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Divers problèmes théoriques et numériques liés à la simulation de fluides non newtoniens

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THÈSE

présentée pour l'obtention du titre de
Docteur de l'Université Paris-Est
Spécialité : Mathématiques Appliquées

par **David BENOIT**

Sujet : Divers problèmes théoriques et numériques
liés à la simulation de fluides non newtoniens

Soutenue le 22 janvier 2014
devant le jury composé de :

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Divers problèmes théoriques et numériques liés à la simulation de fluides non newtoniens

Résumé : Cette thèse est consacrée à la modélisation, à l'analyse mathématique et à la simulation de fluides non newtoniens. Certains fluides dans un état intermédiaire entre un liquide et un solide sont plus particulièrement considérés : les fluides vieillissants. Les échelles de modélisation sont macroscopique et mésoscopique. Le chapitre 1 introduit les modèles et donne les principaux résultats obtenus. Dans le chapitre 2, on présente des simulations numériques d'un modèle macroscopique en deux dimensions. La méthode de discrétisation par éléments finis utilisée est décrite. Pour le cas test de l'écoulement autour d'un cylindre, les phénomènes en jeu dans les fluides vieillissants sont observés. Le chapitre 3 concerne l'étude mathématique de la version unidimensionnelle du système d'équations aux dérivées partielles utilisé pour les simulations. On montre que le problème est bien posé et on examine le comportement en temps long de la solution. Dans le dernier chapitre, des équations macroscopiques sont dérivées à partir d'une équation mésoscopique. L'analyse mathématique de cette équation mésoscopique est également menée.

Mots-clés : fluides non newtoniens ; modèle micro-macro ; équations aux dérivées partielles ; méthodes des éléments finis ; écoulement autour d'un cylindre ; écoulement de Couette ; comportement en temps long.

Various theoretical and numerical issues related to the simulation of non-Newtonian fluids

Abstract : This thesis is devoted to the modelling, the mathematical analysis and the simulation of non-Newtonian fluids. Some fluids in an intermediate liquid-solid phase are particularly considered : aging fluids. Modelling scales are macroscopic and mesoscopic. In Chapter 1, we introduce the models and give the main results obtained. In Chapter 2, we present numerical simulations of a macroscopic two-dimensional model. The finite element method used for discretization is described. For the flow past a cylinder test-case, phenomena at play in aging fluids are observed. The Chapter 3 contains a mathematical analysis of the one-dimensional version of the system of partial differential equations used for the simulations. We show well-posedness and investigate the longtime behaviour of the solution. In the last chapter, macroscopic equations are derived from a mesoscopic equation. The mathematical analysis of this mesoscopic equation is also carried out.

Keywords : Non-Newtonian fluids ; micro-macro model ; partial differential equations ; finite element method ; flow past a cylinder ; Couette flow ; longtime behaviour.

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Introduction

1.1 Généralités sur la modélisation des fluides non newtoniens

1.1.1 Les fluides non newtoniens

Les fluides regroupent l'ensemble des liquides et des gaz. Parmi eux, on trouve les liquides dits usuels, l'eau, le lait par exemple. Ceux-ci peuvent être considérés, dans des conditions standards comme des fluides newtoniens : les contraintes sont proportionnelles aux vitesses de déformation (voir équation (1.3) ci-dessous).

Au-delà de ce cadre newtonien, on trouve des fluides tout autant utilisés mais qui présentent des comportements non linéaires variés. On les nomme fluides non newtoniens ou fluides complexes. Ces comportements non linéaires sont par exemple ceux de la mayonnaise qui devient solide lorsque battue de manière appropriée ou du dentifrice qui sort du tube lorsqu'on appuie assez fort.

Ces fluides peuvent être le fait de la nature, le miel, le sang par exemple et leur étude motive la recherche biologique. Mais nombreux sont ceux issus des diverses industries agro-alimentaires, cosmétiques et de la construction notamment. La modélisation des fluides non newtoniens vise à améliorer la connaissance des divers processus industriels ou biologiques, connaissance qui reste cependant souvent empirique.

Les non-linéarités des fluides sont le résultat d'une micro-structure intrinsèque, de particules en concentration assez grande dans le solvant. Le type de ces particules divise les fluides non newtoniens en grandes classes, entre autres les mousses, les émulsions, les solutions de polymères, les systèmes granulaires. La micro-structure peut ou non intervenir dans la modélisation des fluides non newtoniens.

On considère dans cette thèse des fluides dans un état intermédiaire entre un liquide et un solide. Un comportement non newtonien important est la viscoélasticité : sont présents à la fois l'aspect visqueux (newtonien) d'un liquide et l'aspect élastique (lorsque les contraintes sont proportionnelles aux déformations) d'un solide. Le comportement qui nous intéressera plus particulièrement est le vieillissement. L'état du fluide dépend du passé de la façon suivante : laissé longtemps au repos, le fluide devient de plus en plus solide ; en l'agitant, il redevient liquide. On pense au ketchup ou au béton par exemple.

1.1.2 Modèles macroscopiques et microscopiques pour les fluides non newtoniens

Le comportement des fluides est régi par les lois de conservation des quantités de masse, de moment et d'énergie. On supposera par la suite la température constante de sorte que les effets thermiques sont négligeables et que la loi de conservation de l'énergie peut être laissée de côté.

On suppose de plus le fluide incompressible, sa densité ρ est une constante (scalaire) indépendante du temps et de l'espace. La loi de conservation de masse se réduit alors à

$$\operatorname{div} u = 0, \quad (1.1)$$

où u est le vecteur vitesse du fluide.

La loi de conservation de moment du fluide s'écrit

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = \operatorname{div} \mathcal{T} - \nabla p. \quad (1.2)$$

Dans cette équation, en plus de u , deux autres inconnues ont été introduites : le tenseur des extra-contraintes \mathcal{T} (partie déviatorique du tenseur des contraintes internes) et la pression p . Le membre gauche de l'équation représente la force d'inertie, c'est-à-dire la densité fois l'accélération d'une particule de fluide et le tenseur \mathcal{T} représente les forces développées en réponse à la déformation du matériau.

Les trois inconnues u , \mathcal{T} et p sont des fonctions dépendant du temps $t \geq 0$ et de l'espace $x \in \Omega$ où Ω est un domaine borné régulier de \mathbb{R}^3 . Elles sont à valeurs respectivement dans \mathbb{R}^3 , $\mathbb{R}^{3 \times 3}$ et \mathbb{R} .

Afin d'achever la formulation mathématique, il faut une relation supplémentaire entre le tenseur \mathcal{T} et la vitesse u ; une telle loi est appelée relation constitutive.

Pour les fluides newtoniens, la relation constitutive est

$$\mathcal{T} = \eta \left(\nabla u + \nabla u^T \right), \quad (1.3)$$

où la constante de proportionnalité η est appelée viscosité. Le tenseur gradient de vitesse ∇u est à valeurs dans les matrices 3×3 de composante $\left(\frac{\partial u_i}{\partial x_j} \right)_{1 \leq i, j \leq 3}$ et ∇u^T est sa transposée. Dans ce cas, la loi entre la contrainte \mathcal{T} et le taux de déformation $\frac{\nabla u + \nabla u^T}{2}$ est linéaire. Le système d'équations (1.1)-(1.2)-(1.3) constitue les équations de Navier-Stokes incompressibles.

Au-delà du comportement newtonien, il existe toute une zoologie de modèles qui remplacent la relation constitutive (1.3). On peut séparer les modèles en deux catégories :

- macroscopique. La relation constitutive fait alors intervenir, comme dans le cas newtonien, les variables macroscopiques u, \mathcal{T}, p (complété éventuellement d'autres grandeurs macroscopiques) mais aux moyens d'équations plus compliquées impliquant des opérateurs fonctionnels, différentiels ou intégraux.
- micro-macro. La relation constitutive est remplacée par une formule de calcul du tenseur des contraintes \mathcal{T} en fonction de variables décrivant des échelles plus petites ; l'échelle moléculaire ou plus communément une échelle intermédiaire appelée mésoscopique.

L'échelle mésoscopique est celle de la structure interne du fluide, qui est responsable du comportement non newtonien. Pour cette catégorie de modèle, les deux échelles de modélisation, macroscopique et microscopique, sont couplées.

1.2 Modèles pour les fluides viscoélastiques et vieillissants

On présente ici des modèles pour les fluides viscoélastiques et vieillissants. Pour une description plus approfondie des modèles pour les fluides viscoélastiques, on pourra se reporter par exemple à l'ouvrage de référence [BCAH91] d'une part et à la monographie [Ren00] et aux articles [LBL09, LBL12] qui présentent des revues mathématiques d'autre part. Pour une description physique approfondie des modèles de fluides vieillissants, on renvoie aux chapitres d'ouvrage [Ajd03, BL11] et aux thèses [Pic04, Goy08, Tsa09].

Dans les modèles de fluide non newtonien, la contribution newtonienne est souvent mise à part et la loi constitutive (1.3) est remplacée par la décomposition

$$\mathcal{T} = \eta (\nabla u + \nabla u^T) + \tau \quad (1.4)$$

et une autre équation reliant τ et u .

De plus, les modèles rhéologiques ont été historiquement d'abord introduits pour des inconnues scalaires dépendant d'une seule variable d'espace. Cette situation correspond à l'écoulement bidimensionnel de Couette, proche des conditions expérimentales d'un fluide à l'intérieur d'un rhéomètre. Cet appareil est couramment utilisé pour mesurer les propriétés physiques des fluides. Dans un rhéomètre, le fluide est entre deux cylindres, l'un fixe et l'autre tournant à une vitesse a . On simplifie cette situation en considérant un fluide entre deux plaques infinies, l'une fixe et l'autre avançant à la vitesse a , voir figure 1.1. La géométrie de l'écoulement impose à la

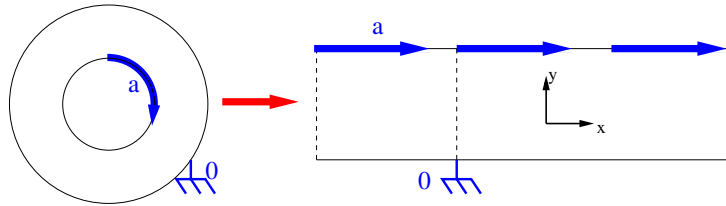


Figure 1.1. Schéma de l'écoulement de Couette.

vitesse d'avoir la forme $u = u(y)\mathbf{e}_x$. La composante de la contrainte à laquelle on s'intéresse est celle de cisaillement $\tau = \tau_{xy}$. Le système d'équations (1.1)-(1.2)-(1.4) se réduit alors à

$$\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}. \quad (1.5)$$

La relation constitutive est quant à elle une équation faisant intervenir la contrainte τ et la vitesse u , ou plutôt la contrainte τ et le taux de déformation défini par

$$\dot{\gamma} = \frac{\partial u}{\partial y}. \quad (1.6)$$

1.2.1 Modèles macroscopiques

1.2.1.1 Modèles de Maxwell et d'Oldroyd-B

On présente le modèle unidimensionnel de Maxwell et sa généralisation en plusieurs dimensions, le modèle d'Oldroyd-B. Ces modèles s'appliquent à des fluides viscoélastiques tels que les liquides polymériques dilués.

Dans le cadre unidimensionnel, la viscoélasticité peut être linéaire notamment dans le modèle de Maxwell. Celui-ci consiste à additionner les déformations visqueuses et élastiques. Une représentation graphique (figure 1.2) est l'association en série

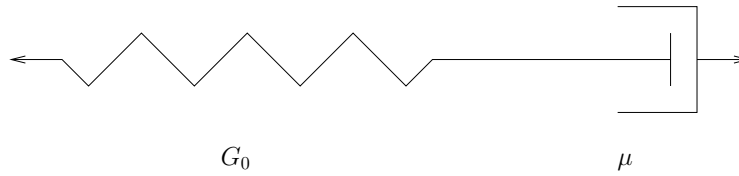


Figure 1.2. Modèle de Maxwell pour un fluide viscoélastique.

- d'un ressort élastique de module G_0 , la contrainte τ est proportionnelle à la déformation γ : $\tau = G_0\gamma$,
- d'un amortisseur de viscosité μ , la contrainte τ est proportionnelle au taux de déformation $\dot{\gamma}$: $\tau = \mu\dot{\gamma}$. La viscosité μ est différente de la viscosité η apparaissant dans la décomposition (1.4). D'ailleurs, dans le modèle initial de Maxwell, $\eta = 0$. Ces deux viscosités représentent typiquement les viscosités propres respectivement au solvant et à la micro-structure qu'il contient.

On obtient l'équation

$$\dot{\gamma} = \frac{d\gamma_{\text{visc}}}{dt} + \frac{d\gamma_{\text{elas}}}{dt} = \frac{1}{G_0}\dot{\tau} + \frac{1}{\mu}\tau \quad (1.7)$$

qu'on réécrit

$$\lambda \frac{\partial \tau}{\partial t} = -\tau + \mu \frac{\partial u}{\partial y}. \quad (1.8)$$

On a utilisé la définition (1.6) de $\dot{\gamma}$. On a également fait apparaître le temps de relaxation $\lambda = \frac{\mu}{G_0}$, qui est une mesure du temps durant lequel le système se rappelle de son écoulement passé. L'analogie en électricité du modèle de Maxwell est l'association en série d'une résistance et d'un condensateur. Le temps de relaxation λ est alors l'analogie du temps caractéristique RC.

Le point de départ de la généralisation du modèle de Maxwell en dimension 2 ou 3 est le modèle d'Oldroyd-B

$$\lambda \left(\frac{\partial \tau}{\partial t} + (u \cdot \nabla)\tau - (\nabla u)\tau - \tau(\nabla u)^t \right) = -\tau + \mu (\nabla u + \nabla u^t). \quad (1.9)$$

La dérivée $\frac{\partial \tau}{\partial t} + (u \cdot \nabla)\tau - (\nabla u)\tau - \tau(\nabla u)^t$ est appelée dérivée sur-convectée. Elle permet de respecter le principe d'invariance par changement de repère. D'autres dérivées respectent ce principe mais la dérivée sur-convectée est celle qui apparaît naturellement lors de la dérivation à partir du modèle microscopique d'haltères. Ceci est détaillé dans la section 1.2.2.1.

Remarque 1.1. *L'équation constitutive (1.9) est non linéaire non seulement du fait de la dérivée matérielle $(u \cdot \nabla)\tau$ mais aussi des termes supplémentaires $-(\nabla u)\tau - \tau(\nabla u)^t$. Ces non-linéarités n'apparaissent pas en une dimension et rendent difficile l'étude et la discrétisation de ce type de modèle.*

Le système d'équations complet modélisant les fluides viscoélastiques de type Oldroyd-B est ainsi constitué respectivement des équations (1.5)-(1.8) en une dimension et (1.1)-(1.2)-(1.4)-(1.9) en plusieurs dimensions.

1.2.1.2 Modèles avec fluidité

Pour les fluides vieillissants, les modèles présentés dans [DAL01, PABL02] ont été introduits en une dimension comme des variations du modèle de Maxwell. Les physiciens introduisent une nouvelle inconnue : un champ scalaire positif, la fluidité f , rendant le temps de relaxation de la contrainte variable, dépendant du temps et de chaque point du fluide. La fluidité caractérise le vieillissement. Le fluide est solide là où $f = 0$ et d'autant plus liquide que la fluidité est grande. L'équation de Maxwell (1.8) est remplacée par (système adimensionné (1) dans [PABL02])

$$\begin{cases} \lambda \frac{\partial \tau}{\partial t} = -f\tau + \mu \frac{\partial u}{\partial y} & (1.10a) \\ \frac{\partial f}{\partial t} = -U(f) + V\left(f, \tau, \frac{\partial u}{\partial y}\right), & (1.10b) \end{cases}$$

où U et V sont deux fonctions positives.

La fluidité f apparaît comme l'inverse d'un temps de relaxation dans l'équation (1.10a) sur τ et satisfait l'équation aux dérivées ordinaires (1.10b). Dans l'équation (1.10b), les deux termes $-U(f)$ et $V\left(f, \tau, \frac{\partial u}{\partial y}\right)$ sont de signe opposé et représentent la compétition caractéristique des fluides vieillissants. Le vieillissement en tant que tel, c'est-à-dire la solidification spontanée du fluide, est modélisé par le terme négatif. Ce terme fait décroître la fluidité, de sorte que le phénomène de relaxation est de plus en plus lent. L'effet opposé, le rajeunissement par l'écoulement est modélisé par le terme positif, qui fait croître la fluidité.

Même si les phénomènes physiques que représentent les termes $U(f)$ et $V\left(f, \tau, \frac{\partial u}{\partial y}\right)$ dans l'équation (1.10b) sont identifiés, leur caractérisation n'est pas précisément connue, le modèle reste phénoménologique. Les articles [DAL01, PABL02] proposent ainsi plusieurs choix possibles pour les fonctions U et V , typiquement des polynômes. Ces choix sont basés sur les caractéristiques du fluide connues par l'expérience.

Par exemple, le monôme dominant en 0 de U est typiquement f^2 car des expériences (e.g. [Str78] sur un PVC) montrent que la fluidité décroît en $\frac{1}{t}$ dans un régime standard.

Lors de notre étude et des simulations de modèle macroscopique, nous retiendrons le choix suivant pour sa simplicité :

$$U(f) = f^2 + \nu f^3 \text{ et } V\left(f, \tau, \frac{\partial u}{\partial y}\right) = \xi |\tau| f^2, \quad (1.11)$$

où ξ et ν sont deux coefficients empiriques positifs. Ce choix est représentatif dans la classe de modèles (1.10).

A notre connaissance, il n'existe pas de généralisation en dimension supérieure à un du modèle de vieillissement (1.10). Cependant, dans une première approche, on peut introduire la variable scalaire fluidité comme temps de relaxation dans le modèle d'Oldroyd-B (1.9). On obtient le système

$$\left\{ \begin{array}{l} \lambda \left(\frac{\partial \tau}{\partial t} + (u \cdot \nabla) \tau - (\nabla u) \tau - \tau (\nabla u)^t \right) = -f \tau + \mu (\nabla u + \nabla u^t) \\ \frac{\partial f}{\partial t} + (u \cdot \nabla) f = -U(f) + V(f, \tau, \nabla u). \end{array} \right. \quad (1.12a)$$

$$\frac{\partial f}{\partial t} + (u \cdot \nabla) f = -U(f) + V(f, \tau, \nabla u). \quad (1.12b)$$

On conserve ainsi les avantages modèle d'Oldroyd-B : la dérivée sur-convectée respecte le principe d'objectivité. On a de plus introduit une dérivée matérielle pour la fluidité. Cependant, le modèle (1.12) reste une approche simplifiée. Il ne fait notamment apparaître qu'un temps de relaxation identique pour toutes les composantes du tenseur des contraintes τ ; une généralisation plus "physique" devrait prendre en compte des temps de relaxation différents.

Le système d'équations complet modélisant les fluides vieillissants est constitué respectivement des équations (1.5)-(1.10) en une dimension et (1.1)-(1.2)-(1.4)-(1.12) en plusieurs dimensions avec le choix (1.11) pour l'équation sur la fluidité (la norme de τ étant la norme euclidienne).

1.2.1.3 Quelques simulations numériques

On résume dans cette section les résultats du chapitre 2 qui présente des simulations numériques pour le système d'équations aux dérivées partielles (EDP) (1.1)-(1.2)-(1.4)-(1.12) avec le choix (1.11) pour l'équation sur la fluidité. En version adimensionnelle, ce système s'écrit

$$\left\{ \begin{array}{l} \text{Re} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + (1 - \alpha) \Delta u + \text{div}(\tau) \\ \text{div}(u) = 0 \\ \frac{\partial \tau}{\partial t} + (u \cdot \nabla) \tau - (\nabla u) \tau - \tau (\nabla u)^t = -\frac{f}{\text{We}} \tau + \frac{\alpha}{\text{We}} (\nabla u + \nabla u^t) \\ \frac{\partial f}{\partial t} + (u \cdot \nabla) f = -f^2 - \nu f^3 + \xi f^2 |\tau|, \end{array} \right. \quad (1.13a)$$

$$\text{div}(u) = 0 \quad (1.13b)$$

$$\frac{\partial \tau}{\partial t} + (u \cdot \nabla) \tau - (\nabla u) \tau - \tau (\nabla u)^t = -\frac{f}{\text{We}} \tau + \frac{\alpha}{\text{We}} (\nabla u + \nabla u^t) \quad (1.13c)$$

$$\frac{\partial f}{\partial t} + (u \cdot \nabla) f = -f^2 - \nu f^3 + \xi f^2 |\tau|, \quad (1.13d)$$

où on a introduit trois constantes : le nombre de Reynolds $\text{Re} > 0$, le nombre de Weissenberg $\text{We} > 0$ et $\alpha \in (0, 1)$ le rapport des viscosités μ et η . De plus, les paramètres ν et ξ de l'équation sur la fluidité sont adimensionnés en étant multipliés par le temps caractéristique de l'écoulement.

Comme expliqué dans la section 1.2.1.2, lorsque $f \equiv 1$, le système (1.13) se réduit au modèle d’Oldroyd-B. La discrétisation de ce modèle a été largement étudiée, voir par exemple [Keu89, Keu00]. On utilise donc une méthode de discrétisation du modèle d’Oldroyd-B qu’on adapte dans le cas où la fluidité f est variable.

La méthode de discrétisation retenue pour le modèle d’Oldroyd-B est décrite dans [BLM09]. Cette méthode fait intervenir le tenseur de conformation

$$\Sigma = \text{Id} + \frac{We}{\alpha} \tau. \quad (1.14)$$

Elle a l’avantage de conserver au niveau discret la propriété de dissipation de l’énergie libre

$$\frac{\text{Re}}{2} \int_{\Omega} |u|^2 + \frac{\alpha}{2We} \int_{\Omega} \text{tr}(\Sigma - \ln \Sigma - \text{Id}), \quad (1.15)$$

pour des conditions de Dirichlet homogènes. C’est une méthode d’éléments finis. On retient le choix $\mathbb{P}_2/\mathbb{P}_1/\mathbb{P}_1$ pour la vitesse, la pression et la contrainte. Un terme de Temam [Tem84, Section III.5] est ajouté pour la convection de la vitesse et la méthode des caractéristique est utilisée pour la convection de la contrainte.

L’équation (1.13d) sur la fluidité ne contient pas de dérivée en espace à part le terme de convection. Elle est discrétisée point par point avec la méthode des caractéristiques.

Le cas test retenu est l’écoulement autour d’un cylindre (*flow past a cylinder*). Par symétrie de translation, le domaine de simulation est 2D, on en représente un maillage sur la figure 1.3. Le cas test *flow past a cylinder* a été largement étudié et simulé avec le modèle d’Oldroyd-B.

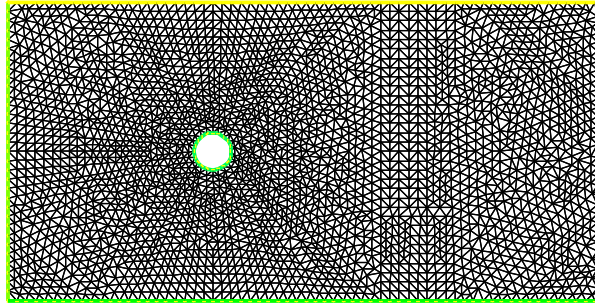


Figure 1.3. Maillage du cas test *flow past a cylinder*

Les articles [SSAB99, BACMP04] donnent par exemple des résultats de référence.

Le “cylindre” se réduit en 2D à un cercle qu’on dénomme bille par la suite. Dans ce cas test, la bille est fixe et le fluide avance autour d’elle. Mais c’est une simplification, les expériences à reproduire à terme par la simulation sont en fait des expériences de chute de bille dans un fluide non newtonien, voir par exemple [GTH⁺06]. Les tests conduits ici concernent uniquement une bille fixe.

Les conditions de bord du cas test *flow past a cylinder* sont les suivantes :

- en entrée (bord gauche), on impose des conditions de Dirichlet sur la vitesse $u = u_i e_x$, la contrainte $\tau = \tau_i \text{Id}$ et la fluidité $f = f_i$, où u_i , τ_i et f_i sont des constantes scalaires positives,

- sur les murs (bords inférieur et supérieur), on impose du glissement sur la vitesse

$$u \cdot n = 0 \text{ et } (((1 - \alpha)D(u) + \tau) \cdot n) \cdot t = 0,$$

où $D(u) = \frac{\nabla u + \nabla u^t}{2}$ est le taux de déformation et n et t sont les vecteurs normal et tangent,

- en sortie (bord droit), on impose des conditions de Neumann homogènes sur la vitesse :

$$(-p\text{Id} + (1 - \alpha)D(u) + \tau) \cdot n = 0,$$

- sur la bille, on impose du collement sur la vitesse ($u = 0$).

Les paramètres physiques sont $\text{Re} = 10^{-10}$, $\alpha = 0.41$ et $\text{We} = 0.1$ pour les nombres sans dimension et $\xi = \nu = 1$ pour les paramètres de l'équation (1.13d) sur la fluidité. Les conditions de bord en entrée sont $u_i = 1.5$, $\tau_i = 0$ et $f_i = 1$. On présente les résultats à l'état stationnaire, en ce sens que les variables varient numériquement très peu d'un pas de temps à l'autre à son voisinage.

La figure 1.4 représente le champ de vitesse pour le modèle d'Oldroyd-B ($f \equiv 1$) à gauche et le modèle avec fluidité à droite. L'écoulement se fait de gauche à droite. La vitesse du fluide est faible au voisinage de la bille et grande près des murs à son niveau. Le champ de vitesse est légèrement différent dans les deux cas. La différence est plus nette sur le champ de contrainte, sur la composante τ_{xx} par exemple (figure 1.5). Les lignes de niveau de la fluidité (du modèle avec

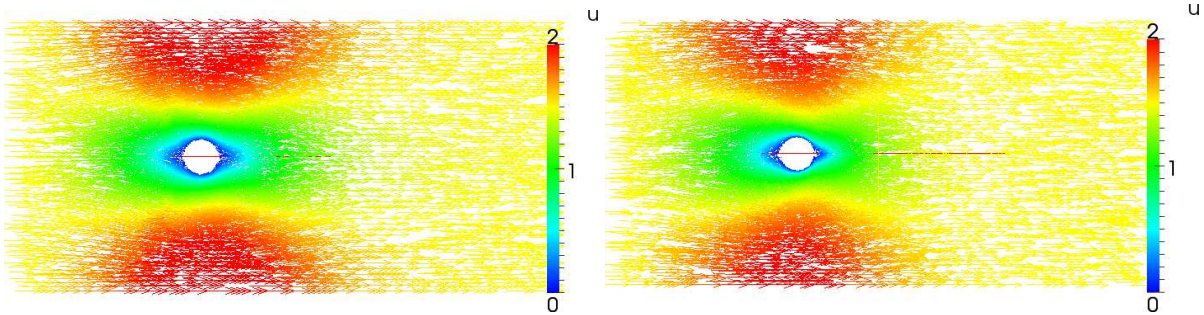


Figure 1.4. Champ de vitesse pour le modèle d'Oldroyd-B (à g.) et le modèle avec fluidité (à d.)

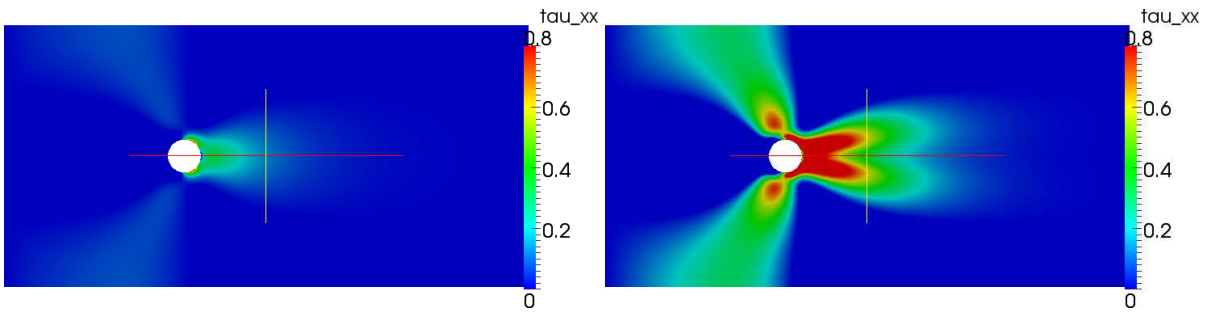


Figure 1.5. Composante τ_{xx} de la contrainte pour le modèle d'Oldroyd-B (à g.) et le modèle avec fluidité (à d.)

fluidité variable) sont représentées sur la figure 1.6. On observe que la fluidité décroît lorsque le fluide avance, de gauche à droite ; le fluide vieillit. La fluidité augmente autour de la bille ; les variations de contrainte qu'elle induit font rajeunir le fluide.

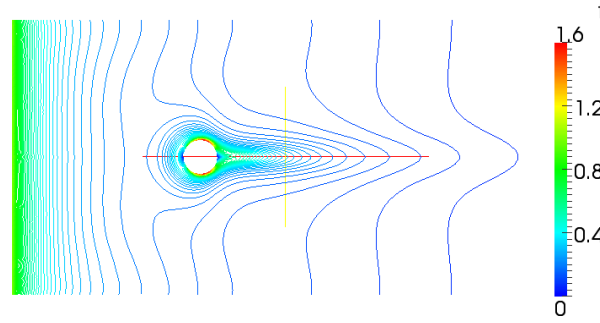


Figure 1.6. Lignes de niveau de la fluidité

Ainsi, sur un cas test assez complexe, on observe le phénomène de compétition vieillissement/rajeunissement introduits par le modèle avec fluidité.

1.2.2 Les modèles mésoscopiques

Dans les modèles micro-macro, en prenant en compte la décomposition (1.4), la relation constitutive est remplacée par une formule de calcul du tenseur τ en fonction de variables décrivant des échelles plus petites, dites mésoscopiques.

Plus précisément, dans les modèles considérés, on introduit une variable mésoscopique Z évoluant dans un espace \mathcal{D}_Z . Le milieu est décrit par la densité de probabilité $p(t, x, Z)$ de la variable Z au point macroscopique x et à l'instant t . Cette densité de probabilité suit une équation dite de Fokker-Planck. Le tenseur des contraintes τ est alors donné par une certaine espérance sous la densité de probabilité p . On obtient formellement le système

$$\begin{cases} \frac{\partial p}{\partial t}(t, x, Z) = \mathcal{H}(p(t, x, Z)) & (1.16a) \\ \tau(t, x) = \int h(Z)p(t, x, Z)dZ, & (1.16b) \end{cases}$$

où \mathcal{H} est un opérateur différentiel et h est une fonction de \mathcal{D}_Z à valeur dans l'espace $\mathbb{R}^{3 \times 3}$ des matrices. La première équation est l'équation de Fokker-Planck et la deuxième est une formule de moyennisation. On couple ensuite le système (1.16) aux équations macroscopiques du mouvement (1.1)-(1.2) (avec la décomposition (1.4)).

1.2.2.1 Modèle des haltères

On présente le plus simple des modèles mésoscopiques pour les fluides polymériques : le modèle des haltères (*dumbbells* en anglais). Pour une description plus approfondie de ce modèle (ou d'autres modèles de fluides polymériques), on pourra se reporter à [BCAH91]. De plus,

la monographie [Ren00] et les articles [LBL09, LBL12] discutent les hypothèses nécessaires à l'obtention de l'équation de Fokker-Planck du modèle des haltères, on ne les redonne pas ici.

L'avantage de ce modèle mésoscopique pour les fluides polymériques est qu'il permet de retrouver le modèle macroscopique d'Oldroyd-B comme expliqué à la fin de cette sous-section et donc de donner une nouvelle interprétation physique aux paramètres des modèles, basée sur la structure moléculaire des polymères.

Dans le modèle des haltères, la molécule de polymère est modélisée par deux billes connectées par un ressort (voir figure 1.7). La variable mésoscopique introduite est le vecteur bout-à-bout X de la molécule. Il relie les deux billes dans la modélisation. Le vecteur X vit dans \mathbb{R}^3 et décrit l'élongation et l'orientation de la molécule.

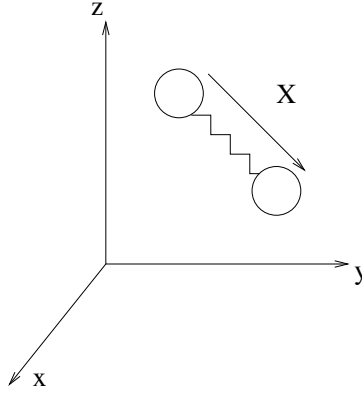


Figure 1.7. Modèle des haltères pour une chaîne polymérique.

La variation de la densité de probabilité $p(t, x, X)$ du vecteur bout-à-bout est régie par trois phénomènes

- (i) la force de friction qui résulte du mouvement des deux billes dans le solvant ; la différence de vitesse entre le solvant et les deux billes donne une force $\zeta \nabla u X$ où ζ est le coefficient de friction,
- (ii) la force du ressort $F(X)$ connectant les deux billes et modélisant les forces intramoléculaires dans la chaîne polymérique,
- (iii) la force brownienne, modélisant les collisions permanentes de la chaîne polymérique avec les molécules du solvant ; cette force donne une diffusion dans l'équation sur p .

La conservation du moment sur chacune des deux billes donne l'équation de Fokker-Planck (en coordonnées eulériennes)

$$\begin{aligned} \frac{\partial p}{\partial t}(t, x, X) + u(t, x) \cdot \nabla_x p(t, x, X) \\ = -\operatorname{div}_X \left(\left(\nabla_x u(t, x) X - \frac{2}{\zeta} F(X) \right) p(t, x, X) \right) + \frac{2kT}{\zeta} \Delta_X p(t, x, X), \end{aligned} \quad (1.17)$$

où T est la température du fluide et k la constante de Boltzmann.

La formule de calcul du tenseur des contraintes τ en fonction de la densité est donnée par la formule dite de Kramers

$$\tau(t, x) = -nkT\text{Id} + n \int_{\mathbb{R}^3} (X \otimes F(X)) p(t, x, X) dX, \quad (1.18)$$

où \otimes dénote le produit tensoriel ($X \otimes F(X)$ est la matrice de coordonnées $(X_i F_j(X))_{i,j}$) et n est le nombre de chaînes polymériques par unité de volume.

Dans le modèle des haltères, la forme précise du système formel (1.16) est ainsi donnée par l'équation de Fokker-Planck (1.17) et la formule (1.18).

Par ailleurs, on n'a jusqu'ici pas donné d'expression précise pour la force du ressort $F(X)$, décrivant les forces intramoléculaires du polymère. Il existe toute une variété de formules pour cette force, on choisit ici l'hypothèse la plus simple : une relation linéaire $F(X) = HX$, où $H > 0$ est une constante scalaire. On parle de modèle hookéen. Dans ce cas, en multipliant l'équation de Fokker-Planck (1.17) par $X \otimes X$ et en intégrant, on obtient avec la formule (1.18)

$$\frac{\partial \tau}{\partial t} + (u \cdot \nabla) \tau - (\nabla u) \tau - \tau (\nabla u)^t = -\frac{4H}{\zeta} \tau + nkT (\nabla u + \nabla u^t). \quad (1.19)$$

Cette équation est précisément celle d'Oldroyd-B (1.9), moyennant des relations simples entre les différents coefficients. On a dérivé l'équation d'Oldroyd-B à partir du modèle d'haltères hookéen.

1.2.2.2 Modèles élastoplastiques

Au niveau mésoscopique, les fluides vieillissants considérés sont amorphes (ils n'ont pas de structure cristalline). On les nomme aussi verres mous (*Soft Glassy rheology*) car ces modèles sont basés sur une analogie à la transition vitreuse (transition du verre entre un liquide et un solide). Les boues, les mousses, les émulsions sont par exemple des verres mous.

Les modèles pour les verres mous sont plus récents et n'ont pas bénéficié d'autant de développements physiques et mathématiques que les modèles pour les fluides polymériques. Notamment, ils restent essentiellement unidimensionnels. On se place donc dans cette section dans le cadre de l'écoulement de Couette (figure 1.1). On rappelle que dans ce cadre, la variable d'espace est notée y .

Les modèles qui nous intéressent sont dits élastoplastiques. La variable mésoscopique qu'ils introduisent est la contrainte locale (scalaire) σ . L'échantillon est divisé en "blocs" fixes à la position y . Les blocs sont de taille mésoscopique et portent chacun en chaque instant t une contrainte $\sigma(t, y)$.

La variation de la densité de probabilité $p(t, y, \sigma)$ de la contrainte locale découle de trois phénomènes (figure 1.8)

- (i) la contrainte évolue élastiquement suivant le taux de déformation $\dot{\gamma} = \frac{\partial u}{\partial y}$ ce qui donne le terme $b \frac{\partial p}{\partial \sigma}$, en notant

$$b(t, y) = G_0 \dot{\gamma}(t, y). \quad (1.20)$$

On rappelle que G_0 , paramètre de l'équation de Maxwell (1.7), est le module élastique.

- (ii) lorsque la contrainte locale dépasse un certain seuil $\sigma_c > 0$, un scalaire fixé, il se produit un "évènement plastique". Autrement dit, la contrainte se relaxe vers 0 avec un temps de

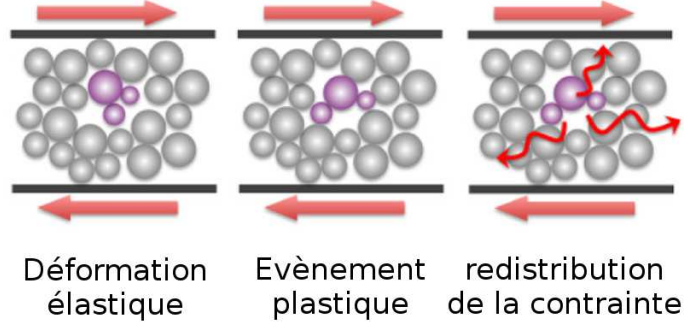


Figure 1.8. Schéma de la modélisation élastoplastique [BCA09].

relaxation T_0 donnant un terme $-\frac{1}{T_0}\chi(\sigma)p + \frac{1}{T_0}(\int \chi p)\delta_0(\sigma)$ où on a noté χ la fonction caractéristique

$$\chi = \mathbb{1}_{\mathbb{R} \setminus [-\sigma_c, \sigma_c]} \quad (1.21)$$

et δ_0 la fonction de Dirac en 0. Dans la suite, on prend $T_0 = 1$.

- (iii) chaque contrainte locale est soumise aux effets des évènements de relaxations voisins. Cette redistribution est modélisée par un terme $\mathcal{G}(p(t, y, \sigma), \{p(t, y', \sigma')\}_{(y', \sigma')})$. L'expression de \mathcal{G} fait ainsi en principe intervenir non seulement le couple (y, σ) mais aussi tous les voisins (y', σ') de (y, σ) . On donne plusieurs expressions possibles de \mathcal{G} plus loin.

On obtient l'équation de Fokker-Planck

$$\begin{aligned} \frac{\partial p}{\partial t}(t, y, \sigma) + b(t, y) \frac{\partial p}{\partial \sigma}(t, y, \sigma) \\ = -\chi(\sigma)p(t, y, \sigma) + \left(\int \chi(\sigma')p(t, y, \sigma')d\sigma' \right) \delta_0(\sigma) + \mathcal{G}(p(t, y, \sigma), \{p(t, y', \sigma')\}_{(y', \sigma')}). \end{aligned} \quad (1.22)$$

Dans cette équation, on voit apparaître le nombre de réorganisation par unité de temps (proportion de blocs dépassant la contrainte seuil σ_c)

$$f(t, y) = \int \chi(\sigma)p(t, y, \sigma)d\sigma. \quad (1.23)$$

Cette variable macroscopique est en fait interprétée comme la fluidité qui avait été introduite dans les modèles macroscopiques de vieillissement (1.10).

La forme précise du système formel (1.16) pour les modèles élastoplastiques est donnée par l'équation de Fokker-Planck (1.22) et la formule de moyenne

$$\tau(t, y) = \int \sigma p(t, y, \sigma)d\sigma. \quad (1.24)$$

La contrainte macroscopique τ est définie comme la moyenne des contraintes mésoscopiques.

Il reste à donner une expression pour le terme de redistribution \mathcal{G} . Dans une première approche, Hébraud et Lequeux [HL98] ont fait une approximation de "champ moyen". La redistribution des contraintes est prise en compte dans l'espace des contraintes σ uniquement comme

un processus de diffusion. Le coefficient de diffusion est supposé proportionnel à la fluidité; on note $\alpha > 0$ la constante de proportionnalité. La variable d'espace y n'est alors qu'un paramètre, on note l'inconnue $p(t, \sigma)$. L'équation (1.22) s'écrit, pour le modèle d'Hébraud-Lequeux,

$$\frac{\partial p}{\partial t} + b(t) \frac{\partial p}{\partial \sigma} = -\chi p + \left(\int \chi p \right) \delta_0(\sigma) + \alpha \left(\int \chi p \right) \frac{\partial^2 p}{\partial \sigma^2}. \quad (1.25)$$

Le modèle d'Hébraud-Lequeux a été étudié mathématiquement dans les articles [BAJ07, CCG05, CLB06, CCGLB05]. Les articles [Oli10, OR11] étudient des régimes asymptotiques du modèle stationnaire et une généralisation multidimensionnelle respectant le principe d'objectivité a été proposée dans [OR13].

Dans la thèse [Pic04] et l'article plus récent [BCA09], la redistribution due aux évènements de relaxation voisins est prise en compte non seulement en contrainte mais aussi en espace selon le terme

$$\begin{aligned} \mathcal{G} \left(p(t, y, \sigma), \{p(t, y', \sigma')\}_{(y', \sigma')} \right) \\ = \int \int \chi(\sigma') p(t, y', \sigma') (p(t, y, \sigma - G(y, y')\sigma') - p(t, y, \sigma)) dy' d\sigma', \end{aligned} \quad (1.26)$$

où $G(y, y')$ est appelé propagateur et est relié à une fonction de Green d'un opérateur de type Stokes. Des formes explicites pour G sont données dans [PALB04]. L'expression (1.26) est expliquée physiquement de la façon suivante : l'évènement de relaxation en y' , d'amplitude $-\sigma'$, fait passer le bloc y de la contrainte σ à $\sigma - G(y, y')\sigma'$.

Par ailleurs, il est possible de dériver un terme de redistribution \mathcal{G} à partir d'un système discret de "particules". Les calculs formels d'une telle dérivation sont présentés dans la section 1.5.2. Le terme \mathcal{G} alternatif à (1.26) est donné par la formule (1.46).

Dans l'article [BCA09], les auteurs proposent un lien entre le modèle mésoscopique (1.22)-(1.26) et des équations macroscopiques faisant intervenir la fluidité. Le raisonnement est présenté pour les équations stationnaires et implique notamment un développement limité de l'opérateur \mathcal{G} défini par (1.26).

Dans cette thèse, nous étudions le modèle simplifié sans redistribution $\mathcal{G} \equiv 0$. Dans ce cas, un de nos objectifs est de faire un lien avec les équations macroscopiques appartenant à la classe (1.10) pour les fluides vieillissants et décrites dans la section 1.2.1.2.

1.3 Analyse d'un modèle macroscopique

On résume dans cette section le chapitre 3 qui contient l'étude d'un système d'équations macroscopiques unidimensionnelles. On montre que le système est bien posé dans les espaces fonctionnels appropriés et le comportement en temps long est étudié.

1.3.1 Système d'équations

On considère l'écoulement de Couette (figure 1.1) d'un fluide vieillissant. Le modèle est présenté dans la section 1.2.1.2. Il est constitué des équations (1.5)-(1.10) avec le choix (1.12) pour l'équation sur la fluidité. On reproduit le système pour plus de clarté

$$\begin{cases} \rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, & (1.27a) \\ \lambda \frac{\partial \tau}{\partial t} = -f\tau + \mu \frac{\partial u}{\partial y}, & (1.27b) \\ \frac{\partial f}{\partial t} = -f^2 - \nu f^3 + \xi |\tau| f^2. & (1.27c) \end{cases}$$

Les trois inconnues scalaires, la vitesse u , la contrainte de cisaillement τ et la fluidité f sont des fonctions de l'espace $y \in [0, 1]$ et du temps $t \geq 0$. Les six coefficients $\rho, \eta, \lambda, \mu, \xi, \nu$ sont des constantes scalaires strictement positives.

On rappelle que le vieillissement du fluide est caractérisé par la fluidité f , variable positive. Le fluide est solide où $f = 0$ et d'autant plus liquide que la fluidité est grande.

1.3.2 Existence et unicité

Les conditions de bord imposées au système (1.27) sont $u(t, 0) = 0$ et $u(t, 1) = a$ pour $t \geq 0$ (avec a une constante scalaire). Alors, le résultat précis d'existence-unicité d'une solution forte et globale en temps est le suivant.

Théorème 1.1. *Soient Ω le domaine unidimensionnel $[0, 1]$ et $T > 0$. Étant donnée la condition initiale*

$$(u_0, \tau_0, f_0) \in H^1(\Omega)^3 \text{ avec } f_0 \geq 0, \quad (1.28)$$

il existe une unique solution globale (u, τ, f) au système (1.27) telle que,

$$(u, \tau, f) \in \left(C([0, T]; H^1) \cap L^2([0, T]; H^2) \right) \times C([0, T]; H^1) \times C([0, T]; H^1). \quad (1.29)$$

De plus, on a $f(t, y) \geq 0$ pour tout $y \in \Omega$ et $t \in [0, T]$ et

$$\left(\frac{\partial u}{\partial t}, \frac{\partial \tau}{\partial t}, \frac{\partial f}{\partial t} \right) \in L^2([0, T]; L^2) \times C([0, T]; L^2) \times C([0, T]; L^2), \quad (1.30)$$

si bien que les équations de (1.27) sont toutes vérifiées dans le sens fort en temps.

Le théorème 1.1 contient aussi la propriété de conservation de la positivité de la fluidité f .

Les arguments de la preuve d'existence sont standards en mécanique des fluides. Des estimées *a priori* sont dérivées formellement, puis une suite d'approximations est mise place. Sur les approximations, les estimées sont établies rigoureusement afin de montrer la convergence. La preuve utilise le résultat d'existence pour le modèle d'Oldroyd-B ($f \equiv 1$) donné dans [GS90].

Dans l'état actuel, les non-linéarités présentes dans le système (1.27) nous empêchent d'étendre ce résultat d'existence-unicité en dimension supérieure à 1. Notamment, la preuve utilise à plusieurs reprises le résultat technique suivant : les fonctions de H^1 sont des fonctions L^∞ . Ce résultat est une particularité de la dimension un.

1.3.3 Comportement en temps long

Le comportement en temps long du système (1.27) dépend de la nature des conditions de bord. Ces conditions de bord sont $u(t, 0) = 0$ et $u(t, 1) = a$ pour $t \geq 0$ (avec a une constante scalaire). On décrit d'abord le cas des conditions de bord homogènes ($a = 0$) puis les conditions de bord non homogènes ($a \neq 0$).

- *Conditions de bord homogènes.* Dans ce cas, on établit la convergence vers l'état stationnaire et un taux pour cette convergence. Les états stationnaires du système (1.27) satisfaisant $f \geq 0$ sont les états $(u \equiv 0, \tau \equiv c, f \equiv 0)$ où c est une constante indépendante de l'espace. Le comportement en temps long diffère à la fois en terme d'état stationnaire et de taux de convergence selon les cas $f_0 \equiv 0$ ou $f_0 \neq 0$.

Dans le cas où $f_0 = 0$, la fluidité f vaut 0 pour tout temps vu la structure de l'équation (1.27c). Le matériau solide à l'instant initial le reste. On montre de plus que la vitesse tend vers 0 et que la contrainte tend vers une constante en temps long. Les convergences sont exponentielles en temps. Ce comportement de retour à l'équilibre est conforme à l'intuition physique.

Dans le cas plus intéressant où $f_0 \neq 0$, on montre que la solution (u, τ, f) converge en temps long vers l'état stationnaire $(0, 0, 0)$ et les taux de convergence sont polynomiaux. Le phénomène physique de vieillissement est validé, au sens où, laissé au repos, le fluide devient de plus en plus solide (la fluidité tend vers 0). Pour la contrainte et la vitesse, les taux de convergence dépendent de la taille de la région où la fluidité f_0 est nulle à l'instant initial. La pertinence physique de la dépendance en cette taille de région n'est pas complètement claire pour nous. Les taux de convergence théoriques sont des majorations et les expériences numériques montrent qu'ils sont optimaux. La preuve consiste à établir une minoration sur la fluidité (en fait sur sa moyenne); minoration qui est insérée dans des estimées sur l'évolution d'une énergie du type

$$E(t) = \mu\rho \|u(t, \cdot)\|_{L^2} + \lambda \|\tau(t, \cdot)\|_{L^2}.$$

- *Conditions de bord non homogènes.* Dans ce cas plus compliqué, nous n'avons obtenu que des résultats partiels dans le cas où la fluidité stationnaire f_∞ satisfait

$$f_\infty(y) > 0 \text{ pour tout } y \in \Omega. \quad (1.31)$$

Dans ce cas il existe un unique état stationnaire. Nous avons obtenu deux résultats. D'une part, pour des conditions initiales qui sont des perturbations petites de l'état stationnaire, on montre (une fois encore en utilisant des inégalités d'énergie) la convergence exponentielle en temps long vers l'état stationnaire. D'autre part, on établit la convergence en temps long pour des conditions initiales satisfaisant $f_0 > 0$ sans condition de petitesse, mais seulement dans un cadre simplifié où le gradient de vitesse, la contrainte et la fluidité sont homogènes (indépendants de l'espace). Dans ce cadre simplifié, le système (1.27) se réduit à un système d'équations différentielles ordinaires (EDO) sur la contrainte et la fluidité. On utilise alors la théorie de Poincaré-Bendixson. Numériquement, on observe la convergence vers l'état stationnaire sans l'hypothèse restrictive (1.31) et sans condition de petitesse.

Malgré leurs limitations, nos résultats montrent que le modèle (1.27) représente correctement un fluide vieillissant. Cependant notre étude fait apparaître un défaut : la compétition

entre le vieillissement et le rajeunissement est en faveur du vieillissement. En effet lorsque le fluide est solide à l'instant initial ($f_0 = 0$), il le reste ($f = 0$). Et l'état solide n'est jamais atteint en temps fini en partant d'un état liquide. Ceci limite l'ensemble des matériaux représentés par le modèle (par exemple, le ketchup convient mais pas le béton). Ce défaut est lié à la structure de l'équation (1.27c) pour laquelle la théorie de Cauchy-Lipschitz s'applique. Afin de pallier ce défaut, la modification suivante peut être envisagée : fixer une constante $f_s > 0$ et identifier les régions solides aux régions où la fluidité f satisfait $f \leq f_s$.

La forme de l'équation (1.27c) est issue d'un choix spécifique des fonctions U et V dans la classe de modèle (1.10). Dans la section suivante, nous dérivons des équations macroscopiques remplaçant les équations (1.10) à partir d'un modèle mésoscopique.

1.4 Passage méso-macro

Cette section est un résumé du chapitre 4 dans lequel des équations macroscopiques sont dérivées à partir d'une équation mésoscopique. Cette équation mésoscopique est d'abord analysée : l'existence-unicité et le comportement en temps long sont étudiés. L'étude de cette section est comme la précédente unidimensionnelle.

1.4.1 Le modèle

Les modèles élastoplastiques, présentés dans la section 1.2.2.2, caractérisent à l'échelle mésoscopique certains fluides non newtoniens. L'équation unidimensionnelle (1.22) régit les variations de la densité de probabilité $p(t, y, \sigma)$ de trouver la contrainte locale σ à l'instant t au point y .

