

Palaiseau, 21 November 2008

Thesis Defence of **Gloria FACCANONI**

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# MODELLING AND SIMULATION OF LIQUID-VAPOR PHASE TRANSITION

A CONTRIBUTION TO THE STUDY OF BOILING CRISIS

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Supervisor at the École Polytechnique: G. Allaire

Supervisor at the CEA-Saclay: S. Kokh

Supervisor at the University of Trento: E. Toro



# OUTLINE

- 1 **Context**
- 2 **Model**
- 3 **Numerical Approximation**
- 4 **Conclusion**



# OUTLINE

## 1 Context

## 2 Model

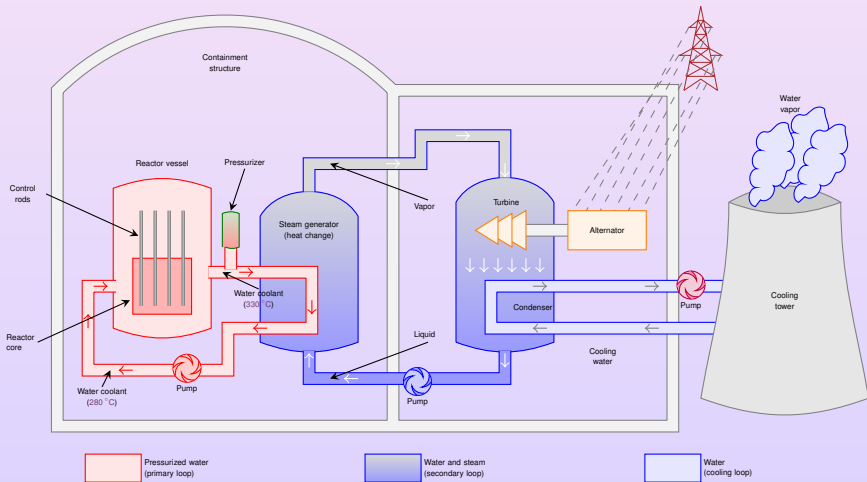
- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
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## 3 Numerical Approximation

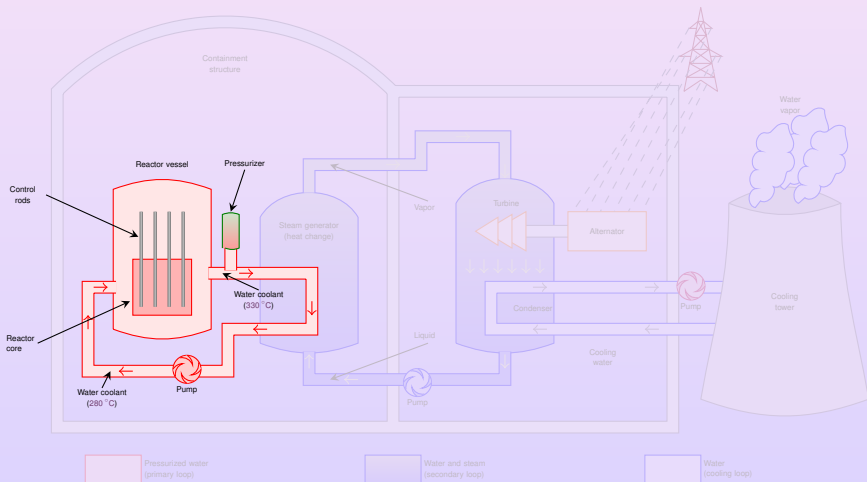
- Numerical Method
- Numerical Examples

## 4 Conclusion

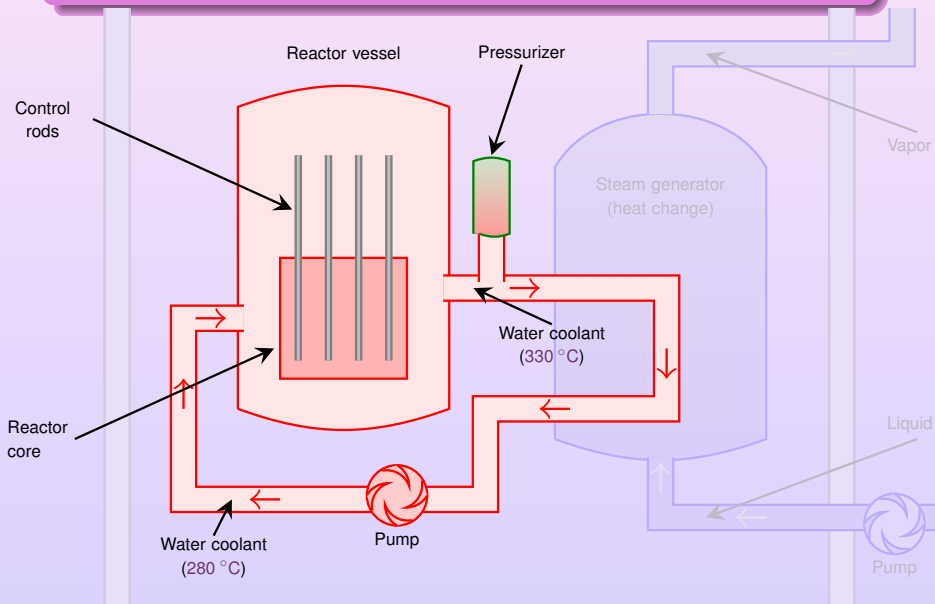
# PRESSURIZED WATER REACTOR



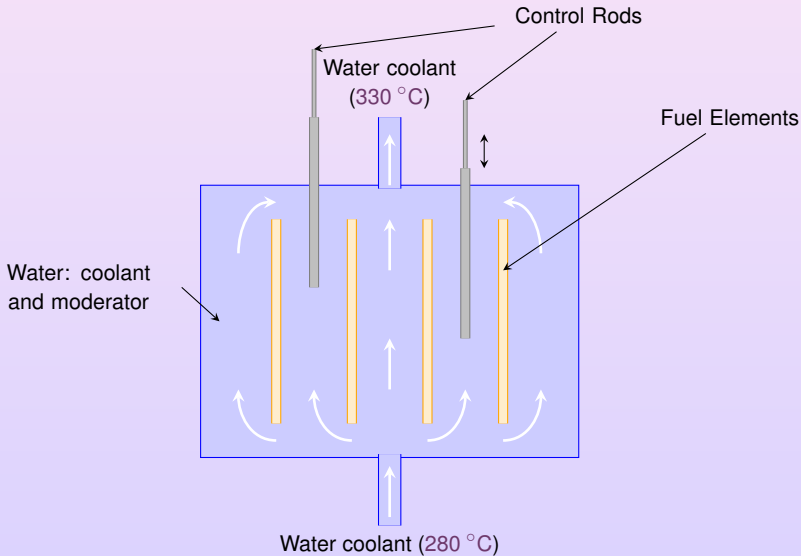
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# CORE OF A PRESSURIZED WATER REACTOR

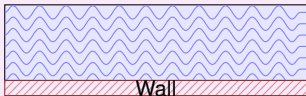


# BOILING CRISIS

## PHENOMENON

Liquid phase heated by a wall at a fixed temperature  $T^{\text{wall}}$ .

When  $T^{\text{wall}}$  increases, we switch from a **Nucleate Boiling** to a **Film Boiling**.



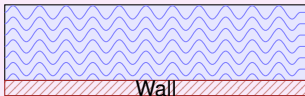


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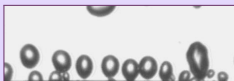
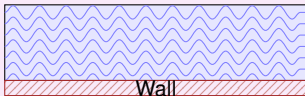
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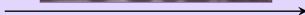
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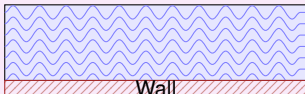
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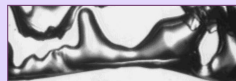
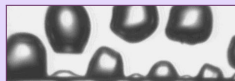
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**Film Boiling**

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# OMEGA - CEA GRENOBLE



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# OUTLINE

## 1 Context

## 2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- Conservation Laws

## 3 Numerical Approximation

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- Numerical Examples

## 4 Conclusion

# “INGREDIENTS” OF THE MODEL

- ✓ **Simulating all bubbles,**
  - System of PDEs for the fluid flow (monophasic or diphasic),
  - Phase transition (pressure and/or temperature variations),
  - Heat Diffusion,
  - Surface Tension,
  - Gravity.

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# EULER SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I}) = \mathfrak{V}_{\text{vf}} - \mathfrak{S}_{\text{sf}}, \\ \partial_t\left(\rho\left(\frac{|\mathbf{u}|^2}{2} + \varepsilon\right)\right) + \operatorname{div}\left(\rho\left(\frac{|\mathbf{u}|^2}{2} + \varepsilon\right)\mathbf{u} + P \mathbf{u}\right) = (\mathfrak{V}_{\text{vf}} - \mathfrak{S}_{\text{sf}}) \cdot \mathbf{u} - \operatorname{div}(q). \end{cases}$$

- $(\mathbf{x}, t) \mapsto \rho$  specific density,
- $(\mathbf{x}, t) \mapsto \varepsilon$  specific internal energy,
- $(\mathbf{x}, t) \mapsto \mathbf{u}$  velocity;
- $(\rho, \varepsilon) \mapsto \mathfrak{V}_{\text{vf}}$  volumic forces,
- $(\rho, \varepsilon) \mapsto \mathfrak{S}_{\text{sf}}$  surface forces,
- $(\rho, \varepsilon) \mapsto \operatorname{div}(q)$  heat transfer.

$(\rho, \varepsilon) \mapsto P$  pressure law.

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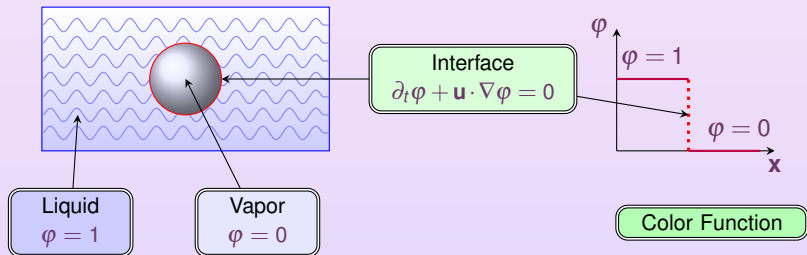
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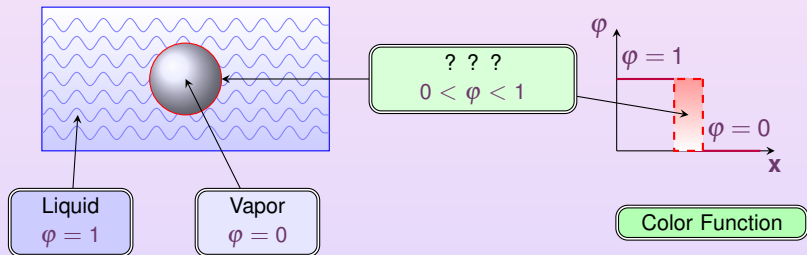


# LIQUID-VAPOR INTERFACE



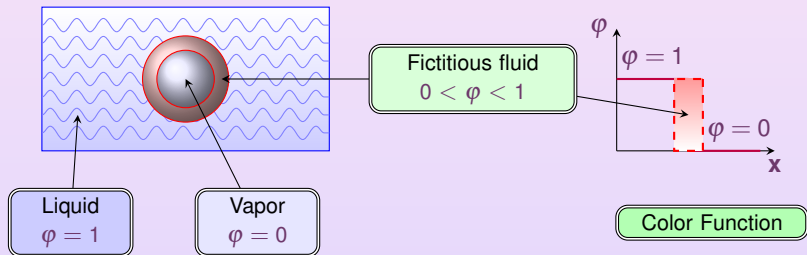
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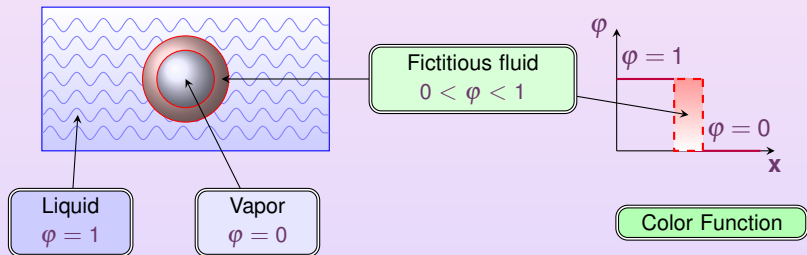
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# LIQUID-VAPOR INTERFACE



➡ Goal: define a global pressure law such that

- $(\rho, \varepsilon, \mathbf{u}, P)$  are continuous (3 zones)
- the interface position and the phase change are implicit (i.e.  $\varphi$ )
- coherence with classical thermodynamics [H. CALLEN]

# EOS OF EACH PHASE $\alpha = 1, 2$

$$\left. \begin{array}{l} \tau_\alpha \text{ specific volume} \\ \varepsilon_\alpha \text{ specific internal energy} \end{array} \right\} \Rightarrow \mathbf{w}_\alpha \stackrel{\text{def}}{=} (\tau_\alpha, \varepsilon_\alpha);$$

$\mathbf{w}_\alpha \mapsto s_\alpha$  specific entropy (Hessian matrix neg. def.);

$$\rightarrow \left\{ \begin{array}{l} T_\alpha \stackrel{\text{def}}{=} \left( \frac{\partial s_\alpha}{\partial \varepsilon_\alpha} \Big|_{\tau_\alpha} \right)^{-1} > 0 \quad \text{temperature,} \\ P_\alpha \stackrel{\text{def}}{=} T_\alpha \frac{\partial s_\alpha}{\partial \tau_\alpha} \Big|_{\varepsilon_\alpha} > 0 \quad \text{pressure,} \\ g_\alpha \stackrel{\text{def}}{=} \varepsilon_\alpha + P_\alpha \tau_\alpha - T_\alpha s_\alpha \quad \text{free enthalpy (Gibbs potential).} \end{array} \right.$$

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# EOS OF THE MIXTURE

- $\mathbf{w} \stackrel{\text{def}}{=} y\mathbf{w}_1 + (1 - y)\mathbf{w}_2$ ;
- $y$  mass fraction;
- $z$  volume fraction s.t.  $y\tau_1 = z\tau$ ;
- $\psi$  energy fraction s.t.  $y\varepsilon_1 = \psi\varepsilon$ .

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$$\sigma \stackrel{\text{def}}{=} y s_1(\mathbf{w}_1) + (1 - y) s_2(\mathbf{w}_2) = y s_1\left(\frac{z}{y}\tau, \frac{\psi}{y}\varepsilon\right) + (1 - y) s_2\left(\frac{1-z}{1-y}\tau, \frac{1-\psi}{1-y}\varepsilon\right)$$

$$P = \left( \frac{\partial \sigma}{\partial \varepsilon} \Big|_{\tau; y, z, \psi} \right)^{-1} \frac{\partial \sigma}{\partial \tau} \Big|_{\varepsilon; y, z, \psi}$$

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# EOS WITH PHASE CHANGE

## ENTROPY WITHOUT PH.CH.

$$(\mathbf{w}, z, y, \psi) \mapsto \sigma$$



## ENTROPY AT EQUILIBRIUM

$$\mathbf{w} \mapsto \mathbf{s}^{\text{eq}}$$

DEFINITION [H. CALLEN, PH. HELLUY ...]

Optimization Problem:

$$s^{\text{eq}}(\mathbf{w}) \stackrel{\text{def}}{=} \max_{z, y, \psi \in [0, 1]^3} \sigma(\mathbf{w}, z, y, \psi)$$

Optimality Condition: 
$$\begin{cases} T_1(z, y, \psi) = T_2(z, y, \psi) \\ P_1(z, y, \psi) = P_2(z, y, \psi) \\ g_1(z, y, \psi) = g_2(z, y, \psi) \\ z, y, \psi \in ]0, 1[^3 \end{cases}$$

Solution:  $(z^*, y^*, \psi^*)$

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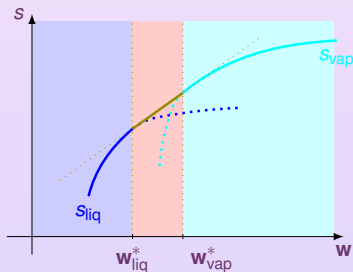
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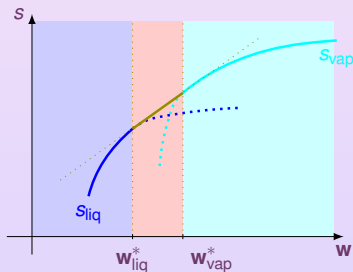
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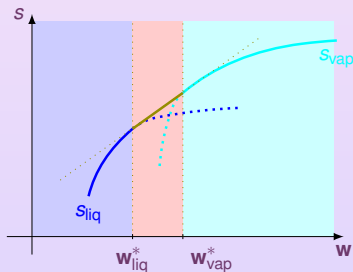
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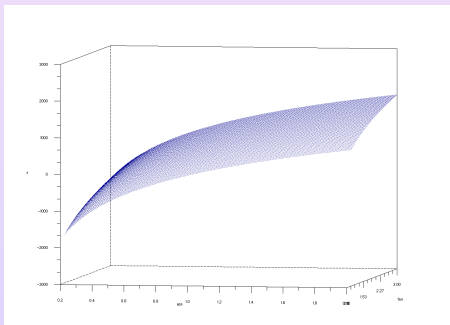
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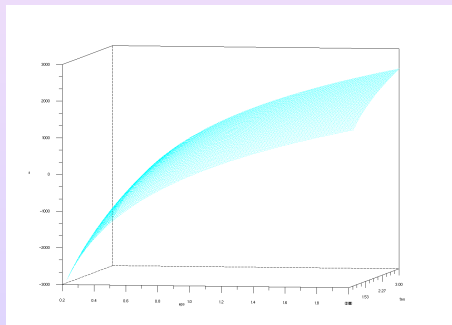
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# CONCAVE HULL WITH TWO PERFECT GASES

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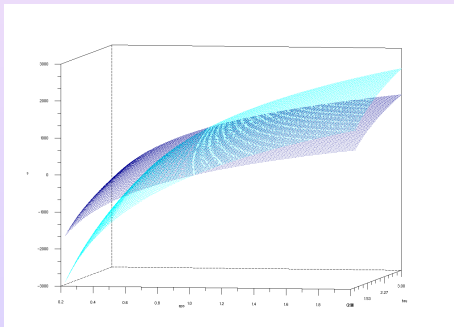


$$(\tau, \varepsilon) \mapsto s_{\text{vap}}$$



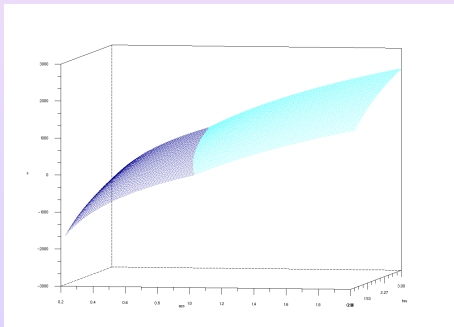
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$$(\tau, \varepsilon) \mapsto \max\{s_{\text{liq}}, s_{\text{vap}}\}$$



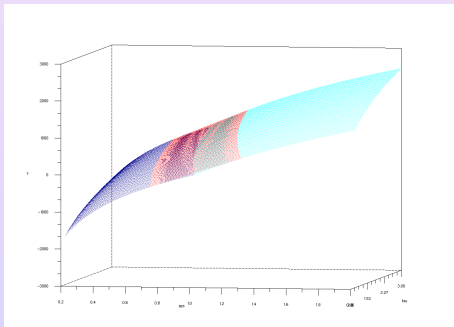
# CONCAVE HULL WITH TWO PERFECT GASES

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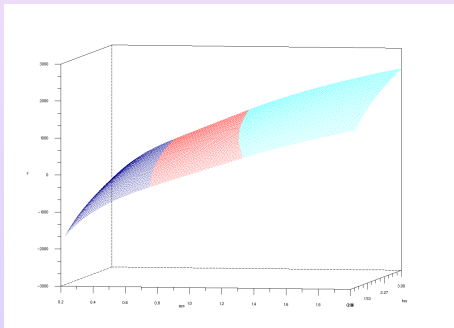
# CONCAVE HULL WITH TWO PERFECT GASES

$$(\tau, \varepsilon) \mapsto \text{co} \left\{ \max \{ s_{\text{liq}}, s_{\text{vap}} \} \right\}$$



# CONCAVE HULL WITH TWO PERFECT GASES

$$(\tau, \varepsilon) \mapsto \text{co} \left\{ \max \{ s_{\text{liq}}, s_{\text{vap}} \} \right\}$$



# FROM $\mathbf{w} \mapsto \mathbf{s}^{\text{eq}}$ TO $\mathbf{w} \mapsto P^{\text{eq}}$

For all  $\tilde{\mathbf{w}}$  fixed, we seek  $(\mathbf{w}_{\text{liq}}^*, \mathbf{w}_{\text{vap}}^*, y^*)$  as the solution of the system

$$\begin{cases} P_{\text{liq}}(\mathbf{w}_{\text{liq}}) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ T_{\text{liq}}(\mathbf{w}_{\text{liq}}) = T_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ g_{\text{liq}}(\mathbf{w}_{\text{liq}}) = g_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ \tilde{\mathbf{w}} = y\mathbf{w}_{\text{liq}} + (1-y)\mathbf{w}_{\text{vap}} \end{cases}$$

- if  $y^* \in ]0, 1[$  then  $\tilde{\mathbf{w}}$  is an **equilibrium mixture state**

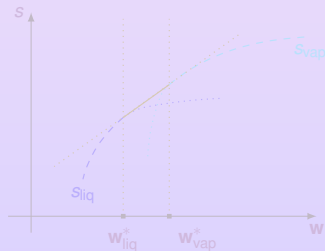
$$\mathbf{s}^{\text{eq}}(\tilde{\mathbf{w}}) = y^* \mathbf{s}_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) + (1-y^*) \mathbf{s}_{\text{vap}}(\mathbf{w}_{\text{vap}}^*),$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}^*);$$

- if the system has no solution or  $y^* \notin ]0, 1[$  then  $\tilde{\mathbf{w}}$  is a **monophasic pure state**

$$\mathbf{s}^{\text{eq}}(\tilde{\mathbf{w}}) = \max\{\mathbf{s}_{\text{liq}}(\tilde{\mathbf{w}}), \mathbf{s}_{\text{vap}}(\tilde{\mathbf{w}})\},$$

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# FROM $\mathbf{w} \mapsto \mathbf{s}^{\text{eq}}$ TO $\mathbf{w} \mapsto \mathbf{P}^{\text{eq}}$

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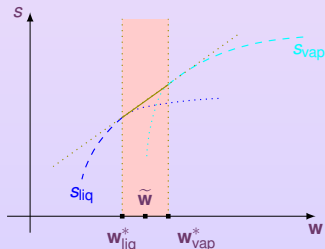
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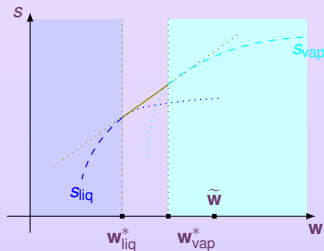
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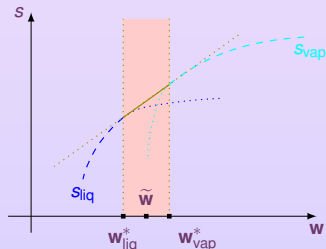
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# FROM $\mathbf{w} \mapsto S^{\text{eq}}$ TO $\mathbf{w} \mapsto P^{\text{eq}}$

For all  $\tilde{\mathbf{w}}$  fixed, we seek  $(\mathbf{w}_{\text{liq}}^*, \mathbf{w}_{\text{vap}}^*, y^*)$  as the solution of the system

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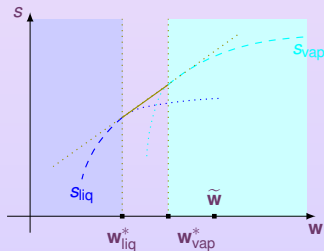
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# SUMMARY OF THE MODEL

$$\mathbf{w} \mapsto S^{\text{eq}}$$

$$\begin{cases} g_1(\mathbf{w}_1) = g_2(\mathbf{w}_2) \\ P_1(\mathbf{w}_1) = P_2(\mathbf{w}_2) \\ T_1(\mathbf{w}_1) = T_2(\mathbf{w}_2) \\ \mathbf{w} = y\mathbf{w}_1 + (1-y)\mathbf{w}_2 \end{cases}$$

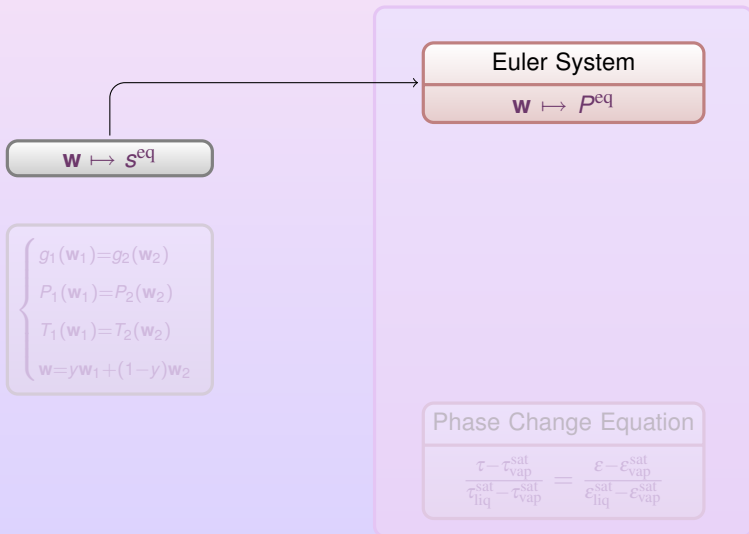
Euler System

$$\mathbf{w} \mapsto P^{\text{eq}}$$

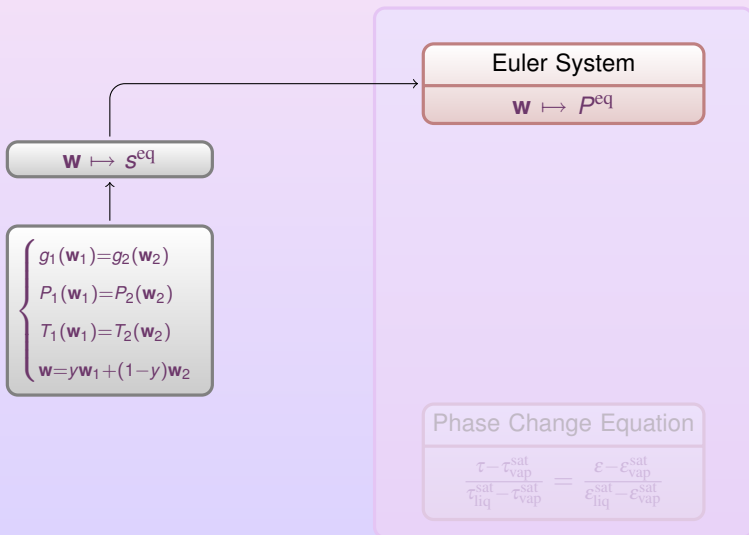
Phase Change Equation

$$\frac{\tau - \tau_{\text{vap}}^{\text{sat}}}{\tau_{\text{liq}}^{\text{sat}} - \tau_{\text{vap}}^{\text{sat}}} = \frac{\varepsilon - \varepsilon_{\text{vap}}^{\text{sat}}}{\varepsilon_{\text{liq}}^{\text{sat}} - \varepsilon_{\text{vap}}^{\text{sat}}}$$

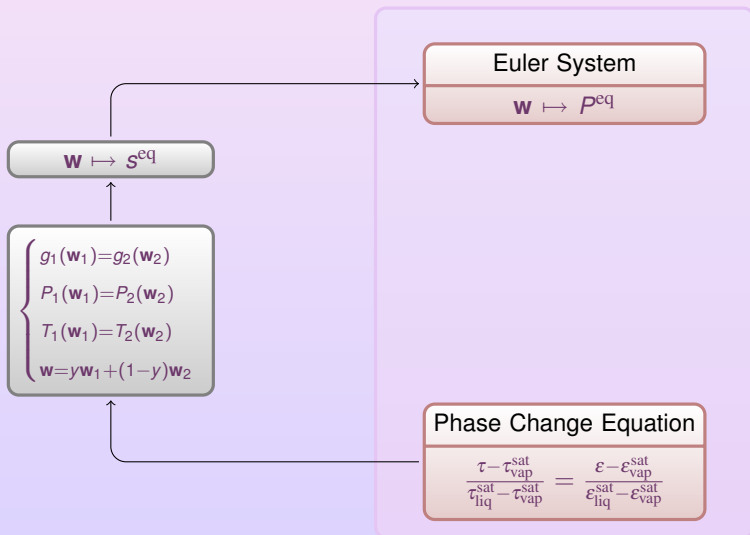
# SUMMARY OF THE MODEL



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# OUTLINE

## 1 Context

## 2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- **The Phase Change Equation**
- Conservation Laws

## 3 Numerical Approximation

- Numerical Method
- Numerical Examples

## 4 Conclusion

## ANALYTICAL EOS

 $(\tau, \varepsilon)$  fixed $(\tau_1, \varepsilon_1, \tau_2, \varepsilon_2, y)$  SOLUTION OF

$$\begin{cases} g_1(\tau_1, \varepsilon_1) = g_2(\tau_2, \varepsilon_2) \\ P_1(\tau_1, \varepsilon_1) = P_2(\tau_2, \varepsilon_2) \\ T_1(\tau_1, \varepsilon_1) = T_2(\tau_2, \varepsilon_2) \\ \tau = y\tau_1 + (1-y)\tau_2 \\ \varepsilon = y\varepsilon_1 + (1-y)\varepsilon_2 \end{cases}$$

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$$\begin{cases} g_1(P, T) = g_2(P, T) \\ \frac{\tau - \tau_2(P, T)}{\tau_1(P, T) - \tau_2(P, T)} = \frac{\varepsilon - \varepsilon_2(P, T)}{\varepsilon_1(P, T) - \varepsilon_2(P, T)} \end{cases}$$

$$T \mapsto P = P^{\text{sat}}(T) \approx P^{\text{sat}}(T)$$

 $T$  SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{where} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \stackrel{\text{def}}{=} \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(P^{\text{sat}}(T), T)$$

\* More examples

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\* New Example

## ANALYTICAL EOS

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\* next slide

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least square approximation

$$T \mapsto P = \hat{P}^{\text{sat}}(T) \approx P^{\text{sat}}(T)$$

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$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{where} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \stackrel{\text{def}}{=} \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(\hat{P}^{\text{sat}}(T), T)$$

▶ Water Example

# TABULATED EOS

$(\tau, \varepsilon)$  fixed

**T SOLUTION OF**

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{with} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \quad \text{tabulated}$$

$\gg$

$$\frac{\tau - \widehat{\tau}_2^{\text{sat}}(T)}{\widehat{\tau}_1^{\text{sat}}(T) - \widehat{\tau}_2^{\text{sat}}(T)} = \frac{\varepsilon - \widehat{\varepsilon}_2^{\text{sat}}(T)}{\widehat{\varepsilon}_1^{\text{sat}}(T) - \widehat{\varepsilon}_2^{\text{sat}}(T)} \quad \text{with} \quad \begin{pmatrix} \widehat{\tau} \\ \widehat{\varepsilon} \end{pmatrix}_{\alpha}^{\text{sat}}(T)$$

★ Water Examples



# TABULATED EOS

$(\tau, \varepsilon)$  fixed

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least square approximations

★ Water Examples

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least square approximations

► Water Examples

# PHASE CHANGE EQUATION: SUMMARY

## PHASE CHANGE EQUATION

$$\frac{\tau - \tau_{\text{vap}}^{\text{sat}}}{\tau_{\text{liq}}^{\text{sat}} - \tau_{\text{vap}}^{\text{sat}}} = \frac{\varepsilon - \varepsilon_{\text{vap}}^{\text{sat}}}{\varepsilon_{\text{liq}}^{\text{sat}} - \varepsilon_{\text{vap}}^{\text{sat}}}$$

with

$$T \mapsto \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) = \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(T, P^{\text{sat}}(T))$$

or

$$P \mapsto \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(P) = \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(T^{\text{sat}}(P), P)$$

# PHASE CHANGE EQUATION: SUMMARY

How to compute saturation functions  $\tau_\alpha^{\text{sat}}$  and  $\varepsilon_\alpha^{\text{sat}}$

- Analytical EOS: we compute the saturation functions  $\tau_\alpha^{\text{sat}}$  and  $\varepsilon_\alpha^{\text{sat}}$  by the **Coexistence Curve**:

- Exact:  $T \mapsto P^{\text{sat}}(T)$  or  $P \mapsto T^{\text{sat}}(P)$

$$\begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha^{\text{sat}}(T) = \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha(T, P^{\text{sat}}(T)) \quad \text{e.g. Perfect Gases}$$

$$\begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha^{\text{sat}}(P) = \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha(T^{\text{sat}}(P), P) \quad \text{e.g. Simplified Stiffened Gases}$$

- Approximated:  $T \mapsto \hat{P}^{\text{sat}}(T) \approx P^{\text{sat}}(T)$

$$\begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha^{\text{sat}}(T) \approx \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha(T, \hat{P}^{\text{sat}}(T)) \quad \text{e.g. General Stiffened Gases}$$

- Tabulated EOS: the saturation functions  $\tau_\alpha^{\text{sat}}$  and  $\varepsilon_\alpha^{\text{sat}}$  **are given** by experiments and we pose

$$\begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha^{\text{sat}}(T \text{ or } P) \approx \begin{pmatrix} \hat{\tau} \\ \hat{\varepsilon} \end{pmatrix}_\alpha^{\text{sat}}(T \text{ or } P)$$

# OUTLINE

## 1 Context

## 2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- **Conservation Laws**

## 3 Numerical Approximation

- Numerical Method
- Numerical Examples

## 4 Conclusion

# DYNAMIC LIQUID-VAPOR PHASE CHANGE

## EULER SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t \left( \rho \left( \frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \right) + \operatorname{div} \left( \rho \left( \frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \mathbf{u} + P^{\text{eq}} \mathbf{u} \right) = 0 \end{cases} \quad \text{with} \quad P^{\text{eq}} \stackrel{\text{def}}{=} \frac{S_\tau^{\text{eq}}}{S_\varepsilon^{\text{eq}}}.$$

## PROPERTIES

If  $\tau_1^* \neq \tau_2^*$  and  $\varepsilon_1^* \neq \varepsilon_2^*$  (first order phase transition) then

$$\textcircled{1} c(w) > 0, \quad \textcircled{2} s_{\tau\varepsilon}^{\text{eq}}(w) > 0$$

- ① Euler system: strict hyperbolicity ( $\neq$  p-system),
- ② Riemann problem: multitude of entropy (Lax) solutions [R. MENIKOFF, B. J. PLOHR], uniqueness of Liu solution.

