Front propagation methods

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Résumé. Ce travail porte sur la résolution de problèmes faisant intervenir des mouvements d'interfaces. Dans les différentes parties de cette thèse, on cherche à déterminer ces mouvements d'interfaces en résolvant des modèles approchés consistant en des équations ou des systèmes d'équations sur des champs. Les problèmes obtenus sont des équations paraboliques et des systèmes hyperboliques.

Dans la première partie (chapitre 2), on étudie un modèle simplifié pour la propagation d'une onde de souffle en dynamique des fluides compressibles. Ce modèle peut s'écrire sous la forme d'un système hyperbolique, et on construit un algorithme résolvant numériquement ce système par une méthode de type Fast-Marching. On mène également une étude théorique de ce système pour déterminer des solutions de référence et tester la validité de l'algorithme. Dans la deuxième partie (chapitres 3 à 5), les équations approchées sont de type parabolique, et on cherche à montrer l'existence de solutions de type régime permanent à ces équations. Dans les chapitres 3 et 4, on étudie une équation générique en une dimension associée à des phénomènes de réaction-diffusion. Dans le chapitre 3, on montre l'existence de solutions quasi-planes pour un terme de réaction (terme non-linéaire) assez général, et dans le chapitre 4 on utilise ces résultats pour montrer l'existence d'ondes pulsatoires progressives dans le cas spécifique d'une non-linéarité bistable. Le modèle étudié dans le chapitre 5 est un modèle de champ de phase approchant un modèle de dynamique des dislocations dans un cristal, dans un domaine correspondant physiquement à une source de Frank-Read.

Mots-clés : Système de lois de conservation, modèle GSD, méthode Fast-Marching, problème de Riemann, équation parabolique, équation semi-linéaire, milieu périodique, bistable, onde progressive pulsatoire.

Abstract. This work is about the resolution of problems associated with the motion of interfaces. In each part of this thesis, the goal is to determine the motion of interfaces by the use of approached models consisting of equations or systems of equation on fields. The problems we get are parabolic equations and hyperbolic systems.

In the first part (Chapter 2), we study a simplified model for the propagation of a shock wave in compressible fluid dynamics. This model can be written as a hyperbolic system, and we construct an algorithm to solve it numerically by a Fast-Marching like method. We also conduct a theoretical study of this system to determine reference solutions and test the algorithm. In the second part (Chapters 3 to 5), the approached models yield parabolic equations, and our goal is to show the existence of permanent regime solutions for these equations. Chapter 3 and 4 are dedicated to the study of a generic one-dimensional equation modelling reaction-diffusion phenomena. In Chapter 3, we show the existence of plane-like solutions for a general reaction term, and in Chapter 4 we use this result to show the existence of pulsating travelling waves in the specific case of a bistable nonlinearity. In Chapter 5, we study a phase-field model approaching a model for the dynamics of dislocations in a crystal, in a domain corresponding to a Frank-Read source.

Keywords : System of conservation laws, GSD model, Fast-Marching method, Riemann problem, parabolic equation, semilinear equation, periodic medium, bistable, pulsating travelling wave.

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Chapitre 1

Introduction

1.1 Méthodes de propagation de front

Beaucoup de phénomènes physiques sont modélisés par des équations aux dérivées partielles (EDP) ou des systèmes d'EDP portant sur des champs qui représentent des grandeurs physiques (on peut citer comme exemples l'équation de la chaleur modélisant la conduction thermique dans un solide, ou les modèles de dynamique des fluides). Dans un certain nombre de cas, ces champs sont définis sur des domaines qui restent inchangés au cours du temps. D'autres classes de problèmes modélisent des évolutions géométriques, c'est le cas par exemple des problèmes de mouvements d'interfaces et des problèmes à frontière libre. Dans le premier cas, on cherche à calculer l'évolution au cours du temps d'une interface pour laquelle la vitesse de déplacement de chaque point peut être une fonction de la position de l'interface, ou encore être la solution d'une EDP. Dans un problème à frontière libre, le domaine d'étude lui-même dépend du temps et son bord évolue avec une vitesse dépendant des valeurs du champ solution de l'équation à l'intérieur du domaine. Les travaux sur ces problèmes sont abondants mais leur étude est souvent compliquée par le fait qu'un grand nombre de méthodes et de résultats connus pour les équations portant sur des champs ne peuvent être utilisés pour les mouvements d'interfaces. C'est pour cette raison qu'il est courant de chercher à transformer des équations de mouvements d'interfaces en équations sur des champs, souvent en considérant l'interface dont on souhaite déterminer la dynamique comme une ligne de niveau d'un champ pour lequel on peut écrire une EDP d'évolution. Selon les problèmes étudiés, plusieurs approches sont possibles. Dans le chapitre 2 de ma thèse, consacré à l'étude théorique et numérique d'un modèle approché pour la propagation d'une onde de souffle, le temps d'arrivée du front en chaque point est donné par un système d'EDP couplant ce temps d'arrivée et la vitesse normale du front. Le système d'EDP qu'on étudie est alors stationnaire, et on utilise la structure eikonale de l'équation reliant les variations spatiales du temps d'arrivée et la vitesse pour construire un schéma numérique nouveau. Le système obtenu peut également être réécrit sous la forme d'un système de lois de conservation. De tels systèmes ont été l'objet de nombreux travaux théoriques et numériques, et cette forme suggère en particulier de rechercher des solutions simples du système en étudiant le problème de Riemann. Les chapitres 3, 4 et 5 font intervenir des approches similaires. Dans les deux cas (les chapitres 3 et 4 sont consacrés à l'étude de la même équation, le raisonnement du chapitre 4 s'appuyant fortement sur les résultats du chapitre 3), les mouvements d'interfaces sont modélisés par des équations paraboliques. La première équation étudiée est une équation classique de type diffusion-réaction, utilisée pour modéliser par exemple des phénomènes de combustion et de propagation de flammes. La deuxième équation, étudiée dans le chapitre 5, est plus spécifique ; c'est l'équation d'un modèle de type champ de phases approximant un modèle de dynamique des dislocations dans un cristal. Dans ce chapitre on étudie ce modèle sur la configuration géométrique d'une source de Frank-Read, un des principaux phénomènes de création de dislocations.

1.2 Calcul rapide de l'évolution d'une onde de souffle avec le modèle de Whitham

Dans le chapitre 2, on s'intéresse à la propagation d'une onde de souffle dans l'air. Ce type de phénomène est généralement modélisé par les équations d'Euler avec fluide compressible. La résolution numérique du système d'Euler est très coûteuse en temps de calcul. L'objectif de cette partie est d'obtenir par des calculs rapides de bonnes approximations des caractéristiques importantes de l'onde de souffle, à savoir principalement en chaque point le premier temps d'arrivée de l'onde, la valeur de la pression et le temps de phase positive (temps pendant lequel la surpression est positive en un point).

Le modèle GSD (Geometrical shock dynamics) de Whitham ([37],[36]) est une approximation du modèle d'Euler pour les fluides compressibles qui permet des calculs plus rapides. Les équations du modèle GSD modélisent uniquement la propagation du front de l'onde de souffle et permettent de déterminer le temps d'arrivée du front de l'onde et la vitesse de ce front en chaque point. Les équations du modèle GSD de Whitham s'écrivent

$$\begin{cases} M|\nabla\alpha| & = 1 \\ \operatorname{div}\left(\dfrac{n}{A(M)}\right) & = 0 \\ n & = \dfrac{\nabla\alpha}{|\nabla\alpha|} \end{cases} \tag{1.1}$$

où le vecteur $n = \dfrac{\nabla\alpha}{|\nabla\alpha|}$ est le vecteur unitaire normal au front.

La quantité physique A qui intervient dans l'équation sur la vitesse M permet de prendre en compte les effets des phénomènes physiques ayant lieu derrière le front tout en limitant les calculs à la seule propagation de ce front. La pression en arrière du front est obtenue à partir de la solution des équations du modèle GSD en utilisant les équations de Friedlander. Ce post-traitement permet d'approcher correctement les caractéristiques de l'onde de souffle (pression derrière le front, durée de pression positive) qui ne sont pas directement données par les équations du modèle GSD.

Notre algorithme résout numériquement les équations du modèle GSD en calculant les temps d'arrivée α par une méthode de type Fast-Marching. La méthode Fast-Marching permet de calculer en un seul passage les temps d'arrivée d'un front en tous les points d'un domaine en partant de la position initiale du front et en progressant dans le domaine dans l'ordre croissant des temps d'arrivée. Plus précisément, les points sont répartis dans trois ensembles. Dans le premier ensemble, l'ensemble des points acceptés, la valeur du temps d'arrivée α est calculée définitivement. Dans le deuxième ensemble, qu'on appelle traditionnellement la "narrow-band" et qui est constitué des points voisins des points acceptés, on dispose de valeurs d'essai de α. Enfin, dans le dernier ensemble, aucune valeur de α n'a été calculée. A chaque itération de l'algorithme, on met à jour les valeurs d'essai de α dans le deuxième ensemble en utilisant la discrétisation traditionnelle de l'équation eikonale (voir [29]), le point pour lequel la valeur de α est la plus petite est accepté et on remet à jour les trois ensembles.

A la différence des méthodes Fast-Marching habituelles, dans notre algorithme, la vitesse n'est pas donnée a priori mais elle dépend localement des valeurs de α et de ses dérivées, et doit donc être calculée simultanément. Plus précisément, à un temps donné, on calcule en chaque point de la narrow-band un couple candidat "temps d'arrivée-vitesse" par la résolution d'un système donné par le schéma discret. On fige ensuite ces valeurs au point pour lequel le temps d'arrivée est minimal. Ce point est ajouté au front, on actualise la narrow-band et on réitère le calcul à partir du nouveau front. Cette approche consistant à utiliser une méthode de Fast Marching pour résoudre un système comportant une équation de type eikonale et une équation donnant l'évolution de la vitesse en fonction de la forme du front est nouvelle à notre connaissance. Si la méthode Fast Marching fournit un schéma naturel pour l'équation eikonale, la construction du schéma en différences finies pour l'équation sur la vitesse est plus délicate. En effet, il est difficile d'utiliser la forme conservative de cette équation en différences finies, et la forme non conservative (1.2) fait intervenir la normale au front n et la courbure du front κ qui sont compliquées à évaluer et peuvent être génératrices d'importantes erreurs numériques (tout particulièrement la courbure du front).

$$n.\nabla M = -\frac{M^2 - 1}{\lambda(M)}\kappa \tag{1.2}$$

Nous avons donc réécrit la deuxième équation de sorte que n'interviennent que des fonctions de α dont la discrétisation serait naturelle, plutôt que la normale et la courbure :

$$\nabla M.\nabla\alpha = \mathcal{S}(M)\Delta\alpha \tag{1.3}$$

Pour vérifier l'intérêt et la correction de notre algorithme, il faut pouvoir vérifier que les solutions calculées sont assez proches des solutions exactes. Quand il n'est pas possible de connaître analytiquement les solutions exactes du système GSD, on peut aussi confronter les solutions obtenues à des solutions de référence calculées avec des algorithmes beaucoup plus coûteux en temps mais dont la fiabilité est connue. Dans le cas des équations du modèle GSD, les solutions numériques pourraient aussi être confrontées aux solutions de référence obtenues avec un algorithme discrétisant les équations d'Euler avec des pas d'espace et de temps très faibles ; une telle comparaison testerait à la fois la correction de notre algorithme et la validité de l'approximation par le modèle GSD pour le cas étudié. Une partie importante de ma contribution à ce travail a été la détermination de solutions de référence pouvant être calculées, soit analytiquement, soit très rapidement au moyen de la résolution d'une simple équation différentielle ordinaire (EDO). Parmi les configurations géométriques simples se prêtant au calcul de ces solutions de référence, on trouve par exemple les fronts plans de vitesse uniforme, ou encore le cas d'une source à symétrie sphérique (en dimension 2 ou 3) ou à symétrie cylindrique en dimension 3.

Il est possible d'exhiber des solutions analytiques pour un autre ensemble, plus important et plus riche, de configurations géométriques : le problème de Riemann. Le problème de Riemann est une configuration géométrique pour laquelle la condition initiale est constituée de deux fronts plans, de vitesses m_ℓ et m_r et de directions θ_ℓ et θ_r se joignant à l'origine (figure 1.1).

Les solutions du problème de Riemann font en général apparaître des états intermédiaires, les états étant séparés entre eux par des transitions ne pouvant prendre que des formes bien précises. Dans le cadre général des systèmes hyperboliques, on sait résoudre le problème de Riemann quand les états de gauche et de droite (ici caractérisés par les couples (m_ℓ, θ_ℓ) et (m_r, θ_r)) sont suffisamment proches, mais dans un cas particulier appelé p-système, le problème de Riemann peut être résolu pour des états initiaux quelconques, et les solutions sont alors constituées de deux transitions au plus, les transitions étant alors soit des raréfactions, soit des chocs (sauf pour certains p-systèmes pour lesquels certaines conditions initiales n'admettent pas de solution dans cet ensemble, ces résultats sont étudiés notamment dans [24]). Même si l'écriture standard du p-système fait intervenir un espace à une dimension et une variable de temps, les équations du système GSD de Whitham (qui sont stationnaires dans un espace à deux dimensions) peuvent s'écrire sous une forme se rapprochant de celle du p-système. On montre alors le théorème suivant :

Theorème 1.1. Existence de solutions au problème de Riemann *On considère le système* (1.1) *et des états* (m_ℓ, θ_ℓ) *et* (m_r, θ_r)*, avec* $m_\ell, m_r > 1$*. Le problème de Riemann de conditions initiales*

$$\begin{cases} (m,\theta) & = (m_\ell,\theta_\ell) \quad \textit{pour } \chi = \theta_\ell - \dfrac{\pi}{2} \\ (m,\theta) & = (m_r,\theta_r) \quad \textit{pour } \chi = \theta_r + \dfrac{\pi}{2} \end{cases}, \tag{1.4}$$

χ *désignant l'angle en coordonnées polaires, admet alors une solution autosimilaire comportant au plus deux transitions (chocs ou raréfactions) dès que la condition*

$$\theta_r - \theta_\ell \leq \theta^*(m_\ell) + \theta^*(m_r)$$

est vérifiée, θ^* *étant une fontion croissante explicite de* m *pour laquelle* $\theta^*(1) = 0$.

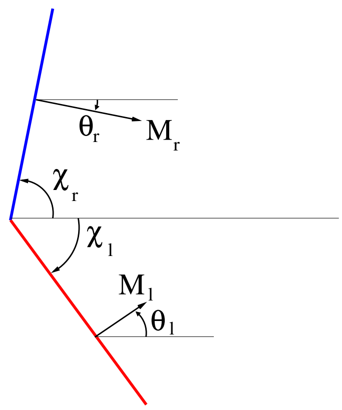

FIGURE 1.1 – Conditions initiales du problème de Riemann

Le système de Riemann peut donc être complètement résolu en dehors d'une zone critique dans l'espace (m, θ) pour les équations de Whitham. Si Schwendeman [28] mentionne que l'obtention des solutions du problème de Riemann est simple pour ce modèle, aucune description complète n'existait jusqu'alors.

Pour comprendre les solutions du problème de Riemann pour le système GSD, il faut connaître la forme des transitions possibles, qui peuvent être ou des chocs, ou des raréfactions. Ces chocs et ces raréfactions ont été décrits par Whitham dans son livre [37] et son article [36], mais les solutions de type raréfaction sont exprimées dans le système des coordonnées (α, β) données par la normale et la tangente au front, ce qui rend difficile leur représentation géométrique. J'ai donc choisi de reprendre complètement la description géométrique de ces solutions en donnant les résultats en coordonnées cartésiennes ou polaires. Avec ces résultats, la géométrie d'un choc est donnée de manière analytique, et la gémoétrie d'une raréfaction peut être trouvée en résolvant une EDO. On peut donc confronter directement les résultats issus des simulations à ces solutions.

Une application directe de ces résultats est le cas d'un front plan comprimé par une rampe, qui est un cas-test simple pour évaluer les performances de l'algorithme et surtout évaluer le traitement des chocs. En effet, le schéma de différences finies n'étant pas a priori conservatif, il n'y a pas de garantie que les solutions numériques ne comporteront que des chocs corrects (i.e. satisfaisant la forme conservative de l'équation sur la vitesse au sens des distributions) et donc que l'algorithme donnera des résultats corrects.

Le cas de la rampe et de manière plus générale le cas des solutions au problème de Riemann sont des tests intéressants pour l'algorithme car ils permettent de vérifier à la fois que les résultats sont bons en l'absence de discontinuités (pour les raréfactions) et que l'algorithme calcule correctement les solutions comportant des discontinuités.

1.3 Existence de solutions quasi-planes pour une équation parabolique semilinéaire

L'objet du chapitre 3 est de montrer l'existence de solutions quasi-planes à l'équation parabolique semi-linéaire

$$u_t(x,t) = u_{xx}(x,t) + f(x, u(x,t)). \tag{1.5}$$

pour $(x,t) \in \mathbb{R} \times \mathbb{R}$ avec une non-linéarité f satisfaisant les propriétés de régularité et de $(\mathbb{Z} \times \mathbb{Z})$-périodicité suivantes :

$$\begin{cases} f \in Lip(\mathbb{R}^2; \mathbb{R}) \\ f(x+k, v+l) = f(x,v) \quad \text{pour tout} \quad (k,l) \in \mathbb{Z}^2, \quad (x,v) \in \mathbb{R} \times \mathbb{R} \end{cases} \tag{1.6}$$

où $Lip(\cdot, \cdot)$ désigne l'ensemble des fonctions Lipschitz.

On parle de solutions quasi-planes dans le sens où on recherche des solutions pour lesquelles les lignes de niveau $\{u = a\}$ pour $a \in \mathbb{R}$ évoluent comme le front d'une onde plane progressive. De telles solutions peuvent donc être des modèles pour des phénomènes physiques impliquant des propagations de fronts.

Plus précisément, pour un scalaire p donné, satisfaisant $p \neq 0$, les solutions quasi-planes que l'on recherche doivent vérifier, pour un certain scalaire $\lambda \in \mathbb{R}$, l'encadrement

$$|u - px - \lambda t| \leq C \tag{1.7}$$

pour une certaine constante $C > 0$ et, quand $\lambda \neq 0$, les relations de périodicité suivantes

$$u\left(x + \frac{1}{p}, t\right) = u(x,t) + 1 \tag{1.8}$$

$$u(x+1,t) = u\left(x, t + \frac{p}{\lambda}\right). \tag{1.9}$$

Ces deux relations permettent de réduire l'étude à un domaine borné car des telles solutions sont entièrement déterminées par leur comportement sur le rectangle $\left[0, \frac{1}{p}\right] \times \left[0, \frac{1}{\lambda}\right]$.

Plus précisément, on démontre le théorème suivant :

Theorème 1.2. (Existence de solutions quasi-planes et périodicité) *On suppose que f vérifie l'hypothèse* (1.6) *et on fixe $p > 0$ avec $p^{-1} \in \mathbb{N}$. Alors il existe un unique réel $\lambda \in \mathbb{R}$ tel qu'il existe une constante $C > 0$ indépendante de p et une solution u de* (1.5) *sur $\mathbb{R} \times \mathbb{R}$ satisfaisant*

$$\begin{cases} |u(x,t) - px - \lambda t| \leq C & (i) \\ u\left(x + \frac{1}{p}, t\right) = 1 + u(x,t) & (ii) \\ \lambda u_t \geq 0 & (iii) \\ u(x+1,t) \geq u(x,t) & (iv). \end{cases} \tag{1.10}$$

De plus,

$$\begin{cases} u\left(x+1, t - \frac{p}{\lambda}\right) = u(x,t) & si \quad \lambda \neq 0 \quad (v) \\ u_t = 0 & si \quad \lambda = 0 \quad (v)'. \end{cases} \tag{1.11}$$

Enfin, les constantes $\lambda(p)$ sont bornées indépendamment de p.

On montre également dans ce chapitre un résultat un peu plus spécifique :

Theorème 1.3. *Soit $\mathcal{Z}$ l'ensemble des zeros de f indépendemment de x (i.e. pour $a \in \mathcal{Z}$, $f(x,a) = 0$ pour tout $x \in [0,1]$). Il existe une solution u de* (1.5) *satisfaisant* (1.10),(1.11) *telle que*

$$\forall a \in \mathcal{Z}, \quad (u(x,t) < a < u(y,t)) \Rightarrow (x < y). \tag{1.12}$$

Cette propriété est cruciale dans le chapitre 4, où l'on montre l'existence de solutions de type ondes progressives pulsatoires dans le cas d'une nonlinéarité f bistable en dimension 1 (voir la section 1.4 et le chapitre 4).

Les travaux sur ces solutions quasi-planes sont relativement peu nombreux. Les principaux résultats sont des propriétés d'existence et de régularité de minimiseurs quasi-plans pour un certain nombre de fonctionnelles d'énergie (voir [4],[11],[8],[7]). Ces travaux ne font pas intervenir d'évolution temporelle, on peut donc les comparer à nos résultats dans le cas où $\lambda = 0$ et où les solutions que nous construisons sont stationnaires. Dans ce cas, la solution stationnaire u est un minimiseur pour la fonctionnelle

$$\mathcal{I}(u) = \int |u_x|^2 - F(x,u)$$

avec $F_u(x,u) = 2f(x,u)$ (F_u désignant la dérivée de F par rapport à la variable u). Puisque nous montrons l'existence de solutions stationnaires en particulier quand f est à moyenne nulle, notre résultat implique l'existence de minimiseurs pour $\mathcal{I}$ lorsque F_u est à moyenne nulle. L'article se rapprochant le plus de notre équation parabolique est celui de Blass, De La Llave et Valdinoci [3]. En effet, cet article étudie des problèmes de flot de gradient, introduisant ainsi effectivement une variable de temps dans le problème. L'équation (1.5) peut d'ailleurs être vue comme un cas particulier de l'équation (10) de [3]. Cependant, cet article ne s'intéresse pas aux mêmes résultats que nous, il montre un principe de comparaison pour des flots plus généraux et pour des données moins régulières.

Pour montrer le théorème 1.2, on construit d'abord les solutions du problème de Cauchy

$$\begin{cases} u_t = u_{xx} + f(x,u) \\ u(0,x) = px \end{cases} \tag{1.13}$$

et on montre qu'elles vérifient l'encadrement (1.7) et la propriété de périodicité (1.8). On considère ensuite le comportement en temps long de ces solutions et on montre ainsi que l'on peut obtenir une solution de (1.5) satisfaisant les conditions du théorème 1.2.

L'étude des solutions du problème de Cauchy et surtout l'obtention de l'encadrement (1.7) nécessitent l'étude préalable d'une équation non locale approchant l'équation (1.5). Le terme non local permet de borner les oscillations spatiales de la solution du problème de Cauchy mais engendre un certain nombre de complications qui rendent ce chapitre assez technique.

Une fois que l'on a montré que les solutions du problème de Cauchy (1.13) vérifiaient (1.7),(1.8), on démontre le théorème 1.2 en étudiant leur comportement en temps long. La régularité de la solution du problème de Cauchy et l'uniformité de ses bornes permettent de placer cette solution dans un espace compact et ainsi de donner un sens précis à ce comportement en temps long sous la forme d'une limite.

Dans le cas où $\lambda \neq 0$, on montre ensuite que la solution limite u_∞ satisfait le théorème 1.2. Quand $\lambda = 0$, on souhaite de plus construire une solution stationnaire. Pour cela, on construit d'abord, à partir de la limite en temps long, une sous-solution et une sur-solution stationnaires, et on déduit l'existence d'une solution stationnaire par la méthode de Perron. On montre ensuite que cette solution vérifie (1.7).

Pour montrer le théorème 1.3, on montre d'abord que la propriété (1.12) est vérifiée pour la solution du problème de Cauchy en montrant que les ensembles $\{(x,t) \in \mathbb{R} \times [0,T] \mid u(x,t) < a\}$ et $\{(x,t) \in \mathbb{R} \times [0,T] \mid u(x,t) > a\}$ sont nécessairement connexes. Une fois la propriété démontrée pour la solution du problème de Cauchy, un passage à la limite permet de montrer le théorème 1.3 dans le cas $\lambda \neq 0$. Quand $\lambda = 0$, on montre que la sous-solution et la sur-solution stationnaires construites vérifient (1.12), puis que cette propriété est conservée par la méthode de Perron.

1.4 Existence et unicité d'ondes progressives pulsatoires dans le cas bistable

L'objet du chapitre 4 est de montrer l'existence de solutions de type onde progressive pulsatoire (en anglais Pulsating Travelling Waves) à l'équation (1.5)

$$u_t(x,t) = u_{xx}(x,t) + f(x, u(x,t))$$

pour une non-linéarité f bistable (en plus des propriétés de régularité et de périodicité). On suppose que la fonction f satisfait l'hypothèse

$$\begin{cases} f \in C^1(\mathbb{R} \times [0,1]; \mathbb{R}) \\ f(x+k, v) = f(x,v) \quad \text{pour tout} \quad k \in \mathbb{Z}, \quad (x,v) \in \mathbb{R} \times [0,1] \end{cases} \tag{1.14}$$

et vérifie également des hypothèses de bistabilité : Il existe $\theta \in (0,1)$ tel que

$$\begin{cases} f(x,0) = f(x,\theta) = f(x,1) = 0 & \text{pour tout} \quad x \in \mathbb{R} \\ f(x,v) < 0 & \text{pour tout} \quad v \in (0,\theta), \quad x \in \mathbb{R} \\ f(x,v) > 0 & \text{pour tout} \quad v \in (\theta,1), \quad x \in \mathbb{R} \\ f'_u(x,\theta) > 0 & \text{pour tout} \quad x \in \mathbb{R} \\ \exists \eta_0 > 0 & \text{tel que} \\ \forall x \in \mathbb{R}, \text{ la fonction } v \mapsto f(x,v) & \text{est strictement décroissante sur} \\ & [0,\eta_0] \cup [1-\eta_0, 1] \end{cases} \tag{1.15}$$

Sous ces hypothèses, on peut montrer l'existence de solutions de type ondes progressives pulsatoires (dans les cas où il n'y a pas de solution stationnaire) :

Theorème 1.4. (Existence et unicité d'ondes progressives pulsatoires) *On suppose* (1.6),(1.15). *Alors il existe* $c_0 \in \mathbb{R}$ *et* u *solution de* (1.5) *sur* $\mathbb{R}\times\mathbb{R}$ *tels que*

$$\begin{cases} c_0 u_t \geq 0 \\ 0 \leq u \leq 1 \\ \liminf\limits_{x+c_0t \to +\infty} u(x,t) = 1 \\ \limsup\limits_{x+c_0t \to -\infty} u(x,t) = 0 \end{cases} \tag{1.16}$$

et

$$\begin{cases} u\left(x+1, t-\dfrac{1}{c_0}\right) = u(x,t), & \textit{si } c_0 \neq 0 \\ u(x,t) \textit{ ne dépend pas de } t, & \textit{si } c_0 = 0. \end{cases} . \tag{1.17}$$

De plus, si $c_0 \neq 0$, la solution u satisfaisant (1.16)-(1.17) *est unique à translation en temps près.*

Sous une hypothèse un peu plus contraignante, on peut montrer l'unicité de la vélocité c :

$$\begin{cases} f'_u(x,0) = -\delta_0 < 0 & \text{pour tout} \quad x \in \mathbb{R} \\ f'_u(x,1) = -\delta_1 < 0 & \text{pour tout} \quad x \in \mathbb{R} \end{cases} \tag{1.18}$$

On montre alors le théorème suivant :

Theorème 1.5. (Unicité de la vélocité)
On suppose vérifiées les hypothèses (1.6),(1.15),(1.18)*. Alors la vélocité $c_0 \in \mathbb{R}$ donnée dans le théorème 1.4 est unique.*

L'étude d'équations paraboliques semi-linéaires telles que (1.5) est l'objet de nombreux travaux, avec en particulier la recherche de solutions de type ondes progressives (travelling waves).

Les premières études ont été menées sur l'équation de réaction-diffusion homogène

$$u_t = u_{xx} + u(1-u)$$

par Kolmogorov, Petrovsky et Piskunov [11] et Fisher [10]. Cette équation trouve alors des applications en écologie dans l'étude des modèles proie-prédateur. Des recherches ont ensuite été consacrées à des cas plus généraux, avec l'ajout d'un terme d'advection, ou la non-homogénéité du terme de réaction. Dans ce contexte général, l'existence de solutions de type fronts progressifs pulsatoires a été démontrée par Berestycki et Hamel dans [2] dans le cas d'une non-linéarité f positive.

Il y a moins de résultats dans le cas d'une non-linéarité bistable comme celle que nous étudions. Un résultat de Xin [17] montre l'existence d'ondes progressives pulsatoires quand la dépendance en espace de la non-linéarité est petite. Un autre résultat plus général est celui de Giletti, Ducrot et Matano [9]. Ces trois auteurs montrent l'existence de fronts progressifs pulsatoires pour des non-linéarités très générales, qui doivent seulement vérifier des conditions implicites assez raisonnables. Par certains points, les chapitres 3 et 4 font appel à des arguments un peu similaires. Néanmoins, l'utilisation cruciale du théorème 1.3 nous permet de nous affranchir de telles conditions implicites, et notre étude pourrait couvrir des cas non traités dans cet article.

Pour démontrer le résultat d'existence du théorème 1.4, on utilise fortement les résultats démontrés dans le chapitre 3. En effet, on considère les solutions $u^{(p)}$ construites dans le théorème 1.2 pour $p \in \mathbb{N}^{-1}$ et on considère leur limite pour construire la fonction u satisfaisant le théorème 1.4.

Les propriétés de monotonie et de périodicité de la limite u se déduisent des propriétés des solutions $u^{(p)}$ pour $p \neq 0$. Pour poursuivre l'analyse, on définit les fonctions

$$\begin{array}{lll} \overline{u}(x) & = \lim\limits_{t \to +\infty} u(x,t) \\ \underline{u}(x) & = \lim\limits_{t \to -\infty} u(x,t) \\ u^+(x,t) & = \lim\limits_{n \to +\infty} u(x+n,t) \\ u^-(x,t) & = \lim\limits_{n \to -\infty} u(x+n,t) \end{array} \tag{1.19}$$

et on montre les encadrements $0 \leq u^+ - u^- \leq 1$ et $0 \leq |\overline{u} - \underline{u}| \leq 1$. Par des translations en temps appropriées on peut s'assurer que u prend la valeur θ, puis, en utilisant les fonctions auxiliaires définies en (1.19), montrer que $0 \leq u \leq 1$. Il faut ensuite distinguer les cas selon que la vélocité est nulle ou non nulle. Dans le cas $c \neq 0$ on montre que si u est solution de (1.5), alors soit u est une pulsating travelling wave joignant les états 0 et 1, soit u est constante et égale à θ. On utilise ensuite le fait que $f'_u(x,\theta) > 0$ pour tout x, et donc que l'état θ est un équilibre instable pour montrer que les solutions que l'on construit ne peuvent être constantes et égales à θ. De manière similaire, dans le cas $c = 0$, on peut montrer que soit on peut exhiber une solution stationnaire satisfaisant $u^+ = 1$, $u^- = 0$, soit la solution u construite est constante et égale à θ. On exclut finalement ce cas $u \equiv \theta$ pour achever la démonstration du théorème 1.4.

A vélocité c fixée, on démontre l'unicité de la solution u en considérant deux solutions de (1.5) u_1 et u_2 satisfaisant (1.16),(1.17) pour la même vélocité c. On peut alors montrer que par une translation en temps appropriée on a $u_2(\cdot, \cdot + \tau) \geq u_1$, et on conclut que u_1 et u_2 sont égales à une translation en temps près en utilisant un principe du maximum fort.

On montre l'unicité de c sous la condition plus restrictive (1.18) en considérant deux solutions u_1 et u_2 de vélocités c_1 et c_2 avec $c_1 > c_2$. On peut alors montrer qu'à une translation en temps près, $u_1(x,0) < u_2(x,0)$ et aboutir à une contradiction en utilisant le principe de comparaison et la propriété (1.17). Cependant, comme u_1 et u_2 ont des vélocités différentes, leurs périodicités sont différentes et il est plus difficile de les ordonner que dans la preuve de l'unicité du profil. Pour contourner cette difficulté, on utilise la condition (1.18) pour construire pour chaque vélocité c des sous-solutions et des sur-solutions explicites de (1.5) dans les demi-espaces $\{x + ct > A\}$ et $\{x + ct < -A\}$ et on compare d'abord u_1 et u_2 à ces sous-solutions et sur-solutions explicites avant de les ordonner au temps 0. Un autre moyen de résoudre cette difficultés et d'affaiblir la condition (1.18) serait d'utiliser le changement de variable proposé par Berestycki et Hamel dans [2]. Un tel traitement pourrait pourrait permettre de montrer simultanément l'unicité de la vitesse et l'unicité du profil.

1.5 Modèle approché pour la dynamique de dislocations autour d'une source de Frank-Read

L'objet de ce chapitre est l'étude d'un modèle approché de type champ de phase pour le mouvement de dislocations autour d'une source de Frank-Read. Une dislocation est un défaut dans la structure d'un cristal. Ainsi, autour d'une dislocation, les atomes d'un cristal ne forment pas un réseau parfait. Une dislocation est de codimension 2 (c'est donc un défaut linéique dans un cristal de dimension 3) et elle est caractérisée par son vecteur de Burgers b. Si on considère un chemin formant une boucle sur un réseau cristallin parfait, alors ce même chemin n'est plus une boucle sur le réseau cristallin autour de la dislocation, et le vecteur de Burgers b est le déplacement nécessaire pour fermer ce chemin autour de la dislocation (le chemin étant orienté dans le sens direct par rapport à la direction de la dislocation).

Une source de Frank-Read est une configuration qui permet la création de dislocations. Dans cette configuration, une dislocation est attachée à deux points fixes et évolue dans un plan, le plan de la dislocation (on peut déduire de la définition d'une dislocation qu'une dislocation est nécessairement une boucle fermée, dans le cas d'une source de Frank-Read cette boucle se referme en dehors du plan de la dislocation et les deux points d'attache restent fixes quand celle-ci évolue). Le mouvement de la dislocation, induit par le potentiel interne et les contraintes externes peut amener cette dislocation d'origine à se recouper elle-même, occasionnant des changements de topologie et la création d'une dislocation indépendante formant une boucle autour de la source. Quand ce phénomène se répète, la source de Frank-Read devient une source régulière de dislocations concentriques qui s'en éloignent.

Physiquement, une dislocation et les contraintes induites se concentrent sur quelques mailles du réseau, et quand on cherche à étudier son comportement à plus grande échelle on la considère comme un objet linéique. Seulement, l'étude directe du mouvement des dislocations présente des difficultés et, pour faciliter cette étude, on peut tenter d'etudier leur comportement à l'aide d'un modèle approché de type champ de phase. Dans un tel modèle, les dislocations sont considérées comme les variations d'un champ v, qui modélise le décalage entre le réseau cristallin réel et le réseau idéal. Au lieu d'être concentrées sur des lignes, les dislocations sont alors plus étalées. Dans le cas limite où l'on autorise seulement des valeurs entières pour v, les dislocations sont concentrées sur des lignes. Dans un tel modèle, l'évolution de ce champ de phase est donnée par une EDP et est plus facile à étudier que le mouvement des lignes de dislocation.

Dans notre étude, le domaine spatial Ω doit correspondre à la géométrie d'une source de Frank-Read. Pour cela, le domaine le plus naturel serait l'espace $\mathbb{R}^2$ privé de deux points, mais pour faciliter l'étude des solutions, et en particulier pour obtenir des estimations uniformes et avoir des propriétés de compacité, nous avons choisi un domaine borné et à frontière régulière (figure 1.2) :

$$\Omega = B(0,R) \setminus (\overline{B}(P^+,\varepsilon) \cup \overline{B}(P^-,\varepsilon))$$

avec $B(a,r)$ (respectivement $\overline{B}(a,r)$) la boule ouverte (respectivement fermée) de centre a et de rayon r. Les deux points constituant la source de Frank-Read sont les points $P^+ = (1,0)$ et $P^- = (-1,0)$. Pour que ce domaine soit bien défini et connexe, on impose aux paramètres R et ε les bornes $R > 2$ et $0 < \varepsilon < 1$. Pour obtenir le domaine d'étude idéal pour la source de Frank-Read, à savoir le domaine $\mathbb{R}^2 \setminus \{P^-, P^+\}$, il faut considérer les limites $R \to \infty$ et $\varepsilon \to 0$. S'il est envisageable que les résultats démontrés dans ce chapitre restent valables pour ce domaine limite, les estimations utilisées dans les preuves sont fortement dépendantes du domaine, c'est pourquoi l'extension de nos résultats à $\mathbb{R}^2 \setminus \{P^-, P^+\}$ n'est pas simple a priori.

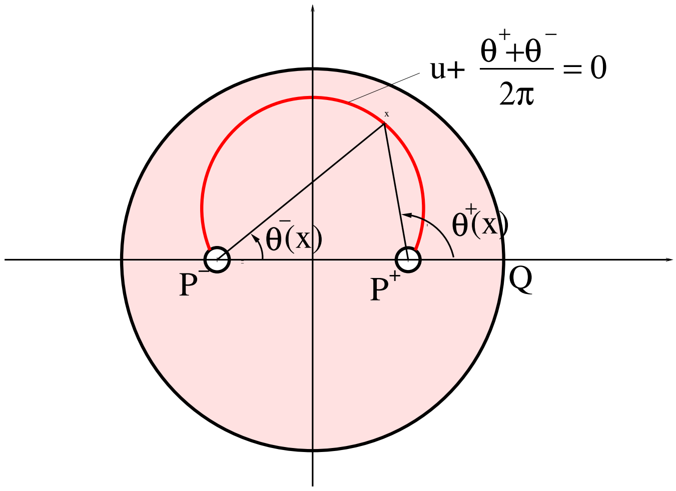

FIGURE 1.2 – Domaine Ω pour l'étude de la source de Frank-Read

.

L'équation du modèle approché est la suivante :

$$\begin{cases} u_t &= \Delta u - W'\left(u + \dfrac{\theta^+ - \theta^-}{2\pi}\right) + \sigma \quad \text{pour } x \in \Omega \\ \dfrac{\partial u}{\partial n} &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{pour } x \in \partial\Omega \end{cases} \tag{1.20}$$

et la fonction W vérifie les hypothèses

$$\begin{cases} W \in C^2(\mathbb{R}) \\ W(v+k) = W(v) \quad \text{pour } k \in \mathbb{Z} \end{cases} . \tag{1.21}$$

L'équation (1.20) est une équation parabolique de type équation de champ de phase où la fonction $v = u + \dfrac{\theta^+ - \theta^-}{2\pi}$ représente la distance (dans la direction du vecteur de Burgers de la dislocation) entre la position des atomes du réseau cristallin et leur position dans la configuration de base. La fonction W représente un potentiel électronique, et la périodicité de W' correspond à celle du réseau cristallin : la contrainte supplémentaire causée par le déplacement du réseau ne dépend que de la valeur de v modulo 1. Le scalaire $\sigma \geq 0$ représente une sollicitation extérieure. Dans la limite où le potentiel W est de plus en plus piqué, cette équation donne l'évolution des dislocations autour de la source de Frank-Read. Plus précisément, avec un potentiel pour lequel W' devient très grand en dehors des valeurs entières, les lignes de niveau $\left\{ u = n + \dfrac{1}{2} \right\}$ (avec $n \in \mathbb{Z}$) donnent l'évolution des positions des dislocations au cours de temps.

L'objet de ce chapitre est la recherche de solutions ayant un régime permanent. Le domaine temporel naturel de ces solutions est $\mathbb{R}$. Pour obtenir ces solutions, nous étudierons le problème de Cauchy pour l'équation (1.20). On considérera donc dans ce cas le domaine temporel $[0, +\infty)$, avec la condition initiale

$$u(x, 0) = u_0(x) \quad \text{pour } x \in \Omega. \tag{1.22}$$

On cherche donc à montrer l'existence de solutions ayant un régime permanent, c'est-à-dire plus précisément des solutions satisfaisant, pour un certain $T > 0$,

$$u(x, t + T) = u(x, t) + 1 \quad \text{pour tout } x \in \Omega, \quad t \in \mathbb{R}. \tag{1.23}$$

On qualifie cette propriété de propriété de régime permanent car entre des temps t et $t + T$, la solution n'est modifiée que par une translation entière. Or, pour ce modèle de champ de phase la contrainte W' ne dépend que de la partie fractionnaire de u, et dans la limite d'un potentiel W piqué donnant le comportement des dislocations autour de la source de Frank-Read, la position des dislocations ne dépend elle aussi que de la partie fractionnaire de u. En conséquence, pour ce type de solution, une translation temporelle d'une période ne modifie pas les propriétés de la solution et préserve en particulier l'interprétation physique.

Pour construire ces solutions possédant un régime permanent, on considère le comportement en temps long des solutions du problème de Cauchy (1.20),(1.22). Pour cela, on montre l'existence de bornes pour ces solutions dans le théorème 1.6.

Theorème 1.6. Contrôle de la solution de problème de Cauchy *Soit u la solution du problème de Cauchy* (1.20),(1.22) *avec condition initiale $u_0 = 0$. Pour tout $\sigma \in \mathbb{R}$, il existe un unique scalaire $\omega \in \mathbb{R}$ tel qu'il existe $C \in \mathbb{R}$ tel que*

$$|u(x, t) - \omega t| \leq C. \tag{1.24}$$

Cet encadrement permet de montrer l'existence d'un régime permanent :

Theorème 1.7. Existence d'un régime permanent
Dans le cas où $\omega \neq 0$, il existe une solution u de (1.20) *satisfaisant*

$$u(x, t+T) = u(x,t) + 1 \qquad \textit{pour tout } x \in \Omega, \quad t \in \mathbb{R} \tag{1.25}$$

avec une période temporelle $T = \frac{1}{\omega}$.
De plus, cette solution u est strictement croissante et unique à translation en temps près.
Dans le cas où $\omega = 0$, il existe une solution stationnaire u de (1.20) *sur $\Omega \times \mathbb{R}$.*

Il est également possible de montrer que la pente ω (qui correspond physiquement à la fréquence de création de dislocations par la source) est une fonction croissante de la contrainte extérieure σ.

Theorème 1.8. Monotonicité de ω
La fréquence ω est une fonction croissante de σ.

L'idée de l'existence de défauts ayant cette structure de dislocation dans les solides cristallins fut présentée en 1905 par Volterra (voir [34] pour une référence en français, la construction de dislocations par la méthode dite de Volterra est décrite dans le chapitre 2). En 1934, Orowan ([23]), Polanyi ([24]) et Taylor ([32]) expliquent presque simultanément, et indépendamment, le mécanisme des déformations plastiques dans les cristaux par le déplacement de dislocations. Les premières observations de dislocations ont été réalisées en 1956 par Hirsch, Horne, Whelan ([18]) et Bollmann ([4]). Le mécanisme de la source de Frank-Read a été présenté en 1950 par Frank et Read ([12],[13]). La source de Frank-Read est le principal mécanisme de création de dislocations et a fait l'objet d'études sur le plan expérimental (voir par exemple [20],[21]) et sur le plan théorique.

Les études théoriques de la source de Frank-Read se sont beaucoup attachées au calcul des forces s'exerçant sur une dislocation possédant un point d'attache ([6]) et à la détermination de la contrainte critique déclenchant la création de dislocations ([31],[10]). Dans le chapitre 5, notre ambition est différente car nous cherchons à déterminer les caractéristiques d'un régime permanent de création de dislocations, ce qui n'a pas été fait à notre connaissance, dans le cadre d'une approche de type champ de phase avec un domaine régulier qui réduit les difficultés techniques. Les difficultés qui apparaîssent dans l'étude de dislocations attachées en un point dans un domaine non borné sont par exemple traitées par Forcadel, Imbert et Monneau dans [2] pour le cas de l'évolution d'une spirale. Dans le cas de domaines bornés, Sato et Giga dans [3], [27] et [4] étudient l'évolution de dislocations avec des conditions aux limites de type Neumann. Ces études suggèrent des voies de prolongement possibles de notre recherche.

Des approches de type champ de phases pour les dislocations ont été proposées dans [13], [8], [16] et [15]. On peut se référer à [26] où à [22] pour des analyses de comportements limites de ces modèles de champ de phase, le comportement limite étant un front se déplaçant avec une vitesse proportionnelle à sa courbure moyenne plus une constante.

Pour prouver les théorèmes 1.6, 1.7 et 1.8, on peut d'abord affirmer que le problème de Cauchy (1.20),(1.21) admet une unique solution et que celle-ci est suffisamment régulière. La principale difficulté de la démonstration du théorème 1.6 est l'encadrement des oscillations spatiales de la solution. Pour cela, on étudie seulement les oscillations de la solution, c'est-à-dire la différence entre la solution et sa valeur moyenne. On écrit l'équation vérifiée par cette différence, et on exprime la solution de cette équation par la formule de Duhamel. On encadre ensuite l'intégrande en utilisant des éléments d'analyse spectrale et un résultat d'estimation intérieure, pour finalement borner les oscillations de la solution indépendamment du temps. On obtient ensuite l'encadrement (1.24) en construisant des pentes candidates $\omega^+(\tau)$ et $\omega^-(\tau)$ pour $\tau > 0$ telles que pour tout $t > 0$,

$$\omega^-(\tau)\tau \leq u(0, t + \tau) - u(0, t) \leq \omega^+(\tau)\tau.$$

On montre ensuite que ces deux fonction ω^+ et ω^- ont la même limite ω dans la limite $\tau \to +\infty$, puis que cette limite ω vérifie la propriété (1.24).

Pour démontrer le théorème 1.7, on distingue les cas $\omega = 0$ et $\omega \neq 0$. Quand $\omega \neq 0$, la solution u est obtenue en considérant le comportement en temps long des solutions du problème de Cauchy. Le théorème 1.6 permet de donner un sens à ce comportement en temps long comme limite de translatées de la solution du problème de Cauchy en plaçant ces translatées dans un espace compact. On montre ensuite que la limite u satisfait la propriété de régime permanent en considérant les translatées $u^b = u(\cdot + b) - 1$ et en les comparant à u par le principe de maximum fort. Cette méthode de démonstration, appelée "sliding method", a été décrite par Berestycki et Nirenberg dans [1]. La monotonie de u et son unicité sont également obtenues par la sliding method.

Quand $\omega = 0$, on peut montrer l'existence d'une solution stationnaire par un argument d'énergie. En effet, on peut exhiber une énergie $E(u)$ telle que $E \geq 0$ et telle que $E(u(\cdot, t))$ est une fonction décroissante de t si u est solution de (1.20). On en déduit ensuite qu'une valeur d'adhérence en temps long de la solution du problème de Cauchy $u_\infty = \lim_{n\to\infty} u(\cdot, t_n)$ (avec $t_n \to +\infty$) est une solution stationnaire de (1.20).

Pour finir, le théorème 1.8 découle directement du principe de comparaison.

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Chapitre 2

A fast-marching like algorithm for Geometrical Shock Dynamics

The content of this chapter has been submitted to the Journal of Computational Physics. It has been written in collaboration with Nicolas Lardjane (CEA, DAM, DIF), Régis Monneau (CERMICS, ENPC) and Youness Noumir (LRC-MESO, CMLA, ENS Cachan).

Abstract

We develop a new algorithm for the computation of the geometrical shock dynamics model (GSD). The method relies on the fast-marching paradigm and enables the discrete evaluation of the first arrival time of a shock wave and its local velocity on a Cartesian grid. The proposed algorithm is based on a second order upwind finite difference scheme and reduces to a local nonlinear system of two equations solved by an iterative procedure. Reference solutions are built for a smooth radial configuration and for the 2D Riemann problem. The link between the GSD model and p-systems is given. Numerical experiments demonstrate the accuracy and the ability of the scheme to handle singularities.