On simplifie cette équation en omettant le terme de redistribution : $\mathcal{G} \equiv 0$. L'équation simplifiée est alors paramétrée par la variable d'espace y . La dépendance paramétrique en y n'est par la suite pas explicitée. L'équation (1.22) s'écrit alors

$$\frac{\partial p}{\partial t}(t, \sigma) + b(t) \frac{\partial p}{\partial \sigma}(t, \sigma) = -\chi(\sigma)p(t, \sigma) + \left(\int \chi(\sigma)p(t, \sigma) d\sigma \right) \delta_0(\sigma). \quad (1.32)$$

On rappelle que δ_0 est la fonction de Dirac en 0 et χ est la fonction caractéristique définie par (1.21). On rappelle également que $b(t)$ est défini par $b = \mu\dot{\gamma}$ (équation (1.20)).

Dans l'étude de l'équation (1.32), on considère le taux de déformation $\dot{\gamma} = \frac{\partial u}{\partial y}$ et donc b donnés. Cependant l'équation (1.32) est en fait couplée à l'équation du mouvement (1.5) sur la vitesse u .

Afin de dériver des équations macroscopiques à partir de l'équation mésoscopique (1.32), on utilise les formules (1.23) et (1.24) définissant respectivement la fluidité f et la contrainte τ . Afin d'avoir des notations consistantes dans cette section, on réécrit ces formules en omettant la dépendance paramétrique en l'espace :

$$f(t) = \int \chi(\sigma)p(t, \sigma) d\sigma \quad (1.33)$$

$$\tau(t) = \int \sigma p(t, \sigma) d\sigma. \quad (1.34)$$

Les équations macroscopiques issues de la dérivation font intervenir la fluidité f , la contrainte τ et le taux de déformation représenté par b mais pas la densité de probabilité p . Comme l'équation mésoscopique, elles sont à coupler à l'équation du mouvement (1.5).

1.4.2 Existence-unicité et comportement en temps long

Nous avons établi le théorème d'existence-unicité suivant.

Théorème 1.2. *Soit b une fonction du temps satisfaisant*

$$b \in L^1_{\text{loc}}(0, \infty) \text{ et } b \geq m_b \text{ p.p. où } m_b > 0 \text{ est un scalaire fixé.} \quad (1.35)$$

Soit $p_0 \in L^1(\mathbb{R})$. Alors, pour tout $T > 0$, il existe un unique p dans $C^0([0, T]; L^1)$ tel que $p(0, \sigma) = p_0(\sigma)$ pour presque tout $\sigma \in \mathbb{R}$ et tel que l'équation (1.32) est satisfaite au sens des distributions sur $(0, T) \times \mathbb{R}$.

En plus de ce résultat, on montre que si la condition initiale p_0 est positive et normalisée, alors cette propriété est préservée en temps pour la solution p . On fait cette hypothèse par la suite de sorte que pour tout $t \geq 0$, $p(t, \cdot)$ est bien une densité de probabilité. Les preuves sont basées sur la construction d'une formule explicite de la solution p . En utilisant cette formule, l'hypothèse $b \geq m_b$ est fondamentale. Bien sûr, on a des résultats équivalents sous l'hypothèse $b \leq -m_b < 0$. L'extension du résultat dans le cas où b vaut éventuellement 0 n'est pas claire.

Dans le cas $b \equiv 0$, on montre de plus qu'il existe une infinité de densités de probabilité solutions stationnaires de l'équation (1.32). Dans le cas $b \equiv b_\infty \neq 0$ où b_∞ est une constante non nulle, il existe une unique densité de probabilité solution stationnaire de (1.32). Lorsque $b \equiv b_\infty \neq 0$, on établit de plus que la solution de (1.32) converge exponentiellement vite en temps vers la solution stationnaire presque partout en $\sigma \in \mathbb{R}$. On obtient une majoration du taux de convergence exponentiel ; des expériences numériques montrent que cette majoration est optimale. La preuve de la convergence en temps long est basée encore une fois sur la formule explicite de p . Dans ce cas b constant, la variable f définie par (1.23) intervenant dans la formule explicite est solution d'une équation différentielle à retard à coefficients constants. Le comportement en temps long pour ce type d'équations est bien connu (voir par exemple [BC63, HL93]).

La preuve du comportement en temps long précédemment décrite et détaillée dans le chapitre 4 est purement déterministe. Alternativement, des arguments probabilistes peuvent aussi être utilisés. Ceux-ci sont liés à l'étude du processus stochastique associé à l'équation de Fokker-Planck (1.32). Nous donnons une idée de ces arguments dans la section 1.5.1.

1.4.3 Passage méso-macro

Comme dans la sous-section précédente, on suppose la fonction b représentant le taux de déformation $\dot{\gamma} = \frac{\partial u}{\partial y}$ donnée. Les autres variables macroscopiques que sont la fluidité f et la contrainte τ sont respectivement définies en fonction de p solution de (1.32) par (1.33) et (1.34). Ainsi, dans une première approche, on peut multiplier l'équation (1.32) respectivement par $\chi(\sigma)$

et σ et intégrer en σ . On obtient alors formellement des équations sur τ et f qui ne sont pas fermées : elle contiennent encore la densité p .

Afin de dériver des équations macroscopiques fermées, on suppose que la fonction b varie lentement comparativement au temps caractéristique de l'équation (1.32). Ainsi, on introduit un paramètre ϵ supposé petit dans la dépendance de la fonction b en le temps t . Considérer la limite macroscopique de l'équation (1.32) revient alors à considérer la limite $\epsilon \rightarrow 0$.

L'équation (1.32) paramétrée par ϵ s'écrit alors

$$\frac{\partial p_\epsilon}{\partial t}(t, \sigma) + b(\epsilon t) \frac{\partial p_\epsilon}{\partial \sigma}(t, \sigma) = -\chi(\sigma) p_\epsilon(t, \sigma) + \left(\int \chi(\sigma) p_\epsilon(t, \sigma) d\sigma \right) \delta_0(\sigma). \quad (1.36)$$

On introduit de plus un temps macroscopique, noté θ , relié au temps mésoscopique t par la formule $\theta = \epsilon t$. Une limite macroscopique est un système d'EDO dont la solution est une approximation à l'ordre $O(\epsilon)$ des variables $f_\epsilon \left(\frac{\theta}{\epsilon} \right)$ et $\tau_\epsilon \left(\frac{\theta}{\epsilon} \right)$. Les variables f_ϵ et τ_ϵ sont définies en fonction de p_ϵ comme dans les formules (1.33) et (1.34) en remplaçant p par p_ϵ . Dans la limite $\epsilon \rightarrow 0$, le temps mésoscopique $t = \frac{\theta}{\epsilon}$ devient un temps long. On est ainsi en mesure d'utiliser le résultat de temps long décrit dans la section précédente par une fonction b constante, résultat qu'il faut préalablement adapter au cas $b(\epsilon t)$.

Plusieurs limites macroscopiques sont possibles, on présente ici l'une d'entre elles. On introduit $(\tau_\epsilon^{**}, f_\epsilon^{**})$ solution du système d'EDO en le temps macroscopique θ :

$$\begin{cases} \epsilon \frac{d\tau_\epsilon^{**}}{d\theta}(\theta) = -\kappa(\theta) f_\epsilon^{**}(\theta) \tau_\epsilon^{**}(\theta) + b(\theta), & (1.37a) \\ \epsilon \frac{df_\epsilon^{**}}{d\theta}(\theta) = -f_\epsilon^{**}(\theta) + \frac{b(\theta)}{\sigma_c + b(\theta)}, & (1.37b) \end{cases}$$

où κ est défini par

$$\kappa(\theta) = \frac{2}{1 + \frac{1}{\left(1 + \frac{\sigma_c}{b(\theta)}\right)^2}}. \quad (1.38)$$

Alors, il existe une constante C indépendante de θ et ϵ telle que

$$\left| \tau_\epsilon \left(\frac{\theta}{\epsilon} \right) - \tau_\epsilon^{**}(\theta) \right| + \left| f_\epsilon \left(\frac{\theta}{\epsilon} \right) - f_\epsilon^{**}(\theta) \right| \leq C \left(\frac{1}{\theta} + 1 \right) \epsilon. \quad (1.39)$$

Ce résultat est établi sous l'hypothèse technique $\frac{\theta}{\epsilon} > 2 \frac{\sigma_c}{m_b}$ où on rappelle que σ_c est la contrainte seuil intervenant dans la fonction χ définie par (1.21) et m_b est la borne inférieure de b dans (1.35). Cette hypothèse n'est pas restrictive dans la limite $\epsilon \rightarrow 0$.

Dans le chapitre 4, on considère une autre limite macroscopique faisant intervenir uniquement la contrainte, proche elle aussi à l'ordre $O(\epsilon)$ de la variable $\tau_\epsilon \left(\frac{\theta}{\epsilon} \right)$ issue du problème mésoscopique (1.36). Toutefois, la limite macroscopique (1.37) est physiquement plus pertinente. En effet, en changeant le coefficient $\kappa(\theta)$ en une constante, le système (1.37) appartient à la classe de modèle (1.10) introduite dans les articles [DAL01, PABL02]. Cette classe d'équations macroscopiques modélise les fluides vieillissants comme décrit dans la section 1.2.1.2. Étant donnée la définition (1.38) de $\kappa(\theta)$, l'hypothèse $\kappa(\theta)$ constante (égale à 2) est raisonnable lorsque b est petit. Ceci est confirmé par des expériences numériques montrant que les solutions du système (1.37) et de son approximation où $\kappa(\theta)$ est remplacé par 2 sont effectivement proches.

1.5 Annexes

Dans cette section, on traite formellement deux sujets que nous n'avons pas eu le temps d'approfondir. Les preuves rigoureuses n'ont pas été rédigées par ailleurs et pourraient donner lieu à des travaux ultérieurs.

1.5.1 Méthode alternative à l'étude du comportement en temps long de l'équation mésoscopique

On donne ici le cadre des arguments probabilistes à mettre en oeuvre afin d'établir une preuve alternative du temps long de l'équation (1.32). En effet la preuve que nous avons menée et résumée dans la section 1.4.2 est purement déterministe. Nous sommes reconnaissants à Florent Malrieu de nous avoir indiqué la méthode probabiliste décrite ci-dessous.

Il existe un processus stochastique $(\Sigma_t)_{t \geq 0}$ associé à l'équation de Fokker-Planck (1.32). Son générateur infinitésimal est donné par

$$\mathcal{L}\phi(t, \sigma) = b(t) \frac{\partial \phi}{\partial \sigma}(t, \sigma) + \mathbb{1}_{\mathbb{R} \setminus [-\sigma_c, \sigma_c]}(\sigma) (\phi(t, 0) - \phi(t, \sigma)). \quad (1.40)$$

En effet, la densité de probabilité p solution de (1.32) satisfait, pour toute fonction $\phi \in \mathcal{D}(\mathbb{R}_+^* \times \mathbb{R})$,

$$\begin{aligned} - \int_{\mathbb{R}_+} \int_{\mathbb{R}} p \frac{\partial \phi}{\partial t} &= \left\langle -b(t) \frac{\partial p}{\partial \sigma} - \chi(\sigma) p(t, \sigma) + \left(\int_{\mathbb{R}} \chi(\sigma) p(t, \sigma) d\sigma \right) \delta_0(\sigma), \phi \right\rangle_{\mathcal{D}', \mathcal{D}} \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(b(t) \frac{\partial \phi}{\partial \sigma}(t, \sigma) + \chi(\sigma) (\phi(t, 0) - \phi(t, \sigma)) \right) p(t, \sigma) d\sigma dt. \end{aligned}$$

Étant donnée la définition (1.21) de la fonction χ , l'équation précédente se réécrit

$$- \int_{\mathbb{R}_+} \int_{\mathbb{R}} p \frac{\partial \phi}{\partial t} = \int_{\mathbb{R}_+} \int_{\mathbb{R}} p \mathcal{L}\phi. \quad (1.41)$$

Le générateur (1.40) donne l'évolution suivante pour le processus stochastique $(\Sigma_t)_{t \geq 0}$: il évolue de manière déterministe avec une dérive $b(t)$ lorsque $|\Sigma_t| \leq \sigma_c$ et saute en zéro avec un taux exponentiel 1 lorsque $|\Sigma_t| \geq \sigma_c$.

Ce processus appartient à la classe des processus de Markov déterministes par morceaux (*piecewise-deterministic Markov processes*, PDMP). Afin d'étudier le comportement en temps long de ce type de processus, des arguments de couplage ont été proposés (voir par exemple [BCG⁺13]). Plus précisément, on montre d'abord que le processus Σ admet une unique mesure de probabilité invariante μ . Ensuite, on couple le processus Σ au processus stationnaire Σ' de mesure μ . Dans le cas homogène ($b = b_\infty$ avec b_∞ une constante non nulle) de notre processus Σ , le générateur infinitésimal du processus de couplage sur $\mathbb{R} \times \mathbb{R}$ est donné par

$$\begin{aligned} \mathcal{L}\phi(\sigma, \sigma') &= b_\infty \partial_1 \phi(\sigma, \sigma') + b_\infty \partial_2 \phi(\sigma, \sigma') + \mathbb{1}_{\mathbb{R} \setminus [-\sigma_c, \sigma_c] \times \mathbb{R} \setminus [-\sigma_c, \sigma_c]}(\sigma, \sigma') (\phi(0, 0) - \phi(\sigma, \sigma')) \\ &\quad + \mathbb{1}_{\mathbb{R} \setminus [-\sigma_c, \sigma_c] \times [-\sigma_c, \sigma_c]}(\sigma, \sigma') (\phi(0, \sigma') - \phi(\sigma, \sigma')) + \mathbb{1}_{[-\sigma_c, \sigma_c] \times \mathbb{R} \setminus [-\sigma_c, \sigma_c]}(\sigma, \sigma') (\phi(\sigma, 0) - \phi(\sigma, \sigma')). \end{aligned}$$

Montrer que le temps de couplage $T = \inf_t \{\Sigma_t = \Sigma'_t\}$ est presque sûrement fini (en étudiant le processus de couplage) permet d'obtenir la convergence du processus Σ vers son état stationnaire. La vitesse de convergence découle de plus d'une majoration stochastique du temps de couplage. Une telle majoration s'écrit

$$\mathbb{P}(T > t) \leq e^{-ct}$$

pour une convergence exponentielle par exemple.

1.5.2 Dérivation probabiliste de la redistribution

Dans la section 1.2.2.2, nous avons présenté le modèle élastoplastique (1.22). Nous donnons ici les calculs formels afin de dériver un terme de redistribution \mathcal{G} dans ce modèle mésoscopique à partir d'un système de particules.

L'article [BAJ07] prouve une telle dérivation pour l'équation d'Hébraud-Lequeux (1.25), cas particulier du modèle (1.22) pour lequel la redistribution est prise en compte uniquement dans l'espace des contraintes.

Nous considérons ici un système de n particules pour lequel la redistribution est prise en compte dans l'espace physique y et l'espace des contraintes σ . Les trois phénomènes moteurs ont été décrits dans [Pic04, BCA09] et la section 1.2.2.2. Ainsi, la particule i , correspondant à la composante position-contrainte $\begin{pmatrix} Y^i \\ \Sigma_t^i \end{pmatrix}$ du processus du système sur $\mathbb{R}^n \times \mathbb{R}^n$, se comporte de la manière suivante :

- (i) (Σ_t^i) évolue selon la dérive $b(t, Y^i)$,
- (ii) lorsque $|\Sigma_t^i| > \sigma_c$, (Σ_t^i) saute en 0 avec un taux exponentiel 1,
- (iii) ce saut d'amplitude $-\Sigma_t^i$ influe les autres particules. Plus précisément, la particule j reçoit la part $-\frac{1}{n}\Sigma_t^i$ de l'amplitude du saut pondérée par $G(Y^i, Y^j)$. On rappelle que la fonction G a été introduite dans la formule (1.26); G est la fonction de Green d'un opérateur de type Stokes.

On introduit les notations $\begin{pmatrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{pmatrix} = \begin{pmatrix} y_i \\ \sigma_i \end{pmatrix}_{1 \leq i \leq n}$ et pour $\phi \in \mathcal{D}(\mathbb{R}_+^* \times \mathbb{R}^n \times \mathbb{R}^n)$, $\phi(t, \mathbf{y}^n, \boldsymbol{\sigma}^n) = \phi_t \begin{pmatrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{pmatrix}$. Alors le générateur infinitésimal du processus sur $\mathbb{R}^n \times \mathbb{R}^n$ soumis aux trois comportements précédents est donné par $\sum_{i=1}^n \mathcal{L}_1^i$ où

$$\begin{aligned} \mathcal{L}_1^i \phi_t \begin{pmatrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{pmatrix} &= b(t, y_i) \frac{\partial \phi_t}{\partial \sigma_i} \begin{pmatrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{pmatrix} \\ &+ \chi(\sigma_i) \left(\phi_t \begin{pmatrix} y_1 & \cdots & y_j & \cdots & y_i & \cdots & y_n \\ \sigma_1 + \frac{1}{n}G(y_1, y_i)\sigma_i & \cdots & \sigma_j + \frac{1}{n}G(y_j, y_i)\sigma_i & \cdots & 0 & \cdots & \sigma_n + \frac{1}{n}G(y_n, y_i)\sigma_i \end{pmatrix} - \phi_t \begin{pmatrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{pmatrix} \right). \end{aligned} \quad (1.42)$$

Ce générateur infinitésimal est lié à la densité de probabilité par la formule (1.41) où la fonction test ϕ est défini non pas sur $\mathbb{R}_+^* \times \mathbb{R}$ mais sur $\mathbb{R}_+^* \times \mathbb{R}^n \times \mathbb{R}^n$. Avec un développement de Taylor à l'ordre 1, on obtient, dans la limite $n \rightarrow \infty$, $\mathcal{L}_1^i \simeq \mathcal{L}_2^i$ où \mathcal{L}_2^i est donné par

$$\mathcal{L}_2^i \phi_t \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right) = b(t, y_i) \frac{\partial \phi_t}{\partial \sigma_i} \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right) + \chi(\sigma_i) \left(\phi_t \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n - \sigma_i \mathbf{e}_i \end{matrix} \right) - \phi_t \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right) \right) + \sum_{j=1}^n \frac{1}{n} \chi(\sigma_j) G(y_j, y_i) \sigma_j \frac{\partial \phi_t}{\partial \sigma_j} \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right). \quad (1.43)$$

En introduisant les notations

$$\begin{aligned} \beta \left(\begin{matrix} y & y' \\ \sigma & \sigma' \end{matrix} \right) \phi &= \chi(\sigma') G(y, y') \sigma' \frac{\partial \phi}{\partial \sigma} \\ \beta \left[\left(\begin{matrix} y \\ \sigma \end{matrix} \right), p \right] \phi &= \int \beta \left(\begin{matrix} y & y' \\ \sigma & \sigma' \end{matrix} \right) \phi p \left(\begin{matrix} y' \\ \sigma' \end{matrix} \right) d \left(\begin{matrix} y' \\ \sigma' \end{matrix} \right) \\ \overline{\mu_{y, \sigma}^n} &= \frac{1}{n} \sum_{i=1}^n \delta \left(\begin{matrix} y_i \\ \sigma_i \end{matrix} \right) \end{aligned}$$

le générateur infinitésimal $\mathcal{L}_2^n = \sum_i \mathcal{L}_2^i$ se réécrit

$$\begin{aligned} \mathcal{L}_2^n \phi_t \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right) &= \sum_{i=1}^n b(t, y_i) \frac{\partial \phi_t}{\partial \sigma_i} \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right) + \sum_{i=1}^n \chi(\sigma_i) \left(\phi_t \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n - \sigma_i \mathbf{e}_i \end{matrix} \right) - \phi_t \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right) \right) + \sum_{j=1}^n \beta \left[\left(\begin{matrix} y_j \\ \sigma_j \end{matrix} \right), \overline{\mu_{y, \sigma}^n} \right] \phi_t \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right). \end{aligned} \quad (1.44)$$

Dans la limite $n \rightarrow \infty$, la mesure empirique $\overline{\mu_{Y, \Sigma}^n}$ associée à la configuration $\left(\begin{matrix} \mathbf{Y}_t^n \\ \boldsymbol{\Sigma}_t^n \end{matrix} \right)$ converge vers la mesure de densité de probabilité $p(t, y, \sigma)$ associé au processus limite $\left(\begin{matrix} Y \\ \Sigma_t \end{matrix} \right)$. Afin de montrer ce résultat, on prouve en fait la propagation du chaos (voir le cours de Sznitman [Szn91, Proposition 2.2, p. 177]). La propagation du chaos s'écrit, pour toutes fonctions test $\psi_1, \psi_2 : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$,

$$\int \psi_1 \left(\begin{matrix} y_1 \\ \sigma_1 \end{matrix} \right) \psi_2 \left(\begin{matrix} y_2 \\ \sigma_2 \end{matrix} \right) p^n \left(t, \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right) \right) d \left(\begin{matrix} \mathbf{y}^n \\ \boldsymbol{\sigma}^n \end{matrix} \right) \xrightarrow{n \rightarrow \infty} \left(\int \psi_1 p(t, \cdot) \right) \left(\int \psi_2 p(t, \cdot) \right),$$

où p^n est la densité de probabilité associé à la configuration $\left(\begin{matrix} \mathbf{Y}_t^n \\ \boldsymbol{\Sigma}_t^n \end{matrix} \right)$.

Le générateur infinitésimal limite de (1.44) est alors

$$\mathcal{L}_2 \phi(t, y, \sigma) = b(t, y) \frac{\partial \phi}{\partial \sigma}(t, y, \sigma) + \chi(\sigma) (\phi(t, y, 0) - \phi(t, y, \sigma)) + \beta \left[\left(\begin{matrix} y \\ \sigma \end{matrix} \right), p(t, \cdot) \right] \phi(t, y, \sigma). \quad (1.45)$$

Ce générateur est non linéaire au sens de McKean : il fait intervenir la densité de probabilité associée p (dans le dernier terme). L'EDP associée au générateur infinitésimal (1.45) est donnée par l'équation élastoplastique (1.22) introduite dans la section 1.2.2.2 en définissant le terme de redistribution \mathcal{G} par

$$\begin{aligned}
\mathcal{G}\left(p(t, y, \sigma), \{p(t, y', \sigma')\}_{(y', \sigma')}\right) &= -\beta \left[\begin{pmatrix} y \\ \sigma \end{pmatrix}, p(t, \cdot) \right] p(t, y, \sigma) \\
&= - \left(\int_{\Omega} \int_{\mathbb{R}} \chi(\sigma') G(y, y') \sigma' p(t, y', \sigma') dy' d\sigma' \right) \frac{\partial p}{\partial \sigma}(t, y, \sigma).
\end{aligned} \tag{1.46}$$

On a ainsi obtenu une expression pour le terme de redistribution \mathcal{G} de l'équation (1.22) à partir d'un système de particules. On souligne encore une fois que les calculs précédents sont formels. Notamment l'erreur introduite par le développement de Taylor dans l'approximation $\mathcal{L}_1^i \simeq \mathcal{L}_2^i$, où \mathcal{L}_1^i et \mathcal{L}_2^i sont respectivement définis par (1.42) et (1.43) n'a pas été quantifiée. De plus, le passage à la limite $n \rightarrow \infty$ nécessite des hypothèses qui restent à vérifier.

Simulations numériques d'un modèle macroscopique de fluide vieillissant en deux dimensions

2.1 Introduction

Nous présentons des simulations numériques d'un système d'équations aux dérivées partielles modélisant certains fluides non newtoniens à l'état intermédiaire entre un liquide et un solide : les fluides vieillissants. Nous considérons la modélisation phénoménologique présentée dans [DAL01, PABL02] et dans laquelle une nouvelle inconnue est introduite : un champ scalaire positif, la fluidité f . La fluidité caractérise le caractère liquide du fluide : le fluide est d'autant plus liquide que la fluidité est grande. Lorsque le fluide vieillit, c'est-à-dire se solidifie, la fluidité diminue. À l'inverse, la fluidité augmente quand le fluide est agité.

Même si le modèle a été introduit en une dimension, les simulations qui suivent concernent une version de ce modèle en dimension $d = 2$, pour laquelle la fluidité reste scalaire. Le modèle ainsi obtenu se généralise en dimension $d = 3$. Nous considérons ainsi un fluide confiné dans un domaine borné Ω de \mathbb{R}^d et régit par les équations suivantes

$$\left\{ \begin{array}{l} \operatorname{Re} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + (1 - \alpha) \Delta u + \operatorname{div}(\tau) \quad (2.1a) \\ \operatorname{div}(u) = 0 \quad (2.1b) \\ \frac{\partial \tau}{\partial t} + (u \cdot \nabla) \tau - (\nabla u) \tau - \tau (\nabla u)^t = -\frac{f}{\operatorname{We}} \tau + \frac{\alpha}{\operatorname{We}} (\nabla u + \nabla u^t) \quad (2.1c) \\ \frac{\partial f}{\partial t} + (u \cdot \nabla) f = -f^2 - \nu f^3 + \xi f^2 |\tau|. \quad (2.1d) \end{array} \right.$$

Les inconnues sont le vecteur vitesse u , la pression p , le tenseur contraintes τ et la fluidité f . Elles sont fonctions du temps $t \geq 0$ et de l'espace $x \in \Omega$ à valeurs respectivement dans \mathbb{R}^d , \mathbb{R} , $\mathbb{R}^{d \times d}$ et \mathbb{R}_+ . Les paramètres suivants sont constants, strictement positifs et sans dimension : le nombre de Reynolds Re , le nombre de Weissenberg We , le rapport de viscosité $\alpha \in (0, 1)$ et les paramètres de fluidité ξ et ν . Le système (2.1) est un système de $d + 2 + \frac{d(d+1)}{2}$ équations. Dans le cas où f est une constante égale à 1, le système (2.1) constitue le modèle d'Oldroyd-B. La variable f est introduite comme l'inverse du temps de relaxation de la contrainte dans l'équation (2.1c). L'équation (2.1d) régit l'évolution de la fluidité. Au membre gauche, une dérivée matérielle est présente. Quant au membre droit, sa forme précise est un choix dans la classe proposée dans [DAL01, PABL02]. L'ingrédient important dans la forme du membre droit est la présence de deux termes de signe opposé, ici $-f^2 - \nu f^3$ et $\xi f^2 |\tau|$, représentant la compétition

caractéristique des fluides vieillissants. Le vieillissement en tant que tel, c'est-à-dire la solidification spontanée du fluide, est modélisé par le terme négatif. Ce terme fait décroître la fluidité, de sorte que le phénomène de relaxation est de plus en plus lent. L'effet opposé, le rajeunissement par l'écoulement est modélisé par le terme positif, qui fait croître la fluidité.

Dans la version multidimensionnelle précédemment décrite, la fluidité reste scalaire. Dans ce cas, le temps de relaxation est le même pour toutes les composantes de la contrainte. Dans une généralisation multidimensionnelle plus physique, les temps de relaxation devraient être différents.

La discrétisation du modèle d'Oldroyd-B a été largement étudiée, voir par exemple [Keu89, Keu00]. On utilise donc une méthode de discrétisation du modèle d'Oldroyd-B qu'on adapte dans le cas où la fluidité f est variable. La méthode de discrétisation retenue pour le modèle d'Oldroyd-B est décrite dans [BLM09]. Cette méthode fait intervenir le tenseur de conformation

$$\Sigma = \text{Id} + \frac{We}{\alpha} \tau. \quad (2.2)$$

En introduisant Σ , le système (2.1) se réécrit

$$\left\{ \begin{array}{l} \text{Re} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + (1 - \alpha) \Delta u + \frac{\alpha}{We} \text{div}(\Sigma) \\ \text{div}(u) = 0 \\ \frac{\partial \Sigma}{\partial t} + (u \cdot \nabla) \Sigma - (\nabla u) \Sigma - \Sigma (\nabla u)^t = -\frac{f}{We} (\Sigma - I) \\ \frac{\partial f}{\partial t} + (u \cdot \nabla) f = -f^2 - \nu f^3 + \xi f^2 \left| \frac{\alpha}{We} (\Sigma - I) \right| \end{array} \right. \quad \begin{array}{l} (2.3a) \\ (2.3b) \\ (2.3c) \\ (2.3d) \end{array}$$

Le système (2.3) est complété par des conditions initiales u_0, p_0, Σ_0 et f_0 fonctions de l'espace. Concernant les conditions de bord, la méthode de discrétisation est présentée pour des conditions de bord de Dirichlet homogène sur la vitesse

$$u = 0 \text{ sur } \partial\Omega$$

même si d'autres conditions sont utilisées dans les tests numériques. Pour le modèle d'Oldroyd-B (cas $f \equiv 1$) et des conditions de Dirichlet homogènes, la méthode de discrétisation retenue a l'avantage de conserver au niveau discret la propriété de décroissance de l'énergie libre :

$$\frac{d}{dt} \left[\frac{\text{Re}}{2} \int_{\Omega} |u|^2 + \frac{\alpha}{2We} \int_{\Omega} \text{tr}(\Sigma - \ln \Sigma - \text{Id}) \right] \leq 0. \quad (2.4)$$

Afin de tester le modèle (2.3), on choisit un écoulement dans une géométrie assez complexe : l'écoulement autour d'un cylindre (*flow past a cylinder*). Le cas test *flow past a cylinder* est classique en mécanique des fluides. Par symétrie de translation, le domaine de simulation est 2D. Le "cylindre" se réduit alors à un cercle qu'on dénomme bille par la suite. Dans ce cas test, la bille est fixe et le fluide avance autour d'elle. Mais c'est une simplification, les expériences à reproduire à terme par la simulation sont en fait des expériences de chute d'une bille dans un fluide non newtonien sous l'effet de la gravité, voir par exemple [GTH⁺06]. Les tests conduits ici concernent uniquement une bille fixe.

Une autre raison de l'importance de ce cas test est sa place au coeur d'un problème numérique encore largement incompris : l'apparition d'instabilités numériques lorsque le nombre de Weissenberg We devient grand dans le modèle d'Oldroyd-B. On appelle ce problème *High Weissenberg Number Problem* (HWNP), voir l'article [Keu00] et les références qu'il contient.

La suite de notre étude se compose de deux sections. Dans la section 2.2, on détaille une méthode de discrétisation du système (2.3). C'est une méthode d'éléments finis. La section 2.3 présente les résultats numériques sur le cas test *flow past a cylinder*. Les simulations présentées concernent d'une part le modèle d'Oldroyd-B (cas $f \equiv 1$) et d'autre part le modèle avec fluidité. Pour le modèle d'Oldroyd-B, les résultats numériques sur la traînée en fonction du nombre de Weissenberg sont en accord avec les résultats de référence [SSAB99, BACMP04]. De plus, le problème numérique HWNP est constaté et on en présente quelques manifestations. On compare ensuite les résultats du modèle d'Oldroyd-B et du modèle avec fluidité. On constate que les phénomènes en jeu dans les fluides vieillissants sont qualitativement reproduits. Pour le modèle avec fluidité, on utilise un nombre de Weissenberg faible et alors on vérifie qu'aucune instabilité numérique n'apparaît.

2.2 Discrétisation

Afin de discrétiser le système (2.3), on utilise une méthode d'éléments finis. Pour simplifier la présentation, on décrit la méthode pour des conditions de bord de Dirichlet homogène sur la vitesse même si d'autres conditions sont utilisées dans les tests numériques.

2.2.1 Formulation variationnelle

La méthode des éléments finis fait partie des méthodes variationnelles. La formulation variationnelle considérée ici pour la discrétisation du système (2.3) concerne uniquement les équations (2.3a)-(2.3b)-(2.3c). En effet, l'équation (2.3d) sur la fluidité f a une structure particulière : il n'y a pas de dérivée spatiale sur f à part le terme de convection.

On introduit une fonction test $(v, q, z) : \Omega \rightarrow \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{\frac{d(d+1)}{2}}$. Alors, en supposant les solutions (u, p, Σ, f) de (2.3) régulières, on multiplie les équations (2.3a), (2.3b) et (2.3c) respectivement par v , q et z et on intègre sur Ω . La somme des trois équations obtenues couplée à l'équation (2.3d) divisée par $f^2 > 0$ constitue le problème :

Trouver (u, p, Σ, f) tel que pour tout (v, q, z) ,

$$\left\{ \begin{array}{l} 0 = \int_{\Omega} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) \cdot v + (1 - \alpha)\nabla u : \nabla v + \frac{\alpha}{We} \Sigma : \nabla v - p \operatorname{div}(v) - \operatorname{div}(u)q \\ \quad + \left(\frac{\partial \Sigma}{\partial t} + (u \cdot \nabla)\Sigma \right) : z - (\nabla u)\Sigma : z - \Sigma(\nabla u)^t : z + \frac{f}{We}(\Sigma - I) : z \quad (2.5a) \\ \quad \frac{\partial}{\partial t} \left(\frac{1}{f} \right) + (u \cdot \nabla) \left(\frac{1}{f} \right) = -1 - \nu f + \xi \left| \frac{\alpha}{We}(\Sigma - I) \right|, \quad (2.5b) \end{array} \right.$$

où on a introduit la notation $y : z$ du double-produit contracté pour les tenseurs défini par $y : z = \operatorname{tr}(yz^t)$.

2.2.2 Schéma numérique

Afin de décrire le schéma numérique basé sur la formulation (2.5), on introduit quelques notations. L'intervalle de temps $[0, T]$ est découpé en N_T intervalles de temps $[t^{k-1}, t^k]$ de taille constante $\Delta t = \frac{T}{N_T}$, avec $t^k = k\Delta t, k = 0 \dots N_T$. On suppose le domaine Ω polyédral et muni d'un maillage tétraédrique conforme. Pour tout $k = 0 \dots N_T - 1$, on dénote $(u_h^k, p_h^k, \Sigma_h^k, f_h^k)$ la valeur au temps t^k de la solution discrète $(u_h, p_h, \Sigma_h, f_h)$ dans l'espace des éléments finis.

Les deux équations du problème (2.5) sont résolues de manière découplée. La formulation (2.5a) est proche de celle d'Oldroyd-B, on lui associe un schéma analogue à celui proposé dans [BLM09]. On retient le choix des éléments finis de Taylor-Hood $\mathbb{P}_2/\mathbb{P}_1$ pour la vitesse et la pression. Un terme de correction de Temam [Tem84, Section III.5] est ajouté, cette modification spécifique à l'équation de Navier-Stokes permet de stabiliser la discrétisation en présence du terme de convection sur la vitesse. Des éléments finis \mathbb{P}_1 sont utilisés pour le tenseur de conformation. La convection de celui-ci est traitée par la méthode des caractéristiques. Quant à l'équation (2.5b) sur la fluidité, étant donnée sa structure particulière, elle est traitée point par point en espace dans le même espace d'éléments finis \mathbb{P}_1 que le tenseur de conformation. La méthode des caractéristiques est utilisée pour la convection.

Le schéma numérique suivant est ainsi retenu :

pour $(u_h^{k-1}, p_h^{k-1}, \Sigma_h^{k-1}, f_h^{k-1})$ donné, trouver $(u_h^k, p_h^k, \Sigma_h^k, f_h^k) \in \mathbb{P}_2^d \times \mathbb{P}_1 \times \mathbb{P}_1^{\frac{d(d+1)}{2}} \times \mathbb{P}_1$ tel que, pour toute fonction test $(v, q, z) \in \mathbb{P}_2^d \times \mathbb{P}_1 \times \mathbb{P}_1^{\frac{d(d+1)}{2}}$,

$$\left\{ \begin{array}{l} 0 = \int_{\Omega} \operatorname{Re} \left(\frac{u_h^k - u_h^{k-1}}{\Delta t} + \int (u_h^{k-1} \cdot \nabla) u_h^k \right) \cdot v + \frac{\operatorname{Re}}{2} \operatorname{div} (u_h^{k-1}) u_h^k \cdot v \\ \quad + (1 - \alpha) \nabla u_h^k : \nabla v + \frac{\alpha}{\operatorname{We}} \Sigma_h^k : \nabla v - p_h^k \operatorname{div} (v) - \operatorname{div} (u_h^k) q \\ \quad + \frac{\Sigma_h^k - \Sigma_h^{k-1} \circ \chi_h^k(t)}{\Delta t} : z - (\nabla u_h^{k-1}) \Sigma_h^k : z - \Sigma_h^k (\nabla u_h^{k-1})^t : z + \frac{f_h^{k-1}}{\operatorname{We}} (\Sigma_h^k - I) : z \\ \quad \frac{1}{\Delta t} \left(\frac{1}{f_h^k} - \frac{1}{f_h^{k-1} \circ \chi_h^k(t)} \right) = -1 - \nu f_h^k + \xi \left| \frac{\alpha}{\operatorname{We}} (\Sigma_h^k - I) \right|. \end{array} \right. \quad (2.6a)$$

$$\frac{1}{\Delta t} \left(\frac{1}{f_h^k} - \frac{1}{f_h^{k-1} \circ \chi_h^k(t)} \right) = -1 - \nu f_h^k + \xi \left| \frac{\alpha}{\operatorname{We}} (\Sigma_h^k - I) \right|. \quad (2.6b)$$

La condition initiale est donnée par $(u^0, p^0, \Sigma^0, f^0) \in \mathbb{P}_2^d \times \mathbb{P}_1 \times \mathbb{P}_1^{\frac{d(d+1)}{2}} \times \mathbb{P}_1$. La fonction $\chi^k(t) : x \in \Omega \mapsto \chi^k(t, x)$ est la caractéristique associée à la vitesse u_h^{k-1} . Elle satisfait, pour tout $x \in \Omega$,

$$\left\{ \begin{array}{l} \frac{d\chi^k}{dt}(t, x) = u_h^{k-1}(\chi^k(t, x)), \quad \forall t \in [t^{k-1}, t^k] \\ \chi^k(t^k, x) = x. \end{array} \right.$$

Dans le cas $f_h^{k-1} \equiv 1$, le schéma numérique (2.6a) est l'un de ceux proposés dans [BLM09] pour le modèle d'Oldroyd-B aux deux modifications suivantes près

- l'espace d'éléments finis sur le tenseur de conformation Σ n'est pas \mathbb{P}_1 comme ici mais \mathbb{P}_0 ou $\mathbb{P}_{1,\text{disc}}$.

- le terme sur-convecté dans la dérivée du tenseur de conformation est discrétisé implicitement dans [BLM09] :

$$\int (\nabla u^k) \Sigma^k : z - \Sigma^k (\nabla u^k)^t : z.$$

Afin de traiter ce terme non linéaire, une méthode possible est l’algorithme itératif de Picard. Linéariser ce terme comme dans la formulation (2.6) revient à considérer l’algorithme de Picard convergé en une itération. On fait ici cette approximation car on s’intéresse à la solution stationnaire.

Dans le cas $f_h^{k-1} \equiv 1$ et à ces deux modifications près, il a été démontré dans [BLM09] que pour des conditions de bord de Dirichlet homogène, l’énergie libre discrète d’Oldroyd-B

$$E_h^k = \frac{\text{Re}}{2} \int_{\Omega} |u_h^k|^2 + \frac{\alpha}{2\text{We}} \int_{\Omega} \text{tr} \left(\Sigma_h^k - \ln \Sigma_h^k - \text{Id} \right) \quad (2.7)$$

est décroissante : $E_h^{k+1} \leq E_h^k$.

2.3 Résultats numériques

Cette section rassemble les résultats des simulations réalisées à l’aide de la discrétisation précédemment présentée. On décrit d’abord le cas test *flow past a cylinder* et les paramètres physiques et numériques qu’on lui associe. Dans le cas du modèle d’Oldroyd-B ($f \equiv 1$), on compare ensuite les résultats numériques obtenus à ceux de la littérature existante. Des résultats sur le modèle avec fluidité sont enfin présentés et confrontés à ceux du modèle d’Oldroyd-B.

2.3.1 Cas test *Flow past a cylinder*, paramètres physiques et numériques

L’écoulement bidimensionnel autour d’une bille placée entre deux parois fixes que constitue le cas test *flow past a cylinder* est un problème standard pour la simulation des fluides. Les résultats pour un ratio entre le diamètre de la bille et la largeur du canal de $\frac{1}{8}$ ont été rapportés dans [SSAB99, BACMP04]. Ce sont ces dimensions que nous retenons ici.

Le domaine est représenté sur la figure 2.1. La bille a un diamètre $\varnothing = 2$ et la largeur du canal est $H = 16$. Les distances du cylindre au bord entrant (à gauche) et sortant (à droite) sont respectivement 10 et 20. Le repère d’espace est centré sur la bille. Les axes horizontal et vertical sont respectivement dirigés par les vecteurs e_x et e_y .

Les conditions de bord du cas test *flow past a cylinder* sont les suivantes :

- sur les murs (bords inférieur et supérieur), les conditions de bord sont différentes dans les deux sections suivantes. D’une part, on impose du collement sur la vitesse ($u = 0$) dans la section 2.3.2. D’autre part, on impose du glissement sur la vitesse dans la section 2.3.3 :

$$u \cdot n = 0 \text{ et } (((1 - \alpha)D(u) + \tau) \cdot n) \cdot t = 0$$

où

$$D(u) = \frac{\nabla u + \nabla u^t}{2} \quad (2.8)$$

est le taux de déformation et n et t sont les vecteurs normal et tangent,

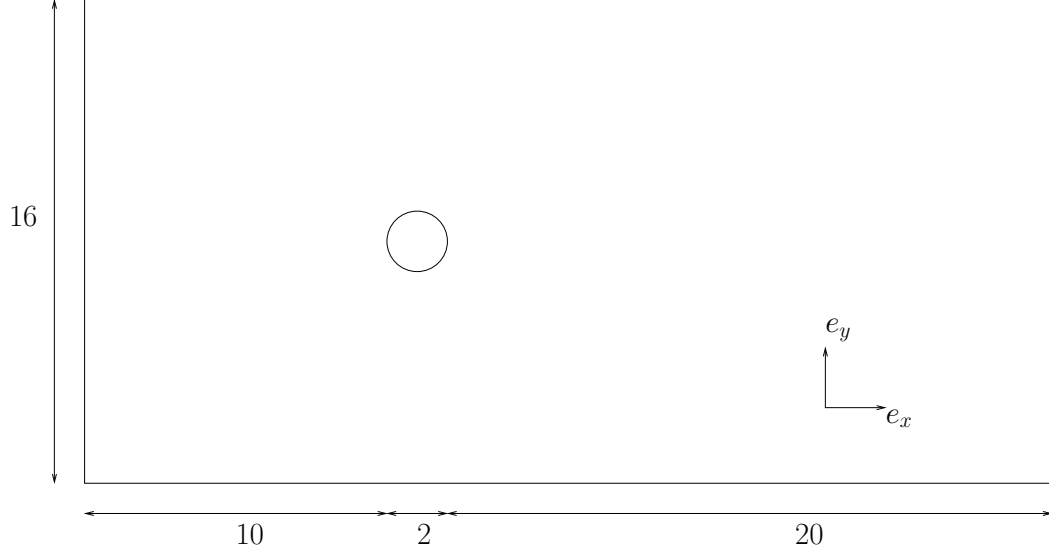


Figure 2.1. Description du cas test *flow past a cylinder*

- en entrée (bord gauche), on impose des conditions de Dirichlet sur la vitesse $u = u_i(y)e_x$, la contrainte $\tau = 0$ et la fluidité $f = 1$. La valeur de la fonction $u_i(y)$ dépend des conditions de bord sur les murs précédemment décrites. Dans le cas du collement sur les murs, un profil parabolique $u_i(y) = 1.5(1 - (2y)^2/H^2)$ est imposé sur la vitesse. Dans le cas du glissement sur les murs, la fonction u_i est constante $u_i \equiv 1.5$.
- en sortie (bord droit), on impose des conditions de Neumann homogènes sur la vitesse :

$$(-p\text{Id} + (1 - \alpha)D(u) + \tau) \cdot n = 0,$$

- sur la bille, on impose du collement sur la vitesse ($u = 0$).

Les conditions initiales sont constituées d'une fonction $u_h^0 \in \mathbb{P}_2^2$ satisfaisant les conditions de bord précédemment décrites et des fonctions constantes $p_h^0 = 0$, $\Sigma_0^h \equiv \text{Id}$ et $f_0^h \equiv 1$.

Les paramètres physiques sont $\text{Re} = 10^{-10}$, $\alpha = 0.41$ et $\xi = \nu = 1$. Le nombre de Weissenberg We varie selon les simulations. Dans la section 2.3.2, il varie entre 0.1 et 2. Dans la section 2.3.3, il est fixé à 0.1.

Les simulations présentées ont été réalisées avec trois maillages différents $m1$, $m2$ et $m3$ contenant respectivement 13803, 18760 et 24448 points. Le maillage $m1$ est représenté sur la figure 2.2. Les figures suivantes correspondent toujours au maillage le plus fin $m3$ sauf quand elles contiennent une comparaison entre les différents maillages.

Le système (2.3) est simulé sur l'intervalle de temps $[0, T]$ avec $T = 50$. Alors, les résultats présentés sont proches de l'état stationnaire, en ce sens que les variables varient numériquement très peu d'un pas de temps à l'autre à son voisinage. Le nombre de pas de temps effectué est $N_T = 1000$.

Le code de calcul utilisé est **FreeFem++** [Fre].

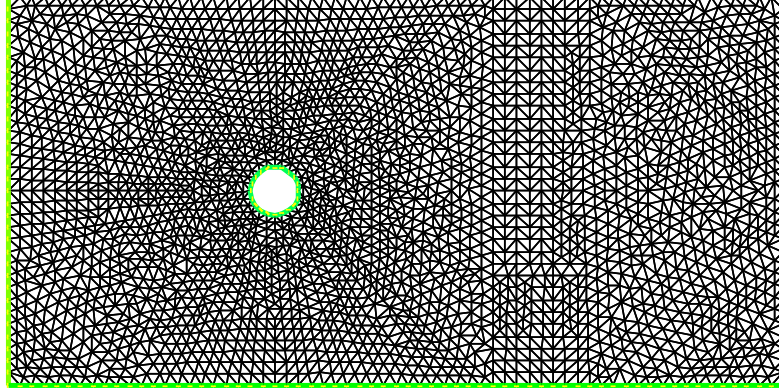


Figure 2.2. Maillage $m1$ du cas test *flow past a cylinder*

2.3.2 Modèle d'Oldroyd-B, comparaison aux résultats existants

Dans cette section, on considère uniquement le cas du modèle d'Oldroyd-B : $f \equiv 1$ dans le système (2.3).

Une quantité caractéristique de l'écoulement autour de la bille est la force de traînée exercée par le fluide sur la bille et ses variations en fonction du nombre de Weissenberg. La formule de la traînée est donnée par l'intégrale

$$F_h^k = - \int_{\Gamma} n \cdot (-p_h^k I + (1 - \alpha) D(u_h^k) + \tau_h^k) \cdot e_x \quad (2.9)$$

où $D(u_h^k)$ est défini comme dans (2.8), Γ est la surface de la bille et n est le vecteur unité normal sortant.

Des résultats sur la traînée ont été rapportés dans [SSAB99, BACMP04]. Les résultats de ces deux références ont été obtenus avec des méthodes d'éléments finis. La méthode de Sun *et al.* [SSAB99] est dite *Discrete Elastic Viscous Split Stress* (DEVSS). Dans cette méthode, la contrainte visqueuse est séparée en deux contributions associées aux gradients de vitesse continu et discontinu. De plus, un opérateur elliptique de stabilisation est ajouté dans la forme faible de l'équation de moment. Ce terme supplémentaire représente la différence entre les deux gradients de vitesse continu et discontinu. La méthode de Behr *et al.* [BACMP04] est une méthode *Galerkin/Least-Squares* (GLS). Dans cette méthode, des termes de stabilisation sont ajoutés à la forme faible rassemblant les trois équations du système d'Oldroyd-B, dans les éléments intérieurs.

On compare les résultats obtenus sur la traînée définie par (2.9) avec les méthodes de Sun *et al.*, Behr *et al.* et le schéma conservatif (2.6a) (cas $f_h^k \equiv 1, \forall k \in \mathbb{N}$). La traînée à l'état stationnaire est tracée en fonction du nombre de Weissenberg ($We \in [0.1, 2]$) sur la figure 2.3. Les valeurs obtenues sont très proches. Les différences sont les plus importantes pour les nombres de Weissenberg les plus grands. C'est le cas pour la différence entre nos résultats et ceux existants mais aussi pour la différence entre les résultats existants. On observe également la forme caractéristique de 'smile'.

Sur les figures 2.4a et 2.4b, l'énergie libre et la traînée respectivement définies par (2.7) et (2.9) sont tracées en fonction du temps. Les résultats concernent différents maillages et $We =$

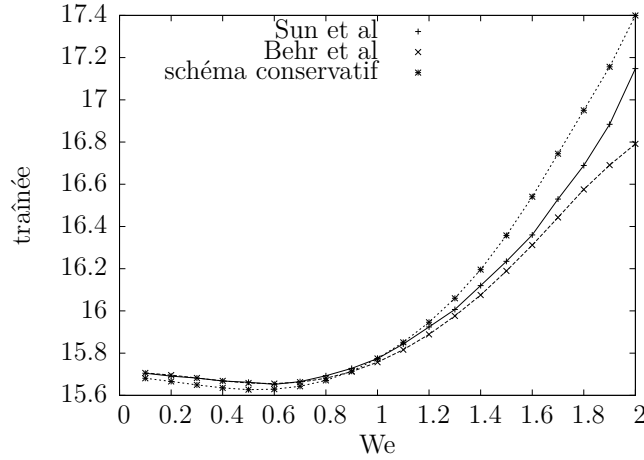


Figure 2.3. Modèle d'Oldroyd-B : traînée calculée pour différents We

{0.1, 2}. L'énergie libre n'est pas décroissante, comme annoncé précédemment ; les conditions de bord ne sont en effet pas homogènes, de l'énergie est apportée au système. Toutefois, l'énergie atteint une valeur stationnaire. C'est également le cas pour la traînée. Les valeurs de la traînée à l'instant final pour $We = \{0.1, 2\}$ correspondent aux valeurs reportés sur la courbe de la figure 2.3. On observe de plus sur les figures 2.4a et 2.4b une première manifestation du problème numérique HWNP. Les courbes de l'énergie libre et de la traînée se superposent pour les différents maillages lorsque $We = 0.1$ est petit. Cependant, quand le nombre de Weissenberg est plus grand $We = 2$, l'énergie libre et la traînée augmentent en raffinant le maillage.

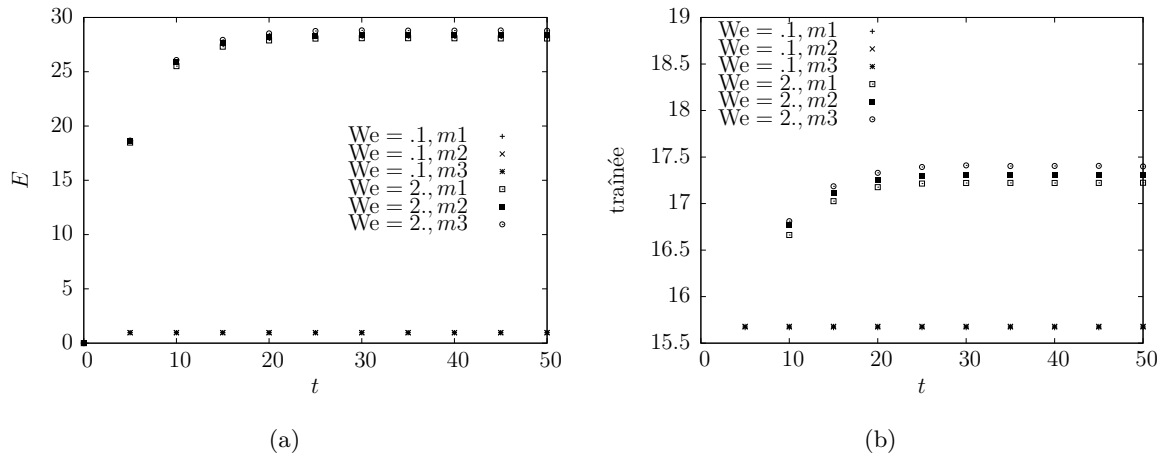


Figure 2.4. Modèle d'Oldroyd-B : énergie libre et traînée en fonction du temps

On représente un autre point de vue sur la figure 2.5. A l'état stationnaire, la différence entre les contraintes normales $\tau_{xx} - \tau_{yy}$ (*first normal stress difference*) est tracée le long de l'axe de symétrie horizontal pour $We \in \{0.1, 2\}$ et différents maillages. On observe d'abord les pics caractéristiques à l'amont et l'aval de la bille. Pour $We = 0.1$, les courbes des différents maillages

se superposent. Pour $We = 2$, le pic aval est de plus en plus accentué en raffinant le maillage. Ce comportement est typique du problème numérique HWNP.

