Extensions of the Itô formula through Malliavin calculus and application to a variational problem

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August 22, 2012

Résumé

Ce travail de thèse est consacré à l'extension de la formule d'Itô au cas de chemins à variations bornées à valeurs dans l'espace des distributions tempérées composés par des processus réguliers au sens de Malliavin. On s'attache en particulier à faire des hypothèses minimales de régularité, ce qui donne accès à un certain nombre d'applications de notre principal résultat, en particulier à l'étude d'un problème variationnel.

Notre principal outil est le calcul de Malliavin, le premier chapitre est donc consacré à des rappels sur ce sujet; la plupart des résultats sont classiques. Le deuxième chapitre est consacré à l'étude de la classe de Schwartz et de l'espace des distributions tempérées. On explique en particulier comment elles peuvent être exprimées, respectivement, comme intersection et comme union d'espaces obtenus comme les domaines des puissances d'un opérateur différentiel simple. Cela nous permet en particulier de comprendre la structure des chemins à valeurs dans l'espace des distributions tempérées. Ces résultats sont connus mais très peu documentés, nous avons donc écrit des preuves détaillées.

Dans le troisième chapitre, on donne un certain nombre de résultats optimaux qui expliquent sous quelles conditions l'on peut définir la quantité $T \circ X$ pour une distribution tempérée T et une variable aléatoire X et quelle est la régularité de l'objet obtenu. On utilise pour cela des arguments d'interpolation qui permettent d'étendre une formule d'intégration par parties au sens de Malliavin au cas fractionnaire. Les résultats sont fortement inspirés de l'article de Shinzo Watanabe, "Fractional order spaces on the Wiener space", [83] mais sont nouveaux. On donne également une version pour le cas où la distribution T est elle-même stochastique; ce résultat est fondé sur une extension de la formule d'intégration par parties qui est elle-même originale.

Ces résultats nous permettent d'écrire, au chapitre 4, une formule d'Itô qui constitue notre principal résultat. Elle s'écrit, pour une semimartingale générale, sous des hypothèses beaucoup plus faibles que d'autres résultats existant dans la littérature. On donne également un résultat plus précis pour le cas où la semimartingale provient de la solution d'une équation différentielle stochastique. On donne enfin deux extensions simples de notre formule: une version anticipative et

une version de type Itô-Wentzell.

Dans le chapitre 5, on utilise notre formule d'Itô pour établir l'existence et l'unicité de la solution d'un problème variationnel simple. En particulier, on montre que l'on peut affaiblir considérablement une hypothèse d'ellipticité faite par la plupart des auteurs.

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Introduction

0.1 Cadre: la formule d'Itô

La formule d'Itô constitue la pierre angulaire du calcul stochastique. L'objet principal de ce travail de thèse est de la généraliser au cadre des distributions tempérées (au lieu des fonctions C^2) composées par des processus réguliers au sens de Malliavin sous des hypothèses minimales. Nous proposons également des applications, notamment à l'étude d'un problème variationnel.

0.1.1 Formule "de base"

Avant de préciser notre démarche, rappelons, à titre illustratif, une version "élémentaire" de la formule d'Itô:

Théorème 0.1.1.1 (Formule d'Itô). *Soient $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé; W un mouvement brownien à valeurs dans $\mathbb{R}^d$ sur cet espace; X une $\mathbb{P}$-semimartingale continue à valeurs dans $\mathbb{R}^N$, adaptée à la filtration de W:*

$$X_t = X_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s \tag{1}$$

où b est un processus adapté à valeurs dans $\mathbb{R}^N$ et σ est un processus adapté à

valeurs dans $\mathbb{R}^d \otimes \mathbb{R}^N$. *Soit enfin* $f \in C^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}\right)$. *Alors on a:*

$$
\begin{aligned}
f(t, X_t) &= f(0, X_0) \\
&+ \int_0^t \partial_t f(s, X_s) ds \\
&+ \sum_{i=1}^N \partial_i f(s, X_s) \cdot b_i(s) ds \\
&+ \frac{1}{2} \cdot \sum_{i,j=1}^N \partial_{ij}^2 f(s, X_s) \cdot a_{ij}(s) ds \\
&+ \sum_{i=1}^N \partial_i f(s, X_s) \cdot \sigma_i(s) dW_s \qquad (2)
\end{aligned}
$$

avec $a = \sigma \cdot^t \sigma$.

De plus l'intégrale stochastique dans (2) définit une martingale locale, donc $(f(X_t), t \geq 0)$ *est une semimartingale dans la filtration de* W.

Les notions définies dans le théorème précédent sont classiques; on renvoie, pour leurs définitions, à [60], [17], [31], [28]... ou bien aux articles originaux [29] et [30].

0.1.2   Quelques généralisations connues

De nombreux auteurs se sont attachés à généraliser la formule (2) dans diverses directions. La première est de renoncer à l'hypothèse de continuité sur la semi-martingale; on obtient alors, en paticulier, une formule d'Itô pour les processus de Lévy, voir, par exemple, [11] ou la thèse de doctorat [85] et les références qu'elle cite. Nous n'avons pas eu le temps d'explorer cette théorie au cours de notre travail. Une seconde direction possible est de généraliser la formule d'Itô à des processus qui ne sont pas des semimartingales. Parmi les classes de processus pour lesquelles il existe des résultats, citons par exemple: le mouvement brownien fractionnaire et les intégrales stochastiques construites sur ce processus (voir par exemple [16]), le mouvement brownien multi-fractionnaire (voir [43]), ou certains processus de Markov. Nous n'avons pas non plus travaillé dans cette direction.

Notre travail s'inscrit en effet dans une seconde "famille" de généralisations de la formule d'Itô qui relâchent les conditions portant sur la fonction f dans (2) plutôt que sur le processus X. Il existe de nombreux résultats sur ce sujet. Le plus célèbre d'entre eux est sans doute la formule de Tanaka (coir [60] par exemple), qui permet de prendre f convexe lorsque $N = d = 1$:

Théorème 0.1.2.1 (Formule de Tanaka). *Soit X une semimartingale continue en dimension 1; pour tout $a \in \mathbb{R}$ il existe un unique processus croissant $L_t^a(X)$ tel que:*

$$|X_t - a| = |X_0 - a| + \int_0^t sgn(X_s - a)dX_s + \frac{1}{2} \cdot L_t^a(X) \tag{3}$$

$L_t^a(X)$ est le temps local de la semimartingale X.

Il vient alors:

Théorème 0.1.2.2 (Formule d'Itô-Tanaka). *Soit X une semiartingale continue en dimension 1 et $f : \mathbb{R} \to \mathbb{R}$ pouvant s'écrire comme différence de deux fonctions convexes, de sorte que f admet une dérivée à gauche f_g' et qu'au sens des distributions, f'' est une mesure de Radon signée. On a la généralisation suivante de la formule d'Itô:*

$$f(X_t) = f(X_0) + \int_0^t f_g'(X_s)dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a(X)f''(da) \tag{4}$$

De nombreux auteurs se sont également intréssés à la possibilité de remplacer la condition $f \in C^2$ par une condition du type $f \in W^{p,1}$, un espace de Sobolev. Par exemple pour le cas particulier du mouvement brownien en dimension 1, [13] expose les propriétés de semimartingale de $x \mapsto L_t^x(X)$ à t fixé et établit le résultat suivant:

Théorème 0.1.2.3 (Formule de Bouleau-Yor). *Soit f une fonction dérivable sur $\mathbb{R}$, de dérivée localement bornée et W un mouvement brownien de dimension 1. On a:*

$$f(W_t) = f(0) + \int_0^t f'(W_s)dW_s - \frac{1}{2} \cdot \int_{\mathbb{R}} f'(x)d_x L_t^x(X) \tag{5}$$

Cette construction faisant intervenir le temps local du mouvement brownien, elle est spécifque à la dimension 1. Cependant, [20] propose une généralisation sous une hypothèse encore plus faible qui, quant à elle, s'étend au cas $N = d$:

Théorème 0.1.2.4 (Formule de Föllmer-Protter-Shiryaev). *Soient f dérivable telle que $Df \in L_{loc}^2\left(\mathbb{R}^N, \mathbb{R}^N\right)$ et W un mouvement brownien en dimension N. On a:*

$$f(W_t) = f(0) + \sum_{i=1}^{N} \partial_i f(W_s)dW_s + \frac{1}{2} \cdot [f'(W), W]_t \tag{6}$$

où la covariation quadratique $[f'(W), W]$ est définie comme une limite de sommes de Riemann:

$$[f'(W), W]_t = \sum_{i=1}^{N} \lim_{n \to \infty} \sum_{k=1}^{n} \left(f'(W_{\frac{k}{n}}) - f'(W_{\frac{k-1}{n}})\right) \cdot \left(W_{\frac{k}{n}} - W_{\frac{k-1}{n}}\right) \tag{7}$$

Nous établirons des liens directs entre ces trois derniers résultats (formules de Tanaka, de Bouleau-Yor et de Föllmer-Protter-Shiryaev) et les nôtres; nous reviendrons sur ce point plus tard. Mentionnons également un dernier résultat du même type, voir [22] ou [12], bien qu'il soit moins directement lié à notre étude:

Théorème 0.1.2.5. *Soient W un mouvement brownien à valeurs dans $\mathbb{R}^d$ et $f \in W^{2,1}\left(\mathbb{R}^d\right)$. Alors il existe une unique décomposition:*

$$f(W_t) = f(0) + M_t(f) + N_t(f) \tag{8}$$

où $M(f)$ est une martingale locale de carré intégrable et $N(f)$ est un processus continu, additif, d'énergie nulle.

0.1.3 Le cas des EDS

Il existe également de nombreux résultats permettant d'écrire une formule d'Itô pour $f \in W^{p,1}$ pour des processus plus généraux que le seul mouvement brownien. En général il est nécessaire de faire une hypothèse d'ellipticité sur le processus X. Pour le résultat suivant, on se donne deux fonctions b et σ pour lesquelles il existe une constante C telle que pour tous $x, y \in \mathbb{R}^N$:

$$\sup_{t \geq 0} |b(t,x) - b(t,y)| \quad \leq C \quad \cdot |x-y| \tag{9}$$

$$\sup_{t \geq 0} |\sigma(t,x) - \sigma(t,y)| \quad \leq C \quad \cdot |x-y| \tag{10}$$

Alors, il existe une unique solution forte à l'équation diff'erentielle stochastique:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \tag{11}$$

On note enfin A le générateur du processus; si $a = \sigma \cdot^t \sigma$:

$$A = \sum b_i \cdot \partial_i + \frac{1}{2} \cdot \sum a_{ij} \cdot \partial_{ij} \tag{12}$$

Avec ces notations, [38] établit:

Théorème 0.1.3.1. *Supposons que b et σ soient bornés presque sûrement et que A soit uniformément elliptique, c'est à dire qu'il existe une constante C telle que pour tout $\xi \in \mathbb{R}^N$:*

$$(A \cdot \xi, \xi) = \sum_{i,j} a_{ij}\xi_i\xi_j \geq C \cdot \|\xi\|^2 \tag{13}$$

Alors, pour tout $f \in W^{2,2}$, on peut écrire la formule d'Itô pour $f(X_t)$ comme dans le cas C^2.

L'idée de la preuve, que nous réutiliserons, est que les fonctions de $W^{2,2}$ peuvent être approchées par des fonctions C^2. En combinant ces idées avec celles de la formule de Föllmer-Protter-Shiryaev, certain auteurs (voir [6], [7] ou [5], par exemple) prouvent des formules d'Itô pour des processus elliptiques et des fonctions $W^{1,p}$. Il existe également des résultats pour le cas où X est une semimartingale.

0.2 Apports du calcul de Malliavin

Nous nous proposons d'établir un résultat encore plus général: en effet nous remplacerons la fonction par une distribution tempérée. Avant d'énoncer notre théorème, il nous faut faire quelques rappels de calcul de Malliavin, théorie qui sera au centre de la plupart des preuves de ce document. Les résultats ci-dessous sont l'objet de rappels plus détaillés au cours du chapitre 1; par ailleurs nous nous sommes beaucoup inspirés, notamment, de [45], [53], [25], [66], [55], [56], [75] et [44].

0.2.1 Rappels de calcul de Malliavin

Plaçons-nous sur l'espace de Wiener $\mathcal{W}$ des fonctions continues sur $[0, 1]$, nulles en 0, à valeurs dans un espace de Hilbert X. Munissons $\mathcal{W}$ de sa structure habituelle d'espace de Banach et de la mesure de Wiener μ, loi du mouvement brownien sur X. L'objet du calcul de Malliavin est la construction d'une analyse de type Sobolev, et en particulier d'un gradient au sens L^p sur (W, μ).

Nous nous intéressons aux dérivées aux sens faible le long de l'espace H, dit espace de Cameron-Martin, des primitives de fonctions $L^2([0, 1], X)$. En effet H est un sous-espace dense de $\mathcal{W}$, et le théorème de Girsanov assure que la mesure de Wiener est quasi-invariante par la translation par $h \in \mathcal{W}$ si et seulement si $h \in H$. Soient alors W un mouvement brownien sur X, $0 \leq t_1 < \cdots < t_n \leq 1$, Y un autre espace de Hilbert et $f : X^n \mapsto Y$ dérivable. Nous introduisons la fonctionnelle dite cylindrique $F = f(B_{t_1}, \cdots B_{t_n})$ ainsi que sa dérivée au sens de Malliavin:

$$\nabla F = h \mapsto \sum_{i=1}^n (\partial_i f)(B_{t_1}, \cdots B_{t_n}) \otimes \dot{h}(t_i) \tag{14}$$

où $\otimes$ designe le produit tensoriel entre espaces de Hilbert, si bien que:

$$\nabla F \in Y \otimes H \approx \mathcal{L}(H, Y) \tag{15}$$

est la différentielle au sens L^p de F. La densité dans L^p des fonctionnelles cylindriques et la propriété de quasi invariance de μ permettent de montrer la fermabilité de ∇ ainsi que la densité de ses domaines de fermeture $\mathbb{D}^{p,1} := dom_{L^p}(\nabla)$.

Ces domaines de fermeture sont les espaces de Sobolev d'ordre 1; les constructions étant valables pour des espaces de Hilbert quelconques on définit de même des gradients itérés et des espaces de Sobolev d'ordre supérieur. On les munit de la norme:

$$\|F\|_{\mathbb{D}^{p,k}} = \|F\|_{dom_{L^p}(\nabla^k)} = \|F\|_{L^p(X)} + \|\nabla^k F\|_{L^p(X \otimes H^{\otimes k})} \tag{16}$$

On peut également construire la divergence comme adjoint du gradient:

$$\delta = \nabla^* : \mathbb{D}^{p,1}(H) \to L^p(X) \tag{17}$$

Notons au passage que la divergence d'un processus régulier et adapté coïncide avec son intégrale stochastique, ce qui permet une extension de la formule d'Itô à des semimartingales non-adaptées (cf [54] ou [53]); nous reviendrons sur ces notions.

Puis, on introduit l'opérateur d'Ornstein-Uhlenbeck $\mathcal{L} = \delta \circ \nabla$. L'opposé de cet opérateur est le générateur d'un semi groupe de contraction sur L^p, dit d'Ornstein-Uhlenbeck. L'étude de ce semi-groupe permet de construire des puissances fractionnaires de tout ordre réel de $Id + \mathcal{L}$ et d'établir les inégalités de Meyer qui assurent que la norme usuelle et la norme $\left\|(Id + \mathcal{L})^{k/2} F\right\|_{L^p}$ sont équivalentes. Ceci permet d'étendre la notion d'espace de Sobolev à des ordres de dérivation fractionnaires ou négatifs. On prouve en outre que le dual de $\mathbb{D}^{p,k}$ est $\mathbb{D}^{p^*,-k}$.

0.2.2   Intégrations par parties et formule d'Itô faible

Introduisons maintenant $\mathbb{D}$ l'espace des variables aléatoires appartenant à tous les espaces de Sobolev: il s'agit d'un espace de variables aléatoires test, très régulières. Son dual $\mathbb{D}'$ est l'espace des distributions de Meyer (formes linéaires sur au moins l'un des espaces de Sobolev). Il est possible d'étendre de nombreux objets du calcul de Malliavin (gradient, divergence...) à l'espace des distributions de Meyer, de même qu'on étend le calcul différentiel sur $\mathbb{R}^n$ aux distributions tempérées par exemple. L'analogie vaut également pour les méthodes de démonstration: des résultats sont établis sur l'espace des fonctions tests et étendus par densité à l'espace des distributions.

On s'intéressera en particulier à la formule d'intégration par parties suivante:

Théorème 0.2.2.1. *Soient $X, Y \in \mathbb{D}$ et $f \in C_b^1$. On définit la matrice de Malliavin Σ de X par:*

$$\Sigma_{ij}(X) = (\nabla X_i, \nabla X_j)_H \tag{18}$$

et on suppose que X est non-dégénérée, c'est à dire que:

$$\gamma(X) := \Sigma(X)^{-1} \in \bigcap_{1 < p < \infty} L^p \tag{19}$$

Alors:

$$E\left[\partial_i f(X) \cdot Y\right] = E\left[f(X) \cdot l_i(Y, X)\right] \tag{20}$$

où

$$l_i(Y, X) = \delta\left(Y \cdot \sum_{i=1}^{N} \gamma_{ij}(X) \cdot X_j\right) \tag{21}$$

Ce résultat permet de définir $f'(X) \in \mathbb{D}'$ par dualité pour $X \in \mathbb{D}$, variable aléatoire non-dégénérée sur l'espace de Wiener et f une fonction telle que f' n'est définie qu'au sens des distributions. Plus précisément, par une récurrence simple fondée sur le théorème précédent et le fait que l'ensemble des distributions tempérées s'obtient, pour tout $1 < p < \infty$, comme:

$$\mathcal{S}' = \bigcup_{k \in \mathbb{N}} \left(Id + X^2 - \Delta\right)^{-k}(L^p) \tag{22}$$

il a été établi dans [81] que l'on peut définir $T \circ X \in \mathbb{D}'$ pour tout $X \in \mathbb{D}$ non-dégénérée et $T \in \mathcal{S}'$. Cela a permis d'obtenir l'extension suivante de la formule d'Itô, voir [78]:

Théorème 0.2.2.2. *Soient b et σ deux processus stochastiques adaptés à la filtration du mouvement brownien W; on suppose, pour tous p et k:*

$$\int_0^t \|b_s\|_{\mathbb{D}_{p,k}}^p \, ds \quad < \quad \infty \tag{23}$$

$$\int_0^t \|\sigma_s\|_{\mathbb{D}_{p,k}}^p \, ds \quad < \quad \infty \tag{24}$$

et on introduit:

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s \tag{25}$$

Soit Σ_s la matrice de Malliavin de X_s; on suppose en outre:

$$\int_0^t \|\Sigma_s^{-1}\|_{L^p}^p \, ds < \infty \tag{26}$$

ou, ce qui revient au même:

$$\int_0^t \left\|\frac{1}{\det \Sigma_s}\right\|_{L^p}^p \, ds < \infty \tag{27}$$

Alors pour tout $\epsilon > 0$ on a:

$$T(t) \circ X_t - T(\epsilon) \circ X_\epsilon \;\; = \;\; M_t^{(\epsilon)} + \sum_i \int_\epsilon^t b_i(u) \cdot (\partial_i T_u) \circ X_u du$$

$$+ \frac{1}{2} \sum_{i,j} \int_\epsilon^t a_{ij}(u) \cdot (\partial_{ij} T_u) \circ X_u du$$

$$+ \int_\epsilon^t T(du) \circ X_u \qquad (28)$$

avec, en utilisant la notation des intégrales stochastiques pour la divergence:

$$M_t^{(\epsilon)} = \sum_j \int_\epsilon^t \sum_i \left[b_i(u) \cdot (\partial_i T_u) \circ X_u \right] dW_u^j \qquad (29)$$

La preuve de [78] est fondée sur la stucture d'espace de Fréchet nucléaire de $\mathcal{S}'$. En effet, on peut alors appliquer le théorème de Grothendieck pour obtenir une représentation du chemin T_t du type:

$$T_t = \sum_{n=1}^{\infty} \lambda_n \cdot V_n(t) \cdot T_n \qquad (30)$$

où $(\lambda_n) \in l^1$, (V_n) est une famille bornée de fonctions à variations bornées sur $\mathbb{R}$ et (T_n) est une famille uniformément continue de $\mathcal{S}'$. On voit alors que l'on peut choisir k tel que $\phi_t := (Id + X^2 - \delta)^{-k} T_t$ soit une fonction C^2 en espace de façon indépendante de t; de plus, la fonction:

$$(t, x) \mapsto \phi_t(x) = \left(\left(Id + X^2 - \delta \right)^{-k} T_t \right)(x) \qquad (31)$$

est alors à variation bornée en temps (et C^2 en espace). Le résultat est finalement obtenu en appliquant la formule d'Itô classique à la fonction ϕ et en "inversant" les intégrations par parties.

0.3   Apports de ce travail

Si l'intérêt théorique des résultats du paragraphe précédent est clair, on voit également que les hypothèses faites (les variables aléatoires doivent être dans $\mathbb{D}$) sont trop fortes pour les applications pratiques. Au cours de ce travail, on s'est donc attaché à les affaiblir. Les deux premiers chapitres de cette thèse fournissent les outils nécessaires à la poursuite de cet objectif. Au chapitre 1, on fait des rappels de calcul de Malliavin, cette théorie intervenant dans la majorité de nos preuves. D'autres

pans des mathématiques interviennent également: espaces vectoriels topologiques, intégration de fonctions à valeurs Banach, interpolation, théorie des semi-groupes, équations au dérivées partielles... On fait les rappels nécessaires à notre étude dans les appendices correspondants, où l'on fournit également des références. Dans le chapitre 2, on étudie la structure de la classe de Schwartz et de l'espace des distributions tempérées en tant qu'espaces vectoriels topologiques (voir par exemple [65], [23], [61], [59], [69]...) En particulier, on construit les puissances fractionnaires de l'opérateur:

$$\mathcal{K} := Id + X^2 - \Delta \tag{32}$$

par des méthodes d'interpolation de semi-groupes et on introduit les espaces:

$$\mathcal{S}_{p,s} = dom_{L^p}\left(\mathcal{K}^{s/2}\right) \tag{33}$$

Leur intérêt est que l'on a:

$$\mathcal{S} = \bigcap_{p,s} \mathcal{S}_{p,s} \tag{34}$$

et:

$$\mathcal{S}' = \bigcup_{p,s} \mathcal{S}_{p,s} \tag{35}$$

Ces unions ensemblistes définissent également une topologie d'espace de Fréchet sur $\mathcal{S}$ et de dual d'espace de Fréchet sur $\mathcal{S}'$ comme il est expliqué dans le théorème 2.3.0.3.

Ce sont sur les éléments développés dans les deux premiers chapitres et les appendices que nous nous appuyons pour établir, au cours du chapitre 3, de nombreux lemmes techniques qui ont un intérêt propre. Nous pouvons ainsi établir des extensions de la formule d'Itô dans le chapitre 4 et en développer une application à un problème variationnel au chapitre 5.

0.3.1 Résultats d'intégration par parties

Les outils ue nous venons de présenter permettent de prouver le théorème ci-dessous, qui est le résultat principal du chapitre 3; il s'agit du théorème 3.2.0.13 que nous reproduisons ici:

Théorème 0.3.1.1. *Soit $F \in \mathbb{D}_{\infty^-, 1+\delta}$ avec $\delta > 0$ une variable aléatoire non-dégénérée à valeurs dans $\mathbb{R}^N$. On suppose que X admet une densité bornée p_X. Alors pour tout $\delta' < \delta$ and $1 < p < p' < \infty$ on peut trouver $q, r > 1$ et $\theta > 0$ tels qu'on ait l'estimation suivante pour tous $\phi \in \mathcal{S}(\mathbb{R}^N)$:*

$$\|\phi \circ F\|_{\mathbb{D}_{p,-\delta}} \leq C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right]^{2m} \cdot \|F\|_{\mathbb{D}_{r,1+\delta'}}^\theta \cdot \|p_F\|_\infty \cdot \|\phi\|_{\mathcal{S}_{p',-\delta'}} \tag{36}$$

Ici C est une constante universelle dépendant de $p, \delta, \delta', q, r, \theta$... mais pas de X ou de ϕ. En particulier pour tout $T \in \mathcal{S}_{p',-\delta'}$, $T \circ F$ peut être défini dans $\mathbb{D}_{p,-\delta}$ et on a encore le contrôle ci-dessus.

Pour prouver ce résultat, on s'appuie en particulier sur trois lemmes. Le premier permet de contrôler les normes de Sobolev de l'inverse de la matrice de Malliavin d'une variable aléatoire suffisement régulière:

Lemme 0.3.1.1. *Pour un certain $s > 1$, considérons une variable aléatoire non dégénérée $F \in \mathbb{D}_{\infty-,s}(\mathbb{R}^{\mathbb{N}})$ et sa matrice de Malliavin Σ. Alors pour tout $1 \leq s' < s$,*

$$\frac{1}{\det \Sigma} \in \mathbb{D}_{\infty^-,s'-1}(\mathbb{R}) \tag{37}$$

De plus, pour tout $p' > p$ il existe un $q > 1$ et un $\alpha > 0$ tels qu'on ait le contrôle suivant:

$$\left\| \frac{1}{det\Sigma} \right\|_{\mathbb{D}_{p,s'-1}} \leq C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^{p'}}^{p} + \left\| \frac{1}{det\Sigma} \right\|_{L^{p'}}^{p} \right] \cdot \|F\|_{\mathbb{D}_{q,s}}^{\alpha} \tag{38}$$

où $k = [s]$ et C est une constante universelle ne dépendant que de p, p', s, s', q.

Le deuxième lemme permet de contrôler les opérateurs d'intégration par parties:

Lemme 0.3.1.2. *Pour tous $p > 1$ et $s > 0$, pour tous $p' > p$ et $s' > s$, il existe $q, r > 1$, $\theta > 0$ et une constante universelle ne dépendant que des paramètres ci-dessus tels que l'on ait le contrôle ci-dessous pour toute variable aléatoire $G \in \mathbb{D}_{\infty^-,s+1}$ et toute variable aléatoire non-dégénérée $F \in \mathbb{D}_{\infty^-,s'+2}$:*

$$\|l_i(G,F)\|_{\mathbb{D}_{p,s}} \leq C \cdot \|G\|_{\mathbb{D}_{p',s+1}} \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^{q} + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^{q} \right] \cdot \|F\|_{\mathbb{D}_{r,s'+2}}^{\theta} \tag{39}$$

où $k = [s]$.

En pratique, nous en utiliserons la généralisation suivante. Soit m_i l'opérateur de multiplication par le monôme X_i. Considérons une famille d'opérateurs μ_j pour $j = 1, \ldots, m$ tels que chaque μ_i soit ou bien l'un des l_i, ou bien l'un des m_i, $i = 1, \ldots, N$. On définit:

$$\lambda(G, F) = \mu_m(G, \mu_{m-1}(G, \mu_{m-2}(G, \ldots, F) \ldots)) \tag{40}$$

pour une variable aléatoire suffisement régulière G et une variable a'léatoire suffisemment régulière et no dégénérée F. On peut alors énoncer le:

Corollaire 0.3.1.1. *Soient $p' > p > 1$ et $s' > s > 0$. On considère λ comme ci-dessus, en supposant qu'au plus n des μ_j soit l'un des l_i. Alors l existe $q, r > 1$, $\theta > 0$ et une constante universelle C tels qu'on ait le contrôle ci-dessous pour G assez régulière et F assez régulière et non-dégénérée:*

$$\|\lambda(G, F)\|_{\mathbb{D}_{p,s}} \leq C \cdot \|G\|_{\mathbb{D}_{p',s+m}} \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right]^m \cdot \|F\|_{\mathbb{D}_{r,s'+m+1}}^\theta$$
(41)

ouù $k = [s]$ et σ est la matrice de Malliavin de F.

Enfin, on utilise le lemme d'interpolation suivant:

Corollaire 0.3.1.2. *Supposons que $F \in \mathbb{D}_{\infty^-,1}$ ait une densité bornée p_F. Alors pour tous $0 \leq \rho < \rho' \leq 1$ et $1 < p < p' < \infty$ il existe une constante universelle $C(\rho, \rho', p, p')$ telle que pour tout $\phi \in \mathcal{S}$ on ait le contrôle ci-dessous:*

$$\|\phi \circ F\|_{\mathbb{D}_{p,\rho}} \leq C \cdot \|p_F\|_\infty \cdot (1 + \|\nabla F\|_{L^q}) \cdot \|\phi\|_{\mathcal{S}_{p',\rho'}}$$
(42)

Au paragraphe 3.2, on explique comment combiner ces lemmes pour obtenir le résultat principal. L' idée est d' écrire $\phi = \mathcal{K}^{-l}\mathcal{K}^l\phi$ pour un l entier choisi proche de s dans un sens que l'on précise. Les deux premiers lemmes permettent alors de faire un premier contrôle en faisant $[s]$ intégrations par parties et le dernier lemme permet de raffiner afin d'obtenir la partie fractionnaire du contrôle.

On donne également des résultats sur l'existence et la régularité de la densité p_X afin de simplifier ce résultat. On obtient en particulier:

Lemme 0.3.1.3. *Soit $\delta > 1$. Soit alors $X \in \mathbb{D}_{\infty^-,1+\delta}$ une variable aléatoire non-dégénérée. Alors X admet une densité continue et bornée; on a plus précisément $X \in \mathcal{S}_{p,\delta}$.*

On a donc la gééralisation suivante du théorème 0.3.1.1:

Théorème 0.3.1.2. *Soit $F \in \mathbb{D}_{\infty^-,1+\delta}$ avec $\delta > 1$ une variable aléatoire non-dégénérée à valeurs dans $\mathbb{R}^N$. Alors pour tout $\delta' < \delta$ and $1 < p < p' < \infty$ on peut trouver $q, r > 1$ et $\theta > 0$ tels qu'on ait l'estimation suivante pour tous $\phi \in \mathcal{S}(\mathbb{R}^N)$:*

$$\|\phi \circ F\|_{\mathbb{D}_{p,-\delta}} \leq C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right]^{2m} \cdot \|F\|_{\mathbb{D}_{r,1+\delta'}}^\theta \cdot \|p_F\|_\infty \cdot \|\phi\|_{\mathcal{S}_{p',-\delta'}}$$
(43)

Ici C est une constante universelle dépendant de $p, \delta, \delta', q, r, \theta$... mais pas de X ou de ϕ. En particulier pour tout $T \in \mathcal{S}_{p',-\delta'}$, $T \circ F$ peut être défini dans $\mathbb{D}_{p,-\delta}$ et on a encore le contrôle ci-dessus.

On établit également une généralisation du théorème 0.3.1.1 pour les espaces $\mathcal{S}_{\infty,s}$:

Théorème 0.3.1.3. *Soit $F \in \mathbb{D}_{\infty^-, 1+\delta}$, pour un certain $\delta > 0$ une variable aléatoire non-dégénérée à valeurs dans $\mathbb{R}^N$. Alors pour tous $\delta' < \delta$ et $1 < p < \infty$ il existe une constante universelle C et des réels $q, r > 1$ et $\theta > 0$ tels qu'on ait le contrôle suivant pour tous les $\phi \in \mathcal{S}(\mathbb{R}^N)$:*

$$\|\phi \circ F\|_{\mathbb{D}_{p,-\delta}} \leq C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right]^{2m} \cdot \|F\|_{\mathbb{D}_{r,1+\delta'}}^{\theta} \cdot \|\phi\|_{\mathcal{S}_{\infty,-\delta'}} \quad (44)$$

En particulier pour tout $T \in \mathcal{S}_{p',-\delta'}$, $T \circ F$ peut être définie dans $\mathbb{D}_{p,-\delta}$ et on a encore le contrôle ci-dessus.

0.3.2   Une formule d'Itô faible

Le théorème 2.3.2.2, que nous reproduisons ci-dessous, donne la structure des chemins à variations bornées à valeurs dans la casse de Schwartz:

Théorème 0.3.2.1. *Soit $T \in BV([0,1], \mathcal{S}')$. Pour tout $p \in [1, \infty]$ il existe $s \in \mathbb{R}$ tel que $T \in BV([0,1], \mathcal{S}_{p,s})$.*

Les théorèmes 0.3.1.2 (ou 0.3.1.3) et 0.3.2.1 résolvent les principales difficultés de la preuve (au chapitre 4) de la formule d'Itô ci-dessous, qui est le centre de cette thèse:

Théorème 0.3.2.2. *Soit W le mouvement brownien canonique de dimension d sur l'espace de Wiener. Soit X le processus d'Itô de en dimension N tel que:*

$$dX_t \;\; = \;\; b_t dt + \sigma_t dW_t \tag{45}$$
$$X_0 \;\; = \;\; x \tag{46}$$

où b est un processus adapté de dimension N et σ est un processus matriciel adapté de dimension $N \otimes d$. Nous notons $a = \sigma^t \sigma$.

Soit aussi $t \in [0,1] \to T(t) \in \mathcal{S}'(\mathbb{R}^N)$ un chemin à variations bornées; il existe alors un $s > 0$ tel que:

$$T \in BV(\mathcal{S}_{p,-s}) \tag{47}$$

On fait également les hypothèses suivantes:

$$\int_0^t \|b_u\|_{\mathbb{D}_{r,s'+3}}^r du \;\; < \;\; \infty \tag{48}$$

$$\int_0^t \|\sigma_u\|_{\mathbb{D}_{r,s'+3}}^r du \;\; < \;\; \infty \tag{49}$$

$$\sup_{\epsilon \leq u \leq t} \left\| \frac{1}{\Sigma(u)} \right\|_{L^r} \;\; < \;\; \infty \tag{50}$$

pour tout $0 < \epsilon < 1$, où $\Sigma(u)$ est la matrice de Malliavin de X_u. On a alors:

$$
\begin{aligned}
T(t) \circ X_t - T(\epsilon) \circ X_\epsilon \; = \; & M_t^{(\epsilon)} + \sum_i \int_\epsilon^t b_i(u) \cdot (\partial_i T_u) \circ X_u du \\
& + \frac{1}{2} \sum_{i,j} \int_\epsilon^t a_{ij}(u) \cdot (\partial_{ij} T_u) \circ X_u du \\
& + \int_\epsilon^t T(du) \circ X_u
\end{aligned}
\tag{51}
$$

où $\left(M_t^{(\epsilon)} \right)_{\epsilon \le t \le 1}$ est une martingale faible sur l'espace de Wiener et au sens des divergences on a:

$$
M_t^{(\epsilon)} = \sum_j \int_\epsilon^t \sum_i \left[b_i(u) \cdot (\partial_i T_u) \circ X_u \right] dW_u^j
\tag{52}
$$

L'idée de la preuve est que, grâce au thórème 0.3.2.1, l'on peut trouver un entier l indépendant de t tels que les $\mathcal{K}^{-l} T_t$ soient des fonction C^2. Alors on écrit $T_t = \mathcal{K}^l \mathcal{K}^{-l} T_t$, on applique la formule d'Itô classique aux fonctions $\mathcal{K}^{-l} T_t$ et on applique le théorème 0.3.1.2 (ou 0.3.1.3) pour relever la formule d'Itô classique en notre extension.

Comme annoncé notre formule est beaucoup plus facile à appliquer que les résultats existants; en particuler on n'a plus besoin de supposer $X_t \in \mathbb{D}$.

0.3.3 Le cas d'une EDS

Nous avons souhaité spécialiser notre formule d'Itô au cas des semimartingales qui sont solution d'une équation différentielle stochastique qui est sans doute le plus utile pour les applications. C'est l'objet du théorème 4.4.3.1. Pour l'obtenir, nous avons d'une part prouvé un résultat de dérivabilité - essentiellement, si les coefficients de l'EDS appartiennent à l'espace de Hölder Λ^s alors sa solution a s dérivées fractionnaires au sens de Malliavin - et d'autre part nous avons emprunté un résultat de [15] pour la non-dégénérescence. Nous reproduisons ces résulats ci-dessous.

