

# Infinite dimensional stochastic calculus via regularization with some financial perspectives

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July 5, 2010

This is based on Di Girolami C. and Russo F. **Infinite dimensional stochastic calculus via regularization and applications** available at

HAL-INRIA, Preprint

<http://hal.archives-ouvertes.fr/inria-00473947/fr/>

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$$\langle W \rangle_t = t.$$

- If  $h \in L^2(\Omega)$ , the martingale representation theorem states the existence of a predictable process  $\xi \in L^2(\Omega \times [0, T])$  such that

$$h = \mathbb{E}[h] + \int_0^T \xi_s dW_s$$

- If  $h \in \mathbb{D}^{1,2}$  in the sense of Malliavin, [Clark-Ocone formula](#) implies that  $\xi_s = \mathbb{E}[D^m h | \mathcal{F}_s]$ , so that

$$h = \mathbb{E}[h] + \int_0^T \mathbb{E}[D^m h | \mathcal{F}_s] dW_s \quad (1)$$

where  $D^m$  is the Malliavin gradient.

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# Examples of processes with finite quadratic variation

- 1)  $S$  is an  $(\mathcal{F}_t)$ -**semimartingale** with decomposition  $S = M + V$ ,  $M$   $(\mathcal{F}_t)$ -local martingale and  $V$  bounded variation process. So  $[S] = [M]$ .
- 2)  $D$  is a  $(\mathcal{F}_t)$ -**Dirichlet** process with decomposition  $D = M + A$ ,  $M$   $(\mathcal{F}_t)$ -local martingale and  $A$  an  $(\mathcal{F}_t)$ -adapted zero quadratic variation process.  $[D] = [M]$ . Föllmer (1981).

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- 3)  $D$  is a  $(\mathcal{F}_t)$ -**weak-Dirichlet** process with decomposition  $D = M + A$ ,  $M$   $(\mathcal{F}_t)$ -local martingale and  $A$  such that  $[A, N] = 0$  for any continuous  $(\mathcal{F}_t)$ -local martingale  $N$ . Errami-Russo (2003), Gozzi-Russo (2005)
  - ① In general  $D$  does not have finite quadratic variation
  - ② If  $A$  is a finite quadratic variation process  $[D] = [M] + [A]$
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- 4)  $B^{H,K}$  bifractional Brownian motion with parameters  $H \in ]0, 1[$ ,  $K \in ]0, 1]$  such that  $HK \geq 1/2$
- If  $HK > 1/2$ ,  $[B^{H,K}] = 0$ .
  - If  $HK = 1/2$ , then
    - $[B^{H,K}]_t = 2^{1-K}t$
    - If  $K = 1$  and if  $H = 1/2$ ,  $B^{H,K}$  is a Brownian motion
    - If  $K \neq 1$ ,  $B^{H,K}$  is not a semimartingale (not even a Dirichlet with respect to its own filtration).
- 5) Skorohod integrals. If  $(u_t)$  is in  $L^{1,2}$ , under reasonable conditions on  $Du$ ,  $[\int_0^t u_s \delta W_s]_t = \int_0^t u_s^2 ds$ .

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- 6) For fixed  $k \geq 1$ , Föllmer Wu Yor construct a weak  $k$ -order Brownian motion  $X$ , which in general is not even Gaussian.  $X$  is a **weak  $k$ -order Brownian motion** if for every  $0 \leq t_1 \leq \dots \leq t_k < +\infty$ ,  $(X_{t_1}, \dots, X_{t_k})$  is distributed as  $(W_{t_1}, \dots, W_{t_k})$ . If  $k \geq 4$  then  $[X]_t = t$ .

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# Notation

## Definition

Let  $T > 0$  and  $X = (X_t)_{t \in [0, T]}$  be a real continuous process prolonged by continuity.

Process  $X(\cdot)$  defined by

$$X(\cdot) = \{X_t(u) := X_{t+u}; u \in [-T, 0]\}$$

will be called **window process**.

- $X(\cdot)$  is a  $C([-T, 0])$ -valued stochastic process.
- $C([-T, 0])$  is a typical non-reflexive Banach space.

## A representation problem

We suppose  $X_0 = 0$  and  $[X]_t = t$ .

The main task will consist in looking for classes of functionals

$$H : C([-T, 0]) \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits representation

$$h = H_0 + \int_0^T \xi_s dX_s$$

- Moreover we look for an explicit expression for
  - $H_0 \in \mathbb{R}$
  - $\xi$  adapted process with respect to the canonical filtration of  $X$

## Idea

We will obtain the representation formula by expressing  $h = H(X_T(\cdot))$  as

$$h = H(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot))$$

where  $u \in C^{1,2}([0, T[ \times C([-T, 0]))$  solves an infinite dimensional PDE, if previous limit exists.

Representation of  $h = H(X_T(\cdot))$ 

Then

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s \quad (2)$$

where  $D^{\delta_0} u(s, \eta) = D u(s, \eta)(\{0\})$  is the *projection* of the Fréchet derivative  $Du(t, \eta)$  on the linear space generated by Dirac measure  $\delta_0$ , we recall that

$$D u : [0, T] \times C([-T, 0]) \longrightarrow C^*([-T, 0]) = \mathcal{M}([-T, 0]).$$

## Definition

Let  $X$  (resp.  $Y$ ) be a continuous (resp. locally integrable) process. Suppose that the random variables

$$\int_0^t Y_s d^-X_s := \lim_{\epsilon \rightarrow 0} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds$$

exists in probability for every  $t \in [0, T]$  and the limiting process admits a continuous modification, then the limiting process denoted by  $\int_0^\cdot Y d^-X$  is called the **(proper) forward integral of  $Y$  with respect to  $X$** .