Keywords : geometrical shock dynamics fast-marching method level-set method Riemann problem finite difference method shock wave p-system

2.1 Introduction

In 1957, when G.B. Whitham published the Geometrical Shock Dynamics (GSD) model [28], he qualified it as âa relatively simple approximate method developed for treating problems of the diffraction and stability of shock wavesâ. The simplicity comes from the fact that the shock front is seen as a surface evolving under its own local speed and curvature, independently of the post-shock flow. The shock adjusts itself to changes in the geometry only [29]. As explained by Best [8], Whitham considered the motion of a shock into a uniform gas at rest, down a tube of slowly varying cross sectional area, A, and under some physically grounded hypothesis, he obtained an expression relating the local shock Mach number, M, to A, now known as the A-M relation [31] (see also (2.9) and (2.4)). The GSD model reads

$$M(\boldsymbol{x})|\boldsymbol{\nabla}\alpha(\boldsymbol{x})| = 1, \quad \mathrm{Div}\left(\frac{\boldsymbol{n}(\boldsymbol{x})}{A(M(\boldsymbol{x}))}\right) = 0 \tag{2.1}$$

where α gives the shock position [29] and $\boldsymbol{n} = \frac{\boldsymbol{\nabla}\alpha}{|\boldsymbol{\nabla}\alpha|}$ is the local normal to the front. This model is hyperbolic provided that $A'(M) < 0$, and can thus develop disturbances on the front which are the trace of waves, not modeled, behind the shock. In practice, GSD has proven to be fairly accurate for diffraction around a corner, non-regular Mach reflection [29], or accelerating shocks and shown only little deviation for expanding decelerating flows [5]. Whitham's model has been extended to take into account unsteady flow behind the shock [8, 9, 10], non-uniform gases properties [20], and has been applied, among others, to imploding shock waves [11, 1], atmospheric propagation [7], detonation in explosives [2, 3, 6], supersonic engine unstart [27] and astrophysics [14].

This success, linked to the compact model formulation and the dimensional reduction, was supported by the development of three kinds of algorithms. (i) Lagrangian, or front-tracking, methods have first been experimented [15, 18]. In such an approach, the shock front is explicitly discretized by markers evolved in time and regularly resampled. This method is natural and quite accurate but difficult to implement in three dimensions, mainly when surface merging or breaking is expected. (ii) Eulerian conservative algorithms [19, 20] reduce this difficulty but do not take any advantage of the front locality. Furthermore, they rely on the definition of an a priori propagation direction, not always easy to determine. (iii) Localized level-set methods are a good compromise since they handle any kind of surface deformation but in an implicit way. The front shock is obtained from a table of arrival time, also called burn table. A 3D unsteady algorithm, based on the Hamilton-Jacobi form of the GSD system (2.1) [17], is described in [2, 3, 4] for Detonation Shock Dynamics. It compares well with reactive Eulerian model results at a much lower CPU time. Nevertheless, due to the nonlinear nature of GSD equations, unphysical shocks can form away from the front position and a frequent resampling of the signed distance is mandatory [23].

In this article we propose an alternative approach based on the level-set fast-marching paradigm [22], which combines the flexibility of (iii) and the locality of (i) while remaining easy to implement. The first and second ordrer specific algorithms are decribed in section 2.2. Reference solutions for a smooth radial problem and the GSD Riemann problem are derived in section 2.3. In section 2.4, the comparison to numerical results indicates that a second order scheme is mandatory for non smooth problems. At last, conclusions are summarized in section 2.5.

2.2 A fast-marching like GSD scheme

In 1988 Osher and Sethian [17] introduced the Eulerian level-set method to solve Hamilton-Jacobi equations and the eikonal equation in particular. Unlike the Lagrangian approach, the level-set method is simple to implement in 3D, high-order extensions are readily derived and topological properties of the front, as the curvature, are easily calculated. However, the level-set mehod, of complexity $O(N^3)$, can be quite time consuming when the number of grid point, N, is large. In the past two decades, several improvements have been proposed in order to reduce this complexity and at same time to enhance the accuracy. Among them, the most popular approach is the Fast-Marching Method (FMM) developed by Sethian [22] and used with success in a large variety of applications. Assuming a single pass front, the complexity of the algorithm reduces to $O(N \log N)$ and even to $O(N)$ under some further assumptions [30].

In this section, we introduce a second-order method to solve the GSD model (2.1), on a Cartesian grid, based on the fast-marching paradigm. We first reformulate the model as a coupled eikonal-transport system to facilitate its discretization. The numerical method, boundary treatment and implementation details are then given.

2.2.1 A modified transport equation

As in the work of Besset [7] or Aslam [2], the GSD model (2.1) is rewritten under the local form

$$\begin{cases} M|\boldsymbol{\nabla}\alpha| = 1 & \text{(2.2a)} \\ M\dfrac{\boldsymbol{\nabla}\alpha}{|\boldsymbol{\nabla}\alpha|} \cdot \boldsymbol{\nabla} M = \dot{M}(M,\kappa) & \text{(2.2b)} \end{cases}$$

with the initial boundary conditions

$$\begin{cases} \alpha_{|\Gamma_0} = \alpha_0, & \text{(2.3a)} \\ M_{|\Gamma_0} = M_0, & \text{(2.3b)} \end{cases}$$

where α_0 and M_0 are given functions on the hypersurface Γ_0, the shock initial position. The GSD closure, linking the local Mach number, M, and the mean curvature of the front, $\kappa = \mathrm{Div}(\boldsymbol{n})$, reads

$$\dot{M}(M,\kappa) = -\frac{M^2-1}{\lambda(M)}\kappa,$$

where $M \geq 1$ and

$$\begin{cases} 0 < \lambda(M) = \left(1 + \dfrac{2}{\gamma+1}\dfrac{1-\mu^2}{\mu}\right)\left(1 + 2\mu + \dfrac{1}{M^2}\right), \\ 1 \geq \mu = \sqrt{\dfrac{(\gamma-1)M^2+2}{2\gamma M^2 - (\gamma-1)}}, \end{cases} \tag{2.4}$$

$\gamma > 1$ being the gas polytropic coefficient, see [29] for details.

In practice, the discretization of the mean curvature is difficult, due to the mixed derivatives of α, especially in the context of the fast-marching method when neighboring points are not yet assigned a value. For this reason we choose to reformulate the transport equation on the Mach number as a convection–diffusion one.

By combining the eikonal equation, $M|\boldsymbol{\nabla}\alpha| = 1$, and the normal definition, $\boldsymbol{n} = \boldsymbol{\nabla}\alpha/|\boldsymbol{\nabla}\alpha|$, one checks that the mean curvature of the front reads $\kappa = \boldsymbol{\nabla}M \cdot \boldsymbol{\nabla}\alpha + M\Delta\alpha$. Transport equation (2.2b) is then reformulated as

$$\boldsymbol{\nabla}M \cdot \boldsymbol{\nabla}\alpha = \mathcal{S}(M)\Delta\alpha,$$

where $\mathcal{S}(M) = -\frac{M(M^2-1)}{M^2(\lambda(M)+1)-1}$ is nonpositive and smooth and for $M > 1$. The reformulated boundary value problem to be solved is then

$$\begin{cases} M|\boldsymbol{\nabla}\alpha| = 1 & \text{(2.5a)} \\ \boldsymbol{\nabla}M \cdot \boldsymbol{\nabla}\alpha = \mathcal{S}(M)\Delta\alpha & \text{(2.5b)} \end{cases}$$

where the source term is now easier to discretize.

Note that this system is not in a conservative form, which could raise difficulties to handle front discontinuities, but as we shall see in section 2.4.3 the numerical scheme performs well in practice.

2.2.2 Level-set method for the eikonal equation

The level-set function α is solution of the eikonal equation (2.5a) on Ω together with the initial condition (2.3a) on Γ_0. We solve it on a uniform Cartesian mesh of the domain $\Omega \equiv [0, Lx] \times [0, Ly] \times [0, Lz] \subset \mathbb{R}^3$ with grid spacings $\triangle x$, $\triangle y$ and $\triangle z$. Let $\alpha_{i,j,k}$ and $M_{i,j,k}$ be the approximate solution at a grid point, *i.e.* $\alpha_{i,j,k} = \alpha(x_i, y_j, z_k)$ and $M_{i,j,k} = M(x_i, y_j, z_k)$ where $x_i = i\triangle x$, $y_j = j\triangle y$ and $z_k = k\triangle z$. We denote by u_l, v_l et w_l (*resp.* u_r, v_r

et w_r) the backward (*resp.* forward) approximation of the derivatives of α at (x_i, y_j, z_k) along x, y and z respectively.

The discrete form of (2.5a) reads

$$M_{i,j,k}\hat{H}(u_l, u_r, v_l, v_r, w_l, w_r) = 1, \tag{2.6}$$

where $\hat{H}$ is chosen as the numerical Hamiltonian of Godunov [25] :

$$\hat{H}(u_l, u_r, v_l, v_r, w_l, w_r) = \sqrt{\max^2(u_l^+, u_r^-) + \max^2(v_l^+, v_r^-) + \max^2(w_l^+, w_r^-)},$$

with the notations : $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$ for $x \in \mathbb{R}$. This upwind scheme has been successfully used in several applications (*cf.* [16], [22]). It has the ability to capture viscosity solutions of the eikonal equation [12] and do not smear out sharp discontinuities excessively [25], which are desirable features in the case of GSD where front singularities may appear.

At first order, the discrete derivatives of α are written :

$$\left.\begin{array}{rcl|rcl}
u_l &=& (D_x^-\alpha)_{i,j,k} := \dfrac{\alpha_{i,j,k} - \alpha_{i-1,j,k}}{\triangle x} & u_r &=& (D_x^+\alpha)_{i,j,k} := \dfrac{\alpha_{i+1,j,k} - \alpha_{i,j,k}}{\triangle x} \\
v_l &=& (D_y^-\alpha)_{i,j,k} := \dfrac{\alpha_{i,j,k} - \alpha_{i,j-1,k}}{\triangle y} & v_r &=& (D_y^+\alpha)_{i,j,k} := \dfrac{\alpha_{i,j+1,k} - \alpha_{i,j,k}}{\triangle y} \\
w_l &=& (D_z^-\alpha)_{i,j,k} := \dfrac{\alpha_{i,j,k} - \alpha_{i,j,k-1}}{\triangle z} & w_r &=& (D_z^+\alpha)_{i,j,k} := \dfrac{\alpha_{i,j,k+1} - \alpha_{i,j,k}}{\triangle z}
\end{array}\right. .$$

The extension to arbitrary higher orders is possible, we restrict ourselves to second-order in this work.

At second order, the discrete derivatives of α are written :

$$\left.\begin{array}{l|l}
u_l = (D_x^-\alpha)_{i,j,k} + \dfrac{\triangle x}{2}(D_x^-(D_x^-\alpha))_{i,j,k} & u_r = (D_x^+\alpha)_{i,j,k} - \dfrac{\triangle x}{2}(D_x^+(D_x^+\alpha))_{i,j,k} \\
v_l = (D_y^-\alpha)_{i,j,k} + \dfrac{\triangle y}{2}(D_y^-(D_y^-\alpha))_{i,j,k} & v_r = (D_y^+\alpha)_{i,j,k} - \dfrac{\triangle y}{2}(D_y^+(D_y^+\alpha))_{i,j,k} \\
w_l = (D_z^-\alpha)_{i,j,k} + \dfrac{\triangle z}{2}(D_z^-(D_z^-\alpha))_{i,j,k} & w_r = (D_z^+\alpha)_{i,j,k} - \dfrac{\triangle z}{2}(D_z^+(D_z^+\alpha))_{i,j,k}
\end{array}\right. .$$

We point out that there is no limiting function. In the context of the FMM, a switching mechanism is rather used as we shall see later.

2.2.3 The standard fast-marching method

Following Sethian [22], the classical FMM is now outlined. In this method, the velocity of propagation is assumed to be known and of constant sign. The front is thus "*single pass*" and one needs to calculate the value of α only in the vicinity of it. The CPU time is then dramatically reduced and, interrelated, the monotonicity of the scheme is guaranteed by following the direction in which the information flows. This is done by propagating the solution from lower to higher values of the level-set function α. To this end, the grid points are partitioned into three groups, namely :

❶ **Known** is the set of vertices where the values of α are known, *i.e.* the vertices already intercepted by the front ;
❷ **NarrowBand** is the set of all neighboring vertices of **Known**, *i.e.* the vertices that are about to be intercepted by the front ;
❸ **Far** is the set of vertices that are neither in **Known** nor in **NarrowBand**. In the **Far** set, α is assigned a huge initial value `INF`.

Think of the NarrowBand set as a buffer zone that serves to start the calculation from the vertices of Known and to spread the information to the vertices of the Far set. For every element of the NarrowBand, test values of α are calculated **from the points of the Known set only**, and then the point of the NarrowBand corresponding to the minimal test value is validated. This means that the value of α at this point is now defined as the minimal test value. A second step consists in including this point in the Known set and adding its close Far neighbors as new points of the NarrowBand. By repeating the previous steps, as long as the NarrowBand is not-empty, the level-set function is calculated in the whole computational domain. The starting values of this process are set by the initial conditions on Γ_0. It is worth mentioning that the most time consuming step in this algorithm is the search of the point with the minimal test value and thus the performance of the FMM depends on it.

At a point (i, j, k) of interest of the NarrowBand, we can deduce from (2.6) that the test value ϑ of α is calculated by solving the following quadratic equation :

$$\sum_{\nu=1}^{3} \max{}^2 \left(\frac{\vartheta - t_{i,j,k}^{\nu^-}}{\triangle_\nu^-}, \frac{\vartheta - t_{i,j,k}^{\nu^+}}{\triangle_\nu^+}, 0 \right) = \frac{1}{M_{i,j,k}^2}, \tag{2.7}$$

with the notations

$$\triangle_1^\pm = \triangle x \left(1 - \frac{s_{i\pm1,j,k}^{\pm x}}{3} \right), \ \triangle_2^\pm = \triangle y \left(1 - \frac{s_{i,j\pm1,k}^{\pm y}}{3} \right), \ \triangle_3^\pm = \triangle z \left(1 - \frac{s_{i,j,k\pm1}^{\pm z}}{3} \right),$$

and with the following generic switches

$$\left|\begin{array}{l} s^{\pm x}_{i,j,k} = \begin{cases} 1 & \text{if} \quad \alpha_{i\pm1,j,k} < \alpha_{i,j,k} \\ 0 & \text{otherwise} \end{cases} \\ \\ s^{\pm y}_{i,j,k} = \begin{cases} 1 & \text{if} \quad \alpha_{i,j\pm1,k} < \alpha_{i,j,k} \\ 0 & \text{otherwise} \end{cases} \\ \\ s^{\pm z}_{i,j,k} = \begin{cases} 1 & \text{if} \quad \alpha_{i,j,k\pm1} < \alpha_{i,j,k} \\ 0 & \text{otherwise} \end{cases} \end{array}\right. ,$$

where the coefficients $\left(t^{\nu\pm}_{i,j,k}\right)_{1\leq\nu\leq3}$ are given by :

$$\left|\begin{array}{l} t^{1\pm}_{i,j,k} = \alpha_{i\pm1,j,k} + \dfrac{s^{\pm x}_{i\pm1,j,k}}{3}\left(\alpha_{i\pm1,j,k} - \alpha_{i\pm2,j,k}\right) \\ \\ t^{2\pm}_{i,j,k} = \alpha_{i,j\pm1,k} + \dfrac{s^{\pm y}_{i,j\pm1,k}}{3}\left(\alpha_{i,j\pm1,k} - \alpha_{i,j\pm2,k}\right) \\ \\ t^{3\pm}_{i,j,k} = \alpha_{i,j,k\pm1} + \dfrac{s^{\pm z}_{i,j,k\pm1}}{3}\left(\alpha_{i,j,k\pm1} - \alpha_{i,j,k\pm2}\right) \end{array}\right. .$$

The switches take into account the local direction of propagation. They ensure the causality condition by setting a zero value when the information at the second order is not available in that direction.

Remark 2.1. *At first-order of accuracy, one can take all switches equal to zero and one has* $\triangle^{\pm}_1 = \triangle x$, $\triangle^{\pm}_2 = \triangle y$, $\triangle^{\pm}_3 = \triangle z$, *and* $t^{1\pm}_{i,j,k} = \alpha_{i\pm1,j,k}$, $t^{2\pm}_{i,j,k} = \alpha_{i,j\pm1,k}$, $t^{3\pm}_{i,j,k} = \alpha_{i,j,k\pm1}$.

2.2.4 Full discretisation of the GSD system

Following the fast-marching paradigm, a method for solving the coupled system of equations (2.5) is now explained. Since M is an unknown of the problem, one has to calculate test values of α (*i.e.* ϑ) and of M (*i.e.* m), **at the same time**, for each point of the NarrowBand, leading to a local nonlinear system in ϑ and m. The discrete velocity $M_{i,j,k}$ is now replaced by m in the equation (2.7) and the discretization of the transport equation (2.5b) is done as follows.

The advection part of equation (2.5b) reads $\boldsymbol{\nabla} M \cdot \boldsymbol{\nabla}\alpha$. Following the upwind direction of discretization for $\boldsymbol{\nabla}\alpha$ ensures that only receivable points are used in the computation of $\boldsymbol{\nabla} M$. More precisely, we have :

$$(\boldsymbol{\nabla} M \cdot \boldsymbol{\nabla}\alpha)\,|_{i,j,k} = \sum_{\nu=1}^{3}\left[\left(\frac{\vartheta - t^{\nu-}_{i,j,k}}{\triangle^{-}_{\nu}}\right)^{+}\frac{m - \ell^{\nu-}_{i,j,k}}{\triangle^{-}_{\nu}} - \left(\frac{t^{\nu+}_{i,j,k} - \vartheta}{\triangle^{+}_{\nu}}\right)^{-}\frac{\ell^{\nu+}_{i,j,k} - m}{\triangle^{+}_{\nu}}\right]$$

where the coefficients $\left(\ell^{\nu\pm}_{i,j,k}\right)_{1\leq\nu\leq 3}$ take the form :

$$\left|\begin{array}{l}\ell^{1\pm}_{i,j,k} = M_{i\pm1,j,k} + \dfrac{s^{\pm x}_{i\pm1,j,k}}{3}\left(M_{i\pm1,j,k} - M_{i\pm2,j,k}\right) \\[2ex] \ell^{2\pm}_{i,j,k} = M_{i,j\pm1,k} + \dfrac{s^{\pm y}_{i,j\pm1,k}}{3}\left(M_{i,j\pm1,k} - M_{i,j\pm2,k}\right) \\[2ex] \ell^{3\pm}_{i,j,k} = M_{i,j,k\pm1} + \dfrac{s^{\pm z}_{i,j,k\pm1}}{3}\left(M_{i,j,k\pm1} - M_{i,j,k\pm2}\right)\end{array}\right. .$$

Concerning the source term of equation (2.5b), $\mathcal{S}(M)\Delta\alpha$, we evaluate $\mathcal{S}$ implicitly and discretize the Laplacian operator at second order. From our experience, it is crucial to use a centered scheme as much as possible. This is done in practice by **taking into account all finite value neighbours of a point of interest**, including those of the NarrowBand, in the calculation of the test value. This feature also affects the switches evaluation in (2.7). In one space direction, if a left and right values are available, the second derivative is chosen centered, no matter the causality condition in the NarrowBand (i.e. we also use points which are not in the Known set). For this reason, the algorithm is not a fast-marching method in the strictest sense, and we say it has fast-marching like properties.

More precisely, $(\mathcal{S}(M)\Delta\alpha)\,|_{i,j,k} = \mathcal{S}(m)\sum_{\nu=1}^{3}\hat{\Delta}_\nu$, where we express only the first term for the sake of simplicity

if $(\alpha_{i-1,j,k} < \texttt{INF}$ **and** $\alpha_{i+1,j,k} < \texttt{INF})$ **then**
  $\hat{\Delta}_1 = \hat{\Delta}_{1_c}$
else
  if $(\vartheta \geq t^{1^-}_{i,j,k}$ **or** $\vartheta \geq t^{1^+}_{i,j,k})$ **then**
    if $\left(\max\left(\dfrac{\vartheta - t^{1^+}_{i,j,k}}{\triangle^+_1}, 0\right) \geq \max\left(\dfrac{\vartheta - t^{1^-}_{i,j,k}}{\triangle^-_1}, 0\right)\right)$ **then**
      $\hat{\Delta}_1 = \hat{\Delta}_{1_r}$
    else
      $\hat{\Delta}_1 = \hat{\Delta}_{1_\ell}$
    end if
  end if
end if

with

- $\hat{\Delta}_{1_c} = \dfrac{\alpha_{i+1,j,k} - 2\vartheta + \alpha_{i-1,j,k}}{\triangle x^2}$

- $\hat{\Delta}_{1_r} = s^{+x}_{i,j,k} s^{+x}_{i+1,j,k} \left[\dfrac{\vartheta - 2\alpha_{i+1,j,k} + \alpha_{i+2,j,k}}{\triangle x^2} \right.$
  $\left. + s^{+x}_{i+2,j,k} \left(\dfrac{\vartheta - 2\alpha_{i+1,j,k} + \alpha_{i+2,j,k}}{\triangle x^2} - \dfrac{\alpha_{i+3,j,k} - 2\alpha_{i+2,j,k} + \alpha_{i+1,j,k}}{\triangle x^2} \right) \right]$

- $\hat{\Delta}_{1_\ell} = s^{-x}_{i,j,k} s^{-x}_{i-1,j,k} \left[\dfrac{\vartheta - 2\alpha_{i-1,j,k} + \alpha_{i-2,j,k}}{\triangle x^2} \right.$
  $\left. + s^{-x}_{i-2,j,k} \left(\dfrac{\vartheta - 2\alpha_{i-1,j,k} + \alpha_{i-2,j,k}}{\triangle x^2} - \dfrac{\alpha_{i-3,j,k} - 2\alpha_{i-2,j,k} + \alpha_{i-1,j,k}}{\triangle x^2} \right) \right]$

where we recall that `INF` denotes the huge initial positive value given to points in the Far set.

The final form of the nonlinear system on ϑ and m is then

$$\left\{ \begin{aligned} &\sum_{\nu=1}^{3} \max{}^2 \left(\frac{\vartheta - t^{\nu-}_{i,j,k}}{\triangle^-_\nu}, \frac{\vartheta - t^{\nu+}_{i,j,k}}{\triangle^+_\nu}, 0 \right) = \frac{1}{m^2} && \text{(2.8a)} \\ &\sum_{\nu=1}^{3} \left[\left(\frac{\vartheta - t^{\nu-}_{i,j,k}}{\triangle^-_\nu} \right)^+ \frac{m - \ell^{\nu-}_{i,j,k}}{\triangle^-_\nu} - \left(\frac{t^{\nu+}_{i,j,k} - \vartheta}{\triangle^+_\nu} \right)^- \frac{\ell^{\nu+}_{i,j,k} - m}{\triangle^+_\nu} \right] = \mathcal{S}(m) \sum_{\nu=1}^{3} \hat{\Delta}_\nu && \text{(2.8b)} \end{aligned} \right.$$

Remark 2.2. *As in remark 2.1, the first-order accurate version of the equation (2.8b) uses the following identities : for the advection part, we simply take* $\ell^{1\pm}_{i,j,k} = M_{i\pm1,j,k}$, $\ell^{2\pm}_{i,j,k} = M_{i,j\pm1,k}$, $\ell^{3\pm}_{i,j,k} = M_{i,j,k\pm1}$, *and the Laplacien is discretized taking* $s^{\pm x}_{i\pm2,j,k} = 0$, $s^{\pm y}_{i,j\pm2,k} = 0$ *and* $s^{\pm z}_{i,j,k\pm2} = 0$ *in the expressions of* $\hat{\Delta}_1$, $\hat{\Delta}_2$ *and* $\hat{\Delta}_3$ *respectively.*

Two strategies have been tested for the numerical resolution of the local nonlinear system (2.8) on ϑ and m. Given the current approximation $\left(\vartheta^{(p)}, m^{(p)}\right)$, the first approach is a fixed point method which consists in solving the quadratic equation (2.8a) on ϑ with $m = m^{(p)}$ to get $\vartheta^{(p+1)}$ which is injected in the equation (2.8b). This gives us a new value $m^{(p+1)}$ by resolution with either a new fixed-point iteration or Newton's method. This procedure is repeated until a desired tolerance is met (10^{-6} in practice). The second approach is to solve the full coupled system of equations (2.8) by Newton's procedure. Indeed, the system (2.8) can be written in the generic form $\mathcal{G}(\mathcal{W}) = 0$, where $\mathcal{G}$ is a nonlinear function in $\mathcal{W} = (m, \vartheta)^t$ which depends on the values of α and M around the point (i, j, k) of the NarrowBand. The resolution of (2.8) is then done by the following iterative algorithm, for $p \geq 0$,

$$\mathcal{W}^{(p+1)} = \mathcal{W}^{(p)} - \left(\mathrm{D}_{\mathcal{W}} \mathcal{G}\left(\mathcal{W}^{(p)}\right) \right)^{-1} \mathcal{G}\left(\mathcal{W}^{(p)}\right),$$

when the computation of the gradient $D_{\mathcal{W}}\mathcal{G}$ is allowed, and where the starting point $\mathcal{W}^{(0)}$ is determined from the values of M and α on neighboring points. Both approaches have been successfully employed, we did not observe any robustness problem.

2.2.5 Boundary conditions

Two kinds of boundary conditions are commonly required : outgoing and rigid walls. In practice, fictious points are added outside of the computational domain, on which the phase and the Mach number are set. The number of fictious cells depends on the order of the interior numerical scheme.

The outgoing conditions are used on the artificial limits of a free boundary. We impose a huge positive value of α on the fictious points to make the scheme upwind.

When a boundary of the computational domain is bounded by a rigid body Γ_R, a wall condition is used. In this case, a Neumann condition applies : $(\boldsymbol{n} \cdot \boldsymbol{\nabla}\alpha)_{|\Gamma_R} = 0$. The numerical implementation is done by reflecting the values of α and M, by symmetry, on the fictious points. Let us recall here that Whitham's model is able to predict only the irregular shock reflection, namely the Mach reflection or *shock-shock* singularity in the terminology of Whitham.

2.2.6 Summary of the algorithm

The numerical scheme we designed to solve the GSD model (2.1) is now summarized. Whitham's model is first rewritten to avoid the direct discretisation of the curvature. The proposed algorithm is an Eulerian approach similar to the fast-marching method performed on a Cartesian grid. This algorithm is based on a second order finite difference method and more precisely on an upwind scheme in the direction of the shock propagation with a priority queue acceptance of the local solution. The arrival time and the Mach number are obtained by solving a local non-linear system.

The overall structure of our algorithm is close to the FMM one but has two main differences : utilization of points not yet accepted and resolution of the eikonal equation together with the transport equation on the propagation velocity. The fast-marching like algorithm contains two main steps, namely initialisation and iteration, and is organized as follows :

- **Initialization step**
  - Set all points of the computational domain in the **Far** set by assigning a huge `INF` value to α.
  - Add each point of the user initial condition in the **Known** set.
  - Create the **NarrowBand** from first neighbours of the **Known** set.
- **Iterative step until the NarrowBand is empty**

- Apply boundary conditions.
- For each point in the **NarrowBand**, compute the test values ϑ and m by solving the local non-linear system (2.8), by a Newton or fixed point method.
- Pick $P_{min} \equiv (i_{min}, j_{min}, k_{min}) \in$ **NarrowBand** such that

$$\alpha(i_{min}, j_{min}, k_{min}) = \min_{(i,j,k)\in\textbf{NarrowBand}} \alpha(i,j,k).$$

  Add P_{min} in the **Known** set/Delete it in **NarrowBand**.
- Add the close neighbors of P_{min} in **NarrowBand** if they were in the **Far** set.

2.3 Reference solutions to GSD

In this section, we provide reference solutions to the GSD equations, that is geometrical configurations where the exact solution can be either calculated analytically or approximated with a quick by-calculation (typically the resolution of a one-dimensional ODE). The numerical solutions of our fast-marching algorithm can then be compared to these reference solutions to evaluate its precision. Ideally, they should cover a wide array of cases, to test the algorithm in the most diverse configurations. Conversely, such reference solutions are often hard to provide, and we only expect to determine them in very simple geometrical configurations.

In the case of the GSD equations, it can be noted that infinitely large plane fronts of any normalized velocity $M > 1$ are exact solutions of the system in the whole space. Reference solutions can also be found in the radial case, when the configuration is invariant under rotations, and more interestingly, for the Riemann problem in $\mathbb{R}^2$. Such solutions enable to test the behaviour of the algorithm on smooth solutions as well as in cases where the velocity has discontinuities. The simplest test case for discontinuous solutions is the compression wedge (see Subsections 2.3.2 and 2.4.2).

In the remainder of this section we will use the conservative form (2.1) of the GSD system, with the integral A-M relation

$$A(M) = A_0 \exp\left(-\int_{M_0}^{M} \frac{m\lambda(m)}{m^2-1} dm\right), \tag{2.9}$$

where $M_0 > 1$ is the initial condition and A_0 the corresponding section area. We also introduce the notation

$$\beta(M) = \frac{M^2-1}{\lambda(M)}.$$

The quantity β allows to write the results of this section in a simpler manner.

2.3.1 Radial solutions

The simplest configuration in which a solution of the GSD equations can be computed is the radial case where the solution depends only on the radial coordinate r. More precisely, in dimension $d = 2, 3$, the velocity M is a solution of a one-dimensional ODE in the r variable :

$$\begin{cases} M(r)\partial_r \alpha(r) = 1 & \text{(2.10a)} \\ \\ M(r)\partial_r M(r) = -\beta(M(r))\dfrac{d-1}{r} & \text{(2.10b)} \end{cases}$$

For a given initial condition, this equation can be solved by a high order algorithm to provide a solution of reference, which is helpful for convergence studies. Analytical solutions are easily obtained in the strong shock limit, $M >> 1$.

2.3.2 Solutions of the Riemann problem in dimension 2

The solutions of the Riemann problem for the GSD equations are of prime importance since they can exhibit discontinuous velocities and develop new intermediate states. The behaviour of the algorithm can then be evaluated on this difficult problem. The simplest test case for discontinuities is the compression wedge, which will be developed in more details later.

For general hyperbolic systems, the theory for the resolution of the Riemann problem only covers the case where the two initial states are not far from each other. Solutions of the Riemann problem for far away initial states are known only for very particular cases, such as the p-system. While the GSD equations are in a setting that differs from the usual p-system, they can be linked to the p-system and indeed the full resolution of the Riemann problem is available.

In his works [28] and [29], Whitham described the form of simple shocks and simple rarefactions. They are studied respectively as the results of the compression of a plane front by a wedge and the diffraction of a plane front by a corner. While these works fully describe the structure of shocks and rarefactions, the rarefaction solution is only given with implicit coordinates. In this part, we describe more explicitly the geometry of the solution. From our knowledge, this is the first time that a comprehensive solution of the GSD Riemann problem is given, together with a link to the p-system. Nevertheless, one can mention the works of Henshaw, Smyth and Schwendeman [15] and Schwendeman [20] where the Whitham GSD equations are rewritten in conservative form. Schwendeman [20] also mentions the fact that the Riemann problem admits simple solutions, but does not give full details.

In this section the GSD equations are recasted as a system of M and θ where θ parametrizes the front normal as $\boldsymbol{n} = (\cos\theta)\boldsymbol{e_x} + (\sin\theta)\boldsymbol{e_y}$, with $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$ the unit vectors along the x and y axes respectively. For the Riemann problem, the initial condition is made of two planar fronts as sketched in figure 2.1 : a shock of velocity M_ℓ and angle θ_ℓ interacts with another one of velocity M_r and angle θ_r. We work in polar coordinates (ρ, χ), taking $\chi = 0$ for the horizontal axis (Ox). Note that the initial position of the first front is then the half-line $\left\{\chi = \chi_\ell = \theta_\ell - \dfrac{\pi}{2}\right\}$. Similarly, the initial position of the second front is the half-line $\left\{\chi = \chi_r = \theta_r + \dfrac{\pi}{2}\right\}$. With these notations, the left front comes before the right one when the angle χ increases. This is coherent with the standard notations for the theory of the Riemann problem and explains why the left front is on the bottom of the picture.

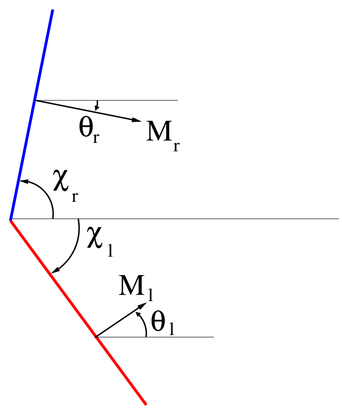


FIGURE 2.1 – Sketch of the initial condition of the GSD Riemann problem.

The p-system

The full resolution of the Riemann problem is known for the p-system (see [24], [21]), which can be written in the form

$$\begin{cases} \partial_t v - \partial_x u & = 0 \\ \partial_t u + \partial_x p(v) & = 0 \end{cases}, \tag{2.11}$$

with $t > 0$, $x \in \mathbb{R}$, along with the properties $p' < 0$ (and $p'' > 0$, but the sign of p'' plays no role in the resolution of the Riemann problem and is only a result of the usual physical interpretation in terms of isentropic gas dynamics).

As one can note, the p-system is one-dimensional in space, and we want to solve the stationary GSD equations in a two-dimensional space. Nevertheless, it can be argued that the level-set function α, is a disguised time variable. We then introduce the (σ, τ) system of coordinates where $\boldsymbol{e_\sigma} = \boldsymbol{n} = \dfrac{\boldsymbol{\nabla}\alpha}{|\boldsymbol{\nabla}\alpha|}$ is the unit vector normal to the front and $\boldsymbol{e_\tau}$ the unit vector tangent to the front such that $(\boldsymbol{e_\sigma}, \boldsymbol{e_\tau})$ is direct. One can note that Whitham [29] already introduced a similar system of coordinates, α and β, proportional to our σ and τ, but with a non normalized gradient, leading to a slightly different system.

Using the fact that (2.1) implies

$$\begin{cases} \operatorname{curl}\left(\dfrac{\boldsymbol{n}}{M}\right) = 0, \\ \\ \operatorname{Div}\left(\dfrac{\boldsymbol{n}}{A(M)}\right) = 0, \end{cases} \tag{2.12}$$

one can rewrite the GSD system in our system of coordinates as

$$\begin{cases} \partial_\sigma\theta + \partial_\tau \log(M) & = 0 \\ \partial_\tau\theta - \partial_\sigma \log(A(M)) & = 0 \end{cases},$$

which bears a strong similarity with the p-system. Keeping in mind that the setting is different, to write it as a p-system we can take

$$\begin{cases} u & = \theta \\ v & = \log(A(M)) \end{cases}.$$

Given that A is a smooth function in M with $A'(M) < 0$, it is invertible. Noting its inverse by A^{-1}, the expression for the function p is

$$p(v) = \log\left(A^{-1}\left(\mathrm{e}^v\right)\right).$$

We can then check that

$$p'(v) = \frac{\mathrm{e}^v}{A'(A^{-1}(\mathrm{e}^v))A^{-1}(\mathrm{e}^v)} < 0,$$

because $A' < 0$, which guarantees that the results obtained in the case of the p-system still hold.

Geometrical structure of simple rarefactions

A rarefaction is a smooth transition between two constant states (M_ℓ, θ_ℓ) and (M_r, θ_r) (see for instance figure 2.2). As a consequence of our analysis, we will show that admissible solutions have to satisfy $\partial_\chi\theta \geq 0$ and $\theta_r \geq \theta_\ell$, even if we do not assume it at the beginning of our analysis.

Writing (2.12) in polar coordinates (ρ, χ), we get

$$\begin{cases} \dfrac{\partial}{\partial\rho}\left(\dfrac{\rho\sin(\theta-\chi)}{M}\right) - \dfrac{\partial}{\partial\chi}\left(\dfrac{\cos(\theta-\chi)}{M}\right) = 0 \\ \\ \dfrac{\partial}{\partial\rho}\left(\dfrac{\rho\cos(\theta-\chi)}{A(M)}\right) + \dfrac{\partial}{\partial\chi}\left(\dfrac{\sin(\theta-\chi)}{A(M)}\right) = 0 \end{cases} \tag{2.13}$$

Keeping in mind that we are looking for solutions that only depend on the angular variable χ, we see that equations (2.13) can be rewritten as

$$\begin{cases} (\partial_\chi\theta)\sin(\theta-\chi) & = -\dfrac{(\partial_\chi M)}{M}\cos(\theta-\chi) \\ \\ (\partial_\chi\theta)\cos(\theta-\chi) & = (\partial_\chi M)\dfrac{A'(M)}{A(M)}\sin(\theta-\chi) \quad = -(\partial_\chi M)\dfrac{M}{\beta(M)}\sin(\theta-\chi) \end{cases} \tag{2.14}$$

Combining the equations of system (2.14), we get the following fundamental relation

$$(\partial_\chi\theta)\left(M\sin^2(\theta-\chi) - \frac{\beta(M)}{M}\cos^2(\theta-\chi)\right) = 0.$$

From this equation we deduce that either θ is constant (and then M is also constant), which corresponds to the case where the front is planar, or M and θ satisfy the relation

$$\tan(\theta-\chi) = \sigma\frac{\sqrt{\beta(M)}}{M} \quad \text{with} \quad \sigma = \pm 1. \tag{2.15}$$

This relation is satisfied for any rarefaction, as soon as θ is not constant. In particular, we have $\frac{\pi}{2} > \sigma(\theta-\chi) > 0$ with $\sigma = 1$ for 1-rarefactions, and with $\sigma = -1$ for 2-rarefactions.

The combination of the equations of system (2.14) also leads to the relation

$$(\partial_\chi\theta)^2 = \frac{(\partial_\chi M)^2}{\beta(M)}, \tag{2.16}$$

which gives

$$\partial_\chi\theta = \varepsilon\frac{(\partial_\chi M)}{\sqrt{\beta(M)}} \quad \text{with} \quad \varepsilon = \pm 1. \tag{2.17}$$

Plugging equations (2.17) and (2.15) in system (2.14) leads to $\sigma\varepsilon = -1$. We then have

$$\sigma\omega(M) + \theta = const \tag{2.18}$$

with

$$\omega(M) = \int_{M_0}^{M}\sqrt{\frac{-A'(m)}{A(m)m}}dm = \int_{M_0}^{M}\frac{1}{\sqrt{\beta(m)}}dm \tag{2.19}$$

It can then be shown with other lengthy calculations involving derivations of (2.15) and (2.18) with respect to χ, that M satisfies the ODE

$$\frac{\mathrm{d}M(\chi)}{\mathrm{d}\chi} = -\sigma\mathcal{F}(M(\chi)) \tag{2.20}$$

along the rarefaction, with $\mathcal{F}(M) = \dfrac{2\sqrt{\beta(M)}(M^2+\beta(M))}{2M^2+M\beta'(M)} > 0$ (when it is well defined). It can be checked that the denominator of $\mathcal{F}(M)$ is positive when $A''(M) > 0$ and this last condition can be checked at least numerically on some values of γ, or in general for large M.
In particular for $M(\chi)$ and $\theta(\chi)$ solutions of (2.20) and (2.15), the quantity

$$\sigma\omega(M(\chi)) + \theta(\chi) = const \tag{2.21}$$

is independent on χ (it is indeed a Riemann invariant here). This shows in particular that in any case we have $\partial_\chi \theta \geq 0$ and then $\theta_r \geq \theta_l$. We also deduce from (2.21) that

$$\omega(M_\ell) - \omega(M_r) = \sigma(\theta_r - \theta_l). \tag{2.22}$$

We can then give the geometrical structure of 1-rarefactions and 2-rarefactions in detail.

Proposition 2.3. (Structure of simple rarefactions)
Assume that $2M + \beta'(M) > 0$. Let $\pi > \theta_r - \theta_l > 0$. We set $\chi_\ell = \theta_\ell - \frac{\pi}{2}$ and $\chi_r = \theta_r + \frac{\pi}{2}$.
i) 1-rarefaction
Let $M_\ell > M_r > 1$ satisfying (2.23) for $\sigma = 1$. Then there exist two angles χ_1^{r-} and χ_1^{r+}, cf. figure 2.3.2, satisfying $\chi_\ell < \chi_1^{r-} < \chi_1^{r+} < \chi_r - \frac{\pi}{2}$ and such that :

- ❶ *In the sector $\{\chi_\ell \leq \chi \leq \chi_1^{r-}\}$, we have $M = M_\ell$ and $\theta = \theta_\ell$.*
- ❷ *In the sector $\{\chi_1^{r-} \leq \chi \leq \chi_1^{r+}\}$, M satisfies (2.20) and θ is given by (2.15) for $\sigma = 1$.*
- ❸ *In the sector $\{\chi_1^{r+} \leq \chi \leq \chi_r\}$, we have $M = M_r$ and $\theta = \theta_r$.*

The angles χ_1^{r-} and χ_1^{r+} are given by (2.15) for $\sigma = 1$, so

$$\tan(\theta_\ell - \chi_1^{r-}) = \frac{\sqrt{\beta(M_\ell)}}{M_\ell} \quad \textit{and} \quad \tan(\theta_r - \chi_1^{r+}) = \frac{\sqrt{\beta(M_r)}}{M_r}.$$

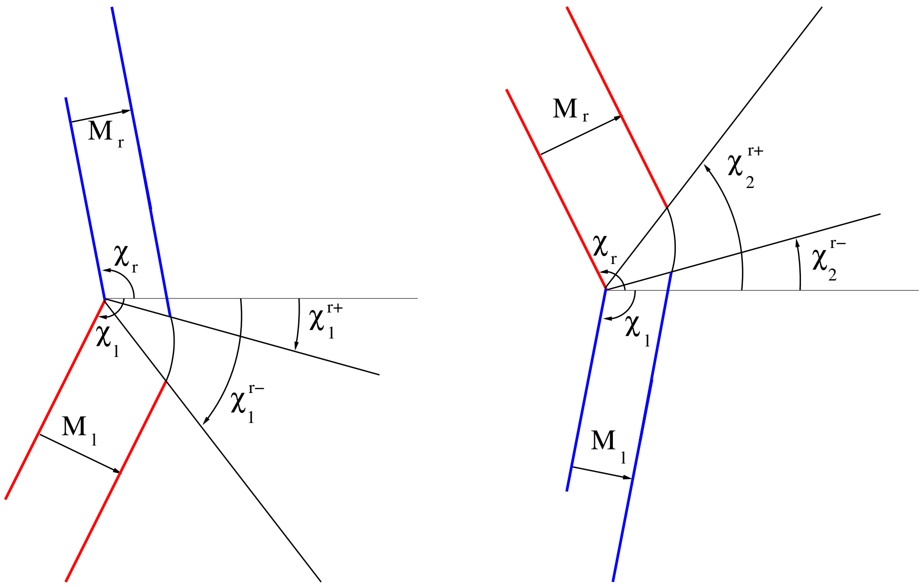
Moreover (2.21) holds true for $\sigma = 1$ for all $\chi \in [\chi_\ell, \chi_r]$.
ii) 2-rarefaction
Let $M_r > M_\ell > 1$ satisfying (2.23) for $\sigma = -1$. Then there exist two angles χ_2^{r-} and χ_2^{r+}, cf. figure 2.3.2, satisfying $\chi_\ell + \frac{\pi}{2} < \chi_2^{r-} < \chi_2^{r+} < \chi_r$ and such that :

- ❶ *In the sector $\{\chi_\ell \leq \chi \leq \chi_2^{r-}\}$, we have $M = M_\ell$ and $\theta = \theta_\ell$.*
- ❷ *In the sector $\{\chi_2^{r-} \leq \chi \leq \chi_2^{r+}\}$, M satisfies (2.20) and θ is given by (2.15) for $\sigma = -1$.*
- ❸ *In the sector $\{\chi_2^{r+} \leq \chi \leq \chi_r\}$, we have $M = M_r$ and $\theta = \theta_r$.*

The angles χ_2^{r-} and χ_2^{r+} are given by (2.15) for $\sigma = -1$, so

$$\tan(\theta_\ell - \chi_2^{r-}) = -\frac{\sqrt{\beta(M_\ell)}}{M_\ell} \quad \textit{and} \quad \tan(\theta_r - \chi_2^{r+}) = -\frac{\sqrt{\beta(M_r)}}{M_r}.$$

Moreover (2.21) holds true for $\sigma = -1$ for all $\chi \in [\chi_\ell, \chi_r]$.



(a) 1-rarefaction (b) 2-rarefaction

FIGURE 2.2 – Geometry of simple rarefaction waves.

Because M satisfies the ODE (2.20) for $\sigma = 1$ along a 1-rarefaction and for $\sigma = -1$ along a 2-rarefaction, independently of θ, the shape of the front along a rarefaction is universal. More precisely if we fix a level-set value $\alpha = t_0$, associated with a time $t_0 > 0$, there exists a spiral $\mathcal{S}_{t_0}$ of polar equation $\rho = t_0\rho_1(\chi)$ such that the level set $\{\alpha = t_0\}$ on a rarefaction fan is the image of a portion of $\mathcal{S}_{t_0}$ by an isometry (a direct isometry for 1-rarefaction, and an indirect isometry for a 2-rarefaction). Moreover, when the time goes on, the shape of the spiral is simply changed by dilation. Finally this spiral is locally a convex curve, because we have shown that $\partial_\chi \theta \geq 0$.

Geometrical structure of simple shocks

A shock is a discontinuity between two constant states (M_ℓ, θ_ℓ) and (M_r, θ_r). Physically, we expect to have $\theta_r < \theta_\ell$ for a simple shock and indeed, while it is possible to construct functions with a simple shock that are formal solutions of (2.12) with $\theta_r < \theta_\ell$, such solutions do not satisfy the entropy condition and are thus not physical (see Remark 2.4). Both quantities M and θ are discontinuous through the shock, but the level-set α is continuous, albeit not smooth (see figure 2.3).

The trajectory of the (punctual) shock is the half-line $\chi = \chi^s$ with $\chi_\ell <$

$\chi^s < \chi_r$. In order to satisfy the divergence equation in (2.1), we need to have $\chi^s - \chi_\ell < \pi$ and $\chi_r - \chi^s < \pi$ and to have either $\chi_\ell < \chi^s < \chi_r - \dfrac{\pi}{2}$ or $\chi_\ell + \dfrac{\pi}{2} < \chi^s < \chi_r$. Otherwise, the flux of $\frac{\boldsymbol{n}}{A}$ would be entering the shock from both sides, which would contradict the fact that $\frac{\boldsymbol{n}}{A}$ is divergence free.

When $\chi_\ell < \chi^s < \chi_r - \dfrac{\pi}{2}$, we have a 1-shock and the continuity of α forces $M_r > M_\ell$. Using system (2.12),

we can get the relations between M_ℓ, M_r, θ_ℓ, θ_r and $\chi^s = \chi_1^s$ with $\theta_\ell, \theta_r > \chi_1^s$. The first equation and the continuity of α lead to

$$\frac{M_\ell}{\cos(\theta_\ell - \chi^s)} = \frac{M_r}{\cos(\theta_r - \chi^s)}. \tag{2.23}$$

The second equation gives that the flux of $\dfrac{\boldsymbol{n}}{A(M)}$ through the boundary of a closed domain is zero. If we consider the flux for a tube going through the interface between the two states, we get

$$\frac{A(M_\ell)}{\sin(\theta_\ell - \chi^s)} = \frac{A(M_r)}{\sin(\theta_r - \chi^s)}. \tag{2.24}$$

From (2.23) and (2.24) we deduce

$$\tan(\theta_r - \chi^s) = \frac{A(M_r)\sin(\theta_\ell - \theta_r)}{A(M_\ell) - A(M_r)\cos(\theta_\ell - \theta_r)} = -\frac{M_\ell - M_r\cos(\theta_\ell - \theta_r)}{M_r\sin(\theta_\ell - \theta_r)}, \tag{2.25}$$

and then by symmetry r/ℓ, we get

$$\tan(\theta_\ell - \chi^s) = -\frac{A(M_\ell)\sin(\theta_\ell - \theta_r)}{A(M_r) - A(M_\ell)\cos(\theta_\ell - \theta_r)} = \frac{M_r - M_\ell\cos(\theta_\ell - \theta_r)}{M_\ell\sin(\theta_\ell - \theta_r)}. \tag{2.26}$$

Notice that this implies

$$\cos(\theta_\ell - \theta_r) = \frac{A(M_\ell)M_\ell + A(M_r)M_r}{A(M_r)M_\ell + A(M_\ell)M_r}. \tag{2.27}$$

We notice in particular that the general reordering inequality holds : $a^+b^+ + a^-b^- \geq a^+b^- + a^-b^+$ for all $a^+ \geq a^- \geq 0$ and $b^+ \geq b^- \geq 0$. Therefore, the fact that $A(M)$ is decreasing in M implies in particular that

$$0 < \frac{A(M_\ell)M_\ell + A(M_r)M_r}{A(M_r)M_\ell + A(M_\ell)M_r} \leq 1$$

which makes sense for equality (2.27).