On considère l'équation différentielle stochastique:

$$
dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t
\tag{53}
$$

On introduit l'espace $C^{0,s}$ des fonctions continues en temps, admettant $[s]$ dérivées en espace, dont la dérivée d'ordre $[s]$ en espace est $\{s\}$-holderienne. On a alors le:

Théorème 0.3.3.1. *On considère l'EDS ci-dessus et on suppose que b et σ sont dans $C^{0,s}$ jusqu'à un certain temps T, pour un certain $s \geq 1$. Alors pour tous $p > 1$, $s' < s$ et pour tout $t \geq 0$, on a $X_t \in \mathbb{D}_{p,s'}$ et il existe des constantes A et B ne dépendant que de p, s, s' et des normes $C^{0,s}$ de b et σ telles que:*

$$\|X_t\|_{\mathbb{D}_{p,s'}} \leq A \exp(B(\|b\|_{C^{0,s}} + \|\sigma\|_{C^{0,s}}) \cdot t) \tag{54}$$

Supposons maintenant que $\sigma \in C^{0,s}$ pour un certain $s \geq 2$ et que $b \in C^{0,1}$. Alors notre EDS a une unique solution. On note $[s] = k$. Pour $i \leq j \leq d$, on introduit le champ de vecteurs: $V_j = \sigma_j \partial_{x_j}$. Soit L_0 l'ensemble de tous les V_j. On note $[\cdot, \cdot]$ le commutateur de deux champs de vecteurs. Alors, on définit par récurrence L_i comme l'algèbre de Lie engendrée par les $[V_j, Z]$, où $Z \in L_{i-1}$. On a le:

Théorème 0.3.3.2. *Supposons que pour un certain entier $n \leq k-2$ il existe $c > 0$ tel que pour tout $\xi \in \mathbb{S}^{N-1}$ on ait:*

$$\sum_{i=0}^{n} \sum_{Z \in L_k} \langle \xi, Z \rangle (0, x) > c \tag{55}$$

Supposons aussi que $\sigma \in C^{(\beta, n+2)}$ pour un certain $\beta > 0$ (c'est à dire que l'on suppose une régularité holderienne en temps). Alors, si Σ_t est la matrice de Malliavin de X_t, pour tout $t > 0$ on a des contrôles du type suivant:

$$\left\| \frac{1}{\Sigma_t} \right\|_{L^p} \leq C \cdot \frac{1}{t^\nu} \cdot e^{Kt} \tag{56}$$

Ici C, ν et K sont des constantes qui ne dépendent que de t. En particulier:

$$\int_{\epsilon}^{T} \left\| \frac{1}{\Sigma_t} \right\|_{L^p}^{p} dt < \infty \tag{57}$$

En combinant ces deux théorèmes, on obtient:

Théorème 0.3.3.3. *Supposons que les coefficients de notre EDS $C^{(0,s)}$ for some $s > 2$ and let $k = [s]$. Supposons que la condition de Hormander du théorème précédent soit vérifiée pour un certain $n \leq k - 2$ et que $\sigma \in C^{(\beta, n+2)}$. Alors si T est un chemin à variations bornées dans $\mathcal{S}_{p,-(s-1)}$ on a la formule d'Itô ci-dessous:*

$$
\begin{aligned}
T_t \circ X_t - T_\epsilon \circ X_\epsilon \;=\;\; & M_t^\epsilon + \sum_i \int_\epsilon^t b_i(u, X_u) \cdot (\partial_i T_u) \circ X_u du \\
& + \frac{1}{2} \sum_{i,j} \int_\epsilon^t a_{ij}(u, X_u) \cdot (\partial_{ij} T_u) \circ X_u du \\
& + \int_\epsilon^t T(du) \circ X_u
\end{aligned}
\tag{58}
$$

et au sens des divergences:

$$M_t^{(\epsilon)} = \sum_j \int_\epsilon^t \sum_i [b_i(u, X_u) \cdot (\partial_i T_u) \circ X_u] \, dW_u^j \tag{59}$$

Pour les applications au chapitre 5, nous utiliserons l'extension suivante:

Théorème 0.3.3.4. *Soient X et T comme dans le théorème précédent. Considérons deux fonctions $b_Y, \sigma_Y \in C^{(0,s)}$ et soit Y l'unique solution forte de l'EDS:*

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dW_t \tag{60}$$

$$Y_0 = y \tag{61}$$

Soit enfin $f \in BV(C^s)$. Alors on peut écrire une formule d'Itô pour:

$$f(t, Y_t) \cdot T_t \circ X_t \tag{62}$$

0.3.4 Autres extensions

Nous avons également étendu nos résultats dans diverses directions. Tout d'abord, nous avons déjà signalé que noous donnons au chapitre 2 des résultats sur les espaces $\mathcal{S}_{\infty,s}$ qui nous permettent d'obtenir, au chapitre 3, une version du théorème 3.2.0.13 faisant intervenir ces espaces au lieu des $\mathcal{S}_{p,s}$: c'est le théorème 3.2.0.15. Ce résultat a un intérêt double. D'abord, comme nous l'avons déjà dit, il permet de ne pas faire d'hypothèse a priori sur l'existence d'une densité pour la variable aléatoire par laquelle on relève. Ensuite, il nous a permis d'étendre le théorème 3.2.0.15 au cas où la distribution que l'on relève est elle-même aléatoire. Nous donnons ces résultats dans le paragraphe 3.5; pour les établir, nous avons étendu la formule d'intégration par parties 0.2.2.1 au cas où la fonction est aléatoire. Dans ce cas la formule d'intégration par parties devient:

Théorème 0.3.4.1. *Soient $p, q, r > 1$ tels que $1/p + 1/q = 1/r$. Supposons que l'on ait:*

- $f \in L^p(C_b^1)$

- $f \in \mathbb{D}_{p,1}(C_b^0)$

- $X \in \mathbb{D}_{q,1}(\mathbb{R}^N)$

Alors, on a, $f(X) \in \mathbb{D}_{r,1}$ et la règle des chaînes suivante:

$$\nabla(f(X)) = \sum_{k=1}^N \partial_k f(X) \nabla X_k + (\nabla f)(X) \tag{63}$$

De plus, pour tout $\phi \in \mathbb{D}_{p^*,1}$ on a:

$$E\left[\partial_k f(X) \cdot \phi\right] = E\left[f(X) \cdot l_k(X, \phi)\right] - E\left[\phi \cdot T_k(f, X)(X)\right] \qquad (64)$$

En suivant la même démarche que dans le début du chapitre 3, on en déduit en particulier le:

Théorème 0.3.4.2. *Pour un $\delta > 0$, soit $X \in \mathbb{D}_{\infty^-,1+\delta}$ une variable aléatoire non-dégénérée à valeurs dans $\mathbb{R}^N$. Alors, pour tous $\delta' < \delta$ et $1 < p < \infty$ il existe une constante universelle C et des réels $p_1, p_2, p_3 > p$ et $\theta > 0$ tels que l'on ait le contrôle suivant pour tout $\phi \in \mathcal{S}(\mathbb{D})$:*

$$\begin{aligned}
&\|\phi \circ X\|_{\mathbb{D}_{p,-\delta}} \\
\leq \quad & C \cdot \left[\left\|\frac{1}{(det\Sigma)^{2(k+1)}}\right\|_{L^{p_1}}^{p_1} + \left\|\frac{1}{det\Sigma}\right\|_{L^{p_1}}^{p_1}\right]^\theta \cdot \|X\|_{\mathbb{D}_{p_2,1+\delta'}}^\theta \\
\times \quad & \sum_{k=0}^{[\delta]} \left[\|\phi\|_{\mathbb{D}_{p_3,k}(\mathcal{S}_{\infty,-(\delta'-k)})} + \|\phi\|_{\mathbb{D}_{p_3,k}(W_{\alpha,-(\delta'-k-\frac{N}{\alpha})})}\right]
\end{aligned}$$

$$(65)$$

En particulier $T \in \mathbb{D}_{r,k}(\mathcal{S}_{p',-\delta'})$, $T \circ X$ peut être définie dans $\mathbb{D}_{p,-\delta}$ et dans ce cas on a encore le contrôle ci-dessus.

Cela nous a permis de prouver une version faible de la formule d'Itô-Wentzell. En effet, considérons deux processus:

$$dX_t = b_t dt + \sigma_t dW_t \qquad (66)$$

Ici X vérifie les mêmes hypothèses que dans le théorème 0.3.2.2. On s'intéresse aux $T_t \circ X_t$ où T est un flux à valeurs dans $\mathcal{S}'$. Plus précisément, on introduit des processus:

$$D_t, V_t \in \bigcap_{k<[s]} \left(\mathbb{D}_{\infty^-,k}\left(\mathcal{S}_{\infty,-(s-k)}\right) \cap \mathbb{D}_{\infty^-,k}\left(W_{\alpha,-(s-k-\frac{N}{\alpha})}\right)\right) \qquad (67)$$

Ici, on suppse que α est tel que $\{s\} > \frac{N}{\alpha}$; si s est entier on suppose seulement:

$$D_t, V_t \in \bigcap_{k<[s]} \mathbb{D}_{\infty^-,k}\left(\mathcal{S}_{\infty,-(s-k)}\right) \qquad (68)$$

On suppose que ces processus sont adaptés dans la filtration de W. On peut alors définir une semimartingale à valeurs dans $\mathcal{S}_{\infty,-s}$ par une valeur initiale $T_0 \in \mathcal{S}_{\infty,-s}$ et l'équation:

$$dT_t = D_t dt + V_t dW_t \qquad (69)$$

C'est à dire que our tout $\phi \in \mathcal{S}$:

$$\langle T_t, \phi \rangle = \langle T_0, \phi \rangle + \int_0^t \langle D_s, \phi \rangle \, ds + \int_0^t \langle V_s, \phi \rangle \, dW_s \tag{70}$$

qui est une semimartingale réelle. On prouve alors:

Théorème 0.3.4.3. *Sous les hypothèses ci-dessus, pour tous $p' > p$ et $s' > s$, pour tout $\epsilon > 0$, on a la formule suivante au sens de l'espace: $\mathbb{D}_{p' \vee 2, -(s'+2)}$:*

$$
\begin{aligned}
T(t) \circ X_t - T(\epsilon) \circ X_\epsilon \;=\; & M_t^{(\epsilon)} + \int_\epsilon^t b(u) \cdot (\partial_x T_u) \circ X_u \, du \\
& + \frac{1}{2} \int_\epsilon^t \sigma(u)^2 \cdot (\partial_{xx} T_u) \circ X_u \, du \\
& + \int_\epsilon^t D_u \circ X_u \, du \\
& + \int_\epsilon^t \sigma(u) \cdot \partial_x V_u \circ X_u \, du
\end{aligned}
\tag{71}
$$

où $\left(M_t^{(\epsilon)} \right)_{\epsilon \le t \le 1}$ est une martingale faible sur l'espace de Wiener, et au sens des divergences on a:

$$M_t^{(\epsilon)} = \int_\epsilon^t \left[b(u) \cdot (\partial_x T_u) \circ X_u + V_u \circ X_u \right] dW_u^j \tag{72}$$

Nous prouvons également une version anticipative de notre formule d'Itô faible (on utilise les notations classiques du calcul stochastique anticipatif; ces notations sont précisées dans le chapitre 4). On considère un processus anticipatif:

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s \tag{73}$$

Ici X_0 est une variable aléatoire et b et σ ne sont pas nécessairement adaptés. On a alors le:

Théorème 0.3.4.4. *Soit $T \in \mathcal{S}_{q, -\delta}$. On suppose que pour tout p:*

$$\int\int_{s \le t} E\left[\left| (Id + \mathcal{L})^{\frac{\delta+3}{2}} D_s \sigma_t \right|^p \right] ds \, dt \;<\; \infty \tag{74}$$

$$\int\int_{s \le t} E\left[\left| (Id + \mathcal{L})^{\frac{\delta+2}{2}} D_s b_t \right|^p \right] ds \, dt \;<\; \infty \tag{75}$$

$$\int_0^1 \left\| \frac{1}{\Sigma_t} \right\|^p dt \;<\; \infty \tag{76}$$

où Σ est la matrice de Malliavin de X_t. Alors, au sens de l'espace $\mathbb{D}_{q,-\delta}$, on a:

$$
\begin{aligned}
T \circ X_t \;=\;& T \circ X_0 \\
&+ \int_0^t \left(b_s \cdot T' \circ X_s + \frac{1}{2}\sigma_s^2 \cdot T'' \circ X_s \right) ds \\
&+ \int_0^t \sigma_s \cdot T' \circ X_s dW_s \\
&+ \int_0^t (D^- X)_s \sigma_s \cdot T'' \circ X_s ds
\end{aligned}
\tag{77}
$$

0.3.5 Application au temps local

Nous comparons notre formule d'Itô à d'autres extensions déjà connues, notamment celles que nous avons évoquées plus haut. Ce faisant, nous montrons en particulier comment le temps local de certains processus peut se désintégrer en une intégrale de Pettis sur $\mathbb{D}'$. En effet, en dimension 1, considérons la solution de l'EDS:

$$
dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t
\tag{78}
$$

On écrit alors, sous des hypothèses (sur X) qui sont précisées au chapitre 4:

$$
\begin{aligned}
&|X_t - K| - |X_\epsilon - K| \\
&= \int_\epsilon^t sgn(X_s - K)b(s, X_s)ds + \int_\epsilon^t sgn(X_s - K)\sigma(s, X_s)dW_s \\
&+ \frac{1}{2}\int_\epsilon^t \sigma(s, X_s)^2 \delta_K \circ X_s ds
\end{aligned}
\tag{79}
$$

On peut alors décomposer le temps local ainsi:

$$
L_t^K(X) = \int_0^t \sigma(s, X_s)^2 \delta_K \circ X_s ds
\tag{80}
$$

On peut généraliser cette idée à des dimensions supérieures en remplaçant la fonction valeur absolue par la solution fondamentale de l'équation de la chaleur en dimension quelconque. Cela permet de définir (au sens faible) une notion de temps local multi-dimensionnel.

Ces idées permettent également d'étudier la régularité du temps local, en espace et au sens de Malliavin.

0.4 Application à un problème variationnel

Le cinquième et dernier chapitre est consacré à l'application principale que nous donnons à la théorie développée dans cette thèse, à savoir l'étude d'un problème

variationnel. On considère l'équation différentielle stochastique:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{81}$$

On note A_t son générateur et on se donne deux fonctions continues et bornées f et g et une fonction continue et positive r et on considère le problème:

$$
\begin{aligned}
K_t u - ru &\leq -g \tag{82}\\
u &\geq f \tag{83}\\
(K_t u - ru)(f - u) &= 0 \tag{84}\\
u(T, \cdot) &= f \tag{85}
\end{aligned}
$$

Ce problème a été étudié par de nombreux auteurs, voir en particulier [8], [9], [38], [33]... On montre dans ces ouvrages que sous certaines hypothèses (régularité des coefficients et ellipticité du générateur), l'unique solution à ce problème est donnée par la fonction:

$$
\begin{aligned}
u(t, x) = \sup_{\tau \in Z_{t,T}} \quad E \quad & \left[f(X_\tau^t(x)) \exp\left(-\int_t^\tau r(u, X_u^t(x))du \right) \right.\\
&+ \left. \int_t^\tau g(s, X_s^t(x)) \exp\left(-\int_t^s r(u, X_u^t(x))du \right) ds \right] \tag{86}
\end{aligned}
$$

Notre contribution est de montrer que l'on peut affaiblir ces hypothèses, c'est à dire les réduire à celles qui permettent d'appliquer la formule d'Itô du théorème 4.4.3.1. En pratique il suffit de supposer que les coefficients de l'EDS ainsi que f, g et r sont trois fois dérivables en espace, que les coefficients de l'EDS sont holderiens en temps et que X vérifie une condition de type Hormander. Cette dernière hypothèse est plus faible que les hypothèses d'ellipticité que l'on fait habituellement. L'idée principale de la preuve est que l'on peut appliquer notre formule d'Itô faible à la fonction u, qui, en général, n'est dérivable ni en temps ni en espace.

Chapter 1

Introduction to Malliavin calculus

Malliavin calculus will be our main tool throughout the whole of this document. For the sake of completeness and in order to introduce some notation, we provide a brief introduction to the topic. Most of the results in this chapter are classic, and we followed [45], [53], [25], [66], [55], [56], [75] and [44].

1.1 Malliavin analysis in finite dimension

We start with results from finite dimensional gaussian analysis. These will lift and extend to the infinite-dimensionnal case of the Wiener space. For this section we followed [44] and [55].

1.1.1 Gaussian random variables

Throughout this document, we will make an intensive use of the normal gaussian law, which we note μ. It is characterized by its density with regard to the Lebesgue measure on $\mathbb{R}$:

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot dx \tag{1.1}$$

or, equivalently, by its Fourier transform:

$$\int_{\mathbb{R}} e^{i\xi x} d\mu(x) = e^{-\frac{\xi^2}{2}} \tag{1.2}$$

The following is straightforward but noteworthy:

Proposition 1.1.1.1. *Let $p \in [1, \infty[$ and $Q \in \mathbb{R}[X]$; then $Q \in L^p(\mathbb{R}, \mu)$.*

There also is:

Theorem 1.1.1.1. *For any $p \in]1, \infty[$, polynomials are dense in $L^p(\mu)$.*

Proof. Let $p \in]1, \infty[$. We have already observed that any polynomial is in $L^p(\mu)$. Now let $f \in L^{p^*}$, so by Hölder's theorem all the $x \mapsto x^n f(x)$ are μ-integrable and suppose that for all $n \in \mathbb{N}$:

$$\int x^n f(x) d\mu(x) = 0 \tag{1.3}$$

Then, by a version of Fubini's theorem:

$$\int f(f) e^{i\xi x} d\mu(x) = \sum_{n=0}^{\infty} \frac{(i\xi)^n}{n!} \int x^n f(x) d\mu(x) = 0 \tag{1.4}$$

so the Fourier transform of the measure $f \cdot d\mu$ is 0, hence by injectivity of the Fourier transform $f = 0$ μ-a.s. By duality this proves that polynomials are dense in $(L^{p^*})' = L^p$. $\qquad\square$

1.1.2 Gradient and divergence operators on $L^p(\mathbb{R}, \mu)$

In the sequel p is any real number in $]1, \infty[$.

A weak (ie: distributional) derivative operator is classically defined on $L^p(\mathbb{R}, \mu)$ as it is on $L^p(\mathbb{R}, \lambda)$; see, for example, [1], [70] or [71]. We will indifferently note f', df or ∇f for the (weak or strong) derivative of f, whenever it exists. We introduce $\mathbb{D}_{p,k}(\mathbb{R})$ the space of those functions in L^p which admit k weak derivatives in L^p; in operator language theory:

$$\mathbb{D}_{p,k}(\mathbb{R}) = dom_{L^p}\left(\nabla^k\right) \tag{1.5}$$

We equip $\mathbb{D}_{p,k}(\mathbb{R})$ with the graph norm:

$$\|f\|_{\mathbb{D}_{p,k}(\mathbb{R})} = \|f\|_{L^p} + \|\nabla^k f\|_{L^p} \tag{1.6}$$

The (gaussian) divergence operator is defined as the transpose of the derivative in the sense of the duality induced by the measure μ. To get a flavour of this, consider $f, g \in \mathcal{S}$. Then, integrating by parts, one notices that:

$$
\begin{aligned}
\int_{\mathbb{R}} f'(x) g(x) d\mu(x) &= \int_{\mathbb{R}} f'(x) g(x) \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \\
&= -\int_{\mathbb{R}} f(x) \frac{d}{dx}\left(g(x) \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\right) dx \\
&= -\int_{\mathbb{R}} f(x) e^{\frac{x^2}{2}} \frac{d}{dx}\left(g(x) e^{\frac{-x^2}{2}}\right) d\mu(x) \\
&= \int_{\mathbb{R}} f(x) \left(-g'(x) + x g(x)\right) d\mu(x) \tag{1.7}
\end{aligned}
$$

This leads us to the following definition:

Definition 1.1.2.1 (Gaussian divergence operator). *It is formally defined on its domain $dom_{L^p}(\delta)$ through the formula:*

$$\delta f(x) = -f'(x) + xf(x) = -e^{\frac{x^2}{2}}\frac{d}{dx}\left(f(x)e^{\frac{-x^2}{2}}\right) \tag{1.8}$$

Remark 1.1.2.1. *The domain of the divergence is dense in L^p. Divergences coincide when defined in the sense of two or more p so we omit the index.*

1.1.3   Hermite polynomials

The purpose of this paragraph is to introduce the:

Definition 1.1.3.1 (Hermite polynomials).

$$H_n(x) = (\delta^n 1)(x) = (-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}\left(e^{\frac{-x^2}{2}}\right) \tag{1.9}$$

By a straightforward recurrence one checks that these indeed are polynomial functions and that H_n has degree n. It is easy to compute the first few Hermite polynomials:

$$
\begin{aligned}
H_0 &= 1 \\
H_1 &= X \\
H_2 &= X^2 - 1 \\
H_3 &= X^3 - 3X \\
H_4 &= X^4 - 6X^2 + 3 \\
&\cdots
\end{aligned}
$$

We will also need the following extension of the Hermite polynomials:

$$h_{n,a}(x) := \sqrt{a}H_n\left(\frac{x}{\sqrt{a}}\right) \tag{1.10}$$

The Hermite polynomials arise from a generating series:

$$(-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}\left(e^{\frac{-x^2}{2}}\right) = e^{\frac{x^2}{2}}\frac{d^n}{dt^n}_{|t=0}\left(e^{\frac{-(x-t)^2}{2}}\right) \tag{1.11}$$

and therefore we indeed have:

$$\exp\left(xt - \frac{t^2}{2}\right) = \exp\left(\frac{x^2}{2} - \frac{1}{2}(x-t)^2\right) = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x) \tag{1.12}$$

as well as:

$$\exp\left(xt - a\frac{t^2}{2}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_{n,a}(x) \tag{1.13}$$

The following result will be crucial to our purpose:

Theorem 1.1.3.1. $\left(\frac{H_n}{\sqrt{n!}}\right)$ *is an orthonormal basis of* $L^2(\mu)$.

Proof. We have seen that polynomials are dense, in particular, in $L^2(\mu)$. Also, we know that H_n has degree n, therefore the Hermite polynomials are a base of $\mathbb{R}[X]$ (resp. $\mathbb{C}[X]$). Therefore we only need to prove that the family we consider is orthonormal.

First let $n > m$ be two integers. Then:

$$E_\mu[H_n H_m] = E_\mu[(\delta^n 1) H_m] = E_\mu[H_m^{(n)}] = 0 \tag{1.14}$$

so we have proven the orthogonality. For the normality, considering the generating series for the Hermite polynomials and taking the orthogonality into account, one sees that:

$$\int_{\mathbb{R}} \left(\exp\left(xt - \frac{t^2}{2}\right)\right)^2 d\mu(x) = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!^2} \int_{\mathbb{R}} H_n(x)^2 d\mu(x) \tag{1.15}$$

It is then classic to compute the integral of the LHS which is e^{t^2}, therefore identifying coefficients in the Taylor series expansion leads to $E_\mu[H_n] = n!$ as expected. $\quad\square$

We provide two other characterizations of the Hermite polynomials:

Proposition 1.1.3.1 (Recursive construction of the Hermite polynomials)**.** *The Hermite polynomials verify* $H_0 = 0$, $H_1 = X$ *and:*

$$H_{n+1} = XH_n - nH_{n-1} \tag{1.16}$$

Proposition 1.1.3.2 (ODE verified by the Hermite polynomials)**.** *The Hermite polynomials verify:*

$$H_n'' - XH_n' + nH_n = 0 \tag{1.17}$$

1.1.4   The Ornstein Uhlenbeck operator and the associated semigroup

It is straightforward calculus to derive the following equation, which is familiar to quantum physicists:

$$d\delta f - \delta df = f \tag{1.18}$$

We want to apply this formula to any one of the H_n. First, consider H'_{n+1}. This has to be a polynomial of degree n. Let us compute it explicitly by projecting it on the H_m, $m \leq n$.

$$E_\mu[H'_{n+1}H_n] = E_\mu[H_{n+1}\delta H_n] = E_\mu[H^2_{n+1}] = (n+1)! \tag{1.19}$$

and similarly, for $m < n$:

$$E_\mu[H'_{n+1}H_m] = 0 \tag{1.20}$$

Therefore by applying formula (1.18) one obtains:

$$\delta dH_n = nH_n \tag{1.21}$$

This leads us to introducing the:

Definition 1.1.4.1 (Ornstein Uhlenbeck operator)**.**

$$\mathcal{L}f = \delta df = d^*df \tag{1.22}$$

Of course it is straightforward to compute:

$$\mathcal{L}f(x) = -f''(x) + xf'(x) \tag{1.23}$$

This operator is symmetrical and positive by construction; it is diagonalized by the orthonormal basis of the scaled Hermite polynomials which is relevant to our study. $-\mathcal{L}$ is the generator of a contraction semigroup (see appendix C) on $L^p(\mu)$: the Ornstein-Uhlenbeck semigroup, which we note P_t. By definition of a contraction semigroup P_t enjoys the two following properties:

$$P_tP_sf = P_{t+s}f \tag{1.24}$$

and

$$\|P_tf\|_{L^p} \leq \|f\|_{L^p} \tag{1.25}$$

Also, semigroup theory tells us that P_tf is the unique solution of the Cauchy problem:

$$\frac{\partial}{\partial t}g(t,x) = \mathcal{L}g(x) \tag{1.26}$$
$$g(0,\cdot) = f \tag{1.27}$$

From the unicity of the solution to such a Cauchy problem one sees that for $f \in L^{p_1} \cap L^{p_2}$, the two potential definitions of P_tf coincide and we will no longer make a distinction. Again because of the unicity and of the formula $\mathcal{L}H_n = H_n$, one sees that:

$$P_tH_n = e^{-nt}H_n \tag{1.28}$$

We shall now prove the following:

Theorem 1.1.4.1 (Mehler's formula). *The Ornstein Uhlenbeck semigroup is defined by a density:*

$$P_t f(x) = \int f(y) p_t(x, y) d\mu(y) \tag{1.29}$$

with, more precisely:

$$p_t(x, y) = \sum_{n=0}^{\infty} \frac{e^{-nt}}{n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1 - e^{-2t}}} \exp\left(-\frac{e^{-2t}x^2 - 2xye^{-t} + e^{-2t}y^2}{1 - e^{-2t}}\right) \tag{1.30}$$

This is equivalent to:

$$P_t f(x) = \int f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) d\mu(y) = E\left[f(e^{-t}x + \sqrt{1 - e^{-2t}}N)\right] \tag{1.31}$$

where N is a normal gaussian random variable.

Proof. We prove the result in the case where $p = 2$ and extend it by density to the other cases. Indeed if $f \in L^2$, one has an expansion of the type:

$$f = \sum c_n(f) \frac{H_n}{\sqrt{n!}} \tag{1.32}$$

where

$$c_n(f) = \int_{\mathbb{R}} f(x) \frac{H_n(x)}{\sqrt{n!}} dx \tag{1.33}$$

and therefore:

$$P_t f = \sum e^{-nt} c_n(f) \frac{H_n}{\sqrt{n!}} \tag{1.34}$$

So if we introduce the function:

$$p_t(x, y) = \sum_{n=0}^{\infty} \frac{e^{-nt}}{n!} H_n(x) H_n(y) \tag{1.35}$$

which is easily verified to be well defined in $L^2(\mu^{\otimes 2})$, we see that:

$$\begin{aligned}
\int_{\mathbb{R}} f(y) p_t(x, y) dy &= \sum e^{-nt} \frac{H_n(x)}{\sqrt{n!}} \int_{\mathbb{R}} f(y) \frac{H_n(y)}{\sqrt{n!}} dy \\
&= \sum e^{-nt} c_n(f) \frac{H_n}{\sqrt{n!}} \\
&= P_t f(x)
\end{aligned} \tag{1.36}$$

The explicit expression for p_t is then computed through inverse Fourier transform. Indeed, recall that:

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right) \tag{1.37}$$

so since:

$$e^{-\frac{x^2}{2}} = \int e^{isx - \frac{x^2}{2}} \frac{ds}{\sqrt{2\pi}} \tag{1.38}$$

there comes:

$$H_n(x) = (-i)^n e^{\frac{x^2}{2}} \int s^n e^{isx - \frac{x^2}{2}} \frac{ds}{\sqrt{2\pi}} \tag{1.39}$$

Therefore:

$$
\begin{aligned}
p_t(x,y) &= \sum_{n=0}^{\infty} \frac{e^{-nt}}{n!} H_n(x) H_n(y) \\
&= e^{\frac{x^2+y^2}{2}} \int\int \sum_{n=0}^{\infty} \frac{(-\sigma\tau t)^n}{n!} e^{i\sigma x - \frac{\sigma^2}{2}} e^{i\tau y - \frac{\tau^2}{2}} \frac{d\sigma}{\sqrt{2\pi}} \frac{d\tau}{\sqrt{2\pi}} \\
&= e^{\frac{x^2+y^2}{2}} \int\int \exp\left\{ -\sigma\tau e^{-t} + i\sigma x - \frac{\sigma^2}{2} + i\tau y - \frac{\tau^2}{2} \right\} \frac{d\sigma}{\sqrt{2\pi}} \frac{d\tau}{\sqrt{2\pi}}
\end{aligned}
\tag{1.40}
$$

Then we do the following change of variable, which separates the double integral:

$$(\sigma, \tau) \mapsto \left(\frac{\sigma+\tau}{\sqrt{2}}, \frac{\sigma-\tau}{\sqrt{2}} \right) \tag{1.41}$$

and we obtain:

$$
\begin{aligned}
&p_t(x,y) \\
&= e^{\frac{x^2+y^2}{2}} \int\int \exp\left\{ -\frac{\sigma^2-\tau^2}{2} e^{-t} + i\frac{\sigma+\tau}{\sqrt{2}} - \frac{(\sigma+\tau)^2}{4} + i\frac{\sigma-\tau}{\sqrt{2}} - \frac{(\sigma-\tau)^2}{4} \right\} \frac{d\sigma}{\sqrt{2\pi}} \frac{d\tau}{\sqrt{2\pi}} \\
&= e^{\frac{x^2+y^2}{2}} \int \exp\left\{ -\frac{1}{2}(1+e^{-t})\sigma^2 + i\frac{x+y}{\sqrt{2}}\sigma \right\} \frac{d\sigma}{\sqrt{2\pi}} \cdot \int \exp\left\{ -\frac{1}{2}(1-e^{-t})\tau^2 + i\frac{x-y}{\sqrt{2}}\tau \right\} \frac{d\tau}{\sqrt{2\pi}} \\
&= e^{\frac{x^2+y^2}{2}} \cdot \frac{e^{-\frac{(x+y)^2}{2(1+e^{-t})}}}{\sqrt{1+e^t}} \cdot \frac{e^{-\frac{(x-y)^2}{2(1-e^{-t})}}}{\sqrt{1-e^{-t}}} \\
&= \frac{1}{\sqrt{1-e^{-2t}}} \cdot \exp\left(-\frac{1-e^{-t}}{1+e^{-t}} \cdot \frac{(x+y)^2}{4} - -\frac{1+e^{-t}}{1-e^{-t}} \cdot \frac{(x-y)^2}{4} \right)
\end{aligned}
\tag{1.42}
$$

so we obtain our expression of p_t. Finally, the last identity arises from a simple changle of variables. $\qquad\square$

Remark 1.1.4.1. *The computation of these densities may also be done through the SDE associated to the Ornstein-Uhlenbeck process:*

$$dX_t = -X_t dt + \sqrt{2} dW_t \tag{1.43}$$

which solves as:

$$X_t = X_0 + \sqrt{2} \int_0^t e^{-(t-s)} dW_s \tag{1.44}$$

and one may check that X_t has density p_t if $X_0 = 0$.

Finally the following result is simple but important:

Theorem 1.1.4.2. *The P_t and $\mathcal{L}$ are symmetrical operators.*

1.1.5   Fractional Sobolev spaces

Fractional powers of $Id - \mathcal{L}$ are defined as is explained in appendix C; then one gets the:

Theorem 1.1.5.1. *Let $p \in]1, \infty[$ and $k \in \mathbb{N}$; the two following norms are equivalent on $\mathbb{D}_{p,k}$:*

$$\|f\|_{\mathbb{D}_{p,k}(\mathbb{R})} = \|f\|_{L^p} + \|\nabla^k f\|_{L^p} \tag{1.45}$$

and:

$$\|f\| = \|(Id - \mathcal{L})^{k/2} f\|_{L^p} \tag{1.46}$$

This allows to define fractional Sobolev spaces built on the gaussian measure in a natural way; we omit the details as we may recover them as a special case of the infinite-dimensionnal results which are more directly relevant to our study - see the next section.

1.1.6   Extension to d-dimensional spaces

If $n = (n_1, \cdots, n_d) \in \mathbb{N}^N$, we note:

$$n! = \prod_{i=1}^{N} n_i! \tag{1.47}$$

We also define N-dimensionnal Hermite polynomials as:

$$H_n(x) = \prod_{i=1}^{N} H_{n_i}(x_i) \tag{1.48}$$

where we abusively used the same notation for 1-dimensionnal and N-dimensionnal Hermite polynomials. We note $\mu^{\otimes N}$ for the N-dimensionnal normal gaussian measure on $\mathbb{R}^N$. Then one proves:

Theorem 1.1.6.1. $\left(\frac{H_n}{\sqrt{n!}}\right)$ *is an orthonormal basis of $L^2(\mu^{\otimes N})$.*

Proof. Since every monomial X_i^k may be written as a linear combination of the $H_n(X_i)$, $n \in \mathbb{N}$, it is clear that every monomial X^k on $\mathbb{R}^N$ may be written as a linear combination of elements in our family. Therefore we will be done as soon

that we prove that our family is orthogonal and this last fact is straightforward as soon as one notices that:

$$\int_{\mathbb{R}^N} H_n(x) H_m(x) d\mu^{\otimes N}(x) = \prod_{i=1}^{N} \int_{\mathbb{R}} H_{m_i}(x_i) H_{n_i}(x_i) d\mu(x_i) \qquad (1.49)$$

$$\square$$

This fact all the results we have just seen to the N-dimensionnal case; as the proofs are straightforard we omit them.

1.2 Malliavin calculus

In thi section we will construct Malliavin calculus as an infinite dimensionnal extension of the results in the previous section.

1.2.1 The Wiener and Cameron-Martin spaces

Definition 1.2.1.1 (Wiener space). *The Wiener space is the following vector space of continuous functions:*

$$\mathcal{W} = \{f \in C\left([0,1], \mathbb{R}\right) s.t. f(0) = 0\} \qquad (1.50)$$

equipped with its natural Banach space structure arising from the following norm:

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)| \qquad (1.51)$$

We recall the:

Theorem 1.2.1.1 (Radon). *The strong dual space $\mathcal{W}'$ of $\mathcal{W}$ is $\mathcal{M}/(\mathbb{R} \cdot \delta_0)$, where $\mathcal{M}$ is the set of all Radon measures on $[0,1]$ and δ_0 is the Dirac mass at 0. It is a Banach space when equipped with the dual norm of $\|\cdot\|_\infty$, which is:*

$$\|\nu\|_{\mathcal{W}^*} = \sup_{\|f\|_\infty = 1} \int f d\nu \qquad (1.52)$$

We will consider the following evaluation functionals:

Definition 1.2.1.2 (Canonical functionals on the Wiener space).

$$W_t = w \in \mathcal{W} \mapsto w(t) \in \mathbb{R} \qquad (1.53)$$

Clearly, these are continuous functionals on the Wiener space: they actually coincide with the Dirac masses. The cornerstone of Malliavin calculus is the following:

Theorem 1.2.1.2. *There exists a unique measure μ on the Wiener space such that the stochastic process $(W_t)_{0 \leq t \leq 1}$ defined on the probability space $(\mathcal{W}, \mu)$ is a Brownian motion. We call that measure the Wiener measure. The coordinate process X considered under the Wiener measure is the canonical Brownian motion.*

Proof. We do not recall how to construct a Brownian motion; instead, we simply show how, being given a Brownian motion Y on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we may map it to the evaluation functionals and construct the Wiener measure. Indeed let us define a measurable application:

$$\Phi = \omega \in \Omega \mapsto \Phi(\omega) \in \mathcal{W} \tag{1.54}$$

by putting:
$$\Phi(\omega) = t \in [0, 1] \mapsto Y_t(\omega) = W_t(Y(\omega)) \in \mathbb{R} \tag{1.55}$$

Clearly the $\Phi(\omega)$ are a.s. in $\mathcal{W}$ as the Brownian motion a.s. has continuous trajectories. Then Φ is measurable because on one hand Y is and on the other hand the evaluation functionals on the Wiener space are continuous, hence borelian for the natural topology. We may therefore define μ as the image measure of $\mathbb{P}$ by Φ and we then have, for any measurable application f on $\mathcal{W}$:

$$E_\mu \left[f\left(W_t, 0 \leq t \leq 1 \right) \right] = E_\mathbb{P} \left[f\left((W \circ Y)_t, 0 \leq t \leq 1 \right) \right] = E_\mathbb{P} \left[f\left(Y_t, 0 \leq t \leq 1 \right) \right] \tag{1.56}$$

Since Y is a Brownian motion under $\mathbb{P}$ this proves that W is a Brownian motion under μ. $\qquad\square$

It is interesting to note that for $\mu - a.s.$ $w_1, w_2 \in \mathcal{W}$ one has:

$$W_t(w_1 + w_2) = W_t(w_1) + W_t(w_2) \tag{1.57}$$

The following space of functions is closely related to the Wiener space:

Definition 1.2.1.3 (Cameron-Martin space)**.**

$$H = \left\{ t \in [0, 1] \mapsto \int_0^t f(s)ds \right\} \tag{1.58}$$

That is: H is the space of primitives of $L^2([0, 1]$ $\lambda)$ functions taking the value 0 at 0. Equivalently, H is the space of those L^2 functions which cancel at 0 and have L^2 weak derivatives. We will usually note $\dot{h}$ for the derivative of $h \in H$.

We will make use of the following classic results:

Theorem 1.2.1.3 (Topological structure of the Cameron-Martin space)*. The following quantity is a norm on the vector space H:*

$$|h|_H = \sqrt{\int_0^1 \dot{h}^2(s)ds} \tag{1.59}$$

It defines a Hilbert space structure on H. Also, we have the following injections:

$$\mathcal{W}^* \hookrightarrow H \hookrightarrow^i \mathcal{W} \tag{1.60}$$

with i dense into $\|\cdot\|_\infty$ and compact into $|\cdot|_H$.

Proof. The Hilbert structure and the RHS injection are obvious. The compactness of i is a special case of the Rellich theorem. The density may be obtained, for example, via the Stone-Weierstrass theorem.

