

Ultracold Bose gases in random potentials: elementary excitations and localization effects

Pierre Lugan

PhD defense

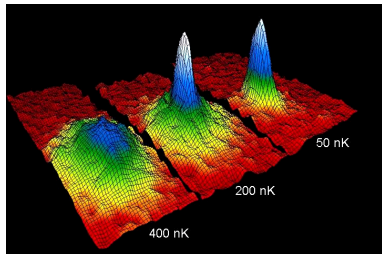
January 25th, 2010

Supervision: Philippe Bouyer & Laurent Sanchez-Palencia
Groupe d'Optique Atomique, Laboratoire Charles Fabry, Institut d'Optique

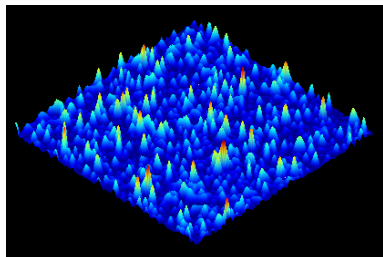


Ultracold gases in random potentials

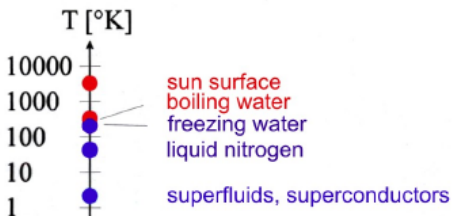
Ultracold gases



Disorder

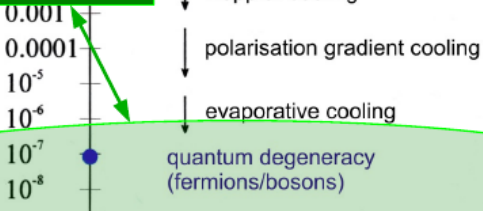


Ultracold gases



$$\lambda_{dB} \sim d$$

Ultra-cold Gases
 $10^{12}-10^{15}$ at/cm³



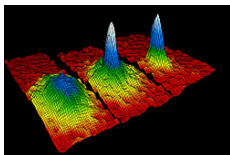
$$\lambda_{dB} \sim d$$

Source: A. Recati

- Ultra-cold/quantum gases: $T < 1 \mu\text{K} \leftrightarrow \lambda_{\text{dB}} > 300 \text{ nm}$
⇒ atomic matter-waves

Ultracold gases

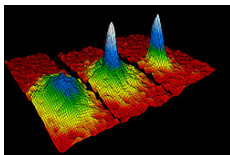
- Ultra-cold/quantum gases: $T < 1 \mu\text{K} \leftrightarrow \lambda_{\text{dB}} > 300 \text{ nm}$
⇒ atomic matter-waves
- Bosons
⇒ Bose-Einstein condensate:



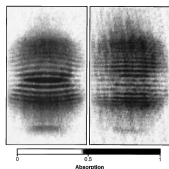
Anderson *et al.*
Science 1995

- Ultra-cold/quantum gases: $T < 1 \mu\text{K} \leftrightarrow \lambda_{\text{dB}} > 300 \text{ nm}$
⇒ atomic matter-waves

- Bosons
⇒ **Bose-Einstein condensate:**



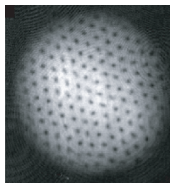
Anderson *et al.*
Science 1995



Andrews *et al.* Science 1997

← Giant coherent **matter-wave**

Interactions (⇒ superfluidity) →



Abo-Shaer *et al.* Science 2001

Gross-Pitaevskii equation (analogous to light in Kerr medium):

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi + g|\psi|^2\psi$$

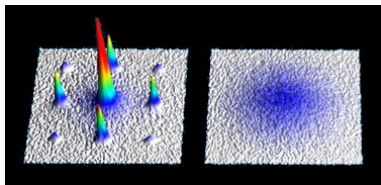
- **Versatility**

- quantum statistics, dimensionality, tunable interactions, shape, bulk/lattice...

- **Variety of diagnostic tools**

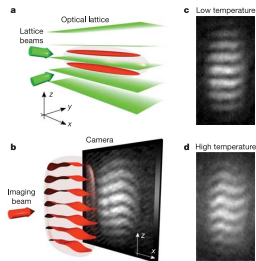
- absorption imaging, collective oscillations, Bragg spectroscopy, noise correlation...

Mott-insulator transition



Greiner *et al.* Nature 2002

BKT cross-over



Hadzibabic *et al.* Nature 2006

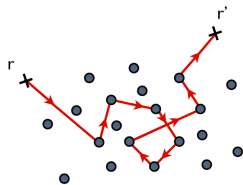
- Model systems → direct comparison with theory
 - dilute gases: simple interaction potential
 - textbook Hamiltonians, parameters known ab-initio

- Controlled disorder
 - optical speckle, impurity atoms, radio-frequency fields, quasi-periodic potentials...

⇒ Good laboratory for physics of disordered quantum systems

Multiple scattering and interference

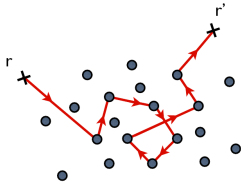
Classical particles



Multiple scattering \Rightarrow diffusion

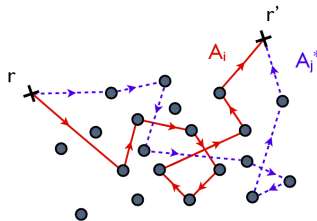
Multiple scattering and interference

Classical particles



Multiple scattering \Rightarrow diffusion

Quantum particles and waves



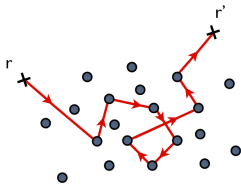
$$P(\mathbf{r}, \mathbf{r}') = |\sum_i A_i|^2 = \sum_i A_i A_i^* + \sum_{i \neq j} A_i A_j^*$$

Weak disorder: $l_e \gg \lambda$ (l_e mean free path)

$$P(\mathbf{r}, \mathbf{r}') \simeq \sum_i A_i A_i^*$$

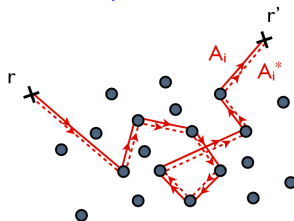
Multiple scattering and interference

Classical particles



Multiple scattering \Rightarrow diffusion

Quantum particles and waves

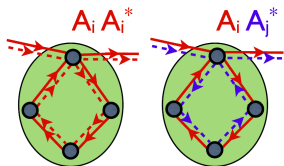


$$P(\mathbf{r}, \mathbf{r}') = |\sum_i A_i|^2 = \sum_i A_i A_i^* + \sum_{i \neq j} A_i A_j^*$$

Weak disorder: $l_e \gg \lambda$ (l_e mean free path)

$$P(\mathbf{r}, \mathbf{r}') \simeq \sum_i A_i A_i^*$$

However ...