# DYNAMIC LIQUID-VAPOR PHASE CHANGE



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# OUTLINE

## 1 Context

## 2 Model

- Equation of State WITHOUT Phase Change
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# HOW TO SIMULATE THE “LIU SOLUTION”

- Exact Riemann Solver  
like [A. VOSS] for Van der Waals EOS
- Viscous Solver (the Liu solution is the only solution that has a viscous profile)  
like [S. JAOUEN] for Perfect Gas EOS with  $c_{V\text{liq}} = c_{V\text{vap}}$
- Solver(s) based on **Relaxation Approach**  
[F. COQUEL, B. PERTHAME],  
[Th. BARBERON, Ph. HELLUY],  
[Ph. HELLUY, N. SEGUIN],  
[F. COQUEL, F. CARO, D. JAMET, S. KOKH],  
[R. ABGRALL, R. SAUREL],  
[V. GUILLEMAUD, J.-M. HÉRARD, S. KOKH],  
...

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# RELAXATION APPROACH

$$\partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

# RELAXATION APPROACH

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu} \mathbf{R}(\mathbf{V}) \quad \xrightarrow[\mu \rightarrow 0]{\text{Formally}} \quad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

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## HOW TO BUILD THE AUGMENTED SYSTEM

### ● Lagrangian:

$$\mathcal{L}(\rho, \mathbf{u}, \sigma, y, z, \psi) \stackrel{\text{def}}{=} \rho \left( \frac{|\mathbf{u}|^2}{2} - \varepsilon(\rho, \sigma, y, z, \psi) \right)$$

### Action:

$$\mathcal{A}(\mathbf{v}) \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \int_{\widehat{\Omega}(t; \mathbf{v})} \mathcal{L}(\widehat{\rho}, \widehat{\rho \mathbf{u}}, \widehat{\sigma}, \widehat{y}, \widehat{z}, \widehat{\psi})(\widehat{\mathbf{x}}, t; \mathbf{v}) \, d\widehat{\mathbf{x}} \, dt$$

Minimization of the Action:  $\frac{d\mathcal{A}}{d\mathbf{v}}(\mathbf{v} = 0) = 0$

### ● Energy: $\varepsilon \stackrel{\text{def}}{=} \sum_{\alpha} y_{\alpha} \varepsilon_{\alpha} \left( \frac{z_{\alpha}}{y_{\alpha}} \frac{1}{\rho}, \frac{\psi_{\alpha}}{y_{\alpha}} \sigma \right)$

### ● Positive Entropy Production: $D_t \sigma \geq 0$

## EQUILIBRIUM SYSTEM

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$$P(\rho, \varepsilon, z, y, \psi) = \frac{\sigma_r}{\sigma_e}$$

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In the interface

$$\begin{cases} \partial_t z + \mathbf{u} \cdot \operatorname{grad} z = \\ \partial_t y + \mathbf{u} \cdot \operatorname{grad} y = \\ \partial_t \psi + \mathbf{u} \cdot \operatorname{grad} \psi = \end{cases}$$

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Formally  
 $\mu_j \rightarrow 0$

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NOTE: we can replace an EDP by an algebraic closure, for example

~~$$\partial_t \psi + \mathbf{u} \cdot \operatorname{grad} \psi = \frac{1}{\mu_\psi} \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \varepsilon \rightsquigarrow T_1 = T_2.$$~~

# NUMERICAL SCHEME

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V}) + \frac{1}{\mu} \mathbf{R}(\mathbf{V})$$

$$\mathbf{V}_i^n \longrightarrow \mathbf{V}_i^{n+1}$$

$$\textcircled{1} \mu_j \rightarrow +\infty$$



$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V})$$

Aug. System:	5-eq. Iso-T
Num. Scheme:	op. splitting
$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = 0$	[G. ALLAIRE and all.]
$\mathbf{S}(\mathbf{V}) = \mathbf{S}_{\text{ad}}(\mathbf{V}) + \mathbf{S}_{\text{trans}}(\mathbf{V}) + \mathbf{S}_T(\mathbf{V})$	
$\partial_t \mathbf{V} = \mathbf{S}_{\text{ad}}(\mathbf{V})$	[J. U. BRACKBILL and all.]
$\partial_t \mathbf{V} = \mathbf{S}_{\text{trans}}(\mathbf{V})$	2D implicit
$\partial_t \mathbf{V} = \mathbf{S}_T(\mathbf{V})$	Euler

$$\textcircled{2} \mu_j \rightarrow 0$$



$$\mathbf{R}(\mathbf{V}) = 0$$

update fractions  
( $y, z, \psi$ )  
solving the  
Phase Change  
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# NUMERICAL SCHEME

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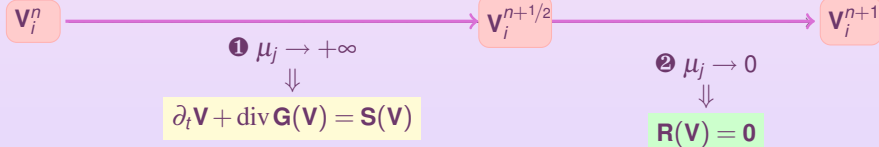
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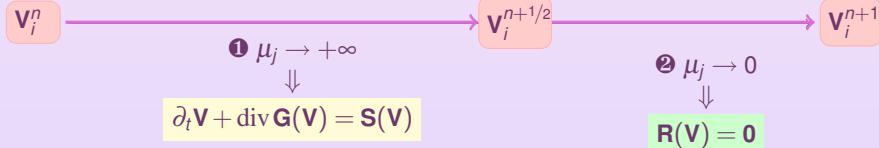


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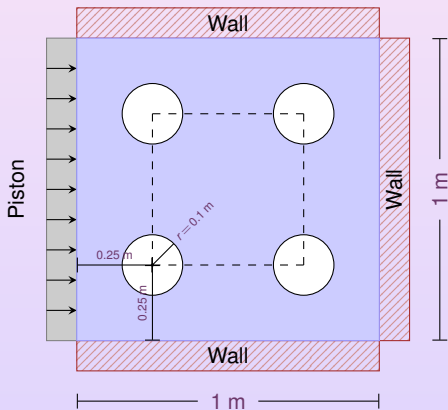
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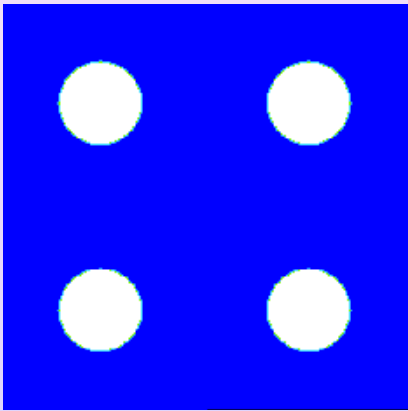
# COMPRESSION OF VAPOR BUBBLES



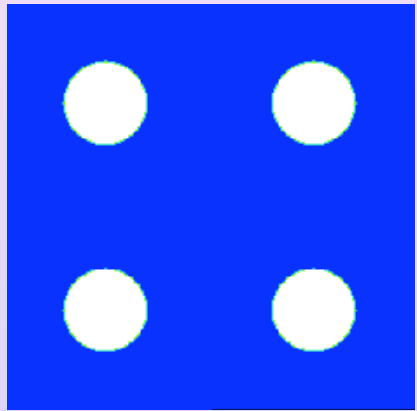
Compression of 4 Vapor Bubbles involving two Stiffened Gases for water and steam. The piston moves towards right at constant speed  $u_p = 30$  m/s.

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



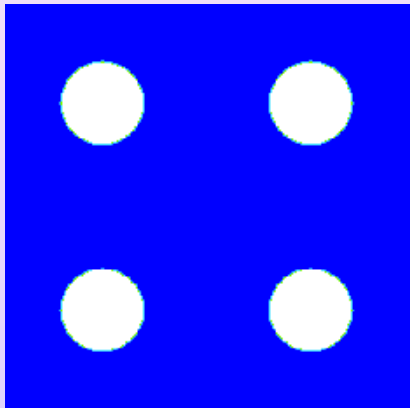
◀ Geometry

▶ Play

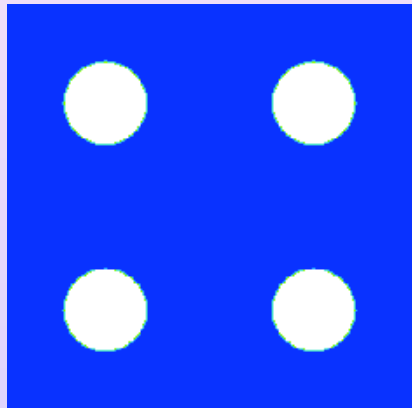
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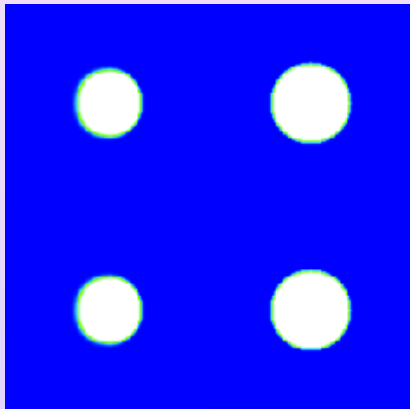
◀ Geometry

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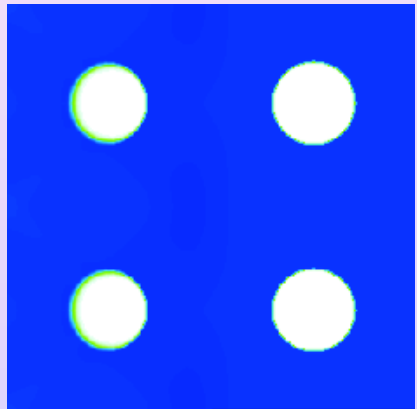
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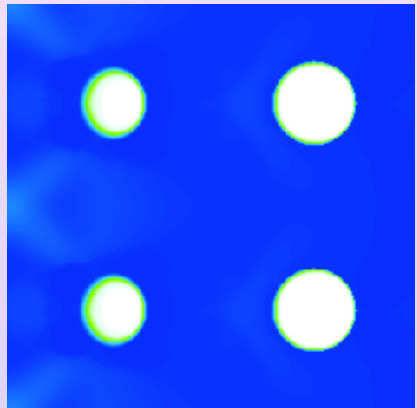
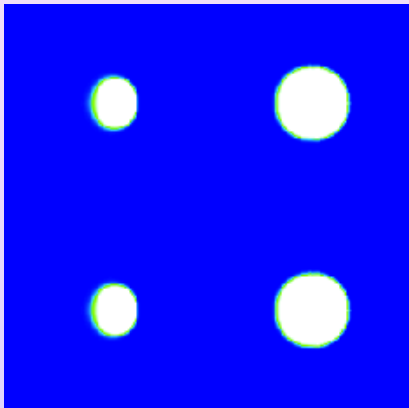
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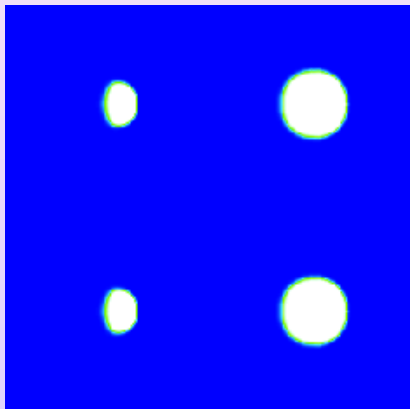
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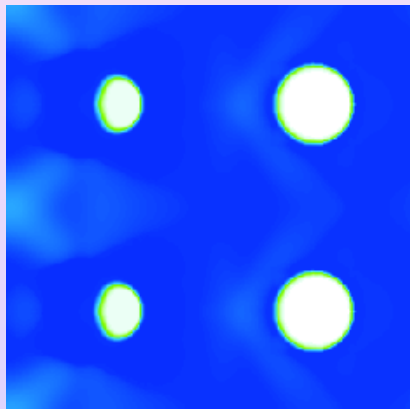
◀ Geometry ▶ Play ▶▶ Skip

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Mass Fraction  $y$



Density  $\rho$



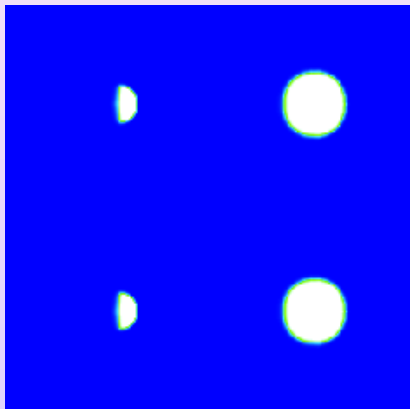
◀ Geometry

▶ Play

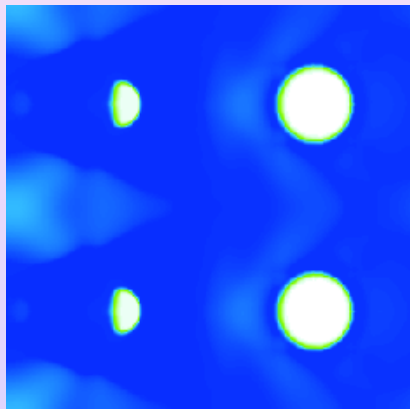
▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



◀ Geometry

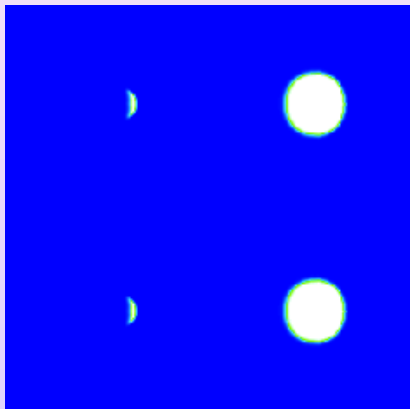
▶ Play

▶▶ Skip

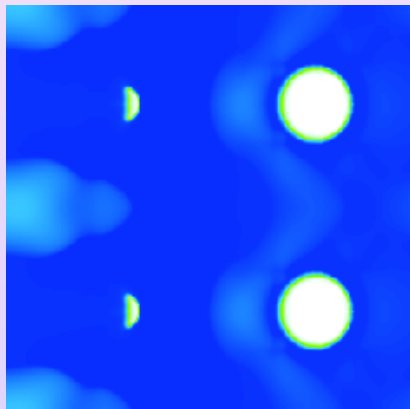


# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



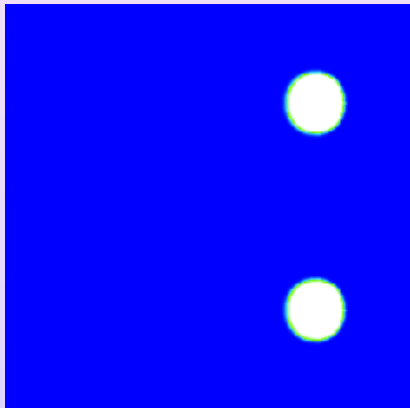
◀ Geometry

▶ Play

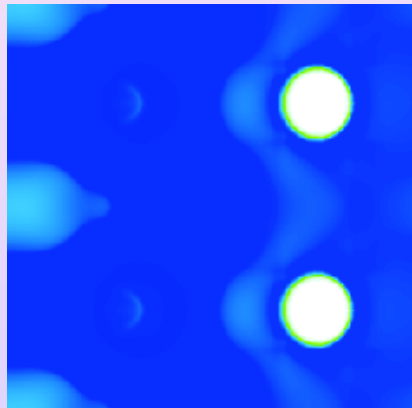
▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



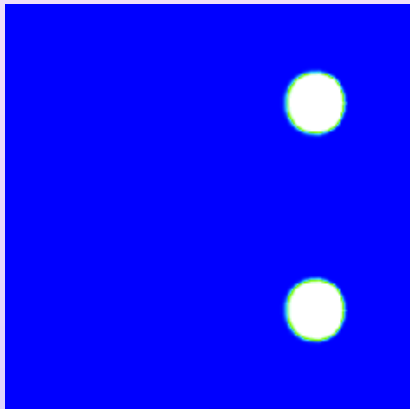
◀ Geometry

▶ Play

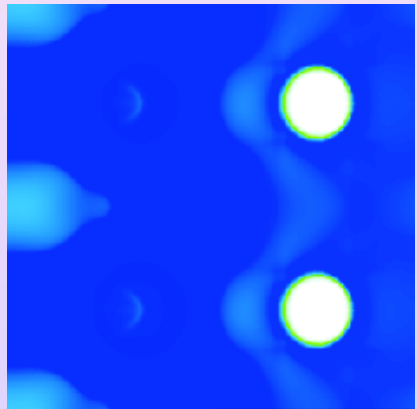
▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



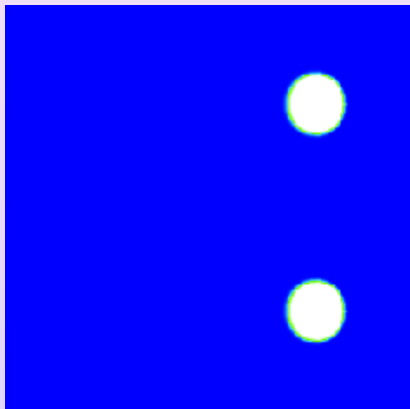
◀ Geometry

▶ Play

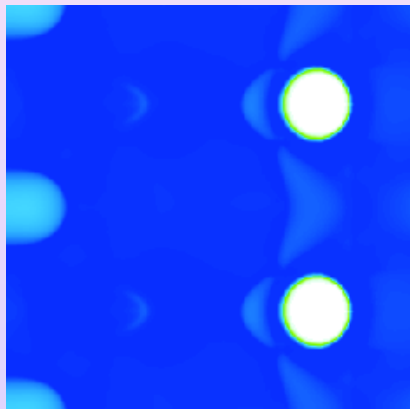
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# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



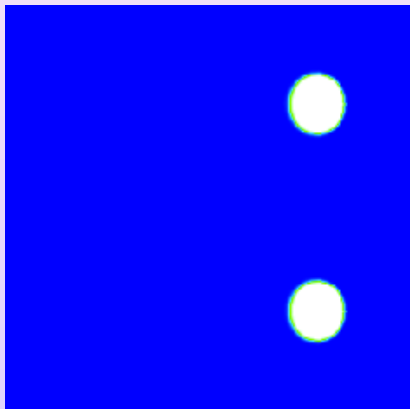
◀ Geometry

▶ Play

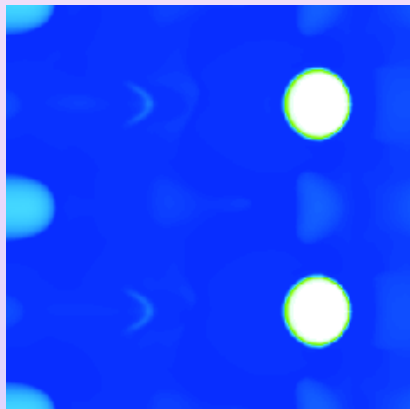
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# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



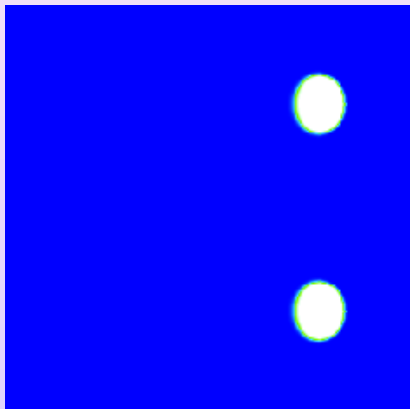
◀ Geometry

▶ Play

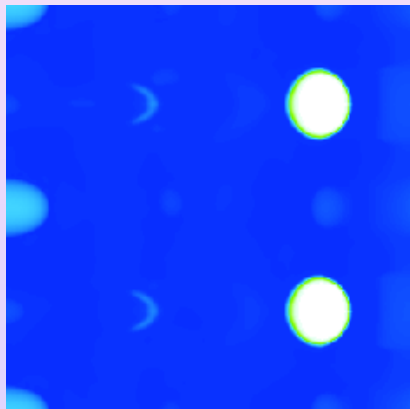
▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



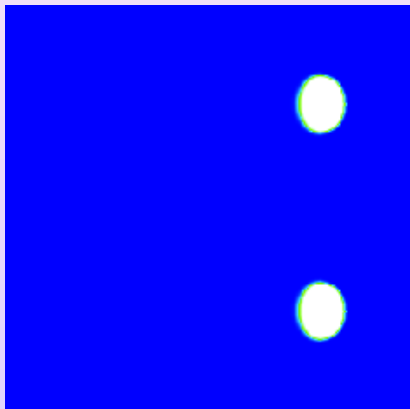
◀ Geometry

▶ Play

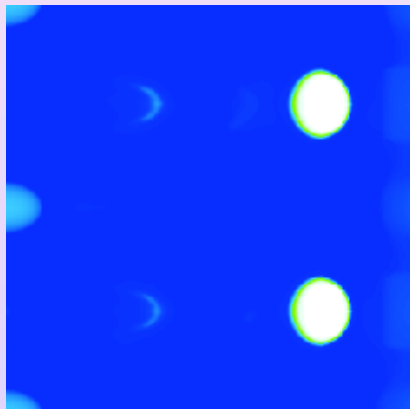
▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



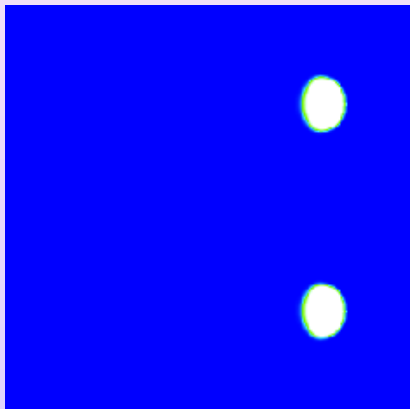
◀ Geometry

▶ Play

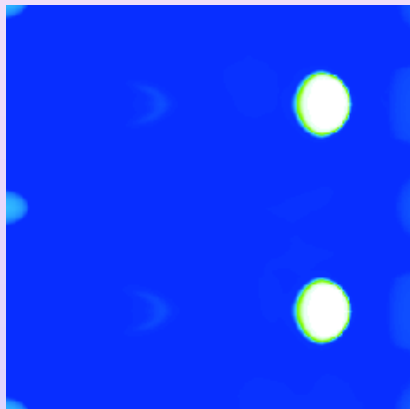
▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



◀ Geometry

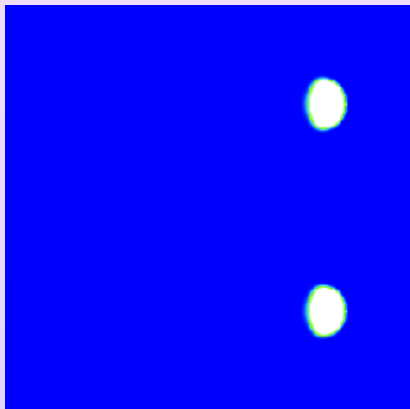
▶ Play

▶▶ Skip

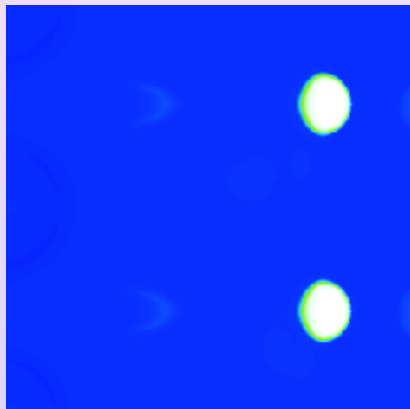


# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



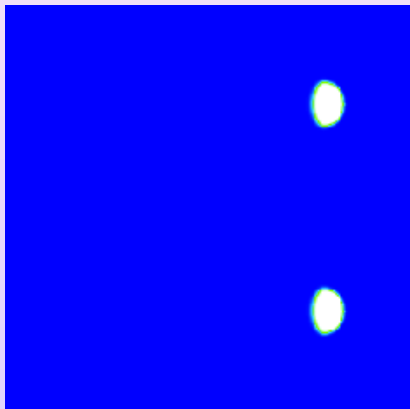
◀ Geometry

▶ Play

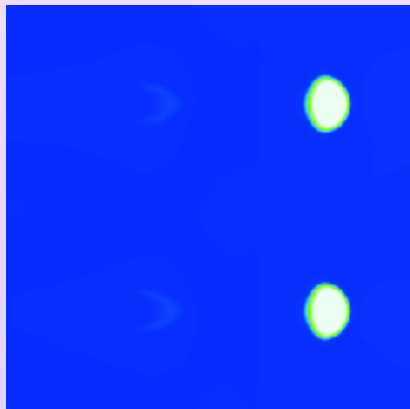
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# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



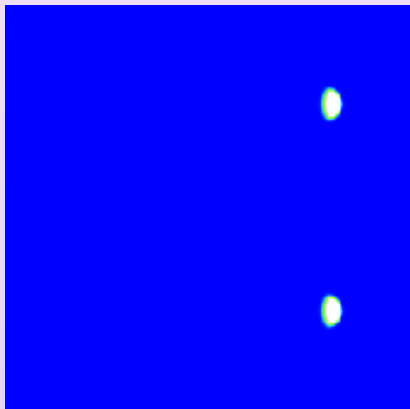
◀ Geometry

▶ Play

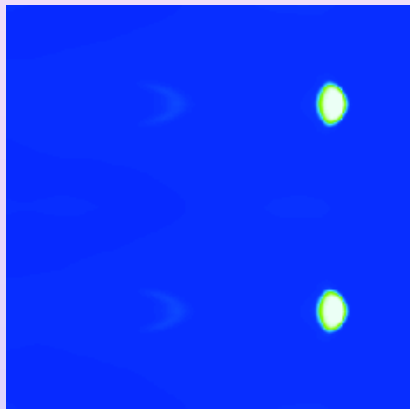
▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



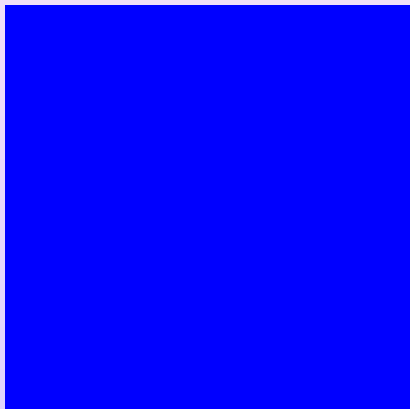
◀ Geometry

▶ Play

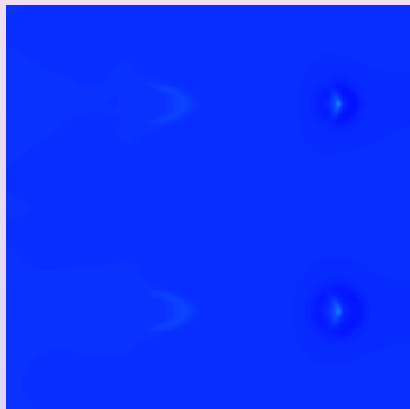
▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



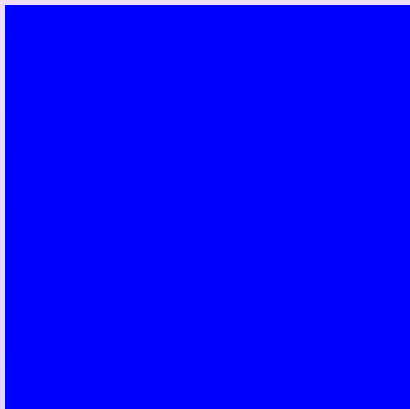
◀ Geometry

▶ Play

▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



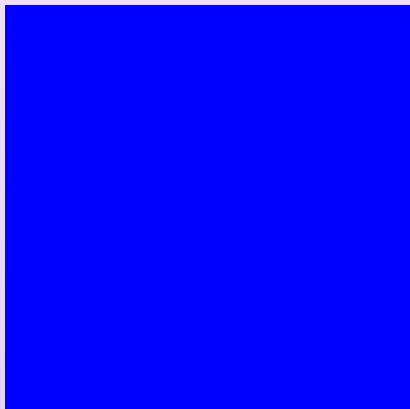
◀ Geometry

▶ Play

▶▶ Skip

# COMPRESSION OF VAPOR BUBBLES

Mass Fraction  $y$



Density  $\rho$



◀ Geometry

▶ Play

▶▶ Skip

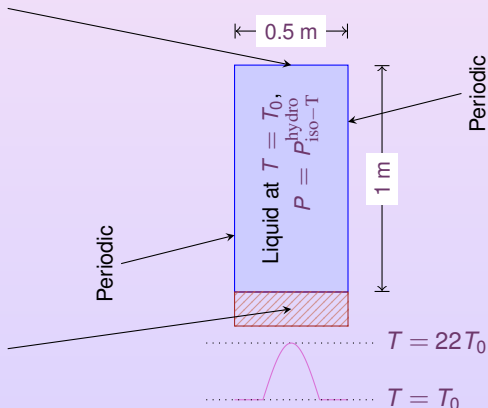
# NUCLEATING BUBBLE

Pressure and  
temperature  
imposed

$$P = P^{\text{ref}} > P^{\text{sat}}(T_0),$$

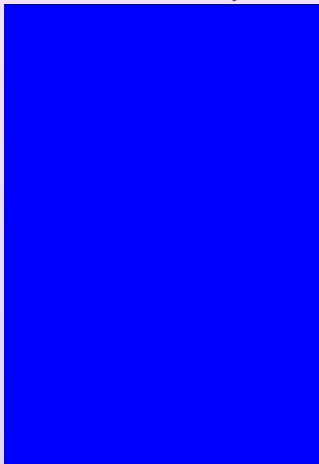
$$T = T_0$$

Wall,  
temperature imposed

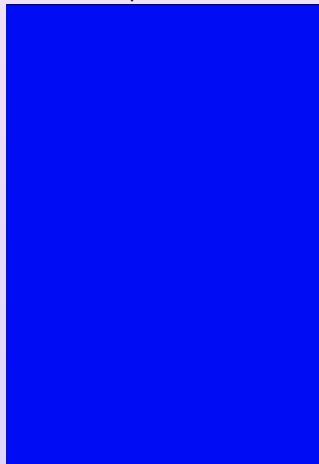


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

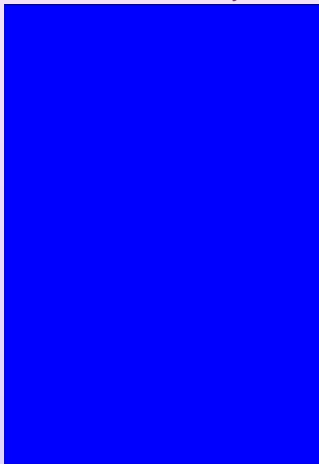
▶ Play

▶▶ Skip

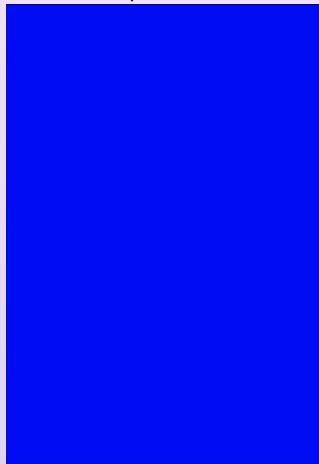


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

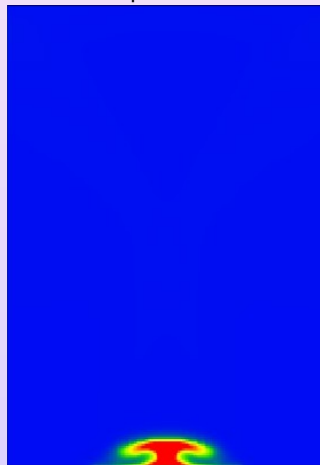
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



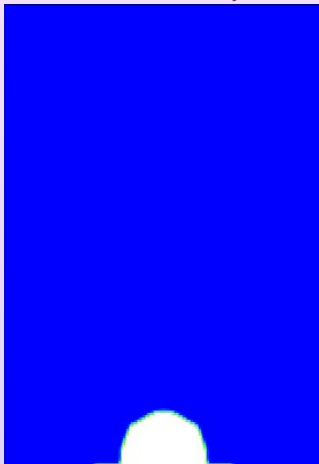
◀ Geometry

▶ Play

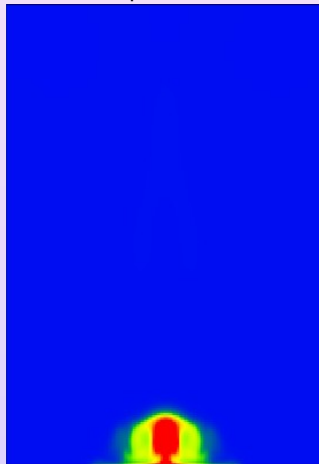
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



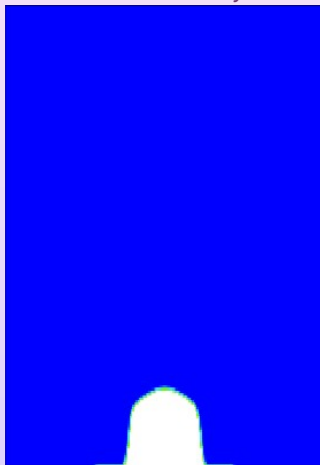
◀ Geometry

▶ Play

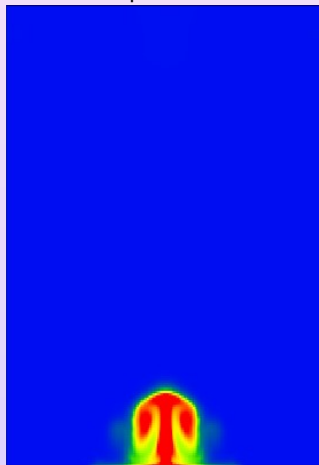
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



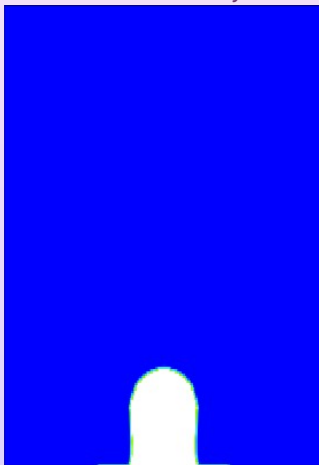
◀ Geometry

▶ Play

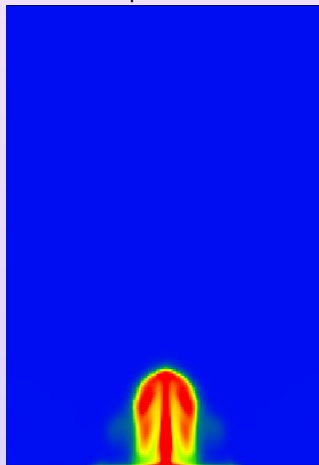
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



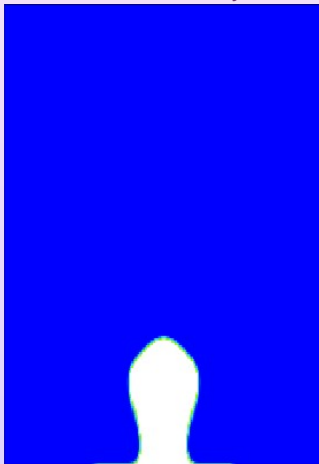
◀ Geometry

▶ Play

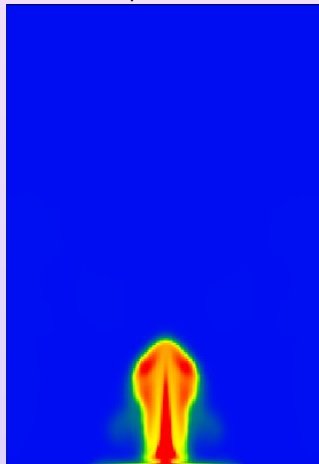
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



