|}+|\varepsilon B|\frac{|\xi|}{|\xi_{3}|}+\varepsilon^{2}F|B|^{2}\Big)|h(\xi)| \end{aligned}$$

with

$$(B-A)(B-\overline{B})\varepsilon^{2}F^{2} = \frac{|\xi|_{F}^{2}}{|\xi|^{2}} + 3\varepsilon^{2}F^{2}B^{2} - 2(\nu-\nu')\varepsilon^{2}F^{2}|\xi|^{2}B$$

and if, in addition $h(\xi)$ is orthogonal to $(-\xi_1, \xi_2, 0, -F\xi_3)$ then:

$$\begin{aligned} \|\mathcal{P}_{2}(\xi,\varepsilon)(h(\xi))\| \\ &\leq \frac{1}{(A-B)(A-\overline{B})\varepsilon^{2}F^{2}} \Big(\frac{\varepsilon^{2}F^{2}|\xi|^{2}A^{2}+|\xi|_{F}^{2}}{\varepsilon^{2}|\xi|^{2}A^{2}+\xi_{3}^{2}}\Big)^{\frac{1}{2}}\varepsilon\big(|A|\frac{|\xi|}{|\xi_{3}|}+\varepsilon FA^{2}\big)|h(\xi)| \end{aligned}$$

with

$$(A-B)(A-\overline{B})\varepsilon^{2}F^{2} = \frac{|\xi|_{F}^{2}}{|\xi|^{2}} + 3\varepsilon^{2}F^{2}A^{2} - 2(\nu-\nu')\varepsilon^{2}F^{2}|\xi|^{2}A$$

Proof: Let us begin with K_2 , using the change of basis we have:

$$\begin{split} K_2 &= \frac{1}{|B|^2 - AB + A^2 - A\overline{B}} \Big(|B|^2 \frac{\xi_2 X - \xi_1 Y}{\xi_3(\xi_1^2 + \xi_2^2)} - (B + \overline{B}) \frac{\xi_1 X + \xi_2 Y}{\varepsilon \xi_3(\xi_1^2 + \xi_2^2)} \\ &+ \frac{T}{F \varepsilon^2 |\xi|^2} - \frac{\xi_3(\xi_2 X - \xi_1 Y)}{\varepsilon^2 |\xi|^2 (\xi_1^2 + \xi_2^2)} \Big) \end{split}$$

Before any computation let us recall some algebraic relations: A, B, and \overline{B} are the roots of the following polynomial (which is nothing but the translated of P by the change of variable giving A in terms of μ):

$$Q_2 = Y^3 - (\nu - \nu')|\xi|^2 Y^2 + \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2} Y - (\nu - \nu')\frac{\xi_3^2}{\varepsilon^2}.$$

The classical root-coefficient relations and the derivation of the polynomial give that : A, B, and \overline{B} satisfy the following relations:

$$\begin{cases} A+B+\overline{B} &= (\nu-\nu')|\xi|^2 \\ AB\overline{B} &= (\nu-\nu')\frac{\xi_3^2}{\varepsilon^2} \\ AB+A\overline{B}+B\overline{B} &= \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2} \\ (A-B)(A-\overline{B}) &= 3A^2 - 2(\nu-\nu')|\xi|^2A + \frac{|\xi|_F^2}{\varepsilon^2 F^2 |\xi|^2} \end{cases}$$

If we use it in the expression of K_2 we obtain that:

$$K_{2} = \frac{1}{(A-B)(A-\overline{B})\xi_{3}(\xi_{1}^{2}+\xi_{2}^{2})} \Big((\xi_{2}X-\xi_{1}Y) \Big(\frac{\xi_{1}^{2}+\xi_{2}^{2}}{\varepsilon^{2}F^{2}|\xi|^{2}} - A\big((\nu-\nu')|\xi|^{2}-A\big) \Big) \\ - (\xi_{1}X+\xi_{2}Y) \frac{(\nu-\nu')|\xi|^{2}-A}{\varepsilon} + \frac{T}{F\varepsilon^{2}|\xi|^{2}}\xi_{3}(\xi_{1}^{2}+\xi_{2}^{2}) \Big)$$

Then we use Lemma 4.9 from [4] i.e the last two coordinates of the eigenvectors W_2 , W_3 , and W_4 are linked by the following relations (which are nothing but a disguised formulation of the fact that A, B, and \overline{B} are the roots of the polynomial Q_2):

$$\begin{split} W_2^3 &= \varepsilon F(A - (\nu - \nu')|\xi|^2)W_2^4, \\ W_3^3 &= \varepsilon F(B - (\nu - \nu')|\xi|^2)W_3^4, \\ W_4^3 &= \varepsilon F(\overline{B} - (\nu - \nu')|\xi|^2)W_4^4. \end{split}$$

We can deduce from this that:

$$(\nu - \nu')|\xi|^2 - A = \frac{A(\xi_1^2 + \xi_2^2)}{F^2(\varepsilon^2|\xi|^2 A^2 + \xi_3^2)}$$

so we can get rid of the annoying $(\xi_1^2 + \xi_2^2)$ in the denominator, which allows us to obtain:

$$K_{2} = \frac{1}{(A-B)(A-\overline{B})\varepsilon^{2}F^{2}(\varepsilon^{2}|\xi|^{2}A^{2}+\xi_{3}^{2})} \left(-\frac{\xi_{3}}{|\xi|^{2}}(\xi_{1}Y-\xi_{2}X-F\xi_{3}T) -\frac{\varepsilon A}{\xi_{3}}(\xi_{1}X+\xi_{2}Y)+\varepsilon^{2}FA^{2}T\right)$$

A simple computation gives that the norm in K^4 :

$$||W_2||^2 = (\varepsilon^2 |\xi|^2 A^2 + \xi_3^2) (\varepsilon^2 F^2 |\xi|^2 A^2 + |\xi|_F^2)$$

so that we finally get, if h is also orthogonal to $(-\xi_2, \xi_1, 0, -F\xi_3)$:

$$||K_2 W_2|| \le \frac{1}{(A-B)(A-\overline{B})\varepsilon^2 F^2} \Big(\frac{\varepsilon^2 F^2 |\xi|^2 A^2 + |\xi|_F^2}{\varepsilon^2 |\xi|^2 A^2 + \xi_3^2}\Big)^{\frac{1}{2}} \varepsilon \Big(|A| \frac{|\xi|}{|\xi_3|} + \varepsilon F A^2\Big) |h(\xi)|$$

with

$$(A-B)(A-\overline{B})\varepsilon^{2}F^{2} = \frac{|\xi|_{F}^{2}}{|\xi|^{2}} + 3\varepsilon^{2}F^{2}A^{2} - 2(\nu-\nu')\varepsilon^{2}F^{2}|\xi|^{2}A$$

The very same argument (with no hypothesis of orthogonality to $(-\xi_2, \xi_1, 0, -F\xi_3)$) gives:

$$\|K_3W_3\| \le \frac{1}{|(A-B)(B-\overline{B})|\varepsilon^2 F^2} \Big(\frac{\varepsilon^2 F^2 |\xi|^2 |B|^2 + |\xi|_F^2}{\varepsilon^2 |\xi|^2 |B|^2 + \xi_3^2}\Big)^{\frac{1}{2}} \Big(\frac{|\xi_3|}{|\xi|}|\varepsilon B| + \frac{|\xi|}{|\xi_3|} + \varepsilon^2 F|B|^2\Big)|h(\xi)|$$

with

$$(B-A)(B-\overline{B})\varepsilon^{2}F^{2} = \frac{|\xi|_{F}^{2}}{|\xi|^{2}} + 3\varepsilon^{2}F^{2}B^{2} - 2(\nu-\nu')\varepsilon^{2}F^{2}|\xi|^{2}B$$

And we have just proved Lemma 4.5.2. We do the same work for K_4W_4 .

Asymptotic expansions

Now that we have vectorial estimates we will use asymptotic expansions in order to get estimates on the norms of the projectors.

Recall the different steps when we compute the roots of Q (see (4.5.41)): first we define the discriminant $D = \frac{q^2}{4} + \frac{p^3}{27}$ which is strictly positive if $\varepsilon \leq \varepsilon_0$, then we define:

$$\alpha = \left(-\frac{q}{2} + D^{\frac{1}{2}}\right)^{\frac{1}{3}}$$
 and $\beta = \left(-\frac{q}{2} - D^{\frac{1}{2}}\right)^{\frac{1}{3}}$

and then the roots are given by (4.5.43). So we will have to compute precisely the asymptotic expansions of $(1+x)^{\frac{1}{2}}$ and $(1+x)^{\frac{1}{3}}$ when x is near 0 : there exist two functions f and g such that for all $x \in [-1, 1]$,

$$\begin{cases} (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + x^3 f(x) \\ (1+x)^{\frac{1}{3}} = 1 + \frac{x}{3} - \frac{x^2}{9} + x^3 g(x) \\ |f(x)| \le 1 & \text{and} & |g(x)| \le 1 \end{cases}$$

After meticulous computations (we write precisely the majorations of the quantities depending on ξ and take advantage of the fact that $|log\varepsilon| \ll |\varepsilon|^{\alpha}$), we finally obtain the following result:

Lemma 4.5.3 There exist a number $\varepsilon_1 = \varepsilon_1(F, \nu - \nu')$ and two functions $F(\xi, \varepsilon)$ and $G(\xi, \varepsilon)$, such that for all $\varepsilon \leq \varepsilon_1$ and all $\xi \in C_{r_{\varepsilon}, R_{\varepsilon}}$ the following asymptotic expansions hold:

$$\begin{split} \mu &= -(\nu\xi_1^2 + \nu\xi_2^2 + \nu'F^2\xi_3^2)\frac{|\xi|^2}{|\xi|_F^2} + \varepsilon^{\frac{1}{2}}F(\xi,\varepsilon) \\ \lambda &= -\tau(\xi)|\xi|^2 + i\frac{|\xi|_F}{\varepsilon F|\xi|} + \varepsilon^{\frac{1}{2}}G(\xi,\varepsilon) \\ \overline{\lambda} &= -\tau(\xi)|\xi|^2 - i\frac{|\xi|_F}{\varepsilon F|\xi|} + \varepsilon^{\frac{1}{2}}\overline{G}(\xi,\varepsilon) \\ A &= (\nu - \nu')F^2\xi_3^2\frac{|\xi|^2}{|\xi|_F^2} + \varepsilon^{\frac{1}{2}}F(\xi,\varepsilon) \\ B &= i\frac{|\xi|_F}{\varepsilon F|\xi|} + \frac{\nu - \nu'}{2}\frac{|\xi|^2}{|\xi|_F^2}(\xi_1^2 + \xi_2^2)\varepsilon^{\frac{1}{2}}G(\xi,\varepsilon) \end{split}$$

with

$$\tau(\xi) = \frac{\nu}{2} \left(1 + \frac{F^2 \xi_3^2}{|\xi|_F^2} \right) + \frac{\nu'}{2} \left(1 - \frac{F^2 \xi_3^2}{|\xi|_F^2} \right) \ge \nu_0 \quad \forall \xi,$$

and the uniform estimates $\forall \varepsilon \leq \varepsilon_1$ and $\forall \xi \in \mathcal{C}_{r_{\varepsilon},R_{\varepsilon}}, \quad F(\xi,\varepsilon) \leq 1, G(\xi,\varepsilon) \leq 1.$