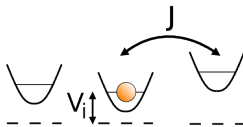
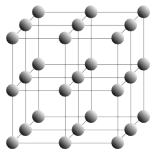


\Rightarrow Interference corrections

- enhanced probability of return
- diffusion constant $D = D_B - \delta D$
- conductivity $\sigma = \sigma_B - \delta \sigma$

Anderson (strong) localization

Anderson, Phys. Rev. 1958



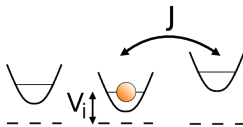
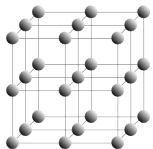
Strong disorder

- ⇒ absence of diffusion
- ⇒ exponential decay of wave function in space

Anderson localization

Anderson (strong) localization

Anderson, Phys. Rev. 1958



Strong disorder

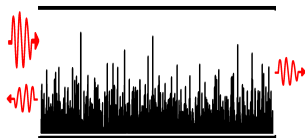
- ⇒ absence of diffusion
- ⇒ exponential decay of wave function in space

Anderson localization

Signatures of localization

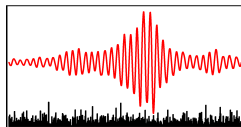
(open systems) transport properties

e.g. transmission T across a 1D sample of length L



$$\frac{1}{2} \langle \ln T(L) \rangle \sim -L/L_{\text{loc}} \quad (L \rightarrow \infty)$$

(closed systems) localized eigenstates



$$\langle \ln |\psi(\mathbf{r})| \rangle \sim -|\mathbf{r}|/L_{\text{loc}} \quad (|\mathbf{r}| \rightarrow \infty)$$

L_{loc} : localization length ↔

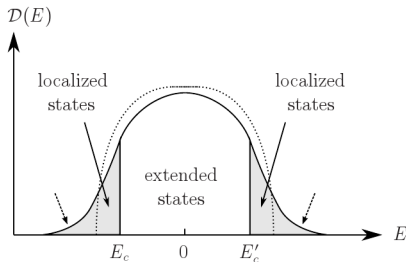
$\gamma = 1/L_{\text{loc}}$: Lyapunov exponent

The role of dimensionality

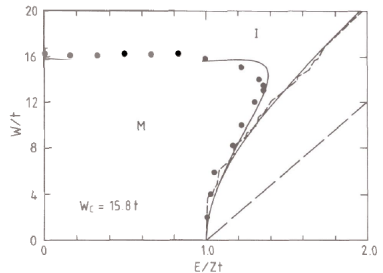
Scaling theory of localization: [Abrahams et al. PRL 1979](#)

- 1D, 2D: all states localized for arbitrary amplitude of disorder
- 3D: metal-insulator phase transition

3D Anderson model (presence of a lattice):



E_c, E'_c mobility edges



[Vollhardt and Wölfle, in *Electronic phase transitions*, Elsevier \(1992\)](#)

Experimental results

Recent reviews: [A. Legendijk, B. van Tiggelen, and D. S. Wiersma, Physics Today **62**, 24 \(2009\)](#)
[A. Aspect and M. Inguscio, Physics Today **62**, 30 \(2009\)](#)
[L. Sanchez-Palencia and M. Lewenstein, Nat. Phys. \(2009\), in press; arXiv:0911.0629](#)

Electronic systems (80s)

Coulomb interaction and phonons

Experimental results

Recent reviews: A. Lagendijk, B. van Tiggelen, and D. S. Wiersma, *Physics Today* **62**, 24 (2009)
A. Aspect and M. Inguscio, *Physics Today* **62**, 30 (2009)
L. Sanchez-Palencia and M. Lewenstein, *Nat. Phys.* (2009), in press; arXiv:0911.0629

Electronic systems (80s)

Coulomb interaction and phonons

Classical waves (90s-today): acoustic waves, microwaves, light...

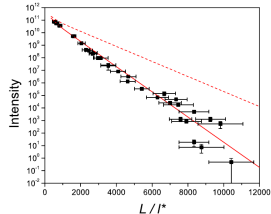
- transmission experiments:

Wiersma *et al.* *Nature* 1997 (light);
Chabanov *et al.* *Nature* 2000 (microwaves);
Störzer *et al.* *PRL* 2006 (light);
Hu *et al.* *Nat. Phys.* 2008 (ultrasound) ...

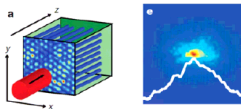
- localized wave functions:

Schwartz *et al.* *Nature* 2007, Lahini *et al.* *PRL* 2008

Aegerter *et al.* *Europhys. Lett.* 2006



Schwartz *et al.* *Nature* 2007



Experimental results

Recent reviews: A. Lagendijk, B. van Tiggelen, and D. S. Wiersma, *Physics Today* **62**, 24 (2009)
A. Aspect and M. Inguscio, *Physics Today* **62**, 30 (2009)
L. Sanchez-Palencia and M. Lewenstein, *Nat. Phys.* (2009), in press; arXiv:0911.0629

Electronic systems (80s)

Coulomb interaction and phonons

Classical waves (90s-today): acoustic waves, microwaves, light...

- transmission experiments:

Wiersma *et al.* *Nature* 1997 (light);
Chabanov *et al.* *Nature* 2000 (microwaves);
Störzer *et al.* *PRL* 2006 (light);
Hu *et al.* *Nat. Phys.* 2008 (ultrasound) ...

- localized wave functions:

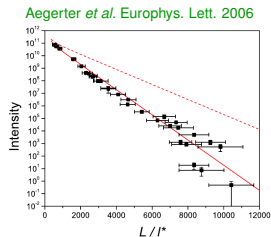
Schwartz *et al.* *Nature* 2007, Lahini *et al.* *PRL* 2008

Cold atoms (90s-today)

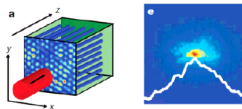
Localization in momentum space

Ultracold atoms

Localized atomic wave functions



Schwartz *et al.* *Nature* 2007



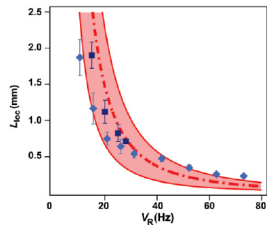
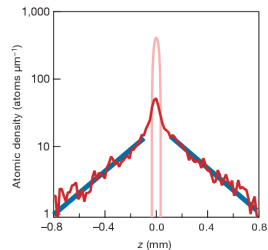
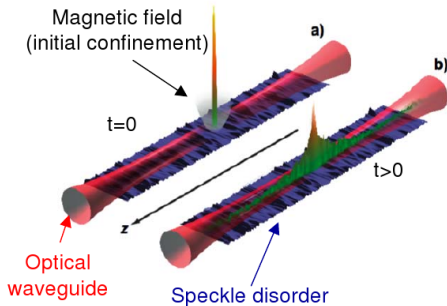
Moore *et al.* *PRL* 1995, Chabé *et al.* *PRL* 2008

Billy *et al.* *Nature* 2008, Roati *et al.* *Nature* 2008

1D Anderson localization of a BEC in a speckle potential

Theory: L. Sanchez-Palencia *et al.* PRL **98**, 210401 (2007); NJP **10**, 045019 (2008)

Experiment: J. Billy *et al.*, Nature **453**, 891 (2008)



Measured localization lengths in agreement with theory ↗

Interplay of disorder and interactions

- Anderson localization in the presence of interactions ?
- Remarkable properties of disorder-free interacting systems:
 - Superconductivity of weakly-attractive fermions
 - Superfluidity of repulsive bosons

⇒ **extended** coherent many-body states

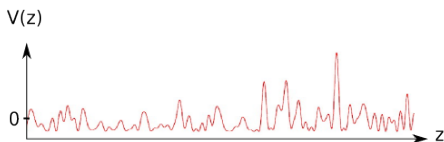
To what extent do **disorder and localization phenomena** alter those properties ?