For the LHS injection, simply consider the application:

$$\Phi = \nu \mapsto \left(t \mapsto \int_0^t \nu(]s,1])ds \right) \tag{1.61}$$

$\square$

It was natural to introduce Φ because the following is valid μ-a.s. through a simple integration by parts which can be justified properly by working on Riemann sums and taking limits in probability:

$$
\begin{aligned}
_{\mathcal{W}^*}\langle \nu, w \rangle_{\mathcal{W}} &= \int_0^1 w(s)d\nu(s) \\
&= \int_0^1 W_s(w)d\nu(s) \\
&= W_1(w)\nu([0,1]) - \left(\int_0^1 \nu([0,s])dW_s \right)(w) \\
&= \left(\int_0^1 \nu(]s,1])dW_s \right)(w)
\end{aligned}
\tag{1.62}
$$

Also we now see that, in addition to $\|\cdot\|_{\mathcal{W}^*}$, there is another natural norm on $\mathcal{W}^*$:

$$\|\nu\|_{\mathcal{W}^*,2} = \sqrt{\int_0^1 \nu(]s,1])^2 ds} = \sqrt{E\left[\left(\int_0^1 \nu(]s,1])dW_s \right)^2 \right]} \tag{1.63}$$

It clearly arises from a scalar product, and provides an isometry between the Radon measures and a subset of the stochastic integrals on deterministic $L^2(\lambda)$ functions. It is not a complete space: indeed, the set of the primitives of Radon measures, ie the image of Φ, is easily seen to be dense in H, therefore the set of the stochastic integrals $\int \dot{h}dW$, where h describes H, is isometric to the completion of the set of Radon measures $\mathcal{W}^*$ for the norm $\|\cdot\|_{\mathcal{W}^*,2}$.

We finish this paragraph with two simple computations related to equation 1.57.

Proposition 1.2.1.1. *Let $h \in L^2$. For a.s. $w_1, w_2 \in \mathcal{W}$:*

$$\left(\int h(s)dW_s\right)(w_1 + w_2) = \left(\int h(s)dW_s\right)(w_1) + \left(\int h(s)dW_s\right)(w_2) \quad (1.64)$$

Proof. Write the stocastic integral on the LHS as the limit of a Riemann sum and apply equation 1.57. $\qquad\square$

With similar techniques, one proves:

Proposition 1.2.1.2. *Let $g, h \in H$. Then*

$$\left(\int \dot{h}(s)dW_s\right)(g) = (g, h)_H \quad (1.65)$$

1.2.2   Chaos decomposition on the Wiener space

We recall that W denotes the canonical Brownian motion on $(\mathcal{W}, \mu)$ and we start with the following:

Lemma 1.2.2.1. *Let (h_i) be any ONB of H. Then:*

$$\sigma(W_t, 0 \le t \le 1) = \sigma\left(\int_0^1 \dot{h}_i(s)dW_s,\ i \in \mathbb{N}\right) \quad (1.66)$$

Proof. By elementary stochastic calculus, any stochastic integral $\int_0^1 \dot{h}_i(s)dW_s$ is measurable with respct to the σ-field $\sigma(W_t, 0 \le t \le 1)$, therefore we have the inclusion:

$$\sigma(W_t, 0 \le t \le 1) \supset \sigma\left(\int_0^1 \dot{h}_i(s)dW_s\right) \quad (1.67)$$

Conversely since (h_i) is an ONB of H, $(\dot{h}_i)$ is an ONB of $L^2(\lambda)$, hence the indicator of an interval has a decomposition:

$$1_{[0,t]} = \sum_{i=1}^{\infty} \alpha_i \dot{h}_i \quad (1.68)$$

Therefore, one sees that, in the L^2 sense:

$$W_t = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i \cdot \int_0^1 \dot{h}_i(s) dW_s \tag{1.69}$$

from which the converse inclusion arises. □

The main interest of this result is that it shows that the information in the canonical Brownian filtration is encompassed in a countable filtration. This will let us define more hilbertian structures. In particular, we introduce the:

Definition 1.2.2.1 (Polynomials on the Wiener space)**.** *Suppose that an ONB* (h_i) *of H has been fixed. We call polynomial on the Wiener space any random variable of the type:*

$$P\left(\int_0^1 \dot{h}_1(s) dW_s, \ldots, \int_0^1 \dot{h}_n(s) dW_s\right) \tag{1.70}$$

where n is an integer and P is a polynomial function on $\mathbb{R}^n$ (resp $\mathbb{C}^n$). We will note $\mathcal{P}$ the space of all polynomial random variables on the Wiener space.

It is noteworthy that since the $\dot{h}_i$ are orthogonal to each other in $L^2(\lambda)$ and have norm 1, there is the identity in law:

$$\left(\int_0^1 \dot{h}_1(s) dW_s, \ldots, \int_0^1 \dot{h}_n(s) dW_s\right) = \mathcal{N}(0, Id_n) \tag{1.71}$$

and in particular polynomial random variables have moments of all orders. Polynomial functions are therefore easy to work with, and our strategy in many cases will be to build an object or prove a result first for the polynomial functions, and then to extend it to more general random variables on the Wiener space. This will be possible because of the:

Theorem 1.2.2.1 (density of the polynomials on the Wiener space)**.** *Let X in $L^p(\mathcal{W}, \mu)$ and $\epsilon > 0$. Then there exists an integer n and a polynomial function in n variables such that:*

$$\left\| X - P\left(\int_0^1 \dot{h}_1(s) dW_s, \ldots, \int_0^1 \dot{h}_n(s) dW_s\right) \right\|_{L^p(\mathcal{W},\mu)} < \epsilon \tag{1.72}$$

Proof. Let us define a discrete martingale as:

$$M_n = E\left[X | \int_0^1 \dot{h}_i(s) dW_s, i \le n\right] \tag{1.73}$$

Then, by lemma 1.2.2.1 M_n converges in L^p to X, ie for big enough n:

$$\|M_n - X\|_{L^p} < \epsilon \qquad (1.74)$$

Also, there exists a function f_n such that:

$$M_n = f_n \left(\int_0^1 \dot{h}_1(s)dW_s, \ldots, \int_0^1 \dot{h}_n(s)dW_s \right) \qquad (1.75)$$

Now since $M_n \in L^p(\mathcal{W}, \mu)$ and $\left(\int_0^1 \dot{h}_1(s)dW_s, \ldots, \int_0^1 \dot{h}_n(s)dW_s \right)$ is $\mathcal{N}(0, Id_n)$ under μ, necessarily $f_n \in L^p(\mu_n)$ and:

$$\|M_n\|_{L^p(\mathcal{W}, \mu)} = \|f_n\|_{L^p(\mu_n)} \qquad (1.76)$$

Finally by theorem 1.1.1.1 there exists a polynomial P_n such that:

$$\|f_n - P_n\|_{L^p(\mu_n)} < \epsilon \qquad (1.77)$$

and of course:

$$\left\| P_n \left(\int_0^1 \dot{h}_1(s)dW_s, \ldots, \int_0^1 \dot{h}_n(s)dW_s \right) \right\|_{L^p(\mathcal{W}, \mu)} = \|P_n\|_{L^p(\mu_n)} \qquad (1.78)$$

so we are done. $\qquad \square$

As we consider polynomial random variables on the Wiener space, since these have the same law as the underlying polynomial evaluated at a normal gaussian random variable, it is natural to decompose the underlying polynomial in the Hermite ONB and to study the hilbertian properties which thus arise. Indeed, straightforward computations show that:

Proposition 1.2.2.1. *If $p, q = (p_1, \ldots, p_{n_1}), (q_1, \ldots, q_{n_1})$ are two different multi-indices:*

$$E \left[H_p \left(\int_0^1 \dot{h}_{p_1}(s)dW_s, \ldots, \int_0^1 \dot{h}_{n_p}(s)dW_s \right) H_q \left(\int_0^1 \dot{h}_{q_1}(s)dW_s, \ldots, \int_0^1 \dot{h}_{n_q}(s)dW_s \right) \right] = 0 \qquad (1.79)$$

and

$$E \left[H_p \left(\int_0^1 \dot{h}_{p_1}(s)dW_s, \ldots, \int_0^1 \dot{h}_{n_p}(s)dW_s \right)^2 \right] = p! \qquad (1.80)$$

By combining theorem 1.2.2.1 and proposition 1.2.2.1 , we immediately obtain the:

Theorem 1.2.2.2 (Wiener chaos decomposition - first statement)**.** *The countable family of the* $\frac{1}{p!} \cdot H_p \left(\int_0^1 \dot{h}_{p_1}(s)dW_s, \ldots, \int_0^1 \dot{h}_{n_p}(s)dW_s \right)$*, where p describes all multi-indices, the H_p are the Hermite polynomials and (h_i) is any ONB of H constitutes an ONB of $L^2(\mathcal{W}, \mu)$.*

The terminology "Wiener chaos decomposition" is justified by the following:

Definition 1.2.2.2 (Wiener chaos)**.** *The nth order Wiener chaos, which we note $\mathcal{C}_n$, is the linear span of the $H_p \left(\int_0^1 \dot{h}_{p_1}(s)dW_s, \ldots, \int_0^1 \dot{h}_{n_p}(s)dW_s \right)$, where the H_p are the Hermite polynomials with total degree n.*

We will characterize the Wiener chaoses thanks to the following result:

Proposition 1.2.2.2. *For any $n \in \mathbb{N}$ and $h \in H$ such that $|h|_H = 1$:*

$$H_n \left(\int_0^1 \dot{h}(s)dW_s \right) = \int \cdots \int_{0 < s_1 < \cdots < s_n < 1} \dot{h}(s_1) \cdots \dot{h}(s_n)dW_{s_1} \cdots dW_{s_n} \quad (1.81)$$

Proof. The statement is trivial for $n = 0, 1$. It is then proved by a simple recursion as the Hermite polynomials and the multiple integrals verify the same recursion relationship. $\square$

From this we deduce:

Theorem 1.2.2.3 (Caracterization of the nth Wiener chaos)**.** *$\mathcal{C}_n$ is the linear span of the order n multiple Wiener integrals described in proposition 1.2.2.2, where h describes any given ONB of H.*

$\mathcal{C}_n$ also is the eigenspace of the operators P_t and $\mathcal{L}$, associated to the eigenvalues e^{-nt} and n respectively. In particular, the definition of the Wiener chaoses does not depend on the choice of one specific ONB of H.

Now the Wiener chaos decomposition theorem may be rephrased:

Theorem 1.2.2.4 (Wiener chaos decomposition - second statement)**.**

$$L^2(\mathcal{W}, \mu) = \bigoplus^{\perp} \mathcal{C}_n \quad (1.82)$$

1.2.3 Gradient and divergence on the Wiener space

We want to define a functional derivative on the Wiener space. The only way to do this is through partial, ie directional derivatives, and we will want to consider, when they exist, L^p-limits of the type:

$$\nabla_{\tilde{w}} X(w) = \lim_{\alpha \to 0} \frac{X(w + \alpha \tilde{w}) - X(w)}{\alpha} \quad (1.83)$$

Of course, for the theory to make sense, it is necessary that, if $X = Y$ almost surely and X is differentiable, then Y is also differentiable and $\nabla X = \nabla Y$ a.s. in every direction. The cases where this is possible are explained in the following paragraph:

The Girsanov and Cameron-Martin theorems

First we recall the classic:

Theorem 1.2.3.1 (Girsanov). *Assume that u is a measurable process on the Wiener space adapted to the canonical Brownian filtration, such that μ-a.s.*

$$\int_0^1 u_s^2 ds < \infty \tag{1.84}$$

and let

$$\Lambda_t = \exp\left\{-\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds\right\} \tag{1.85}$$

Assume that $E[\Lambda_1] = 1$. Then, the process $(W_t + \int_0^t u_s ds, 0 \le t \le 1)$ is a Brownian motion under the probability $\Lambda_1 \cdot \mu$.

Proof. Omitted, see [60]. $\qquad\square$

Remark 1.2.3.1. *Two archetypal cases where $E[\Lambda_1] = 1$ are, first, when the Novikov condition is verified:*

$$E\left[\exp\left(\frac{1}{2}\int_0^1 u_s^2 ds\right)\right] < \infty \tag{1.86}$$

and, second, when the Kazamaki condition is verified:

$$E\left[\exp\left(\frac{1}{2}\int_0^1 u_s dW_s\right)\right] < \infty \tag{1.87}$$

Using either one of the two criteria above, it is immediate to obtain the following corollary of the Girsanov theorem:

Theorem 1.2.3.2 (Cameron, Martin). *Let $h \in H$. Then $W + h$ is a Brownian motion under the probability measure with density with regard to the Wiener measure given by:*

$$\exp\left\{-\int_0^1 \dot{h}_s dW_s - \frac{1}{2}\int_0^1 h_s^2 ds\right\} \tag{1.88}$$

Equivalently, this means that for any $F \in L^p(\mathcal{W}, \mu)$, $p > 1$, and any $h \in H$, one has:

$$E_\mu\left[F(w + h)\exp\left\{-\int \dot{h}_s dW_s - \frac{1}{2}\int_0^1 \dot{h}_s^2 ds\right\}\right] = E_\mu[F] \tag{1.89}$$

The Cameron-Martin theorem was actually proved prior to the Girsanov theorem. It admits various intuitive proofs, one of which is based on characteristic functions and suits our setting well:

Proof. First recall equation (1.62): for any $\nu \in \mathcal{W}^*$:

$$_{\mathcal{W}^*} \langle \nu, w \rangle_{\mathcal{W}} = \left(\int_0^1 \nu(]t, 1]) dW_t \right)(w) \tag{1.90}$$

Therefore the characterisitc function of the Wiener measure is:

$$E[e^{i\langle \nu, w \rangle}] = E[e^{i \int_0^1 \nu(]t,1]) dW_t}] = e^{-\frac{1}{2} \int_0^1 \nu(]t,1])^2 dt} \tag{1.91}$$

and similarly:

$$
\begin{aligned}
& E[\exp\left(i \langle \nu, w \rangle (w + h)\right) \cdot \Lambda_1] \\
= {} & E\left[\exp\left\{ i \int_0^1 \nu(]t,1]) dW_t + i \int_0^1 \nu(]t,1]) \dot{h}(t) dt - \int_0^1 \dot{h}(t) dW_t - \frac{1}{2} \int_0^1 \dot{h}(t)^2 dt \right\}\right] \\
= {} & E\left[\exp\left\{ \int_0^1 \left(i\nu(]t,1]) - \dot{h}(t)\right) dW_t \right\}\right] \exp\left\{ i \int_0^1 \nu(]t,1]) \dot{h}(t) dt - \frac{1}{2} \cdot \int_0^1 \dot{h}(t)^2 dt \right\} \\
= {} & \exp\left\{ \int_0^1 \left(i\nu(]t,1]) - \dot{h}(t)\right)^2 dt \right\} \exp\left\{ i \int_0^1 \nu(]t,1]) \dot{h}(t) dt - \frac{1}{2} \cdot \int_0^1 \dot{h}(t)^2 dt \right\} \\
= {} & \exp\left(-\frac{1}{2} \cdot \int_0^1 \nu(]t,1])^2 dt \right) \\
= {} & E[\exp\left(i \langle \nu, w \rangle\right)]
\end{aligned} \tag{1.92}
$$

so we are done. $\qquad\square$

By the Cameron-Martin theorem, it is clear that if $X = Y$ a.s. then for any $h \in H$ one also has $X(\cdot + h) = Y(\cdot + h)$ a.s. Therefore it is reasonable to try and consider derivatives in this direction. Also, the following result which we copy from [41] is a converse to the Cameron-Martin theorem which leads us not to consider other directions than the space H for our derivatives:

Theorem 1.2.3.3. *Let $w \in W - H$. Then: $\mu(\cdot + w)$ is **not** absolutely continuous with regards to μ.*

Proof. Omitted as it is based on elaborate Fourier analysis considerations and we will not make direct use of the results; see [41] for the details. $\qquad\square$

Construction of the Gross-Sobolev derivative on the Wiener space

Let $1 < p < \infty$.

Definition 1.2.3.1 (differentiability in a direction). *We will say that $X \in L^p$ is differentiable in the direction $h \in H$ when the following limit exists in the L^p sense:*

$$\nabla_h X(w) = \lim_{\alpha \to 0} \frac{X(w + \alpha h) - X(w)}{\alpha} \tag{1.93}$$

Suppose that $X \in L^p$ is differentiable in every direction $h \in H$. Then it is clear that a.s. $h \in H \mapsto \nabla_h X \in \mathbb{R}$ is linear and we would like to consider the "whole" gradient ∇X of X as a random variable with values in $\mathcal{L}(H, \mathbb{R}) \approx H$. For this to be a tractable framework, we need an L^p structure for H-valued random variables. More generally, if Ξ is some separable Hilbert space, one may define:

$$\|X\|_{L^p(\mathcal{W}, \mu, \Xi)} = \||X|_\Xi\|_{L^p(\mu)} = \|\sup_{|x|_\Xi = 1} (x, X)_\Xi\|_{L^p(\mu)} \tag{1.94}$$

This leads us to the following definition of differentiability, which, because we work in infinite dimensions, demands slightly more than just having a partial derivative in each direction:

Definition 1.2.3.2 (differentiable random variables). *We say that $X \in L^p$ is differentiable and we will note $X \in \mathbb{D}_{p,1}$, if X is L^p-differentiable in every direction $h \in H$ and if $\nabla X \in L^p(\mathcal{W}, \mu, H)$, ie:*

$$\left\| \sup_{|g|_H = 1} \nabla_g X \right\|_{L^p} < \infty \tag{1.95}$$

Remark 1.2.3.2. *In operator theory language $\mathbb{D}_{p,1}$ is the domain of the operator ∇.*

Let us now give a few archetypal examples of differentiable random variables together with their derivatives:

Proposition 1.2.3.1. *Let $g, h \in H$. Then $\int \dot{h} dW \in \mathbb{D}_{p,1}$ and*

$$\nabla_g \int \dot{h} dW = (g, h)_H \tag{1.96}$$

Proof. $\int \dot{h} dW \in L^p$ because it is gaussian. That it has a derivative in every direction, and the computation of these, are straightforward consequences of the linearity, see proposition 1.2.1.2. Then by the Cauchy-Schwarz inequality it is clear that, μ-a.s:

$$\sup_{|g|_H = 1} \nabla_g \int \dot{h} dW = \sup_{|g|_H = 1} (g, h)_H \leq |h|_H \tag{1.97}$$

so $\nabla \int \dot{h} dW$ is well defined in $L^p(\mathcal{W}, \mu, H)$ and we are done. $\qquad\square$

Let us also note that, as a consequence of the previous proposition, $W_t \in \mathbb{D}_{p,1}$ and:

$$\nabla_g W_t = \nabla_g \int 1_{[0,t]} dW = \int 1_{[0,t]} \dot{g} = g(t) \tag{1.98}$$

Another crucial example is the following:

Proposition 1.2.3.2. *Let $g, h \in H$ and let $P \in \mathbb{C}[X]$. Then $P(\int \dot{h} dW) \in \mathbb{D}_{p,1}$, and:*

$$\nabla_g P \left(\int \dot{h} dW \right) = P' \left(\int \dot{h} dW \right) \cdot (g, h)_H \tag{1.99}$$

or equivalently, by the Riesz theorem:

$$\nabla \int \dot{h} dW = h \tag{1.100}$$

Proof. Since $\int \dot{h} dW$ is gaussian it has moments of all orders and $P(\int \dot{h} dW) \in L^p$. Now, using proposition 1.2.1.2 and a Taylor expansion, we obtain, μ-a.s:

$$
\begin{aligned}
P\left(\left(\int \dot{h} dW \right)(w + \alpha g) \right) &= P\left(\left(\int \dot{h} dW \right)(w) + \alpha(g, h)_H \right) \\
&= \sum_{n=0}^{degP} \frac{P^{(n)}((\int \dot{h} dW)(w))}{n!} \alpha^n (h, g)_H^n \tag{1.101}
\end{aligned}
$$

Since the $P^{(n)}((\int \dot{h} dW)(w))$ are in L^p as well it is clear that the Newton ratio converges in L^p as α tends to 0 and we obtain the existence if derivatives in every direction. Now, similarly to what we have done in the previous proof, a.s:

$$\sup_{|g|_H = 1} \nabla_g P \left(\int \dot{h} dW \right) \le \left| P' \left(\int \dot{h} dW \right) \right| \cdot |h|_H \tag{1.102}$$

so we obtain $\nabla P \left(\int \dot{h} dW \right) \in L^p(\mathcal{W}, \mu, H)$ and

$$\left\| \nabla P \left(\int \dot{h} dW \right) \right\|_{L^p} \le \left\| P' \left(\int \dot{h} dW \right) \right\| \cdot |h|_H \tag{1.103}$$

$\square$

It is straightforward to generalize this to polynomials in several variables, and therefore we obtain the following:

Proposition 1.2.3.3 (differentiability of the polynomials on the Wiener space).
*Suppose that an ONB (h_i) of H has been fixed. Then, any polynomial on the
Wiener space is in $\mathbb{D}_{p,1}$, and for any $g \in H$:*

$$\nabla_g P\left(\int \dot{h}_1 dW, \cdots, \int \dot{h}_n dW\right) = \sum_{i=1}^n \partial_i P\left(\int \dot{h}_1 dW, \cdots, \int \dot{h}_n dW\right)(g, h_i)_H$$
$$(1.104)$$

or, equivalently:

$$\nabla P\left(\int \dot{h}_1 dW, \cdots, \int \dot{h}_n dW\right) = \sum_{i=1}^n \partial_i P\left(\int \dot{h}_1 dW, \cdots, \int \dot{h}_n dW\right) h_i \quad (1.105)$$

We now prove the:

Theorem 1.2.3.4. *The Gross-Sobolev derivative is a closed operator, ie if (X_n)
is a sequence in $\mathbb{D}_{p,1}$ which converges to 0 in L^p then if (∇X_n) converges in
$L^p(\mathcal{W}, \mu, H)$ its limit has to be 0.*

Proof. Since we know that $\lim \nabla X_n$ exists it is enough to prove that it is 0 in every
direction, ie for any $h \in H$, $\nabla_h X_n \to 0$. Since the polynomial random variables
on the Wiener space are dense in L^{p^*}, by duality it is enough to prove that for any
polynomial ϕ: $E[\phi \nabla_h X_n] \to 0$. To do this, we note that:

$$E[\nabla_h X_n \phi] = \frac{d}{d\alpha}_{|\alpha=0} E[X_n(\cdot + \alpha h) \cdot \phi]$$

$$= \frac{d}{d\alpha}_{|\alpha=0} E\left[X_n \phi(\cdot - \alpha h) \exp\left\{\alpha \int_0^1 \dot{h}(s) dW_s - \frac{\alpha^2}{2} \int_0^1 \dot{h}^2(s) ds\right\}\right]$$

$$= E\left[X_n\left(-\nabla_h \phi + \phi \int_0^1 \dot{h}(s) dW_s\right)\right] \quad (1.106)$$

so we are done by Hölder's inequality. $\square$

Theorem 1.2.3.5. $\mathbb{D}_{p,1}$ *is a Banach space if equipped with the norm:*

$$\|X\|_{\mathbb{D}_{p,1}} = \|X\|_{L^p} + \|\nabla X\|_{L^p(\mathcal{W}, \mu, H)} \quad (1.107)$$

*More precisely, $\mathbb{D}_{p,1}$ is the completion of the set of polynomials on the Wiener space
for that norm.*

Proof. For the first part, an operator is closed if and only if its domain is a Banach
space for its graph norm so the previous theorem gives the conclusion.

Alternatively we may obtain the result in a more "computational" way by
considering a Cauchy sequence (X_n), and by noting that since the Lebesgue spaces

are Banach spaces, (X_n) has to converge in L^p, to, say, X, and (∇X_n) has to converge in $L^p(H)$, to, say, ξ. Now we just need to prove that $X \in \mathbb{D}_{p,1}$ and that $\nabla X = \xi$. To do this, let $h \in H$ and (α_k) a sequence of real numbers decreasing to 0. Then, since L^p convergence implies the almost sure convergence of a subsequence, by a diagonal extraction procedure there exists a subsequence $(X_{\psi(n)})$ such that a.s. for every k:

$$X_{\psi(n)}(\cdot + \alpha_k h) \to X(\cdot + \alpha_k h) \tag{1.108}$$

Now we simply write:

$$
\left| \frac{X_{\psi(m+n)}(\cdot + \alpha_k h) - X_{\psi(m+n)}}{\alpha_k} - (\xi, h)_H \right|
$$
$$
\leq \left| \frac{X_{\psi(m+n)}(\cdot + \alpha_k h) - X_{\psi(m+n)}}{\alpha_k} - \frac{X_{\psi(n)}(\cdot + \alpha_k h) - X_{\psi(n)}}{\alpha_k} \right|
$$
$$
\leq \left| \frac{X_{\psi(n)}(\cdot + \alpha_k h) - X_{\psi(n)}}{\alpha_k} - \nabla_h X_{\psi(n)} \right|
$$
$$
\leq \left| \nabla_h X_{\psi(n)} - (\xi, h)_H \right| \tag{1.109}
$$

Let $\epsilon > 0$. There exists N such that for any $n \geq N$ the third term is controlled by ϵ. Take $n = N$ from now on. There exists K such that for any $k \geq K$, at $n = N$, the second term is controlled by ϵ. Finally, at $n = N$ and $k = K$, by the Cauchy property for big enough m the first term is controlled by ϵ. Therefore if we let m tend to infinity we obtain:

$$\left| \frac{X(\cdot + \alpha_K h) - X}{\alpha_K} - (\xi, h)_H \right| < 3\epsilon \tag{1.110}$$

This proves that X has a differential in direction h and that this differential is $(\xi, h)_H$. Then since $\xi \in L^p(H)$ we automatically obtain $X \in \mathbb{D}_{p,1}$ and $\nabla X = \xi$ so we are done.

For the second part, we note that the polynomials are dense in any L^p, and actually by using the same technique as before one may prove that they are dense in $\mathbb{D}_{p,1}$ as well. □

Remark 1.2.3.3. *The construction is sometimes done by defining the derivative on the polynomials only, by proving a version of theorem 1.2.3.4 for polynomials only, and by defining $\mathbb{D}_{p,1}$ as the set of those random variables X for which there exists an approximating sequence of polynomials (P_n) such that the gradients (∇P_n) converge. Then by theorem 1.2.3.4 it makes sense to define $\nabla X = \lim \nabla P_n$. In this construction $\mathbb{D}_{p,1}$ is automatically the completion of the polynomials for the operator norm. Of course it is possible (but tedious) to prove that both constructions are equivalent.*

Extension to Hilbert space valued random variables and higher order derivatives

If Ξ is a separable Hilbert space, then there is no difficulty in extending our results to $L^p(\mathcal{W}, \mu, \Xi)$, the Lebesgue space of random variables taking values in Ξ. If (x_i) is an ONB of Ξ and if (h_i) is an ONB of H then we define the polynomials on the Wiener space as random variables of the type:

$$\sum_{j=1}^{n} P_j \left(\int h_1 dW, \cdots, \int h_{n_j} dW \right) \cdot x_j \tag{1.111}$$

and all the ideas and proofs may be adapted in a straightforward way.

We are especially interested in the case where $\Xi = H^{\hat{\otimes}k}$, the kth order symmetrized tensor power of H (see appendix A). Indeed, this case allows us to construct higher order Gross-Sobolev spaces via an easy recurrence:

$$\mathbb{D}_{p,k} = \left\{ X \in \mathbb{D}_{p,k-1} \middle| \nabla^{k-1} X \in \mathbb{D}_{p,1} \left(H^{\hat{\otimes}(k-1)} \right) \right\} \tag{1.112}$$

and

$$\nabla^k X = \nabla \left(\nabla^{k-1} X \right) \tag{1.113}$$

in the sense of $\mathbb{D}_{p,1} \left(H^{\hat{\otimes}(k-1)} \right)$.

1.2.4   The divergence operator

As in the finite dimensionnal case, the divergence operator will be defined as the adjoint of the gradient, although on te Wiener space the formal definition is slightly more involved:

Definition 1.2.4.1 (Divergence on the Wiener space)**.** *Let $\xi \in L^p(\mathcal{W}, \mu, H)$. We say that $\xi \in dom_p(\delta)$ if for any $\phi \in \mathbb{D}_{p^*,1}$ we have:*

$$E[(\nabla \phi, \xi)_H] \le c \|\phi\|_{\mathbb{D}_{p^*,1}} \tag{1.114}$$

and in this case we define $\delta \xi$ via the Riesz theorem by:

$$E[\phi \delta \xi] = E[(\xi, \nabla \phi)_H] \tag{1.115}$$

The following results will be useful:

Proposition 1.2.4.1 (divergence of a product)**.** *Let $\xi \in dom_p(\delta)$ and $X \in \mathbb{D}_{q,1}$ for some $q > p^*$. Define $r > 1$ such that:*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \tag{1.116}$$

Then $X\xi \in dom_r(\delta)$ and:

$$\delta(X\xi) = X\delta\xi - (\nabla X, \xi)_H \tag{1.117}$$

Proof. The choice of r justifies that both expressions on the RHS are well defined in L^r by Hölder's theorem. Also, let ϕ be any polynomial on the Wiener space. Then ϕ has moments of all orders, and especially of order r^*. Therefore we may write:

$$
\begin{aligned}
E[\delta(X\xi)\phi] &= E[(X\xi, \nabla\phi)_H] \\
&= E[(\xi, X\nabla\phi)_H] \\
&= E[(\xi, \nabla(X\xi) - \phi\nabla X)_H] \\
&= E[(\phi(X\delta\xi - (\nabla X, \xi)_H)]
\end{aligned}
\tag{1.118}
$$

so by duality the proof is finished. $\square$

Proposition 1.2.4.2. *The divergence of a deterministic $h \in H$ is the stochastic integral of its derivative:*

$$\delta h = \int_0^1 \dot{h}(s)dW_s \tag{1.119}$$

Proof. By the definition of the divergence, for any polynomial ϕ:

$$E[\phi\delta h] = E[\nabla_h\phi] \tag{1.120}$$

On the other hand, simplifying formula (1.106) we obtain:

$$E[\nabla_h\phi] = E\left[\phi \cdot \int \dot{h}dW\right] \tag{1.121}$$

so by the usual duality argument we are done. $\square$

1.2.5   Ornstein-Uhlenbeck operator on the Wiener space

Definition and first properties

Let us define a family of operators on the polynomials on the Wiener space by the formula:

$$P_t\left(f\left(\int h_1 dW, \ldots, \int h_n dW\right)\right) = \left(P_t^{(n)}f\right)\left(\int h_1 dW, \ldots, \int h_n dW\right) \tag{1.122}$$

where $P_t^{(n)}$ is the Ornstein Uhlenbeck operator on $\mathbb{R}^n$. The results for the finite dimensionnal case are automatically lifted since the $(\int h_1 dW, \ldots, \int H_n dW)$ are

normal centered gaussians; in particular (P_t) defines a contractive L^p semigroup on the polynomials on the Wiener space, and for any polynomial X, the Mehler formula holds:

$$P_t X(w) = \tilde{E}\left[X(e^{-t}w + \sqrt{1 - e^{-2t}}\tilde{w})\right] \tag{1.123}$$

It is straightforward to extend (P_t) to a contraction semigroup on all L^p by density; Mehler's formula is then verified on all L^p and, as in the finite dimensionnal case, there is no ambiguity about the choice of p. As a semigroup (P_t) admits a generator which we denote $\mathcal{L}$. As in the finite dimensionnal case, the P_t and $\mathcal{L}$ are symmetrical. Similarly to equation 1.122 we have:

$$\mathcal{L}\left(f\left(\int h_1 dW, \ldots, \int h_n dW\right)\right) = (\mathcal{L}^{(n)} f)\left(\int h_1 dW, \ldots, \int h_n dW\right) \tag{1.124}$$

We will also prove that:

Theorem 1.2.5.1. *Whenever both expressions are well defined:*

$$\mathcal{L} = \delta \circ \nabla \tag{1.125}$$

Proof. It is enough to prove the equation for the polynomials, and indeed:

$$\nabla\left(P\left(\int h_1 dW, \ldots, \int h_n dW\right)\right) = \sum_{i=1}^{n} \partial_i P\left(\int h_1 dW, \ldots, \int h_n dW\right) h_i \tag{1.126}$$

and therefore

$$
\begin{aligned}
&\delta\nabla\left(P\left(\int h_1 dW, \ldots, \int h_n dW\right)\right)\\
&= \sum_{i=1}^{n} \partial_i P\left(\int h_1 dW, \ldots, \int h_n dW\right)\delta h_i - \sum_{i=1}^{n} \partial_{ii}^2 P\left(\int h_1 dW, \ldots, \int h_n dW\right)|h_i|^2\\
&= (\mathcal{L}^{(n)} P)\left(\int h_1 dW, \ldots, \int h_n dW\right)\\
&= \mathcal{L}\left(P\left(\int h_1 dW, \ldots, \int h_n dW\right)\right)
\end{aligned} \tag{1.127}
$$

$\square$

Finally the following result is an immediate consequence of formula (1.122):

Theorem 1.2.5.2. *For any $X \in \mathcal{C}_n$:*

$$P_t X = e^{-nt} X \tag{1.128}$$

and

$$\mathcal{L}X = -nX \tag{1.129}$$

In other words, the Wiener chaos decomposition diagonalizes the Ornstein Uhlenbeck semigroup.

Subordination of the Ornstein-Uhlenbeck semigroup

In order to prove some interpolation results for the Gross-Sobolev spaces, we will need to make an intensive use of a semigroup which is closely related to the Ornstein-Uhlenbeck semigroup. More precisely, one may introduce T_t be the semigroup generated by $-(Id - \mathcal{L})^{1/2}$; then we know that $Id - \mathcal{L}$ is the generator of the submarkovian semigroup $e^{-t}P_t$ and that T_t is its 1/2-subordination (see appendix C for details), therefore one has the following explicit expression for T_t:

$$T_t = \int_0^\infty e^{-s}P_s\lambda_t^{1/2}(ds) \tag{1.130}$$

where the probability measures λ are characterized through their Laplace transforms: for $0 < \beta < 1$

$$\int_0^\infty e^{-\gamma s}\lambda_t^\beta(ds) = e^{-\gamma^\beta t} \tag{1.131}$$

and one may compute explicitely by inverting the Laplace transform:

$$\lambda_t^{1/2}(ds) = \frac{1}{2\sqrt{\pi}}ts^{-3/2}e^{-\frac{t^2}{4s}} \cdot ds \tag{1.132}$$

and therefore:

$$T_t = \frac{1}{2\sqrt{\pi}}t\int_0^\infty s^{-3/2}e^{-s}e^{-\frac{t^2}{4s}}P_s ds \tag{1.133}$$

We note that $T_t \cdot 1 = e^{-t} \cdot 1$ and that more generally:

Proposition 1.2.5.1. *If $F \in L^p$ then*

$$\|T_tF\|_{L^p} \leq e^{-t}\|F\|_{L^p} \tag{1.134}$$

Also:

Proposition 1.2.5.2. *If $F \in L^p$ then*

$$T_tF \in \bigcap_s \mathbb{D}_{p,s} \tag{1.135}$$

1.2.6   Fractional spaces and the Meyer inequalities

We would like to define fractionnal powers of the operator $Id + \mathcal{L}$ as opeators on a subspace of L^p. In the case where $p = 2$, it is natural to use the Wiener chaos representation and to set:

$$(Id + \mathcal{L})^{\alpha/2} I_n(f_n) = (1 + n)^{\alpha/2} \cdot I_n(f_n) \tag{1.136}$$

Then, the definition is completed by linearity and density. Of course, for a general p this strategy is no longer valid, but we inspire from it and the equation:

$$(1 + x)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty e^{-t(1+x)} t^{-\alpha} dt \tag{1.137}$$

which is valid for $x, \alpha > 0$ if Γ denotes the usual Gamma function and we define an operator on L^p as:

$$A_\alpha X = \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty e^{-t} t^{-\alpha} P_t X dt \tag{1.138}$$

It is easily verified that the integral above is well defined on L^p in the sense of Bochner and that A_α is a contraction on L^p. Also, by a simple change of variables:

$$A_\alpha I_n(f_n) = (1 + n)^{-\alpha} I_n(f_n) \tag{1.139}$$

Therefore, on one hand our definition makes intuitive sense and on the other hand we note that:

Proposition 1.2.6.1. A_α *maps the polynomials on the polynomials.*

and

Corollary 1.2.6.1.

$$A_\alpha(L^p) \supset L^p \tag{1.140}$$

We will now prove that:

Proposition 1.2.6.2. A_α *is injective.*

Proof. First suppose that $p = 2$, and let $X \in L^2$ be such that $A_\alpha X = 0$. Then if we write a decomposition in Wiener chaoses:

$$X = \sum I_n(f_n) \tag{1.141}$$

we have

$$A_\alpha X = \sum \frac{I_n(f_n)}{(1 + n)^\alpha} \tag{1.142}$$

so the orthogonality of the Wiener chaoses yield the result. Now if $p > 2$, $L^p \subset L^2$ and we also get the result.