When $\chi_\ell + \dfrac{\pi}{2} < \chi^s < \chi_r$, we have a 2-shock and the continuity of α forces $M_r < M_\ell$. Moreover, we have $\theta_\ell, \theta_r < \chi^s = \chi_2^s$. The same reasoning leads to (2.23) and (2.24) which gives the same relations as the ones above.

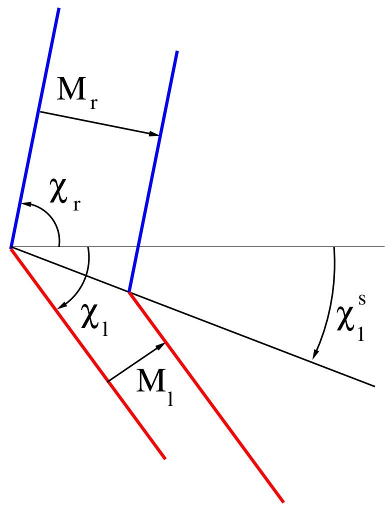
Remark 2.4. *Recall that admissible shocks have to satisfy an entropy condition in order to be stable : this is the Liu E-condition (which implies the Lax E-condition), see paragraph 8.4 in [13]. It is also known that Lax E-condition selects only half of the shock curve (see page 244, paragraph 8.3 in [13]). We can see it on figure 2.4. This shows that only the case $\theta_r < \theta_\ell$ is selected for shocks.*

With these informations in mind, we can give the structure of simple shocks :

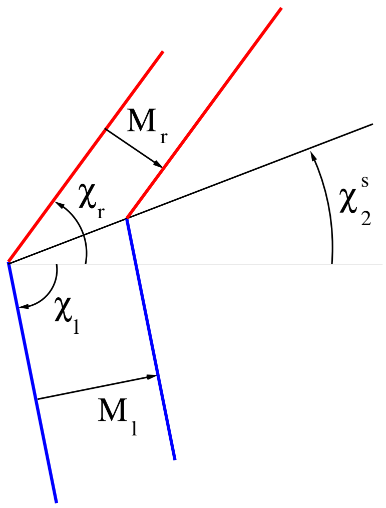
Proposition 2.5. (Structure of simple shocks)
Let $0 < \theta_\ell - \theta_r < \frac{\pi}{2}$ and $M_r, M_\ell > 1$ satysfying (2.27).
i) **1-shock**

If $M_r > M_\ell > 1$, then there exists $\chi^s = \chi_1^s < \theta_\ell, \theta_r$ given by (2.25) (or equivalently by (2.26)), such that the situation is pictured on figure 2.3.2.
ii) **2-shock**

If $M_\ell > M_r > 1$, then there exists $\chi^s = \chi_2^s > \theta_\ell, \theta_r$ given by by (2.25) (or equivalently by (2.26)), such that the situation is pictured on figure 2.3.2.



(a) 1-shock



(b) 2-shock

FIGURE 2.3 – Geometry of simple shock waves.

Remark 2.6. *We note that by replacing* $\cos(\theta_\ell - \theta_r)$ *by its expression* (2.27) *in* (2.26)*, one checks that* (2.26) *is equivalent to the relation given in [29]* *for a compression wedge.*

$$\tan|\theta_\ell - \chi^s| = \frac{A_\ell}{M_\ell}\left(\frac{M_r^2 - M_\ell^2}{A_\ell^2 - A_r^2}\right)^{1/2} \quad \textit{with} \quad A_b = A(M_b) \quad \textit{for} \quad b = \ell, r. \tag{2.28}$$

Remark 2.7. *Here, we have privileged the geometric interpretation of the Riemann problem. However, one can use the conservative form (2.13) of the GSD model in polar coordinates to obtain in similar manner as in [29] the equations (2.26)-(2.27) and (2.25)-(2.27).*

Complete solutions of the Riemann problem

The relations satisfied by the velocities M_ℓ and M_r and the angles θ_ℓ and θ_r for rarefactions and for shocks lead us to conclude that, as in the case of the p-system, there always exists a unique solution of the Riemann problem constituted of at most two transitions (a transition being a rarefaction or a shock). This happens to be true in a whole domain except in a region where the model ceases to be physical (see figure 2.4).

Supposing the left state (M_ℓ, θ_ℓ) is fixed, we can define the 1-shock and 2-shock curves functions, $S^1_{(M_\ell,\theta_\ell)}(M)$ and $S^2_{(M_\ell,\theta_\ell)}(M)$, by stipulating that for $M > M_\ell$, $\theta = S^1_{(M_\ell,\theta_\ell)}(M)$ is the only angle such that (M_ℓ, θ_ℓ) and (M, θ) satisfy the shock relation (2.27) (which implies that there is an acceptable 1-shock between the states (M_ℓ, θ_ℓ) and (M, θ)) and for $M < M_\ell$, $\theta = S^2_{(M_\ell,\theta_\ell)}(M)$ is the only angle such that (M_ℓ, θ_ℓ) and (M, θ) satisfy the shock relation (2.27) (which implies that there is an acceptable 2-shock between the states (M_ℓ, θ_ℓ) and (M, θ)). We define $R^1_{(M_\ell,\theta_\ell)}(M)$ and $R^2_{(M_\ell,\theta_\ell)}(M)$ in a similar manner. For $M < M_\ell$, $\theta = R^1_{(M_\ell,\theta_\ell)}(M)$ is the only angle such that (M_ℓ, θ_ℓ) and (M, θ) satisfy the 1-rarefaction relation (2.23) for $\sigma = 1$, and for $M > M_\ell$, $\theta = R^2_{(M_\ell,\theta_\ell)}(M)$ is the only angle such that (M_ℓ, θ_ℓ) and (M, θ) satisfy the 2-rarefaction relation (2.23) for $\sigma = -1$. These functions are drawn in a phase diagram on figure 2.4.

Nine cases, depending on the position of (M_r, θ_r) relatively to (M_ℓ, θ_ℓ) in this diagram, are then defined. The first four ones are the general cases, containing two transitions and an intermediate state, as in the study of the p-system (see [24], pp. 306–320). The other five cases are limit cases with at most one transition.

Case 1 : (M_r, θ_r) is in zone I
The solution contains a 1-rarefaction between states (M_ℓ, θ_ℓ) and (M^*, θ^*), and a 2-shock between states (M^*, θ^*) and (M_r, θ_r).

Case 2 : (M_r, θ_r) is in zone II
The solution contains a 1-shock between states (M_ℓ, θ_ℓ) and (M^*, θ^*), and a 2-shock between states (M^*, θ^*) and (M_r, θ_r).

Case 3 : (M_r, θ_r) is in zone III
The solution contains a 1-shock between states (M_ℓ, θ_ℓ) and (M^*, θ^*), and a 2-rarefaction between states (M^*, θ^*) and (M_r, θ_r).

Relations (2.23) and (2.27) ensure that solutions of this form always exist when (M_r, θ_r) is in zone I, II or III.

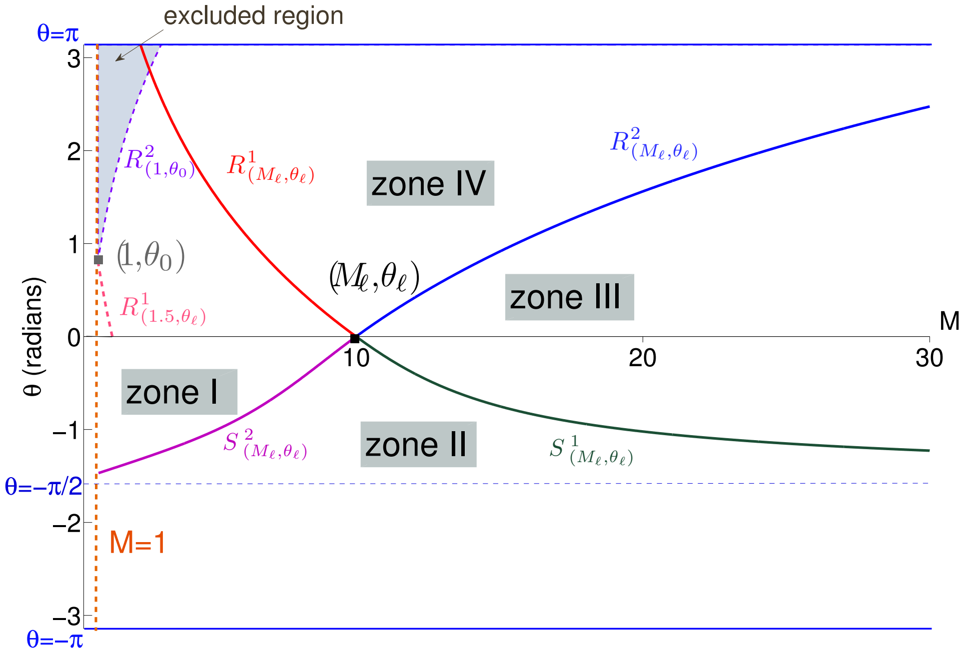


FIGURE 2.4 – Example of a phase diagram for $(M_\ell, \theta_\ell) = (10, 0)$ $\big($curves $R^1_{(M_\ell,\theta_\ell)}$, $R^2_{(M_\ell,\theta_\ell)}$, $S^1_{(M_\ell,\theta_\ell)}$ and $S^2_{(M_\ell,\theta_\ell)}\big)$, for $(M_\ell, \theta_\ell) = (1.5, 0)$ $\big($curve $R^1_{(1.5,0)}\big)$ and for $(M_\ell, \theta_\ell) = (1, \theta_0)$ $\big($curve $R^2_{(1,\theta_0)}\big)$.

Case 4 : (M_r, θ_r) is in zone IV
Except when (M_r, θ_r) is in a particular, excluded region, there exists a solution containing a 1-rarefaction between states (M_ℓ, θ_ℓ) and (M^*, θ^*), and a 2-rarefaction between states (M^*, θ^*) and (M_r, θ_r). The curve $R^1_{(M_\ell,\theta_\ell)}$ intersects the vertical axis at the point $(1, \theta_0)$. The boundary of the excluded region is then given by the curve $\theta = R^2_{(1,\theta_0)}(M)$.

In zone IV, as the shape of the front along a rarefaction is universal, it is possible to find the shape of the front (for example the level set $\{\alpha = t_0\}$) by noticing that the intermediate plane front must be both tangent to the (sole) image by a direct isometry of $\mathcal{S}_{t_0}$ tangent to the front of the state (M_ℓ, θ_ℓ) and tangent to the sole image by an indirect isometry of $\mathcal{S}_{t_0}$ tangent to the front of the state (M_r, θ_r), where we recall that $\mathcal{S}_{t_0}$ was defined at the end of subsection 2.3.2. In particular the front is constructed by a simple geometrical procedure (see figure 2.5).

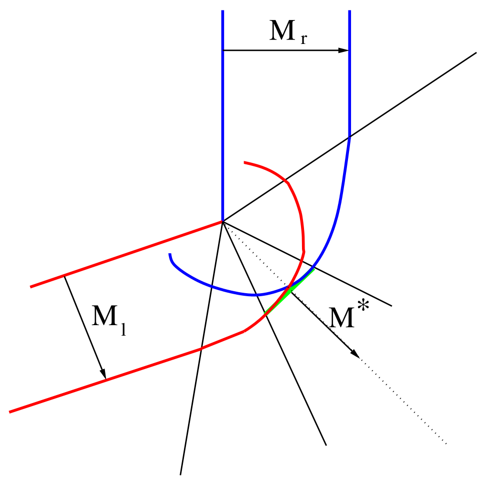


FIGURE 2.5 – Geometrical determination of the intermediate state for a double rarefaction.

We also remark that as in the analysis of [24], some states in the excluded region (as illustrated in figure 2.4) can not be joined with (M_ℓ, θ_ℓ) by the means of two rarefactions. The reason is that $\lim\limits_{M\to 1} \omega(M) < +\infty$, which means that all curves R^1 intersect the line $M = 1$ at a finite point $(1, \theta_0)$ depending on (M_ℓ, θ_ℓ). When M_ℓ is high enough, $\theta_0 \geq \theta_\ell + \pi$ and there is a solution for every physically possible initial data (M_r, θ_r) (that is with $\theta_r < \theta_\ell + \pi$). But when M_ℓ is small, $\theta_0 < \theta_\ell + \pi$ and in this case, when (M_r, θ_r) is above the curve $R^2_{(1,\theta_0)}$, the Riemann problem does not admit a solution with two rarefactions. The physical interpretation of this is less clear than in the usual interpretation of the p-system in terms of isentropic gas dynamics, because the GSD approximation does not work as well when M is near 1.

Case 5 : $M_r > M_\ell$ **and** $\theta_r = S^1_{(M_\ell,\theta_\ell)}(M_r)$
The solution only contains a 1-shock between states (M_ℓ, θ_ℓ) and (M_r, θ_r).

Case 6 : $M_r < M_\ell$ **and** $\theta_r = S^2_{(M_\ell,\theta_\ell)}(M_r)$
The solution only contains a 2-shock between states (M_ℓ, θ_ℓ) and (M_r, θ_r).

Case 7 : $M_r < M_\ell$ **and** $\theta_r = R^1_{(M_\ell,\theta_\ell})(M_r)$
The solution only contains a 1-rarefaction between states (M_ℓ, θ_ℓ) and (M_r, θ_r).

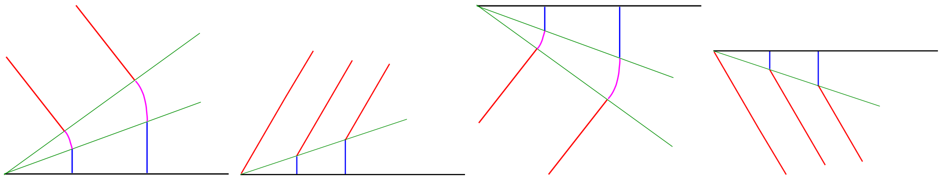
Case 8 : $M_r > M_\ell$ **and** $\theta_r = R^2_{(M_\ell,\theta_\ell)}(M_r)$
The solution only contains a 2-rarefaction between states (M_ℓ, θ_ℓ) and (M_r, θ_r).

Case 9 : $M_r = M_\ell$ **and** $\theta_r = \theta_\ell$
The solution is a uniform planar front of characteristics (M_ℓ, θ_ℓ).

Geometrical interpretation

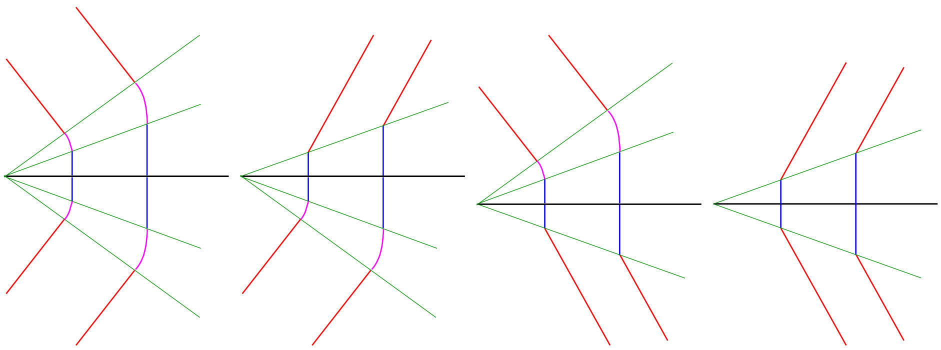
In the preceding subsection we have constructed the complete solutions of the Riemann problem in the phase space before describing their geometry. There is a more geometrical way to view this reconstruction. Indeed, as it is described by Whitham in [29], a simple shock is the result of the compression of a plane front by a wedge, and a rarefaction the result of the diffraction of a plane front by a corner. Such solutions involve a wall boundary, and in both cases the solution is a plane front near the wall, with a normal vector $\boldsymbol{n}$ parallel to the wall. As a consequence, two such solutions can be glued together along the wall boundary to form a complete solution of (2.1), provided they have the same velocity M^* at the wall. The solution thus constructed is naturally the solution of a Riemann problem.

Conversely, the solution of any Riemann problem can be seen as the reunion of two such half-solutions involving a wedge or a corner. For a typical solution constituted of two transitions with an intermediary state, we can split the solution by a virtual boundary at the angle χ_w for which $\theta = \chi$. Then the restrictions of the solution on both sectors $\{\chi < \chi_w\}$ and $\{\chi > \chi_w\}$ are solutions which arise from the interaction of a plane front with a wedge or with a corner. Figures 2.6 and 2.7 illustrate this construction.



(a) 2-rarefaction (b) 2-shock (c) 1-rarefaction (d) 2-shock

FIGURE 2.6 – Half-solutions (with a horizontal wall).



(a) (b) (c) (d)

FIGURE 2.7 – Complete solutions of the Riemann problem.

2.4 Algorithm Validation

The algorithm summarized in section 2.2.6 is now evaluated on a set of test cases of increasing complexity. For each of them a reference solution is known or has been introduced previously in section 2.3. We begin by checking the numerical convergence order of the scheme by comparison with the smooth radial solution, before looking for discontinuous solutions in the case of a compression wedge (shock-shock). The analysis of the Riemann problem completes this experimental study of the numerical scheme. For these reference solutions in the discontinuous case, explicit expressions are developed in the strong shock limit (*i.e.* $M >> 1$) using the simplified A-M relation

$$\frac{A}{A_0} = \left(\frac{M_0}{M}\right)^{\lambda_\infty}, \quad \text{with } \lambda_\infty = 5.0743 \text{ for } \gamma = 1.4. \tag{2.29}$$

All numerical experiments are performed with the full A-M relation (2.9) but for a large enough Mach number value such that the infinite limit is asymptotically reached.

2.4.1 Order of the scheme

The reference solution is obtained by numerically integrating the radial system (2.10) with a fourth-order Runge-Kutta scheme in cylindrical and spherical coordinates. The convergence study is made on the discrete L^∞ norm.

• **Cylindrical shock** For this two dimensional case, the computational domain is reduced to the square $[0, 50] \times [0, 50]$ due to the symmetry of the problem. The initial conditions on α and M are given by the radial solution (2.10) on the quarter-circle of radius 4.9 centered at (0.0). We impose $M = 10$ for $r < 1$ to avoid the singularity at the origin. The boundary conditions are of outgoing type on the edges $\{x = 50\}$ and $\{y = 50\}$, and of rigid wall type on the edges $\{x = 0\}$ and $\{y = 0\}$. For this test case, the determination of test values ϑ and m is made by the fixed-point algorithm.

The tables (2.1) give the norm of the error versus the grid spacing for both the first-order and second-order schemes respectively. Note that N is the number of nodes in each direction and that $\triangle x = \triangle y$. The discrete order is defined by :

$$\text{order} = \frac{\log(E_2/E_1)}{\log(N_1/N_2)},$$

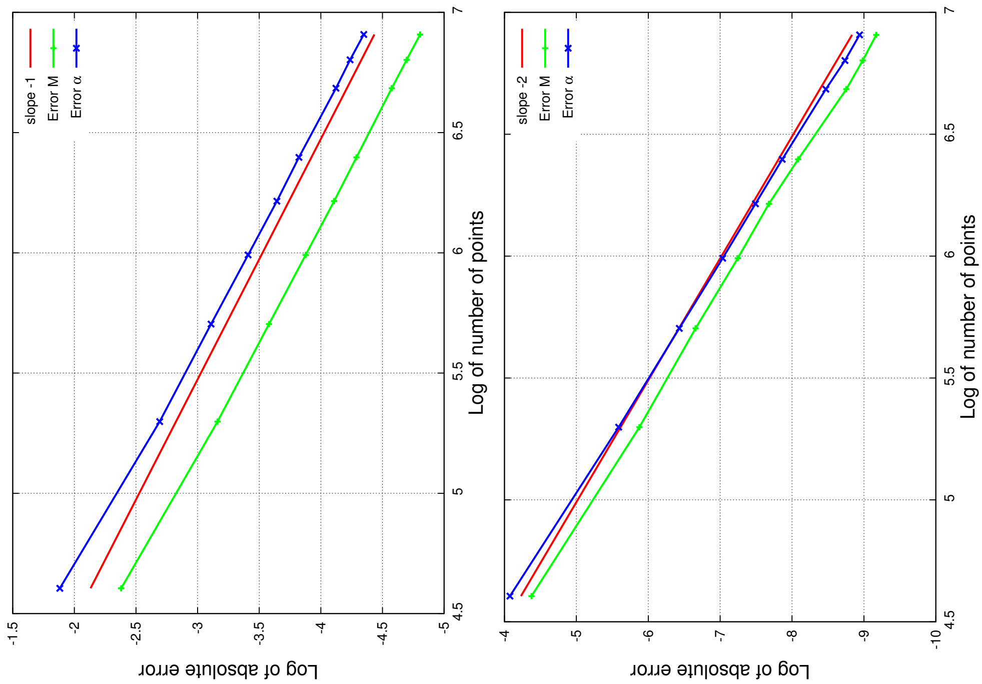
where E_1 and E_2 are respectively the errors associated to the meshes with $N_1 \times N_1$ and $N_2 \times N_2$ elements. A logarithmic representation is given in figures 2.4.1-2.4.1. One can note that the expected order of convergence is recovered and as expected, the second-order scheme provides a better accuracy level.

Mesh	Error M	Order M	Error α	Order α
100×100	9.2577e-02		1.5226e-01	
200×200	4.2359e-02	1.1280	6.7709e-02	1.1691
300×300	2.7900e-02	1.0298	4.4638e-02	1.0276
400×400	2.0718e-02	1.0346	3.3047e-02	1.0451
500×500	1.6447e-02	1.0346	2.6159e-02	1.0475
600×600	1.3723e-02	0.9930	2.1880e-02	0.9797
800×800	1.0302e-02	0.9968	1.6198e-02	1.0452
900×900	9.1333e-03	1.0222	1.4450e-02	0.9692
1000×1000	8.1830e-03	1.0427	1.2935e-02	1.0514
E.C.O. [1]	1.0445		1.0597	

Mesh	Error M	Order M	Error α	Order α
100×100	1.2571e-02		1.6974e-02	
200×200	2.8049e-03	2.1641	3.7407e-03	2.1819
300×300	1.2828e-03	1.9295	1.6097e-03	2.0797
400×400	7.1314e-04	2.0409	8.8050e-04	2.0971
500×500	4.6273e-04	1.9383	5.5824e-04	2.0422
600×600	3.0824e-04	2.2284	3.8439e-04	2.0466
800×800	1.5767e-04	2.3303	2.0961e-04	2.1080
900×900	1.2552e-04	1.9358	1.6077e-04	2.2522
1000×1000	1.0399e-04	1.7857	1.3119e-04	1.9299
E.C.O.	2.0819		2.1047	

TABLE 2.1 – Mesh convergence in the L^∞ norm for the cylindrical case.

1. E.C.O. denotes the Experimental Convergence Order given by the linear fit of the error in the logarithmic scale.



(a) first-order (b) second-order

FIGURE 2.8 – Mesh convergence in the L^∞ norm for the cylindrical case in the logarithmic scale.

• **Spherical shock** The analysis of the numerical order of the scheme is also carried on in 3D. Using symmetry conditions, the computational domain reduces to the cube $[0, 10] \times [0, 10] \times [0, 10]$. The initial front is given by the radial solution on the sphere of radius 2 centered at the origin. As in the 2D case, we impose $M = 10$ for $r < 1$ to avoid the singularity at the origin. The boundary conditions are of outgoing type on the edges $\{x = 10\}$, $\{y = 10\}$, $\{z = 10\}$, and of rigid wall type on the edges $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$. Here, the resolution of the nonlinear system (2.8) is made by Newton's method. The results of the second-order numerical scheme are compared with the semi-analytical radial solution considered previously. The table 2.2 gives the error in the discrete L^∞ norm between the two solutions, the logarithms of these values are displayed in figure 2.9. The second-order of convergence is also obtained in 3D.

Mesh	Error M	Order M	Error α	Order α
$40 \times 40 \times 40$	0.00557		0.00521	
$60 \times 60 \times 60$	0.00234	2.1389	0.00232	1.9953
$80 \times 80 \times 80$	0.00129	2.0700	0.00126	2.1220
$100 \times 100 \times 100$	0.000791	2.1919	0.000808	1.9911
$160 \times 160 \times 160$	0.000309	1.9999	0.000308	2.0520
E.C.O.	2.0895		2.2873	

TABLE 2.2 – Mesh convergence in the L^∞ norm for the spherical case.

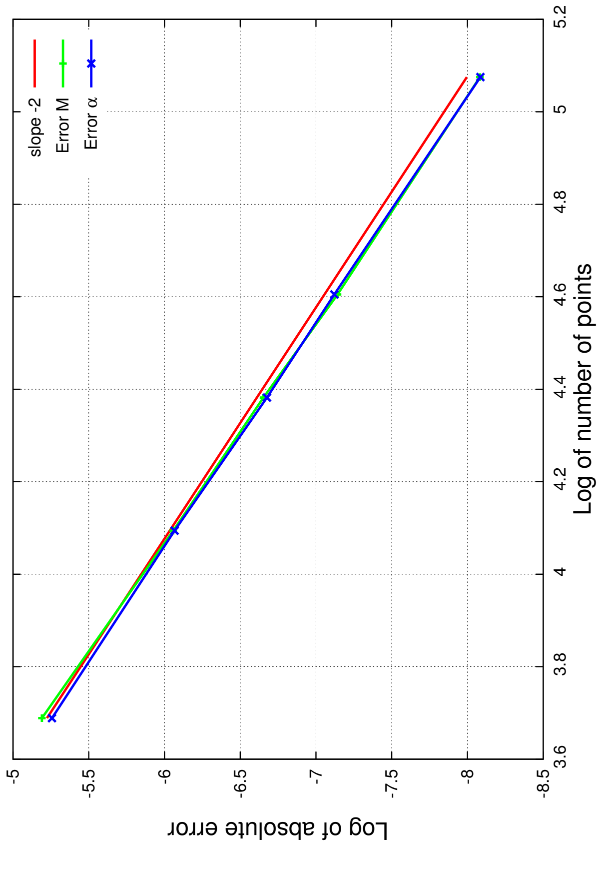


FIGURE 2.9 – Mesh convergence in the L^∞ norm in the logarithmic scale.

These smooth test cases have proven that our fast-marching like algorithm doesn't suffer from using points in the NarrowBand and that the expected theoretical order of convergence is reached.

2.4.2 Compression wedge test case

Attention is now paid to discontinuous solutions. We deal here with a well-know test case, studied by many authors [29, 2], consisting in the diffraction of an oblique shock by a wedge. It can be seen as an application of the study of simple shocks. For an incoming front of velocity M_0 and angle $\theta_0 = \beta$ interacting with a wedge of angle $\theta_w = 0$, we are looking for a 2-shock joining the states $(M_\ell, \theta_\ell) = (M_w, 0)$ and $(M_r, \theta_r) = (M_0, \beta)$, the velocity M_w of the Mach front and the angle χ_2^s of the trajectory of the triple point being unknown. The analysis of the structure of 2-shocks then gives us that M_w is governed by the relation (2.27), and χ_2^s by (2.25).

In the strong shock limit, the equations (2.28) and (2.27) can be simplified in the following implicit relationship between β and χ_2^s

$$\begin{cases} \tan \chi_2^s &= m^{\lambda_\infty} \left(\dfrac{1 - m^2}{1 - m^{2\lambda_\infty}} \right)^{1/2} \\ \cos \beta &= \dfrac{m + m^{\lambda_\infty}}{1 + m^{1+\lambda_\infty}} \end{cases} , \tag{2.30}$$

where $m = \dfrac{M_0}{M_w} < 1$ denotes the Mach number ratio on either side of the shock-shock and $\lambda_\infty = 5.0743$. The subscripts w and 0 characterize the quantities on the wall and those of the incident (or initial) shock respectively, χ_2^s is the angle of the shock-shock line with the horizontal axis (Ox). This analytical solution is displayed as a solid curve in the figure 2.10.

Here, the wedge is chosen aligned with the (Ox) axis to avoid a difficult boundary treatment. The rigid wall condition is then imposed on the edge $\{y = 0\}$. The computation was performed on the domain $[0, 3] \times [0, 3]$ with a 300×300 mesh. The initial shock position is given by

$$\alpha_0(x, y) = \frac{(x - 0.5)\cos\beta - y \sin\beta}{M_0},$$

where $\beta \in [0, \pi/2]$ is the angle between the front normal and the wedge, and $M_0 = 10$. The values of M and α for all points inside the half space on the left of the initial shock position (*i.e.* $\alpha_0(x, y) \leq 0$) are already intercepted by the front and easily known since the shock is plane. An outflow condition is imposed on the $\{x = 3\}$ edge. Special care must be taken for the $\{y = 3\}$ edge since it corresponds to an inflow condition, so we set the exact solution on it.

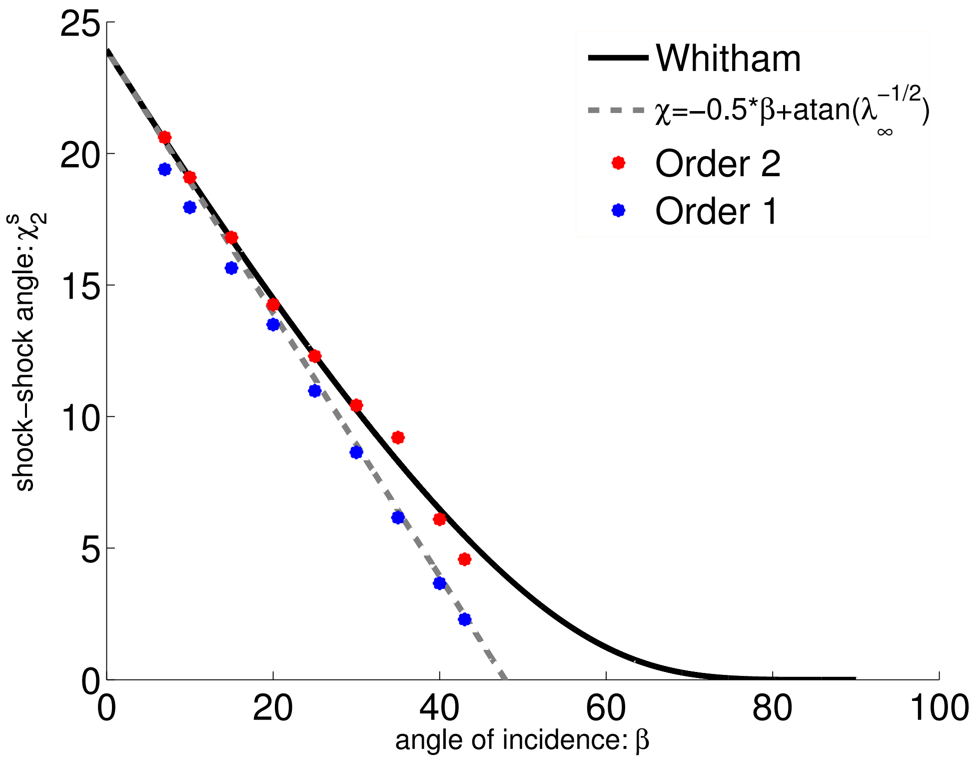


FIGURE 2.10 – Shock-shock angle versus angle of incidence for the first and second order algorithm in comparison to Whitham's analytical solution.

For a range of incident shock angle, a comparison between this theoretical shock-shock locus and the numerical ones obtained by the first and second order schemes is shown in figure 2.10. It is worth noting the very good agreement between the theoretical position and the results of the second-order version. This result is in agreement with [2] where a classical level-set algorithm was used. One can also notice that the first-order numerical scheme underestimates the value of χ_2^s by several degrees, and follows a nearly linear behaviour, lying on the tangent (dashed line) to the theoretical curve at $M = 1$. A mesh refinement does not improve these results, which demonstrates the necessity to use the second-order scheme when a shock singularity is expected. Above nearly 45 degrees, the second order algorithm is also in default and doesn't show any shock-shock angle. This may be due to the switching procedure, applied as it on the wall, which reduces locally the order of the scheme. A better boundary treatment could help improve this behaviour. Nevertheless, this is not a severe limitation in practice since the GSD model is not expected to give accurate results in such an extreme case [29].

2.4.3 Test cases for the Riemann problem

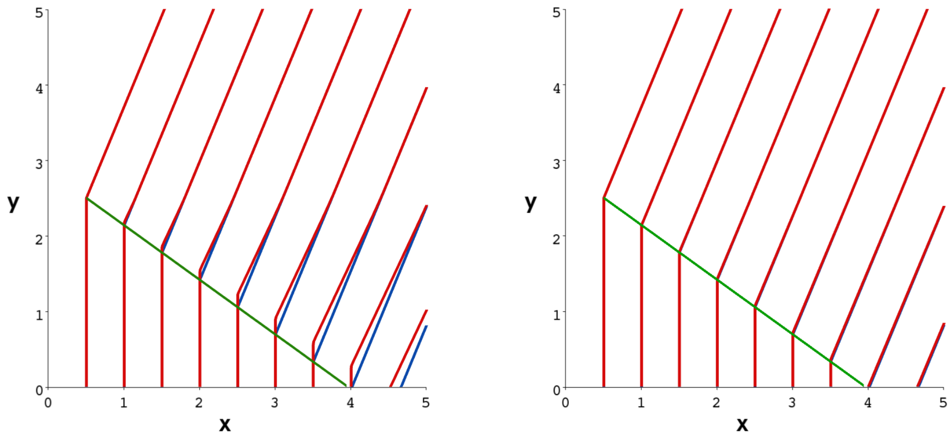
A series of numerical experiments is now conducted to assess in more details the ability of the numerical scheme to solve the two dimensional GSD Riemann problem. In section 2.3 we have given the mathematical solutions and we refer the reader to 2.6 for the detail of their explicit construction in the strong shock limit. We compare these reference solutions to the numerical ones, by taking varying values of the initial constant states to cover all configurations of paragraph 2.3.2. The first tests use the left state (M_ℓ, θ_ℓ) connected directly to the right state (M_r, θ_r) by an elementary wave (either a shock or a rarefaction), *i.e.* (M_r, θ_r) lies on one of the curves R^1, R^2, S^1, S^2 issued from (M_ℓ, θ_ℓ) (see figure 2.4). More general initial datas are then considered to cover the different zones of the figure 2.4 : zone I, zone II, zone III and zone IV, *i.e.* solutions with an intermediate state.

The numerical solution is obtained on the domain $[0, 5] \times [0, 5]$ with a 500×500 grid. In all cases, the left state $(M_\ell, \theta_\ell) = (10, 0)$ is given and we make the right state (M_r, θ_r) vary in the plane (M, θ). We display a comparison between the isolines of α, obtained by the first and second order schemes, and the exact solution in each configuration. In these figures, the red curve represents the numerical solution, the blue one represents the exact solution and the green lines correspond to the characteristic curves. The following figures gather all the results obtained for the different solutions of the Riemann problems.

Simple waves

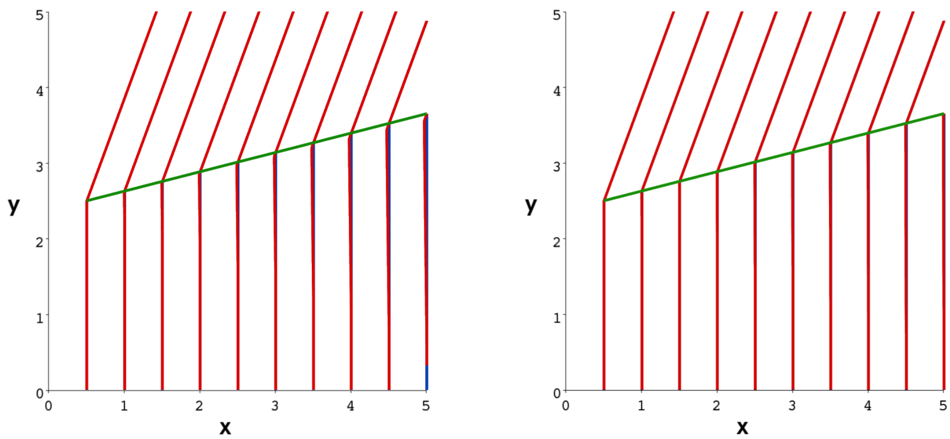
Simple waves correspond to cases 5 (1-shock), 6 (2-shock), 7 (1-rarefaction) and 8 (2-rarefraction) of paragraph 2.3.2. They are difficult to compute in practice since numerical roundoff on the initial condition may lead to the emergence of an intermediate state.

The initial right state for the 1-shock and the 2-shock cases are $(M_r, \theta_r) = (12, -22.45)$ and $(M_r, \theta_r) = (8.5, -20.24)$ respectively. Results are drawn in figures 2.11 and 2.12. As expected from the compression wedge study, one can note the excellent behaviour of the second order scheme for the 2-shock case, but also for the 1-shock case. A severe deviation is present for the first order algorithm applied on the 1-shock case.



(a) 1st order (b) 2nd order

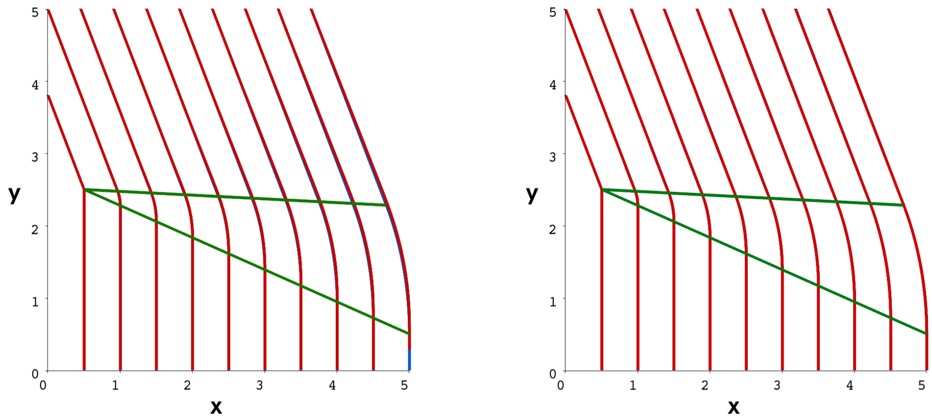
FIGURE 2.11 – **1-shock** case : $(M_\ell, \theta_\ell) = (10, 0)$ and $(M_r, \theta_r) = (12, -22.45)$. Comparison of the numerical solution — and the exact solution —. The characteristic curves — are displayed for convenience.



(a) 1st order (b) 2nd order

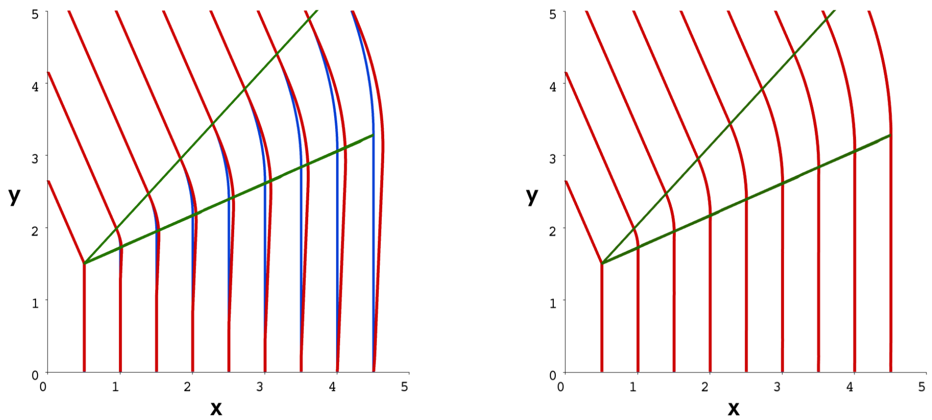
FIGURE 2.12 – **2-shock** case : $(M_\ell, \theta_\ell) = (10, 0)$ and $(M_r, \theta_r) = (8.5, -20.24)$. Same drawing convention.

The initial right state for the 1-rarefaction and the 2-rarefaction cases are $(M_r, \theta_r) = (8.5, 20.97)$ and $(M_r, \theta_r) = (12, 23.53)$ respectively. Figures 2.13 and 2.14 exhibit a different behaviour for the first order version : the 1-rarefaction wave is well captured while the 2-rarefaction one shows large deviation. The second order scheme is excellent in both cases, nearly superimposed with the reference solution.



(a) 1st order (b) 2nd order

FIGURE 2.13 – **1-rarefaction** case : $(M_\ell, \theta_\ell) = (10, 0)$ and $(M_r, \theta_r) = (8.5, 20.97)$. Same drawing convention.



(a) 1st order (b) 2nd order

FIGURE 2.14 – **2-rarefaction** case : $(M_\ell, \theta_\ell) = (10, 0)$ and $(M_r, \theta_r) = (12, 23.53)$. Same drawing convention.

1-rarefaction, 2-shock

This problem corresponds to case 1 of paragraph 2.3.2. The solution consists of a 1-rarefaction connecting the left state $(M_\ell, \theta_\ell) = (10, 0)$ to a constant intermediate state (M^*, θ^*) followed by a 2-shock connecting (M^*, θ^*) to the right state $(M_r, \theta_r) = (5, 30)$, *i.e.* (M_r, θ_r) is within zone I on figure 2.4. Numerical results are shown on figure 2.15. One can note that the higher order scheme allows a much better capture of the intermediate state, retrieving a nearly plane front, while the first order scheme exhibit an unexpected wave pattern.

1-shock, 2-shock configuration

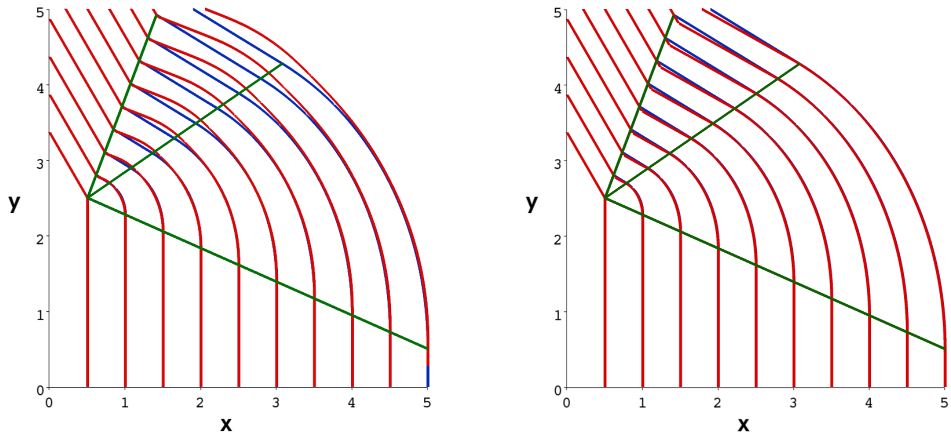
This problem corresponds to case 2 of paragraph 2.3.2. The solution consists of a 1-shock connecting the left state $(M_\ell, \theta_\ell) = (10, 0)$ to a constant intermediate state (M^*, θ^*) followed by a 2-shock connecting (M^*, θ^*) to the right state $(M_r, \theta_r) = (10, -50)$, *i.e.* (M_r, θ_r) is within zone II on figure 2.4. Numerical results are shown on figure 2.16. As previously noticed, the higher order scheme allows a better capture of the intermediate state. Here, the first order scheme leads to an underestimation of the intermediate state's Mach number.

1-shock, 2-rarefaction configuration

This problem corresponds to case 3 of paragraph 2.3.2. The solution consists of a 1-shock connecting the left state $(M_\ell, \theta_\ell) = (10, 0)$ to a constant intermediate state (M^*, θ^*) followed by a 2-rarefaction connecting (M^*, θ^*) to the right state $(M_r, \theta_r) = (20, 10)$, *i.e.* (M_r, θ_r) is within zone III on figure 2.4. Numerical results are shown on figure 2.17. As expected from the simple wave results, the 1-shock is badly captured by the first order scheme and the overall solution behaviour is dramatically affected. Here again, the second order scheme performs well.

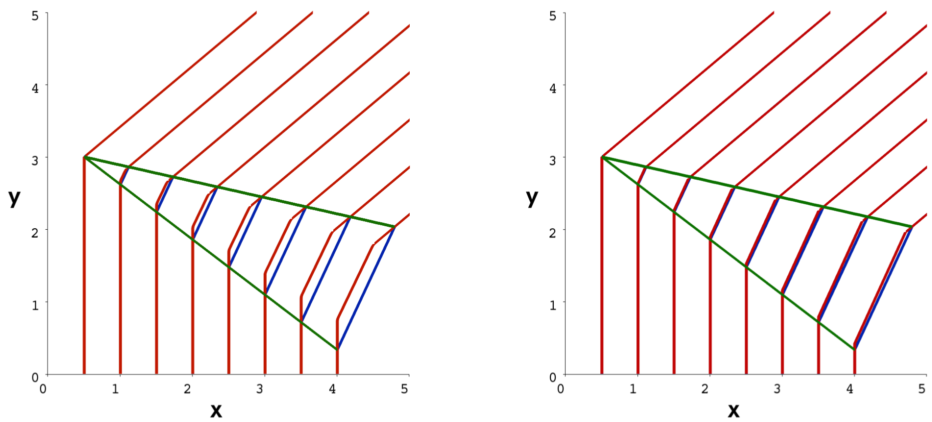
1-rarefaction, 2-rarefaction configuration

This problem corresponds to case 4 of paragraph 2.3.2. The solution consists of a 1-rarefaction connecting the left state $(M_\ell, \theta_\ell) = (10, 0)$ to a constant intermediate state (M^*, θ^*) followed by a 2-rarefaction connecting (M^*, θ^*) to the right state $(M_r, \theta_r) = (10, 50)$, *i.e.* (M_r, θ_r) is within zone IV on figure 2.4. Numerical results are shown on figure 2.18. The superiority and the necessity of a higher order scheme is clearly stated on this case.



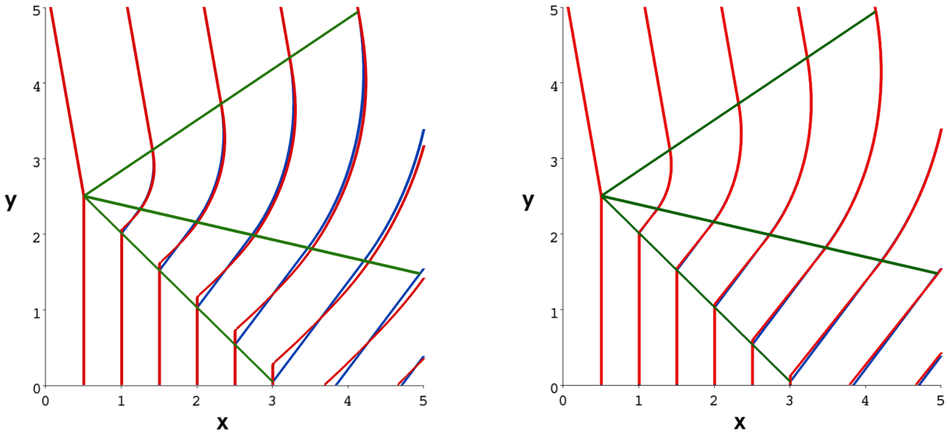
(a) 1st order (b) 2nd order

FIGURE 2.15 – **1-rarefaction + 2-shock** : $(M_\ell, \theta_\ell) = (10, 0)$ and $(M_r, \theta_r) = (5, 30)$. Same drawing convention.



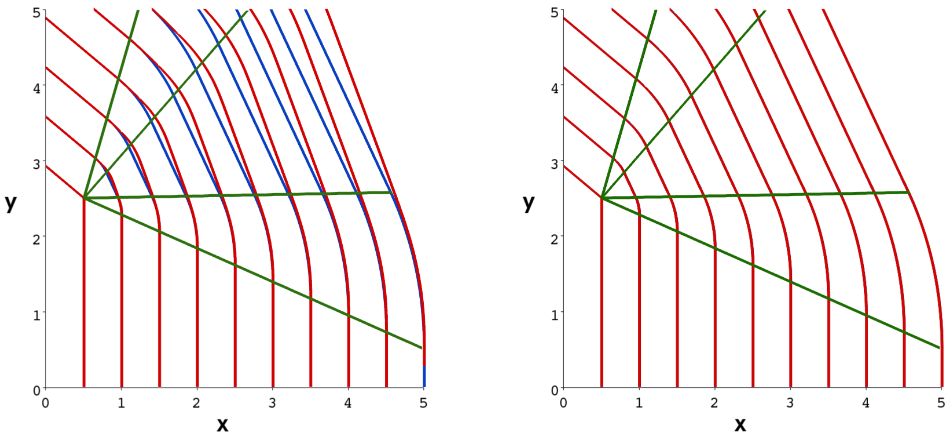
(a) 1st order (b) 2nd order

FIGURE 2.16 – **1-shock + 2-shock** : $(M_\ell, \theta_\ell) = (10, 0)$ and $(M_r, \theta_r) = (10, -50)$. Same drawing convention.