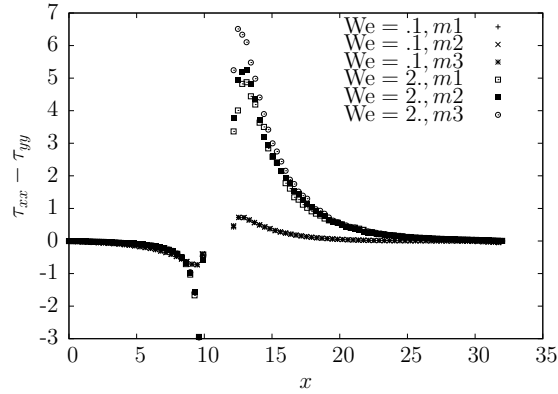


Figure 2.5. Modèle d’Oldroyd-B : différence des contraintes normales le long de l’axe de symétrie horizontal

Ainsi, pour la discrétisation du modèle d’Oldroyd-B, les résultats obtenus sur la traînée sont en accord avec les résultats existants. De plus, les manifestations du problème numérique HWNP sont reproduites.

2.3.3 Modèle avec fluidité

Dans cette section, on montre des simulations du modèle avec fluidité en les confrontant à celles du modèle d’Oldroyd-B (fluidité $f \equiv 1$). Le nombre de Weissenberg est dorénavant fixé à une valeur faible $We = 0.1$.

Les simulations présentées dans la section précédente pour le modèle d’Oldroyd-B concernent des conditions au bord de collement sur les murs. En utilisant le modèle avec fluidité, on observe que le cisaillement près des murs modifie la fluidité de manière significative autour de la bille. Afin que les murs ne modifient pas trop la fluidité autour de la bille, les conditions de collement sur les murs sont remplacées par des conditions de glissement. L’élargissement de la largeur de l’écoulement est une autre solution envisageable que nous n’avons pas retenue.

La figure 2.6 représente le champ de vitesse stationnaire pour le modèle d’Oldroyd-B ($f \equiv 1$) à gauche et le modèle avec fluidité à droite. L’écoulement se fait de gauche à droite. La vitesse du fluide est faible au voisinage de la bille et grande près des murs à son niveau. Le champ de vitesse est légèrement différent dans les deux cas. Afin d’observer plus précisément la différence, on représente sur la figure 2.7 la norme de la vitesse le long de l’axe de symétrie horizontal pour le modèle d’Oldroyd-B et le modèle avec fluidité. On observe que la vitesse en aval de la bille est plus faible dans le cas du modèle avec fluidité. Ce comportement est conforme à la physique du modèle avec fluidité.

Sur la figure 2.8, on représente la composante τ_{xx} du champ de contrainte stationnaire pour le modèle d’Oldroyd-B et le modèle avec fluidité. L’intensité du champ dans le second cas est plus importante. C’est également le cas sur les autres composantes de la contrainte non représentées

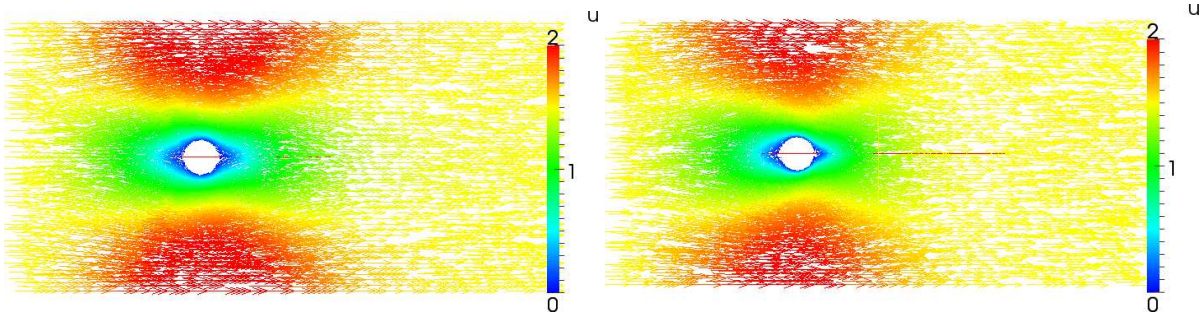


Figure 2.6. Champ de vitesse pour le modèle d'Oldroyd-B (à g.) et le modèle avec fluidité (à d.)

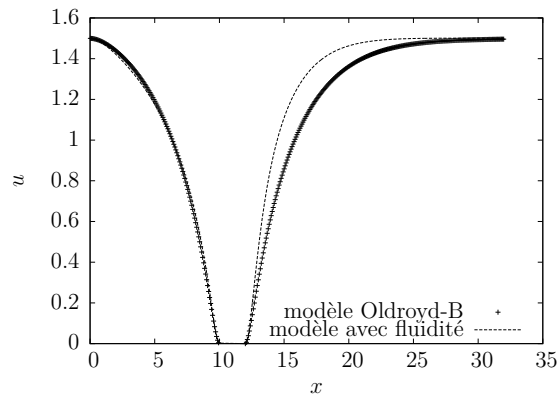


Figure 2.7. Champ de vitesse le long de l'axe de symétrie horizontal

ici. De même, la traînée est plus importante dans le cas du modèle avec fluidité ($F_h^{Nr} \simeq 29$) que dans le cas du modèle d'Oldroyd-B ($F_h^{Nr} \simeq 13$). Ces comparaisons sur la contrainte et la traînée sont valables pour le jeu de paramètres $\xi = \nu = 1$ choisi mais les conclusions sont différentes en changeant ces paramètres.

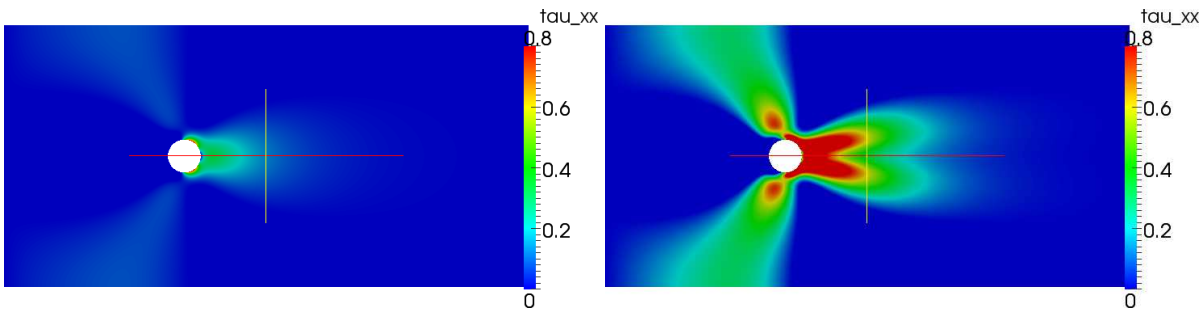


Figure 2.8. Composante τ_{xx} de la contrainte pour le modèle d'Oldroyd-B (à g.) et le modèle avec fluidité (à d.)

Les figures restantes concernent uniquement le modèle avec fluidité. Les lignes de niveau de la fluidité stationnaire sont représentées sur la figure 2.9. La compétition entre le vieillissement et le rajeunissement est observée. En effet, la fluidité décroît lorsque le fluide avance, de gauche

à droite; le fluide vieillit. La fluidité augmente autour de la bille; les variations de contrainte qu'elle induit font rajeunir le fluide.

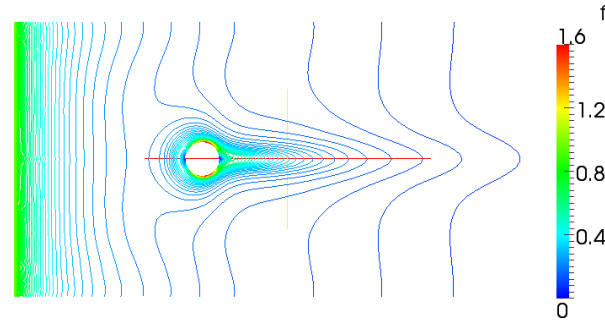


Figure 2.9. Lignes de niveau de la fluidité

Sur les figures 2.10a et 2.10b, l'énergie libre et la traînée sont tracées en fonction du temps pour différents maillages ($m4$ et $m5$ sont des maillages raffinés comportant respectivement 31057 et 37998 points). L'énergie libre et la traînée atteignent un état stationnaire. Les valeurs augmentent en raffinant le maillage. Ceci dit, elles semblent converger. Sur la figure 2.11, la

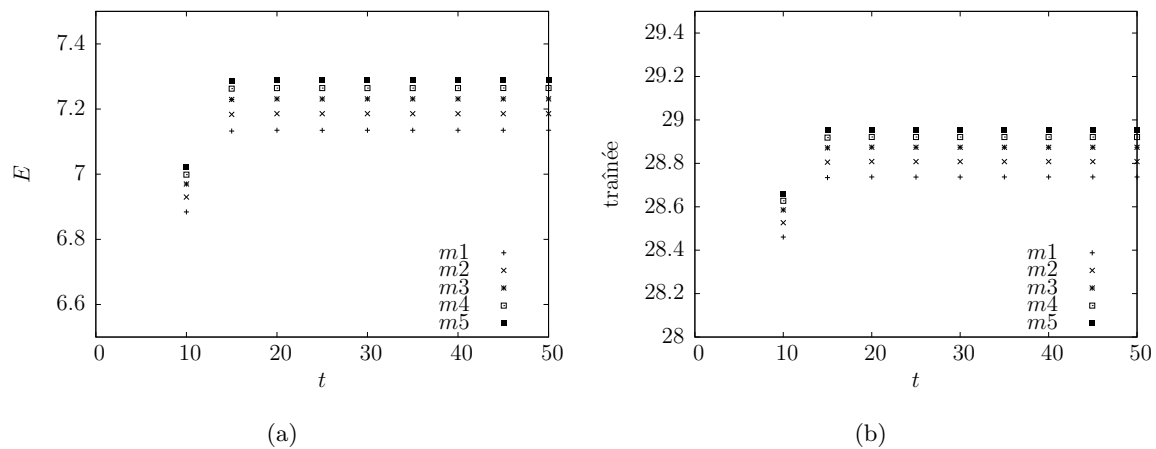


Figure 2.10. Modèle avec fluidité : énergie libre et traînée en fonction du temps

différence des contraintes normales est représentée le long de l'axe de symétrie horizontal. Les pics en amont et aval de la bille sont présents. Ceux-ci n'explorent pas en raffinant le maillage.

Ainsi, sur un cas test assez complexe, le modèle avec fluidité donne des résultats différents du modèle d'Oldroyd-B. Ces différences sont en accord avec la physique des fluides vieillissants. Notamment, le phénomène de compétition vieillissement/rajeunissement du modèle avec fluidité est observé. De plus, pour un nombre de Weissenberg faible, le modèle avec fluidité semble aussi stable que le modèle d'Oldroyd-B original, à fluidité constante.

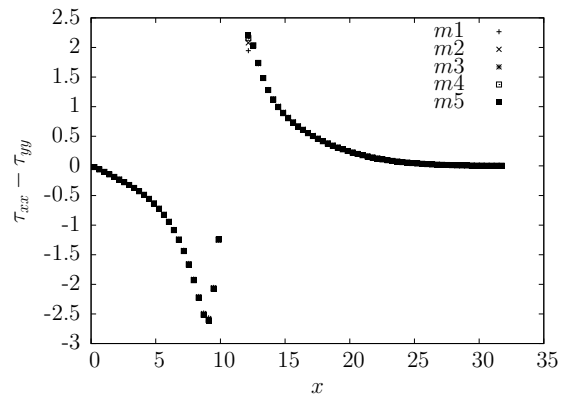


Figure 2.11. Modèle avec fluidité : différence des contraintes normales le long de l'axe de symétrie horizontal

Analyse d'un modèle macroscopique unidimensionnel de fluide vieillissant

Les résultats présentés dans ce chapitre ont fait l'objet d'un article paru dans *Mathematical Models and Methods in Applied Sciences*. Ils ont été obtenus en collaboration avec Lingbing He, Claude Le Bris et Tony Lelièvre. Un système d'équations macroscopiques unidimensionnelles est étudié. On montre que le système est bien posé dans les espaces fonctionnels appropriés et le comportement en temps long est étudié.

Par ailleurs, l'annexe 3.A de ce chapitre ne fait pas partie de la publication et contient la preuve d'une extension de l'un des résultats.

Mathematical analysis of a one-dimensional model for an aging fluid

David Benoit, Lingbing He, Claude Le Bris and Tony Lelièvre

Abstract. We study mathematically a system of partial differential equations arising in the modelling of an aging fluid, a particular class of non Newtonian fluids. We prove well-posedness of the equations in appropriate functional spaces and investigate the longtime behaviour of the solutions.

Key words. Non-Newtonian fluids; Couette flow; longtime behaviour.

3.1 Introduction

Our purpose is to study mathematically a system of partial differential equations arising in the modelling of some particular non Newtonian fluids. These fluids are often called *aging fluids*. Two physical phenomena are indeed permanently competing within the flow of such fluids. On the one hand, the fluid *ages* in the sense that it solidifies. On the other hand, aging is counterbalanced by a flow-induced rejuvenation.

The specific modelling we consider has been proposed in Ref. [DAL01] on the basis of phenomenological arguments and experimental observations. A coefficient f , called the *fluidity* encodes aging for all times and at every location within the fluid. The fluid is solid where $f = 0$, and behaves all the more as a liquid when f grows. Our mathematical study aims to contribute to better understand how well such a model captures the essential phenomena at play in fluid aging.

For our study, we proceed in a one-dimensional setting corresponding physically to the consideration of a laminar Couette flow. This type of flow appropriately models flows in shear rheometers. Our three unknown fields, the velocity u , the shear stress τ and the fluidity f are defined as functions of a space variable y varying in the interval $[0, 1]$. They are also, of course, functions of the time $t \geq 0$. The specific system we choose for our study reads

$$\begin{cases} \rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, & (3.1a) \\ \lambda \frac{\partial \tau}{\partial t} = G \frac{\partial u}{\partial y} - f \tau, & (3.1b) \\ \frac{\partial f}{\partial t} = (-1 + \xi |\tau|) f^2 - \nu f^3. & (3.1c) \end{cases}$$

Six dimensionless coefficients, all positive, constant in time and throughout the domain, are present in the system: the density ρ , the viscosity η , the characteristic relaxation time λ , the elastic modulus G , and two coefficients ξ and ν specifically related to the equation for the evolution of the fluidity f . System (3.1) is a fully coupled system of three equations. The first two equations are classical in nature. The first one is the equation of conservation of momentum for u . The second equation rules the evolution of the shear stress τ . The non classical ingredient therein (as opposed, say, to an Oldroyd-B type equation) is the presence of an extra parameter,

the fluidity f , the role of which is formally similar to that of an inverse time in a relaxation phenomenon. We note that when f is a constant in time and throughout the domain, the equation agrees with the one-dimensional Oldroyd-B equation considered *e.g.* in Ref. [BLM09] and [GS90]. The third equation is of the form of one of the many such evolution equations suggested in Ref. [DAL01]. We hope it is, in this respect, prototypical of a general class. It models the evolution of the fluidity f in function of the stress tensor. The right-hand side of (3.1c) may differ from one model to another. The important ingredient is the presence of two competitive terms: a negative term modelling aging and a positive term modelling rejuvenation. For mathematical convenience, we have taken two particular instances of these two terms.

We examine well-posedness and longtime behaviour for system (3.1). Because we provide self-contained proofs, our study is rather long. Similar questions on a different, although related model for a viscoelastic fluid, have been examined in Ref. [Ren09a] and [Ren09b].

Our article is articulated as follows.

To start with, we prove in Section 2 that the system under consideration admits a global-in-time solution in appropriate functional spaces. The solution is shown to be unique, and indeed strong. System (3.1) is thus satisfied in a classical sense. Our precise statement is the object of Theorem 1. The bulk of Section 2 consists of our proof. The arguments are standard arguments of mathematical fluid dynamics: formal *a priori* estimates, approximation, rigorous *a priori* estimates, convergence. The many nonlinearities present in system (3.1) however prevent us, in the current state of our understanding, from extending our analysis to settings in dimensions higher than or equal to two. Technically, this is related to the fact we repeatedly use, in our arguments, that H^1 functions are L^∞ functions, a specificity of the one-dimensional setting of course.

In Sections 3 and 4, we study the long time behaviour of the solution. Section 3 deals with return to equilibrium. We supply the system with *homogeneous* Dirichlet boundary conditions for the velocity and investigate whether the flow converges to a steady state. For homogeneous Dirichlet boundary conditions, the steady states are $(u \equiv 0, \tau \equiv c, f \equiv 0)$, where c is a constant throughout the domain. The long-time convergence to these steady-states sensitively depends, in system (3.1), of the fluidity f . The situation is qualitatively different depending on the fluidity f_0 at initial time. The more delicate, but of course more interesting, case mathematically is the case where the fluidity f_0 at initial time does not vanish everywhere: a part of the material, possibly the whole of it, is originally fluid. Section 3.3.1 addresses this case. We show (and the proof is quite substantial even in the one-dimensional setting we consider) that the flow converges to the null steady-state in suitable functional norms. The precise statement is the purpose of Theorem 2. The convergence is then shown to be polynomial in time, for all three fields u , τ and f . The rates of convergence are made precise in Theorem 3. Numerical simulations we perform in Section 5 will show these rates are indeed sharp. It is interesting to emphasize the physical signification of our mathematical results. With regard to modelling, the convergence of the fluidity f to zero that we establish, under homogeneous Dirichlet boundary conditions, means that when left at rest, the fluid progressively solidifies, a certainly intuitive fact. In addition, for u and τ , the rate of convergence sensitively depends on the size of the region where, originally, the material is liquid (a size measured by our parameter β defined in (3.63) and present in the right-hand sides of the estimates of Theorem 3). The larger the liquid region the quicker the

convergence of both the velocity and the shear stress to zero. It is not completely clear to us whether the latter qualitative behaviour is or not compatible with experimental observations or physical intuition.

If the material is entirely solid at initial time, that is $f_0 \equiv 0$ everywhere, the behaviour is quite different. Then the material stays solid for all times, while the velocity vanishes and the shear stress converges to a constant exponentially fast. We present the simple analysis of this behaviour in Section 3.3.2. Note that the result agrees with simple physical intuition.

Non-homogeneous boundary conditions, studied throughout Section 3.4, are, as always for questions related to long-time behaviours, significantly more intricate to address. We adopt constant boundary conditions, respectively $u = 0$ and $u = a > 0$ at $y = 0$ and $y = 1$. The sign of a is chosen positive without loss of generality because both (u, τ) and $(-u, -\tau)$ are solutions to the first two equations of system (3.1) for fixed f . We begin by showing in Section 3.4.1 that, when we impose that the fluidity is strictly positive everywhere, there exists a *unique* steady state. We next show in Section 3.4.2 that this steady state is stable under small perturbations. Our precise result is stated in Theorem 5. When the perturbations of the state are not small, analyzing return to equilibrium is, in general, beyond our reach. We are however able to show that, when we assume a particular form of the initial condition (namely linear velocity, constant shear stress, constant positive fluidity), then return to equilibrium does hold true even if the initial condition is not close to the steady state. Some suitable assumptions relating the size of the parameters in system (3.1) and the non-zero boundary condition a are also needed (see condition (3.115)). Our precise result is Theorem 6. The reason why we have to assume this specific form of the initial conditions is purely technical (and our numerical simulations will actually show that these restrictions are, in practice, unnecessary). In that case, system (3.1) reduces to a two-dimensional system of ordinary differential equations, for which Poincaré-Bendixson Theory allows us to understand the longtime behaviour. Our study is performed in Section 3.4.3.

As briefly mentioned above, Section 5 presents some numerical simulations. We first show that the rates of convergence estimated by our various mathematical arguments in the various regimes considered in Sections 3.3 and 3.4 are indeed sharp. We also investigate numerically the stability of the steady state. Our simulations show that, irrespective of the size of the initial perturbation (and thus in a more general regime than that for our mathematical arguments), the fluid returns to equilibrium, or more generally converges to the suitable steady state. The rates of convergence are also examined.

We conclude this introduction by mentioning that, despite their limitations, our results show that the model derived in Ref. [DAL01] does adequately account for aging and rejuvenation. However, two shortcomings need to be emphasized. Both originate from the mathematical nature of equation (3.1c) (and are actually related to the fact that the Cauchy Lipschitz theory applies to this equation). First, when f vanishes, then f remains zero for all subsequent times. This property, present everywhere in our mathematical study, prevents fluidification to occur after solidification. This clearly limits the range of materials covered by the modelling (compare muds and concrete, say). Second, f can only vanish asymptotically and never in finite time unless it is already zero before. Otherwise stated, solidification can occur, but never in finite time: again a modelling limitation. The one usefulness, if any, of our study, is therefore to point out that a

mathematically well founded model where fluidification *and* solidification compete on an equal footing is still to be derived. Our study implicitly points to suitable directions to this end.

Further mathematical investigations on models for aging fluids will be presented in Ref. [Benon].

3.2 Global existence and uniqueness

In this section, we establish the following global existence and uniqueness result for system (3.1) supplied with initial conditions u_0, τ_0, f_0 and the boundary conditions $u(t, 0) = 0$ and $u(t, 1) = a \geq 0$ for all time $t \in [0, T]$ (where a is a constant scalar).

Theorem 1 *Recall that Ω is the one-dimensional domain $[0, 1]$ and that $T > 0$ is fixed. Consider the initial data*

$$(u_0, \tau_0, f_0) \in H^1(\Omega)^3 \text{ with } f_0 \geq 0. \quad (3.2)$$

Then there exists a unique global solution (u, τ, f) to system (3.1) such that for any $T > 0$,

$$(u, \tau, f) \in \left(C([0, T]; H^1) \cap L^2([0, T]; H^2) \right) \times C([0, T]; H^1) \times C([0, T]; H^1) \quad (3.3)$$

and $f \geq 0$ for all $x \in \Omega$ and $t \in [0, T]$.

In addition, we have

$$\left(\frac{\partial u}{\partial t}, \frac{\partial \tau}{\partial t}, \frac{\partial f}{\partial t} \right) \in L^2([0, T]; L^2) \times C([0, T]; L^2) \times C([0, T]; L^2), \quad (3.4)$$

so that the equations in (3.1) are all satisfied in the strong sense in time.

Before we get to the proof, we eliminate the non-homogeneous Dirichlet boundary condition, introducing the auxiliary velocity field

$$u(t, y) - ay.$$

This velocity field, which we still denote by u , solves the system

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \end{array} \right. \quad (3.5a)$$

$$\left\{ \begin{array}{l} \lambda \frac{\partial \tau}{\partial t} = G \frac{\partial u}{\partial y} - f\tau + Ga, \end{array} \right. \quad (3.5b)$$

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} = (-1 + \xi|\tau|)f^2 - \nu f^3, \end{array} \right. \quad (3.5c)$$

supplied with the *homogeneous* Dirichlet boundary conditions $u(t, 0) = 0$ and $u(t, 1) = 0$ for all time $t \in [0, T]$ and initial conditions $u_0, \tau_0, f_0 \in H^1(\Omega)$. The proof of Theorem 1 will actually be completed on system (3.5). The result on (3.1) then immediately follows.

Proof. The proof falls in eight steps. The first five steps consist in deriving *formal* a priori estimates. These estimates are next made rigorous for a sequence of approximate solution in Step 6. The convergence of this sequence is proven in Step 7, thereby establishing existence of a solution to (3.5). Step 8 addresses uniqueness.

Step 1: Non-negativity of the fluidity. Let us first formally prove that $f \geq 0$. Fix $y \in \Omega$ and introduce

$$E_0 = \{y \in \Omega, f_0(y) > 0\}.$$

For $y \in \Omega \setminus E_0$, we have $f_0(y) = 0$ and thus $f(t, y) = 0$ for all time $t \in [0, T]$ because of (3.5c). On the other hand, for $y \in E_0$, we now show that $f(t, y) > 0$ for all time $t \in [0, T]$. We argue by contradiction and suppose, by continuity of $f(\cdot, y)$, that

$$t_m = \inf \{t \in (0, T], f(t, y) = 0\} < T.$$

The Cauchy-Lipschitz Theorem applied to (3.5c) with zero as initial condition at time t_m implies that $f(t, y) = 0$ for $t \in (t_m - \varepsilon, t_m + \varepsilon)$ for $\varepsilon > 0$, which contradicts the definition of t_m .

We have therefore shown that f stays zero where it is zero, and stays positive where it is positive, which in particular implies non-negativity everywhere.

Step 2: Formal first energy estimates. We again argue formally. We first multiply the evolution equation (3.5a) on u by u itself and integrate over the domain. This gives a first estimate

$$\frac{1}{2}\rho \frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 + \eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 = \int_{\Omega} \left(\frac{\partial \tau}{\partial y} u \right) (t, \cdot). \quad (3.6)$$

Similarly, we multiply the evolution equation (3.5b) by τ and integrate over Ω to find

$$\frac{1}{2}\lambda \frac{d}{dt} \|\tau(t, \cdot)\|_{L^2}^2 + \left\| \sqrt{f}\tau(t, \cdot) \right\|_{L^2}^2 = G \int_{\Omega} \left(\frac{\partial u}{\partial y} \tau \right) (t, \cdot) + G a \bar{\tau}(t), \quad (3.7)$$

where we denote by

$$\bar{q}(t) = \int_{\Omega} q(t, y) dy \quad (3.8)$$

the average over Ω of a function $q : (t, y) \in [0, T] \times \Omega \rightarrow \mathbb{R}$.

Combining estimates (3.6) and (3.7) and using integration by parts and the fact that u vanishes on the boundary, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(G\rho \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2 \right) + G\eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| (\sqrt{f}\tau)(t, \cdot) \right\|_{L^2}^2 \\ = G a \bar{\tau}(t). \end{aligned} \quad (3.9)$$

We now turn to (3.5c). Integrating (3.5c) over Ω yields

$$\frac{d}{dt} \|f(t, \cdot)\|_{L^1} + \|f(t, \cdot)\|_{L^2}^2 + \nu \|f(t, \cdot)\|_{L^3}^3 = \xi \int_{\Omega} (|\tau|f^2)(t, \cdot). \quad (3.10)$$

The Young inequality

$$\xi |\tau|f^2 = \sqrt{\nu} f^{\frac{3}{2}} \cdot \frac{\xi}{\sqrt{\nu}} |\tau|f^{\frac{1}{2}} \leq \frac{\nu}{2} f^3 + \frac{\xi^2}{2\nu} f\tau^2$$

then yields

$$\frac{d}{dt} \|f(t, \cdot)\|_{L^1} + \|f(t, \cdot)\|_{L^2}^2 + \frac{\nu}{2} \|f(t, \cdot)\|_{L^3}^3 \leq \frac{\xi^2}{2\nu} \left\| (\sqrt{f\tau})(t, \cdot) \right\|_{L^2}^2. \quad (3.11)$$

Collecting (3.9) and (3.11), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(G\rho \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2 + \frac{2\nu}{\xi^2} \|f(t, \cdot)\|_{L^1} \right) \\ & + G\eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{1}{2} \left\| (\sqrt{f\tau})(t, \cdot) \right\|_{L^2}^2 \leq C a \|\tau(t, \cdot)\|_{L^2} \end{aligned} \quad (3.12)$$

where C , here and throughout our text, denotes a constant, the actual value of which is independent from T and only depends on the domain Ω and the coefficients $\rho, \eta, \lambda, G, \xi, \nu$ in (3.5).

Using the inequality $\|\tau(t, \cdot)\|_{L^2} \leq \frac{\|\tau(t, \cdot)\|_{L^2}^2 + 1}{2}$ and applying the Gronwall Lemma to (3.12), we obtain

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 + \|f(t, \cdot)\|_{L^1} \right) \\ & + \int_0^T \left(\|u(t, \cdot)\|_{H^1}^2 + \left\| (\sqrt{f\tau})(t, \cdot) \right\|_{L^2}^2 \right) dt \leq C_{0, T}, \end{aligned} \quad (3.13)$$

where $C_{0, T}$ is a constant depending not only on $\Omega, \rho, \eta, \lambda, G, \xi, \nu$, but also on the boundary condition a , the initial data u_0, τ_0, f_0 and the time T .

Remark 1 For homogeneous boundary conditions, that is $a = 0$, we mention that the right-hand sides of (3.9) and thus (3.12) vanish. The constant $C_{0, T}$ in (3.13) therefore does not depend on T and we get a bound uniform in time.

Step 3: A priori estimates on an auxiliary function. Denote

$$g(t, y) = \int_0^y (\tau(t, x) - \bar{\tau}(t)) dx.$$

This function g satisfies *homogeneous* Dirichlet boundary conditions and formally solves

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial \tau}{\partial y}.$$

Using (3.5a) and (3.5b), which respectively imply

$$\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2}{\partial y^2} \left(u + \frac{1}{\eta} g \right),$$

and

$$\lambda \frac{\partial g}{\partial t} = - \int_0^y (f\tau - \bar{f}\bar{\tau}) + Gu,$$

we remark that the auxiliary function

$$\begin{aligned} U &= u + \frac{1}{\eta} \int_0^y (\tau - \bar{\tau}) \\ &= u + \frac{1}{\eta} g \end{aligned} \quad (3.14)$$

solves:

$$\frac{\partial U}{\partial t} = \frac{\eta}{\rho} \frac{\partial^2 U}{\partial y^2} - \frac{1}{\lambda \eta} \int_0^y (f\tau - \bar{f}\bar{\tau}) + \frac{G}{\lambda \eta} u. \quad (3.15)$$

Multiplying equation (3.15) by $\frac{\partial^2 U}{\partial y^2}$ and integrating over Ω yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{\rho} \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \\ &\leq C \left(\|(f\tau)(t, \cdot)\|_{L^1} \int_{\Omega} \left| \frac{\partial^2 U}{\partial y^2} \right|(t, \cdot) + \int_{\Omega} \left| u \frac{\partial^2 U}{\partial y^2} \right|(t, \cdot) \right), \end{aligned}$$

after elementary manipulations in the right-hand side. Then using the Young and the Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{2\rho} \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \\ &\leq C \left(\|f(t, \cdot)\|_{L^1} \left\| (\sqrt{f}\tau)(t, \cdot) \right\|_{L^2}^2 + \|u(t, \cdot)\|_{L^2}^2 \right). \end{aligned} \quad (3.16)$$

Using (3.13), we know that the right-hand side is $L^1(0, T)$. In view of our regularity assumptions on the initial conditions, we have $\frac{\partial U}{\partial y} \Big|_{t=0} = \frac{\partial u_0}{\partial y} + \frac{1}{\eta}(\tau_0 - \bar{\tau}_0) \in L^2(\Omega)$. We therefore deduce from (3.16) that

$$U \in L^\infty([0, T], H_0^1) \cap L^2([0, T], H^2). \quad (3.17)$$

Step 4: L^∞ estimates. We are now in position to obtain (again formal) L^∞ -bounds on τ and f . We consider the evolution equation (3.5b), which we rewrite in terms of U defined by (3.14) and using $\bar{\tau}$ defined for τ as in (3.8):

$$\lambda \frac{\partial \tau}{\partial t} = G \frac{\partial U}{\partial y} - \left(f + \frac{G}{\eta} \right) \tau + \frac{G}{\eta} \bar{\tau} + Ga.$$

Multiplying this equation by τ , we obtain

$$\frac{\lambda}{2} \frac{d}{dt} |\tau|^2 + \left(f + \frac{G}{\eta} \right) |\tau|^2 \leq C \left(|\tau| \cdot \left| \frac{\partial U}{\partial y} \right| + |\tau| \cdot \|\tau\|_{L^2} + a |\tau| \right),$$

so that, repeatedly applying the Young inequality,

$$\frac{\lambda}{2} \frac{d}{dt} |\tau|^2 + \left(f + \frac{G}{2\eta} \right) |\tau|^2 \leq C \left(\left| \frac{\partial U}{\partial y} \right|^2 + \|\tau\|_{L^2}^2 + a \right). \quad (3.18)$$

We apply the Gronwall Lemma to (3.18) and use $\frac{\partial U}{\partial y} \in L^2([0, T], L^\infty)$ because of (3.17), estimate (3.13), $\tau_0 \in H^1(\Omega)$ and we work in a one-dimensional setting, to obtain

$$\|\tau(t, \cdot)\|_{L^\infty} \leq C_{0,T} \quad (3.19)$$

that is, $\tau \in L^\infty([0, T], L^\infty)$.

As for the function f , using the Duhamel formula for the evolution equation (3.5c) rewritten as

$$\frac{\partial f}{\partial t} = (-f - \nu f^2)f + \xi|\tau|f^2,$$

we obtain, for almost all $y \in \Omega$,

$$\begin{aligned} f(t, y) &= e^{-\int_0^t (f+\nu f^2)(s,y)ds} f_0(y) + \xi \int_0^t e^{-\int_s^t (f+\nu f^2)(t',y)dt'} (|\tau|f^2)(s, y)ds \\ &\leq f_0(y) + \frac{\xi}{\nu} \|\tau\|_{L_T^\infty(L^\infty)} \int_0^t e^{-\int_s^t \nu f^2(t',y)\nu dt'} \nu f^2(s, y)ds, \end{aligned}$$

where we have used the non-negativity of f and the previously derived L^∞ -bound on τ to obtain the second line. The above equation leads to

$$\begin{aligned} f(t, y) &\leq f_0(y) + \frac{\xi}{\nu} \|\tau\|_{L_T^\infty(L^\infty)} \left(1 - e^{-\int_0^t \nu f^2(s,y)ds}\right) \\ &\leq f_0(y) + \frac{\xi}{\nu} \|\tau\|_{L_T^\infty(L^\infty)}. \end{aligned} \quad (3.20)$$

Using that $f_0 \in H^1$ and that we work in a one-dimensional setting, we obtain that $f \in L^\infty([0, T], L^\infty)$.

Remark 2 For homogeneous boundary conditions, the Gronwall Lemma applied to (3.18) implies

$$\|\tau(t, \cdot)\|_{L^\infty}^2 \leq \|\tau_0\|_{L^2}^2 e^{-\frac{G}{\lambda\eta}t} + \int_0^t \left\| \frac{\partial U}{\partial y}(s, \cdot) \right\|_{L^\infty}^2 ds + \sup_{t \in [0, T]} \|\tau(t, \cdot)\|_{L^2}^2 \int_0^t e^{\frac{G}{\lambda\eta}(s-t)} ds. \quad (3.21)$$

Moreover, as explained at the end of Step 2, the constant $C_{0,T}$ in (3.13) does not depend on T . Hence, the right-hand side of (3.16) and the bound in $L^2([0, T]; L^\infty)$ -norm for $\frac{\partial U}{\partial y}$, deduced from (3.17), also do not depend on T . It follows from (3.21) that the L^∞ -bound (3.19) on τ is uniform in time. Equation (3.20) yields a similar conclusion for the bound on f .

Step 5: Second a priori estimates. In order to get estimates on higher order derivatives, we now differentiate with respect to y the evolution equation (3.5b) and obtain

$$\begin{aligned} \lambda \frac{\partial}{\partial t} \left(\frac{\partial \tau}{\partial y} \right) &= G \frac{\partial^2 u}{\partial y^2} - f \frac{\partial \tau}{\partial y} - \frac{\partial f}{\partial y} \tau \\ &= G \frac{\partial^2 U}{\partial y^2} - \frac{G}{\eta} \frac{\partial \tau}{\partial y} - f \frac{\partial \tau}{\partial y} - \frac{\partial f}{\partial y} \tau. \end{aligned} \quad (3.22)$$

Likewise, we differentiate with respect to y the evolution equation (3.5c) and get

$$\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) = \xi \frac{\partial |\tau|}{\partial y} f^2 + 2(\xi |\tau| - 1) f \frac{\partial f}{\partial y} - 3\nu f^2 \frac{\partial f}{\partial y}. \quad (3.23)$$

Multiplying equations (3.22) and (3.23) respectively by $\frac{\partial \tau}{\partial y}$ and $\frac{\partial f}{\partial y}$, integrating over the domain, summing up and using that both τ and f are in $L^\infty([0, T], L^\infty)$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\lambda \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \\ & \leq C_{0,T} \int_{\Omega} \left(\frac{\partial^2 U}{\partial y^2} \frac{\partial \tau}{\partial y} + \left(\frac{\partial \tau}{\partial y} \right)^2 + \frac{\partial f}{\partial y} \frac{\partial \tau}{\partial y} + \frac{\partial |\tau|}{\partial y} \frac{\partial f}{\partial y} + \left(\frac{\partial f}{\partial y} \right)^2 \right) (t, \cdot). \end{aligned}$$

Repeatedly applying the Young inequality and using that τ belongs to $L^2([0, T], H^1)$ (which implies $\left| \frac{\partial |\tau|}{\partial y}(t, y) \right| = \left| \frac{\partial \tau}{\partial y}(t, y) \right|$ for almost all $t \in [0, T]$ and $y \in \Omega$), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\lambda \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \\ & \leq C_{0,T} \left(\left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \right). \end{aligned} \quad (3.24)$$

We apply the Gronwall Lemma to (3.24), use that $\tau_0, f_0 \in H^1(\Omega)$ and the estimate (3.17) to obtain that $\tau, f \in L^\infty([0, T], H^1)$.

It follows from the definition (3.14) and the estimate (3.17) that $u \in L^\infty([0, T], H^1) \cap L^2([0, T], H^2)$.

Step 6: Construction of an approximate solution. Now that we have established all the necessary formal a priori estimates, we turn to the construction of a sequence of approximating solutions to (3.5) on which we will rigorously derive these a priori estimates. We introduce, for $n \geq 1$, the sequence of systems

$$\begin{cases} \rho \frac{\partial u_n}{\partial t} = \eta \frac{\partial^2 u_n}{\partial y^2} + \frac{\partial \tau_n}{\partial y}, & (3.25a) \end{cases}$$

$$\begin{cases} \lambda \frac{\partial \tau_n}{\partial t} = G \frac{\partial u_n}{\partial y} - f_{n-1} \tau_n + Ga, & (3.25b) \end{cases}$$

$$\begin{cases} \frac{\partial f_n}{\partial t} = (-1 + \xi |\tau_n|) f_{n-1} f_n - \nu f_{n-1} f_n^2, & (3.25c) \end{cases}$$

supplied with the *homogeneous* Dirichlet boundary conditions $u_n(t, 0) = 0$ and $u_n(t, 1) = 0$ for all time $t \in [0, T]$ and initial conditions $(u_{n0}, \tau_{n0}, f_{n0}) = (u_0, \tau_0, f_0)$. We actually use the initial condition (u_0, τ_0, f_0) also to initialize the iterations in n , thus the coincidence of notation.

We argue by induction. Consider

$$(u_{n-1}, \tau_{n-1}, f_{n-1}) \in \left(C([0, T]; H^1) \cap L^2([0, T]; H^2) \right) \times C([0, T]; H^1) \times C([0, T]; H^1)$$

and $f_{n-1} \geq 0$. We first show that there exists a unique solution (u_n, τ_n, f_n) to (3.25) belonging to the same functional spaces and such that $f_n \geq 0$. For this purpose, we decompose (3.25) into

two subsystems: the linear (Oldroyd-B) type model coupling the evolution equations (3.25a) on u_n and (3.25b) on τ_n on the one hand and the ordinary differential equation (3.25c) on f_n satisfied for all $y \in \Omega$ on the other hand. The existence and uniqueness of a solution (u_n, τ_n) in the space $(C([0, T]; H^1) \cap L^2([0, T]; H^2)) \times C([0, T]; H^1)$ for the former system is obtained using a classical approach (see for instance Ref. [GS90] for a very close system). We now turn to f_n . We show that f_n exists in $C([0, T]; H^1)$ and $f_n \geq 0$. The equation (3.25c) writes

$$\begin{aligned} \frac{\partial f_n}{\partial t} &= \psi(t, f_n, y), \\ f_n|_{t=0} &= f_0 \end{aligned} \quad (3.26)$$

where ψ is a function from $[0, T] \times \mathbb{R} \times \Omega$ to \mathbb{R} .

We first fix $y \in \Omega$ and show that the function $f_n(\cdot, y)$ is continuous in time and non-negative. The function ψ is continuous in its first two variables and locally Lipschitz in its second variable. The Cauchy-Lipschitz Theorem shows there exists a unique local solution with $f_0(y)$ as initial condition. Let $[0, T^*)$ be the interval of existence of the maximal solution for positive time. For all $t \in [0, T^*)$, we have $f_n \geq 0$, using Step 1. In addition, since f_{n-1} and f_n are both non-negative, (3.25c) implies for all $t \in [0, T^*)$,

$$\begin{aligned} \frac{\partial f_n}{\partial t} &\leq \xi |\tau_n| f_{n-1} f_n \\ &\leq \xi \|\tau_n\|_{C_T(L^\infty)} \|f_{n-1}\|_{C_T(L^\infty)} f_n, \end{aligned} \quad (3.27)$$

using that both τ_n and f_{n-1} belong to $C([0, T]; H^1)$. The Gronwall Lemma then proves that f_n remains bounded on $[0, T^*]$ and thus we have established existence and uniqueness on $[0, T]$.

We now turn to the local property of continuity of f_n as a function of y . We use that the function ψ is continuous in y , because both τ_n and f_{n-1} are continuous in y in our one-dimensional setting and the theorem on the continuous dependence on a parameter for ordinary differential equations of the form (3.26) (see *e.g.* Ref. [Car67], Theorem 1.11.1, p. 126). We now show that $\frac{\partial f_n}{\partial y}$ belongs to $C([0, T]; L^2)$. We consider, for almost all $y \in \Omega$, the following linear ordinary differential equation on $\frac{\partial f_n}{\partial y}$

$$\frac{\partial}{\partial t} \left(\frac{\partial f_n}{\partial y} \right) = A \frac{\partial f_n}{\partial y} + B, \quad (3.28)$$

where we have introduced the functions

$$A = \xi |\tau_n| f_{n-1} - f_{n-1} - 2\nu f_{n-1} f_n \in C([0, T]; L^\infty), \quad (3.29)$$

$$B = \xi \frac{\partial |\tau_n|}{\partial y} f_{n-1} f_n + (\xi |\tau_n| - 1) f_n \frac{\partial f_{n-1}}{\partial y} - \nu f_n^2 \frac{\partial f_{n-1}}{\partial y} \in C([0, T]; L^2). \quad (3.30)$$

The Cauchy-Lipschitz Theorem then guarantees the existence of $\frac{\partial f_n}{\partial y}(\cdot, y)$ continuous in time, for almost all $y \in \Omega$. The Duhamel formula applied to (3.28) yields, for all $t \in [0, T]$ and almost all $y \in \Omega$,

$$\frac{\partial f_n}{\partial y}(t, y) = \frac{\partial f_0}{\partial y}(y) e^{\int_0^t A(\cdot, y)} + \int_0^t B(s, y) e^{\int_s^t A(\cdot, y)} ds,$$

so that, using (3.2), (3.29) and (3.30), $\frac{\partial f_n}{\partial y}$ belong to $C([0, T]; L^2)$. As (3.28) is the derivative with respect to y of (3.25c), this yields $f_n \in C([0, T]; H^1)$.

Now that we have established, for all n , the existence of a solution (u_n, τ_n, f_n) to (3.25) in the appropriate functional spaces (as in (3.3)-(3.4)), we derive, for (u_n, τ_n, f_n) , the a priori estimates formally established on (u, τ, f) in the previous steps. Estimate (3.9) now reads

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(G\rho \|u_n(t, \cdot)\|_{L^2}^2 + \lambda \|\tau_n(t, \cdot)\|_{L^2}^2 \right) + G\eta \left\| \frac{\partial u_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \left(\sqrt{f_{n-1}\tau_n} \right) (t, \cdot) \right\|_{L^2}^2 \\ = G a \bar{\tau}_n(t). \end{aligned} \quad (3.31)$$

Likewise, (3.11) is now replaced by

$$\begin{aligned} \frac{d}{dt} \|f_n(t, \cdot)\|_{L^1} + \int_{\Omega} (f_{n-1}f_n)(t, \cdot) + \frac{\nu}{2} \int_{\Omega} (f_{n-1}f_n^2)(t, \cdot) \\ \leq \frac{\xi^2}{2\nu} \left\| \left(\sqrt{f_{n-1}\tau_n} \right) (t, \cdot) \right\|_{L^2}^2, \end{aligned} \quad (3.32)$$

Collecting (3.31) and (3.32) yields the following estimate, analogous to (3.12),

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(G\rho \|u_n(t, \cdot)\|_{L^2}^2 + \lambda \|\tau_n(t, \cdot)\|_{L^2}^2 + \frac{2\nu}{\xi^2} \|f_n(t, \cdot)\|_{L^1} \right) \\ + G\eta \left\| \frac{\partial u_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{1}{2} \left\| \left(\sqrt{f_{n-1}\tau_n} \right) (t, \cdot) \right\|_{L^2}^2 \leq C a \|\tau_n(t, \cdot)\|_{L^2}, \end{aligned} \quad (3.33)$$

and therefore, (3.13) holds with (u_n, τ_n, f_n) instead of (u, τ, f) .

The arguments given in Step 3 to derive (3.16) and in Step 4 for the L^∞ estimates can be mimicked for the approximate system in (u_n, τ_n, f_{n-1}) instead of (u, τ, f) , and the corresponding auxiliary functions g_n and U_n .

At this point, we have rigorously established on (u_n, τ_n, f_n) and our formal estimates of steps 2 to 4:

$$\sup_n \sup_{t \in [0, T]} (\|u_n(t, \cdot)\|_{L^2} + \|\tau_n(t, \cdot)\|_{L^2} + \|f_n(t, \cdot)\|_{L^1}) \leq C_{0, T}, \quad (3.34)$$

and

$$\sup_n \sup_{t \in [0, T]} (\|U_n(t, \cdot)\|_{H^1} + \|\tau_n(t, \cdot)\|_{L^\infty} + \|f_n(t, \cdot)\|_{L^\infty}) + \|U_n\|_{L_T^2(H^2)} \leq C_{0, T}, \quad (3.35)$$

where we recall that $C_{0, T}$ denotes various constants which depend on the coefficients in system (3.5), the initial data u_0, τ_0, f_0 and the time T .

We now turn to the a priori estimates of Step 5. Using arguments similar to the formal arguments of Step 5, we obtain

$$\begin{aligned} \frac{d}{dt} \left(\lambda \left\| \frac{\partial \tau_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \\ \leq C_{0, T} \left(\left\| \frac{\partial \tau_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f_{n-1}}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial^2 U_n}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \right). \end{aligned} \quad (3.36)$$

We now observe that showing H^1 bounds on τ_n and f_n is less straightforward than in our formal Step 5. We introduce

$$Y_n(t) = \left\| \frac{\partial \tau_n}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f_n}{\partial y}(t, \cdot) \right\|_{L^2}^2.$$

Integrating (3.36) from 0 to $t \leq T$, we see that Y_n satisfies

$$Y_n(t) \leq C_{0,T} \int_0^t Y_n + C_{0,T} \int_0^t Y_{n-1} + C_{0,T} \|U_n\|_{L_T^2(H^2)}^2 + Y_0. \quad (3.37)$$

Applying the Gronwall Lemma to (3.37), we find

$$Y_n(t) \leq \left(C_{0,T} \|U_n\|_{L_T^2(H^2)}^2 + Y_0 \right) e^{C_{0,T}t} + C_{0,T} \int_0^t Y_{n-1}(s) e^{C_{0,T}(t-s)} ds$$

which we rewrite

$$Y_n(t) \leq C_{0,T} + C_{0,T} \int_0^t Y_{n-1}(s) ds.$$

Arguing by induction, one can check that this implies, for all $t \in [0, T]$ and n ,

$$Y_n(t) \leq C_{0,T} \sum_{i=0}^{n-1} \frac{(C_{0,T}t)^i}{i!} + \frac{(C_{0,T}t)^n}{n!} Y_0.$$

It follows that, $C_{0,T}$ denoting various constants,

$$\sup_n \sup_{t \in [0, T]} Y_n(t) \leq C_{0,T} e^{C_{0,T}T}, \quad (3.38)$$

Recalling that $\frac{\partial u_n}{\partial y} = \frac{\partial U_n}{\partial y} - \frac{1}{\eta}(\tau_n - \bar{\tau}_n)$, we use inequalities (3.35) and (3.38) to derive

$$\sup_n \sup_{s \in [0, T]} (\|u_n(s, \cdot)\|_{H^1} + \|\tau_n(s, \cdot)\|_{H^1} + \|f_n(s, \cdot)\|_{H^1}) + \|u_n\|_{L_T^2(H^2)} \leq C_{0,T}. \quad (3.39)$$

This implies

$$\sup_n \left(\left\| \frac{\partial u_n}{\partial t} \right\|_{L_T^2(L^2)} + \left\| \frac{\partial \tau_n}{\partial t} \right\|_{L_T^2(L^2)} + \left\| \frac{\partial f_n}{\partial t} \right\|_{L_T^2(L^2)} \right) \leq C_{0,T}. \quad (3.40)$$

Step 7: Convergence of the sequence of approximate solution. The bounds obtained in the previous steps, namely (3.39) and (3.40) show that, at least up to extraction of a subsequence, we have the weak convergences

$$(u_n, \tau_n, f_n) \rightharpoonup (u, \tau, f) \text{ weakly-} \star \text{ in } L^\infty([0, T]; H^1)^3, \quad (3.41)$$

$$u_n \rightharpoonup u \text{ weakly in } L^2([0, T]; H^2), \quad (3.42)$$

$$\left(\frac{\partial u_n}{\partial t}, \frac{\partial \tau_n}{\partial t}, \frac{\partial f_n}{\partial t} \right) \rightharpoonup \left(\frac{\partial u}{\partial t}, \frac{\partial \tau}{\partial t}, \frac{\partial f}{\partial t} \right) \text{ weakly in } L^2([0, T]; L^2)^3. \quad (3.43)$$

But, in order to pass to the limit in (3.25), we need the convergence of the whole sequence itself because (3.25) involves indices $n - 1$ and n and strong convergence to establish convergence of the product terms $f_{n-1}f_n$, $|\tau_n|f_{n-1}f_n$, $f_{n-1}f_n^2$.

We now establish strong convergence of the whole sequence. We prove this convergence in $(L^\infty([0, T]; L^2(\Omega)))^3$. This will a posteriori imply that all the convergences (3.41), (3.42) and (3.43) actually hold true not only for a subsequence, but the whole sequence itself. And this will provide sufficient information to pass to the limit in our nonlinear terms.