Russo-Vallois 1993



# Covariation for real valued processes

## Definition

The **covariation of  $X$  and  $Y$**  is defined by

$$[X, Y]_t = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) ds$$

if the limit exists in the ucp sense with respect to  $t$ .

Obviously  $[X, Y] = [Y, X]$ .

If  $X = Y$ ,  $X$  is said to be **finite quadratic variation process** and  $[X] := [X, X]$ .

# Connections with semimartingales

- ① Let  $S^1, S^2$  be  $(\mathcal{F}_t)$ -semimartingales with decomposition  $S^i = M^i + V^i$ ,  $i = 1, 2$  where  $M^i$   $(\mathcal{F}_t)$ -local continuous martingale and  $V^i$  continuous bounded variation processes.

Then

- $[S^i]$  classical bracket and  $[S^i] = \langle M^i \rangle$ .
- $[S^1, S^2]$  classical bracket and  $[S^1, S^2] = \langle M^1, M^2 \rangle$ .
- If  $S$  semimartingale and  $Y$  cadlag and predictable

$$\int_0^\cdot Y d^- S = \int_0^\cdot Y dS \quad (\text{It\^o})$$

# Itô formula for finite quadratic variation processes

## Theorem

Let  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $F \in C^{1,2}([0, T] \times \mathbb{R})$  and  $X$  be a finite quadratic variation process. Then

$$\int_0^t \partial_x F(s, X_s) d^- X_s$$

exists in the ucp sense and equals

$$F(t, X_t) - F(0, X_0) - \int_0^t \partial_s F(s, X_s) ds - \frac{1}{2} \int_0^t \partial_{xx} F(s, X_s) d[X]_s$$

Stochastic calculus via regularization is a theory which allows, in many specific cases to manipulate stochastic integrals

$$\int_0^t Y d^-X$$

when  $Y$  is anticipating or  $X$  is not a semimartingale.  
It is particularly efficient when  $X$  is a finite quadratic variation process.

## An infinite dimensional framework

We fix now in a general (infinite dimensional) framework. Let

- $B$  general Banach space
- $\mathbb{X}$  a  $B$ -valued process
- $F : B \rightarrow \mathbb{R}$  be of class  $C^2$  in Fréchet sense.

### An Ito formula for $B$ -valued processes

We would like to have an Itô type expansion of  $F(\mathbb{X})$ , **available also for  $B = C([-T, 0])$ -valued processes, as window processes, i.e. when  $\mathbb{X} = X(\cdot)$ .**

The literature does not apply: several problems appear even in the simple case  $W(\cdot)$ !

# Fréchet derivative and tensor product of Banach spaces

$F : B \rightarrow \mathbb{R}$  be of class  $C^2$  in Fréchet sense, then

- $DF : B \rightarrow L(B; \mathbb{R}) := B^*$ ;
- $D^2F : B \rightarrow L(B; B^*) \cong \mathcal{B}(B \times B) \cong (B \hat{\otimes}_{\pi} B)^*$

where

- $\mathcal{B}(B, B)$  Banach space of real valued bounded bilinear forms on  $B \times B$
- $(B \hat{\otimes}_{\pi} B)^*$  dual of the tensor projective tensor product of  $B$  with  $B$ .
- $B \hat{\otimes}_{\pi} B$  fails to be Hilbert even if  $B$  is a Hilbert space (**is not even a reflexive space**).

## A first attempt to an Itô type expansion of $F(\mathbb{X})$

$$F(\mathbb{X}_t) = F(\mathbb{X}_0) + \int_0^t B^* \langle DF(\mathbb{X}_s), d\mathbb{X}_s \rangle_B + \\ + \frac{1}{2} \int_0^t (B \hat{\otimes}_\pi B)^* \langle D^2 F(\mathbb{X}_s), d[\mathbb{X}]_s \rangle_{B \hat{\otimes}_\pi B}$$

## A formal proof

$$\int_0^t \frac{F(\mathbb{X}_{s+\epsilon}) - F(\mathbb{X}_s)}{\epsilon} ds \xrightarrow[\epsilon \rightarrow 0]{ucp} F(\mathbb{X}_t) - F(\mathbb{X}_0)$$

By a Taylor's expansion the left-hand side equals the sum of

$$\int_0^t B^* \langle DF(\mathbb{X}_s), \frac{\mathbb{X}_{s+\epsilon} - \mathbb{X}_s}{\epsilon} \rangle_B ds +$$

$$\int_0^t (B \hat{\otimes}_\pi B)^* \langle D^2 F(\mathbb{X}_s), \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} \rangle_{B \hat{\otimes}_\pi B} ds + R(\epsilon, t)$$



## Problems related to integration in Banach space

Let  $B$  be a Banach space and  $\mathbb{X}$  a  $B$ -valued process.

The problem arises at two levels

- 1 Stochastic integration with respect to an integrator  $\mathbb{X}$ .
- 2 Quadratic variation of  $\mathbb{X}$ .

Among the principal references about the subject there are:

- 1 Da Prato G. and Zabczyk J. [Stochastic Equations in Infinite Dimensions](#). Cambridge University Press, 1992.
- 2 Métivier M. and Pellaumail J. [Stochastic Integration](#). New York, 1980.
- 3 Dinculeanu N. [Vector Integration and Stochastic Integration in Banach spaces](#). Wiley-Interscience, New York, 2000.