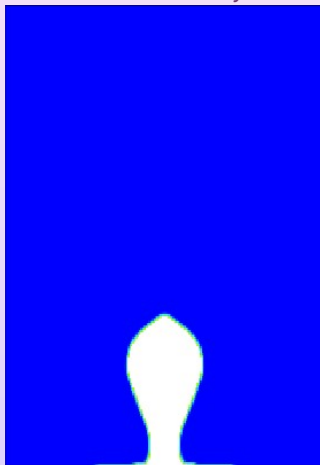
◀ Geometry

▶ Play

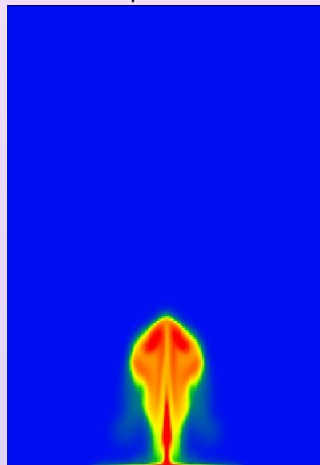
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



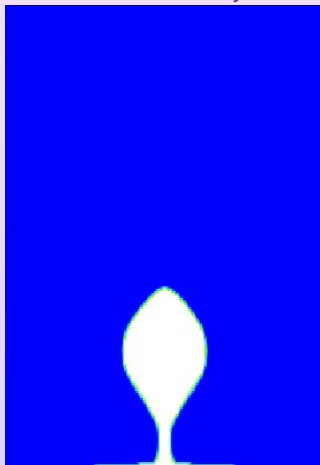
◀ Geometry

▶ Play

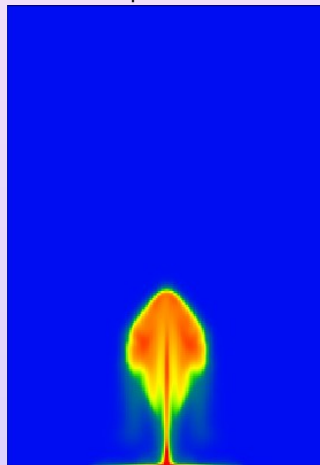
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

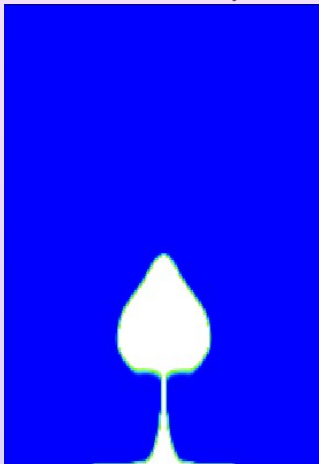
▶ Play

▶▶ Skip

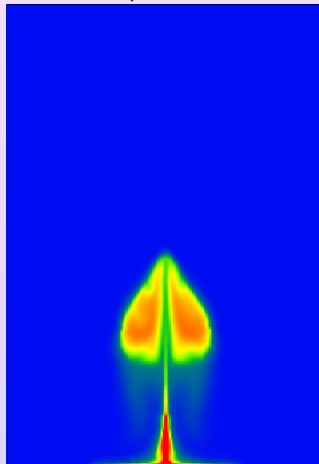


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



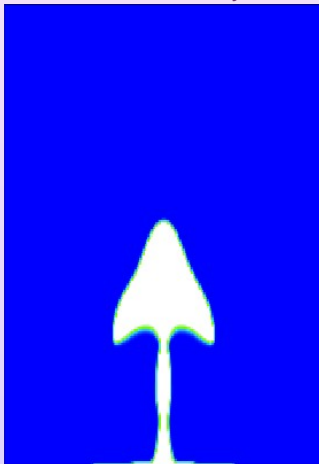
◀ Geometry

▶ Play

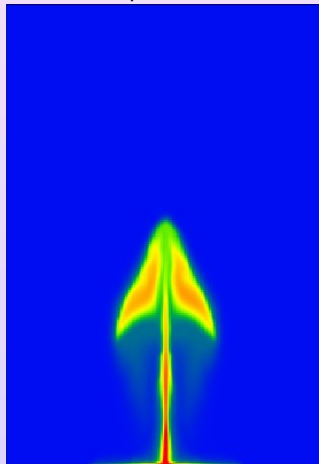
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



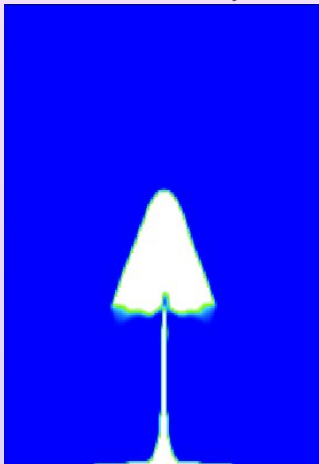
◀ Geometry

▶ Play

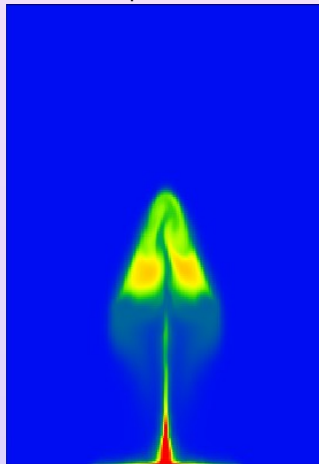
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



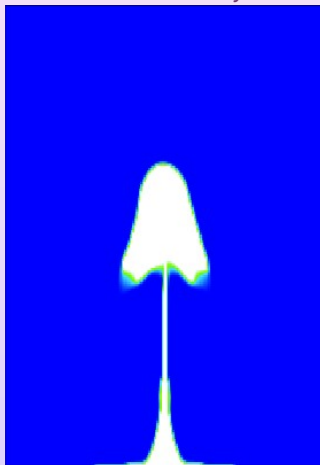
◀ Geometry

▶ Play

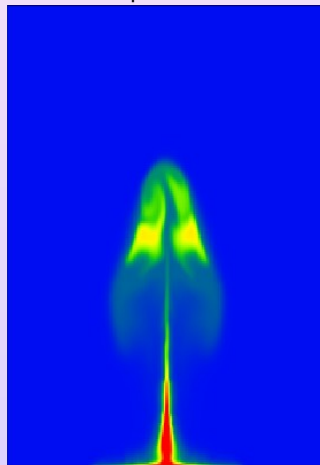
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



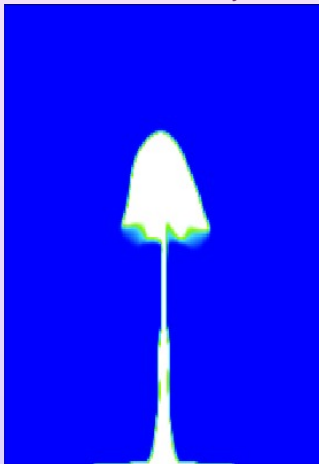
◀ Geometry

▶ Play

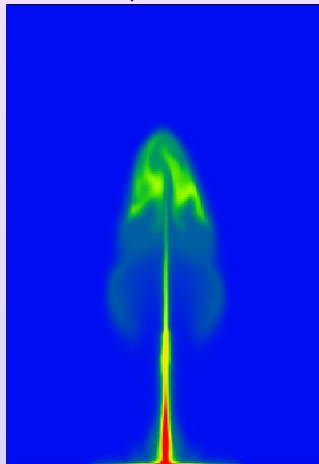
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



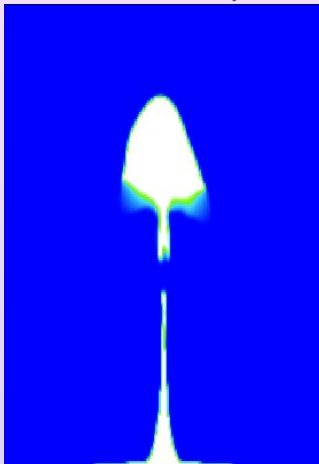
◀ Geometry

▶ Play

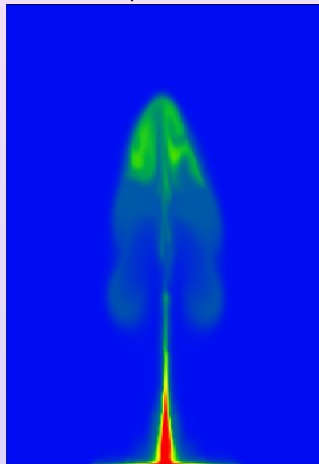
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



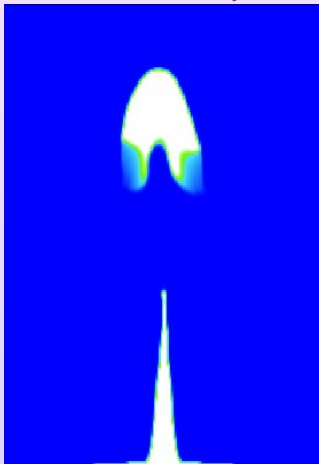
◀ Geometry

▶ Play

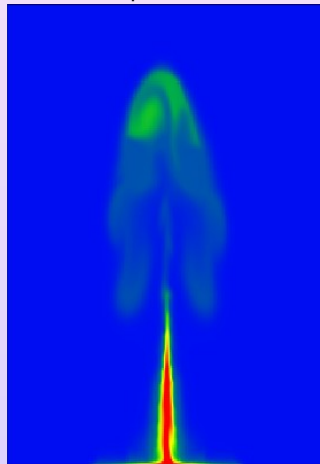
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



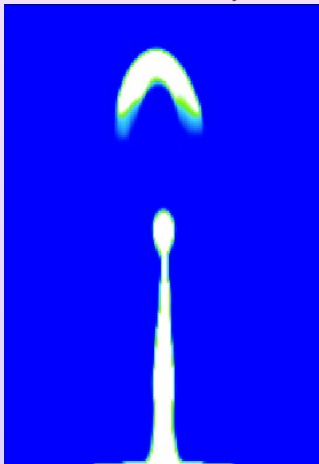
◀ Geometry

▶ Play

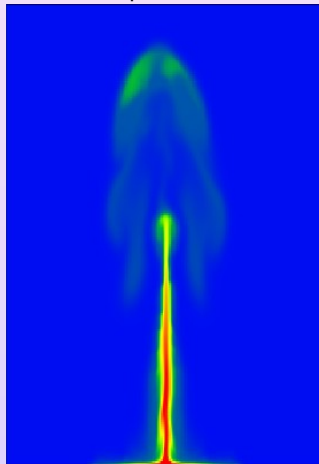
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

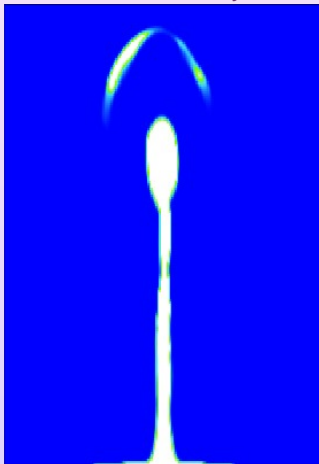
▶ Play

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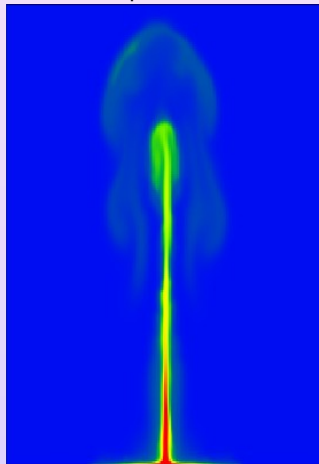


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



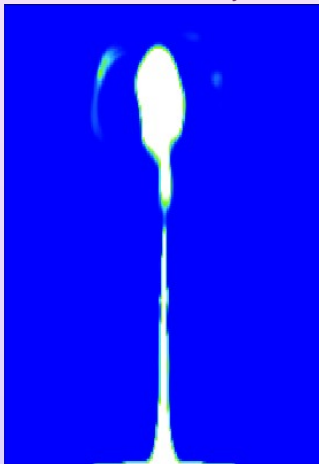
◀ Geometry

▶ Play

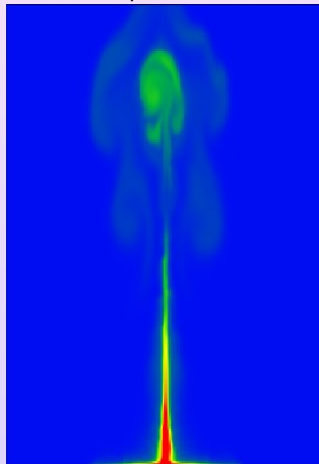
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



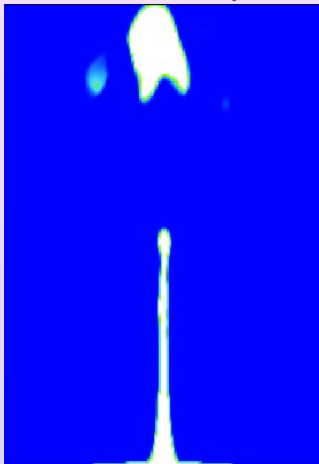
◀ Geometry

▶ Play

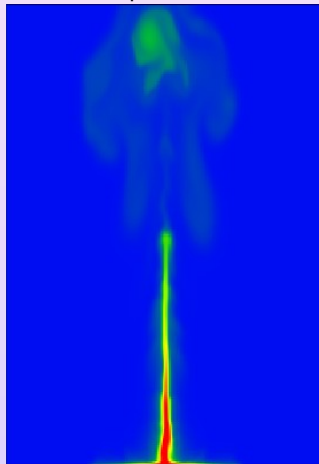
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



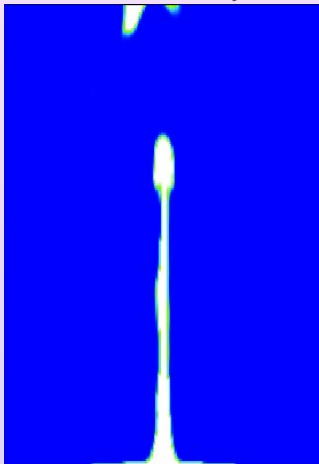
◀ Geometry

▶ Play

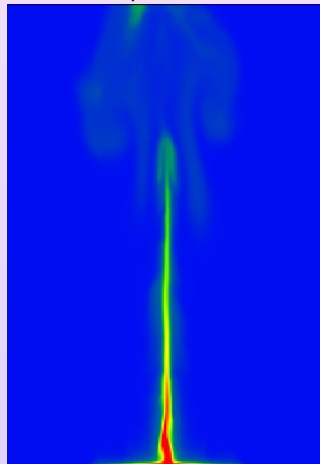
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



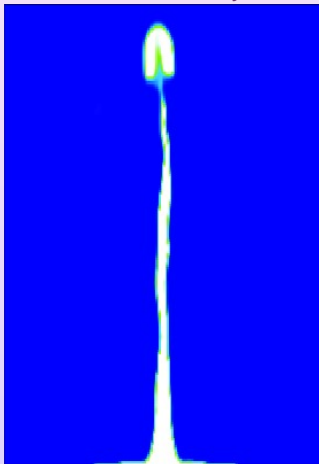
◀ Geometry

▶ Play

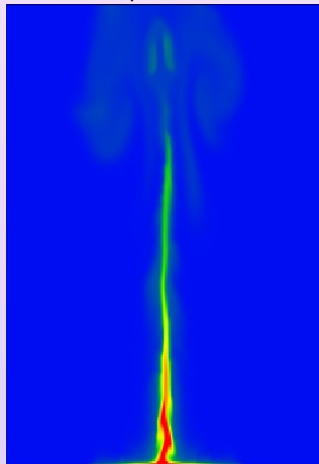
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



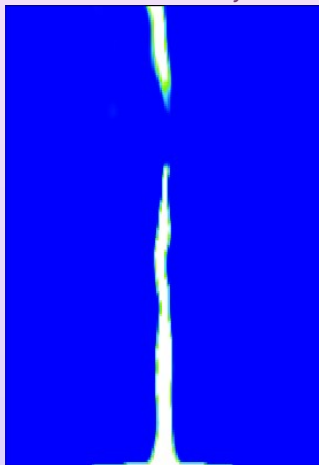
◀ Geometry

▶ Play

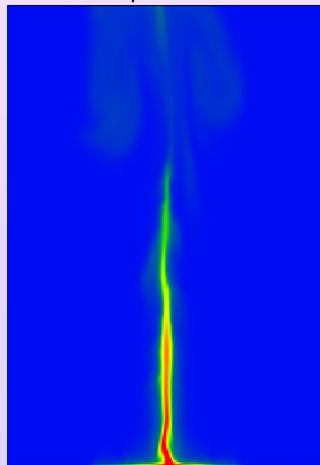
▶▶ Skip

# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



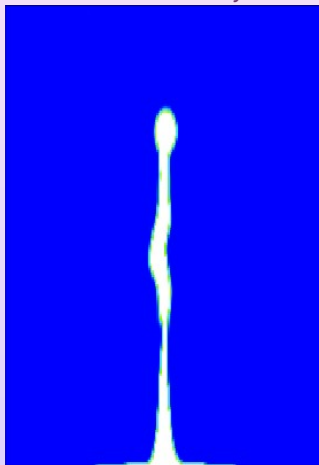
◀ Geometry

▶ Play

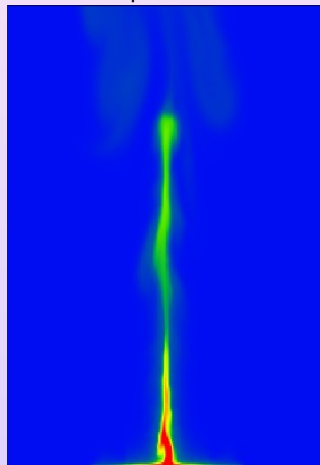
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



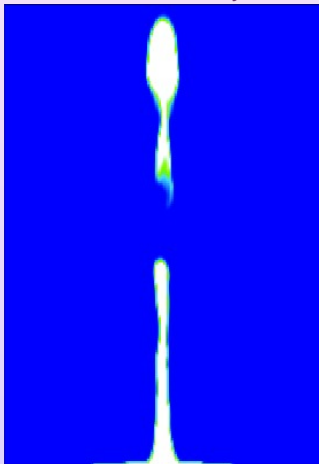
◀ Geometry

▶ Play

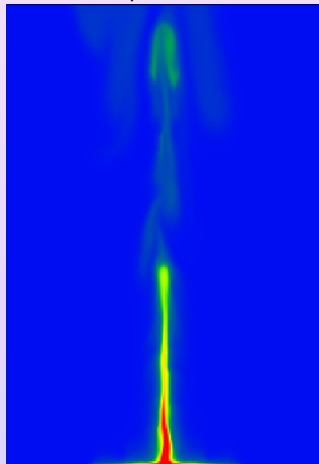
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

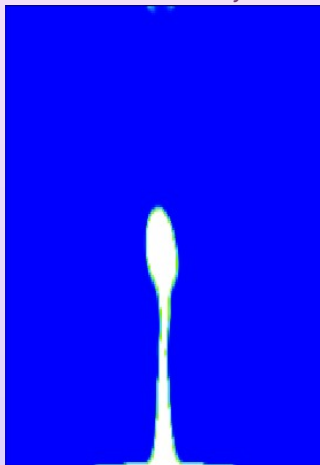
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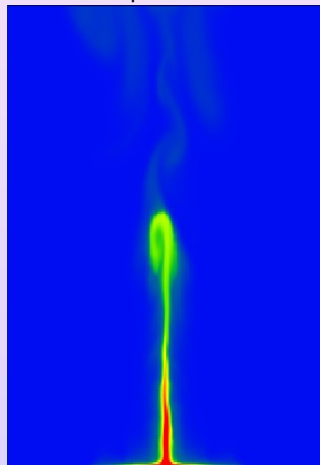


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



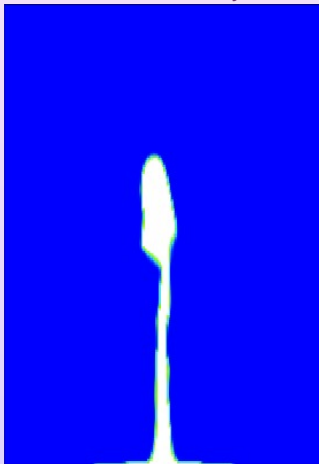
◀ Geometry

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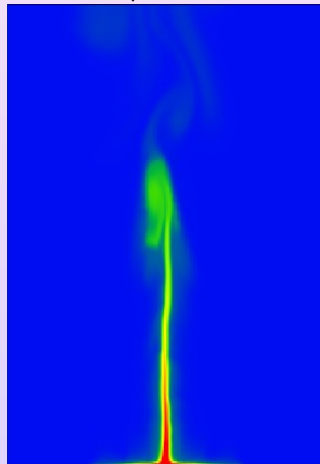
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



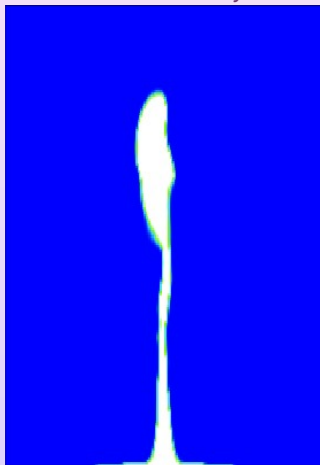
◀ Geometry

▶ Play

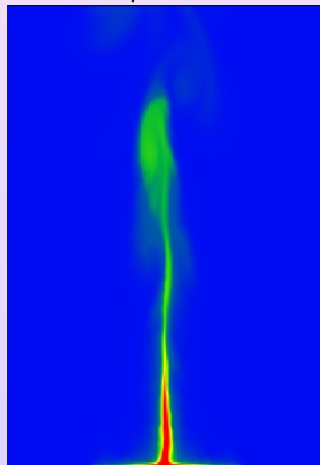
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



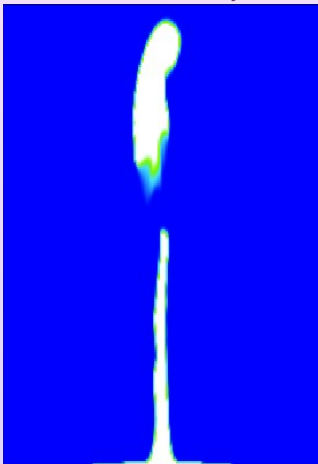
◀ Geometry

▶ Play

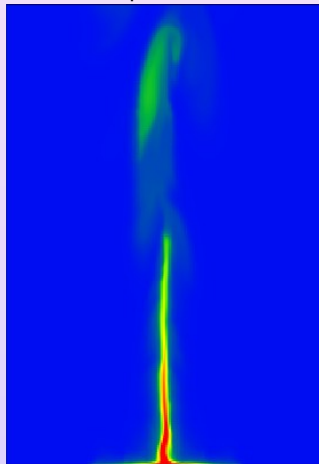
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



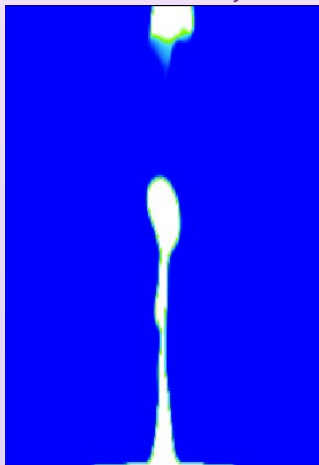
◀ Geometry

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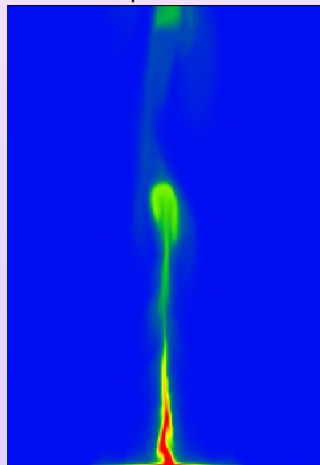
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



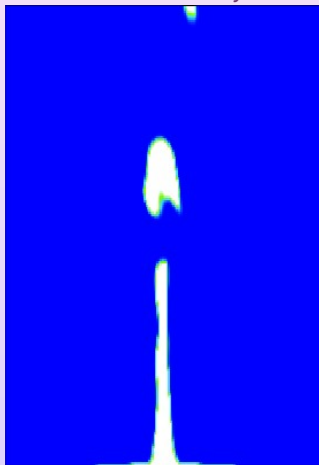
◀ Geometry

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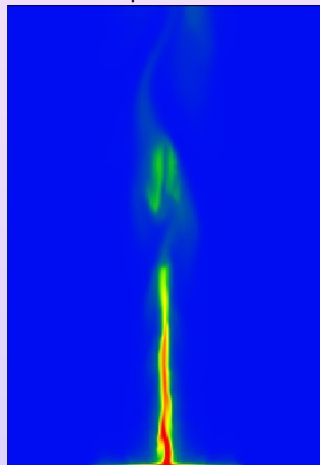
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



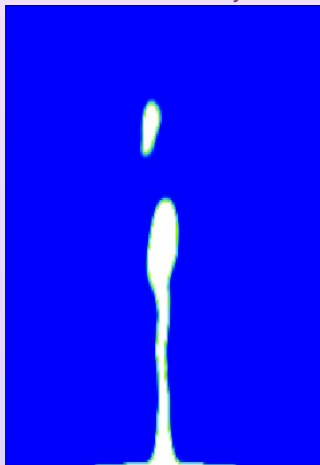
◀ Geometry

▶ Play

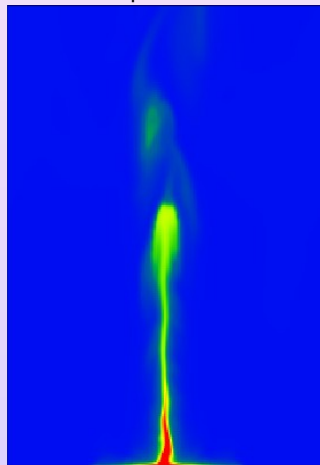
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



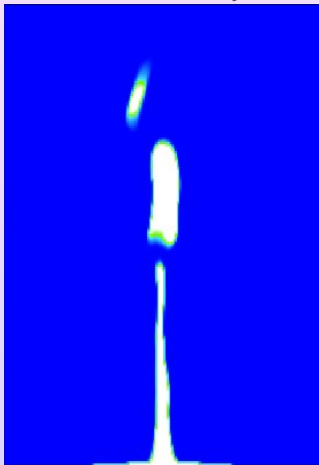
◀ Geometry

▶ Play

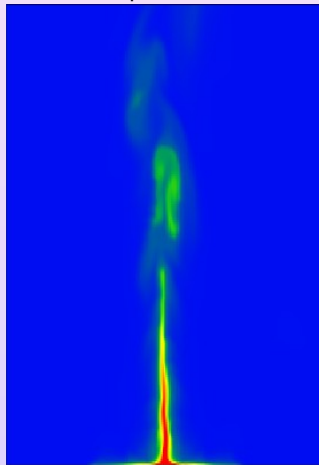
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

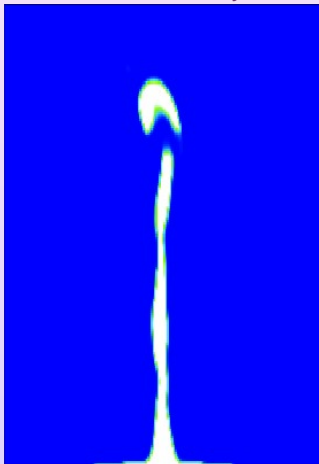
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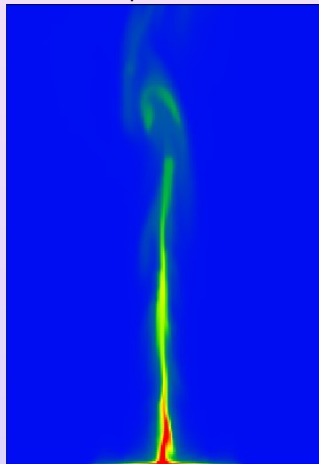


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



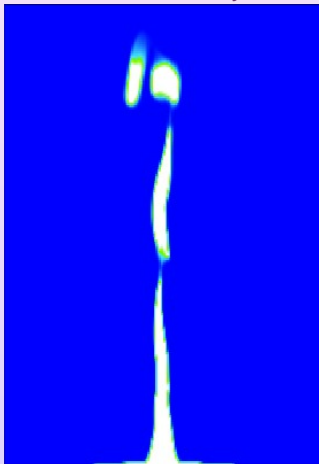
◀ Geometry

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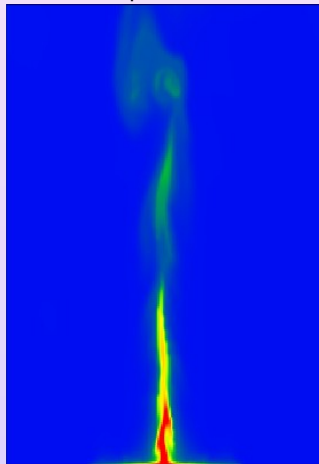
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



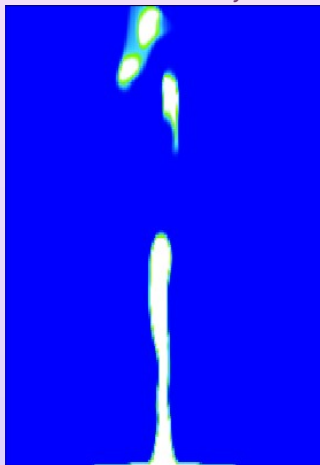
◀ Geometry

▶ Play

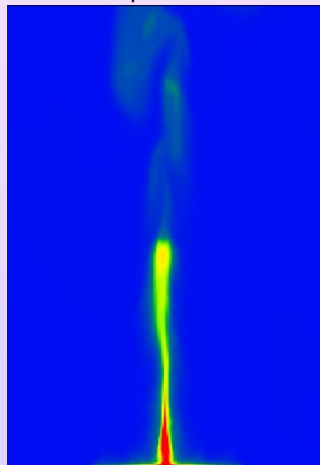
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



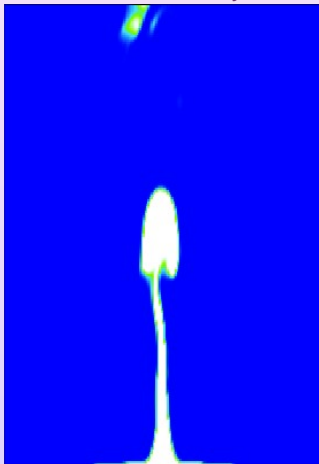
◀ Geometry

▶ Play

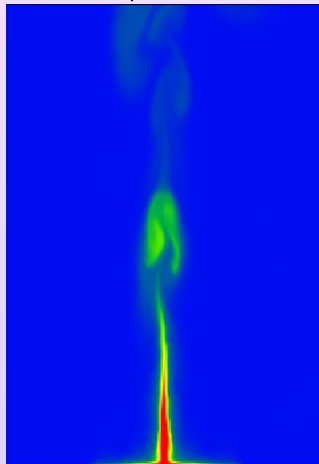
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



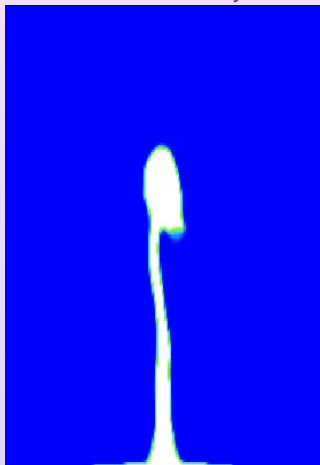
◀ Geometry

▶ Play

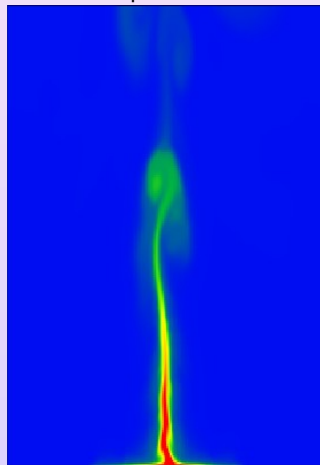
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



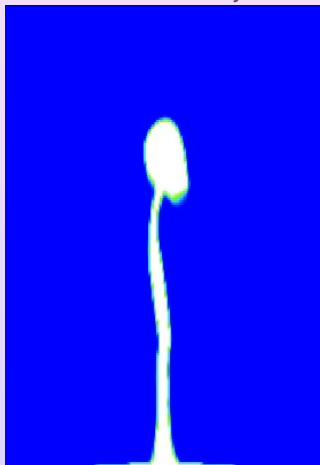
◀ Geometry

▶ Play

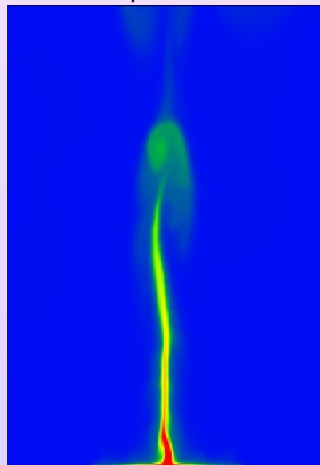
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



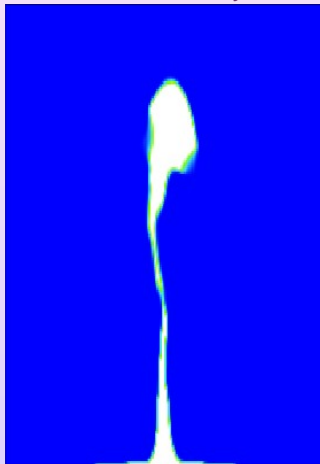
◀ Geometry

▶ Play

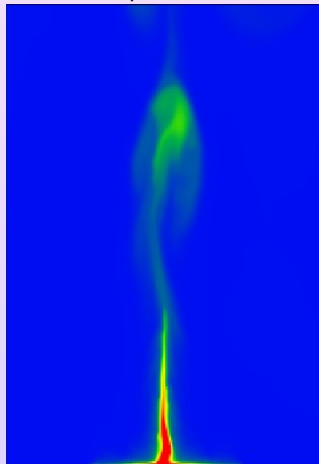
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



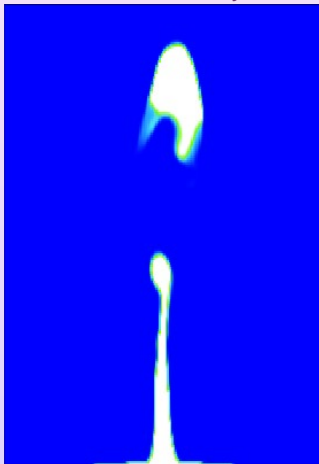
◀ Geometry

▶ Play

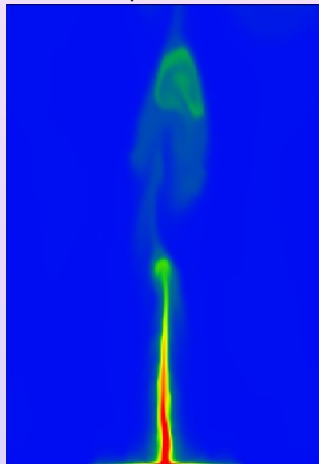
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