Reppy J. Low Temp. Phys. 1992

- 1 Anderson localization of **non-interacting particles** in correlated one-dimensional disorder
- 2 Localization in **weakly-interacting Bose gases** in one-dimensional disorder
 - Fragmentation of the **ground-state** density profile
 - Localization of **elementary excitations**

- 1 Anderson localization of non-interacting particles in correlated one-dimensional disorder
- 2 Localization in weakly-interacting Bose gases in one-dimensional disorder
 - Fragmentation of the ground-state density profile
 - Localization of elementary excitations

Random potential:



- Average: $\langle V \rangle = 0$
- Amplitude: $V_R = \sqrt{\langle V^2 \rangle}$
- Correlation functions:

$$C_2(z_1) = \langle V(z_0) V(z_0 + z_1) \rangle$$

$$C_3(z_1, z_2) = \langle V(z_0) V(z_0 + z_1) V(z_0 + z_2) \rangle$$

$$\dots = \dots$$

- Correlation length : $\sigma_R \leftarrow$ width of C_2

Lyapunov exponent of single particles in 1D

$$-\frac{\hbar^2}{2m}\partial_z^2\psi + V(z)\psi = E\psi$$

- Single-particle states are all localized in 1D

Gertsenshtein and Vasil'ev 1959

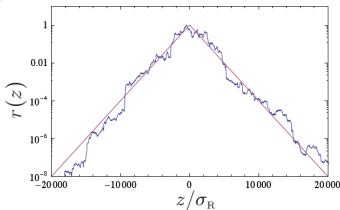
Mott and Twose 1961

Borland 1963

Mathematical proofs e.g. :

Goldsheid, Molchanov, and Pastur 1977

Kotani and Simon 1987



Lyapunov exponent of single particles in 1D

$$-\frac{\hbar^2}{2m}\partial_z^2\psi + V(z)\psi = E\psi$$

- Single-particle states are all localized in 1D

Gertsenshtein and Vasil'ev 1959

Mott and Twose 1961

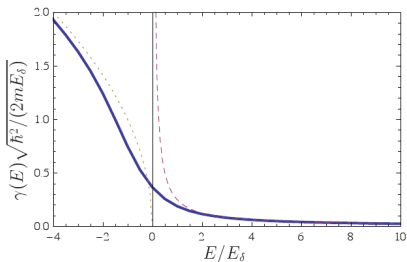
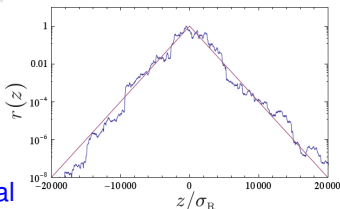
Borland 1963

Mathematical proofs e.g. :

Goldsheid, Molchanov, and Pastur 1977

Kotani and Simon 1987

- Lyapunov exponent in an **uncorrelated potential**



$$C_2(z) = D\delta(z)$$

$$E_\delta = \left(\frac{mD^2}{2\hbar^2}\right)^{1/3}$$

Lyapunov exponent of single particles in 1D

$$-\frac{\hbar^2}{2m}\partial_z^2\psi + V(z)\psi = E\psi$$

- Single-particle states are all localized in 1D

Gertsenshtein and Vasil'ev 1959

Mott and Twose 1961

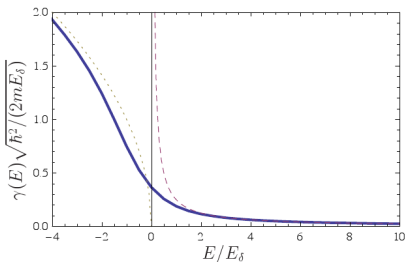
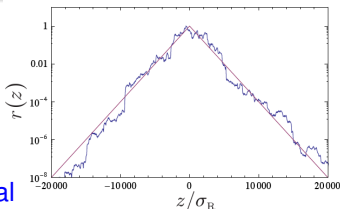
Borland 1963

Mathematical proofs e.g. :

Goldsheid, Molchanov, and Pastur 1977

Kotani and Simon 1987

- Lyapunov exponent in an **uncorrelated potential**



$$C_2(z) = D\delta(z)$$

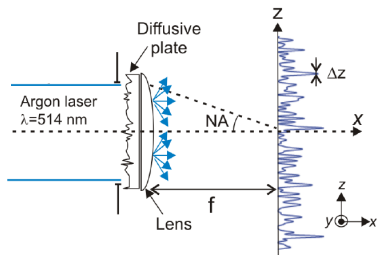
$$E_\delta = \left(\frac{mD^2}{2\hbar^2}\right)^{1/3}$$

- Lyapunov exponent in a **correlated potential**

Born approximation: $\gamma \simeq \frac{\sqrt{2\pi}}{2} \left(\frac{m}{\hbar^2 k}\right)^2 \hat{C}_2(2k)$

$$E = \frac{\hbar^2 k^2}{2m}$$

Speckle potentials



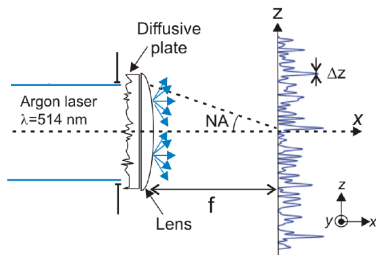
Blue-detuned light

⇒ repulsive potential $V(z) \propto I(z)/\delta$

grain size $\Delta z \propto \sigma_R$ inversely proportional to aperture size

D. Clément *et al.*, *New J. Phys.* **8**, 165 (2006)

- auto-correlation $C_2(z) = V_R^2 \text{sinc}(z/\sigma_R)^2$
- precise experimental control of amplitude V_R and correlation length σ_R



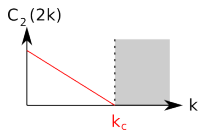
Blue-detuned light

⇒ repulsive potential $V(z) \propto I(z)/\delta$

grain size $\Delta z \propto \sigma_R$ inversely proportional to aperture size

D. Clément *et al.*, *New J. Phys.* **8**, 165 (2006)

- auto-correlation $C_2(z) = V_R^2 \text{sinc}(z/\sigma_R)^2$
- precise experimental control of amplitude V_R and correlation length σ_R
- sharp cut-off in Fourier spectrum: $k_c = 1/\sigma_R$



⇒ Born approximation $\gamma \simeq \frac{\sqrt{2\pi}}{2} \left(\frac{m}{\hbar^2 k}\right)^2 \hat{C}_2(2k)$ vanishes for $k > k_c$

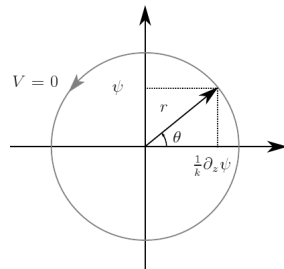
Phase formalism for the Schrödinger equation

$$-\partial_z^2 \psi + V(z)\psi = k^2 \psi \quad (k = \sqrt{E}, \langle V \rangle = 0)$$

- Polar representation:

$$\psi(z) = r(z) \sin[\theta(z)]$$

$$\partial_z \psi(z) = kr(z) \cos[\theta(z)]$$



Phase formalism for the Schrödinger equation

$$-\partial_z^2 \psi + V(z)\psi = k^2 \psi \quad (k = \sqrt{E}, \langle V \rangle = 0)$$

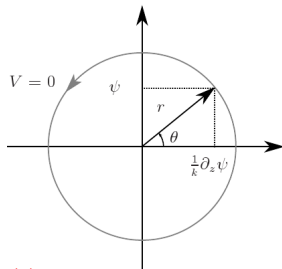
- Polar representation:

$$\psi(z) = r(z) \sin[\theta(z)]$$

$$\partial_z \psi(z) = kr(z) \cos[\theta(z)]$$

- Phase: $\partial_z \theta(z) = k - \frac{V(z)}{k} \sin^2[\theta(z)]$

- Weak-disorder expansion: $\theta = kz + \sum_{n \geq 1} \delta\theta^{(n)}$



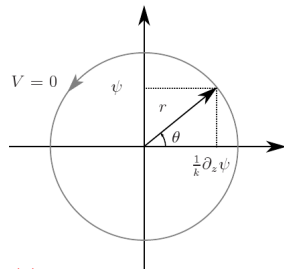
Phase formalism for the Schrödinger equation

$$-\partial_z^2 \psi + V(z)\psi = k^2 \psi \quad (k = \sqrt{E}, \langle V \rangle = 0)$$