(a) 1st order (b) 2nd order

FIGURE 2.17 – **1-shock + 2-rarefaction** : $(M_\ell, \theta_\ell) = (10, 0)$ and $(M_r, \theta_r) = (20, 10)$. Same drawing convention.



(a) 1st order (b) 2nd order

FIGURE 2.18 – **1-rarefaction + 2-rarefaction** : $(M_\ell, \theta_\ell) = (10, 0)$ and $(M_r, \theta_r) = (10, 50)$. Same drawing convention.

Synthesis

In view of the results obtained in this series of test cases, we note that both first-order and second-order numerical schemes give, qualitatively, the expected behaviour of the exact solution to the Riemann problem. However, there are substantial differences between the results obtained by each order. It is observed that the first-order scheme is not accurate enough, which leads to a significant discrepancy on the position of the shock, while the second-order scheme provides a much better approximation and coincides quite strictly with the exact solution for all cases.

2.5 Conclusion

In this paper, we described a new algorithm for solving the Geometrical Shock Dynamics model. In this hyperbolic two equations model, the arrival time, $\alpha(\boldsymbol{x})$, of a shock wave is given by a nonlinear eikonal equation whose local speed, or Mach number $M(\boldsymbol{x})$, is governed by a differential equation depending on flow parameters and the front curvature. By reformulating the transport equation on M as a convection–diffusion one, its approximation is made easier and the lengthy discretization of the mean curvature is avoided. The new algorithm follows the fast-marching method on a Cartesian grid but uses trial values when updating the current node. First derivatives, appearing in the eikonal and transport equations, are upwind sided up to second order depending on the causality condition on α. A key feature of the method is to use a centered discrete Laplacian for the diffusion part, as much as possible, without taking into account the causality condition (*i.e.* using points which are not in the Known set). For this reason, the algorithm is not a fast-marching method in the strictest sense, and we say it has fast-marching like properties. The resulting algebraic nonlinear system is solved by an iterative procedure, fixed point or Newton's method, at each node.

For validation purpose, reference solutions are built for a smooth radial expansion wave and the interaction of two shocks. This latter case is a Riemann problem for which a link is made with the p-system in two dimensions. A detailed analysis of this problem is provided and simple front characterization is given in the strong shock limit for any left and right states. In general, those states are connected by shock or rarefaction waves developing on the front.

Numerical experiments have shown that, as expected, the algorithm is second order for smooth fronts and performs remarkably well on discontinuous solutions, although the non conservative model equations are solved. Differences are however observed between the first and second order algorithm on the Riemann problems. The second order scheme is much better and very close to the reference solution, except when it locally switches to first order. This is particularly noticeable on the compression wedge case where the second order scheme fails above a wedge angle of nearly 45 degrees. This is not a severe limitation in practice since the GSD model is not expected to give accurate results in such a case.

In the future we plan to extend our algorithm to deal with immersed rigid bodies and optimize the solver to achieve very fast computations. At last, we infer that this new algorithm could also be valuable to the detonic community, mainly for Detonation Shock Dynamics [26].

2.6 Appendix : Reference solutions of the Riemann problem in the strong shock limit

Based on the geometric description of section 2.3, we outline the construction of explicit solutions, in the strong shock limit ($M >> 1$), for the 2D Riemann problem. For the sake of simplicity, we give the explicit solution for specific configurations only : simple waves and a particular case with an intermediate state. The key tool is to use the simplified relation (2.29) between the area section and the Mach number.

We consider initial constant left and right states joined at the origin $O = (0,0)$ and use the polar coordinates $x = \rho\cos\chi$ and $y = \rho\sin\chi$.

2.6.1 Simple shock wave

From the limit form (2.29) of the A-M relation in the strong shock limit, the equations (2.25)-(2.27) become (see also (2.30))

$$\left\{\begin{array}{rcl} \cos(\theta_\ell - \theta_r) & = & \dfrac{m^{\lambda_\infty} + m}{1 + m^{\lambda_\infty + 1}} \\ \tan\left(\chi_\nu^s - \theta_\ell\right) & = & \left[-1 + \nu(\nu - 1)\right] m^{\lambda_\infty} \left(\dfrac{1 - m^2}{1 - m^{2\lambda_\infty}}\right)^{1/2} \end{array}\right. ,$$

where $m = \dfrac{M_r}{M_\ell}$ and $\nu = 1, 2$ for the 1-shock or 2-shock cases. Given $0 < \theta_\ell - \theta_r < \frac{\pi}{2}$, from the first equation of this algebraic system, one deduces, by an iterative procedure, the Mach number ratio m on each side of the shock-shock singularity. This singularity is characterized by the angle χ_ν^s which is obtained by injecting m in the second equation.

For this case, the geometric construction is straightforward and proceeds as follows. As shown in figure 2.19, we consider the two angular sectors separated by the shock line extending from the origin O and having the slope $\tan\chi_\nu^s$. At a given time, one starts from a current position $P_1 \equiv (x_1, y_1)$ on the left state and draws the line segment whose normal makes the angle θ_ℓ with the (Ox) axis. The other endpoint of this segment, $P_2 \equiv (x_2, y_2)$, lies on the shock line and so has the polar angle χ_ν^s. This process is repeated to draw the front shape at different times to cover the computational domain.

The scalar function α, that characterizes the successive front positions,

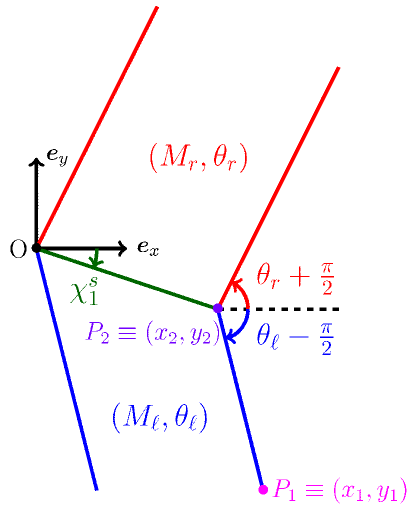


FIGURE 2.19 – Example of geometric construction of 1-shock.

is then (see also figure 2.3)

$$\alpha(x, y) = \begin{cases} \dfrac{x\cos\theta_\ell + y\sin\theta_\ell}{M_\ell}, & \chi \leq \chi_\nu^s \\[2ex] \dfrac{x\cos\theta_r + y\sin\theta_r}{M_r}, & \chi \geq \chi_\nu^s \end{cases}.$$

2.6.2 Simple rarefaction wave

As mentioned in section 2.3.2, the rarefaction locus is a spiral

$$\rho = t_0\rho_1(\chi) \quad \text{where } t_0 > 0,$$

connecting the two constant states. In the strong shock limit ($M >> 1$), this spiral is of logarithmic type, and the angle between its normal and the radial direction is constant (see (2.15)) and is given by

$$\delta_\infty = \arctan\sqrt{\frac{1}{\lambda_\infty}}\,.$$

In other words, the angles in Proposition 2.3 are written

$$\chi_{1,2}^{r-} = \theta_\ell \mp \delta_\infty \quad \text{and} \quad \chi_{1,2}^{r+} = \theta_r \mp \delta_\infty,$$

where the signs $\mp$ refer to the 1-rarefaction and the 2-rarefaction respectively. Moreover equation (2.23) simplifies to give

$$\theta_r - \theta_\ell = \mp\sqrt{\lambda_\infty}\log\left(\frac{M_r}{M_\ell}\right)$$

with $\pi > \theta_r - \theta_\ell > 0$ (see Proposition 2.3).

The scalar function α, that characterizes the successive positions of the shock front, is then written (see figure 2.2)

$$\alpha(x,y) = \begin{cases} \dfrac{x\cos\theta_\ell + y\sin\theta_\ell}{M_\ell}, & \chi \leq \theta_\ell \mp \delta_\infty \\[2ex] K\rho\, \mathrm{e}^{\pm\frac{\chi}{\sqrt{\lambda_\infty}}} & , \theta_\ell \mp \delta_\infty \leq \chi \leq \theta_r \mp \delta_\infty \\[2ex] \dfrac{x\cos\theta_r + y\sin\theta_r}{M_r}, & \chi \geq \theta_r \mp \delta_\infty \end{cases} \cdot$$

with $K = \dfrac{\cos\delta_\infty}{M_\ell} e^{\mp\frac{\chi_{1,2}^{r-}}{\sqrt{\lambda_\infty}}}$.

2.6.3 Complete solutions of the Riemann problem

Whatever the left and right initial conditions, the intermediate state (θ^*, M^*) can be written in the generic form :

$$\begin{cases} \theta^* - \theta_\ell &= \mathcal{G}_{\mathrm{R,S}}\left(\dfrac{M^*}{M_\ell}\right) \\[2ex] \theta_r - \theta^* &= \mathcal{G}_{\mathrm{R,S}}\left(\dfrac{M^*}{M_r}\right) \end{cases}, \tag{2.31}$$

where, in the strong shock limit, the function $\mathcal{G}_{\mathrm{R,S}}$ is given by

$$\mathcal{G}_{\mathrm{R,S}}(m) = \left| \begin{array}{lcll} \mathcal{G}_{\mathrm{R}}(m) &=& -\sqrt{\lambda_\infty}\log m & \text{for a rarefaction} \\[2ex] \mathcal{G}_{\mathrm{S}}(m) &=& -\arccos\left(\dfrac{m + m^{\lambda_\infty}}{1 + m^{1+\lambda_\infty}}\right) & \text{for a shock} \end{array} \right. .$$

For illustration purpose, we consider only the 1-rarefaction followed by 2-rarefaction case, for which system (2.31) can be solved explicitly. One then checks that

$$\begin{cases} \theta^* &= \dfrac{\theta_r + \theta_\ell - \sqrt{\lambda_\infty}\log\left(\frac{M_r}{M_\ell}\right)}{2} \\[2ex] M^* &= \sqrt{M_r M_\ell e^{\frac{\theta_\ell - \theta_r}{\sqrt{\lambda_\infty}}}} \end{cases} .$$

Remark 2.8. *For any other configuration, an iterative procedure is required to determine the intermediate state.*

For the example under consideration, the front is compound of five angular sectors, according to the values of the polar angle χ (see figure 2.5). The Riemann solution consists in two spiral arcs, characterized respectively by the angles $\pm\delta_\infty$, connecting the intermediate state to the plane shocks on the left and on the right respectively. Note that in the region between the two spiral arcs, the shock is plane (*i.e.* $M = M^*$ and $\theta = \theta^*$) and tangent to both curves. More precisely, the function α is written

$$\alpha(x,y) = \begin{cases} \dfrac{x\cos\theta_\ell + y\sin\theta_\ell}{M_\ell} & , \quad \chi \leq \theta_\ell - \delta_\infty \\ K_1\rho\, \mathrm{e}^{\frac{\chi}{\sqrt{\lambda_\infty}}} & , \quad \theta_\ell - \delta_\infty \leq \chi \leq \theta^* - \delta_\infty \\ \dfrac{x\cos\theta^* + y\sin\theta^*}{M^*} & , \quad \theta^* - \delta_\infty \leq \chi \leq \theta^* + \delta_\infty \\ K_2\rho\, \mathrm{e}^{-\frac{\chi}{\sqrt{\lambda_\infty}}} & , \quad \theta^* + \delta_\infty \leq \chi \leq \theta_r + \delta_\infty \\ \dfrac{x\cos\theta_r + y\sin\theta_r}{M_r} & , \quad \chi \geq \theta_r + \delta_\infty. \end{cases} ,$$

with

$$K_1 = \frac{\cos\delta_\infty}{M_\ell} e^{-\frac{(\theta_\ell - \delta_\infty)}{\sqrt{\lambda_\infty}}}, \quad K_2 = \frac{\cos\delta_\infty}{M_r} e^{\frac{(\theta_\ell + \delta_\infty)}{\sqrt{\lambda_\infty}}},$$

such that the function α is obviously continuous globally.

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Chapitre 3

Existence of plane wave like solutions for a nonlinear parabolic equation in dimension one

This work has been written in collaboration with Régis Monneau (CERMICS, ENPC).

Abstract

In this paper we consider a reaction-diffusion equation in a periodic medium. This is a parabolic semilinear equation in dimension one. The nonlinearity is supposed to be periodic in space and in the state variable. Under certain assumptions, we construct plane wave like solutions of this problem, and show that these solutions satisfy a constraining structure property.

AMS Classification : 35K55
Keywords : periodic medium, semilinear equations, plane-like solutions.

3.1 Introduction

3.1.1 Setting of the problem

We consider solutions $u(x,t)$ of the following parabolic semilinear equation for $x \in \mathbb{R}$, $t > 0$:

$$u_t = u_{xx} + f(x,u), \tag{3.1}$$

for some function f which is 1-periodic in x and u.

Our goal is to construct plane-like solutions satisfying a pseudo-periodicity property, that is

$$u\left(x+1, t-\frac{p}{\lambda}\right)=u(x,t)$$

for a couple $(p,\lambda)\in\mathbb{R}^2$. To this end, we assume that the function f satisfies the following conditions :

Lipschitz regularity and $\mathbb{Z}$-periodicity properties :

$$\left\{\begin{array}{l} f\in Lip(\mathbb{R}^2;\mathbb{R}) \\ f(x+k,v+l)=f(x,v) \quad \text{for all} \quad (k,l)\in\mathbb{Z}^2, \quad (x,v)\in\mathbb{R}^2 \end{array}\right. \tag{3.2}$$

3.1.2 Main results

Theorem 3.1. (Existence of plane-like solutions with periodicity property)
Under assumption (3.2), for a given $p>0$ with $p^{-1}\in\mathbb{N}$, there exists a unique $\lambda\in\mathbb{R}$ such that there exists a constant $C>0$ independent of p and a solution u of (3.1) on $\mathbb{R}\times\mathbb{R}$ which satisfy the following properties

$$\left\{\begin{array}{ll} |u(x,t)-px-\lambda t|\le C & (i) \\ u\left(x+\dfrac{1}{p},t\right)=1+u(x,t) & (ii) \\ \lambda u_t\ge 0 & (iii) \\ u(x+1,t)\ge u(x,t) & (iv). \end{array}\right. \tag{3.3}$$

Moreover

$$\left\{\begin{array}{lll} u\left(x+1,t-\dfrac{p}{\lambda}\right)=u(x,t) & \text{if} \quad \lambda\neq 0 & (v) \\ u_t=0 & \text{if} \quad \lambda=0 & (v)'. \end{array}\right. \tag{3.4}$$

Finally, the constants $\lambda(p)$ are bounded independently of p.

Proposition 3.2. *We fix $p>0$ and consider a solution u of (3.1) as in Theorem 3.1. Then λ is such that if $\lambda\neq 0$, then*

$$\int_{[0,1]^2} f\neq 0$$

and

$$\lambda\int_{[0,1]^2} f>0.$$

Remark 3.3.
The hypothesis $p^{-1}\in\mathbb{N}$ is a working hypothesis which is useful to show (ii). It sould be possible to show the same results for $p\in\mathbb{Q}^*$ with the same arguments

Proposition 3.4. (A relaxed monotonicity property for plane-like solutions)

We define $\mathcal{Z}$ the set of values of u which are zeros of f independently of x, that is

$$\mathcal{Z} = \{u \in \mathbb{R} \mid \forall x \in \mathbb{R}, \quad f(x,u) = 0\}.$$

Then for a given $p > 0$, there exists a solution u of (3.1) *and $\lambda \in \mathbb{R}$, $C > 0$ satisfying* (3.3),(3.4) *such that the following property holds :*

$$\forall a \in \mathcal{Z}, \quad (u(x,t) < a < u(y,t)) \Rightarrow (x < y)\,. \tag{3.5}$$

3.1.3 Review of the literature

The homogeneous reaction-diffusion equation

$$u_t = u_{xx} + u(1-u)$$

was first studied in the pioneering papers of Kolmogorov, Petrovsky and Piskunov [11] and Fisher [10]. Many works have been dedicated to the existence of travelling front solutions of this kind of equations of the form $u(x,t) = U(x+ct)$ with c the velocity of the wave. In the case of a periodic medium, when the nonlinearity depends on the space variable, we have the existence of pulsating travelling fronts for particular forms of the nonlinearity f.

Here we prove the existence of plane wave like solutions of this equation, that is solutions that are at a bounded distance of a plane wave. The existence and regularity of plane-like minimizers of functionals in periodic media has been shown for functionals of specific forms in a stationary (that is, time independent) setting (see Caffarelli and De La Llave [4], and then Valdinoci [11], De La Llave and Valdinoci [8], and Davila [7]). When the solutions we construct are time independent, which is the case when the mean of f is zero, this paper gives another proof for the existence of a stationary minimizer in some of the cases they study. It must be noted that contrary to these articles, our study remains limited to smooth solutions.

A paper of Blass, De La Llave and Valdinoci [3] studies the same problem of plane-like minimizers, but it does so by studying a gradient semi-flow on an instationary equation, introducing a parameter that could be viewed as a time variable. As such, our parabolic equation (3.1) could be seen as a particular case of the general equations studied there. However, the article does not try to build plane-like solutions of the instationary equation and focuses on the proof of comparison principles for solutions of relatively low regularity.

3.1.4 Organization of the paper

Section 3.2 gives known results for viscosity solutions and smooth solutions of the parabolic equation (3.1) with the possible addition of a nonlocal term (given by an integral with a smooth kernel). In Section 3.3, we show regularity properties for the viscosity solutions of such equations. In Section 3.4, we introduce the approached equation (3.11) with a nonlocal term, and we show that with suitable initial conditions the solution of the Cauchy problem is periodic in space and satisfies (3.3) (i). This approach is a way to bound the space oscillations of the solutions and obtain the bound (3.3) (i). We then show the existence of plane-like solutions for the approached equation (3.11) in Section 3.5. With the help of this construction, we then show in Section 3.6 the existence of plane-like solutions for equation (3.1). Section 3.7 is devoted to the demonstration of the structure property Proposition 3.4. The short Section 3.8 proves the link between the sign of λ and the mean of f expressed in Proposition 3.2.

3.2 Toolbox

In Section 3.4 and Section 3.5, we will use nonlocal operators defined for $v \in L^\infty_{loc}(\mathbb{R})$ by

$$M[v](x) = \int_{\mathbb{R}} J(y)(v(x+y) - v(x))dy$$

with J a smooth, symmetric, nonnegative function with a compact support $[-\Lambda, \Lambda]$. To simplify notations in the remainder of the paper, we define the operator A_ε for $\varepsilon \geq 0$ by

$$(A_\varepsilon u)(x,t) = f(x, u(x,t)) + \varepsilon M[u(\cdot,t)](x)$$

where we always assume that f satisfies

$$f \in W^{1,\infty}(\mathbb{R}^2) \quad \text{and } f \text{ is } \mathbb{Z}^2\text{-periodic.} \tag{3.6}$$

In particular, when $\varepsilon = 0$, we have

$$(A_0 u)(x,t) = f(x, u(x,t)).$$

In this toolbox, we will write I a general time interval, which we will suppose open. We will consider general equations

$$u_t = u_{xx} + A_\varepsilon u. \tag{3.7}$$

3.2.1 Viscosity solutions

We refer to [8, 1] for usual definitions of viscosity solutions. Here because our nonlocal operator is very regular, we do not need to use the test function in it. (More precisely, we can show that in the case of this nonlocal operator, our definition of viscosity solutions is equivalent to the usual definition with nonlocal operators.)

Therefore, we will say that u is a subsolution to equation (3.7) on $\mathbb{R} \times I$ if for any $(x_0, t_0) \in \mathbb{R} \times I$ and any C^2 test function φ satisfying for some arbitrary $r > 0$

$$u^* \leq \varphi \quad \text{on} \quad \mathbb{R} \times (t_0 - r, t_0 + r), \quad \text{with equality at } P_0 = (x_0, t_0),$$

then

$$\varphi_t \leq \varphi_{xx} + A_\varepsilon u \quad \text{at } P_0.$$

We define supersolutions similarly (with u^* replaced by u_*).

Proposition 3.5. (Existence by Perron's method)
Let u^+ (resp. u^-) be a viscosity supersolution (resp. subsolution) of (3.7) on $\mathbb{R} \times I$, for $\varepsilon \geq 0$, satisfying $u^- \leq u^+$. Then there exists a viscosity solution u of (3.7) on $\mathbb{R} \times I$ such that

$$u^- \leq u \leq u^+.$$

Proposition 3.6. (Global comparison principle for viscosity solutions)
Let $T > 0$. Let u_1 be a viscosity subsolution and u_2 a viscosity supersolution of (3.7) on $Q = \mathbb{R} \times (0, T)$, for $\varepsilon \geq 0$. If $u_1 \leq u_2$ at $t = 0$, then $u_1 \leq u_2$ on Q.

Proposition 3.7. (Additivity of viscosity solutions)
Here $I = \mathbb{R}$ or $(0, +\infty)$. We consider g^i for $i = 1, 2$, two functions in $C^0(\mathbb{R} \times I)$. Let u^i for $i = 1, 2$ be viscosity solutions of the following equations :

$$u^i_t - u^i_{xx} = g^i \quad \text{on} \quad \mathbb{R} \times I.$$

We define $u = u^1 + u^2$ and $g = g^1 + g^2$. Then u is a viscosity solution of

$$u_t - u_{xx} = g \quad \text{on} \quad \mathbb{R} \times I.$$

3.2.2 Classical solutions

Definition 3.8. (Parabolic boundary)
Let Q be a connected subset of $\mathbb{R}_x \times \mathbb{R}_t$. We define its parabolic boundary $\partial^p Q$ (see [10], chapter 2, p. 7) as follows :
$\partial^p Q$ is the set of all points $(x_0, t_0) \in \partial Q$ such that for any $\varepsilon > 0$, the cylinder $(x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \varepsilon, t_0)$ contains points not in Q.

Proposition 3.9. (Local comparison principle)
Let Q be a connected open subset of $\mathbb{R} \times I$ with I a time interval. Let us denote by $\partial^p Q$ its parabolic boundary. For $\varepsilon = 0$, let u_1 be a subsolution and u_2 a supersolution of (3.7) on Q which are assumed to belong to $C^{2,1}(Q) \cap C(\overline{Q})$. If $u_1 \leq u_2$ on $\partial^p Q$, then $u_1 \leq u_2$ on Q.

This proposition is proved in [10], chapter 2, Corollary 2.5 p. 9.

In the next propositions, we will state regularity properties for solutions of (3.7) based on interior estimates. To that effect, we define the parbolic closed cylinders

$$Q_r(x,t) = [x-r, x+r] \times [t-r^2, t] \subset \mathbb{R}^2$$

for $(x,t) \in \mathbb{R}^2$.

Proposition 3.10. (Interior estimates in $W^{2,1;p}$ spaces)
Let u be a solution (in the distribution sense) of (3.7) in $\mathbb{R} \times I$.
We fix $r > 0$ and $1 < p < +\infty$. Then there exists a constant $C > 0$ such that for all $(x,t) \in \mathbb{R} \times I$,

$$\|u\|_{W^{2,1;p}(Q_r(x,t))} \leq C \left(\|A_\varepsilon u\|_{L^p(Q_{2r}(x,t))} + \|u\|_{L^p(Q_{2r}(x,t))} \right)$$

as soon as $Q_{2r}(x,t) \subset \mathbb{R} \times I$.

Proposition 3.11. (Interior estimates in Hölder spaces)
Let u be a solution (in the distribution sense) of (3.7) in $\mathbb{R} \times I$. We fix $r > 0$ and $0 < \alpha < 1$. Then there exists a constant $C > 0$ such that for all $(x,t) \in \mathbb{R} \times I$,

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_r(x,t))} \leq C \left(\|A_\varepsilon u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{2r}(x,t))} + \|u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_{2r}(x,t))} \right)$$

as soon as $Q_{2r}(x,t) \subset \mathbb{R} \times I$.

This proposition derives from [9], Theorem 5.4 p. 448.

Proposition 3.12. (Sobolev injection in a parabolic setting)
Consider $3 < p < +\infty$ and $\alpha < 1 - \dfrac{3}{p}$ and fix $r > 0$. Then for all $(x,t) \in \mathbb{R}^2$ and for all $u \in W^{2,1;p}(Q_r(x,t))$, we have that $u \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_r(x,t))$ and there exists a constant C, independent on (x,t), such that

$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_r(x,t))} \leq C \|u\|_{W^{2,1;p}(Q_r(x,t))}.$$

This proposition is shown in [9], Lemma 3.3 p. 80.

Proposition 3.13. (Strong maximum principle for strong solutions)
Let $Q \subset \mathbb{R} \times I$ be a space-time cylinder, subset of $\mathbb{R} \times I$. Let u_1 and u_2 be two strong solutions of

$$u_t = \Delta u + A_\varepsilon u$$

on Q, satisfying

$$\begin{cases} u_1 \leq u_2 & \text{on} \quad Q \\ u_1(x_0,t_0) = u_2(x_0,t_0) & \text{for some} \quad (x_0,t_0) \in Q. \end{cases}$$

Then $u_1 = u_2$ on Q.

Such a strong version of the strong maximum principle, enabling general nonlinearities, is stated in [1]. Actually, the usual formulation of the strong maximum principle (as given in [1]) only gives that $u_1(x,t) = u_2(x,t)$ for $t < t_0$. Here, as we consider regular solutions and the domain is a cylinder, the comparison principle gives $u_1(x,t) = u_2(x,t)$ also when $t \geq t_0$.

Proposition 3.14. (Existence of a viscosity solution for the Cauchy problem and bounds)
Here $I = [0, +\infty)$. We write $u_0(x) = px$ with $p \in \mathbb{R}$. Then the problem

$$\begin{cases} u_t = u_{xx} + A_\varepsilon u & \textit{on } \mathbb{R} \times I \\ u(x,0) = u_0(x) & \textit{on } \mathbb{R} \end{cases}$$

admits a unique solution u in the viscosity sense.
Moreover, there exists $C > 0$ such that $u_0 - Ct \leq u \leq u_0 + Ct$.

Proof of Proposition 3.14
This proposition follows from the fact that $u_0 + \|f\|_\infty t$ and $u_0 - \|f\|_\infty t$ are respectively a supersolution and a subsolution of (3.7) on $\mathbb{R} \times (0, +\infty)$ with the initial condition u_0.

3.3 Preliminary results

Proposition 3.15. (Hölder regularity of viscosity solutions)
Here $I = (0, +\infty)$ or $I = \mathbb{R}$. Let $p \in \mathbb{R}$ and consider a continuous viscosity solution u of the problem

$$\begin{cases} u_t = \Delta u + A_\varepsilon u & \textit{on} \quad \mathbb{R} \times I \\ u(x,0) = u_0(x) = px & \textit{on} \quad \mathbb{R}. \end{cases} \tag{3.8}$$

when $I = (0, +\infty)$, or a viscosity solution u of

$$u_t = \Delta u + A_\varepsilon u \textit{ on } \quad \mathbb{R} \times I \tag{3.9}$$

when $I = \mathbb{R}$.
Then $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(K) \cap C^{2,1}_{loc}(\mathbb{R} \times I)$ for all compact cylinders $K = Q_r(x_0, t_0)$ (with t_0 large enough in the case $I = (0, +\infty)$) and for some $\alpha > 0$. Moreover, u has the following Hölder bounds :

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(K)} \leq C_{\alpha, r} \left(\|A_\varepsilon u\|_{L^\infty(K')} + \|u\|_{L^\infty(K')} \right)$$

with

$$K' = [x_0 - 2r' - 1, x_0 + 2r' + 1] \times [t_0 - 4r'^2 - 1, t_0 + 1]$$

and

$$r' = \begin{cases} 2r + \Lambda & \textit{when } \varepsilon > 0 \\ 2r & \textit{when } \varepsilon = 0 \end{cases} .$$

Proof of Proposition 3.15

The proof is based on a bootstrap technique and the use of interior estimates in which we derive strong estimates on a solution u in a small box from weaker estimates in a larger box. In addition to the notation

$$Q_r(x_0,t_0) = [x_0 - r, x_0 + r] \times [t_0 - r^2, t_0],$$

we will use another notation for cylinders

$$Q'_r(x_0,t_0) = [x_0 - r - 1, x_0 + r + 1] \times [t_0 - r^2 - 1, t_0 + 1].$$

The second notation allows us to take into account the smearing created by a convolution with a smooth kernel.

Step 1 : Setting of the proof

We consider u a viscosity solution of (3.8) or (3.9).

Now we define $g = f(x,u) + \varepsilon M[u](x)$, where the small quantity ε is set to be zero when $Au = f(x,u)$. The function g is continuous, and if there exists $\lambda \in \mathbb{R}$ such that $u - px - \lambda t$ is bounded, then g is also bounded. Indeed, in that case $g = f(x,u) + \varepsilon M[u - px - \lambda t]$ and

$$\|g\|_\infty \leq \|f\|_\infty + 2\|J\|_{L^1(\mathbb{R})}\|u - px - \lambda t\|_{L^\infty(\mathbb{R}\times I)}.$$

By our definition u is a viscosity solution of

$$u_t - u_{xx} = g \text{ in } \mathbb{R} \times I.$$

Step 2 : Regularization of u

We consider a function $\rho \in C^\infty(\mathbb{R}\times\mathbb{R})$ such that

$$\rho > 0, \quad \rho(x) = \rho(-x), \quad \int_{\mathbb{R}} \rho = 1, \quad supp(\rho) \in B_1(0,0).$$

With these properties, the sequence $(\rho_\eta)_{\eta>0}$, with $\rho_\eta(x,t) = \dfrac{1}{\eta^2}\rho\left(\dfrac{x}{\eta}, \dfrac{t}{\eta}\right)$, is an approximate identity for the convolution.

We define $u_\eta = \rho_\eta \star u$ and $g_\eta = \rho_\eta \star g$. We have $u_\eta, g_\eta \in C^\infty(\mathbb{R}\times I_\eta)$, where $I_\eta = \mathbb{R}$ when $I = \mathbb{R}$ and $I_\eta = [\eta, +\infty)$ when $I = [0, +\infty)$. Viewing g_η and u_η as limits of weighted sums of translations of g and u, we can use the additivity of viscosity solutions (Proposition 3.7) to show that u_η is a viscosity solution of

$$(u_\eta)_t - (u_\eta)_{xx} = g_\eta \text{ in } \mathbb{R} \times I_\eta.$$

Step 3 : Estimates on regular solutions u_η

As the functions u_η and g_η are regular, we can use regularity theories to estimate the derivatives of u_η. For a point $P_0 = (x_0, t_0)$, we have that

$Q_r(P_0) \subset \mathbb{R} \times I$ unconditionally when $I = \mathbb{R}$ and as soon as $t_0 > r^2$ when $I = [0, +\infty)$. We consider $P_0 \in \mathbb{R} \times I$ such that $Q_r(P_0) \subset \mathbb{R} \times I$ and $Q_{2r}(P_0) \subset \mathbb{R} \times I$. Then the interior estimates theorem in $W^{2,1;p}$ classes (Proposition 3.10) gives us that

$$\begin{aligned}&\max\{\|u_\eta\|_{L^p(Q_r(P_0))}, \|(u_\eta)_t\|_{L^p(Q_r(P_0))}, \|(u_\eta)_x\|_{L^p(Q_r(P_0))}, \|(u_\eta)_{xx}\|_{L^p(Q_r(P_0))}\}\\ &\quad \leq C_{p,r}\left(\|g_\eta\|_{L^p(Q_{2r}(P_0))} + \|u_\eta\|_{L^p(Q_{2r}(P_0))}\right),\end{aligned}$$

the constant $C_{p,r}$ being independent of P_0. As g_η and u_η are continuous, we may also write

$$\begin{aligned}&\max\{\|u_\eta\|_{L^p(Q_r(P_0))}, \|(u_\eta)_t\|_{L^p(Q_r(P_0))}, \|(u_\eta)_x\|_{L^p(Q_r(P_0))}, \|(u_\eta)_{xx}\|_{L^p(Q_r(P_0))}\}\\ &\quad \leq C'_{p,r}\left(\|g_\eta\|_{L^\infty(Q_{2r}(P_0))} + \|u_\eta\|_{L^\infty(Q_{2r}(P_0))}\right).\end{aligned}$$

When $\eta < 1$, for a any continuous function $h \in C^0(\mathbb{R} \times I)$, we have

$$\|h_\eta\|_{L^\infty(Q_{2r}(P_0))} \leq \|h\|_{L^\infty(Q'_{2r}(P_0))}$$

as soon as $Q'_{2r}(P_0) = [x_0 - 2r - 1, x_0 + 2r + 1] \times [t_0 - 4r^2 - 1, t_0 + 1] \subset \mathbb{R} \times I$.

Using the Sobolev embedding $W^{2,1;p}(Q_r) \to C^{1+\alpha, \frac{1+\alpha}{2}}(Q_r)$ with $\alpha = 1 - \dfrac{3}{p}$, we have that

$$\|u_\eta\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_r)} \leq C'_{p,r}\left(\|g\|_{L^\infty(Q'_{2r}(P_0))} + \|u\|_{L^\infty(Q'_{2r}(P_0))}\right).$$

To have the Hölder continuity of u_η, it is then sufficient to take $p \geq 4$.

Step 4 : Analysis of the limit $\eta \to 0$ and regularity of u

The remaining estimations are derived directly on u (the usual bootstrap technique does not work directly on u_η because of the convolution). We take the limit as $\eta \to 0$ of the equation

$$(u_\eta)_t - (u_\eta)_{xx} = g_\eta$$

in the distribution sense. Then we recall that from the construction of u_η, u_η converges to u uniformly, so the derivatives converge in the distribution sense. As g is continuous, g is also the limit of the functions g_η for the uniform convergence on every compact subset.

The functions u_η are uniformly bounded in $C^{1+\alpha,\frac{1+\alpha}{2}}(Q_r(P_0))$, the bounds depending only on $\|u\|_{L^\infty(Q'_{2r}(P_0))}$, $\|g\|_{L^\infty(Q'_{2r}(P_0))}$, r and α (in particular they are independent of P_0). As a consequence, for any $\alpha' < \alpha$, $u_\eta \to u$ in $C^{1+\alpha',\frac{1+\alpha'}{2}}(Q_r(P_0))$ as soon as $Q'_{2r}(P_0) \in \mathbb{R} \times I_{\eta'}$ for some $\eta' > 0$ and u has similar bounds in $C^{1+\alpha',\frac{1+\alpha'}{2}}(Q_r(P_0))$.

As $M[u]$ has the same regularity on $[x_1, x_2] \times [t_1, t_2]$ as u on $[x_1 - \Lambda, x_2 + \Lambda] \times [t_1, t_2]$ and f is Lipschitz continuous, the function $g = f(u, x) + \varepsilon M[u]$

is bounded in $C^{1,\frac{1+\alpha'}{2}}(Q_r(P_0)) \subset C^{\beta,\frac{\beta}{2}}(Q_r(P_0))$ for some $\beta < 1$, the bounds depending only on the quantities $\|u\|_{C^{1+\alpha',\frac{1+\alpha'}{2}}([x_0-r-\Lambda,x_0+r+\Lambda]\times[t_0-r^2,t_0])}$, $\|g\|_{C^{1+\alpha',\frac{1+\alpha'}{2}}([x_0-r-\Lambda,x_0+r+\Lambda]\times[t_0-4r^2,t_0])}$, r and β.

For $r' = r+\Lambda$, we have $[x_0 - r - \Lambda, x_0 + r + \Lambda] \times [t_0 - 4r^2, t_0] \subset Q_{r'}(P_0)$, so using the previous Hölder bounds on u and g we get that the bound of g in $C^{\beta,\frac{\beta}{2}}(Q_r(P_0))$ depends only on $\|u\|_{L^\infty(Q'_{2r'}(P_0))}$, $\|g\|_{L^\infty(Q'_{2r'}(P_0))}$, r and β.

We can then use the Hölder interior estimates (Proposition 3.11) to derive stronger bounds on u :

$$\begin{aligned}&\max\{\|u\|_{C^{\beta,\frac{\beta}{2}}(Q_r(P_0))}, \|u_t\|_{C^{\beta,\frac{\beta}{2}}(Q_r(P_0))}, \|u_x\|_{C^{\beta,\frac{\beta}{2}}(Q_r(P_0))}, \|u_{xx}\|_{C^{\beta,\frac{\beta}{2}}(Q_r(P_0))}\}\\ &\leq C^H_{\beta,r}\left(\|g\|_{C^{\beta,\frac{\beta}{2}}(Q_{2r}(P_0))} + \|u\|_{C^{\beta,\frac{\beta}{2}}(Q_{2r}(P_0))}\right).\end{aligned}$$

We conclude that, for $r' = 2r + \Lambda$,

$$\|u\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_r(P_0))} \leq C_{\beta,r}\left(\|g\|_{L^\infty(Q'_{2r'}(P_0))} + \|u\|_{L^\infty(Q'_{2r'}(P_0))}\right).$$

The value of r can be as small as one want (even if the constants become high), so this means that $u \in C^{2,1}(\mathbb{R} \times \mathring{I})$.

When $\varepsilon = 0$, there is no need to take into account the regularity of $M[u]$, and $g = Au$ is bounded in $\mathbb{R} \times I$ by $\|f\|_\infty$ so we have the estimate

$$\|u\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_r(P_0))} \leq C'_{\beta,r}\left(\|f\|_\infty + \|u\|_{L^\infty(Q'_{2r'}(P_0))}\right),$$

now with $r' = 2r$.

Proposition 3.16. (Hölder bounds for solutions of the form "linear+bounded")
We consider a solution u of the problem (3.8) or (3.9) as above. Moreover, we suppose that for some $\lambda \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$|u - px - \lambda t| \leq C.$$

Then u has the following (stricter) bounds in $C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(Q_r(P_0))$ for some $\alpha > 0$:

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_r(P_0))} \leq \|u\|_{C^0(Q_r(P_0))} + C_4$$

uniformly for $P_0 \in \mathbb{R} \times I$ with t_0 large enough.

Proof of Proposition 3.16

If $|u - px - \lambda t| \leq C$, then $A_\varepsilon u = f(x,u) + \varepsilon M[u]$ is uniformly bounded on $\mathbb{R} \times I$, because M is zero for constants and linear functions. Moreover, if u is a solution of (3.8) (without the initial condition) or (3.9), then $u + n$ is also a solution of (3.8) or (3.9) for $n \in \mathbb{Z}$. As a consequence, for all $n \in \mathbb{Z}$, and taking $r' = 2r + \Lambda$, we get

$$\|u-n\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_r(P_0))} \leq C'_{\beta,r}(C_1 + \|u-n\|_{L^\infty([x_0-2r'-1,x_0+2r'+1]\times[t_0-4r'^2-1,t_0+1])}).$$

We note $K_1 = [x_0 - 2r' - 1, x_0 + 2r' + 1] \times [t_0 - 4r'^2 - 1, t_0 + 1]$ and take $n = \lfloor \frac{1}{|K_1|}\int_{K_1} u \rfloor$ where $\lfloor \cdot \rfloor$ is the floor function. The bound on $u - px - \lambda t$ implies that

$$|u-n| \leq 1 + p(4r'+2) + |\lambda|(4r'^2+2) = C_2$$

$$\text{on} \quad [x_0 - 2r' - 1, x_0 + 2r' + 1] \times [t_0 - 4r'^2 - 1, t_0 + 1].$$

Then for a fixed r, there exists a constant C_3 (possibly depending on C, r, β and Λ) such that for all P_0 with t_0 large enough, there exists $n \in \mathbb{Z}$ such that :

$$\|u-n\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_r(P_0))} \leq C_3.$$

But we can remark that

$$\|u-n\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_r(P_0))} = \|u-n\|_{C^0(Q_r(P_0))} + \left(\|u\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_r(P_0))} - \|u\|_{C^0(Q_r(P_0))} \right)$$

because the C^1, C^2 seminorms or the Hölder brackets of $(u-n)$ do not depend on n. As a result, we have that

$$\|u\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_r(P_0))} \leq \|u\|_{C^0(Q_r(P_0))} + C_4$$

uniformly for $P_0 \in \mathbb{R} \times I$ with t_0 large enough (and with $C_4 = 2C_3$).

When $\varepsilon = 0$, we can take $r' = 2r + 1$ and C_4 does not depend on Λ any more.

Corollary 3.17. Regularity of the time derivative of u

In the case $\varepsilon = 0$, consider u a solution of (3.9) *and suppose f is of class C^1 on the range of u. Then u_t is bounded in $C^{2+\alpha,1+\frac{\alpha}{2}}(K)$ for all compacts K and for some $\alpha > 0$, and it is a solution of the equation*

$$(u_t)_t = (u_t)_{xx} + f'_u(x,u)u_t. \tag{3.10}$$

Proof of Corollary 3.17

Taking the derivative of equation (3.9) gives that u_t satisfies (3.10) in the distibution sense.

Proposition 3.15 gives us that u_t is bounded in $C^{\alpha,\frac{\alpha}{2}}(K)$ for all compacts K and as u is regular enough, $f'_u(x,u)$ is Lipschitz in (x,t). Then by Proposition 3.11, u_t is bounded in $C^{2+\alpha,1+\frac{\alpha}{2}}(K)$ for all compacts K, and as a consequence it is a strong solution of (3.10).

3.4 Analysis of the Cauchy problem with a nonlocal term

Our strategy to prove Theorem 3.1 is to solve the Cauchy problem and pass to the limit as the time variable goes to infinity. We do not know how to do it directly. In order to get enough control on the solution, we introduce the following approached problem for some $\varepsilon > 0$:

$$u_t = \Delta u + f(u,x) + \varepsilon M[u(\cdot,t)] \tag{3.11}$$

where for any given $p > 0$ with $p^{-1} \in \mathbb{N}$, the nonlocal operator $M = M_p$ depends on p and is defined by

$$M[v](x) = M_p[v](x) = \int_{\mathbb{R}} J_p(y)(v(x+y) - v(x))dy,$$

with J_p a smooth nonnegative symmetric function satisfying

$$\begin{cases} J_p(x) = 0 & \text{for } |x| > \dfrac{1}{p} \\ J_p(x) > \dfrac{1}{2} & \text{for } |x| < \dfrac{1}{2p}. \end{cases} \tag{3.12}$$

Proposition 3.18. (Control of the solution of the Cauchy problem) *Let u be the solution of* (3.11) *for $\varepsilon > 0$, with initial condition*

$$u(x,0) = px \tag{3.13}$$

where $p > 0$ with $p^{-1} \in \mathbb{N}$. Then u satisfies (3.3) *parts* (*i*) *and* (*ii*) *on $\mathbb{R} \times \mathbb{R}_+$ for some constants C_ε and λ_ε.*

Proof of Proposition 3.18

Step 0 : Construction of barriers

Let us consider the functions

$$u^{\pm}(x,t) = px \pm Ct \quad \text{with} \quad C = ||f||_\infty.$$

Because the operator M vanishes on affine functions, we deduce that u^+ (resp. u^-) is a viscosity supersolution (resp. subsolution) of (3.11) with initial condition (3.13). From Proposition 3.5, we deduce that there exists a viscosity solution u of (3.11), (3.13) satisfying

$$u^- \leq u \leq u^+. \tag{3.14}$$

This means that $(u)^*$ is a subsolution and $(u)_*$ is a supersolution. The comparison principle (Proposition 3.6) implies that $(u)^* \leq (u)_*$ which shows that the solution u is continuous (and is unique).

Step 1 : Proof of (ii) **for the solution of** (3.11)**,** (3.13)

Let

$$\tilde{u}(x,t) = u\left(x + \frac{1}{p}, t\right) - 1.$$

As $p^{-1} \in \mathbb{N}$, $\tilde{u}$ is a solution of (3.11), because this equation is invariant by addition of integers and by integer translations in space. Moreover, $\tilde{u}(x,0) = px = u(x,0)$. By the comparison principle (Proposition 3.6), $\tilde{u} = u$ and then u satisfies (ii).
As $u_0(x+1) > u_0(x)$ and u and $u(\cdot+1,\cdot)$ are solutions of the same equation, by the comparison principle $u(x+1,t) \geq u(x,t)$.

Step 2 : Control of the space oscillations for the solution of (3.11)**,** (3.13)

The goal of this step is to show that there exists $C'_\varepsilon > 0$ such that for all $t > 0$, for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, we have

$$|u(x+y,t) - u(x,t) - py| \leq C'_\varepsilon. \tag{3.15}$$

Substep 2.1 : Inequations on the extrema of v
We define $v(x,t) = u(x,t) - px$. So v satisfies

$$\begin{cases} v_t = \Delta v + f(x, v+px) + \varepsilon M[v(\cdot,t)] \\ v(x,0) = 0. \end{cases}$$

As u is a continuous viscosity solution of (3.11), by Proposition 3.15, u is a regular solution of (3.11), this equation on v is satified in the sense of strong solutions. Notice that (3.14) implies $|v| \leq Ct$. We then define

$$\begin{aligned} \overline{m}(t) &= \sup_{x\in\mathbb{R}} v(x,t) \\ \underline{m}(t) &= \inf_{x\in\mathbb{R}} v(x,t) \\ \sigma(t) &= \overline{m}(t) - \underline{m}(t). \end{aligned}$$

The Lipschitz regularity of v also guarantee that $\overline{m}$ and $\underline{m}$ are at least locally Lipschitz continuous. Using (3.3),(ii), we can remark that

$$v\left(x + \frac{1}{p}, t\right) = u\left(x + \frac{1}{p}, t\right) - px - 1 = u(x,t) + 1 - px - 1 = v(x,t)$$

so v is $\dfrac{1}{p}$-periodic in x so at a given time t both extrema are reached for finite values of x, and we can suppose

$$\begin{aligned} \overline{m}(t) &= v(Y_t, t) \\ \underline{m}(t) &= v(y_t, t) \end{aligned}$$

with $|Y_t|, |y_t| \leq \dfrac{1}{2p}$. We do not know if $\overline{m}$ and $\underline{m}$ are differentiable, but we will write

$$\overline{m}' = \limsup_{h\to 0,\ h\neq 0} \frac{\overline{m}(t+h) - \overline{m}(t)}{h}$$

and

$$\underline{m}' = \liminf_{h\to 0,\ h\neq 0} \frac{\underline{m}(t+h) - \underline{m}(t)}{h}.$$

We will particularly study $\overline{m}'$, the reasoning is the same for $\underline{m}'$.

At a given time t, we note $\mathcal{Y}_t$ the set of maximizers of v at time t in the interval $\left[-\dfrac{1}{2p}, \dfrac{1}{2p}\right]$. Now for all $h \neq 0$, we consider a point $Y_{t+h} \in \mathcal{Y}_{t+h}$. A compactness arument shows that any subsequence of (Y_{t+h}) with $h \to 0$ has a converging subsequence. We consider a sequence $h_n \to 0$ such that Y_{t+h_n} converges and we call Y the limit. As v is continuous, Y is a maximizer of v at time t, i.e. $Y \in \mathcal{Y}_t$.