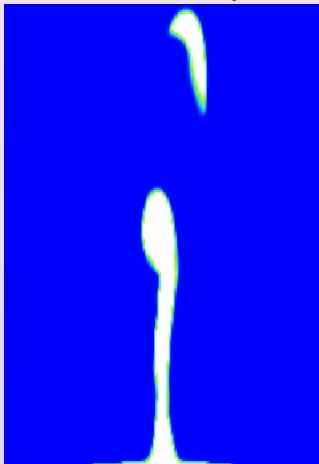
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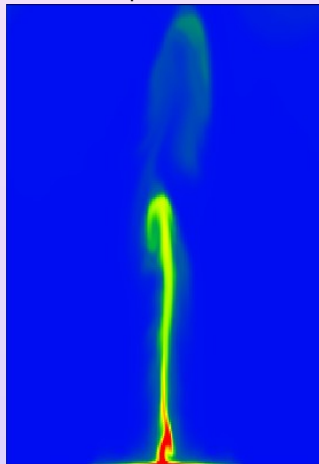


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



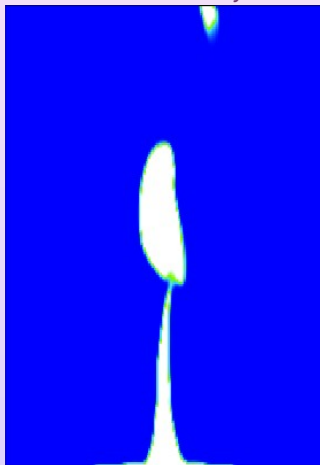
◀ Geometry

▶ Play

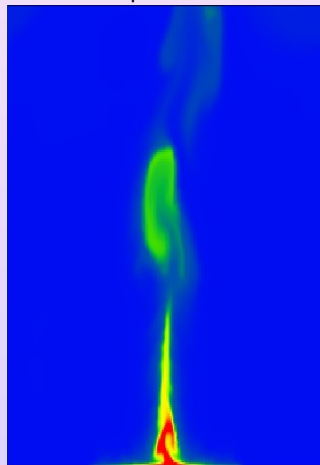
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



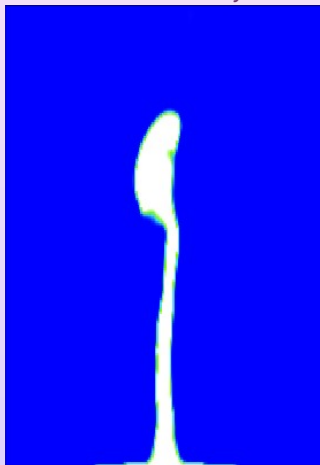
◀ Geometry

▶ Play

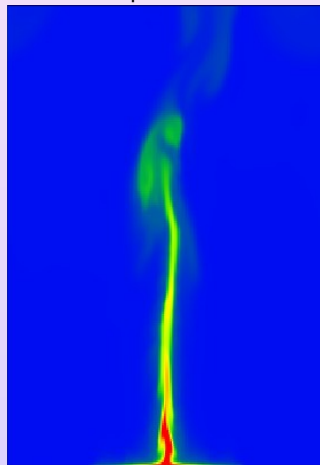
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



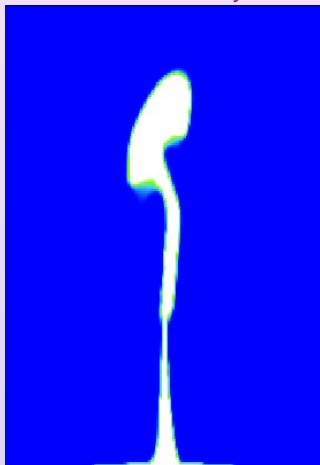
◀ Geometry

▶ Play

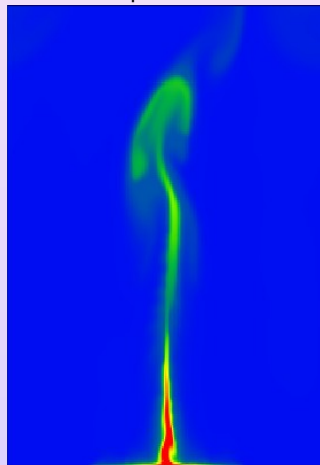
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



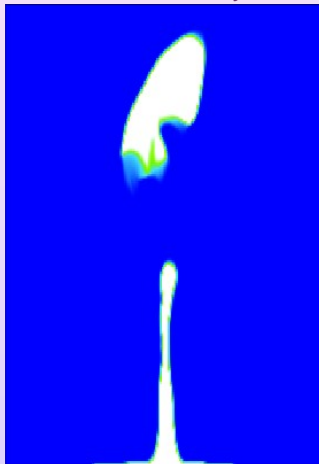
◀ Geometry

▶ Play

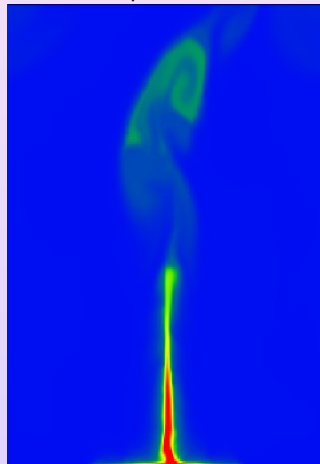
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



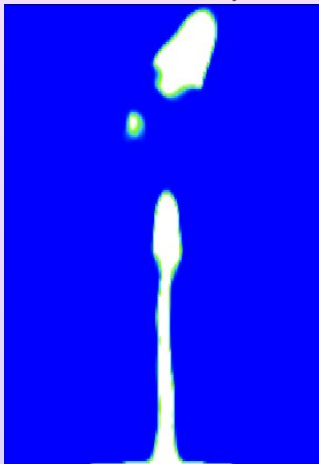
◀ Geometry

▶ Play

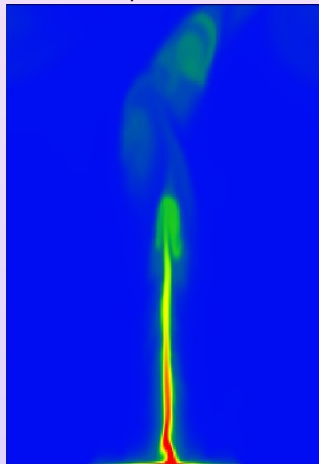
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



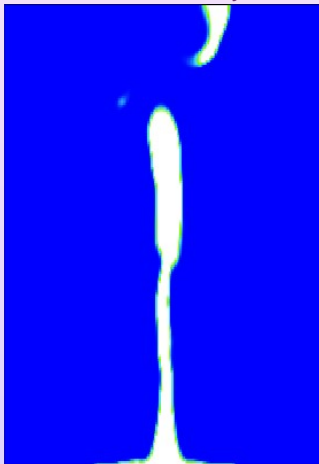
◀ Geometry

▶ Play

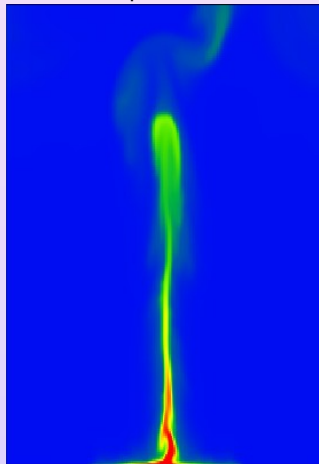
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



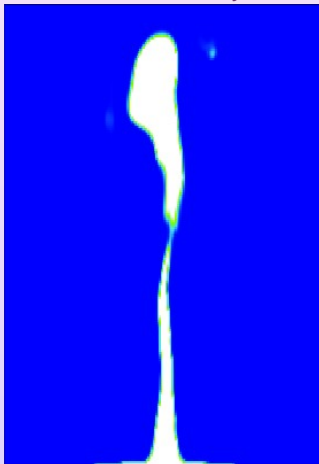
◀ Geometry

▶ Play

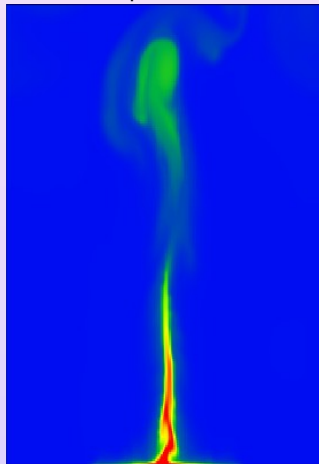
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

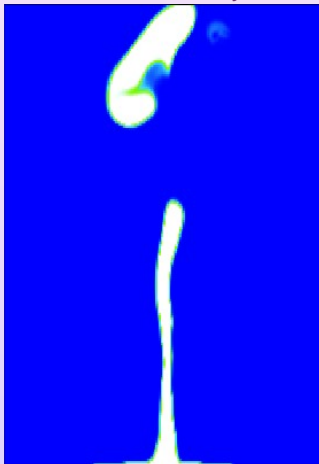
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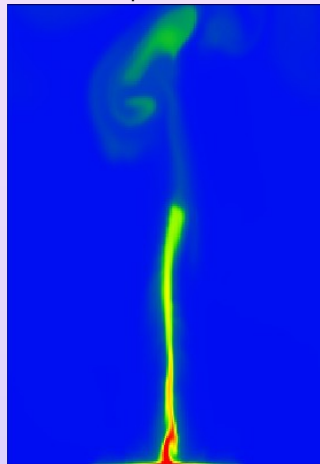


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



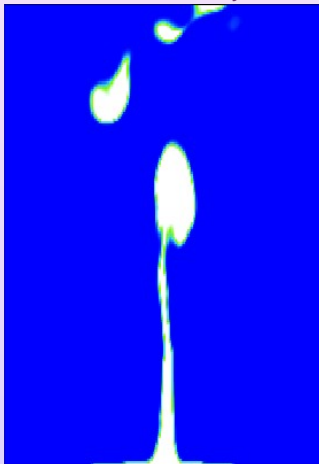
◀ Geometry

▶ Play

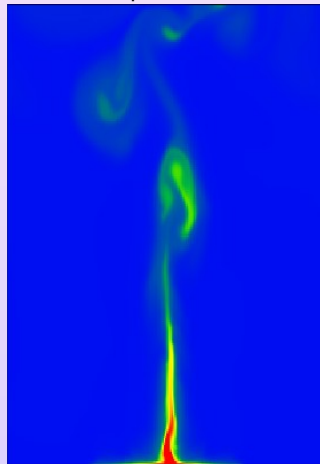
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



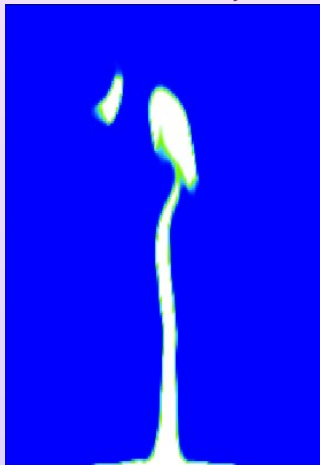
◀ Geometry

▶ Play

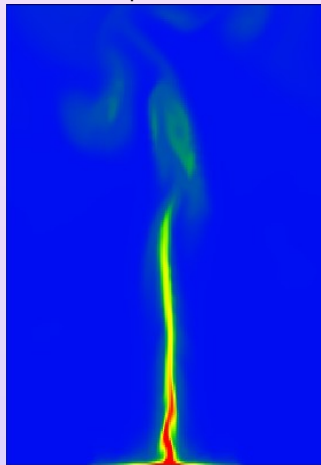
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

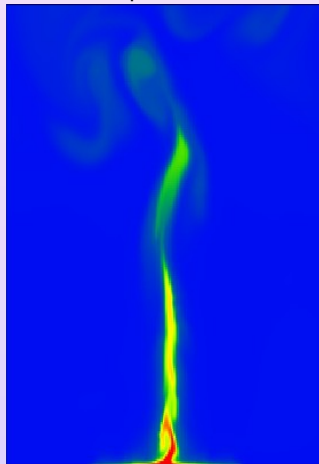
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

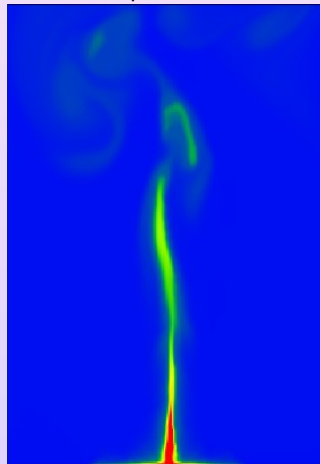
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



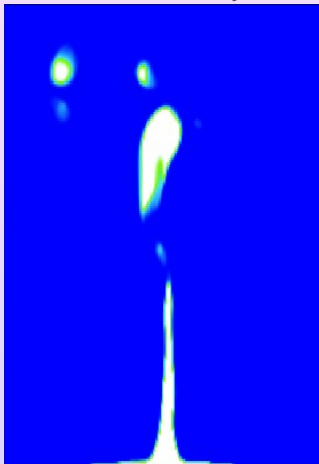
◀ Geometry

▶ Play

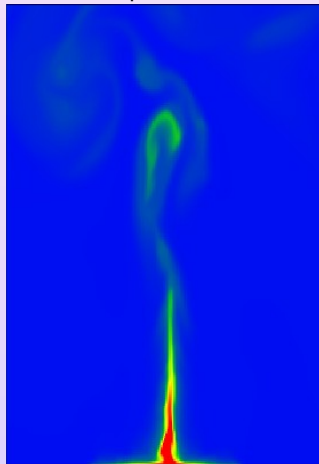
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



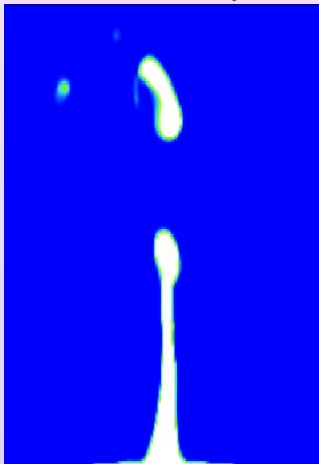
◀ Geometry

▶ Play

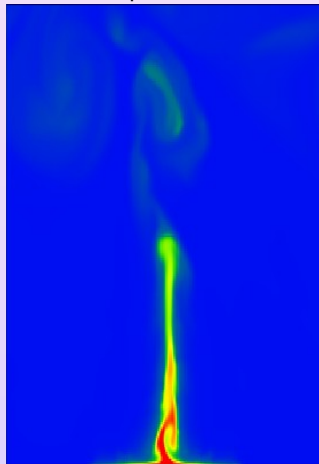
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



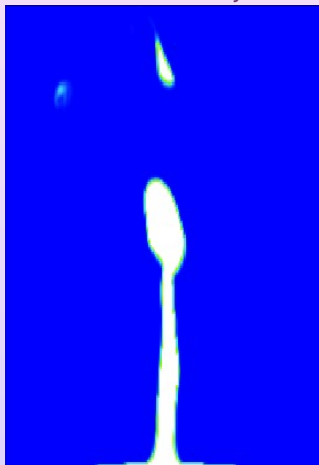
◀ Geometry

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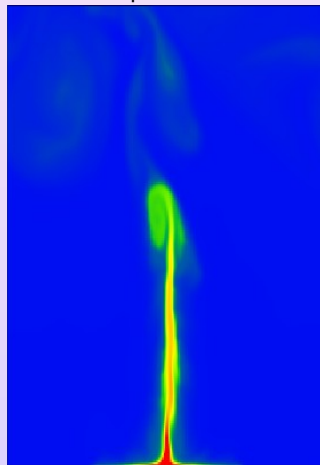
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

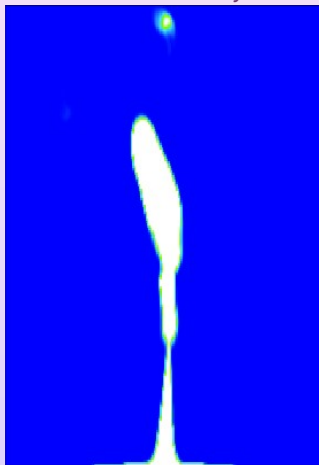
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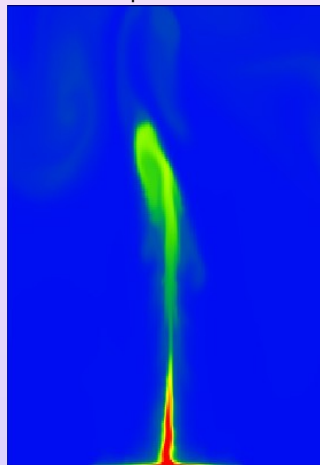


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



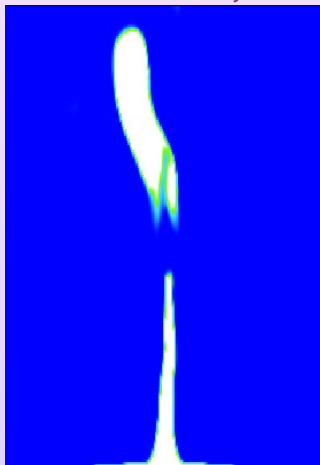
◀ Geometry

▶ Play

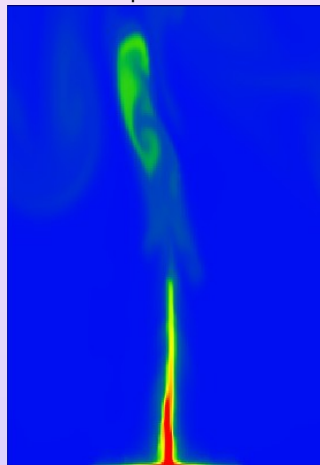
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



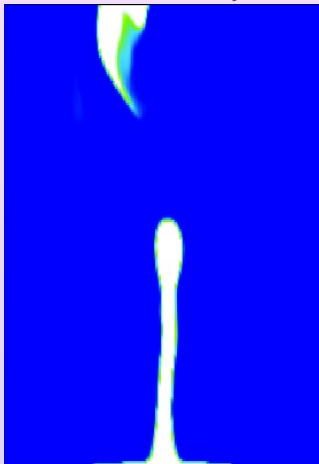
◀ Geometry

▶ Play

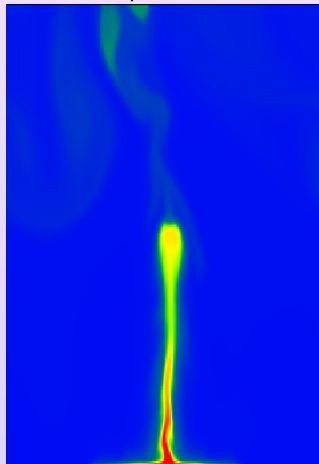
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



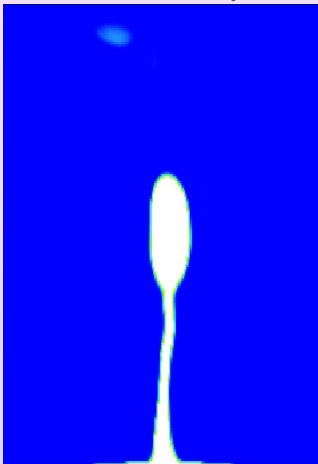
◀ Geometry

▶ Play

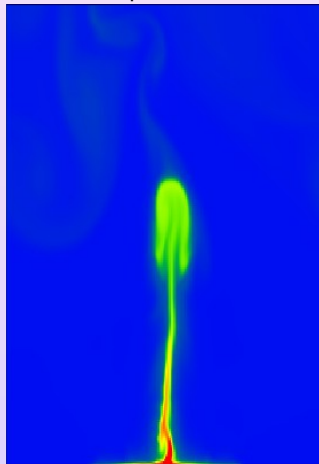
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



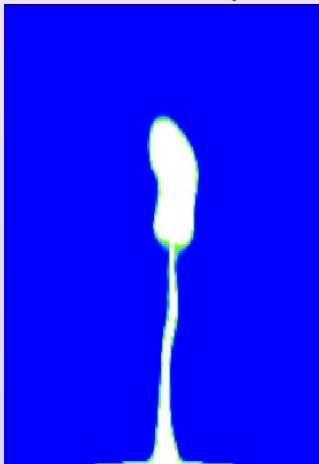
◀ Geometry

▶ Play

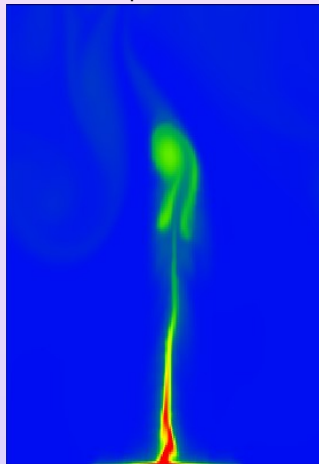
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



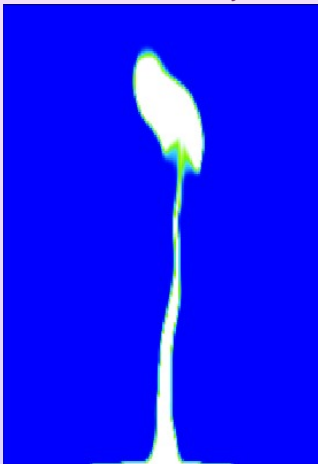
◀ Geometry

▶ Play

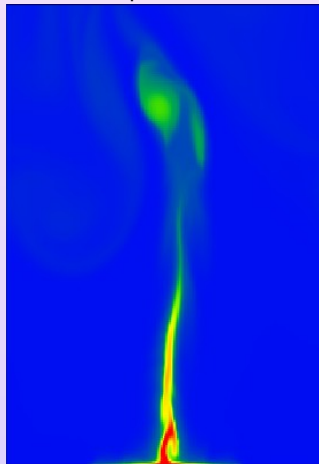
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



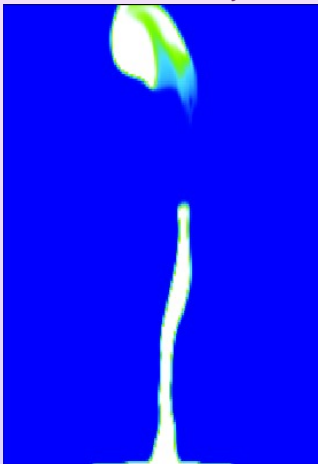
◀ Geometry

▶ Play

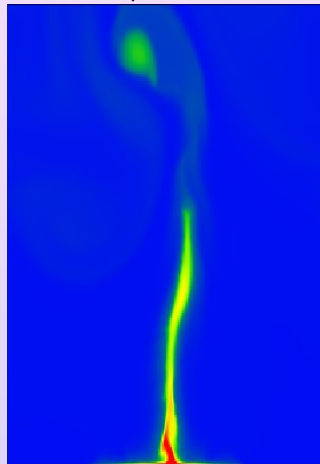
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



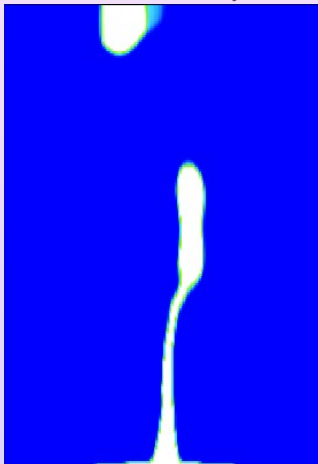
◀ Geometry

▶ Play

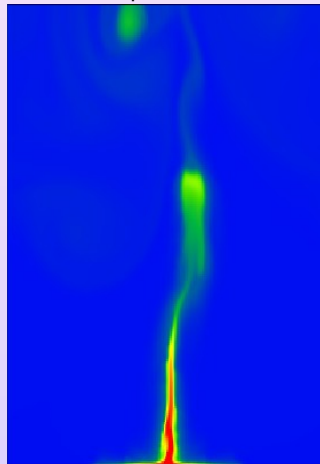
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

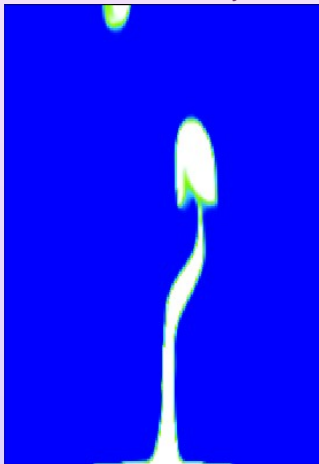
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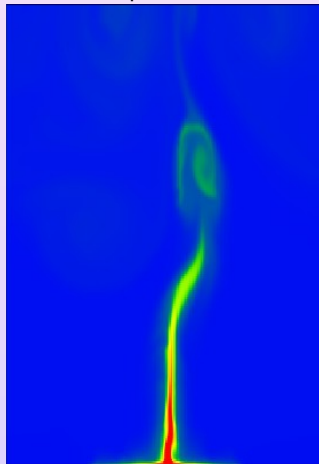


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



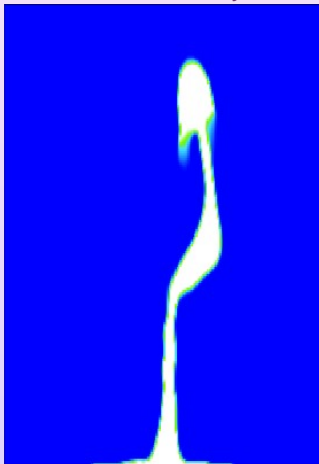
◀ Geometry

▶ Play

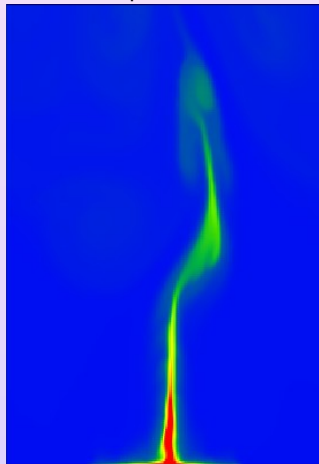
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



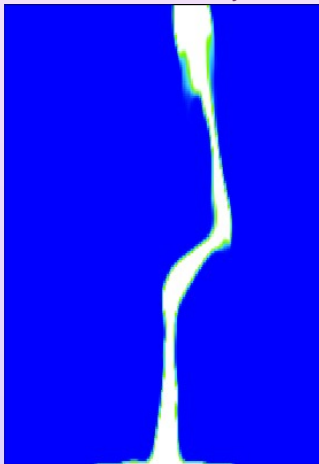
◀ Geometry

▶ Play

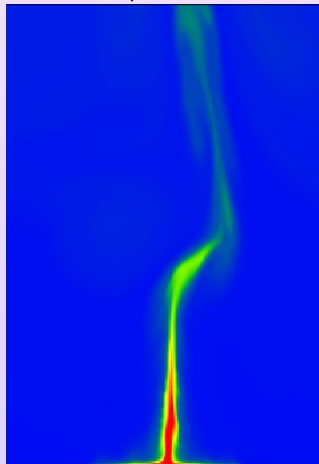
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



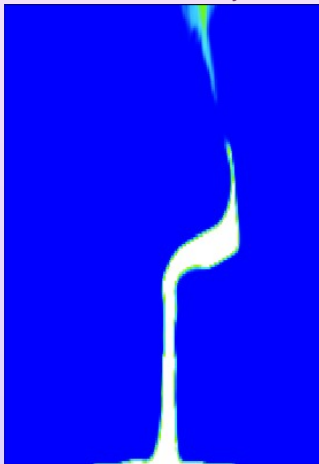
◀ Geometry

▶ Play

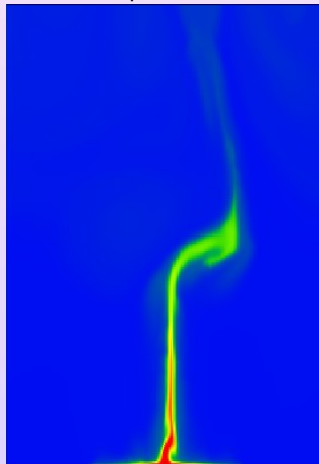
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



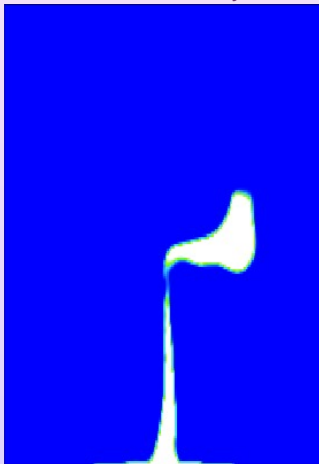
◀ Geometry

▶ Play

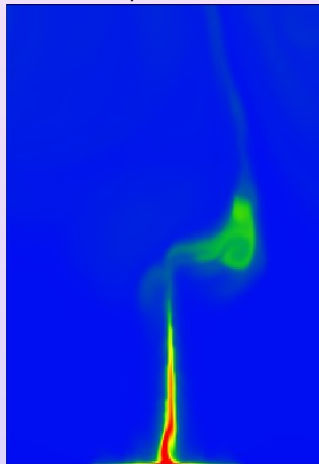
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



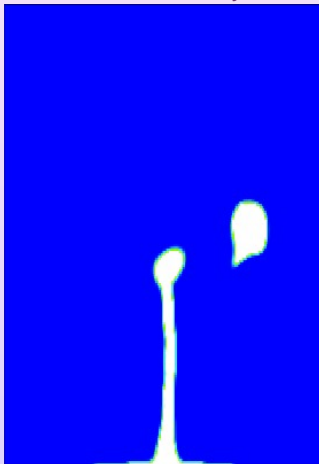
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▶ Play

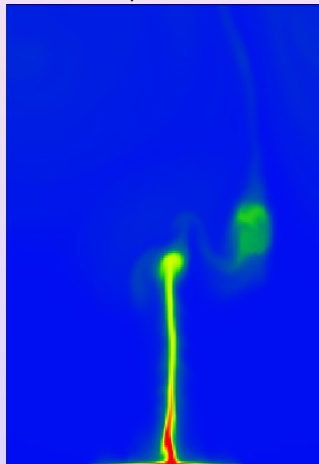
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



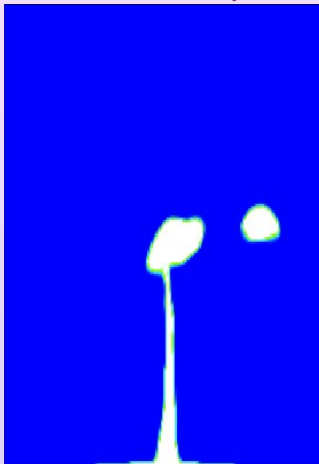
◀ Geometry

▶ Play

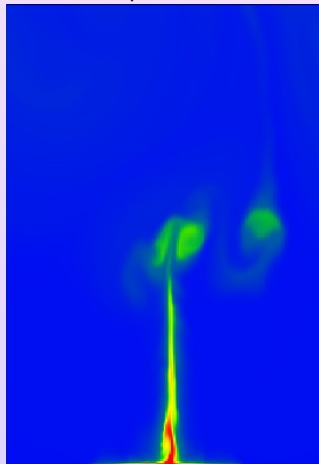
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



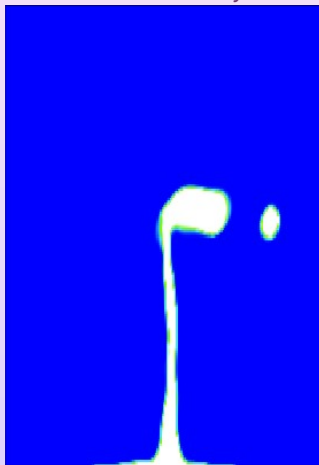
◀ Geometry

▶ Play

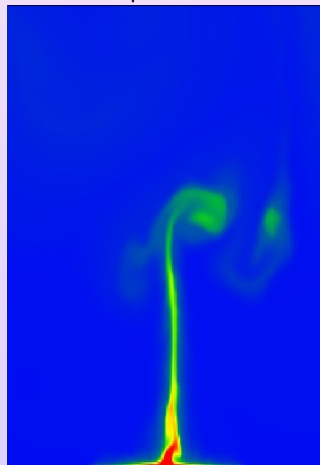
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

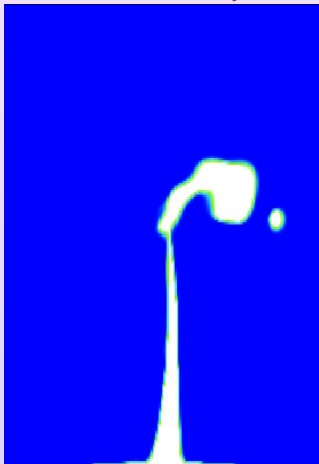
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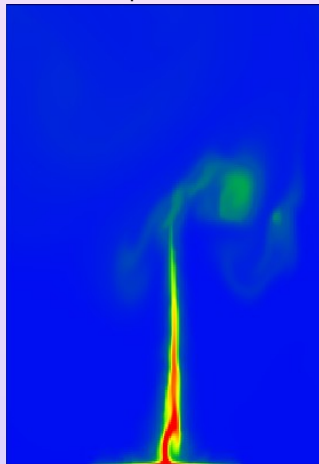


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



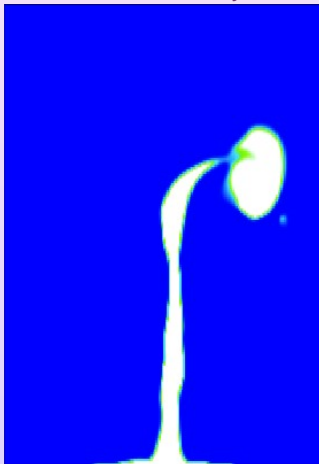
◀ Geometry

▶ Play

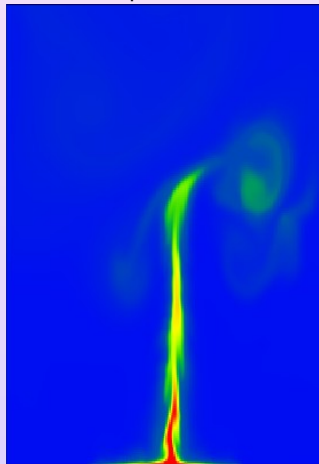
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



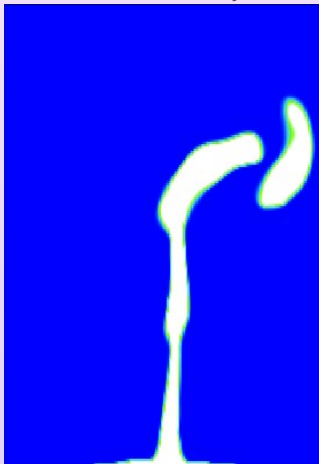
◀ Geometry

▶ Play

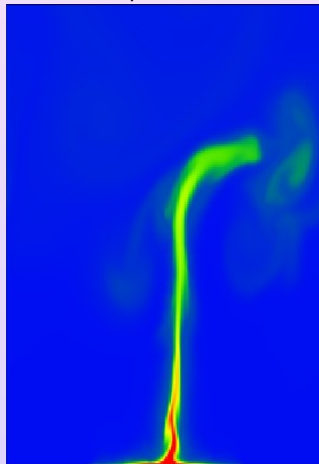
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