- Polar representation:

$$\psi(z) = r(z) \sin[\theta(z)]$$

$$\partial_z \psi(z) = kr(z) \cos[\theta(z)]$$



- Phase: $\partial_z \theta(z) = k - \frac{V(z)}{k} \sin^2[\theta(z)]$
- Weak-disorder expansion: $\theta = kz + \sum_{n \geq 1} \delta \theta^{(n)}$
- Wave function amplitude: $\ln[r(z)/r(0)] = \int_0^z dz' \frac{V(z')}{2k} \sin[2\theta(z')]$
- Lyapunov exponent: $\gamma = + \lim_{|z| \rightarrow \infty} \frac{\langle \ln[r(z)] \rangle}{|z|}$

Phase formalism for the Schrödinger equation

⇒ Perturbation series for the Lyapunov exponent:

$$\gamma(k) = \sum_{n \geq 2} \gamma^{(n)}(k)$$

$$\gamma^{(n)} = \sigma_R^{-1} (\epsilon_R / k \sigma_R)^n f_n(k \sigma_R) \quad \epsilon_R = \frac{2m\sigma_R^2 V_R}{\hbar^2}$$

Phase formalism for the Schrödinger equation

⇒ Perturbation series for the Lyapunov exponent:

$$\gamma(k) = \sum_{n \geq 2} \gamma^{(n)}(k)$$

$$\gamma^{(n)} = \sigma_R^{-1} (\epsilon_R / k \sigma_R)^n f_n(k \sigma_R) \quad \epsilon_R = \frac{2m\sigma_R^2 V_R}{\hbar^2}$$

$$f_2(\kappa) = +\frac{1}{4} \int_{-\infty}^0 du c_2(u) \cos(2\kappa u) \quad [\text{Born}]$$

$$f_3(\kappa) = -\frac{1}{4} \int_{-\infty}^0 du \int_{-\infty}^u dv c_3(u, v) \sin(2\kappa v)$$

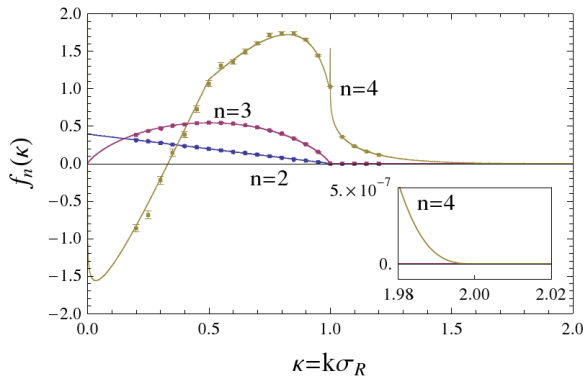
$$f_4(\kappa) = -\frac{1}{8} \int_{-\infty}^0 du \int_{-\infty}^u dv \int_{-\infty}^v dw c_4(u, v, w) \\ \times \{2 \cos(2\kappa w) + \cos[2\kappa(v+w-u)]\}.$$

Reduced correlation functions: $C_n(z_1, \dots, z_{n-1}) = V_R^n c_n(z_1/\sigma_R, \dots, z_{n-1}/\sigma_R)$

Lyapunov exponent in a speckle potential

$$\gamma^{(n)} = \sigma_R^{-1} (\epsilon_R / k \sigma_R)^n f_n(k \sigma_R)$$

f_n functions for usual speckle:

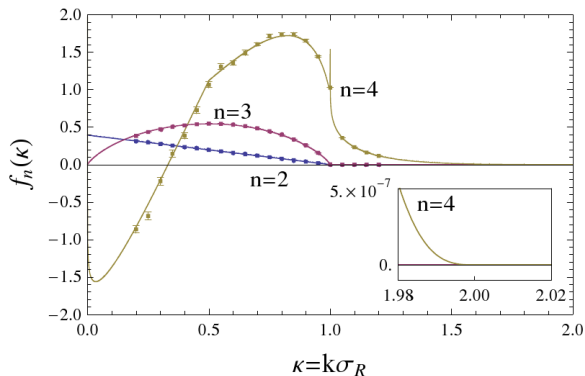


numerics by
D. Delande

Lyapunov exponent in a speckle potential

$$\gamma^{(n)} = \sigma_R^{-1} (\epsilon_R / k \sigma_R)^n f_n(k \sigma_R)$$

f_n functions for usual speckle:



numerics by
D. Delande

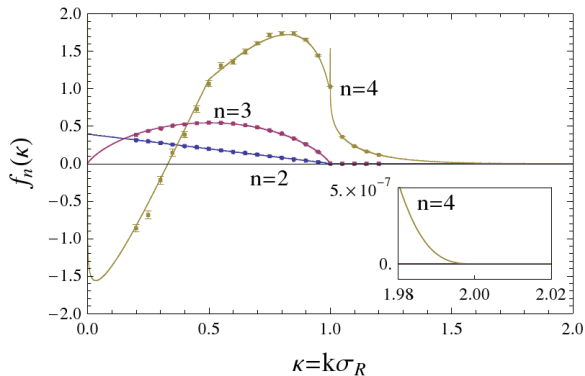
Remarkable features:

- f_3 has same cut-off as Born term $n=2$

Lyapunov exponent in a speckle potential

$$\gamma^{(n)} = \sigma_R^{-1} (\epsilon_R / k \sigma_R)^n f_n(k \sigma_R)$$

f_n functions for usual speckle:



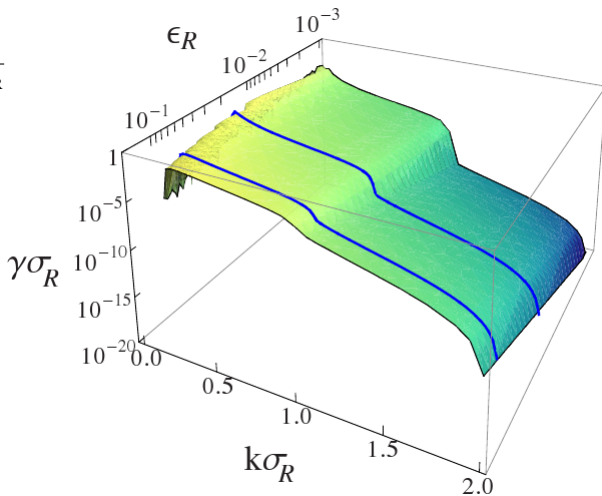
numerics by
D. Delande

Remarkable features:

- f_3 has same cut-off as Born term $n = 2$
- f_4 has cut-off at $k\sigma_R = 2$

Effective mobility edges in speckle potentials

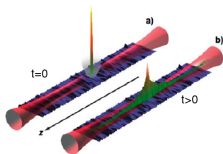
$$\epsilon_R = \frac{V_R}{E_\sigma}$$
$$E_\sigma = \frac{\hbar^2}{2m\sigma_R^2}$$



PL, A. Aspect, L. Sanchez-Palencia, D. Delande, C. A. Müller, B. Grémaud, and C. Miniatura, *PRA* **80**, 023605 (2009)

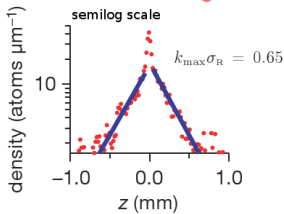
see also L. Tessieri, *J. Phys. A* **35**, 9585 (2002); E. Gurevich and O. Kenneth, *PRA* **79**, 063617 (2009)