Then using this lemma, another series of computations lead to the following lemma:

Lemma 4.5.4 There exists a constant C_F and $\varepsilon_0 > 0$ such that for all h orthogonal to $(\xi_1, \xi_2, \xi_3, 0)$ and $\varepsilon \leq \varepsilon_0$, for all $\xi \in C_{r_{\varepsilon}, R_{\varepsilon}}$ we have the estimate (i=3,4):

$$\|\mathcal{P}_i(\xi,\varepsilon)(h(\xi))\|_{K^4} \le C_F |\log \varepsilon|^{M+m} |h(\xi)|.$$

And if, in addition $h(\xi)$ is orthogonal to $(-\xi_1, \xi_2, 0, -F\xi_3)$ then for all $\xi \in \mathcal{C}_{r_{\varepsilon}, R_{\varepsilon}}$:

$$\|\mathcal{P}_2(\xi,\varepsilon)(h(\xi))\|_{K^4} \le C_F |\nu - \nu'|\varepsilon^{\frac{1}{2}}|h(\xi)|.$$

This concludes the proof of proposition 4.2.1. \blacksquare

4.5.2 Dispersive estimates

Lemma 4.5.5 Let us define

$$K(t,t',\varepsilon,x) = \int_{\mathbb{R}^3} e^{-(t+t')\tau(\xi)|\xi|^2 + i\frac{(t-t')}{\varepsilon}\frac{|\xi|_F}{F|\xi|} + \varepsilon^{\frac{1}{2}}(tG(\xi,\varepsilon) + t'\overline{G}(\xi,\varepsilon))} \big(\chi(\frac{|\xi|}{2R_\varepsilon})(1-\chi(\frac{2|\xi_3|}{r_\varepsilon})\big)^2 d\xi$$

Then there exist $\varepsilon_0(\nu, \nu', F)$ and $C = C(F, \nu, \nu')$ such that for all $\varepsilon \leq \varepsilon_0$,

$$\|K(t,t',\varepsilon,.)\|_{L^{\infty}} \leq C \frac{\varepsilon^{\frac{1}{2}}}{|t-t'|} |\log \varepsilon|^{4m+6M} e^{-\nu_0(t+t')|\log \varepsilon|^{-2m}}.$$

Proof: we refer to [4], section 4.3.6. The only new work being to precise the constants depending on the radiuses.

4.5.3 Proof of the Strichartz estimates

In order to simplify, we assume that the initial data and the forcing terms only depend on the eigen $\lambda(\xi, \varepsilon)$. Let us begin by localizing the equation, the Duhamel formula gives then:

$$\widehat{\Delta_q f}(t,\xi) = e^{t\lambda(\xi,\varepsilon)} \widehat{\Delta_q f_0}(\xi) + \int_0^t e^{(t-\tau)\lambda(\xi,\varepsilon)} \widehat{\Delta_q g^b}(\tau,\xi) d\tau + \int_0^t e^{(t-\tau)\lambda(\xi,\varepsilon)} \widehat{\Delta_q g^l}(\tau,\xi) d\tau$$

We then follow the very same lines as in section 4.5 from [5]: the only difference is that we have more precise values for the estimates on $C_{r_{\varepsilon},R_{\varepsilon}}$ also depending on ε . Separating the homogeneous, inhomogeneous linear and inhomogeneous bilinear cases like in [5], we obtain that:

$$\|e^{t\lambda(\xi,\varepsilon)}\widehat{\Delta_q f_0}\|_{L^2(\mathbb{R}_+,L^\infty(\mathbb{R}^3))} \le C_F \frac{\varepsilon^{\frac{1}{8}}}{\nu_0^{\frac{3}{8}}} |\log\varepsilon|^{5(m+M)} \|\Delta_q f_0\|_{L^2(\mathbb{R}^3)}$$

$$\|\int_0^t e^{(t-\tau)\lambda(\xi,\varepsilon)}\widehat{\Delta_q g^b}(\tau,\xi)d\tau\|_{L^2(\mathbb{R}_+,L^\infty(\mathbb{R}^3))} \le C_F \frac{\varepsilon^{\frac{1}{8}}}{\nu_0^{\frac{3}{8}}} |\log\varepsilon|^{5(m+M)} \|\Delta_q g^b\|_{L^1(\mathbb{R}_+,L^2(\mathbb{R}^3))}$$

$$\|\int_0^t e^{(t-\tau)\lambda(\xi,\varepsilon)}\widehat{\Delta_q g^l}(\tau,\xi)d\tau\|_{L^2(\mathbb{R}_+,L^\infty(\mathbb{R}^3))} \le C_F \frac{\varepsilon^{\frac{1}{4}}}{\nu_0^{\frac{3}{4}}} |\log\varepsilon|^{6(m+M)} \|\Delta_q g^l\|_{L^2(\mathbb{R}_+,L^2(\mathbb{R}^3))}$$

So, summing for $q = -1...\infty$, we obtain that if ε is small enough (and thanks to a Minkowski inequality),

$$\begin{split} \|f\|_{L^{2}L^{\infty}} &\leq \|f\|_{L^{2}B^{0}_{\infty,1}} \leq \|f\|_{\widetilde{L^{2}}B^{0}_{\infty,1}} \leq C_{F}(\nu_{0}^{-\frac{3}{8}} + \nu_{0}^{-\frac{3}{4}})\varepsilon^{\frac{1}{8}} |\log\varepsilon|^{5(m+M)} \\ &\sum_{q} \left(\|\Delta_{q}f_{0}\|_{L^{2}} + \|\Delta_{q}g^{b}\|_{L^{1}L^{2}} + \|\Delta_{q}g^{l}\|_{L^{2}L^{2}} + \right) \end{split}$$

Where we recall the notation $\|f\|_{\widetilde{L}^2 B^0_{\infty,1}} = \sum_q \|\|\Delta_q\|_{L^\infty}\|_{L^2_T}$. Taking advantage of the fact that the initial data and the forcing terms are localized

in frequency in $\mathcal{C}_{r_{\varepsilon},R_{\varepsilon}}$, we easily obtain, thanks to Hölder inequalities:

- $\sum_{q} \|\Delta_q f_0\|_{L^2} \leq CR_{\varepsilon} \|f_0\|_{L^2}$
- $\sum_{q} \|\Delta_q g^b\|_{L^1 L^2} \leq C R_{\varepsilon} \|g^b\|_{L^1 L^2}$
- $\sum_{q} \|\Delta_{q}g^{l}\|_{L^{2}L^{2}} \leq CR_{\varepsilon}\|g^{l}\|_{L^{2}L^{2}}$

So finally:

$$\|f\|_{L^{2}L^{\infty}} \leq C_{F,\nu_{0}}\varepsilon^{\frac{1}{8}} |\log\varepsilon|^{6(m+M)} \left(\|f_{0}\|_{L^{2}} + \|g^{b}\|_{L^{1}L^{2}} + \|g^{l}\|_{L^{2}L^{2}}\right)$$

Application to W_{ε}^{T} 4.5.4

The previous formula gives:

$$\begin{split} \|W_{\varepsilon}^{t}\|_{L^{2}L^{\infty}} &\leq C_{F,\nu_{0}}\varepsilon^{\frac{1}{8}} |\log\varepsilon|^{6(m+M)} \big(\|\mathbb{P}_{3+4}\chi(\frac{|D|}{R_{\varepsilon}})(1-\chi(\frac{|D_{3}|}{r_{\varepsilon}}))U_{0,\varepsilon,osc}\|_{L^{2}} \\ &+ \|\mathbb{P}_{3+4}\chi(\frac{|D|}{R_{\varepsilon}})(1-\chi(\frac{|D_{3}|}{r_{\varepsilon}}))G^{b}\|_{L^{1}L^{2}} + \|\mathbb{P}_{3+4}\chi(\frac{|D|}{R_{\varepsilon}})(1-\chi(\frac{|D_{3}|}{r_{\varepsilon}}))G^{l}\|_{L^{2}L^{2}} \big). \end{split}$$

Then, using the estimates from proposition 4.2.1, and the Bernstein lemma which gives:

$$\|\mathbb{P}_{3+4\chi}(\frac{|D|}{R_{\varepsilon}})(1-\chi(\frac{|D_3|}{r_{\varepsilon}}))U_{0,\varepsilon,osc}\|_{L^2} \le C_F |\log \varepsilon|^{M+\frac{3}{2}m} \|U_{0,\varepsilon,osc}\|_{\dot{H}^{\frac{1}{2}}}$$

and using the estimates on the initial data, there exists $\varepsilon_0(F,\nu_0,m,M)$ such that for all $\varepsilon \leq \varepsilon_0$, denoting by K_0 a constant depending on F, ν_0 , $U_{0,\varepsilon,osc}$,

$$\|W_{\varepsilon}^{T}\|_{L^{2}(\mathbb{R}_{+},L^{\infty}(\mathbb{R}^{3}))} \leq K_{0}\varepsilon^{\frac{1}{16}}.$$

Appendix 2 : Strichartz estimates when $\nu = \nu'$ 4.6

Before stating any results, let us see the simplifications involved by the fact that $\nu = \nu'$. If we denote by $\mathbb{B}(\xi, \varepsilon)$ the following matrix (in Fourier variables):

$$\mathbb{B}(\xi,\varepsilon) \stackrel{\text{def}}{=} \mathcal{F}\left(-\nu\Delta - \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}\right) = \begin{pmatrix} -\nu|\xi|^2 + \frac{\xi_1\xi_2}{\varepsilon|\xi|^2} & \frac{\xi_2^2 + \xi_3^2}{\varepsilon|\xi|^2} & 0 & \frac{\xi_1\xi_3}{\varepsilon F|\xi|^2} \\ -\frac{\xi_1^2 + \xi_3^2}{\varepsilon|\xi|^2} & -\nu|\xi|^2 - \frac{\xi_1\xi_2}{\varepsilon|\xi|^2} & 0 & \frac{\xi_2\xi_3}{\varepsilon F|\xi|^2} \\ \frac{\xi_2\xi_3}{\varepsilon|\xi|^2} & -\frac{\xi_1\xi_3}{\varepsilon|\xi|^2} & -\nu|\xi|^2 & -\frac{\xi_1^2 + \xi_2^2}{\varepsilon F|\xi|^2} \\ 0 & 0 & \frac{1}{\varepsilon F} & -\nu|\xi|^2 \end{pmatrix}.$$