Then for all n, as Y is a maximizer at time t,

$$v(Y_{t+h_n}, t+h_n) - v(Y,t) \leq v(Y_{t+h_n}, t+h_n) - v(Y_{t+h_n}, t). \tag{3.16}$$

But we have

$$\limsup_{h\in(h_n)} \frac{v(Y_{t+h}, t+h) - v(Y_{t+h}, t)}{h} \leq \Delta v(Y,t) + f(v(Y,t) - pY) + \varepsilon M[v(\cdot,t)](Y). \tag{3.17}$$

We note that because Y is a maximizer and $v \in C^{2,1}(\mathbb{R}\times(0,+\infty))$, $\Delta v(Y,t) \leq 0$. Combining (3.16) and (3.17), we get

$$\limsup_{h\in(h_n)} \frac{\overline{m}(t+h) - \overline{m}(t)}{h} \leq f(v(Y,t) - pY) + \varepsilon M[v(\cdot,t)](Y).$$

As this is true for all converging subsequence (h_n) and for all accumulation value Y, we get that

$$\limsup_{h\to 0,\ h\neq 0} \frac{\overline{m}(t+h) - \overline{m}(t)}{h} \leq \max_{Y\in\mathcal{Y}_t} \left(f(v(Y,t) - pY) + \varepsilon M[v(\cdot,t)](Y)\right).$$

In particular there exists $Y_t \in \mathcal{Y}_t$ such that

$$\overline{m}'(t) \leq f(v(Y_t,t) - pY_t) + \varepsilon M[v(\cdot,t)](Y_t). \tag{3.18}$$

Similarly, we can show that there exists $y_t \in \mathcal{Y}'_t$ (where $\mathcal{Y}'_t$ is the set of minimizers of v) such that

$$\underline{m}'(t) \geq f(v(y_t,t) - py_t) + \varepsilon M[v(\cdot,t)](y_t). \tag{3.19}$$

Substep 2.2 : Construction of an inequation on the oscillation and conclusion

We can take the difference of the inequalities (3.18) and (3.19), and if we note

$$\sigma' = \limsup_{h\to 0} \frac{\sigma(t+h)-\sigma(t)}{h},$$

we get

$$\begin{aligned}\sigma' \leq\ & f(v(Y_t,t)+pY_t,Y_t) - f(v(y_t,t)+py_t,y_t)\\ &+\varepsilon\int_{\mathbb{R}} J(z)\left(v(Y_t+z,t)-v(Y_t,t)-v(y_t+z,t)+v(y_t,t)\right)dz.\end{aligned}$$

We define $c_t = \dfrac{Y_t+y_t}{2}$, $\delta_t = \dfrac{Y_t-y_t}{2}$, $z' = z+\delta_t$ and $z'' = z-\delta_t$. We note that $Y_t+z = c_t+\delta_t+z = c_t+z'$ and $y_t+z = c_t-\delta_t+z = c_t+z''$. Then we have

$$\begin{aligned}\sigma' \leq\ \ 2\|f\|_\infty\ \ &+\varepsilon\int_{\mathbb{R}} J(z'-\delta_t)\big(v(c_t+z',t)-v(Y_t,t)\big)\,dz'\\ &+\varepsilon\int_{\mathbb{R}} J(z''+\delta_t)\big(v(y_t,t)-v(c_t+z'',t)\big)\,dz''.\end{aligned}$$

By the construction of Y_t and y_t, we have

$$v(c_t+z',t)-v(Y_t,t) \leq 0$$

and

$$v(y_t,t)-v(c_t+z'',t) \leq 0.$$

If we replace the silent variables z' and z'' by the same z, we have

$$\begin{aligned}&\sigma' \leq 2\|f\|_\infty\\ &+\varepsilon\int_{\mathbb{R}}\inf(J(z-\delta_t),J(z+\delta_t))\left(v(y_t,t)-v(c_t+z,t)+v(c_t+z,t)-v(Y_t,t)\right)dz\\ &\quad\leq 2\|f\|_\infty+\varepsilon\int_{\mathbb{R}}\inf(J(z-\delta_t),J(z+\delta_t))(-\sigma(t))dz\\ &\quad\leq 2\|f\|_\infty-\varepsilon\sigma K,\end{aligned}$$

where $K = \displaystyle\int_{\mathbb{R}}\inf(J(z+\delta_t),J(z-\delta_t))dz$. Note that with the periodicity of v we can ensure that $\dfrac{-1}{2p} \leq \delta_t \leq \dfrac{1}{2p}$, so K is positive for $J = J_p$ as defined in (3.12) (using the continuity of J_p).

So finally

$$\sigma \leq \frac{2\|f\|_\infty}{\varepsilon K},$$

which implies (3.15) and then concludes the proof of this second step.

Step 3 : Control of the space-time oscillations (i)

The goal of this step is to show that u satisfies part (i) of (3.3) with constants C_ε and λ_ε possibly dependent of ε.

For $T > 0$,we define

$$\lambda^+(T) = \sup_{t\geq 0} \frac{u(0,t+T)-u(0,t)}{T}$$

and

$$\lambda^-(T) = \inf_{t\geq 0} \frac{u(0,t+T)-u(0,t)}{T}.$$

In the following, we still use the notation $v(x,t) = u(x,t) - px$.M

Substep 3.1 : Bounds on $\lambda^+(T)$ and $\lambda^-(T)$

We first prove that there exists a constant C_1 independent on $T \geq 1$ such that

$$-C_1 \leq \lambda^-(T) \leq \lambda^+(T) \leq C_1. \tag{3.20}$$

It is clear that for all T, $\lambda^+(T) \geq \lambda^-(T)$.

Fix a time $t_0 \geq 0$. We know that $v(\cdot,t_0) \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v(\cdot,t_0)$ is periodic. Then there exists $k \in \mathbb{Z}$ such that $\min v(\cdot,t_0) \in [k,k+1[$. By the oscillation estimate (3.15), it follows that $\max v(\cdot,t_0) \leq (k+1) + C'_\varepsilon$. Equivalently, $0 \leq v(\cdot,t_0) - k \leq C'_\varepsilon + 1$.

Now define the functions

$$v' : \tau \mapsto C'_\varepsilon + 1 + \tau \|f\|_\infty$$

and

$$u'(\tau) = v'(x,\tau) + px.$$

We then have

$$u'_t \geq \Delta u' + f(x,u') + \varepsilon M[u'].$$

This implies that u' is a supersolution of (3.11). We also have that $(x,\tau) \mapsto u(x,t_0+\tau) - 2k$ is a solution of (3.11), since the equation is invariant by time translation and addition of integers. At time $\tau = 0$, we have

$$u'(x,0) = C'_\varepsilon + 1 + px \geq u(x,t_0) - k.$$

From the comparison principle (Proposition 3.6), it follows that for all $\tau \geq 0$,

$$u'(x,\tau) \geq u(x,t_0+\tau) - k.$$

In particular, for $\tau = T$, we have

$$C'_\varepsilon + 1 + T\|f\|_\infty + px \geq u(x,t_0+T) - k.$$

Combined with the fact that $u(0,t_0) - k \geq 0$, this implies that

$$\frac{u(0,t_0+T) - u(0,t_0)}{T} \leq \frac{C'_\varepsilon + 1}{T} + \|f\|_\infty.$$

This inequality remains valid for all $t_0 \geq 0$, so, for $T \geq 1$,

$$\lambda^+(T) \leq C'_\varepsilon + 1 + \|f\|_\infty = C_1. \tag{3.21}$$

Similarly, we can derive the lower bound (3.20) for $\lambda^-(T)$ when $T \geq 1$.

Substep 3.2 : Upper bound on $\lambda^+(T) - \lambda^-(T)$

Consider a positive real number $T > 0$.

We will first study the case where both the infimum $\lambda^-(T)$ and the supremum $\lambda^+(T)$ are reached in $[0, +\infty[$, which means that there exists $t_1 \geq 0$ and $t_2 \geq 0$ such that

$$\lambda^+(T) = \frac{u(0, t_1 + T) - u(0, t_1)}{T}$$

$$\lambda^-(T) = \frac{u(0, t_2 + T) - u(0, t_2)}{T}.$$

Consider the function

$$u'(x, t) = u(x, t + t_2 - t_1) + K$$

with $K = \lfloor u(0, t_1) - u(0, t_2) + 2C'_\varepsilon \rfloor + 1$, where the function $\lfloor \cdot \rfloor$ is the floor function. Then u and u' are both solutions of (3.11). Now for $t = t_1$,

$$u'(x, t_1) = u(x, t_2) + ([u(0, t_1) - u(0, t_2) + 2C'_\varepsilon] + 1) \geq u(x, t_1)$$

where we have used the estimate on space oscillations (3.15).
Then, by the comparison principle, for all $t \geq t_1$,

$$u'(x, t) \geq u(x, t).$$

In particular, for $t = t_1 + T$, we have

$$u(x, t_2 + T) - u(0, t_2) + 1 + 2C'_\varepsilon \geq u(x, t_1 + T) - u(0, t_1),$$

so, taking $x = 0$, we get

$$\lambda^-(T) + \frac{1 + 2C'_\varepsilon}{T} \geq \lambda^+(T),$$

which we can rewrite

$$0 \leq \lambda^+(T) - \lambda^-(T) \leq \frac{1 + 2C'_\varepsilon}{T}. \tag{3.22}$$

Even if the infimum or the supremum is not reached, we have that, for $\eta > 0$, there exists $t_1 \geq 0$ and $t_2 \geq 0$ such that

$$\lambda^+(T) < \frac{u(0, t_1 + T) - u(0, t_1)}{T} + \eta$$

$$\lambda^-(T) > \frac{u(0,t_2+T)-u(0,t_2)}{T} - \eta.$$

The same line of reasoning then leads to

$$u(x,t_2+T)-u(0,t_2)+1+2C'_\varepsilon \geq u(x,t_1+T)-u(0,t_1)$$

so

$$\lambda^-(T)+\eta+\frac{1+2C'_\varepsilon}{T} \geq \lambda^+(T)-\eta.$$

As this is true for all $\eta > 0$, (3.22) still holds in that case.

Substep 3.3 : Evaluation of the variations of λ^+ and λ^-

Now, let $s \in \mathbb{N} \setminus \{0\}$ an integer.

We first assume that there exists $t_1 \geq 0$ such that

$$\lambda^+(pT) = \frac{u(0,t_1+pT)-u(0,t_1)}{pT}.$$

It is clear that

$$sT\lambda^+(sT) = u(0,t_1+sT)-u(0,t_1) \leq sT\lambda^+(T),$$

so

$$\lambda^+(sT) \leq \lambda^+(T).$$

If the supremum $\lambda^+(sT)$ is not reached, then for all $\eta > 0$ there exists a time t_η such that

$$sT\lambda^+(sT)-\eta \leq u(0,t_\eta+sT)-u(0,t_\eta).$$

As above,

$$u(0,t_\eta+sT)-u(0,t_\eta) \leq sT\lambda^+(T),$$

so we have, for all $\eta > 0$,

$$sT\lambda^+(sT)-\eta \leq sT\lambda^+(T),$$

which implies again that $\lambda^+(sT) \leq \lambda^+(T)$. Similarly, it can be proved that $\lambda^-(sT) \geq \lambda^-(T)$.

Now take $T_1 > 0$ and $T_2 > 0$ and assume that $sT_1 = rT_2$, with $s \in \mathbb{N}\setminus\{0\}$ and $r \in \mathbb{N} \setminus \{0\}$. Using the preceding conclusions, we have the following inequalities :

$$\lambda^+(T_1) \geq \lambda^+(sT_1) = \lambda^+(rT_2) \geq \lambda^-(rT_2) \geq \lambda^-(T_2) \geq \lambda^+(T_2) - \frac{1+2C'_\varepsilon}{T_2}$$

so

$$\lambda^+(T_2)-\lambda^+(T_1) \leq \frac{1+2C'_\varepsilon}{T_1}.$$

Exchanging the indices 1 and 2, we see that this implies

$$\left|\lambda^+(T_1) - \lambda^+(T_2)\right| \leq (1+2C'_\varepsilon) \max\left(\frac{1}{T_1}, \frac{1}{T_2}\right) \tag{3.23}$$

for all $T_1, T_2 > 0$ such that $\dfrac{T_1}{T_2} \in \mathbb{Q}$.

The solution u being in $C^{2,1}(\mathbb{R} \times (0, +\infty))$, the function $u(0, \cdot)$ is uniformly continuous; as a consequence the function λ^+ is continuous in T, which implies that the inequalities above remain true for any $T_1, T_2 > 0$.

Using the same method we derive the same property for λ^- : for all $T_1, T_2 > 0$,

$$\left|\lambda^-(T_1) - \lambda^-(T_2)\right| \leq (1+2C'_\varepsilon) \max\left(\frac{1}{T_1}, \frac{1}{T_2}\right). \tag{3.24}$$

Substep 3.4 : Limit of λ^+ and λ^- and end of the proof

Then, if $(T_n)_{n\in\mathbb{N}}$ is an increasing sequence with limit $+\infty$, both sequences $\lambda^+(T_n)$ and $\lambda^-(T_n)$ are Cauchy sequences and then converge. As this is true for any sequence (T_n) with limit $+\infty$, both $\lambda^+(T)$ and $\lambda^-(T)$ have a limit for $T \to \infty$. Furthermore, the inequality (3.22) shows these two limits are equal. Define

$$\lambda = \lim_{T\to\infty} \lambda^+(T) = \lim_{T\to\infty} \lambda^-(T).$$

We consider inequalities (3.23) and (3.24) and take the limit $T_2 \to \infty$ to deduce the following inequalities, for $T > 0$

$$\left|\lambda^+(T) - \lambda\right| \leq \frac{1+2C'_\varepsilon}{T}$$

and

$$\left|\lambda^-(T) - \lambda\right| \leq \frac{1+2C'_\varepsilon}{T}.$$

Now, for all $T > 0$, we deduce

$$\lambda T - (1+2C'_\varepsilon) \leq u(0,T) - u(0,0) \leq \lambda T + (1+2C'_\varepsilon).$$

Combining this with (3.15), we can write

$$\lambda T - (1+3C'_\varepsilon) \leq u(x,T) - u(0,0) \leq \lambda T + (1+3C'_\varepsilon).$$

As we have taken $u(0,0) = 0$, noting $C' = 1 + 3C'_\varepsilon$, we have then proved that, for all $t \geq 0$

$$|u(x,t) - px - \lambda t| \leq C'. \tag{3.25}$$

This proves this third step and finishes the proof of Proposition 3.18.

3.5 Construction of global solutions of the approached problem

We can show that when there exists a solution of the Cauchy problem (3.11),(3.13) with these properties, we can construct a global solution satisfying (3.3) and (3.4) if $\lambda_\varepsilon \neq 0$. This implication remains true when $\varepsilon = 0$, i.e. for the equation (3.1).

Proposition 3.19 (Existence of a global solution and uniform bounds). *We consider a given $p > 0$ with $p^{-1} \in \mathbb{N}$ and $\varepsilon \geq 0$. We suppose that there exists a solution u of the Cauchy problem* (3.11),(3.13) *(or* (3.1),(3.13) *in the case $\varepsilon = 0$) satisfying* (3.3), (i), (ii), (iv).
Then there exists a unique $\lambda_\varepsilon \in \mathbb{R}$ such that there exists a global solution for equation (3.11) *on $\mathbb{R} \times \mathbb{R}$ satisfying* (3.3) *and* (3.4).
Moreover, the constants λ_ε remain bounded uniformly in $\varepsilon \geq 0$ and p, and there exists C independent on ε such that the estimate (3.3) (i) *holds for all $\varepsilon \geq 0$. This result remain true when $\varepsilon = 0$ (in which case u is a global solution of* (3.1)*) and in this case the constant C is also independent on p.*

Proof of Proposition 3.19

The following proof stands for the solutions of both (3.1) and (3.11). Therefore, when we use the notations λ_ε or u_ε, the results stated remain true when $\varepsilon = 0$.

Step 1 : Construction of a global solution of (3.11) **satifying** (3.3), (i), (ii) **and** (iv)

By the regularity result from Proposition 3.16, we have that u is bounded in $C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(Q_r(P))$ for a compact cylinder $Q_r(P)$, and for some $\alpha > 0$, and the bound only depends on the C^0 norm of u and on r. Moreover, u can be extended to a function on $\mathbb{R} \times \mathbb{R}$ which still satisfies the control property

$$|u(x,t) - px - \lambda_\varepsilon t| \leq C_\varepsilon$$

and the same property of bounds on the Hölder norms on compacts, the bounds being uniform with respect to the shape of the compact (the extension can be arbitrary except for the preservation of these two properties).
We define

$$u_n(x,t) = u(x, t+n) - \lfloor \lambda_\varepsilon n \rfloor.$$

We note that for all $n > 0$,

$$\begin{aligned} \|u_n\|_{C^0(B_R(0,0))} &= \|u\|_{C^0(B_R(0,n))} - \lfloor \lambda_\varepsilon n \rfloor \\ &\leq \|u\|_{C^0(B_R(0,0))} + \lambda_\varepsilon n + 2C_\varepsilon - \lfloor \lambda_\varepsilon n \rfloor \\ &\leq \|u\|_{C^0(B_R(0,0))} + 2C_\varepsilon + 1. \end{aligned}$$

Then for all $R > 0$, the u_n have a uniform bound in $C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(B_R(0,0))$ for some $\alpha > 0$, so they can be included in a compact subset of $C^{2+\beta,1+\frac{\beta}{2}}_{x,t}(B_R(0,0))$ for some $\beta < \alpha$, and by a method of diagonal extraction we can construct a function u_∞ and an increasing function $\phi \colon \mathbb{N} \to \mathbb{N}$ such that, for all $R > 0$, $u_\infty \in C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(B_R(0,0))$ and $u_{\phi(n)} \to u_\infty$ in $C^{2+\beta,1+\frac{\beta}{2}}_{x,t}(B_R(0,0))$. As the convergence is in $C^{2+\beta,1+\frac{\beta}{2}}_{x,t}(B_R(0,0))$ for all $R > 0$, u_∞ satisfies (3.11) (or (3.1) when $\varepsilon = 0$), and also (i), (ii) and (iv), which conludes the proof of this first step. In the following, we will note this global solution u.

Step 2 : Proof of (3.4)**,** (v) **for the global solution when** $\lambda_\varepsilon \neq 0$**.**
Case 1 : $\lambda_\varepsilon > 0$

This proof uses the sliding method described in [6]. Figure 3.1 illustrates the principle of the sliding method, which is to compare $u + 1$ and time translations of u.

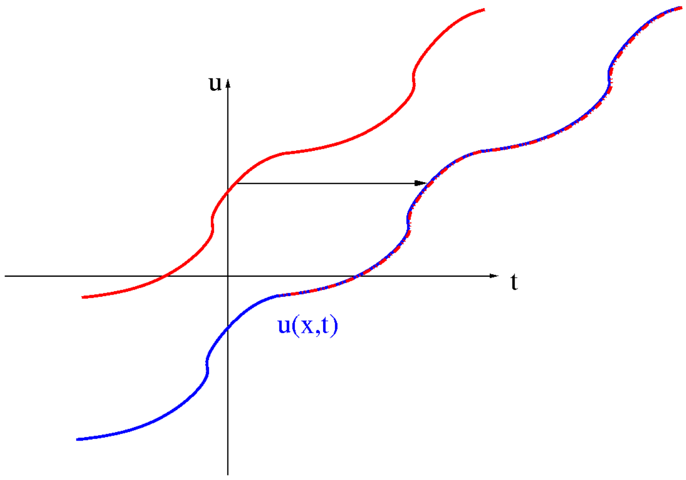

FIGURE 3.1 – Illustration of the sliding method : graphs of $u(x,t)$ and $u(x,t) + 1$ for a fixed value of x.

As $\lambda_\varepsilon > 0$, by (i) we know that there exists $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$, $u(x, t + \alpha) \geq u(x + 1, t)$. In the following we will note $u^\alpha(x,t) = u(x, t + \alpha)$. We define

$$\alpha^* = \inf \left\{ \alpha_0 \geq 0 \mid \forall \alpha \geq \alpha_0, \quad u^\alpha(x,t) \geq u(x+1,t) \right\}.$$

In particular $u^{\alpha^*}(x,t) \geq u(x+1,t)$ and for all $\delta > 0$, there exists $P_\delta = (x_\delta, t_\delta) \in \mathbb{R}^2$ such that $u^{\alpha^*}(x_\delta, t_\delta) < u(x_\delta + 1, t_\delta) + \delta$ (if it was not the case, because u is Lipschitz continuous, we could build $\alpha^{*\prime} < \alpha^*$ such that for all $\alpha \geq \alpha^{*\prime}$, $u^\alpha(x,t) \geq u(x+1,t)$). Here we will distinguish two subcases :
Subcase 1 : There exists $P_0 \in \mathbb{R}^2$ such that $P_\delta \to P_0$ up to a subsequence.

In this case we have

$$\begin{cases} u^{\alpha^*} \geq u(\cdot+1,\cdot) & \text{on } \mathbb{R}^2 \\ u^{\alpha^*} = u(\cdot+1,\cdot) & \text{at } P_0. \end{cases}$$

We write $w = u^{\alpha^*} - u(\cdot+1,\cdot)$. The function w satisfies

$$\begin{aligned} w_t &= \Delta w + f(x,u^{\alpha^*}) - f(x+1,u(x+1,t)) \\ &\quad + M[u^{\alpha^*}](x,t) - M[u(\cdot+1,\cdot)](x,t) \\ &= \Delta w + \mathcal{R}(x,t)w \end{aligned}$$

where

$\mathcal{R}(x,t) =$

$$\frac{f(x,u(x,t+\alpha^*)) - f(x+1,u(x+1,t)) + M[u^{\alpha^*}](x,t) - M[u(\cdot+1,\cdot)](x,t)}{u(x,t+\alpha^*) - u(x+1,t)}.$$

The function 0 is a solution of this equation, and we know that w is non negative and reaches the value 0, so by the strong maximum principle $w=0$ and $u(x+1,t-\alpha^*) = u(x,t)$.

Subcase 2 : $|P_\delta| \to +\infty$ up to a subsequence.

We define $v_n(x,t) = u((x-\lfloor x_\delta \rfloor, t-t_\delta) - P_{\frac{1}{n}})$. Then for all $n \in \mathbb{N}$, there exists $x_n \in [0,1)$ such that $v_n^{\alpha^*}(x_n,0) < v_n(x_n+1,0)+\dfrac{1}{n}$, and v_n is a solution of (3.1). As the v_n are uniformly bounded in $C^{2+\gamma,1+\frac{\gamma}{2}}(B_R(0,0))$ for all $R>0$, for some $\gamma>0$, so by a process of diagonal extraction, one shows that they converge up to a subsequence to a function $v \in C^{2+\gamma,1+\frac{\gamma}{2}}(B_R(0,0))$ for all R, the convergence being in $C^{2+\gamma',1+\frac{\gamma'}{2}}(B_R(0,0))$ for some $\gamma' < \gamma$. Then v is also a solution of (3.1), and if we write $w = v^{\alpha^*} - v(\cdot+1,\cdot)$, the same line of reasoning as in subcase 1 applies, $w=0$ and $v(x+1,t-\alpha^*) = v(x,t)$.

In both cases there exists a global solution u such that $u(x+1,t-\alpha^*) = u(x,t)$, so it only remains to show that $\alpha^* = \dfrac{p}{\lambda_\varepsilon}$. For that, we observe that if we inject $u(x+1,t-\alpha^*) = u(x,t)$ in the bounds (i), we have for all $k \in \mathbb{Z}$,

$$|u(x+k,t-k\alpha^*) - p(x+k) - \lambda_\varepsilon(t-k\alpha^*)| < C.$$

Adding (i) and using the inequality $|a-b| \leq ||a|+|b||$ we get, for all $k \in \mathbb{Z}$,

$$|pk - \lambda_\varepsilon \alpha^* k| = |k|\,|p - \lambda_\varepsilon \alpha^*| < 2C.$$

which is only possible if $\alpha^* = \dfrac{p}{\lambda_\varepsilon}$, which concludes the proof of this step.

Case 2 : $\lambda_\varepsilon < 0$

The same line of reasoning applies when $\lambda_\varepsilon > 0$, the difference being that the quantity α^* is negative.

Step 3 : Monotonicity of the global solution when $\lambda_\varepsilon \neq 0$: proof of (iii)

Having used the sliding method to show the space-time periodicity property, we can use it as well to show monotonicity. We suppose $\lambda_\varepsilon > 0$. As above, we note $u^\alpha(x,t) = u(x, t+\alpha)$ and this time we define

$$\alpha^* = \inf\left\{\alpha_0 \geq 0 \mid \forall \alpha \geq \alpha_0, \quad u^\alpha(x,t) \geq u(x,t)\right\}.$$

This definition is valid because the set we consider is non empty (due to property (i) and $\lambda_\varepsilon > 0$). We remark that u is nondecreasing if and only if $\alpha^* = 0$. As above, $u^{\alpha^*}(x,t) \geq u(x,t)$ and for all $\delta > 0$, there exists $P_\delta = (x_\delta, t_\delta) \in \mathbb{R}^2$ such that $u^{\alpha^*}(x_\delta, t_\delta) < u(x_\delta + 1, t_\delta) + \delta$ (if it was not the case, because u is Lipschitz continuous, we could build $\alpha^{*\prime} < \alpha^*$ such that for all $\alpha \geq \alpha^{*\prime}$, $u^\alpha(x,t) \geq u(x,t)$). Again we have two cases :

Case 1 : There exists $P_0 \in \mathbb{R}^2$ such that $P_\delta \to P_0$ up to a subsequence

In this case we have

$$\begin{cases} u^{\alpha^*} \geq u & \text{on } \mathbb{R}^2 \\ u^{\alpha^*} = u & \text{at } P_0. \end{cases}$$

As u is a solution of (3.1), by the strong maximum principle $u^{\alpha^*} = u$. As u can not be periodic in time (due to (i) and $\lambda_\varepsilon > 0$), this means that $\alpha^* = 0$ and u is nondecreasing.

Case 2 : $|P_\delta| \to +\infty$ up to a subsequence

We define $v_n(x,t) = u\Big((x - \lfloor x_\delta \rfloor, t - t_\delta) - P_{\frac{1}{n}}\Big)$. Then for all $n \in \mathbb{N}$, there exists $x_n \in [0,1)$ such that $v_n^{\alpha^*}(x_n, 0) < v_n(x_n+1, 0) + \dfrac{1}{n}$, and v_n is a solution of (3.1). As the v_n are uniformly bounded in $C^{2+\gamma, 1+\frac{\gamma}{2}}(K)$ for all compact $K \subset \mathbb{R}^2$, for some $\gamma > 0$, by a diagonal extraction they converge up to a subsequence to a function $v \in C^{2+\gamma,1+\frac{\gamma}{2}}(K)$ for all compact K, the convergence being in $C^{2+\gamma', 1+\frac{\gamma'}{2}}(K)$ for some $\gamma' < \gamma$. Then v is also a solution of (3.1), so the same reasoning as in case 1 applies to v, so $v^{\alpha^*} = v$, which implies that $\alpha^* = 0$ and u is nondecreasing.

In both cases u is nondecreasing, which concludes this step.

Step 4 : Existence of a constant C independent of ε

In this step we show the following results :

The constant λ_ε remains bounded when $\varepsilon \to 0$. Moreover, there exists a constant $C > 0$ independent of ε such that the estimate (3.3), (i), holds for all ε.

As in Step 2 of the proof of Proposition 3.18 we write $v(x,t) = u(x,t) - px$, and use the same notation σ for the oscillation. Then we kwow that λ_ε is such that for the solution u_c of the Cauchy problem,

$$\|u_c - px - \lambda_\varepsilon t\| \leq C_\varepsilon.$$

As $px + \|f\|_\infty t$ and $px - \|f\|_\infty t$ are respectively a super- and subsolution of the Cauchy problem, we have that λ_ε is necessarily lesser than $\|f\|_\infty$ independently of ε and p.

For the existence of a bound C' independent on ε, we define

$$w(x) = \sup_{t\in\mathbb{R}} (u(x,t) - \lambda_\varepsilon t).$$

The relations $u\left(x, t + \dfrac{p}{\lambda_\varepsilon}\right) = u(x+1,t)$ and $u\left(x + \dfrac{1}{p}, t\right) = u(x,t)+1$ with $p^{-1} \in \mathbb{N}$ imply that

$$u\left(x, t + \frac{1}{\lambda_\varepsilon}\right) = u(x,t) + 1,$$

so, for a fixed value of x, $u(x,t) - \lambda_\varepsilon t$ is a periodic function in t and its supremum is reached for some $t(x) \in \left[0, \dfrac{1}{\lambda}\right]$.

Moreover, for all $x \in \mathbb{R}$, at the point $(x, t(x))$, $u_t - \lambda_\varepsilon = 0$, so

$$\lambda_\varepsilon = \Delta u + f(x,u) + \varepsilon M[u].$$

As $w(x)$ is the maximum of $u(x,t) - \lambda t$, we have

$$w(x) = u(x, t(x)) - \lambda_\varepsilon t(x)$$

and

$$w(y) \geq u(y, t(x)) - \lambda_\varepsilon t(x)$$

for y in a neighbourhood of x, so

$$\Delta w(x) \geq \Delta(u(\cdot, t(x)) - \lambda_\varepsilon t(x))(x) = \Delta u(\cdot, t(x))(x),$$

and

$$\lambda_\varepsilon \leq \Delta w + f(x,u) + \varepsilon M[u].$$

We also note that

$$\begin{aligned} w(x+1) &= \sup_{t\in\mathbb{R}} (u(x+1,t) - \lambda_\varepsilon t) \\ &= \sup_{t\in\mathbb{R}} \left(u\left(x, t + \frac{p}{\lambda_\varepsilon}\right) - \lambda_\varepsilon \left(t + \frac{p}{\lambda_\varepsilon}\right)\right) + p = w(x) + p, \end{aligned}$$

so $\tilde{w} = w - px$ is 1-periodic and

$$\Delta \tilde{w} = \Delta w \geq \lambda_\varepsilon + f(x,u) + \varepsilon M[u].$$

We know that $|\lambda_\varepsilon| \leq C$, $f \in L^\infty(\mathbb{R}\times\mathbb{R})$ and $|\varepsilon M[u]| \leq \left(\int_{\mathbb{R}} J(z)dz\right) \dfrac{2\varepsilon \|f\|_\infty}{K}$, so there exists $C' > 0$ such that $\Delta \tilde{w} = \tilde{w}_{xx} \geq -C'$. When $\varepsilon = 0$, the constant C' does not depend on p.

But $\tilde{w}$ is 1-periodic, so $\tilde{w}(0) = \tilde{w}(1)$; by Rolle's theorem, there exists $x_0 \in [0, 1]$ such that $\tilde{w}'(x_0) = 0$. But then, for all $y \in \mathbb{R}$ we can write

$$y = k + x_0 + r \text{ with } k \in \mathbb{Z} \text{ and } 0 \leq r < 1.$$

As a consequence, $\tilde{w}'(y) \geq \tilde{w}'(x_0 + k) - C' = -C'$ and thus for all $z, z' \in \mathbb{R}$, we can write $z' = z + k + r$, $k \in \mathbb{Z}$, $0 \leq r < 1$, so

$$\tilde{w}(z') \geq w(z + k) - C',$$

and

$$\tilde{w}(z) = \tilde{w}(z + k + 1) \geq \tilde{w}(z') - C'.$$

Finally, we can conclude that $|\tilde{w}(z) - \tilde{w}(z')| \leq C'$. Up to the addition of integers, we can suppose $|u(0, 0)| \leq 1$. For $t \in \left[0, \dfrac{1}{\lambda_\varepsilon}\right]$,

$$u(x, 0) \leq u(x, t) \leq u\left(x, \frac{1}{\lambda_\varepsilon}\right) = u(x + p^{-1}, 0) = u(x, 0) + 1,$$

so

$$u(x, 0) - 1 \leq u(x, t) - \lambda_\varepsilon t \leq u(x, 0) + 1.$$

In particular, for $t = t(x)$,

$$u(x, 0) - 1 \leq w(x) \leq u(x, 0) + 1,$$

which implies that, for all $t \in \left[0, \dfrac{1}{\lambda_\varepsilon}\right]$,

$$w(x) - 2 \leq u(x, t) - \lambda_\varepsilon t \leq w(x),$$

and

$$\tilde{w}(x) - 2 \leq u(x, t) - \lambda_\varepsilon t - px \leq \tilde{w}(x).$$

By the pseudo-periodicity property of u, this relation holds for all $t \in \mathbb{R}$. Now we observe that $\tilde{w}(0) - 2 \leq u(0, 0) \leq \tilde{w}(0)$ so $-1 \leq \tilde{w}(0) \leq 3$, and for all $(x, t) \in \mathbb{R}^2$,

$$u(x, t) - \lambda_\varepsilon t - px \geq \tilde{w}(x) - 2 \geq \tilde{w}(0) - C' - 2 \geq -C' - 3,$$

and

$$u(x, t) - \lambda_\varepsilon t - px \leq \tilde{w}(x) \leq \tilde{w}(0) + C' \leq C' + 3,$$

so there exists a constant C'' independent of ε such that

$$|u(x, t) - \lambda_\varepsilon t - px| \leq C''.$$

Given its construction, this constant C'' may depend on p, via the function J_p and the constant K_p. However, in the case $\varepsilon = 0$, C'' does not depend on p.

This fourth step concludes the proof of Proposition 3.19 in the case $\lambda_\varepsilon \neq 0$.

Step 5 : Existence of a stationary solution when $\lambda_\varepsilon = 0$

In the case $\lambda_\varepsilon = 0$, the function $u - px$ is uniformly bounded in time, so we can define

$$\overline{v} = \sup_{t\in\mathbb{R}}(u(x,t) - px)$$

$$\underline{v} = \inf_{t\in\mathbb{R}}(u(x,t) - px).$$

As $u - px$ is Lipschitz continuous, both $\overline{v}$ and $\underline{v}$ are Lipschitz continuous. Using the fact that u satisfies (ii), we deduce that $\overline{v}$ and $\underline{v}$ are $\begin{pmatrix} 1 \\ \overline{p} \end{pmatrix}$-periodic. We also remark that because u satisfies (3.3) (iv), $\overline{v}$ is such that $\overline{v}(x+1)+p \geq \overline{v}(x)$, and so is $\underline{v}$. Finally, $\overline{v}$ is a subsolution (both in the weak sense and in the viscosity sense) of

$$0 = \Delta v + f(x, v + px) + \varepsilon M[v] \tag{3.26}$$

on $\mathbb{R}$ and $\underline{v}$ is a supersolution of the same equation.

Using the fact that $|u - px| \leq C$, we also have that

$$\overline{v} \leq \underline{v} + 2\lceil C\rceil,$$

and $\underline{v} + 2\lceil C\rceil$ is still a supersolution of the equation (3.26) (here $\lceil\cdot\rceil$ is the ceiling function). Then by Perron's method there exists a viscosity solution $\tilde{v}$ of (3.26), and a stationary viscosity solution $\tilde{u}$ of (3.11). By construction, $\tilde{u}$ still satisfies (i), and as $\overline{v}$ and $\underline{v}$ are both periodic, we can apply Perron's method in a way that ensures that $\tilde{u}$ still satisfies (ii).

Finally, we can show by contradiction that $\tilde{u}(x+1) \geq \tilde{u}(x)$. Suppose there exists $x_0 \in \mathbb{R}$ such that $\tilde{u}(x_0+1) < \tilde{u}(x_0)$, which can be written equivalently $\tilde{v}(x_0+1) + p < \tilde{v}(x_0)$. Then $\tilde{v}(\cdot - 1) - p$ is also a subsolution of (3.26), and for all $x \in \mathbb{R}$, we have that

$$\tilde{v}(x-1) - p \leq \underline{v}(x-1) + 2\lceil C\rceil - p \leq \underline{v}(x) + 2\lceil C\rceil.$$

As a consequence, $\max(\tilde{v}, \tilde{v}(\cdot - 1) - p)$ is a subsolution of (3.26) and is bounded between $\overline{v}$ and $\underline{v} + 2\lceil C\rceil$. But this contradicts the maximality of $\tilde{v}$ since $\tilde{v}(x_0) - p > \tilde{v}(x_0+1)$.

This concludes the proof of Proposition 3.19.

We can use the uniform bound C on the global solution to show that there also exists uniform bounds (that is, independent on ε) on the solution of the Cauchy problem (3.11),(3.13) :

Proposition 3.20 (Uniform bounds on the solution of the Cauchy problem).
We note u_ε the solution of the Cauchy problem (3.11),(3.13). *Then there exists $C > 0$ such that for all $\varepsilon > 0$, there exists $\lambda_\varepsilon \in \mathbb{R}$ such that for all $(x,t) \in \mathbb{R} \times (0,+\infty)$,*

$$|u_\varepsilon - px - \lambda_\varepsilon t| \leq C.$$

Remark 3.21 (Identification of λ_ε).
The real number λ_ε is the same for the solution of the Cauchy problem and for the global solution.

Proof of Proposition 3.20
We note U_ε and u_ε the solutions of the global problem and the Cauchy problem respectively, and we know that for all x, t,

$$|U_\varepsilon - px - \lambda t| \leq C,$$

where C is independent of ε. Then, in particular, for all $x \in \mathbb{R}$,

$$-C + px \leq U_\varepsilon(x,0) \leq C + px.$$

As a consequence, we have that

$$U_\varepsilon(x,0)(x,0) - C \leq px = u_\varepsilon(x,0)(x,0) \leq U(x,0) + C.$$

But we also know that $U_\varepsilon - \lceil C \rceil$ and $U_\varepsilon + \lceil C \rceil$ are solutions of (3.11) on $\mathbb{R} \times (0,+\infty)$, so we have that

$$U_\varepsilon - \lceil C \rceil \leq u_\varepsilon \leq U_\varepsilon + \lceil C \rceil$$

on $\mathbb{R} \times (0,+\infty)$. Using the bounds on U_ε, we have that

$$-C + px + \lambda t - \lceil C \rceil \leq u_\varepsilon \leq C + px + \lambda t C \lceil C \rceil,$$

so

$$|u_\varepsilon - px - \lambda_\varepsilon t| \leq C + \lceil C \rceil,$$

which is what we wanted to show and concludes the proof of Proposition 3.20.

3.6 Construction of plane-like solutions of the initial problem

Having proved Proposition 3.19 and Proposition 3.20, we can now conclude the proof of Theorem 3.1.

Proof of Theorem 3.1

Step 1 : Construction of the limit solution of the Cauchy problem

The constants λ_ε converge to a constant λ up to an extraction. The solutions u_ε of (3.11) with initial condition (3.13) among the extracted subsequence converge to a solution u of (3.1),(3.13) up to another extraction. Indeed, as the convergence is in $C^{2+\beta,1+\frac{\beta}{2}}(K)$ for all compact ball $K \subset \mathbb{R} \times [0,+\infty)$, we have that $(u_\varepsilon)_t(x,t) \to u_t(x,t)$ and $\Delta u_\varepsilon(x,t) \to \Delta u(x,t)$ uniformly on $\mathbb{R} \times [0,+\infty)$. By the continuity of f, we also have $f(x,u_\varepsilon(x,t)) \to f(x,u(x,t))$ uniformly on $\mathbb{R} \times [0,+\infty)$. Finally,

$$\begin{aligned} M[u^\varepsilon] &= \int_{\mathbb{R}} e^{-|z|}(u^\varepsilon(x+z) - u^\varepsilon(x))dz \\ &= \int_{\mathbb{R}} e^{-|z|}(u^\varepsilon(x+z) - u^\varepsilon(x) - px)dz \\ &\leq 2C \int_{\mathbb{R}} e^{-|z|}dz \end{aligned}$$

Thus $\varepsilon M[u^\varepsilon] \to 0$, so u is a solution of (3.1),(3.13).

Furthermore, u satisfies (3.3), (i) and (ii), with the constant λ and the constant $C = C(p) = \sup_{\varepsilon>0} C_\varepsilon$.

Step 2 : Conclusion

As the Cauchy problem (3.1),(3.13) admits a solution u satisfying (3.3), (i) and (ii), using Proposition 3.19 in the case $\varepsilon = 0$, we have that there exists a global solution of (3.1), which we will still write u for convenience, which satisfies (3.3) and (3.4).

Moreover, Proposition 3.19 also shows that (3.3) (i) holds with a constant $C' > 0$ independent of p, and that the constants $\lambda(p)$ are uniformly bounded in p.

This last step concludes the proof of Theorem 3.1.

3.7 A structural property for the plane-like solutions

In this section we show that the plane-like solutions constructed in Section 3.6 satisfy the structure property (3.5), thereby showing Proposition 3.4.

Proposition 3.22. (A relaxed monotonicity property for the global solution)
Let u be the solution of (3.1) *on $\mathbb{R} \times \mathbb{R}$ satisfying* (3.3), (3.4) *constructed in the proof of Theorem 3.1. Then u satisfies* (3.5).

Proof of Lemma 3.22

To prove this result, we introduce the sets

$$\Omega_a^+ := \{(x,t) \mid u(x,t) > a\}$$

and

$$\Omega_a^- := \{(x,t) \mid u(x,t) < a\}.$$

We first prove the property for the solution of the Cauchy problem (3.1),(3.13), that we still write u for commodity.

Step 1 : A connexity property

For the Cauchy problem, we first prove that for all $T > 0$, the set

$$\Omega_{a,T}^+ = \Omega_a^+ \cap (\mathbb{R} \times [0,T])$$

is connected. As $u(x,0) = px$, we know that $\Omega_a^+ \cup (\mathbb{R} \times \{0\}) = \left]\frac{a}{p}, +\infty\right[\times \{0\}$. Moreover, as $|u(x,t) - px - \lambda t| \leq C$, we have the following inclusions :

$$\{(x,t) \mid px + \lambda t > C + a\} \subset \Omega_a^+ \subset \{(x,t) \mid px + \lambda t > -C + a\}. \qquad (3.27)$$

Now suppose by contradiction that $\Omega_{a,T}^+$ is not connected, and consider Ω_1 a connected component of $\Omega_{a,T}^+$ that does not contain (and hence is not connected with) the half-line $\left]\frac{a}{p}, +\infty\right[\times \{0\}$. Because of (3.27), Ω_1 is included in the bounded band

$$\{(x,t) \mid -C + a < px + \lambda t \leq C + a\} \cup (\mathbb{R} \times [0,T]),$$

so it is a bounded subset of $\mathbb{R} \times [0,T]$ that is open in $\mathbb{R} \times [0,T]$. Now consider its parabolic boundary $\partial^p \Omega_1$ (see Definition 3.8), and a point $(x,t) \in \partial^p \Omega_1$. Now we have several cases :
if $0 < t \leq T$, then by continuity $u(x,t) = a$ (see Figure 3.2).
if $t = 0$, we recall that Ω_1 is not connected to the half-line $\left]\frac{a}{p}, +\infty\right[\times \{0\}$, so $u(x,t) \leq a$, and we also get $u(x,t) = a$ by continuity.

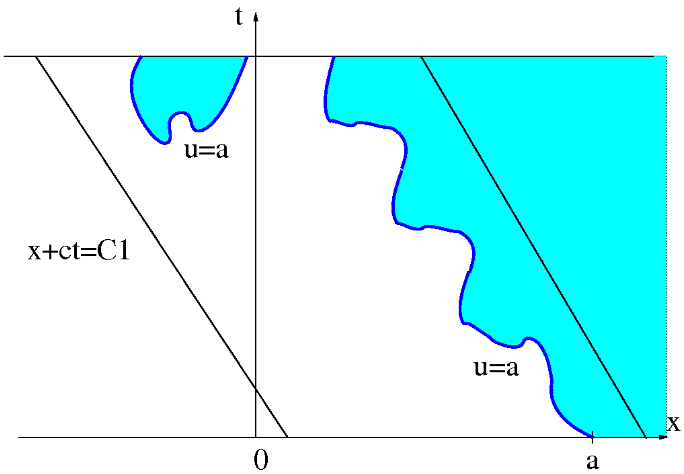

FIGURE 3.2 – Illustration of Ω_a^+ and Ω_1 : $u = a$ on the parabolic boundary of Ω_1.

So Ω_1 is a bounded set for which $u = a$ on the parabolic boundary, and u is a solution of (3.1) on Ω_1, with Dirichlet boundary condition a on $\partial^p\Omega_1$. On the other hand, the constant a is also a solution of this problem. By the local comparison principle (Proposition 3.9), $u = a$ on Ω_1, which is a contradiction. Therefore, $\Omega^+_{a,T}$ is connected.

In the same way, we can show that $\Omega^-_{a,T}$ is connected.

Step 2 : Proof of the structure property for the Cauchy problem (3.1),(3.13)

Now suppose that property (3.5) is false for the solution of the Cauchy problem (3.1) with initial condition (3.13). Then there exists a time $t_0 > 0$ and two points (x_1, t_0) and (x_2, t_0) with $x_1 < x_2$ such that

$$u(x_2, t_0) < a < u(x_1, t_0).$$

As Ω^+_{a,t_0} is connected, there is a path in Ω^+_{a,t_0} between (x_1, t_0) and $\left(\frac{a}{p} + 1, 0\right)$. Similarly, Ω^-_{a,t_0} is connected, so there is a path in Ω^-_{a,t_0} between (x_2, t_0) and $\left(\frac{a}{p} - 1, 0\right)$. But these two paths must cross each other (see Figure 3.3), which brings a contradiction, so the structure property holds for the solution of the Cauchy problem.

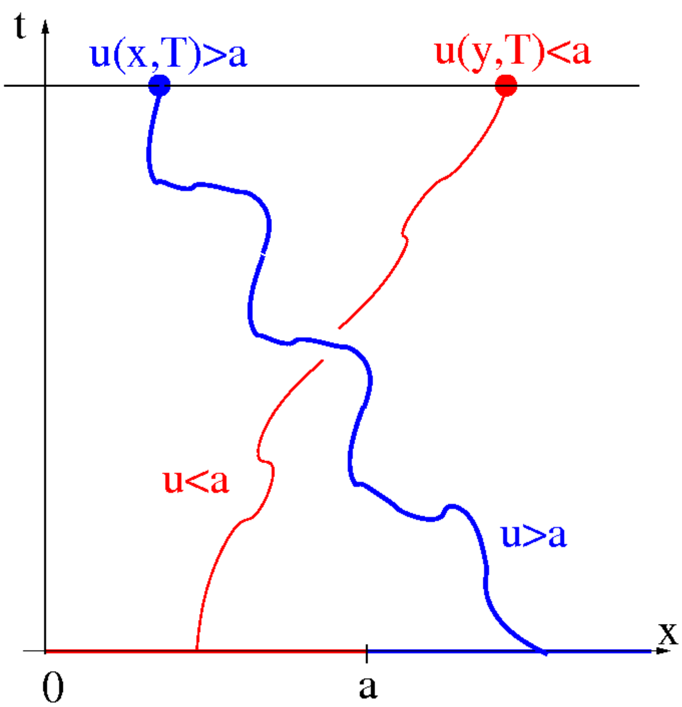

FIGURE 3.3 – Illustration of the proof of Lemma 3.22 for the solution of the Cauchy problem.

Step 3 : Conclusion

We still write u the solution of the Cauchy problem (3.1),(3.13), and define

$$u_n = u(x, t+n) - \lfloor \lambda n \rfloor$$

as in Step 1 of the proof of Proposition 3.19. The functions u_n converge strongly to a solution u_∞ of (3.1), and it is easy to see that u_∞ still satisfies (3.5).

In the case when $\lambda \neq 0$, this solution u_∞ is the solution we construct to prove Theorem 3.1, and as u_∞ satisfies (3.5), Lemma 3.22 holds in this case.

Remains the case where $\lambda = 0$, where we use the solution u_∞ to construct another solution u which, in addition to the properties of u_∞, is also stationary.

In that case, the solution u is constructed in Step 5 of the proof of Proposition 3.19 with $\varepsilon = 0$. So we define

$$\begin{cases} \overline{u}_\infty(x) = \sup\limits_t (u_\infty(x,t)) \\ \underline{u}_\infty(x) = \inf\limits_t (u_\infty(x,t)) \end{cases}$$

and we know that $\underline{u}_\infty$ is a stationary supersolution of (3.1) in the viscosity sense, and $\overline{u}_\infty$ is a stationary subsolution of (3.1) in the viscosity sense. As u_∞ is globally Lipschitz continuous, both $\underline{u}_\infty$ and $\overline{u}_\infty$ are Lipschitz continuous. Moreover $0 \leq \overline{u}_\infty - \underline{u}_\infty \leq 2C+1$ and both $\underline{u}_\infty$ and $\overline{u}_\infty$ satisfy (3.3). We can also show they satisfy the structure property (3.5). We do the demonstration for $\overline{u}_\infty$, it is similar for $\underline{u}_\infty$. Take $x_0 \in \mathbb{R}$ such that $\overline{u}_\infty(x_0) > a$; then, by the property of the supremum, there exists $t_0 \in \mathbb{R}$ such that $u_\infty(x_0, t_0) > a$. As u_∞ satisfies (3.5), for all $x > x_0$, $u_\infty(x, t_0) \geq a$ and thus $\overline{u}_\infty(x) > a$, which shows that $\overline{u}_\infty$ satisfy (3.5).

By Perron's method (Proposition 3.5), there exist a stationary solution u of (3.1) in the viscosity sense and $\overline{u}_\infty \leq u \leq \underline{u}_\infty + \lceil 2C \rceil + 1$. By the regularity result of Proposition 3.16, $u \in C^{2,1}_{loc}(\mathbb{R}^2)$ and in particular u is Lipschitz. We show by contradiction that this solution u constructed by Perron's method satisfies the structure property.