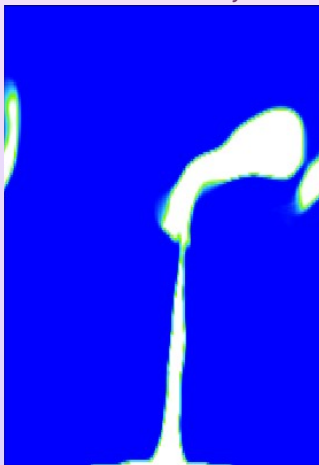
◀ Geometry

▶ Play

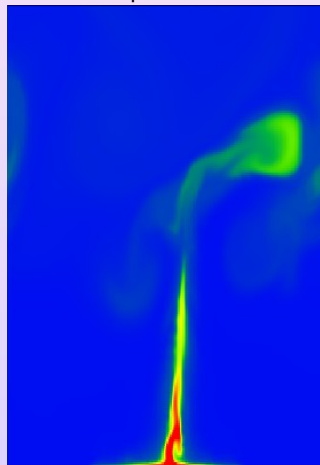
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



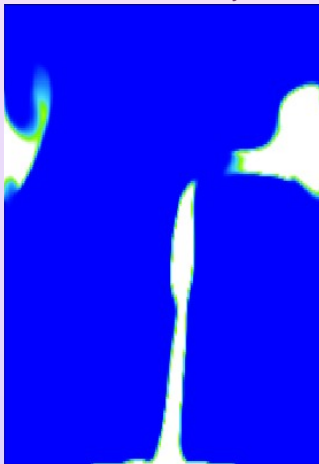
◀ Geometry

▶ Play

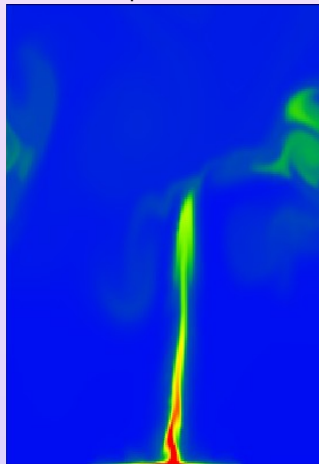
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

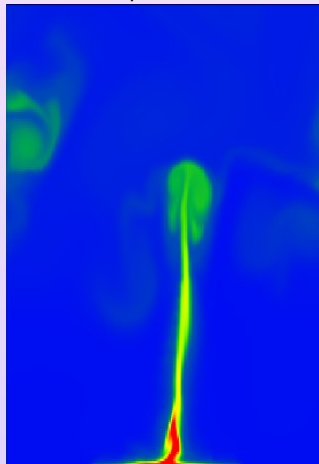
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



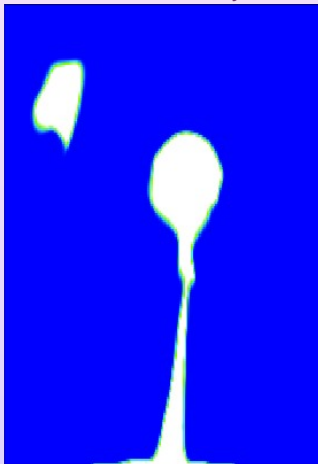
◀ Geometry

▶ Play

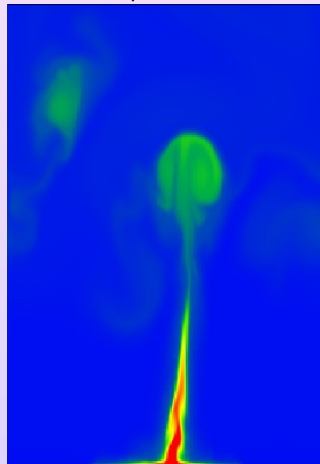
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



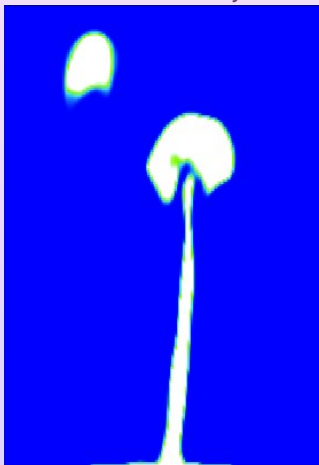
◀ Geometry

▶ Play

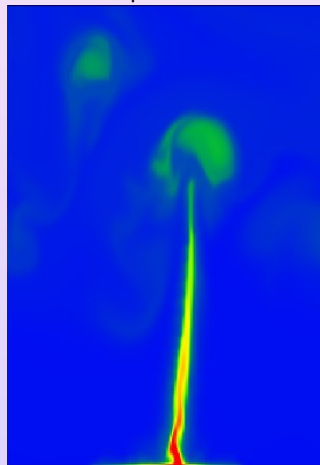
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

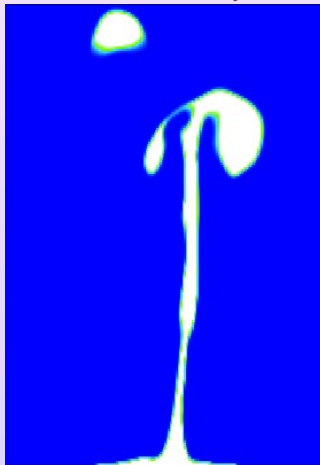
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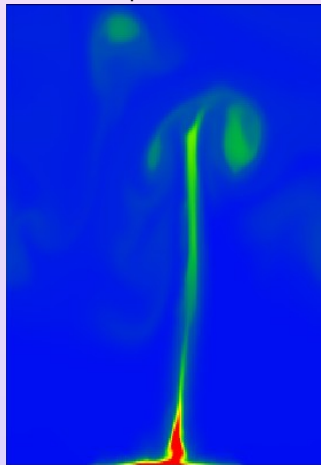


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

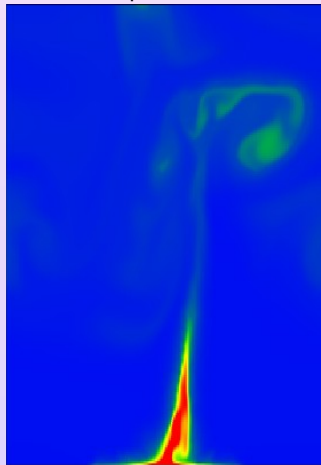
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

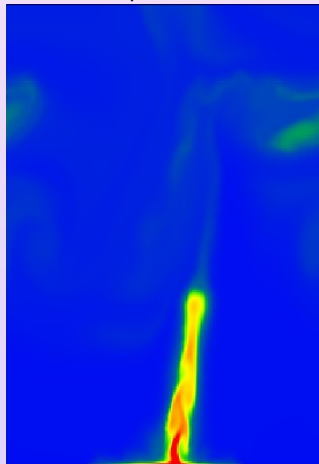
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



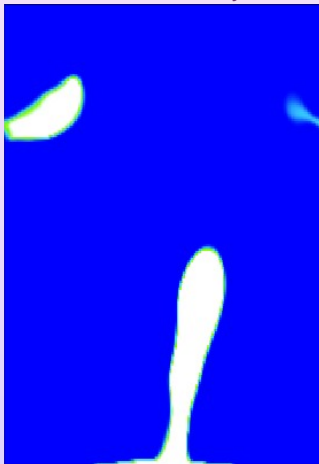
◀ Geometry

▶ Play

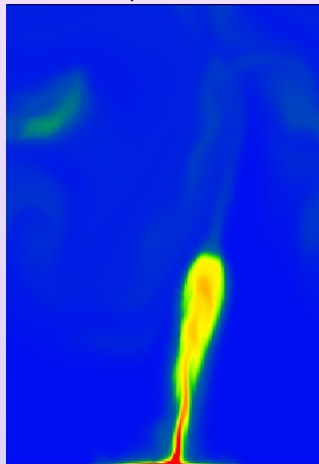
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



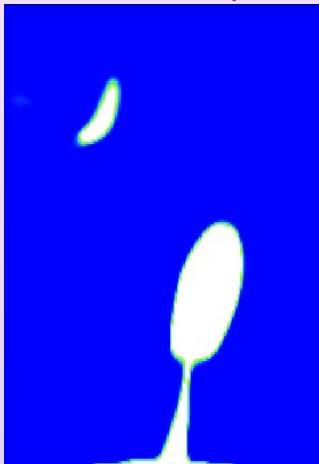
◀ Geometry

▶ Play

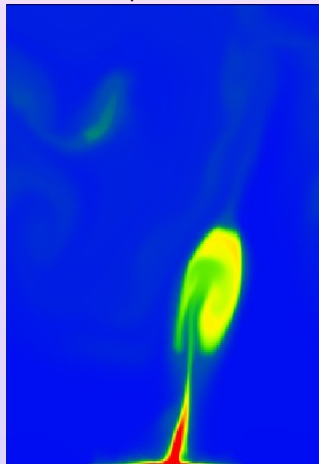
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

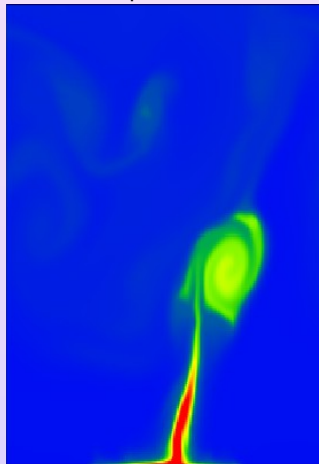
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

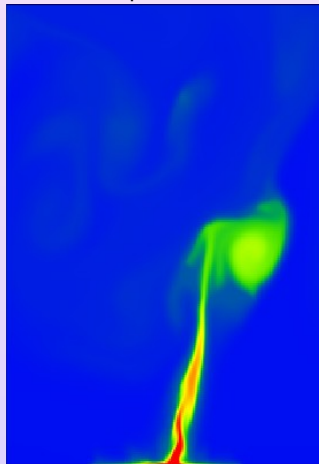
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

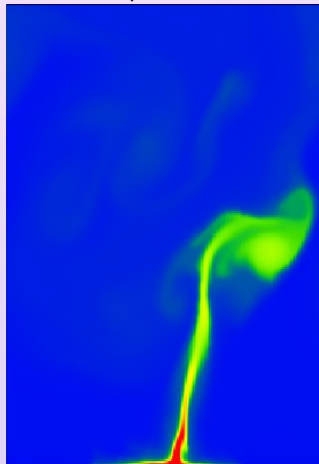
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

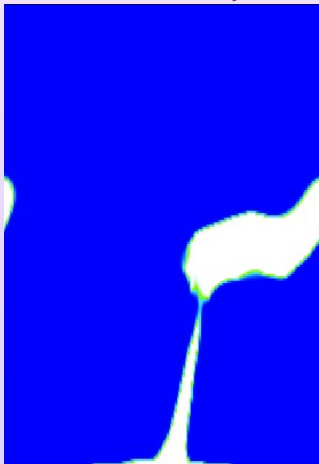
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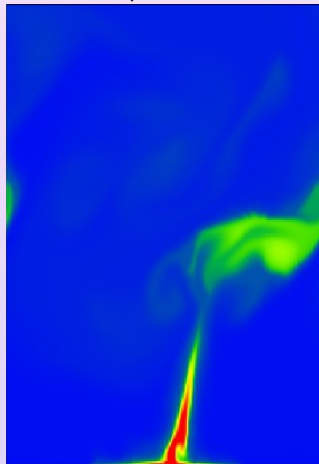


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

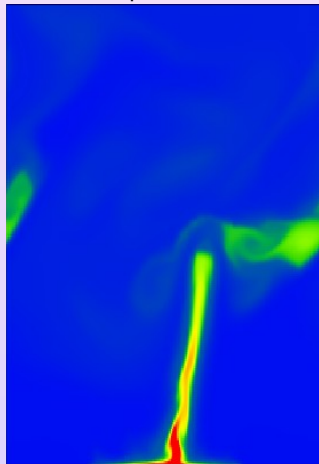
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



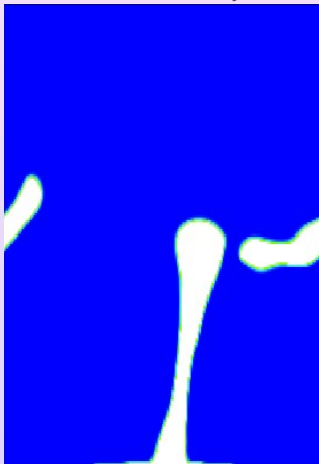
◀ Geometry

▶ Play

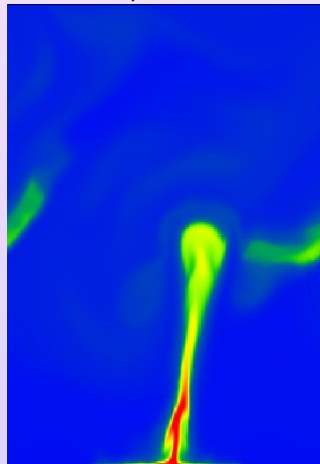
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



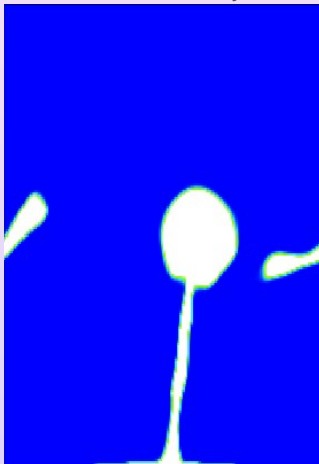
◀ Geometry

▶ Play

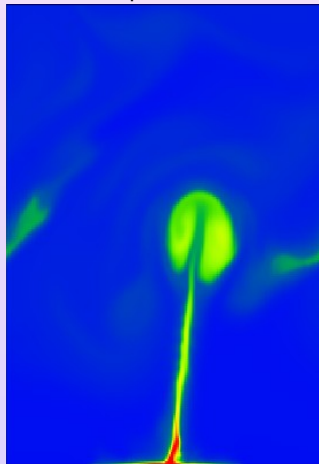
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

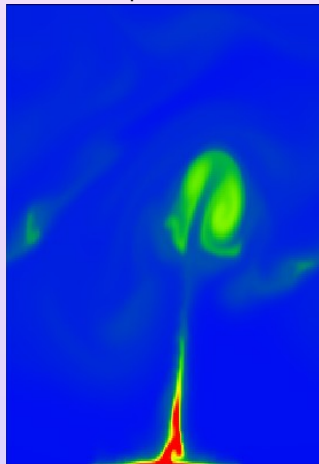
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



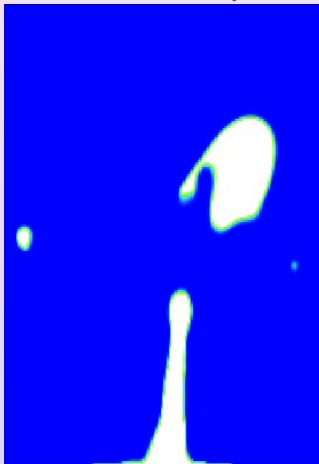
◀ Geometry

▶ Play

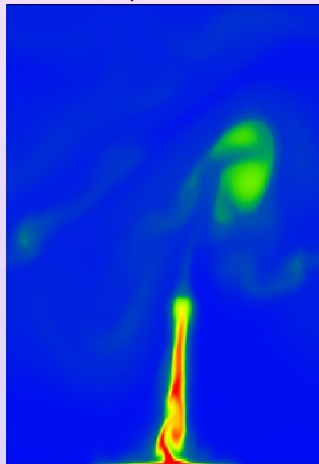
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



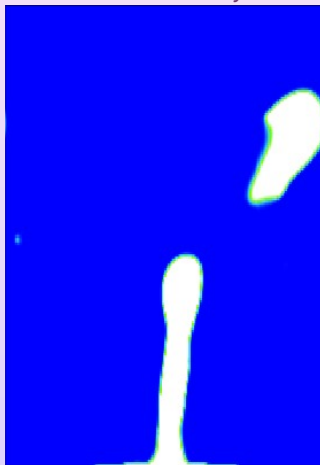
◀ Geometry

▶ Play

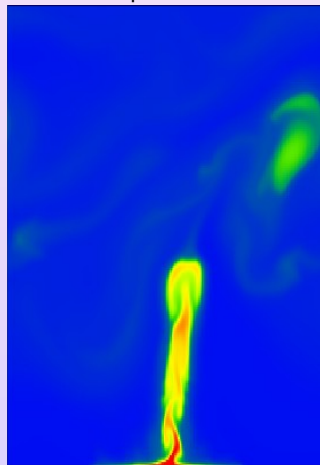
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



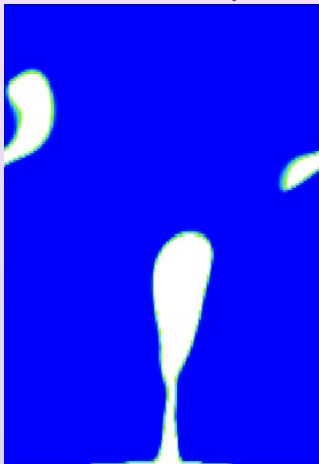
◀ Geometry

▶ Play

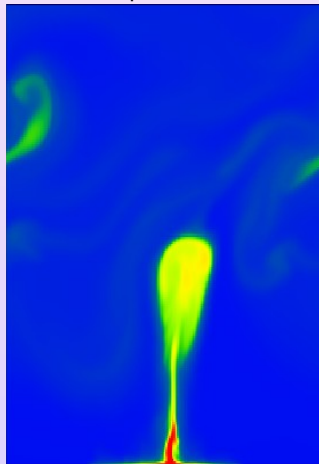
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

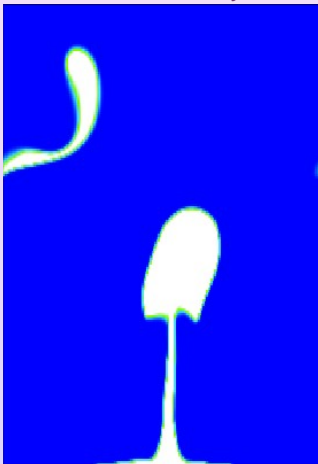
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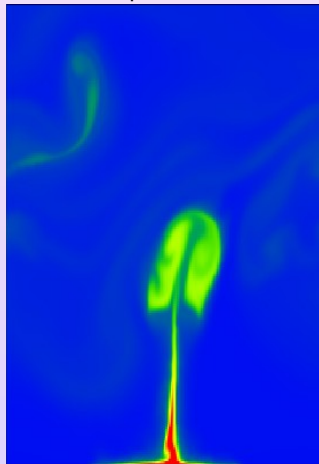


# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



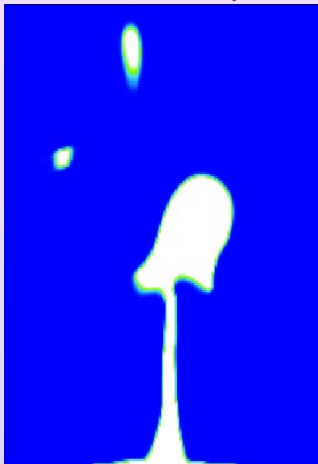
◀ Geometry

▶ Play

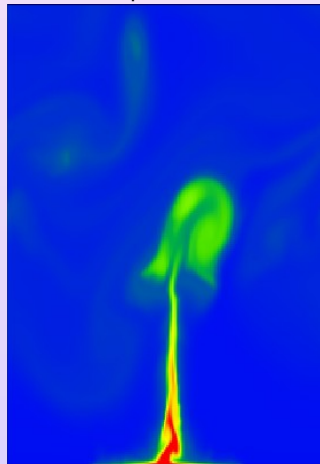
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



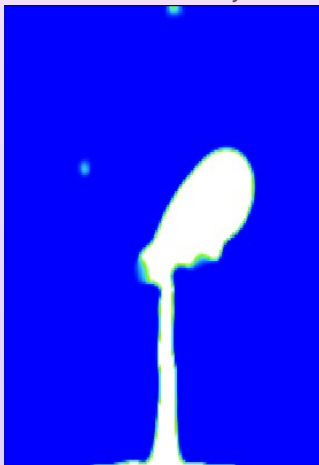
◀ Geometry

▶ Play

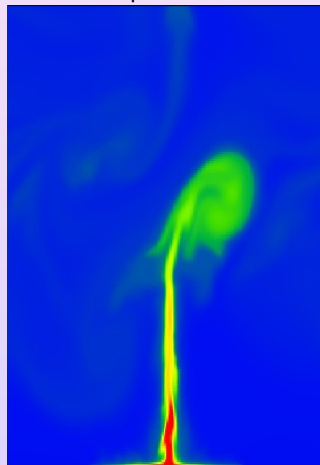
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



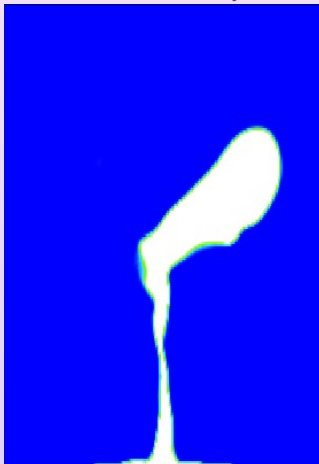
◀ Geometry

▶ Play

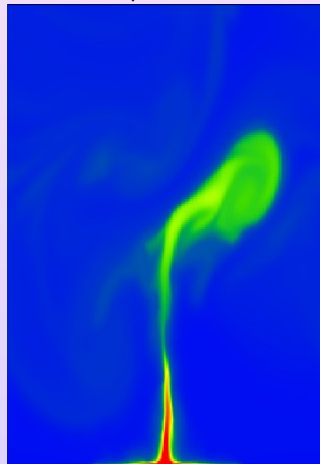
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



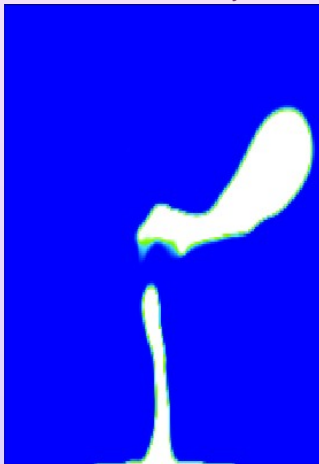
◀ Geometry

▶ Play

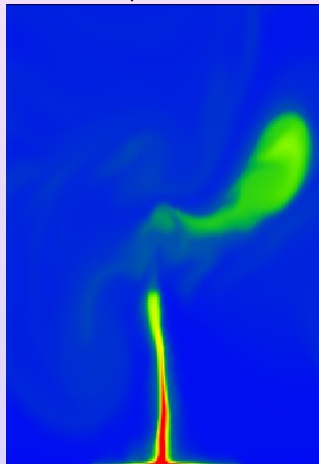
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$



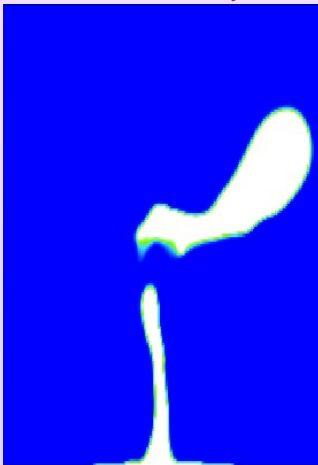
◀ Geometry

▶ Play

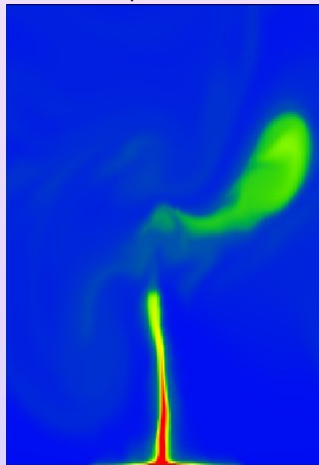
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# NUCLEATING BUBBLE

Mass Fraction  $y$



Temperature  $T$

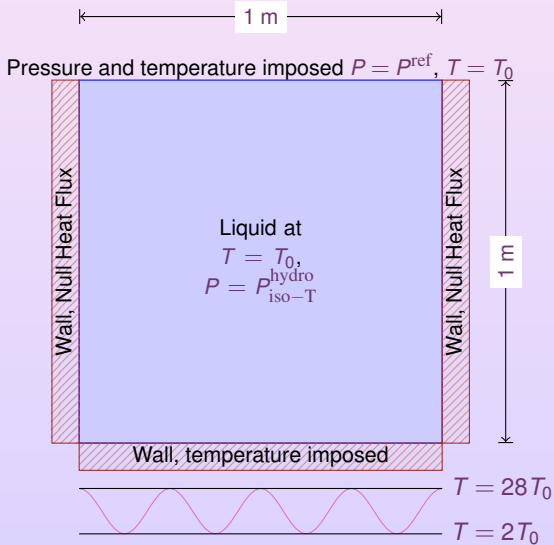


◀ Geometry

▶ Play

▶▶ Skip

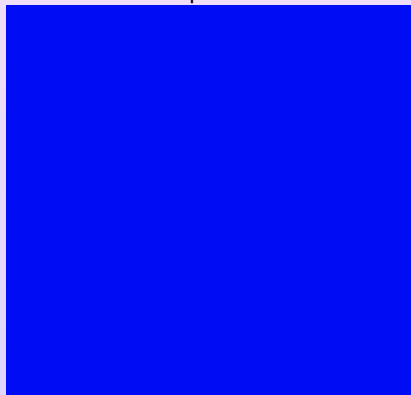
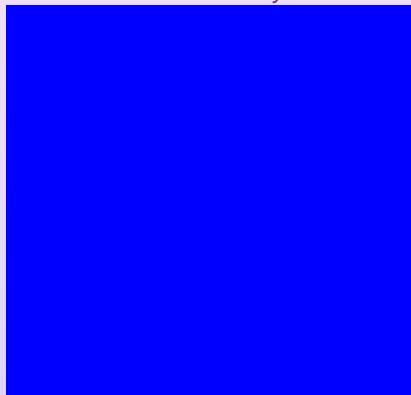
# FILM



# FILM

Mass Fraction  $y$

Temperature  $T$



◀ Geometry

▶ Play

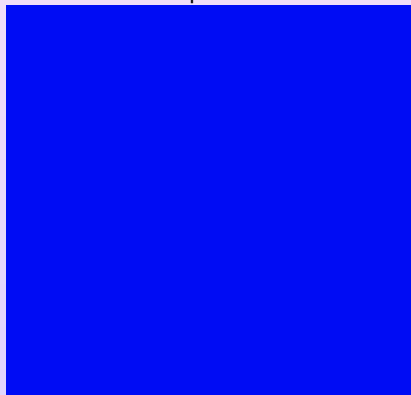
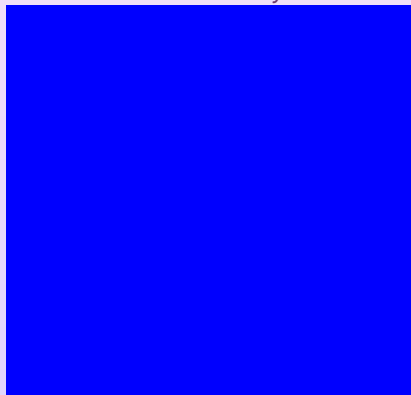
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# FILM

Mass Fraction  $y$

Temperature  $T$



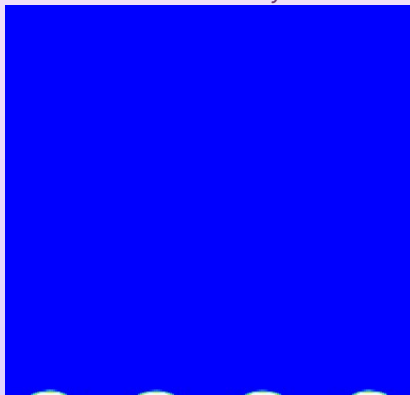
◀ Geometry

▶ Play

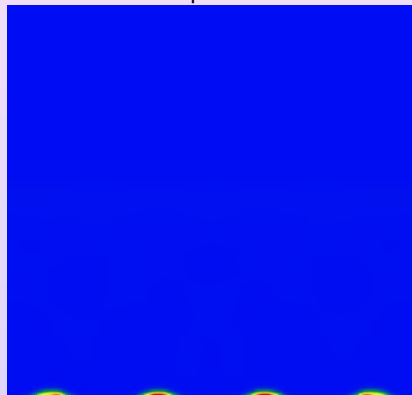
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# FILM

Mass Fraction  $y$



Temperature  $T$



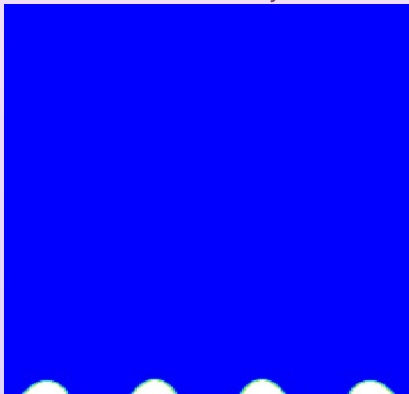
◀ Geometry

▶ Play

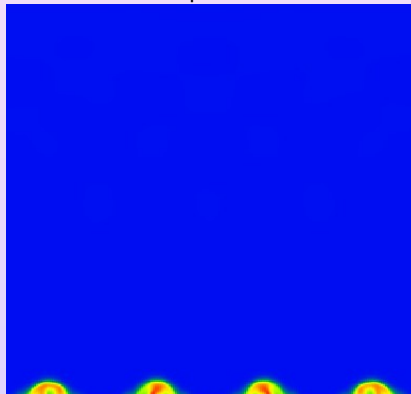
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# FILM

Mass Fraction  $y$



Temperature  $T$



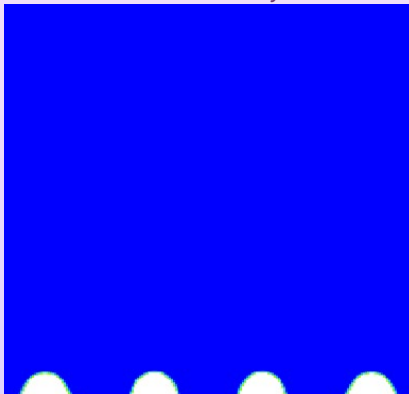
◀ Geometry

▶ Play

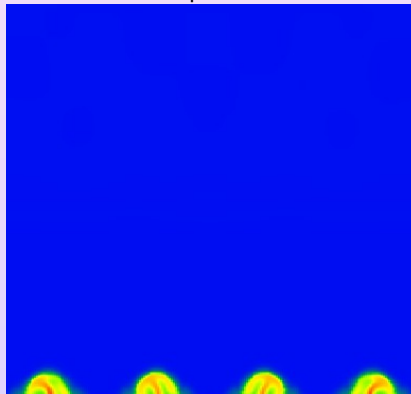
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# FILM

Mass Fraction  $y$



Temperature  $T$



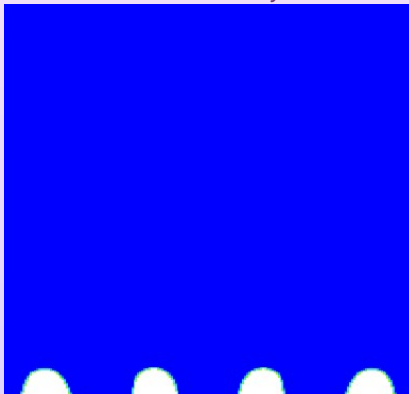
◀ Geometry

▶ Play

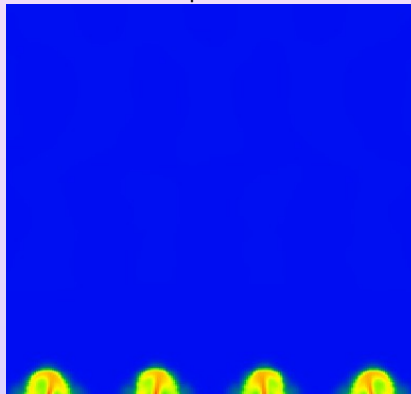
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

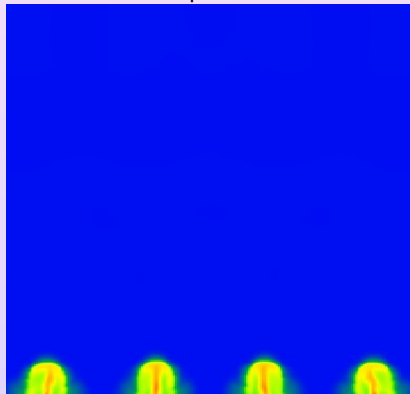
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Mass Fraction  $y$



Temperature  $T$



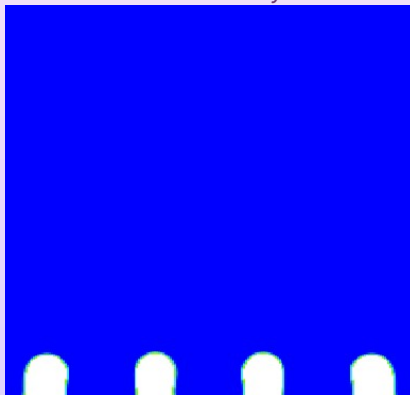
◀ Geometry

▶ Play

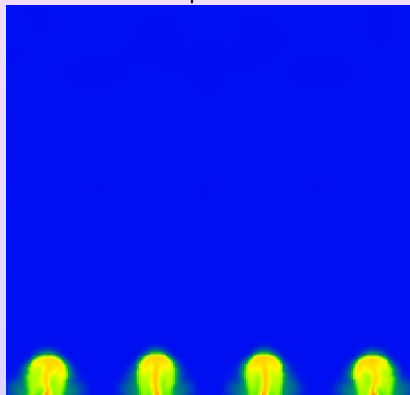
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# FILM

Mass Fraction  $y$



Temperature  $T$



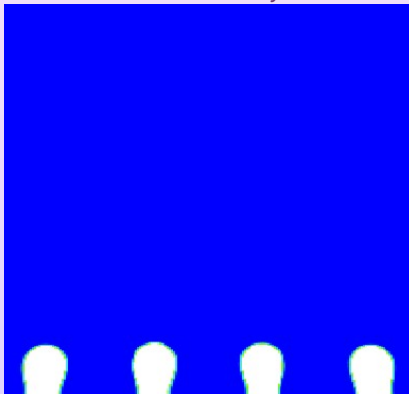
◀ Geometry

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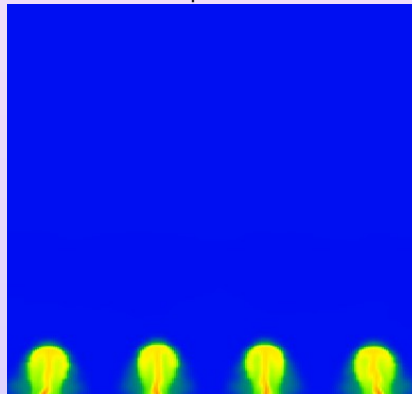
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

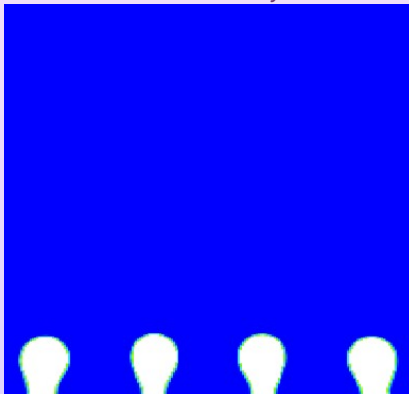
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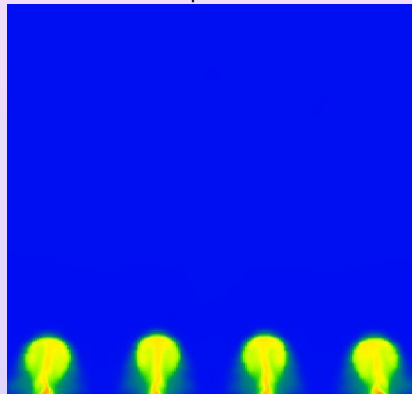