In this case we have explicit simple expressions for the eigenelements of this matrix: it has three distinct eigenvalues:

$$\begin{cases} -\nu |\xi|^2, & \text{which is double} \\ -\nu |\xi|^2 \pm \frac{i}{\varepsilon} \frac{|\xi|_F}{F|\xi|}, \end{cases}$$

corresponding respectively to the following eigenvectors:

$$W_{2} = \begin{pmatrix} \xi_{2}\xi_{3} \\ -\xi_{1}\xi_{3} \\ 0 \\ F\xi_{3}^{2} \end{pmatrix} \quad W_{3} = \begin{pmatrix} \xi_{3}(\xi_{2} + i\xi_{1}\frac{|\xi|_{F}}{F|\xi|}) \\ \xi_{3}(-\xi_{1} + i\xi_{2}\frac{|\xi|_{F}}{F|\xi|}) \\ -i\frac{|\xi|_{F}}{F|\xi|}(\xi_{1}^{2} + \xi_{2}^{2}) \\ -\frac{(\xi_{1}^{2} + \xi_{2}^{2})}{F} \end{pmatrix} \quad W_{4} = \overline{W_{3}}.$$

The matrix $\mathbb{B}(\xi,\varepsilon)$ is still diagonalizable on the subspace of vector fields orthogonal to vector $(\xi_1,\xi_2,\xi_3,0)$ (which we will call "divergence-free" vector fields, so $W_2(\xi,\varepsilon),...,W_4(\xi,\varepsilon)$ is still a basis of the "divergence-free" vector fields then we are sure that $\widehat{U_{\varepsilon}}$ depends only on the last three eigenvectors : $\widehat{U_{\varepsilon,osc}} = K_2W_2 + K_3W_3 + K_4W_4$. The simplification is due to two facts :

- first these three vectors are "divergence-free" and they are pairwise orthogonal, so their respectives projectors in Fourier variable (\mathbb{P}_i , i = 2, 3, 4) are now orthogonal projectors, whose norms are less than 1.
- Second, the fact that W_2 is colinear to $(-\xi_2, \xi_1, 0, -F\xi_3 \text{ and } W_3, W_4 \text{ are orthogonal}$ to this vector allow us to write that $\mathcal{PP} = \mathbb{P}_3 + \mathbb{P}_4$ and $\mathcal{QP} = \mathbb{P}_2$ so $\widehat{U_{\varepsilon,QG}}$ depends only on W_2 and $\widehat{U_{\varepsilon,osc}}$ depends only on W_3 and W_4 .

The object of this section is to prove the following result :

Lemma 4.6.1 Assume that $f_0 \in L^2(\mathbb{R}^3) \cap B^{s+\frac{3}{4}}_{2,q}$, $g \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^1(\mathbb{R}_+, B^{s+\frac{3}{4}}_{2,q})$, where $q \in [1, 2]$ and $s \in \mathbb{R}$, and that f solves the following system :

$$\begin{cases} \partial_t f - \nu \Delta f + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} f = g \\ f_{/t=0} = f_0, \end{cases}$$

Assume also that f_0 and g(t) for all $t \ge 0$ have a zero potential viscosity.

Then there exists a constant C_F only depending on the Froude number such that we have the following estimate : $\forall t \geq 0$ and $\forall q \in [1, 2]$

$$\|f\|_{L^{2}(\mathbb{R}_{+},B^{s}_{\infty,q})} \leq C_{F} \frac{\varepsilon^{\frac{1}{8}}}{\nu^{\frac{3}{8}}} (\|f_{0}\|_{L^{2}(\mathbb{R}^{3})} + \|f_{0}\|_{B^{s+\frac{3}{4}}_{2,q}} + \|g\|_{L^{1}(\mathbb{R}_{+},L^{2}(\mathbb{R}^{3}))} + \|g\|_{L^{1}(\mathbb{R}_{+},B^{s+\frac{3}{4}}_{2,q})})$$

4.6.1 Preliminary remarks

The fact that the initial data and the right-hand side g have a zero potential vorticity ensures us that their Fourier transforms only depend on the last two eigenvectors of matrix $\mathbb{B}(\xi,\varepsilon)$ so we just have to apply projectors \mathbb{P}_3 and \mathbb{P}_4 to the equation to have the complete diagonalization :

$$\begin{cases} \partial_t \mathbb{P}_i f - \nu \Delta \mathbb{P}_i f + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \mathbb{P}_i f = \mathbb{P}_i g\\ \mathbb{P}_i f_{/t=0} = \mathbb{P}_i f_0, \end{cases}$$

With a wish of simplification we will consider that f_0 , g(t) (then f(t)) only depend on the third eigenvector (ie $\mathbb{P}_3 f_0 = f_0...$)

The idea is to localize in frequency (ie we will apply $\dot{\Delta}_j$ (homogeneous) to the equation). Thanks to the Duhamel formula we obtain :

$$\widehat{\dot{\Delta}_j f}(t,\xi) = e^{-\nu t |\xi|^2 + i \frac{t}{\varepsilon} \frac{|\xi|_F}{F|\xi|}} \widehat{\dot{\Delta}_j f_0}(\xi) + \int_0^t e^{-\nu (t-\tau)|\xi|^2 + i \frac{t-\tau}{\varepsilon} \frac{|\xi|_F}{F|\xi|}} \widehat{\dot{\Delta}_j g}(\tau,\xi) d\tau$$

We will begin with the homogeneous case, ie when g = 0. If ϕ and χ are the two functions introduced in [7] to construct the Littlewood-Paley theory, the Lebesgue theorem says that :

$$\sum_{k=-\infty}^{j+1} \phi(2^{-k}\xi_3)\widehat{\dot{\Delta}_j f}(\xi) = \widehat{\dot{\Delta}_j f}(\xi) \quad \text{in} \quad L^1$$

so, using the inverse Fourier transform

$$\sum_{k=-\infty}^{j+1} \dot{\Delta}_k^v \dot{\Delta}_j f \quad \text{converges to} \quad \dot{\Delta}_j f \quad \text{in} \quad L^{\infty}$$

and,

$$\|\dot{\Delta}_j f\|_{L^{\infty}} \le \sum_{k=-\infty}^{j+1} \|\dot{\Delta}_k^v \dot{\Delta}_j f\|_{L^{\infty}}$$

such that, even if we didn't localize in $C_{r,R}$ in the first place, we have for each j and k a localization which is exactly the same thing because $|\xi|$ is bounded, and $|\xi_3|$ is bounded from below :

$$A_{j,k} \stackrel{\text{def}}{=} \begin{cases} c2^{j} \le |\xi| \le C2^{j} \\ c2^{k} \le |\xi_{3}| \le C2^{k} \end{cases}$$
(4.6.45)

4.6.2 Duality argument

In this section we will describe the duality method (also called TT^*) to precicely determine the kernel on which we will work to get dispersive estimates. Let us begin with the definition $\mathcal{B} = \{\psi \in \mathcal{C}_0^{\infty}(\mathbb{R}_+ \times \mathbb{R}^3) / \|\psi\|_{L^2(\mathbb{R}_+, L^1(\mathbb{R}^3)) \leq 1}\}$ then for all j and k:

$$\|\dot{\Delta}_k^v \dot{\Delta}_j f\|_{L^2 L^\infty} = \sup_{\psi \in \mathcal{B}} \int_0^\infty \int_{\mathbb{R}^3} \dot{\Delta}_k^v \dot{\Delta}_j f(t, x) \psi(t, x) dx dt$$

using the Plancherel theorem we get :

Poches de tourbillons

$$\|\dot{\Delta}_k^v \dot{\Delta}_j f\|_{L^2 L^\infty} = C \sup_{\psi \in \mathcal{B}} \int_0^\infty \int_{\mathbb{R}^3} e^{-\nu t |\xi|^2 + i\frac{t}{\varepsilon} \frac{|\xi|_F}{F|\xi|}} \phi(2^{-k}|\xi_3|) \phi(2^{-j}|\xi|) \widehat{f_0}(\xi) \widehat{\psi}(t,\xi) d\xi dt$$

If ϕ_1 is a C_0^{∞} function whose support is a neighbourhood of those of ϕ and equal to 1 on $supp\phi$ then we can write :

using the Fubini theorem, we get :

$$\begin{split} \|\dot{\Delta}_{k}^{v}\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} &= C\sup_{\psi\in\mathcal{B}}\int_{\mathbb{R}^{3}}\phi(2^{-j}|\xi|)\widehat{f}_{0}(\xi)\\ & \left(\int_{0}^{\infty}e^{-\nu t|\xi|^{2}+i\frac{t}{\varepsilon}\frac{|\xi|_{F}}{F|\xi|}}\phi(2^{-k}|\xi_{3}|)\phi_{1}(2^{-j}|\xi|)\widehat{\psi}(t,\xi)dt\right)d\xi \end{split}$$

A use of the Hölder inequality gives :

$$\begin{split} \|\dot{\Delta}_{k}^{v}\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} &= C\sup_{\psi\in\mathcal{B}}\|\widehat{\Delta}_{j}\widehat{f_{0}}(\xi)\|_{L^{2}}\Big(\int_{\mathbb{R}^{3}}\int_{0}^{\infty}\int_{0}^{\infty}e^{-\nu(t+s)|\xi|^{2}+i\frac{t-s}{\varepsilon}\frac{|\xi|_{F}}{F|\xi|}}\phi(2^{-k}|\xi_{3}|)^{2}\\ &\phi_{1}(2^{-j}|\xi|)^{2}\widehat{\psi}(t,\xi)\overline{\widehat{\psi}(s,\xi)}dtds)d\xi\Big)^{\frac{1}{2}} \end{split}$$

and thanks to, successively, Fubini and Plancherel we finally obtain :

$$\begin{split} \|\dot{\Delta}_{k}^{v}\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} &= C\sup_{\psi\in\mathcal{B}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} \Big(\int_{0}^{\infty}\int_{0}^{\infty} \|K(\nu(t+s),\frac{t-s}{\varepsilon},.)\|_{L^{\infty}} \\ &\|\psi(t)\|_{L^{1}}\|\psi(s)\|_{L^{1}}dtds)^{\frac{1}{2}}, \end{split}$$

where

$$K(\theta,\tau,x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} e^{ix.\xi-\theta|\xi|^2 + i\tau\lambda(\xi)} \phi(2^{-k}|\xi_3|)^2 \phi_1(2^{-j}|\xi|)^2 d\xi$$

with $\lambda(\xi) = \frac{|\xi|_F}{F|\xi|}$ (4.6.46)

Then the method consists in estimating in two different ways the kernel. As we have to deal with a summation from $-\infty$ to j + 1 we will get two kinds of estimates in terms of k: some depending as 2^k (summable in $-\infty$ but no ε) and some as $2^{-k}\varepsilon$ (not summable).