Finally let us suppose that $p < 2$. Then, we have the following:

Lemma 1.2.6.1. *Let $X \in L^p$ and $Y \in L^{p^*}$. Then:*

$$E\left[A_\alpha X \cdot Y\right] = E\left[X \cdot A_\alpha Y\right] \tag{1.143}$$

Now let E be the set of those polynomials with L^{p^*}-norm 1. Then:

$$\|A_\alpha X\|_{L^p} = \sup_{Y \in E} E\left[A_\alpha X \cdot Y\right] = \sup_{Y \in E} E\left[X \cdot (1+n)^{-\alpha} Y\right] \tag{1.144}$$

therefore $\|A_\alpha X\|_{L^p} = 0$ if and only if X is orthogonal to every polynomial, ie iff $X = 0$ so we are done. $\qquad\square$

This justifies that we introduce:

$$(Id + \mathcal{L})^\alpha = \begin{array}{ll} A_\alpha & if \ \alpha < 0 \\ A_\alpha^{-1} & if \ \alpha > 0 \end{array} \tag{1.145}$$

This definition of our fractional power coincides with the intuitive one in the L^2 case (because the two operators coincide on polynomials). Also, we have the following crucial:

Theorem 1.2.6.1 (Meyer inequalities). *Let $1 < p < \infty$ and $n \in \mathbb{N}$. Then, the two following norms are equivalent:*

$$\|X\| = \|X\|_{L^p} + \|\nabla^n X\|_{L^p(H^{\otimes n})}$$

and

$$\|X\| = \left\|(Id + \mathcal{L})^{n/2} X\right\|_{L^p}$$

In the $p = 2$ case it is very easy to prove the result by comparing the respective actions of ∇ and $(Id + \mathcal{L})^{1/2}$ on a polynomial. Unfortunately, this idea does not extend to a general p for the lack of an orthogonality property. We omit the proper proof which, although classic, is quite involved and may be found in [53] or [66] for example. The same references also provide a slightly more precise result which is a consequence of Meyer's multiplier theorem:

Theorem 1.2.6.2. *Let $1 < p < \infty$, $s \in \mathbb{R}$ and $\alpha > 0$. Then the two following norms are equivalent:*
$$\|X\| = \left\|(Id + \mathcal{L})^{s/2} X\right\|_{L^p}$$
and
$$\|X\| = \left\|(\alpha \cdot Id + \mathcal{L})^{s/2} X\right\|_{L^p}$$

The Meyer inequalities allow us to extend the definition of the $\mathbb{D}_{p,k}$ spaces for any $1 < p < \infty$ and any real number k. Because the A_α are contractions, one may see that:

Proposition 1.2.6.3. *Let $1 < p_1 < p_2 < \infty$ and $s_1 < s_2$. Then:*

$$\mathbb{D}_{p_1,s_1} \subset \mathbb{D}_{p_2,s_2} \tag{1.146}$$

In particular, elements of $\mathbb{D}_{p,-s}$ for a positive s are not necessarily random variables and we refer to them as distributions on the Wiener space. Also, an immediate corollary of lemma 1.2.6.1 is:

Theorem 1.2.6.3. *Let $1 < p < \infty$ and $s > 0$. Then:*

$$\mathbb{D}_{p,s}^* = \mathbb{D}_{p^*,-s} \tag{1.147}$$

Also, for $p > 1$, $\mathbb{D}_{p,s}$ is uniformly convex as it is naturally isometric to L^p which in turn is uniformly convex; in particular this proves that $\mathbb{D}_{p,s}$ is isomorphic to its bidual and:

Corollary 1.2.6.2.

$$\mathbb{D}_{p^*,-s}^* = \mathbb{D}_{p,s} \tag{1.148}$$

Another consequence of the reflexivity of $\mathbb{D}_{p,s}$ is the very convenient:

Lemma 1.2.6.2. *Let $p > 1$ and $s > 0$. Let (X_n) be a sequence of random variables which converges to X in L^p and which is bounded in $\mathbb{D}_{p,s}$. Then (X_n) converges to X in $\mathbb{D}_{p,s}$.*

Proof. Since $\mathbb{D}_{p,s}$ is a reflexive Banach space, as a consequence of the Banach-Alaoglu theorem there exists a subsequence $X_{\phi(n)}$ which converges weakly in $\mathbb{D}_{p,s}$. In particular, for any polynomial random variable P:

$$\lim_{n\to\infty} E\left[(Id + \mathcal{L})^{s/2} X_{\phi(n)} \cdot P \right] = E\left[(Id + \mathcal{L})^{s/2} X_n \cdot P \right] \tag{1.149}$$

so we are done. $\qquad\square$

We finish this paragraph with a result which we will often use:

Proposition 1.2.6.4. *Let $p \in]1, \infty[$.*

- *Let $h \in L^p(H)$. Then:*

$$P_t \int_0^1 \dot{h}_u dW_u = e^{-t} \cdot \int_0^1 P_t \dot{h}_u dW_u \tag{1.150}$$

- *Let $\alpha > 0$, $s \in \mathbb{R}$ and $h \in \mathbb{D}_{p,s}(H)$. Then, $\int_0^1 \dot{h}_s dW_s \in \mathbb{D}_{p,s}$ and:*

$$(\alpha \cdot Id + \mathcal{L})^{s/2} \int_0^1 \dot{h}_u dW_u = \int_0^1 ((1+\alpha) \cdot Id + \mathcal{L})^{s/2} \dot{h}_u dW_u \tag{1.151}$$

1.2.7 Distributions on the Wiener space

First we introduce:

Definition 1.2.7.1 (Smooth random variables). *The space $\mathbb{D}$ of the smooth random variables on the Wiener space is defined as:*

$$\mathbb{D} = \bigcap_{p,s} \mathbb{D}_{p,s} \tag{1.152}$$

We equip it with its natural Fréchet space topology (see appendix A), ie the one given by the distance:

$$d(x,y) = \sum_{p=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2^{p+k}} \cdot \frac{\|X - Y\|_{\mathbb{D}_{p,k}}}{1 + \|X - Y\|_{\mathbb{D}_{p,k}}} \tag{1.153}$$

Then it is natural to consider the:

Definition 1.2.7.2 (Distributions on the Wiener space). *The vector space $\mathbb{D}'$ of the distributions on the Wiener space is defined as the strong dual of the space $\mathbb{D}$ of smooth random variables:*

$$\mathbb{D}' = \bigcup_{p>1, s \in \mathbb{R}} \mathbb{D}_{p,s} \tag{1.154}$$

This space is naturally equipped with its strong dual topology.

The interest of the space $\mathbb{D}'$ is that the objects of the Malliavin calculus, which were defined on $\mathbb{D}$, extend to $\mathbb{D}'$ by duality, just as operators on spaces of test functions extend to distributions. We exemplify this idea with the gradient and the divergence, starting with the:

Theorem 1.2.7.1 (Continuity of the gradient). *For every $p \in]1, \infty[$ and $s \in \mathbb{R}$, ∇ admits a continuous extension:*

$$\nabla : \mathbb{D}_{p,s} \to \mathbb{D}_{p,s-1}(H) \tag{1.155}$$

Proof. Let $X \in \mathbb{D}$. Then there is:

$$
\begin{aligned}
\|\nabla X\|_{\mathbb{D}_{p,s-1}(H)} &\leq \left\| (Id + \mathcal{L})^{\frac{s-1}{2}} \nabla X \right\|_{L^p(H)} \\
&= \left\| \nabla \left(2 \cdot Id + \mathcal{L} \right)^{\frac{s-1}{2}} X \right\|_{L^p(H)} \\
&\leq C \cdot \left\| (\cdot Id + \mathcal{L}) \cdot (2 \cdot Id + \mathcal{L})^{\frac{s-1}{2}} X \right\|_{L^p)} \\
&\leq C \cdot \left\| (Id + \mathcal{L})^{\frac{s}{2}} X \right\|_{L^p} \\
&= C \cdot \|X\|_{\mathbb{D}_{p,s}} \tag{1.156}
\end{aligned}
$$

Here we have used the Meyer inequalities (and their extension). The prove is then finished as $\mathbb{D}$ is dense in any of the $\mathbb{D}_{p,s}$. $\qquad\square$

As an immediate consequence of this we get:

Corollary 1.2.7.1 (Weak gradient). ∇ *admits a continuous extension:*

$$\nabla : \mathbb{D}' \to \mathbb{D}'(H) \tag{1.157}$$

Also by duality one deduces:

Corollary 1.2.7.2 (Weak divergence). *The divergence admits continuous extensions:*

$$\delta : \mathbb{D}_{p,s}(H) \to \mathbb{D}_{p,s-1} \tag{1.158}$$

and:

$$\delta : \mathbb{D}' \to \mathbb{D}' \tag{1.159}$$

1.2.8   Weak martingales

In this section we define a notion of martingality for weak processes. The results in this paragraph perhaps are less classic than those in the rest of this chapter, and rather than giving the most general possible statements we content ourselves with notions which will fulfill our purpose later on. More general results may be found in [77] or [49].

First we need to define a notion of weak measurability and a weak conditionnal expectation. To do so we show that in relevant cases the strong conditionnal expectations commutes with operators related to the Ornstein-Uhlenbeck operator:

Lemma 1.2.8.1. *If $X \in L^p(\mu)$, $p \geq 1$ is $\mathcal{F}_t$-measurable and $s \geq 0$ then $P_s X$ is $\mathcal{F}_t$-measurable as well.*

Proof. First we obtain the result if X is in the nth Wiener chaos since then $P_s X = e^{-ns} X$. The result then holds for a polynomial function on W, and finally for any X in L^p by density of the polynomials. $\qquad\square$

Lemma 1.2.8.2. *If $X \in L^p(\mu)$, $p \geq 1$ and $s \geq 0$ then*

$$P_s E\left[X | \mathcal{F}_t\right] = E\left[P_s X | \mathcal{F}_t\right] \tag{1.160}$$

Proof. Indeed if $Y \in L^{p'}$ and $Y \in \mathcal{F}_t$ we have:

$$E\left[P_s E\left[X | \mathcal{F}_t\right] Y\right] = E\left[E\left[X | \mathcal{F}_t\right] P_s Y\right] = E\left[X P_s Y\right] = E\left[P_s X Y\right] \tag{1.161}$$

and $P_s E\left[X | \mathcal{F}_t\right] \in \mathcal{F}_t$ from the previous result, which completes the proof. $\qquad\square$

Similar results for $\mathcal{L}$ or for $(Id + \mathcal{L})^{k/2}$, $k \in \mathbb{Z}$ are obtained by the same method if X is regular enough. This leads to:

Lemma 1.2.8.3. *If $X \in \mathbb{D}_{p,k}$, $p > 1$, $k \geq 0$ then $E[X|\mathcal{F}_t] \in \mathbb{D}_{p,k}$.*

Proof. Indeed, $Y := (Id + \mathcal{L})^{k/2} X \in L^p$, so:

$$E[X|\mathcal{F}_t] = E\left[(Id + \mathcal{L})^{-k/2} Y | \mathcal{F}_t\right] = (Id + \mathcal{L})^{-k/2} E[Y|\mathcal{F}_t] \tag{1.162}$$

$\square$

We may now define a notion of weak measurability:

Definition 1.2.8.1 (Weak measurability). *We will say that $T \in \mathbb{D}'$ is weakly $\mathcal{F}_t$-measurable if $\forall X \in \mathbb{D}$,*

$$\langle T, X \rangle = \langle T, E[X|\mathcal{F}_t] \rangle \tag{1.163}$$

If X is a proper random variable this is equivalent to being measurable. Also we have:

Lemma 1.2.8.4. *$T \in \mathbb{D}_{p,-k}$ is weakly $\mathcal{F}_t$-measurable if and only if $(Id + \mathcal{L})^{-k/2} X$ is measurable.*

Proof. Indeed if T is weakly measurable the measurability rewrites, $\forall X \in \mathbb{D}$:

$$
\begin{aligned}
E\left[(Id + \mathcal{L})^{-k/2} T \cdot (Id + \mathcal{L})^{k/2} X\right] &= E\left[(Id + \mathcal{L})^{-k/2} T \cdot (Id + \mathcal{L})^{k/2} E[X|\mathcal{F}_t]\right] \\
&= E\left[(Id + \mathcal{L})^{-k/2} T \cdot E\left[(Id + \mathcal{L})^{k/2} X | \mathcal{F}_t\right]\right] \\
&= E\left[E\left[(Id + \mathcal{L})^{-k/2} T | \mathcal{F}_t\right] \cdot (Id + \mathcal{L})^{k/2} X\right]
\end{aligned}
\tag{1.164}
$$

and $\mathbb{D}$ is stable by $(Id + \mathcal{L})^{k/2}$ and dense in $L^{p'}$, so we get:

$$(Id + \mathcal{L})^{-k/2} T = E\left[(Id + \mathcal{L})^{-k/2} T | \mathcal{F}_t\right] \tag{1.165}$$

and therefore $(Id + \mathcal{L})^{-k/2} T$ is $\mathcal{F}_t$-measurable. Conversely if $(Id + \mathcal{L})^{-k/2} T$ is $\mathcal{F}_t$-measurable $(Id + \mathcal{L})^{-k/2} T = E\left[(Id + \mathcal{L})^{-k/2} T | \mathcal{F}_t\right]$ holds and we can do the above computations backwards to find that T is weakly $\mathcal{F}_t$-measurable. $\square$

We also define the conditionnal expectation of $T \in \mathbb{D}'$ by:

$$\langle E\left[T | \mathcal{F}_t\right], X \rangle = \langle T, E\left[X | \mathcal{F}_t\right]\rangle \tag{1.166}$$

It is easily checked that this definition extends the usual conditionnal expectation on L^p and that it is equivalent to defining:

$$E\left[T | \mathcal{F}_t\right] = \left(Id + \mathcal{L}\right)^{k/2} E\left[\left(Id + \mathcal{L}\right)^{-k/2} T | \mathcal{F}_t\right] \tag{1.167}$$

and hence that the negative Gross-Sobolev spaces also are left stable by the weak conditionnal expectation.

We now may define a weak martingale as a weak adapted process M verifying the martingale equality:

$$E\left[M_t | \mathcal{F}_s\right] = M_s \tag{1.168}$$

for $s \leq t$. If M is a weak martingale and $M_t \in \mathbb{D}_{p,-k}$, we note that:

$$E\left[\left(Id + \mathcal{L}\right)^{-k/2} M_t | \mathcal{F}_s\right] = \left(Id + \mathcal{L}\right)^{-k/2} M_s \tag{1.169}$$

from which we deduce the two following results:

Lemma 1.2.8.5. *If M is a weak martingale and $M_t \in \mathbb{D}_{p,-k}$ then $\forall s \leq t$, $M_s \in \mathbb{D}_{p,-k}$ and $\|M_s\|_{\mathbb{D}_{p,-k}} \leq \|M_t\|_{\mathbb{D}_{p,-k}}$.*

Lemma 1.2.8.6. *A weak process $(M_t)_{0 \leq t \leq 1}$ such that $M_1 \in \mathbb{D}_{p,-k}$ is a weak martingale if and only if the process $\left(\left(Id + \mathcal{L}\right)^{-k/2} M_t\right)_{0 \leq t \leq 1}$ is an L^p martingale.*

We now show how we may define Brownian integrals of weak adapted processes and how these provide an example of weak martingales. First we recall that the divergence as defined on $\mathbb{D}_{p,1}(H)$ extends the stochastic integrals to non adapted processes and that the divergence extends by duality to $\mathbb{D}'(H)$. We will use the following notation: if $f \in \mathbb{D}'(H)$, we note $f_{|[0,t]}$ the element of $\mathbb{D}'(H)$ whose time derivative is $\dot{f}$ before t and 0 after t and we introduce: $\int_0^t \dot{f}(u) dW_u := \delta\left(f_{|[0,t]}\right)$.

Now let us suppose that $f \in \mathbb{D}_{p,-k}(H)$. Then $f_{|[0,t]} \in \mathbb{D}_{p,-k}(H)$ and we know that:

$$\left(Id + \mathcal{L}\right)^{-k/2} \delta\left(f_{|[0,t]}\right) = \delta\left(\left(2 \cdot Id + \mathcal{L}\right)^{-k/2} f_{|[0,t]}\right) \tag{1.170}$$

Let us note M_t for this quantity. Clearly, if f is weakly adapted,

$$M_t = \int_0^t \left[\left(2 \cdot Id + \mathcal{L}\right)^{-k/2} \dot{f}(u)\right] dW_u \tag{1.171}$$

is a martingale, so we have proved:

Proposition 1.2.8.1. *If $f \in \mathbb{D}'(H)$ is a weakly adapted process, then the weak process $\left(\int_0^t \dot{f}(u) dW_u\right)_t$ is a weak martingale.*

1.2.9   Malliavin calculus in Banach spaces

It is possible to extend the construction of most of the mathematical objects we have reviewed in this chapter to the case of random variables taking values in a Banach space X if that space enjoys the so-called uniform martingale-difference property, UMD for short:

Definition 1.2.9.1 (UMD property). *A Banach space X is said to have the UMD property if for some $p \in]1, \infty[$, there exists a universal constant $C(p, X)$ such that for every (discrete) X-valued, L^p martingale, for every $\epsilon \in \{-1, 1\}^{\mathbb{N}}$ and every $N \in \mathbb{N}$ there is:*

$$E\left[\left\|\sum_{n=1}^{N} \epsilon_n (M_n - M_{n-1})\right\|_X^p\right] \leq C(p, X) \cdot E\left[\|M_N - M_0\|_X^p\right] \tag{1.172}$$

We will not provide a full review of UMD spaces and their applications; instead we suggest the survey [58] and the references therein. We will content ourelves with using the fact that the most important results in Malliavin calculus, especially the construction of the gradient and the Meyer inequalities, hold for UMD valued random variables. The intuition behind this is that the UMD property is a necessary and sufficient condition on a Banach space for the Hilbert transform to be bounded on that space.

We also mention examples of Banach spaces enjoying the UMD property: Hilbert spaces on one hand (we have already contructed the Maliavin calculus on Hilbert spaces) and on the other hand the Sobolev spaces $W_{p,s}$ for $1 < p < \infty$, which will be our case of application. The C_b^k, however, are not UMD.

Chapter 2

Topological structure of $\mathcal{S}$ and $\mathcal{S}'$

In this section, we recall the results about the topology of $\mathcal{S}$ and $\mathcal{S}'$ which we will need in the sequel. First, several families of semi-norms are introduced and it is proved that they all endow the Schwartz class with the same (usual) Fréchet space topology. We then recall some important properties of this topology.

2.1 Two standard families of semi norms on $\mathcal{S}$

For a multi-index $\alpha \in \mathbb{N}^N$ and $x \in \mathbb{R}^N$, we will use the following notation:

$$\alpha! = \alpha_1 \cdots \alpha_N$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$$
$$D^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$$

Then there is a classic family of semi-norms defined by the following equation for $\alpha, \beta \in \mathbb{N}^N$:

$$\|f\|_{\alpha,\beta}^{(\infty)} = \sup_{x \in \mathbb{R}^N} |x^\alpha D^\beta f(x)| \tag{2.1}$$

We recall that the Schwartz class, for which we note $\mathcal{S}(\mathbb{R}^N)$, is defined as the space of those functions on $\mathbb{R}^N$ such that for every $\alpha, \beta \in \mathbb{N}^N$: $\|f\|_{\alpha,\beta}^{(\infty)} < \infty$. Then this family of semi-norms naturally define a Fréchet space topology (see appendix A) through the distance:

$$d(f,g) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{1}{2^{\alpha+\beta}} \cdot \frac{\|f - g\|_{\alpha,\beta}^{(\infty)}}{1 + \|f - g\|_{\alpha,\beta}^{(\infty)}} \tag{2.2}$$

We also introduce the following, for $\alpha, \beta \in \mathbb{N}^N$:

$$\|f\|_{\alpha,\beta}^{(p)} = \|x \mapsto x^\alpha D^\beta f(x)\|_{L^p} \tag{2.3}$$

and we have the:

Proposition 2.1.0.1. *For $p, q \in [1, \infty]$, the families $(\|\cdot\|_{\alpha,\beta}^{(p)}, \alpha, \beta \in \mathbb{N}^N)$ and $(\|\cdot\|_{\alpha,\beta}^{(q)}, \alpha, \beta \in \mathbb{N}^N)$ define the same topology on $\mathcal{S}\left(\mathbb{R}^N\right)$.*

Proof. Indeed, on one hand, multiplying and dividing by $(1 + |x|^2)^s$ yields:

$$\|f\|_{L^p} \leq \left[\int \frac{dx}{(1 + |x|^2)^s}\right]^{1/p} \cdot \sup_{x \in \mathbb{R}^N} |(1 + |x|^2)^{s/p} f(x)| \tag{2.4}$$

and the integral above is finite for big enough $s(N)$. p and such an s being fixed, the exists a polynomial dominating $(1 + |x|^2)^{s/p}$. Hence there exists a constant $C(N, p)$ and an integer $k(N, p)$ such that:

$$\|f\|_{\alpha,\beta}^{(p)} \leq C \sum_{|\gamma| \leq k} \|f\|_{\alpha+\gamma,\beta}^{(\infty)} \tag{2.5}$$

On the other hand, one can use the following (non-optimal, but sufficiently precise for our use) Sobolev embedding result; see, for example, [1]:

Proposition 2.1.0.2 (Sobolev embedding). *If $k > N/p$, $W^{k,p} \hookrightarrow C^0$.*

which, choosing a big enough integer k at fixed p and N and using the Leibniz formula, proves that there exists constants $C_i(p, N)$ such that:

$$\|f\|_{\alpha,\beta}^{(\infty)} \leq C_1 \cdot \|x^\alpha D^\beta f\|_{W^{k,p}} \leq C_2 \cdot \sum_{|\gamma| \leq k, \delta \leq \alpha} \|f\|_{\delta,\beta+\gamma}^{(p)} \tag{2.6}$$

This concludes our proof. $\qquad\square$

2.2 The operator $\mathcal{K}$ and associated semi norms

In this section, we introduce families of semi-norms which define the same topology on $\mathcal{S}$ than the ones we have studied in the previous paragraph. The purpose of these is that these less classical families will be more convenient for the study of $\mathcal{S}'$. This will be useful as we turn to the lifting of random variables in the next chapter. Such results are well-known (cf, for example, [81] or [25]) but not very documented, so we provide details for the sake of completeness.

All our study will be based on the following differential operator:

$$\mathcal{K} = Id - \Delta + |x|^2 \tag{2.7}$$

ie:

$$\mathcal{K}f(x) = f(x) - \Delta f(x) + |x|^2 f(x) \tag{2.8}$$

We also introduce:

$$\tilde{\mathcal{K}} = \mathcal{K} - Id \tag{2.9}$$

Let us note that $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are continuous operators on $\mathcal{S}$. Our agenda is to formally define fractional powers of $\mathcal{K}$ and to study their regularity.

2.2.1 Semi norms built on L^2

We start with the following observation: $\mathcal{K}$ is symmetric and positive for the Hilbert space structure induced by L^2 on $\mathcal{S}$, ie for every $f, g \in \mathcal{S}$:

$$\int (\mathcal{K}f) \cdot g = \int f \cdot (\mathcal{K}g) \tag{2.10}$$

and:

$$\int (\mathcal{K}f) \cdot f = \int \left((1 + |x|^2) \cdot |f|^2 + |Df|^2 \right) \geq \int |f|^2 \tag{2.11}$$

It is therefore natural to try and diagonalize $\mathcal{K}$ and indeed we prove that there exists an orthogonal basis of L^2 made of elements of $\mathcal{S}(\mathbb{R}^N)$ which are eigenvectors of $\mathcal{K}$. These eigenvectors are closely related to the so-called Hermite functions. We have recalled the facts we need on Hermite polynomials in chapter 1 and now, for $n \in \mathbb{N}^N$, we introduce:

$$\tilde{f}_n(x) = H_n(\sqrt{2}x) \exp(-|x|^2/2) \tag{2.12}$$

It is noteworthy that, abusively using the same notation for the Hermite functions in dimensions 1 and N:

$$\tilde{f}_n(x) = \tilde{f}_{n_1}(x_1) \ldots \tilde{f}_{n_N}(x_N) \tag{2.13}$$

These belong to the Schwartz class and as can be checked using the PDE verified by the Hermite polynomials, they are eigenfunctions of the differential operator $\mathcal{K}$ associated to the eigenvalue:

$$\lambda_n = 2|n| + N + 1 \geq 1 > 0 \tag{2.14}$$

respectively. In dimension 1, the $\tilde{f}_n$ form an orthogonal family since each $\tilde{f}_n$ is an eigenvector associated to a different eigenvalue of the symmetric operator $\mathcal{K}$. The orthogonality extends to multiple dimensions as the relevant multiple integral can be separated into a product of multiple integrals using (2.13). Using this fact it is easy to compute the L^2 norm of $\tilde{f}_n$, hence the following functions form an orthonormal family:

$$f_n(x) = \frac{\tilde{f}_n(x)}{\pi^{N/4}\sqrt{n!}} = \frac{H_n(\sqrt{2}x) \exp(-|x|^2/2)}{\pi^{N/4}\sqrt{n!}} \tag{2.15}$$

Finally, (f_n) is an orthogonal basis of L^2, which can be checked by using the fact the the normalized Hermite polynomials are an orthogonal basis of $L^2(\mu_N)$ and that the following is an isometric isomorphism mapping f_n on $H_n/\sqrt{n!}$:

$$\begin{array}{rcc} L^2 & \longrightarrow & L^2(\mu) \\ f & \longmapsto & x \mapsto \pi^{N/4} e^{|x|^2/4} f(x/\sqrt{2}) \end{array} \tag{2.16}$$

and we have proved the:

Theorem 2.2.1.1 (Reduction of $\mathcal{K}$). (f_n) *is an orthonormal basis of L^2 made of elements of $\mathcal{S}(\mathbb{R}^N)$. Moreover f_n is an eigenvector of $\mathcal{K}$ associated to the eigenvalue $\lambda_n = 2|n| + N + 1$.*

Now, we may decompose any $f \in \mathcal{S}$ (or more generally: any $f \in L^2$) as:

$$f = \sum_{n \in \mathbb{N}^N} c_n(f) \cdot f_n \tag{2.17}$$

and then:

$$\|f\|_{L^2} = \sqrt{\sum_{n \in \mathbb{N}^N} c_n(f)^2} \tag{2.18}$$

This allows us to define the fractional power $\mathcal{K}^s$ for $s \in \mathbb{R}$ in the following way:

$$\mathcal{K}^s f = \sum_{n \in \mathbb{N}^N} c_n(f) \cdot \lambda_n^s \cdot f_n \tag{2.19}$$

on the vector space $dom_{L^2}(\mathcal{K}^J)$ of those f for which this quantity is well defined (see appendix C for the details). We also introduce the norm:

$$\|f\|_{\mathcal{S}_{2,s}} = \|\mathcal{K}^{s/2} f\|_{L^2} \tag{2.20}$$

and the space $\mathcal{S}_{2,s} = dom_{L^2}(\mathcal{K}^{s/2})$ is the completion of $\mathcal{S}$ for this norm. When this is defined, there is:

$$\|f\|_{\mathcal{S}_{2,s}} = \sqrt{\sum_{n \in \mathbb{N}^N} c_n(f)^2 \cdot \lambda_n^s} \tag{2.21}$$

We now study the effect of deriving or multiplying by a monomial an element of our ONB; let us note ϵ_i for the N-tuple with a 1 in the ith coordinate and 0 elsewhere, then:

$$x_i f_n(x) = -\sqrt{\frac{n_i}{2}} f_{n-\epsilon_i}(x) + \sqrt{\frac{n_i + 1}{2}} f_{n+\epsilon_i}(x) \tag{2.22}$$

and

$$\partial_i f_n(x) = \frac{3\sqrt{2n_i}}{2} f_{n-\epsilon_i}(x) - \sqrt{\frac{n_i + 1}{2}} f_{n+\epsilon_i}(x) \tag{2.23}$$

Once again the details of the computations rely on the recursion verified by the Hermite polynomials. Using the decomposition (2.17) in combination with the two equations above one may explicitly compute the coefficients m_n and d_n such that:

$$x \cdot f(x) = \sum_{n \in \mathbb{N}^N} m_n f_n(x) \tag{2.24}$$

and

$$\partial_i f(x) = \sum_{n \in \mathbb{N}^N} d_n f_n(x) \tag{2.25}$$

Then one obtains the following:

Theorem 2.2.1.2. *There exists a universal constant $C(N)$ such that the following controls hold:*

$$\|x_i \cdot f\|_{L^2} \ \leq \ C \cdot \|f\|_{\mathcal{S}_{2,1}} \tag{2.26}$$
$$\|\partial_i \cdot f\|_{L^2} \ \leq \ C \cdot \|f\|_{\mathcal{S}_{2,1}} \tag{2.27}$$

From which on deduces the:

Theorem 2.2.1.3. *The families of semi norms $(\|\cdot\|_{\alpha,\beta}^{(2)}, \alpha, \beta \in \mathbb{N}^N)$ and $(\|\cdot\|_{\mathcal{S}_{2,s}}, s \in \mathbb{N})$ define the same topology on $\mathcal{S}\left(\mathbb{R}^N\right)$.*

We do not work out the details of the proof, first they may be found in [59], second because in the next paragraph we will prove results in the L^p framework which are an extension of the L^2 case we studied in this paragrph.

2.2.2 Semi norms built on L^p, $1 < p < \infty$

In order to define the fractional powers of $\mathcal{K}$, we will prove that this operator may be understood as the generator of a semigroup on L^p; then we will be able to use the results in appendix C. The following results are detailed in [25], for example.

First we note that $\mathcal{K}$ is the generator of a semigroup K_t on L^p; the transition density k_t of K_t is given by:

$$k_t(x,y) = \sum_{n \in \mathbb{N}^N} \exp(-\lambda_n \cdot t) \cdot f_n(x) \cdot f_n(y) \tag{2.28}$$

In the case of dimension 1, by Mehler's formula this is:

$$
\begin{aligned}
& k_t^{(1)}(x,y) \\
=\ & \exp(-2t) \cdot \sum_{n=0}^{\infty} (\exp(-2t))^n \cdot f_n(x) \cdot f_n(y) \\
=\ & \sqrt{\frac{e^{-4t}}{\pi(1-e^{-4t})}} \cdot \exp\left(\frac{4xye^{-2t} - (x^2+y^2)(1+e^{-4t})}{2(1-e^{-4t})}\right) \\
=\ & \frac{e^{-t}}{\sqrt{2\sinh(2t)}} \cdot \exp\left(-\frac{1}{2} \cdot \coth(2t) \cdot \left(x^2 - \frac{2xy}{\cosh(2t)} + y^2\right)\right) \quad (2.29)
\end{aligned}
$$

and in dimension N there is:

$$k_t(x,y) = \prod_{i=1}^{N} p_t^{(1)}(x_i, y_i) \tag{2.30}$$

The fractional powers of $\mathcal{K}$ may therefore be defined as domain operators on L^p as is explained in apendix C. We may therefore define the spaces $\mathcal{S}_{p,s}$ as $dom_{L^p}(\mathcal{K}^{s/2})$, ie the completion of $\mathcal{S}$ for the norm:

$$\|f\|_{\mathcal{S}_{p,s}} = \|\mathcal{K}^{s/2}f\|_{L^p} \tag{2.31}$$

Similarly, $\tilde{\mathcal{K}}$ is the generator of a semigroup $\tilde{K}_t$ with transition density:

$$\tilde{k}_t^{(1)}(x,y) = \exp(t) \cdot k_t^{(1)}(x,y) \tag{2.32}$$

and it is shown in [25] that there exists a universal constant $C(N)$ such that:

$$\tilde{k}_t(x,y) \leq C \cdot q_t(x,y) \tag{2.33}$$

where q_t is the transition density of the heat kernel. Therefore there is the:

Proposition 2.2.2.1. *For $s > 0$, and $f \in L^p$, there is:*

$$\|f\|_{\mathcal{S}_{p,-s}} \leq C \cdot \left\| (Id - \Delta)^{-s/2} |f| \right\|_{L^p} \leq C \cdot \|f\|_{L^p} \tag{2.34}$$

and conversely:

$$\|f\|_{L^p} \leq C^{-1} \cdot \|f\|_{\mathcal{S}_{p,s}} \tag{2.35}$$

Proof. According to the results in appendix C let us write:

$$\begin{aligned}
\mathcal{K}^{-s/2}f(x) &= \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{-t} \tilde{K}_t f(x) dt \\
&= \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{-t} \int_{\mathbb{R}^N} \tilde{k}_t(x,y) f(y) dy dt
\end{aligned} \tag{2.36}$$

Then by Hölder's inequality and because $\tilde{k}$ is controlled by the heat kernel:

$$\begin{aligned}
|\mathcal{K}^{-s/2}f(x)| &\leq \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{-t} \int_{\mathbb{R}^N} \tilde{k}_t(x,y)|f|(y) dy dt \\
&\leq \frac{C}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{-t} \int_{\mathbb{R}^N} q_t(x,y)|f|(y) dy dt \\
&= C \cdot \left\| (Id - \Delta)^{-s/2} |f| \right\|_{L^p} \\
&\leq C \cdot \||f|\|_{L^p} \\
&= C \cdot \|f\|_{L^p}
\end{aligned} \tag{2.37}$$

Finally, the second inequality is obtained by applying the first one to $\mathcal{K}^{s/2}f$. $\qquad\square$

Remark 2.2.2.1. *We could not directly obtain a result such as:*

$$\|f\|_{\mathcal{S}_{p,-s}} \leq C \cdot \|(Id - \Delta)^{-s/2} f\|_{L^p} \tag{2.38}$$

ie getting rid of the absolute value inside the second term is not trivial; it is the point of the next result.

We define continuous operators X_i and X^2 on $\mathcal{S}$ as:

$$(X_i f)(x) = x_i f(x) \tag{2.39}$$

and:

$$(X^2 f)(x) = |x|^2 f(x) \tag{2.40}$$

Then we prove the:

Theorem 2.2.2.1. *There exists a universal constant C, depending only on $1 < p < \infty$ and N, such that the following estimates hold:*

$$\|\partial_i u\|_{L^p} \leq C\|\mathcal{K}^{1/2} u\|_{L^p} \tag{2.41}$$

$$\|X_i u\|_{L^p} \leq C\|\mathcal{K}^{1/2} u\|_{L^p} \tag{2.42}$$

Proof. All the notation in this proof is that of appendix E. We will apply theorem E.0.0.26 to the operator $\mathcal{K}$; to do so, first notice that one gets a good weight function by setting:

$$\rho(x) = \left(1 + |x|^2\right)^{\frac{1}{3}} \tag{2.43}$$

Then take $m = 1$, $b_0 = 1$, $b_1 = -1$, $a_\beta = 0$, $\mu = 0$ and $\nu = 3$, so that $\chi_0 = \nu = 3$ and $\chi_2 = \mu = 0$. Hence,

$$A = \rho^\nu - \rho^\mu \Delta = \mathcal{K} \tag{2.44}$$

and we have proved the:

Lemma 2.2.2.1. $\mathcal{K} \in \mathcal{A}^1_{0,3}\left(\mathbb{R}^N, \rho\right)$.

Therefore we may apply theorem E.0.0.26 (in this case with $\chi = 0$); for a generic constant C_i and some λ with small enough real part we obtain:

$$
\begin{aligned}
\|\Delta u\|_{L^p} &\leq \|u\|_{W^2_p} \\
&\leq C_1 \|u\|_{W^2_p(1,\rho^{3p})} \\
&\leq C_2 \|\mathcal{K}u - \lambda u\|_{L^p} \\
&\leq C_2 \cdot \left(\|\mathcal{K}u\|_{L^p} + |\lambda| \cdot \|u\|_{L^p}\right) \\
&\leq C_3 \cdot \|\mathcal{K}u\|_{L^p}
\end{aligned} \tag{2.45}
$$

For the last inequality we used proposition 2.2.2.1.

Then by complex interpolation one gets:

$$\|\sqrt{-\Delta}u\|_{L^p} \leq C\|\sqrt{\mathcal{K}}u\|_{L^p} = C\|u\|_{\mathcal{S}_{p,1}} \tag{2.46}$$

so we obtain the first part of the result through the well-known (cf, for example, [67]) inequality:

$$\|\partial_i u\|_{L^p} \leq C\|\sqrt{-\Delta}u\|_{L^p} \tag{2.47}$$

Also, one sees that:

$$\|X^2 u\|_{L^p} \leq \|\mathcal{K}u\|_{L^p} + \|u\|_{L^p} + \|\Delta u\|_{L^p} \leq C\|\mathcal{K}u\|_{L^p} \tag{2.48}$$

and similarly, one gets the second part of the result by interpolation.   □

We finish this paragraph with a couple of results related to the interpolation of the $\mathcal{S}_{p,s}$ spaces. First as an application of the results for semigroup interpolation which may be found in appendix D there is the:

Theorem 2.2.2.2. *We consider $p_1, p_2 > 1$, $s_1, s_2 \in \mathbb{R}$ and $0 < \theta < 1$. Let p and s be such that: $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $s = \theta s_1 + (1-\theta)s_2$. Then $[\mathcal{S}_{p_1,s_1}, \mathcal{S}_{p_2,s_2}]_\theta = \mathcal{S}_{p,s}$.*

Also we obtain the following through classic semigroup interpolation arguments, cf appendix C:

Theorem 2.2.2.3 (Dual space of $\mathcal{S}_{p,s}$). *For $1 < p < \infty$ and $s \in \mathbb{R}$ there is:*

$$\mathcal{S}'_{p,s} = \mathcal{S}_{p^*,-s} \tag{2.49}$$

where $1/p^* + 1/p = 1$.