Effective mobility edge in the experiment

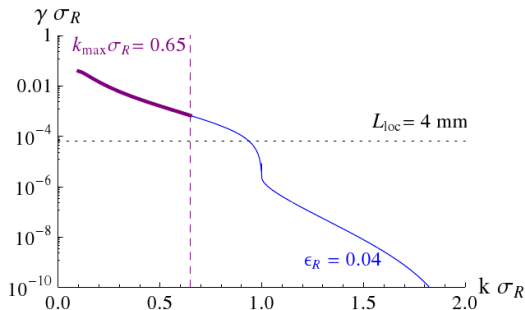


Theory: L. Sanchez-Palencia *et al.* PRL **98**, 210401 (2007)

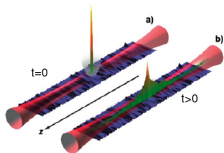
Experiment: J. Billy *et al.*, Nature **453**, 891 (2008)



Beyond the Born approximation:

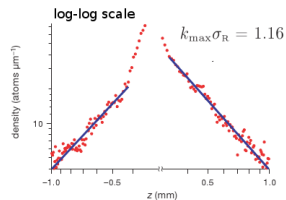
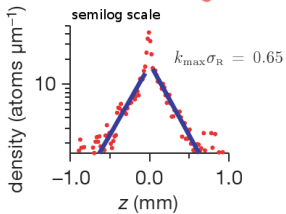


Effective mobility edge in the experiment

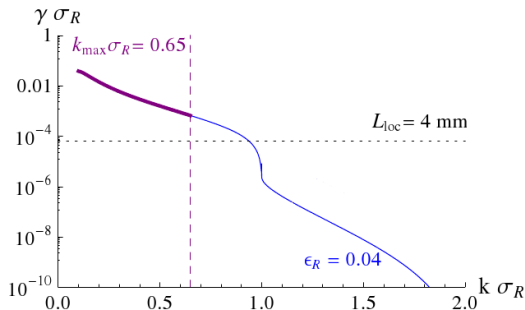


Theory: L. Sanchez-Palencia *et al.* PRL **98**, 210401 (2007)

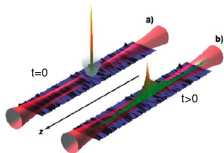
Experiment: J. Billy *et al.*, Nature **453**, 891 (2008)



Beyond the Born approximation:

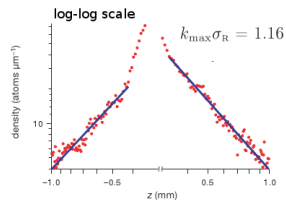
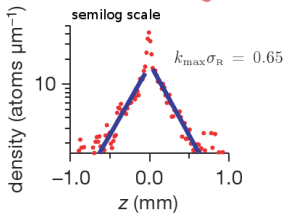


Effective mobility edge in the experiment

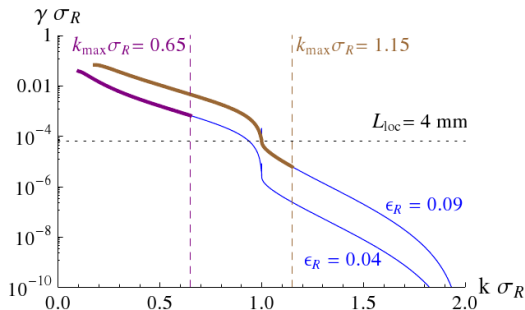


Theory: L. Sanchez-Palencia *et al.* PRL **98**, 210401 (2007)

Experiment: J. Billy *et al.*, Nature **453**, 891 (2008)



Beyond the Born approximation:

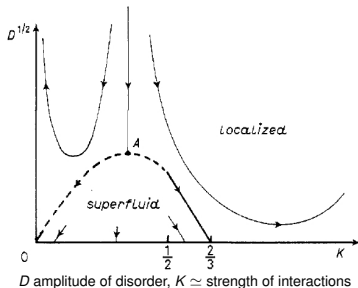


- 1 Anderson localization of **non-interacting particles** in correlated one-dimensional disorder
- 2 **Localization in weakly-interacting Bose gases in one-dimensional disorder**
 - Fragmentation of the **ground-state** density profile
 - Localization of **elementary excitations**

The boson localization problem

Hertz *et al.* PRL **43**, 942 (1979)

Ground-state quantum-phase diagram of disordered repulsive bosons in 1D

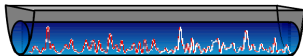


Giamarchi and Schulz, *Europhys. Lett.* **3**, 1287 (1987)

See also DMRG calculations: Rapsch *et al.*, *Europhys. Lett.* **46**, 559 (1999)

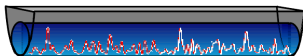
Localization-delocalization scenario in the weakly-interacting regime ?

Disordered 1D Bose gases at $T = 0$



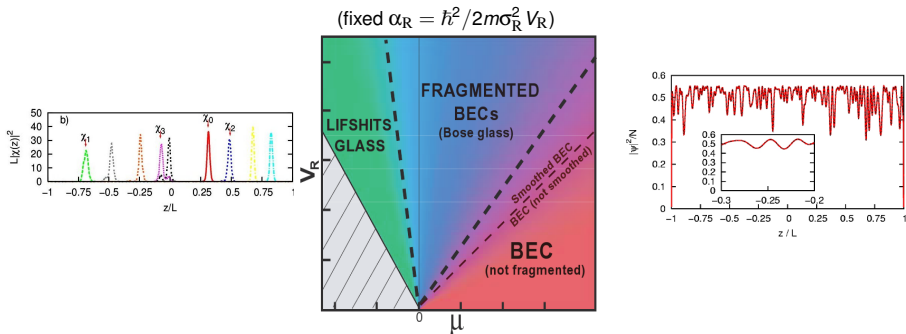
$$n^{1-2/d} \ll \hbar^2/mg \quad (d=1)$$
$$V(z) = V_R V(z/\sigma_R)$$

Disordered 1D Bose gases at $T = 0$



$$n^{1-2/d} \ll \hbar^2/mg \quad (d=1)$$
$$V(z) = V_R V(z/\sigma_R)$$

Ground-state quantum-state diagram of a weakly-interacting Bose gas in 1D disorder



L. Sanchez-Palencia, PRA **74**, 053625 (2006)

PL, D. Clément, P. Bouyer, A. Aspect, M. Lewenstein, and L. Sanchez-Palencia, PRL **98**, 170403 (2007)

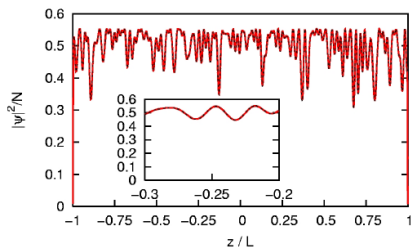
Density profile in the BEC regime

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(z) + gn_c(z) \right] \sqrt{n_c(z)} = \mu \sqrt{n_c(z)}$$

healing length ξ : $\mu = \frac{\hbar^2}{4m\xi^2}$

- Thomas-Fermi regime: $\xi \ll \sigma_R$

$$n_c(\mathbf{r}) \simeq [\mu - V(z)]/g$$



Density profile in the BEC regime

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(z) + gn_c(z) \right] \sqrt{n_c(z)} = \mu \sqrt{n_c(z)}$$

healing length ξ : $\mu = \frac{\hbar^2}{4m\xi^2}$

- Thomas-Fermi regime: $\xi \ll \sigma_R$

$$n_c(\mathbf{r}) \simeq [\mu - V(z)]/g$$

- Smoothing regime: $\xi \gtrsim \sigma_R$

$$n_c(\mathbf{r}) \simeq [\mu - \tilde{V}(z)]/g$$

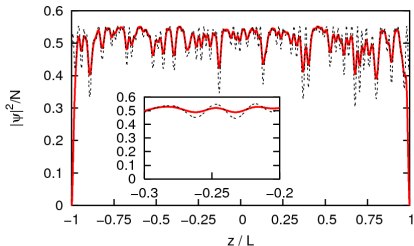
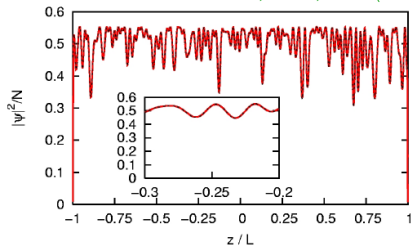
\tilde{V} smoothed potential