## Da Prato - Zabczyk

- $B$  separable Hilbert space.
- $\mathbb{X}$   $B$ -valued Itô process.

but...

- $C([-T, 0])$  is not a Hilbert space.
- $W(\cdot)$  is not a  $C([-T, 0])$ -valued semimartingale.

## Metivier-Pellaumail and Dinculeanu

- $B$  is a general Banach space.
- $\mathbb{X}$  essentially semimartingale.
- The natural generalization concept of quadratic variation for Banach valued processes  $\mathbb{X}$  is a  $(B \hat{\otimes}_{\pi} B)$ -valued process denoted by  $[\mathbb{X}]^{\otimes}$  and called tensor quadratic variation.

but...

$W(\cdot)$  does not admit a tensor quadratic variation. In fact it is not possible to control the limit for  $\epsilon$  going to zero of

$$\frac{1}{\epsilon} \int_0^t \|W_{s+\epsilon}(\cdot) - W_s(\cdot)\|_{C([-T,0])}^2 ds$$

# Stochastic calculus via regularization for Banach valued processes

## We will define

- a stochastic integral for  $B^*$ -valued integrand with respect to  $B$ -valued integrators, which are not necessarily semimartingale.
- a new concept of quadratic variation which generalizes the tensor quadratic variation and which involves a Banach subspace  $\chi$  of  $(B \hat{\otimes}_\pi B)^*$ . It will be called  $\chi$ -quadratic variation of  $\mathbb{X}$ .

## Definition

Let  $\mathbb{X}$  and  $\mathbb{Y}$  be respectively a  $B$ -valued and a  $B^*$ -valued continuous stochastic processes.

If the process defined for every fixed  $t \in [0, T]$  by

$$\int_0^t B^* \langle \mathbb{Y}_s, d^- \mathbb{X}_s \rangle_B := \lim_{\epsilon \rightarrow 0} \int_0^t B^* \langle \mathbb{Y}(s), \frac{\mathbb{X}(s + \epsilon) - \mathbb{X}(s)}{\epsilon} \rangle_B ds$$

in probability admits a continuous version, then process

$$\left( \int_0^t B^* \langle \mathbb{Y}_s, d^- \mathbb{X}_s \rangle_B \right)_{t \in [0, T]}$$

will be called **forward stochastic integral of  $\mathbb{Y}$  with respect to  $\mathbb{X}$** .

## Connection with Da Prato-Zabczyk integral

If  $B = H$  separable Hilbert space

### Theorem

Let  $\mathbb{W}$  be a  $H$ -valued  $Q$ -Brownian motion with  $Q \in L^1(H)$  and  $\mathbb{Y}$  be  $H^*$ -valued process such that  $\int_0^t \|\mathbb{Y}_s\|_{H^*}^2 ds < \infty$  a.s. Then, for every  $t \in [0, T]$ ,

$$\int_0^t \langle \mathbb{Y}_s, d^- \mathbb{W}_s \rangle_H = \int_0^t \mathbb{Y}_s \cdot d\mathbb{W}_s^{dz} \quad (\text{Da Prato-Zabczyk integral})$$

# Definition of Chi-subspace

## Definition

A Banach subspace  $\chi$  continuously injected into  $(B \hat{\otimes}_{\pi} B)^*$  will be called a **Chi-subspace of  $(B \hat{\otimes}_{\pi} B)^*$** .

In particular it holds

$$\|\cdot\|_{\chi} \geq \|\cdot\|_{(B \hat{\otimes}_{\pi} B)^*}.$$

## Notion of $\chi$ -quadratic variation

Let

- $\mathbb{X}$  be a  $B$ -valued continuous process,
- $\chi$  a Chi-subspace of  $(B \hat{\otimes}_{\pi} B)^*$ ,
- $\mathcal{C}([0, T])$  space of real continuous processes equipped with the ucp topology.



## Notion of $\chi$ -quadratic variation

Let  $[\mathbb{X}]^\epsilon$  be the application

$$[\mathbb{X}]^\epsilon : \chi \longrightarrow \mathcal{C}([0, T])$$

defined by

$$\phi \mapsto \left( \int_0^t \chi \langle \phi, \frac{J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2)}{\epsilon} \rangle_{\chi^*} ds \right)_{t \in [0, T]}$$

where  $J : B \hat{\otimes}_\pi B \rightarrow (B \hat{\otimes}_\pi B)^{**}$  is the canonical injection a Banach space and its bidual, in the sequel will be omitted.

Let  $E$  be a Banach space and  $J : E \rightarrow E^{**}$ :

- $J$  is an isometry with respect to the strong topology,
- $J(E)$  is weak star dense in  $E^{**}$ ,
- If  $E$  is not reflexive,  $J(E) \subsetneq E^{**}$ .

# Definition of Chi-quadratic variation

## Definition

$\mathbb{X}$  admits a  $\chi$ -quadratic variation if

H1 For all  $(\epsilon_n) \downarrow 0$  it exists a subsequence  $(\epsilon_{n_k})$  such that

$$\sup_k \int_0^T \frac{\left\| (\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes^2 \right\|_{\chi^*}}{\epsilon_{n_k}} ds < \infty \quad \text{a.s.}$$

H2 There exists  $[\mathbb{X}] : \chi \rightarrow \mathcal{C}([0, T])$  such that

$$[\mathbb{X}]^\epsilon(\phi) \xrightarrow[\epsilon \rightarrow 0]{ucp} [\mathbb{X}](\phi) \quad \forall \phi \in \chi$$

H3 There is a  $\chi^*$ -valued bounded variation process  $\widetilde{[\mathbb{X}]}$ , such that  $\widetilde{[\mathbb{X}]}_t(\phi) = [\mathbb{X}](\phi)_t$  a.s. for all  $\phi \in \chi$ .