4.6.3 Dispersive estimates

The aim of this section is the proof of the following lemma :

Lemma 4.6.2 There exists a constant C_F such that $\forall j \in \mathbb{Z}, \forall k \leq j + 1, \forall \theta > 0$ and $\forall \tau > 0$ we have :

$$||K(\theta,\tau,.)||_{L^{\infty}} \le C_F e^{-c\theta 2^{2j}} 2^{3j} \min(2^{k-j}, \frac{2^{j-k}}{\sqrt{\tau}}),$$

where c > 0 is the constant from (4.6.45).

Proof: using the Plancherel theorem we obtain (see (4.6.45) for the definition of $A_{i,k}$):

$$\|K(\theta,\tau,.)\|_{L^2} \le \left(\int_{A_{j,k}} e^{-2\theta|\xi|^2} \phi(2^{-k}|\xi_3|)^4 \phi_1(2^{-j}|\xi|)^4 d\xi\right)^{\frac{1}{2}}.$$

Which, using the change of variable $\xi = 2^{j}\eta$, we turn into :

$$\|K(\theta,\tau,.)\|_{L^2} \le \left(2^{3j} \int_{A_{0,k-j}} e^{-2\theta 2^{2j}|\eta|^2} \phi(2^{j-k}|\eta_3|)^4 \phi_1(|\eta|)^4 d\eta\right)^{\frac{1}{2}}$$

then, using that ϕ and ϕ_1 are bounded by 1 and the fact that $c \leq |\eta| \leq C$, allow us to estimate roughly the integrals in η_1 and η_2 . If we use the change of variable $\eta_3 = 2^{k-j}\xi_3$ we finally get :

$$||K(\theta, \tau, .)||_{L^2} \le C 2^{\frac{3j}{2}} 2^{\frac{k-j}{2}} e^{-c\theta 2^{2j}}.$$

Then, as $K(\theta, \tau, .)$ is localized in frequency, a use of the Bernstein lemma gives :

$$\|K(\theta,\tau,.)\|_{L^{\infty}} \le ((2^{j})^{2}(2^{k}))^{\frac{1}{2}-\frac{1}{\infty}} \|K(\theta,\tau,.)\|_{L^{2}}$$

so we have :

$$\|K(\theta,\tau,.)\|_{L^{\infty}} \le C 2^{3j} 2^{k-j} e^{-c\theta 2^{2j}}$$
(4.6.47)

Now let us estimate explicitely the kernel : before that, the change of variable $\xi = 2^j \eta$ gives us :

$$K(\theta, \tau, x) = 2^{3j} \widetilde{K}(2^{2j}\theta, \tau, 2^j x),$$

with

$$\widetilde{K}(\theta,\tau,x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} e^{ix\cdot\eta - \theta|\eta|^2 + i\tau\lambda(\eta)} \phi(2^{j-k}|\eta_3|)^2 \phi_1(|\eta|)^2 d\eta$$

In order to use as in [4] a stationnary phase method, let us define the operator :

$$\mathcal{L} = \frac{1}{1 + \tau \Lambda(\eta)^2} (1 + i\Lambda(\eta)\partial_{\eta_2})$$

$$\mathcal{L}(\eta) = \frac{1}{1 + \tau \Lambda(\eta)^2} (1 + i\Lambda(\eta)\partial_{\eta_2})$$

with $\lambda(\eta) = \frac{|\eta|_F}{F|\eta|}$ and $\Lambda(\eta) = -\partial_{\eta_2}\lambda(\eta) = \frac{F^2 - 1}{F} \frac{\eta_2 \eta_3^2}{|\eta|_F |\eta|^3}$

Remark 4.6.1 We can assume $\tau > 0$: if not just take instead of \mathcal{L} :

$$\frac{1}{1-\tau\Lambda(\eta)^2}(1-i\Lambda(\eta)\partial_{\eta_2})$$

The well-know invariance under any rotation around the third axis of coordinates allows us to assume that $x_2 = 0$ so that $e^{ix.\eta}$ doesn't depend on η_2 :

$$\mathcal{L}(e^{ix.\eta + i\tau\lambda\eta}) = e^{ix.\eta}\mathcal{L}(e^{i\tau\lambda\eta}) = e^{ix.\eta + i\tau\lambda\eta}$$

As in [9], the transposed of operator \mathcal{L} writes:

$$\mathcal{L}^{T}(g) = \left(\frac{1}{1+\tau\Lambda^{2}} - i\partial_{\eta_{2}}\Lambda \frac{1-\tau\Lambda^{2}}{(1+\tau\Lambda^{2})^{2}}\right)g - \frac{i\Lambda}{1+\tau\Lambda^{2}}\partial_{\eta_{2}}g,$$

so that we obtain, estimating ϕ , ϕ_1 and their derivatives by a constant C' independant of the parameters :

$$|\widetilde{K}(\theta,\tau,x)| \leq \int_{A_{0,k-j}} \frac{C'}{1+\tau\Lambda^2} \Big(1+|\partial_{\eta_2}\Lambda|)e^{-\theta|\eta|^2} + |\Lambda||\eta_2|\theta e^{-\theta|\eta|^2} + |\Lambda|e^{-\theta|\eta|^2}\Big)d\eta.$$

An elementary computation gives that there exists a constant $C' = 2e^{-1}$ such for all $x \ge 0$, $xe^{-x} \le C'e^{\frac{x}{2}} \le e^{\frac{x}{2}}$, so for all η , $|\eta_2|\theta e^{-\theta|\eta|^2} \le \frac{1}{|\eta|}e^{\frac{-\theta|\eta|^2}{2}}$.

We use the facts that $c \leq |\eta| \leq C$ and that $c_F \leq \frac{|\eta|_F}{|\eta|} \leq C_F$ to estimate Λ and $\partial_{\eta_2} \Lambda$ by a universal constant depending only on c, C and F:

$$|\widetilde{K}(\theta,\tau,x)| \leq C'_F e^{-c\theta} \int_{A_{0,k-j}} \frac{1}{1 + \tau \left(\frac{C_F \eta_2 \eta_3^2}{|\eta|_F |\eta|^3}\right)^2} d\eta.$$

Taking advantage of the localizations, we can get rid of the integrals in the variables η_1 and η_3 , and also write that

$$\left(\frac{C_F \eta_3^2}{|\eta|_F |\eta|^3}\right)^2 \ge c_F 2^{4(k-j)}$$

So that

$$|\widetilde{K}(\theta,\tau,x)| \le C_F e^{-c\theta}(C)(C2^{k-j}) \int_{|\eta_2| \le C} \frac{1}{1 + \tau C_F 2^{4(k-j)} \eta_2^2} d\eta.$$

The change of variable $y = \eta_2 \sqrt{\tau C_F} 2^{2(k-j)}$ and the estimate $\int_0^\infty \frac{1}{1+y^2} dy \leq C$ finally give that :

$$\widetilde{K}(\theta, \tau, x)| \le C_F e^{-c\theta} \frac{2^{j-k}}{\sqrt{\tau}}$$

so, returning to the original variables,

$$\|K(\theta,\tau,.)\|_{L^{\infty}} \le C_F e^{-c\theta 2^{2j}} \frac{2^{j-k}}{\sqrt{\tau}} 2^{3j}$$
(4.6.48)

Finally, (4.6.47) and (4.6.48) give that :

$$\|K(\theta, \tau, .)\|_{L^{\infty}} \le C_F e^{-c\theta 2^{2j}} 2^{3j} \min(2^{k-j}, \frac{2^{j-k}}{\sqrt{\tau}}).$$

4.6.4 End of the proof

Let us go back to (4.6.46): using the dispersive estimate we can write that :

$$\begin{split} \|\dot{\Delta}_{k}^{v}\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} &\leq C \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} \Big(\int_{0}^{\infty} \int_{0}^{\infty} C_{F}e^{-c\nu(t+s)2^{2j}}2^{3j}\min(2^{k-j},\frac{2^{j-k}}{\sqrt{\frac{|t-s|}{\varepsilon}}}) \\ &\|\psi(t)\|_{L^{1}}\|\psi(s)\|_{L^{1}}dtds)^{\frac{1}{2}} \end{split}$$

And, as we have :

$$\|\dot{\Delta}_j f\|_{L^2 L^{\infty}} \leq \sum_{k=-\infty}^{j+1} \|\dot{\Delta}_k^v \dot{\Delta}_j f\|_{L^2 L^{\infty}},$$

we can write that :

$$\begin{split} \|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} &\leq \sum_{k=-\infty}^{j+1} C_{F} \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} \Big(\int_{0}^{\infty} \int_{0}^{\infty} \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} \\ &e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, \frac{2^{j-k}\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) dt ds \Big)^{\frac{1}{2}}. \end{split}$$

As we want a norm in $L^2(\mathbb{R}_+, L^{\infty})$ we will have to loose on the viscosity (i.e a negative power of the viscosity, due to integration of the $e^{-c\nu t 2^{2j}}$) and then we don't need to be too precise on the integrations, so using the fact that $\int min(f,g) \leq min(\int f, \int g)$ we obtain that :

$$\begin{split} \|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} &\leq \sum_{k=-\infty}^{j+1} C_{F} \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} \\ &\min\left(\int_{0}^{\infty} \int_{0}^{\infty} 2^{3j} e^{-c\nu(t+s)2^{2j}} 2^{k-j} \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds, \\ &\int_{0}^{\infty} \int_{0}^{\infty} 2^{3j} e^{-c\nu(t+s)2^{2j}} 2^{j-k} \varepsilon^{\frac{1}{2}} \frac{\|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}}}{|t-s|^{\frac{1}{2}}} dt ds \right)^{\frac{1}{2}} \end{split}$$

Using, the fact that the variables are separated in the first integral, and in the second one the Hardy-Littlewood theorem (with coefficients $\frac{4}{3}$, $\frac{4}{3}$ and $\frac{1}{2}$):

$$\begin{aligned} \|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} &\leq \sum_{k=-\infty}^{j+1} C_{F} \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} \\ \min\left(2^{3j}2^{k-j} \left(\int_{0}^{\infty} e^{-c\nu t 2^{2j}} \|\psi(t)\|_{L^{1}} dt\right)^{2}, \quad 2^{3j}2^{j-k}\varepsilon^{\frac{1}{2}} \|e^{-c\nu t 2^{2j}} \|\psi(t)\|_{L^{1}} \|_{L^{\frac{4}{3}}_{t}(\mathbb{R}_{+})}^{2}\right)^{\frac{1}{2}}. \end{aligned}$$

The Hölder lemma and the fact that $\psi \in \mathcal{B}$ give :

$$\left(\int_0^\infty e^{-c\nu t 2^{2j}} \|\psi(t)\|_{L^1} dt\right)^2 \le C \frac{2^{-2j}}{\nu},$$
and

$$\|e^{-c\nu t 2^{2j}}\|\psi(t)\|_{L^1}\|_{L^{\frac{4}{3}}_t(\mathbb{R}_+)}^2 \le C\frac{2^{-j}}{\nu^{\frac{1}{2}}}$$