We introduce the notation: $\tilde{h}_n = h_n - h_{n-1}$ and derive the evolution equations for $(\tilde{u}_n, \tilde{\tau}_n, \tilde{f}_n)$

$$\left\{ \begin{array}{l} \rho \frac{\partial \tilde{u}_n}{\partial t} = \eta \frac{\partial^2 \tilde{u}_n}{\partial y^2} + \frac{\partial \tilde{\tau}_n}{\partial y}, \\ \lambda \frac{\partial \tilde{\tau}_n}{\partial t} = G \frac{\partial \tilde{u}_n}{\partial y} - f_{n-1} \tilde{\tau}_n - \tau_{n-1} \tilde{f}_{n-1}, \\ \frac{\partial \tilde{f}_n}{\partial t} = (-1 + \xi |\tau_{n-1}|)(f_{n-1} \tilde{f}_n + f_{n-1} \tilde{f}_{n-1}) \\ \quad - \nu f_{n-1}(f_n + f_{n-1}) \tilde{f}_n - \nu f_{n-1}^2 \tilde{f}_{n-1} + \xi |\widetilde{\tau_n}| f_{n-1} f_n. \end{array} \right. \quad \begin{array}{l} (3.44a) \\ (3.44b) \\ (3.44c) \end{array}$$

Since (u_n, τ_n, f_n) belong to the spaces that appear in (3.3) and (3.4), the same holds for $(\tilde{u}_n, \tilde{\tau}_n, \tilde{f}_n)$. We multiply equations (3.44a), (3.44b) and (3.44c), respectively by \tilde{u}_n , $\tilde{\tau}_n$ and \tilde{f}_n , integrate over Ω , sum up and use the non-negativity of f_{n-1} and f_n to find

$$\begin{aligned} \frac{d}{dt} \left(G\rho \|\tilde{u}_n(t, \cdot)\|_{L^2}^2 + \lambda \|\tilde{\tau}_n(t, \cdot)\|_{L^2}^2 + \|\tilde{f}_n(t, \cdot)\|_{L^2}^2 \right) &\leq - \int_{\Omega} \tau_{n-1} \tilde{f}_{n-1} \tilde{\tau}_n(t, \cdot) \\ &+ \int_{\Omega} (-1 + \xi |\tau_{n-1}|)(f_{n-1} \tilde{f}_n^2 + f_{n-1} \tilde{f}_{n-1} \tilde{f}_n) - \nu f_{n-1}^2 \tilde{f}_{n-1} \tilde{f}_n + \xi f_{n-1} f_n |\widetilde{\tau_n}| \tilde{f}_n(t, \cdot). \end{aligned}$$

The presence of two indices $n - 1$ and n again makes an additional step necessary. We introduce $X_n(t) = \|\tilde{u}_n(t, \cdot)\|_{L^2}^2 + \|\tilde{\tau}_n(t, \cdot)\|_{L^2}^2 + \|\tilde{f}_n(t, \cdot)\|_{L^2}^2$. Repeatedly using the L^∞ -bounds (3.35) on $\{\tau_n, f_n, \tau_{n-1}, f_{n-1}\}$ and the Young inequality, we see that X_n satisfies

$$\dot{X}_n(t) \leq C_{0,T}(X_n(t) + X_{n-1}(t)). \quad (3.45)$$

Applying the Gronwall Lemma to (3.45), we find

$$X_n(t) \leq C_{0,T} \int_0^t X_{n-1}(s) e^{C_{0,T}(t-s)} ds \leq C_{0,T} e^{C_{0,T}T} \int_0^t X_{n-1}(s) ds,$$

which implies that

$$X_n(t) \leq \frac{(C_{0,T} e^{C_{0,T}T} t)^{n-1}}{(n-1)!} \sup_{s \in [0, T]} X_1(s).$$

The sequence (u_n, τ_n, f_n) is therefore a Cauchy sequence in $(L^\infty([0, T]; L^2(\Omega)))^3$. The sequence converges in this space.

Now that we have strong convergence of the whole sequence, we show how to pass to the limit in all the terms of (3.25), including the nonlinear ones. We only consider $|\tau_n|f_{n-1}f_n$.

The other terms can be treated using similar arguments. We use a classical compactness result (see *e.g.* Ref. [Lio69], Theorem 5.1, p. 58) to deduce from (3.41) and (3.43) that τ_n and f_n strongly converge respectively to τ and f in $L^2([0, T]; L^4)^3$. Moreover, f_{n-1} strongly converges to f in $L^\infty([0, T]; L^2)$. We thus have convergence for $|\tau_n|f_{n-1}f_n$ in $L^1([0, T]; L^1)$.

The triple (u, τ, f) thus satisfies system (3.25), at least in the weak sense. We now derive further regularity. We have

$$u \in L^2([0, T]; H^2) \text{ with } \frac{\partial u}{\partial t} \in L^2([0, T]; L^2),$$

and therefore, by interpolation (see *e.g.* Ref. [Tem79], Chapter 3, Lemma 1.2),

$$u \in C([0, T]; H^1) \cap L^2([0, T]; H^2).$$

Moreover, we have

$$\left(\frac{\partial \tau}{\partial t}, \frac{\partial f}{\partial t} \right) \in L^2([0, T]; L^2)^2$$

and, using the second a priori estimate (3.24),

$$\left(\frac{\partial}{\partial t} \frac{\partial \tau}{\partial y}, \frac{\partial}{\partial t} \frac{\partial f}{\partial y} \right) \in L^2([0, T]; L^2)^2,$$

so that,

$$(\tau, f) \in C([0, T]; H^1)^2.$$

We have obtained (3.3) and therefore (3.4), using system (3.5). The non-negativity of the fluidity is preserved, passing to the limit. This completes the existence proof.

Step 8: Uniqueness. Consider (u_1, τ_1, f_1) and (u_2, τ_2, f_2) satisfying (3.3) and solutions to system (3.5) supplied with the same initial condition $(u_0, \tau_0, f_0) \in H^1(\Omega)$. We introduce $(\tilde{u} = u_2 - u_1, \tilde{\tau} = \tau_2 - \tau_1, \tilde{f} = f_2 - f_1)$ which therefore satisfies

$$\begin{cases} \rho \frac{\partial \tilde{u}}{\partial t} = \eta \frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{\partial \tilde{\tau}}{\partial y}, & (3.46a) \end{cases}$$

$$\begin{cases} \lambda \frac{\partial \tilde{\tau}}{\partial t} = G \frac{\partial \tilde{u}}{\partial y} - f_2 \tilde{\tau} - \tau_1 \tilde{f}, & (3.46b) \end{cases}$$

$$\begin{cases} \frac{\partial \tilde{f}}{\partial t} = -(f_1 + f_2) \tilde{f} + \xi f_1^2 |\tilde{\tau}| + \xi |\tau_2| (f_1 + f_2) \tilde{f} - \nu (f_1^2 + f_1 f_2 + f_2^2) \tilde{f}, & (3.46c) \end{cases}$$

supplied with *homogeneous* boundary conditions and $(0, 0, 0)$ as initial data. Multiplying equations (3.46a), (3.46b) and (3.46c), respectively by \tilde{u} , $\tilde{\tau}$ and \tilde{f} , integrating over Ω , summing up, using the L^∞ -bounds established in Step 4 for terms involving τ_1, τ_2, f_1, f_2 and repeatedly applying the Young inequality, we find

$$\frac{1}{2} \frac{d}{dt} \left(\rho G \|\tilde{u}(t, \cdot)\|_{L^2}^2 + \lambda \|\tilde{\tau}(t, \cdot)\|_{L^2}^2 + \|\tilde{f}(t, \cdot)\|_{L^2}^2 \right) \leq C_{0,T} \left(\|\tilde{\tau}(t, \cdot)\|_{L^2}^2 + \|\tilde{f}(t, \cdot)\|_{L^2}^2 \right).$$

The Gronwall Lemma then implies uniqueness. This concludes the proof of Theorem 1. \square

3.3 Longtime behaviour for *homogeneous* boundary conditions

In this section, we study the longtime behaviour of system (3.1) supplied with *homogeneous* boundary conditions. We will show convergence to a steady state and establish a rate for this convergence. For *homogeneous* boundary conditions, the H^1 -steady states of (3.1) such that $f \geq 0$ are exactly the states $(u \equiv 0, \tau \equiv c, f \equiv 0)$, where c is a constant throughout the domain.

Indeed, such a steady state $(u_\infty, \tau_\infty, f_\infty)$ satisfies, combining equation (3.1a) integrated over the domain and (3.1b),

$$\tau_\infty \left(\frac{\eta}{G} f_\infty + 1 \right) = c, \quad (3.47)$$

where c is a constant over the domain. We now distinguish between two cases. Either $c = 0$, in which case $\tau_\infty \equiv 0$. The *homogeneous* boundary conditions on u and (3.1b) imply that $u_\infty \equiv 0$ and (3.1c) that $f_\infty \equiv 0$. Or $c \neq 0$ and it follows from (3.47) that τ_∞ is non-zero and has a constant sign and from (3.1b) that $\frac{\partial u_\infty}{\partial y}$ has a constant sign. Because of the *homogeneous* boundary conditions on the velocity, we obtain that $u_\infty \equiv 0$. Therefore, (3.1b) yields $f_\infty \tau_\infty = 0$ and $f_\infty \equiv 0$, because τ_∞ is non-zero in this case.

We will show that the longtime behaviour differs both in terms of steady state and rate of convergence, depending whether $f_0 \neq 0$ or $f_0 \equiv 0$. When $f_0 \neq 0$, a case studied in subsection 3.3.1, the solution (u, τ, f) converges to the steady state $(0, 0, 0)$ in the longtime and the rates of convergence are power-laws of the time. In the case $f_0 \equiv 0$, the fluidity f vanishes for all time, as easily seen on (3.1c). In subsection 3.3.2, we show that (u, τ, f) then converges to $(0, \bar{\tau}_0, 0)$ in the longtime at an exponential rate, $\bar{\tau}_0$ being the average of τ_0 over Ω . Evidently, the former case $f_0 \neq 0$ require more efforts than the latter case $f_0 \equiv 0$ where $f \equiv 0$ for all times.

3.3.1 Case $f_0 \neq 0$

In this subsection, we consider the case $f_0 \geq 0$, $f_0 \neq 0$. We first establish the convergence in the longtime.

Theorem 2 *Supply system (3.1) with homogeneous boundary conditions and initial conditions that satisfy (3.2) and $f_0 \neq 0$. The solution (u, τ, f) , the existence and uniqueness of which have been established in Theorem 1, converges to the steady state $(0, 0, 0)$ in $H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$ in the longtime:*

$$\|u(t, \cdot)\|_{H^1} + \|\tau(t, \cdot)\|_{L^\infty} + \|f(t, \cdot)\|_{L^\infty} \rightarrow 0.$$

Proof. The proof falls in three steps. In the first step, we establish a lower bound for the average of the fluidity f , which, in Step 2, is useful to prove convergence in the longtime in $L^2(\Omega)$. In the third step, we show convergence of $(u(t, \cdot), \tau(t, \cdot), f(t, \cdot))$ in $H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$.

In this section, C_0 denotes various constants that are independent from time, while C_i , $i=1, \dots, 4$ denote some fixed constants independent from time. These constants C_0 and C_i used to be denoted $C_{0,T}$ in the previous section. The subscript T is now omitted because, as explained in Remarks 1 and 2, the constants are independent from T for *homogeneous* boundary conditions.

Step 1: A lower bound for the average of f . We first derive a lower bound on \bar{f} , defined as in (3.8), and not directly on f because the latter may vanish (since f_0 may vanish) on some part of the domain. Since $f_0 \not\equiv 0$, there exists, by continuity of f_0 (assumed in H^1), a non-empty closed interval Ω_0 in Ω where f_0 does not vanish. Arguing as in Step 1 of the proof of Theorem 1, the fluidity f does not vanish for all $t > 0$ and $y \in \Omega_0$. The evolution equation (3.1c) on f rewrites, for all $t > 0$ and $y \in \Omega_0$,

$$\frac{\partial}{\partial t} \frac{1}{f} = 1 - \xi|\tau| + \nu f. \quad (3.48)$$

As explained in Remark 2, the L^∞ -bounds on τ and f are uniform in time for *homogeneous* boundary conditions. The equation (3.48) thus implies, for all $y \in \Omega_0$ and $t > 0$,

$$\frac{\partial}{\partial t} \frac{1}{f} \leq C_0,$$

and therefore,

$$\begin{aligned} f(t, y) &\geq \frac{1}{\frac{1}{f_0(y)} + C_0 t} \\ &\geq \frac{1}{\left\| \frac{1}{f_0} \right\|_{L^\infty(\Omega_0)} + C_0 t}. \end{aligned} \quad (3.49)$$

Since $\bar{f} \geq \int_{\Omega_0} f$, this yields the lower bound

$$\bar{f}(t) \geq \frac{C_1}{1 + C_0 t}. \quad (3.50)$$

Step 2: Longtime convergence in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. We now show the longtime convergence in L^2 . Estimates (3.9) and (3.16) respectively rewrite, for *homogeneous* boundary conditions,

$$\frac{1}{2} \frac{d}{dt} \left(G\rho \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2 \right) + G\eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| (\sqrt{f}\tau)(t, \cdot) \right\|_{L^2}^2 = 0 \quad (3.51)$$

and

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{2\rho} \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \\ &\leq C_2 \left(\|f(t, \cdot)\|_{L^1} \left\| (\sqrt{f}\tau)(t, \cdot) \right\|_{L^2}^2 + \|u(t, \cdot)\|_{L^2}^2 \right), \end{aligned} \quad (3.52)$$

where U is defined by (3.14). The evolution equation on $\bar{\tau}$ writes

$$\lambda \frac{d\bar{\tau}}{dt} + \bar{f}\bar{\tau} = -\overline{f(\tau - \bar{\tau})}. \quad (3.53)$$

We introduce the positive scalar ϵ , to be fixed later on. We use the Cauchy-Schwarz and Young inequalities

$$\begin{aligned} |\bar{\tau}| \left| \overline{f(\tau - \bar{\tau})} \right| &\leq \sqrt{\bar{f}} |\bar{\tau}| \left\| \left(\sqrt{f}(\tau - \bar{\tau}) \right) (t, \cdot) \right\|_{L^2} \\ &\leq \epsilon \bar{f} |\bar{\tau}|^2 + \frac{1}{4\epsilon} \left\| \left(\sqrt{f}(\tau - \bar{\tau}) \right) (t, \cdot) \right\|_{L^2}^2, \end{aligned}$$

so that, multiplying evolution equation (3.53) by $\bar{\tau}$, we obtain

$$\frac{\lambda}{2} \frac{d|\bar{\tau}|^2}{dt}(t) + (1 - \epsilon) \bar{f} |\bar{\tau}|^2(t) \leq \frac{1}{4\epsilon} \|f(t, \cdot)\|_{L^\infty} \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2. \quad (3.54)$$

The evolution equation on $\tau - \bar{\tau}$ reads

$$\lambda \frac{\partial}{\partial t}(\tau - \bar{\tau}) + \frac{G}{\eta}(\tau - \bar{\tau}) = -(f\tau - \overline{f\tau}) + G \frac{\partial U}{\partial y}. \quad (3.55)$$

Multiplying evolution equation (3.55) by $\tau - \bar{\tau}$, integrating over Ω and repeatedly using the Young inequality, we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\lambda \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2) + \frac{G}{2\eta} \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 \\ \leq C \left(\|f\tau(t, \cdot)\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right), \end{aligned}$$

so that, using the uniform in time L^∞ -bound on f ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\lambda \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2) + \frac{G}{2\eta} \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 \\ \leq C_3 \left(\|f(t, \cdot)\|_{L^\infty} \left\| \left(\sqrt{f}\tau \right) (t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right). \end{aligned} \quad (3.56)$$

We introduce some positive scalars m_1, m_2, m_3 and the energy function

$$\begin{aligned} E(t) = m_1 (G\rho \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2) + m_2 \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \\ + m_3 \lambda \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 + \lambda |\bar{\tau}(t)|^2, \end{aligned} \quad (3.57)$$

which therefore satisfy, combining (3.51), (3.52), (3.54) and (3.56),

$$\begin{aligned} \frac{1}{2} \frac{dE}{dt}(t) + (m_1 - C_2 m_2 \|f(t, \cdot)\|_{L^1} - C_3 m_3 \|f(t, \cdot)\|_{L^\infty}) \left\| \left(\sqrt{f}\tau \right) (t, \cdot) \right\|_{L^2}^2 \\ + (m_1 G\eta - C_2 m_2 C_p) \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left(\frac{\eta}{2\rho} m_2 - C_3 m_3 C_p \right) \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \\ + \left(\frac{G}{2\eta} m_3 - \frac{1}{4\epsilon} \|f(t, \cdot)\|_{L^\infty} \right) \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 + (1 - \epsilon) \bar{f} |\bar{\tau}|^2(t) \leq 0, \end{aligned} \quad (3.58)$$

where C_p is the Poincaré constant.

The coefficients m_1, m_2, m_3 are chosen sufficiently large so that, for all time $t > 0$, every term in the left-hand side of (3.58) is positive. The conditions

$$\begin{aligned} \frac{G}{2\eta} m_3 &> \frac{1}{4\epsilon} \sup_{t>0} \|f(t, \cdot)\|_{L^\infty}, \\ \frac{\eta}{2\rho} m_2 &> C_3 m_3 C_p, \\ m_1 &> \max \left(\frac{C_2 m_2 C_p}{G\eta}, C_2 m_2 \sup_{t>0} \|f(t, \cdot)\|_{L^1} + C_3 m_3 \sup_{t>0} \|f(t, \cdot)\|_{L^\infty} \right) \end{aligned}$$

are sufficient. Using in addition the lower bound (3.49) and the Poincaré inequality, (3.58) becomes

$$\begin{aligned} \frac{1}{2} \frac{dE}{dt}(t) + C_0 \left(\|u(t, \cdot)\|_{L^2}^2 + \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \\ + (1 - \epsilon) \frac{C_1}{1 + C_0 t} |\bar{\tau}|^2(t) \leq 0. \end{aligned}$$

Using the triangle inequality

$$\frac{\epsilon}{2m_1} m_1 \|\tau(t, \cdot)\|_{L^2}^2 \leq \epsilon |\bar{\tau}|^2(t) + \epsilon \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2, \quad (3.59)$$

we find, for t sufficiently large,

$$\frac{1}{2} \frac{dE}{dt} + \frac{1}{\lambda} \min \left(1 - 2\epsilon, \frac{\epsilon}{2m_1} \right) \frac{C_1}{1 + C_0 t} E \leq 0. \quad (3.60)$$

We take $\epsilon < \frac{1}{2}$ and apply the Gronwall Lemma to (3.60) to obtain that E goes to zero in the longtime limit. In particular, we have

$$\lim_{t \rightarrow \infty} \left(\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) = 0. \quad (3.61)$$

Step 3: Longtime convergence in $H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$. Combining (3.16) and (3.18), using the uniform in time L^∞ -bound on f and the Poincaré inequality on $\frac{\partial U}{\partial y}$, the spatial average of which is zero, we obtain

$$\begin{aligned} \frac{d}{dt} \left(\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \lambda |\tau(t, y)|^2 \right) + \left(\left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 + |\tau(t, y)|^2 \right) \\ \leq C_0 \left(\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right). \end{aligned}$$

We apply the Gronwall Lemma and use the convergence (3.61) that is uniform in space to derive

$$\lim_{t \rightarrow \infty} \|\tau(t, \cdot)\|_{L^\infty} = 0.$$

Using the convergence of $\|\tau(t, \cdot)\|_{L^\infty}$, the evolution equation (3.1c) on f implies, for t sufficiently large,

$$\frac{\partial f}{\partial t} \leq -\frac{1}{2} f^2. \quad (3.62)$$

This yields the convergence of $\|f(t, \cdot)\|_{L^\infty}$ to zero in the longtime.

Additionally, using the definition (3.14) and (3.61), $\frac{\partial u}{\partial y}$ converges to zero in $L^2(\Omega)$. This ends the proof. \square

We now turn to making precise the rates of convergence to the steady-state. We introduce the non-negative scalar

$$\beta = \text{meas} \{y \in \Omega | f_0(y) > 0\}. \quad (3.63)$$

By assumption in this section, we have $\beta > 0$. The following result establishes the convergence rates in function of β . In Section 3.5.1, we will check using numerical simulations that these rates are indeed sharp.

Theorem 3 *Supply system (3.1) with homogeneous boundary conditions and initial conditions that satisfy (3.2) and $f_0 \not\equiv 0$. The solution (u, τ, f) , the existence and uniqueness of which have been established in Theorem 1, satisfies the following convergence estimates: for any arbitrarily small $\alpha > 0$, there exists a constant κ_α independent from time and there exists a time t_0 , both depending on the domain, the initial data, the coefficients in the system and α , such that, for all $t > t_0$,*

$$\|u(t, \cdot)\|_{H^1} + \|\tau(t, \cdot)\|_{L^2} \leq \kappa_\alpha (1+t)^{-\frac{\beta}{\lambda}(1-\alpha)}, \quad (3.64)$$

where β is defined by (3.63) and for all $t > t_0$ and $y \in \Omega$, we have

$$\frac{1}{\frac{1}{f(t_0, y)} + (1+\alpha)(t-t_0)} \leq f(t, y) \leq \frac{1}{\frac{1}{f(t_0, y)} + (1-\alpha)(t-t_0)}. \quad (3.65)$$

In addition, there exists another constant κ_α , such that, for all $t > t_0$,

$$\|u(t, \cdot)\|_{H^1} + \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2} \leq \kappa_\alpha (1+t)^{-1-\frac{\beta}{\lambda}(1-\alpha)}, \quad (3.66)$$

$$\left\| \left(\eta \frac{\partial u}{\partial y} + \tau - \bar{\tau} \right) (t, \cdot) \right\|_{L^2} \leq \kappa_\alpha (1+t)^{-2-\frac{\beta}{\lambda}(1-\alpha)}, \quad (3.67)$$

where the function $\bar{\tau}$ is the spatial average of τ , defined as in (3.8).

Proof. The proof falls in four steps. We first consider the fluidity, then derive first convergence rates for the velocity and the stress. A study of the auxiliary function defined by (3.14) next allows to conclude on the convergence estimates (3.66) and (3.67).

We fix ϵ an arbitrarily small positive scalar, actually equal to $\frac{\alpha}{4}$, where α is the constant that appears in the statement of the Theorem. The constants κ_ϵ depend on ϵ and have value that may vary from one instance to another, the actual value being irrelevant.

Step 1: Convergence rate for the fluidity. In view of Theorem 2, $\|\tau(t, \cdot)\|_{L^\infty}$ and $\|f(t, \cdot)\|_{L^\infty}$ vanish in the longtime. This implies that there exists a time t_0 , such that, for all $y \in \Omega$ and $t > t_0$, the evolution equation (3.1c) on f leads to

$$-(1 + \epsilon)f^2(t, y) \leq \frac{\partial f}{\partial t}(t, y) \leq -(1 - \epsilon)f^2(t, y). \quad (3.68)$$

For all $y \in \Omega$ such that $f_0(y) > 0$, we have $f(t_0, y) > 0$, as shown in Step 1 of the proof of Theorem 1. The equation (3.68) becomes, for such y and $t > t_0$,

$$\frac{1}{\frac{1}{f(t_0, y)} + (1 + \epsilon)(t - t_0)} \leq f(t, y) \leq \frac{1}{\frac{1}{f(t_0, y)} + (1 - \epsilon)(t - t_0)}. \quad (3.69)$$

and hence (3.65). This inequality is also valid for all y such that $f_0(y) = 0$, that is $f(t_0, y) = 0$, and therefore for all $y \in \Omega$.

Step 2: First convergence rates for the velocity and the stress. We first make more precise the lower bound on \bar{f} . As f_0 is continuous, there exists a closed set Ω_ϵ such that $f_0(\Omega_\epsilon) > 0$ and $\text{meas}\{\Omega_\epsilon\} = \beta(1 - \epsilon)$. As shown in Step 1 of the proof of Theorem 1, we also have $f(t_0, \Omega_\epsilon) > 0$. Furthermore, as $f(t_0, \cdot)$ is continuous, we obtain $f(t_0, \Omega_\epsilon) > \kappa_\epsilon$. The inequality (3.69) thus becomes, for all $y \in \Omega_\epsilon$ and $t > t_0$,

$$f(t, y) > \frac{1}{1 + \epsilon} \frac{1}{\kappa_\epsilon + t}.$$

It follows from $\bar{f} \geq \int_{\Omega_\epsilon} f$ that, for all $t > t_0$,

$$\bar{f}(t) \geq \beta \frac{1 - \epsilon}{1 + \epsilon} \frac{1}{\kappa_\epsilon + t}. \quad (3.70)$$

We now use the energy E introduced in (3.57). As $\|f(t, \cdot)\|_{L^\infty}$ vanishes in the longtime, the coefficients m_1, m_2, m_3 can be chosen arbitrarily small in (3.58), independently from ϵ , for $t > t_0$ sufficiently large. We insert (3.70) in (3.58) so that, for sufficiently large t ,

$$\begin{aligned} \frac{1}{2} \frac{dE}{dt}(t) + C_0 \left(\|u(t, \cdot)\|_{L^2}^2 + \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \\ + (1 - \epsilon) \beta \frac{1 - \epsilon}{1 + \epsilon} \frac{1}{\kappa_\epsilon + t} |\bar{\tau}|^2(t) \leq 0. \end{aligned}$$

Using the triangle inequality (3.59), we obtain, for sufficiently large t ,

$$\frac{1}{2} \frac{dE}{dt} + \frac{\beta}{\lambda} \min \left(1 - 2\epsilon, \frac{\epsilon}{2m_1} \right) \frac{1 - \epsilon}{1 + \epsilon} \frac{1}{\kappa_\epsilon + t} E \leq 0.$$

and therefore, using that $1 - 2\epsilon < \frac{\epsilon}{2m_1}$ as m_1 is arbitrarily small,

$$\frac{1}{2} \frac{dE}{dt} + \frac{\beta}{\lambda} (1 - 4\epsilon) \frac{1}{\kappa_\epsilon + t} E \leq 0. \quad (3.71)$$

Applying the Gronwall Lemma to (3.71), we find,

$$\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 + \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1 + t)^{-2\frac{\beta}{\lambda}(1 - 4\epsilon)}. \quad (3.72)$$

where we recall that κ_ϵ denotes various constants. We have obtained (3.64).

Step 3: Convergence rate for the auxiliary function U . We recall that the function U is defined by (3.14). We first prove that U is more regular than claimed in (3.17). We rewrite (3.15)

$$\frac{\partial U}{\partial t} - \frac{\eta}{\rho} \frac{\partial^2 U}{\partial y^2} = -\frac{1}{\lambda\eta} \int_0^y (f\tau - \overline{f\tau}) + \frac{G}{\lambda\eta} u. \quad (3.73)$$

We deduce that $\frac{\partial U}{\partial y}$ satisfies the heat equation with a right-hand side in $L^2_{loc}((t_0, +\infty), L^2)$ and initial condition $\frac{\partial U}{\partial y}(t_0, \cdot) \in H^1(\Omega)$ at time t_0 (up to a possible modification on a set of times of measure zero). Therefore, we have $\frac{\partial U}{\partial y} \in H^1_{loc}((t_0, +\infty), L^2)$, so that

$$U \in H^1_{loc}((t_0, +\infty), H^1_0). \quad (3.74)$$

We next differentiate (3.15) with respect to t , insert (3.1a) and find,

$$\frac{\partial^2 U}{\partial t^2} - \frac{\eta}{\rho} \frac{\partial^2}{\partial y^2} \left(\frac{\partial U}{\partial t} \right) - \frac{G}{\lambda\rho} \frac{\partial^2 U}{\partial y^2} = I, \quad (3.75)$$

where I is the function defined by

$$I(t, y) = -\frac{1}{\lambda\eta} \int_0^y \left(\frac{\partial f\tau}{\partial t} - \frac{\partial \overline{f\tau}}{\partial t} \right). \quad (3.76)$$

We now regularize I as follows. We consider a sequence of functions I_m such that for all m , I_m is infinitely differentiable from $(t_0, +\infty)$ to $L^2(\Omega)$ and as $m \rightarrow \infty$,

$$I_m \rightarrow I \text{ in } L^2_{loc}((t_0, +\infty), L^2). \quad (3.77)$$

Consider a solution $U_m \in C^\infty((t_0, +\infty), H^2 \cap H^1_0)$ to

$$\frac{\partial^2 U_m}{\partial t^2} - \frac{\eta}{\rho} \frac{\partial^2}{\partial y^2} \left(\frac{\partial U_m}{\partial t} \right) - \frac{G}{\lambda\rho} \frac{\partial^2 U_m}{\partial y^2} = I_m. \quad (3.78)$$

Equation (3.78) has been studied in Ref. [HX10] and [KS92]. Inspired by arguments from these references, we introduce the energy functions H_m and F_m depending on a constant $\delta \in (0, 1)$ to be determined later

$$\begin{aligned} H_m(t) &= \left\| \frac{\partial U_m}{\partial t}(t, \cdot) \right\|_{L^2}^2 + \frac{G}{\lambda\rho} \left\| \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{\rho} \delta \left\| \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 \\ &+ \delta \int_\Omega \left(\frac{\partial U_m}{\partial t} U_m \right) (t, \cdot) + 2 \frac{\eta}{\rho} \delta^2 \left\| \frac{\partial}{\partial t} \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 \\ &+ 2 \frac{G}{\lambda\rho} \delta^2 \int_\Omega \left(\frac{\partial}{\partial t} \left(\frac{\partial U_m}{\partial y} \right) \frac{\partial U_m}{\partial y} \right) (t, \cdot) \end{aligned}$$

and

$$\begin{aligned} F_m(t) &= \left(\frac{\eta}{\rho} - \delta^2 \frac{G}{\lambda\rho} \right) \left\| \frac{\partial}{\partial t} \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 - \delta \left\| \frac{\partial U_m}{\partial t}(t, \cdot) \right\|_{L^2}^2 \\ &+ \delta \frac{G}{\lambda\rho} \left\| \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \delta^2 \left\| \frac{\partial^2 U_m}{\partial t^2}(t, \cdot) \right\|_{L^2}^2. \end{aligned}$$

We multiply (3.78) by $\frac{\partial U_m}{\partial t} + \delta U_m + \delta^2 \frac{\partial^2 U_m}{\partial t^2}$, integrate over Ω and find

$$\frac{1}{2} \frac{dH_m}{dt}(t) + F_m(t) = \int_{\Omega} I_m(t, y) \left(\frac{\partial U_m}{\partial t} + \delta U_m + \delta^2 \frac{\partial^2 U_m}{\partial t^2} \right) (t, y) dy.$$

We use the Poincaré inequality and choose δ sufficiently small, depending on the domain and the coefficients in (3.1) such that, for suitable constants c_1 , c_2 and c_3 ,

$$c_1 \left(\left\| \frac{\partial}{\partial t} \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \leq H_m(t) \leq c_2 \left(\left\| \frac{\partial}{\partial t} \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \quad (3.79)$$

and

$$F_m(t) \geq c_3 \left(\left\| \frac{\partial}{\partial t} \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial U_m}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial^2 U_m}{\partial t^2}(t, \cdot) \right\|_{L^2}^2 \right). \quad (3.80)$$

Using the upper bound in (3.79) and (3.80) and the Young and the Poincaré inequalities, we obtain

$$\frac{dH_m}{dt}(t) + CH_m(t) \leq \|I_m(t, \cdot)\|_{L^2}^2. \quad (3.81)$$

We multiply the above equation by e^{Ct} , integrate from t_0 to t and find

$$H_m(t)e^{Ct} \leq H_m(t_0)e^{Ct_0} + \int_{t_0}^t \|I_m(s, \cdot)\|_{L^2}^2 e^{Cs} ds. \quad (3.82)$$

Equation (3.75) is linear so that by (3.74) and (3.77), we can pass to the limit $m \rightarrow \infty$ in (3.82) and find, for all $t > t_0$,

$$H(t)e^{Ct} \leq H(t_0)e^{Ct_0} + \int_{t_0}^t \|I(s, \cdot)\|_{L^2}^2 e^{Cs} ds. \quad (3.83)$$

where H is defined as H_m with U instead of U_m .

The study of (3.75) reduces to the understanding of (3.83). We now make precise the behaviour of I or more precisely at the one of $\frac{\partial f\tau}{\partial t}$. We combine equations (3.1b) and (3.1c) to find

$$\frac{\partial f\tau}{\partial t} = f \left(-\frac{1}{\lambda} f\tau + \frac{G}{\lambda} \frac{\partial u}{\partial y} \right) + \tau \left(-f^2 - \nu f^3 + \xi|\tau|f^2 \right). \quad (3.84)$$

Multiplying the evolution equation (3.84) by $f\tau$ and integrating over Ω yields

$$\begin{aligned} \frac{1}{2} \left\| \frac{\partial f\tau}{\partial t}(t, \cdot) \right\|_{L^2}^2 &= \int_{\Omega} \left(-\frac{1}{\lambda} - 1 - \nu f + \xi|\tau| \right) f^3 \tau^2 + \frac{G}{\lambda} \int_{\Omega} f^2 \tau \frac{\partial u}{\partial y} \\ &\leq C_0 \|f(t, \cdot)\|_{L^\infty}^2 \left(\left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 \right), \end{aligned} \quad (3.85)$$

where we have used the L^∞ -bounds on both τ and f and the Cauchy-Schwarz inequality to derive the second line. Inserting (3.69) which gives the convergence in $\frac{1}{t}$ of $\|f(t, \cdot)\|_{L^\infty}$ and (3.72), equation (3.85) implies

$$\left\| \frac{\partial f \tau}{\partial t}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1+t)^{-2-2\frac{\beta}{\lambda}(1-4\epsilon)}. \quad (3.86)$$

Since the L^2 -norm of $\frac{\partial f \tau}{\partial t}$ controls the L^2 -norm of I , we insert (3.86) in (3.83) so that, for all $t > t_0$,

$$H(t)e^{Ct} \leq H(t_0)e^{Ct_0} + \kappa_\epsilon \int_{t_0}^t \frac{e^{Cs}}{(1+s)^{2+2\frac{\beta}{\lambda}(1-4\epsilon)}} ds. \quad (3.87)$$

Moreover, for $q > 0$, for all $t > t_0$, we integrate by parts to obtain

$$\int_{t_0}^t \frac{e^{Cs}}{(1+s)^q} ds \leq \frac{q}{C(1+t_0)} \int_{t_0}^t \frac{e^{Cs}}{(1+s)^q} ds + \frac{e^{Ct}}{C(1+t)^q}. \quad (3.88)$$

We insert (3.88) with $q = 2 + 2\frac{\beta}{\lambda}(1-4\epsilon)$ in (3.87), so that for t sufficiently large

$$H(t) \leq \kappa_\epsilon \frac{1}{(1+t)^{2+2\frac{\beta}{\lambda}(1-4\epsilon)}}. \quad (3.89)$$

Using the lower bound in (3.79), we have therefore obtained

$$\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1+t)^{-2-2\frac{\beta}{\lambda}(1-4\epsilon)}. \quad (3.90)$$

Step 4: Convergence rates (3.66) and (3.67). Using (3.56) rewritten as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\lambda \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2) + \frac{G}{2\eta} \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 \\ & \leq C \left(\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \|f(t, \cdot)\|_{L^\infty}^2 \|\tau(t, \cdot)\|_{L^2}^2 \right), \end{aligned}$$

and convergence estimates (3.69), (3.72), (3.90), we obtain

$$\|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}^2 \leq \kappa_\epsilon (1+t)^{-2-2\frac{\beta}{\lambda}(1-4\epsilon)},$$

and eventually

$$\left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1+t)^{-2-2\frac{\beta}{\lambda}(1-4\epsilon)}.$$

We thus obtain (3.66) with $\alpha = 4\epsilon$ and conclude establishing (3.67) as follows: we return to (3.85) and improve the convergence estimate for $\frac{\partial f \tau}{\partial t}$, namely

$$\left\| \frac{\partial f \tau}{\partial t}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1+t)^{-4-2\frac{\beta}{\lambda}(1-4\epsilon)}.$$

This implies, mimicking (3.87) and using (3.88) with $q = 4 + 2\frac{\beta}{\lambda}(1-4\epsilon)$, that for t sufficiently large,

$$\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \leq \kappa_\epsilon (1+t)^{-4-2\frac{\beta}{\lambda}(1-4\epsilon)}, \quad (3.91)$$

that is (3.67) with $\alpha = 4\epsilon$. \square

3.3.2 Case $f_0 \equiv 0$

In the case $f_0 \equiv 0$, f vanishes for all time. System (3.1) then reads

$$\begin{cases} \rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \\ \lambda \frac{\partial \tau}{\partial t} = G \frac{\partial u}{\partial y}. \end{cases} \quad (3.92a)$$

$$\quad (3.92b)$$

The existence and uniqueness of a regular solution to (3.92) is easy to establish. The longtime behaviour of system (3.92) is now made precise.

Theorem 4 *Supply system (3.92) with homogeneous boundary conditions. Consider a solution (u, τ) in the space*

$$\left(C([0, +\infty); H^1) \cap L^2_{loc}([0, +\infty); H^2) \right) \times C([0, +\infty); H^1)$$

Then, the solution converges exponentially fast to the steady state $(0, \bar{\tau}_0)$ in $H^1(\Omega) \times L^2(\Omega)$ in the longtime: there exist two constants C , independent from time and initial data, and C_0 , independent from time, such that, for t sufficiently large ,

$$\left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2} + \|\tau(t, \cdot) - \bar{\tau}_0\|_{L^2} \leq C_0 e^{-Ct}. \quad (3.93)$$

Proof. We perform the same manipulations as those used to obtain equation (3.75) in Step 3 of the proof of Theorem 3. Since we deal here with the case $f \equiv 0$, we have $I = 0$ in (3.75). We have proven that studying the longtime behaviour to (3.75) amounts to proving (3.83). We therefore find, for t sufficiently large,

$$\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2} \leq C_0 e^{-Ct}.$$

We next differentiate equation (3.92a) with respect to t and insert (3.92b) to obtain

$$\frac{\partial^2 u}{\partial t^2} - \frac{\eta}{\rho} \frac{\partial^2}{\partial y^2} \left(\frac{\partial u}{\partial t} \right) - \frac{G}{\rho \lambda} \frac{\partial^2 u}{\partial y^2} = 0.$$

The function u satisfies the same equation as U and thus has the same convergence rate. Applying the Gronwall Lemma to (3.55) therefore implies, for t sufficiently large,

$$\|(\tau - \bar{\tau})(t, \cdot)\|_{L^2} \leq C_0 e^{-Ct}. \quad (3.94)$$

Integrating (3.92b) over Ω , we have

$$\lambda \frac{d}{dt} \bar{\tau} = 0,$$

so that $\bar{\tau}(t) = \bar{\tau}_0$ for all times. We thus have the convergence estimate (3.93). \square

3.4 Longtime behaviour for non-homogeneous boundary conditions in a simple case

In this section, we study the longtime behaviour of the system (3.1) supplied with *non-homogeneous* boundary conditions $u(t, 0) = 0$ and $u(t, 1) = a$ (where a is a constant scalar different from zero and chosen positive, without loss of generality, $a > 0$).

We denote $(u_\infty, \tau_\infty, f_\infty)$ a stationary state to the system (3.1). We only consider the simplified case

$$f_\infty > 0 \text{ everywhere.} \quad (3.95)$$

The only stationary state that satisfies (3.95) is made explicit in subsection 3.4.1. In subsection 3.4.2, we show convergence in the longtime to this stationary state for small initial perturbations. In subsection 3.4.3, we study the longtime behaviour for initial data that satisfy $f_0 > 0$ without any smallness conditions, but only in a simplified case that reduces system (3.1) to a system of ordinary differential equations.

We do not state any result for the convergence to stationary states when fluidity vanishes on some part of Ω .

3.4.1 Stationary state

The following lemma makes precise the stationary state that satisfies the condition (3.95).

Lemma 1 (Stationary state) *Supply system (3.1) with non-homogeneous boundary conditions $u_\infty(0) = 0$ and $u_\infty(1) = a > 0$. The unique stationary solution $(u_\infty, \tau_\infty, f_\infty)$ in $(H^1(\Omega))^3$ satisfying (3.95) reads*

$$(u_\infty, \tau_\infty, f_\infty)(y) = \left(ay, \frac{\sqrt{1 + 4\nu\xi Ga} + 1}{2\xi}, \frac{\sqrt{1 + 4\nu\xi Ga} - 1}{2\nu} \right). \quad (3.96)$$

Remark 3 *It is easy to extend the above result to stationary solutions $(u_\infty, \tau_\infty, f_\infty)$ in $H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$ that satisfy $f_\infty \not\equiv 0$. Introducing $\Omega_\infty = \{y \in \Omega, f_\infty(y) > 0\}$ and $\beta_\infty = \text{meas}(\Omega_\infty)$, the set of such stationary solutions reads*

$$\left(\frac{\partial u_\infty}{\partial y}, \tau_\infty, f_\infty \right) (y) = \begin{cases} \left(\frac{a}{\beta_\infty}, \tau_L, \frac{-1 + \xi\tau_L}{\nu} \right) & \text{on } \Omega_\infty \\ \left(0, \eta \frac{a}{\beta_\infty} + \tau_L, 0 \right) & \text{on } \Omega \setminus \Omega_\infty, \end{cases}$$

with $\tau_L = \frac{1}{2\xi} \left(1 + \sqrt{1 + 4\nu\xi Ga/\beta_\infty} \right)$.

Proof. The stationary states $(u_\infty, \tau_\infty, f_\infty) : \Omega \rightarrow \mathbb{R}$ of the system (3.1) that satisfy (3.95) are solutions of the following system

$$\begin{cases} 0 = \eta \frac{\partial^2 u_\infty}{\partial y^2} + \frac{\partial \tau_\infty}{\partial y}, & (3.97a) \\ 0 = G \frac{\partial u_\infty}{\partial y} - f_\infty \tau_\infty, & (3.97b) \\ f_\infty = \frac{-1 + \xi |\tau_\infty|}{\nu}. & (3.97c) \end{cases}$$

We now show that such a steady state is unique and explicitly identify it. Since $\tau_\infty \in H^1(\Omega)$, (3.97a) shows that u_∞ belongs to $H^2(\Omega)$. We integrate (3.97a) and (3.97b) over Ω and obtain

$$K = \eta \frac{\partial u_\infty}{\partial y} + \tau_\infty, \quad (3.98)$$

where K is a constant and, using the boundary conditions on u_∞ ,

$$\int_\Omega f_\infty \tau_\infty = Ga. \quad (3.99)$$

We combine (3.97b) and (3.98) to obtain

$$\left(\frac{\eta}{G} f_\infty + 1 \right) \tau_\infty = K \quad (3.100)$$

so that τ_∞ has the constant sign of K . Equation (3.99) then implies that τ_∞ , thus K are positive. We now claim that τ_∞ is constant over Ω : inserting (3.97c) in (3.100), we obtain that τ_∞ satisfies

$$\tau_\infty \left(1 + \eta \frac{-1 + \xi \tau_\infty}{G\nu} \right) = K.$$

It is easy to see that this equation has a unique positive solution τ_∞ . It follows from (3.98) that $\frac{\partial u_\infty}{\partial y}$ is constant throughout Ω so that, using the boundary conditions, $u_\infty(y) = ay$. We rewrite equation (3.97b) as

$$Ga = \frac{-1 + \xi \tau_\infty}{\nu} \tau_\infty,$$

to find the value of

$$\tau_\infty = \frac{1}{2\xi} (1 + \sqrt{1 + 4\nu\xi Ga}).$$

The stationary state reads $\left(ay, \tau_\infty, \frac{-1 + \xi \tau_\infty}{\nu} \right)$, that is (3.96). \square

3.4.2 Longtime behaviour with smallness assumption

The following theorem states the convergence in the longtime to the stationary state (3.96) for small initial perturbations.

Theorem 5 *Supply system (3.1) with non-homogeneous boundary conditions $u(t, 0) = 0$ and $u(t, 1) = a > 0$. Consider the solution (u, τ, f) the existence and uniqueness of which have been established in Theorem 1 and the associated stationary state $(u_\infty, \tau_\infty, f_\infty)$ defined by (3.96). There exists $\epsilon > 0$ (sufficiently small so that at least τ_0 and f_0 are positive), such that, if the initial data (u_0, τ_0, f_0) for (3.1) satisfy*

$$\|u_0 - u_\infty\|_{H^1}^2 + \|\tau_0 - \tau_\infty\|_{L^\infty}^2 + \|f_0 - f_\infty\|_{L^\infty}^2 \leq \epsilon^2,$$

then the solution (u, τ, f) of system (3.1) converges, as t goes to infinity, to $(u_\infty, \tau_\infty, f_\infty)$ in $H^1(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$.

More precisely, there exist a constant C independent from ϵ , time and initial data and a constant κ_ϵ independent from time such that, for t sufficiently large,

$$\|u(t, \cdot) - u_\infty(\cdot)\|_{H^1} + \|\tau(t, \cdot) - \tau_\infty(\cdot)\|_{L^\infty} + \|f(t, \cdot) - f_\infty(\cdot)\|_{L^\infty} \leq \kappa_\epsilon e^{-(C-\epsilon)t}. \quad (3.101)$$

Remark 4 *It is indeed possible, under the same assumptions, to prove that both τ and f converge to zero in $H^1(\Omega)$ and not only in $L^\infty(\Omega)$. The proof is more tedious, see Section 3.A.*

Before we get to the proof, we note that we will return to system (3.1), and not (3.5) since of course boundary conditions will play a crucial role throughout the section. We also rewrite system (3.1) as

$$\begin{cases} \rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2}{\partial y^2} ((u - u_\infty) + u_\infty) + \frac{\partial}{\partial y} ((\tau - \tau_\infty) + \tau_\infty), \\ \lambda \frac{\partial \tau}{\partial t} = G \frac{\partial}{\partial y} ((u - u_\infty) + u_\infty) - ((f - f_\infty) + f_\infty) ((\tau - \tau_\infty) + \tau_\infty), \\ \frac{\partial f}{\partial t} = (-1 + \xi |(\tau - \tau_\infty) + \tau_\infty|) ((f - f_\infty) + f_\infty)^2 - \nu ((f - f_\infty) + f_\infty)^3. \end{cases}$$

To lighten the notation, we henceforth denote (u, τ, f) instead of $(u - u_\infty, \tau - \tau_\infty, f - f_\infty)$ and consider

$$\begin{cases} \rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial y^2} + \frac{\partial \tau}{\partial y}, \end{cases} \quad (3.102a)$$

$$\begin{cases} \lambda \frac{\partial \tau}{\partial t} = G \frac{\partial u}{\partial y} - f_\infty \tau - \tau_\infty f - f \tau, \end{cases} \quad (3.102b)$$

$$\begin{cases} \frac{\partial f}{\partial t} = -\nu (f + f_\infty)^2 f + \xi (f + 2f_\infty) f \tau + \xi f_\infty^2 \tau, \end{cases} \quad (3.102c)$$

supplied with *homogeneous* boundary conditions on u and initial data that satisfy

$$\|u_0\|_{H^1}^2 + \|\tau_0\|_{L^\infty}^2 + \|f_0\|_{L^\infty}^2 \leq \epsilon^2.$$

Proof. The proof is divided into three steps. The first step establishes a priori estimates on system (3.102). In the second step, we show that the solution remains small for sufficiently small perturbations. In Step 3, we show that, still for small perturbations, the solution converges to the steady state and that the rate of convergence is exponential.

As in the previous proofs, C and κ_ϵ denote various constants the value of which may vary from one instance to another, the actual value being irrelevant.

Step 1: A priori energy estimates. We argue as in Step 2 of the proof of Theorem 1. We multiply (3.102a), (3.102b) and (3.102c) respectively by u , τ and f , integrate over Ω and find

$$\begin{aligned} -\frac{\rho}{2} \frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 + \eta \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 &= \int_{\Omega} u \frac{\partial \tau}{\partial y}(t, \cdot), \\ \frac{\lambda}{2} \frac{d}{dt} \|\tau(t, \cdot)\|_{L^2}^2 + \int_{\Omega} (f + f_{\infty}) \tau^2(t, \cdot) &= G \int_{\Omega} \tau \frac{\partial u}{\partial y}(t, \cdot) - \tau_{\infty} \int_{\Omega} f \tau(t, \cdot), \\ \frac{1}{2} \frac{d}{dt} \|f(t, \cdot)\|_{L^2}^2 + \nu \int_{\Omega} (f + f_{\infty})^2 f^2(t, \cdot) &= \xi f_{\infty}^2 \int_{\Omega} f \tau(t, \cdot) + \xi \int_{\Omega} (f + 2f_{\infty}) f^2 \tau(t, \cdot). \end{aligned}$$

Combining these estimates leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\rho G \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2 + \frac{\tau_{\infty}}{\xi f_{\infty}^2} \|f(t, \cdot)\|_{L^2}^2 \right) \\ + \eta G \left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + f_{\infty} \|\tau(t, \cdot)\|_{L^2}^2 + \frac{\tau_{\infty} \nu}{\xi} \|f(t, \cdot)\|_{L^2}^2 \\ - C \|f\|_{L_T^{\infty}(L^{\infty})} (\|f\|_{L_T^{\infty}(L^{\infty})} + 2f_{\infty}) \left(\|\tau(t, \cdot)\|_{L^2}^2 + \|f(t, \cdot)\|_{L^2}^2 \right) \leq 0, \end{aligned} \quad (3.103)$$

using the L^{∞} -estimate on $[0, T]$ on f established in the proof of Theorem 1 and the Young inequality. We now use Step 3 of the proof of Theorem 1 and more precisely the estimate (3.16) on U defined by (3.14). We have

$$\frac{d}{dt} \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\eta}{\rho} \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \leq C \left(\|u(t, \cdot)\|_{L^2}^2 + \|f\|_{L_T^{\infty}(L^{\infty})}^2 \|\tau(t, \cdot)\|_{L^2}^2 \right). \quad (3.104)$$

Now that we have estimates in Sobolev spaces, we turn to point wise estimates on τ and f . We refine our argument in Step 4 of the proof of Theorem 1. We rewrite the evolution equation (3.102b) as

$$\lambda \frac{\partial \tau}{\partial t} + \left(f + f_{\infty} + \frac{G}{\eta} \right) \tau = G \frac{\partial U}{\partial y} + \frac{G}{\eta} \bar{\tau} - \tau_{\infty} f,$$

multiply it by τ , apply the Young inequality and obtain

$$\frac{\lambda}{2} \frac{d}{dt} |\tau|^2 + \left(f + f_{\infty} + \frac{G}{2\eta} \right) |\tau|^2 \leq \eta \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^{\infty}}^2 + \|\tau(t, \cdot)\|_{L^2}^2 - \tau_{\infty} f \tau. \quad (3.105)$$

Similarly, we multiply (3.102c) by f and find

$$\frac{1}{2} \frac{d}{dt} |f|^2 + \nu (f + f_{\infty})^2 |f|^2 = \xi (f + 2f_{\infty}) \tau |f|^2 + \xi f_{\infty}^2 f \tau. \quad (3.106)$$

We combine (3.105) and (3.106) and use the Poincaré inequality on $\frac{\partial U}{\partial y}$, the spatial average of which is zero, to obtain

$$\begin{aligned} \frac{\lambda}{2} \frac{d}{dt} |\tau|^2 + \frac{\tau_{\infty}}{2\xi f_{\infty}^2} \frac{d}{dt} |f|^2 + \left(f + f_{\infty} + \frac{G}{2\eta} \right) |\tau|^2 \\ + \frac{\tau_{\infty}}{\xi f_{\infty}^2} \left(\nu (f + f_{\infty})^2 - \xi (\|f\|_{L_T^{\infty}(L^{\infty})} + 2f_{\infty}) \|\tau\|_{L_T^{\infty}(L^{\infty})} \right) |f|^2 \\ \leq \eta \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2. \end{aligned} \quad (3.107)$$

Step 2: Smallness of the solution for small perturbations. We now prove that, for $\epsilon \in (0, 1)$ to be fixed later on and an initial condition satisfying

$$\|u_0\|_{H^1}^2 + \|\tau_0\|_{L^\infty}^2 + \|f_0\|_{L^\infty}^2 \leq \epsilon^2, \quad (3.108)$$

we have, for all time $t > 0$,

$$\|u(t, \cdot)\|_{H^1}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^\infty}^2 \leq \epsilon. \quad (3.109)$$

We argue by contradiction and suppose

$$T_M = \inf \left\{ t \in \mathbb{R}^+ \mid \left(\|u(t, \cdot)\|_{H^1}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^\infty}^2 \right) \geq \epsilon \right\} \text{ is finite.}$$

For all $t \leq T_M$, we use the estimates from the previous step. For ϵ sufficiently small such that all the terms in the left-hand side of (3.103) are positive (this gives one condition on ϵ), we have, integrating (3.103) from 0 to t ,

$$\begin{aligned} & \rho G \|u(t, \cdot)\|_{L^2}^2 + \lambda \|\tau(t, \cdot)\|_{L^2}^2 + \frac{\tau_\infty}{\xi f_\infty^2} \|f(t, \cdot)\|_{L^2}^2 \\ & + \int_0^t \left(\eta G \left\| \frac{\partial u}{\partial y}(s, \cdot) \right\|_{L^2}^2 + \frac{f_\infty}{2} \|\tau(s, \cdot)\|_{L^2}^2 + \frac{\tau_\infty \nu}{\xi} \|f(s, \cdot)\|_{L^2}^2 \right) ds \leq C\epsilon^2. \end{aligned}$$

Integrating (3.104) from 0 to t then yields

$$\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \int_0^t \left\| \frac{\partial^2 U}{\partial y^2}(s, \cdot) \right\|_{L^2}^2 ds \leq C\epsilon^2. \quad (3.110)$$

We now integrate (3.107) from 0 to t and get

$$\frac{\lambda}{2} |\tau|^2 + \frac{\tau_\infty}{2\xi f_\infty^2} |f|^2 \leq C\epsilon^2. \quad (3.111)$$

For all $t \leq T_M$, (3.110) and (3.111) imply

$$\left\| \frac{\partial u}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^\infty}^2 \leq C\epsilon^2.$$

Choosing ϵ sufficiently small such that $C\epsilon^2 < \epsilon$ (which gives another condition on ϵ) contradicts the definition of T_M , and so $T_M = \infty$. It follows that (3.109) holds for all time $t > 0$, the solution remains small.