For every fixed  $\phi \in \chi$ , processes  $\widetilde{[\mathbb{X}]}_t(\phi)$  and  $[\mathbb{X}](\phi)_t$  are indistinguishable.

## Definition of Chi-quadratic variation

### Definition

When  $\mathbb{X}$  admits a  $\chi$ -quadratic variation, the  $\chi^*$ -valued process  $[\widetilde{\mathbb{X}}]$  (and even the application  $[\mathbb{X}]$ ) will be called  **$\chi$ -quadratic variation** of  $\mathbb{X}$

### Remark

- 1  $[\widetilde{\mathbb{X}}]$  is the quadratic variation intervening in the second order derivative term of **Itô's formula**.

## Global quadratic variation concept

### Definition

We say that  $\mathbb{X}$  admits a **global quadratic variation (g.q.v.)** if it admits a  $\chi$ -quadratic variation with  $\chi = (B \hat{\otimes}_{\pi} B)^*$ .

When  $\chi = (B \hat{\otimes}_{\pi} B)^*$

- H2 requires a weak\* convergence in  $(B \hat{\otimes}_{\pi} B)^{**}$ , i.e.

$$[\widetilde{\mathbb{X}}]^{\epsilon} \xrightarrow[\epsilon \rightarrow 0]{w^*} [\widetilde{\mathbb{X}}],$$

where  $[\widetilde{\mathbb{X}}]^{\epsilon}$  is a  $\chi^*$ -valued bounded variation process associated to  $[\mathbb{X}]^{\epsilon}$ , defined by  $[\widetilde{\mathbb{X}}]_t^{\epsilon}(\phi) := [\mathbb{X}]^{\epsilon}(\phi)_t$  a.s.

- The g.q.v.  $[\widetilde{\mathbb{X}}]$  is  $(B \hat{\otimes}_{\pi} B)^{**}$ -valued.

## Connection with other concepts of quadratic variation

Global quadratic variation permit us to recover quadratic variation concepts in literature.

- 1  $B = \mathbb{R}^n$ .  $\mathbb{X} = (X^1, \dots, X^n)$  admits a covariations matrix  $[\mathbb{X}^*, \mathbb{X}] = ([X^i, X^j])_{1 \leq i, j \leq n}$  if and only if  $\mathbb{X}$  admits g.q.v.  $[\widetilde{\mathbb{X}}] = [\mathbb{X}^*, \mathbb{X}]$ .
- 2  $B = H$  Hilbert separable. If  $\mathbb{X}$  has  $L^1(H)$ -valued quadratic variation  $[\mathbb{X}]^{dz}$  then  $\mathbb{X}$  admits g.q.v.  $[\widetilde{\mathbb{X}}] = [\mathbb{X}]^{dz}$ .
- 3  $B$  general. If  $\mathbb{X}$  admits tensor quadratic variation  $[\mathbb{X}]^{\otimes}$  then  $\mathbb{X}$  admits g.q.v.  $[\widetilde{\mathbb{X}}] = [\mathbb{X}]^{\otimes}$ .

In all the recovered cases we have a strong convergence, so the g.q.v.  $[\widetilde{\mathbb{X}}]$  is always  $B \hat{\otimes}_{\pi} B$ -valued.

We recall  $B \hat{\otimes}_{\pi} B \subseteq (B \hat{\otimes}_{\pi} B)^{**}$ .

## Infinite dimensional Itô's formula

Let  $B$  a separable Banach space

### Theorem (Itô's formula)

Let  $\mathbb{X}$  a  $B$ -valued continuous process admitting a  $\chi$ -quadratic variation.

Let  $F : [0, T] \times B \rightarrow \mathbb{R}$  be  $C^{1,2}$  Fréchet such that

$$D^2F : [0, T] \times B \rightarrow \chi \subset (B \hat{\otimes}_{\pi} B)^* \quad \text{continuously}$$

Then for every  $t \in [0, T]$  the forward integral

$$\int_0^t B^* \langle DF(s, \mathbb{X}_s), d^- \mathbb{X}_s \rangle_B$$

exists and following formula holds.

## Ito's formula

$$\begin{aligned} F(t, \mathbb{X}_t) &= F(0, \mathbb{X}_0) + \int_0^t \partial_s F(s, \mathbb{X}_s) ds + \\ &+ \int_0^t B^* \langle DF(s, \mathbb{X}_s), d^- \mathbb{X}_s \rangle_B + \\ &+ \frac{1}{2} \int_0^t \chi \langle D^2 F(s, \mathbb{X}_s), d[\widetilde{\mathbb{X}}]_s \rangle_{\chi^*} \end{aligned}$$

# Window processes

- We fix attention now on  $B = C([-T, 0])$ -valued window processes.
- $X$  continuous real valued process and  $X(\cdot)$  its window process.
- $\mathbb{X} = X(\cdot)$



## Evaluations of $\chi$ -quadratic variation for window processes

- If  $X$  has Hölder continuous paths of parameter  $\gamma > 1/2$ , then  $X(\cdot)$  has a zero g.q.v.  
For instance:
  - $X = B^H$  fractional Brownian motion with parameter  $H > 1/2$ .
  - $X = B^{H,K}$  bifractional Brownian motion with parameters  $H \in ]0, 1[$ ,  $K \in ]0, 1]$  s.t.  $HK > 1/2$ .
- $W(\cdot)$  does not admit a g.q.v.