# FILM

Mass Fraction  $y$



Temperature  $T$



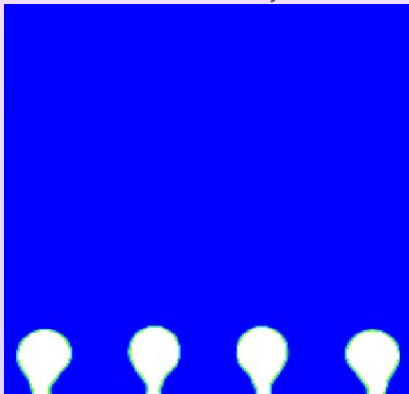
◀ Geometry

▶ Play

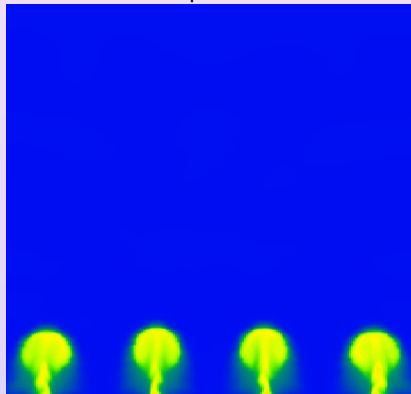
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# FILM

Mass Fraction  $y$



Temperature  $T$



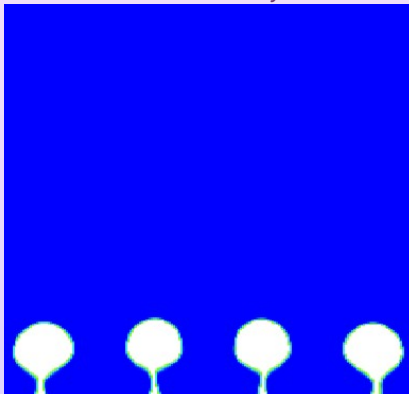
◀ Geometry

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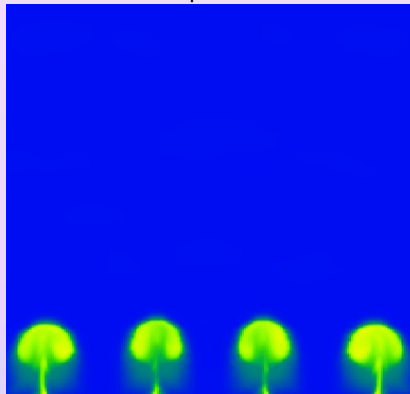
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# FILM

Mass Fraction  $y$



Temperature  $T$



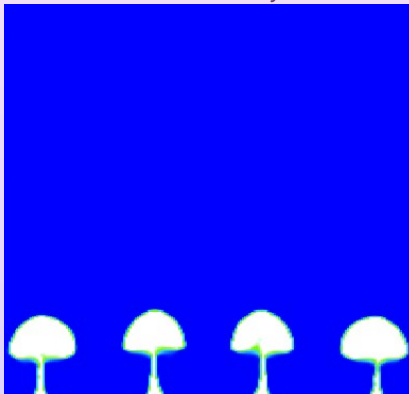
◀ Geometry

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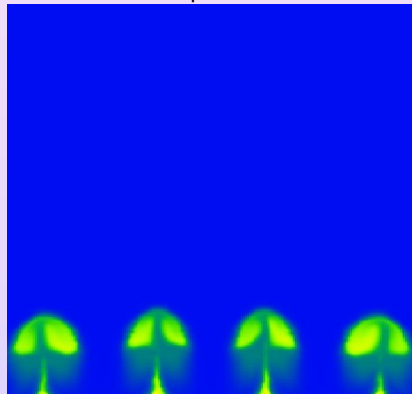
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

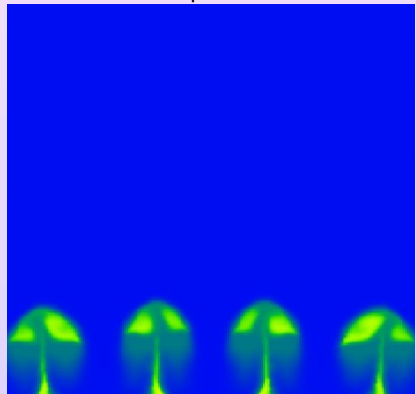
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# FILM

Mass Fraction  $y$



Temperature  $T$



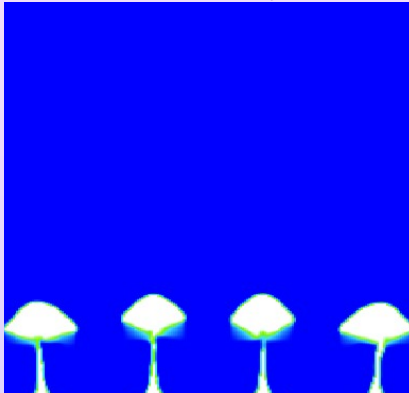
◀ Geometry

▶ Play

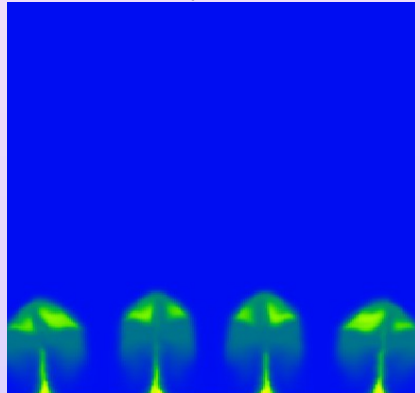
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# FILM

Mass Fraction  $y$



Temperature  $T$



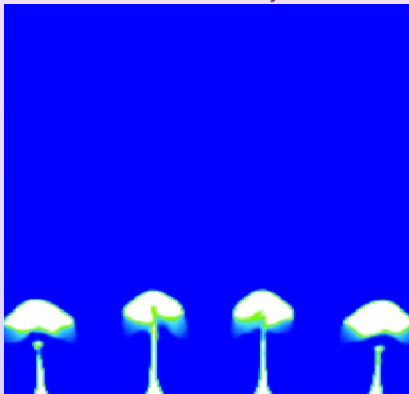
◀ Geometry

▶ Play

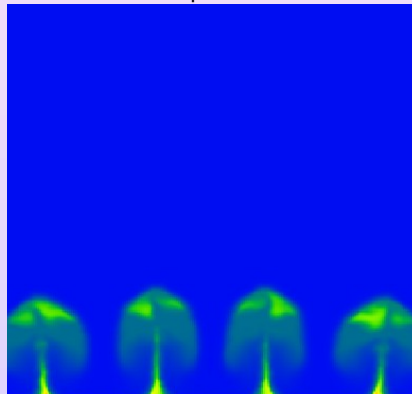
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# FILM

Mass Fraction  $y$



Temperature  $T$



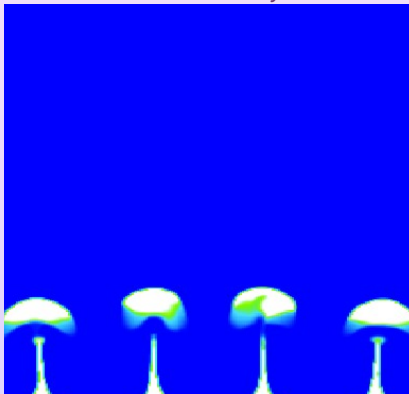
◀ Geometry

▶ Play

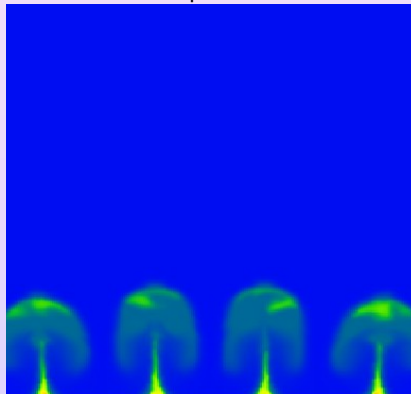
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

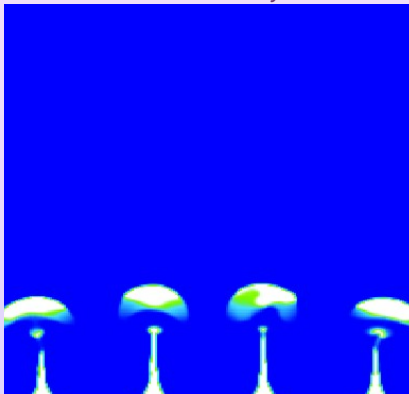
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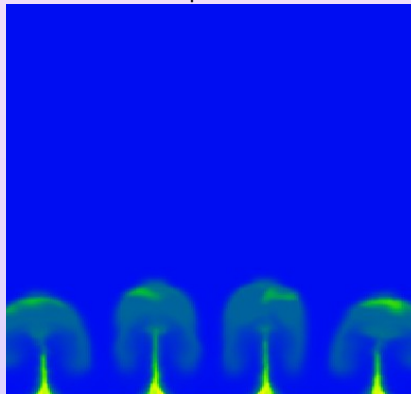


# FILM

Mass Fraction  $y$



Temperature  $T$



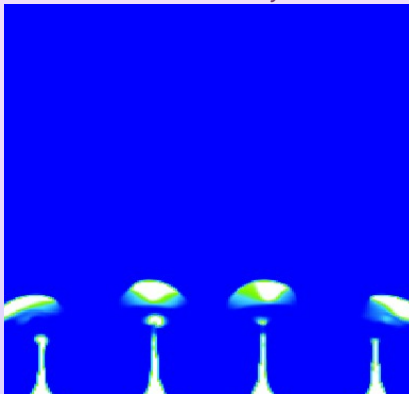
◀ Geometry

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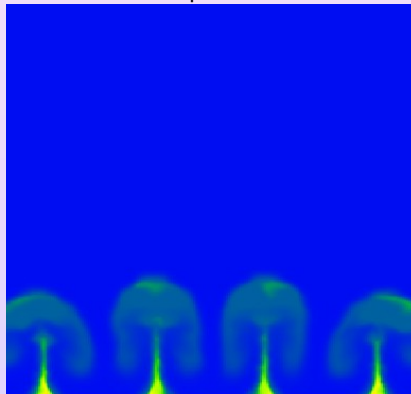
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# FILM

Mass Fraction  $y$



Temperature  $T$



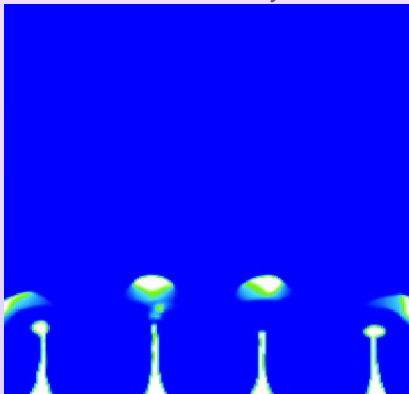
◀ Geometry

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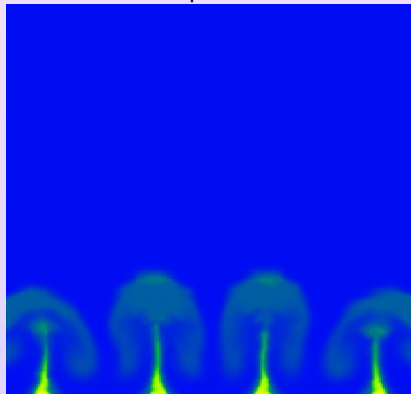
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# FILM

Mass Fraction  $y$



Temperature  $T$



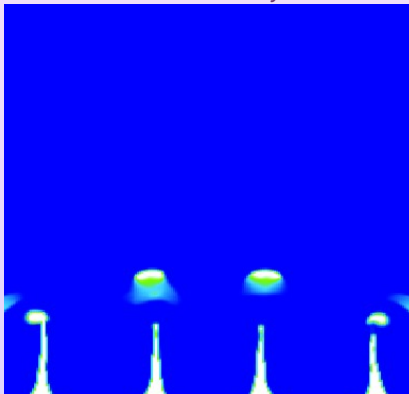
◀ Geometry

▶ Play

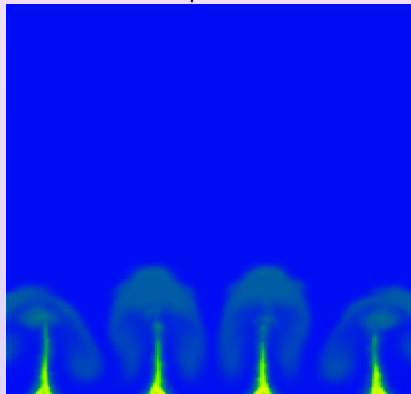
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

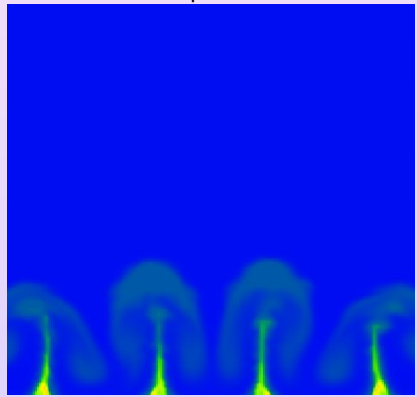
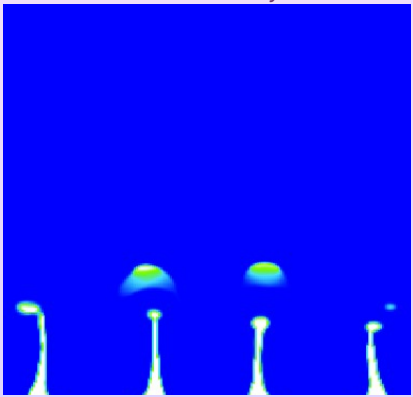
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# FILM

Mass Fraction  $y$

Temperature  $T$

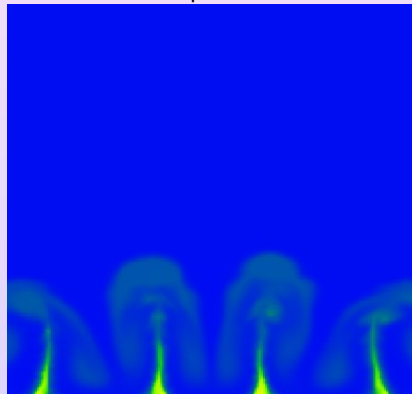
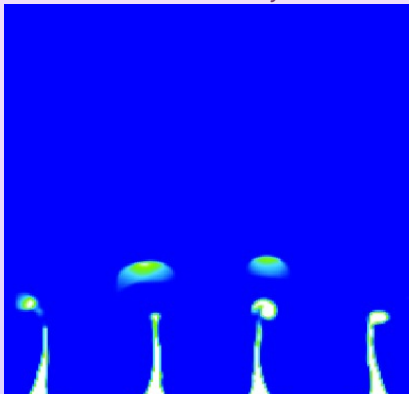


◀ Geometry ▶ Play ▶▶ Skip

# FILM

Mass Fraction  $y$

Temperature  $T$



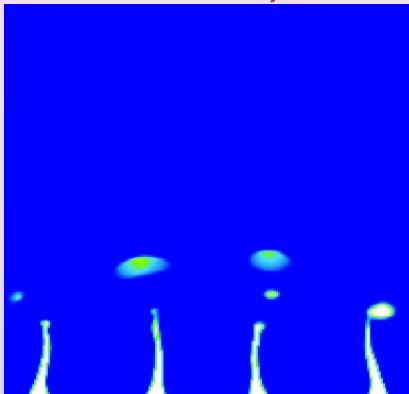
◀ Geometry

▶ Play

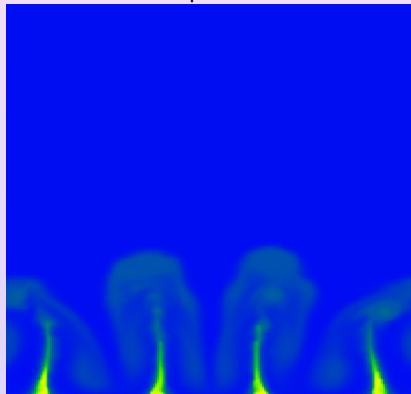
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# FILM

Mass Fraction  $y$



Temperature  $T$



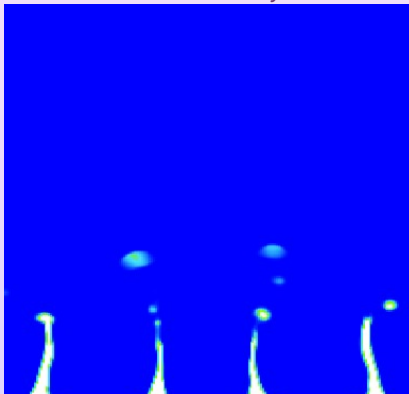
◀ Geometry

▶ Play

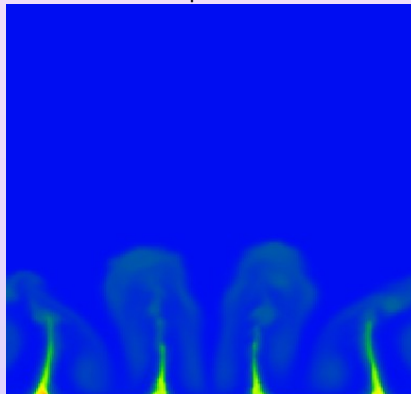
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

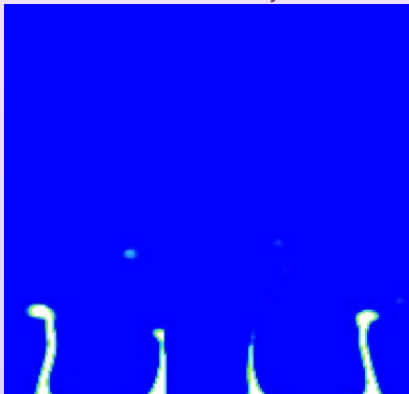
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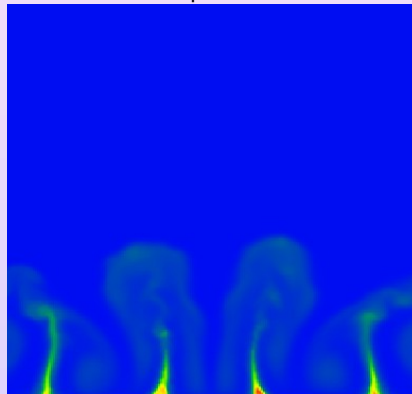


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Mass Fraction  $y$



Temperature  $T$



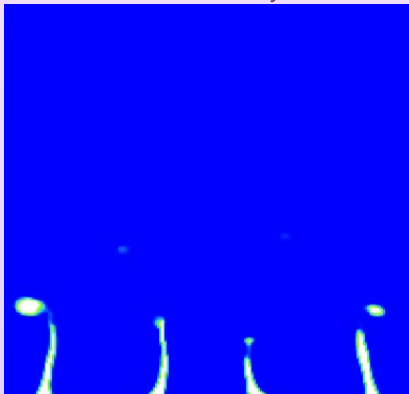
◀ Geometry

▶ Play

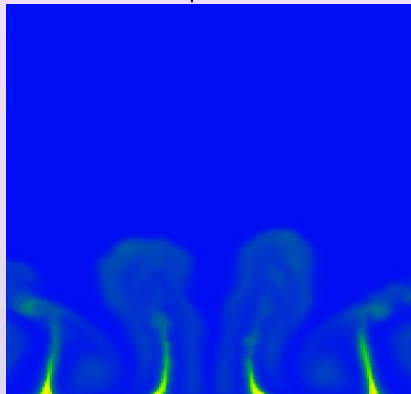
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# FILM

Mass Fraction  $y$



Temperature  $T$



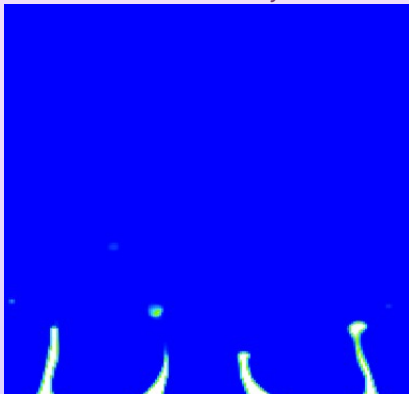
◀ Geometry

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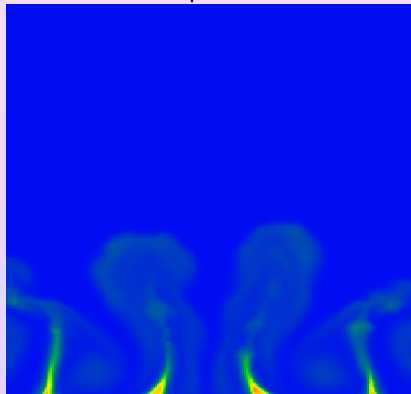
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

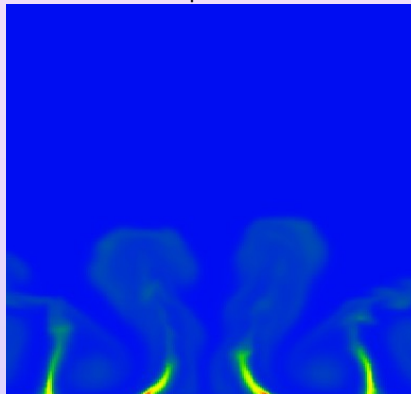
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# FILM

Mass Fraction  $y$



Temperature  $T$



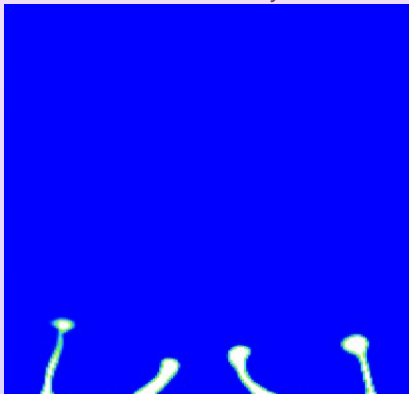
◀ Geometry

▶ Play

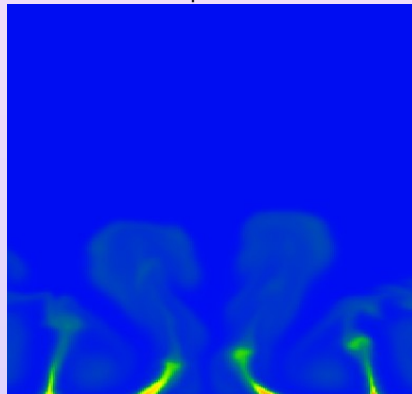
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# FILM

Mass Fraction  $y$



Temperature  $T$



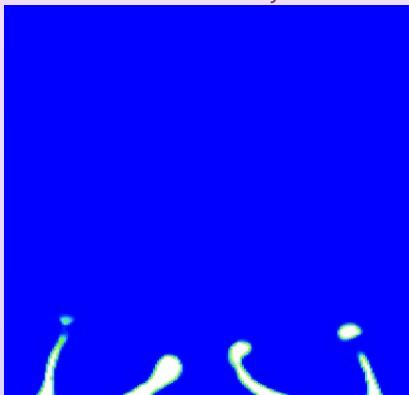
◀ Geometry

▶ Play

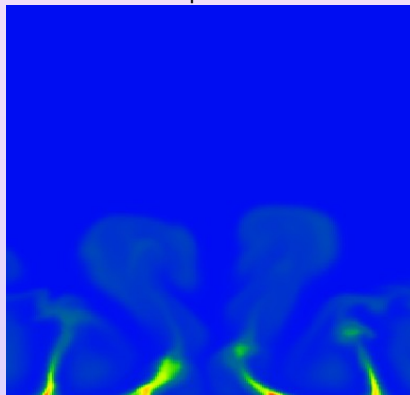
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

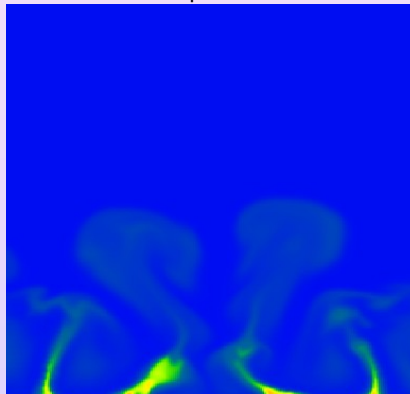
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

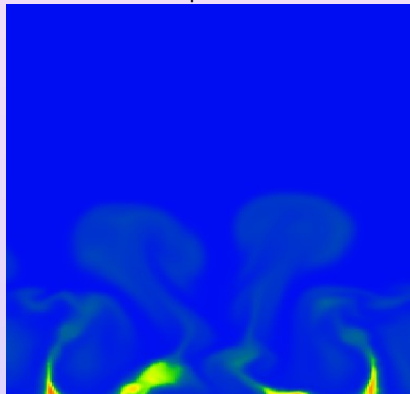
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

▶▶ Skip

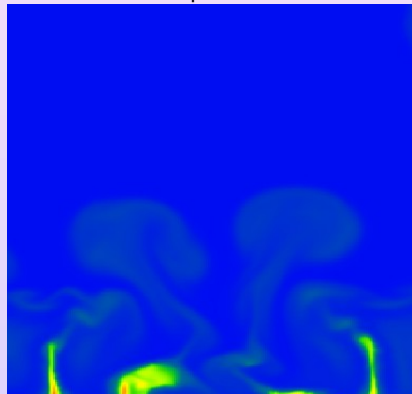


# FILM

Mass Fraction  $y$



Temperature  $T$



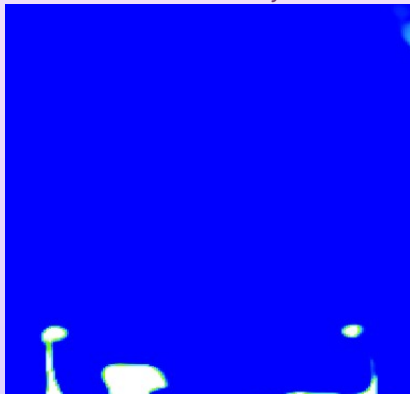
◀ Geometry

▶ Play

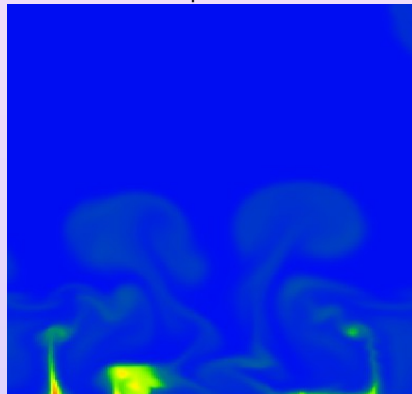
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# FILM

Mass Fraction  $y$



Temperature  $T$



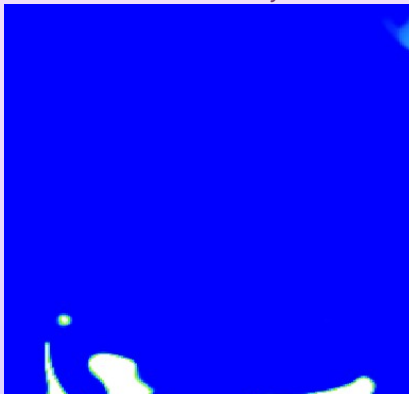
◀ Geometry

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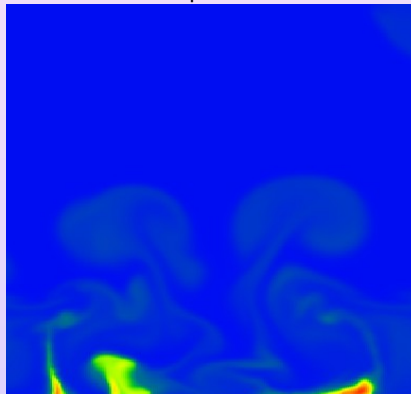
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# FILM

Mass Fraction  $y$



Temperature  $T$



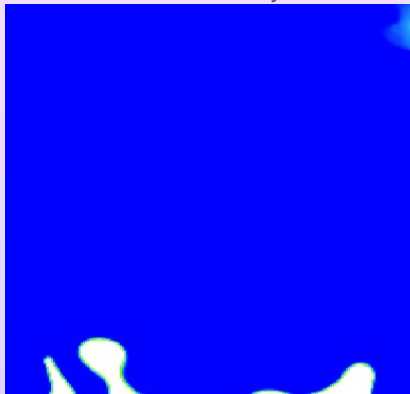
◀ Geometry

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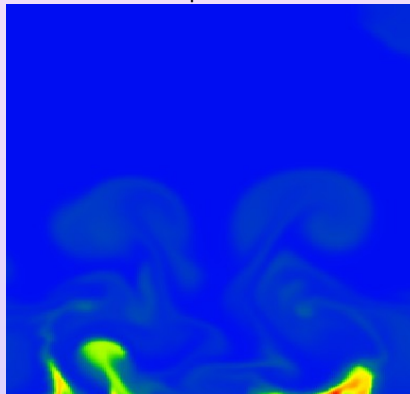
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

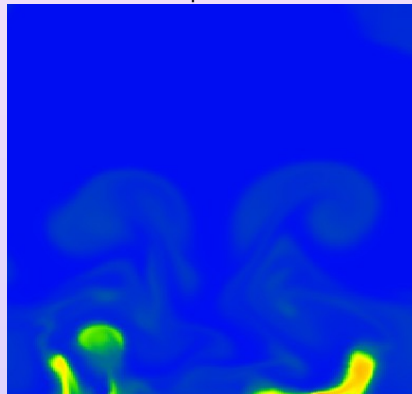
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

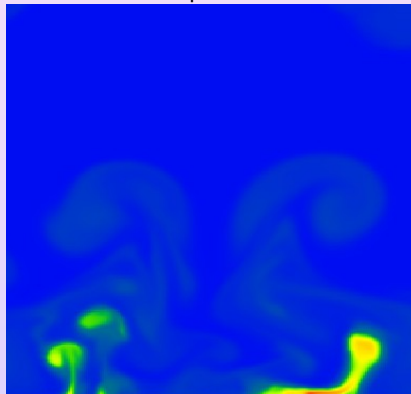
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

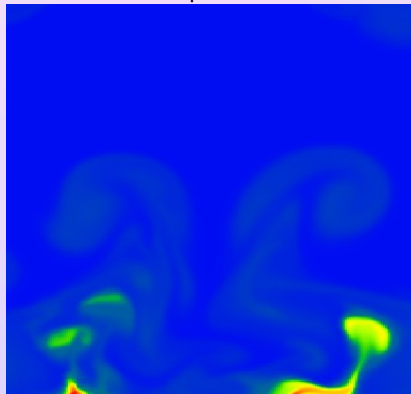
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# FILM

Mass Fraction  $y$



Temperature  $T$



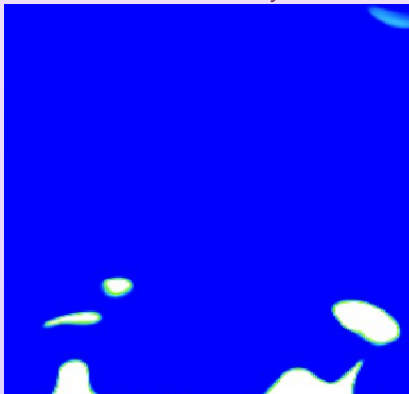
◀ Geometry

▶ Play

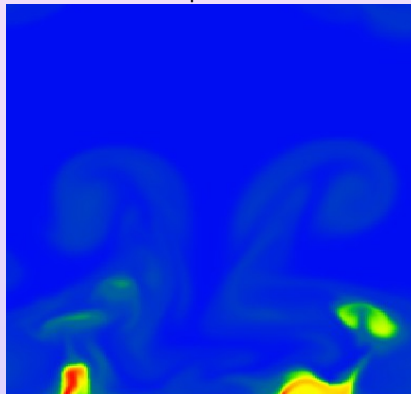
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

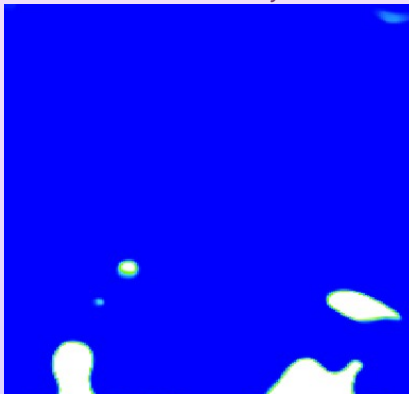
▶ Play

▶▶ Skip

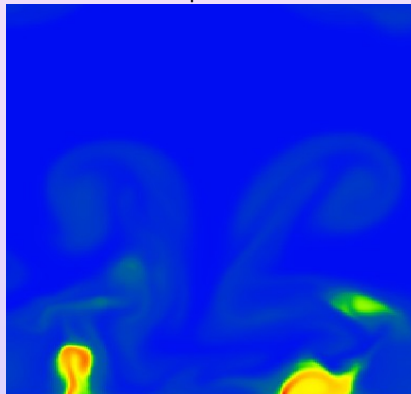


# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

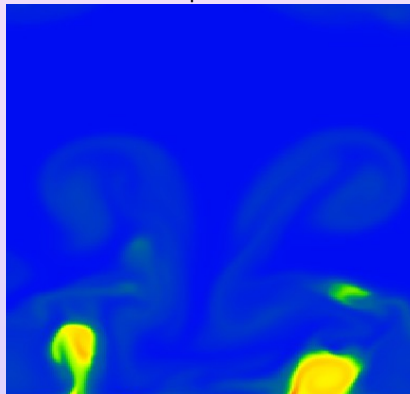
▶▶ Skip

# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

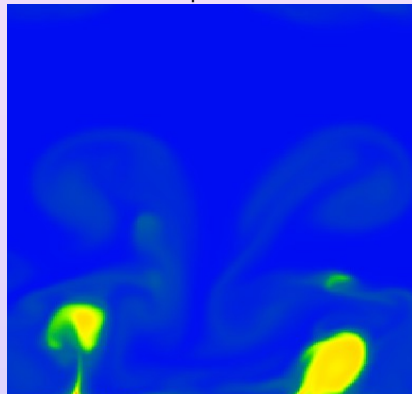
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

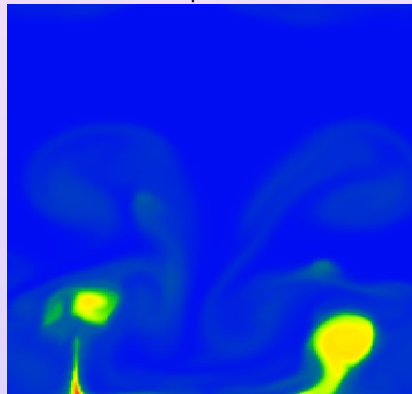
▶▶ Skip

# FILM

Mass Fraction  $y$



Temperature  $T$



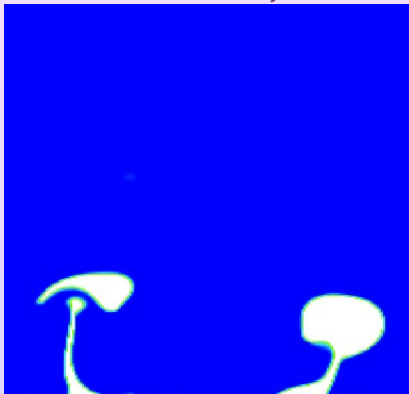
◀ Geometry

▶ Play

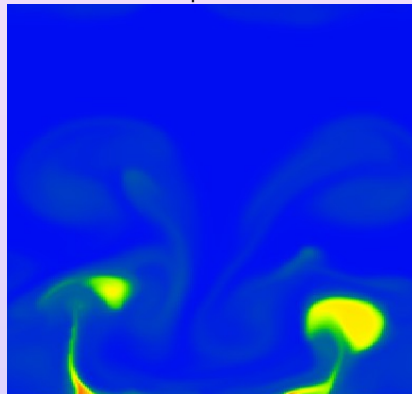
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# FILM