Step 3: Convergence to the stationary state. We now prove that, if the initial data satisfy (3.108), then the solution converges exponentially fast to the stationary state in the longtime. For t sufficiently large, (3.103) implies that

$$\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 + \|f(t, \cdot)\|_{L^2}^2 \leq \kappa_\epsilon e^{-(C-\epsilon)t}. \quad (3.112)$$

Adding (3.104) multiplied by 2ρ to (3.107) leads to

$$\begin{aligned}
 & \frac{d}{dt} \left(2\rho \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\lambda}{2} |\tau|^2 + \frac{\tau_\infty}{2\xi f_\infty^2} |f|^2 \right) + \eta \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 + \left(f + f_\infty + \frac{G}{2\eta} \right) |\tau|^2 \\
 & + \frac{\tau_\infty}{\xi f_\infty^2} \left(\nu(f + f_\infty)^2 - \xi(\|f\|_{L_T^\infty(L^\infty)} + 2f_\infty) \|\tau\|_{L_T^\infty(L^\infty)} \right) |f|^2 \\
 & \leq C \left(\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 + \|f(t, \cdot)\|_{L^2}^2 + \|f\|_{L_T^\infty(L^\infty)}^2 \|\tau(t, \cdot)\|_{L^2}^2 \right).
 \end{aligned}$$

We use the Poincaré inequality on $\frac{\partial U}{\partial y}$, the spatial average of which is zero, apply the Gronwall Lemma, insert (3.112) and find

$$\left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^\infty}^2 + \|f(t, \cdot)\|_{L^\infty}^2 \leq \kappa_\epsilon e^{-(C-\epsilon)t}.$$

This convergence estimate is equivalent to (3.101) and we have exponential convergence. \square

3.4.3 Longtime behaviour without smallness assumption (simplified case)

We now examine the longtime behaviour of system (3.1) supplied with *not necessarily small* initial data (u_0, τ_0, f_0) . We are unable to prove a general result and focus our attention to the particular case where the initial condition is $u_0 = ay$ (a positive constant), $\tau_0 = \text{constant} = \overline{\tau_0}$, $f_0 = \text{constant} = \overline{f_0} > 0$. In such a case, a substantial simplification occurs. Indeed, (3.1) reduces to the following system of ordinary differential equations:

$$\begin{cases} \lambda \frac{\partial \tau}{\partial t} = -f\tau + Ga & (3.113a) \\ \frac{\partial f}{\partial t} = (-1 + \xi|\tau|)f^2 - \nu f^3, & (3.113b) \end{cases}$$

supplied with initial conditions $\tau_0, f_0 \in \mathbb{R}$ with $f_0 > 0$.

System (3.113) has a unique steady state such that $f_\infty > 0$ and it reads

$$(\tau_\infty, f_\infty) = \left(\frac{\sqrt{1 + 4\nu\xi Ga} + 1}{2\xi}, \frac{\sqrt{1 + 4\nu\xi Ga} - 1}{2\nu} \right). \quad (3.114)$$

Indeed, such a steady state (τ_∞, f_∞) satisfies $f_\infty \tau_\infty = Ga$ (so that $\tau_\infty > 0$) and $\nu f_\infty = -1 + \xi \tau_\infty$. Combining these equations implies $(-1 + \xi \tau_\infty) \tau_\infty = \nu Ga$. This equation has a unique solution given in (3.114).

In addition, we introduce the condition

$$-\frac{1}{\lambda} - 2 + 2\xi(1 + Ga) \left(\frac{1}{\sigma} + \frac{\lambda\xi}{2Ga} \left(\frac{\nu\sigma + 1}{\xi} + \frac{4}{\xi} \right)^2 \right) < 0 \quad (3.115)$$

with

$$\sigma = \min \left\{ \frac{3Ga}{Gav + 4\tau_\infty}, \frac{\sqrt{1 + 4\nu\xi Ga} - 1}{3\nu} \right\}. \quad (3.116)$$

We are unable to perform our proof without this additional assumption. The numerical simulations we perform (see Figure 3.3) however show convergence holds even when (3.115) is not satisfied.

Theorem 6 *Supply system (3.113) with initial conditions $\tau_0, f_0 \in \mathbb{R}$ with $f_0 > 0$. Then the solution (τ, f) remains bounded.*

In addition, under assumption (3.115), the solution (τ, f) converges to (τ_∞, f_∞) in the longtime and the rate of convergence is exponential: for t sufficiently large,

$$|\tau(t) - \tau_\infty| + |f(t) - f_\infty| \leq C_0 e^{-C_r t}, \quad (3.117)$$

where C_0 is a constant independent from time and C_r reads

$$C_r = \begin{cases} \frac{1}{2} \left(\frac{f_\infty}{\lambda} + \nu f_\infty^2 \right) - \frac{1}{2} \sqrt{\Delta}, & \text{if } \Delta \geq 0, \\ \frac{1}{2} \left(\frac{f_\infty}{\lambda} + \nu f_\infty^2 \right), & \text{if } \Delta < 0, \end{cases} \quad (3.118)$$

with

$$\Delta = f_\infty^2 \left(\left(\frac{1}{\lambda} + \nu f_\infty \right)^2 - 4 \left(\frac{\nu}{\lambda} f_\infty + \frac{\xi}{\lambda} \tau_\infty \right) \right). \quad (3.119)$$

Proof. The proof is divided into seven steps. Step 1 introduces simplifications on the initial data and the system, that are not restrictive for the longtime behaviour. Some notation is given in Step 2. A lower bound on f is derived in Step 3 and is used in Step 4 to prove that the solution is bounded. Further restrictions are made in Step 5 still without loss of generality. Step 6 establishes the convergence, which is proven to be exponential in Step 7.

We consider until Step 4 the maximal solution to (3.113) although the solution a posteriori exists for all times because of boundedness.

Step 1: Simplifications on the initial data. We show that τ and f solution to (3.113) remain positive, possibly after some time for τ . We first remark that, since $f_0 > 0$, $f > 0$ for all times, arguing as in Step 1 of the proof of Theorem 1. On the other hand, if $\tau \leq 0$ on some time interval, evolution equation (3.113a) thus implies that τ increases strictly on this time interval (recalling that $a > 0$). Hence, there exist a time T_0 such that $\tau(T_0) > 0$. Moreover, for all $t > T_0$, τ remains positive (since if τ is zero at one time $T_1 > T_0$, $\frac{d\tau}{dt}(T_1) = Ga > 0$, which is in contradiction with $\tau > 0$ for $t < T_1$.)

For the purpose of studying the longtime limit, we may always consider, without loss of generality, the system

$$\begin{cases} \lambda \frac{\partial \tau}{\partial t} = -f\tau + Ga & (3.120a) \\ \frac{\partial f}{\partial t} = (-1 + \xi\tau)f^2 - \nu f^3, & (3.120b) \end{cases}$$

supplied with positive initial conditions τ_0, f_0 .

Step 2: Some notation. We consider the three subdomains:

$$\begin{aligned} A_1 &= \left\{ (\tau, f) \mid f \geq \frac{\xi\tau - 1}{\nu}, f \leq \sigma \right\}, \\ A_2 &= \left\{ (\tau, f) \mid f \leq \frac{\xi\tau - 1}{\nu}, f \leq \sigma \right\}, \\ A_3 &= \{ (\tau, f) \mid f \geq \sigma \}, \end{aligned}$$

where we recall that σ is defined by (3.116). We also introduce their intersections:

$$\begin{aligned}\Gamma_{13} &= \left\{ (\tau, \sigma) \mid \tau \leq \frac{\xi\sigma - 1}{\nu} \right\}, \\ \Gamma_{12} &= \left\{ (\tau, f) \mid f = \frac{\xi\tau - 1}{\nu}, f \leq \sigma \right\}, \\ \Gamma_{23} &= \left\{ (\tau, \sigma) \mid \tau \geq \frac{\xi\sigma - 1}{\nu} \right\}.\end{aligned}$$

See Figure 3.1 for a graphical description.

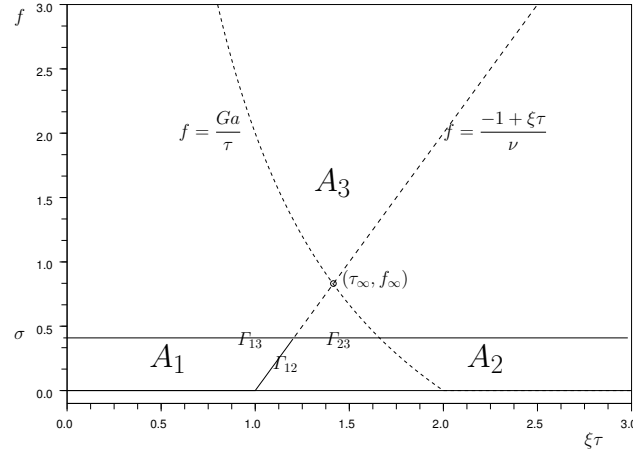


Fig. 3.1. Notation on $(0, +\infty) \times (0, +\infty)$

Step 3: Lower bound on f . We now establish a lower bound for the fluidity f in each domain. In the cases $(\tau_0, f_0) \in A_2$ or $(\tau_0, f_0) \in A_3$, we have

$$f \geq \min \{f_0, \sigma\}.$$

The case $(\tau_0, f_0) \in A_1$ requires more developments. The evolution equations (3.120a) and (3.120b) respectively rewrite

$$\begin{aligned}\frac{d}{dt} \frac{1}{f} &= 1 - \xi\tau + \nu f, \\ \frac{\lambda}{2} \frac{d}{dt} \left(\tau - \frac{4}{\xi} \right)^2 &= -f\tau \left(\tau - \frac{4}{\xi} \right) + Ga \left(\tau - \frac{4}{\xi} \right).\end{aligned}$$

We combine these two equations and obtain

$$\begin{aligned}\lambda\xi \frac{d}{dt} \left(\frac{Ga}{\lambda\xi} \frac{1}{f} + \frac{1}{2} \left(\tau - \frac{4}{\xi} \right)^2 \right) &= -3Ga + Gavf - \xi f\tau^2 + 4f\tau \\ &\leq -3Ga + (Gav + 4\tau_\infty)\sigma \\ &\leq 0,\end{aligned}\tag{3.121}$$

where we have used firstly that $0 \leq f \leq \sigma$ and $\tau \leq \tau_\infty$ in A_1 and secondly (3.116). Integrating (3.121) yields

$$\frac{Ga}{\lambda\xi} \frac{1}{f(t)} \leq \frac{Ga}{\lambda\xi} \max\left\{\frac{1}{f_0}, \frac{1}{\sigma}\right\} + \frac{1}{2} \left(\frac{\nu\sigma+1}{\xi} + \frac{4}{\xi}\right)^2,$$

using that $\tau \leq \frac{\nu\sigma+1}{\xi}$. We introduce

$$m_f = \left(\max\left\{\frac{1}{f_0}, \frac{1}{\sigma}\right\} + \frac{\lambda\xi}{2Ga} \left(\frac{\nu\sigma+1}{\xi} + \frac{4}{\xi}\right)^2 \right)^{-1}, \quad (3.122)$$

which is therefore a lower bound for f in the region A_1 . This lower bound also holds for initial conditions that belong to A_2 and A_3 and we thus have, for all $t > 0$,

$$f(t) \geq m_f, \quad (3.123)$$

with m_f defined by (3.122).

Step 4: Boundedness. The purpose of this step is to prove that the solution (τ, f) remains bounded.

Applying the Duhamel formula on (3.120a) yields

$$\tau(t) = e^{-\int_0^t \frac{f(s)}{\lambda} ds} \tau_0 + \frac{Ga}{\lambda} \int_0^t e^{-\int_s^t \frac{f(s')}{\lambda} ds'} ds,$$

so that, using the lower bound (3.123) on f ,

$$\tau(t) \leq e^{-\frac{m_f t}{\lambda}} \tau_0 + \frac{Ga}{m_f}.$$

Therefore, τ is bounded, and there exists a time t_0 such that, for all $t > t_0$,

$$\tau(t) \leq \frac{Ga+1}{m_f}. \quad (3.124)$$

We now turn to the boundedness of f . We introduce $M_\tau = \frac{Ga+1}{m_f}$ and $M_f = \frac{2}{\nu}(-1 + \xi M_\tau)$.

We will show that, for all $t > t_0$,

$$f(t) < \max(f(t_0), M_f). \quad (3.125)$$

We distinguish between two cases. Let us first suppose that $\frac{\partial f}{\partial t}(t_0) \geq 0$. In this case, $f(t_0) \leq \frac{1}{\nu}(-1 + \xi\tau(t_0)) < M_f$ because of (3.124). Moreover, for all $t > t_0$, $f(t) < M_f$. Indeed, by contradiction, if

$$t_1 = \inf\{t > t_0, f(t_1) = M_f\} < +\infty,$$

then, by continuity, $\frac{\partial f}{\partial t}(t_1) \geq 0$. On the other hand, we have

$$\begin{aligned}\frac{\partial f}{\partial t}(t_1) &= f^2(t_1)(-1 + \xi\tau(t_1) - \nu f(t_1)) \\ &< f^2(t_1)(-1 + \xi M_\tau - \nu M_f) \\ &< 0,\end{aligned}$$

hence the contradiction.

In the other case $\frac{\partial f}{\partial t}(t_0) < 0$, f strictly decreases until (possibly) equality occurs at a later time t_3 ($\frac{\partial f}{\partial t}(t_3) = 0$), which leads to the previous case with t_3 instead of t_0 . In any case, we have obtained (3.125) for all $t > t_0$.

Step 5: Further simplifications on the initial data. Table 1 first summarizes how (τ, f) behaves when it touches an intersection line. We use Table 1 to show that the solution enters region A_3

starting line	$\frac{\partial f}{\partial t}$	$\frac{\partial \tau}{\partial t}$	entering region
Γ_{13}	-	+	A_1
Γ_{12}	0	+	A_2
Γ_{23}	+		A_3

Table 3.1. Motion on intersection lines

at some time.

In region A_1 , we have $\frac{\partial \tau}{\partial t} > 0$, so that there does not exist any periodic orbit inside region A_1 . There is also no steady state in this region. Using the Poincaré-Bendixson Theorem on the bounded solution of ordinary differential equation system (3.120) (see *e.g.* Ref. [Per00], Theorem 1, p. 245), the solution leaves region A_1 at some time. According to Table 1, it enters region A_2 .

Applying similar arguments on region A_2 where $\frac{\partial f}{\partial t} > 0$, the solution enters region A_3 at some time.

We can therefore restrict the studying of the longtime limit to initial data (τ_0, f_0) that belongs to region A_3 , without loss of generality.

The bounds (3.123) and (3.124) become

$$f(t) \geq \left(\frac{1}{\sigma} + \frac{\lambda\xi}{2Ga} \left(\frac{\nu\sigma + 1}{\xi} + \frac{4}{\xi} \right)^2 \right)^{-1} \quad (3.126)$$

and

$$\tau(t) \leq (Ga + 1) \left(\frac{1}{\sigma} + \frac{\lambda\xi}{2Ga} \left(\frac{\nu\sigma + 1}{\xi} + \frac{4}{\xi} \right)^2 \right). \quad (3.127)$$

Step 6: Convergence. We introduce

$$\mathcal{G}(\tau, f) = -\frac{1}{\lambda}f\tau + \frac{1}{\lambda}Ga$$

and

$$\mathcal{F}(\tau, f) = (-1 + \xi\tau)f^2 - \nu f^3.$$

We have

$$\begin{aligned} \frac{\partial \mathcal{G}(\tau, f)}{\partial \tau} + \frac{\partial \mathcal{F}(\tau, f)}{\partial f} &= f \left(-\frac{1}{\lambda} - 2 + 2\xi\tau - 3\nu f \right), \\ &< f \left(-\frac{1}{\lambda} - 2 + 2\xi(1 + Ga) \left(\frac{1}{\sigma} + \frac{\lambda\xi}{2Ga} \left(\frac{\nu\sigma + 1}{\xi} + \frac{4}{\xi} \right)^2 \right) \right), \end{aligned}$$

using (3.127) and the positivity of f . Because of our assumption (3.115), system (3.120), which writes

$$\begin{cases} \frac{\partial \tau}{\partial t} = \mathcal{G}(\tau, f) \\ \frac{\partial f}{\partial t} = \mathcal{F}(\tau, f), \end{cases}$$

satisfies

$$\frac{\partial \mathcal{G}(\tau, f)}{\partial \tau} + \frac{\partial \mathcal{F}(\tau, f)}{\partial f} < 0.$$

According to the Bendixson-Dulac Criterion (see *e.g.* Ref. [Per00], Theorem 1, p. 264), there does not exist any periodic orbit for (3.120). Since it has only one steady state (τ_∞, f_∞) , the solution converges to it:

$$\lim_{t \rightarrow \infty} (|\tau(t) - \tau_\infty| + |f(t) - f_\infty|) = 0.$$

Step 7: Exponential convergence. Now that we have convergence to the steady state, we can use linear stability. System (3.120) linearized around the stationary state (τ_∞, f_∞) reads

$$\frac{d}{dt} \begin{pmatrix} \tau_l \\ f_l \end{pmatrix} = \begin{pmatrix} -\frac{f_\infty}{\lambda} & -\frac{\tau_\infty}{\lambda} \\ \xi f_\infty^2 & -\nu f_\infty^2 \end{pmatrix} \begin{pmatrix} \tau_l \\ f_l \end{pmatrix}.$$

The eigenvalues of the associated matrix depend on the sign of Δ defined by (3.119). If $\Delta < 0$, the eigenvalues are complex and their real part is $-\frac{1}{2} \left(\frac{f_\infty}{\lambda} + \nu f_\infty^2 \right)$. If $\Delta \geq 0$, the eigenvalues are real negative, the smaller one in absolute value is $-\frac{1}{2} \left(\frac{f_\infty}{\lambda} + \nu f_\infty^2 \right) + \frac{1}{2} \sqrt{\Delta}$. The real part of the eigenvalues gives the rate of convergence and hence of values of C_r in (3.118).

□

3.5 Numerical results

In this section, we present numerical simulations that complement the theoretical results on the behaviour of the previous sections.

We simulate numerically (3.1) in the interval $\Omega = [0, 1]$ and the interval $[0, T]$ for $T = 10000$. The system is supplied either with *homogeneous* boundary conditions or non-homogeneous boundary conditions $u(t, 0) = 0$ and $u(t, 1) = a$ for all time $t \in [0, T]$. In the latter case, we

take $a = 1$. As for the initial conditions, we take sinusoidal functions for all three fields. The values of u_0 oscillate between -0.002 and 0.002 for *homogeneous* boundary conditions and between 0 and a otherwise. The values of τ_0 and f_0 oscillate between -0.5 and 0.5 .

We use the following set of physical parameters. The density $\rho = 0.001$ and the viscosity $\eta = 1$ so that the Reynolds number is low. The elastic modulus G and the coefficients ξ and ν are equal to one. The characteristic relaxation time λ is 0.5 unless otherwise stated.

System (3.1) is solved using a constant time step $\Delta t = 0.005$ with the following time scheme:

$$\begin{cases} \frac{\rho}{\Delta t} (u_n - u_{n-1}) = \eta \frac{\partial^2 u_n}{\partial y^2} + \frac{\partial \tau_{n-1}}{\partial y}, & (3.128a) \end{cases}$$

$$\begin{cases} \frac{\lambda}{\Delta t} (\tau_n - \tau_{n-1}) = G \frac{\partial u_n}{\partial y} - f_{n-1} \tau_{n-1}, & (3.128b) \end{cases}$$

$$\begin{cases} \frac{1}{\Delta t} (f_n - f_{n-1}) = (-1 + \xi |\tau_n|) f_{n-1} f_n - \nu f_{n-1} f_n^2. & (3.128c) \end{cases}$$

For the space variable, we use linear $\mathbb{P}1$ finite elements for u and piecewise constant finite elements for both τ and f . Note that, in contrast to the approximating system (3.25) we used for our theoretical proof, we take τ_{n-1} instead of τ_n in the right-hand sides of (3.128a) and (3.128b). This allows us to solve each equation separately. This choice is made for simplicity. Other approaches could have been employed. For our tests, we use elements of constant size $h = 0.002$ and perform the computations using Scilab.[Sci]

3.5.1 Homogeneous boundary conditions

We first focus on the *homogeneous* boundary conditions on u considered in Section 3.3. The case $f_0 \equiv 0$, that implies $f \equiv 0$ for all times, is uninteresting numerically. We therefore only show results for $f_0 \neq 0$. In this case, we have convergence to the stationary state $(0, 0, 0)$ as proven in Theorem 2. The convergence estimates are established in Theorem 3. We recall the parameter

$$\beta = \text{meas} \{y \in \Omega \mid f_0(y) > 0\}$$

and these convergence rates: for α arbitrarily small, there exist various constants κ_α and a time t_0 , such that, for all $t > t_0$,

$$\begin{aligned} \|\tau(t, \cdot)\|_{L^2} &\leq \kappa_\alpha (1+t)^{-\frac{\beta}{\lambda}(1-\alpha)}, \\ \frac{1}{\frac{1}{f(t_0, y)} + (1+\alpha)(t-t_0)} &\leq f(t, y) \leq \frac{1}{\frac{1}{f(t_0, y)} + (1-\alpha)(t-t_0)}, \\ \|u(t, \cdot)\|_{H^1} + \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2} &\leq \kappa_\alpha (1+t)^{-1-\frac{\beta}{\lambda}(1-\alpha)}, \\ \left\| \left(\eta \frac{\partial u}{\partial y} + \tau - \bar{\tau} \right) (t, \cdot) \right\|_{L^2} &\leq \kappa_\alpha (1+t)^{-2-\frac{\beta}{\lambda}(1-\alpha)}. \end{aligned}$$

Note that the last three estimates are exactly the same as in Theorem 3, the first estimate is an immediate consequence of (3.64) and (3.69). We now check that these estimates are sharp. We begin with the case $f_0 > 0$ on Ω that is $\beta = 1$. The evolutions of

$\|\tau(t, \cdot)\|_{L^2}, \|f(t, \cdot)\|_{L^2}, \|u(t, \cdot)\|_{H^1} + \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}, \left\| \left(\eta \frac{\partial u}{\partial y} + \tau - \bar{\tau} \right) (t, \cdot) \right\|_{L^2}$ are represented in Figure 3.2(a). We use a log-log representation. The slopes s , which correspond to a decrease as t^s , are fitted on the numerical results and indicated on Figure 3.2(a): the numerical convergence rates, obtained with $\lambda = 0.5$, are in good agreement with the estimates.

We next consider cases where $f_0 = 0$ on some part of the domain. In Figure 3.2(b), we show simulations obtained with different values of β . For each simulation, that is for each value of β considered, the convergence rates are fitted and represented as a function of β . The numerical and theoretical convergence rates s agree.

We have extended these results to the other values of λ than $\lambda = 0.5$ and other values of the parameters ρ, η, G, ξ, ν to check that the convergence estimates of Theorem 3 depend only on λ and β and are indeed sharp.

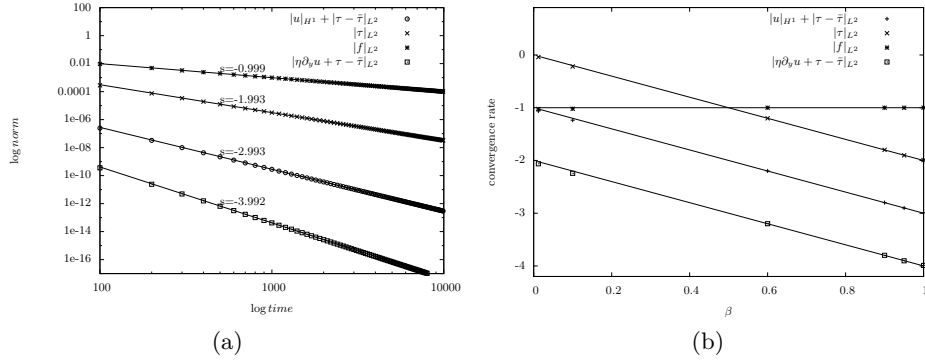


Fig. 3.2. (a) Time evolution in log-log scale for *homogeneous* boundary conditions; the points are the simulated trajectories; the lines and the corresponding slopes s are fitted. (b) Fitted convergence rates s for $\beta = 0, 0.01, 0.1, 0.6, 0.9, 0.99$; the lines are the theoretical convergence rates function of β .

3.5.2 Non-homogeneous boundary conditions

The longtime behaviour for non-homogeneous boundary conditions has been studied in Section 3.4. We consider only stationary states $(u_\infty, \tau_\infty, f_\infty)$ that satisfy $f_\infty > 0$ everywhere. We have shown that such a steady state (3.96) is unique. We established in Theorem 5 that we have convergence to this steady state for small perturbations. To have convergence, we of course need to assume $f_0 > 0$ everywhere. We observe numerically that no other condition, and specifically non assumption on the smallness of the data, is required. We consider the perturbations $(u - u_\infty, \tau - \tau_\infty, f - f_\infty)$ to equilibrium and show that they vanish in the longtime, see Figure 3.3. The evolution is plotted in semi-logarithmic scale. The convergences of the various norms $\|\tau(t, \cdot)\|_{L^2}, \|f(t, \cdot)\|_{L^2}, \|u(t, \cdot)\|_{H^1} + \|(\tau - \bar{\tau})(t, \cdot)\|_{L^2}, \left\| \left(\eta \frac{\partial u}{\partial y} + \tau - \bar{\tau} \right) (t, \cdot) \right\|_{L^2}$ are indeed exponential. Note that the slope of the curves breaks to zero at some time because machine precision is reached.

In section 3.4, in order to establish a result without any smallness assumption, we have considered a particular initial data that reduces (3.1) to the ordinary differential equation system (3.113). We have obtained convergence to the stationary state (3.114) and explicit formula for the rate of convergence. Numerically, we observe convergence even when the condition (3.115),

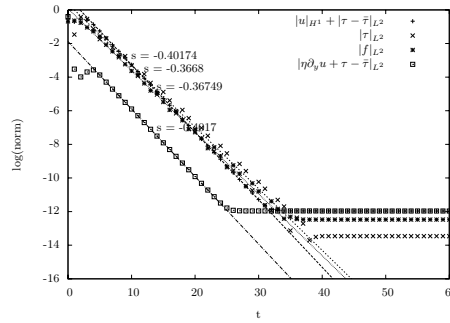


Fig. 3.3. Time evolution of the perturbation to equilibrium in semi-logarithmic scale for non-homogeneous boundary conditions ; the points are the simulated trajectories and the line and the corresponding slope are fitted.

which was assumed for the proof of Theorem 6, is not satisfied. The time evolution is shown in the space of (τ, f) and the convergence are represented in Figure 3.4(a). We check that the convergence is exponential as observed numerically in the general case of (3.1) (see Figure 3.3). Again the slope of the curves breaks to zero because machine precision is reached. Moreover, we compute the convergence rate and compare it to the theoretical rate C_r defined by (3.118). The evolution of the perturbation function $|\tau(t) - \tau_\infty| + |f(t) - f_\infty|$ is plotted as a function of time in semi-logarithmic scale in Figure 3.4(b). The first case $\lambda = 0.5$ correspond to the case when the eigenvalues of the associated linearized system are complex, the expected value of C_r is 0.8090; the other case $\lambda = 0.1$ is when the eigenvalues are real negative, the expected value of C_r is 1.7895. The theoretical and numerical value agree.

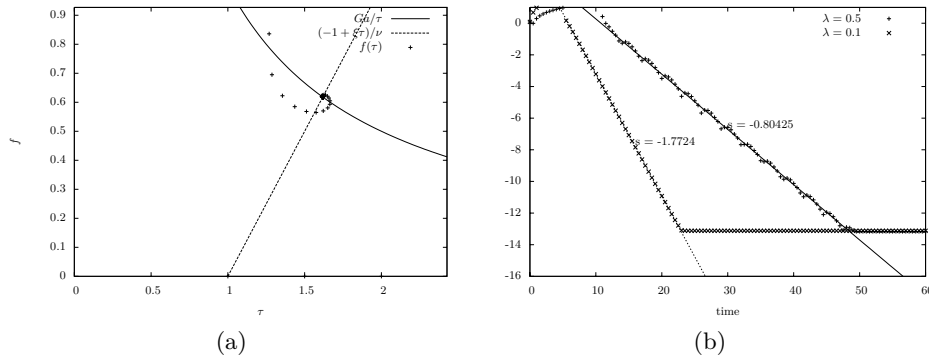


Fig. 3.4. For the system of ordinary differential equations (3.113), time evolution (a) in the space (τ, f) for $\lambda = 0.5$; (b) of the perturbation to equilibrium in semi-logarithmic scale for $\lambda = 0.5, 0.1$; the points are the simulated trajectories and the line and the corresponding slope are fitted.

Acknowledgement.

The authors are grateful to François Lequeux (ESPCI Paris) for many stimulating and enlightening discussions on the rheology of complex fluids.

3.A Addendum to Chapter 3 : Proof of Remark 4

In this section, we present an additional result that was not published in the article [BHBL13]. This result is mentioned in the Remark 4 of this article. We prove that under the assumptions of Theorem 5, τ and f converge to zero in $H^1(\Omega)$ and not only in $L^\infty(\Omega)$ as stated in the theorem.

We first derive second a priori estimates on τ and f . We differentiate evolution equations (3.102b) and (3.102c)

$$\begin{aligned} \lambda \frac{\partial}{\partial t} \left(\frac{\partial \tau}{\partial y} \right) &= G \frac{\partial^2 U}{\partial y^2} - \frac{G}{\eta} \frac{\partial \tau}{\partial y} - f_\infty \frac{\partial \tau}{\partial y} - \tau_\infty \frac{\partial f}{\partial y} - \tau \frac{\partial f}{\partial y} - f \frac{\partial \tau}{\partial y} \\ \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) &= -\nu(f + f_\infty)^2 \frac{\partial f}{\partial y} - 2\nu(f + f_\infty)f \frac{\partial f}{\partial y} \\ &\quad + \xi(f + 2f_\infty) \left(\tau \frac{\partial f}{\partial y} + f \frac{\partial \tau}{\partial y} \right) + \xi f \tau \frac{\partial f}{\partial y} + \xi f_\infty^2 \frac{\partial \tau}{\partial y}. \end{aligned}$$

We multiply the above equations respectively by $\frac{\partial \tau}{\partial y}$ and $\frac{\partial f}{\partial y}$, integrate over Ω and combine the resulting equations to obtain

$$\begin{aligned} &\frac{d}{dt} \left(\lambda \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\tau_\infty}{\xi f_\infty^2} \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) + \left(\frac{G}{\eta} + f_\infty \right) \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|_{L^2} + \frac{\tau_\infty \nu}{\xi f_\infty^2} \left\| (f + f_\infty) \frac{\partial f}{\partial y}(t, \cdot) \right\|^2 \\ &\leq G \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 + C \left((\|f(t, \cdot)\|_{L^\infty} + 1) (\|\tau(t, \cdot)\|_{L^\infty} + \|f(t, \cdot)\|_{L^\infty}) \left(\left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \right), \end{aligned} \tag{3.129}$$

after manipulations in the right-hand side that use the L^∞ -estimates on τ and f . Combining (3.129) with (3.104) and using the convergence of the L^∞ -norms of τ and f leads to

$$\begin{aligned} &\frac{d}{dt} \left(\lambda \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{\tau_\infty}{\xi f_\infty^2} \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|_{L^2}^2 + \frac{2\rho G}{\eta} \left\| \frac{\partial U}{\partial y}(t, \cdot) \right\|_{L^2}^2 \right) \\ &\quad + \left(\frac{G}{\eta} + f_\infty \right) \left\| \frac{\partial \tau}{\partial y}(t, \cdot) \right\|_{L^2} + \frac{\tau_\infty \nu}{\xi} \left\| \frac{\partial f}{\partial y}(t, \cdot) \right\|^2 + G \left\| \frac{\partial^2 U}{\partial y^2}(t, \cdot) \right\|_{L^2}^2 \leq C \left(\|u(t, \cdot)\|_{L^2}^2 + \|\tau(t, \cdot)\|_{L^2}^2 \right) \end{aligned}$$

We use the Poincaré inequality on $\frac{\partial U}{\partial y}$, the spatial average of which is zero, apply the Gronwall Lemma, insert (3.112) and find

$$\|\tau(t, \cdot)\|_{H^1}^2 + \|f(t, \cdot)\|_{H^1}^2 \leq \kappa_\epsilon e^{-(C-\epsilon)t}.$$

The functions τ and f converge to zero in $H^1(\Omega)$.

Analyse d'un modèle mésoscopique de fluide vieillissant et passage à la limite macroscopique

Dans ce chapitre, on présente des résultats obtenus en collaboration avec Claude Le Bris et Tony Lelièvre, et qui sont soumis pour publication dans une revue à comité de lecture. On étudie une équation unidimensionnelle modélisant à l'échelle mésoscopique un fluide non newtonien. Il est démontré que cette équation est bien posée et le comportement en temps long est déterminé. De plus, des équations macroscopiques sont dérivées à partir de cette équation mésoscopique.

Analysis and macroscopic limit of a one-dimensional model for aging fluids

David Benoit, Claude Le Bris and Tony Lelièvre

Abstract. We study a one-dimensional equation arising in the multiscale modeling of some non-Newtonian fluids. At a given shear rate, the equation provides the instantaneous mesoscopic response of the fluid, allowing to compute the corresponding stress. In a simple setting, we study the well-posedness of the equation and next the long-time behavior of its solution. In the limit of a response of the fluid much faster than the time variations of the ambient shear rate, we derive some equivalent macroscopic differential equations that relate the shear rate and the stress. Our analytical conclusions are confronted to some numerical experiments. The latter quantitatively confirm our derivations.

Key words. Non-Newtonian fluids; micro-macro model; longtime behavior.

4.1 Introduction

Generalities

The present contribution is devoted to the mathematical analysis of the equation

$$\frac{\partial p}{\partial t}(t, \sigma) + \gamma(t) \frac{\partial p}{\partial \sigma}(t, \sigma) = -\chi(\sigma)p(t, \sigma) + \left(\int \chi(\sigma)p(t, \sigma) d\sigma \right) \delta_0(\sigma). \quad (4.1)$$

Equation (4.1) arises in the modeling of some non-Newtonian fluid flows. Some details on the modeling will be given below. The variable σ is one-dimensional, varies on the real line \mathbb{R} , and models a quantity homogeneous to a stress (actually to a certain entry of the stress tensor). The variable t of course denotes the time, and equation (4.1) is supplied with some initial condition $p(t = 0, \cdot) = p_0(\cdot)$. The unknown real-valued function $p(t, \sigma)$, solution to (4.1), satisfies the two properties: it is nonnegative

$$p(t, \sigma) \geq 0, \quad \text{for all } t \geq 0 \quad \text{and} \quad \sigma \in \mathbb{R}, \quad (4.2)$$

and normalized to one

$$\int_{-\infty}^{+\infty} p(t, \sigma) d\sigma = 1, \quad (4.3)$$

for all times $t \geq 0$. The function p models the density of probability to have a certain (elementary) stress σ , at time t , at a macroscopic space position x . Equation (4.1) is thus implicitly parameterized by this position x (thus the multiscale nature of the problem, as will be seen below). The function γ , also a function of time, is assumed given. It models the shear rate at the position x , under which we wish to compute, using (4.1), the mesoscopic response of the fluid. Note that the traditional Fluid Mechanics notation for the shear rate is rather $\dot{\gamma}$ than γ but we will adopt γ in order to keep notation simple. In (4.1), we denote by χ the characteristic function

$$\chi = \mathbb{1}_{\mathbb{R} \setminus [-\sigma_c, \sigma_c]}$$

where $\sigma_c > 0$ a scalar positive parameter, fixed once and for all. It models some local threshold value of the stress, which plays a crucial role in the modeling. As is usual, we denote by δ_0

the Dirac mass at zero. From the definitions of χ and δ_0 , it is immediately seen that, at least formally (and this will indeed be made rigorous, see Lemmata 4.2 and 4.3 in Section 4.4 below), equation (4.1) preserves in time the two properties (4.2) and (4.3).

Two quantities are typically computed using the solution p to (4.1): first the so-called *fluidity*

$$f(t) = \int \chi(\sigma)p(t, \sigma) d\sigma \quad (4.4)$$

and next the (real-valued) stress

$$\tau(t) = \int \sigma p(t, \sigma) d\sigma. \quad (4.5)$$

Our purpose in this article is to mathematically study equation (4.1) (in terms of existence and uniqueness of the solution p , properties and long-time behavior of that solution) and to derive a macroscopic equation equivalent to this equation. By *macroscopic* equation, we mean an equation (actually a differential equation, or a system of differential equations) that directly relates the shear rate γ , the fluidity f and the stress τ without the explicit need to compute p . In these macroscopic equations, the scalar $f(t)$ will be the inverse of the mechanical relaxation time, thus its name “fluidity”. We will be able, in particular, to obtain a macroscopic equation which is close to models that have been proposed for aging fluids [DAL01, PABL02], see the discussion at the end of Section 4.7.

Some elements on the modeling

Equation (4.1) is the simplest possible form of an equation describing the mesoscopic behavior of a complex fluid, such as a concentrated suspension, or more generically a soft amorphous material, with properties intermediate between those of a fluid and those of a solid. These materials exhibit a highly non-Newtonian behavior and may give rise to a macroscopic yield stress.

At low stress, such a material behaves in an elastic way. But above a certain stress threshold, here denoted by the critical value σ_c , one observes a relaxation toward a completely relaxed state. This behavior is modeled by equation (4.1). The probability of finding the fluid in the state of stress σ at time t evolves in time for two different reasons: the term $\gamma(t)\frac{\partial p}{\partial \sigma}$ models the modification of the stress induced by the existence of the shear rate, while the term $-\chi p + \left(\int \chi p\right)\delta_0$ encodes the relaxation toward zero of the part of the stress above the threshold σ_c . From a probabilistic viewpoint, the stochastic process $(\Sigma_t)_{t \geq 0}$ associated to the Fokker-Planck equation (4.1) evolves deterministically when $|\Sigma_t| \leq \sigma_c$ and jumps to zero with an exponential rate 1 when $|\Sigma_t| \geq \sigma_c$. The process $(\Sigma_t)_{t \geq 0}$ belongs to the class of *piecewise-deterministic Markov processes*, which have been introduced in the probabilistic literature in the 1980’s for biological modeling for example. In particular, coupling arguments have been proposed to study the longtime behavior of such processes (see [BCG⁺13]). We argue on the Fokker-Planck equation and proceed differently. The argument we are using here to study the longtime behavior is purely deterministic in nature, and is based on a delay equation related to the Fokker-Planck equation (4.1).

We would like to mention that for more realistic models, a third phenomenon is typically at play, in addition to the stress induced by the ambient fluid, and to the relaxation to zero. All states of stress are not independent of one another, and they may also depend on the state of stress at neighboring points within the fluid. A certain redistribution of the stress therefore always occurs. This redistribution can be encoded in various ways, depending on some more detailed elements of modeling. In the so-called Hébraud-Lequeux model introduced in the seminal article [HL98] (and then extensively studied mathematically in the works [CCG05, CLB06, CCGLB05]), the redistribution is performed by a diffusion term in the stress space, at the given location x in the ambient physical space, and the complete equation thus writes

$$\frac{\partial p}{\partial t} + \gamma(t) \frac{\partial p}{\partial \sigma} = -\chi p + \left(\int \chi p \right) \delta_0(\sigma) + \alpha \left(\int \chi p \right) \frac{\partial^2 p}{\partial \sigma^2} \quad (4.6)$$

where $\alpha > 0$ is some parameter. In an alternative model introduced by Bocquet and coll. in [BCA09], the redistribution is achieved by some type of local “convolution” in the physical space. The equation (we recall, set at the physical location x) writes

$$\begin{aligned} \frac{\partial p}{\partial t}(t, x, \sigma) + \gamma(t, x) \frac{\partial p}{\partial \sigma}(t, x, \sigma) &= -\chi(\sigma)p(t, x, \sigma) + \left(\int d\sigma' \chi(\sigma') p(t, x, \sigma') \right) \delta_0(\sigma). \\ &+ \int \int dx' d\sigma' \chi(\sigma') (p(t, x', \sigma') p(t, x, \sigma - G(x, x')\sigma') - p(t, x', \sigma') p(t, x, \sigma)). \end{aligned} \quad (4.7)$$

with a function $G(x, x')$ related to the Green function of some local Stokes-type problem.

The equation (4.1) which we study in the present article ignores the redistribution phenomenon, which amounts to taking $\alpha = 0$ in (4.6) or $G \equiv 0$ in (4.7). Some more detailed comments on the modeling, as well as some formal foundations of the model based on a system of interacting particles are presented in [Benon].

Plan of our contribution

Our article is organized as follows. To start with, we study in Section 4.2 the stationary solutions to (4.1). We next show in Section 4.3 existence and uniqueness of the solutions to the time-dependent equation (4.1). Our result is stated in Theorem 4.1. Section 4.4 follows, establishing some useful properties of the solution. In order to understand the macroscopic equivalent of equation (4.1) for a given shear rate $\gamma(t)$, which we assume varies slowly as compared to the characteristic time of equation (4.1), we need to understand the long-time behavior of the solution to (4.1). We therefore study this behavior in Sections 4.5 and 4.6, respectively in the case of a constant shear rate γ , and in the case of a slowly varying shear rate $\gamma(\epsilon t)$. The results are stated in Theorems 4.2 and 4.3. We are then in position to derive, in Section 4.7, the macroscopic differential equations equivalent to (4.1) in this limit, namely system (4.102). Our final section, Section 4.8, presents some numerical experiments which confirm and illustrate our theoretical results.

4.2 Stationary states

We study in this section the stationary states of (4.1). We therefore assume that $\gamma(t) \equiv \gamma_\infty$ is a fixed scalar and consider the solutions $p_\infty : \mathbb{R} \rightarrow \mathbb{R}$ to the following equation

$$\gamma_\infty \frac{dp_\infty}{d\sigma} = -\chi p_\infty + \left(\int \chi p_\infty \right) \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}). \quad (4.8)$$

Here and in the following, for a subset $I \subset \mathbb{R}^d$, $\mathcal{D}'(I)$ denotes the set of distributions on I . By convention, since the time-dependent version of the equation is linear and formally preserves positiveness and the integral over the real line, we are only interested in the stationary solutions p_∞ that additionally satisfy (4.2) and (4.3), that is,

$$p_\infty \in L^1(\mathbb{R}), \quad p_\infty \geq 0, \quad \text{a.e. and} \quad \int p_\infty = 1. \quad (4.9)$$

We have the following result:

Lemma 4.1. *When $\gamma_\infty = 0$, the solutions to (4.8)-(4.9) are exactly all nonnegative normalized densities with compact support in $[-\sigma_c, \sigma_c]$. When $\gamma_\infty \neq 0$, there exists a unique solution p_∞ to (4.8)-(4.9).*

Proof. In the case $\gamma_\infty = 0$, the equation (4.8) implies $p_\infty = 0$ in $\mathcal{D}'(\mathbb{R} \setminus [-\sigma_c, \sigma_c])$, hence the result. Up to a change of p_∞ into $\sigma \rightarrow p_\infty(-\sigma)$, we may, without loss of generality, consider only the case $\gamma_\infty > 0$ for our proof. We first note that p_∞ defined by

$$p_\infty(\sigma) = \begin{cases} 0 & \text{if } \sigma < 0 \\ \frac{1}{\sigma_c + \gamma_\infty} & \text{if } 0 < \sigma \leq \sigma_c \\ \frac{1}{\sigma_c + \gamma_\infty} e^{-(\sigma - \sigma_c)/\gamma_\infty} & \text{if } \sigma_c < \sigma. \end{cases} \quad (4.10)$$

is a solution to (4.8), hence the existence result.

We now show uniqueness. By linearity, we assume that $p_\infty \in L^1(\mathbb{R})$ is a solution of (4.8) with $\int_{\mathbb{R}} p_\infty = 0$ and show that $p_\infty = 0$. Equation (4.8) implies $\gamma_\infty \frac{dp_\infty}{d\sigma} + p_\infty = 0$ in $\mathcal{D}'(\mathbb{R} \setminus [-\sigma_c, \sigma_c])$. Because $p_\infty \in L^1(\mathbb{R})$, this leads to $p_\infty = \varpi e^{-\frac{\sigma}{\gamma_\infty}}$ a.e. on (σ_c, ∞) with ϖ a scalar and $p_\infty = 0$ a.e. on $(-\infty, -\sigma_c)$.

In the case $\varpi = 0$, this implies $\chi p_\infty = 0$. Consequently, (4.8) rewrites $\frac{dp_\infty}{d\sigma} = 0$ on the whole real line. This readily implies $p_\infty = 0$ since $\int_{\mathbb{R}} p_\infty = 0$.

In the case $\varpi \neq 0$, we obtain that (4.8) writes

$$\gamma_\infty \frac{dp_\infty}{d\sigma} = \varpi \left(\int_{\sigma_c}^{\infty} e^{-\frac{\sigma'}{\gamma_\infty}} d\sigma' \right) \delta_0 \text{ in } \mathcal{D}'(-\infty, \sigma_c)$$

so that $p_\infty = \frac{\varpi}{\gamma_\infty} \left(\int_{\sigma_c}^{\infty} e^{-\frac{\sigma'}{\gamma_\infty}} d\sigma' \right) \mathbb{1}_{\mathbb{R}_+}(\sigma)$ a.e. on $(-\infty, \sigma_c)$. Using that $p_\infty = \varpi e^{-\frac{\sigma}{\gamma_\infty}}$ a.e. on (σ_c, ∞) and $\int p_\infty = 0$, we find $\varpi = 0$ and thus $p_\infty = 0$. This concludes the proof. \square

4.3 Existence and uniqueness

This section is devoted to the proof of the following result:

Theorem 4.1 (Existence and uniqueness). *Consider γ a function of time that satisfies*

$$\gamma \in L^1_{\text{loc}}(0, \infty) \text{ and } \gamma \geq m_\gamma \text{ a.e. where } m_\gamma > 0 \text{ is a fixed scalar.} \quad (4.11)$$

Denote by $\Gamma(t) = \int_0^t \gamma(s) ds$. Consider $p_0 \in L^1(\mathbb{R})$. Then, for any $T > 0$, there exists a unique p in $C^0([0, T]; L^1)$ such that $p(0, \sigma) = p_0(\sigma)$ for almost all $\sigma \in \mathbb{R}$ and such that (4.1) holds for p in the sense of distributions on $(0, T) \times \mathbb{R}$. In addition, introduce

$$A(t) = \int \chi(\sigma) p_0(\sigma - \Gamma(t)) e^{-\int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} d\sigma \quad (4.12)$$

and ϕ defined by induction on $k \in \mathbb{N}^*$ as follows

$$\begin{aligned} \phi(t) &= A(t) \\ &+ \begin{cases} 0 & \text{when } t \in (0, \Gamma^{-1}(\sigma_c)) \\ \int_0^{\Gamma^{-1}(\Gamma(t) - \sigma_c)} \phi(s) e^{-t + \Gamma^{-1}(\Gamma(s) + \sigma_c)} ds & \text{when } t \in (\Gamma^{-1}(k\sigma_c), \Gamma^{-1}((k+1)\sigma_c)). \end{cases} \end{aligned} \quad (4.13)$$

Both A and ϕ belong to $L^\infty(0, T)$. Then, the solution p to (4.1) is explicitly given by

$$\begin{aligned} p(t, \sigma) &= p_0(\sigma - \Gamma(t)) e^{-\int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} \\ &+ \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} e^{-\int_{\Gamma^{-1}(\Gamma(t) - \sigma)}^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} \mathbb{1}_{(0, \Gamma(t))}(\sigma) \end{aligned} \quad (4.14)$$

and f defined by (4.4) is equal to ϕ :

$$f = \phi \text{ a.e. on } (0, T). \quad (4.15)$$

Remark 4.1. *The above results also hold for $\gamma < -m_\gamma$ negative. However, it is unclear how to extend these results if γ is allowed to vanish.*

Remark 4.2. *We will see in the next section that if the initial condition p_0 is nonnegative and normalized, then this property is preserved in time for the solution p .*

Proof. We first note that

$$\Gamma(t) = \int_0^t \gamma(s) ds$$

is a strictly increasing continuous function of the time because of (4.11). Throughout this proof, we assume $T > \Gamma^{-1}(\sigma_c)$. When $T \leq \Gamma^{-1}(\sigma_c)$, the arguments are similar and actually simpler. We first show uniqueness, then p given by (4.14) belongs to $C^0([0, T]; L^1)$ and is a solution of (4.1) in $\mathcal{D}'((0, T) \times \mathbb{R})$.