We suppose there exists $a \in \mathbb{R}$ with $f(x,a) = 0$ and a couple $(x_1, x_2) \in \mathbb{R}^2$ such that $x_1 \leq x_2$ and $u(x_2) < a < u(x_1)$. Using (3.3) (ii), up to redefining x_1 and x_2, we can suppose $x_2 - x_1 < p^{-1}$. We then consider $x_3 = x_1 + p^{-1}$, and we have $u(x_3) = u(x_1) + 1 > a$. As $\underline{u}_\infty(x_1) + \lceil 2C \rceil + 1 \geq u(x_1) > a$ and $\underline{u}_\infty + \lceil 2C \rceil + 1$ satisfies the structure property (3.5), we know that

$$\underline{u} + \lceil 2C \rceil + 1 \geq a \text{ on } [x_1, +\infty) \supset [x_1, x_3].$$

Now we observe that the constant a is a subsolution of (3.1) on $[x_1, x_3]$, so if we note

$$\tilde{u} = \begin{cases} u(x) & \text{when } x \notin [x_1, x_3] \\ \max(u(x), a) & \text{when } x \in [x_1, x_3] \end{cases},$$

then $\tilde{u}$ is also a subsolution of (3.1) on $\mathbb{R}$. Indeed, $\tilde{u}$ is a supremum of subsolutions on $(-\infty, x_1)$, (x_1, x_3) and $(x_3, +\infty)$. As u is Lipschitz, $\tilde{u} = u$ in a neighbourhood of x_1 and x_3 so $\tilde{u}$ is a subsolution of (3.1) on $\mathbb{R}$.

Moreover, $\overline{u} \leq \tilde{u} \leq \underline{u} + \lceil 2C \rceil + 1$, and $\tilde{u}(x_2) = a > u(x_2)$, so this contradicts the maximality of the subsolution u constructed by Perron's method. This contradiction concludes the proof of Lemma 3.22.

3.8 Analysis of the sign of λ

Proof of Proposition 3.2

In this proof we consider $p > 0$, and we suppose, for simplicity, that $\lambda > 0$ (the proof is completely similar when $\lambda < 0$).

Using (3.3), (ii) and (3.4), we have the relation

$$u\left(x, t + \frac{1}{\lambda}\right) = u(x,t) + 1.$$

The process of this proof is then to multiply (3.1) by the quantity u_t and integrate on the box $\left[0, \frac{1}{p}\right] \times \left[0, \frac{1}{\lambda}\right]$, which gives the following equality

$$\int_0^{\frac{1}{p}} \int_0^{\frac{1}{\lambda}} u_t^2 = \int_0^{\frac{1}{p}} \int_0^{\frac{1}{\lambda}} u_{xx} u_t + \int_0^{\frac{1}{p}} \int_0^{\frac{1}{\lambda}} f(x, u(x,t)) u_t(x,t)$$

We analyse each term separately :

$$\int_0^{\frac{1}{p}} \int_0^{\frac{1}{\lambda}} u_t^2 > 0.$$

This term is clearly non-negative, and as $\lambda > 0$, it can not be zero because of (3.3). An integration by parts shows that the second term is zero :

$$\begin{aligned}
&\int_0^{\frac{1}{p}} \int_0^{\frac{1}{\lambda}} u_{xx} u_t \\
&\quad = \int_0^{\frac{1}{\lambda}} \left(-\int_0^{\frac{1}{p}} u_x u_{xt} + u_x\left(\frac{1}{p}, t\right) u_t\left(\frac{1}{p}, t\right) - u_x(0,t)\, u_t(0,t) \right) dt \\
&\quad = -\int_0^{\frac{1}{p}} \left(u_x^2\left(x, \frac{1}{\lambda}\right) - u_x^2(x, 0) \right) dx = 0.
\end{aligned}$$

Finally, we compute the last term using the change of variable $v = u(x,t)$:

$$\begin{aligned}
\int_0^{\frac{1}{p}} \int_0^{\frac{1}{\lambda}} f(x, u(x,t)) u_t(x,t) dt dx &= \int_0^{\frac{1}{p}} \int_{u(x,0)}^{u(x,0)+1} f(x,v) dv dx \\
&= \frac{1}{p} \int_0^1 \int_0^1 f(x,v) dv dx.
\end{aligned}$$

As a consequence, we have

$$\int_{[0,1]^2} f(x,v)\,dv\,dx > 0,$$

which proves Proposition 3.2 in the case $\lambda > 0$.

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Chapitre 4

Pulsating travelling waves for the bistable case in dimension one

This work has been written in collaboration with Régis Monneau (CERMICS, ENPC).

Abstract

In this paper we consider a reaction-diffusion equation in a periodic medium. This is a parabolic semilinear equation in dimension one. The nonlinearity is bistable and periodic in space with fixed zeros (with respect to space variable). Under certain assumptions, we construct pulsating travelling waves which are unique (up to translations) when their velocity is non zero. We also show the uniqueness of the velocity. Our method of proof is new and is related to the construction of correctors in homogenization problems.

AMS Classification : 35K55
Keywords : bistable, pulsating travelling wave, periodic medium, semilinear equations, homogenization.

4.1 Introduction

4.1.1 Setting of the problem

We consider solutions $u(x,t)$ of the following parabolic semilinear equation for $x \in \mathbb{R}$, $t > 0$:

$$u_t = u_{xx} + f(x,u), \tag{4.1}$$

for some function f which is 1-periodic in x and is of bistable type in u. Our goal is to construct pulsating travelling waves moving with a suitable velocity c_0, i.e. satisfying (when $c_0 \neq 0$) :

$$u\left(x+1, t-\frac{1}{c_0}\right) = u(x,t).$$

To this end, we assume that the function f satisfies the following conditions :

Regularity and $\mathbb{Z}$-periodicity properties :

$$\left\{\begin{array}{l} f \in C^1(\mathbb{R}\times[0,1];\mathbb{R}) \\ f(x+k,v) = f(x,v) \quad \text{for all} \quad k\in\mathbb{Z}, \quad (x,v)\in\mathbb{R}\times[0,1] \end{array}\right. \tag{4.2}$$

Bistability : There exists $\theta \in (0,1)$ such that

$$\left\{\begin{array}{lll} f(x,0)=f(x,\theta)=f(x,1)=0 & \text{for all} & x\in\mathbb{R} \\ f(x,v)<0 & \text{for all} & v\in(0,\theta), \quad x\in\mathbb{R} \\ f(x,v)>0 & \text{for all} & v\in(\theta,1), \quad x\in\mathbb{R} \\ f'_u(x,\theta)>0 & \text{for all} & x\in\mathbb{R} \\ \exists\eta_0>0 \text{ such that }, \forall x\in\mathbb{R} & & \\ \text{ the map } v\mapsto f(x,v) \text{ is decreasing on} & [0,\eta_0]\cup[1-\eta_0,1] & \end{array}\right. \tag{4.3}$$

where $f'_u := \dfrac{\partial f}{\partial u}$.

We refer to Figure 4.1 for an illustration of the function f with three zeros which are independent on x : two stable zeros at 0 and 1 and one unstable zero at θ.

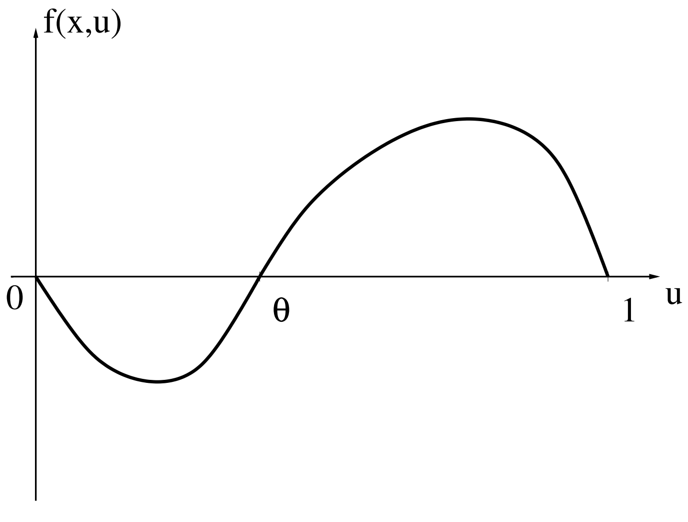

FIGURE 4.1 – The bistable nonlinearity f.

In order to prove the uniqueness of the velocity of propagation of the pulsating travelling wave (which we will then sometimes write PTW), we will need the following additional assumption :

Strong stability of 0 **and** 1 :

$$\begin{cases} f_u'(x,0) = -\delta_0 < 0 & \text{for all} \quad x \in \mathbb{R} \\ f_u'(x,1) = -\delta_1 < 0 & \text{for all} \quad x \in \mathbb{R} \end{cases} \tag{4.4}$$

where δ_0, δ_1 are some positive constants.

4.1.2 Main results

Theorem 4.1. (Existence and uniqueness of pulsating travelling waves)
Assume (4.2),(4.3). *Then there exists* $c_0 \in \mathbb{R}$ *and there exists* u *solution of* (4.1) *on* $\mathbb{R} \times \mathbb{R}$ *such that*

$$\begin{cases} c_0 u_t \geq 0 \\ 0 \leq u \leq 1 \\ \liminf\limits_{x+c_0t \to +\infty} u(x,t) = 1 \\ \limsup\limits_{x+c_0t \to -\infty} u(x,t) = 0 \end{cases} \tag{4.5}$$

and

$$\begin{cases} u\left(x+1, t-\dfrac{1}{c_0}\right) = u(x,t), & \text{if } c_0 \neq 0 \\ u(x,t) \text{ does not depend on } t, & \text{if } c_0 = 0. \end{cases} \tag{4.6}$$

Moreover, if (c_0, u) *is a solution of* (4.1) *satisfying* (4.5),(4.6) *and if* $c_0 \neq 0$, *then the profile* u *is unique up to a time translation.*

Remark 4.2. *Up to our knowledge, the uniqueness or non-uniqueness of the profile* u *is unknown for* $c_0 = 0$.

Theorem 4.3. (Uniqueness of the velocity c_0**)**
Under assumptions (4.2),(4.3),(4.4), *the velocity* $c_0 \in \mathbb{R}$ *given in Theorem 4.1 is unique.*

Remark 4.4. *Condition* (4.4) *is a technical assumption, and it seems reasonable to think that the uniqueness of* c_0 *could hold under weaker conditions. In particular, it should be possible to adapt the uniqueness proof devised by Berestycki and Hamel in [2] to our case, which would enable us to lift assumption* (4.4) *completely.*

4.1.3 Review of the literature

The homogeneous reaction-diffusion equation

$$u_t = u_{xx} + u(1-u)$$

was first studied in the pioneering papers of Kolmogorov, Petrovsky and Piskunov [11], and Fisher [10]. This homogeneous equation admits solutions that have the form of planar fronts (travelling fronts) $u(x,t) = U(x+ct)$ with c the velocity of the wave. If we add a heterogeneous advection term and work in higher dimension, the existence of travelling fronts has been proved by Berestycki and Nirenberg [6] and Berestycki, Larrouturou and Lions [5] when the advection term does not depend on the variable of the direction of propagation. In [15], Shigesada, Kawasaki and Teramoto introduced the notion of pulsating travelling fronts, which generalizes the notion of travelling fronts when the functional operator or the nonlinearity f depend on x and are periodic.

Existence of pulsating travelling fronts has been proved in the case of space-periodic advection and nonnegative nonlinearity f by J. Xin in [16] and the result has been extended to a fully space periodic environment, when the equation has the form

$$u_t = \operatorname{div}(a(x)\nabla u) + b(x).\nabla u + f(u,x),$$

with a positive nonlinearity f, by Berestycki and Hamel in [2], among other more general results. More recently, existence of pulsating travelling fronts has been proved in a more general context, where a, b and f also depend on time, see Nadin [14].

Until recently there were few results for the existence of pulsating travelling waves in the case of more general nonlinearities f. Continuing the results mentioned above, Berestycki, Hamel and Roques **??** addressed the case of a monostable nonlinearity taking negative values. In the bistable case, X. Xin [17] showed in 1991 the existence of pulsating travelling waves in a periodic medium for a bistable nonlinearity f, when the dependence in the space variable is a small perturbation of the homogeneous case, and f only depends on u. In [7], Chen, Guo and Wu showed the existence of travelling waves in a discrete periodic setting with bistable nonlinearity, as well as uniqueness and stability results.

In [9] Giletti, Ducrot and Matano show the existence of pulsating travelling waves under mild assumptions in a very general context, where the nonlinearity can be monostable, bistable, or even more complex. In the particular case of a bistable nonlinearity, they show the existence of pulsating travelling waves between the stable state 0 and a positive stable state $p(x)$ under two implicit conditions :

- **Convergence of a solution with compactly supported initial condition**
  There exists a solution u of the Cauchy problem with compactly supported initial data $0 \leq u_0(x) < p(x)$ converging locally uniformly to p as $t \to +\infty$.
- **Nonexistence of intermediate stable from below solutions**
  There exists no 1-periodic stationary solution q satisfying $0 < q(x) <$

$p(x)$ and such that it is isolated and stable from below with respect to the equation.

The demonstration of Giletti, Matano and Ducrot is based on the study in large time of solutions of the Cauchy problem

$$\begin{cases} u_t = u_{xx} + f(x,u) \\ u(x,0) = u_0(x) \end{cases} \tag{4.7}$$

with an initial data of Heaviside type. A strong maximum principle is used to show the pulsating travelling wave property.

Our conditions on the function f are more specific, but they could possibly allow cases where there exists an intermediate periodic stationary solution between 0 and 1 that is stable ; if such intermediate stable solutions were to exist, this result would not be contained in [9]. The result of Proposition 4.10 enables us to exclude the possibility that our solution u pins on a possible intermediate stable solution, a process that is new up to our knowledge. In contrast to [9], our proof relies on the construction of plane like solutions, described in Chapter 3, before constructing pulsating travelling waves bounded between 0 and 1 as limits of these plane-like solutions when the slope of the plane goes to zero. Like in [9], we use a strong maximum principle to show the pulsating travelling wave property. We hope this method could be a guideline to study the case of time-dependent nonlinearities and more general nonlinear problems.

4.1.4 Organization of the paper

This chapter will be organized as follows. Section 4.2 is a Toolbox in which we recall classical results (regularity properties, comparison principle and strong maximum principle) and two propositions proved in Chapter 3 : existence of plane-like solutions for (4.1) and a structure property on these solutions (akin to a weak monotonicity property in space). The first proposition is the starting point to prove the results of this paper, and the structure property is crucial to characterize the bounded solutions we obtain. The existence of pulsating travelling waves solutions of (4.1) is shown in Section 4.3. They are obtained as limits of the plane-like solutions whose existence is recalled in Section 4.2. Most of the properties of the limit follow from the properties of the plane-like solutions. Particular care must be taken to show that the limit cannot be the constant solution θ. Uniqueness results (uniqueness of the profile for non zero velocities and uniqueness of the velocity) are shown in Section 4.4.

4.1.5 Notation

The function f can be extended to a $\mathbb{Z}^2$-periodic function $\tilde{f}$ by the following definition

$$\tilde{f}(x,v) = f(x, v - \lfloor v \rfloor)$$

where $\lfloor \cdot \rfloor$ is the floor integer function. The function $\tilde{f}$ is then $\mathbb{Z}^2$-periodic and Lipschitz continuous on $\mathbb{R}^2$, and thus satisfies property (3.2) given in Chapter 3. In the remainder of the paper, $\tilde{f}$ will be simply denoted by f.

4.2 Toolbox

4.2.1 Classical properties

In this subsection we recall, without proof, results on classical solutions of parabolic equations that were shown in Chapter 3.

Proposition 4.5. (Local comparison principle)
We assume f satisfies (4.2),(4.3). *Let Q be a connected open subset of $\mathbb{R} \times I$ with I a time interval. Let us denote by $\partial^p Q$ its parabolic boundary. For $\varepsilon = 0$, let u_1 be a subsolution and u_2 a supersolution of (4.1) on Q which are assumed to belong to $C^{2,1}(Q) \cap C(\overline{Q})$. If $u_1 \leq u_2$ on $\partial^p Q$, then $u_1 \leq u_2$ on Q.*

Proposition 4.6. (Strong maximum principle for strong solutions)
We assume f satisfies (4.2),(4.3). *Let $Q \subset \mathbb{R} \times I$ be a connected open subset of $\mathbb{R} \times I$. Let u_1 and u_2 be two solutions of*

$$u_t = \Delta u + f(x,u)$$

on Q, satisfying

$$\begin{cases} u_1 \leq u_2 & \text{on} \quad Q \\ u_1(x_0,t_0) = u_2(x_0,t_0) & \text{for some} \quad (x_0,t_0) \in Q. \end{cases}$$

Then $u_1 = u_2$ on Q.

Proposition 4.7. (Hölder bounds for solutions of the form "linear+ bounded")
We assume f satisfies (4.2),(4.3). *We consider a solution u of the problem*

$$u_t = \Delta u + f(x,u) \ \text{on} \quad \mathbb{R} \times I. \tag{4.8}$$

We also suppose that for some $\lambda \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$|u - px - \lambda t| \leq C.$$

Then u has the following bounds in $C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(Q_r(P_0))$ for some $\alpha > 0$:

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_r(P_0))} \leq \|u\|_{C^0(Q_r(P_0))} + C_4$$

uniformly for $P_0 \in \mathbb{R} \times I$.

Corollary 4.8. (Regularity of the time derivative of u) *We assume f satisfies* (4.2),(4.3). *Consider u a solution of* (4.8). *Then u_t is bounded in $C^{2+\alpha,1+\frac{\alpha}{2}}(K)$ for all compacts $K \subset \mathbb{R}^2$ and for some $\alpha > 0$, and it is a solution of the equation*

$$(u_t)_t = (u_t)_{xx} + f'_u(x,u)u_t. \tag{4.9}$$

4.2.2 Existence of plane-like solution and structural properties

In this subsection we recall the results shown in Chapter 3 in the case of our bistable nonlinearity f satisfying (4.2),(4.3). The first result is the existence of plane-like solutions of (4.1) :

Theorem 4.9. (Existence of a global solution with plane like properties)
Under assumption (4.2)*, for a given* $p > 0$ *with* $p^{-1} \in \mathbb{N}$*, there exists a unique* $\lambda \in \mathbb{R}$ *such that there exists a constant* $C > 0$ *independent of* p *and a solution* u *of* (4.1) *on* $\mathbb{R} \times \mathbb{R}$ *which satisfy the following properties*

$$\begin{cases} |u(x,t) - px - \lambda t| \leq C & (i) \\ u\left(x + \dfrac{1}{p}, t\right) = 1 + u(x,t) & (ii) \\ \lambda u_t \geq 0 & (iii) \\ u(x+1,t) \geq u(x,t) & (iv). \end{cases} \tag{4.10}$$

Moreover

$$\begin{cases} u\left(x+1, t - \dfrac{p}{\lambda}\right) = u(x,t) & \text{if} \quad \lambda \neq 0 \quad (v) \\ u_t = 0 & \text{if} \quad \lambda = 0 \quad (v)'. \end{cases} \tag{4.11}$$

Finally, the constants $\lambda(p)$ *are bounded independently of* p.

Figure 4.2 illustrates the periodicity and the limit properties of a function $u^{(p)}$. Because of the periodicty, all the information s contained in the yellow band, and $u^{(p)}$ has the limit $+\infty$ (respectively $-\infty$) when $x + c_p t \to +\infty$ (respectively $x + c_p t \to -\infty$).

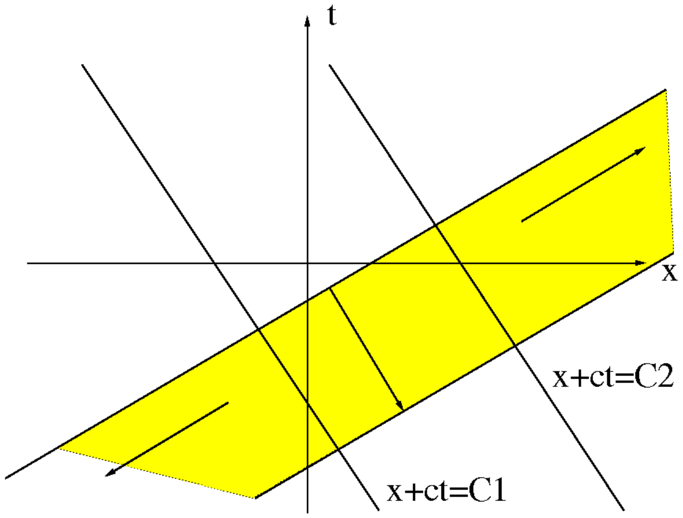

FIGURE 4.2 – Illustration of the periodicity and limits of $u^{(p)}$.

The second proposition is a direct application of Proposition 3.4 in Chapter 3. This property is important in the remainder of the paper to prevent the solution we construct to be equal to an intermediate stationary solution between 0 and 1 (this stationary solution being either stable or unstable).

Proposition 4.10. (A relaxed monotonicity property for the global solution)
There exists a solution u of (4.1) *on* $\mathbb{R}\times\mathbb{R}$ *satisfying* (4.10), (4.11) *as in Theorem 4.9, such that*

$$\forall a \in \mathbb{Z}+\{0,\theta\}\,, \quad (u(x,t)<a<u(y,t)) \Rightarrow (x<y)\,. \tag{4.12}$$

4.3 Construction of Pulsating Travelling Waves as $|p| \to 0$

The goal of this section is to prove the existence part of Theorem 4.1. The idea is to consider the solutions $u^{(p)}$ constructed in Theorem 4.9 and pass to the limit $p \to 0$.

4.3.1 Construction of limit objects as $p \to 0$

Proposition 4.11 (Existence of limit objects). *For $p^{-1} \in \mathbb{N}\setminus\{0\}$, let $u^{(p)}$ be a function given by Theorem 4.9, solution of* (4.1) *on* $\mathbb{R}\times\mathbb{R}$ *satisfying* (4.10),(4.11) *and $\lambda=\lambda(p)$ the constant associated in the space-time oscillation estimate* (4.10) (*i*). *Then we have the following results*
(1) $\lambda(p) \underset{p\to 0}{\longrightarrow} 0$.
(2) There exists $c_0 \in \overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,+\infty\}$ *such that*

$$c_p = \frac{\lambda(p)}{p} \underset{p\to 0}{\longrightarrow} c_0$$

up to a subsequence.
(3) There exists a bounded solution u of (4.1) *such that*

$$u^{(p)} \underset{p\to 0}{\longrightarrow} u$$

in $C^{2,1}_{loc}(\mathbb{R}^2)$ *up to a subsequence.*

Finally u satisfies the conclusion of Proposition 4.10, (4.12), *that is for all* $a \in \mathbb{Z}+\{0,\theta\}$, *we have that*

$$u(x,t)<a<u(y,t) \text{ implies } x<y.$$

Proof of Proposition 4.11
By Theorem 4.9, we have that

$$|u^{(p)} - px - \lambda(p)t| \le C$$

with a bound independent on p. Using the regularity result Proposition 4.7, we have that the solutions $u^{(p)}$ are uniformly bounded in $C^{2+\alpha,1+\frac{\alpha}{2}}(B_R(0,0))$ for all $R>0$ and for some $\alpha \in (0,1)$.

Therefore, they are in a compact subset of $C^{2+\beta,1+\frac{\beta}{2}}(B_R(0,0))$ for some $\beta < \alpha$, and so there exists $c_0 \in \overline{\mathbb{R}}$ and $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(B_R(0,0))$ for all $R>0$ such that $(c_p, u^{(p)})$ converges to (c_0, u) in $\overline{\mathbb{R}} \times C^{2+\beta,1+\frac{\beta}{2}}(B_R(0,0))$ for all $R>0$, up to a subsequence.

As the convergence is in $C^{2+\beta,1+\frac{\beta}{2}}(B_R(0,0))$ for all $R>0$, u is a strong solution of (4.1).

Now let λ_0 be an accumulation value of $\lambda(p)$. Passing to the limit in (4.10)(i), we get $|u(x,t)-\lambda_0 t| \le C$. Moreover, we know that the integers are global solutions of (4.1). Now there exists $k \in \mathbb{N}$ such that

$$-k \le -C \le u(x,0) \le C \le k.$$

Then, by the comparison principle on the evolutionary problem, we get that for all $t>0$,

$$-k \le u(x,t) \le k.$$

This is only possible if $\lambda_0 = 0$.

Finally, as all $u^{(p)}$ satisfy Proposition 4.10, this result also holds for u. This concludes the proof of Proposition 4.11.

Proposition 4.12 (Qualitative properties of u).
With the notations of Proposition 4.11, the limit (c_0,u) of $(c_p,u^{(p)})$ satisfies :

$$\left\{\begin{array}{ll} \overline{u}(x) = \lim_{t\to\infty} u(x,t) & \textit{is a stationary solution} \\ \underline{u}(x) = \lim_{t\to-\infty} u(x,t) & \textit{is a stationary solution} \\ u^+(x,t) = \lim_{n\to\infty} u(x+n,t) & \textit{is a periodic solution} \\ u^-(x,t) = \lim_{n\to\infty} u(x-n,t) & \textit{is a periodic solution} \end{array}\right. \tag{4.13}$$

and $\overline{u}$, $\underline{u}$, u^+ and u^- all satisfy the conclusion of Proposition 4.10, (4.12).

Moreover, we have the following additional properties which depend on the sign of the velocity in three cases.

Positive case

In the case where $c_0>0$ or $c_0=0$ with $c_p>0$ for a subsequence $p\to 0$,

$$\left\{\begin{array}{l} u_t \ge 0 \\ 0 \le \overline{u}-\underline{u} \le 1 \\ u^+ - u^- \le 1. \end{array}\right. \tag{4.14}$$

As u_t itself is the solution of a parabolic equation (see Proposition 4.8) that satisfies the strong maximum principle (Proposition 4.6) and $u_t = 0$ is a solution, we then have the following alternative :

either $u_t > 0$ *on* $\mathbb{R}^2$
or $u_t = 0$ *on* $\mathbb{R}^2$.

Negative case

In the case where $c_0 < 0$ *or* $c_0 = 0$ *with* $c_p < 0$ *for a subsequence* $p \to 0$,

$$\left\{ \begin{array}{l} u_t \leq 0 \\ 0 \leq \underline{u} - \overline{u} \leq 1 \\ u^+ - u^- \leq 1. \end{array} \right. \tag{4.15}$$

As in the preceding case, we have the alternative : either $u_t < 0$ *on* $\mathbb{R}^2$
or $u_t = 0$ *on* $\mathbb{R}^2$.

Constant case

In the case where $c_0 = 0$ *with* $c_p = 0$ *in a neighbourhood of* 0, *then*

$$\left\{ \begin{array}{l} u_t = 0 \\ \underline{u} = \overline{u} = u \\ u^+ - u^- \leq 1. \end{array} \right. \tag{4.16}$$

The limit also has the following travelling wave property

Lemma 4.13. (Travelling wave property of the limit)
Let u, $\overline{u}$ *and* $\underline{u}$ *the functions defined in Proposition 4.12. Then*

$$\begin{array}{l} \text{if } c_0 = 0, \text{ then } \overline{u}(x) \leq \underline{u}(x+1) \text{ and } \underline{u}(x) \leq \overline{u}(x+1), \\ \text{if } c_0 \neq 0, \text{ then } u\left(x+1, t - \dfrac{1}{c_0}\right) = u(x,t), \end{array} \tag{4.17}$$

with the convention that $\dfrac{1}{c_0} = 0$ *if* $c_0 = \pm\infty$.

Remark 4.14. (Note on the negative case)
We can remark that the positive case and the negative case above are very similar to each other.
Indeed, if we consider a function f_1 *such that the solutions* $u_1^{(p)}$ *are in the negative case, we can build the function*

$$f_2(x,u) = -f_1(x, 1-u)$$

The function f_2 *satisfies the conditions* (4.2),(4.3),(4.4), *with* $\theta' = 1 - \theta$. *Then the corresponding solutions* $u_2^{(p)}$ *are*

$$u_2^{(p)}(x,t) = 1 - u_1(x,t)$$

and they are in the positive case, because $\lambda_2(p) = -\lambda_1(p)$.
In the remainder of the paper, we will only work on the positive case and the constant case.

Proof of proposition 4.12

Step 1 : Preliminaries

Since $\lambda = \lambda(p) \to 0$ as $|p| \to 0$, then passing to the limit in the inequality $\left|u^{(p)} - px - \lambda t\right| \leq C$, we get that u is bounded.
As signalled above, in this proof we only treat the positive case in addition to the constant case. As the arguments are simpler in the constant case, we will treat it as a first step of the proof and will focus on arguments specific to the positive case in the remainder.

Step 2 : Proof for the constant case and properties of u^+ and u^-

In the constant case, for all $p > 0$ small enough, $c_p = 0$, which means that $\lambda(p) = 0$ and thus that the solution $u^{(p)}$ is stationary for all p. As a strong limit of these solutions, u is also stationary, and we have $u = \underline{u} = \overline{u}$. Besides, for all p, $u^{(p)}$ satisfies

$$u^{(p)}(x+1,t) - u^{(p)}(x,t) \geq 0, \tag{4.18}$$

so this inequality remains true for u. As a consequence, as u is bounded, u^+ and u^- are well defined and bounded, and as u is bounded in $C^{2+\alpha,1+\frac{\alpha}{2}}(K)$ for all compacts K, u^+ and u^- have the same regularity and are strong solutions of (4.1). We also note that property (4.12) is preserved when we pass to the limit, so u^+ and u^- satisfy (4.12). By construction, u^+ and u^- are also periodic in space. For all $p \in \mathbb{N}^{-1}$, we have

$$u^{(p)}(x,t) + 1 = u^{(p)}\left(x + \frac{1}{p}, t\right),$$

so

$$u^{(p)}\left(x + \frac{1}{2p}, t\right) - u^{(p)}\left(x - \frac{1}{2p}, t\right) = 1.$$

Then if we note $N = \dfrac{1}{2p}$ when the fraction is an integer, for all $n \leq N$, we have that

$$u^{(p)}(x+n,t) - u^{(p)}(x-n,t) \leq 1$$

because of the relation (4.18). If we take the limit $p \to 0$ (which is equivalent to $N \to \infty$), we get for all $n \in \mathbb{N}$,

$$u(x+n,t) - u(x-n,t) \leq 1,$$

hence, passing to the limit $n \to \infty$, we get

$$u^+(x,t) - u^-(x,t) \leq 1.$$

This finishes the proof in the constant case, but one can note that these arguments on u^+ and u^- do not depend on the sign of c_0, so they are also valid in the positive case. The remaining steps finish the proof for the positive case.

Step 3 : Monotonicity of u, existence of $\overline{u}$ and $\underline{u}$

First, $u^{(p)}$ is nondecreasing when $\lambda(p) > 0$, so it is clear that u is nondecreasing.

As a consequence, $\overline{u}$, $\underline{u}$ are also well defined and bounded in $C^{2+\alpha,1+\frac{\alpha}{2}}(K)$ for all compacts K, and they are strong solutions of (4.1). They also satisfy (4.12) because this property is preserved when we pass to the limit. By construction, $\overline{u}$ and $\underline{u}$ are stationary.

Step 4 : Properties of $\overline{u}$ and $\underline{u}$

Using the relation

$$u^{(p)}\left(x+1, t-\frac{p}{\lambda(p)}\right) = u^{(p)}(x,t),$$

we have

$$u^{(p)}(x,t) = u^{(p)}\left(x+\frac{1}{p}, t-\frac{1}{\lambda(p)}\right) = 1 + u^{(p)}\left(x, t-\frac{1}{\lambda(p)}\right),$$

so

$$u^{(p)}\left(x, t+\frac{1}{2\lambda(p)}\right) - u^{(p)}\left(x, t-\frac{1}{2\lambda(p)}\right) = 1.$$

As u is nondecreasing, we deduce that for all $\tau \leq \dfrac{1}{2\lambda(p)}$,

$$u^{(p)}\left(x, t+\tau\right) - u^{(p)}\left(x, t-\tau\right) \leq 1.$$

Taking the limit $p \to 0$, we have $\lambda(p) \to 0$ so for all $\tau > 0$,

$$u\left(x, t+\tau\right) - u\left(x, t-\tau\right) \leq 1.$$

As a consequence,

$$\overline{u} - \underline{u} \leq 1.$$

We also have

$$0 \leq \overline{u} - \underline{u}$$

because u is nondecreasing in t.

This concludes the proof of Proposition 4.12.

Proof of Lemma 4.13

Lemma 4.13 is obtained by considering what (4.11) (v), namely

$$u^{(p)}\left(x+1, t-\frac{p}{\lambda}\right) = u^{(p)}(x,t)$$

implies in the limit $p \to 0$.

In the case $c_p \to \infty$, it is clear that $\lambda(p) \neq 0$ for $|p|$ small enough and passing to the limit $p \to 0$ in (4.11) (v), we get

$$u(x+1,t) = u(x,t).$$

In the case $c_p \to c_0 \in (0,+\infty)$, it is also clear that $\lambda(p) > 0$ for $|p|$ small enough (even if 0 is an accumulation point of $\lambda(p)$), and in the limit we get

$$u\left(x+1, t-\frac{1}{c_0}\right) = u(x,t).$$

In the case $c_p \to 0$, in the positive case we have that $c_p > 0$ for a subsequence $p \to 0$, and for these values of p we have :

$$u^{(p)}(x,t) = u^{(p)}\left(x+1, t-\frac{1}{c_p}\right).$$

As $c_p \to 0$, if we fix $A > 0$ and $t \in \mathbb{R}$ we have $\dfrac{1}{c_p} > A$ for $|p|$ small enough, so as $u^{(p)}$ is a monotonous function of t, we have

$$u^{(p)}(x,t) \leq u^{(p)}(x+1, t-A),$$

so, in the limit,

$$u(x,t) \leq u(x+1, t-A).$$

As this is true for all $A > 0$ and for all $(x,t) \in \mathbb{R}\times\mathbb{R}$, and as u is monotonous in t with $\lim\limits_{t\to-\infty} u(x,t) = \underline{u}(x)$, we have that for all $(x,t) \in \mathbb{R}\times\mathbb{R}$;

$$u(x,t) \leq \underline{u}(x+1)$$

Now as $u(x,t)$ converges to $\overline{u}(x)$ when $t \to +\infty$ and the above relation holds for all t, we can conclude

$$\overline{u}(x) \leq \underline{u}(x+1).$$

In the case $c_p \to 0$ in the constant case, we have $\overline{u} = \underline{u} = u$ and Lemma 4.13 holds because $u(x+1) \geq u(x)$.

This concludes the proof of Lemma 4.13.

The following property will also be useful, as stationary periodic solutions appear everywhere as limits of solutions.

Lemma 4.15 (Identification of the stationary periodic solutions). *If u is a stationary 1-periodic solution of* (4.1) *satisfying the property* (4.12) *of Proposition 4.10, then u is constant and $u \in \{0,\theta\} + \mathbb{Z}$.*

Proof of Lemma 4.15

The fact that u satisfies (4.12) means that if $a \in \{0, \theta\} + \mathbb{Z}$, for any $t \in \mathbb{R}$, $u(x_0, t) > a$ implies $u(x, t) \geq a$ for any $x \geq x_0$, and $u(x_0, t) < a$ implies $u(x, t) \leq a$ for any $x \leq x_0$. Because of the periodicity of u, it is only possible if either u is constant (and its value is a zero of f), or u is bounded from above and from below by two consecutive zeros of f, which means there exists $k \in \mathbb{Z}$ such as either $k \leq u \leq k + \theta$ or $k + \theta \leq u \leq k + 1$.

Stationary solutions satisfy

$$0 = \Delta u + f(x, u)$$

Integrating this equation over a period ($[0, 1]$ for example) yields

$$\int_{[0,1]} f(x, u(x))dx = 0.$$

But if $k \leq u \leq k + \theta$, $f(x, u) \leq 0$, and $f(x, u)$ is identically equal to 0 only if u itself is constant and equal to k or $k + \theta$. Then if u is not constant we have

$$\int_{[0,1]} f(x, u(x))dx < 0,$$

which brings a contradiction. By the same argument, $k + \theta \leq u \leq k + 1$ is also only possible if u is constant, so the only possibility is that u is constant and its value is a zero of f.

4.3.2 Analysis of the case $c_p \to c_0 \in (0, +\infty)$

Proposition 4.16 (Properties of the solution when $c_0 \in (0, +\infty)$). *If $c_0 \in (0, +\infty)$, then any u limit of the $u^{(p)}$ given in Proposition 4.11 satisfies $\underline{u} = 0$ and $\overline{u} = 1$, with $\underline{u}$ and $\overline{u}$ defined in* (4.13).

Proof of Proposition 4.16

We can suppose $c_p > 0$ for $|p|$ small enough.

Step 1 : Identification of $\underline{u}$ and $\overline{u}$

We know that u is a solution of (4.1) which satisfies Proposition 4.10. By appropriate time translations we can ensure that $u^{(p)}(0, 0) = \theta$ for all p, and thus $u(0, 0) = \theta$.

Taking the limit of

$$u(x, t) = u\left(x + 1, t - \frac{1}{c}\right)$$

as $|t| \to \infty$, we get that $\overline{u}$ and $\underline{u}$ are 1-periodic. By Lemma 4.15, $\overline{u}$ and $\underline{u}$ are constant and their values are in $\{0, \theta\} + \mathbb{Z}$.

We recall that $u(0,0) = \theta$, which implies $\overline{u} \geq \theta$ because u is nondecreasing in time. Now we have two cases ($\overline{u} = \theta$ and $\overline{u} > \theta$).

Case 1 : $\overline{u} = \theta$
If $\overline{u} = \theta$, as $u(0,0) = \theta$ and u is nondecreasing in t, then we have $u_t(0,t) = 0$ for $t > 0$, and by the strong maximum principle (Proposition 4.6), $u_t \equiv 0$, which implies that u is stationary and $u = \overline{u} = \theta$.
Case 2 : $\overline{u} > \theta$
If $\overline{u} > \theta$, (4.14) implies that $\underline{u} > \theta - 1$.

By the same line of reasoning on $\underline{u}$, if u is non stationary, then $\underline{u} < \theta$ and $\overline{u} < \theta + 1$.
To summarize, we have the inequalities

$$\theta - 1 < \underline{u} < \theta < \overline{u} < \theta + 1$$

and we can conclude that when u is non stationary we have $\overline{u} = 1$ and $\underline{u} = 0$.

Step 2 : Exclusion of the case $u \equiv \theta$
By (4.3), we know that there exists θ_1, θ_2 with $0 < \theta_1 < \theta < \theta_2 < 1$ such that for all $x \in \mathbb{R}$, for $u \in [\theta_1, \theta_2]$,

$$f'_u(x,u) \geq \eta > 0. \tag{4.19}$$

Substep 2.1 : Minoration of u_t **in the set** $\{u = \theta_1\}$
We define

$$\begin{aligned} A = \{ \quad & a \in \mathbb{R} \mid \exists (p_n) \in (\mathbb{N}^{-1})^{\mathbb{N}} \text{ with } p_n \to 0, \\ & \exists ((x^{p_n}, t^{p_n}))_{p_n}, u^{(p_n)}(x^{p_n}, t^{p_n}) = \theta_1, u_t^{(p_n)}(x^{p_n}, t^{p_n}) \to a \quad \} \end{aligned}$$

the set of accumulation values of u_t at points where $u = \theta_1$. To make the proof more readable, we will use the general notations (p) and (x^p, t^p) for the sequences associated to elements of A.

We note that if $a \in \mathbb{R}$ is an accumulation point of A, we can show by a process of diagonal extraction that $a \in A$, so A is closed in $\mathbb{R}$.

We consider $a \in A$ and $((x^p, t^p))_p$ a corresponding subsequence. Using the relation

$$u^{(p)}(x,t) = u^{(p)}\left(x + 1, t - \frac{1}{c_p}\right)$$

we can suppose $0 \leq x^p < 1$, and we define $u^{(p)'} = u^{(p)}(\cdot, \cdot + t^p)$. With that definition, we have $u^{(p)'}(x^p, 0) = \theta_1$. The sequence $u^{(p)'}$ converges to a solution u' of (4.1) up to a subsequence. If we take $x_0 \in [0,1]$ an accumulation value of the x^p for this subsequence, we have that $u'(x_0, 0) = \theta_1$.

Now we can remark that if u'_t reaches the value 0, by the strong maximum principle (Proposition 4.6), $u'_t \equiv 0$ and u' is stationary. But as $\overline{u'}$ and $\underline{u'}$ are constant (as in Step 1), this would imply that u' is constant, and thus takes the value θ_1 everywhere, which brings a contradiction with the fact that it is a solution of (4.1).

As the convergence of $u^{(p)'}$ to u' is in $C^{2+\alpha,1+\frac{\alpha}{2}}(K)$ for all K compact, in particular ${u^{(p)}}'_t$ converges uniformly to u'_t, and so $u'_t(x_0,0)=a$ and $a>0$. As A is closed, $A\cap(-\infty,0]=\emptyset$ implies $\min\{A\}=\delta_A>0$.

Substep 2.2 : Minoration of u_t in the set $\{u=\theta_2\}$

If we define in similar fashion the set B of accumulation values of u_t at points where $u=\theta_2$, we can use the same method to show that $\min\{B\}=\delta_B>0$.

Substep 2.3 : Conclusion

We note $\delta=\min(\delta_A,\delta_B)$. The preceding results indicate that there exists $p_0>0$ such that for all p such that $0<p<p_0$, $u^{(p)}(x,t)\in\{\theta_1,\theta_2\}$ implies $u_t^{(p)}(x,t)>\dfrac{\delta}{2}$.

Now we suppose that a subsequence of $u^{(p)}$ converges to a constant solution $u\equiv\theta$. We then have $u_t\equiv 0$, so there exists $p_{cut}>0$ such that for all $p<p_{cut}$, we have $u_t^{(p)}(0,0)<\dfrac{\delta}{2}$ (with $u^{(p)}(0,0)=\theta$). We consider the subset $\Omega^p\subset\mathbb{R}^2$ defined by

$$\Omega^p=\left\{(x,t)\in\mathbb{R}^2 \mid \theta_1\le u^{(p)}(x,t)\le\theta_2\right\}.$$

It is a closed subset and by continuity of $u^{(p)}$ its boundary $\partial\Omega^p$ is included in the subset

$$\left\{(x,t)\in\mathbb{R}^2 \mid u^{(p)}(x,t)\in\{\theta_1,\theta_2\}\right\}.$$

Hence, $u_t^{(p)}>\dfrac{\delta}{2}$ on $\partial\Omega^p$. We remark that by (4.10) (i), we have

$$\Omega^p\subset\{|x+c_pt|\le K_p\}$$

for some constant $K_p>0$, and by (4.11) (v), we also have (see Figure 4.3) :

$$(x,t)\in\Omega^p\Rightarrow(x,t)+\left(1,-\frac{1}{c_p}\right)\in\Omega^p \text{ and } u^{(p)}(x,t)=u^{(p)}\left(x+1,t-\frac{1}{c_p}\right).$$

With these properties we can conclude that

$$u^{(p)}(\Omega^p)=u^{(p)}\left(\Omega^p\cap\{x\in[0,1]\}\right),$$

the set $\Omega^p\cap\{x\in[0,1]\}$ being bounded in $\mathbb{R}^2$.

As $(0,0)\in\Omega^p$ and $u_t^{(p)}(0,0)<\dfrac{\delta}{2}$ for p small enough, we can deduce that u_t reaches a minimum for a point $(x^p_{min},t^p_{min})\in\mathring{\Omega}^p$ (Figure 4.3) and we can

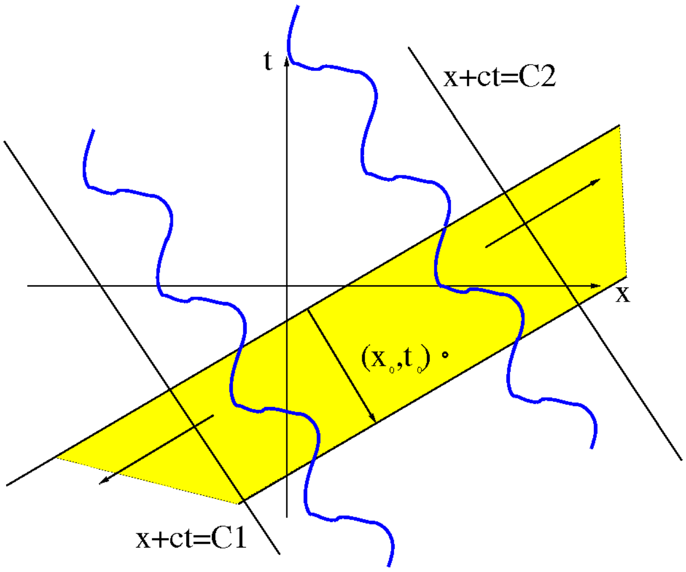

FIGURE 4.3 – Illustration of the periodicity of Ω^p.

suppose $x^p_{min} \in [0,1)$. If we write $v^{(p)} = u^{(p)}_t$, by Proposition 4.8 we know that $v^{(p)}$ is in $C^{2+\alpha;1+\frac{\alpha}{2}}(Q_r)$ for all parabolic cylinder $Q_r$and satisfies the following equation in the strong sense :

$$v^{(p)}_t = \Delta v^{(p)} + f'_u(x, u^{(p)})v^{(p)}. \tag{4.20}$$

With this equation we can first show that $v^{(p)} > 0$. Indeed, if $v^{(p)}$ reaches the value 0, by the strong maximum principle (Proposition 4.6), $v^{(p)} = 0$ everywhere, but the inequality $\left|u^{(p)} - px - \lambda t\right| \leq C$ forbids that $u^{(p)}$ is stationary.

As $v^{(p)}$ reaches a minimum on (x^p_{min}, t^p_{min}), at that point $v^{(p)}_t = 0$ and $\Delta v \geq 0$, and at that point $u^{(p)} \in [\theta_1, \theta_2]$ so $f'_u(x^p_{min}, u^{(p)}) > 0$, but we also have $v^{(p)} > 0$, so we have a contradiction with (4.20).

As a consequence, u can not be stationary with value θ.

4.3.3 Analysis of the case $c_p \to +\infty$

Proposition 4.17 (**Exclusion of the case** $c_p \to \infty$)**.**
The sequence c_p can not diverge to $+\infty$.

Proof of Proposition 4.17

Step 1 : Reduction to the case $u \equiv \theta$

The same reasoning as the one in the Step 1 of the proof of Proposition 4.16 (where $c \in (0, +\infty)$) guarantees that $\overline{u}$ and $\underline{u}$ are constant, and that either $u \equiv \theta$ (the constant case), or $\overline{u} = 1$ and $\underline{u} = 0$ (the non stationary case).

From (4.17) with $c_0 = +\infty$, we have that for all t, $u(\cdot, t)$ is 1-periodic. We know that $u(0,0) = \theta$, so if $u \leq \theta$, or $u \geq \theta$, by the strong maximum principle (Proposition 4.6), $u \equiv \theta$ in both cases, and we are in the constant case.

But if $u(\cdot, 0) \leq \theta$, the comparison principle (Proposition 4.5) guarantees $u(\cdot, t) \leq \theta$ for $t > 0$ and the monotonicity of u implies $u(\cdot, t) \leq \theta$ for $t < 0$. So we have $u \leq \theta$ on $\mathbb{R}^2$, which we have already reduced to the constant case. Similarly, we can't have $u(\cdot, 0) \geq \theta$ in the non stationary case.

So there exist $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ such that $u(x_1, 0) > \theta$ and $u(x_2, 0) < \theta$. Because of the 1-periodicity of u in x, we can suppose $-1 \leq x_1 < 0 < x_2 \leq 1$. But this contradicts the fact that u satisfies (4.12).

Step 2 : Exclusion of the case $u \equiv \theta$

The line of reasoning given in Step 2 of the proof of Proposition 4.16 (where $c \in (0, +\infty)$) is also valid when $c_0 = +\infty$, using the convention $\frac{1}{c} = 0$.

4.3.4 Analysis of the case $c_p \to 0$, with a stationary $c_p = 0$ for $|p|$ small enough

Proposition 4.18. (Properties of the solution when $c_0 = 0$ and $c_p = 0$ for a subsequence (the constant case))
If $c_p = 0$ for $|p|$ small enough, then u is stationary and is a transition between $u^- = 0$ and $u^+ = 1$.

Proof of Proposition 4.18

In this case, for the subsequence mentioned above, the solutions $u^{(p)}$ are all stationary, and as a consequence u is stationary. In the following of this proof we will use the auxiliary function $w^{(p)}(x) = u^{(p)}(x+1) - u^{(p)}(x)$ and $w(x) = u(x+1) - u(x)$. As we know that $u^{(p)}(x+1) \geq u^{(p)}(x)$, these functions $w^{(p)}$ and w are non-negative. We begin this proof by showing that they satisfy a strong maximum principle, that is if w (or $w^{(p)}$) reaches the value 0, then w (or $w^{(p)}$) is constant with value 0.