## Some examples of Chi-subspaces

- $\chi$  Chi-subspace of  $(B \hat{\otimes}_\pi B)^*$ . For instance:
  - $\mathcal{M}([-T, 0]^2)$  equipped with the total variation norm.
  - $L^2([-T, 0]^2)$ .
  - $\mathcal{D}_{0,0} = \{\mu(dx, dy) = \lambda \delta_0(dx) \otimes \delta_0(dy)\}$ .
  - $(\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2$   
 $= \mathcal{D}_{0,0} \oplus L^2([-T, 0]) \hat{\otimes}_h \mathcal{D}_0 \oplus \mathcal{D}_0 \hat{\otimes}_h L^2([-T, 0]) \oplus L^2([-T, 0]^2)$ .
  - $Diag := \{\mu(dx, dy) = g(x) \delta_y(dx) dy; g \in L^\infty([-T, 0])\}$ .

## Evaluations of $\chi$ -quadratic variation for window processes

- $W(\cdot)$  does not admit a  $\mathcal{M}([-T, 0]^2)$ -quadratic variation.
- If  $X$  is a **real finite quadratic variation** process, then
  - $X(\cdot)$  has zero  $L^2([-T, 0]^2)$ -quadratic variation.
  - $X(\cdot)$  has  $\mathcal{D}_{0,0}$ -quadratic variation

$$[X(\cdot)] : \mathcal{D}_{0,0} \longrightarrow \mathcal{C}[0, T], \quad [X(\cdot)]_t(\mu) = \mu(\{0, 0\})[X]_t$$

- $X(\cdot)$  has  $(\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2$ -quadratic variation

$$[X(\cdot)] : (\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2 \longrightarrow \mathcal{C}[0, T], \quad [X(\cdot)]_t(\mu) = \mu(\{0, 0\})[X]_t$$

- $X(\cdot)$  has *Diag*-quadratic variation

$$[X(\cdot)] : \text{Diag} \longrightarrow \mathcal{C}[0, T], \quad [X(\cdot)]_t(\mu) = \int_0^t g(-x)[X]_{t-x} dx$$

where  $\mu(dx, dy) = g(x)\delta_y(dx)dy$ .

## Robustness of Black-Scholes formula

Let  $(S_t)$  be the price of a financial asset of the type

$$S_t = \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right), \quad \sigma > 0.$$

Let  $h = \tilde{f}(S_T) = f(W_T)$  where  $f(y) = \tilde{f}\left(\exp(\sigma y - \frac{\sigma^2}{2} T)\right)$ .

Let  $\tilde{u} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  solving

$$\begin{cases} \partial_t \tilde{u}(t, x) + \frac{1}{2} \partial_{xx} \tilde{u}(t, x) = 0 \\ \tilde{u}(T, x) = \tilde{f}(x) \end{cases} \quad x \in \mathbb{R}$$

Applying classical Itô formula we obtain

$$h = \tilde{u}(0, S_0) + \int_0^T \partial_x \tilde{u}(s, S_s) dS_s = u(0, W_0) + \int_0^T \partial_x u(s, W_s) dW_s$$

for a suitable  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ .

Does one have a similar formula if  $W$  is replaced by a finite quadratic variation  $X$  such that  $[X]_t = t$ ? The answer is YES!

Let  $X$  such that  $[X]_t = t$

A1  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous and polynomial growth

A2  $v \in C^{1,2}([0, T] \times \mathbb{R}) \cap C^0([0, T] \times \mathbb{R})$  such that

$$\begin{cases} \partial_t v(t, x) + \frac{1}{2} \partial_{xx} v(t, x) = 0 \\ v(T, x) = f(x) \end{cases}$$

Then

$$h := f(X_T) = v(0, X_0) + \underbrace{\int_0^T \partial_x v(s, X_s) d^- X_s}_{\text{improper forward integral}}$$

Schoenmakers-Kloeden (1999) Coviello-Russo (2006)

Bender-Sottinen-Valkeila (2008)

## Natural question

Is it possible to express generalization of it where the option is path dependent? As first step we revisit the toy model.

## The toy model revisited

### Proposition

We set  $B = C([-T, 0])$  and  $\eta \in B$  and we define

- $H : B \rightarrow \mathbb{R}$ , by  $H(\eta) := f(\eta(0))$
- $u : [0, T] \times B \rightarrow \mathbb{R}$ , by  $u(t, \eta) := v(t, \eta(0))$

Then

$$u \in C^{1,2}([0, T[ \times B; \mathbb{R}) \cap C^0([0, T] \times B; \mathbb{R})$$

and solves

$$\begin{cases} \partial_t u(t, \eta) + \frac{1}{2} \langle D^2 u(t, \eta), \mathbf{1}_D \rangle = 0 \\ u(T, \eta) = H(\eta) \end{cases}$$

## Proof.

- $u(T, \eta) = v(T, \eta(0)) = f(\eta(0)) = H(\eta)$
- $\partial_t u(t, \eta) = \partial_t v(t, \eta(0))$
- $Du(t, \eta) = \partial_x v(t, \eta(0)) \delta_0$
- $D^2 u(t, \eta) = \partial_{xx}^2 v(t, \eta(0)) \delta_0 \otimes \delta_0$
- $\partial_t u(t, \eta) + \frac{1}{2} D^2 u(t, \eta)(\{0, 0\}) = 0$



And, let  $X$  such that  $[X]_t = t$ , we have

$$h := H(X_T(\cdot)) = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s$$



# Notation

We set  $B = C([-T, 0])$  and  $\eta \in B$ .