Step 1: Uniqueness Equation (4.1) is linear, we therefore consider a solution $p \in C^0([0, T]; L^1)$ associated to the zero initial condition $p_0 = 0$ and we intend to show that $p = 0$. Denote by

$$\tilde{p}(t, \xi) = p(t, \xi + \Gamma(t))e^{\int_0^t \chi(\xi + \Gamma(u))du}.$$

We now show

$$\frac{\partial \tilde{p}}{\partial t}(t, \xi) = f(t)\delta_{-\Gamma(t)}(\xi)e^{\int_0^t \chi(-\Gamma(t) + \Gamma(u))du} \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}) \quad (4.16)$$

with f defined from p by (4.4). We have, for all $\psi \in \mathcal{D}((0, T) \times \mathbb{R})$, (where $\mathcal{D}((0, T) \times \mathbb{R})$ denotes the set of C^∞ functions with compact support in $(0, T) \times \mathbb{R}$)

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}} \tilde{p} \frac{\partial \psi}{\partial t} &= - \int_0^T \int_{\mathbb{R}} p(t, \xi + \Gamma(t))e^{\int_0^t \chi(\xi + \Gamma(u))du} \frac{\partial \psi}{\partial t}(t, \xi) d\xi dt \\ &= - \int_0^T \int_{\mathbb{R}} p(t, \sigma) \frac{\partial \psi}{\partial t}(t, \sigma - \Gamma(t))e^{\int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u))du} d\sigma dt. \end{aligned} \quad (4.17)$$

For $n \in \mathbb{N}$, denote now by ρ^n a mollifier, $\chi^n = \rho^n * \chi$ and

$$\eta^n(t, \sigma) = \psi(t, \sigma - \Gamma(t))e^{\int_0^t \chi^n(\sigma - \Gamma(t) + \Gamma(u))du}.$$

The fact that p is solution to (4.1) in $\mathcal{D}'((0, T) \times \mathbb{R})$ yields

$$- \int_0^T \int_{\mathbb{R}} p \left(\frac{\partial \eta^n}{\partial t} + \gamma \frac{\partial \eta^n}{\partial \sigma} - \chi \eta^n \right) = \int_0^T f(t) \eta^n(t, 0) dt.$$

This rewrites

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}} p(t, \sigma) \frac{\partial \psi}{\partial t}(t, \sigma - \Gamma(t))e^{\int_0^t \chi^n(\sigma - \Gamma(t) + \Gamma(u))du} d\sigma dt \\ + \int_0^T \int_{\mathbb{R}} p(t, \sigma) \eta^n(t, \sigma) (\chi - \chi^n)(\sigma) d\sigma dt \\ = \int_0^T f(t) \psi(t, -\Gamma(t))e^{\int_0^t \chi^n(-\Gamma(t) + \Gamma(u))du} dt. \end{aligned} \quad (4.18)$$

As n goes to infinity, χ^n converges to χ in $L^1_{\text{loc}}(\mathbb{R})$ and for almost all $t \in (0, T)$, $\sigma \in \mathbb{R}$,

$$\int_0^t \chi^n(\sigma - \Gamma(t) + \Gamma(u))du \rightarrow \int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u))du.$$

Because p and ψ respectively belong to $L^\infty((0, T), L^1)$ and $\mathcal{D}((0, T) \times \mathbb{R})$, all terms of (4.18) are bounded from below and from above by an integrable function independent of n . We apply the dominated convergence theorem to pass to the limit in (4.18)

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}} p(t, \sigma) \frac{\partial \psi}{\partial t}(t, \sigma - \Gamma(t))e^{\int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u))du} d\sigma dt \\ = \int_0^T f(t) \psi(t, -\Gamma(t))e^{\int_0^t \chi(-\Gamma(t) + \Gamma(u))du} dt, \end{aligned}$$

hence (4.16), using (4.17).

Define $t^* = \Gamma^{-1}(\sigma_c)$. We now show that $\tilde{p} = 0$ in $L^\infty(0, t^*; L^1)$, this will prove that $p = 0$ in $L^\infty(0, t^*; L^1)$.

From (4.16), we deduce that $\frac{\partial \tilde{p}}{\partial t} = 0$ in $\mathcal{D}'(\tilde{\Omega})$ with

$$\tilde{\Omega} = (0, t^*) \times ((-\infty, -\Gamma(t^*)) \cup (0, \infty)).$$

Using that $p_0 = 0$, we find $\tilde{p} = 0$ in $\mathcal{D}'(\tilde{\Omega})$ and therefore in $L^\infty(0, t^*; L^1((-\infty, -\Gamma(t^*)) \cup (0, \infty)))$. This implies $p = 0$ in $L^\infty(0, t^*; L^1((-\infty, -\Gamma(t^*)) \cup (\Gamma(t^*), \infty)))$.

In particular, since $\Gamma(t^*) = \sigma_c$, we have, for all $t \in (0, t^*)$, $f(t) = \int \chi p(t, \cdot) = 0$ and thus, the equation (4.16) reads $\frac{\partial \tilde{p}}{\partial t} = 0$ in $\mathcal{D}'((0, t^*) \times \mathbb{R})$. We deduce that $\tilde{p} = 0$ and therefore that $p = 0$ in $L^\infty(0, t^*; L^1)$.

Taking t^* as initial time, we find $p = 0$ in $L^\infty(t^*, 2t^*; L^1)$ with the previous arguments. Iterating, we obtain $p = 0$ in $L^\infty(0, T; L^1)$. This concludes the proof of uniqueness.

Our next two steps are respectively devoted to proving that p defined by (4.14) belongs to $C([0, T]; L^1)$ and that it satisfies equation (4.1).

Step 2: Regularity of expression (4.14) First, the function A defined by (4.12) belongs to $L^\infty(0, T)$ (with $\|A\|_{L^\infty(0, T)} \leq \|p_0\|_{L^1}$) and therefore ϕ defined by the recurrence relation (4.13) also belongs to $L^\infty(0, T)$. This implies, for almost all $t \in [0, T)$,

$$\int |p(t, \sigma)| d\sigma \leq \|p_0\|_{L^1} + \frac{\Gamma(T)}{m_\gamma} \|\phi\|_{L^\infty(0, T)},$$

that is p defined by (4.14) belongs to $L^\infty(0, T; L^1)$. Denote

$$\begin{aligned} p_{11}(t, \sigma) &= p_0(\sigma - \Gamma(t)) \\ p_{12}(t, \sigma) &= \int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du \\ p_{21}(t, \sigma) &= \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} \mathbb{1}_{(0, \Gamma(t))}(\sigma) \mathbb{1}_{\mathbb{R}_+^*}(t) \\ p_{22}(t, \sigma) &= \int_{\Gamma^{-1}(\Gamma(t) - \sigma)}^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du \end{aligned}$$

so that $p = p_{11}e^{-p_{12}} + p_{21}e^{-p_{22}}$. We now check that p_{11} and p_{21} belong to $C^0([0, T]; L^1)$ and p_{12} and p_{22} belong to $C^0([0, T]; L^\infty)$. Using that $x \mapsto e^{-x}$ is 1-Lipschitz on $[0, \infty)$, this will prove that $p \in C^0([0, T]; L^1)$.

Consider $\epsilon > 0$, $t \in [0, T)$ and h such that $t + h \in [0, T)$. By density of $\mathcal{D}(\mathbb{R})$ in $L^1(\mathbb{R})$, there exists $p_\epsilon \in \mathcal{D}(\mathbb{R})$ such that

$$\|p_\epsilon - p_0\|_{L^1} < \epsilon.$$

We obtain

$$\begin{aligned} \|p_{11}(t + h, \cdot) - p_{11}(t, \cdot)\|_{L^1} &= \int_{\mathbb{R}} |p_0(\sigma - \Gamma(t + h)) - p_0(\sigma - \Gamma(t))| d\sigma \\ &\leq 2 \|p_\epsilon - p_0\|_{L^1} + \int_{\mathbb{R}} |p_\epsilon(\sigma - \Gamma(t + h)) - p_\epsilon(\sigma - \Gamma(t))| d\sigma \\ &\leq 2\epsilon + \int_{\mathbb{R}} g_\epsilon^h(\sigma) d\sigma \end{aligned}$$

with $g_\epsilon^h(\sigma) = |p_\epsilon(\sigma - \Gamma(t+h)) - p_\epsilon(\sigma - \Gamma(t))|$. Moreover, by continuity of p_ϵ and Γ , we have, for all $\sigma \in \mathbb{R}$,

$$g_\epsilon^h(\sigma) \rightarrow 0 \text{ as } h \rightarrow 0$$

and $g_\epsilon^h \leq 2 \|p_\epsilon\|_{L^\infty}$ on a bounded interval (depending on the support of p_ϵ and on $\Gamma(T)$). Using Lebesgue dominated convergence Theorem, we deduce that there exists $\eta_\epsilon > 0$ such that for all $h \in (-\eta_\epsilon, \eta_\epsilon)$,

$$\int_{\mathbb{R}} g_\epsilon^h(\sigma) d\sigma \leq \epsilon$$

so that

$$\|p_{11}(t+h, \cdot) - p_{11}(t, \cdot)\|_{L^1} \leq 3\epsilon.$$

This yields $p_{11} \in C^0([0, T]; L^1)$.

We now turn to p_{12} . For almost all $\sigma \in \mathbb{R}$, we have

$$\begin{aligned} & |p_{12}(t+h, \sigma) - p_{12}(t, \sigma)| \\ & \leq \int_0^t |\chi(\sigma - \Gamma(t+h) + \Gamma(u)) - \chi(\sigma - \Gamma(t) + \Gamma(u))| du \\ & \quad + \int_t^{t+h} \chi(\sigma - \Gamma(t+h) + \Gamma(u)) du \\ & \leq \frac{1}{m_\gamma} \int_\sigma^{\sigma + \Gamma(t)} |\chi(v - \Gamma(t+h)) - \chi(v - \Gamma(t))| dv + h \\ & \leq \frac{1}{m_\gamma} \int_{\mathbb{R}} |\chi(v - \Gamma(t+h)) - \chi(v - \Gamma(t))| dv + h. \end{aligned}$$

This leads to

$$\|p_{12}(t+h, \cdot) - p_{12}(t, \cdot)\|_{L^\infty} \leq \frac{2}{m_\gamma} (\Gamma(t+h) - \Gamma(t)) + h$$

which yields $p_{12} \in C^0([0, T]; L^\infty)$.

For any $t \in (0, T)$ and h such that $t+h \in [0, T)$, we have

$$\begin{aligned} & \int_{\mathbb{R}} |p_{21}(t+h, \sigma) - p_{21}(t, \sigma)| d\sigma \\ & \leq \int_0^{\Gamma(t)} \left| \frac{\phi \circ \Gamma^{-1}(\Gamma(t+h) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t+h) - \sigma)} - \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} \right| d\sigma \\ & \quad + \left| \int_{\Gamma(t)}^{\Gamma(t+h)} \frac{\phi \circ \Gamma^{-1}(\Gamma(t+h) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t+h) - \sigma)} d\sigma \right|. \end{aligned}$$

Since ϕ belongs to $L^\infty(0, T)$, $\phi \circ \Gamma^{-1}$ belongs to $L^\infty(0, \Gamma(T)) \subset L^1(0, \Gamma(T))$ and we introduce a sequence θ_n in $\mathcal{D}(0, \Gamma(T))$ such that

$$\theta_n \rightarrow \phi \circ \Gamma^{-1} \text{ in } L^1(0, \Gamma(T))$$

and obtain

$$\begin{aligned}
& \int_{\mathbb{R}} |p_{21}(t+h, \sigma) - p_{21}(t, \sigma)| d\sigma \\
& \leq \frac{2}{m_\gamma} \left\| \theta_n - \phi \circ \Gamma^{-1} \right\|_{L^1(0, \Gamma(T))} \\
& \quad + \int_0^{\Gamma(t)} \left| \frac{\theta_n(\Gamma(t+h) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t+h) - \sigma)} - \frac{\theta_n(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} \right| d\sigma \\
& \quad + \frac{\Gamma(t+h) - \Gamma(t)}{m_\gamma} \|\phi\|_{L^\infty(0, T)}.
\end{aligned}$$

Using the dominated convergence theorem for the second term, this implies that $\int_{\mathbb{R}} |p_{21}(t+h, \sigma) - p_{21}(t, \sigma)| d\sigma$ vanishes with h . We have obtained that p_{21} belongs to $C^0((0, T); L^1)$. The continuity holds also at $t = 0$ because for all $h \in (0, T)$

$$\begin{aligned}
\int_{\mathbb{R}} |p_{21}(h, \sigma) - 0| d\sigma & \leq \int_0^{\Gamma(h)} \left| \frac{\phi \circ \Gamma^{-1}(\Gamma(h) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(h) - \sigma)} \right| d\sigma \\
& \leq \frac{\Gamma(h)}{m_\gamma} \|\phi\|_{L^\infty(0, T)}.
\end{aligned}$$

Finally, the function $p_3 = p_{12} - p_{22}$ is in $C^0((0, T); L^\infty)$ (and so is p_{22}). Indeed, for any $t \in [0, T)$ and h such that $t+h \in [0, T)$, for almost all $\sigma \in \mathbb{R}$,

$$\begin{aligned}
& |p_3(t+h, \sigma) - p_3(t, \sigma)| \\
& \leq \int_0^{\Gamma^{-1}(\Gamma(t) - \sigma)} |\chi(\sigma - \Gamma(t+h) + \Gamma(u)) - \chi(\sigma - \Gamma(t) + \Gamma(u))| du \\
& \quad + \int_{\Gamma^{-1}(\Gamma(t) - \sigma)}^{\Gamma^{-1}(\Gamma(t+h) - \sigma)} \chi(\sigma - \Gamma(t+h) + \Gamma(u)) du
\end{aligned}$$

so that, with the change of variable $v = \sigma + \Gamma(u)$,

$$\begin{aligned}
& \|p_3(t+h, \cdot) - p_3(t, \cdot)\|_{L^\infty} \\
& \leq \frac{1}{m_\gamma} \left(\int_{\mathbb{R}} |\chi(v - \Gamma(t+h)) - \chi(v - \Gamma(t))| dv + \int_{\Gamma(t)}^{\Gamma(t+h)} dv \right) \\
& \leq \frac{3}{m_\gamma} (\Gamma(t+h) - \Gamma(t)).
\end{aligned}$$

This concludes the proof of the continuity in time of p , with values in L^1 .

Step 3: Expression (4.14) satisfies (4.1) We first show that p defined by (4.14) satisfies

$$\frac{\partial p}{\partial t} + \gamma(t) \frac{\partial p}{\partial \sigma} = -\chi p + \phi \delta_0(\sigma) \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}). \quad (4.19)$$

For all $\psi \in \mathcal{D}((0, T) \times \mathbb{R})$,

$$\begin{aligned}
& - \int_0^T \int_{\mathbb{R}} p \left(\frac{\partial \psi}{\partial t} + \gamma \frac{\partial \psi}{\partial \sigma} - \chi \psi \right) \\
& = - \int_0^T \int_{\mathbb{R}} p_0(\sigma - \Gamma(t)) e^{-\int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} \left(\frac{\partial \psi}{\partial t} + \gamma \frac{\partial \psi}{\partial \sigma} - \chi \psi \right) (t, \sigma) d\sigma dt \\
& \quad - \int_0^T \int_0^{\Gamma(t)} \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} e^{-\int_{\Gamma^{-1}(\Gamma(t) - \sigma)}^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} \\
& \quad \quad \times \left(\frac{\partial \psi}{\partial t} + \gamma \frac{\partial \psi}{\partial \sigma} - \chi \psi \right) (t, \sigma) d\sigma dt \\
& = - \int_0^T \int_{\mathbb{R}} p_0(\xi) e^{-\int_0^t \chi(\xi + \Gamma(u)) du} \left(\frac{\partial \psi}{\partial t} + \gamma \frac{\partial \psi}{\partial \sigma} - \chi \psi \right) (t, \xi + \Gamma(t)) d\xi dt \\
& \quad - \int_0^T \int_0^t \phi(v) e^{-\int_v^t \chi(-\Gamma(v) + \Gamma(u)) du} \\
& \quad \quad \times \left(\frac{\partial \psi}{\partial t} + \gamma \frac{\partial \psi}{\partial \sigma} - \chi \psi \right) (t, -\Gamma(v) + \Gamma(t)) dv dt
\end{aligned}$$

where we have made the changes of variables $\xi = \sigma - \Gamma(t)$ and $v = \Gamma^{-1}(\Gamma(t) - \sigma)$. Introducing

$$\begin{aligned}
\nu(t, \xi) &= \psi(t, \xi + \Gamma(t)) e^{-\int_0^t \chi(\xi + \Gamma(u)) du} \\
\text{and } \mu(t, v) &= \psi(t, -\Gamma(v) + \Gamma(t)) e^{-\int_v^t \chi(-\Gamma(v) + \Gamma(u)) du}
\end{aligned}$$

this rewrites

$$\begin{aligned}
& - \int_0^T \int_{\mathbb{R}} p \left(\frac{\partial \psi}{\partial t} + \gamma \frac{\partial \psi}{\partial \sigma} - \chi \psi \right) \\
& = - \int_{\mathbb{R}} p_0(\xi) \left[\int_0^T \frac{\partial \nu}{\partial t}(t, \xi) dt \right] d\xi - \int_0^T \phi(v) \left[\int_v^T \frac{\partial \mu}{\partial t}(t, v) dt \right] dv \\
& = \int_{\mathbb{R}} p_0(\xi) [\nu(0, \xi) - \nu(T, \xi)] d\xi \\
& \quad + \int_0^T \phi(v) [\mu(v, v) - \mu(T, v)] dv = \int_0^T \phi(v) \psi(v, 0) dv
\end{aligned}$$

thus (4.19).

We finally show that $\phi = \int \chi p$ a.e. on $(0, T)$, where p is defined by (4.14). This will prove that (4.1) holds in $D'((0, T) \times \mathbb{R})$. First, for almost all $t \in (0, \Gamma^{-1}(\sigma_c))$, definition (4.14) of p implies $\int \chi(\sigma) p(t, \sigma) d\sigma = A(t)$. The definition (4.13) of ϕ implies $\phi(t) = A(t)$, thus $\phi(t) = \int \chi(\sigma) p(t, \sigma) d\sigma$ for such a time t . We next take $t \in (\Gamma^{-1}(\sigma_c), T)$. We have

$$\begin{aligned}
f(t) &= A(t) + \int_{\sigma_c}^{\Gamma(t)} \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} e^{-\int_{\Gamma^{-1}(\Gamma(t) - \sigma)}^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} d\sigma \\
&= A(t) + \int_{\sigma_c}^{\Gamma(t)} \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} e^{-t + \Gamma^{-1}(\Gamma(t) - \sigma + \sigma_c)} d\sigma \\
&= A(t) + \int_0^{\Gamma^{-1}(\Gamma(t) - \sigma_c)} \phi(s) e^{-t + \Gamma^{-1}(\Gamma(s) + \sigma_c)} ds \\
&= \phi(t),
\end{aligned}$$

where we have respectively simplified the exponential term, made the change of variables $s = \Gamma^{-1}(\Gamma(t) - \sigma)$ and used the definition (4.13) of ϕ . This concludes the proof of this step, and thus that of Theorem 4.1. \square

4.4 Properties of the solution

In this section we prove various properties of the solution of (4.1) the existence and uniqueness of which has been established in Theorem 4.1. We therefore assume throughout this section that, as for Theorem 4.1, p_0 in $L^1(\mathbb{R})$ and γ satisfies (4.11).

Lemma 4.2 (Maximum principle). *If $p_0(\sigma) \geq 0$ for almost all $\sigma \in \mathbb{R}$ then $p(t, \sigma) \geq 0$ for almost all $t \in [0, T], \sigma \in \mathbb{R}$.*

Proof. Using the definition (4.12) of A , we first have $A(t) \geq 0$ for almost all $t \in [0, T]$. Because of recurrence relation (4.13) on ϕ , we then find that $\phi(t) \geq 0$ for almost all $t \in [0, T]$. Consequently, the expression (4.14) on p gives the result. \square

Lemma 4.3 (Mass conservation). *If $\int_{\mathbb{R}} p_0 = 1$ then $\int_{\mathbb{R}} p(t, \cdot) = 1$ for all $t \in [0, T]$.*

Proof. Denote

$$h(t) = \int_{\mathbb{R}} p(t, \cdot) \quad (4.20)$$

which is continuous since p belongs to $C^0([0, T]; L^1)$. Using (4.14), we obtain that h reads

$$\begin{aligned} h(t) &= \int_{\mathbb{R}} p_0(\sigma - \Gamma(t)) e^{-\int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} d\sigma \\ &\quad + \int_0^{\Gamma(t)} \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} e^{-\int_{\Gamma^{-1}(\Gamma(t) - \sigma)}^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} d\sigma. \end{aligned}$$

For all $t \in [0, \Gamma^{-1}(\sigma_c)]$, h rewrites

$$\begin{aligned} h(t) &= \int_{\mathbb{R}} p_0(\xi) e^{-\int_0^t \chi(\xi + \Gamma(u)) du} d\xi + \int_0^{\Gamma(t)} \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} d\sigma \\ &= \int_{\mathbb{R}} p_0(\xi) e^{-\int_0^t \chi(\xi + \Gamma(u)) du} d\xi + \int_0^t \phi(s) ds. \end{aligned}$$

Differentiating h in the sense of distributions in time, we obtain

$$\begin{aligned} \dot{h}(t) &= - \int_{\mathbb{R}} \chi(\xi + \Gamma(t)) p_0(\xi) e^{-\int_0^t \chi(\xi + \Gamma(u)) du} d\xi + \phi(t) \\ &= -A(t) + \phi(t) \\ &= 0, \end{aligned}$$

using the definitions (4.12) of A and (4.13) of ϕ . Using that h is continuous on $[0, T]$ with $h(0) = 1$, we find $h = 1$ on $[0, \Gamma^{-1}(\sigma_c)]$. For all $t \in (\Gamma^{-1}(\sigma_c), T)$, h rewrites

$$\begin{aligned} h(t) &= \int_{\mathbb{R}} p_0(\xi) e^{-\int_0^t \chi(\xi + \Gamma(u)) du} d\xi + \int_0^{\sigma_c} \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} d\sigma \\ &\quad + \int_{\sigma_c}^{\Gamma(t)} \frac{\phi \circ \Gamma^{-1}(\Gamma(t) - \sigma)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma)} e^{-t + \Gamma^{-1}(\Gamma(t) - \sigma + \sigma_c)} d\sigma \\ &= \int_{\mathbb{R}} p_0(\xi) e^{-\int_0^t \chi(\xi + \Gamma(u)) du} d\xi + \int_{\Gamma^{-1}(\Gamma(t) - \sigma_c)}^t \phi(s) ds \\ &\quad + \int_0^{\Gamma^{-1}(\Gamma(t) - \sigma_c)} \phi(s) e^{-t + \Gamma^{-1}(\Gamma(s) + \sigma_c)} ds. \end{aligned}$$

Differentiating h in $\mathcal{D}'(\Gamma^{-1}(\sigma_c), T)$, we obtain

$$\begin{aligned} \dot{h}(t) &= -A(t) + \phi(t) - \int_0^{\Gamma^{-1}(\Gamma(t) - \sigma_c)} \phi(s) e^{-t + \Gamma^{-1}(\Gamma(s) + \sigma_c)} ds \\ &= 0, \end{aligned}$$

because of definitions (4.12) of A and (4.13) of ϕ . This implies $h = 1$ on $[0, T]$, hence the result. \square

Lemma 4.4 (L^∞ -bounds). *Assume p_0 satisfies*

$$p_0 \in L^1(\mathbb{R}), \quad p_0 \geq 0.$$

Then, ϕ defined by (4.13) (or, equivalently, f , given (4.15)) satisfies

$$\|\phi\|_{L^\infty(0, T)} \leq \int_{\mathbb{R}} p_0. \quad (4.21)$$

If in addition p_0 satisfies

$$p_0 \in L^\infty(\mathbb{R}),$$

then p belongs to $L^\infty(0, T; L^\infty)$ and there exists a constant C_∞ which depends only on $\|p_0\|_{L^1 \cap L^\infty}$ and the bound m_γ in (4.11) such that

$$\|p\|_{L_T^\infty(L^\infty)} \leq C_\infty. \quad (4.22)$$

Proof. First, we notice that

$$f(t) = \int_{\mathbb{R}} \chi p(t, \cdot) \leq \int_{\mathbb{R}} p(t, \cdot) = \int_{\mathbb{R}} p_0, \quad (4.23)$$

successively using the definition (4.4) of f , Lemma 4.2 and Lemma 4.3. This immediately implies

$$\|\phi\|_{L^\infty(0, T)} \leq \int_{\mathbb{R}} p_0.$$

Using the lower bound (4.11) on γ , the L^∞ -bound on p_0 and the expression (4.14) of p in terms of ϕ , we find that p belongs to $L^\infty(0, T; L^\infty)$ and

$$\|p\|_{L_T^\infty(L^\infty)} \leq \|p_0\|_{L^\infty} + \frac{\int_{\mathbb{R}} p_0}{m_\gamma}. \quad \square$$

Lemma 4.5 (Delay differential equation). *Assume $p_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, A and $\phi = f$ respectively defined by (4.12), (4.13) and (4.4), belong to $W^{1,1}(0, T)$ thus are continuous in time. Moreover, A and ϕ satisfy, for almost all $t \in (0, T)$,*

$$\begin{aligned} \dot{A}(t) + A(t) &= \gamma(t) \left[p_0(\sigma_c - \Gamma(t)) e^{-\int_0^t \chi(\sigma_c - \Gamma(t) + \Gamma(u)) du} \right. \\ &\quad \left. - p_0(-\sigma_c - \Gamma(t)) e^{-\int_0^t \chi(-\sigma_c - \Gamma(t) + \Gamma(u)) du} \right] \end{aligned} \quad (4.24)$$

and for almost all $t \in (\Gamma^{-1}(\sigma_c), T)$,

$$\dot{\phi}(t) + \phi(t) - \frac{\gamma(t)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma_c)} \phi \circ \Gamma^{-1}(\Gamma(t) - \sigma_c) = \dot{A}(t) + A(t). \quad (4.25)$$

Proof. As stated in Theorem 4.1, the functions A and ϕ belong to $L^\infty(0, T) \subset L^1(0, T)$. We show that both \dot{A} and $\dot{\phi}$ belong to $L^1(0, T)$.

The expression (4.12) on A rewrites

$$A(t) = \int_{\mathbb{R} \setminus [-\sigma_c - \Gamma(t), \sigma_c - \Gamma(t)]} p_0(\xi) e^{-\int_0^t \chi(\xi + \Gamma(u)) du} d\xi.$$

Differentiating the above expression in the sense of distributions in time, we obtain equation (4.24). Since p_0 belongs to $L^\infty(0, T)$ and γ and A belong to $L^1(0, T)$, \dot{A} belongs to $L^1(0, T)$ and the equation (4.24) holds for almost all $t \in (0, T)$.

Since $\phi = A$ on $(0, \Gamma^{-1}(\sigma_c))$, $\dot{\phi}$ belongs to $L^1(0, \Gamma^{-1}(\sigma_c))$. Differentiating the recurrence relation (4.13) in the sense of distributions on $(\Gamma^{-1}(\sigma_c), T)$, we have

$$\begin{aligned} \dot{\phi}(t) &= \dot{A}(t) + \frac{\gamma(t)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma_c)} \phi \circ \Gamma^{-1}(\Gamma(t) - \sigma_c) \\ &\quad - \int_0^{\Gamma^{-1}(\Gamma(t) - \sigma_c)} \phi(s) e^{-t + \Gamma^{-1}(\Gamma(s) + \sigma_c)} ds, \\ &= \dot{A}(t) + \frac{\gamma(t)}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma_c)} \phi \circ \Gamma^{-1}(\Gamma(t) - \sigma_c) + A(t) - \phi(t). \end{aligned}$$

Using that A , γ and ϕ respectively belong to $W^{1,1}(0, T)$, $L^1(0, T)$ and $L^\infty(0, T)$, the right-hand side of the above equation and thus $\dot{\phi}$ belong to $L^1(\Gamma^{-1}(\sigma_c), T)$. Moreover, the equation (4.25) holds for almost all $t \in (\Gamma^{-1}(\sigma_c), T)$. This ends the proof. \square

Lemma 4.6 (Existence of τ). *Assume that p_0 satisfies*

$$p_0 \in L^1(\mathbb{R}), \quad p_0 \geq 0 \quad \int |\sigma| p_0 < \infty.$$

Then σp belongs to $L^\infty(0, T; L^1)$ so that the average stress τ defined by (4.5) belongs to $L^\infty(0, T)$. Moreover, if there exists a scalar M_γ independent from T such that $\gamma \leq M_\gamma$, then there exists a constant C_τ independent from T such that

$$\left\| \int |\sigma| p(t, \sigma) d\sigma \right\|_{L^\infty(0, T)} \leq C_\tau (1 + M_\gamma T). \quad (4.26)$$

Proof. We multiply the expression (4.14) on p by $|\sigma|$ and integrate in σ . This implies

$$\begin{aligned} & \int |\sigma| p(t, \sigma) d\sigma \\ & \leq \int |\sigma| p_0(\sigma - \Gamma(t)) d\sigma + \int_0^{\Gamma(t)} \sigma \frac{\|\phi\|_{L^\infty}}{m_\gamma} e^{-\int_{\Gamma^{-1}(\Gamma(t)-\sigma)}^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} d\sigma \\ & \leq \int |\sigma| p_0 + \left(\int p_0 \right) \left(\Gamma(T) + \int_0^{\Gamma(t)} \frac{\sigma}{m_\gamma} e^{-\int_{\Gamma^{-1}(\Gamma(t)-\sigma)}^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du} d\sigma \right), \end{aligned} \quad (4.27)$$

using that $\gamma \geq m_\gamma$ and then the upper bound (4.21) on $\|\phi\|_{L^\infty(0, T)}$. We deduce

$$\int |\sigma| p(t, \sigma) d\sigma \leq \int |\sigma| p_0 + \left(\int p_0 \right) \left(\Gamma(T) + \int_0^{\Gamma(T)} \frac{\sigma}{m_\gamma} d\sigma \right),$$

so that σp belongs to $L^\infty(0, T; L^1)$. Moreover, from (4.27), we obtain, for all $t > \Gamma^{-1}(\sigma_c)$,

$$\begin{aligned} \int |\sigma| p(t, \sigma) d\sigma & \leq \max \left(\int p_0, \int |\sigma| p_0 \right) \\ & \left(1 + M_\gamma T + \int_0^{\sigma_c} \frac{\sigma}{m_\gamma} d\sigma + \frac{1}{m_\gamma} \int_{\sigma_c}^{\Gamma(t)} \sigma e^{-t + \Gamma^{-1}(\Gamma(t) - \sigma + \sigma_c)} d\sigma \right). \end{aligned} \quad (4.28)$$

Additionally, with the change of variable $v = t - \Gamma^{-1}(\Gamma(t) - \sigma + \sigma_c)$, the last integral satisfies

$$\begin{aligned} \int_{\sigma_c}^{\Gamma(t)} \sigma e^{-t + \Gamma^{-1}(\Gamma(t) - \sigma + \sigma_c)} d\sigma & = \int_0^{t - \Gamma^{-1}(\sigma_c)} (\Gamma(t) - \Gamma(t - v) + \sigma_c) e^{-v} \frac{1}{\gamma \circ \Gamma^{-1}(\Gamma(t) - \sigma + \sigma_c)} dv \\ & \leq \int_0^{t - \Gamma^{-1}(\sigma_c)} \left(\int_{t-v}^t M_\gamma du + \sigma_c \right) \frac{e^{-v}}{m_\gamma} dv \\ & \leq \int_0^\infty (M_\gamma v + \sigma_c) \frac{e^{-v}}{m_\gamma} dv < \infty. \end{aligned}$$

Inserting the above inequality in (4.28) yields (4.26). This concludes the proof. \square

Throughout our article, we now assume that

$$p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad p_0 \geq 0, \quad \int_{\mathbb{R}} p_0 = 1 \quad \text{and} \quad \int_{\mathbb{R}} |\sigma| p_0 < \infty, \quad (4.29)$$

so that the five Lemmata 4.2, 4.3, 4.4, 4.5 and 4.6 hold.

4.5 Longtime behavior in the case $\gamma(t) = \gamma_\infty$

In this section, we assume that $\gamma(t) = \gamma_\infty$ where $\gamma_\infty > 0$ is a given *fixed* constant and we study the longtime convergence for equation (4.1). We prove the following.

Theorem 4.2. *Assume that $\gamma(t) = \gamma_\infty$ with $\gamma_\infty > 0$ a given constant. Supply equation (4.1) with an initial condition p_0 that satisfies (4.29). Consider p the solution to (4.1) and p_∞ the associated stationary solution, the existence and uniqueness of those has been respectively established in Theorem 4.1 and Lemma 4.1. Then p converges exponentially fast in time to p_∞ for almost all $\sigma \in \mathbb{R}$. In addition, f defined by (4.4) converges exponentially fast in time to $\int \chi p_\infty$.*

More precisely, there exist $b, C_1 > 0$ such that, for all t and for almost all $\sigma \in \mathbb{R}$,

$$|p(t, \sigma) - p_\infty(\sigma)| \leq C_1 \left(e^{-t} + \left(e^{-bt} e^{\frac{b\sigma}{\gamma_\infty}} + e^{-t} e^{\frac{\sigma}{\gamma_\infty}} \right) \mathbb{1}_{(0, \gamma_\infty t)}(\sigma) \right). \quad (4.30)$$

In addition, there exists a positive continuous function C_2 of $\omega \in \mathbb{R}_+^*$ such that for all $t \geq 0$,

$$\left| f(t) - \int \chi p_\infty \right| \leq C_2(\omega) \left(e^{-bt} + e^{-t} \right). \quad (4.31)$$

In the two estimates (4.30) and (4.31), the rate $b > 0$ can be chosen as

$$b = -\max\{x < 0 \text{ s.t. } x + 1 - e^{-\omega x} \cos(\omega \sqrt{e^{-2\omega x} - (x+1)^2}) = 0\} - \eta \quad (4.32)$$

for any $\eta > 0$, where ω is defined by:

$$\omega = \frac{\sigma_c}{\gamma_\infty}. \quad (4.33)$$

Notice that the estimates (4.30) and (4.31) rely on the two natural timescales of the original problem: the exponential rate 1 of the jump process to zero (the hidden coefficient 1 multiplying the right-hand side of the Fokker-Planck equation (4.1)), and the typical time ω required for the process to leave the domain $(-\sigma_c, \sigma_c)$, when the shear rate is γ_∞ .

Let us notice from (4.32) that we may assume in the following, without loss of generality, that

$$b \neq 1. \quad (4.34)$$

This will be an assumption on b in the forthcoming sections. This assumption is exclusively technical. We require it to simplify some proofs where convolutions with exponential kernels are involved, in particular to use estimates such as $\int_0^t e^{-(t-s)} e^{-bs} ds \leq C(e^{-bt} + e^{-t})$ (which is only true if $b \neq 1$).

Before we get to the proof of Theorem 4.2, to which the rest of this section is devoted, we need to introduce some notation and make some preliminaries.

Denote by $q(t, \sigma) = p(t, \sigma) - p_\infty(\sigma)$ and $q_0(\sigma) = p_0(\sigma) - p_\infty(\sigma)$. By linearity, and in place of A and ϕ introduced in Theorem 4.1, we similarly introduce

$$B(t) = \int \chi(\xi + \gamma_\infty t) q_0(\xi) e^{-\frac{1}{\gamma_\infty} \int_\xi^{\xi + \gamma_\infty t} \chi(v) dv} d\xi \quad (4.35)$$

and

$$g(t) = \begin{cases} B(t) & \text{for almost all } t \in (0, \omega) \\ B(t) + \int_0^{t-\omega} g(s) e^{-t+s+\omega} ds & \text{for almost all } t \in (k\omega, (k+1)\omega). \end{cases} \quad (4.36)$$

We then have from Theorem 4.1

$$q(t, \sigma) = q_0(\sigma - \gamma_\infty t) e^{-\frac{1}{\gamma_\infty} \int_{\sigma - \gamma_\infty t}^\sigma \chi(v) dv} + \frac{1}{\gamma_\infty} g\left(t - \frac{\sigma}{\gamma_\infty}\right) e^{-\frac{1}{\gamma_\infty} \int_0^\sigma \chi(v) dv} \mathbb{1}_{(0, \gamma_\infty t)}(\sigma). \quad (4.37)$$

Using Lemma 4.5, g satisfies, for almost all $t > \omega$,

$$\dot{g}(t) + g(t) - g(t - \omega) = \dot{B}(t) + B(t). \quad (4.38)$$

The equation (4.38) on g is a delay differential equation with constant coefficients. The proof of Theorem 4.2 is based upon three lemmata for such a delay differential equation, denoted in generality by

$$\begin{cases} \dot{u}(t) + u(t) - u(t - \omega) = \mu(t) & \text{for } t \geq \omega \\ u(t) = \nu(t) & \text{for } t \in (0, \omega) \end{cases} \quad (4.39)$$

where $\omega > 0$ is a constant and μ is a locally integrable function. Equation (4.39) is understood in the sense of distribution in time. To such a delay differential equation is classically associated the unique function $k(t)$ satisfying the following properties:

1. $k(t) = 0, \quad \forall t < 0;$
2. $k(0) = 1;$
3. $k(t)$ is continuous on $[0, \infty);$
4. $k(t)$ satisfies for all $t > 0,$

$$\dot{k}(t) + k(t) - k(t - \omega) = 0. \quad (4.40)$$

The three lemmata useful for the proof of Theorem 4.2 are Lemma 4.9, itself proved using Lemma 4.7, and Lemma 4.8. The latter two lemmata, Lemma 4.7 and Lemma 4.8, are borrowed respectively from [HL93] and [BC63]. They are valid for more general delay differential equations, but for simplicity, we state them here for our specific delay differential equation (4.39).

Lemma 4.7. [HL93, Equation (5.10), p. 22] Consider k defined by (4.40) and the associated properties above. Denote by

$$h(\lambda) = \lambda + 1 - e^{-\omega\lambda}. \quad (4.41)$$

Then, for all $\alpha_m \in \mathbb{R}$ such that no root of h has real part equal to α_m , the function k writes, for all $t > 0,$

$$k(t) = \sum_{j=1}^{k_m} \text{Res}_{\lambda=\lambda_j} \left(\frac{e^{\lambda t}}{h(\lambda)} \right) + \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{-T}^T \frac{e^{(\alpha_m + iu)t}}{h(\alpha_m + iu)} du. \quad (4.42)$$

where $\lambda_1, \dots, \lambda_{k_m}$ are the roots of h such that $\text{Re}(\lambda_j) > \alpha_m$ and $\text{Res}_{\lambda=\lambda_j}$ denotes the residue at $\lambda = \lambda_j.$

Intuitively, (4.42) is obtained as follows. The function k solves $\dot{k}(t) + k(t) - k(t - \omega) = \delta_0$ on $\mathcal{D}'(\mathbb{R})$, which by Laplace transform and using the notation (4.41), yields $h(s)\mathcal{L}(k)(s) = 1$ (where $\mathcal{L}(k)$ denotes the Laplace transform of k). It remains then to divide by h and apply a reverse Laplace transform to finally obtain k . The difficulty is of course related to the zeros of the function h .

Lemma 4.8. [BC63, Theorem 3.7, p. 75] *Consider two functions $\nu \in C^0[0, \omega]$ and $\mu \in L^1_{\text{loc}}(\omega, \infty)$. Then, there exists a unique solution $u(t) \in C^0(\mathbb{R}_+)$ verifying (4.39) in the sense of distribution, and $u(t)$ satisfies: for $t \geq 0$,*

$$u(t) = \nu(\omega)k(t - \omega) - \int_0^\omega \nu(t_1)k(t - t_1 - \omega)dt_1 + \int_\omega^t \mu(t_1)k(t - t_1)dt_1. \quad (4.43)$$

Remark 4.3. *The result of Lemma 4.8 is stated in [BC63] for μ continuous but holds for $\mu \in L^1_{\text{loc}}(\omega, \infty)$. Indeed, the existence of a unique solution is still valid in this more general setting (see [HL93, p.14]) and expression (4.43) satisfies (4.39) almost everywhere.*

As announced above, we first use Lemma 4.7 to prove the following

Lemma 4.9. *Assume*

$$m_0 < \omega < M_0. \quad (4.44)$$

Then there exist $b > 0$ and $C_0 > 0$ which depend only on m_0 and M_0 and such that for all $t > 0$

$$k(t) = \frac{1}{1 + \omega} + k_1(t) \quad (4.45)$$

with

$$|k_1(t)| \leq C_0 e^{-bt}. \quad (4.46)$$

Moreover, b can be chosen as (4.32), for any $\eta > 0$.

We immediately emphasize that the point in Lemma 4.9 is to show that the prefactor C_0 and the exponent b appearing in (4.46) do not depend on ω itself, but can be chosen locally uniformly, that is, depend only on the bounds m_0 and M_0 of the interval where ω lies. Proving (4.46) for a fixed ω is a simple consequence of the classical results contained e.g. in [BC63, HL93].

Proof of Lemma 4.9. The proof falls in three steps. We first derive an upper bound on the real part of the nonzero roots of the function h defined by (4.41). This upper bound actually yields the exponent b in the exponential estimates of Lemma 4.9 and thus of Theorem 4.2. In the second step, we apply Lemma 4.7. In the third and final step, we conclude.

Step 1: Upper bound on the real part of the nonzero roots of h The roots of the function h defined by (4.41) are 0 and the complex numbers $\lambda = \alpha + i\beta$ (with $\alpha, \beta \in \mathbb{R}$) that satisfy

$$\alpha + 1 = e^{-\omega\alpha} \cos(\omega\beta), \quad (4.47)$$

$$\beta = -e^{-\omega\alpha} \sin(\omega\beta). \quad (4.48)$$

It is easy to check that $\alpha = 0$ implies $\beta = 0$ and conversely, so that in the following, we assume $\alpha \neq 0$ and $\beta \neq 0$. The equation (4.48) rewrites $-\frac{\sin(\omega\beta)}{\beta} = e^{\omega\alpha}$. Since the function $x \mapsto -\frac{\sin(\omega x)}{x}$ is non-positive on $[-\frac{\pi}{\omega}, \frac{\pi}{\omega}]$, β satisfies

$$|\beta| > \frac{\pi}{\omega}. \quad (4.49)$$

Moreover, we combine (4.47) and (4.48) and obtain

$$e^{2\omega\alpha} \left((\alpha + 1)^2 + \beta^2 \right) = 1. \quad (4.50)$$

This implies that α is negative and therefore, using the bounds (4.44) and (4.49) respectively on ω and β ,

$$e^{2M_0\alpha} \left((\alpha + 1)^2 + \frac{\pi^2}{M_0^2} \right) < 1.$$

The function

$$\zeta : x \mapsto e^{2M_0x} \left((x + 1)^2 + \frac{\pi^2}{M_0^2} \right)$$

is continuous, satisfies $\zeta(0) > 1$ and $\lim_{-\infty} \zeta = 0$ so that by the intermediate value theorem, there exists

$$b > 0 \text{ such that } \zeta(-b) = 1 \text{ and } \zeta(x) \geq 1 \text{ on } [-b, 0].$$

The scalar $b > 0$ depends only on M_0 . Additionally, the real part α of the nonzero roots of h satisfies

$$\alpha < -b$$

and, combining (4.47) and (4.50),

$$\alpha + 1 - e^{-\omega\alpha} \cos(\omega\sqrt{e^{-2\omega\alpha} - (\alpha + 1)^2}) = 0$$

Therefore b can be chosen as (4.32), for any $\eta > 0$.

Step 2: Applying Lemma 4.7 From the previous step, we know that the only root of h with real part strictly above $-b$ is 0. We now apply Lemma 4.7 with $\alpha_m = -b$. Since the root 0 is a simple root of h , the residue of $\frac{e^{\lambda t}}{h(\lambda)}$ at 0 is $\frac{1}{\dot{h}(0)} = \frac{1}{1 + \omega}$. Equation (4.42) therefore writes

$$k(t) = \frac{1}{1 + \omega} + \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{-T}^T \frac{e^{(-b+iu)t}}{h(-b+iu)} du. \quad (4.51)$$

Proving that there exists $C_0 > 0$ which depends only on m_0 and M_0 , such that for all $t > 0$,

$$\lim_{T \rightarrow \infty} \left| \int_{-T}^T \frac{e^{(-b+iu)t}}{h(-b+iu)} du \right| \leq C_0 e^{-bt}.$$

therefore amounts to concluding the proof of Lemma 4.9. Actually, we will show that this holds up to changing b to $b - \eta$ in the right hand side, for any positive η . This will conclude the proof.

Step 3: Exponential bound We first show, for all $t > 0$,

$$\lim_{T \rightarrow \infty} \int_{-T}^T \frac{e^{(-b+iu)t}}{h(-b+iu)} du = \lim_{T \rightarrow \infty} \frac{1}{t} \int_{-T}^T e^{(-b+iu)t} \frac{\dot{h}}{h^2}(-b+iu) du. \quad (4.52)$$

By integration by parts, we have

$$\int_{-T}^T \frac{e^{(-b+iu)t}}{h(-b+iu)} du = \frac{1}{t} \int_{-T}^T e^{(-b+iu)t} \frac{\dot{h}}{h^2}(-b+iu) du + \frac{1}{it} \left[\frac{e^{(-b+iu)t}}{h(-b+iu)} \right]_{-T}^T. \quad (4.53)$$

Introduce $T_0 > 0$ such that for all $|T| \geq T_0$,

$$\left(1 + \frac{b^2}{T^2} \right)^{\frac{1}{2}} - \frac{1}{|T|} (1 + e^{M_0 b}) \geq \frac{1}{2}.$$

Then, for all $|T| > T_0$,

$$\begin{aligned} |h(-b+iT)| &= \left| -b + iT + 1 - e^{-\omega(-b+iT)} \right| \\ &\geq \sqrt{b^2 + T^2} - 1 - e^{\omega b} \geq \sqrt{b^2 + T^2} - (1 + e^{M_0 b}) \geq \frac{|T|}{2} \end{aligned}$$

so that, for all $t > 0$,

$$\left| \frac{1}{it} \left[\frac{e^{(-b+iu)t}}{h(-b+iu)} \right]_{-T}^T \right| \leq \frac{4e^{-bt}}{|T|t}. \quad (4.54)$$

By passing to the limit $T \rightarrow \infty$ in (4.53), we thus obtain (4.52).

Now, for all $u \in \mathbb{R}$,

$$\begin{aligned} \left| \frac{\dot{h}}{h^2} \right|(-b+iu) &= \frac{|1 + \omega e^{-\omega(-b+iu)}|}{(1 - b - e^{\omega b} \cos(\omega u))^2 + (u + e^{\omega b} \sin(\omega u))^2} \\ &< \frac{1 + M_0 e^{M_0 b}}{(1 - b)^2 + u^2 - 2e^{\omega b} ((1 - b) \cos(\omega u) - u \sin(\omega u))}. \end{aligned}$$

Introduce $u_0 > 0$ which we may take depending only on M_0 , such that for all $|u| \geq u_0$,

$$(1-b)^2 + u^2 - 2e^{M_0 b}((1-b) + |u|) > 0$$

so that

$$\left| \frac{\dot{h}}{h^2} \right| (-b + iu) < \frac{1 + M_0 e^{M_0 b}}{(1-b)^2 + u^2 - 2e^{M_0 b}((1-b) + |u|)}.$$

For $T > u_0$ and $t > 0$, this implies

$$\begin{aligned} & \frac{1}{t} \left| \int_{-T}^T e^{(-b+iu)t} \frac{\dot{h}}{h^2} (-b + iu) du \right| \\ & \leq \frac{e^{-bt}}{t} \left(\int_{-u_0}^{u_0} \left| \frac{\dot{h}}{h^2} \right| (-b + iu) du + 2 \int_{u_0}^{\infty} \frac{(1 + M_0 e^{M_0 b}) du}{(1-b)^2 + u^2 - 2e^{M_0 b}((1-b) + |u|)} \right). \end{aligned} \quad (4.55)$$

The function $\omega \mapsto \int_{-u_0}^{u_0} \left| \frac{\dot{h}}{h^2} \right| (-b + iu) du$ is continuous for $\omega \in [m_0, M_0]$ and is therefore bounded by a constant that only depends on m_0 and M_0 . From (4.52) and the bound (4.55), we deduce that there exists a constant $C_0 > 0$ that also only depends on m_0 and M_0 such that

$$\lim_{T \rightarrow \infty} \left| \int_{-T}^T \frac{e^{(-b+iu)t}}{h(-b + iu)} du \right| \leq C_0 e^{-(b-\eta)t},$$

for any positive η . This concludes the proof. \square

We are now in position to turn to the

Proof of Theorem 4.2. The proof proceeds in five steps. In step 1, we apply the above lemmata to the delay differential equation (4.38). In steps 2 and 3, we derive some estimates that will be useful, in the last two steps, to show convergence of g , and eventually q .

Step 1: Applying Lemmata 4.9 and 4.8 The function g defined by (4.36) satisfies (4.39) in the particular case $\mu = \dot{B} + B$ and $\nu(t) = B(t)$. Notice that, by the Lemma 4.5, the function B defined by (4.35) belongs to $W^{1,1}(0, T)$ and thus, in particular, is continuous in time. We apply Lemma 4.8 and find (by integration by parts): for all $t > \omega$,

$$\begin{aligned} g(t) &= B(\omega)k(t - \omega) + \int_0^\omega B(t_1)k(t - t_1 - \omega)dt_1 + \int_\omega^t (\dot{B} + B)(t_1)k(t - t_1)dt_1 \\ &= B(\omega)k(t - \omega) + \int_0^\omega B(t_1)(\dot{k} + k)(t - t_1)dt_1 + \int_\omega^t (\dot{B} + B)(t_1)k(t - t_1)dt_1 \\ &= B(0)k(t) + \int_0^t (\dot{B} + B)(t_1)k(t - t_1)dt_1. \end{aligned} \quad (4.56)$$

We now recall that, in this section, the value of ω is fixed by (4.33) at $\omega = \frac{\sigma_c}{\gamma_\infty}$. We can apply Lemma 4.9 and insert the decomposition (4.45) of k into the previous equation on g . We obtain

$$\begin{aligned}
g(t) &= \frac{1}{1+\omega} \left(B(0) + \int_0^t (\dot{B} + B)(t_1) dt_1 \right) \\
&\quad + B(0)k_1(t) + \int_0^t (\dot{B} + B)(t_1)k_1(t-t_1) dt_1.
\end{aligned} \tag{4.57}$$

where k_1 satisfies (4.46) with $b, C_0 > 0$ only depending on $\omega = \frac{\sigma_c}{\gamma_\infty}$. Moreover, b can be chosen as (4.32) for any positive η as stated in Lemma 4.9.

Our next two steps consist in deriving a couple of estimates (see (4.59) and (4.60) below) on the terms of (4.57).