$$\tilde{V}(z) = \int dz' G_\xi(z - z') V(z')$$

$$\tilde{V}(q) = \frac{V(q)}{1 + (q\xi)^2}$$

$$\tilde{V}_R \sim (\sigma_R/\xi)^{1/2} V_R \quad [\xi \gg \sigma_R]$$

L. Sanchez-Palencia, PRA 74, 053625 (2006)



Scaling of the fragmentation threshold

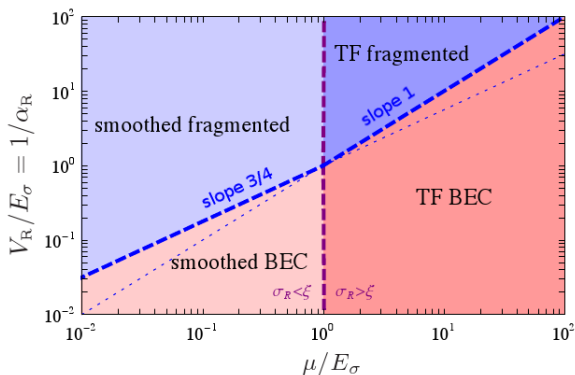
Fragmentation threshold:

- Thomas-Fermi regime ($\sigma_R \gg \xi$): $V_R \propto \mu$
- Smoothing regime ($\sigma_R \ll \xi$): $\tilde{V}_R \propto \mu \Rightarrow V_R \propto \mu^{3/4}$

Scaling of the fragmentation threshold

Fragmentation threshold:

- Thomas-Fermi regime ($\sigma_R \gg \xi$): $V_R \propto \mu$
- Smoothing regime ($\sigma_R \ll \xi$): $\tilde{V}_R \propto \mu \Rightarrow V_R \propto \mu^{3/4}$



Smoothing regime: same scaling as in numerical study of Bose-glass transition of [Fontanesi et al. PRL 103, 030403 \(2009\)](#)

See also [Aleiner et al. arXiv:0910.4534](#)

- Grand-canonical Hamiltonian

$$\hat{K} = \hat{H} - \mu \hat{N} = \int dr \left\{ \hat{\Psi}^\dagger \left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) - \mu \right] \hat{\Psi} + \frac{g}{2} \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \right\}$$

- Density-phase picture: $\hat{\Psi} = \exp(i\hat{\theta}) \sqrt{\hat{n}}$
- Quasi-BEC in 1D (suppressed density fluctuations):

$$\hat{n}(r) = n_c(r) + \delta\hat{n}(r)$$

- Ground state: Gross-Pitaevskii equation (GPE)

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V + gn_c \right] \sqrt{n_c} = \mu \sqrt{n_c}$$

- Excitations: density and phase fluctuations

$$\hat{\theta}(r) = \frac{-i}{2\sqrt{n_c(r)}} \sum_{\mathbf{v}} [f_{\mathbf{v}}^+(r) \hat{b}_{\mathbf{v}} - H.c.]$$
$$\delta \hat{n}(r) = \sqrt{n_c(r)} \sum_{\mathbf{v}} [f_{\mathbf{v}}^-(r) \hat{b}_{\mathbf{v}} + H.c.]$$

- Coherence terms f^+ , f^- :
'wave functions' of phase and density fluctuations respectively

- Ground state: Gross-Pitaevskii equation (GPE)

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V + gn_c \right] \sqrt{n_c} = \mu \sqrt{n_c}$$

- Excitations: Bogolyubov-de Gennes equations (BdGEs)

$$\begin{aligned} \left[-\frac{\hbar^2}{2m} \nabla^2 + V + gn_c - \mu \right] f_v^+ &= \varepsilon_v f_v^- \\ \left[-\frac{\hbar^2}{2m} \nabla^2 + V + 3gn_c - \mu \right] f_v^- &= \varepsilon_v f_v^+ \end{aligned}$$

⇒ Bogolyubov quasi-particles

Develop 1st order perturbation theory around $V = 0$:

- GPE:

$$gn_c(r) \simeq \mu - \tilde{V}(r)$$

- BdGEs:

$$\begin{aligned} \left[-\frac{\hbar^2}{2m} \nabla^2 + V - \tilde{V} \right] f_v^+ &= \epsilon_v f_v^- \\ \left[-\frac{\hbar^2}{2m} \nabla^2 + 2\mu + V - 3\tilde{V} \right] f_v^- &= \epsilon_v f_v^+ \end{aligned}$$

Develop 1st order perturbation theory around $V = 0$:

- GPE:

$$gn_c(r) \simeq \mu - \tilde{V}(r)$$

- BdGEs:

$$\begin{aligned} \left[-\frac{\hbar^2}{2m} \nabla^2 + V - \tilde{V} \right] f_v^+ &= \epsilon_v f_v^- \\ \left[-\frac{\hbar^2}{2m} \nabla^2 + 2\mu + V - 3\tilde{V} \right] f_v^- &= \epsilon_v f_v^+ \end{aligned}$$

- Introduce

$$g_{\mathbf{k}}^+ = \sqrt{\rho_{\mathbf{k}}} f_{\mathbf{k}}^+ + \frac{1}{\sqrt{\rho_{\mathbf{k}}}} f_{\mathbf{k}}^-$$

$$g_{\mathbf{k}}^- = -\frac{1}{\sqrt{\rho_{\mathbf{k}}}} f_{\mathbf{k}}^+ + \sqrt{\rho_{\mathbf{k}}} f_{\mathbf{k}}^-$$

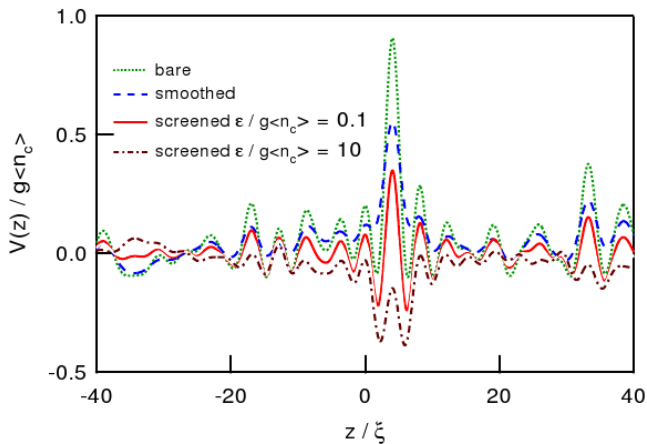
$$\rho_{\mathbf{k}} = \sqrt{1 + 1/(k\xi)^2}$$

$$-\frac{\hbar^2}{2m}\nabla^2 g_{\mathbf{k}}^+ + \mathcal{V}_{\mathbf{k}}(\mathbf{r})g_{\mathbf{k}}^+ \simeq \frac{\hbar^2 k^2}{2m} g_{\mathbf{k}}^+$$

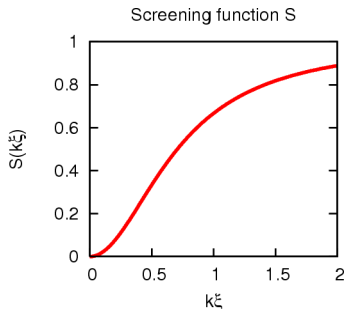
$$\mathcal{V}_{\mathbf{k}}(\mathbf{r}) = V(\mathbf{r}) - \frac{1 + 4(k\xi)^2}{1 + 2(k\xi)^2} \tilde{V}(\mathbf{r})$$

Formally equivalent to a Schrödinger equation
for a **non-interacting particle**
with **energy** $\hbar^2 k^2 / 2m$ ($\neq \epsilon_k$)
in a **potential** $\mathcal{V}_{\mathbf{k}}(\mathbf{r})$.