We may also introduce some real interpolation spaces:

Definition 2.2.2.1 (Real interpolation of the $\mathcal{S}_{p,s}$). *Let $p > 1$ and $s \in \mathbb{R}$. We define a family of Banach spaces $\mathcal{T}_{p,s}$ as follows:*

- *if $s \in \mathbb{Z}$ then $\mathcal{T}_{p,s} = \mathcal{S}_{p,s}$*

- *Otherwise let $k = [s]$, $\sigma = \{s\}$. Then $\mathcal{T}_{p,s} = (\mathcal{S}_{p,k+1}, \mathcal{S}_{p,k})_{1-\sigma,p}$.*

and there is the:

Theorem 2.2.2.4. *For any $\epsilon > 0$, there is:*

$$\mathcal{T}_{p,s-\epsilon} \hookrightarrow \mathcal{S}_{p,s} \hookrightarrow T_{p,s+\epsilon} \tag{2.50}$$

Proof. Omitted as it is very similar to that of theorem 3.1.0.9, which we prove in detail.   □

We also have a duality result for the real interpolation spaces:

Theorem 2.2.2.5 (Dual space of $\mathcal{T}_{p,s}$). *For $1 < p < \infty$ and $s \in \mathbb{R}$ there is:*

$$\mathcal{T}'_{p,s} = \mathcal{T}_{p^*,-s} \tag{2.51}$$

where $1/p^* + 1/p = 1$.

2.2.3   Semi norms built on L^∞

In this section we extend the L^p results of the previous section to the L^∞ case. We were not able to find the exact results we needed in the literature, so we built our own proofs. More precisely, we will prove the:

Theorem 2.2.3.1. *There exists a universal constant C such that the following estimates hold:*

$$\|\sqrt{-\Delta}u\|_\infty \;\; \leq \;\; C\|\mathcal{K}^{1/2}u\|_\infty \tag{2.52}$$

$$\|X_i u\|_\infty \;\; \leq \;\; C\|\mathcal{K}^{1/2}u\|_\infty \tag{2.53}$$

Proof. We prove the theorem only in dimension 1; results generalize to higher dimension with no other difficulty than tedious notation. We start with the second part of the theorem. We recall that $\mathcal{K}$ is invertible on any L^p and its inverse has a kernel Π which is given by:

$$\mathcal{K}^{-1}f(x) \;\; = \;\; \int_{-\infty}^{\infty} \Pi(x,y)f(y)dy \tag{2.54}$$

$$\Pi(x,y) \;\; = \;\; \sum_{n=0}^{\infty} \frac{f_n(x)f_n(y)}{2n+2} \tag{2.55}$$

where the f_n are the Hermite functions. Then by Fubini's theorem:

$$\Pi(x,y) = \sum_{n=0}^{\infty} f_n(x)f_n(y) \int_0^1 r^{2n+1}dr = \int_0^1 r \cdot \sum_{n=0}^{\infty} r^{2n} f_n(x)f_n(y)dr \tag{2.56}$$

so by Mehler's formula one obtains:

$$\Pi(x,y) = \int_0^1 \sqrt{\frac{r^2}{\pi(1-r^4)}} \cdot \exp\left(\frac{4xyr^2 - (x^2+y^2)(1+r^4)}{2(1-r^4)}\right) dr \tag{2.57}$$

and hence:

$$
\begin{aligned}
\mathcal{K}^{-1}f(x) &= \int_0^1 \sqrt{\frac{r^2}{\pi(1-r^4)}} \int_{-\infty}^{\infty} \exp\left(\frac{4xyr^2 - (x^2+y^2)(1+r^4)}{2(1-r^4)}\right) f(y)dr \\
&= \int_0^1 \sqrt{\frac{r^2}{\pi(1-r^4)}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{1+r^4}{1-r^4}\left[x^2+y^2-\frac{4r^2}{1+r^4}xy\right]\right) f(y)dr \\
&= \int_0^1 \sqrt{\frac{r^2}{\pi(1-r^4)}} \exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2\right) \\
&\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{1+r^4}{1-r^4}\left(y-\frac{2r^2}{1+r^4}x\right)^2\right) f(y)dydr \\
&= \int_0^1 \sqrt{\frac{r^2}{\pi(1-r^4)}} \exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2\right) \\
&\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{1+r^4}{1-r^4}y^2\right) f\left(y+\frac{2r^2}{1+r^4}x\right) dydr \quad\quad (2.58)
\end{aligned}
$$

So we obtain:

$$
\begin{aligned}
&|\mathcal{K}^{-1}f(x)| \\
&\leq \frac{\|f\|_\infty}{\sqrt{\pi}} \cdot \int_0^1 \sqrt{\frac{r^2}{\pi(1-r^4)}} \exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{1+r^4}{1-r^4}y^2\right) dydr \\
&= \sqrt{2}\cdot\|f\|_\infty \cdot \int_0^1 \sqrt{\frac{r^2}{\pi(1+r^4)}} \exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2\right) dr \\
&= \sqrt{2}\cdot\|f\|_\infty \cdot e^{\frac{x^2}{2}} \cdot \int_0^1 \sqrt{\frac{r^2}{\pi(1+r^4)}} \exp\left(-\frac{1}{1+r^4}x^2\right) dr \\
&\leq \sqrt{2}\cdot\|f\|_\infty \cdot e^{\frac{x^2}{2}} \cdot \int_0^1 \sqrt{\frac{1}{\pi(1+r^4)}} \exp\left(-\frac{1}{1+r^4}x^2\right) dr \quad\quad (2.59)
\end{aligned}
$$

First, we notice the following majorization:

$$
|\mathcal{K}^{-1}f(x)| \leq \sqrt{\frac{2}{\pi}} \cdot \|f\|_\infty \cdot e^{\frac{x^2}{2}} \quad\quad (2.60)
$$

From this, we deduce that $\mathcal{K}^{-1}f$ is controlled by $\|f\|_\infty$ on any compact. Then we do the following changes of variables in the last integral:

$$
\begin{aligned}
s &= \frac{1}{1+r^4} & (2.61) \\
t &= sx^2 & (2.62)
\end{aligned}
$$

and these yield:

$$
\begin{aligned}
|\mathcal{K}^{-1}f(x)| &\leq \frac{\sqrt{2}}{4}\cdot\|f\|_\infty\cdot e^{\frac{x^2}{2}}\cdot\int_{1/2}^1 [s(1-s)]^{-3/4}\,e^{-sx^2}ds \\
&= \frac{\sqrt{2}}{4}\cdot\|f\|_\infty\cdot xe^{\frac{x^2}{2}}\cdot\int_{\frac{x^2}{2}}^{x^2}\left[t(x^2-t)\right]^{-3/4}e^{-t}dt \qquad (2.63)
\end{aligned}
$$

From this one may already see that $\mathcal{K}f$ is continuous and bounded on any compact. Now we cut this last integral into two parts:

$$
\int_{\frac{x^2}{2}}^{x^2}\cdots = \int_{\frac{x^2}{2}}^{\frac{3x^2}{4}}\cdots + \int_{\frac{3x^2}{4}}^{x^2}\cdots \qquad (2.64)
$$

For the first integral, we note that on the domain of integration, there is:

$$
t(x^2-t) \geq \frac{3}{16}x^4 \qquad (2.65)
$$

so

$$
\int_{\frac{x^2}{2}}^{\frac{3x^2}{4}}\left[t(x^2-t)\right]^{-3/4}e^{-t}dt \leq \left(\frac{3}{16}\right)^{-\frac{3}{4}}x^{-3}\cdot\int_{\frac{x^2}{2}}^{\frac{3x^2}{4}}e^{-t}dt \leq \left(\frac{3}{16}\right)^{-\frac{3}{4}}x^{-3}e^{-\frac{x^2}{2}} \quad (2.66)
$$

and for the second one:

$$
\begin{aligned}
\int_{\frac{3x^2}{4}}^{x^2}\left[t(x^2-t)\right]^{-3/4}e^{-t}dt &\leq \left(\frac{3}{4}\right)^{-\frac{3}{4}}x^{-3/2}e^{-\frac{3}{4}x^2}\cdot\int_{\frac{3x^2}{4}}^{x^2}(x^2-t)^{-3/4}dt \\
&= \left(\frac{3}{4}\right)^{-\frac{3}{4}}x^{-1}e^{-\frac{3}{4}x^2}\int_{3/4}^1(1-\tau)^{-\frac{3}{4}}d\tau \qquad (2.67)
\end{aligned}
$$

and this last integral is finite (its value is $4/\sqrt{2}$). Overall, we obtain a majorization of the form:

$$
|\mathcal{K}^{-1}f(x)| \leq \left[C_1 x^{-3}e^{-\frac{x^2}{2}} + C_2 x^{-1}e^{-\frac{3}{4}x^2}\right]xe^{\frac{x^2}{2}}\|f\|_\infty \qquad (2.68)
$$

Now we have on one hand, because of (2.60):

$$
\sup_{|x|\leq 1}|\mathcal{K}^{-1}f(x)| \leq \sqrt{\frac{2e}{\pi}}\cdot\|f\|_\infty \qquad (2.69)
$$

and on the other hand, because of (2.68):

$$
\sup_{|x|\geq 1}|\mathcal{K}^{-1}f(x)| \leq \left(\sup_{|x|\geq 1}(C_1 x^{-2} + C_2 xe^{-\frac{1}{4}x})\right)\cdot\|f\|_\infty \qquad (2.70)
$$

Also because of (2.68):

$$\sup_{|x|\geq 1}\left(|x^2|\cdot|\mathcal{K}^{-1}f(x)|\right)\leq\left(\sup_{|x|\geq 1}(C_1+C_2x^3e^{-\frac{1}{4}x^2})\right)\cdot\|f\|_\infty \qquad (2.71)$$

ie:

$$\|X^2\mathcal{K}^{-1}f\|_\infty\leq C\cdot\|f\|_\infty \qquad (2.72)$$

The proof is then finished as in the L^p case. $\qquad\square$

We note that we could not get a control of the type: $\|\partial_i u\|_\infty\leq C\cdot\|\mathcal{K}^{1/2}u\|_\infty$. This is because $\|\partial_i u\|_\infty\leq C\cdot\|\sqrt{-\Delta}u\|_\infty$ does not hold in general. However, such an inequality does hold for the Hölder spaces Λ^γ, $\gamma\in\mathbb{R}_+-\mathbb{N}$, cf [67]. We shall therefore prove the:

Theorem 2.2.3.2. *For $\gamma\in\mathbb{R}_+-\mathbb{N}$, there exists a universal constant C such that the following estimates hold:*

$$\|\partial_i u\|_{\Lambda^\gamma}\leq\|\sqrt{-\Delta}u\|_{\Lambda^\gamma}\quad\leq\quad C\|\mathcal{K}^{1/2}u\|_{\Lambda^\gamma} \qquad (2.73)$$

$$\|X_i u\|_{\Lambda^\gamma}\quad\leq\quad C\|\mathcal{K}^{1/2}u\|_{\Lambda^\gamma} \qquad (2.74)$$

Proof. We do the proof for $0<\gamma<1$. Once again, we work with the operator $\widetilde{\mathcal{K}}$. Let $0<\epsilon<1$; we want to control the difference:

$$(x+\epsilon)^2\widetilde{\mathcal{K}}^{-1}f(x+\epsilon)-x^2\widetilde{\mathcal{K}}^{-1}f(x)$$
$$=\quad\left((x+\epsilon)^2-x^2\right)\widetilde{\mathcal{K}}^{-1}f(x+\epsilon)+x^2\left(\widetilde{\mathcal{K}}^{-1}f(x+\epsilon)-\widetilde{\mathcal{K}}^{-1}f(x)\right)\quad (2.75)$$

The first term of the sum on the right hand side simply is: $(2x\epsilon+\epsilon^2)\widetilde{\mathcal{K}}^{-1}f(x+\epsilon)$; using the previous theorem one easily controls this by $C\epsilon\|f\|_\infty$. For the second term, write:

$$\widetilde{\mathcal{K}}^{-1}f(x+\epsilon)-\widetilde{\mathcal{K}}^{-1}f(x)$$
$$=\int_0^1\sqrt{\frac{1}{\pi(1-r^4)}}\left[\exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}(x+\epsilon)^2\right)-\exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2\right)\right]$$
$$\times\int_{-\infty}^\infty\exp\left(-\frac{1}{2}\frac{1+r^4}{1-r^4}y^2\right)f\left(y+\frac{2r^2}{1+r^4}x\right)dydr$$
$$-\int_0^1\sqrt{\frac{1}{\pi(1-r^4)}}\exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2\right)$$
$$\times\int_{-\infty}^\infty\exp\left(-\frac{1}{2}\frac{1+r^4}{1-r^4}y^2\right)\left[f\left(y+\frac{2r^2}{1+r^4}(x+\epsilon)\right)-f\left(y+\frac{2r^2}{1+r^4}x\right)\right]dydr$$
$$(2.76)$$

To control the first integral, one may use the following simple calculus lemma:

Lemma 2.2.3.1. *Let $0 < \eta < 1$. Then one has a uniform control of the type:*

$$\left| \exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}(x+\epsilon)^2 \right) - \exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2 \right) \right|$$

$$\leq \; C(\eta)\frac{1-r^4}{1+r^4}\epsilon \exp\left(-\frac{1-\eta}{2}\frac{1-r^4}{1+r^4}x^2 \right) \tag{2.77}$$

and therefore the first integral is controled by:

$$C\epsilon\|f\|_\infty \int_0^1 \sqrt{\frac{1}{1-r^4}\frac{1-r^4}{1+r^4}} \exp\left(-\frac{1-\eta}{2}\frac{1-r^4}{1+r^4}x^2 \right) \int_{-\infty}^\infty \exp\left(-\frac{1}{2}\frac{1+r^4}{1-r^4}y^2 \right) dy dr$$

$$= \; C\epsilon\|f\|_\infty \int_0^1 \frac{(1-r^4)^{3/2}}{(1+r^4)^2} \exp\left(-\frac{1-\eta}{2}\frac{1-r^4}{1+r^4}x^2 \right) dr$$

$$\leq \; \epsilon\|f\|_\infty \left(C_1(\eta)x^{-2} + C_2(\eta)e^{-\frac{1-\eta}{4}x^2}x^2 \right) \tag{2.78}$$

by methods similar to those of the proof of the previous theorem, noting that: $\frac{(1-r^4)^{3/2}}{(1+r^4)^2} \leq \frac{1}{1+r^4}$. It is possible, although non-necessary, to optimize the above in η. Then, the second integral is controled by:

$$C\epsilon^\gamma\|f\|_{\Lambda^\gamma} \int_0^1 \sqrt{\frac{1}{\pi(1-r^4)}} \left(\frac{2r^2}{1+r^4}\right)^\gamma \exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2 \right) \int_{-\infty}^\infty \exp\left(-\frac{1}{2}\frac{1+r^4}{1-r^4}y^2 \right) dy dr$$

$$\leq C\epsilon^\gamma\|f\|_{\Lambda^\gamma} \int_0^1 \sqrt{\frac{1}{(1-r^4)}} \exp\left(-\frac{1}{2}\frac{1-r^4}{1+r^4}x^2 \right) dr \tag{2.79}$$

because $\frac{2r^2}{1+r^4} \leq 1$. Now the last integral on the right hand side has already been encountered and majorized in the proof of the previous theorem, so we are done as, putting the parts together, we have proved:

$$\left| (x+\epsilon)^2\widetilde{\mathcal{K}}^{-1}f(x+\epsilon) - x^2\widetilde{\mathcal{K}}^{-1}f(x) \right| \leq C\epsilon^\gamma\|f\|_{\Lambda^\gamma} \tag{2.80}$$

which means that:

$$\|X^2\widetilde{\mathcal{K}}^{-1}f\|_{\Lambda^\gamma} \leq \|f\|_{\Lambda^\gamma} \tag{2.81}$$

and we finish the proof by the usual method. $\qquad\qquad\square$

We also relate the $\mathcal{S}_{\infty,s}$ spaces to the $\mathcal{S}_{p,s}$ spaces in the following way:

Proposition 2.2.3.1 (Some Sobolev type injections)**.** *The following injections hold:*

- $\mathcal{S}_{p,s} \hookrightarrow C_b^0$ *for* $s > \frac{N}{p}$

- $\mathcal{S}_{\infty,s} \hookrightarrow L^p$ *for* $s > \frac{N}{2p}$

Proof. For the first injection, simply observe that $\mathcal{S}_{p,s} \hookrightarrow W_{p,s}$. For the second one, write:

$$\|f\|_{L^p}^p \leq \int \frac{dx}{(1+|x|)^s} \cdot \left[\sup(1+|x|)^{\frac{s}{p}}|f(x)|\right]^p \tag{2.82}$$

and notice that the integral if finite iff $s > N$ and that according to theorem 2.2.3.2:

$$\sup(1+|x|)^{\frac{s}{p}}|f(x)| \leq C \cdot \|\mathcal{K}^{\frac{s}{2p}}f\|_\infty = C\|f\|_{\mathcal{S}_{\infty,\frac{s}{2p}}} \tag{2.83}$$

<div align="right">□</div>

Now since $C_b^0 = \mathcal{S}_{\infty,0}$ and $L^p = \mathcal{S}_{p,0}$ we immediately obtain the:

Corollary 2.2.3.1. *For any $\epsilon > 0$, the following injections hold:*

- $\mathcal{S}_{p,s+\frac{N}{p}+\epsilon} \hookrightarrow \mathcal{S}_{\infty,s}$

- $\mathcal{S}_{\infty,s+\frac{N}{2p}+\epsilon} \hookrightarrow \mathcal{S}_{p,s}$

We may now deduce the:

Proposition 2.2.3.2 (Dual space for $\mathcal{S}_{\infty,s}$). *For any $\epsilon > 0$, the following injection holds:*

$$\mathcal{S}'_{\infty,s} \hookrightarrow \mathcal{S}_{\infty,-s-N-\epsilon} \tag{2.84}$$

Proof. We apply the first inclusion of the above corollary twice in a row and obtain:

$$\mathcal{S}'_{\infty,s} \hookrightarrow \mathcal{S}'_{p,s+\frac{N}{p}+\epsilon} = \mathcal{S}_{p^*,-s-\frac{N}{p}-\epsilon} \hookrightarrow \mathcal{S}_{\infty,-s-N(\frac{1}{p}+\frac{1}{p^*})-\epsilon'} \tag{2.85}$$

<div align="right">□</div>

2.3   The topologies on $\mathcal{S}$ and $\mathcal{S}'$

The results in the previous paragraphs allow us to understand how the topology on $\mathcal{S}$ is defined. More precisely, there is the:

Theorem 2.3.0.3 (Equivalence of some topologies on $\mathcal{S}$). *Any one of the following families of semi norms equips $\mathcal{S}$ with the same Fréchet space topology:*

- *for some fixed $1 \leq p \leq \infty$: $\|\cdot\|_{\alpha,\beta}^{(p)}$, where α, β describe $\mathbb{N}^N$;*

- *for some fixed $1 < p \leq \infty$: $\|\cdot\|_{\mathcal{S}_{p,s}}$, where s describes $\mathbb{N}$ (or $\mathbb{Z}$);*

- *for some fixed $1 < p \leq \infty$: $\|\cdot\|_{\mathcal{T}_{p,s}}$, where s describes $\mathbb{N}$ (or $\mathbb{Z}$).*

It will be useful to note that for any $1 < p \leq \infty$:

$$\mathcal{S} = \bigcap_{s \in \mathbb{R}} \mathcal{S}_{p,s} \tag{2.86}$$

and because of theorems 2.2.2.1 and 2.2.3.2:

$$\mathcal{S}' = \bigcup_{s \in \mathbb{R}} \mathcal{S}_{p,s} \tag{2.87}$$

The topology on $\mathcal{S}'$ is then obtained as a strong dual-Fréchet space topology.

In the final paragraphs of this chapter, we list a number of useful properties of the topology described in theorem 2.3.0.3. In the rest of this document we will always consider that $\mathcal{S}$ is equipped with this topology and that $\mathcal{S}'$ is equipped with the associated strong dual space topology.

2.3.1 Properties of the topologies on $\mathcal{S}$ and $\mathcal{S}'$

The results in this sections are consequences of more general theorems from the theory of topological vector spaces; we refer to this appendix A for the relevant definitions and results.

First, for some $f \in L^2$, recall that there is a decomposition:

$$\mathcal{K}^{-s} f(x) = \sum \frac{c_n(f)}{\lambda_n^s} f_n(x) \tag{2.88}$$

and therefore:

Proposition 2.3.1.1. *For $s > N\frac{N}{2}$, $\mathcal{K}^-s$ is a Hilbert-Schmidt operator, and:*

$$\|\mathcal{K}^{-s}\|_{HS} = \left[\sum_{n \in \mathbb{N}^N} \frac{1}{\lambda_n^{2s}} \right]^{\frac{1}{2}} \tag{2.89}$$

ie the inclusion $\mathcal{S}_{2,k} \hookrightarrow \mathcal{S}_{2,k+s}$ is Hilbert-Schmidt (nuclear) for any k.

We deduce from this and theorem A.4.4.2 that:

Theorem 2.3.1.1. *$\mathcal{S}$ equipped with its standard Fréchet space topology is a nuclear topological vector space.*

Also, because of theorem A.4.4.3 there is the:

Theorem 2.3.1.2. *$\mathcal{S}'$ equipped with its stong dual topology is a nuclear space.*

There also is the:

Theorem 2.3.1.3. $\mathcal{S}$ *is a Montel space.*

Proof. $\mathcal{S}$ is a nuclear Fréchet space.   □

and therefore:

Theorem 2.3.1.4. $\mathcal{S}'$ *is a Montel space.*

Proof. $\mathcal{S}'$ is the strong dual of a Montel space.   □

Let us also note that $\mathcal{S}$ is barelled as a Fréchet space and that $\mathcal{S}'$ is barelled as a Montel space.

2.3.2 $\mathcal{S}'$-valued paths

In this section we mention those results of topological nature which will provide us with an understanding of the structure of regular $\mathcal{S}'$-valued paths. We refer to appendix A and to the references therein for details.

More precisely we would like to understand the structure of continuous paths and of paths with bounded variation (BV for short) taking values in $\mathcal{S}'$. First, we note that since $\mathcal{S}(\mathbb{R}^N)$ is a Montel space, every weakly convergent sequence in $\mathcal{S}'(\mathbb{R}^N)$ also is strongly convergent. Moreover, we recall that a function from a metric space to a topological space is continuous if and only if it is sequentially continuous. Therefore the notion of a continuous path on $\mathcal{S}'(\mathbb{R}^N)$ will not depend on which topology we pick, and we introduce the following:

Definition 2.3.2.1 (Continuous $\mathcal{S}'$-valued paths)**.** *The path:*

$$T : [0,1] \longrightarrow \mathcal{S}'(\mathbb{R}^N) \tag{2.90}$$

is said to be continuous if and only if for all $\phi \in \mathcal{S}(\mathbb{R}^N)$, $t \mapsto \langle T(t), \phi \rangle$ is a function in $C\big([0,1], \mathbb{R}^N\big)$.

A similar definition holds for a bounded variation path.

We now turn on a representation of bounded variation paths on $\mathcal{S}'(\mathbb{R}^N)$ which will be convenient to our purpose. Let T be a bounded variation path.

Theorem 2.3.2.1 (Grothendieck representation of BV paths)**.** *Consider $T \in BV([0,1], \mathcal{S}')$. Then, there exist sequences $(\lambda_n) \in l^1$, (V_n) bounded in BV, and (F_n) an equicontinuous family of $\mathcal{S}'(\mathbb{R}^N)$ depending only on T such that $\forall \phi \in \mathcal{S}(\mathbb{R}^N)$:*

$$\langle T(t), \phi \rangle = \sum_{n=1}^{\infty} \lambda_n \cdot V_n(t) \cdot \langle F_n, \phi \rangle \tag{2.91}$$

An analogue result holds for continuous paths.

Proof. We consider the following linear mapping:

$$l = \begin{array}{ccc} \mathcal{S}'(\mathbb{R}^N) & \to & BV \\ \phi & \mapsto & (t \mapsto \langle T(t), \phi \rangle) \end{array} \tag{2.92}$$

Now let $(\phi_n) \in \mathcal{S}(\mathbb{R}^N)^{\mathbb{N}}$ and $\phi \in \mathcal{S}(\mathbb{R}^N)$ such that $\phi_n \xrightarrow{\mathcal{S}} \phi$ and there exists $g \in BV$ such that $l(\phi) \xrightarrow{BV} g$. Then since BV convergence implies pointwise convergence it is immediately checked that $g = l(\phi)$, hence the graph of l is closed. Now since $\mathcal{S}(\mathbb{R}^N)$ is a Fréchet space and BV is a Banach space we can use a version of the closed graph theorem to conclude that l is continuous. Finally, since $\mathcal{S}(\mathbb{R}^N)$ is a nuclear space l is a nuclear mapping. Therefore we finish the proof by using the Grothendieck theorem A.4.3.1. $\qquad\square$

The representation in the previous theorem will be useful because of the following fact:

Proposition 2.3.2.1. *Consider some $p > 1$. Any equicontinuous set $\mathcal{E} \subset \mathcal{S}(\mathbb{R}^N)$ is included in one of the $\mathcal{S}_{p,-\nu}$ for some $\nu > 0$.*

Proof. The equicontinuity of $\mathcal{E}$ in $\mathcal{S}$ means that the following set:

$$\left\{ \phi \in \mathcal{S}(\mathbb{R}^N) | \forall T \in \mathcal{E}, |\langle T, \phi \rangle| < \epsilon \right\} \tag{2.93}$$

is a neighbourhood of 0. Therefore it contains a ball for one of the seminorms $\| \cdot \|_{\mathcal{S}_{k,2}}$, ie there exists $k \in \mathbb{Z}$ and $\eta > 0$ such that:

$$\|\phi\|_{\mathcal{S}_{k,2}} < \eta \Rightarrow \forall T \in \mathcal{E} \, |\langle T, \phi \rangle| < \epsilon \tag{2.94}$$

We now introduce:

$$g_n := \frac{\eta}{2\lambda_n^k} \cdot f_n \tag{2.95}$$

Clearly:

$$\|g_n\|_{\mathcal{S}_{k,2}} = \frac{\eta}{2} < \eta \tag{2.96}$$

and if $T \in \mathcal{E}$:

$$\left\langle T, \tilde{f}_n \right\rangle = \frac{\eta}{2\lambda_n^k} \cdot c_n(T) \tag{2.97}$$

so we eventually get:

$$|c_n(T)| < \frac{2\epsilon}{\eta} \cdot \lambda_n^k \tag{2.98}$$

and therefore:

$$\|T\|_{\mathcal{S}_{k,2}}^2 < \frac{2\epsilon}{\eta} \cdot \sum_n \lambda_n^{k-\nu} \tag{2.99}$$

and this is a finite sum for big enough ν, so $\mathcal{E} \subset \mathcal{S}_{2,-\nu}$. The proof is finished by using the equivalence of the Fréchet space topologies on $\mathcal{S}$ and on its dual as was stated in theorem 2.3.0.3. $\qquad\square$

As an immediate consequence of the two previous results, there is the:

Theorem 2.3.2.2. *Let $T \in BV([0,1], \mathcal{S}')$. For every $p \in [1, \infty]$ there exists $s \in \mathbb{R}$ such that $T \in BV([0,1], \mathcal{S}_{p,s})$.*

and once again there is a similar result for continuous paths.

Chapter 3

Lifting of random variables by distributions

The lifting of smooth random variables is a well documented topic, see for example: [81], [25], [75] or [83]. Here we try to improve the usual results by using the interpolation theory to extend them to fractional indices for the distribution space and the Gross-Sobolev space. To do so we follow the ideas in [83] but we extend them in several ways.

First we will consider any tempered distribution instead of just elements in a negative Bessel potential space. We achieve this by working in the spaces $\mathcal{S}_{p,\delta}$; indeed:

$$\mathcal{S} = \bigcap \mathcal{S}_{p,\delta} \tag{3.1}$$

so:

$$\mathcal{S}' = \bigcup \mathcal{S}_{p,\delta} \tag{3.2}$$

while the union of all Bessel potential spaces is strictly included in $\mathcal{S}'$.

Second we pay extra attention to some constants in our majorizations as we will need that much precision further on.

Third, we give results for the lifting of the $\mathcal{S}_{\infty^-,-\delta}$ spaces. The results are more complicated to state and prove but do not require a priori assumptions on the existence of a density for the random variable we consider. This allows us to give results for the lifting of regular enough random variables by regular enough random distributions.

For a primer in interpolation theory, we refer to appendix D and to the references therein.

3.1   Interpolation of Gross-Sobolev spaces

We start by recalling some results for the interpolation of Gross-Sobolev spaces. We copy the first one from [82]:

Theorem 3.1.0.3 (Complex interpolation of Gross-Sobolev spaces). *We consider $p_1, p_2 > 1$, $s_1, s_2 \in \mathbb{R}$ and $0 < \theta < 1$. Let p and s be such that: $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $s = \theta s_1 + (1 - \theta)s_2$. Then $[\mathbb{D}_{p_1,s_1}, \mathbb{D}_{p_2,s_2}]_\theta = \mathbb{D}_{p,s}$.*

It is interesting to note that this is related to the following result which we copy from [66]:

Theorem 3.1.0.4. *Let $1 < p < \infty$ and $s_1 < s_2 \in \mathbb{R}$; let $\lambda \in [0,1]$ and set:*

$$s = \lambda \cdot s_1 + (1 - \lambda) \cdot s_2 \tag{3.3}$$

Then there exists a universal constant $C(p, s_1, s_2, \lambda)$ such that for every $\phi \in \mathbb{D}$ the following holds:

$$\|X\|_{\mathbb{D}_{p,s}} \leq C \cdot \|X\|_{\mathbb{D}_{p,s_1}}^{\lambda} \cdot \|X\|_{\mathbb{D}_{p,s_2}}^{1-\lambda} \tag{3.4}$$

Proof. This is an immediate consequence of the previous theorem and the definition of an interpolation functor. We also refer to [66] for an explcit proof which does not directly use interpolation theory, although the ideas are the same. □

Theorem 3.1.0.4 has an obvious but very useful corollary:

Corollary 3.1.0.1. *Let p, s_1, s_2 and λ be as in theorem 3.1.0.4 and let (X_n) a sequence of random variables such that (X_n) converges in $\mathbb{D}_{p,s_1}$ and is bounded in $\mathbb{D}_{p,s_2}$. Then for every $s < s_2$ (X_n) converges in $\mathbb{D}_{p,s}$.*

Remark 3.1.0.1. *It is interesting to compare this last result to lemma 1.2.6.2.*

As another consequence of the complex interpolation of Gross-Sobolev spaces we give two extensions of Hölder's theorem which we also take from [82]:

Theorem 3.1.0.5 (Hölder theorem for Gross-Sobolev spaces). *Consider $s > 0$ and $1 < p, q, r < \infty$ such that $1/p = 1/q + 1/r$. Then there exists a universal constant $C(s, q, r)$ such that for every $X, Y \in \mathbb{D}$:*

$$\|X \cdot Y\|_{\mathbb{D}_{p,s}} \leq C \cdot \|X\|_{\mathbb{D}_{q,s}} \cdot \|Y\|_{\mathbb{D}_{r,s}} \tag{3.5}$$

Proof. First we prove the result for $s \in \mathbb{N}$. Indeed $\nabla(X \cdot Y)$ is computed by the Leibniz formula and then the result is just a consequence of the usual Hölder theorem. The general result is then obtained by complex interpolation. □

Theorem 3.1.0.6 (Hölder theorem for Gross-Sobolev distribution spaces)**.** *Let* $1 < p, q, r < \infty$ *such that* $1/p = 1/q + 1/r$ *and* $s > 0$*. Then there exists a universal constant* $C(s, q, r)$ *such that for every* $X, Y \in \mathbb{D}$*:*

$$\|X \cdot Y\|_{\mathbb{D}_{p,-s}} \leq C \cdot \|X\|_{\mathbb{D}_{q,s}} \cdot \|Y\|_{\mathbb{D}_{r,-s}} \tag{3.6}$$

Proof. By the definition of the duality on the Gross-Sobolev spaces, there is on one hand:

$$\|X \cdot Y\|_{\mathbb{D}_{p,-s}} = \sup_{\|\phi\|_{\mathbb{D}_{p^*,s}}=1} \langle X \cdot Y, \phi \rangle \tag{3.7}$$

and on the other hand:

$$\langle X \cdot Y, \phi \rangle = \langle Y, X \cdot \phi \rangle \leq \|Y\|_{\mathbb{D}_{r,-s}} \cdot \|X \cdot \phi\|_{\mathbb{D}_{r^*,s}} \tag{3.8}$$

Now since $1/r^* = 1/p^* + 1/q$, applying the previous theorem there is:

$$\|X \cdot \phi\|_{\mathbb{D}_{r^*,s}} \leq \|X\|_{\mathbb{D}_{q,s}} \cdot \|\phi\|_{\mathbb{D}_{p^*,-s}} \tag{3.9}$$

and therefore:

$$\langle X \cdot Y, \phi \rangle \leq \|X\|_{\mathbb{D}_{q,s}} \cdot \|Y\|_{\mathbb{D}_{r,-s}} \cdot \|\phi\|_{\mathbb{D}_{p^*,-s}} \tag{3.10}$$

so we are done. $\square$

Often we will simply refer to these last two theorems as Hölder's inequqlity; hopefully the context will always clarify which one we are applying.

Going further with the notion of interpolation, it is interesting to consider the action of real interpolation functors on Gross-Sobolev spaces. Therefore following [83] we introduce:

Definition 3.1.0.2 (Real interpolation of Gross-Sobolev spaces)**.** *Let* $p > 1$ *and* $s \in \mathbb{R}$*. We define a family of Banach spaces* $\mathcal{E}_{p,s}$ *as follows:*

- *if* $s \in \mathbb{Z}$ *then* $\mathcal{E}_{p,s} = \mathbb{D}_{p,s}$

- *Otherwise let* $k = [s]$, $\sigma = \{s\}$*. Then* $\mathcal{E}_{p,s} = (\mathbb{D}_{p,k+1}, \mathbb{D}_{p,k})_{1-\sigma,p}$*.*

We specialize the definition of the real interpolation spaces to our case and obtain a caracterization of the space $\mathcal{E}_{p,s}$ and an expression for its norm; we refer to appendix D and the references therein or to [83] for the details. In practice, we will mostly rely on the following result:

Theorem 3.1.0.7 (semigroup caracterization of the space $\mathcal{E}_{p,s}$)**.**

$$\|F\|_{\mathcal{E}_{p,s}}^p = \|F\|_{\mathbb{D}_{p,k}}^p + \int_0^1 t^{-1-p\sigma} \|F - T_t F\|_{\mathbb{D}_{p,k}}^p dt$$

Proof. This is a specialization of a general result on semigroup interpolation to the operator $(Id + \mathcal{L})^{1/2}$. See appendix D for the details. $\qquad\square$

Also as an application of the duality theorem in D we get the:

Theorem 3.1.0.8 (Duality for the $\mathcal{E}_{p,s}$ spaces)**.** *If $1 < p < \infty$ and $s \in \mathbb{R}$ then:*

$$\mathcal{E}^*_{p,s} = \mathcal{E}_{p^*,-s} \tag{3.11}$$

The following result how closely related the complex and real interpolated Gross-Sobolev spaces are. It will also turn out to be very convenient as it will allow us to prove a number of results for whichever family of spaces is more convenient to use and to automatically obtain them for the other family "up to ϵ".

Theorem 3.1.0.9 (inclusions between the real and complex spaces)**.** *For every $p > 1$, $s \in \mathbb{R}$ and $\epsilon > 0$ one has the following continuous injections:*

$$\mathcal{E}_{p,s+\epsilon} \hookrightarrow \mathbb{D}_{p,s} \hookrightarrow \mathcal{E}_{p,s-\epsilon} \tag{3.12}$$

Proof. We could use a general result from semigroup interpolation theory but due to the importance of this theorem for the sake of completeness we provide a full proof following [83].

We note that we only need to prove the first inclusion; indeed the second one then automatically follows by the duality theorem.