Step 2: Longtime convergence of $B(0) + \int_0^t (\dot{B} + B)(t_1) dt_1$ Using (4.24), the function B defined by (4.35) satisfies, for almost all $t > 0$,

$$\begin{aligned}
\dot{B}(t) + B(t) &= \gamma_\infty \left(-q_0(-\sigma_c - \gamma_\infty t) e^{-\frac{1}{\gamma_\infty} \int_{-\sigma_c - \gamma_\infty t}^{-\sigma_c} \chi(\sigma) d\sigma} \right. \\
&\quad \left. + q_0(\sigma_c - \gamma_\infty t) e^{-\frac{1}{\gamma_\infty} \int_{\sigma_c - \gamma_\infty t}^{\sigma_c} \chi(\sigma) d\sigma} \right).
\end{aligned} \tag{4.58}$$

Computing $B(0)$ and integrating (4.58) from 0 to t yield

$$\begin{aligned}
B(0) + \int_0^t (\dot{B} + B) &= \int \chi q_0 - \gamma_\infty \int_0^t q_0(-\sigma_c - \gamma_\infty t_1) e^{-\frac{1}{\gamma_\infty} \int_{-\sigma_c - \gamma_\infty t_1}^{-\sigma_c} \chi(\sigma) d\sigma} dt_1 \\
&\quad + \gamma_\infty \int_0^t q_0(\sigma_c - \gamma_\infty t_1) e^{-\frac{1}{\gamma_\infty} \int_{\sigma_c - \gamma_\infty t_1}^{\sigma_c} \chi(\sigma) d\sigma} dt_1
\end{aligned}$$

so that, respectively with the changes of variables $v = -\sigma_c - \gamma_\infty t$ and $v = \sigma_c - \gamma_\infty t$ in the last two integrals, we obtain

$$\begin{aligned}
B(0) + \int_0^t (\dot{B} + B) &= \int \chi q_0 - \int_{-\sigma_c - \gamma_\infty t}^{-\sigma_c} q_0(v) e^{-\frac{1}{\gamma_\infty} \int_v^{-\sigma_c} \chi(\sigma) d\sigma} dv \\
&\quad + \int_{\sigma_c - \gamma_\infty t}^{\sigma_c} q_0(v) e^{-\frac{1}{\gamma_\infty} \int_v^{\sigma_c} \chi(\sigma) d\sigma} dv.
\end{aligned}$$

For $t > 2\omega$, this implies that

$$\begin{aligned}
B(0) + \int_0^t (\dot{B} + B)(t_1) dt_1 &= \int \chi q_0 - \int_{-\sigma_c - \gamma_\infty t}^{-\sigma_c} q_0(v) e^{-\frac{1}{\gamma_\infty} \int_v^{-\sigma_c} \chi(\sigma) d\sigma} dv + \int_{-\sigma_c}^{\sigma_c} q_0 \\
&\quad + \int_{\sigma_c - \gamma_\infty t}^{\sigma_c} q_0(v) e^{-\frac{1}{\gamma_\infty} \int_v^{\sigma_c} \chi(\sigma) d\sigma} dv \\
&= \int q_0 - \int_{-\sigma_c - \gamma_\infty t}^{\sigma_c - \gamma_\infty t} q_0(v) e^{\frac{\sigma_c + v}{\gamma_\infty}} dv \\
&= -\gamma_\infty \int_{-\omega - t}^{\omega - t} q_0(\gamma_\infty v) e^{\omega + v} dv
\end{aligned}$$

using that $\int q_0 = 0$. We deduce

$$\left| B(0) + \int_0^t (\dot{B} + B)(t_1) dt_1 \right| \leq \gamma_\infty \|q_0\|_{L^\infty} e^{2\omega - t}. \tag{4.59}$$

Step 3: Longtime convergence of $B(0)k_1(t) + \int_0^t (\dot{B} + B)(t_1)k_1(t-t_1)dt_1$ Using (4.58) on $\dot{B} + B$ and the estimate (4.46) on k_1 , we have, for $t > 2\omega$,

$$\begin{aligned}
 & \left| B(0)k_1(t) + \int_0^t (\dot{B} + B)(t_1)k_1(t-t_1)dt_1 \right| \\
 & \leq C_0 \left(e^{-bt} \int \chi |q_0| + \int_0^t \gamma_\infty |q_0(-\sigma_c - \gamma_\infty t_1)| e^{-t_1-b(t-t_1)} dt_1 \right. \\
 & \quad \left. + \int_0^t \gamma_\infty |q_0(\sigma_c - \gamma_\infty t_1)| e^{2\omega} e^{-t_1-b(t-t_1)} dt_1 \right) \\
 & \leq C_0 \gamma_\infty \|q_0\|_{L^\infty} (1 + e^{2\omega}) \left(e^{-bt} + e^{-bt} \int_0^t e^{(b-1)t_1} dt_1 \right) \\
 & \leq C_0 \gamma_\infty \|q_0\|_{L^\infty} (1 + e^{2\omega}) \left(\frac{1}{|b-1|} e^{-t} + \frac{b}{|b-1|} e^{-bt} \right). \tag{4.60}
 \end{aligned}$$

Here, we have used the assumption (4.34) on b .

Step 4: Longtime convergence of $g(t)$ Using the decomposition (4.57) and the estimates (4.59) and (4.60) derived in steps 2 and 3, we have, for $t > 2\omega$,

$$|g(t)| \leq \gamma_\infty \|q_0\|_{L^\infty} \left(\frac{e^{2\omega}}{1 + \omega} e^{-t} + C_0 (1 + e^{2\omega}) \left(\frac{1}{|b-1|} e^{-t} + \frac{b}{|b-1|} e^{-bt} \right) \right). \tag{4.61}$$

Recall that g defined by (4.36) satisfies by linearity $g = \phi - \int \chi p_\infty$ and that $f = \phi$ a.e. (see (4.15)). We have obtained estimate (4.31) on $f - \int \chi p_\infty$.

Step 5: Longtime convergence of q We now turn to $q(t, \sigma) = p(t, \sigma) - p_\infty(\sigma)$. Using (4.37), we have

$$|q(t, \sigma)| \leq \|q_0\|_{L^\infty} e^{2\omega-t} + \frac{1}{\gamma_\infty} \left| g \left(t - \frac{\sigma}{\gamma_\infty} \right) \right| \mathbb{1}_{(0, \gamma_\infty t)}(\sigma).$$

In view of the estimate (4.31) on g , we deduce, for almost all $\sigma \in \mathbb{R}$ and $t > 0$,

$$|q(t, \sigma)| \leq \|q_0\|_{L^\infty} e^{2\omega-t} + \frac{C_2}{\gamma_\infty} \left(e^{-(t-\frac{\sigma}{\gamma_\infty})} + e^{-b(t-\frac{\sigma}{\gamma_\infty})} \right) \mathbb{1}_{(0, \gamma_\infty t)}(\sigma).$$

This concludes the proof of Theorem 4.2. □

4.6 Longtime convergence in the case $\gamma(\epsilon t)$

Our main result in this section is the following.

Theorem 4.3. *Consider γ a Lipschitz function with Lipschitz constant L_γ , which satisfies, for all $t \geq 0$,*

$$m_\gamma \leq \gamma(t) \leq M_\gamma, \text{ for some } m_\gamma, M_\gamma > 0 \text{ constant scalars.} \tag{4.62}$$

Consider an initial condition p_0 which satisfies (4.29). For $\theta > 0$, $\epsilon > 0$ such that $\frac{\theta}{\epsilon} > 2\frac{\sigma_c}{m_\gamma}$, consider the functions $p_\epsilon(t, \sigma)$ and $p_\infty(\theta, \sigma)$ respectively solutions to

$$\begin{cases} \frac{\partial p_\epsilon}{\partial t}(t, \sigma) + \gamma(\epsilon t) \frac{\partial p_\epsilon}{\partial \sigma}(t, \sigma) = -\chi(\sigma)p_\epsilon(t, \sigma) + \left(\int \chi(\sigma)p_\epsilon(t, \sigma) d\sigma \right) \delta_0(\sigma) \\ p_\epsilon(0, \sigma) = p_0(\sigma) \end{cases} \quad (4.63)$$

$$\gamma(\theta) \frac{\partial p_\infty(\theta, \sigma)}{\partial \sigma} = -\chi(\sigma)p_\infty(\theta, \sigma) + \left(\int \chi(\sigma)p_\infty(\theta, \sigma) d\sigma \right) \delta_0(\sigma) \quad (4.64)$$

the existence and uniqueness of which have been respectively established in Theorem 4.1 and Lemma 4.1.

Then, there exist constants $b, C_3, C_4 > 0$ independent from θ and ϵ (satisfying $\frac{\theta}{\epsilon} > 2\frac{\sigma_c}{m_\gamma}$) such that,

$$\left| \int \chi(\sigma) \left(p_\epsilon \left(\frac{\theta}{\epsilon}, \sigma \right) - p_\infty(\theta, \sigma) \right) d\sigma \right| \leq C_3 \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right) \quad (4.65)$$

and, for almost all $\sigma \in \mathbb{R}$ such that $\sigma \leq \Gamma \left(\frac{\theta}{\epsilon} \right)$,

$$\left| p_\epsilon \left(\frac{\theta}{\epsilon}, \sigma \right) - p_\infty(\theta, \sigma) \right| \leq C_4 \left[\left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right) + \epsilon \mathbb{1}_{(\sigma_c, \infty)}(\sigma) (\sigma - \sigma_c)^2 \right]. \quad (4.66)$$

In order to prove Theorem 4.3, we need the following technical lemma.

Lemma 4.10. Consider γ a function of time that satisfies (4.11). Denote by $\Gamma(t) = \int_0^t \gamma(s) ds$. Then, for all $t > \frac{2\sigma_c}{m_\gamma}$ and almost all $\sigma \in \mathbb{R}$, we have

$$\int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du \geq t - \frac{2\sigma_c}{m_\gamma}. \quad (4.67)$$

Proof. Denote by $Z(t, \sigma) = \int_0^t \chi(\sigma - \Gamma(t) + \Gamma(u)) du$. For all $t > \frac{2\sigma_c}{m_\gamma}$ and almost all $\sigma \in \mathbb{R}$, we have

$$\begin{aligned}
 Z(t, \sigma) &= \int_0^t \left(\mathbb{1}_{(-\infty, -\sigma_c)} + \mathbb{1}_{(\sigma_c, \infty)} \right) (\sigma - \Gamma(t) + \Gamma(u)) du \\
 &= \int_0^t \left(\mathbb{1}_{(-\infty, \Gamma^{-1}(-\sigma_c + \Gamma(t) - \sigma))} + \mathbb{1}_{(\Gamma^{-1}(\sigma_c + \Gamma(t) - \sigma), \infty)} \right) (u) du \\
 &= \begin{cases} t & \sigma < -\sigma_c \\ \Gamma^{-1}(-\sigma_c + \Gamma(t) - \sigma) & -\sigma_c < \sigma < \Gamma(t) - \sigma_c \\ 0 & \Gamma(t) - \sigma_c < \sigma \end{cases} \\
 &\quad + \begin{cases} 0 & \sigma < \sigma_c \\ t - \Gamma^{-1}(\sigma_c + \Gamma(t) - \sigma) & \sigma_c < \sigma < \Gamma(t) + \sigma_c \\ t & \Gamma(t) + \sigma_c < \sigma \end{cases} \\
 &= \begin{cases} \Gamma^{-1}(-\sigma_c + \Gamma(t) - \sigma) & -\sigma_c < \sigma \leq \sigma_c \\ t + \Gamma^{-1}(-\sigma_c + \Gamma(t) - \sigma) - \Gamma^{-1}(\sigma_c + \Gamma(t) - \sigma) & \sigma_c < \sigma \leq \Gamma(t) - \sigma_c \\ t - \Gamma^{-1}(\sigma_c + \Gamma(t) - \sigma) & \Gamma(t) - \sigma_c < \sigma \leq \Gamma(t) + \sigma_c \\ t & \sigma \leq -\sigma_c \text{ or } \sigma > \Gamma(t) + \sigma_c \end{cases}
 \end{aligned}$$

using that $\Gamma(t) > 2\sigma_c$. We now estimate the above expression depending on σ . For almost all $\sigma \in (-\sigma_c, \sigma_c)$, the function Z is decreasing in σ so that $Z(t, \sigma) \geq \Gamma^{-1}(\Gamma(t) - 2\sigma_c)$. Moreover, because of (4.11), the function γ satisfies

$$\Gamma(t) - \Gamma\left(t - \frac{2\sigma_c}{m_\gamma}\right) = \int_0^t \gamma - \int_0^{t - \frac{2\sigma_c}{m_\gamma}} \gamma \geq 2\sigma_c$$

so that $\Gamma^{-1}(\Gamma(t) - 2\sigma_c) \geq t - \frac{2\sigma_c}{m_\gamma}$, hence (4.67) for almost all $\sigma \in (-\sigma_c, \sigma_c)$.

Additionally, because of (4.11), the function Γ satisfies, for all $v > u \geq 0$,

$$\Gamma(v) - \Gamma(u) = \int_u^v \gamma \geq m_\gamma (v - u).$$

This yields that Γ^{-1} is Lipschitz with a Lipschitz constant $\frac{1}{m_\gamma}$ on $[0, \infty)$. Therefore, for almost $\sigma \in (\sigma_c, \Gamma(t) - \sigma_c)$,

$$t - Z(t, \sigma) = \Gamma^{-1}(\sigma_c + \Gamma(t) - \sigma) - \Gamma^{-1}(-\sigma_c + \Gamma(t) - \sigma) \leq \frac{2\sigma_c}{m_\gamma},$$

hence (4.67).

For almost all $\sigma \in (\Gamma(t) - \sigma_c, \Gamma(t) + \sigma_c)$, the function Z is increasing in σ so that $Z(t, \sigma) \geq t - \Gamma^{-1}(2\sigma_c)$. Moreover,

$$2\sigma_c = \int_0^{\frac{2\sigma_c}{m_\gamma}} m_\gamma \leq \int_0^{\frac{2\sigma_c}{m_\gamma}} \gamma(u) du = \Gamma\left(\frac{2\sigma_c}{m_\gamma}\right)$$

so that $\Gamma^{-1}(2\sigma_c) \leq \frac{2\sigma_c}{m_\gamma}$, hence (4.67) for almost all $\sigma \in (\Gamma(t) - \sigma_c, \Gamma(t) + \sigma_c)$.

The result (4.67) also holds in the case $\sigma \in \mathbb{R} \setminus [-\sigma_c, \Gamma(t) + \sigma_c]$ where $Z(t, \sigma) = t$. This ends the proof. \square

Now that we have proved the technical Lemma 4.10, we turn to the

Proof of Theorem 4.3. The proof is divided into six steps. The first step establishes a delay differential equation on a function g_ϵ , for which an explicit decomposition is known thanks to the Lemma 4.8. We then rewrite g_ϵ in a different form whose terms are estimated in Steps 3 and 4. In the last two steps, we use these estimates to obtain (4.65) and then (4.66).

Before we get to the proof we introduce some notation. The scalars $\theta > 0$, $\epsilon > 0$ are fixed and satisfy $\frac{\theta}{\epsilon} > 2\frac{\sigma_c}{m_\gamma}$. In Section 4.5, we have introduced

$$\omega_\theta = \frac{\sigma_c}{\gamma(\theta)}$$

which from bounds (4.62) on γ satisfies

$$\frac{\sigma_c}{M_\gamma} < \omega_\theta < \frac{\sigma_c}{m_\gamma}. \quad (4.68)$$

We can therefore apply Lemma 4.9 to the function k_θ satisfying (4.40) with $\omega = \omega_\theta$, so that there exist $b, C_0 > 0$ which depend only on σ_c, m_γ and M_γ such that (4.45) and (4.46) hold for all $t > 0$. Notably, b and C_0 are independent from θ (and ϵ).

Step 1: Applying Lemma 4.8 For a fixed θ , denote by

$$A_\theta(t) = \int \chi(\sigma) p_0(\sigma - \gamma(\theta)t) e^{-\frac{1}{\gamma(\theta)} \int_{\sigma - \gamma(\theta)t}^\sigma \chi(v) dv} d\sigma \quad (4.69)$$

and by ϕ_θ the solution to

$$\dot{\phi}_\theta(t) + \phi_\theta(t) - \phi_\theta(t - \omega_\theta) = \dot{A}_\theta(t) + A_\theta(t)$$

with the initial condition $\phi_\theta(t) = A_\theta(t)$, $0 < t < \omega_\theta$. Consistently with (4.4), let us also introduce $f_\epsilon(t) = \int \chi(\sigma) p_\epsilon(t, \sigma) d\sigma$ where p_ϵ satisfies (4.63). Then

$$g_\epsilon(t) = f_\epsilon(t) - \phi_\theta(t) \quad (4.70)$$

belongs to $W^{1,1}(0, T)$ (because f_ϵ and ϕ_θ do, see Lemma 4.5) and satisfies, for almost all $t > \omega_\theta$

$$\dot{g}_\epsilon(t) + g_\epsilon(t) - g_\epsilon(t - \omega_\theta) = \dot{f}_\epsilon(t) + f_\epsilon(t) - \dot{A}_\theta(t) - A_\theta(t) - f_\epsilon(t - \omega_\theta).$$

Introduce $s > 2\frac{\sigma_c}{m_\gamma}$. We apply Lemma 4.8 and obtain (using the same computations as in (4.56) above and the fact that $f_\epsilon(0) = \phi_\theta(0)$),

$$\begin{aligned} g_\epsilon(s) &= (f_\epsilon - \phi_\theta)(\omega_\theta) k_\theta(s - \omega_\theta) + \int_0^{\omega_\theta} (f_\epsilon - \phi_\theta)(t) k_\theta(s - t - \omega_\theta) dt \\ &\quad + \int_{\omega_\theta}^s \left(\dot{f}_\epsilon(t) + f_\epsilon(t) - \dot{A}_\theta(t) - A_\theta(t) - f_\epsilon(t - \omega_\theta) \right) k_\theta(s - t) dt \\ &= \int_0^s \left(\dot{f}_\epsilon(t) + f_\epsilon(t) - \dot{A}_\theta(t) - A_\theta(t) \right) k_\theta(s - t) dt - \int_{\omega_\theta}^s f_\epsilon(t - \omega_\theta) k_\theta(s - t) dt. \end{aligned} \quad (4.71)$$

Step 2: Rewriting g_ϵ In order to rewrite g_ϵ we show that, for almost all $t \in (0, s)$,

$$\dot{f}_\epsilon(t) + f_\epsilon(t) = \gamma(\epsilon t) (p_\epsilon(t, \sigma_c) - p_\epsilon(t, -\sigma_c)). \quad (4.72)$$

First, the function p_ϵ solution to (4.63) with p_0 as initial condition satisfies, for all $\eta \in \mathcal{D}([0, s] \times \mathbb{R})$,

$$- \int_0^s \int_{\mathbb{R}} p_\epsilon \left(\frac{\partial \eta}{\partial t} + \gamma(\epsilon t) \frac{\partial \eta}{\partial \sigma} - \chi \eta \right) = \int_{\mathbb{R}} p_0(\sigma) \eta(0, \sigma) d\sigma + \int_0^s f_\epsilon(t) \eta(t, 0) dt. \quad (4.73)$$

Denote ρ a function of $\mathcal{D}([0, s])$, ρ^n a mollifier on \mathbb{R} and $\chi^n = \rho^n * \chi$. Inserting $\eta^n(t, \sigma) = \chi^n(\sigma) \rho(t)$ in (4.73) yields, for n sufficiently large such that $\chi^n(0) = 0$,

$$- \int_0^s \int_{\mathbb{R}} p_\epsilon \left(\frac{\partial \eta^n}{\partial t} + \gamma(\epsilon t) \frac{\partial \eta^n}{\partial \sigma} - \chi \eta^n \right) = 0$$

which rewrites

$$\begin{aligned} - \int_0^s \dot{\rho}(t) \int \chi^n p_\epsilon(t, \cdot) dt - \int \dot{\chi}^n(\sigma) \int_0^s \rho(t) \gamma(\epsilon t) p_\epsilon(t, \sigma) dt d\sigma + \int_0^s \rho(t) \int \chi \chi^n p_\epsilon(t, \cdot) dt \\ = 0. \end{aligned} \quad (4.74)$$

The function $t \mapsto (\sigma \mapsto p_\epsilon(t, \sigma))$ belongs to $C([0, s], L^1)$, see Theorem 4.1, so that, by the dominated convergence theorem, for all $t \in [0, s]$, $\int \chi^n p_\epsilon(t, \cdot)$ and $\int \chi \chi^n p_\epsilon(t, \cdot)$ converge to $f_\epsilon(t)$ defined by (4.4) as n goes to infinity. Moreover $\sigma \mapsto (t \mapsto p_\epsilon(t, \sigma))$ belongs to $C(\mathbb{R}, L^1(0, s))$ (the proof is similar to the one in Step 2 of Theorem 4.1) so that $\int_0^s \rho(t) \gamma(\epsilon t) p_\epsilon(t, \sigma) dt$ is continuous in σ . Passing to the limit $n \rightarrow \infty$ in the above equation yields

$$- \int_0^s \dot{\rho}(t) f_\epsilon(t) dt - \int_0^s \rho(t) \gamma(\epsilon t) (p_\epsilon(t, \sigma_c) - p_\epsilon(t, -\sigma_c)) dt + \int_0^s \rho(t) f_\epsilon(t) dt = 0,$$

hence (4.72) since f_ϵ and $p_\epsilon(\cdot, \pm\sigma_c)$ belong to $L^1(0, s)$.

We then denote

$$Q_\epsilon(t) = \gamma(\epsilon t) (p_\epsilon(t, \sigma_c) - p_\epsilon(t, -\sigma_c)) - \dot{A}_\theta(t) - A_\theta(t) - f_\epsilon(t - \omega_\theta) \mathbb{1}_{(\omega_\theta, s)}(t) \quad (4.75)$$

so that the expression (4.71) on g_ϵ rewrites

$$\begin{aligned} g_\epsilon(s) &= \int_0^s Q_\epsilon(t) k_\theta(s-t) dt \\ &= \frac{1}{1+\omega_\theta} \int_0^s Q_\epsilon(t) dt + \int_0^s Q_\epsilon(t) k_{\theta,1}(s-t) dt, \end{aligned} \quad (4.76)$$

using the decomposition $k_\theta = \frac{1}{1+\omega_\theta} + k_{1,\theta}$ (see (4.45)) that was established in Lemma 4.9. We now derive estimates on the two terms of the above expression, when $s = \frac{\theta}{\epsilon}$.

Step 3: Estimate of $\int_0^{\frac{\theta}{\epsilon}} Q_\epsilon$ Introduce

$$\eta_-(t, \sigma) = \mathbb{1}_{[-\sigma_c - \gamma(\theta)(s-t), -\sigma_c]}(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} \quad (4.77)$$

which satisfies, in $\mathcal{D}'([0, s] \times \mathbb{R})$,

$$\frac{\partial \eta_-}{\partial \sigma} = -\delta_{-\sigma_c}(\sigma) + \delta_{-\sigma_c - \gamma(\theta)(s-t)}(\sigma) e^{t-s} + \frac{1}{\gamma(\theta)} \mathbb{1}_{[-\sigma_c - \gamma(\theta)(s-t), -\sigma_c]}(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}}$$

and

$$\begin{aligned} -\frac{\partial \eta_-}{\partial t} - \gamma(\epsilon t) \frac{\partial \eta_-}{\partial \sigma} + \chi \eta_- &= \gamma(\epsilon t) \delta_{-\sigma_c}(\sigma) + (\gamma(\theta) - \gamma(\epsilon t)) \delta_{-\sigma_c - \gamma(\theta)(s-t)}(\sigma) e^{t-s} \\ &\quad + \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} \mathbb{1}_{[-\sigma_c - \gamma(\theta)(s-t), -\sigma_c]}(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}}. \end{aligned}$$

For $n, m \in \mathbb{N}$, take as a test function

$$\eta = \eta_-^{n,m}(t, \sigma) = \rho^n * \mathbb{1}_{[-\sigma_c - \gamma(\theta)(s-t), -\sigma_c]}(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} \zeta_{[0,s]}^m(t)$$

in (4.73) and pass to the limit in n and then m . Here and in the following, $\zeta_{[0,s]}^m$ denotes a $C^\infty([0, s], \mathbb{R})$ function with compact support in $[0, s]$, such that $\zeta_{[0,s]}^m$ converges pointwise to $\mathbb{1}_{[0,s]}$.

We omit the details, the arguments being similar to those in (4.74). We obtain

$$\begin{aligned} \int_0^s \gamma(\epsilon t) p_\epsilon(t, -\sigma_c) dt + \int_0^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, -\sigma_c - \gamma(\theta)(s-t)) e^{t-s} dt \\ + \int_0^s \int_{-\sigma_c - \gamma(\theta)(s-t)}^{-\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} d\sigma dt \\ = \int_{-\sigma_c - \gamma(\theta)s}^{-\sigma_c} p_0(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} d\sigma. \end{aligned} \quad (4.78)$$

With changes of variable $\sigma = -\sigma_c - \gamma(\theta)(u-t)$ and $\sigma = -\sigma_c - \gamma(\theta)t$ in the last two integrals, this rewrites

$$\begin{aligned} \int_0^s \gamma(\epsilon t) p_\epsilon(t, -\sigma_c) dt + \int_0^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, -\sigma_c - \gamma(\theta)(s-t)) e^{t-s} dt \\ + \int_0^s \int_t^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, -\sigma_c - \gamma(\theta)(u-t)) e^{t-u} du dt \\ = \gamma(\theta) \int_0^s p_0(-\sigma_c - \gamma(\theta)t) e^{-t} dt. \end{aligned} \quad (4.79)$$

Let us assume that $s > 2\omega_\theta$ and introduce

$$\begin{aligned} \eta_+(t, \sigma) &= -\mathbb{1}_{[\sigma_c - \gamma(\theta)(s-t), -\sigma_c]}(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} \mathbb{1}_{(0, s-2\omega_\theta)}(t) \\ &\quad + \mathbb{1}_{[-\sigma_c, \sigma_c - \gamma(\theta)(s-t)]}(\sigma) \mathbb{1}_{(s-2\omega_\theta, s)}(t) - \mathbb{1}_{[-\sigma_c, \sigma_c]}(\sigma) \end{aligned} \quad (4.80)$$

which satisfies, in $\mathcal{D}'((0, s) \times \mathbb{R})$,

$$\begin{aligned} \frac{\partial \eta_+}{\partial \sigma} &= \delta_{\sigma_c} - \delta_{\sigma_c - \gamma(\theta)(s-t)}(\sigma) e^{2\frac{\sigma_c}{\gamma(\theta)} + t-s} \mathbb{1}_{(0, s-2\omega_\theta)}(t) \\ &\quad - \frac{1}{\gamma(\theta)} e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} \mathbb{1}_{[\sigma_c - \gamma(\theta)(s-t), -\sigma_c]}(\sigma) \mathbb{1}_{(0, s-2\omega_\theta)}(t) - \delta_{\sigma_c - \gamma(\theta)(s-t)}(\sigma) \mathbb{1}_{(s-2\omega_\theta, s)}(t) \end{aligned}$$

and

$$\begin{aligned}
& -\frac{\partial \eta_+}{\partial t} - \gamma(\epsilon t) \frac{\partial \eta_+}{\partial \sigma} + \chi \eta_+ \\
& = -\gamma(\epsilon t) \delta_{\sigma_c} - (\gamma(\theta) - \gamma(\epsilon t)) \delta_{\sigma_c - \gamma(\theta)(s-t)}(\sigma) e^{2\frac{\sigma_c}{\gamma(\theta)} + t-s} \mathbb{1}_{(0, s-2\omega_\theta)}(t) \\
& \quad - \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} \mathbb{1}_{[\sigma_c - \gamma(\theta)(s-t), -\sigma_c]}(\sigma) \mathbb{1}_{(0, s-2\omega_\theta)}(t) \\
& \quad - (\gamma(\theta) - \gamma(\epsilon t)) \delta_{\sigma_c - \gamma(\theta)(s-t)}(\sigma) \mathbb{1}_{(s-2\omega_\theta, s)}(t).
\end{aligned}$$

We again use a regularization

$$\begin{aligned}
\eta_+^{n,m} & = -\rho^n * \mathbb{1}_{[\sigma_c - \gamma(\theta)(s-t), -\sigma_c]}(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} \zeta_{[0, s-2\omega_\theta]}^m(t) \\
& \quad + \rho^n * \mathbb{1}_{[-\sigma_c, \sigma_c - \gamma(\theta)(s-t)]}(\sigma) \zeta_{(s-2\omega_\theta, s)}^m(t) \\
& \quad - \rho^n * \mathbb{1}_{[-\sigma_c, \sigma_c]}(\sigma) \zeta_{[0, s]}^m(t)
\end{aligned}$$

and pass to the limit in (4.73)

$$\begin{aligned}
& -\int_0^s \gamma(\epsilon t) p_\epsilon(t, \sigma_c) dt - \int_0^{s-2\omega_\theta} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, \sigma_c - \gamma(\theta)(s-t)) e^{2\frac{\sigma_c}{\gamma(\theta)} + t-s} dt \\
& \quad - \int_0^{s-2\omega_\theta} \int_{t+2\omega_\theta}^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, \sigma_c - \gamma(\theta)(u-t)) e^{2\frac{\sigma_c}{\gamma(\theta)} + t-u} dudt \\
& \quad \quad - \int_{s-2\omega_\theta}^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, \sigma_c - \gamma(\theta)(s-t)) dt \\
& = -\gamma(\theta) \int_0^s p_0(\sigma_c - \gamma(\theta)t) e^{-\frac{1}{\gamma(\theta)} \int_{\sigma_c - \gamma(\theta)t}^{\sigma_c} \chi dt} - \int_0^{s-\frac{\sigma_c}{\gamma(\theta)}} f_\epsilon. \tag{4.81}
\end{aligned}$$

In addition, from its definition (4.69), we know that A_θ satisfies (see (4.24))

$$\begin{aligned}
\int_0^s \dot{A}_\theta + A_\theta & = -\gamma(\theta) \int_0^s p_0(-\sigma_c - \gamma(\theta)t) e^{-t} dt \\
& \quad + \gamma(\theta) \int_0^s p_0(\sigma_c - \gamma(\theta)t) e^{-\frac{1}{\gamma(\theta)} \int_{\sigma_c - \gamma(\theta)t}^{\sigma_c} \chi dt}. \tag{4.82}
\end{aligned}$$

Summing up expressions (4.79), (4.81) and (4.82), we obtain

$$\begin{aligned}
\int_0^s Q_\epsilon & = \int_0^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, -\sigma_c - \gamma(\theta)(s-t)) e^{t-s} dt \\
& \quad + \int_0^s \int_t^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, -\sigma_c - \gamma(\theta)(u-t)) e^{t-u} dudt \\
& \quad - \int_0^{s-2\omega_\theta} \int_{t+2\omega_\theta}^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, \sigma_c - \gamma(\theta)(v-t)) e^{2\omega_\theta + t-v} dvdt \\
& \quad - \int_0^{s-2\omega_\theta} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, \sigma_c - \gamma(\theta)(s-t)) e^{2\omega_\theta + t-s} dt \\
& \quad - \int_{s-2\omega_\theta}^s (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, \sigma_c - \gamma(\theta)(s-t)) dt.
\end{aligned}$$

Taking $s = \frac{\theta}{\epsilon}$ and summing up the second and the third term (with the change of variable $u = v - 2\omega_\theta$ in the third term), this rewrites

$$\begin{aligned}
\int_0^{\frac{\theta}{\epsilon}} Q_\epsilon &= \int_0^{\frac{\theta}{\epsilon}} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, -\sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) e^{t - \frac{\theta}{\epsilon}} dt \\
&+ \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} \int_t^{\frac{\theta}{\epsilon}} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, -\sigma_c - \gamma(\theta)(u - t)) e^{t-u} dudt \\
&+ \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon(t, -\sigma_c - \gamma(\theta)(u - t)) e^{t-u} dudt \\
&- \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, \sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) e^{2\omega_\theta + t - \frac{\theta}{\epsilon}} dt \\
&- \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, \sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) dt.
\end{aligned} \tag{4.83}$$

Using the Lipschitz property of γ and L^∞ -bound (4.22) on p_ϵ , this implies

$$\begin{aligned}
\left| \int_0^{\frac{\theta}{\epsilon}} Q_\epsilon \right| &\leq C_\infty L_\gamma \int_0^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) e^{t - \frac{\theta}{\epsilon}} dt \\
&+ C_\infty L_\gamma \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} \int_t^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) e^{t-u} dudt \\
&+ C_\infty L_\gamma \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) e^{t-u} dudt \\
&+ C_\infty L_\gamma \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} (\theta - \epsilon t) e^{2\omega_\theta + t - \frac{\theta}{\epsilon}} dt \\
&+ C_\infty L_\gamma \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) dt.
\end{aligned}$$

For a constant $\alpha < 0$, we have

$$\begin{aligned}
\int_0^{\frac{\theta}{\epsilon}} (\theta - \epsilon v) e^{\alpha \left(\frac{\theta}{\epsilon} - v \right)} dv &= \frac{\epsilon}{\alpha^2} \int_0^{-\frac{\alpha\theta}{\epsilon}} u e^{-u} du \\
&< \frac{\epsilon}{\alpha^2} \int_0^\infty u e^{-u} du.
\end{aligned} \tag{4.84}$$

Using estimate (4.84) or variants, we find

$$\begin{aligned}
\left| \int_0^{\frac{\theta}{\epsilon}} Q_\epsilon \right| &\leq C_\infty L_\gamma \epsilon \left(\int_0^\infty u e^{-u} du + 4\omega_\theta^2 + (1 + e^{2\omega_\theta}) \int_0^\infty u e^{-u} du \right. \\
&\quad \left. + e^{2\omega_\theta} \int_0^\infty u e^{-u} du + 2\omega_\theta^2 \right) \\
&\leq C_\infty L_\gamma \epsilon \left(2 + 6\omega_\theta^2 + 2e^{2\omega_\theta} \right).
\end{aligned}$$

Since $\omega_\theta < \frac{\sigma_c}{m_\gamma}$, we obtain

$$\left| \int_0^{\frac{\theta}{\epsilon}} Q_\epsilon \right| \leq KC_\infty L_\gamma \epsilon, \tag{4.85}$$

with K a constant that is independent from θ and ϵ . Throughout the rest of the proof below, we will likewise denote by K such a constant, whose precise value may change from one occurrence to another.

Step 4: Estimate of $\int_0^{\frac{\theta}{\epsilon}} Q_\epsilon(t) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt$ Inserting expression (4.69) of A_θ (see (4.82) above for a similar computation), Q_ϵ defined by (4.75) satisfies

$$\begin{aligned} \int_0^{\frac{\theta}{\epsilon}} Q_\epsilon(t) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt &= \int_0^{\frac{\theta}{\epsilon}} \gamma(\epsilon t) (p_\epsilon(t, \sigma_c) - p_\epsilon(t, -\sigma_c)) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt \\ &+ \gamma(\theta) \int_0^{\frac{\theta}{\epsilon}} \left(p_0(-\sigma_c - \gamma(\theta)t) e^{-t} - p_0(\sigma_c - \gamma(\theta)t) e^{-\frac{1}{\gamma(\theta)} \int_{\sigma_c - \gamma(\theta)t}^{\sigma_c} \chi} \right) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt \\ &- \int_{\omega_\theta}^{\frac{\theta}{\epsilon}} f_\epsilon(t - \omega_\theta) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt. \end{aligned} \quad (4.86)$$

For further use, notice that the integral in the second line above can be rewritten as:

$$\begin{aligned} &\int_0^{\frac{\theta}{\epsilon}} \left(p_0(-\sigma_c - \gamma(\theta)t) e^{-t} - p_0(\sigma_c - \gamma(\theta)t) e^{-\frac{1}{\gamma(\theta)} \int_{\sigma_c - \gamma(\theta)t}^{\sigma_c} \chi} \right) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt \\ &= \int_{-\sigma_c - \gamma(\theta)\frac{\theta}{\epsilon}}^{-\sigma_c} p_0(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} k_{\theta,1} \left(\frac{\theta}{\epsilon} + \frac{\sigma + \sigma_c}{\gamma(\theta)} \right) dt \\ &- \int_{\sigma_c - \gamma(\theta)\frac{\theta}{\epsilon}}^{\sigma_c} p_0(\sigma) e^{-\frac{1}{\gamma(\theta)} \int_{\sigma}^{\sigma_c} \chi} k_{\theta,1} \left(\frac{\theta}{\epsilon} + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) dt. \end{aligned} \quad (4.87)$$

In order to rewrite the first term of the right-hand side, we use again the functions η_- and η_+ respectively defined by (4.77) and (4.80). Using a regularization of $\eta_-(t, \sigma) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma + \sigma_c}{\gamma(\theta)} \right)$ as test function in (4.73) (with $s = \theta/\epsilon$) and then passing to the limit, we obtain

$$\begin{aligned} &\int_0^{\frac{\theta}{\epsilon}} \int_{\mathbb{R}} \left(-\frac{\partial \eta_-}{\partial t} - \gamma(\epsilon t) \frac{\partial \eta_-}{\partial \sigma} + \chi \eta_- \right) (t, \sigma) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma + \sigma_c}{\gamma(\theta)} \right) p_\epsilon(t, \sigma) dt d\sigma \\ &+ \int_0^{\frac{\theta}{\epsilon}} \int_{\mathbb{R}} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} \eta_-(t, \sigma) \dot{k}_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma + \sigma_c}{\gamma(\theta)} \right) p_\epsilon(t, \sigma) dt d\sigma \\ &= \int_{\mathbb{R}} p_0(\sigma) \eta_-(0, \sigma) k_{\theta,1} \left(\frac{\theta}{\epsilon} + \frac{\sigma + \sigma_c}{\gamma(\theta)} \right) d\sigma. \end{aligned}$$

This rewrites (using similar computations as in (4.78) above)

$$\begin{aligned} &\int_0^{\frac{\theta}{\epsilon}} \gamma(\epsilon t) p_\epsilon(t, -\sigma_c) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt \\ &+ k_{\theta,1}(0) \int_0^{\frac{\theta}{\epsilon}} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, -\sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) e^{t - \frac{\theta}{\epsilon}} dt \\ &+ \int_0^{\frac{\theta}{\epsilon}} \int_{-\sigma_c - \gamma(\theta)\left(\frac{\theta}{\epsilon} - t\right)}^{-\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} (k_{\theta,1} + \dot{k}_{\theta,1}) \left(\frac{\theta}{\epsilon} - t + \frac{\sigma + \sigma_c}{\gamma(\theta)} \right) d\sigma dt \\ &= \int_{-\sigma_c - \gamma(\theta)\frac{\theta}{\epsilon}}^{-\sigma_c} p_0(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} k_{\theta,1} \left(\frac{\theta}{\epsilon} + \frac{\sigma + \sigma_c}{\gamma(\theta)} \right) d\sigma. \end{aligned} \quad (4.88)$$

Similarly, using a regularization of $\eta_+(t, \sigma)k_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right)$ as test function in (4.73), we obtain

$$\begin{aligned} & \int_0^{\frac{\theta}{\epsilon}} \int_{\mathbb{R}} \left(-\frac{\partial \eta_+}{\partial t} - \gamma(\epsilon t) \frac{\partial \eta_+}{\partial \sigma} + \chi \eta_+ \right) (t, \sigma) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) p_\epsilon(t, \sigma) dt d\sigma \\ & \quad + \int_0^{\frac{\theta}{\epsilon}} \int_{\mathbb{R}} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} \eta_+(t, \sigma) \dot{k}_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) p_\epsilon(t, \sigma) dt d\sigma \\ & = \int_{\mathbb{R}} p_0(\sigma) \eta_+(0, \sigma) k_{\theta,1} \left(\frac{\theta}{\epsilon} + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma + \int_0^{\frac{\theta}{\epsilon}} f_\epsilon(t) \eta_+(t, 0) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t - \frac{\sigma_c}{\gamma(\theta)} \right) dt, \end{aligned}$$

so that (using similar computations as in (4.81) above)

$$\begin{aligned} & - \int_0^{\frac{\theta}{\epsilon}} \gamma(\epsilon t) p_\epsilon(t, \sigma_c) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt \\ & - k_{\theta,1}(0) \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, \sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) e^{2\omega_\theta + t - \frac{\theta}{\epsilon}} dt \\ & - k_{\theta,1}(0) \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, \sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) dt \\ & - \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} \int_{\sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right)}^{-\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} (k_{\theta,1} + \dot{k}_{\theta,1}) \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma dt \\ & - \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} \int_{\sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right)}^{-\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) \dot{k}_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma dt \\ & - \int_0^{\frac{\theta}{\epsilon}} \int_{-\sigma_c}^{\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) \dot{k}_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma dt \\ & = - \int_{\sigma_c - \gamma(\theta) \frac{\theta}{\epsilon}}^{-\sigma_c} p_0(\sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} k_{\theta,1} \left(\frac{\theta}{\epsilon} + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma - \int_{-\sigma_c}^{\sigma_c} p_0(\sigma) k_{\theta,1} \left(\frac{\theta}{\epsilon} + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma \\ & \quad - \int_0^{\frac{\theta}{\epsilon} - \omega_\theta} f_\epsilon(t) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t - \omega_\theta \right) dt. \quad (4.89) \end{aligned}$$

We now perform the linear combinations: (4.86) - ((4.88) + (4.89)). The last two term of the right-hand side of (4.86) cancel out with the right-hand sides of (4.88) and (4.89) (using in particular (4.87)) so that

$$\begin{aligned}
\int_0^{\frac{\theta}{\epsilon}} Q_\epsilon(t) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt &= k_{\theta,1}(0) \int_0^{\frac{\theta}{\epsilon}} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, -\sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) e^{t - \frac{\theta}{\epsilon}} dt \\
&+ \int_0^{\frac{\theta}{\epsilon}} \int_{-\sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right)}^{-\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} (k_{\theta,1} + \dot{k}_{\theta,1}) \left(\frac{\theta}{\epsilon} - t + \frac{\sigma + \sigma_c}{\gamma(\theta)} \right) d\sigma dt \\
&\quad - k_{\theta,1}(0) \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, \sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) e^{2\omega_\theta t - \frac{\theta}{\epsilon}} dt \\
&\quad - k_{\theta,1}(0) \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} (\gamma(\theta) - \gamma(\epsilon t)) p_\epsilon \left(t, \sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right) \right) dt \\
&- \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} \int_{\sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right)}^{-\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) e^{\frac{\sigma + \sigma_c}{\gamma(\theta)}} (k_{\theta,1} + \dot{k}_{\theta,1}) \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma dt \\
&\quad - \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} \int_{\sigma_c - \gamma(\theta) \left(\frac{\theta}{\epsilon} - t \right)}^{-\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) \dot{k}_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma dt \\
&\quad - \int_0^{\frac{\theta}{\epsilon}} \int_{-\sigma_c}^{\sigma_c} \frac{\gamma(\theta) - \gamma(\epsilon t)}{\gamma(\theta)} p_\epsilon(t, \sigma) \dot{k}_{\theta,1} \left(\frac{\theta}{\epsilon} - t + \frac{\sigma - \sigma_c}{\gamma(\theta)} \right) d\sigma dt.
\end{aligned}$$

Using Lemma 4.9, $k_{\theta,1}$ satisfies $\dot{k}_{\theta,1}(t) + k_{\theta,1}(t) - k_{\theta,1}(t - \omega_\theta) = 0$ (this is a consequence of (4.40) and (4.45)) so that (using (4.46)):

$$|\dot{k}_{\theta,1}(t)| \leq C_0 \left(1 + e^{\frac{b\sigma_c}{m\gamma}} \right) e^{-bt}. \quad (4.90)$$

Using the Lipschitz property of γ , the L^∞ -bound (4.22) of p_ϵ and estimates (4.46) and (4.90) of $k_{\theta,1}$ and $\dot{k}_{\theta,1}$, we obtain

$$\begin{aligned}
&\left| \int_0^{\frac{\theta}{\epsilon}} Q_\epsilon(t) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt \right| \\
&\leq C_0 C_\infty L_\gamma \int_0^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) e^{t - \frac{\theta}{\epsilon}} dt \\
&\quad + C_0 \left(2 + e^{\frac{b\sigma_c}{m\gamma}} \right) C_\infty L_\gamma \int_0^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) \frac{e^{b(t - \frac{\theta}{\epsilon})} - e^{t - \frac{\theta}{\epsilon}}}{1 - b} dt \\
&\quad + C_0 C_\infty L_\gamma \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} (\theta - \epsilon t) e^{2\omega_\theta t - \frac{\theta}{\epsilon}} dt \\
&\quad + C_0 C_\infty L_\gamma \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) dt \\
&\quad + C_0 \left(2 + e^{\frac{b\sigma_c}{m\gamma}} \right) C_\infty L_\gamma \int_0^{\frac{\theta}{\epsilon} - 2\omega_\theta} (\theta - \epsilon t) \frac{e^{b(t - \frac{\theta}{\epsilon} + 2\omega_\theta)} - e^{t - \frac{\theta}{\epsilon} + 2\omega_\theta}}{1 - b} dt \\
&\quad + C_0 \left(1 + e^{\frac{b\sigma_c}{m\gamma}} \right) C_\infty L_\gamma \int_{\frac{\theta}{\epsilon} - 2\omega_\theta}^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) \frac{e^{-2\omega_\theta b} + e^{b(t - \frac{\theta}{\epsilon})}}{b} dt \\
&\quad + C_0 \left(1 + e^{\frac{b\sigma_c}{m\gamma}} \right) C_\infty L_\gamma \int_0^{\frac{\theta}{\epsilon}} (\theta - \epsilon t) e^{b(t - \frac{\theta}{\epsilon})} \frac{1 + e^{-2\omega_\theta b}}{b} dt.
\end{aligned}$$

Using the estimate (4.84) or variants and the bounds (4.68) on ω_θ , one concludes

$$\left| \int_0^{\frac{\theta}{\epsilon}} Q_\epsilon(t) k_{\theta,1} \left(\frac{\theta}{\epsilon} - t \right) dt \right| \leq KC_\infty C_0 L_\gamma \epsilon. \quad (4.91)$$

Step 5: Estimate of $g_\epsilon \left(\frac{\theta}{\epsilon} \right)$ Using the decomposition (4.76) and estimates (4.85) and (4.91), we find

$$\left| g_\epsilon \left(\frac{\theta}{\epsilon} \right) \right| \leq KC_\infty (1 + C_0) L_\gamma \epsilon.$$

Moreover, the estimate (4.31) established in Theorem 4.2 yields, using the bounds (4.68) on ω_θ

$$\left| \phi_\theta \left(\frac{\theta}{\epsilon} \right) - \int \chi p_\infty(\theta, \cdot) \right| \leq \tilde{C}_3 \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} \right)$$

where $\tilde{C}_3 > 0$ is independent from ϵ and θ . By the definition (4.70) of g_ϵ , we have $f_\epsilon - \int \chi p_\infty(\theta, \cdot) = g_\epsilon + (\phi_\theta - \int \chi p_\infty(\theta, \cdot))$ so that

$$\begin{aligned} \left| f_\epsilon \left(\frac{\theta}{\epsilon} \right) - \int \chi p_\infty(\theta, \cdot) \right| &\leq \tilde{C}_3 \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} \right) + KC_\infty (1 + C_0) L_\gamma \epsilon \\ &\leq C_3 \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right) \end{aligned}$$

where $C_3 > 0$ is independent from ϵ and θ . This concludes the proof of (4.65).

Step 6: Estimate of $p_\epsilon \left(\frac{\theta}{\epsilon}, \sigma \right) - p_\infty(\theta, \sigma)$ Recall that the scalars $\theta > 0$, $\epsilon > 0$ are fixed and satisfy $\frac{\theta}{\epsilon} > 2\frac{\sigma_c}{m_\gamma}$. For all $\sigma \in \mathbb{R}$, denote

$$u_{\epsilon,\theta,\sigma} = \Gamma^{-1}(\Gamma(\theta) - \epsilon\sigma)$$

where $\Gamma(t) = \int_0^t \gamma(s) ds$. The expression (4.14) of p , that was established in Theorem 4.1, reads, for almost all $\sigma \in \mathbb{R}$ such that $\sigma \leq \Gamma \left(\frac{\theta}{\epsilon} \right)$,

$$\begin{aligned} p_\epsilon \left(\frac{\theta}{\epsilon}, \sigma \right) &= p_0 \left(\sigma - \frac{\Gamma(\theta)}{\epsilon} \right) e^{-\int_0^{\frac{\theta}{\epsilon}} \chi \left(\sigma - \frac{\Gamma(\theta)}{\epsilon} + \frac{\Gamma(\epsilon v)}{\epsilon} \right) dv} \\ &\quad + \frac{f_\epsilon \left(\frac{u_{\epsilon,\theta,\sigma}}{\epsilon} \right)}{\gamma(u_{\epsilon,\theta,\sigma})} \times \begin{cases} 0 & \text{if } \sigma < 0 \\ 1 & \text{if } 0 < \sigma \leq \sigma_c \\ e^{-\frac{\theta}{\epsilon} + \frac{1}{\epsilon} \Gamma^{-1}(\Gamma(\theta) - \epsilon\sigma + \epsilon\sigma_c)} & \text{if } \sigma_c < \sigma. \end{cases} \end{aligned} \quad (4.92)$$

Note that the condition $\sigma \leq \Gamma \left(\frac{\theta}{\epsilon} \right)$ is not restrictive because we are interested in the limit $\epsilon \rightarrow 0$ for a fixed σ . The rest of the proof depends on the value of σ .

Let us start with the case $\sigma < 0$. We have

$$\left| p_\epsilon \left(\frac{\theta}{\epsilon}, \sigma \right) \right| \leq \|p_0\|_{L^\infty} e^{-\int_0^{\frac{\theta}{\epsilon}} \chi \left(\sigma - \frac{\Gamma(\theta)}{\epsilon} + \frac{\Gamma(\epsilon v)}{\epsilon} \right) dv}.$$

We now apply Lemma 4.10 with $t = \frac{\theta}{\epsilon} > 2\frac{\sigma_c}{m_\gamma}$ and $\Gamma_\epsilon(t) = \frac{1}{\epsilon} \Gamma(\epsilon t)$ and obtain

$$\left| p_\epsilon \left(\frac{\theta}{\epsilon}, \sigma \right) \right| \leq \|p_0\|_{L^\infty} e^{\frac{2\sigma_c - \theta}{m_\gamma} - \frac{\theta}{\epsilon}} \leq K e^{-\frac{\theta}{\epsilon}}. \quad (4.93)$$

Notice that (4.93) actually holds for all $\sigma \in \mathbb{R}$ (this will be used below). From the expression (4.10) of p_∞ , $p_\infty(\theta, \sigma) = 0$ when $\sigma < 0$. This gives (4.66) for almost all $\sigma < 0$.

Let us now consider the case $\sigma \in (0, \sigma_c]$. Let us introduce the notation $f_\infty(\theta) = \int \chi(\sigma) p_\infty(\theta, \sigma) d\sigma$. We have, for all positive σ ,

$$\begin{aligned} & \left| \frac{f_\epsilon \left(\frac{u_{\epsilon, \theta, \sigma}}{\epsilon} \right)}{\gamma(u_{\epsilon, \theta, \sigma})} - \frac{1}{\sigma_c + \gamma(\theta)} \right| \\ & \leq \frac{1}{m_\gamma} \left| f_\epsilon \left(\frac{u_{\epsilon, \theta, \sigma}}{\epsilon} \right) - f_\infty(u_{\epsilon, \theta, \sigma}) \right| + \frac{1}{m_\gamma} |f_\infty(u_{\epsilon, \theta, \sigma}) - f_\infty(\theta)| \\ & \quad + \left| \frac{f_\infty(\theta)}{\gamma(u_{\epsilon, \theta, \sigma})} - \frac{1}{\sigma_c + \gamma(\theta)} \right| \\ & \leq K(C_3 + 1) \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right), \end{aligned} \quad (4.94)$$

using the estimate (4.65) on $f_\epsilon(\cdot) - f_\infty$, the Lipschitz property and the boundedness of γ and the expression (4.10) which gives $f_\infty(\theta) = \frac{\gamma(\theta)}{\sigma_c + \gamma(\theta)}$. Here, we also used the fact that $u_{\epsilon, \theta, \sigma} \geq \theta - \frac{\epsilon\sigma}{m_\gamma}$, which is a consequence of the Lipschitz property of Γ^{-1} : $|u_{\epsilon, \theta, \sigma} - \theta| \leq \frac{\epsilon\sigma}{m_\gamma}$. For almost all $\sigma \in (0, \sigma_c]$, we deduce (using again (4.93))

$$\begin{aligned} & \left| p_\epsilon \left(\frac{\theta}{\epsilon}, \sigma \right) - \frac{1}{\sigma_c + \gamma(\theta)} \right| \\ & \leq \left| p_0 \left(\sigma - \frac{\Gamma(\theta)}{\epsilon} \right) e^{-\int_0^{\frac{\theta}{\epsilon}} \chi \left(\sigma - \frac{\Gamma(\theta)}{\epsilon} + \frac{\Gamma(\epsilon v)}{\epsilon} \right) dv} \right| + \left| \frac{f_\epsilon \left(\frac{u_{\epsilon, \theta, \sigma}}{\epsilon} \right)}{\gamma(u_{\epsilon, \theta, \sigma})} - \frac{1}{\sigma_c + \gamma(\theta)} \right| \\ & \leq K e^{-\frac{\theta}{\epsilon}} + K(C_3 + 1) \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right). \end{aligned}$$

Moreover, we have, from (4.10), $p_\infty(\theta, \sigma) = \frac{1}{\sigma_c + \gamma(\theta)}$ for almost all $\sigma \in (0, \sigma_c]$. This proves (4.66) in this region of σ .