Valid in any dimension.



$$\Gamma_k = [S(k\xi)]^2 \gamma_k \quad S(k\xi) = \frac{2(k\xi)^2}{1 + 2(k\xi)^2}$$

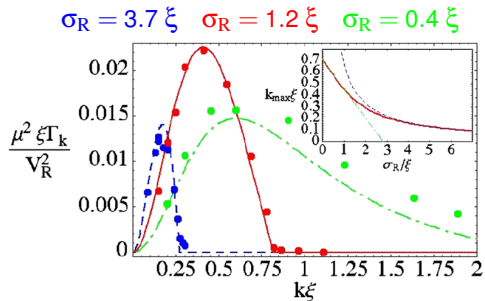
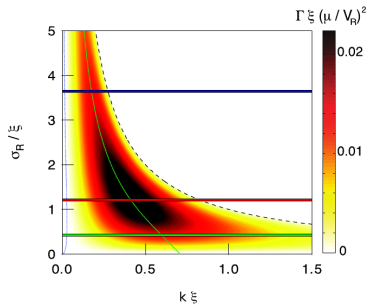


S drops to zero in the phonon regime:
screening of the external potential.

Lyapunov exponent of Bogolyubov quasi-particles in 1D

$$\Gamma_k \simeq \frac{\sqrt{2\pi}}{8} \left(\frac{V_R}{\mu} \right)^2 \frac{k^2 \sigma_R}{[1 + 2(k\xi)^2]^2} \hat{c}_2(2k\sigma_R)$$

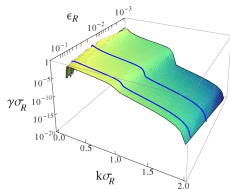
Speckle potential:



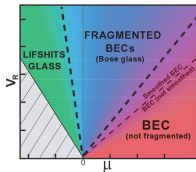
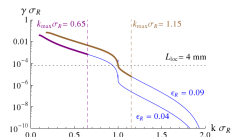
PL, D. Clément, P. Bouyer, A. Aspect, and L. Sanchez-Palencia, PRL **99**, 180402 (2007)

White noise limit: N. Bilas and N. Pavloff, EPJD **40**, 387 (2006)

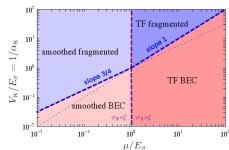
Conclusion



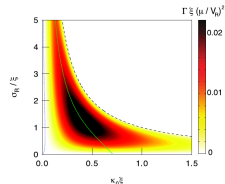
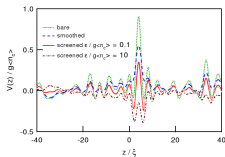
1D Anderson localization of single-particles in correlated potentials



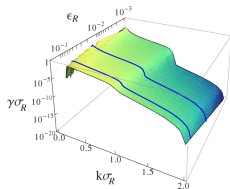
Ground state of weakly-interacting Bose gases in 1D disorder



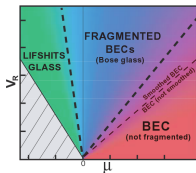
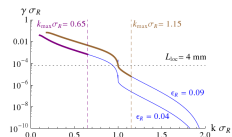
Anderson localization of Bogolyubov quasi-particles in 1D



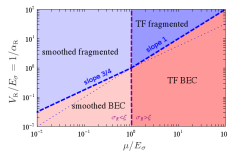
Conclusion



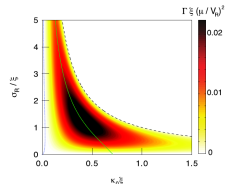
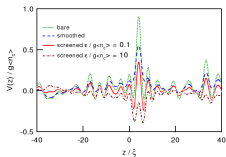
1D Anderson localization of single-particles in correlated potentials
effective mobility edges



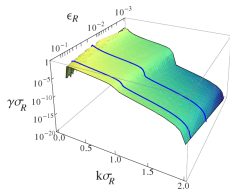
Ground state of weakly-interacting Bose gases in 1D disorder



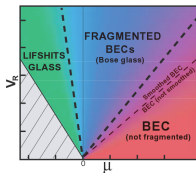
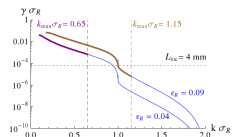
Anderson localization of Bogolyubov quasi-particles in 1D



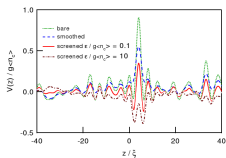
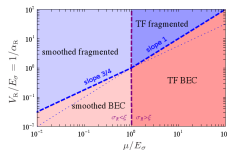
Conclusion



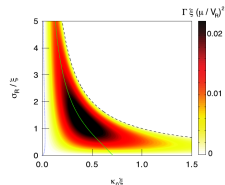
1D Anderson localization of single-particles in correlated potentials
effective mobility edges



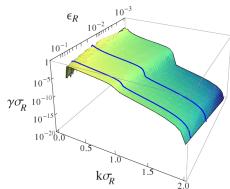
Ground state of weakly-interacting Bose gases in 1D disorder
smoothing effect and scaling of fragmentation threshold



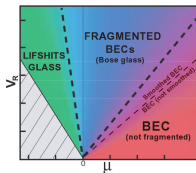
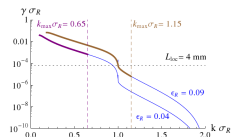
Anderson localization of Bogolyubov quasi-particles in 1D



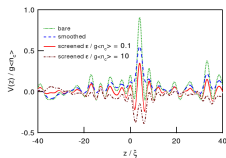
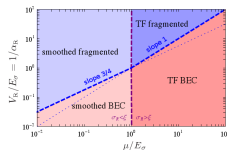
Conclusion



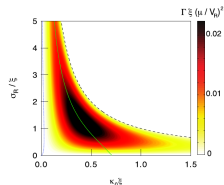
1D Anderson localization of single-particles in correlated potentials
effective mobility edges



Ground state of weakly-interacting Bose gases in 1D disorder
smoothing effect and scaling of fragmentation threshold



Anderson localization of Bogolyubov quasi-particles in 1D
competition of correlation length and healing length



Anderson localization of non-interacting particles

- Localization in potentials with exotic correlation functions
- Localization properties in a trap
- Consequences of special correlations on localization in 2D, 3D

Ground state of weakly-interacting disordered Bose gases

- Experiments at fragmentation threshold Deissler *et al.* arXiv:0910.5062v1
- Beyond-mean-field description of fragmented regime

Anderson localization of Bogolyubov quasi-particles

- Experimental observation with Bragg spectroscopy
- Higher dimensions (2D, 3D), mobility edges in 3D