If $s \in \mathbb{Z}$ there is nothing to prove. Otherwise let us write: $[s] = k \in \mathbb{Z}$ and $\{s\} = \sigma \in]0, 1[$ and let $X \in \mathbb{D}$. Then, as per the fractionnal power results provided in appendix C there is the following relation, where the RHS is a Bochner integral:

$$(I - \mathcal{L})^{\sigma/2} X = \frac{\sigma}{\Gamma(1 - \sigma)} \cdot \int_0^\infty t^{-1+\sigma} \cdot [X - T_t X] \, dt \tag{3.13}$$

In the remainder of this proof, C will denote some universal constant possibly depending on p, k, σ, ϵ, etc. but not on X. There is:

$$
\begin{aligned}
& \|X\|^p_{\mathbb{D}_{p,s}} \\
= \ & \left\| (I - \mathcal{L})^{\sigma/2} X \right\|^p_{\mathbb{D}_{p,k}} \\
\leq \ & C \cdot \left\{ \int_0^\infty t^{-1+\sigma} \cdot \|X - T_t X\|^p_{\mathbb{D}_{p,k}} \, dt \right\}^p \\
= \ & C \cdot \left\{ \int_0^1 t^{-1+\sigma} \cdot \|X - T_t X\|^p_{\mathbb{D}_{p,k}} \, dt + \int_1^\infty t^{-1+\sigma} \cdot \|X - T_t X\|^p_{\mathbb{D}_{p,k}} \, dt \right\}^p
\end{aligned}
$$

$$\tag{3.14}$$

In the above, the $\int_1^\infty$ integral is dominated by $C \cdot \|X\|_{\mathbb{D}_{p,k}}$ and for the $\int_0^1$ integral we write:

$$
\begin{aligned}
\int_0^1 t^{-1+\sigma} \cdot \|X - T_t X\|_{\mathbb{D}_{p,k}}^p \, dt &= \int_0^1 t^{-\frac{1}{p^*}-\epsilon} \cdot t^{-\frac{1}{p}+\epsilon+\sigma} \cdot \|X - T_t X\|_{\mathbb{D}_{p,k}}^p \, dt \\
&\leq C \cdot \left\{ t^{-(1+(\sigma+\epsilon)p)} \cdot \|X - T_t X\|_{\mathbb{D}_{p,k}}^p \, dt \right\}^{\frac{1}{p}}
\end{aligned}
\tag{3.15}
$$

by Hölder's theorem. We finish the proof by injecting these two majorizations in (3.14) and using theorem 3.1.0.7. $\qquad \square$

Now let us introduce the following quantities: for $F \in L^p$ we set:

- $f(0) = F$

- $f(t) = \frac{1}{t} \cdot \int_0^t T_\tau F d\tau$ if $t > 0$

Since $(I - \mathcal{L})^{1/2}$ commutes to any T_τ it is easily verified that the $f(t)$ are smooth for $t > 0$ and:

Proposition 3.1.0.2. *If $F \in \mathbb{D}_{p,s}$ then:*

- $\|f(t)\|_{\mathbb{D}_{p,s}} \leq \|F\|_{\mathbb{D}_{p,s}}$

- $\|f(t) - F\|_{\mathbb{D}_{p,s}} \to 0$

We will need the following estimates on f:

Lemma 3.1.0.1. *Let $F \in \mathcal{E}_{p,s}$ for some $p > 1$ and $s > 0$. We note $k = [s]$ and $\sigma = \{s\}$. Then there exist constants C_i depending only on p and s such that:*

$$
\int_0^1 t^{-1+p(1-\sigma)} \|f(t)\|_{\mathbb{D}_{p,k}}^p dt \leq C_1 \|F\|_{\mathcal{E}_{p,s}}^p
\tag{3.16}
$$

$$
\int_0^1 t^{-1+p(1-\sigma)} \|f'(t)\|_{\mathbb{D}_{p,k}}^p dt \leq C_2 \|F\|_{\mathcal{E}_{p,s}}^p
\tag{3.17}
$$

$$
\int_0^1 t^{-1+p(1-\sigma)} \|f(t)\|_{\mathbb{D}_{p,k+1}}^p dt \leq C_3 \|F\|_{\mathcal{E}_{p,s}}^p
\tag{3.18}
$$

Proof. The first point is obvious because f is bounded in $\mathbb{D}_{p,k}$ as was stated in the previous lemma and because of theorem 3.1.0.9.

The proof of the second point will require the following simple result which we borrow from [1]:

Lemma 3.1.0.2. *Let ϕ be a scalar-valued function defined on $\mathbb{R}_+$ and let*

$$\psi(t) = \frac{1}{t} \int_0^t \phi(s) ds \tag{3.19}$$

Then for $1 \le p < \infty$ and for ν, θ such that $\nu + 1/p = \theta < 1$ one has:

$$\int_0^\infty t^{\nu p} |\psi(t)|^p dt \le \left(\frac{1}{1-\theta}\right)^p \cdot \int_0^\infty t^{\nu p} |\phi(t)|^p dt \tag{3.20}$$

Then we notice that for $t > 0$:

$$f'(t) = -\frac{1}{t^2} \cdot \int_0^t T_\tau F d\tau + \frac{1}{t} T_t F = \frac{1}{t}(T_t F - F) - \frac{1}{t^2} \cdot \int_0^t (T_\tau F - F) d\tau \tag{3.21}$$

therefore:

$$
\begin{aligned}
\int_0^1 t^{-1+p(1-\sigma)} \|f'(t)\|_{\mathbb{D}_{p,k}}^p dt \;\le\;& C \cdot \Bigg[\int_0^1 t^{-1-p\sigma} \|T_t F - F\|_{\mathbb{D}_{p,k}}^p dt \\
&+\; \int_0^1 t^{-1-p-p\sigma} \left\| \int_0^t (T_\tau F - F) d\tau \right\|_{\mathbb{D}_{p,k}}^p dt \Bigg]
\end{aligned} \tag{3.22}
$$

The first term in this sum is controlled by using the semigroup characterization in theorem 3.1.0.7. For the second term we apply lemma 3.1.0.2 with $\nu = -1/p - \sigma$, hence $1 - \theta = 1 + \sigma$ and we obtain:

$$\int_0^1 t^{-1-p\sigma} \left\| \int_0^t \frac{T_\tau F - F}{t} d\tau \right\|_{\mathbb{D}_{p,k}}^p dt \le \frac{1}{(1+\sigma)^p} \int_0^1 t^{-1-p\sigma} \|T_t F - F\|_{\mathbb{D}_{p,k}}^p dt \tag{3.23}$$

so then again the semigroup caracterization controls the second term as was desired.

For the third point one notices that:

$$(I + \mathcal{L})^{1/2} f(t) = \frac{1}{t} \int_0^t (I + \mathcal{L})_\tau^{1/2} F d\tau = -\frac{1}{t} \int_0^1 T'\tau F d\tau = -\frac{1}{t}(T_t F - F) \tag{3.24}$$

therefore

$$\|f(t)\|_{\mathbb{D}_{p,k+1}} = \left\| \frac{1}{t}(T_t F - F) \right\|_{\mathbb{D}_{p,k+1}} \tag{3.25}$$

and as previously the semigroup characterization concludes. $\qquad\square$

We also need estimates on $1/f(t)$:

Lemma 3.1.0.3. *Let $F \in \mathbb{D}_{\infty^-,k}$ for some $k \in \mathbb{N}$. If $F > 0$ a.s. and $1/F \in L^{\infty^-}$ then for any $p > 1$ and any $p' > p$ there exists $q > 1$ depending only on p, p' and k such that the following estimate holds:*

$$\sup_{0 \leq t \leq 1} \left\| \frac{1}{f(t)} \right\|_{\mathbb{D}_{p,k}} \leq \left[\left(\frac{1}{1 - e^{-1}} \right)^{k+1} \cdot \left\| \frac{1}{F^{k+1}} \right\|_{L^{p'}} + \frac{1}{1 - e^{-1}} \cdot \left\| \frac{1}{F} \right\|_{L^p} \right] \cdot \|F\|_{\mathbb{D}_{q,k}}^k \tag{3.26}$$

Proof. First, applying the Malliavin calculus version of Faa di Bruno's formula, we see that:

$$\nabla^k \frac{1}{f(t)} = \frac{1}{f(t)^{k+1}} \cdot \Pi \tag{3.27}$$

where Π is a linear combination of the $\nabla^{i_1} f(t) \otimes \cdots \otimes \nabla^{i_n} f(t)$ with $i_1 + \cdots + i_n = k$. Then, successive applications of Hölder's inequality and Meyer's inequalities lead to:

$$\left\| \frac{1}{f(t)} \right\|_{\mathbb{D}_{p,k}} \leq \left\| \frac{1}{f(t)} \right\|_{L^p} + \left\| \frac{1}{f(t)^{k+1}} \right\|_{L^{p'}} \cdot \|F\|_{\mathbb{D}_{q,k}}^k \tag{3.28}$$

We may take k as the exponent because every term in Π is of order at most k in terms of tensor products and by applying Hölder's inequality. A possibile value for q is: $1/p = 1/p' + k/q$. Now we only need to control $\|1/f(t)^{k+1}\|_{L^{p'}}$. To do this we notice that we define a family of markovian kernels (and of contractions) on the L^p spaces by setting:

$$G(t)F = \frac{t}{1 - e^{-t}} f(t) = \frac{1}{1 - e^{-t}} \cdot \int_0^t T_\tau F d\tau \tag{3.29}$$

and applying Jensen's inequality leads to:

$$\frac{1}{G(t)F} \leq G(t) \left[\frac{1}{F} \right] \tag{3.30}$$

Returning to f one gets:

$$\frac{1}{f(t)} \leq \frac{t}{(1 - e^{-t}) \cdot G(t)F} \leq \frac{t}{1 - e^{-t}} \cdot G(t) \left[\frac{1}{F} \right] \leq \frac{1}{1 - e^{-1}} \cdot G(t) \left[\frac{1}{F} \right] \tag{3.31}$$

and we conclude since the G_t are contractions. $\qquad \square$

Remark 3.1.0.2. *In the case where $k = 0$ the polynomial Π in the above proof is 1 and therefore the result improves to:*

$$\sup_{0 \leq t \leq 1} \left\| \frac{1}{f(t)} \right\|_{L^p} \leq \frac{1}{1 - e^{-1}} \cdot \left\| \frac{1}{F} \right\|_{L^p} \tag{3.32}$$

We may now state the following:

Theorem 3.1.0.10. *Let $s > 0$ and $F \in \mathcal{E}_{\infty^-,s}$ such that $F > 0$ a.s. and $1/F \in L^{\infty^-}$. Then for every $s' < s$ $1/F \in \mathcal{E}_{\infty^-,s'}$ and for any $p' > p$ there exists $q(p,p',s) > 1$ and a constant $C(p,p',s)$ such that the following estimate holds:*

$$\left\| \frac{1}{F} \right\|_{\mathcal{E}_{p,s'}} \leq C \cdot \left[\left\| \frac{1}{F^{2(k+1)}} \right\|_{L^{p'}}^p + \left\| \frac{1}{F} \right\|_{L^{p'}}^p \right] \cdot \|F\|_{\mathcal{E}_{q,s}}^{kp+\frac{p}{p'}} \tag{3.33}$$

where $k = [s]$.

Once the theorem is proved the following is an immediate consequence of theorem 3.1.0.9:

Corollary 3.1.0.2. *A similar statement holds for the spaces $\mathbb{D}_{\infty^-,s}$*

Proof. With the notation of the theorem, we set $k = [s]$ and $\sigma = \{s\}$. Let us consider $s' < s$; if s is an integer the result is included in lemma 3.1.0.3 so we ay suppose $[s'] = k$ as well and we set $\sigma' = \{s'\}$. We now introduce $g(t) = 1/f(t)$ for $t \in [0,1]$ and we extend g to a smooth function with comapct support. Of course $g(0) = 1/F$ so by the definition of the real interpolation functor, to obtain a majorization of $\|1/F\|_{\mathcal{E}_{p,s'}}$ it is enough to control the two following terms:

$$\int_0^1 t^{\nu p} \|g(t)\|_{\mathbb{D}_{p,k+1}}^p dt$$

and

$$\int_0^1 t^{\nu p} \|g'(t)\|_{\mathbb{D}_{p,k}}^p dt$$

where ν is such that $p^{-1} + \nu = 1 - \sigma'$. In the sequel C_i will always be a universal constant, possibly depending on some specified parameters.

We start with the second term. Since $g'(t) = -f'(t)/f(t)^2$ by replacing, applying Hölder's inequality and lemma 3.1.0.3 we obtain, for any $\delta > 0$:

$$\int_0^1 t^{\nu p} \|g'(t)\|_{\mathbb{D}_{p,k}}^p dt = \int_0^1 t^{\nu p} \left\| \frac{f'(t)}{f(t)^2} \right\|_{\mathbb{D}_{p,k}}^p dt$$

$$\leq C_1(p,k,\delta) \cdot \int_0^1 t^{\nu p} \left\| \frac{1}{f(t)^2} \right\|_{\mathbb{D}_{p(1+\delta)/\delta,k}}^p \cdot \|f'(t)\|_{\mathbb{D}_{p(1+\delta),k}}^p dt$$

$$\leq C_2(p,k,\delta) \cdot \left[\left\| \frac{1}{F^{2(k+1)}} \right\|_{L^{p\frac{1+\delta'}{\delta'}}}^p + \left\| \frac{1}{F} \right\|_{L^{p\frac{1+\delta}{\delta}}}^p \right] \cdot \|F\|_{\mathbb{D}_{q,k}}^{Nk} \cdot \int_0^1 t^{\nu p} \|f'(t)\|_{\mathbb{D}_{p(1+\delta),k}}^p dt \tag{3.34}$$

for any $0 < \delta' < \delta$ and for some q as in lemma 3.1.0.3. Now by Jensen's inequality:

$$
\begin{aligned}
\int_0^1 t^{\nu p} \|f'(t)\|^p_{\mathbb{D}_{p(1+\delta),k}} dt &\leq \left[\int_0^1 t^{\nu p(1+\delta)} \|f'(t)\|^{p(1+\delta)}_{\mathbb{D}_{p(1+\delta),k}} dt \right]^{\frac{1}{1+\delta}} \\
&\leq \|F\|^{\frac{1}{1+\delta}}_{\mathcal{E}_{p(1+\delta),s}}
\end{aligned}
\tag{3.35}
$$

since $\nu p(1+\delta) = -1 + p(1+\delta)(1-\sigma)$. Finally:

$$
\int_0^1 t^{\nu p} \|g(t)\|^p_{\mathbb{D}_{p,k+1}} dt \leq C_3 \cdot \left[\left\| \frac{1}{F^{2(k+1)}} \right\|^p_{L^p \frac{1+\delta'}{\delta'}} + \left\| \frac{1}{F} \right\|^p_{L^p \frac{1+\delta}{\delta}} \right] \cdot \|F\|^{kp}_{\mathbb{D}_{q,k}} \cdot \|F\|^{\frac{1}{1+\delta}}_{\mathcal{E}_{p(1+\delta),s}}
\tag{3.36}
$$

where $C_3 = C_3(p,k,\delta,\delta',\nu)$.

The first term is dealt with in a similar way. First, by Meyer's inequalities:

$$
\begin{aligned}
\|g(t)\|_{\mathbb{D}_{p,k+1}} &\leq C_4(p,k) \left[\|g(t)\|_{\mathbb{D}_{p,k}} + \|\nabla g(t)\|_{\mathbb{D}_{p,k}} \right] \\
&= C_4(p,k) \left[\|g(t)\|_{\mathbb{D}_{p,k}} + \left\| \frac{\nabla f(t)}{f(t)^2} \right\|_{\mathbb{D}_{p,k}} \right]
\end{aligned}
\tag{3.37}
$$

We consider the two terms in the sum separately. The first term is dealt with by using lemma 3.1.0.3. For the second term Hölder's inequality leads to:

$$
\begin{aligned}
\left\| \frac{\nabla f(t)}{f(t)^2} \right\|_{\mathbb{D}_{p,k}} &\leq C_5(p,k) \cdot \left\| \frac{1}{f(t)^2} \right\|_{\mathbb{D}_{p\frac{1+\delta}{\delta},k}} \cdot \|\nabla f(t)\|_{\mathbb{D}_{p(1+\delta),k}} \\
&\leq \left\| \frac{1}{f(t)^2} \right\|_{\mathbb{D}_{p\frac{1+\delta}{\delta},k}} \cdot \|f(t)\|_{\mathbb{D}_{p(1+\delta),k+1}}
\end{aligned}
\tag{3.38}
$$

and we proceed as previously to obtain, for some $C_6(p,k,\delta,\delta',\nu)$:

$$
\int_0^1 t^{\nu p} \|g(t)\|^p_{\mathbb{D}_{p,k+1}} dt \leq C_6 \cdot \left[\left\| \frac{1}{F^{2(k+1)}} \right\|^p_{L^p \frac{1+\delta'}{\delta'}} + \left\| \frac{1}{F} \right\|^p_{L^p \frac{1+\delta}{\delta}} \right] \cdot \|F\|^{kp}_{\mathbb{D}_{q,k}} \cdot \|F\|^{\frac{1}{1+\delta}}_{\mathcal{E}_{p(1+\delta),s}}
\tag{3.39}
$$

We then finish the proof of the theorem by putting the terms together and setting $p' = p(1+\delta)$. $\qquad\square$

3.2   Integration by parts

First we introduce a crucial quantity:

Definition 3.2.0.3 (Malliavin matrix). *Let $F \in \mathbb{D}_{\infty-,1}(\mathbb{R}^{\mathbb{N}})$. We introduce the Malliavin matrix of F as the random matrix with coefficients:*

$$
\Sigma_{ij} = (\nabla F_i, \nabla F_j)_H
\tag{3.40}
$$

We note that the Malliavin matrix is a.s. symmetric and positive as a Gram matrix. Also, if $F \in \mathbb{D}_{\infty-,s}(\mathbb{R}^{\mathbb{N}})$ then $\Sigma_{ij} \in \mathbb{D}_{\infty-,s-1}(\mathbb{R})$ and by Hölder's theorem:

$$\|\Sigma_{ij}\|_{\mathbb{D}_{p,s-1}} \leq C(p,s) \cdot \|F\|_{\mathbb{D}_{2Np,s}}^{2N} \tag{3.41}$$

We are interested in the invertibility of Σ and in the regularity of its potential inverse. We will need another standard definition:

Definition 3.2.0.4 (non-degeneracy)**.** *The random variable $F \in \mathbb{D}_{\infty-,1}(\mathbb{R}^{\mathbb{N}})$ is non-degenerate if:*

$$\frac{1}{\det \Sigma} \in L^{\infty-} \tag{3.42}$$

If a random variable is non-negenerate then its Malliavin matrix Σ is invertible and we will often note γ for this inverse. The following is a straightforward consequence of theorem 3.1.0.10 and Hölder's inequality but will be crucial to our purpose:

Proposition 3.2.0.3. *We consider a non-degenerate $F \in \mathbb{D}_{\infty-,s}(\mathbb{R}^{\mathbb{N}})$ for some $s > 1$ and Σ its Malliavin matrix. Then for any $1 \leq s' < s$,*

$$\frac{1}{\det \Sigma} \in \mathbb{D}_{\infty-,s'-1}(\mathbb{R}) \tag{3.43}$$

Also, for all $p' > p$ there exists $q > 1$ and $\alpha > 0$ such that the following estimate holds:

$$\left\| \frac{1}{det\Sigma} \right\|_{\mathbb{D}_{p,s'-1}} \leq C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^{p'}}^{p} + \left\| \frac{1}{det\Sigma} \right\|_{L^{p'}}^{p} \right] \cdot \|F\|_{\mathbb{D}_{q,s}}^{\alpha} \tag{3.44}$$

where $k = [s]$ and C is a universal constant depending only on p, p', s, s', q.

Proof. Applying theorem 3.1.0.10 yields:

$$\left\| \frac{1}{det\Sigma} \right\|_{\mathbb{D}_{p,s'-1}} \leq C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^{p'}}^{p} + \left\| \frac{1}{det\Sigma} \right\|_{L^{p'}}^{p} \right] \cdot \|det\Sigma\|_{\mathbb{D}_{q,s-1}}^{kp+\frac{p}{p'}} \tag{3.45}$$

Then since $det\Sigma$ is a polynomial in the ∇F_i we finish the proof with Hölder's inequality. $\square$

Remark 3.2.0.3. *A similar result holds for real interpolation spaces.*

We now recall the following classical result:

Theorem 3.2.0.11 (Integration by parts on the Wiener space). *Consider the random variables $F \in \mathbb{D}_{\infty^-,2}(\mathbb{R}^{\mathbb{N}})$ and $G \in \mathbb{D}_{\infty^-,1}(\mathbb{R}^{\mathbb{N}})$. Suppose that F is nondegenerate. Then for any $i = 1,\ldots,N$ the following quantity is well defined:*

$$l_i(G,F) = \delta\left(G \cdot \sum_{j=1}^{N} \gamma_{ij} \nabla F_j\right) \tag{3.46}$$

and if $\phi \in \mathcal{C}^1(\mathbb{R}^{\mathbb{N}})$ then:

$$E[(\partial_i \phi) \circ F \cdot G] = E[\phi \circ F \cdot l_i(G,F)] \tag{3.47}$$

Proof. First we prove that $l_i(G,F)$ is well defined in L^p. Indeed, noting C for a generic constant:

$$\|l_i(G,F)\|_{L^p} \leq \left\|G \cdot \sum_{i=1}^{N} \gamma_{ij} \nabla F_j\right\|_{\mathbb{D}_{p,1}} \leq C \cdot \|G\|_{\mathbb{D}_{q,1}} \cdot \left\|\sum_{i=1}^{N} \gamma_{ij} \nabla F_j\right\|_{\mathbb{D}_{r,1}} \tag{3.48}$$

Then we use the fact that:

$$\gamma = \frac{1}{det\Sigma}{}^{t}(comat\Sigma) \tag{3.49}$$

therefore by Hölder's inequality:

$$\left\|\sum_{i=1}^{N} \gamma_{ij} \nabla F_j\right\|_{\mathbb{D}_{r,1}} \leq C \cdot \left\|\frac{1}{det\Sigma}\right\|_{\mathbb{D}_{r_1,1}} \cdot \left\|\sum_{i=1}^{N} {}^{t}(comat\Sigma)_{ij} \nabla F_j\right\|_{\mathbb{D}_{r_2,1}} \tag{3.50}$$

Now the term with the determinant may be controlled in some $\mathbb{D}_{r,2}$ by proposition 3.2.0.3, and the other terms may be controlled in some $\mathbb{D}_{r,2}$ because the ${}^{t}(comat\Sigma)_{ij} \nabla F_j$ are multivariate polynomials in ∇F.

Now we prove the actual integration by parts formula. By the Malliavin calculus chain rule, $\phi(F) \in \mathbb{D}_{p,1}$ and:

$$\nabla(\phi(F)) = \sum_{i=1}^{N} (\partial_i \phi)(F) \cdot \nabla F_i \tag{3.51}$$

therfore:

$$(\nabla(\phi(F)), \nabla F_j)_H = \sum_{i=1}^{N} (\partial_i \phi)(F) \cdot \sigma_{ij} \tag{3.52}$$

and since γ is the inverse of the Malliavin matrix, elementary linear algebra yields:

$$(\partial_i \phi)(F) = \sum_{j=1}^{N} \gamma_{ij} (\nabla(\phi(F)), \nabla F_j)_H \tag{3.53}$$

The result is finally obtained by integrating by parts. $\qquad\square$

It is straightforward but noteworthy that the operators l_i are linear in the first variable but not in the second. This is the main difficulty in the study of their regularity. Some results are well known for Gross-Sobolev spaces with an integer order of derivability. Here, following the logic of Watanabe we state the more precise:

Theorem 3.2.0.12 (Fractional regularity of the operator l_i)**.** *For any $p > 1$ and $s > 0$, for any $p' > p$ and $s' > s$ there exists $q, r > 1$, $\theta > 0$ and a universal constant depending only on the above parameters such that the following estimate holds for any $G \in \mathbb{D}_{\infty^-, s+1}$ and non-degenerate $F \in \mathbb{D}_{\infty^-, s'+2}$:*

$$\|l_i(G,F)\|_{\mathbb{D}_{p,s}} \le C \cdot \|G\|_{\mathbb{D}_{p',s+1}} \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right] \cdot \|F\|_{\mathbb{D}_{r,s'+2}}^\theta \quad (3.54)$$

where $k = [s]$.

Remark 3.2.0.4. *In the case where s is an integer, the usual, simpler result lets one take s instead of s' in the majorization of the above type.*

Proof. Use the exact same technique than to prove the existence of $l_i(G,F)$ in the previous theorem, only with higher order spaces. $\qquad\square$

In the sequel, it will be convenient to use x_i the operator mapping $\mathcal{S}$ to itself, defined by $(x_i f)(x) = x_i f(x)$. Here is a continuity result analogue to the previous one which is longer to state than to prove:

Lemma 3.2.0.4. *For any random variables F and G and for any measurable function f one has:*

$$E[(x_i f)(F) \cdot G] = E[f(F) \cdot m_i(G,F)]$$

with

$$m_i(G,F) = F_i \cdot G$$

and for any $p' > p > 1$ if $1/p = 1/p' + 1/q$:

$$\|m_i(G,F)\|_{L^{p'}} \le \|G\|_{L^q} \|F\|_{L^{p'}}$$

Now let us consider a family of functionals μ_j for $j = 1, \ldots, m$ such that every μ_i is either one of the l_i or one of the m_i, $i = 1, \ldots, N$. We define

$$\lambda(G,F) = \mu_m(G, \mu_{m-1}(G, \mu_{m-2}(G, \ldots, F)\ldots)) \quad (3.55)$$

for regular enough G and non-degenerate F. Then, combining theorem 3.2.0.12 and lemma 3.2.0.4 with a straightorward recurrence one obtains:

Proposition 3.2.0.4. *Let $p' > p > 1$ and $s' > s > 0$. Suppose that λ is as above, with at most n of the μ_j being an l_i. Then there exists $q, r > 1$, $\theta > 0$ and a universal constant C such that the following estimate holds for any regular enough F, G, F being nondegenerate:*

$$\|\lambda(G, F)\|_{\mathbb{D}_{p,s}} \leq C \cdot \|G\|_{\mathbb{D}_{p',s+m}} \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right]^m \cdot \|F\|_{\mathbb{D}_{r,s'+m+1}}^{\theta}$$

$$(3.56)$$

where $k = [s]$ and σ is the Malliavin matrix of F.

We now state a first result related to the interpolation of Gross-Sobolev spaces and the spaces introduced in section 2:

Lemma 3.2.0.5. *Suppose that $F \in \mathbb{D}_{\infty^-,1}$ has a bounded density p_F. Then for every $0 \leq \rho < \rho' \leq 1$ and $1 < p < p' < \infty$ there exists a universal constant $C(\rho, \rho', p, p')$ such that for any $\phi \in \mathcal{S}$ the following estimate holds:*

$$\|\phi \circ F\|_{\mathbb{D}_{p,\rho}} \leq C \cdot \|p_F\|_{\infty} \cdot (1 + \|\nabla F\|_{L^q}) \cdot \|\phi\|_{\mathcal{S}_{p',\rho'}} \qquad (3.57)$$

Proof. First we note that:

$$\|\phi \circ F\|_{L^p} \leq \|p_F\|_{\infty} \cdot \|\phi\|_{L^p} \qquad (3.58)$$

and for q such that $1/p' + 1/q = 1/p$, by Meyer's inequalities and Hölder's theorem:

$$
\begin{aligned}
\|\phi \circ F\|_{\mathbb{D}_{p,1}} &\leq c(p) \cdot \left[\|\phi \circ F\|_{L^p} + \sum_{i=1}^{N} \|\partial_i \phi \circ F \cdot \nabla F_i\|_{L^p} \right] \\
&\leq c(p) \cdot \left[\|\phi \circ F\|_{L^p} + \sum_{i=1}^{N} \|\partial_i \phi \circ F\|_{L^{p'}} \cdot \|\nabla F\|_{L^q} \right] \\
&\leq c(p) \cdot [\|p_F\|_{\infty} \cdot \|\phi\|_{L^p} + \|p_F\|_{\infty} \cdot \|\nabla \phi\|_{L^{p'}} \cdot \|\nabla F\|_{L^q}] \\
&\leq c(p) \|p_F\|_{\infty} (1 + \|\nabla F\|_{L^q}) \cdot |\phi|_{\mathcal{S}_{p,1}} \qquad (3.59)
\end{aligned}
$$

For the last inequality we used theorem 2.2.2.1. We see that we have proved our result for the cases $\rho = 0$ or $\rho' = 1$. We now suppose $0 < \rho < \rho' < 1$. We may then choose $p < p_1 < p'$ and $\rho < \rho_1 < \rho_2 < \rho'$ such that:

$$1 - \rho_2 - \frac{1}{p'} = 1 - \rho_1 - \frac{1}{p_1}$$

Then by real interpolation one obtains:

$$\|\phi \circ F\|_{\mathcal{E}_{p_1,\rho_1}} \leq c(p) \|p_F\|_{\infty} (1 + \|\nabla F\|_{L^q}) \cdot \|\phi\|_{\mathcal{T}_{p_1,\rho_1}} \qquad (3.60)$$

The proof is then finished by combining the above with theorems 3.1.0.9 and 2.2.2.4. □

We now prove a result which will let us lift non-degenerate random variables by tempered distributions with minimal conditions:

Theorem 3.2.0.13. *Let $F \in \mathbb{D}_{\infty^-, 1+\delta}$ for some $\delta > 0$ be a non-degenerate $\mathbb{R}^N$-valued random variable admitting a bounded density. Then for every $\delta' < \delta$ and $1 < p < p' < \infty$ there exists a universal constant C and some $q, r > 1$ and $\theta > 0$ such that the following estimate holds for every $\phi \in \mathcal{S}(\mathbb{R}^N)$:*

$$\|\phi \circ F\|_{\mathbb{D}_{p,-\delta}} \leq C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right]^{2m} \cdot \|F\|_{\mathbb{D}_{r,1+\delta'}}^\theta \cdot \|p_F\|_\infty \cdot \|\phi\|_{\mathcal{S}_{p',-\delta'}}$$

$$(3.61)$$

Hence for any $T \in \mathcal{S}_{p',-\delta'}$, $T \circ F$ may be defined in $\mathbb{D}_{p,-\delta}$ and the above estimate still holds.

Proof. We still note Σ for the Malliavin matrix of F. Also, we note k and σ for $[\delta]$ and $\{\delta\}$ respectively. If δ is an integer, the result is classical, so we only prove the case where $\sigma \neq 0$. In this case δ' may be chosen such that $[\delta'] = k$. If k is odd, we write $k = 2l - 1$ and $\phi = \mathcal{K}^l \mathcal{K}^{-l} \phi$. We then expand $\mathcal{K}^l$:

$$\mathcal{K}^l = \sum A_{i_1} \circ \cdots \circ A_{i_{2m}}$$

where, for each term in the sum, $m \leq l$ and each A_{i_j} is either a partial derivative ∂_k or a multiplication by a coordinate x_k. An application of proposition 3.2.0.4 to a regular enough ψ leads to:

$$E[(A_{i_1} \circ \cdots \circ A_{i_{2m}} \psi) \circ F \cdot G] = E[\psi \circ F \lambda(G, F)] \qquad (3.62)$$

for some λ as in proposition 3.2.0.4, and if $1/\pi + 1/p = 1$ and $1 < \pi' < \pi$ then for some q, r, θ one has:

$$\|\lambda(G, F)\|_{\mathbb{D}_{\pi',\delta'-k-1}} \leq C \cdot \|G\|_{\mathbb{D}_{\pi,\delta}} \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right]^{2m} \cdot \|F\|_{\mathbb{D}_{r,1+\delta'}}^\theta$$

$$(3.63)$$

Now we recall that:

$$\|(A_{i_1} \circ \cdots \circ A_{i_{2m}} \phi) \circ F\|_{\mathbb{D}_{p,-\delta}} = \sup\left\{ E[(A_{i_1} \circ \cdots \circ A_{i_{2m}} \phi) \circ F \cdot G], \|G\|_{\mathbb{D}_{\pi,\delta}} \leq 1 \right\}$$
$$= \sup\left\{ E[\psi \circ F \cdot \lambda(G, F)], \|G\|_{\mathbb{D}_{\pi,\delta}} \leq 1 \right\} \qquad (3.64)$$

so by taking linear combinations and replacing ψ by $\mathcal{K}^{-l}\phi$ we obtain:

$$\|\phi \circ F\|_{\mathbb{D}_{p,-\delta}} \leq C \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^q + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^q \right]^{2m} \|F\|_{\mathbb{D}_{r,1+\delta'}}^\theta \|(\mathcal{K}^{-l}\phi) \circ F\|_{\mathbb{D}_{p',k+1-\delta'}}$$

$$(3.65)$$

and since $0 < k + 1 - \delta' < 1$ lemma 3.2.0.5 applies to the last factor of the right hand side so the proof is finished in this case since the $\|\nabla F\|_{L^q}$ term may be "included" in the $\|F\|_{\mathbb{D}_{r,1+\delta'}}^{\theta}$ term by Meyer's inequalities.

If k is even, let us write $k = 2l$. As in the previous case, we expand $\mathcal{K}^{l+1}$ so we may write $Id = \mathcal{K}^{l+1}\mathcal{K}^{-(l+1)}$ as a linear combination of factors of the following type:

$$
\begin{aligned}
A_{i_1} \circ \cdots \circ A_{i_{2m+2}} \circ \mathcal{K}^{-(l+1)} &= A_{i_1} \circ \cdots \circ A_{i_{2m+1}} \circ \mathcal{K}^{-(l+1)} \circ A_{2m+2} \\
&+ A_{i_1} \circ \cdots \circ A_{i_{2m+1}} \circ [\mathcal{K}^{-(l+1)}, A_{2m+2}] \quad (3.66)
\end{aligned}
$$

where $[,]$ is the commutator of two linear operators. We may then proceed exactly as in the odd case, noting that the operators $\mathcal{K}^{-(l+1)} \circ A$ and $[\mathcal{K}^{-(l+1)}, A]$ map L^p to itself when A is a partial derivative or a multiplication by a coordinate, see [70] or [71]. $\qquad\square$

In order to simplify the above theorem, we recall that regular enough random variables always admit a bounded (and even continuous) density. We copy the following result from [66]:

Theorem 3.2.0.14. *Suppose that $p > N$ and that $F \in \mathbb{D}_{8Np,2}$ is non-degenerate. Then F admits a continuous and bounded density p_f and:*

$$
\|p_F\|_{\infty} \leq C \cdot \|\nabla F\|_{\mathbb{D}_{8Np,1}} \cdot \left\| \frac{1}{\Sigma} \right\|_{L^{4p}}^{2N} \quad (3.67)
$$

Combining these two theorems we obtain:

Theorem 3.2.0.15. *Let $F \in \mathbb{D}_{\infty^-,1+\delta}$ for some $\delta > 1$ be a non-degenerate $\mathbb{R}^N$-valued random variable. Then for every $\delta' < \delta$ and $1 < p < p' < \infty$ there exists a universal constant C and some $q, r > 1$ and $\theta > 0$ such that the following estimate holds for every $\phi \in \mathcal{S}(\mathbb{R}^N)$:*

$$
\|\phi \circ F\|_{\mathbb{D}_{p,-\delta}} \leq C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^q}^{q} + \left\| \frac{1}{det\Sigma} \right\|_{L^q}^{q} \right]^{2m} \cdot \|F\|_{\mathbb{D}_{r,1+\delta'}}^{\theta} \cdot \|\phi\|_{\mathcal{S}_{p',-\delta'}} \quad (3.68)
$$

Hence for any $T \in \mathcal{S}_{p',-\delta'}$, $T \circ F$ may be defined in $\mathbb{D}_{p,-\delta}$ and the above estimate still holds.

Remark 3.2.0.5. *Once again there are similar results for the spaces $\mathcal{E}$ and $\mathcal{T}$.*

To conclude this section, we give another result for the existence of densities. This is more precise that theorem 3.2.0.14 but it only works in dimension 1 as of now.