So that we can write :

$$\|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} \leq \frac{C_{F}}{\sqrt{\nu}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} \sum_{k=-\infty}^{j+1} \min(2^{\frac{k}{2}}, (\varepsilon\nu)^{\frac{1}{4}} 2^{\frac{3j}{2}-\frac{k}{2}}).$$

It is obvious that:

$$2^{\frac{k}{2}} \le (\varepsilon\nu)^{\frac{1}{4}} 2^{\frac{3j}{2} - \frac{k}{2}} \Longleftrightarrow k \le \frac{1}{4} \log_2(\varepsilon\nu) + \frac{3j}{2}$$

We have to compare it with j + 1 as the summation index goes from $-\infty$ to j + 1:

$$j+1 \le \frac{1}{4}\log_2(\varepsilon\nu) + \frac{3j}{2} \Longleftrightarrow j \ge 2 - \frac{1}{2}\log_2(\varepsilon\nu) \Longleftrightarrow 2^j \ge \frac{4}{(\varepsilon\nu)^{\frac{1}{2}}}$$

So if $2^j \ge \frac{4}{(\varepsilon\nu)^{\frac{1}{2}}}$ we obtain that

$$\|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} \leq \frac{C_{F}}{\sqrt{\nu}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} \sum_{k=-\infty}^{j+1} 2^{\frac{k}{2}} \leq \frac{C_{F}}{\sqrt{\nu}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} 2^{\frac{j}{2}}$$

And if $2^j \leq \frac{4}{(\varepsilon\nu)^{\frac{1}{2}}}$ we have to cut the summation for $j+1 \geq \frac{1}{4}\log_2(\varepsilon\nu) + \frac{3j}{2}$:

$$\|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} \leq \frac{C_{F}}{\sqrt{\nu}} \|\dot{\Delta}_{j}f_{0}(\xi)\|_{L^{2}} \Big(\sum_{k=-\infty}^{\frac{1}{4}\log_{2}(\varepsilon\nu)+\frac{3j}{2}} 2^{\frac{k}{2}} + \sum_{k=\frac{1}{4}\log_{2}(\varepsilon\nu)+\frac{3j}{2}}^{j+1} (\varepsilon\nu)^{\frac{1}{4}} 2^{\frac{3j}{2}-\frac{k}{2}} \Big),$$

which gives that:

$$\|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} \leq \frac{C_{F}}{\sqrt{\nu^{\frac{3}{8}}}} \varepsilon^{\frac{1}{8}} 2^{\frac{3j}{4}} \|\dot{\Delta}_{j}f_{0}\|_{L^{2}}$$

using that $2^j \ge \frac{4}{(\varepsilon\nu)^{\frac{1}{2}}}$ we can write $2^{\frac{j}{2}} = 2^{\frac{-j}{4}} 2^{\frac{3j}{4}} \le \frac{4^{\frac{1}{4}}}{(\varepsilon\nu)^{\frac{1}{8}}}$ and then for all $j \in \mathbb{Z}$ we have:

$$\|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} \leq \frac{C_{F}}{\nu^{\frac{3}{8}}} \varepsilon^{\frac{1}{8}} 2^{\frac{3j}{4}} \|\dot{\Delta}_{j}f_{0}\|_{L^{2}}.$$

The argument for the inhomogeneous case is the same so we obtain: for all $j \in \mathbb{Z}$:

$$\|\dot{\Delta}_{j}f\|_{L^{2}L^{\infty}} \leq \frac{C_{F}}{\nu^{\frac{3}{8}}} \varepsilon^{\frac{1}{8}} 2^{\frac{3j}{4}} (\|\dot{\Delta}_{j}f_{0}\|_{L^{2}} + \|\dot{\Delta}_{j}g\|_{L^{1}((R)_{+},L^{2})}).$$

Then as we will use Sobolev injections, we want to get estimates in inhomogeneous spaces so we will get estimates for Δ_q with $q \ge -1$. We know that there exists a constant C such that

$$\sum_{j \le -1} \|\dot{\Delta}_j f_0\|_{L^2}^2 \le C \|\Delta_{-1} f_0\|_{L^2}^2 + C \|f_0\|_{L^2}^2$$

and from that we can estimate in terms of ε this quantity and finally obtain that $\forall t \ge 0$ and $\forall q \in [1, 2]$

$$\|f\|_{L^{2}(\mathbb{R}_{+},B^{s}_{\infty,q})} \leq C_{F} \frac{\varepsilon^{\frac{1}{8}}}{\nu^{\frac{3}{8}}} (\|f_{0}\|_{L^{2}(\mathbb{R}^{3})} + \|f_{0}\|_{B^{s+\frac{3}{4}}_{2,q}} + \|g\|_{L^{1}(\mathbb{R}_{+},L^{2}(\mathbb{R}^{3}))} + \|g\|_{L^{1}(\mathbb{R}_{+},B^{s+\frac{3}{4}}_{2,q})})$$

4.6.5 Application to W_{ε}

We use that there exists a constant C such that $||u||_{B_{2,1}^{s+\frac{3}{4}}} \leq C||u||_{H^{s+1}}$, so:

$$\|W_{\varepsilon}\|_{L^{2}(\mathbb{R}_{+},B^{s}_{\infty,1})} \leq C_{F} \frac{\varepsilon^{\frac{1}{8}}}{\nu^{\frac{3}{8}}} \left(\|U_{0,\varepsilon,osc}\|_{H^{s+1}} + \|G\|_{L^{1}(\mathbb{R}_{+},H^{s+1})} \right)$$

precisely:

$$\|W_{\varepsilon}\|_{L^{2}(\mathbb{R}_{+},L^{\infty})} \leq \|W_{\varepsilon}\|_{L^{2}(\mathbb{R}_{+},B^{0}_{\infty,1})} \leq C_{F} \frac{\varepsilon^{\frac{1}{8}}}{\nu^{\frac{3}{8}}} \big(\|U_{0,\varepsilon,osc}\|_{H^{1}} + \|G\|_{L^{1}(\mathbb{R}_{+},H^{3})}\big),$$

and as $||G||_{L^1H^{s+1}} \leq C ||\widetilde{U_{QG}}||_{L^2H^{s+2}}^2 \leq \frac{C}{\nu} ||U_{0,QG}||_{H^{s+1}}^2$ and with the estimates of the initial data given in the assumptions of the Theorem we end the proof of lemma 4.3.4.

4.7 Appendix 3 : a priori estimates

In this section we will establish a priori estimates on the solution U_{ε} . As the method is exactly the same as in [11] we will not give many details.

Lemma 4.7.1 Let $s \in]0,1[$ or s > 1. There exists a constant C_s such that we have for all $t \in [0, T_{\varepsilon}^*$:

$$\|U_{\varepsilon}(t)\|_{H^s} \leq \|U_{\varepsilon}(0)\|_{H^s} e^{C_s \int_0^t \|U_{\varepsilon}(t')\|_{Lip} dt'}$$

4.7.1 A general lemma

First of all, projecting on the divergence free vector fields (ie the Leray projector \mathbb{P}) and using the fact that $\mathbb{P}U_{\varepsilon} = U_{\varepsilon}$ gives us that U_{ε} satisfies the pressure-free primitive equations :

$$\begin{cases} \partial_t U_{\varepsilon} + v_{\varepsilon} \cdot \nabla U_{\varepsilon} - LU_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \mathbb{P} U_{\varepsilon} &= \Pi(U_{\varepsilon}, U_{\varepsilon}) \\ U_{\varepsilon/t=0} = U_{\varepsilon,0} \end{cases}$$
(4.7.49)
with $\Pi(U_{\varepsilon}, U_{\varepsilon}) = (Id - \mathbb{P}(v_{\varepsilon} \cdot \nabla U_{\varepsilon})) = \Delta^{-1} \nabla \sum_{i,j} \partial_i \partial_j (v_{\varepsilon}^i \cdot v_{\varepsilon}^j).$

So let us now state the lemma :

Lemma 4.7.2 Let $\sigma \in]0,1[, s > 1, and f,g,v : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^4$ such that for all $t \in [0,T]$,

$$\begin{cases} \partial_t f + v \cdot \nabla f - Lf + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \mathbb{P} f &= g \\ divv = 0 \\ f_{/t=0} = f_0 \end{cases}$$
(4.7.50)

Suppose that $f_0 \in H^{\sigma}(\mathbb{R}^3)$, $g \in L^1([0,T], H^{\sigma}(\mathbb{R}^3))$ and $v \in L^1([0,T]Lip(\mathbb{R}^3))$, then $f \in L^{\infty}([0,T], H^{\sigma}(\mathbb{R}^3))$ and there exists a constant C_{σ} such that for all $t \in [0,T]$:

$$\|f(t)\|_{H^{\sigma}} \le \|f_0\|_{H^{\sigma}} + \int_0^t (\|g(t')\|_{H^{\sigma}} + C_{\sigma}\|f(t')\|_{H^{\sigma}}\|v(t')\|_{Lip})dt'$$

Suppose that $f_0 \in H^s(\mathbb{R}^3)$, $g \in L^1([0,T], H^s(\mathbb{R}^3))$ and $v \in L^1([0,T]Lip(\mathbb{R}^3))$, then $f \in L^\infty([0,T], H^s(\mathbb{R}^3))$ and there exists a constant C_s such that for all $t \in [0,T]$:

$$\|f(t)\|_{H^s} \le \|f_0\|_{H^s} + \int_0^t (\|g(t')\|_{H^s} + C_s(\|f(t')\|_{H^s}\|v(t')\|_{Lip} + \|f(t')\|_{Lip}\|v(t')\|_{H^s})dt'$$

Proof: : Let us start with the case $\sigma \in]0, 1[$. The first step is to apply a localisation operator Δ_q to system (4.7.50) and in order to keep the advection term we introduce a commutator :

$$\begin{cases} \partial_t \Delta_q f + v \cdot \nabla \Delta_q f - L \Delta_q f + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \mathbb{P} \Delta_q f &= \Delta_q g + [v \cdot \nabla, \Delta_q] f \\ \operatorname{div} v = 0 \\ \Delta_q f_{/t=0} &= \Delta_q f_0 \end{cases}$$
(4.7.51)

A scalar product in L^2 of this equation with $\Delta_q f$ and the use of the skewsymmetry of \mathcal{A} give that for all $t \in [0, T]$,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q f\|_{L^2}^2 + \nu_0 \|\nabla\Delta_q f\|_{L^2}^2 \le \|\Delta_q f\|_{L^2} (\|\Delta_q g\|_{L^2} + \|[v.\nabla,\Delta_q]f\|_{L^2})$$

Let us the state the classical following commutation lemma (one can see [7] or [11]):