Laboratoire Charles Fabry
P. Chavel

Atom Optics Group
A. Aspect
P. Bouyer and L. Sanchez-Palencia

Collaborations
M. Lewenstein, D. Delande, C. Müller, C. Miniatura, B. Grémaud

D. Clément, J. Billy, V. Josse, Z. Zuo, A. Bernard

B. Hambrecht, L. Pezzé, M. Piraud, T.-L. Dao

F. Moron and A. Villing

... and all the members of the Atom Optics Group



Teaching

F. Goudail, S. Lebrun, C. Garrido-Alzar, F. Kröger, D. Boiron

Staff of Institut d'Optique

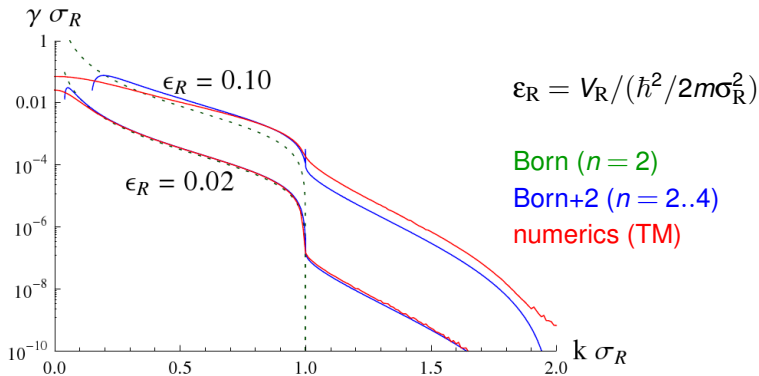
Friends and family

Thank you
for your attention

Additional slides

Comparison with transfer matrix calculations

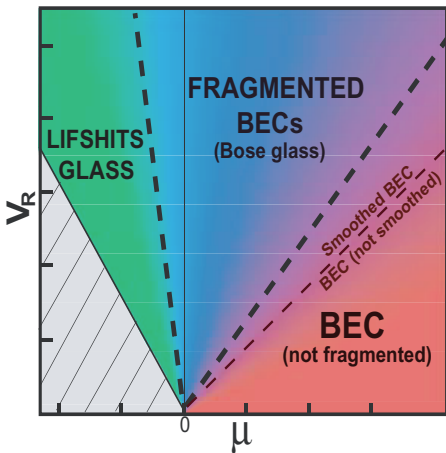
Lyapunov exponent in a speckle potential



Transfer matrix calculations by D. Delande

Boundaries in the quantum state diagram

$\alpha_R = \hbar^2 / 2m\sigma_R^2 V_R$ is fixed



- pure T-F BEC versus smoothing :

$$\sigma_R \simeq \xi \text{ i.e. } \mu = \alpha_R V_R$$

- single BEC versus fragmentation :

$$\mu \simeq \tilde{V}_R = V_R \sqrt{\Sigma^0(\sigma_R/\xi)}$$

$$\mu/V_R = F(\sqrt{\mu/V_R}/\sqrt{\alpha_R})$$

- Lifshits regime :

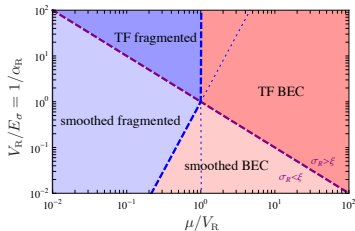
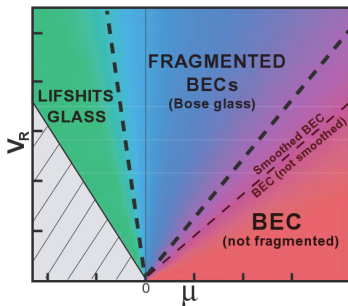
$$\mathcal{N}_{2L}(\mu) \leq v^{\max}$$

Scaling arguments:

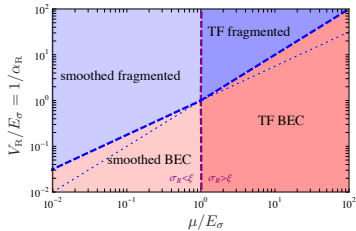
$$v^{\max} = \frac{L}{\sigma_R} G(\alpha_R)$$

$$\mathcal{N}_{2L}(\varepsilon) = \frac{L}{\sigma_R} \eta(\alpha_R, \varepsilon/V_R)$$

Smoothing and fragmentation in the BEC regime



(a) Slopes: $-1, 3, \infty$



(b) Slopes: $3/4, 1, \infty$

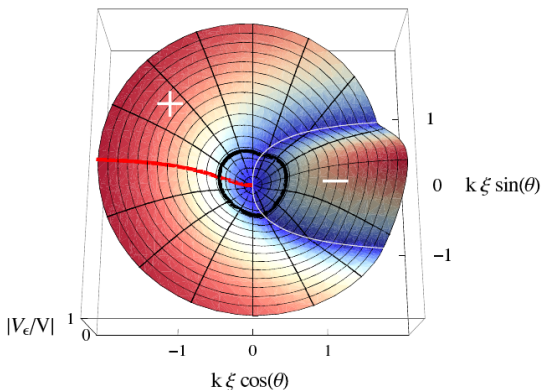
$$\begin{aligned} -\frac{\hbar^2}{2m} \nabla^2 g_{\mathbf{k}}^+ &= \frac{\hbar^2 k^2}{2m} g_{\mathbf{k}}^+ \\ -\frac{\hbar^2}{2m} \nabla^2 g_{\mathbf{k}}^- &= -\rho_k^2 \frac{\hbar^2 k^2}{2m} g_{\mathbf{k}}^- \end{aligned}$$

Boundary conditions (BC): zero Dirichlet or periodic.

- **Oscillating** modes $g_{\mathbf{k}}^+$ are **quantized** by BC.
- **Evanescent** modes $g_{\mathbf{k}}^-$ are **forbidden** by BC in the absence of an external potential.

Elastic scattering of BQPs: envelope function

$$\mathcal{V}'_{\mathbf{k}}(\mathbf{k}' - \mathbf{k})/V(\mathbf{k}' - \mathbf{k}) \text{ with } \mathbf{k}' \cdot \mathbf{k} = k^2 \cos(\theta)$$



single particles: s-wave scattering pattern
phonons: p-wave scattering pattern

First studied by [C. Gaul and C.A. Müller, Europhys. Lett. 83, 10006 \(2008\)](#)

Lyapunov exponent of BQPs in the Born approximation

- Lyapunov exponent: $\langle \ln |g_k^+(z)| \rangle \sim -\Gamma_k |z|$
- Known:

$$\Gamma_k = \frac{\sqrt{2\pi}}{8k^2} \left(\frac{2m}{\hbar^2} \right)^2 \hat{C}_k(2k)$$

$\hat{C}_k(q)$ Fourier transform of the autocorrelation function of $\mathcal{V}_k(r)$

- Wiener-Khinchin:

$$\hat{C}_k(q) \propto \langle |\hat{\mathcal{V}}_k(q)|^2 \rangle$$

- Here:

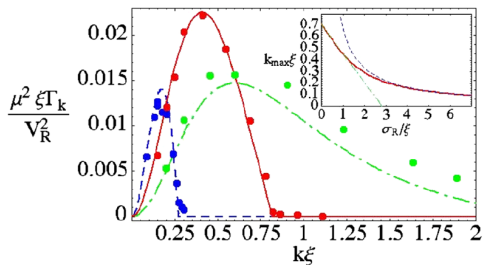
$$\hat{\mathcal{V}}_k(z) = V(z) - \frac{1 + 4(k\xi)^2}{1 + 2(k\xi)^2} \tilde{V}(z), \quad \tilde{V}(q) = \frac{V(q)}{1 + (q\xi)^2}$$

$$\hat{\mathcal{V}}_k(2k) = S(k\xi) \hat{V}(2k), \quad S(k\xi) = \frac{2(k\xi)^2}{1 + 2(k\xi)^2}$$

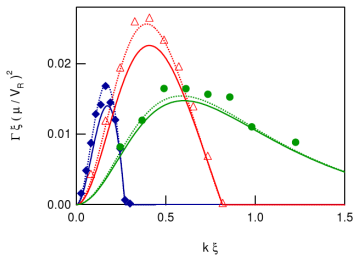
- Conclusion:

$$\Gamma_k = [S(k\xi)]^2 \gamma_k \quad (\gamma_k: \text{free particles})$$

Lyapunov exponent of BQPs: numerical calculations



with smoothed potential



with exact density profile