- $X$  real continuous stochastic process
- $X_0 = 0$ ,
- $[X]_t = t$ .

# A representation problem

The main task will consist in looking for classes of functionals

$$H : B \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits representation

$$h = H_0 + \int_0^T \xi_s d^- X_s$$

- Moreover we look for an explicit expression for
  - $H_0 \in \mathbb{R}$
  - $\xi$  adapted process with respect to the canonical filtration of  $X$

# Idea

Obtain the representation formula by expressing  $h = H(X_T(\cdot))$  as

$$h = H(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot))$$

where  $u \in C^{1,2}([0, T[ \times B)$  solves an infinite dimensional PDE, if previous limit exists.

# An infinite dimensional PDE

Let  $H : B \rightarrow \mathbb{R}$ , we will show the existence of a function  $u : [0, T] \times B \rightarrow \mathbb{R}$  of class  $C^{1,2}([0, T] \times B) \cap C^0([0, T] \times B)$  solving

## Infinite dimensional PDE

$$\begin{cases} \partial_t u(t, \eta) + \int_{-t}^0 D^{ac} u(t, \eta) d\eta + \frac{1}{2} \langle D^2 u(t, \eta), \mathbb{1}_D \rangle = 0 \\ u(T, \eta) = H(\eta) \end{cases} \quad (3)$$

where

- $\mathbb{1}_D(x, y) := \begin{cases} 1 & \text{if } x = y, x, y \in [-T, 0] \\ 0 & \text{otherwise} \end{cases}$
- $D^{ac} u(t, \eta)$  absolute continuous part of measure  $Du(t, \eta)$
- If  $x \mapsto D_x^{ac} u(t, \eta)$  has bounded variation, previous integral is defined by an integration by parts.

## Representation of $h = H(X_T(\cdot))$

Then

$$h = H_0 + \int_0^T \xi_s d^- X_s \quad (4)$$

with

- $H_0 = u(0, X_0(\cdot))$
- $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$

## Methodology: two steps

- We will choose a functional  $u : [0, T] \times B \rightarrow \mathbb{R}$  which solves the infinite dimensional PDE (3) with final condition  $H$ .
- Using Itô formula we establish a representation form (4).

### Toy models

- 1  $H(\eta) = f(\eta(0))$  where  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous and polynomial growth  $\Rightarrow u$  such that  $D^2u(t, \eta) \in \mathcal{D}_{0,0}$
- 2  $H(\eta) = \left( \int_{-T}^0 \eta(s) ds \right)^2 \Rightarrow u$  such that  $D^2u(t, \eta) \in (\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2$
- 3  $H(\eta) = \int_{-T}^0 \eta(s)^2 ds \Rightarrow u$  such that  $D^2u(t, \eta) \in (\text{Diag} \oplus \mathcal{D}_{0,0})$

# A general representation theorem

## Theorem

- $H : B \rightarrow \mathbb{R}$
- $u \in C^{1,2}([0, T[ \times B) \cap C^0([0, T] \times B)$
- $x \mapsto D_x^{ac} u(t, \eta)$  has bounded variation
- $D^2 u(t, \eta) \in (\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2$
- $u$  solves

$$\begin{cases} \partial_t u(t, \eta) + \int_{] -t, 0]} D^{ac} u(t, \eta) d\eta + \frac{1}{2} D^2 u(t, \eta)(\{0, 0\}) = 0 \\ u(T, \eta) = H(\eta) \end{cases} \quad (5)$$

then  $h$  has representation (4).

The proof follows immediately applying the Itô's formula.

## Sufficient conditions to solve (5)

- 1 When  $X$  general process such that  $[X]_t = t$ .
  - $H$  has a smooth Fréchet dependence on  $L^2([-T, 0])$ .
  - $h := H(X_T(\cdot)) = f\left(\int_0^T \varphi_1(s) d^- X_s, \dots, \int_0^T \varphi_n(s) d^- X_s\right)$ ,
    - $f : \mathbb{R}^n \rightarrow \mathbb{R}$  measurable and with linear growth
    - $(\varphi_i) \in C^2([0, T]; \mathbb{R})$
- 2 When  $X = W$  if Clark-Ocone formula does not apply.  
For instance when  $h \notin \mathbb{D}^{1,2}$ , or  $h \notin L^2(\Omega)$  (even not in  $L^1(\Omega)$ ).



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Thank you!!!

# A stochastic flow

## Definition

For  $0 < s < t < T$  and  $\eta \in B$  the **stochastic flow** is defined

$$Y_t^{s,\eta}(x) = \begin{cases} \eta(x + t - s) & x \in [-T, s - t] \\ \eta(0) + W_t(x) - W_s & x \in [s - t, 0] \end{cases}$$

where  $W$  standard Brownian motion.

## Remark

- $(Y_t^{s,\eta})_{0 \leq s \leq t \leq T, \eta \in B}$  is a  $B$ -valued random field

- 

$$Y_r^{s,\eta} = Y_r^t, Y_t^{s,\eta} \quad \text{for } 0 < s < t < r < T$$

# Theorem

Let  $H : L^2([-T, 0]) \rightarrow \mathbb{R}$

- $H \in C^3(L^2[-T, 0])$  with  $D^2H \in L^2([-T, 0]^2)$  and  $D^3H$  polynomial growth
- $DH(\eta) \in H^1([-T, 0])$  and other technical assumptions

$$u(t, \eta) := \mathbb{E} [H(Y_T^{t, \eta})]$$

Then

- $u \in C^{1,2}([0, T] \times B)$
- $u$  solves (5)