Mass Fraction  $y$



Temperature  $T$



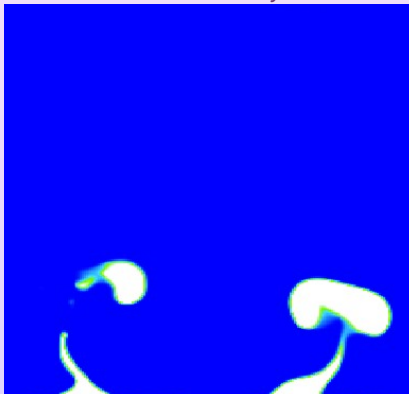
◀ Geometry

▶ Play

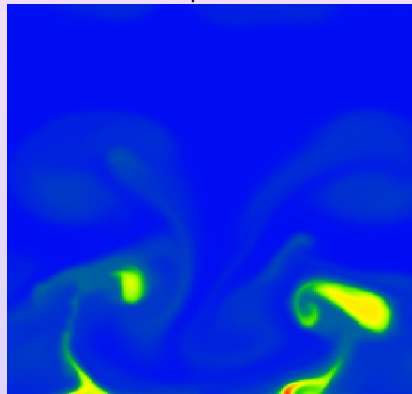
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

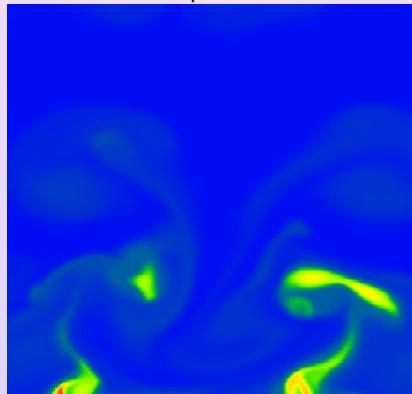
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

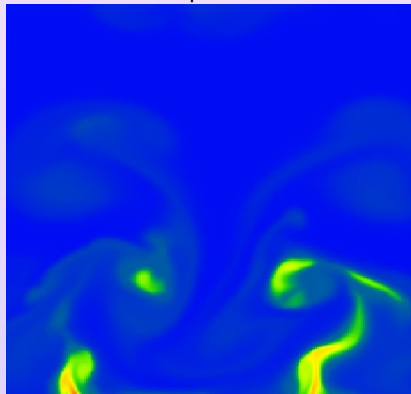
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

▶▶ Skip

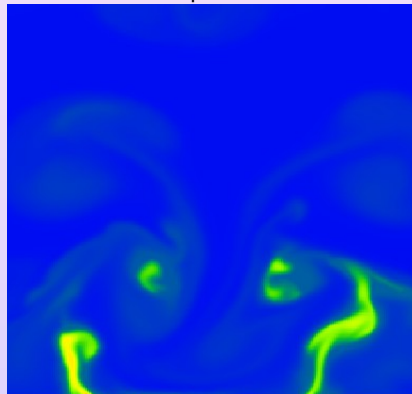


# FILM

Mass Fraction  $y$



Temperature  $T$



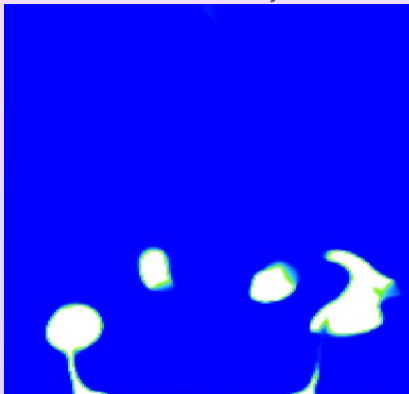
◀ Geometry

▶ Play

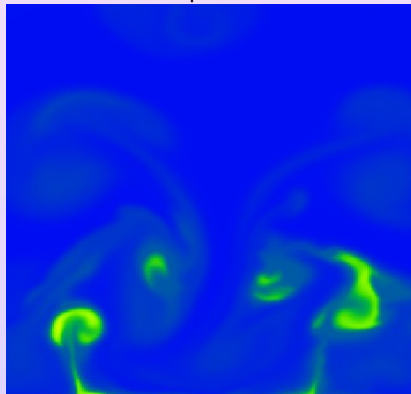
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

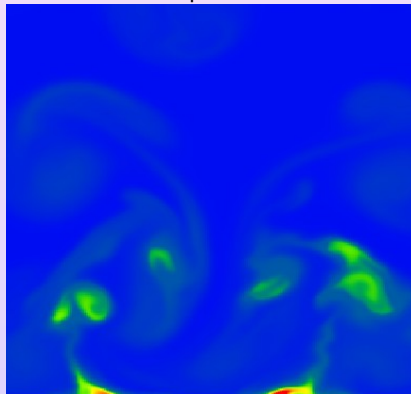
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

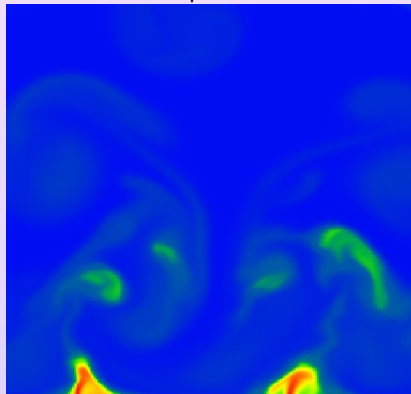
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

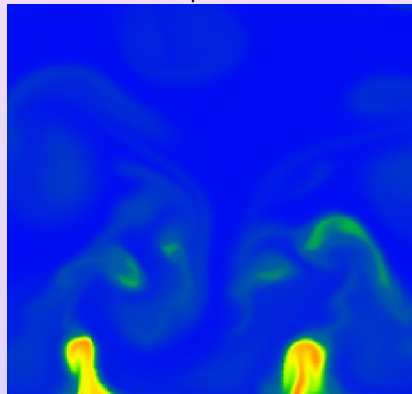
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# FILM

Mass Fraction  $y$



Temperature  $T$



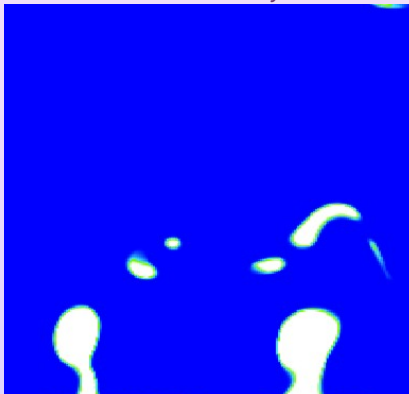
◀ Geometry

▶ Play

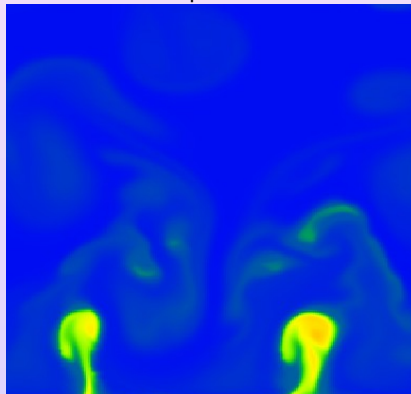
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# FILM

Mass Fraction  $y$



Temperature  $T$



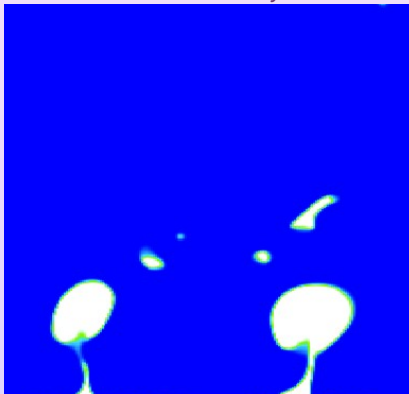
◀ Geometry

▶ Play

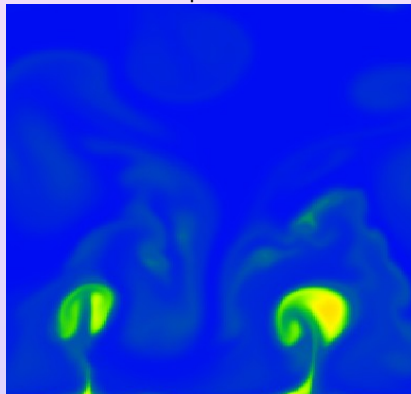
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# FILM

Mass Fraction  $y$



Temperature  $T$



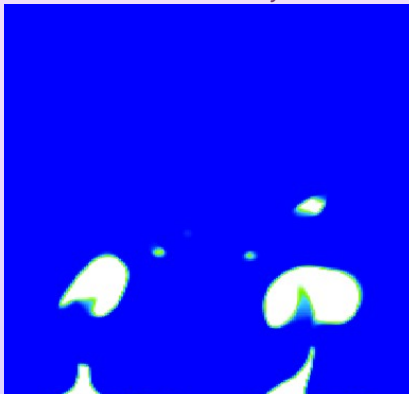
◀ Geometry

▶ Play

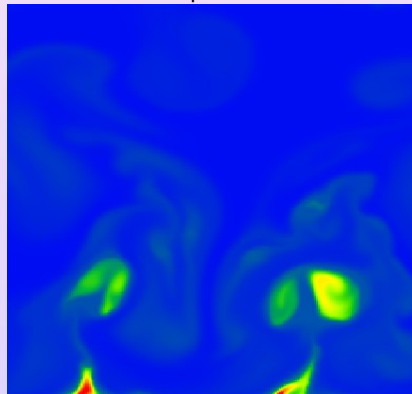
▶▶ Skip

# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

▶▶ Skip

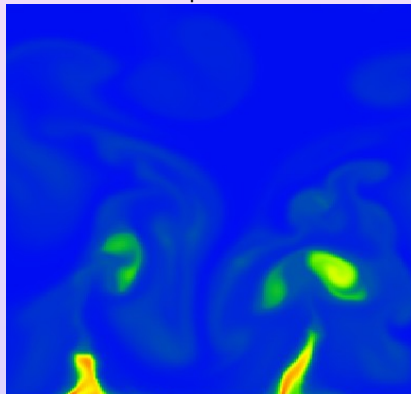


# FILM

Mass Fraction  $y$



Temperature  $T$



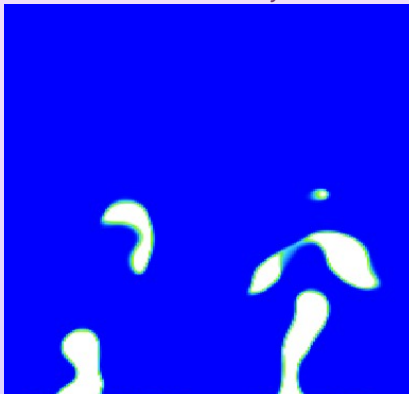
◀ Geometry

▶ Play

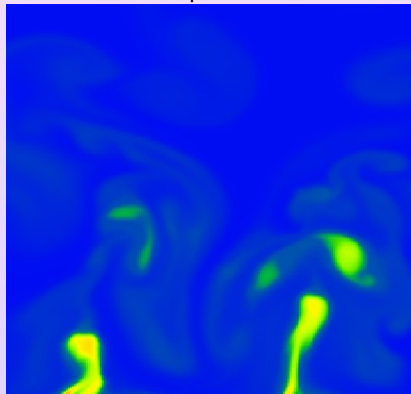
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# FILM

Mass Fraction  $y$



Temperature  $T$



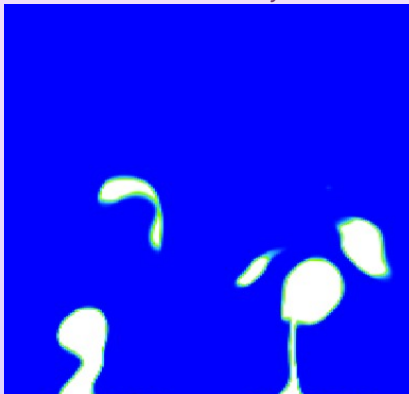
◀ Geometry

▶ Play

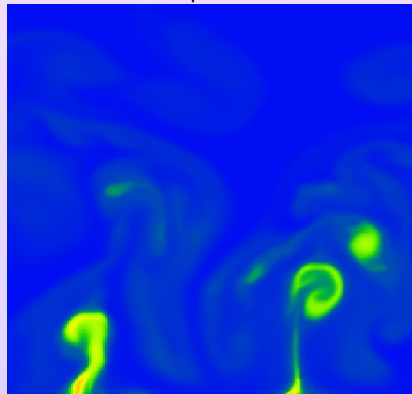
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# FILM

Mass Fraction  $y$



Temperature  $T$



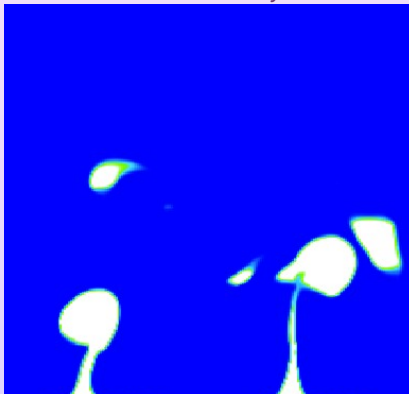
◀ Geometry

▶ Play

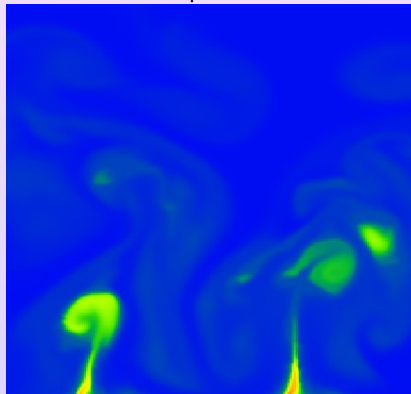
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# FILM

Mass Fraction  $y$



Temperature  $T$



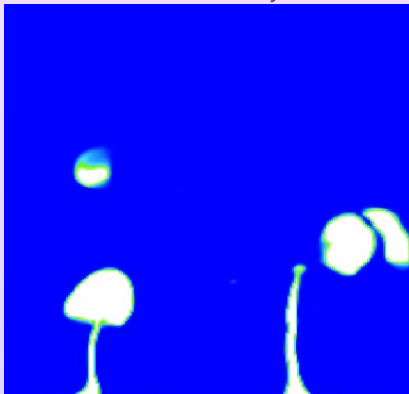
◀ Geometry

▶ Play

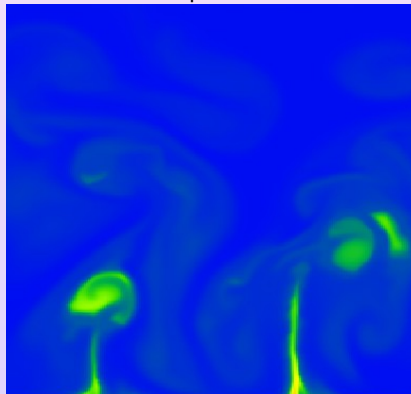
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

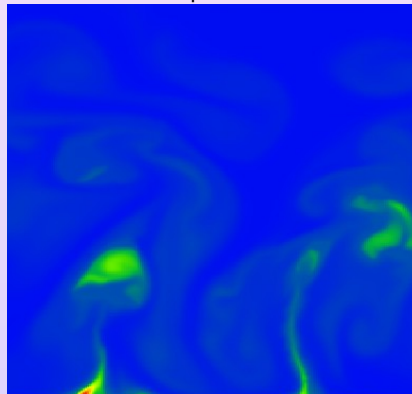
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

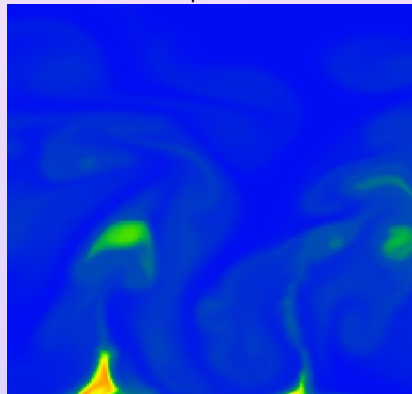
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# FILM

Mass Fraction  $y$



Temperature  $T$



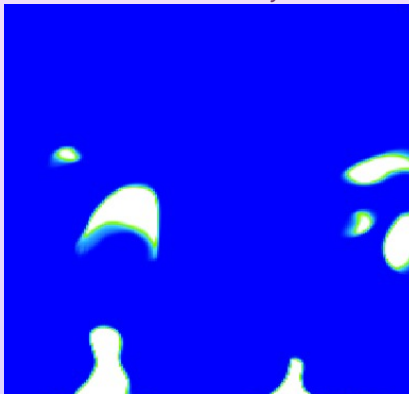
◀ Geometry

▶ Play

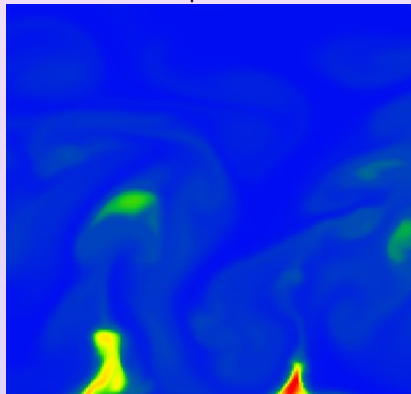
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

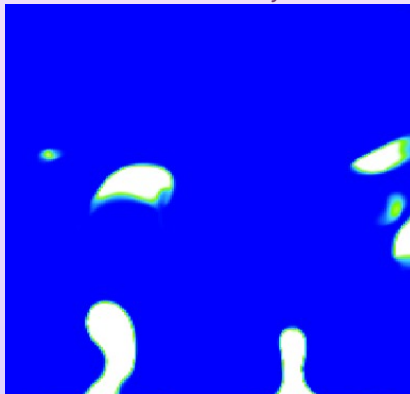
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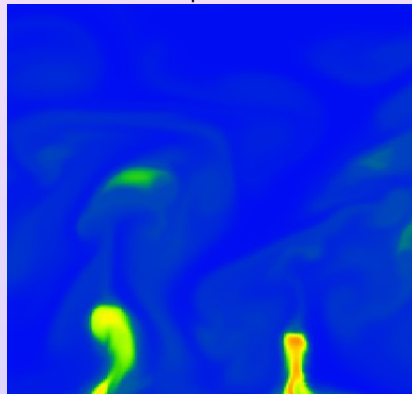


# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

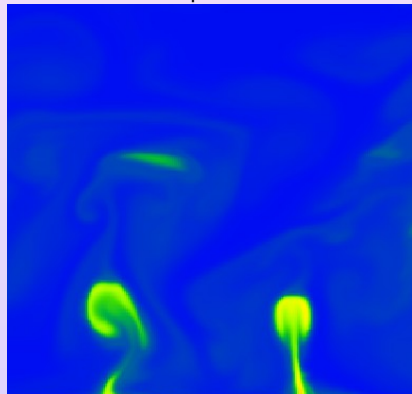
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

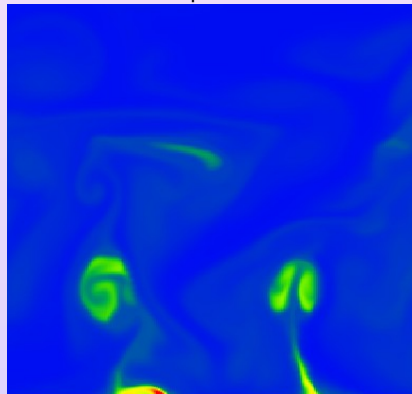
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# FILM

Mass Fraction  $y$



Temperature  $T$



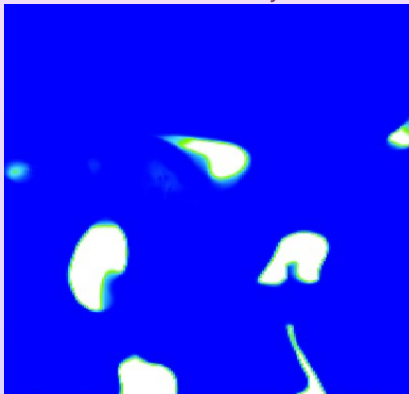
◀ Geometry

▶ Play

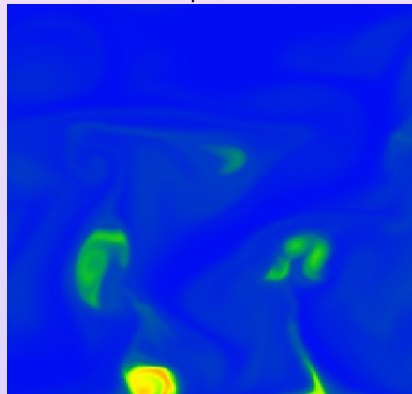
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

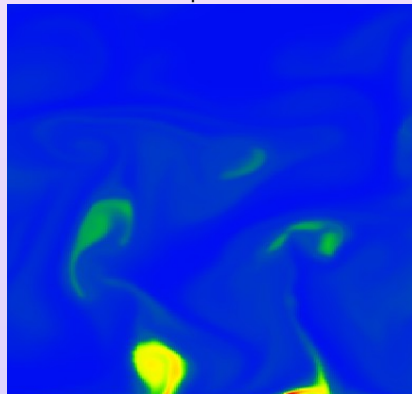
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

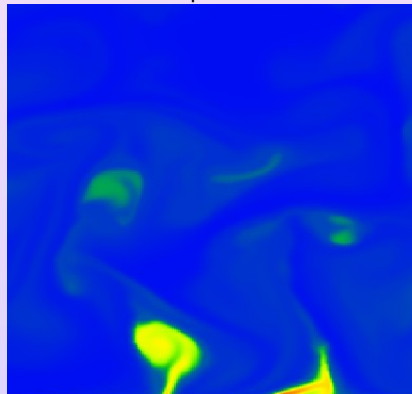
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# FILM

Mass Fraction  $y$



Temperature  $T$



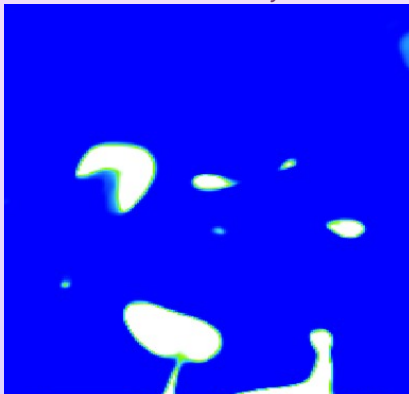
◀ Geometry

▶ Play

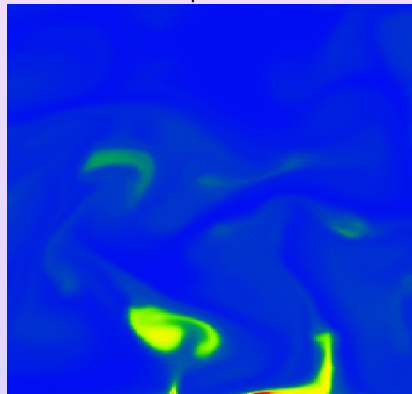
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

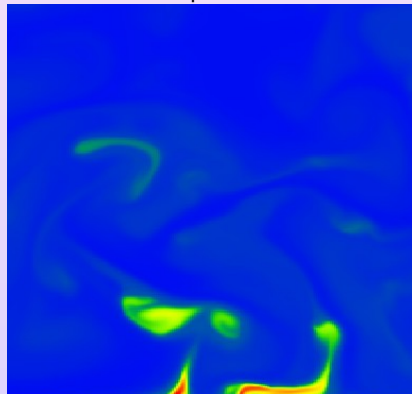
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

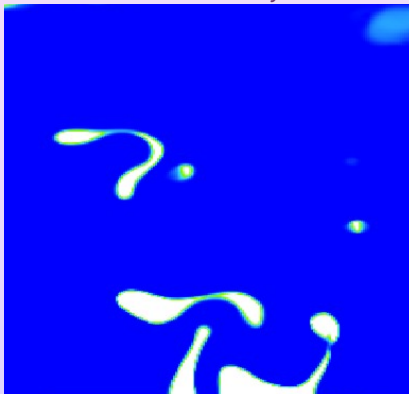
▶ Play

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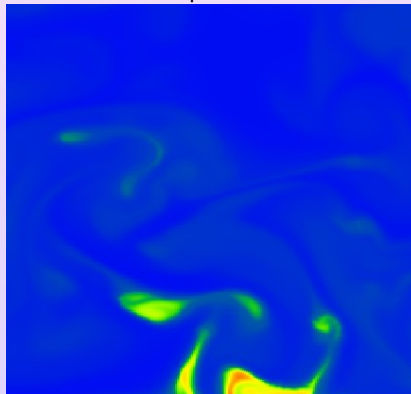


# FILM

Mass Fraction  $y$



Temperature  $T$



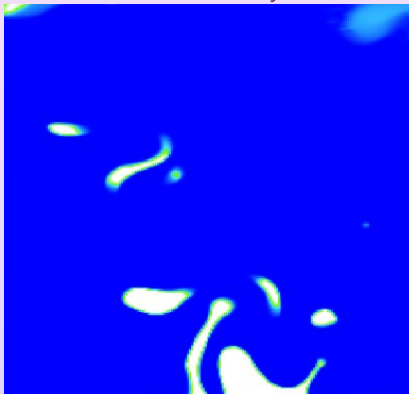
◀ Geometry

▶ Play

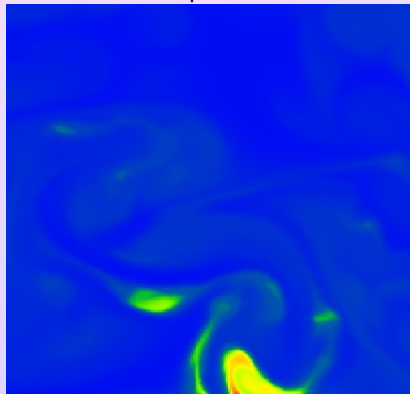
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# FILM

Mass Fraction  $y$



Temperature  $T$



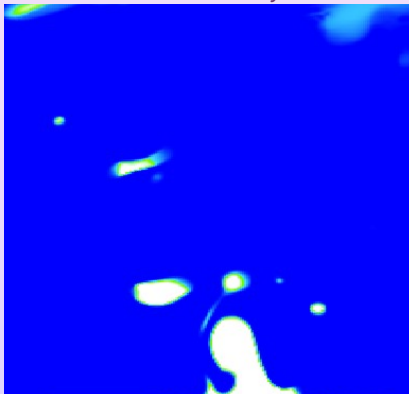
◀ Geometry

▶ Play

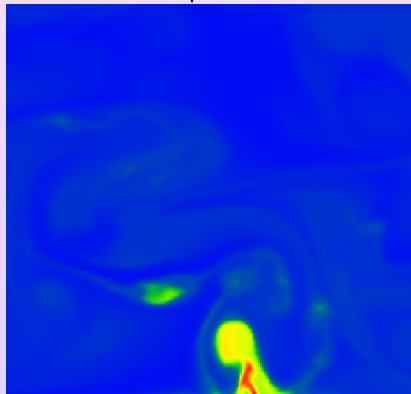
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# FILM

Mass Fraction  $y$



Temperature  $T$



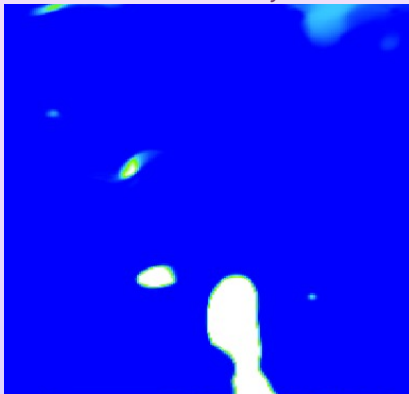
◀ Geometry

▶ Play

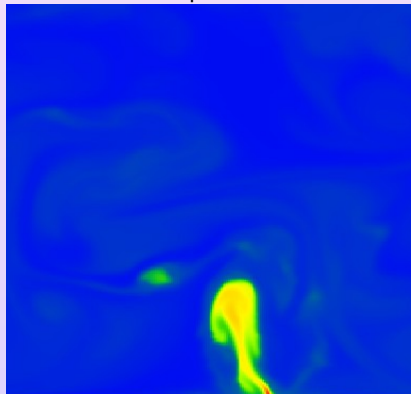
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

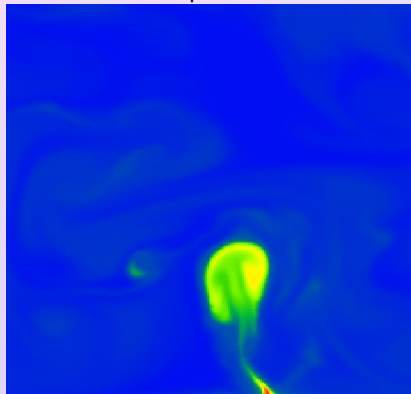
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# FILM

Mass Fraction  $y$



Temperature  $T$

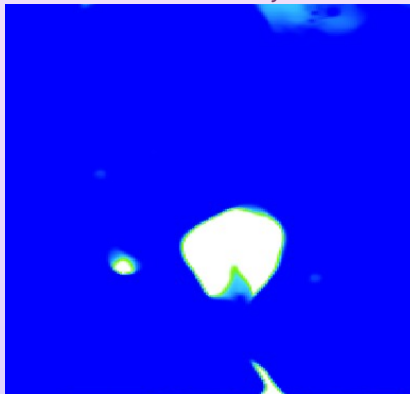
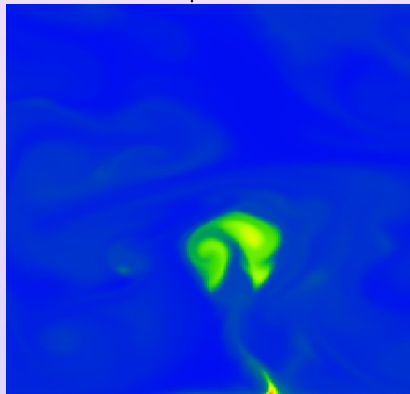


◀ Geometry

▶ Play

▶▶ Skip

## FILM

Mass Fraction  $y$ Temperature  $T$ 

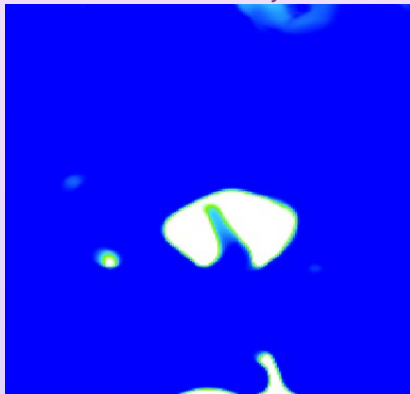
◀ Geometry

▶ Play

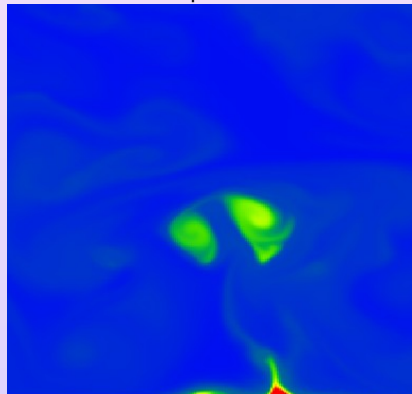
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

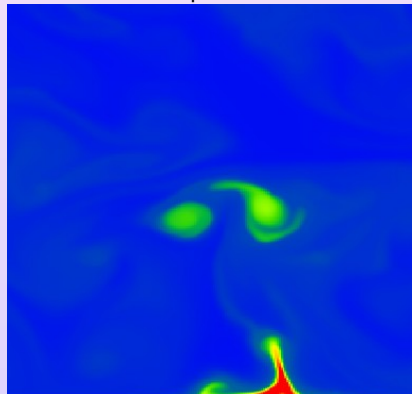
▶▶ Skip

# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

▶▶ Skip

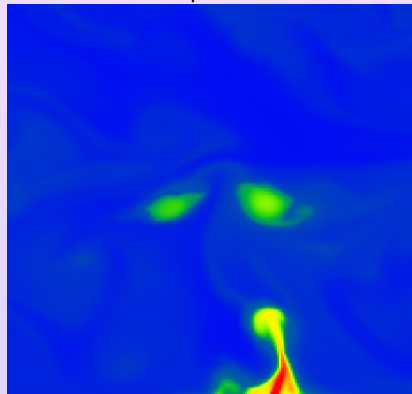


# FILM

Mass Fraction  $y$



Temperature  $T$



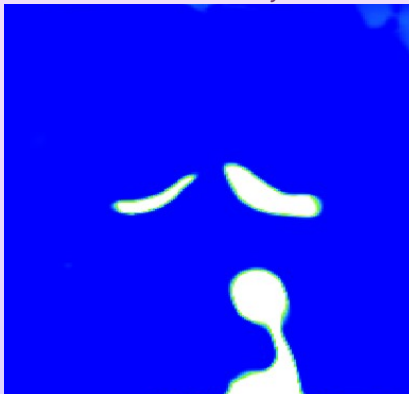
◀ Geometry

▶ Play

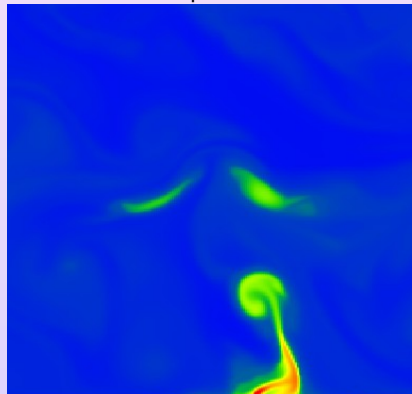
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

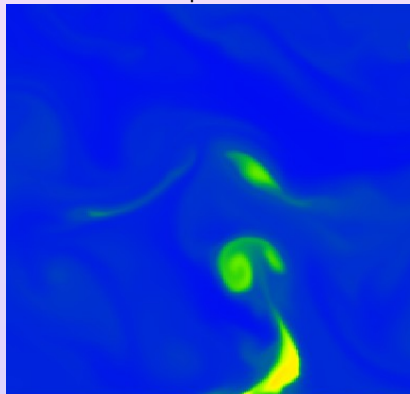
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

▶ Play

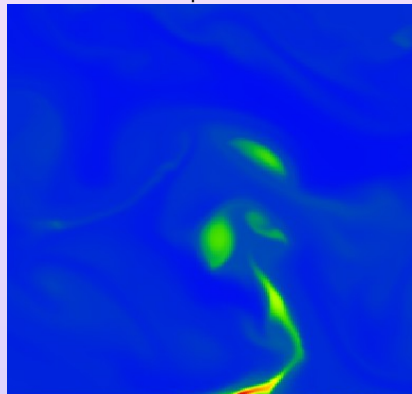
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# FILM