Lemma 4.7.3 There exist a constant C_{σ} and a sequence $(c_q) \in l^2$ such that

$$||[v.\nabla, \Delta_q]f||_{L^2} \le C_\sigma c_q 2^{-q\sigma} ||v||_{Lip} ||f||_{H^\sigma}$$

Proof: The idea is to use the Bony decomposition :

$$[v \cdot \nabla, \Delta_q]f = \sum_{i=1}^3 \left((T_{v^i} \partial_i \Delta_q f + T_{\partial_i \Delta_q f} v^i + R(v^i, \partial_i \Delta_q f)) - \Delta_q (T_{v^i} \partial_i f + T_{\partial_i f} v^i + R(v^i, \partial_i f)) \right)$$

and group terms in order to take advantage of the convolution form of the expression of Δ_q :

$$\begin{cases} [v.\nabla, \Delta_q]f &= \sum_{i=1}^3 \left((T_{v^i}\partial_i \Delta_q f - \Delta_q T_{v^i}\partial_i f) \right. \\ &+ (T_{\partial_i \Delta_q f}\chi(D)v^i + R(\chi(D)v^i, \partial_i \Delta_q f)) - \Delta_q(T_{\partial_i f}\chi(D)v^i + R(\chi(D)v^i, \partial_i f)) \right. \\ &+ (T_{\partial_i \Delta_q f}(1 - \chi(D))v^i + R((1 - \chi(D))v^i, \partial_i \Delta_q f)) \\ &- (\Delta_q(T_{\partial_i f}(1 - \chi(D))v^i + R((1 - \chi(D))v^i, \partial_i f))) \right) \end{cases}$$

The use of a second truncation of the speed allows us to treat separately low frequencies. Then the estimates of each group gives the lemma. \blacksquare

Let us go back to the proof of lemma (4.7.2): using the estimate on the commutator, a Gronwall lemma, and then taking the l^2 -norm give the first part of the lemma.

When s > 1, as $||g||_{H^s} \leq \sigma_{|\alpha| \leq [s]} ||\partial^{\alpha}g||_{H^{\sigma}}$ we will estimate each derivative ∂^{α} in H^{σ} where $\sigma = s - [s] \in]0, 1[$: for that we derive the localized system, which makes appear a new commutator :

$$\begin{cases} \partial_t \partial^{\alpha} f + v \cdot \nabla \partial^{\alpha} f - L \partial^{\alpha} f + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \mathbb{P} \partial^{\alpha} f &= \partial^{\alpha} g + [v \cdot \nabla, \partial^{\alpha}] f \\ \operatorname{div} v = 0 \\ \partial^{\alpha} f_{/t=0} &= \partial^{\alpha} f_0 \end{cases}$$
(4.7.52)

And before applying the first result of lemma (4.7.2) we have to estimate the commutator which is the aim of the following classical lemma (see [11]) :

Lemma 4.7.4 There exists a constant C_{σ} such that :

$$\| [v \cdot \nabla, \partial^{\alpha}] f \|_{H^{\sigma}} \le C_{\sigma} (\|v\|_{Lip} \|f\|_{H^{\sigma}} + \|v\|_{H^{\sigma}} \|f\|_{Lip})$$

then using this estimate concludes the proof of lemma (4.7.2).

4.7.2 Application to U_{ε}

As we want to apply the result of the previous section the only thing to see is to estimate the operator Π and that is the object of the following classical lemma (see [11]) :

Lemma 4.7.5 If s > 0 there exists a constant C_s such that :

$$\|\Pi(V,W)\|_{H^s} \le C_s(\|V\|_{Lip}\|W\|_{H^s} + \|W\|_{Lip}\|V\|_{H^s})$$

Returning to the primitive system, using lemmas (4.7.2) and (4.7.5) give :

$$\|U_{\varepsilon}(t)\|_{H^s} \le \|U_{\varepsilon}(0)\|_{H^s} + C_s \int_0^t \|U_{\varepsilon}(t')\|_{Lip} \|U_{\varepsilon}(t')\|_{H^s} dt'.$$

Then a use of the Gronwall lemma gives the wanted result.

4.8 Appendix 4 : stable Strichartz estimates

The term "stable" is used to emphasize the fact that, contrary to the Strichartz estimates obtained in the second appendix, the following ones are uniform with respect to the viscosity, allowing a vanishing viscosity limit. On counterpart we are forced to be local in time.

4.8.1 Statement of the results

This section is devoted to the proof of the following estimates:

Lemma 4.8.1 Let U_{ε} be the solution of PE_{ε} , whose lifespan is $T_{\varepsilon}^* > 0$, $s \in]0, 1[$, and fix a time $0 < T < T_{\varepsilon}^*$. If $\nu = \nu'$, there exist constants C > 0 and $C_{F,s} > 0$ such that for all $\varepsilon > 0$,

$$\|U_{\varepsilon,osc}\|_{L^{8}([0,T],Lip/C^{s+1})} \leq C_{F,s}\varepsilon^{\frac{1}{8}-\alpha}e^{CV_{\varepsilon}(T)}, \quad \text{with} \quad V_{\varepsilon}(T) = \int_{0}^{T} \|v_{\varepsilon}(\tau)\|_{Lip}d\tau.$$

4.8.2 Proof of the Strichartz estimates when $\nu = \nu'$

preliminary remarks

In particular, thanks to the diagonalization and the orthogonality of the eigenvectors, each $\mathbb{P}_i U_{\varepsilon}$ satisfies the following system:

$$\begin{cases} \partial_t \mathbb{P}_i U_{\varepsilon} - \nu \Delta \mathbb{P}_i U_{\varepsilon} + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \mathbb{P}_i U_{\varepsilon} = \mathbb{P}_i \mathbb{P}(v_{\varepsilon} \cdot \nabla U_{\varepsilon}) \\ \mathbb{P}_i U_{\varepsilon/t=0} = \mathbb{P}_i U_{\varepsilon,0}, \end{cases}$$
(4.8.53)

general estimates

The object of this section is to prove the following result :

Lemma 4.8.2 Assume that $f_0 \in L^2(\mathbb{R}^3) \cap B_{2,q}^{s+\frac{3}{2}}$ and $g \in L^1([0,T], L^2(\mathbb{R}^3)) \cap L^1([0,T], B_{2,q}^{s+\frac{3}{2}})$ with $q \in [1,2]$ and for s > 0, and that f solves on [0,T] the system

$$\begin{cases} \partial_t f - \nu \Delta f + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} f = g \\ f_{/t=0} = f_0, \end{cases}$$

Assume also that f_0 and g(t) for all $t \in [0, T]$ have a null potential viscosity.

Then there exists a constant $C_{F,s}$ only depending on F and s such that we have the following estimate $\forall p \geq 8, 1 \leq q \leq p$:

$$\|f\|_{L_T^p B_{\infty,q}^s} \le C_{F,s} \varepsilon^{\frac{1}{p}} \Big(\|f_0\|_{L^2} + \|f_0\|_{B_{2,q}^{s+\frac{3}{2}}} + \|g\|_{L_T^1 L^2} + \|g\|_{L_T^1 B_{2,q}^{s+\frac{3}{2}}} \Big)$$

where the notation $L^p_T L^q$ means $L^p([0,T], L^q(\mathbb{R}^3))$

Exactly like in section 4.6 we have a complete diagonalization.

$$\begin{cases} \partial_t \mathbb{P}_i f - \nu \Delta \mathbb{P}_i f + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} \mathbb{P}_i f = \mathbb{P}_i g\\ \mathbb{P}_i f_{/t=0} = \mathbb{P}_i f_0, \end{cases}$$

With a wish of simplification we will consider that f_0 , g(t) (then f(t)) only depend on the third eigenvector (ie $\mathbb{P}_3 f_0 = f_0...$)

The idea is to localize in frequency (ie we will apply $\dot{\Delta}_j$ (homogeneous) to the equation). Thanks to the Duhamel formula we obtain :

$$\widehat{\dot{\Delta}_j f}(t,\xi) = e^{-\nu t|\xi|^2 + i\frac{t}{\varepsilon}\frac{|\xi|_F}{F|\xi|}} \widehat{\dot{\Delta}_j f_0}(\xi) + \int_0^t e^{-\nu (t-\tau)|\xi|^2 + i\frac{t-\tau}{\varepsilon}\frac{|\xi|_F}{F|\xi|}} \widehat{\dot{\Delta}_j g}(\tau,\xi) d\tau$$

We will begin with the homogeneous case, ie when g = 0.

Duality argument

In this section we will describe the duality method (also called TT^*) to precicely determine the kernel on which we will work to get dispersive estimates. Let us begin with the definition of the space $\mathcal{B} = \{\psi \in \mathcal{C}_0^{\infty}([0,T] \times \mathbb{R}^3) / \|\psi\|_{L^{\overline{p}}([0,T],L^1(\mathbb{R}^3)) \leq 1}\}$ then for all j:

$$\|\dot{\Delta}_j f\|_{L^p_T L^\infty} = \sup_{\psi \in \mathcal{B}} \int_0^T \int_{\mathbb{R}^3} \dot{\Delta}_j f(t, x) \psi(t, x) dx dt$$

using the Plancherel theorem we get :

$$\|\dot{\Delta}_j f\|_{L^p_T L^\infty} = C \sup_{\psi \in \mathcal{B}} \int_0^T \int_{\mathbb{R}^3} e^{-\nu t |\xi|^2 + i\frac{t}{\varepsilon} \frac{|\xi|_F}{F|\xi|}} \phi(2^{-j}|\xi|) \widehat{f_0}(\xi) \widehat{\psi}(t,\xi) d\xi dt$$

If ϕ_1 is a C_0^{∞} function whose support is a neighbourhood of the one of ϕ and equal to 1 on $supp\phi$ then we can write :

$$\|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} = C \sup_{\psi \in \mathcal{B}} \int_{0}^{T} \int_{\mathbb{R}^{3}} e^{-\nu t|\xi|^{2} + i\frac{t}{\varepsilon} \frac{|\xi|_{F}}{F|\xi|}} \phi(2^{-j}|\xi|) \widehat{f_{0}}(\xi) \phi_{1}(2^{-j}|\xi|) \widehat{\psi}(t,\xi) d\xi dt$$

using the Fubini theorem, we get :

$$\|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} = C \sup_{\psi \in \mathcal{B}} \int_{\mathbb{R}^{3}} \phi(2^{-j}|\xi|) \widehat{f}_{0}(\xi) \Big(\int_{0}^{T} e^{-\nu t|\xi|^{2} + i\frac{t}{\varepsilon}\frac{|\xi|_{F}}{F|\xi|}} \phi_{1}(2^{-j}|\xi|) \widehat{\psi}(t,\xi) dt \Big) d\xi$$