Step 1 : A strong maximum principle for w

Suppose that there exists $x_0 \in \mathbb{R}$ such that $w(x_0) = 0$, that is $u(x_0) = u(x_0 + 1) = d_0$. As $u(x+1) \geq u(x)$ everywhere, this implies that there exists $d_1 \in \mathbb{R}$ such that $u_x(x_0) = u_x(x_0 + 1) = d_1$. As f is 1-periodic in space, the functions u and $\tilde{u} = u(\cdot + 1)$ are both solutions of the Cauchy problem

$$\begin{cases} 0 = u_{xx} + f(x, u) & \text{on } (x_0, +\infty) \\ u(x_0) = d_0 \\ u'(x_0) = d_1 \end{cases}$$

As f is Lipschitz-continuous, we have that u and $\tilde{u}$ coincide on $(x_0, +\infty)$. In the same manner, we can show they coincide on $(-\infty, x_0)$, so u and $\tilde{u}$ are equal. To sum up, we have that if w reaches the value 0, then it is necessarily constant.

Step 2 : Identification of u^- and u^+

The solution u is stationary, so both u^+ and u^- are stationary 1-periodic solutions of (4.1). As a consequence, they are constant and their value is a zero of f.

By suitable space translations we can ensure that for all $p > 0$ there exists $x^p \in [0, 1)$ such that $u^{(p)}(x^p) = \theta$. If we note x_0 an accumulation values of the sequence x^p as $p \to 0$, we have $u(x_0) = \theta$.

As $u^- \leq u \leq u^+$ and $u^+ - u^- \leq 1$, u^- is either $\theta - 1$, 0 or θ, and u^+ is either θ, 1 or $\theta + 1$. If u^+ or u^- is equal to θ, then u is constant (and equal to θ). Indeed, $u(0) = u^+ = \theta$ and $u(x + 1) \geq u(x)$ imply that $u(n) = \theta$ for $n \in \mathbb{N}$, because $u^+ = \lim u(x+n)$. It means that $w = u(x+1) - u(x)$, which is non-negative, reaches the value 0 and by the strong maximum principle (see Step 1 of this proof) it is constant and equal to 0, which in turn implies that u is constant. The only other case is $u^- = 0$ and $u^+ = 1$.

Step 3 : Exclusion of the case $u \equiv \theta$

In a similar way as for the above cases, we know that for all $x \in \mathbb{R}$, $f(x, u)$ is increasing in u for $u \in [\theta_1, \theta_2]$, where $0 < \theta_1 < \theta < \theta_2 < 1$. We consider $\theta_3 \in (\theta, \theta_2)$ and remark that for any $p > 0$, if $\theta_1 \leq u^{(p)}(x) \leq \theta_3$ and $w^{(p)}(x) < \theta_2 - \theta_3 := \delta_\theta$, then $\theta_1 \leq u^{(p)}(x + 1) \leq \theta_2$.

Substep 3.1 : Minoration of w on the set $\{u = \theta_1\}$

We define

$$\begin{aligned} A' = \{ \quad & a \in \mathbb{R} \mid \exists (p_n) \in (\mathbb{N}^{-1})^{\mathbb{N}} \text{ with } p_n \to 0, \\ & \exists (x^{p_n})_{p_n}, u^{(p_n)}(x^{p_n}) = \theta_1, u_t^{(p_n)}(x^{p_n}) \to a \quad \} \end{aligned}$$

the set of accumulation values of w at points where $u = \theta_1$. As before, we will use the simplified notations (p) and (x^p) for subsequences associated to elements of A'. We note that if $a \in \mathbb{R}$ is an accumulation point of A', we can show by a process of diagonal extraction that $a \in A'$, so A' is closed in $\mathbb{R}$.

We consider $a \in A'$ and $(x_p)_p$ a corresponding subsequence, and we define

$$u^{(p)'} = u^{(p)}(\cdot + \lfloor x_p \rfloor).$$

Then $u^{(p)'}$ is a solution of (4.1) with $u^{(p)'}(x_p - \lfloor x_p \rfloor) = \theta_1$. The sequence $u^{(p)'}$ converges to a solution u' of (4.1) up to a subsequence, and if we take $x_0 \in [0, 1]$ an accumulation value of $x_p - \lfloor x_p \rfloor$, we have that $u'(x_0) = \theta_1$.

Now we remark that if $w'(x) = u'(x + 1) - u'(x)$ reaches the value 0, by the strong maximum principle, it is constant with value 0, which implies that u' is 1-periodic, $u' = u'^{+} = u'^{-}$, and u' is a constant and its value is a zero of f, which contradicts $u'(x_0) = \theta_1$. But $w^{(p)'}$ converges uniformly to w', so

$$w'(x_0) = \lim_{p \to 0} \left(w^{(p)'}(x_p - \lfloor x_p \rfloor) \right) = a,$$

so $a > 0$, and as A' is closed, $\min A' = \delta_{A'} > 0$.

Substep 3.2 : Minoration of w on the set $\{u = \theta_3\}$

If we define in similar fashion the set B' of accumulation values of w at points where $u = \theta_3$, we can use the same method to show that $\min \{B'\} = \delta_{B'} > 0$.

Substep 3.3 : Conclusion

We note $\delta' = \min(\delta_{A'}, \delta_{B'}, \delta_\theta)$. The preceding results indicate that there exists $p'_0 > 0$ such that for all $p > 0$ such that $p < p'_0$, $u^{(p)}(x,t) \in \{\theta_1, \theta_3\}$ implies $w^{(p)}(x,t) > \dfrac{\delta'}{2}$.

Now we suppose that a subsequence of $u^{(p)}$ converges to a constant solution $u \equiv \theta$ (this implies $w \equiv 0$). For all $p > 0$, the bounds

$$|u^{(p)}(x) - px| \leq C$$

mean that there exists $x^p \in \mathbb{R}$ such that $u^{(p)}(x^p) = \theta$. Up to a translation $u'^{(p)}(x) = u^{(p)}(x + \lfloor x^p \rfloor)$, which preserves the solution, we can suppose $x^p \in [0,1)$. As a consequence, there exists a sequence $p_n \to 0$, a sequence $(x_n) \in [0,1)^{\mathbb{N}}$ and $p'_{cut} > 0$ such that for $|p_n| < p'_{cut}$, $w^{(p_n)}(x_n) < \dfrac{\delta'}{2}$ (with $u^{(p_n)}(x_n) = \theta$). Now for $p \in (p_n)_n$, $p < \min(p'_0, p'_{cut})$, we consider the subset $\Omega^p \subset \mathbb{R}$ defined as above :

$$\Omega^p = \left\{ x \in \mathbb{R} \mid \theta_1 \leq u^{(p)}(x) \leq \theta_3 \right\}.$$

It is a closed subset and its boundary $\partial\Omega^p$ is included in the subset

$$\left\{ x \in \mathbb{R} \mid u^{(p)}(x) \in \{\theta_1, \theta_3\} \right\}.$$

Hence, $w(p) > \dfrac{\delta'}{2}$ on $\partial\Omega^p$. As $(x^p) \in \Omega^p$ and $w^{(p)}(x^p) < \dfrac{\delta'}{2}$, we can deduce that w reaches a minimum for a point $(x^p_{min}) \in \mathring{\Omega}^p$. Now $w^{(p)}$ satisfies the following equation

$$0 = \Delta w^{(p)}(x) + (f(x, u^{(p)}(x+1)) - f(x, u^{(p)}(x)). \tag{4.21}$$

As $w^{(p)}$ reaches a minimum on (x^p_{min}), at that point

$$\Delta w^{(p)}(x^p_{min}) \geq 0. \tag{4.22}$$

Moreover, $w^{(p)} > 0$ because if it reaches the value 0, by the strong maximum principle (see Step 1 of this proof), $w^{(p)} = 0$ everywhere, but the inequality $\left|u^{(p)} - px\right| \le C$ forbids that. As a consequence, we have $u^{(p)}(x^p_{min} + 1) > u^{(p)}(x^p_{min})$. By construction, $\theta_1 \le u^{(p)}(x^p_{min}) \le \theta_3$, and we have specified $w^{(p)}(x^p_{min}) \le \frac{\delta'}{2} < \delta_\theta$, so $\theta_1 \le u^{(p)}(x^p_{min}) \le \theta_3$. As for every value of x, f is an increasing function of u when u is in the interval $[\theta_1, \theta_2]$, we have

$$f(x^p_{min}, u^{(p)}(x^p_{min} + 1)) - f(x^p_{min}, u^{(p)}(x^p_{min})) > 0. \tag{4.23}$$

Combining (4.21),(4.22) and (4.23), we get a contradiction.

As a consequence, u can not be stationary with value θ.

4.3.5 Analysis of the case $c_p \to 0$, with $c_p > 0$

Proposition 4.19 (Properties of the solution when $c_0 = 0$ in the positive case). *If $c_p \to 0$ and the sequence is not stationary, with $c_p > 0$, then there exists a stationary solution u of* (4.1) *which satisfies* (4.5),(4.6), *and is a transition between $u^- = 0$ and $u^+ = 1$.*

Proof of Proposition 4.19
Step 1 : Behaviour of $\overline{u}$ and $\underline{u}$ for the limit u of $u^{(p)}$

As $c_p > 0$ for $p > 0$, we can suppose that $u^{(p)}(0,0) = \theta$, and so $u(0,0) = \theta$. We know that $\overline{u}$ and $\underline{u}$ are two stationary solutions and we recall that

$$\underline{u}(x+1) \ge \overline{u}(x).$$

We note that as $n \to \infty$, both $\overline{u}(x+n)$ and $\underline{u}(x+n)$ converge to periodic stationary solutions, that is constants in $\{0, \theta\} + \mathbb{Z}$. But as $\underline{u}(x) \le \overline{u}(x) \le \underline{u}(x+1)$, these limits are necessary the same, and $\underline{u}^+ = \overline{u}^+ = u^+$. In the same way, $\underline{u}^- = \overline{u}^- = u^-$.

As $u(0,0) = \theta$, we have $u^- = \underline{u}^- \le \theta \le \overline{u}^+ = u^+$, and we know that $u^+ - u^- \le 1$.

Now suppose $u^+ = \theta$. As $u(0,0) = \theta$, $\overline{u}(0) \ge \theta$ because u is increasing in time. Combining this with $u^+ = \theta$, it implies $\overline{u}(0) = \theta$ and as $\overline{u}(n)$ is nondecreasing, $\overline{u}(n) = \theta$ for all $n \in \mathbb{N}$. But in this case, $\overline{w}(x) = \overline{u}(x+1) - \overline{u}(x)$ reaches the value 0, and by the strong maximum principle (see Step 1 of the proof of Proposition 4.18), it is necessary constant and equal to 0. This implies that $\overline{u}$ is constant and equal to θ, and as

$$0 = \overline{u}^+ - \underline{u}^+ = \lim \overline{u}(x+n) - \underline{u}(x+n)$$

and

$$0 = \overline{u}^- - \underline{u}^- = \lim \overline{u}(x-n) - \underline{u}(x-n),$$

$\underline{u}$ is also constant with value θ, which in turn implies that u is constant with value θ, a case which we will exclude below.

The same happens if $u^- = \theta$.

The only remaining case is $u^- = 0$, and $u^+ = 1$. In that case, both $\underline{u}$ and $\overline{u}$ are stationary solutions of (4.1), with $\overline{u}^- = \underline{u}^- = 0$ and $\overline{u}^+ = \underline{u}^+ = 1$, which is what we wanted to prove. Note that these two stationary solutions may be equal, in which case u itself is stationary.

Step 2 : Exclusion of the case $u \equiv \theta$

This proof given in the case $c_p \to c > 0$ still holds here. Actually, in the proof in the case $c > 0$, we never use the fact that the limit c is positive, only the fact that for $p > 0$, $c_p > 0$. So in our case u can not be stationary with value θ either.

4.3.6 Proof of the main existence result

In this subsection we use the preceding results to prove the existence part of Theorem 4.1

Proof of Theorem 4.1

Keeping in mind that the negative case is similar to the positive case, Proposition 4.17 indicates that $c_0 \in \mathbb{R}$. Proposition 4.12 guarantees that the solution u we have constructed satisfies (4.6) and $c_0 u_t \geq 0$. Finally, Proposition 4.16 shows that the remainder of (4.5) holds when $c_0 \neq 0$, and Proposition 4.18 and Proposition 4.19 show that (4.5) also holds when $c_0 = 0$, which concludes the proof of the existence part of Theorem 4.1.

4.4 Uniqueness of the profile and of the velocity

In this section we show the uniqueness part of Theorem 4.1 (uniqueness of the profile u) and the result of Theorem 4.3 (uniqueness of the velocity c_0).

4.4.1 Uniqueness of the profile

In this first subsection we prove the uniqueness of the solution for a given speed c_0 :

Proposition 4.20.
For a given $c_0 > 0$, there exists a unique (up to time translations) solution u of (4.1) in $\mathbb{R}^2$ satisfying the properties (4.5) with $\underline{u} = 0$ and $\overline{u} = 1$ and $u\left(x+1, t - \dfrac{1}{c_0}\right) = u(x,t)$ (with $0 < c_0 < +\infty$).

Proof of proposition 4.20
Step 1 : Setting of the proof

Assumption (4.3) guarantees that there exist $\theta_1, \theta_2 \in]0,1[$ such that for all $x \in \mathbb{R}$, $f(x,\cdot)$ is decreasing on $[0,\theta_1]$ and on $[\theta_2, 1]$. We consider two solutions u and v satisfying the properties above. We use the relation

$$u\left(x+1, t-\frac{1}{c_0}\right) = u(x,t)$$

and deduce that there exists a band $\Sigma = \{-M \le x + ct \le M\}$ of width R such that for all $(x,t) \in \Sigma^- = \{x + ct < -M\}$, $u(x,t) < \theta_1$ and $v(x,t) < \theta_1$, and for all $(x,t) \in \Sigma^+ = \{x + ct > M\}$, $u(x,t) > \theta_2$ and $v(x,t) > \theta_2$.

Step 2 : Comparison of u and v

First, we can show that there exists $\tau > 0$ such that

$$\left\{\begin{array}{ll} v(\cdot,\cdot+\tau) \ge u & \text{on } \Sigma \\ v(\cdot,\cdot+\tau) \ge \theta_2 & \text{on } \Sigma^+ \\ u \le \theta_1 & \text{on } \Sigma^- \end{array}\right. \tag{4.24}$$

We already know that $u \le \theta_1$ on Σ^-.

Moreover, we know that for all $x \in \mathbb{R}$, $\lim_{t\to+\infty} v(x,t) = 1$. The solution u is continuous on the compact set $\Sigma_p = \Sigma \cap ([0,1]\times\mathbb{R})$, so it has a maximum u_{max}, and as $0 < u < 1$, $u_{max} < 1$. Note that by construction of Σ, $\max_{\Sigma} \ge \theta_2$, and by periodicity $u_{max} > \theta_2$. Now the limit $v(x,t) \to 1$ as $t \to \infty$ is uniform on $x \in [0,1]$ (by compactness), so there exists $t_m > 0$ such that for all $t > t_m$, $x \in [0,1]$, $v(x,t) > u_{max}$. So for τ high enough ($\tau = t_{max} + \dfrac{M+1}{c}$ is sufficient), $v(\cdot,\cdot+\tau) > u_{max}$ on Σ_p, so $v(\cdot,\cdot+\tau) > u$ on Σ_p. By the periodicity property of u and v, this holds on all Σ.

Finally, $v(\cdot,\cdot+\tau) > u_{max}$ on $\Sigma \cup \Sigma^+$, so in particular $v > \theta_2$ on Σ^+, which proves (4.24).

We note $w = v(\cdot,\cdot+\tau)$. With the properties above, we can show that $w \ge u$ on $\mathbb{R}^2$. We already have that $w \ge u$ on Σ.

In the following part of the proof we use the vectors

$$e_1 = \frac{1}{1+c^2}(1,c) \text{ and } e_2 = \frac{1}{1+c^2}(-c,1).$$

We define $m = \sup_{\Sigma^+}(u-w)$. We want to show by contradiction $m \le 0$, so we suppose $m > 0$. As $(w-u)((x,t)+se_1) \to 0$ as $s \to +\infty$ and $(w-u)\left((x,t)+\left(-\frac{1}{c}e_2\right)\right) = (w-u)(x,t)$, this supremum m is reached for a finite point $P_0 = (x_0,t_0)$ (and we can for example suppose $0 \le x_0 < 1$). We have the equation

$$(u-w)_t = \Delta(u-w) + f(x,u) - f(x,w)$$

and at the point P_0, $(u-w)$ reaches a maximum, so $(u-w)_t = 0$, and $\Delta(u-w) \leq 0$. As $m > 0$, we also have $u(P_0) > w(P_0) \geq \theta_2$ and for all $x \in \mathbb{R}$, $f(x,\cdot)$ is decreasing on $[\theta_2, 1]$, so $f(x_0, u(P_0)) < f(x_0, w(P_0))$, which brings a contradiction. This proves that $m \leq 0$.

In the same way we prove that $m' = \sup\limits_{\Sigma^-}(u-w) \leq 0$, and we can conclude that $w \geq u$.

Step 3 : Conclusion by the sliding method

The functions u and w are two solutions of (4.1) satisfying (4.5) with $\overline{u} = \overline{w} = 1$, $\underline{u} = \underline{w} = 0$, and $w \geq u$. We can conclude the proof by the sliding method.

We define $\alpha^* = \sup\{\alpha \geq 0 \mid \forall \tau \leq \alpha, u(x, t+\tau) \leq w(x,t)\}$. We then have that $u(x, t+\alpha^*) \leq w(x,t)$ and $\inf\limits_{\mathbb{R}^2}(w(x, t-\alpha^*) - u(x,t)) = 0$ (if it was positive, as u and w are Lipschitz, we could construct $\alpha' > \alpha^*$ such that for all $\tau \leq \alpha'$, $u(x, t+\tau) \leq w(x,t)$, which would contradict the definition of α^*).

More precisely, if we define the widened band

$$\overline{\Sigma} = \{(x,t) \mid -A < x + ct < A\}$$

such that for some $\varepsilon > 0$, for all $\alpha \in [0, \alpha^* + \varepsilon]$, $w(\cdot, \cdot - \alpha) \geq \theta_2$ on $\overline{\Sigma}^+ = \{(x,t) \mid x + ct \geq A\}$ (in fact by the monotonicity of w it is sufficient to have this property for $\alpha = \alpha^* + \varepsilon$), we even have $\inf\limits_{\overline{\Sigma}}(w(x, t-\alpha^*) - u(x,t)) = 0$.

Indeed, if it was positive, there would exist $\delta > 0$ such that $w(\cdot, \cdot - \alpha^*) - u \geq \delta > 0$ on $\overline{\Sigma}$. As u and w are both Lipschitz, there would then exist $\varepsilon' \in [0, \varepsilon]$ such that for all $\alpha \in [\alpha^*, \alpha^* + \varepsilon']$, $w(\cdot, \cdot - \alpha) - u \geq \dfrac{\delta}{2}$. But as for $\alpha \in [0, \alpha^* + \varepsilon']$ we also have $w(\cdot, \cdot - \alpha) \geq \theta_2$ on $\overline{\Sigma}^+$, we deduce as in Step 2 that $w(\cdot, \cdot - \alpha) \geq u$ in $\mathbb{R}^2$ for $\alpha \in [0, \alpha^* + \varepsilon']$, which contradicts the definition of α^*.

Now we know that $\inf\limits_{\overline{\Sigma}}(w(x, t-\alpha^*) - u(x,t)) = 0$, and by the periodicity property of $w(\cdot, \cdot - \alpha^*) - u$, the infimum is reached at a finite point $P_0 = (x_0, t_0) \in \overline{\Sigma}$. In that case we have $w(x,t) \geq u(x, t+\alpha^*)$ and $w(x_0, t_0 - \alpha^*) = u(x_0, t_0)$. By the strong maximum principle (Proposition 4.6) $w = u(\cdot, \cdot + \alpha^*)$, and u and v are equal up to a time translation.

4.4.2 Uniqueness of the velocity

Proof of Theorem 4.3

We recall that we have supposed

$$\begin{cases} f'_u(x,0) = -\delta_0 < 0 & \text{for all} \quad x \in \mathbb{R} \\ f'_u(x,1) = -\delta_1 < 0 & \text{for all} \quad x \in \mathbb{R} \end{cases}.$$

We show the uniqueness of c by contradiction. We suppose there exist $c_1, c_2 \geq 0$ with $c_1 > c_2$ and solutions u_1 and u_2 of (4.1) satisfying (4.5) and (4.6) with respective velocities c_1 and c_2. The objective is to show that $u'_2 \geq u'_1$ for time translations u'_1 of u_1 and u'_2 of u_2 and deduce that there is a contradiction. The change of variable

$$\varphi(x,t) = u(x - ct, t)$$

described by Berestycki and Hamel in [2] could be adapted to build a more generic proof than the present one and lift the restricting assuption (4.4).

In the remainder of this subsection the generic velocity c and the velocities c_1 and c_2 are always the velocities as $p \to 0$.

Step 1 : Setting

As above, for $i = 1, 2$, we have bands $\Sigma_{M,c_i} = \{-M \leq x + c_i t \leq M\}$ of width R_i such that for all $(x,t) \in \Sigma^-_{M,c_i} = \{x + c_i t < -M\}$, $u_i(x,t) < \theta_1$, and for all $(x,t) \in \Sigma^+_{M,c_i} = \{x + c_i t > M\}$, $u_i(x,t) > \theta_2$.

Step 2 : Construction of subsolutions and supersolutions in the limits $x + ct \to +\infty$ and $x + ct \to -\infty$.

We define

$$u^*(x,t) = e^{\bar{\mu}(x+ct)}$$

and look for a condition on $\bar{\mu}$ for u^* to be a subsolution or a supersolution of (4.1) on $\Sigma^-_{A,c} = \{(x,t), x + ct < -A\}$ (the same work can be done on $\Sigma^+_{A,c} = \{(x,t), x + ct > A\}$). We calculate

$$u^*_t = \bar{\mu} c e^{\bar{\mu}(x+ct)}$$

$$\Delta u^* = \bar{\mu}^2 e^{\bar{\mu}(x+ct)}$$

$$e^{-\bar{\mu}(x+ct)} f(x, u^*) = e^{-\bar{\mu}(x+ct)} f(x, e^{\bar{\mu}(x+ct)}) = -\delta_0 + O_0(e^{\bar{\mu}(x+ct)})$$

and because of the periodicity of f in x, $e^{-\bar{\mu}(x+ct)} f(x,u^*) + \delta_0 = O_0(e^{\bar{\mu}(x+ct)})$ uniformly in x, which means that there exists $C_f > 0$ and $\varepsilon > 0$ such that $e^{\bar{\mu}(x+ct)} < \varepsilon$ implies

$$\left| e^{-\bar{\mu}(x+ct)} f(x,u^*) + \delta_0 \right| < C_f e^{\bar{\mu}(x+ct)}.$$

So u^* is a supersolution on $\overline{\Sigma}^-_{A,c}$ if $\bar{\mu} c \geq \bar{\mu}^2 - \delta_0 + C'$ where

$$C' = \sup_{\overline{\Sigma}^-_{A,c}} \left| e^{-\bar{\mu}(x+ct)} f(x, e^{\bar{\mu}(x+ct)}) + \delta_0 \right|.$$

For A large enough, this constant C' can be as small as one wants. If we define $\mu = \dfrac{c + \sqrt{c^2 + 4\delta_0}}{2}$ then $\mu c - \mu^2 = -\delta_0$, and if $\bar{\mu} < \mu$, then

$$\bar{\mu} c - \bar{\mu}^2 + \delta = \bar{\mu} c - \bar{\mu}^2 - (\mu c - \mu^2) = \eta > 0.$$

Then for A large enough,

$$e^{-\bar{\mu}(x+ct)} f(x, e^{\bar{\mu}(x+ct)}) + \delta_0 < \eta,$$

and u^* is a supersolution of (4.1) on $\overline{\Sigma}^-_{A,c}$.

Conversely, u^* is a subsolution on $\overline{\Sigma}^-_{A,c}$ if $\bar{\mu}c - \bar{\mu}^2 \leq -\delta - C'$. As a consequence, if $\bar{\mu} > \mu$, for A large enough, u^* is a subsolution of (4.1) on $\overline{\Sigma}^-_{A,c}$.

We now do the same work on $\overline{\Sigma}^+_{A,c}$. We define $u^* = 1 - e^{-\bar{\mu}(x+ct)}$ and we calculate

$$u^*_t = \bar{\mu} c e^{-\bar{\mu}(x+ct)}$$

$$\Delta u^* = -\bar{\mu}^2 e^{-\bar{\mu}(x+ct)}$$

$$e^{\bar{\mu}(x+ct)} f(x, u^*) = e^{\bar{\mu}(x+ct)} f(x, 1 - e^{-\bar{\mu}(x+ct)}) = \delta_1 + O_0(e^{-\bar{\mu}(x+ct)})$$

and $e^{\bar{\mu}(x+ct)} f(x, u^*) - \delta_1 = O_0(e^{-\bar{\mu}(x+ct)})$ uniformly in x. Then the function u^* is a supersolution of (4.1) in $\Sigma^+_{A,c}$ if $\bar{\mu}c + \bar{\mu}^2 \geq \delta_1 + C''$, where

$$C'' = \sup_{\Sigma^+_{A,c}} (e^{\bar{\mu}(x+ct)} f(x, 1 - e^{-\bar{\mu}(x+ct)}) - \delta_1).$$

We define $\mu' = \dfrac{-c + \sqrt{c^2 + 4\delta}}{2}$. In the same manner as above, if $\bar{\mu} > \mu'$, there exists $A > 0$ such that u^* is a supersolution of (4.1) on $\overline{\Sigma}^+_{A,c}$.

Conversely, u^* is a subsolution on $\overline{\Sigma}^+_{A,c}$ if $-\bar{\mu}c - \bar{\mu}^2 \leq -\delta - C'$. As a consequence, if $\bar{\mu} < \mu'$, for A large enough, u^* is a subsolution of (4.1) on $\overline{\Sigma}^+_{A,c}$.

Step 3 : Comparison of u_1 and u_2

We want to compare time translations of u_1 and u_2 on $]-\infty, -A] \times \{0\}$, and on $[A, \infty[\times\{0\}$ for some A. For that we will compare them to ansatz for solutions on Σ^-_{A,c_i} and Σ^+_{A,c_i} for $i = 1, 2$.

As $c_1 > c_2$, we have

$$\mu_1 = \frac{c_1 + \sqrt{c_1^2 + 4\delta_0}}{2} > \frac{c_2 + \sqrt{c_2^2 + 4\delta_0}}{2} = \mu_2$$

and

$$\mu'_1 = \frac{-c_1 + \sqrt{c_1^2 + 4\delta_1}}{2} < \frac{-c_2 + \sqrt{c_2^2 + 4\delta_1}}{2} = \mu'_2$$

(because $x \mapsto -x + \sqrt{x^2 + 4\delta_1}$ is decreasing for $x > 0$).

Then we note

$$\mu_{2+} = \frac{2\mu_2 + \mu_1}{3}$$
$$\mu_{1-} = \frac{\mu_2 + 2\mu_1}{3}$$
$$\mu'_{2-} = \frac{\mu'_1 + 2\mu'_2}{3}$$
$$\mu'_{1+} = \frac{2\mu'_1 + \mu'_2}{3},$$

and we have the following inequalities

$$\begin{array}{l} \mu_2 < \mu_{2+} < \mu_{1-} < \mu_1 \\ \mu'_1 < \mu'_{1+} < \mu'_{2-} < \mu'_2. \end{array}$$

We consider a real number $A_0 > 0$ such that

$$\begin{cases} e^{\mu_{1-}(x+c_1t)} & \text{is a supersolution of (4.1) on } \overline{\Sigma}^-_{A_0,c_1} \\ e^{\mu_{2+}(x+c_2t)} & \text{is a subsolution of (4.1) on } \overline{\Sigma}^-_{A_0,c_2} \\ 1-e^{-\mu_{1+}(x+c_1t)} & \text{is a supersolution of (4.1) on } \overline{\Sigma}^+_{A_0,c_1} \\ 1-e^{-\mu_{2-}(x+c_2t)} & \text{is a subsolution of (4.1) on } \overline{\Sigma}^+_{A_0,c_2} \end{cases}.$$

The preceding step assures that such a A_0 exists. Then there exists $\tau_0 > 0$ such that

$$\begin{cases} u_2(x,\tau_0) > u_1(x,-\tau_0) & \text{for} \quad x \in [-A_0, A_0] \\ u_2(x,t+\tau_0) > e^{\mu_{2+}(x+c_2t)} & \text{in} \quad \{(x,t) \mid x+c_2t = -A_0\} \\ u_2(x,t+\tau_0) > 1-e^{-\mu_{2-}(x+c_2t)} & \text{in} \quad \{(x,t) \mid x+c_2t = A_0\} \\ u_1(x,t-\tau_0) < e^{\mu_{1-}(x+c_1t)} & \text{in} \quad \{(x,t) \mid x+c_1t = -A_0\} \\ u_1(x,t-\tau_0) < 1-e^{-\mu_{1+}(x+c_1t)} & \text{in} \quad \{(x,t) \mid x+c_1t = A_0\} \end{cases}.$$

There exists τ_0 such that the last four lines hold because

$$\lim_{x+c_1t\to-\infty} u_1(x,t) = 0 \text{ and } \lim_{x+c_2t\to+\infty} u_2(x,t) = 1.$$

Moreover, the first line also holds because

$$\lim_{t\to-\infty} u_1(x,t) = 0 \text{ uniformly in } x \in [-A_0, A_0],$$

and

$$\lim_{t\to+\infty} u_2(x,t) = 1 \text{ uniformly in } x \in [-A_0, A_0].$$

Using a comparison principle in the half-planes (this comparison principle will be justified at the end of the proof (Step 5) in order not to disrupt the reasoning) $\overline{\Sigma}^-_{A_0,c_1}$, $\overline{\Sigma}^-_{A_0,c_2}$, $\overline{\Sigma}^+_{A_0,c_1}$, $\overline{\Sigma}^+_{A_0,c_2}$, we have that

$$\begin{array}{lll} u_1(x,t-\tau_0) \le e^{\mu_{1-}(x+c_1t)} & \text{in} & \overline{\Sigma}^-_{A_0,c_1} \\ u_2(x,t+\tau_0) \ge e^{\mu_{2+}(x+c_2t)} & \text{in} & \overline{\Sigma}^-_{A_0,c_2} \\ u_1(x,t-\tau_0) \le 1-e^{-\mu_{1+}(x+c_1t)} & \text{in} & \overline{\Sigma}^+_{A_0,c_1} \\ u_2(x,t+\tau_0) \ge 1-e^{-\mu_{2-}(x+c_2t)} & \text{in} & \overline{\Sigma}^+_{A_0,c_2}. \end{array}$$

We use these inequalities at time $t = 0$ to obtain

$$\begin{array}{lll} u_1(x,-\tau_0) \leq e^{\mu_1 - x} \leq e^{\mu_2 + x} \leq u_2(x,\tau_0) & \text{for} & x \leq -A_0 \\ u_1(x,\tau_0) \leq 1 - e^{-\mu_1 + x} \leq 1 - e^{-\mu_2 - x} \leq u_2(x,\tau_0) & \text{for} & x \geq A_0. \end{array}$$

We define $u_1'(x,t) = u_1(x, t - \tau_0)$ and $u_2'(x,t) = u_2(x, t + \tau_0)$. As we already knew that $u_1' \leq u_2'$ on $[-A_0, A_0] \times \{0\}$, we can conclude that $u_1'(x,0) \leq u_2'(x,0)$ for all $x \in \mathbb{R}$.

Step 4 : Conclusion

As $u_1' \leq u_2'$ at time $t = 0$, by the comparison principle (Proposition 4.5) we have $u_1' \leq u_2'$ for all $t \geq 0$. But we also have $|u_i - c_i t| \leq C$ for $i = 1, 2$, and $c_1 > c_2$, which implies that for t high enough, $u_1' > u_2'$, which is a contradiction. We have thus proved the uniqueness of c.

Step 5 : Justification of the comparison principle in half-planes

There only remains to justify the use of a comparison principle in the half-planes $\overline{\Sigma}^-_{A_0,c_1}$, $\overline{\Sigma}^-_{A_0,c_2}$, $\overline{\Sigma}^+_{A_0,c_1}$, $\overline{\Sigma}^+_{A_0,c_2}$, which is done in this step. We do it only in the first case, for the half-plane $\overline{\Sigma}^-_{A_0,c_1}$ (which we will note Σ in the remainder of this step). We also note $h_1(x,t) = u_1(x, t - \tau_0)$ and $h_2(x,t) = e^{\mu_1 - (x + c_1 t)}$. We know that

$$\begin{cases} 0 \leq h_1 \leq 1 \\ 0 \leq h_2 \leq 1 \\ \lim\limits_{x + c_1 t \to -\infty} h_1 = \lim\limits_{x + c_1 t \to -\infty} h_2 = 0 \end{cases}$$

and

$$h_1 < h_2 \text{ on } \partial\Sigma = \{(x,t) \mid x + c_1 t = -A_0\}.$$

Because of the bounds on h_1 and h_2, we know that $h_1 - 1 \leq h_2$. We note

$$\alpha^* = \sup\{\alpha \in [0,1] \mid \forall \gamma \leq \alpha \quad h_1 - 1 + \gamma \leq h_2 \text{ on } \Sigma\}.$$

Suppose by contradiction $\alpha^* < 1$. Then by continuity of h_1 and h_2, we have $h_1 - 1 + \alpha^* \leq h_2$, and $\inf\limits_{\Sigma}(h_2 - h_1 + 1 - \alpha^*) = 0$, otherwise we could show that α^* is not maximal.

We consider $P_n = (x_n, t_n)$ a minimizing sequence such that $x_n \in [0,1]$ for all n. As h_1 and h_2 have the same limit 0 as $x - c_1 t \to -\infty$, t_n can not diverge to $-\infty$ because it would imply $\alpha^* = 1$, therefore it remains bounded and by compactness the minimizing sequence has an accumulation point (x_∞, t_∞). As $h_1 < h_2$ on $\partial\Sigma$, $(x_\infty, t_\infty) \notin \partial\Sigma$, so $(x_\infty, t_\infty) \in \mathring{\Sigma}$.

Then $h_1 - 1 + \alpha^* \leq h_2$ in Σ and $h_1 - 1 + \alpha^* = h_2$ at point $(x_\infty, t_\infty) \in \mathring{\Sigma}$. By the strong comparison principle (Proposition 4.6), then $h_1 - 1 + \alpha^* = h_2$. But then $\alpha^* < 1$ is in contradiction with the fact that h_1 and h_2 have the same limit as $x - c_1 t \to -\infty$. Therefore we have a contradiction and $\alpha^* = 1$.

So we have proved that $h_1 \leq h_2$ in Σ.

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Chapitre 5

Existence of a permanent regime for an Allen-Cahn type equation approaching a Frank-Read source model

Abstract

In this chapter we study a phase-field model approximating a model for the dynamics of dislocations around a Frank-Read source on a bounded domain. The equation we study is of parabolic type, its solutions admits uniform bounds in space and its time evolution can also be controlled. We can then construct permanent regime solutions, extending to the Frank-Read source the notion of pulsating travelling spirals on annuli. Finally, we show that the time frequency of this permanent regime has a monotonous dependence on the exterior constraint.

AMS Classification : 35R35.
Keywords : Parabolic equation, semilinear equation.

5.1 Introduction

5.1.1 Setting of the problem and main results

We consider the following domain $\Omega \subset \mathbb{R}^2$:

$$\Omega = B(0,R) \setminus (\overline{B}(P^+,\varepsilon) \cup \overline{B}(P^-,\varepsilon))$$

where we note $B(a,r)$ (respectively $\overline{B}(a,r)$) the open (respectively closed) ball of center a and radius r. The points P^+ and P^- are defined as $P^+ = (1,0)$ and $P^- = (-1,0)$, and the quantities R and ε are such that $R > 2$ and $0 < \varepsilon < 1$.

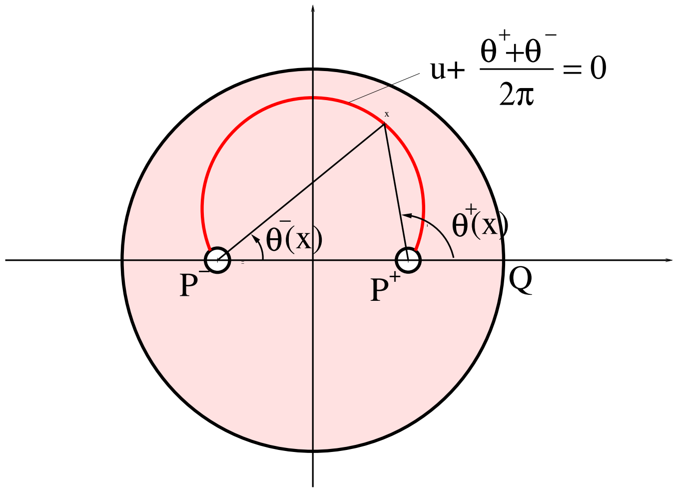

FIGURE 5.1 – The domain Ω

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In this chapter we study the dynamics of dislocations in a Frank-Read source. A Frank-Read source is a dislocation pinned to two fixed points. The evolution of this dislocation can generate changes in its topology, creating free dislocation loops while a dislocation line always remains pinned to the two fixed points which constitute the source. The ideal domain for the study of the Frank-Read source would be $\mathbb{R}^2 \setminus \{P^-, P^+\}$. The domain Ω approaches this ideal domain and has the advantages of being bounded and having no singularity at points P^- and P^+, which makes the mathematical analysis considerably easier.

In the following we will study a phase-field model approximating the dynamics of dislocations (in an appropriate limit, this model enables to recover the movement of dislocations moving with a normal speed equal to the curvature of the front plus a constant).

We consider the following equation, for $t \in \mathbb{R}$:

$$\left\{ \begin{array}{lcll} u_t & = & \Delta u - W'\left(u + \dfrac{\theta^+ - \theta^-}{2\pi}\right) + \sigma & \text{for } x \in \Omega \\ \dfrac{\partial u}{\partial n} & = & 0 & \text{for } x \in \partial\Omega \end{array} \right. \tag{5.1}$$

where σ is a real number, and W satisfies the following conditions :

$$\left\{ \begin{array}{l} W \in C^2(\mathbb{R}) \\ W(v+k) = W(v) \quad \text{for } k \in \mathbb{Z}. \end{array} \right. \tag{5.2}$$

For $x = (x_1, x_2) \in \Omega$, the angles $\theta^+(x)$ and $\theta^-(x)$ are defined as follows : if we note $Q = (R, 0)$, $\theta^+(x)$ is the angle between $\overrightarrow{P^+Q}$ and $\overrightarrow{P^+x}$, and $\theta^-(x)$ the angle between $\overrightarrow{P^-Q}$ and $\overrightarrow{P^-x}$. Both angles are defined modulo 2π, but as W is 1-periodic, the right hand side of the equation is well-defined. The multi-valued function

$$v = u + \frac{\theta^+ - \theta^-}{2\pi}$$

can be considered as a phase transition associated to the potential W. The Neumann boundary conditions are chosen because the Neumann problem is a simple formulation for which we don't have to prescribe boundary values. This choice enables the formation of dislocation loops and the apparition of a permanent regime. In the mathematical analysis, this enable an easy use of the Hopf lemma in combination with the strong maximum principle. Neumann boudary conditions are a frequent choice for the study of dislocation dynamics in a bounded domain, see for example Giga and Sato ([3] and [4]).

The goal of this chapter is to establish the existence of a permanent regime for this equation. The description of such a permanent regime would enable to make a direct link between the constraints applied to the Frank-Read source and the frequency of the creation of dislocations by the source, and ultimately the plastic deformations around it. This permanent regime will be obtained by considering the long time behaviour of solutions of a Cauchy problem, so we will initially be interested in the solution of this model for $t > 0$, with the initial condition

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega. \tag{5.3}$$

with $u_0 \in C^2(\bar{\Omega})$ with $\dfrac{\partial u_0}{\partial n} = 0$ on $\partial\Omega$.

In this chapter, we show the following results :

Theorem 5.1. (Control of the solution)
Consider u the solution of (5.1) with initial condition $u(x,0) = 0$. Then, for any $\sigma \in \mathbb{R}$, there exists a unique $\omega \in \mathbb{R}$ such that there exists $C \in \mathbb{R}$ such that

$$|u(x,t) - \omega t| \le C. \tag{5.4}$$

Theorem 5.2. (Monotonicity of the frequency ω)
The frequency ω is a non-decreasing function of σ.

Theorem 5.3. (Existence of a permanent regime)
If $\omega \neq 0$, there exists a solution u for equation (5.1) of the form

$$u(x,t+T) = u(x,t) + 1 \qquad \text{for all } x \in \Omega, \quad t \in \mathbb{R} \tag{5.5}$$

with $T = \frac{1}{\omega}$.

Furthermore, this solution u is such that ωu is nondecreasing in time, and it is unique up to time translations.

If $\omega = 0$, there exists a stationary solution u of equation (5.1) on $\Omega \times \mathbb{R}$.

Remark 5.4. *From a physical point of view, the existence of a stationary solution corresponds to a situation where the external constraint σ is not high enough to overcome the effects of the curvature and put the dislocation in motion.*

5.1.2 Existing results

We are not aware of works studying the dynamics of dislocations for the specific geometry of the Frank-Read source, but there exist prominent articles studying dislocation dynamics on simpler configurations. In this respect, the study of spirals is particularly interesting in the sense that with appropriate gluing conditions, the movement of a Frank-Read source could possibly be obtained by the study of the movement of two spirals attached to the points P^+ and P^-. The motion of spiral dislocations has been studied with the help of a level-set formulation by Ohtsuka [12] and by Goto, Nakagawa and Ohtsuka [6], who obtained the existence of solutions of the level-set approach, and the uniqueness of the evolution of an initial curve. Smereka [14] introduced another original level-set formulation to simulate the evolution of dislocations on a wider array of geometrical configurations, including the Frank-Read source. Giga, Ishimura and Koshaka [5] showed the existence of spiral-shaped solutions for the motion of dislocations on a compact annulus using a parametrization of the spiral dislocation curves. Forcadel, Imbert and Monneau [2] also used a parametrization of spirals to prove existence and uniqueness results, this time in an unbounded domain with a singularity at the origin. This article describes well the problems encountered with the singularity of the attachment point and with the unbounded character of the domain. Karma and Plapp [7] introduced a phase-field formulation to study and simulate the formation of spiral dislocations. Phase-field approaches were then also treated by Rodney, Le Bouar and Finel [13], Koslowski, Cuitino and Ortiz [8], Wan, Jin, Cuitino and Khachaturyan [16], Xiang, Cheng, Srolovitz and E [15]. In [11], Ogiwara and Nakamura used such a phase-field approach for a theoretical study of the motion of spiral dislocations, in the case of a bounded domain with no singularity, and proved the existence of spiral travelling wave solutions, which exhibit a permanent regime behaviour. The setting of our paper is similar to [11], and some of their considerations can be successfully used in the context of the Frank-Read source, even if the solutions we sonstruct are less constrained than spirals. The use of Neumann boundary conditions is consistent with this approach. We also refer to Giga and Sato ([3] and [4]) for other studies of dislocation dynamics with Neumann boundary conditions.

5.1.3 Organization of the paper

Section 5.2 is devoted to the recalling of classical results for parabolic equations in bounded regular domains. Apart from the necessary existence results, the comparison principle and the strong maximum principle are particularly useful in the remainder of the chapter. The proof of Theorem 5.1 is given in Sections 5.3 and 5.4. In Section 5.3 we show that the space oscillations of the solution of the Cauchy problem are bounded, and in Section 5.4

we control the time oscillations and show the bound (5.4). The existence of a permanent regime (Theorem 5.3) is shown in Section 5.5. It is given by the long-time behaviour of the solutions of the Cauchy problem. Finally, we establish the monotonicity relation of Theorem 5.2 in Section 5.6.

5.2 General results

5.2.1 Useful tools

Proposition 5.5. (Comparison principle for strong solutions)
Consider P a semi-linear operator defined by

$$Pu = -u_t + \Delta u + B(X, u) \tag{5.6}$$

with $X = (x,t) \in Q = \Omega \times (0,T)$.

Consider also the boundary operator M defined by

$$Mu = -\frac{\partial u}{\partial n}, \quad (x,t) \in \partial\Omega \times (0,T)$$

Suppose $B : (X,z) \mapsto B(X,z)$ is continuously differentiable with respect to z.

If u and v are in $C^{2,1}(Q) \cap C^0(\bar{Q})$, if $Pu \geq Pv$ in Q, $Mu \geq Mv$ on $(\partial\Omega \times (0,T))$ and $u \leq v$ on $(\bar{\Omega} \times \{0\})$, then $u \leq v$ in Q.

This result can be inferred from [10], combining the arguments of the proofs of Theorem 2.10 p. 13 and Theorem 9.1 p. 213.

We also have a strong maximum principle :

Proposition 5.6. (Strong maximum principle)
Let u_1 and u_2 be two solutions of

$$\begin{cases} u_t = \Delta u + f(u,x) & (x,t) \in \Omega \times I \\ u_n = 0 & (x,t) \in \partial\Omega \times I \end{cases}$$

with f twice differentiable with respect to u and x, with continuous second derivatives.

If we have

$$\begin{cases} u_1(x,t) \leq u_2(x,t) & (x,t) \in \Omega \times I \\ u_1(x_0,t_0) = u_2(x_0,t_0) & \textit{for a couple } (x_0,t_0) \in \bar{\Omega} \times \mathring{I} \end{cases}$$

then $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \Omega \times I$.

This proposition results a the combination of the strong maximum principle (as it is stated in [1] for example) and the Hopf lemma. The Hopf lemma covers the cases where the point x_0 is on the boundary $\partial\Omega$. Actually, the usual formulation of the strong maximum principle (as in [1]) only gives that $u_1(x,t) = u_2(x,t)$ for $t < t_0$. Here, as our solutions are regular and the space-time domain is a cylinder, the comparison principle gives $u_1(x,t) = u_2(x,t)$ also when $t \geq t_0$.

Proposition 5.7. (Sobolev injections in a parabolic setting)
Let $\Omega \subset \mathbb{R}^2$ a bounded, regular open domain.
(i) Consider $p \leq 2$. Then for all $u \in W_p^{2,1}(\Omega \times (T_1,T_2))$, we have that $u \in L^q(\Omega \times (T_1,T_2))$ with

$$\begin{array}{lll} q \leq p^* & \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{2} & \textit{if } p < 2 \\ q < +\infty & & \textit{if } p = 2, \end{array}$$

and there exists a constant C, independent on u, such that

$$\|u\|_{L^q(\Omega\times(T_1,T_2))} \leq C\|u\|_{W_p^{2,1}(\Omega\times(T_1,T_2))}.$$

(ii) Consider $p > 2$. Then for all $u \in W_p^{2,1}(\Omega \times (T_1,T_2))$, we have that $u \in C^{\alpha,\frac{\alpha}{2}}(\Omega \times (T_1,T_2))$ with

$$\alpha < \alpha^* \quad \alpha^* = \max\left(2 - \frac{4}{p}, 1\right)$$

and there exists a constant C, independent on u, such that

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(T_1,T_2))} \leq C\|u\|_{W_p^{2,1}(\Omega\times(T_1,T_2))}.$$

This result is shown in [9], Lemma 3.3 p. 80.

5.2.2 Existence of strong solutions

Proposition 5.8. (Existence of strong solutions for the Cauchy problem)
Suppose W stisfies 5.2. The Cauchy problem

$$\left\{\begin{array}{lll} u_t & = & \Delta u - W'\left(u + \dfrac{\theta^+ - \theta^-}{2\pi}\right) + \sigma \quad \textit{for } x \in \Omega \\ \dfrac{\partial u}{\partial n} & = & 0 \quad \textit{for } x \in \partial\Omega \\ u & = & 0 \quad \textit{when } t = 0 \end{array}\right. \tag{5.7}$$

admits a unique strong solution $u \in C^{2,1}(\Omega \times (0,+\infty) \cap C(\bar{\Omega} \times [0,+\infty))$

This is a straightforward application of [10], Theorem 14.23 p. 378.