We start with a lemma which is a slightly more general version of a result in [4]:

Lemma 3.2.0.6. *Suppose that $X \in \mathbb{D}_{\infty^-,1}(\mathbb{R})$ has a cumulative distribution function which is γ-Hölder. Then, for any x, for any $p > 1$ and for any $\alpha < \frac{\gamma}{p}$:*

$$1_{X>x} \in \mathbb{D}_{p,\alpha} \tag{3.69}$$

and, for some r and a constant $C(p,\alpha) > 0$, the following inequality then holds:

$$\|1_{X>x}\|_{\mathbb{D}_{p,\alpha}} \leq C(p,\alpha)\left(1 + \|\nabla X\|_{L^r}\right)\|F_X\|_{\Lambda^\gamma} \tag{3.70}$$

Proof. Being inspired by [4] we use the K interpolation method and we write:

$$H = H_\epsilon + H - H_\epsilon \tag{3.71}$$

where H is the Heaviside function and H_ϵ is the piecewise linear function with values 0 on $]-\infty, -\epsilon]$ and 1 on $[0, \infty[$. We note that H_ϵ is $1/\epsilon$-Lipschitz, so $H_\epsilon(X - x) \in \mathbb{D}_{p,1}$ and we apply the K-method to the decomposition:

$$1_{X>x} = H_\epsilon(X - x) + 1_{X>x} - H_\epsilon(X - x) \tag{3.72}$$

First, by the chain rule and Hölder's inequality one has, for any $q > 1$:

$$
\begin{aligned}
\|\nabla H_\epsilon(X - x)\|_{L^p} &\leq \epsilon^{-1}\|\nabla X\|_{L^{\frac{pq}{q-1}}}\|1_{x-\epsilon<X<x}\|_{L^{pq}} \\
&\leq \epsilon^{-1}\|\nabla X\|_{L^{\frac{pq}{q-1}}}\left[P(X<x) - P(X<x-\epsilon)\right]^{\frac{1}{pq}} \\
&\leq \|F_X\|_{\Lambda^\gamma}\|\nabla X\|_{L^{\frac{pq}{q-1}}}\epsilon^{-1+\frac{\gamma}{pq}}
\end{aligned}
\tag{3.73}
$$

and similarly:

$$\|1_{X>x} - H_\epsilon(X - x)\|_{L^p} = P(x - \epsilon < X < x)^{\frac{1}{p}} \leq \|F_X\|_{\Lambda^\gamma}\epsilon^{\frac{\gamma}{p}} \tag{3.74}$$

so overall:

$$K_\epsilon \leq \epsilon\|H_\epsilon(X - x)\|_{\mathbb{D}_{p,1}} + \|1_{X>x} - H_\epsilon(X - x)\|_{L^p} \leq \|F_X\|_{\Lambda^\gamma}\epsilon^{\frac{\gamma}{pq}} \tag{3.75}$$

where $K_\epsilon = K(\epsilon, 1_{X>x}$ following the notation of appendix D for the K method of interpolation and finally:

$$\|1_{X>x}\|_{\mathbb{D}_{p,\alpha}}^p = \int_0^1 \left[\epsilon^{-\alpha}K_\epsilon\right]^p \frac{d\epsilon}{\epsilon} \leq C\int_0^1 \epsilon^{-\alpha p - 1 + \frac{\gamma}{q}}d\epsilon \tag{3.76}$$

and clearly this last integral converges if and only if $\alpha < \frac{\gamma}{pq}$. Since one may choose q arbitrarily close to 1 the proof is complete. $\qquad\square$

Our lemma now allows us to prove the following:

Theorem 3.2.0.16. *Suppose that $X \in \mathbb{D}_{\infty^-, 2-\frac{\gamma}{p}+\epsilon}$ is non-degenerate with a γ-Hölder cumulative distribution function. Then X has a continuous and bounded density given by the formula:*

$$p_X(x) = E\left[\left(\nabla 1_{X>x}, \frac{\nabla X}{|\nabla X|^2}\right)_H\right] = E\left[1_{X>x}\delta\left(\frac{\nabla X}{|\nabla X|^2}\right)\right] \qquad (3.77)$$

Proof. The identity between the two expectations is just an integration by parts. We only need to prove that these quantities are well defined. First, we may apply lemma 3.2.0.6 to see that

$$\nabla 1_{X>x} \in \mathbb{D}_{p,-(1-\alpha)} \qquad (3.78)$$

for any $\alpha < \frac{\gamma}{p}$. Second, by Hölder's inequality and theorem 3.1.0.10 we see that

$$\frac{\nabla X}{|\nabla X|^2} \in \mathbb{D}_{p^*, 1-\frac{\gamma}{p}+\frac{\epsilon}{2}} \qquad (3.79)$$

so once again a version of Hölder's inequality proves that the first expectation is well defined. The second one is dealt with in a similar way and allows one to see that the density is continuous and bounded. $\qquad \square$

Remark 3.2.0.6. *If $\gamma > 1$ then the random variable X has a distribution function in C_b^1, hence we know that it has a continuous and bounded density without any hypothesis on the Malliavin regularity of X. Conversely, if $\gamma = 0$, it is better to use theorem 3.2.0.14.*

3.3 Some results for the $\mathcal{S}_{\infty,s}$ spaces

The results in the previous section were only for $1 < p < \infty$. Here we provide analogue results for $p = \infty$. We start with a result similar to lemma 3.2.0.5:

Lemma 3.3.0.7. *Take $F \in \mathbb{D}_{\infty^-,1}$. Then for $0 < \rho < 1$ and $\epsilon > 0$ there exists a universal constant $C(\rho, p, \epsilon)$ such that for any $\phi \in \mathcal{S}_{\infty,\rho+\frac{1}{p}+\epsilon}$, the following estimate holds:*

$$\|\phi(F)\|_{\mathbb{D}_{p,\rho}} \leq C\left(1 + \|\nabla F\|_{L^p}\right) \|f\|_{\mathcal{S}_{\infty,\rho+\frac{1}{p}+\epsilon}} \qquad (3.80)$$

Before we prove this results let us make a few comments. First, this is easier to apply than lemma 3.2.0.5 because the random variable F is not required to have a density a priori. However, this is interesting only if $\rho + 1/p < 1$: otherwise ϕ is Lipschitz and one may just apply the usual chain rule. Finally, since one may take p as big as one wants on the RHS and since the $\|\cdot\|_{\mathbb{D}_{p,k}}$ increase in p, one gets the obvious:

Corollary 3.3.0.3. *Take $F \in \mathbb{D}_{\infty^-,1}$. Then for $0 < \rho < 1$ and $\epsilon > 0$ there exists some $q(\epsilon) > p$ and a universal constant $C(\rho, \epsilon)$ such that for any $\phi \in \mathcal{S}_{\infty,\rho+\epsilon}$, the following estimate holds:*

$$\|\phi(F)\|_{\mathbb{D}_{p,\rho}} \leq C \left(1 + \|\nabla F\|_{L^{q(\epsilon)}}\right) \|f\|_{\mathcal{S}_{\infty,\rho+\epsilon}} \tag{3.81}$$

Now we turn to the proof of the lemma:

Proof. First let ψ be any smooth function. Then:

$$\|\psi(F)\|_{L^p} \leq \|\psi\|_{\infty} \tag{3.82}$$

and

$$\|\psi(F)\|_{\mathbb{D}_{p,1}} \leq C \left[\|\psi\|_{\infty} + \|\nabla F\|_{L^p}\|\psi\|_{C^1}\right] \leq C \left(1 + \|\nabla F\|_{L^p}\right) \|\psi\|_{\mathcal{S}_{\infty,1+\epsilon}} \tag{3.83}$$

because $\|\psi\|_{C^1} \leq \|\psi\|_{\Lambda^{1+\epsilon}}$ and because of theorem 2.2.3.2. Now let $\psi(t)$ be a smooth real interpolation path for $\|\phi\|$ and let $f(t) = \psi(t)(F)$. Then, setting $\nu = -\frac{1}{p} + 1 - \rho$:

$$\int_0^1 t^{p\nu}\|f(t)\|_{\mathbb{D}_{p,1}}dt \leq C \left[1 + \|\nabla F\|_{L^p}\right]^p \int_0^1 t^{-1+(1-\rho)p}\|\psi(t)\|_{\mathcal{S}_{\infty,1+\epsilon}}^p dt \tag{3.84}$$

and therefore:

$$\left[\int_0^1 t^{p\nu}\|f(t)\|_{\mathbb{D}_{p,1}}dt\right]^{\frac{1}{p}} \leq C \left[1 + \|\nabla F\|_{L^p}\right] \sup_{0 \leq t \leq 1} t^{1-(\rho+\frac{1}{p})}\|\psi(t)\|_{\mathcal{S}_{\infty,1+\epsilon}} \tag{3.85}$$

Then we may control $\int_0^1 t^{p\nu}\|\frac{d}{dt}f(t)\|_{L^p}dt$ in a similar way, and by taking infimums real interpolation yields:

$$\|\phi(F)\|_{\mathcal{E}_{p,\rho}} \leq C \left[1 + \|\nabla F\|_{L^p}\right] \cdot \|\phi\|_{\mathcal{T}_{\infty,(1+\epsilon)(\rho+\frac{1}{p})}} \tag{3.86}$$

Then we are done because this is valid for any $\epsilon > 0$ and because of theorems 3.1.0.9 and 2.2.2.4. □

Finally, by simply copying the proof of theorem 3.2.0.13 and using theorem 2.2.3.2 instead of theorem 2.2.2.1 one obtains the following:

Theorem 3.3.0.17. *Let $F \in \mathbb{D}_{\infty^-,1+\delta}$ for some $\delta > 0$ be a non-degenerate $\mathbb{R}^N$-valued random variable. Then for every $\delta' < \delta$ and $1 < p < \infty$ there exists a universal constant C and some $q, r > 1$ and $\theta > 0$ such that the following estimate holds for every $\phi \in \mathcal{S}(\mathbb{R}^N)$:*

$$\|\phi \circ F\|_{\mathbb{D}_{p,-\delta}} \leq C \cdot \left[\left\|\frac{1}{(det\Sigma)^{2(k+1)}}\right\|_{L^q}^q + \left\|\frac{1}{det\Sigma}\right\|_{L^q}^q\right]^{2m} \cdot \|F\|_{\mathbb{D}_{r,1+\delta'}}^{\theta} \cdot \|\phi\|_{\mathcal{S}_{\infty,-\delta'}} \tag{3.87}$$

Hence for any $T \in \mathcal{S}_{p',-\delta'}$, $T \circ F$ may be defined in $\mathbb{D}_{p,-\delta}$ and the above estimate still holds.

Then again, here we do not need to make an assumption on the existence of a density for the random variable X.

3.4 Existence and regularity of densities

Consider some $\delta > 0$. In this section, we suppose that $X \in \mathbb{D}_{\infty^-,1+\delta}(\mathbb{R}^N)$ is a nondegenerate random variable which admits a continuous and bounded density p_X. As an immediate corollary of theorem 3.2.0.13, we see that the following linear application is continuous:

$$\begin{aligned} \mathcal{S}_{p',-\delta'} &\rightarrow \mathbb{D}_{p,-\delta} \\ T &\mapsto T \circ X \end{aligned} \tag{3.88}$$

for any $p' > p$, $\delta' < \delta$. Now let $\phi \in \mathbb{D}'_{p,-\delta} = \mathbb{D}_{p^*,\delta}$. Clearly the following is a continous linear form:

$$\begin{aligned} \mathcal{S}_{p',-\delta'} &\rightarrow \mathbb{R} \\ T &\mapsto \langle T \circ X, \phi \rangle \end{aligned} \tag{3.89}$$

Therefore by duality, there exists some function $p_{X,\phi} \in \mathcal{S}'_{p',-\delta'} = \mathcal{S}_{(p')^*,\delta}$ such that for any $T \in \mathcal{S}_{p',-\delta'}$ the following holds:

$$\langle T \circ X, \phi \rangle = \langle T, p_{X,\phi} \rangle \tag{3.90}$$

Since p' and δ' may approach p and δ infinitesimally, by density one checks that $p_{X,\phi}$ does not depend on the choice of these parameters and that $p_{X,\phi} \in \mathcal{S}_{(p^*)^-,\delta^-}$. By the Sobolev injections, we deduce that $p_{X,\phi} \in C^{(\delta-\frac{N}{p^*})^-}$ if tht exponent is positive. Also, since the Dirac mass $\delta_x \in \mathcal{S}_{p,-\frac{N}{p^*}-\epsilon}$ for any x and any $\epsilon > 0$, one sees that it makes sense, at least for p small enough, to take $T = \delta_x$ in the expression above. Then there comes an explicit expression for $p_{X,\phi}$:

$$p_{X,\phi}(x) = \langle \delta_x \circ X, \phi \rangle \tag{3.91}$$

In the case where T is a proper function, we note that we have the identity:

$$E\left[T(X)\phi\right] = \int T(x)p_{X,\phi}(x)dx \tag{3.92}$$

Identitying with the usual definitions for the density and the conditionnal density of a random variable, we then obtain:

$$p_{X,1} = p_X \tag{3.93}$$

and

$$p_{X,\phi}(x) = p_X(x) \cdot E\left[\phi|X=x\right] \tag{3.94}$$

We sum up our results in the following:

Theorem 3.4.0.18. *Let $\delta > 0$ and let $X \in \mathbb{D}_{\infty^-, 1+\delta}(\mathbb{R}^N)$ be a nondegenerate random variable which admits a continuous and bounded density p_X. Let $p > 1$ and $\phi \in \mathbb{D}_{p,\delta}$. Then there exists a unique function $p_{X,\phi} \in \mathcal{S}_{p^-, \delta^-}$ such that for any $T \in \mathcal{S}_{p^*, -\delta}$:*

$$\langle T \circ X, \phi \rangle = \langle T, p_{X,\phi} \rangle \tag{3.95}$$

Also, if $\delta > \frac{N}{p}$ then $p_{X,\phi} \in C^{(\delta - \frac{N}{p})^-}$. One has the expression:

$$p_{X,\phi}(x) = p_X(x) \cdot E\left[\phi | X = x\right] = \langle \delta_x \circ X, \phi \rangle \tag{3.96}$$

In particular:

$$p_X(x) = p_{X,1}(x) = \langle \delta_x \circ X, 1 \rangle \tag{3.97}$$

and for all $p > 1$ $p_X \in \mathcal{S}_{p,\delta}$, therefore $p_X \in C^{\delta^-}$.

Very similarly, without assuming a priori that X has a continuous density by using theorem 3.2.0.15 we could have proved:

Theorem 3.4.0.19. *Let $\delta > 1$ and let $X \in \mathbb{D}_{\infty^-, 1+\delta}(\mathbb{R}^N)$ be a nondegenerate random variable. Then X admits a continuous and bounded density p_X; actually $p_X \in \mathcal{S}_{p,\delta}$ for all p and $p_X \in C^{s^-}$, therefore all the conclusions of the previous theorem hold.*

Also, in the case where X, ϕ are in $\mathbb{D}$, it is noteworthy that $p_{X,\phi}$, and especially p_X, are in $\mathcal{S}$, which is much stronger than just C^∞. Finally, the following is an immediate consequence of equation (3.94) which will be useful later on:

Proposition 3.4.0.5. *Under the conditions of either one of the two theorems above, p_X and $p_{X,\phi}$ have the same supports.*

3.5 The case of random fields

In this paragraph we study the Malliavin regularity of random variables of the type:

$$f(X) = f(w, X(w)) \tag{3.98}$$

where X is a regular enough random variable and f is a random flow on $\mathbb{R}^N$ in the sense defined by [40], enjoying some regularity both on its paths and in the sense of Malliavin. It is classic to introduce norms such as:

$$\|f\|_{L^p(C_b^0)} = \left\| \sup_{x \in \mathbb{R}} |f(x)| \right\|_{L^p} \tag{3.99}$$

or:

$$\|f\|_{L^p(C_b^k)} = \left\| \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |f^{(k)}(x)| \right\|_{L^p} \tag{3.100}$$

Accordingly, we note $L^p(C_b^k)$ for the corresponding Banach spaces. For short, we will refer to such flows as C_b^k, L^p flows, or as $L^p(C_b^k)$ flows. Furthermore, the C_b^k are separable Banach spaces, so Gross-Sobolev spaces may be built on them, and following Meyer's inequalities we introduce the following norms:

$$\|f\|_{\mathbb{D}_{p,s}(C_b^0)} = \left\| \sup_{x \in \mathbb{R}} |(Id + \mathcal{L})^{\frac{s}{2}} f(x)| \right\|_{L^p} \tag{3.101}$$

and:

$$\|f\|_{\mathbb{D}_{p,s}(C_b^k)} = \left\| \sup_{x \in \mathbb{R}} |(Id + \mathcal{L})^{\frac{s}{2}} f(x)| + \sup_{x \in \mathbb{R}} |(Id + \mathcal{L})^{\frac{s}{2}} f^{(k)}(x)| \right\|_{L^p} \tag{3.102}$$

as well as the corresponding spaces and obvious notation.

Our first result is the following:

Proposition 3.5.0.6 (Malliavin chain rule for random flows)**.** *Let $p, q, r > 1$ such that $1/p + 1/q = 1/r$ and suppose that the following hypotheses hold:*

- $f \in L^p(C_b^1)$

- $f \in \mathbb{D}_{p,1}(C_b^0)$

- $X \in \mathbb{D}_{q,1}(\mathbb{R}^N)$

Then, $f(X) \in \mathbb{D}_{r,1}$ and the following chain rule holds:

$$\nabla(f(X)) = \sum_{k=1}^{N} \partial_k f(X) \nabla X_k + (\nabla f)(X) \tag{3.103}$$

We borrow the idea behind this proof from [53]:

Proof. Let ψ_n be a smooth approximation of the Dirac mass. Then, almost surely, for any x:

$$f(w, X(w)) = \lim_{n \to \infty} f(w, x) \psi_n(X(w) - x) \tag{3.104}$$

Also, by the usual chain rule:

$$\nabla \left[f(w, x) \psi_n(X(w) - x) \right] = \psi_n(X(w) - x)(\nabla f)(w, x)$$
$$- f(w, x) \cdot \sum_{k=1}^{N} \partial_k \psi_n(X(w) - x) \nabla X_i \tag{3.105}$$

and clearly this converges, at least in the almost sure sense, to the right hand side of equation 3.103. Also, by the hypotheses we made and Hölder's inequality, this quantity is bounded in $L^r(C_b^0)$, so the convergence takes place in L^r and we are done. $\square$

Alternatively an intuitive idea is to "separate the variables" between the randomness in the flow and the randomness in the random variable. We sketch another proof of proposition 3.5.0.6 which is in that spirit and is interesting in its own right:

Proof. Let us take inverse Fourier transforms and write:

$$f(w, X(w)) = \int e^{i\xi X(w)} \hat{f}(w, \xi) d\xi \qquad (3.106)$$

Then by using a Hilbert space version of the Fubini theorem (cf [40] for example) and similar arguments as above:

$$\nabla(f(X)) = \sum_{k=1}^{N} \left(\int i\xi_k e^{i\xi X(w)} \hat{f}(w, \xi) d\xi \right) \nabla X_k(w) + \int e^{i\xi X(w)} (\nabla \hat{f})(w, \xi) d\xi$$
$$(3.107)$$

which is indeed the right hand side of equation 3.103 as can be seen by taking inverse Fourier transforms. □

We now turn to an extension of the integration by parts formula. We start with some preliminary computations. Suppose that the hypotheses of proposition 3.5.0.6 hold and that X is non-degenerate. As usual we note σ for its Malliavin matrix and γ for the inverse of σ. Then for any j we may write:

$$(\nabla(f(X)), \nabla X_j) = \sum_{k=1}^{n} \sigma_{kj} \partial_k f(X) + ((\nabla f)(X), \nabla X_j) \qquad (3.108)$$

therefore by straightforward linear algebra:

$$\partial_k f(X) = \sum_{j=1}^{N} \gamma_{kj} (\nabla(f(X)) - (\nabla f)(X), \nabla X_j) \qquad (3.109)$$

Now we define N new random flows by the equations:

$$T_k(f, X) = x \mapsto \sum_{j=1}^{N} \gamma_{kj} ((\nabla f)(x), \nabla X_j) \qquad (3.110)$$

Note that this depends linearly on f, and depends on X linearly through ∇X_j but also non-linearly through γ. It is not difficult to see that $T_k(f, X)$ is an L^p random flow with the same smoothness as $(\nabla f)(x)$ (at least C_b^0 as per our set of hypotheses). Of course if f is deterministic then $T_k(f, X)$ is null.
    Now we are ready to state the following:

Proposition 3.5.0.7 (Flow integration by parts formula). *Under the hypotheses of proposition 3.5.0.6, for any $\phi \in \mathbb{D}_{p^*,1}$:*

$$E\left[\partial_k f(X) \cdot \phi\right] = E\left[f(X) \cdot l_k(X, \phi)\right] - E\left[\phi \cdot T_k(f, X)(X)\right] \tag{3.111}$$

We note that if f is deterministic this boils down to the usual Malliavin integration by parts formula.

We will now be following the same agenda as in the previous paragraphs. Since (from previous paragraphs in this chapter) we already know how to control $l_k(X, \phi)$ we will focus on $T_k(f, X)$ - and more specifically on $T_k(f, X)(X)$. We start with the following:

Lemma 3.5.0.8. *Let $p > 1$, and f and X be such that the following hypotheses hold:*

- f *is a C_b^n flow*

- *There exists $p < r < \infty$ such that for every $k \geq 1$ and every multi-index j such that $|j| + k = n + 1$, $\nabla^k \partial_{j_1,\dots,j_N} f$ exists in L^r as a C_b^0 flow, ie:*

$$f \in \bigcap_{|j|+k=n+1} \mathbb{D}_{r,k}\left(C_b^j(\mathbb{R}^N)\right) \tag{3.112}$$

- $X \in \mathbb{D}_{\infty^-, n}$

Then, $(\nabla f)(X) \in \mathbb{D}_{p,n}(H)$, and for some $\rho > p$ and some universal constant C:

$$
\begin{aligned}
\left\|\nabla^n\left((\nabla f)(X)\right)\right\|_{L^p(H)} &\leq C \cdot \left(\sum_{|j|+k=n+1; k \geq 1} \left\|(\nabla^k \partial_{j_1,\dots,j_N} f)(X)\right\|_{L^r} \right) \cdot \|X\|_{\mathbb{D}_{\rho,n}}^n \\
&\leq C \cdot \left(\sum_{|j|+k=n+1; k \geq 1} \|f\|_{\mathbb{D}_{r,k}\left(C_b^j(\mathbb{R}^N)\right)} \right) \cdot \|X\|_{\mathbb{D}_{\rho,n}}^n \tag{3.113}
\end{aligned}
$$

Proof. A simple recurrence based on proposition 3.5.0.6 yields that $(\nabla f)(X)$ is n times differentiable. The second bullet point in our hypotheses ensure us that for any j, k:

$$\left\| \sup_x \left|(\nabla^k \partial_{j_1,\dots,j_N} f)(x)\right|_{H^{\otimes k}} \right\|_{L^r} < \infty \tag{3.114}$$

so a fortiori the right hand side in our majorization is finite. To obtain the actual inequality, compute $\nabla^n\left((\nabla f)(X)\right)$ with Faa di Bruno's formula and apply Hölder's inequality. $\qquad \square$

We would like to improve this result into a more precise one for the fractional case. We start with a result which is analogue to lemma 3.3.0.7 and corollary 3.3.0.3:

Lemma 3.5.0.9. *Let $X \in \mathbb{D}_{\infty^-, 1}$. Then, for every $0 < \rho < 1$, for every $\alpha \in]1, \infty[$, for every $\epsilon > 0$, for every $p > 1$, there exist $q, r > 1$ and a universal constant C such that for every $f \in \mathbb{D}_{\infty^-, 1}(\mathcal{S})$ the following estimate holds:*

$$\|f(X)\|_{\mathbb{D}_{p,\rho}} \leq C \cdot (1 + \|\nabla X\|_{L^r}) \cdot \left(\|f\|_{L^q(\mathcal{S}_{\infty, \rho + \epsilon})} + \|f\|_{\mathbb{D}_{p,\rho}(W_{\alpha, \frac{N}{\alpha} + \epsilon})} \right) \quad (3.115)$$

Proof. We will use the Sobolev injection:

$$W_{\alpha, \frac{N}{\alpha} + \epsilon} \hookrightarrow C_b^0 \quad (3.116)$$

Now let us write:

$$\|f(X)\|_{L^p} \leq \|f\|_{L^p(C_b^0)} \leq \|f\|_{L^p(W_{\alpha, \frac{N}{\alpha} + \epsilon})} \quad (3.117)$$

and, using the flow version of the chain rule and Hölder's inequality:

$$
\begin{aligned}
\|f(X)\|_{\mathbb{D}_{p,1}} &\leq C \cdot \left(\|f\|_{L^p(C_b^0)} + \|f'\|_{L^q(C_b^0)} \cdot \|\nabla X\|_{L^r} + \|\nabla f\|_{L^p(C_b^0)} \right) \\
&\leq C \cdot \left(\|f\|_{L^p(W_{\alpha, \frac{N}{\alpha} + \epsilon})} + \|f\|_{L^q(\mathcal{S}_{\infty, 1})} \cdot \|\nabla X\|_{L^r} + \|\nabla f\|_{L^p(W_{\alpha, \frac{N}{\alpha} + \epsilon})} \right)
\end{aligned}
$$
$$(3.118)$$

Now since $W_{\alpha, \frac{N}{\alpha} + \epsilon}$ is a UMD Banach space the Meyer inequalities are valid for $W_{\alpha, \frac{N}{\alpha} + \epsilon}$-valued random variables, so we may finish the proof by interpolation as for lemma 3.3.0.7. $\qquad \square$

Now we get back to controlling $\|(\nabla f)(X)\|_{\mathbb{D}_{p,s}}$:

Lemma 3.5.0.10. *Let $p > 1$, $s > 0$, $\epsilon > 0$ and f and X be such that the following hypotheses hold:*

- *There exists $p < q < \infty$ such that:*

$$f \in \bigcap_{k+l=[s]+1} \mathbb{D}_{q,k} \left(\mathcal{S}_{\infty, \{s\} + l + \epsilon} \right) \quad (3.119)$$

- *There exists $1 < \alpha < \infty$ such that:*

$$f \in \bigcap_{k+l=[s]+1} \mathbb{D}_{p, k+\{s\}} \left(W_{\alpha, \frac{N}{\alpha} + \epsilon + l} \right) \quad (3.120)$$

- $X \in \mathbb{D}_{\infty^-, s}$

Then, $(\nabla f)(X) \in \mathbb{D}_{p,s}(H)$ and and for some $\rho > p$ and some universal constant C the following control holds:

$$\|(\nabla f)(X)\|_{\mathbb{D}_{p,s}}$$
$$\leq \quad C \cdot (1 + \|\nabla^{[s]+1}X\|^{[s]+1}) \cdot \sum_{k+l=[s]+1} \left(\|f\|_{\mathbb{D}_{q,k}(\mathcal{S}_{\infty, \{s\}+l+\epsilon})} + \|f\|_{\mathbb{D}_{p,k+\{s\}}(W_{\alpha, \frac{N}{\alpha}+\epsilon+l})} \right)$$

(3.121)

Proof. By Meyer's inequalities:

$$\|(\nabla f)(X)\|_{\mathbb{D}_{p,s}} \leq \left\| (Id + \mathcal{L})^{\frac{\{s\}}{2}} \left(\nabla^{[s]}(\nabla f)(X) \right) \right\|_{L^p}$$

(3.122)

Now $\nabla^{[s]}(\nabla f)(X)$ may be expanded by Faa di Bruno's formula; each of the terms of the expansion is of the form:

$$(\nabla^k \partial_{l_1} \cdots \partial_{l_N} f)(X) \otimes P$$

(3.123)

where $k + |l| = [s] + 1$ and P is a polynomial in the $\nabla_k X_i$ with total degree lower than $[s]$. Therefore, we finish the proof by separating the terms using the triangle inequality and then using Hölder's inequality and the previous lemma on each term in the sum. $\square$

Our lemma provides us with a control for $T_k(f, X)(X)$:

Proposition 3.5.0.8. *Suppose that f and X verify all the hypotheses in the previous lemma. Suppose in addition that X is nondegenerate. Then for some $s' > s$, $\theta > 0$ and for any p_1, p_2, p_3 such that $1/p_1 + 1/p_2 + 1/p_3 < 1/p$:*

$$\|T_k(f, X)(X)\|_{\mathbb{D}_{p,s}} \leq C \quad \cdot \quad \left[\sum_{k+l=[s]+1; k \geq 1} \|f\|_{\mathbb{D}_{q,k}(\mathcal{S}_{\infty, \{s\}+l+\epsilon})} + \|f\|_{\mathbb{D}_{p,k+\{s\}}(W_{\alpha, \frac{N}{\alpha}+\epsilon+l})} \right]$$
$$\cdot \quad \left[\left\| \frac{1}{\det \Sigma} \right\|_{L^{p_2}}^{p_2} + \left\| \frac{1}{\det \Sigma^{2([s]+1)}} \right\|_{L^{p_2}}^{p_2} \right] \cdot \|X\|_{\mathbb{D}_{p_3, 1+s'}}^{\theta}$$

(3.124)

Proof. We still note γ for the inverse of the Malliavin matrix of X. By Hölder's inequality:

$$\|T_k(f, X)(X)\|_{\mathbb{D}_{p,s}} \leq C \cdot \|(\nabla f)(X)\|_{\mathbb{D}_{q,s}} \cdot \left\| \delta \left(\sum_j \gamma_{kj} \nabla X_j \right) \right\|_{\mathbb{D}_{r,s}}$$

(3.125)

First, we have just seen how to control the first term of the product in the right hand side. Then, we have seen how to control the second term in the proof of theorem 3.2.0.11. Therefore by recombining we are done. $\square$

We may now state the main result in this section:

Theorem 3.5.0.20 (Lifting of a random distribution by a non degenerate RV)**.** *Let $X \in \mathbb{D}_{\infty^-,1+\delta}$ for some $\delta > 0$ be a non-degenerate $\mathbb{R}^N$-valued random variable. Then for every $\delta' < \delta$ and $1 < p < \infty$ there exists a universal constant C and some $p_1, p_2, p_3 > p$ and $\theta > 0$ such that the following estimate holds for every $\phi \in \mathcal{S}(\mathbb{D})$:*

$$
\begin{aligned}
&\|\phi \circ X\|_{\mathbb{D}_{p,-\delta}} \\
\leq\ & C \cdot \left[\left\| \frac{1}{(det\Sigma)^{2(k+1)}} \right\|_{L^{p_1}}^{p_1} + \left\| \frac{1}{det\Sigma} \right\|_{L^{p_1}}^{p_1} \right]^{\theta} \cdot \|X\|_{\mathbb{D}_{p_2,1+\delta'}}^{\theta} \\
\times\ & \sum_{k=0}^{[\delta]} \left[\|\phi\|_{\mathbb{D}_{p_3,k}(\mathcal{S}_{\infty,-(\delta'-k)})} + \|\phi\|_{\mathbb{D}_{p_3,k}(W_{\alpha,-(\delta'-k-\frac{N}{\alpha})})} \right]
\end{aligned}
$$

$$\tag{3.126}$$

Hence for any $T \in \mathbb{D}_{r,k}(\mathcal{S}_{p',-\delta'})$, $T \circ X$ may be defined in $\mathbb{D}_{p,-\delta}$ and the above estimate still holds.

Proof. We follow the same strategy as for the proof of theorem 3.3.0.17 so we omit the details. $\qquad\square$

We conclude this section with two simple remarks: first that in practice it will be best to take α as big as possible; second that if we stick to the case where $s \in \mathbb{N}$ it is possible to write much simpler results which do not involve the $W_{\alpha,\frac{N}{\alpha}+\epsilon}$ spaces.

Chapter 4

A weak Itô formula

In the previous chapter, we have stated precise conditions under which a tempered distribution may be lifted by a regular, non-degenerate random variable, the result being a distribution on the Wiener space. Of course, we could make both the distribution and the random variable depend on time and then the result would be a weak process as per the notion of weak measurability which we defined in section 1.2.8. It is then natural to think of proving an Itô formula for distributions and this had first been done in [78]. In this chapter, we will use somehow different techniques based on the precise estimates we obtained in the previous chapter to obtain an Itô formula which may be applied under less demanding conditions than the one stated in [78]. We shall also prove a formula of the Itô-Wentzell type which holds when a regular enough random process is lifted by a tempered-distribution valued semi-martingale. These results will lead to applications which are detailed in this chapter and the next one, and would have remained out of reach with the previously existing Itô formulae.

At various points in this chapter, we will need some results on the topologies of $\mathcal{S}$ and $\mathcal{S}'$ which are quite involved. For these we refer to chapter 2 and appendix A.

4.1 The formula for a general Itô process

4.1.1 Processes with an absolutely continuous drift

In this paragraph W denotes the canonical d-dimensionnal Brownian motion on the Wiener space. We work with the filtration of W. Let X be the N-dimensionnal Itô process such that:

$$
\begin{aligned}
dX_t &= b_t dt + \sigma_t dW_t & (4.1)\\
X_0 &= x & (4.2)
\end{aligned}
$$

where b is an adapted N-dimensionnal process and σ is an adapted process of $N \otimes d$-dimensionnal random matrices. We will note $a = \sigma^t \sigma$.

Also let $t \in [0,1] \to T(t) \in \mathcal{S}'(\mathbb{R}^N)$ be a (deterministic) path with bounded variations. First we need to find tractable conditions under which it is licit to define the composes $T_t \circ X_t$, $\partial_i T_t \circ X_t$ and $\partial_{ij}^2 \circ X_t$ for $t > 0$. Let $1 < p < \infty$. By the Grothendieck representation, for any $1 < p < \infty$ there exists $s' > 0$ such that T is a path with bounded variations in $\mathcal{S}_{p,-s}$, as is explained in theorem 2.3.2.2 - the case where the T_t are actual functions and not distributions in the strict sense is easier to deal with. Then the $\partial_i T_t$ are and the ∂_{ij}^2 are BV paths taking values in $\mathcal{S}_{p,-(s+1)}$ and $\mathcal{S}_{p,-(s+2)}$ respectively. From now on, since this is not a loss of generality, we will restrict our study to that of a path:

$$T_t \in BV \left(\mathcal{S}_{p,-s} \right) \tag{4.3}$$

In particular, $t \mapsto \|T_t\|_{\mathcal{S}_{p,-s}}$ is bounded. Then if T_t is as in (4.3) according to theorem 3.2.0.15, for all the distributions we are interested in to exist we need that $X_t \in \mathbb{D}_{\infty^-, s'+3}$ for some $s' > s$ and that X_t be non-degenerate for $t > 0$. In fact for technical reasons we need to make uniform assumptions on X; more precisely let us suppose that for all $r > 1$:

$$\int_0^t \|b_u\|_{\mathbb{D}_{r,s'+3}}^r du \quad < \quad \infty \tag{4.4}$$

$$\int_0^t \|\sigma_u\|_{\mathbb{D}_{r,s'+3}}^r du \quad < \quad \infty \tag{4.5}$$

$$\sup_{\epsilon \leq u \leq t} \left\| \frac{1}{\Sigma(u)} \right\|_{L^r} \quad < \quad \infty \tag{4.6}$$

for any $0 < \epsilon < 1$, where $\Sigma(u)$ is the Malliavin matrix of X_u. Then we have:

Proposition 4.1.1.1. *Under the three above hypotheses, one has:*

- $\forall u \geq 0,\ X_u \in \mathbb{D}_{\infty^-, s'+3}$

- $\forall r > 0,\ \forall \epsilon > 0,\ \int_\epsilon^t \|X_u\|_{\mathbb{D}_{r,s'+3}}^r ds < \infty$

- $\forall u > 0,\ X_u$ *is non-degenerate.*

In particular, the $T_t \circ X_t$ are well defined in $\mathbb{D}_{p',-s'}$, the $\partial_i T_t \circ X_t$ are well defined in $\mathbb{D}_{p,-(s'+1)}$ and the $\partial_{ij}^2 T_t \circ X_t$ are well defined in $\mathbb{D}_{p,-(s'+2)}$.

Proof. The first two points are easy consequences of the Hölder inequalities. For the third third point, we notice that condition (4.6) implies that for Lebesgue-a.s. $u \in [\epsilon, 1]$, X_u is non-degenerate; then we use that the inverse Malliavin matrix is an

Itô process, hence it has continuous trajectories to deduce that X_u is nondegenerate for every u. Then the existence of the weak processes is a direct application of the results in the previous chapter. $\qquad\square$

Remark 4.1.1.1. *It is noteworthy that the conditions we take ensure us that for every $u > \epsilon$, X_u has a bounded and continuous density.*

Our strategy in order to obtain a weak Itô formula will now be the following: we introduce the classic sequence of mollifiers:

$$\rho_n(x) = \frac{1}{(2\pi n)^{N/2}} \cdot \exp\left(-\frac{|x|^2}{2n}\right) \tag{4.7}$$

and we regularize our distribution valued-path; more precisely we introduce a sequence of functions $T^{(n)} \in \mathcal{S}$ defined as:

$$T^{(n)} = \rho_n * T \tag{4.8}$$

so we are able to apply the (classic, C^2 version of the) Itô formula:

$$
\begin{aligned}
T^{(n)}(t, X_t) - T^{(n)}(\epsilon, X_\epsilon) \ = \ & \sum_i \int_\epsilon^t b_i(u) \cdot \partial_i T^{(n)}(u, X_u) du \\
& + \ \frac{1}{2} \sum_{i,j} \int_\epsilon^t a_{ij}(u) \cdot \partial_{ij} T^{(n)}(u, X_u) du \\
& + \ \int_\epsilon^t T^{(n)}(du, X_u) \\
& + \ \sum_j \int_\epsilon^t \sum_i \left[b_i(u) \cdot \partial_i T^{(n)}(u, X_u) \right] dW_u^j \quad (4.9)
\end{aligned}
$$

and since by design:

$$T^{(n)} \overset{\mathcal{S}_{p,-s}}{\Rightarrow} T \tag{4.10}$$

we will study the convergence of each term as n tends to infinity. Note that we do not start from 0 as the initial condition x is trivially degenrate, hence there is no hope of defining $T_t \circ x$.

We start with the following:

Lemma 4.1.1.1 (Convergence of the LHS).

$$T^{(n)}(t, X_t) - T^{(n)}(\epsilon, X_\epsilon) \overset{\mathbb{D}_{p',-s'}}{\Rightarrow} T(t) \circ X_t - T(\epsilon) \circ X_\epsilon \tag{4.11}$$

Proof. This is an immediate consequence of theorem 3.2.0.15; let us note that here we only used the $\mathbb{D}_{\infty^-, s'}$ regularity of the X_u. $\qquad\square$

Then we prove:

Lemma 4.1.1.2 (Convergence of the first term).

$$\int_\epsilon^t b_i(u) \cdot \partial_i T^{(n)}(u, X_u) du \stackrel{\mathbb{D}_{p',-(s'+1)}}{\longrightarrow} \int_\epsilon^t b_i(u) \cdot \partial_i T_u \circ X_u du \qquad (4.12)$$

Proof. First let us note that theorem 3.2.0.15 allows us to check that the RHS above is well defined as a Bochner integral in $\mathbb{D}_{p',-(s'+1)}$. Since the techniques involved are the same than those we will use to prove the actual convergence we do not write the details.