A use of the Hölder inequality gives :

$$\begin{split} \|\dot{\Delta}_j f\|_{L^p_T L^\infty} &= C \sup_{\psi \in \mathcal{B}} \|\widehat{\dot{\Delta}_j f_0}\|_{L^2} \Big(\int_{\mathbb{R}^3} \int_0^T \int_0^T e^{-\nu(t+s)|\xi|^2 + i\frac{t-s}{\varepsilon}\frac{|\xi|_F}{F|\xi|}} \\ & \phi_1(2^{-j}|\xi|)^2 \widehat{\psi}(t,\xi) \overline{\widehat{\psi}(s,\xi)} dt ds) d\xi \Big)^{\frac{1}{2}} \end{split}$$

and thanks to, successively, Fubini and Plancherel we finally obtain :

$$\begin{split} \|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} &= C \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_{j}f_{0}\|_{L^{2}} \Big(\int_{0}^{T} \int_{0}^{T} \|K(\nu(t+s), \frac{t-s}{\varepsilon}, .)\|_{L^{\infty}} \\ & \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \Big)^{\frac{1}{2}} \\ & \text{where} \quad K(\theta, \tau, x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^{3}} e^{ix.\xi - \theta|\xi|^{2} + i\tau\lambda(\xi)} \phi_{1}(2^{-j}|\xi|)^{2} d\xi \end{split}$$

with
$$\lambda(\xi) = \frac{|\xi|_F}{F|\xi|}$$

Applying the additional localizations described in (4.6.45) to the kernel K we obtain that :

$$\|K(\theta,\tau,.)\|_{L^{\infty}} \leq \sum_{k=-\infty}^{j+1} \|\dot{\Delta}_k^v K(\theta,\tau,.)\|_{L^{\infty}}$$

Then similarly :

$$\begin{aligned} \|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} &= C \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_{j}f_{0}\|_{L^{2}} \Big(\int_{0}^{T} \int_{0}^{T} \sum_{k=-\infty}^{j+1} \|K_{1}(\nu(t+s), \frac{t-s}{\varepsilon}, .)\|_{L^{\infty}} \\ \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \Big)^{\frac{1}{2}} \end{aligned}$$

with
$$K_1(\theta, \tau, x) = \dot{\Delta}_k^v K(\theta, \tau, x) = \int_{\mathbb{R}^3} e^{ix.\xi - \theta|\xi|^2 + i\tau\lambda(\xi)} \phi_1(2^{-j}|\xi|)^2 \phi(2^{-k}|\xi_3|) d\xi$$
 (4.8.54)

Then the method consists in estimating in two different ways the kernel. Like in section 4.6 as we have to deal with a summation from $-\infty$ to j + 1 we will get two kinds of estimates in terms of k: some depending as 2^k (summable in $-\infty$ but no ε) and some as $2^{-k}\varepsilon$ (not summable). The difference is that here, the summation occurs before integration.

Dispersive estimates

We use the same dispersive estimates as in Lemma 4.6.2:

Lemma 4.8.3 There exists a constant C_F such that $\forall j \in \mathbb{Z}, \forall k \leq j + 1, \forall \theta > 0$ and $\forall \tau > 0$ we have :

$$||K_1(\theta,\tau,.)||_{L^{\infty}} \le C_F e^{-c\theta 2^{2j}} 2^{3j} \min(2^{k-j}, \frac{2^{j-k}}{\sqrt{\tau}}),$$

where c > 0 is the constant from (4.6.45).

End of the proof

Let us go back to (4.8.54): using the dispersive estimate we can write that :

$$\begin{split} \|\dot{\Delta}_{j}f\|_{L^{p}([0,T],L^{\infty})} &= C_{F} \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_{j}f_{0}\|_{L^{2}} \Big(\int_{0}^{T} \int_{0}^{T} \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \\ \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \Big)^{\frac{1}{2}} \end{split}$$

Now we will focus on the summation beneath the integral and precise its expression. A simple computation shows that:

$$2^{k-j} \le 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}} \quad \Longleftrightarrow \quad k \le j + \frac{1}{4} \log_2\left(\frac{\varepsilon}{|t-s|}\right)$$

We have to compare it with j + 1 as the summation index k goes from $-\infty$ to j + 1:

$$j+1 \le j + \frac{1}{4}log_2\left(\frac{\varepsilon}{|t-s|}\right) \quad \Longleftrightarrow \quad \frac{\varepsilon}{|t-s|} \ge 16$$

So we cut, in the integral, \mathbb{R}^3 in two domains:

$$\begin{split} &\int_{0}^{T} \int_{0}^{T} \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds = \\ &\int_{0}^{T} \int_{0}^{T} \sum_{\frac{\varepsilon}{|t-s|} \ge 16}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} 2^{k-j} \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \\ &+ \int_{0}^{T} \int_{0}^{T} \sum_{\frac{\varepsilon}{|t-s|} \le 16}^{j+\frac{1}{4} \log_{2}\left(\frac{\varepsilon}{|t-s|}\right)} 2^{k-j} + \sum_{\substack{k=j+\frac{1}{4} \log_{2}\left(\frac{\varepsilon}{|t-s|}\right)+1}}^{j+1} 2^{j-k} \left(\frac{\varepsilon}{|t-s|}\right)^{\frac{1}{2}} \right) \\ &e^{-c\nu(t+s)2^{2j}} 2^{3j} \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds. \end{split}$$

After computating the summations (and using the fact that $\frac{\varepsilon}{|t-s|} \leq 16$ in order to estimate the last one), we obtain that:

$$\begin{split} \int_{0}^{T} \int_{0}^{T} \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \leq \\ C \int_{0}^{T} \int_{0}^{T} \sum_{\frac{\varepsilon}{|t-s|} \ge 16}^{e^{-c\nu(t+s)2^{2j}}} 2^{3j} \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \\ + C \int_{0}^{T} \int_{0}^{T} \sum_{\frac{\varepsilon}{|t-s|} \le 16}^{e^{-c\nu(t+s)2^{2j}}} 2^{3j} \Big(\frac{\varepsilon}{|t-s|}\Big)^{\frac{1}{4}} \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds. \end{split}$$

And, as $e^{-c\nu(t+s)2^{2j}} \leq 1$ we finally get:

$$\int_{0}^{T} \int_{0}^{T} \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \leq (4.8.55)$$

$$\begin{split} C \int_0^T \int_0^T \sum_{|t-s| \ge 16}^{\varepsilon} 2^{3j} \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds \\ + C \int_0^T \int_0^T \int_0^T \sum_{|t-s| \le 16}^{\varepsilon} 2^{3j} \Big(\frac{\varepsilon}{|t-s|}\Big)^{\frac{1}{4}} \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds. \end{split}$$

This is here that we have to distinguish when p = 8 or p > 8. Let us begin with the second case:

$$\int_0^T \int_0^T \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds \le \frac{1}{2} \int_0^T \int_0^T \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds \le \frac{1}{2} \int_0^T \int_0^T \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds \le \frac{1}{2} \int_0^T \int_0^T \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds \le \frac{1}{2} \int_0^T \int_0^T \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{j-k} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds \le \frac{1}{2} \int_0^T |\psi(s)|^{\frac{1}{2}} dt ds \le \frac{1}{2} \int_0^T |\psi(s)|^{\frac{1}{2}$$

$$C2^{3j} \int_0^T \int_0^T h_{\varepsilon}(t-s) \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds \le C2^{3j} \int_{\mathbb{R}} m(t) (h_{\varepsilon} * m)(t) dt,$$

with

$$\begin{cases} m(t) = \mathbf{1}_{[0,T]}(t) \| \psi(t) \|_{L^1} \\ h_{\varepsilon}(t) = \mathbf{1}_{\frac{\varepsilon}{|t|} \ge 16} + \mathbf{1}_{\frac{\varepsilon}{|t|} \le 16} \left(\frac{\varepsilon}{|t|}\right)^{\frac{1}{4}} = h(\frac{t}{\varepsilon}) \\ h(t) = \mathbf{1}_{|t| \le 16} + \mathbf{1}_{|t| \ge 16} \left(\frac{1}{|t|}\right)^{\frac{1}{4}} \end{cases}$$

All that remains is to estimate, using the Hölder and Young inequalities:

$$\int_{\mathbb{R}} m(t)(h_{\varepsilon} * m)(t)dt \le \|m\|_{L^{\overline{p}}} \|h_{\varepsilon} * m\|_{L^{p}} \le \|m\|_{L^{\overline{p}}} (\|h_{\varepsilon}\|_{L^{\frac{p}{2}}} . \|m\|_{L^{\overline{p}}})$$

As p > 8, $\frac{p}{2} > 4$ and $h_{\varepsilon} \in L^{\frac{p}{2}}$, so we can write:

$$\|h_{\varepsilon}\|_{L^{\frac{p}{2}}} = \varepsilon^{\frac{2}{p}} \|h\|_{L^{\frac{p}{2}}},$$

and using that $||m||_{L^{\overline{p}}} = ||\psi||_{L^{\overline{p}}([0,T],L^1)} \leq 1$ we obtain that:

$$\int_0^T \int_0^T \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^1} \|\psi(s)\|_{L^1} dt ds \le C\varepsilon^{\frac{2}{p}} 2^{3j},$$

which finally gives:

 $\forall p > 8, j \in \mathbb{Z}$:

$$\|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} = C_{F}\varepsilon^{\frac{1}{p}}2^{\frac{3j}{2}}\|\dot{\Delta}_{j}f_{0}\|_{L^{2}}$$

In order to get the limit case p = 8 let us go back to (4.8.55): taking advantage of the fact that in the first integral we have $\frac{\varepsilon}{|t-s|} \ge 16$ we can write $1 = 1^{\frac{1}{4}} \le \frac{1}{2} \left(\frac{\varepsilon}{|t-s|}\right)^{\frac{1}{4}}$ and use it in the same integral to obtain:

$$\int_{0}^{T} \int_{0}^{T} \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \leq C \int_{0}^{T} \int_{0}^{T} 2^{3j} \left(\frac{\varepsilon}{|t-s|}\right)^{\frac{1}{4}} \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds.$$

Since, $p = q = \frac{8}{7}$ and $\alpha = \frac{1}{4}$ fulfill the assumptions of the Hardy-Littlewood lemma, we also obtain that:

 $\forall j \in \mathbb{Z}$:

$$\|\dot{\Delta}_{j}f\|_{L_{T}^{8}L^{\infty}} = C_{F}\varepsilon^{\frac{1}{8}}2^{\frac{3j}{2}}\|\dot{\Delta}_{j}f_{0}\|_{L^{2}}$$

And then $\forall p \geq 8, j \in \mathbb{Z}$:

$$\|\dot{\Delta}_{j}f\|_{L^{p}([0,T],L^{\infty})} = C_{F}\varepsilon^{\frac{1}{p}}2^{\frac{3j}{2}}\|\dot{\Delta}_{j}f_{0}\|_{L^{2}}$$