5.2.3 Interior estimates for the solution of the heat equation

In this section we state an interior estimate property for the solution of the heat equation that will be crucial in the rest of the paper

Proposition 5.9. (Interior estimates for the heat equation)

Consider the heat equation with neumann boundary conditions

$$\begin{cases} u_t = \Delta u + f(x,t) & (x,t) \in \Omega \times (T_1, T_2) \\ u(x,0) = u_0(x) & x \in \Omega \\ \frac{\partial u}{\partial n} = 0 & (x,t) \in \partial\Omega \times (T_1, T_2) \end{cases}.$$

Then, for any $\varepsilon > 0$ (with $\varepsilon \le T_2 - T_1$),
(i) if $u \in L^p(\Omega \times (T_1, T_2))$ is a solution of (5.9)*, and $f \in L^p(\Omega \times (T_1, T_2))$, then u is also in*
$W_p^{2,1}(\Omega \times (T_1 + \varepsilon, T_2))$ and there exists $C > 0$ independent of u_0 such that

$$\|u\|_{W_p^{2,1}(\Omega\times(T_1+\varepsilon,T_2))} \le C\left(\|u\|_{L^p(\Omega\times(T_1,T_2))} + \|f\|_{L^p(\Omega\times(T_1,T_2))}\right).$$

(ii) if $u \in C^{\alpha,\frac{\alpha}{2}}(\Omega \times (T_1, T_2))$ is a solution of (5.9)*, and $f \in C^{\alpha,\frac{\alpha}{2}}(\Omega \times (T_1, T_2))$, then u is also in $C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times (T_1 + \varepsilon, T_2))$ and there exists C' independent of u_0 such that*

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(T_1+\varepsilon,T_2))} \le C'\left(\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(T_1,T_2))} + \|f\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(T_1,T_2))}\right).$$

This result is a consequence of [10], Theorem 7.20 p. 179, using also [9], Theorem 3.3 p. 80 to control the first order space derivatives.

5.3 A bound on the space oscillation of the solution

The goal of this part is to prove that for a solution u of (5.1) with initial data $u(\cdot,0) \equiv 0$, the space oscillation of u remains bounded for all time.

Proposition 5.10. Bound on the space oscillation of the solution
Suppose W satisfies (5.2)*. Let u be the solution of* (5.1) *with initial data $u(\cdot,0) \equiv 0$. Then there exists a positive constant C independent of t such that*

$$\max_{x\in\bar\Omega} u(x,t) - \min_{x\in\bar\Omega} u(x,t) \le C, \qquad \text{for all } t \ge 0. \tag{5.8}$$

To control the space oscillations of a solution u of (5.1) and prove Proposition 5.10, we will use a result controlling the solution of the homogeneous heat equation, stated in the following lemma :

Lemma 5.11. (A decreasing bound for the solution of the homogenous equation)
Consider the equation

$$\left\{\begin{array}{lcll} u_t - \Delta u & = & 0 & \text{on } \Omega \times (0,+\infty) \\ \partial_n u & = & 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(\cdot,0) & = & h_0 & \text{on } \Omega \end{array}\right. \tag{5.9}$$

with $h_0 \in L^\infty(\Omega)$.

Then there exists a time $T_0 > 0$ and a real number $\mu \in]0,1[$, both independent of h_0 and u, such that for all $t > T_0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \mu \|h_0\|_{L^\infty(\Omega)}.$$

Proof of Lemma 5.11

To prove this lemma we use the fact that the operator $(-\Delta)$ on Ω with homogeneous Neumann boundary conditions is a self-adjoint compact operator on the Hilbert space $L^2(\Omega)$, and that its minimum eigenvalue is strictly positive. We write $(\lambda_i)_{i>1}$ the sequence of its eigenvalues in non-decreasing order, and $(\psi_i)_{i>1}$ a corresponding sequence of unitary eigenvectors. We then have $\lambda_1 > 0$.

We observe that as Ω is bounded, there is a continuous injection $L^\infty(\Omega) \to L^2(\Omega)$, so there exists $C > 0$ such that for all $h \in L^\infty(\Omega)$, we have $h \in L^2(\Omega)$ and

$$\|h\|_{L^2(\Omega)} \le C \|h\|_{L^\infty(\Omega)}.$$

Now for a given $h \in L^\infty(\Omega) \subset L^2(\Omega)$, we can write its decomposition on the basis of the eigenvectors (ψ_i) of the operator $(-\Delta)$ with homogeneous Neumann boundary conditions :

$$h = \sum_{i=1}^{+\infty} c_i \psi_i \quad \text{in } L^2(\Omega)$$

with $\sum_{i=1}^{+\infty} |c_i|^2 \le C^2 \|h\|^2_{L^\infty(\Omega)}$.

Then we have

$$u = \sum_{i=1}^{+\infty} c_i e^{-\lambda_i t} \psi_i \quad \text{in } L^2(\Omega).$$

For the following, we use interior estimates (Proposition 5.9) and Sobolev embeddings to transpose the estimates in L^2 to estimates in L^∞.

We remark that the estimate of Proposition 5.9 can be rewritten, considering a shifted domain,

$$\|u\|_{W_p^{2,1}(\Omega\times(2\varepsilon,T_{max}))} \le C' \|u\|_{L^p(\Omega\times(\varepsilon,T_{max}))}$$

for some $\varepsilon > 0$. We consider this inequality for the particular value $p = 2$, and use the embedding $L^\infty((\varepsilon, T_{max}), L^2(\Omega)) \subset L^2(\Omega \times (\varepsilon, T_{max}))$ to obtain

$$\|u\|_{W_2^{2,1}(\Omega\times(2\varepsilon,T_{max}))} \leq C'' \|u\|_{L^\infty((\varepsilon,T),L^2(\Omega))} .$$

The second quantity is clearly finite because the maximum principle implies that $u \in L^\infty(\Omega \times (0, +\infty))$. With the spectral decomposition we also have a bound for the right hand side of this estimate : for all $t > 0$,

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq \sum_{i=0}^{+\infty} |c_i|^2 e^{-2\lambda_i t} \leq C^2 e^{-2\lambda_1 t} \|h\|^2_{L^\infty(\Omega)} ,$$

which implies

$$\|u\|_{L^\infty((\varepsilon,+\infty),L^2(\Omega))} \leq C_2 e^{-\lambda_1\varepsilon} \|h\|_{L^\infty(\Omega)} .$$

We then have the inequality

$$\|u\|_{W_2^{2,1}(\Omega\times(2\varepsilon,T_{max}))} \leq C_3 e^{-\lambda_1\varepsilon} \|h\|_{L^\infty(\Omega)} .$$

To conclude the proof of this lemma, we need to have the same estimation for $\|u\|_{W_p^{2,1}(\Omega\times(2\varepsilon,T_{max}))}$ for some $p > 2$, which would enable us to use a Sobolev embedding to derive an estimation in $C^0(\Omega \times (2\varepsilon, T_{max}))$. The calculations above show that $\|u\|_{W_2^{2,1}((2\varepsilon,T_{max}),\Omega)}$ is finite, so by the Sobolev embedding

$$W_2^{2,1}(\Omega \times (2\varepsilon, T_{max})) \subset L^p(\Omega \times (2\varepsilon, T_{max})) \quad \text{for } p < +\infty,$$

we have that $\|u\|_{L^p(\Omega\times(2\varepsilon,T_{max}))}$ is also finite for some $p > 2$. Using again the interior estimates theorem, we get

$$\|u\|_{W_p^{2,1}(\Omega\times(3\varepsilon,T_{max}))} \leq C_4 \|u\|_{W_2^{2,1}(\Omega\times(2\varepsilon,T_{max}))} ,$$

so

$$\|u\|_{W_p^{2,1}(\Omega\times(3\varepsilon,T_{max}))} \leq C_5 e^{-\lambda_1\varepsilon} \|h\|^2_{L^\infty(\Omega)} .$$

Then we use another Sobolev embedding : for all $T_{max} > 0$, we have the continuous inclusion $C^{\beta,\frac{\beta}{2}}(\Omega \times (3\varepsilon, T_{max})) \subset W_p^{2,1}(\Omega \times (3\varepsilon, T_{max}))$ for some $\beta > 0$ (more precisely this is true for $\beta < 2 - \frac{4}{p}$). Then we have the inequality

$$\begin{aligned} \|u\|_{L^\infty(\Omega\times(3\varepsilon,T_{max}))} &\leq C_6 \|u\|_{C^{\beta,\frac{\beta}{2}}(\Omega\times(3\varepsilon,T_{max}))} \\ &\leq C_7 \|u\|_{W_p^{2,1}(\Omega\times(3\varepsilon,T_{max}))} \\ &\leq C_8 e^{-\lambda_1\varepsilon} \|h\|^2_{L^\infty(\Omega)} , \end{aligned}$$

which concludes the proof of Lemma 5.11.

With this result we can prove Proposition 5.10.

Proof of Proposition 5.10
Step 1 : Elimination of the mean of u on Ω

To study the oscillation of u, it is useful to consider the difference between u and its mean value on Ω. We rrite

$$\bar{u}(x,t) = u(x,t) - \langle u(\cdot,t)\rangle$$

with

$$\langle u(\cdot,t)\rangle = \frac{1}{|\Omega|}\int_\Omega u(y,t)dy.$$

We know that u is a solution of (5.1), so

$$u_t = \bar{u}_t + \partial_t\langle u\rangle = \Delta\bar{u} - W'\left(u + \frac{\theta^+ - \theta^-}{2\pi}\right) + \sigma \quad \text{on } \Omega\times(0,+\infty) \tag{5.10}$$

We also have

$$\partial_n\bar{u} = 0 \qquad \text{on } \partial\Omega\times(0,+\infty)$$

and

$$\bar{u}(x,0) = 0 \qquad \text{for } x\in\Omega$$

because $u(x,0)\equiv 0$. Then, we integrate (5.10) on Ω. Note that

$$\int_\Omega \Delta\bar{u} = \int_{\partial\Omega}\partial_n\bar{u} = 0$$

and

$$\int_\Omega \bar{u}_t = \partial_t\int_\Omega \bar{u} = 0.$$

Therefore, we have

$$\partial_t\langle u\rangle = -\frac{1}{|\Omega|}\left(\int_\Omega W'\left(u + \frac{1}{2\pi}(\theta^+ - \theta^-)\right)\right) + \sigma.$$

So, rewriting (5.10), we find that $\bar{u}$ is a solution of the following equation :

$$\left\{\begin{array}{lcll} \bar{u}_t - \Delta\bar{u} & = & h & \text{on } \Omega\times(0,+\infty)\\ \partial_n\bar{u} & = & 0 & \text{on } \partial\Omega\times(0,+\infty)\\ \bar{u}(x,0) & = & 0 & \text{on } \Omega \end{array}\right. \tag{5.11}$$

with initial condition $\bar{u}(0,x) = 0$, and

$$h(x,t) = W'\left(u(x,t) + \frac{1}{2\pi}(\theta^+ - \theta^-)\right) + \sigma - \partial_t\langle u(\cdot,t)\rangle.$$

The preceding calculations indicate that for all $t > 0$,

$$\int_\Omega h(x,t)dx = 0.$$

We also note that, as W' is bounded, h is bounded, i.e. $h \in L^\infty(\Omega \times (0, +\infty))$.

Step 2 : Bounds for $\bar{u}$

As the article of Ogiwara and Nakamura [11], we define the set

$$X_0 = \left\{ v \in C(\bar{\Omega}) \mid \int_\Omega v(x)dx = 0 \right\}.$$

The restriction of Δ on X_0 defines an analytic semigroup $\left\{e^{t\Delta}\right\}_{t\in(0,\infty]}$ on X_0 and the Duhamel formula gives

$$\bar{u}(\cdot, t) = \int_0^t e^{(t-s)\Delta} h(\cdot, s) ds. \tag{5.12}$$

The bound given in Lemma 5.11 can be written equivalently as

$$\left\|e^{t\Delta} h_0\right\|_{L^\infty(\Omega)} \leq \mu \left\|h_0\right\|_{L^\infty(\Omega)} \quad \text{for all } t > T_0.$$

As a consequence, there exists $C > 0$ and $\lambda \in (0, +\infty)$ such that, for all $t > T_0$

$$\left\|e^{t\Delta} h_0\right\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \left\|h_0\right\|_{L^\infty(\Omega)}.$$

With this, we can derive a bound on $\bar{u}$ for all $t \in (0, +\infty)$

$$\begin{aligned} \|\bar{u}(\cdot, t)\|_{L^\infty(\Omega)} &\leq \int_0^t \left\|e^{(t-s)\Delta} h(\cdot, s)\right\|_{L^\infty(\Omega)} ds \\ &\leq C \int_0^t (\mathbf{1}_{\{t-s<T_0\}} + \mathbf{1}_{\{t-s>T_0\}} e^{-\lambda(t-s)}) \left\|h(\cdot, s)\right\|_{L^\infty(\Omega)} ds \\ &\leq \left(C \int_0^t (\mathbf{1}_{\{t-s<T_0\}} ds + \frac{C}{\lambda} \right) \|h\|_{L^\infty(\Omega \times (0, +\infty))} \\ &\leq \left(CT_0 + \frac{C}{\lambda} \right) \|h\|_{L^\infty(\Omega \times (0, +\infty))}, \end{aligned}$$

which is indeed a uniform bound.

Finally,

$$\max_{x\in\Omega} u(x, t) - \min_{x\in\Omega} u(x, t) \leq 2 \left\|\bar{u}(\cdot, t)\right\|_{L^\infty(\Omega)},$$

which shows that the oscillation of u is indeed bounded.

5.4 Linear growth of the solution in time

Proposition 5.12 (Linear growth of the solution).
Let u be the solution of (5.1) with initial data $u(\cdot, 0) \equiv 0$. Then there exists a constant $\omega \in \mathbb{R}$ and a positive constant C' such that, for all $x \in \Omega$, $t > 0$,

$$|u(x, t) - \omega t| \leq C' \qquad \textit{for all } x \in \Omega, \quad t > 0 \tag{5.13}$$

Proof of Proposition 5.12

This proof is very similar to the one given in Step 3 of the proof of Proposition 3.18 in Chapter 3. It is given here to maintain the internal consistency of the chapter.

We recall that the potential W is a 1-periodic function. Then as u is the solution of (5.1) with initial condition $u \equiv 0$, $u + k$ is also a solution of (5.1) (for another initial condition) for all $k \in \mathbb{Z}$.

Now, for $T > 0$, define

$$\lambda^+(T) = \sup_{t \geq 0} \frac{u(0, t+T) - u(0,t)}{T}$$

and

$$\lambda^-(T) = \inf_{t \geq 0} \frac{u(0, t+T) - u(0,t)}{T}.$$

With this construction, $\lambda^+(T)$ and $\lambda^-(T)$ are estimates of the mean slope of the solution u in time. Heuristically, these estimates should become more accurate as $T \to +\infty$. In the remainder of the proof we show that both function λ^+ and λ^- converge to a quantity ω which is such that (5.13) is satisfied.

Step 1 : Bounds on $\lambda^+(T)$ and $\lambda^-(T)$ when $T > 1$.

We will prove that there exists a constant C_1 independent on $T \geq 1$ such that

$$-C_1 \leq \lambda^-(T) \leq \lambda^+(T) \leq C_1.$$

It is clear that for all T, $\lambda^+(T) \geq \lambda^-(T)$.

Fix a time $t_0 \geq 0$. We know that $u(\cdot, t) \in L^\infty(\Omega)$. Then there exists $k \in \mathbb{Z}$ such that $\min u(\cdot, t_0) \in [k, k+1[$. By the oscillation estimate (5.8), it follows that $\max u(\cdot, t) \leq (k+1) + C$. Equivalently, $0 \leq u(\cdot, t_0) - k \leq C + 1$.

Now define the function $v : \tau \mapsto C + 1 + (|W'|_\infty + \sigma)\tau$. We then have

$$v_t \geq \Delta v - W'(v) + \sigma.$$

This inequality and the boundary conditions imply that v is a supersolution of (5.1). We also have that $(x, \tau) \mapsto u(x, t+\tau) - 2k$ is a solution of (5.1), since the equation is invariant by time translation and addition of integers. At time $\tau = 0$, we have

$$v(x, 0) = C + 1 \geq u(x, t_0) - k.$$

From the comparison principle (Proposition 5.5), it follows that for all $\tau \geq 0$,

$$v(x, \tau) \geq u(x, t_0 + \tau) - k.$$

In particular, for $\tau = T$, we have

$$C + 1 + (\left|W'\right|_\infty + \sigma)T \geq u(x, t_0 + T) - k. \tag{5.14}$$

Combined with the fact that $u(0,t_0)-k \geq 0$ this implies that

$$\frac{u(0,t_0+T)-u(0,t_0)}{T} \leq \frac{C+1}{T} + \left|W'\right|_\infty + \sigma. \tag{5.15}$$

This inequality remains valid for all $t_0 \geq 0$, so, for $T \geq 1$,

$$\lambda^+(T) \leq C + 1 + \left|W'\right|_\infty + |\sigma| = C_1. \tag{5.16}$$

Similarly, we can derive a lower bound for $\lambda^-(T)$.

Step 2 : Upper bound on $\lambda^+(T) - \lambda^-(T)$
Consider a real number $T > 0$.

We will first study the case where both the infimum $\lambda^-(T)$ and the supremum $\lambda^+(T)$ are reached in $[0,+\infty[$, which means that there exist $t_1 \geq 0$ and $t_2 \geq 0$ such that

$$\lambda^+(T) = \frac{u(0,t_1+T)-u(0,t_1)}{T}$$

$$\lambda^-(T) = \frac{u(0,t_2+T)-u(0,t_2)}{T}.$$

Consider the function

$$v(x,t) = u(x,t+t_2-t_1) + K,$$

with $K = \lfloor u(0,t_1) - u(0,t_2) + 2C \rfloor + 1$, where the function $\lfloor \cdot \rfloor$ is the floor function. Then u and v are both solutions of (5.1), and for $t = t_1$,

$$v(x,t_1) = u(x,t_2) + (\lfloor u(0,t_1) - u(0,t_2) + 2C \rfloor + 1) \geq u(x,t_1),$$

where we have used the estimate on space oscillations (5.8). Then, by the comparison principle, for all $t \geq t_1$,

$$v(x,t) \geq u(x,t).$$

In particular, for $t = t_1 + T$, we have

$$u(x,t_2+T) - u(0,t_2) + 1 + 2C \geq u(x,t_1+T) - u(0,t_1), \tag{5.17}$$

so, taking $x = 0$, we get

$$\lambda^-(T) + \frac{1+2C}{T} \geq \lambda^+(T), \tag{5.18}$$

which we can rewrite

$$0 \leq \lambda^+(T) - \lambda^-(T) \leq \frac{1+2C}{T}. \tag{5.19}$$

Even if the infimum or the extremum is not reached, we have, for $\varepsilon > 0$, there exists $t_1 \geq 0$ and $t_2 \geq 0$ such that

$$\lambda^+(T) < \frac{u(0, t_1 + T) - u(0, t_1)}{T} + \varepsilon$$

$$\lambda^-(T) > \frac{u(0, t_2 + T) - u(0, t_2)}{T} - \varepsilon.$$

The same line of reasoning then leads to

$$u(x, t_2 + T) - u(0, t_2) + 1 + 2C \geq u(x, t_1 + T) - u(0, t_1),$$

so

$$\lambda^-(T) + \varepsilon + \frac{1 + 2C}{T} \geq \lambda^+(T) - \varepsilon. \tag{5.20}$$

As this is true for all $\varepsilon > 0$, (5.19) still holds in that case.

Step 3 : Evaluation of the variations of λ^+ and λ^-

Now, let $p \in \mathbb{N} \setminus \{0\}$ be an integer.

We first assume that there exists $t_1 \geq 0$ such that

$$\lambda^+(pT) = \frac{u(0, t_1 + pT) - u(0, t_1)}{pT}.$$

It is clear that

$$pT\lambda^+(pT) = u(0, t_1 + pT) - u(0, t_1) \leq pT\lambda^+(T),$$

so

$$\lambda^+(pT) \leq \lambda^+(T).$$

If the supremum $\lambda^+(pT)$ is not reached, then for all $\varepsilon > 0$ there exists a time t_ε such that

$$pT\lambda^+(pT) - \varepsilon \leq u(0, t_\varepsilon + pT) - u(0, t_\varepsilon). \tag{5.21}$$

As above,

$$u(0, t_\varepsilon + pT) - u(0, t_\varepsilon) \leq pT\lambda^+(T), \tag{5.22}$$

so we have, for all $\varepsilon > 0$

$$pT\lambda^+(pT) - \varepsilon \leq pT\lambda^+(T), \tag{5.23}$$

which implies again that $\lambda^+(PT) \leq \lambda^+(T)$. Similarly, it can be proved that $\lambda^-(pT) \geq \lambda^-(T)$.

Now take $T_1 > 0$ and $T_2 > 0$ and assume that $pT_1 = qT_2$, with $p \in \mathbb{N}\setminus\{0\}$ and $q \in \mathbb{N} \setminus \{0\}$. Using the preceding conclusions, we have the following inequalities :

$$\lambda^+(T_1) \geq \lambda^+(pT_1) = \lambda^+(qT_2) \geq \lambda^-(qT_2) \geq \lambda^-(T_2) \geq \lambda^+(T_2) - \frac{1 + 2C}{T_2},$$

so
$$\lambda^+(T_2) - \lambda^+(T_1) \leq \frac{1+2C}{T_2}. \tag{5.24}$$
Exchanging the indices 1 and 2, we see that this implies
$$\left|\lambda^+(T_1) - \lambda^+(T_2)\right| \leq (1+2C) \max\left(\frac{1}{T_1}, \frac{1}{T_2}\right) \tag{5.25}$$
for all $T_1, T_2 > 0$ such that $\frac{T_1}{T_2} \in \mathbb{Q}$. The solution u being in $C^{2,1}(\Omega \times (0, +\infty)$, the function $u(0, \cdot)$ is uniformly continuous ; as a consequence the function λ^+ is continuous in T, which implies that the inequalities above remain true for any $T_1, T_2 > 0$.

Using the same method we derive the same property for λ^- : for all $T_1, T_2 > 0$,
$$\left|\lambda^-(T_1) - \lambda^-(T_2)\right| \leq (1+2C) \max\left(\frac{1}{T_1}, \frac{1}{T_2}\right). \tag{5.26}$$

Step 4 : Limit of λ^+ and λ^- and end of the proof

Now, if $(T_n)_{n\in\mathbf{N}}$ is an increasing sequence with limit $+\infty$, both sequences $\lambda^+(T_n)$ and $\lambda^-(T_n)$ are Cauchy sequences and then converge. As this is true for any sequence with limit $+\infty$, both $\lambda^+(T)$ and $\lambda^-(T)$ have a limit for $T \to \infty$. Furthermore, the inequality (5.19) shows that these two limits are equal. Define
$$\lambda = \lim_{T\to\infty} \lambda^+(T) = \lim_{T\to\infty} \lambda^-(T).$$
We consider inequalities (5.25) and (5.26) and take the limit $T_2 \to \infty$ to deduce the following inequalities, for $T > 0$
$$\left|\lambda^+(T) - \lambda\right| \leq \frac{1+2C}{T}$$
and
$$\left|\lambda^-(T) - \lambda\right| \leq \frac{1+2C}{T}.$$
Now, for all $T > 0$, we deduce
$$\lambda T - (1+2C) \leq u(0,T) - u(0,0) \leq \lambda T + (1+2C).$$
Combining this with (5.8), we can write
$$\lambda T - (1+3C) \leq u(x,T) - u(0,0) \leq \lambda T + (1+3C),$$
and we recall that $u(0,0) = 0$.

Taking $\omega = \lambda$ and $C' = 1 + 3C$, we have then proved that, for all $t \geq 0$,
$$|u(x,t) - \omega t| \leq C'. \tag{5.27}$$

This concludes the proof of Proposition 5.12, which in turn proves Theorem 5.1.

5.5 Permanent regime solution

Proof of Theorem 5.3 in the case $\omega > 0$
Step 1 : Definition of the auxiliary function w and a priori bounds
In this step we want to prove that there exists a solution u of (5.1) on $\Omega \times \mathbb{R}$ satisfying estimates (5.8) and (5.13).

Let u be the solution of (5.1) with $u(\cdot, 0) = 0$. We define

$$w(x,t) = u(x,t) - \omega t.$$

From Proposition 5.12, we know that

$$|u(x,t) - \omega t| \leq C' \text{ for all } t \geq 0.$$

As a consequence, $w \in L^\infty(\Omega \times (0,+\infty))$ and

$$\|w\|_{L^\infty(\Omega\times(0,+\infty))} \leq C'.$$

The function w satisfies the following equation :

$$w_t = \Delta w + f,$$

with

$$f = -\omega - W'\left(w + \omega t + \frac{\theta^+ - \theta^-}{2\pi}\right) + \sigma.$$

By the interior estimate theorem (Proposition 5.9), we have, for all $k \in \mathbb{N}$,

$$\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(\frac{k+2}{\omega},\frac{k+3}{\omega}))} \leq C_1 \left(\|w\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(\frac{k+1}{\omega},\frac{k+3}{\omega}))} + \|f\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(\frac{k+1}{\omega},\frac{k+3}{\omega}))}\right),$$

the constant C_1 being independent on k. A Sobolev embedding and again the interior estimate theorem give

$$\begin{aligned}
\|w\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(\frac{k+1}{\omega},\frac{k+3}{\omega}))} &\leq C_2 \|w\|_{W^{2,1}_p(\Omega\times(\frac{k+1}{\omega},\frac{k+3}{\omega}))} \\
&\leq C_3 \left(\|w\|_{L^p(\Omega\times(\frac{k}{\omega},\frac{k+3}{\omega}))} + \|f\|_{L^\infty(\Omega\times(\frac{k}{\omega},\frac{k+3}{\omega}))}\right) \\
&\leq C_3' \|w\|_{L^\infty(\Omega\times(\frac{k}{\omega},\frac{k+3}{\omega}))} + C_3''
\end{aligned}$$

for some $p > \dfrac{4}{2-\alpha}$. The constants C_3' and C_3'' are still independent on k. For the other term,

$$\begin{aligned}
&\|f\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times(\frac{k+1}{\omega},\frac{k+3}{\omega}))} \leq |\omega| + \|W'\|_{L^\infty(\mathbb{R})} + |\sigma| + [f]_{\alpha,\frac{\alpha}{2},\Omega\times(\frac{k+1}{\omega},\frac{k+3}{\omega})} \\
&\quad \leq |\omega| + \|W'\|_{L^\infty(\mathbb{R})} + |\sigma| \\
&\quad + \left([w]_{\alpha,\frac{\alpha}{2},\Omega\times(\frac{k+1}{\omega},\frac{k+3}{\omega})} + \frac{1}{2\pi}[\theta^+ - \theta^-]_{\alpha,\Omega} + |\omega|\left|\frac{2}{\omega}\right|^{1-\frac{\alpha}{2}}\right)\|W''\|_{L^\infty(\mathbb{R})} \\
&\quad \leq |\omega| + \|W'\|_{L^\infty(\mathbb{R})} + \sigma + \left(C_4 + C_3' \|w\|_{L^\infty(\Omega\times(\frac{k}{\omega},\frac{k+3}{\omega}))}\right)\|W''\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

As a consequence

$$\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}\left(\Omega\times\left(\frac{2}{\omega},+\infty\right)\right)} \le C_5 + C_6\,\|w\|_{L^\infty(\Omega\times(0,+\infty))} \le C_7$$

for some $0<\alpha<1$, where the constant C_7 may depend only on ω.

Step 2 : Definition of the sequences u_n and w_n and construction of a global solution

Our goal will be to exhibit a permanent regime, which is expected to be reached for large values of t. As w is bounded in a Hölder space, we define

$$w_n(x,t) = w\left(x, t+\frac{n}{\omega}\right) \text{ for } (x,y)\in\Omega\times\left(-\frac{n}{\omega},+\infty\right)$$

and

$$u_n(x,t) = w_n(x,t) + \omega t.$$

As

$$u_n(x,t) = w\left(x,t+\frac{n}{\omega}\right) + \omega\left(t+\frac{n}{\omega}\right) = u\left(x,t+\frac{n}{\omega}\right) + n,$$

u_n is a solution of (5.1) and satisfies the time estimation (5.13).

As w_n is just a time-translation of w_0, it is also in $L^\infty\left(\Omega\times\left(-\frac{n}{\omega},+\infty\right)\right)$ and in $C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}\left(\Omega\times\left(-\frac{n-1}{\omega},+\infty\right)\right)$, with the same bounds C' and C_7. Eventually allowing for time domain extensions to $\mathbb{R}$ with arbitrary values that do not change the Hölder bounds and using a compact injection, we have that all w_n belong to a compact subset of $C^{2+\beta,1+\frac{\beta}{2}}(\Omega\times(-M,M))$ for any $M>0$ and $\beta<\alpha$, with $\beta>0$.

As a consequence of that, for any $M>0$, we can extract a subsequence $w'_{\varphi(n)}$ which converges to a limit w_∞^M in this compact set. Using a diagonal extraction procedure for a sequence $M_n\to\infty$, we have that there exists a function w_∞ in $C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega\times(-M,M))$ for all $M>0$ and such that w_n converges to w in $C^{2+\beta,1+\frac{\beta}{2}}(\Omega\times(-M,M))$ for all $M>0$, up to a subsequence.

Now that we have built w_∞, we define u_∞ by $u_\infty(x,t) = w_\infty(x,t)+\omega t$. Then u_n converges to u_∞ in $C^{2+\beta,1+\frac{\beta}{2}}(\Omega\times(-M,M))$ for all $M>0$. As a consequence, for all $M>0$,

$$\begin{cases} (u_n)_t \to (u_\infty)_t & \text{in } L^\infty(\Omega\times(-M,M)) \\ \Delta u_n \to \Delta u_\infty & \text{in } L^\infty(\Omega\times(-M,M)) \\ W'\left(u_n+\dfrac{\theta^+-\theta^-}{2\pi}\right) \to W'\left(u_\infty+\dfrac{\theta^+-\theta^-}{2\pi}\right) & \text{in } L^\infty(\Omega\times(-M,M)). \end{cases}$$

The last convergence occurs because W is regular enough (it is sufficient for the convergence that W' is Lipschitz). We also have a convergence for the limit conditions :

$$\frac{\partial u_n}{\partial n} \to \frac{\partial u_\infty}{\partial n} \text{ in } L^\infty(\Omega\times(-M,M).$$

As a result u_∞ is a strong solution of (5.1) on $\Omega \times (-\infty, +\infty)$.

Step 3 : Existence of the permanent regime solution

The next step is to prove that this function u_∞ has the form (5.5) with $T = \frac{2\pi}{\omega}$. For commodity we will write it u in the following.

Define

$$b = \max\left\{\tau \geq 0, \quad \forall \tau' < \tau, \quad u(x,t) + 1 \geq u(x, t+\tau')\right\}.$$

This set is nonempty because for τ' small enough, $u(x,t) + 1 \geq u(x,t+\tau')$ because u is Lipschitz. Moreover, b is finite because for t large enough, $u(0,t) > u(0,0) + 1$ (because of (5.13) and $\omega > 0$).

In addition, if we fix $\varepsilon > 0$, there exists $x_\varepsilon \in \Omega$, $t_\varepsilon \in \mathbb{R}$ such that

$$u(x_\varepsilon, t_\varepsilon) + 1 \leq u(x_\varepsilon, t_\varepsilon + b) + \varepsilon$$

because u is Lipschitz (if such a x_ε did not exist, there would exist $\tau > b$ such that for all $\tau' < \tau$, $u(x,t) + 1 \geq u(x, t+\tau')$). Taking $\varepsilon = \dfrac{1}{n}$, we can consider sequences x_n and t_n such that

$$u(x_n, t_n) + 1 \leq u(x_n, t_n + b) + \frac{1}{n}.$$

Case 1 :(x_n, t_n) **have a finite limit when** $n \to \infty$

In this case there exists $x_\infty \in \bar{\Omega}$ and $t_\infty \in \mathbb{R}$ such that $u(x_\infty, t_\infty) + 1 = u(x_\infty, t_\infty + b)$ and we still have for all $x \in \Omega$, $t \in \mathbb{R}$, $u(x,t) + 1 \geq u(x, t + b)$. By the strong maximum principle (Proposition 5.6) we then have that $u(x,t) + 1 = u(x, t+b)$ on $\Omega \times \mathbb{R}$.

Case 2 : $|t_n| \to \infty$

In this case we try to drag the contact point to 0 by defining $v_n(x,t) = u(x, t+t_n) + 1 - \lfloor \omega t_n \rfloor$ and $v_n' = u(x, t + t_n + b) - \lfloor \omega t_n \rfloor$. We note that v_n is still a supersolution of (5.1) and v_n' is a subsolution of (5.1).

If we define $z_n = v_n - \omega t$ and $z_n' = v_n' - \omega t$, both sequences are uniformly bounded in $C^{2+\beta, 1+\frac{\beta}{2}}(\Omega \times (-M, M))$, with the same bounds as those of w (with a minor exception for the C^0 bound, which can be higher due to the use of the floor function, but the difference can not be higher than 1). As a consequence, all those functions belong to a compact subspace of $C^{2+\gamma, 1+\frac{\gamma}{2}}(\Omega \times (-M, M))$. By the same process of diagonal extraction, there exist functions z_∞ and z_∞' such that z_n and z_n' converge to z_∞ and z_∞' respectively, in $C^{2+\gamma, 1+\frac{\gamma}{2}}(\Omega \times (-M, M))$ for all $M > 0$. As a consequence, defining, $v_\infty = z_\infty + \omega t$, $v_\infty' = z_\infty' + \omega t$, we have that v_n and v_n' converge to v_∞ and v_∞' respectively, in $C^{2+\gamma, 1+\frac{\gamma}{2}}(\Omega \times (-M, M))$ for all $M > 0$.

As the convergence is in $C^{2+\gamma, 1+\frac{\gamma}{2}}(\Omega \times (-M, M))$ for all $M > 0$, v_∞ and v_∞' are still solutions of (5.1).

Now we note that for all n, we have $v_n(x,t) \geq v'_n(x,t)$ for all $x \in \Omega$, $t \in \mathbb{R}$ and $v_n(x_n,0) \leq v'_n(x_n,0) + \frac{1}{n}$. As $\bar{\Omega}$ is compact, there exists $x_\infty \in \bar{\Omega}$ such that x_n converges to x_∞ up to a subsequence. Taking the limit as $n \to \infty$, and owing to the fact that all v_n, v'_n, v_∞ and v'_∞ are bounded in $C^{2,1}_{loc}(\Omega \times \mathbb{R})$, we have that

$$v_\infty(x,t) \geq v'_\infty(x,t)$$

for all $x \in \Omega$, $t \in \mathbb{R}$, and

$$v_\infty(x_\infty, 0) = v'_\infty(x_\infty, 0).$$

There again, by the strong maximum principle (Proposition 5.6), we have $v_\infty = v'_\infty$ on $\Omega \times \mathbb{R}$. Moreover, for all $n \in \mathbb{N}$, we have for all $x \in \Omega$, $t \in \mathbb{R}$, $v_n(x,t) - 1 = v'_n(x, t-b)$, so this relation is still true for the limits and $v_\infty(x,t) - 1 = v'_\infty(x,t-b)$. Combining this with $v_\infty = v'_\infty$, we get that, for all $x \in \Omega$, $t \in \mathbb{R}$,

$$v_\infty(x,t+b) = v_\infty(x,t) + 1$$

and v_∞ is also a solution of (5.1). As u satisfies $|u(x,t) - \omega t| \leq C'$, each v_n satisfies $|v_n(x,t) - \omega t| \leq C' + 2$, so we also have $|v_\infty(x,t) - \omega t| \leq C' + 2$.

As a consequence, in both cases we have proved that there exists a solution u of (5.1) such that

$$u(x,t+b) = u(x,t) + 1,$$

and in both cases the relation $|u(x,t) - \omega t| \leq C' + 2$ holds.

Now it can be shown that $b = \frac{1}{\omega}$. Indeed, suppose $b \neq \frac{1}{\omega}$ and note $\varepsilon = |1 - \omega b|$. Take $N \in \mathbb{N}$ such that $N\varepsilon > |u(0,0)| + C' + 2$. Then we have

$$|u(0,bN) - \omega bN| = |u(0,0) + N - \omega bN| \geq ||N - \omega bN| - |u(0,0)||$$

But $|N - \omega bN| = \varepsilon N$, so

$$|u(0,bN) - \omega bN| \geq |\varepsilon N\varepsilon - |u(0,0)|| > C' + 2,$$

which brings a contradiction with the earlier result on time oscillations.

Finally, we can conclude that $u(x,t+T) = u(x,t) + 1$ with $T = \frac{1}{\omega}$.

Step 4 : Monotonicity of the permanent regime solution

We recall that we suppose $\omega > 0$. We consider u the solution of (5.1) built in the preceding steps, and as above, we define

$$b = \max\left\{\tau \geq 0, \quad \forall \tau' \leq \tau, \quad u(x,t) + 1 \geq u(x,t+\tau')\right\}.$$

As $u(x,t)+1-u(x,t+\tau)$ is a T-peridodic function for all $\tau \geq 0$, its minimum is reached for a finite $t \in \mathbb{R}$. For $\tau = b$, this minimum is 0 (if it were positive, as u is Lipschitz, there would exist $\tau > b$ such that $u(x,t) + 1 \geq u(x, t + \tau')$ for all $x \in \Omega$, $t \in \mathbb{R}$, $b \leq \tau' \leq \tau$, which would contradict the definition of b).

As a consequence, there exists $t_0 \in \mathbb{R}$ and $x_0 \in \bar{\Omega}$ such that $u(x_0, t_1)+1 = u(x_0, t_1 + b)$ and $u(x,t) + 1 \geq u(x, t + b)$ for all $x \in \Omega$, $t \in \mathbb{R}$. Again, by the strong maximum principle (Proposition 5.6), we have that $u(x,t) + 1 = u(x, t + b)$ is true for all $x \in \Omega$, $t \in \mathbb{R}$.

Now we show by contradiction that u is nondecreasing in time. Suppose that there exists $x_0 \in \Omega$, $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$ such that $u(x_0, t_1) > u(x_0, t_2)$. As for all $N \in \mathbb{N}$, $u(x_0, t_1 + Nb) = u(x_0, t_1) + N \geq u(x_0, t_1) > u(x_0, t_2)$, we can suppose that $t_1 < t_2 < t_1 + b$. But then, if we note $\tau' = t_1 + b - t_2 < b$, we have that $u(x_0, t_2) + 1 < u(x_0, t_1) + 1 = u(x_0, t_1 + b) = u(x_0, t_2 + \tau')$. This contradicts the definition of b.

We have thus proved that u is nondecreasing.

Step 5 : Uniqueness of the permanent regime solution for a given ω

By the same kind of method, we want to show that the permanent regime solution u is unique up to time translations. So we consider u_1 and u_2 two strong solutions of (5.1) satisfying $u_i(x,t) + 1 = u_i(x, t + T)$, $i = 1, 2$. If we note $w_1 = u_1 - \omega t$ and $w_2 = u_2 - \omega t$, both w_1 and w_2 are bounded in $C^{2+\gamma, 1+\frac{\gamma}{2}}(\Omega \times (-M, M))$ for some $\gamma > 0$, for all $M > 0$. As a consequence, there exists $N \in \mathbb{N}$ such that $w_1 + N \geq w_2$, which implies $u_1 + N \geq u_2$.

As above, we define

$$b' = \max\left\{\tau \geq 0, \quad \forall \tau' \leq \tau, \quad u_1(x,t) + N \geq u_2(x, t + \tau')\right\}.$$

$u_1(x,t) + N - u_2(x, t + b')$ is T-periodic, so its minimum is reached for a finite $t_0 \in \mathbb{R}$, and $x_0 \in \bar{\Omega}$. As before, this minimum is 0, because if it were not the case, as u_1 and u_2 are Lipschitz, there would exist $\tau > b'$ such that for all $\tau' \leq \tau$, $u_1(x,t) + N \geq u_2(x, t + \tau')$ for all $x \in \Omega$, $t \in \mathbb{R}$, which would contradict the definition of b'. So we have

$$\begin{cases} u_1(x,t) + N \geq u_2(x, t + b') & x \in \Omega, \quad t \in \mathbb{R} \\ u_1(x_0, t_0) + N = u_2(x_0, t_0 + b') & \end{cases}$$

By the strong maximum principle (Proposition 5.6), we have that $u_1(x,t) + N = u_2(x, t + b')$ for all $x \in \Omega$, $t \in \mathbb{R}$, so $u_1 = u_2(\cdot, \cdot + b' - NT)$ and u_1 and u_2 are equal up to a time translation, which is what we wanted to prove.

Proof of Theorem 5.3 in the case $\omega < 0$

In the case where $\omega < 0$, it is clear that if we consider u the solution of (5.1) with $u(\cdot, 0) = 0$, then $-u$ is solution of (5.1) with $-u(\cdot, 0) = 0$ up to the replacement of W and σ by their opposites. Then $-u$ satisfies (5.4), this time with $\omega > 0$, so the demonstration given in the case $\omega > 0$ also proves Theorem 5.3 when $\omega < 0$.

Proof of Theorem 5.3 in the case $\omega = 0$

We now cover the case where $\omega = 0$. Let u be the solution of (5.1) with initial condition $u(x, 0) = 0$. As $\omega = 0$, the bound 5.4 implies that u is bounded. We prove the existence of a stationary solution to (5.1) by showing that u admits a limit as $t \to +\infty$ (possibly up to the extraction of a subsequence of times (t_n)), and that this limit is indeed a stationnary solution of (5.1).

Multiplying equation (5.1) on $\Omega \times \mathbb{R}$ by u_t and integrating on Ω, we get

$$\int_\Omega (u_t)^2 + \frac{d}{dt}\left(\int_\Omega \frac{(\nabla u)^2}{2} + \int_\Omega \left(W\left(u + \frac{\theta^+ - \theta^-}{2\pi}\right) - \sigma u\right)\right) = 0.$$

This expression is licit because of the regularity of u and W, which implies that the expression for which we take a time derivative is indeed differentiable. We can define the analog of an energy for u by

$$E(u(\cdot, t)) = \int_\Omega \frac{(\nabla u)^2}{2} + \int_\Omega \left(W\left(u + \frac{\theta^+ - \theta^-}{2\pi}\right) - \sigma u\right).$$

Then we get

$$\frac{dE(u)}{dt} = -\int_\Omega (u_t)^2 \leq 0.$$

As a consequence, $\tilde{E} \colon t \mapsto E(u(\cdot, t))$ is a nonincreasing function, and as u is bounded, it is also bounded. So there exists a constant E_∞ such that $\lim_{t\to\infty} \tilde{E}(t) = E_\infty$. Writing the equality

$$E_\infty - \tilde{E}(t_0) = \int_{t_0}^{+\infty} \int_\Omega |u_t(x, t)|^2 dx dt,$$

we deduce that $u_t \in L^2(\Omega \times (t_0, +\infty))$, and that

$$\lim_{t_0 \to \infty} \|u_t\|_{L^2(\Omega \times (t_0, +\infty))} = 0. \tag{5.28}$$

In addition, the line of reasoning of Step 1 of the proof in the case $\omega > 0$ remain valid, except for the fact that because $\omega = 0$, there is no natural time period. In particular, $u(\cdot, t)$ is bounded in $C^{2+\alpha}(\Omega)$ for t high enough. As a consequence, any sequence $(u(\cdot, t_n))_n$ with $t_n \to +\infty$ is relatively compact in $C^2(\Omega)$ and thus converges to a limit u_∞ in $C^2(\Omega)$ up to the extraction of a subsequence.

We consider $\eta > 0$. From (5.28), there exists $T > 0$ such that for all $t > T$,

$$\|u_t\|_{L^2(\Omega \times (t, +\infty))} < \eta.$$

As a consequence, the set

$$S = \{t > T \mid \|u(\cdot, t)\|_{L^2(\Omega)} > \eta\}$$

has a measure $m = |S| \leq 1$. We can consider a sequence $(t_n)_n$ such that $t_n \to +\infty$ and $t_n \in (T, +\infty) \setminus S$ for all $n \in \mathbb{N}$. As we have shown, the sequence $u_n = u(\cdot, t_n)$ admits an accumulation value u_∞ in $C^2(\Omega)$. If we define, for $v \in C^2(\Omega)$, the quantity

$$\mathcal{F}(v) = \Delta v - W'\left(v + \frac{\theta^+ - \theta^-}{2\pi}\right) + \sigma,$$

then $\mathcal{F}(u_n)$ converges to $\mathcal{F}(u_\infty)$ in $C^0(\Omega)$ (and thus also in $L^2(\Omega)$), and there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\|\mathcal{F}(u_\infty) - \mathcal{F}(u_n)\|_{L^2(\Omega)} \leq \eta.$$

As $t_n \in (T, +\infty) \setminus S$, we also have that

$$\|\mathcal{F}(u_n)\|_{L^2(\Omega)} = \|u_t(\cdot, t_n)\|_{L^2(\Omega)} \leq \eta.$$

As a consequence,

$$\|\mathcal{F}(u_\infty)\|_{L^2(\Omega)} \leq 2\eta.$$

As this is true for all $\eta > 0$, we can conclude that $\|\mathcal{F}(u_\infty)\|_{L^2(\Omega)} = 0$, which implies that u_∞ is a stationary solution of (5.1). This completes the proof of Theorem 5.3.

5.6 Growth of ω in σ

Proof of Theorem 5.2

From what we have seen before, ω can depend on σ and on the initial condition $u(\cdot, 0)$.

First, it is possible to show that ω does not depend on the initial condition. We consider u_1 and u_2 two solutions of (5.1) with the same value of σ, and with initial conditions $u_1(\cdot, 0) = u_{1,0} \in C^{2+\alpha}(\Omega)$ and $u_2(\cdot, 0) = u_{2,0} \in C^{2+\alpha}(\Omega)$, satisfying the compatibility conditions $\dfrac{\partial u_{1,0}}{\partial n} = \dfrac{\partial u_{2,0}}{\partial n} = 0$ on $\partial\Omega$. Theorem 5.1 indicates that there exist $\omega_1 \in \mathbb{R}$ and $\omega_2 \in \mathbb{R}$ such that

$$|u_1(x,t) - \omega_1 t| \leq C_1, \tag{5.29}$$

and

$$|u_2(x,t) - \omega_2 t| \leq C_2. \tag{5.30}$$

As $u_{1,0}$ and $u_{2,0}$ are both bounded on Ω, there exists $M \in \mathbb{Z}$ and $N \in \mathbb{Z}$ such that

$$u_{1,0} + M \leq u_{2,0} \leq u_{1,0} + N.$$

As M and N are integers, $u_1 + M$ and $u_1 + N$ are both solutions of (5.1). Then, by the comparison principle (Proposition 5.5), we have

$$u_1 + M \leq u_2 \leq u_1 + N. \tag{5.31}$$

The relations (5.29),(5.30) and (5.31) imply that $\omega_1 = \omega_2$.

Now we consider u_1 a solution of (5.1) with $\sigma = \sigma_1$, with the initial condition $u_1(\cdot, 0) = u_{1,0} \in C^{2+\alpha}(\Omega)$ satisfying $\dfrac{\partial u_{1,0}}{\partial n} = 0$, and u_2 a solution of (5.1) with $\sigma = \sigma_2$, with the initial condition $u_2(\cdot, 0) = u_{2,0} \in C^{2+\alpha}(\Omega)$ satisfying $\dfrac{\partial u_{2,0}}{\partial n} = 0$. We suppose that $\sigma_1 < \sigma_2$, and our goal is to show that $\omega_1 \le \omega_2$.

We will show this inequality by contradiction, so we suppose that $\omega_1 > \omega_2$. As ω_1 and ω_2 do not depend on $u_{1,0}$ and $u_{2,0}$, we can suppose that $u_{2,0} > u_{1,0}$. Furthermore, we know that there exists $\mathcal{C} > 0$, such that

$$|u_1 - \omega_1 t| \le \mathcal{C},$$

and

$$|u_2 - \omega_2 t| \le \mathcal{C},$$

so, for some $t_0 > \dfrac{2\mathcal{C}}{\omega_1 - \omega_2}$, we have that, for $x \in \Omega$,

$$u_1(t_0, x) \ge \omega_1 t_0 - \mathcal{C} > \omega_2 t_0 + \mathcal{C} \ge u_2(t_0, x)$$

This contradicts the comparison principle (Proposition 5.5), and this contradiction concludes the proof of Theorem 5.2.

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