# Theorem

Let



$$H(\eta) := f \left( \int_{[-T,0]} \varphi_1(u+T) d\eta(u), \dots, \int_{[-T,0]} \varphi_n(u+T) d\eta(u) \right)$$

- $f : \mathbb{R}^n \rightarrow \mathbb{R}$  continuous and with linear growth and
- $(\varphi_i) \in C^2([0, T]; \mathbb{R})$
- Matrix  $\Sigma_t := (\Sigma_t)_{i,j} = \left( \int_t^T \varphi_i(s) \varphi_j(s) ds \right)$ ,  $t \in [0, T]$ .

$$\det(\Sigma_t) > 0 \quad \forall t \in ]0, T[$$

## Remark

$\Sigma_t$  is the Covariance matrix of Gaussian vector

$$G := \left( \int_t^T \varphi_1(s) dW_s, \dots, \int_t^T \varphi_n(s) dW_s \right)$$

# Theorem

$$u(t, \eta) := \Psi \left( t, \int_{[-t,0]} \varphi_1(s+t) d\eta(s), \dots, \int_{[-t,0]} \varphi_n(s+t) d\eta(s) \right)$$

with

$$\Psi(t, y_1, \dots, y_n) = \int_{\mathbb{R}^n} f(z_1, \dots, z_n) p(t, z_1 - y_1, \dots, z_n - y_n) dz_1 \cdots dz_n$$

and  $p \in C^{3,\infty}([0, T] \times \mathbb{R}^n)$  density of Gaussian vector  $G$

Then

- $u \in C^{1,2}([0, T] \times B) \cap C^0([0, T] \times C([-T, 0]))$
- $u$  solves (5)

## Remark

If  $X = W$  an analogous result is true with a weaker condition on  $f$

Let

- $f$  polynomial growth

Then

- $u \in C^{1,2}([0, T[ \times B)$
- 

$$h = u(0, W_0(\cdot)) + \underbrace{\int_0^T D^{\delta_0} u(s, W_s(\cdot)) d^- W_s}_{\text{improper forward integral}}$$

- $u(0, W_0(\cdot)) = \mathbb{E}[h]$
- $f$  Lipschitz then  $D^{\delta_0} u(s, W_s(\cdot)) = \mathbb{E}[D_s^m h | \mathcal{F}_t]$  since  $h \in \mathbb{D}^{1,2}$

# Theorem

$$H : B \longrightarrow \mathbb{R}$$



$$H(\eta) = f \left( \int_{-T}^0 \eta(s) ds \right)$$

- $f : \mathbb{R} \longrightarrow \mathbb{R}$  Borel subexponential (not necessarily continuous)
- $h = f \left( \int_0^T W_s ds \right) \in L^1(\Omega)$

$$u(t, \eta) = \int_{\mathbb{R}} f \left( \int_{-T}^0 \eta(r) dr + \eta(0)(T - t) + x \right) p_{\sigma}(t, x) dx$$

with  $\sigma_t = \sqrt{\frac{(T-t)^3}{3}}$



# Theorem

Then

- $u \in C^{1,2}([0, T[ \times B)$
- 

$$h = u(0, W_0(\cdot)) + \underbrace{\int_0^T D^{\delta_0} u(s, W_s(\cdot)) d^- W_s}_{\text{improper forward integral}}$$

- $u(0, W_0(\cdot)) = \mathbb{E}[h]$

## Remark

Since  $h \notin L^2(\Omega)$ , a priori neither Clark-Ocone formula nor its extensions to Wiener distributions apply

## A toy model for $X$ real valued

Let  $X$  such that  $[X]_t = t$

A1  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous and polynomial growth

A2  $v \in C^{1,2}([0, T] \times \mathbb{R}) \cap C^0([0, T] \times \mathbb{R})$  such that

$$\begin{cases} \partial_t v(t, x) + \frac{1}{2} \partial_{xx} v(t, x) = 0 \\ v(T, x) = f(x) \end{cases}$$

Then

$$h := f(X_T) = v(0, X_0) + \underbrace{\int_0^T \partial_x v(s, X_s) d^- X_s}_{\text{improper forward integral}}$$

Schoenmakers-Kloeden (1999), Coviello-Russo (2006),  
Bender-Sottinen-Valkeila (2008)

# Considerations about previous representation in toy model

- If  $X_t = W_t + tG$ ,  $G$  non-negative r.v.  $\notin L^1(\Omega)$  and  $f(x) = x$  then  $h = f(X_T) \notin L^1(\Omega)$ .
- If  $X = W$ ,
  - 1  $A1 \implies h = f(W_T) \in L^p(\Omega)$ , with  $p \geq 1$ . **not new...but...**
  - 2  $\begin{cases} f \text{ subexponential} \\ f(W_T) \in L^1(\Omega) \end{cases} \implies$

$$h := f(W_T) = v(0, W_0) + \underbrace{\int_0^T \partial_x v(t, W_t) d^- W_s}_{\text{improper forward integral}}$$

## Remark

$f$  not necessarily continuous,  $v \notin C^0([0, T] \times R)$

## A first motivating example

1)

$$H(\eta) = \left( \int_{-T}^0 \eta(s) ds \right)^2$$

$$u(t, \eta) := \left( \int_{-T}^0 \eta(s) ds + \eta(0)(T - t) \right)^2 + \frac{(T - t)^3}{3}$$

solves (3) and  $h$  has representation (4).