The proof of the case when $g \neq 0$ and $f_0 = 0$ is dealt the same way, using the Duhamel formula. Using the same notation as (4.8.54) we have :

$$\begin{aligned} \|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} &= C \sup_{\psi \in \mathcal{B}} \int_{0}^{T} \|\dot{\Delta}_{j}g\|_{L^{2}} \Big(\int_{\tau}^{T} \int_{\tau}^{T} \sum_{k=-\infty}^{j+1} \|K_{1}(\nu(t+s-2\tau), \frac{t-s}{\varepsilon}, .)\|_{L^{\infty}} \\ \|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \Big)^{\frac{1}{2}} d\tau \end{aligned}$$

Lemma 4.6.2 implies :

$$\begin{split} \|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} &= C \sup_{\psi \in \mathcal{B}} \int_{0}^{T} \|\dot{\Delta}_{j}g\|_{L^{2}} \Big(\int_{\tau}^{T} \int_{\tau}^{T} \sum_{k=-\infty}^{j+1} e^{-c\nu(t+s-2\tau)2^{2j}} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \\ &\|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \Big)^{\frac{1}{2}} d\tau. \end{split}$$

And, as $t, s \ge \tau$, $e^{-c\nu(t+s-2\tau)} \le 1$ so we can write :

$$\begin{split} \|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} &= C \sup_{\psi \in \mathcal{B}} \|\dot{\Delta}_{j}g\|_{L^{1}_{T}L^{2}} \Big(\int_{0}^{T} \int_{0}^{T} \sum_{k=-\infty}^{j+1} 2^{3j} \min(2^{k-j}, 2^{j-k} \frac{\varepsilon^{\frac{1}{2}}}{|t-s|^{\frac{1}{2}}}) \\ &\|\psi(t)\|_{L^{1}} \|\psi(s)\|_{L^{1}} dt ds \Big)^{\frac{1}{2}}. \end{split}$$

Identical computations give $\forall p \geq 8$ and $\forall j \in \mathbb{Z}$:

$$\|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} = C_{F}\varepsilon^{\frac{1}{p}}2^{\frac{3j}{2}}\|\dot{\Delta}_{j}g\|_{L^{1}_{T}L^{2}}.$$

So we finally have $\forall p \geq 8$ and $\forall j \in \mathbb{Z}$:

$$\|\dot{\Delta}_{j}f\|_{L^{p}_{T}L^{\infty}} = C_{F}\varepsilon^{\frac{1}{p}}2^{\frac{3j}{2}}(\|\dot{\Delta}_{j}f_{0}\|_{L^{2}} + \|\dot{\Delta}_{j}g\|_{L^{1}_{T}L^{2}}).$$

Remark 4.8.1 This is here, that we really needed to estimate the kernel in terms of the minimum of two quantities: one is sommable when $k = -\infty$ but does not depend on ε whereas the other depends on ε but diverges in $-\infty$.

At this point we only have obtained estimates with homogeneous localization, and the result concerns inhomogeneous localizations. As for every $j \ge 0$, $\Delta_j = \dot{\Delta}_j$, we only have to deal with $\Delta_{-1} = \sum_{j \le -1} \dot{\Delta}_j$, and using the Minkowski inequality :

$$\|\Delta_{-1}f\|_{L^p_T,L^{\infty}} \leq \sum_{j \leq -1} \|\dot{\Delta}_j\|_{L^p_T,L^{\infty}} \leq C_F \varepsilon^{\frac{1}{p}} \sum_{j \leq -1} 2^{\frac{3j}{2}} (\|\dot{\Delta}_j f_0\|_{L^2} + \|\dot{\Delta}_j g\|_{L^1_TL^2}),$$

which we estimate, thanks to Hölder :

$$\|\Delta_{-1}f\|_{L^p_T,L^{\infty}} \le C_F \varepsilon^{\frac{1}{p}} \Big(\sum_{j \le -1} 2^{3j}\Big)^{\frac{1}{2}} \Big((\sum_{j \le -1} \|\dot{\Delta}_j f_0\|_{L^2}^2)^{\frac{1}{2}} + (\sum_{j \le -1} \|\dot{\Delta}_j g\|_{L^1_T L^2}^2)^{\frac{1}{2}} \Big).$$

Another use of Minkowski implies :

$$\|\Delta_{-1}f\|_{L^p_T,L^{\infty}} \le C_F \varepsilon^{\frac{1}{p}} \Big((\sum_{j \le -1} \|\dot{\Delta}_j f_0\|_{L^2}^2)^{\frac{1}{2}} + \int_0^T (\sum_{j \le -1} \|\dot{\Delta}_j g(t)\|_{L^2}^2)^{\frac{1}{2}} dt \Big).$$

It is then about to estimate $\sum_{j \leq -1} \|\dot{\Delta}_j f_0\|_{L^2}^2$: using the fact that $\operatorname{supp} \chi \subset B(0, \frac{4}{3})$ equals 1 on B(0, 1), and $\operatorname{supp} \phi \subset C(0, \frac{3}{4}, \frac{8}{3})$ we obtain that there exists a constant C such that :

$$\sum_{j \le -1} \|\dot{\Delta}_j f_0\|_{L^2}^2 \le C(\|\Delta_{-1} f_0\|_{L^2}^2 + \|f_0\|_{L^2}^2),$$

and

$$\|\Delta_{-1}f\|_{L^p_T,L^{\infty}} \le C_F \varepsilon^{\frac{1}{p}} \Big(C(\|\Delta_{-1}f_0\|_{L^2}^2 + \|f_0\|_{L^2}^2)^{\frac{1}{2}} + \int_0^T (\sum_{j \le -1} (\|\Delta_j g(t)\|_{L^2}^2 + \|g(t)\|_{L^2}^2)^{\frac{1}{2}} dt \Big).$$

On the other hand $\forall j \in \mathbb{Z}$:

$$2^{js} \|\dot{\Delta}_j f\|_{L^p_T L^\infty} = C_F \varepsilon^{\frac{1}{p}} 2^{j(s+\frac{3}{2})} (\|\dot{\Delta}_j f_0\|_{L^2} + \|\dot{\Delta}_j g\|_{L^1_T L^2}),$$

so that we obtain, using the Minkowski inequality :

$$\|f\|_{L^{p}([0,T],B_{\infty,q}^{s})} = \left\| \|2^{js}\|\Delta_{j}f(t)\|_{L^{\infty}} \|_{l^{q}} \right\|_{L^{p}} \le \left\| \|2^{js}\|\Delta_{j}f(t)\|_{L^{\infty}} \|_{L^{p}} \right\|_{l^{q}}$$

$$\|f\|_{L^{p}([0,T],B_{\infty,q}^{s})} \leq C_{F}\varepsilon^{\frac{1}{p}} \Big(\|f_{0}\|_{B_{2,q}^{s+\frac{3}{2}}} + C_{s}\|f_{0}\|_{L^{2}} + \|2^{j(s+\frac{3}{2})}\|\Delta_{j}f(t)\|_{L^{1}_{T}L^{2}}\|_{l^{q}} + C_{s}\|g\|_{L^{1}_{T}L^{2}}\Big).$$

Another use of Minkowski finally gives that $\forall p \geq 8, 1 \leq q \leq p$, and $\forall t \in [0, T]$:

$$\|f\|_{L^{p}([0,T],B_{\infty,q}^{s})} \leq C_{F,s}\varepsilon^{\frac{1}{p}} \Big(\|f_{0}\|_{L^{2}} + \|f_{0}\|_{B_{2,q}^{s+\frac{3}{2}}} + \|g\|_{L^{1}_{T}L^{2}} + \|g\|_{L^{1}([0,T],B_{2,q}^{s+\frac{3}{2}})}\Big)$$

Conclusion : adaptation of the result

In the following, we will need estimates on $||f||_{L^pLip}$ and $||f||_{L^pC^{s+1}}$, and we wish to estimate in terms of Sobolev spaces instead of Besov spaces so we have to adapt Lemma 4.8.2. Thanks to the facts that

$$||f||_{Lip} = ||f||_{L^{\infty}} + ||\nabla f||_{L^{\infty}} \le ||f||_{B^{0}_{\infty,1}} + ||f||_{B^{1}_{\infty,1}} \le 2||f||_{B^{1}_{\infty,1}},$$

$$||f||_{C^{s+1}} = ||f||_{B^{s+1}_{\infty,\infty}} \le ||f||_{B^{s+1}_{\infty,2}},$$

and that, for the right-hand side there exists a constant C such that :

$$||f_0||_{L^2} + ||f_0||_{B^{\frac{5}{2}}_{2,1}} \le C ||f_0||_{H^3},$$

we finally get the two following estimates:

$$\|f\|_{L^8_T,Lip} \le C_F \varepsilon^{\frac{1}{8}} \Big(\|f_0\|_{H^3} + \|g\|_{L^1_T H^3} \Big),$$

and
$$||f||_{L^8_T, C^{s+1}} \le C_F \varepsilon^{\frac{1}{8}} \left(||f_0||_{H^{s+\frac{5}{2}}} + ||g||_{L^1_T H^{s+\frac{5}{2}}} \right)$$

Then apply these estimates to systems (4.8.53):

$$\left\|\mathbb{P}_{i}U_{\varepsilon}\right\|_{L_{T}^{8},C^{s+1}} \leq C_{F}\varepsilon^{\frac{1}{8}}\left(\left\|\mathbb{P}_{i}U_{\varepsilon,0}\right\|_{H^{s+\frac{5}{2}}} + \left\|\mathbb{P}_{i}\mathbb{P}(v_{\varepsilon}.\nabla U_{\varepsilon})\right\|_{L_{T}^{1}H^{s+\frac{5}{2}}}\right)$$

Using that $||uv||_{H^s} \le ||u||_{L^{\infty}} ||v||_{H^s} + ||u||_{H^s} ||v||_{L^{\infty}}$ we easily show that

$$\left\|\mathbb{P}_{i}\mathbb{P}(v_{\varepsilon}.\nabla U_{\varepsilon})\right\|_{H^{s+\frac{5}{2}}} \leq C\left\|U_{\varepsilon}\right\|_{Lip}\left\|U_{\varepsilon}\right\|_{H^{s+\frac{7}{2}}}$$

Then, thanks to the a priori estimates, we can bound the $H^{s+\frac{7}{2}}$ norm with the norms of the initial data and we obtain (the estimate for the Lipschitzian norm is dealt the very same way):

$$\|U_{\varepsilon,osc}\|_{L^8([0,T],Lip/C^{s+1})} \le C_{F,s} \varepsilon^{\frac{1}{8}-\alpha} e^{CV_{\varepsilon}(T)}, \quad \text{with} \quad V_{\varepsilon}(T) = \int_0^T \|v_{\varepsilon}(\tau)\|_{Lip} d\tau.$$

This concludes the proof of lemma 4.8.1 in the case $\nu = \nu'$.

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