We now eventually consider the case $\sigma > \sigma_c$. Applying the Taylor-Lagrange theorem on Γ^{-1} at $\Gamma(\theta)$, we have (using the fact that γ is bounded from above)

$$\left| \frac{\Gamma^{-1}(\Gamma(\theta) - \epsilon(\sigma - \sigma_c)) - \theta}{-\epsilon(\sigma - \sigma_c)} - \frac{1}{\gamma(\theta)} \right| \leq K\epsilon(\sigma - \sigma_c).$$

Using that the function $x \mapsto e^{-x}$ is 1-Lipschitz on $[0, \infty)$, this implies

$$\left| e^{-\frac{\theta}{\epsilon} + \frac{1}{\epsilon} \Gamma^{-1}(\Gamma(\theta) - \epsilon\sigma + \epsilon\sigma_c)} - e^{-\frac{\sigma - \sigma_c}{\gamma(\theta)}} \right| \leq K\epsilon(\sigma - \sigma_c)^2.$$

For almost all $\sigma > \sigma_c$, p_∞ reads (see (4.10))

$$p_\infty(\theta, \sigma) = \frac{1}{\sigma_c + \gamma(\theta)} e^{\frac{\sigma_c - \sigma}{\gamma(\theta)}}$$

and thus, we have (using (4.93) and (4.94))

$$\begin{aligned}
& \left| p_\epsilon \left(\frac{\theta}{\epsilon}, \sigma \right) - p_\infty(\theta, \sigma) \right| \\
& \leq \left| p_0 \left(\sigma - \frac{\Gamma(\theta)}{\epsilon} \right) e^{-\int_0^{\frac{\theta}{\epsilon}} \chi \left(\sigma - \frac{\Gamma(\theta)}{\epsilon} + \frac{\Gamma(\epsilon v)}{\epsilon} \right) dv} \right| \\
& \quad + \left| \frac{f_\epsilon \left(\frac{u_{\epsilon, \theta, \sigma}}{\epsilon} \right)}{\gamma(u_{\epsilon, \theta, \sigma})} - \frac{1}{\sigma_c + \gamma(\theta)} \right| e^{-\frac{\theta}{\epsilon} + \frac{1}{\epsilon} \Gamma^{-1}(\Gamma(\theta) - \epsilon \sigma + \epsilon \sigma_c)} \\
& \quad + \frac{1}{\sigma_c + \gamma(\theta)} \left| e^{-\frac{\theta}{\epsilon} + \frac{1}{\epsilon} \Gamma^{-1}(\Gamma(\theta) - \epsilon \sigma + \epsilon \sigma_c)} - e^{-\frac{\sigma - \sigma_c}{\gamma(\theta)}} \right| \\
& \leq K e^{-\frac{\theta}{\epsilon}} + K(C_3 + 1) \left(e^{-b \frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right) + K \epsilon (\sigma - \sigma_c)^2.
\end{aligned}$$

This proves (4.66) and ends the proof. \square

4.7 Macroscopic limit

The purpose of this section is to obtain a macroscopic limit for the equation we have been studying.

For $\epsilon > 0$ presumably small, we first introduce the notation $\tilde{u}(\theta) = u \left(\frac{\theta}{\epsilon} \right)$, for any real-valued function u of the real variable θ/ϵ . The small parameter ϵ encodes the discrepancy between the typical time of variation of the macroscopic shear rate $\gamma(t)$ and the typical mesoscopic time t with which the solution to (4.63) varies. The parameter ϵ will therefore be the small parameter on which our macroscopic limit is performed. To obtain a macroscopic limit of our equation (4.63), we will look at a specific macroscopic time, denoted by θ , related to the mesoscopic time t by $\theta = \epsilon t$. Letting ϵ vanish, we will obtain the corresponding macroscopic behavior of our mesoscopic quantities. In this latter process, we will of course use the results of the previous section, since formally, given a macroscopic time θ , the corresponding mesoscopic time $t = \theta/\epsilon$ is a long time limit.

We begin by formally multiplying (4.63) respectively by σ and $\chi(\sigma)$ and integrating in σ . We next evaluate at the time $\frac{\theta}{\epsilon}$ the two equations obtained. This gives the following system of equations on $\tilde{\tau}_\epsilon$ and \tilde{f}_ϵ , the quantities associated with p_ϵ solution to (4.63),

$$\left\{ \begin{aligned} \epsilon \frac{d\tilde{\tau}_\epsilon}{d\theta}(\theta) &= - \int \chi(\sigma) \sigma \tilde{p}_\epsilon(\theta, \sigma) d\sigma + \gamma(\theta), \end{aligned} \right. \quad (4.95a)$$

$$\left\{ \begin{aligned} \epsilon \frac{d\tilde{f}_\epsilon}{d\theta}(\theta) &= -\tilde{f}_\epsilon(\theta) + \gamma(\theta) (\tilde{p}_\epsilon(\theta, \sigma_c) - \tilde{p}_\epsilon(\theta, -\sigma_c)). \end{aligned} \right. \quad (4.95b)$$

The difficulty is that this system is not closed in the couple of unknown functions $(\tilde{\tau}_\epsilon, \tilde{f}_\epsilon)$ since \tilde{p}_ϵ still appears. We next intend to “eliminate” \tilde{p}_ϵ from this system, and thereby obtain a system of ordinary differential equations, the solution of which is an approximation, for ϵ sufficiently small, of $\tilde{\tau}_\epsilon$ and \tilde{f}_ϵ . There are indeed many options to do so. We present two sets of equations which we can derive and that, in a sense made precise below, are equivalent to system (4.95). The precise results are contained in two theorems we now successively state and prove, namely Theorem 4.4 and Theorem 4.5.

Theorem 4.4. For $\epsilon > 0$, consider τ_ϵ^* the solution to the following differential equation in “macroscopic” time θ :

$$\epsilon \frac{d\tau_\epsilon^*}{d\theta} = -\tau_\epsilon^* + \frac{\sigma_c^2}{2(\sigma_c + \gamma(\theta))} + \gamma(\theta) \quad (4.96)$$

supplied with any initial condition $\tau^*(0)$ independent of ϵ . Under the assumptions of Theorem 4.3, consider the function p_ϵ solution to (4.63). Then, there exists a constant C , independent from ϵ and θ provided $\frac{\theta}{\epsilon} > 2\frac{\sigma_c}{m_\gamma}$ (with m_γ the lower bound of γ in (4.62)), such that

$$\left| \tau_\epsilon \left(\frac{\theta}{\epsilon} \right) - \tau_\epsilon^*(\theta) \right| \leq C \left(\frac{1}{\theta} + 1 \right) \epsilon, \quad (4.97)$$

where (in accordance with (4.5)) $\tau_\epsilon(t) = \int \sigma p_\epsilon(t, \sigma) d\sigma$ denotes the stress associated to p_ϵ solution to (4.63).

Remark 4.4. The above result holds whatever the initial condition $\tau^*(0)$ for the equation (4.96). Indeed, we are only considering macroscopic times such that $\frac{\theta}{\epsilon} > 2\frac{\sigma_c}{m_\gamma}$ and, in the limit $\epsilon \rightarrow 0$, the boundary layer in time around $\theta = 0$ does not affect the result.

Proof. We first give the arguments to derive system (4.95). In the proof of Theorem 4.2, we established equation (4.72), which reads

$$\dot{f}_\epsilon(t) = -f_\epsilon(t) + \gamma(\epsilon t) (p_\epsilon(t, \sigma_c) - p_\epsilon(t, -\sigma_c)).$$

Denoting ρ a function of $\mathcal{D}((0, s))$, ρ^n a mollifier on \mathbb{R} and $\chi^n = \rho^n * \chi$, we used the test function $\eta^n(t, \sigma) = \chi^n(\sigma) \rho(t)$ in the weak form (4.73) of equation (4.63) and then passed to the limit $n \rightarrow \infty$. We now establish an equation on τ_ϵ with the same method.

Denote $I : \sigma \mapsto \sigma$, $I^n = \rho^n * (I \mathbb{1}_{[-n, n]})$ and use $\eta^n = (t, \sigma) = I^n(\sigma) \rho(t)$ as a test function in (4.73):

$$\begin{aligned} - \int_0^s \dot{\rho}(t) \int I^n p_\epsilon(t, \cdot) dt - \int_0^s \rho(t) \gamma(\epsilon t) \int \dot{I}^n p_\epsilon(t, \cdot) dt + \int_0^s \rho(t) \int \chi I^n p_\epsilon(t, \cdot) dt \\ = I^n(0) \int_0^s f_\epsilon(t) \rho(t) dt. \end{aligned}$$

We pass to the limit $n \rightarrow \infty$, using that the function $t \mapsto (\sigma \mapsto p_\epsilon(t, \sigma))$ belongs to $C([0, s], L^1)$ (see Theorem 4.1), $\int p_\epsilon(t, \cdot) = 1$, the dominated convergence theorem for the terms in the left-hand side, and that $I^n(0) \rightarrow I(0) = 0$ for the right-hand side. We obtain, for all $s > 0$,

$$- \int_0^s \dot{\rho}(t) \tau_\epsilon(t) dt - \int_0^s \rho(t) \gamma(\epsilon t) dt + \int_0^s \rho(t) \int \chi I p_\epsilon(t, \cdot) dt = 0,$$

so that

$$\dot{\tau}_\epsilon(t) = - \int \chi(\sigma) \sigma p_\epsilon(t, \sigma) d\sigma + \gamma(\epsilon t). \quad (4.98)$$

Changing the variable t in $\frac{\theta}{\epsilon}$ in equations (4.72) and (4.98), we obtain system (4.95).

Now that we have established system (4.95), we rewrite (4.95a) in the form

$$\epsilon \frac{d\tilde{\tau}_\epsilon}{d\theta} = -\tilde{\tau}_\epsilon + \int_{-\sigma_c}^{\sigma_c} \sigma \tilde{p}_\epsilon(\theta, \sigma) d\sigma + \gamma(\theta),$$

using the definition of χ . Moreover, we use the expression (4.10) of p_∞ to rewrite (4.96) as follows

$$\epsilon \frac{d\tau_\epsilon^*}{d\theta} = -\tau_\epsilon^* + \int_{-\sigma_c}^{\sigma_c} \sigma p_\infty(\theta, \sigma) d\sigma + \gamma(\theta).$$

Subtracting the above two equations yields

$$\frac{d(\tilde{\tau}_\epsilon - \tau_\epsilon^*)}{d\theta} + \frac{\tilde{\tau}_\epsilon - \tau_\epsilon^*}{\epsilon} = \int_{-\sigma_c}^{\sigma_c} \sigma \frac{\tilde{p}_\epsilon - p_\infty}{\epsilon} d\sigma.$$

Denote $\varsigma = 2\frac{\sigma_c}{m_\gamma}$ so that $\frac{\theta}{\epsilon} > \varsigma$. We next apply the Duhamel formula and find

$$(\tilde{\tau}_\epsilon - \tau_\epsilon^*)(\theta) = (\tilde{\tau}_\epsilon - \tau_\epsilon^*)(\varsigma\epsilon) e^{\varsigma - \frac{\theta}{\epsilon}} + \int_{\varsigma\epsilon}^{\theta} e^{\frac{u-\theta}{\epsilon}} \int_{-\sigma_c}^{\sigma_c} \sigma \frac{\tilde{p}_\epsilon - p_\infty}{\epsilon} d\sigma du. \quad (4.99)$$

Using the estimate (4.26) on τ_ϵ , the bounds (4.62) on γ , we find

$$\begin{aligned} |\tilde{\tau}_\epsilon - \tau_\epsilon^*(\varsigma\epsilon)| &\leq (1 + M_\gamma \varsigma) C_\tau + \left(|\tau_0| e^{-\varsigma} + \int_0^\varsigma e^{u-\varsigma} \left(\frac{\sigma_c^2}{2(\sigma_c + m_\gamma)} + M_\gamma \right) du \right) \\ &\leq K, \end{aligned} \quad (4.100)$$

where we recall that K denotes a constant that is independent from θ and ϵ and whose precise value may change from one occurrence to another. Inserting the above estimate and the estimate (4.66) on $|\tilde{p}_\epsilon - p_\infty|$ in (4.99) yields

$$\begin{aligned} |\tilde{\tau}_\epsilon - \tau_\epsilon^*(\theta)| &\leq K \left[e^{\varsigma - \frac{\theta}{\epsilon}} + \int_0^\theta e^{\frac{u-\theta}{\epsilon}} \left(\frac{e^{-b\frac{u}{\epsilon}}}{\epsilon} + \frac{e^{-\frac{u}{\epsilon}}}{\epsilon} + 1 \right) du \right] \\ &\leq K \left(e^{-\frac{\theta}{\epsilon}} + \frac{1}{1-b} \left(e^{-b\frac{\theta}{\epsilon}} - e^{-\frac{\theta}{\epsilon}} \right) + \frac{\theta}{\epsilon} e^{-\frac{\theta}{\epsilon}} + \epsilon \right) \\ &\leq K \left(\frac{b}{|1-b|} \frac{e^{-1}}{\theta} + \frac{1}{|1-b|} \frac{e^{-1}}{b\theta} + \frac{4e^{-2}}{\theta} + 1 \right) \epsilon, \end{aligned} \quad (4.101)$$

using that the functions $x \mapsto xe^{-x}$ and $x \mapsto x^2e^{-x}$ are respectively bounded by e^{-1} and $4e^{-2}$ on \mathbb{R}_+ in order to derive the last line. This concludes the proof. \square

Theorem 4.5. For $\epsilon > 0$, consider $(\tau_\epsilon^{**}, f_\epsilon^{**})$ satisfying the following system of equations in macroscopic time θ :

$$\left\{ \begin{aligned} \epsilon \frac{d\tau_\epsilon^{**}}{d\theta}(\theta) &= -\kappa(\theta) f_\epsilon^{**}(\theta) \tau_\epsilon^{**}(\theta) + \gamma(\theta), \end{aligned} \right. \quad (4.102a)$$

$$\left\{ \begin{aligned} \epsilon \frac{df_\epsilon^{**}}{d\theta}(\theta) &= -f_\epsilon^{**}(\theta) + \frac{\gamma(\theta)}{\sigma_c + \gamma(\theta)}, \end{aligned} \right. \quad (4.102b)$$

where we have introduced the notation

$$\kappa(\theta) = \frac{2}{1 + \frac{1}{\left(1 + \frac{\sigma_c}{\gamma(\theta)}\right)^2}}, \quad (4.103)$$

and where the equations are supplied with any couple of scalars (independent of ϵ) $(\tau^{**}(0), f^{**}(0))$ as initial conditions. Consider $\theta > 0$ such that $\frac{\theta}{\epsilon} > 2\frac{\sigma_c}{m\gamma}$. Under the assumptions of Theorem 4.3, consider p_ϵ the solution of (4.63). Then, there exists a constant C independent from ϵ and θ such that,

$$\left| \tau_\epsilon \left(\frac{\theta}{\epsilon} \right) - \tau_\epsilon^{**}(\theta) \right| + \left| f_\epsilon \left(\frac{\theta}{\epsilon} \right) - f_\epsilon^{**}(\theta) \right| \leq C \left(\frac{1}{\theta} + 1 \right) \epsilon. \quad (4.104)$$

Remark 4.5. As in Theorem 4.4, the above result holds whatever the set of initial conditions $(\tau^{**}(0), f^{**}(0))$ for the system of differential equations (4.102).

Proof. The proof falls in three steps. We first study f_ϵ , then an auxiliary function β_ϵ and finally τ_ϵ .

*Step 1: Approximation of f_ϵ^{**}* Applying the Duhamel formula to (4.102b) yields

$$\begin{aligned} f_\epsilon^{**}(\theta) &= f_0 e^{-\frac{\theta}{\epsilon}} + \frac{1}{\epsilon} \int_0^\theta \frac{\gamma(v)}{\sigma_c + \gamma(v)} e^{-\frac{v-\theta}{\epsilon}} dv \\ &= f_0 e^{-\frac{\theta}{\epsilon}} + \frac{\gamma(\theta)}{\sigma_c + \gamma(\theta)} - \frac{\gamma(0)}{\sigma_c + \gamma(0)} e^{-\frac{\theta}{\epsilon}} - \int_0^\theta \frac{\sigma_c \dot{\gamma}(v)}{(\sigma_c + \gamma(v))^2} e^{-\frac{v-\theta}{\epsilon}} dv. \end{aligned} \quad (4.105)$$

Using the Lipschitz property of γ and denoting by $f_\infty(\theta) = \int \chi(\sigma) p_\infty(\theta, \sigma) d\sigma = \frac{\gamma(\theta)}{\sigma_c + \gamma(\theta)}$ (following (4.4) and (4.10)), we easily obtain

$$|f_\epsilon^{**} - f_\infty|(\theta) \leq K \left(e^{-\frac{\theta}{\epsilon}} + \epsilon \right). \quad (4.106)$$

Collecting the above equation and the estimate (4.65) established in Theorem 4.3, we obtain

$$\begin{aligned} \left| \tilde{f}_\epsilon - f_\epsilon^{**} \right|(\theta) &\leq K \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right) \\ &\leq K \left(\frac{e^{-1}}{b\theta} + \frac{e^{-1}}{\theta} + 1 \right) \epsilon, \end{aligned} \quad (4.107)$$

using that the function $x \mapsto xe^{-x}$ is bounded by e^{-1} on \mathbb{R}_+ .

Step 2: Introduction of the auxiliary function β_ϵ We now introduce

$$\beta(t) = \int \chi(\sigma) \sigma p(t, \sigma) d\sigma, \quad (4.108)$$

defined for a density probability p such that $\sigma p \in L^1$. Denote ρ a function of $\mathcal{D}((0, s))$, ρ^n a mollifier on \mathbb{R} and $I : \sigma \mapsto \sigma$. Using $\eta^n(t, \sigma) = (I\chi \mathbb{1}_{[-n, n]}) * \rho^n(\sigma) \rho(t)$ as test function in (4.73) and passing to the limit with the same arguments as in (4.74), we obtain

$$\epsilon \frac{d\widetilde{\beta}_\epsilon}{d\theta}(\theta) + \widetilde{\beta}_\epsilon(\theta) = \gamma(\theta) \left(\widetilde{f}_\epsilon(\theta) + \sigma_c (\widetilde{p}_\epsilon(\theta, \sigma_c) + \widetilde{p}_\epsilon(\theta, -\sigma_c)) \right).$$

Consider β_ϵ^* the solution of the ordinary differential equation

$$\epsilon \frac{d\beta_\epsilon^*}{d\theta} + \beta_\epsilon^* = \gamma(\theta) \left(f_\epsilon^{**} + \frac{\sigma_c}{\sigma_c + \gamma(\theta)} \right) \quad (4.109)$$

supplied with a scalar β_0 as initial condition. Subtracting the above two equations and using that (see (4.10)) $p_\infty(\theta, \sigma_c) = \frac{1}{\sigma_c + \gamma(\theta)}$ and $p_\infty(\theta, -\sigma_c) = 0$, we obtain

$$\frac{d(\widetilde{\beta}_\epsilon - \beta_\epsilon^*)}{d\theta} + \frac{\widetilde{\beta}_\epsilon - \beta_\epsilon^*}{\epsilon} = G_\epsilon(\theta) \quad (4.110)$$

with

$$G_\epsilon(\theta) = \gamma(\theta) \frac{\widetilde{f}_\epsilon - f_\epsilon^{**}}{\epsilon}(\theta) + \gamma(\theta) \sigma_c \frac{\widetilde{p}_\epsilon - p_\infty}{\epsilon}(\theta, \sigma_c) + \gamma(\theta) \sigma_c \frac{\widetilde{p}_\epsilon - p_\infty}{\epsilon}(\theta, -\sigma_c).$$

Recall that $\varsigma = 2 \frac{\sigma_c}{m_\gamma}$ and $\theta > \varsigma \epsilon$. Applying the Duhamel formula yields,

$$\left(\widetilde{\beta}_\epsilon - \beta_\epsilon^* \right) (\theta) = \left(\widetilde{\beta}_\epsilon - \beta_\epsilon^* \right) (\varsigma \epsilon) e^{\varsigma - \frac{\theta}{\epsilon}} + \int_{\varsigma \epsilon}^{\theta} e^{\frac{u-\theta}{\epsilon}} G_\epsilon(u) du. \quad (4.111)$$

Using the upper bound (4.26), $\widetilde{\beta}_\epsilon$ satisfies

$$\left| \widetilde{\beta}_\epsilon(\varsigma \epsilon) \right| \leq \int |\sigma| \chi p_\epsilon(\varsigma, \cdot) \leq \int |\sigma| p_\epsilon(\varsigma, \cdot) \leq (1 + M_\gamma \varsigma) C_\tau \leq K.$$

Moreover, β_ϵ^* solution of (4.109) satisfies

$$\begin{aligned} |\beta_\epsilon^*(\varsigma \epsilon)| &\leq |\beta_0| e^{-\varsigma} + \int_0^\varsigma e^{u-\varsigma} \left(\left(|f_0| + \frac{M_\gamma}{\sigma_c + m_\gamma} \right) + \frac{\sigma_c}{\sigma_c + m_\gamma} \right) du \\ &\leq K. \end{aligned}$$

Using (4.107), the boundedness of γ and the estimate (4.66) on $\widetilde{p}_\epsilon - p_\infty$, the right-hand side $G_\epsilon(\theta)$ of (4.110) satisfies,

$$G_\epsilon(\theta) \leq K \left(\frac{e^{-b \frac{\theta}{\epsilon}}}{\epsilon} + \frac{e^{-\frac{\theta}{\epsilon}}}{\epsilon} + 1 \right).$$

Inserting the three above inequalities in (4.111) implies

$$\left| \widetilde{\beta}_\epsilon - \beta_\epsilon^* \right| (\theta) \leq K \left(e^{-b \frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right). \quad (4.112)$$

Additionally, applying the Duhamel formula to (4.109) and using the explicit formula $f_\infty(\theta) = \frac{\gamma(\theta)}{\sigma_c + \gamma(\theta)}$ yield

$$\beta_\epsilon^*(\theta) - \int_0^\theta \frac{\gamma(u)}{\epsilon} e^{\frac{u-\theta}{\epsilon}} du = \beta_0 e^{-\frac{\theta}{\epsilon}} + \int_0^\theta \frac{\gamma(u)}{\epsilon} (f_\epsilon^{**}(u) - f_\infty) e^{\frac{u-\theta}{\epsilon}} du.$$

Using the Lipschitz property of γ and the estimate (4.106), we obtain

$$|\beta_\epsilon^* - \gamma|(\theta) \leq K \left(e^{-\frac{\theta}{\epsilon}} + \epsilon \right). \quad (4.113)$$

Combining (4.112) and (4.113) leads to

$$\left| \widetilde{\beta}_\epsilon - \gamma \right|(\theta) \leq K \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right) \quad (4.114)$$

and eventually,

$$\begin{aligned} |\widetilde{\tau}_\epsilon - \tau_\infty|(\theta) &\leq \left| \int \chi \sigma (\widetilde{p}_\epsilon - p_\infty) \right| + \left| \int (1 - \chi) \sigma (\widetilde{p}_\epsilon - p_\infty) d\sigma \right| \\ &\leq K \left(e^{-b\frac{\theta}{\epsilon}} + e^{-\frac{\theta}{\epsilon}} + \epsilon \right), \end{aligned} \quad (4.115)$$

respectively using (4.114) and (4.66) to estimate the two terms of the right-hand side. Here we have used the notation $\tau_\infty(\theta) = \int \sigma p_\infty(\theta, \sigma) d\sigma$, and the fact that $\int \chi(\sigma) \sigma p_\infty(\theta, \sigma) d\sigma = \gamma(\theta)$.

Step 3: Approximation of τ_ϵ We now turn to $(\widetilde{\tau}_\epsilon - \tau_\epsilon^{**})$. Combining (4.95a) and (4.102a) yields

$$\begin{aligned} \epsilon \frac{d(\widetilde{\tau}_\epsilon - \tau_\epsilon^{**})}{d\theta} &= +\kappa f_\epsilon^{**} \tau_\epsilon^{**} - \int \chi \sigma \widetilde{p}_\epsilon \\ &= -\kappa f_\infty (\widetilde{\tau}_\epsilon - \tau_\epsilon^{**}) - \kappa \tau_\epsilon^{**} (f_\infty - f_\epsilon^{**}) \\ &\quad - \kappa f_\infty (\tau_\infty - \widetilde{\tau}_\epsilon) - \left(-\kappa f_\infty \tau_\infty + \int \chi \sigma p_\infty \right) - \int \chi \sigma (\widetilde{p}_\epsilon - p_\infty). \end{aligned}$$

From the formula (4.10) on p_∞ , we compute $\int \chi \sigma p_\infty(\theta, \cdot) = \gamma(\theta)$, $f_\infty(\theta) = \frac{\gamma(\theta)}{\sigma_c + \gamma(\theta)}$, $\tau_\infty(\theta) = \frac{1}{2} \left(\frac{\gamma(\theta)^2}{\sigma_c + \gamma(\theta)} + \sigma_c + \gamma(\theta) \right)$ so that the term $-\kappa f_\infty \tau_\infty + \int \chi \sigma p_\infty$ cancels out because of the definition (4.103) of κ . We therefore obtain

$$\frac{d(\widetilde{\tau}_\epsilon - \tau_\epsilon^{**})}{d\theta} + \frac{\kappa f_\infty}{\epsilon} (\widetilde{\tau}_\epsilon - \tau_\epsilon^{**}) = H_\epsilon(\theta) \quad (4.116)$$

with

$$H_\epsilon(\theta) = -\kappa \tau_\epsilon^{**} \frac{f_\infty - f_\epsilon^{**}}{\epsilon}(\theta) - \kappa f_\infty \frac{\tau_\infty - \widetilde{\tau}_\epsilon}{\epsilon}(\theta) - \int \chi(\sigma) \sigma \frac{\widetilde{p}_\epsilon - p_\infty}{\epsilon}(\theta, \sigma) d\sigma.$$

We have $\kappa \geq 1$ (see (4.103)) so that

$$\kappa f_\infty \geq \frac{1}{1 + \frac{\sigma_c}{m_\gamma}}. \quad (4.117)$$

The Duhamel formula then implies

$$|\widetilde{\tau}_\epsilon - \tau_\epsilon^{**}|(\theta) \leq |\widetilde{\tau}_\epsilon - \tau_\epsilon^{**}|(\varsigma \epsilon) e^{\frac{1}{1 + \frac{\sigma_c}{m_\gamma}} (\varsigma \epsilon - \frac{\theta}{\epsilon})} + \int_{\varsigma \epsilon}^\theta e^{\frac{1}{1 + \frac{\sigma_c}{m_\gamma}} \frac{u - \theta}{\epsilon}} |H_\epsilon(u)| du \quad (4.118)$$

Using the upper bound (4.26), $\widetilde{\tau}_\epsilon$ satisfies

$$|\tilde{\tau}_\epsilon(\varsigma\epsilon)| \leq (1 + M_\gamma\varsigma) C_\tau \leq K.$$

Moreover, the solution τ_ϵ^{**} of (4.102) satisfies (using the non negativity of κ and f_ϵ^{**}):

$$\begin{aligned} \tau_\epsilon^{**}(\theta) &= e^{-\frac{1}{\epsilon} \int_0^\theta \kappa(s) f_\epsilon^{**}(s) ds} \tau_\epsilon^{**}(0) + \frac{1}{\epsilon} \int_0^\theta \gamma(s) e^{-\frac{1}{\epsilon} \int_s^\theta \kappa(r) f_\epsilon^{**}(r) dr} ds \\ &\leq \tau_\epsilon^{**}(0) + M_\gamma \frac{\theta}{\epsilon} \end{aligned}$$

so that

$$|\tau_\epsilon^{**}(\varsigma\epsilon)| \leq K(1 + \varsigma).$$

Collecting (4.65) established in Theorem 4.3 , (4.114) and (4.115), the right-hand side $H_\epsilon(u)$ of (4.116) satisfies

$$H_\epsilon(u) \leq K \left(\frac{e^{-b\frac{u}{\epsilon}}}{\epsilon} + \frac{e^{-\frac{u}{\epsilon}}}{\epsilon} + 1 \right).$$

Inserting the three above inequalities in (4.118) implies

$$\begin{aligned} |\tilde{\tau}_\epsilon - \tau_\epsilon^{**}|(\theta) &\leq K \left(e^{-\frac{1}{1+\frac{\sigma\epsilon}{m\gamma}} \frac{\theta}{\epsilon}} + \int_0^\theta e^{\frac{1}{1+\frac{\sigma\epsilon}{m\gamma}} \frac{u-\theta}{\epsilon}} \left(\frac{e^{-b\frac{u}{\epsilon}}}{\epsilon} + \frac{e^{-\frac{u}{\epsilon}}}{\epsilon} + 1 \right) du \right) \\ &\leq K \left(\frac{1}{\theta} + 1 \right) \epsilon. \end{aligned}$$

□

We end this section with a discussion on the two macroscopic limits (4.96) and (4.102) we have obtained. First, as mentioned above, there are many ways to close the system (4.95) in the limit $\epsilon \rightarrow 0$. We have proposed here two possible macroscopic limits, which are indeed close up to terms of order $O(\epsilon)$ to the original problem (4.63).

Second, we would like to argue that the system (4.102) derived in Theorem 4.5 is physically more relevant. Indeed, up to changing the coefficient $\kappa(\theta)$ defined by (4.103) by a constant, this system belongs to a class of equations introduced in [DAL01, PABL02] to model the evolution of aging fluids. These equations read (see [PABL02, Eq. (1)])

$$\begin{cases} \frac{\partial \tau}{\partial t} = -f\tau + \gamma, & (4.119a) \\ \frac{\partial f}{\partial t} = -U(f) + V(f, \tau, \gamma), & (4.119b) \end{cases}$$

where U and V are positive functions. The formal similarity between (4.102) and (4.119) is clear.

For this class of systems, the fluidity f appears as the inverse of the relaxation time for the stress τ in equation (4.119a). In equation (4.119b) the evolution results from the competition between the two terms with opposite signs. Aging, meaning solidification of the fluid, is modeled by the negative term. It makes the fluidity decrease so that the relaxation phenomenon is slower with time. The opposite effect, flow-induced rejuvenation, is modeled by the positive term, which makes the fluidity (the inverse relaxation time) increase.

Note that the assumption $\kappa(\theta)$ constant is a reasonable approximation when γ is small. In this case, system (4.102) is close to system (4.121), on which we present numerical results in section 4.8. System (4.121) is indeed of the form (4.119) and the numerical results confirm that the solutions to (4.102) and (4.121) are indeed close when γ is small.

4.8 Numerical experiments

This section is devoted to some numerical experiments. We consider three different situations, depending on the value of the function $\gamma(t)$ for $t \in [0, T]$.

- (i) In our first series of tests, we consider the *constant* shear rate $\gamma(t) \equiv \gamma_\infty$, for different values $0.1, 0.2, \dots, 0.8$ of γ_∞ . In that case, the final time is $T = 40$
- (ii) In our second series of tests, we consider $\gamma(t) = t$, and the final time $T = \frac{\theta}{\epsilon}$ with $\theta = 1$ and ϵ varying between the values 0.005 and 0.05 .
- (iii) In our third and final series of tests, we take $\gamma(t) = 0.01 \cdot t$, and the same values of T , θ , and ϵ as in case (ii).

In all our tests, the reference equation, namely (4.1), is simulated over the time interval $[0, T]$. Since in theory it is posed on the whole real line, we need to truncate the domain and thus actually solve the equation on the bounded interval $\sigma \in [-M_\sigma, M_\sigma]$ (with periodic boundary conditions), for $M_\sigma = 10$, with a constant space step $\Delta\sigma = \frac{M_\sigma}{2 \cdot 10^5}$. The initial condition p_0 is the normal centered Gaussian density, appropriately renormalized on the interval $[-M_\sigma, M_\sigma]$. The threshold value for the stress is $\sigma_c = 2$. The time discretization is performed using a splitting method: over the time interval $[n\Delta t, (n+1)\Delta t]$,

$$\left\{ \begin{array}{l} \frac{\partial p^{n+\frac{1}{2}}}{\partial t} = -\chi p^{n+\frac{1}{2}} + \left(\int \chi p^{n+\frac{1}{2}} \right) \delta_0 \\ \frac{\partial p^{n+1}}{\partial t} + \gamma(\epsilon n \Delta t) \frac{\partial p^{n+\frac{1}{2}}}{\partial \sigma} = 0. \end{array} \right. \quad (4.120a)$$

$$\left\{ \begin{array}{l} \frac{\partial p^{n+\frac{1}{2}}}{\partial t} = -\chi p^{n+\frac{1}{2}} + \left(\int \chi p^{n+\frac{1}{2}} \right) \delta_0 \\ \frac{\partial p^{n+1}}{\partial t} + \gamma(\epsilon n \Delta t) \frac{\partial p^{n+\frac{1}{2}}}{\partial \sigma} = 0. \end{array} \right. \quad (4.120b)$$

Equation (4.120a) is solved explicitly, pointwise for each $\sigma \neq 0$, while the equation for $\sigma = 0$ is indeed solved using the conservation of the total mass of the density p , see [Gat05] for more details on this elementary technique. The advection equation (4.120b) is solved using an upwind finite difference scheme. All computations are performed using C++.

- (i) For our first series of tests, performed for the shear rate $\gamma(t) = \gamma_\infty$, we first check the optimality of the long time convergence result stated in Theorem 4.2. For the various values of γ_∞ indicated above, we simulate (4.1). Using a least-square fit, we then estimate, in function of γ_∞ , the exponent of the exponential rate of convergence of $(\int (p(t, \cdot) - p_\infty)^2)^{1/2}$ to zero as time goes to infinity, more precisely what corresponds to the parameter b of the right-hand side of (4.30). The function p_∞ is of course the stationary solution (4.10), itself function of γ_∞ , explicitly determined in our theoretical study. Note that for this practical experiment we intentionally make a confusion between the rate of convergence of the pointwise difference $p(t, \cdot) - p_\infty$ in (4.30) and its L^2 norm. Figure 4.1 shows, as a function of γ_∞ , a comparison between the rate of convergence fitted on the numerical results and the theoretical value of the inferior bound on this rate of convergence provided by our theoretical estimate (4.32). The two sets of data agree, thereby showing the quality of our estimate (4.32).

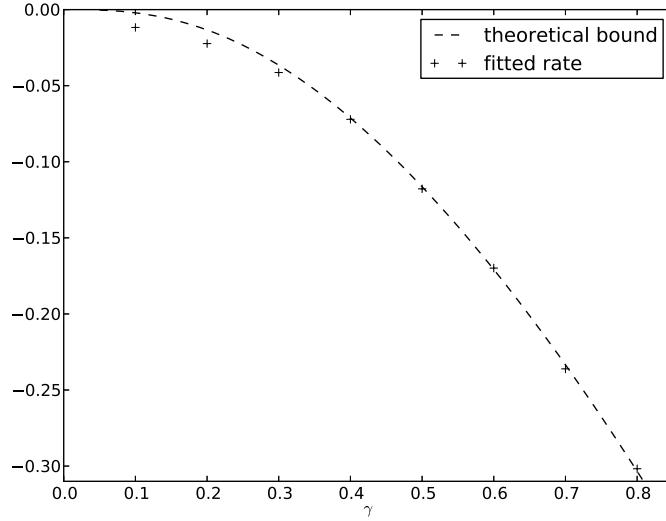


Fig. 4.1. Exponential convergence rate of $(\int (p(t, \cdot) - p_\infty)^2)^{1/2}$ as time goes to infinity function of $\gamma(t) = \gamma_\infty$

(ii) We next consider $\gamma(t) = t$, and implement the change of time scale by taking $\gamma(\epsilon t)$ as input for equation (4.1), that is, we solve equation (4.63). We do this for various values of the small parameter ϵ . In this case (ii), our purpose is twofold. First, we consider the quantities

$$|\tilde{f}_\epsilon - f_\infty|(\theta) = \left| \int \chi p_\epsilon \left(\frac{\theta}{\epsilon}, \cdot \right) - \int \chi p_\infty(\theta, \cdot) \right|$$

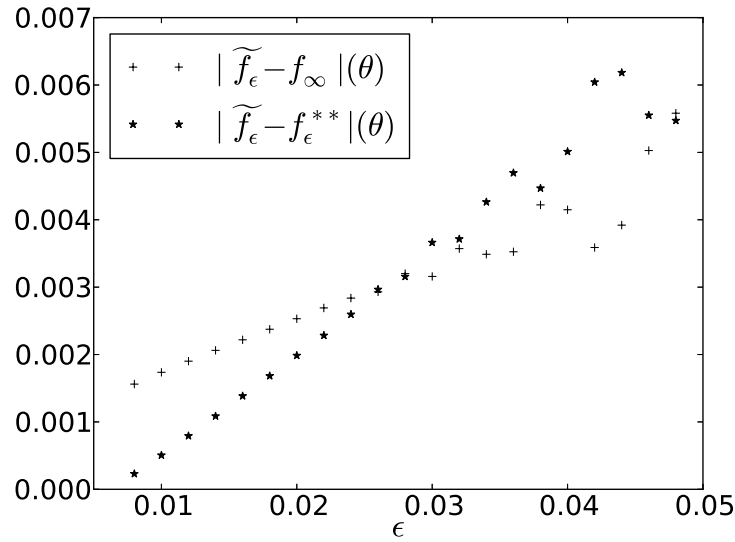
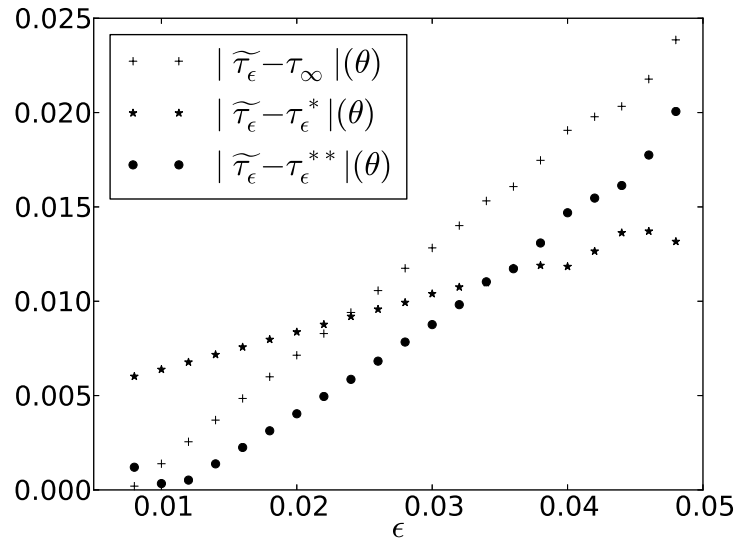
and

$$|\tilde{\tau}_\epsilon - \tau_\infty|(\theta) = \left| \int \sigma p_\epsilon \left(\frac{\theta}{\epsilon}, \cdot \right) - \int \sigma p_\infty(\theta, \cdot) \right|$$

where p_ϵ and p_∞ are the solutions to (4.63) and (4.64), respectively. We wish to check that, as the estimate proved in Theorem 4.3 suggests, these two quantities behave linearly in function of ϵ for small ϵ . The figures 4.2 and 4.3 show this is indeed the case.

Our second purpose is to illustrate that the macroscopic equations obtained in Theorems 4.4 and 4.5 indeed behave similarly to the original mesoscopic equation (4.63), that is, provide a fluidity f and a stress τ comparable to that computed from the solution to the latter equation. For the different values of ϵ mentioned above, we therefore simulate equations (4.96) and (4.102) on the time interval $[0, \theta]$, with the time step $\Delta t = \frac{T}{1.10^4}$. The values of $|\tilde{\tau}_\epsilon - \tau_\epsilon^*|(\theta)$, $|\tilde{f}_\epsilon - f_\epsilon^{**}|(\theta)$ et $|\tilde{\tau}_\epsilon - \tau_\epsilon^{**}|(\theta)$ are displayed on Figures 4.2 and 4.3. We observe that the convergence is linear in ϵ , as predicted by our theoretical results Theorems 4.4 and 4.5. The macroscopic behavior is thus suitably reproduced, up to an error of size $O(\epsilon)$.

(iii) Our final test case addresses the case $\gamma(t) = 0.01 \cdot t$. We again rescale the time and consider $\gamma(\epsilon t)$. A similar experiment as that performed in the previous case (ii) again shows that equation (4.102) reproduces well the stress tensor computed from the solution to equation (4.63), for the different values of ϵ . Simulating (4.63) and (4.102), we compute $|\tilde{\tau}_\epsilon - \tau_\epsilon^{**}|(\theta)$. The results are displayed on Figure 4.4.


 Fig. 4.2. Convergence of f

 Fig. 4.3. Convergence of τ

But our purpose here is also to illustrate another fact. When γ is small, and it is indeed the case for our specific choice of γ in this case (iii), the value of the parameter $\kappa(\theta)$ defined by (4.103) and appearing in the macroscopic system (4.102) is approximately 2. System (4.102) is thus close to the system

$$\begin{cases} \epsilon \frac{d\tau_\epsilon^{***}}{d\theta} = -2f_\epsilon^{***}\tau_\epsilon^{***} + \gamma(\theta) & (4.121a) \end{cases}$$

$$\begin{cases} \epsilon \frac{df_\epsilon^{***}}{d\theta} = -f_\epsilon^{***} + \frac{\gamma(\theta)}{\sigma_c + \gamma(\theta)}. & (4.121b) \end{cases}$$

As explained above, this system of differential equations belongs to the class of systems (4.119) explicitly suggested in [DAL01, PABL02] as a macroscopic system of evolution of f and τ for a non Newtonian aging fluid. Our theoretical results of the previous sections can therefore be interpreted as a *derivation*, from a model at a finer scale, of the macroscopic system (4.121), present in the applicative literature. On Figure 4.4, we indeed check that the stresses solution to the systems (4.102) and (4.121) are close, up to an error of size $O(\epsilon)$, to the stress provided by (4.63),

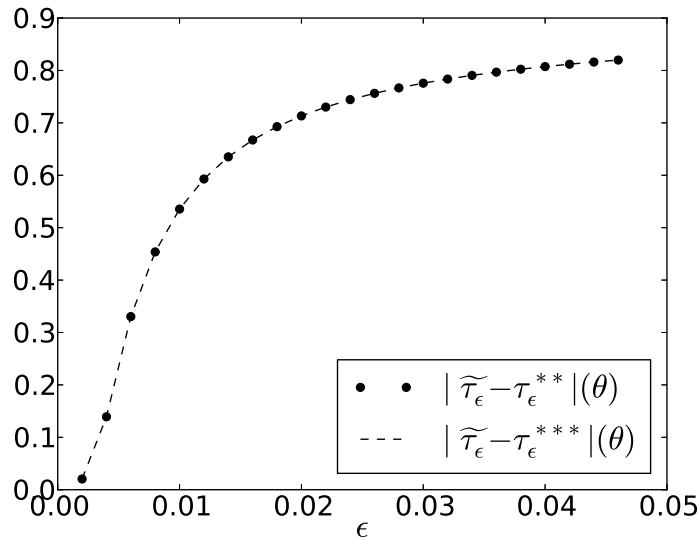


Fig. 4.4. Convergence of τ in the case γ small

Bibliographie

- [Ajd03] A. Ajdari. Course 2 : Mechanical Aging and Non-linear Rheology of Concentrated Colloidal Suspensions : Experimental Facts and Simple Models. In *Slow Relaxations and nonequilibrium dynamics in condensed matter*, Volume 77 of *Les Houches-Ecole d'Eté de Physique Theorique*, pages 41–73. Springer Berlin Heidelberg, 2003.
- [BCG⁺13] J.-B. Bardet, A. Christen, A. Guillin, F. Malrieu and P.-A. Zitt. Total variation estimates for the TCP process. *Electron. J. Probab.*, volume 18 : no. 10, 21, 2013.
- [BL11] J.-L. Barrat and A. Lemaitre. Heterogeneities in amorphous systems under shear. In *Dynamical Heterogeneities in Glasses, Colloids, and Granular Media*. Oxford University Press, Oxford, 2011.
- [BACMP04] M. Behr, D. Arora, O. Coronado-Matutti and M. Pasquali. Stabilized Finite Element Methods of GLS Type for Oldroyd-B Viscoelastic Fluid. In *European Congress on Computational Methods in Applied Science and Engineering ECCO-MAS*, 2004.
- [BC63] R. Bellman and K. L. Cooke. *Differential-difference equations*. Academic Press, New York, 1963.
- [BAJ07] M. Ben Alaya and B. Jourdain. Probabilistic approximation of a nonlinear parabolic equation occurring in rheology. *Journal of applied probability*, volume 44, number 2 : 528–546, 2007.
- [BHBL13] D. Benoit, L. He, C. L. Bris and T. Lelièvre. Mathematical analysis of a one-dimensional model for an aging fluid. *Mathematical Models and Methods in Applied Sciences*, volume 23, number 09 : 1561–1602, 2013.
- [Benon] D. Benoit. *Various theoretical and numerical issues related to the simulation of non-Newtonian fluids*. PhD thesis, Université Paris Est, Ecole des Ponts ParisTech, in preparation.
- [BCAH91] R. Bird, C. Curtiss, R. Armstrong and O. Hassager. *Dynamics of Polymeric Liquids, 2 Volume Set*. Dynamics of Polymeric Liquids. Wiley, 1991.
- [BCA09] L. Bocquet, A. Colin and A. Ajdari. Kinetic Theory of Plastic Flow in Soft Glassy Materials. *Phys. Rev. Lett.*, volume 103 : 036001, 2009.
- [BLM09] S. Boyaval, T. Lelièvre and C. Mangoubi. Free-energy-dissipative schemes for the Oldroyd-B model. *Mathematical Modelling and Numerical Analysis*, volume 43, number 2 : 523–561, 2009.

- [CCG05] E. Cancès, I. Catto and Y. Gati. Mathematical analysis of a nonlinear parabolic equation arising in the modelling of non-Newtonian flows. *SIAM Journal on Mathematical Analysis*, volume 37, number 1 : 60–82, 2005.
- [CLB06] E. Cancès and C. Le Bris. Convergence to equilibrium of a multiscale model for suspensions. *Discrete Contin. Dyn. Syst. Ser. B*, volume 6, number 3 : 449–470, 2006.
- [CCGLB05] E. Cancès, I. Catto, Y. Gati and C. Le Bris. Well-posedness of a multiscale model for concentrated suspensions. *Multiscale Modeling & Simulation*, volume 4, number 4 : 1041–1058, 2005.
- [Car67] H. Cartan. *Calcul différentiel*. Hermann, Paris, 1967.
- [DAL01] C. Derec, A. Ajdari and F. Lequeux. Rheology and aging : a simple approach. *Eur. Phys. J. E*, volume 4, number 3 : 355–361, 2001.
- [Fre] FreeFem++. <http://www.freefem.org>.
- [Gat05] Y. Gati. Numerical simulation of a micro–macro model of concentrated suspensions. *International journal for numerical methods in fluids*, volume 47, number 8-9 : 1019–1025, 2005.
- [Goy08] J. Goyon. *Matériaux amorphes : des solides qui coulent de façon collective*. PhD thesis, Université Bordeaux I, 2008.
- [GTH⁺06] B. Gueslin, L. Talini, B. Herzhaft, Y. Peysson and C. Allain. Flow induced by a sphere settling in an aging yield-stress fluid. *Physics of Fluids*, volume 18 : 103101, 2006.
- [GS90] C. Guillopé and J. C. Saut. Global Existence and One-dimensional Nonlinear Stability of Shearing Motions of Viscoelastic Fluids of Oldroyd Type. *RAIRO-Mathematical Modelling and Numerical Analysis*, volume 24, number 3 : 369–401, 1990.
- [HL93] J. Hale and S. Lunel. *Introduction to Functional Differential Equations*. Number vol. 99 in Applied Mathematical Sciences. Springer, 1993.
- [HX10] L. He and L. Xu. Global Well-Posedness for Viscoelastic Fluid System in Bounded Domains. *SIAM Journal on Mathematical Analysis*, volume 42 : 2610, 2010.
- [HL98] P. Hébraud and F. Lequeux. Mode-Coupling Theory for the Pasty Rheology of Soft Glassy Materials. *Phys. Rev. Lett.*, volume 81 : 2934–2937, 1998.
- [KS92] S. Kawashima and Y. Shibata. Global existence and exponential stability of small solutions to nonlinear viscoelasticity. *Communications in mathematical physics*, volume 148, number 1 : 189–208, 1992.
- [Keu89] R. Keunings. Simulation of viscoelastic fluid flow. *Fundamentals of computer modeling for polymer processing*, pages 402–470, 1989.
- [Keu00] R. Keunings. A survey of computational rheology. In *Proceedings of the XIIIth International Congress on Rheology*, Volume 1, pages 7–14. British Soc. Rheol, 2000.
- [LBL09] C. Le Bris and T. Lelièvre. Multiscale modelling of complex fluids : a mathematical initiation. In *Multiscale modeling and simulation in science*, pages 49–137. Springer, 2009.

- [LBL12] C. Le Bris and T. Lelièvre. Micro-macro models for viscoelastic fluids : modelling, mathematics and numerics. *Science China Mathematics*, volume 55, number 2 : 353–384, 2012.
- [Lio69] J. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Études mathématiques. Gauthier-Villars, 1969.
- [Oli10] J. Olivier. Asymptotic analysis in flow curves for a model of soft glassy fluids. *Zeitschrift für angewandte Mathematik und Physik*, volume 61, number 3 : 445–466, 2010.
- [OR11] J. Olivier and M. Renardy. Glass transition seen through asymptotic expansions. *SIAM Journal on Applied Mathematics*, volume 71, number 4 : 1144–1167, 2011.
- [OR13] J. Olivier and M. Renardy. On the Generalization of the Hébraud–Lequeux Model to Multidimensional Flows. *Archive for Rational Mechanics and Analysis*, pages 1–33, 2013.
- [Per00] L. Perko. *Differential Equations and Dynamical Systems*. Texts in Applied Mathematics. Springer, 2000.
- [Pic04] G. Picard. *Hétérogénéité de l'écoulement de fluides à seuil : approche phénoménologique et modélisation élasto-plastique*. PhD thesis, Université Denis Diderot - Paris VII, 2004.
- [PABL02] G. Picard, A. Ajdari, L. Bocquet and F. Lequeux. Simple model for heterogeneous flows of yield stress fluids. *Phys. Rev. E*, volume 66, number 5 : 051501, Nov 2002.
- [PALB04] G. Picard, A. Ajdari, F. Lequeux and L. Bocquet. Elastic consequences of a single plastic event : A step towards the microscopic modeling of the flow of yield stress fluids. *The European Physical Journal E*, volume 15, number 4 : 371–381, 2004.
- [Ren00] M. Renardy. *Mathematical Analysis of Viscoelastic Flows*. CBMS-NSF regional conference series in applied mathematics. Society for Industrial and Applied Mathematics, 2000.
- [Ren09a] M. Renardy. Global Existence of Solutions for Shear Flow of Certain Viscoelastic Fluids. *Journal of Mathematical Fluid Mechanics*, volume 11, number 1 : 91–99, 2009.
- [Ren09b] M. Renardy. Some Global Stability Results for Shear Flows of Viscoelastic Fluids. *Journal of Mathematical Fluid Mechanics*, volume 11, number 1 : 100–109, 2009.
- [Sci] Scilab. <http://www.scilab.org>.
- [Str78] L. C. E. Struik. *Physical aging in amorphous polymers and other materials*, Volume 106. Elsevier Amsterdam, 1978.
- [SSAB99] J. Sun, M. Smith, R. Armstrong and R. Brown. Finite element method for viscoelastic flows based on the discrete adaptive viscoelastic stress splitting and the discontinuous Galerkin method : DAVSS-G/DG. *Journal of Non-Newtonian Fluid Mechanics*, volume 86, number 3 : 281–307, 1999.
- [Szn91] A.-S. Sznitman. Topics in propagation of chaos. In *Ecole d'Eté de Probabilités de Saint-Flour XIX - 1989*, pages 165–251. Springer, 1991.
- [Tem79] R. Temam. *Navier-Stokes equations : theory and numerical analysis*. North-Holland Pub. Co., 1979.

- [Tem84] R. Temam. *Navier–Stokes Equations*. American Mathematical Soc., 1984.
- [Tsa09] M. Tsamados. *Mechanical response of glassy materials : theory and simulation*. PhD thesis, Université Claude Bernard - Lyon I, 2009.