$$\partial_t u(t, \eta) = -2\eta(0) \left( \int_{-T}^0 \eta(s) ds + \eta(0)(T-t) \right) - (T-t)^2$$

$$D_{dx} u(t, \eta) = 2 \left( \int_{-T}^0 \eta(s) ds + \eta(0)(T-t) \right) \cdot \left( \mathbb{1}_{[-T,0]}(x) dx + (T-t)\delta_0(dx) \right)$$

$$\begin{aligned} D_{dx}^2 \phi(t, \eta) &= 2\mathbb{1}_{[-T,0]^2}(x, y) dx dy + \\ &+ 2(T-t)\mathbb{1}_{[-T,0]}(x) dx \delta_0(dy) + \\ &+ 2(T-t)\delta_0(dx) \mathbb{1}_{[-T,0]}(y) dy + \\ &+ 2(T-t)^2 \delta_0(dx) \delta_0(dy) \end{aligned}$$

$$D^2 u(t, \eta) \in (\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2 \quad \text{and} \quad [X]_t = t$$

## Remark

If  $X = W$

- Forward integral equals Itô integral
- The representation coincides with Clark-Ocone formula
- $H_0 = \mathbb{E}[h]$ .

## An interesting case

2)

$$H(\eta) = \int_{-T}^0 \eta(s)^2 ds$$

$$u(t, \eta) := \int_{-T}^0 \eta^2(s) ds + \eta(0)^2(T - t) + \frac{(T - t)^2}{2}$$

solves (3) and  $h$  has representation (4).

$$\partial_t u(t, \eta) = -\eta^2(0) - (T - t);$$

$$D_{dx} u(t, \eta) = 2\eta(x)dx + 2\eta(0)(T - t) \delta_0(dx)$$

$$D_{dx dy}^2 \phi(t, \eta) = 2\delta_y(dx) dy + 2(T - t)\delta_0(dx)\delta_0(dy) = 2\delta_x(dy) dx + 2(T$$

- $D^2 u(t, \eta) \in (\text{Diag} \oplus \mathcal{D}_{0,0})$  and  $[X]_t = t$
- $D^{ac}$  is not of bounded variation



## Stability result for $\mathbb{R}^n$ valued processes

In the finite dimensional case it holds.

### Theorem

Let  $X$  be a  $\mathbb{R}^n$ -valued process having all its mutual covariations  $([X^*, X]_t)_{1 \leq i, j \leq n} = [X^i, X^j]_t$  and  $F, G \in C^1(\mathbb{R}^n)$ . Then the covariation  $[F(X), G(X)]$  exists and is given by

$$[F(X), G(X)]. = \sum_{i,j=1}^n \int_0^\cdot \partial_i F(X) \partial_j G(X) d[X^i, X^j]$$

Setting  $n = 2$ ,  $F(x, y) = f(x)$ ,  $G(x, y) = g(y)$ ,  $f, g \in C^1(\mathbb{R})$  we have:

$$[f(X), g(Y)]. = \int_0^\cdot f'(X) g'(Y) d[X, Y]$$

# Stability result for $B$ -valued processes

Previous results admit some generalizations in the infinite dimensional framework.

## Theorem

Let  $X$  be a  $B$ -valued continuous stochastic process admitting a  $\chi$ -quadratic variation.

Let  $F^i, F^j : B \rightarrow \mathbb{R}$  be  $C^1$  Fréchet such that for  $i, j = 1, 2$

$$DF^i(\cdot) \otimes DF^j(\cdot) : B \times B \rightarrow \chi \subset (B \hat{\otimes}_\pi B)^*$$

$$(x, y) \mapsto DF^i(x) \otimes DF^j(y) \quad \text{continuous}$$

Then  $[F^i(X), F^j(X)]$  exists and it is given by

$$[F^i(X), F^j(X)]_\cdot = \int_0^\cdot \langle DF^i(X_s) \otimes DF^j(X_s), d[\tilde{X}]_s \rangle$$

# Stability results involving window Dirichlet processes

Let  $D$  a real continuous  $(\mathcal{F}_t)$ -Dirichlet process,

$$D = M + A,$$

- $D$  a real continuous  $(\mathcal{F}_t)$ -Dirichlet process,  $D = M + A$ ,
- $M$  an  $(\mathcal{F}_t)$ -local martingale
- $A$  a zero quadratic variation process with  $A_0 = 0$ .

# Time-homogeneous Stability Theorem

## Theorem

Let

- $F : B \rightarrow \mathbb{R}$  be  $C^1$  Fréchet
- $DF : B \rightarrow \mathcal{D}_0 \oplus L^2$  continuously

Then  $F(D(\cdot))$  is an  $(\mathcal{F}_t)$ -Dirichlet process with local martingale component equal to

$$\tilde{M} = F(D_0(\cdot)) + \int_0^\cdot D^{\delta_0} F(D_s(\cdot)) dM_s$$

where  $D^{\delta_0} F(\eta) := DF(\eta)(\{0\})$ .

# Stability results involving window weak Dirichlet processes

- $D$  a finite quadratic variation  $(\mathcal{F}_t)$ -weak Dirichlet process

$$D = M + A$$

- $M$  is the local martingale

# Stability Theorem

## Theorem

Let

- $F : [0, T] \times B \longrightarrow \mathbb{R}$  be  $C^{0,1}$  Fréchet such that
- $DF : [0, T] \times B \longrightarrow \mathcal{D}_0 \oplus L^2$  continuously

Then  $F(\cdot, D(\cdot))$  is an  $(\mathcal{F}_t)$ -weak Dirichlet process with martingale part

$$\tilde{M}_t^F = F(0, D_0(\cdot)) + \int_0^t D^{\delta_0} F(s, D_s(\cdot)) dM_s .$$