Mass Fraction  $y$



Temperature  $T$



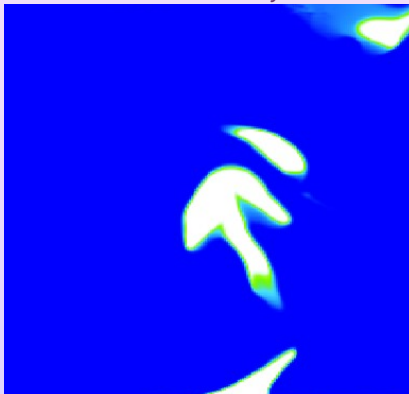
◀ Geometry

▶ Play

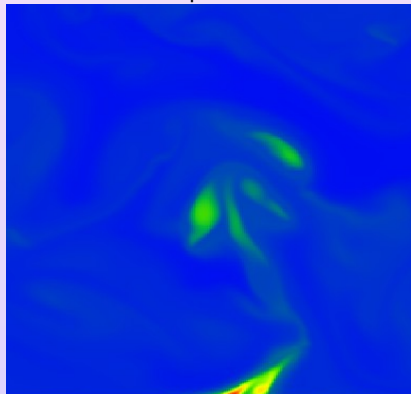
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# FILM

Mass Fraction  $y$



Temperature  $T$



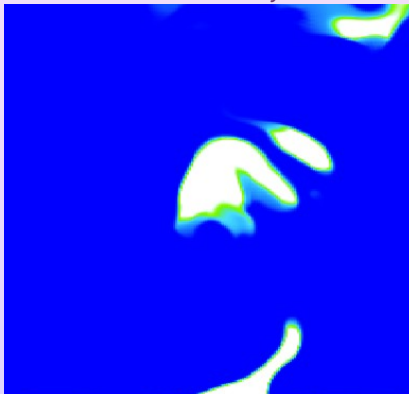
◀ Geometry

▶ Play

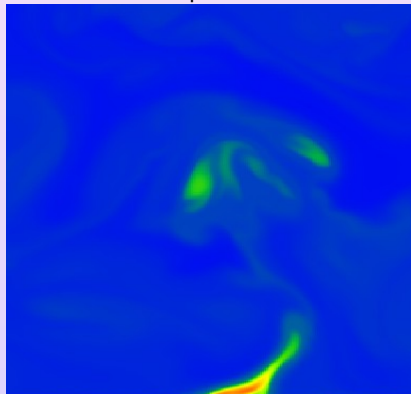
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# FILM

Mass Fraction  $y$



Temperature  $T$



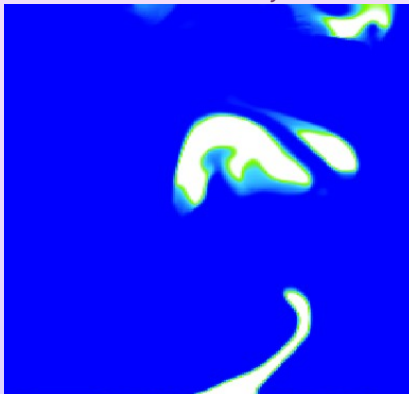
◀ Geometry

▶ Play

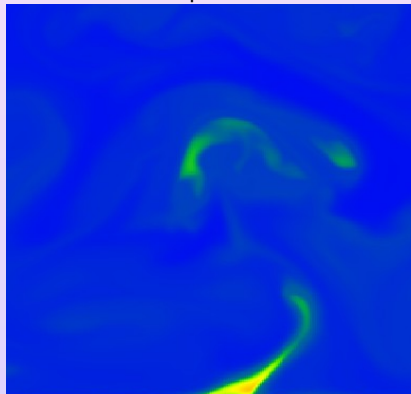
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# FILM

Mass Fraction  $y$



Temperature  $T$



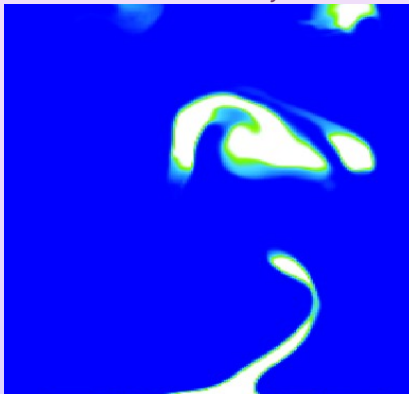
◀ Geometry

▶ Play

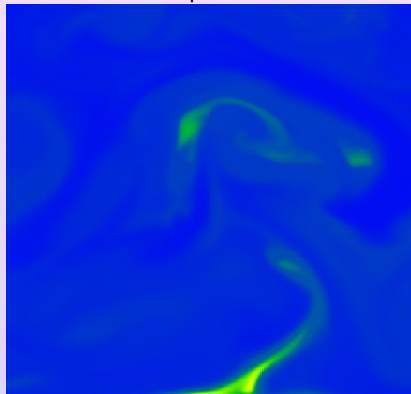
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# FILM

Mass Fraction  $y$



Temperature  $T$



◀ Geometry

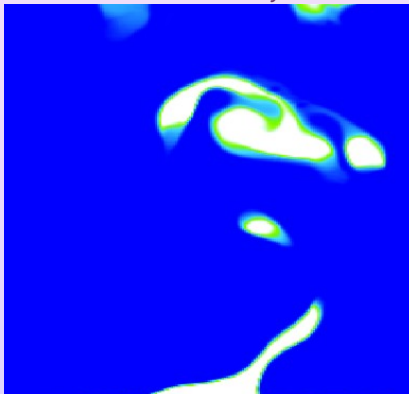
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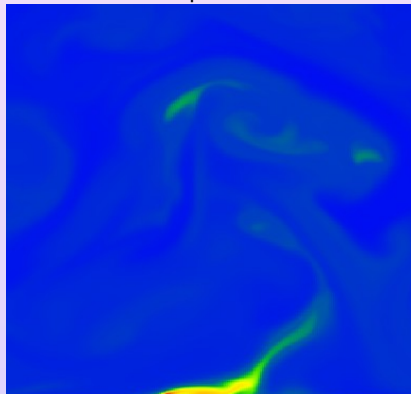


# FILM

Mass Fraction  $y$



Temperature  $T$



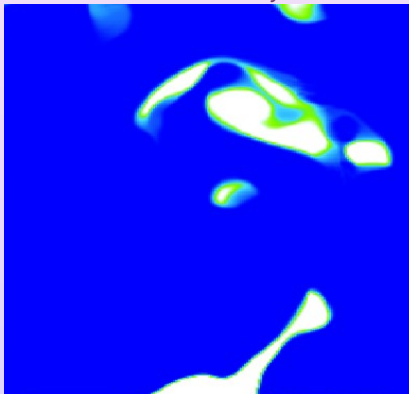
◀ Geometry

▶ Play

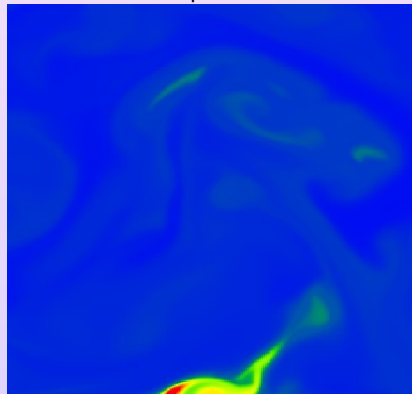
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# FILM

Mass Fraction  $y$



Temperature  $T$



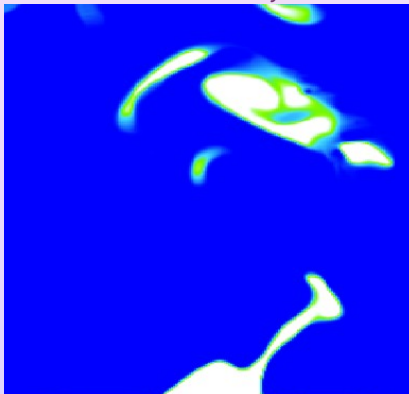
◀ Geometry

▶ Play

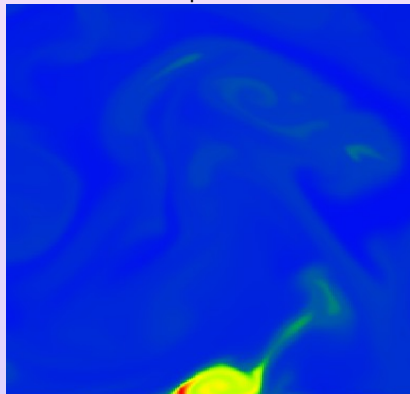
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# FILM

Mass Fraction  $y$



Temperature  $T$



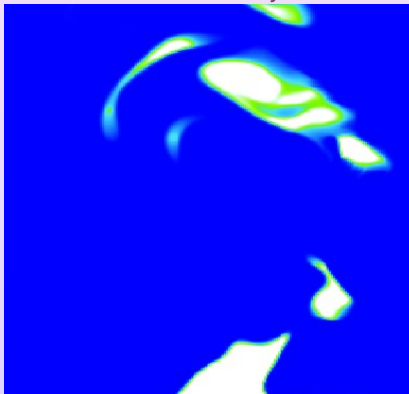
◀ Geometry

▶ Play

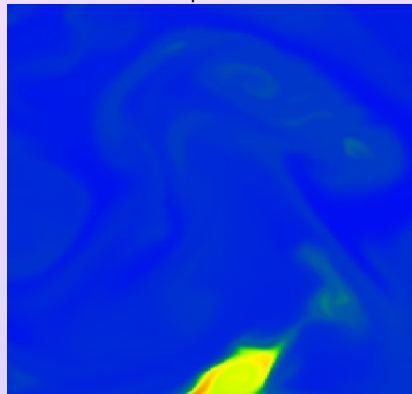
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# FILM

Mass Fraction  $y$



Temperature  $T$



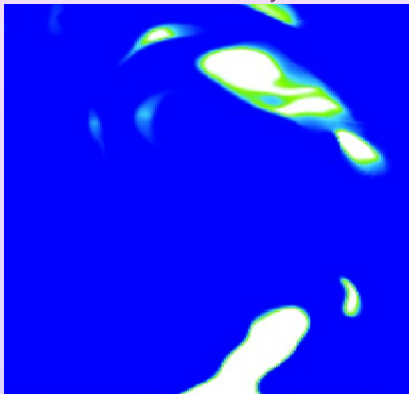
◀ Geometry

▶ Play

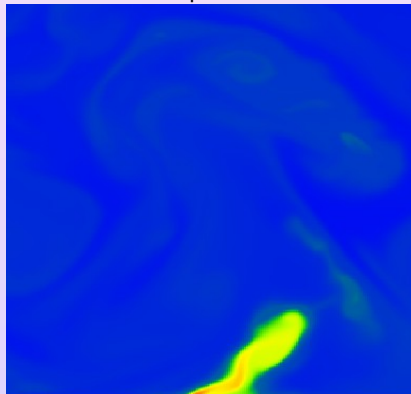
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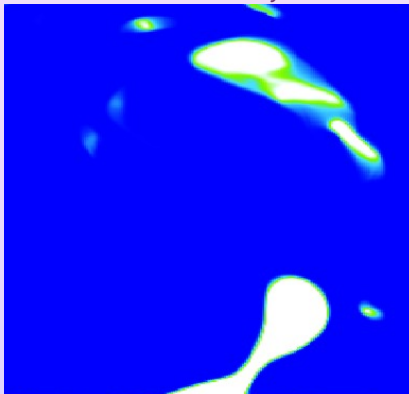
◀ Geometry

▶ Play

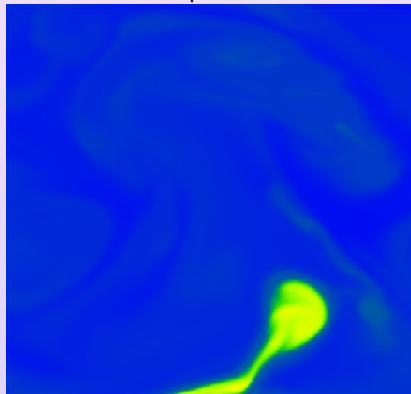
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# FILM

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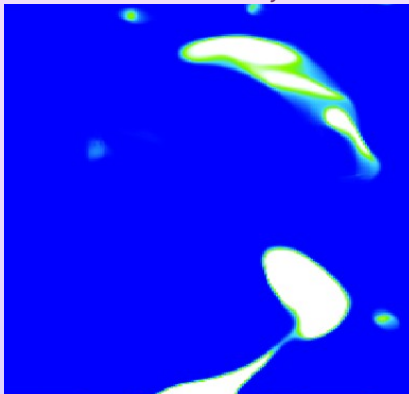
◀ Geometry

▶ Play

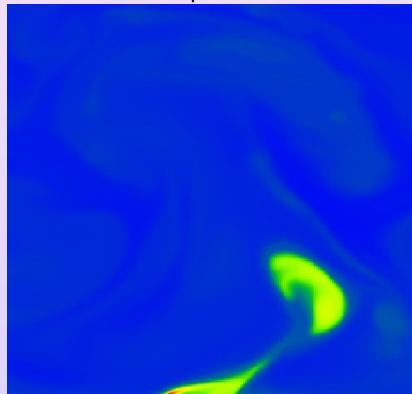
▶▶ Skip

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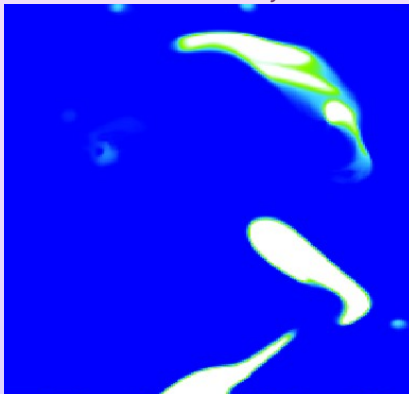
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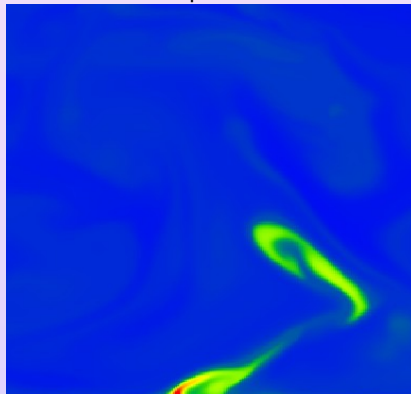
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Temperature  $T$



◀ Geometry

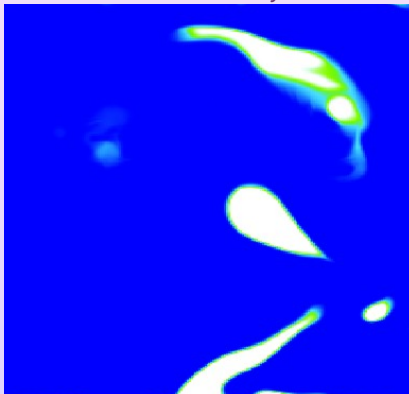
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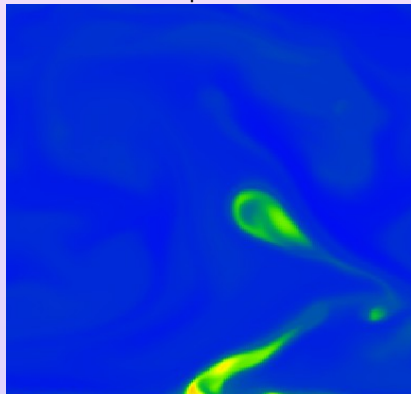


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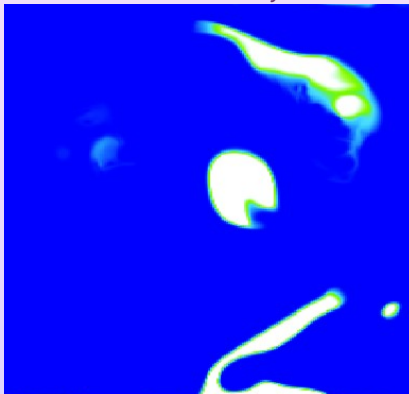
◀ Geometry

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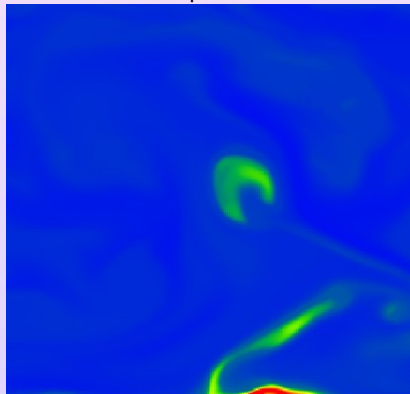
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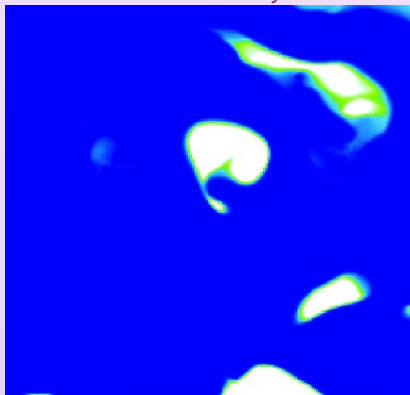
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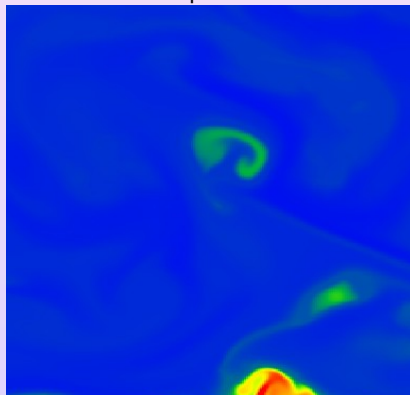
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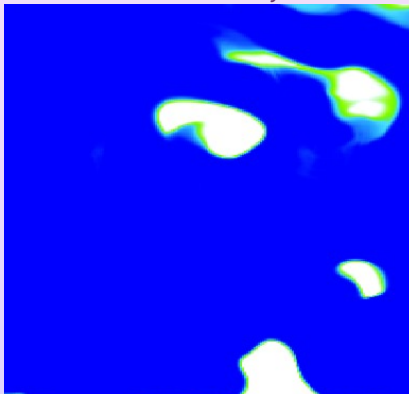
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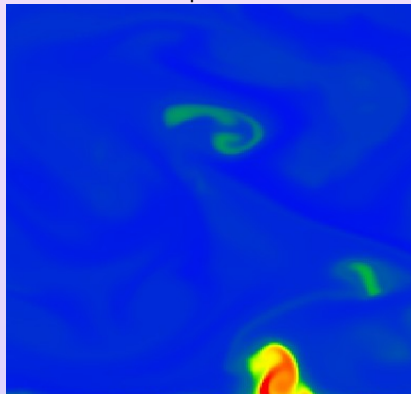
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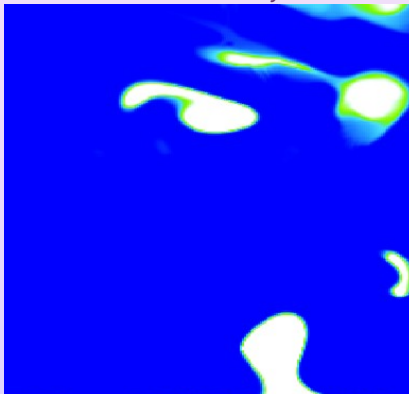
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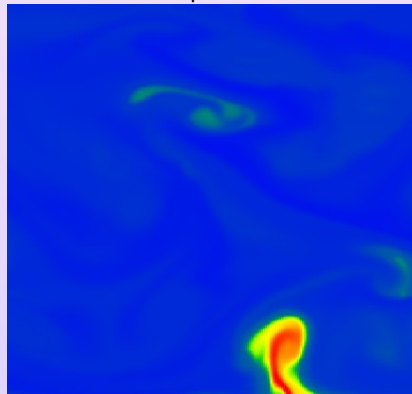
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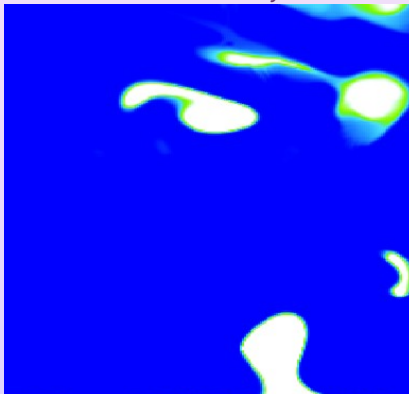
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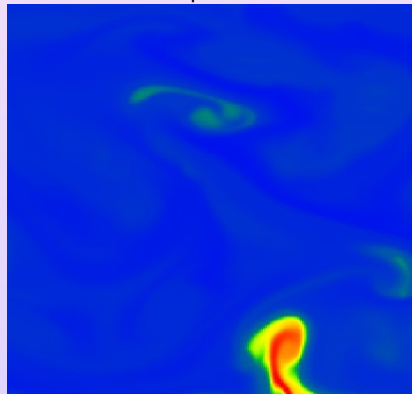
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▶ Play

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# OUTLINE

## 1 Context

## 2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- Conservation Laws

## 3 Numerical Approximation

- Numerical Method
- Numerical Examples

## 4 Conclusion

# SUMMARY & PERSPECTIVES

## ● Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,

## ● Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],

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- ✓ 2D with Stiffened Gas EOS for
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- ✗ **3D simulations.**

# APPENDIX

- ▶ Stiffened Gas for Water
- ▶ Tabulated EOS for Water
- ▶ Speed of sound
- ▶ Isentropic curves
- ▶ Surface Tension
- ▶ Metastability
- ▶ Critical Point

# STIFFENED GAS FOR WATER

Phase	$c_v$ [J/(kg·K)]	$\gamma$	$\pi$ [Pa]	$q$ [J/kg]	$m$ [J/(kg·K)]
Water	1816.2	2.35	$10^9$	$-1167.056 \times 10^3$	-32765.55596
Steam	1040.14	1.43	0	$2030.255 \times 10^3$	-33265.65947

**Table:** Parameters proposed by [O. LE METAYER] for water.

$$(\tau_\alpha, \varepsilon_\alpha) \mapsto s_\alpha = c_{v\alpha} \ln(\varepsilon_\alpha - q_\alpha - \pi_\alpha \tau_\alpha) + c_{v\alpha} (\gamma_\alpha - 1) \ln \tau_\alpha + m_\alpha$$

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# WATER TABULATED EOS

$T$ (K)	$P^{\text{sat}}$ (MPa)	Volume ( $\text{m}^3/\text{kg}$ )		Internal Energy (kJ/kg)	
		$\tau_{\text{liq}}^{\text{sat}}$	$\tau_{\text{vap}}^{\text{sat}}$	$\epsilon_{\text{liq}}^{\text{sat}}$	$\epsilon_{\text{vap}}^{\text{sat}}$
275	0,00069845	0,0010001	181,60	7,7590	2377,5
278	0,00086349	0,0010001	148,48	20,388	2381,6
281	0,0010621	0,0010002	122,01	32,996	2385,7
284	0,0012999	0,0010004	100,74	45,586	2389,8
287	0,0015835	0,0010008	83,560	58,162	2393,9
290	0,0019200	0,0010012	69,625	70,727	2398,0
293	0,0023177	0,0010018	58,267	83,284	2402,1
296	0,0027856	0,0010025	48,966	95,835	2406,2
299	0,0033342	0,0010032	41,318	108,38	2410,3
302	0,0039745	0,0010041	35,002	120,92	2414,4
305	0,0047193	0,0010050	29,764	133,46	2418,4
308	0,0055825	0,0010060	25,403	146	2422,5
...	...	...	...	...	...

Source: <http://webbook.nist.gov/chemistry/fluid/>



# WATER TABULATED EOS

$$\left. \begin{array}{l} T^i = 278\text{K} \dots 610\text{K}, \\ \epsilon_{\alpha}^{\text{sat}}(T^i), \tau_{\alpha}^{\text{sat}}(T^i) \text{ found in the tables} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \mathfrak{A} = \left\{ \left( T_i, \frac{1}{\epsilon_{\text{vap}}^{\text{sat}}(T_i)} \right) \right\}_i \\ \mathfrak{B} = \left\{ \left( T_i, \frac{\epsilon_{\text{liq}}^{\text{sat}}(T_i)}{\epsilon_{\text{vap}}^{\text{sat}}(T_i)} \right) \right\}_i \\ \mathfrak{C} = \left\{ \left( T_i, \frac{1}{\tau_{\text{vap}}^{\text{sat}}(T_i)} \right) \right\}_i \\ \mathfrak{D} = \left\{ \left( T_i, \frac{\tau_{\text{liq}}^{\text{sat}}(T_i)}{\tau_{\text{vap}}^{\text{sat}}(T_i)} \right) \right\}_i \end{array} \right.$$

$\widehat{\epsilon}_{\alpha}^{\text{sat}}$  and  $\widehat{\tau}_{\alpha}^{\text{sat}}$  defined by using a least square approximation of  $\mathfrak{A}$ ,  $\mathfrak{B}$ ,  $\mathfrak{C}$  and  $\mathfrak{D}$ :

$$T \mapsto \epsilon_{\text{vap}}^{\text{sat}} \approx \widehat{\epsilon}_{\text{vap}}^{\text{sat}} \stackrel{\text{def}}{=} \frac{1}{\sum_{k=0}^6 a_k T^k}$$

$$T \mapsto \epsilon_{\text{liq}}^{\text{sat}} \approx \widehat{\epsilon}_{\text{liq}}^{\text{sat}} \stackrel{\text{def}}{=} \widehat{\epsilon}_{\text{vap}}^{\text{sat}}(T) \sum_{k=0}^6 b_k T^k$$

$$T \mapsto \tau_{\text{vap}}^{\text{sat}} \approx \widehat{\tau}_{\text{vap}}^{\text{sat}} \stackrel{\text{def}}{=} \frac{1}{\sum_{k=0}^8 c_k T^k}$$

$$T \mapsto \tau_{\text{liq}}^{\text{sat}} \approx \widehat{\tau}_{\text{liq}}^{\text{sat}} \stackrel{\text{def}}{=} \widehat{\tau}_{\text{vap}}^{\text{sat}}(T) \sum_{k=0}^9 d_k T^k$$

# SPEED OF SOUND

$$c^2 \stackrel{\text{def}}{=} \tau^2 \left( P^{\text{eq}} \frac{\partial P^{\text{eq}}}{\partial \varepsilon} \Big|_{\tau} - \frac{\partial P^{\text{eq}}}{\partial \tau} \Big|_{\varepsilon} \right) = \overset{0}{-\tau^2 \mathcal{T}^{\text{eq}}} \begin{bmatrix} P^{\text{eq}}, & -1 \end{bmatrix} \begin{bmatrix} S_{\varepsilon\varepsilon}^{\text{eq}} & S_{\tau\varepsilon}^{\text{eq}} \\ S_{\tau\varepsilon}^{\text{eq}} & S_{\tau\tau}^{\text{eq}} \end{bmatrix} \begin{bmatrix} P^{\text{eq}} \\ -1 \end{bmatrix} \leq 0$$

## HESSIAN MATRIX OF $w \mapsto s^{\text{eq}}$

- for all  $w$  pure phase state

$$\mathbf{v}^T d^2 s^{\text{eq}}(w) \mathbf{v} < 0 \quad \forall \mathbf{v} \neq 0,$$

- for all  $w$  equilibrium mixture state

$$\exists \mathbf{v}(w) \neq 0 \text{ s.t. } (\mathbf{v}(w))^T d^2 s^{\text{eq}}(w) \mathbf{v}(w) = 0.$$

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## HESSIAN MATRIX OF $\mathbf{w} \mapsto s^{\text{eq}}$

- for all  $\mathbf{w}$  pure phase state

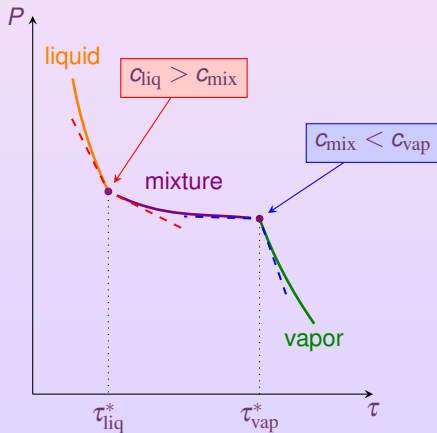
$$\mathbf{v}^T d^2 s^{\text{eq}}(\mathbf{w}) \mathbf{v} < 0 \quad \forall \mathbf{v} \neq 0,$$

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$$\forall \mathbf{w} \text{ equilibrium mixture state, } \mathbf{v}(\mathbf{w}) \not\propto [P^{\text{eq}}(\mathbf{w}), -1]$$

# ISENTROPIC CURVES



$$\gamma \stackrel{\text{def}}{=} - \frac{\tau}{P} \frac{\partial P}{\partial \tau} \Big|_s$$

$$\Gamma \stackrel{\text{def}}{=} \tau \frac{\partial P}{\partial \varepsilon} \Big|_\tau$$

$$\mathcal{G} \stackrel{\text{def}}{=} \frac{\tau^2}{2\gamma P} \frac{\partial^2 P}{\partial \tau^2} \Big|_s$$

- Pure Phases

- (H)  $\gamma > 0$
- (H)  $\Gamma > 0$
- (H)  $\mathcal{G} > 0$

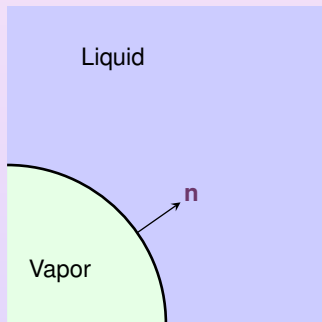
- Mixture

- (P)  $\gamma > 0$
- (P)  $\Gamma > 0$
- (H)  $\mathcal{G} > 0$

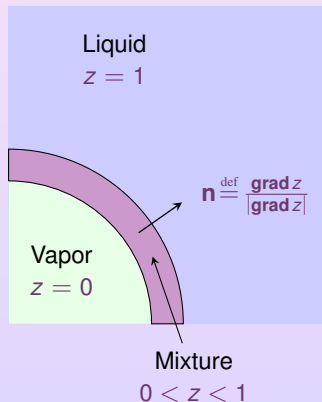
- Regularity: [J. CORREIA, P.G. LEFLOCH, M.D. THANH]
- Loss of convexity: [A. VOSS]

# CONTINUUM SURFACE FORCE (CSF) APPROACH

Physical Interface



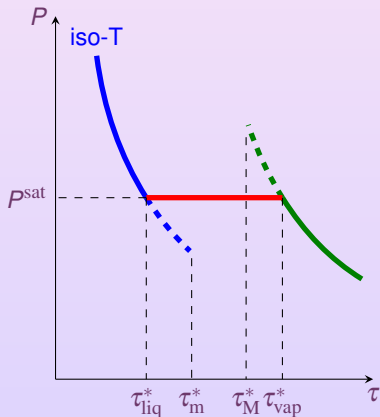
Diffuse Interface



$$\Pi_{\text{tension}} = -\sigma \text{div}(\mathbf{n})\mathbf{n}$$



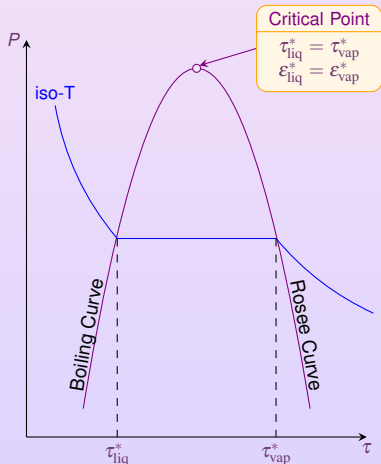
# METASTABILITY



$$P^{\text{eq}} = \begin{cases} P_{\text{liq}}, & \text{if } \tau < \tau_{\text{liq}}^*, \\ P^{\text{sat}}, & \text{if } \tau_{\text{liq}}^* < \tau < \tau_{\text{vap}}^*, \\ P_{\text{vap}}, & \text{if } \tau_{\text{vap}}^* < \tau. \end{cases}$$

$$P^{\text{met}} = \begin{cases} P_{\text{liq}}, & \text{if } \tau < \tau_{\text{liq}}^*, \\ [P^{\text{sat}} \text{ or } P_{\text{liq}}], & \text{if } \tau_{\text{liq}}^* < \tau < \tau_m^*, \\ P^{\text{sat}}, & \text{if } \tau_m^* < \tau < \tau_M^*, \\ [P^{\text{sat}} \text{ or } P_{\text{vap}}], & \text{if } \tau_M^* < \tau < \tau_{\text{vap}}^*, \\ P_{\text{vap}}, & \text{if } \tau_{\text{vap}}^* < \tau, \end{cases}$$

# CRITICAL POINT



## PHYSIC

- 2 Pure Phases EOS  $(\tau, \epsilon) \mapsto P_\alpha$
  - 1 Saturation EOS  $\tau \mapsto P^{\text{sat}}$
- Eq

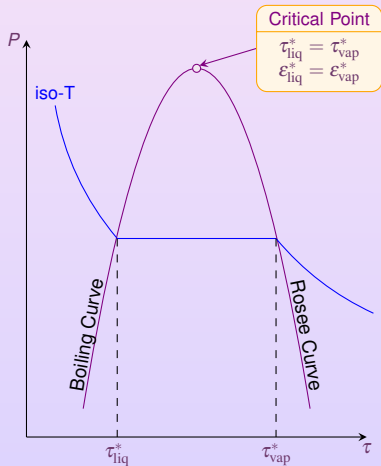
## EOS

**PG**  $\epsilon_{\text{liq}}^* = \epsilon_{\text{vap}}^* \Leftrightarrow c_{V\text{liq}} = c_{V\text{vap}}$  (indip. of  $T$ )

**SG**  $\{\tau_i, P_i^{\text{sat},e}\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha \rightsquigarrow \tau \mapsto P^{\text{sat}}$   
 $\tau_{\text{liq}}^* = \tau_{\text{vap}}^*$  but  $\epsilon_{\text{liq}}^* \neq \epsilon_{\text{vap}}^*$

**TAB**  $\{\tau_i, P_i^{\text{sat},e}\}_i \rightsquigarrow \tau \mapsto P^{\text{sat}}$   
 $\{(\tau_i, \epsilon_i), (P_\alpha^c)_i\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha$

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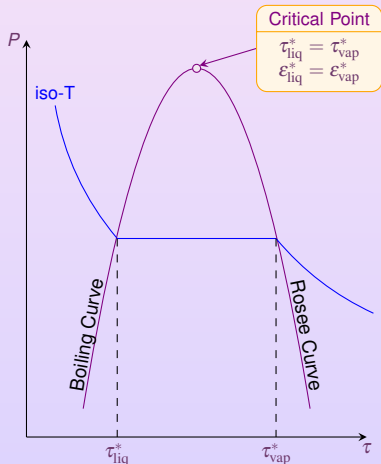
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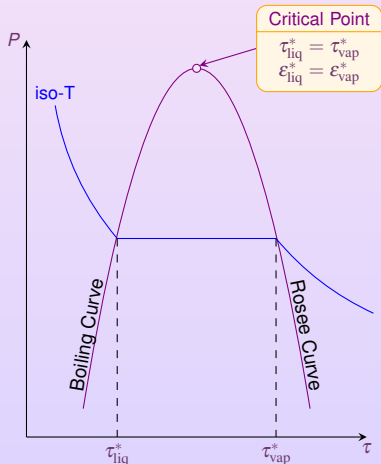
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