Now let us turn to the proof of the convergence; by Hölder's theorem, for some $r > 1$ and $p' > p'' > p > 1$:

$$\|b_i(u) \cdot \partial_i T^{(n)}(u, X_u) - b_i(u) \cdot (\partial_i T_u) \circ X_u\|_{\mathbb{D}_{p,-(s+1)}}$$
$$\leq C\|b_i(u)\|_{\mathbb{D}_{r,s+1}}\|\partial_i T^{(n)}(u, X_u) - (\partial_i T_u) \circ X_u\|_{\mathbb{D}_{p'',-(s'+1)}} \quad (4.13)$$

The $\|b_i(u)\|_{\mathbb{D}_{r,s+1}}$ are bounded in u by hypothesis. By theorem 3.2.0.15, we have a bound of the type:

$$\|\partial_i T^{(n)}(u, X_u) - (\partial_i T_u) \circ X_u\|_{\mathbb{D}_{p'',-(s'+1)}} \leq C(X_u) \cdot \|\partial_i T^{(n)}(u) - \partial_i T(u)\|_{\mathcal{S}_{p',-(s'+1)}}$$
$$(4.14)$$

where $u \mapsto C(X_u)$ is an integrable function on $[0, 1]$ because of the hypotheses on X, and the other term is bounded in n by an integrable function of u because of how the $T^{(n)}$ were chosen. Also, at fixed u, the term under the integral converges to 0 in $\mathbb{D}_{p,-(s'+1)}$ because of theorem 3.2.0.15. Therefore we may conclude to the convergence of the first term by the dominated convergence theorem for Bochner integrals.

Let us note that $C(X_u)$ only involves the $\mathbb{D}_{\infty-,s'+2}$ regularity of the X_u and we did not use more regularity in our proof.   □

Similalry one may prove:

Lemma 4.1.1.3 (Convergence of the second term).

$$\int_\epsilon^t a_{ij}(u) \cdot \partial_{ij} T^{(n)}(u, X_u) du \stackrel{\mathbb{D}_{p',-(s'+2)}}{\longrightarrow} \int_\epsilon^t a_{ij}(u) \cdot \partial_{ij} T_u \circ X_u du \qquad (4.15)$$

Proof. The proof is the exact same, however we note that for this term the $\mathbb{D}_{\infty-,s'+3}$ regularity of the X_u is required indeed.   □

and:

Lemma 4.1.1.4 (Convergence of the fourth term)**.**

$$\sum_j \int_\epsilon^t \sum_i \left[b_i(u) \cdot \partial_i T^{(n)}(u, X_u) \right] dW_u^j \overset{\mathbb{D}_{p', -(s'+1)}}{\rightarrow} M_t^{(\epsilon)} \tag{4.16}$$

where $\left(M_t^{(\epsilon)} \right)_{\epsilon \leq t \leq 1}$ is a weak martingale on the Wiener space, and in the sense of divergences we have:

$$M_t^{(\epsilon)} = \sum_j \int_\epsilon^t \sum_i \left[b_i(u) \cdot (\partial_i T_u) \circ X_u \right] dW_u^j \tag{4.17}$$

Proof. The general idea is the same, except that we need to use the Burkholder Davis Gundy inequalities:

$$\left\| \int_\epsilon^t b_i(u) \cdot \partial_i T^{(n)}(u, X_u) dW_u^j - \int_\epsilon^t b_i(u) \cdot \partial_i T_u \circ X_u dW_u^j \right\|_{\mathbb{D}_{p,s}}$$

$$= \left\| \int_\epsilon^t (2 \cdot Id + \mathcal{L})^{s/2} \left[b_i(u) \cdot \left(\partial_i T^{(n)} - \partial_i T_u \circ X_u(u, X_u) \right) \right] dW_u^j \right\|_{L^p}$$

$$\leq C(p) \cdot \left\| \left(\int_\epsilon^t \left| (2 \cdot Id + \mathcal{L})^{s/2} \left[b_i(u) \cdot \left(\partial_i T^{(n)} - \partial_i T_u \circ X_u(u, X_u) \right) \right] \right|^2 du \right)^{1/2} \right\|_{L^p}$$

$$\leq C(p, s) \left[\int_\epsilon^t \left\| b_i(u) \cdot \left(\partial_i T^{(n)} - \partial_i T_u \circ X_u(u, X_u) \right) \right\|_{\mathbb{D}_{p,s}}^p du \right]^{1/p} \tag{4.18}$$

The $p \geq 2$ restriction was needed to obtain this last inequality. We may then complete the proof with the exact same techniques than for the first and second terms. Once again, we note that only the $\mathbb{D}_{\infty^-, s'+2}$ regularity of the X_u was required. $\qquad \square$

Finally we have to give a result for the third term. This one is the trickiest because the right candidate for the limit of the Stieltjes integrals is less obvious. We start with the following technical result:

Lemma 4.1.1.5. *For any $p > 1$, $s > 0$, for any $n \in \mathbb{N}$ and for any $f \in \mathcal{S}$:*

$$\| \rho_n * f \|_{\mathcal{S}_{p, -s}} \leq \| f \|_{\mathcal{S}_{p, -s}} \tag{4.19}$$

Proof. First:

$$\rho_n * f(x) = \int \frac{1}{(2\pi n)^{N/2}} \cdot \exp\left(-\frac{|y|^2}{2n} \right) f(x - y) dy \tag{4.20}$$

and therefore

$$
\begin{aligned}
\left(\mathcal{K}^{-\frac{s}{2}}\left(\rho_n * f\right)\right)(x) &= \int \frac{1}{(2\pi n)^{N/2}} \cdot \exp\left(-\frac{|y|^2}{2n}\right)\left(\mathcal{K}^{-\frac{s}{2}}f\right)(x-y)dy \\
&= \left(\rho_n * \left(\mathcal{K}^{-\frac{s}{2}}f\right)\right)(x) \tag{4.21}
\end{aligned}
$$

so finally:

$$
\|\rho_n * f\|_{\mathcal{S}_{p,-s}}^p = \|\rho_n * \left(\mathcal{K}^{-\frac{s}{2}}f\right)\|_{L^p}^p \le \|\mathcal{K}^{-\frac{s}{2}}f\|_{L^p}^p = \|f\|_{\mathcal{S}_{p,-s}}^p \tag{4.22}
$$

where the inequality is a classic consequence of Jensen's inequality on the probability space $L^p(\rho_n)$. $\qquad\square$

We are now ready to state:

Lemma 4.1.1.6 (Convergence of the third term). *The following limit exists in* $\mathbb{D}_{p',-s'}$:

$$
\int_\epsilon^t T(du, \circ X_u) := \lim_{n\to\infty} \int_\epsilon^t T^{(n)}(du, X_u) \tag{4.23}
$$

and it is defined by:

$$
\left\langle \int_\epsilon^t T(du, \circ X_u), \phi \right\rangle = \int_\epsilon^t T\left(du, p_{X_u,\phi}\right) \tag{4.24}
$$

Proof. Let us denote (u_i) for the elements of a partition π of $[\epsilon, t]$ arranged in increasing order. Then a Stieltjes integral on $[\epsilon, t]$ is approached by Riemann sums:

$$
\int_\epsilon^t T^{(n)}\left(du, X_u\right) = \lim_{|\pi|\to 0} \sum_i \left[T^{(n)}\left(u_{i+1}, X_{u_i}\right) - T^{(n)}\left(u_i, X_{u_i}\right)\right] \tag{4.25}
$$

where the limit is taken in probabilities. Now, applying theorem 3.2.0.15 as in the previous proofs, one obtains a control of the type:

$$
\begin{aligned}
&\left\|\left[T^{(n)}\left(u_{i+1}, X_{u_i}\right) - T^{(n)}\left(u_i, X_{u_i}\right)\right] - \left[T_{u_{i+1}} \circ X_{u_i} - T_{u_i} \circ X_{u_i}\right]\right\|_{\mathbb{D}_{p',-s'}} \\
&\le B(u_i) \cdot \left\|\left(T_{u_{i+1}}^{(n)} - T_{u_i}^{(n)}\right) - \left(T_{u_{i+1}} - T_{u_i}\right)\right\|_{\mathcal{S}_{p',-s'}} \tag{4.26}
\end{aligned}
$$

where B is a function which is bounded on $[\epsilon, t]$ independently of n. Also by the triangle inequality and lemma 4.1.1.5:

$$
\left\|\left(T_{u_{i+1}}^{(n)} - T_{u_i}^{(n)}\right) - \left(T_{u_{i+1}} - T_{u_i}\right)\right\|_{\mathcal{S}_{p',-s'}} \le 2 \cdot \left\|T_{u_{i+1}} - T_{u_i}\right\|_{\mathcal{S}_{p',-s'}} \tag{4.27}
$$

which does not depend on n either. Now since:

$$\sum_i \left\| T_{u_{i+1}} - T_{u_i} \right\|_{\mathcal{S}_{p',-s'}} \leq Var(T) < \infty \tag{4.28}$$

and this control is independant of π by the dominated convergence theorem one obtains the convergence of the following Riemann sum in $\mathbb{D}_{p',-s'}$:

$$\sum_i \left[T_{u_{i+1}} \circ X_{u_i} - T_{u_i} \circ X_{u_i} \right] \tag{4.29}$$

and we denote its limit by:

$$\int_\epsilon^t T(du, \circ X_u) := \lim_{|\pi| \to 0} \sum_i \left[T_{u_{i+1}} \circ X_{u_i} - T_{u_i} \circ X_{u_i} \right] \tag{4.30}$$

Now let $\phi \in \mathbb{D}$. By theorem 3.4.0.18 for every u, $p_{X_u,\phi}$ exists and has at least $\mathcal{S}_{(p')^*,s'}$ regularity. Therefore by disintegrating each term of the Riemann sum separately one obtains:

$$E\left[\phi \cdot \left[T_{u_{i+1}} \circ X_{u_i} - T_{u_i} \circ X_{u_i} \right] \right] = {}_{\mathcal{S}'} \langle T_{u_{i+1}} - T_{u_i}, p_{X_u,\phi} \rangle_{\mathcal{S}} \tag{4.31}$$

Since $Var(T) < \infty$ in $\mathcal{S}_{p',-s'}$ and $u \mapsto p_{X_u,\phi}$ is bounded in $\mathcal{S}_{(p')^*,s'}$ by similar techniques one shows that this converges to:

$$\int_\epsilon^t T\left(du, p_{X_u,\phi} \right) := \int_\epsilon^t \int_{\mathbb{R}^N} \left(\mathcal{K}^{\frac{s'}{2}} T \right) (du, x) \cdot \left(\mathcal{K}^{-\frac{s'}{2}} p_{X_u,\phi} \right)(x) dx \tag{4.32}$$

so we are done. Once again, we did not use the $\mathbb{D}_{\infty^-,s'+3}$ regularity of the X_u: only $\mathbb{D}_{\infty^-,s'+1}$ was required for this part of the proof. $\qquad\square$

Remark 4.1.1.2. *Let us remember that we restricted ourselves to $BV(\mathcal{S}_{p',-s'})$ because of the Grothendieck theorem. Therefore it is worth mentionning that the convergence of our Stieltjes integral may also be obtained directly from the Grothendieck representation. Indeed since (F_j) is a bounded sequence in $\mathcal{S}_{p',-s'}$ by a method similar to the one used for the first term one has a control of the type:*

$$\| F_j^{(n)}(X_u) - F_j \circ X_u \|_{\mathbb{D}_{p'',-s}} \leq C(X_u) \cdot \| F_j^{(n)} - F_j \|_{\mathcal{S}_{p',-s'}} \tag{4.33}$$

where $F_j^{(n)} = \rho_n * F_j$ and $u \mapsto C(X_u)$ is an integrable function of $u \in [0,1]$ which is independant of n. Therefore one obtains that for every j:

$$\int_\epsilon^1 \| F_j^{(n)}(X_u) - F_j \circ X_u \|_{\mathbb{D}_{p'',-s}} d|V_j|(u) \to 0 \tag{4.34}$$

by the dominated convergence theorem for Bochner integrals. Since the sequence of these integrals also is bounded in j one concludes to the dominated convergence of the third term by the dominated convergence theorem for series. Also, one then has a more explicit expression for the Stieltjes integral:

$$\int_\epsilon^t T(du, \circ X_u) = \sum_{n=0}^\infty \lambda_n \cdot \int_\epsilon^t F_n \circ X_u dV_n(u) \tag{4.35}$$

We are now ready to state the main result in this paragraph:

Theorem 4.1.1.1 (Weak Itô formula)**.** *Under the conditions (4.3), (4.5), (4.6) and (4.6), for any $\epsilon > 0$, for all $t \in [\epsilon, 1]$, the following holds in the space $\mathbb{D}_{p'\vee 2, -(s'+2)}$:*

$$\begin{aligned}
T(t) \circ X_t - T(\epsilon) \circ X_\epsilon \ = \ & M_t^{(\epsilon)} + \sum_i \int_\epsilon^t b_i(u) \cdot (\partial_i T_u) \circ X_u du \\
& + \frac{1}{2} \sum_{i,j} \int_\epsilon^t a_{ij}(u) \cdot (\partial_{ij} T_u) \circ X_u du \\
& + \int_\epsilon^t T(du) \circ X_u
\end{aligned} \tag{4.36}$$

where $\left(M_t^{(\epsilon)} \right)_{\epsilon \le t \le 1}$ is a weak martingale on the Wiener space, and in the sense of divergences we have:

$$M_t^{(\epsilon)} = \sum_j \int_\epsilon^t \sum_i \left[b_i(u) \cdot (\partial_i T_u) \circ X_u \right] dW_u^j \tag{4.37}$$

Proof. We only need to show the convergence of every term, which is done by applying the previous lemmae. Let us also note that the lemmae provide precise estimates of the $\mathbb{D}_{p'\vee 2, -(s'+2)}$ norm of every term - we have $p' \vee 2$ rather than simply p' because of the martingale term; see the proof of the corresponding lemma.   □

Now we make the following observation: if we prove that all but one term in the proof of the Itô formula above converge, then by elementary algebra the last one has to converge too and the convergence takes place in the same space. Therefore one may obtain the three following results which are slightly more precise.

The first one is interesting when T_t is more regular in the continuous sense than in the BV sense.

Corollary 4.1.1.1. *Replace hypothesis (4.3) with:*

$$T_t \in C^0(\mathcal{S}_{p,-\sigma}) \tag{4.38}$$

where possibly $\sigma > s$ and replace s by σ in hypotheses (4.5), (4.6) and (4.6). Then, the formula in theorem 4.1.1.1 still holds in the space $\mathbb{D}_{p' \vee 2, -(\sigma'+2)}$.

Proof. The proof is the same as for theorem 4.1.1.1 except that the part about the Stieltjes integral is not required. $\qquad\square$

The second result allows one to "save" one order of regularity:

Corollary 4.1.1.2. *Suppose that (4.3) holds and replace $s+3$ by $s+2$ in hypotheses (4.5), (4.6) and (4.6). Then the conclusions of theorem 4.1.1.1 still hold in the space $\mathbb{D}_{p' \vee 2, -(s'+1)}$.*

Proof. The proof is the same as for theorem 4.1.1.1 except that the part about the crochet integral (which contain the order 2 derivatives) is not required. $\qquad\square$

Finally the third corollary allow us to obtain controls in the space $\mathbb{D}_{p', -(s'+2)}$ for $p' < 2$:

Corollary 4.1.1.3. *Suppose (4.3), (4.5), (4.6) and (4.6) as in theorem 4.1.1.1. Then the conclusions of theorem 4.1.1.1 hold in the space $\mathbb{D}_{p', -(s'+2)}$ for $p' < 2$.*

Proof. This time it is the part about the martingale term which is not required. $\quad\square$

Finally there is a straightforward extension of the Itô formula in theorem 4.1.1.1, which allows one to multiply $T_t \circ X_t$ by a regular enough process Y_t. More precisely consider a one-dimensionnal Itô process:

$$
\begin{aligned}
dY_t &= b'_t dt + \sigma'_t dW_t & (4.39) \\
Y_0 &= y & (4.40)
\end{aligned}
$$

and make the following hypotheses on Y: for all $r > 1$,

$$
\int_0^1 \|b'_u\|_{\mathbb{D}_{r,s}}^r du < \infty \tag{4.41}
$$

$$
\int_0^1 \|\sigma'_u\|_{\mathbb{D}_{r,s}}^r du < \infty \tag{4.42}
$$

$$
\tag{4.43}
$$

We do not need to suppose that Y is non-degenerate. Then:

Theorem 4.1.1.2. *Suppose that X and T are as in theorem 4.1.1.1 and that Y is as above. Then the following stands in $\mathbb{D}_{p,s}$:*

$$
\begin{aligned}
Y_t \cdot T(t) \circ X_t - Y_\epsilon \cdot T(\epsilon) \circ X_\epsilon \;=\;& M_t^{(\epsilon)} + \sum_i \int_\epsilon^t b_i(u) Y_u \cdot (\partial_i T_u) \circ X_u du \\
&+ \frac{1}{2} \sum_{i,j} \int_\epsilon^t Y_u a_{ij}^2(u) \cdot (\partial_{ij} T_u) \circ X_u du \\
&+ \int_\epsilon^t b'(u) Y_u \cdot T_u \circ X_u du \\
&+ \int_\epsilon^t Y_u T(du) \circ X_u \qquad\qquad (4.44)
\end{aligned}
$$

where

$$
M_t^{(\epsilon)} = \sum_j \int_\epsilon^t Y_u \sum_i \left[b_i(u) \cdot (\partial_i T_u) \circ X_u \right] dW_u^j + \sum_j \int_\epsilon^t \sigma_j'(u) T_u \circ X_u dW_u^j \quad (4.45)
$$

This result is proved exactly as theorem 4.1.1.1. Of course corollaries 4.1.1.1, 4.1.1.2 and 4.1.1.3 also admit similar extensions.

4.1.2   Processes with a general drift

In this paragraph we consider the process X such that:

$$
\begin{aligned}
dX_t &= dA_t + \sigma_t dW_t & (4.46) \\
X_0 &= x & (4.47)
\end{aligned}
$$

where σ is as in the previous paragraph and A is a bounded variation process. We still make the hypotheses (4.6), (4.6) and (4.3) and instead of (4.5) we suppose that for every $r > 1$:

$$
\int_0^t \|A_u\|_{\mathbb{D}_{r,s'+3}}^r du < \infty \qquad (4.48)
$$

which is enough to ensure that the quantities we consider are well defined.

We now state the main result in this paragraph:

Theorem 4.1.2.1. *Under the hypotheses (4.5), (4.6), (4.6) and (4.3), for any*

$\epsilon > 0$, *for all* $t \in [\epsilon, 1]$, *the following holds in the space* $\mathbb{D}_{p' \vee 2, -(s'+2)}$:

$$T(t) \circ X_t - T(\epsilon) \circ X_\epsilon = M_t^{(\epsilon)} + \int_\epsilon^t (\partial_i T_u) \circ dA(u)$$

$$+ \frac{1}{2} \sum_{i,j} \int_\epsilon^t a_{ij}^2(u) \cdot (\partial_{ij} T_u) \circ X_u du$$

$$+ \int_\epsilon^t T(du) \circ X_u \tag{4.49}$$

where $\left(M_t^{(\epsilon)} \right)_{\epsilon \leq t \leq 1}$ *is as in theorem 4.1.1.1 and the integral against* A *is defined as:*

$$\int_\epsilon^t (\partial_i T_u) \circ X_u dA(u) = \lim_{n \to \infty} \int_\epsilon^t (\partial_i (\rho_n * T_u)) (X_u) dA(u) \tag{4.50}$$

Proof. As in the proof of theorem 4.1.1.1 we introduce $T^{(n)} = \rho_n * T$ in order apply the usual Itô formula to that function and we study the convergence of each term in the formula. Except for the integral against A, every term is analogue to that in the proof of theorem 4.1.1.1 and since we have the same regularity for X we obtain the convergence of these terms similarly. Finally the convergence of the integral against A is then automatic. $\qquad\square$

The problem with this idea is that it gives no explicit information about the behaviour of the BV integral against A. This may be partly addressed to if we make one extra assumption, namely that for some $q > (p')^*$:

$$\sup \sum_k \left\| A_{u_{k+1}} - A_{u_k} \right\|_{\mathcal{S}_{q,s'}} < \infty \tag{4.51}$$

where p is given in (4.3) and the sup is taken over all the partitions of $[\epsilon, 1]$. Indeed we then have the:

Lemma 4.1.2.1. *Under hypotheses (4.3), (4.48) and (4.51),*

$$\int_\epsilon^t (\partial_i T_u) \circ X_u dA(u) = \lim_{n \to \infty} \int_\epsilon^t (\partial_i (\rho_n * T_u)) (X_u) dA(u) \tag{4.52}$$

is given as the limit of the following Riemann sum:

$$\sum_k ((\partial_i T_{u_k}) \circ X_{u_k}) \cdot \left(A_{u_{k+1}} - A_{u_k} \right) \tag{4.53}$$

Proof. Same technique as for the proof of lemma 4.1.1.6. $\qquad\square$

Of course, as in the previous paaragraph, one may notice that it is sufficient to prove the convergence of all but one term, one may obtain more precise results.

4.1.3 The case where T_t is random

Let X be as in section 4.1.1 and suppose that T_t is a $\mathcal{S}'$-valued random process with bounded variations, adapted to the filtration of the Brownian motion W, such that for every $r > 1$ and every integer $k \leq [s']$:

$$\int_0^t \|T_s\|_{\mathbb{D}_{r,k}(\mathcal{S}_{\infty,s'-k})}^r ds < \infty \tag{4.54}$$

Then the same weak Itô formula as in theorem 4.1.1.1 holds. This is proved by following the proof of theorem 4.1.1.1 but using theorem 3.5.0.20 instead of theorem 3.2.0.15. Indeed hypothesis 4.54 was designed exactly so that lemma 3.5.0.20 may be applied.

4.2 An Itô - Wentzell formula

In this section we study the case of $T_t \circ X_t$ where X_t is a regular, non-degenerate process of the type (4.1) and T_t is a $\mathcal{S}'$-valued semi-martingale. More precisely, using the results from section 3.5, we will prove an extension of the Itô-Wentzell formula (see, for example, [40]). We stick to the case where the dimension is $N = 1$ for simpler notation but there is no difficulty in extending the results.

We suppose that X verifies hypotheses (4.5), (4.6) and (4.6). We also introduce two time-continous, distribution-valued processes:

$$D_t, V_t \in \bigcap_{k<[s]} \left(\mathbb{D}_{\infty^-,k}\left(\mathcal{S}_{\infty,-(s-k)}\right) \cap \mathbb{D}_{\infty^-,k}\left(W_{\alpha,-(s-k-\frac{N}{\alpha})}\right) \right) \tag{4.55}$$

Here we suppose that α is such that $\{s\} > \frac{N}{\alpha}$; is s is an integer we only suppose:

$$D_t, V_t \in \bigcap_{k<[s]} \mathbb{D}_{\infty^-,k}\left(\mathcal{S}_{\infty,-(s-k)}\right) \tag{4.56}$$

We suppose that these processes are adapted to the filtration of W. Then we may define an $\mathcal{S}_{\infty,-s}$-valued semimartingale by an initial value $T_0 \in \mathcal{S}_{\infty,-s}$ and the equation:

$$dT_t = D_t dt + V_t dW_t \tag{4.57}$$

where the above means that for any $\phi \in \mathcal{S}$:

$$\langle T_t, \phi \rangle = \langle T_0, \phi \rangle + \int_0^t \langle D_s, \phi \rangle \, ds + \int_0^t \langle V_s, \phi \rangle \, dW_s \tag{4.58}$$

which is a regular semimartingale in $\mathbb{R}$. For more details about the construction of such objects, see, for example, [76] or [39]. It is then easily verified that:

$$T_t \in \bigcap_{k<[s]} \mathbb{D}_{\infty-,k}\left(\mathcal{S}_{\infty,-(s-k)}\right) \tag{4.59}$$

and all the $t \mapsto \|T_t\|_{\mathbb{D}_{\infty-,k}\left(\mathcal{S}_{\infty,-(s-k)}\right)}$ are bounded. There also is a similar result for the $\mathbb{D}_{\infty-,k}\left(W_{\alpha,-(s-k-\frac{N}{\alpha})}\right)$ spaces.

We now state our result:

Theorem 4.2.0.1 (Weak Itô-Wentzell formula). *Under the hypotheses (4.55), (4.5), (4.6) and (4.6), for any $p' > p$ and $s' > s$, for any $\epsilon > 0$, the following formula holds in $\mathbb{D}_{p'\vee 2,-(s'+2)}$:*

$$\begin{aligned}
T(t) \circ X_t - T(\epsilon) \circ X_\epsilon \;=\; & M_t^{(\epsilon)} + \int_\epsilon^t b(u) \cdot (\partial_x T_u) \circ X_u du \\
& + \frac{1}{2} \int_\epsilon^t \sigma(u)^2 \cdot (\partial_{xx} T_u) \circ X_u du \\
& + \int_\epsilon^t D_u \circ X_u du \\
& + \int_\epsilon^t \sigma(u) \cdot \partial_x V_u \circ X_u du
\end{aligned} \tag{4.60}$$

where $\left(M_t^{(\epsilon)}\right)_{\epsilon \leq t \leq 1}$ is a weak martingale on the Wiener space, and in the sense of divergences we have:

$$M_t^{(\epsilon)} = \int_\epsilon^t \left[b(u) \cdot (\partial_x T_u) \circ X_u + V_u \circ X_u \right] dW_u^j \tag{4.61}$$

Proof. We use the same method as for the proof of theorem 4.1.1.1, except that we use theorem 3.5.0.20 instead of theorem 3.2.0.15. The condition (4.55) has been written so that theorem 3.5.0.20 always applies directly, so there is no extra difficulty. $\qquad \square$

4.3 An anticipative version

In this section we still consider a $\mathcal{S}'$-valued path T_t and a process:

$$X_t = X_0 + \int_0^t b_u du + \int_0^t \sigma_u dW_u \tag{4.62}$$

but we allow X_0 to be a random variable and we do not suppose that b or σ are adapted to the Brownian filtration anymore (so the stochastic integrals have to be understood as divergences). We study under which hypotheses on the Malliavin regularity of X_0, b and σ we may still write a weak (anticipative) Itô formula as in the previous section.

We will now recall a few points about anticipative stochastic calculus. We do not pretend to provide a full introduction to that topic; instead we only introduce the notation we will need and we refer to [53] and the references therein for details.

First, for $1 < p < \infty$ we introduce the space $\mathbb{L}_{p,1}$ of those processes such that:

$$\|u\|_{\mathbb{L}_{1,p}}^p := E\left[\int_0^1 |u_t|^p dt\right] + E\left[\int\int_{s\leq t} |D_s u_t|^p ds dt\right] < \infty \qquad (4.63)$$

Then for $1 < q \leq p$ we introduce the spaces $\mathbb{L}_{p,1}^{q+}$, resp. $\mathbb{L}_{p,1}^{q-}$ of thoses processes $u \in \mathbb{L}_{p,1}$ such that there exists a process D^+u, resp. D^-u such that for every t:

$$\lim_{n\to\infty} \int_0^1 \sup_{s<t\leq(s+\frac{1}{n})\wedge 1} E\left[|D_s u_t - (D^+u)_s|^p\right] ds = 0 \qquad (4.64)$$

resp:

$$\lim_{n\to\infty} \int_0^1 \sup_{(s-\frac{1}{n})\vee 0\leq t<s} E\left[|D_s u_t - (D^-u)_s|^p\right] ds = 0 \qquad (4.65)$$

Finally we note: $\mathbb{L}_{p,1}^q = \mathbb{L}_{1,p}^{q+} \cap \mathbb{L}_{p,1}^{q-}$. To acquire an intuition about that notion, let us note that if u is a process such that $(s,t) \mapsto D_s u_t$ is continuous on a neighbourhood of the diagonal, then: $(D^+u)_t = D_t u_t = (D^-u)_t$. However that continuity property does not hold for many processes which are of interest; for example there is the:

Theorem 4.3.0.2. *Consider the process (4.62) and suppose that:*

- *$X_0 \in \mathbb{D}_{2,1}$;*

- *$b \in \mathbb{D}_{2,2}(H)$;*

- *$\sigma \in \mathbb{L}_{2,1}$.*

Then $u \in \mathbb{L}_{2,1}^2$ and:

$$(D^-X)_t = D_t X_0 + \int_0^t D_t b_r dr + \int_0^t D_t \sigma_r dW_r \qquad (4.66)$$

$$(D^+X)_t = (D^-X)_t + \sigma_t \qquad (4.67)$$

Also, in [53] (or in the original [54]), it is proved that if F is a C^2 function and if one assumes enough regularity on b and σ, the following Itô formula holds:

$$
\begin{aligned}
f(X_t) \;=\;& f(X_0) \\
&+ \int_0^t \left(b_s f'(X_s) + \frac{1}{2}\sigma_s^2 f''(X_s) \right) ds \\
&+ \int_0^t \sigma_s f'(X_s) dW_s \\
&+ \int_0^t (D^- X)_s \sigma_s f''(X_s) ds
\end{aligned}
\tag{4.68}
$$

Here the Brownian term is a divergence. We also note that the difference between the usual Itô formula and this anticipative one lies only in the last term. Finally it is interesting to notice that this term in the one with least Malliavin regularity. Then by the same methods as in the previous section we prove the:

Theorem 4.3.0.3 (Anticipative weak Itô formula). *Let $T \in \mathcal{S}_{q,-\delta}$. Consider the process (4.62), with the hypotheses that for every p:*

$$
\int \int_{s \leq t} E\left[\left| (Id + \mathcal{L})^{\frac{\delta+3}{2}} D_s \sigma_t \right|^p \right] ds dt \;<\; \infty
\tag{4.69}
$$

$$
\int \int_{s \leq t} E\left[\left| (Id + \mathcal{L})^{\frac{\delta+2}{2}} D_s b_t \right|^p \right] ds dt \;<\; \infty
\tag{4.70}
$$

$$
\int_0^1 \left\| \frac{1}{\Sigma_t} \right\|^p dt \;<\; \infty
\tag{4.71}
$$

where Σ is the Malliavin matrix of X_t. Then the following holds in the space $\mathbb{D}_{q,-\delta}$:

$$
\begin{aligned}
T \circ X_t \;=\;& T \circ X_0 \\
&+ \int_0^t \left(b_s \cdot T' \circ X_s + \frac{1}{2}\sigma_s^2 \cdot T'' \circ X_s \right) ds \\
&+ \int_0^t \sigma_s \cdot T' \circ X_s dW_s \\
&+ \int_0^t (D^- X)_s \sigma_s \cdot T'' \circ X_s ds
\end{aligned}
\tag{4.72}
$$

Proof. The method is exactly the same as in the previous paragraph so we only mention a few notewothy differences.

First since we allow X_0 to be a (nondegenerate) random variable our formula may involve integrals starting from 0 rather than some ϵ.

Second we cannot lift the Brownian (divergence) term into a martingale since σ is not adapted. Therefore instead of the Burkholder-Davies-Gundy inequlities we will need to use the divergence inequailty:

$$\|\delta u\|_{\mathbb{D}_{p,s}} \leq \|u\|_{\mathbb{D}_{p,s+1}(H)} \tag{4.73}$$

Finally we note that our hypotheses have been tailored so that D^-X has enough (ie $\delta + 3$) Malliavin regularity. $\qquad \square$

4.4   Application to the solution of a SDE

In this paragraph we give conditions on the coefficients of a stochastic differential equation for its solution to be regular in the sense of Malliavin calculus and uniformly non-degenerate. This allows us to derive under which conditions on the coefficients of an SDE which are sufficient to apply the weak Itô formula (or its extensions) to its unique strong solution.

4.4.1   Fractional regularity of the solution of an SDE

We introduce the Hölder spaces C^s. If s is an integer, then C^s simply is the space of bounded functions with bounded derivatives of total order less than s. Otherwise write $s = [s] + \{s\}$; then C^s is the space of those functions in $C^{[s]}$ with all derivatives of total order $[s]$ being $\{s\}$-holderian. These spaces are closely related to the Poisson semi-group (Π_t), cf [67]. Let us recall a few facts about this semi-group.

First for $t > 0$ we introduce the functions:

$$
\begin{aligned}
\pi_t(x) &= \int_{\mathbb{R}^N} e^{-2i\pi y \cdot x} \cdot e^{-2\pi|y|t} dy \\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \cdot \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}
\end{aligned}
\tag{4.74}
$$

and for $f \in C_b^0$ we set:

$$(\Pi_t f)(x) = (\pi_t * f)(x) \tag{4.75}$$

Then Π_t defines a semigroup on C_b^0, which enjoys the following property:

Theorem 4.4.1.1. *Let $f \in L^\infty$ and $0 < s < 1$. Then the following are equivalent:*

- $f \in C^s$;

- $\left\| \frac{\partial (\Pi_t f)}{\partial t} \right\| = O(t^{-1+s})$;

- *For every $1 \leq i \leq N$: $\left\|\frac{\partial(\Pi_t f)}{\partial x_i}\right\| = O(t^{-1+s})$*

We also introduce the space $C^{0,s}$ of those functions defined on $[0, T] \times \mathbb{R}^N$ which are C^s in space uniformly in time, with the norm:

$$\|f\|_{C^{0,s}} = \sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{C^s} \tag{4.76}$$

We will consider the following N-dimensionnal stochastic differential equation:

$$dX_t = b(t, X_t)ds + \sigma(t, X_t)dW_t \tag{4.77}$$
$$X_0 = x \tag{4.78}$$

with $b : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N \otimes \mathbb{R}^d$ and W the canonical Brownian motion on the d-dimensionnal Wiener space.

The following result is well known and may be found in [66], [53] or [56]:

Theorem 4.4.1.2. *Consider the stochastic differential equation (4.77) and suppose that b and σ are in $C^{0,k}$ for some T. Then for any $p > 1$ and any $t \geq 0$, $X_t \in \mathbb{D}_{p,k}$ and there exists a constant C depending only on p, k and the $C^{0,k}$ norms of b and σ such that:*

$$\left\| \sup_{0 \leq t \leq T} |(Id + \mathcal{L})^{k/2} X_t| \right\|_{L^p} \leq e^{CT}$$

Remark 4.4.1.1. *In fact it is sufficient that b and σ be $C^{0,k-1}$ with uniformly lipschitz derivatives of order $k - 1$.*

Proof. Classic and omitted, cf, for example, [53]. $\square$

We will generalize this to the following:

Theorem 4.4.1.3. *Consider the stochastic differential equation (4.77) and suppose that b and σ are in $C^{0,s}$ for some T and some $s \geq 1$. Then for any $p > 1$, $s' < s$ and any $t \geq 0$, $X_t \in \mathbb{D}_{p,s'}$ and there exist constants A and B depending only on p, s, s' and the $C^{0,s}$ norms of b and σ such that:*

$$\|X_t\|_{\mathbb{D}_{p,s'}} \leq A \exp(B(\|b\|_{C^{0,s}} + \|\sigma\|_{C^{0,s}}) \cdot t) \tag{4.79}$$

Proof. We stick to the case $d = N = 1$ during the proof, but the general case includes no other difficulty than tedious notation. We note $[s] = k$ and $\{s\} = \alpha$. We suppose $\alpha > 0$, otherwise our result is included in theorem 4.4.1.2. We note that the fact that $s \geq 1$ implies the existence and the unicity of a strong solution to the SDE.

We recall that Π denotes the Poisson kernel; for $\tau > 0$ we introduce the functions σ_τ and b_τ defined by:

$$\sigma_\tau(t, \cdot) = (\Pi_\tau \sigma)(t, \cdot) \tag{4.80}$$

with an analoguous definition for b_τ. For $\tau > 0$ these functions have bounded derivatives of all order; we recall that we have the estimates, cf [67]:

$$\|\sigma_\tau\|_{C^l} \leq \|\sigma\|_{C^l} \tag{4.81}$$

for $l \leq k$ and

$$\|\sigma_\tau\|_{C^{k+1}} \leq \tau^{-1+\alpha}\|\sigma\|_{C^s} \tag{4.82}$$

We also consider X_t^τ the solution of the stochastic differential equation associated to these coefficients. We may apply theorem 4.4.1.2 to X_t for $k \leq s$, and to the X_t^τ for any integer (here: $k + 1$) if $\tau > 0$. Let us write:

$$\nabla X_t^\tau = \int_0^t \sigma_\tau'(s, X_s^\tau)\nabla X_s^\tau dW_s + \int_0^t b_\tau'(s, X_s^\tau)\nabla X_s^\tau dW_s + \int_0^{min(t,\cdot)} \sigma_\tau(s, X_s^\tau)ds \tag{4.83}$$

and by the Malliavin calculus analogue to Faa di Bruno's formula: if $k \geq 1$:

$$\begin{aligned}
\nabla^{k+1} X_t^\tau &= \int_0^t \sigma_\tau^{(k+1)}(s, X_s^\tau)\nabla^{\otimes(k+1)} X_s^\tau dW_s + \int_0^t \sigma_\tau'(s, X_s^\tau)\nabla^{k+1} X_s^\tau dW_s \\
&\quad + \int_0^t b_\tau^{(k+1)}(s, X_s^\tau)\nabla^{\otimes(k+1)} X_s^\tau ds + \int_0^t b_\tau'(s, X_s^\tau)\nabla^{k+1} X_s^\tau ds \\
&\quad + \int_0^t P(s, X)dW_s + \int_0^t Q(s, X)ds + \int_0^{min(t,\cdot)} R(s, X)ds \tag{4.84}
\end{aligned}$$

where P, Q and R are polynomials in the space derivatives of order less than k of b_τ and σ_τ and in the $\nabla^{\otimes l} X_t^\tau$ for $l \leq k$ in the sense of tensor products. Therefore, applying the Burkholder Davis Gundy inequality, theorem 4.4.1.2 and Hölder's theorem one gets:

$$\left\|\sup_{0\leq u\leq t} |\nabla^{k+1} X_u^\tau|\right\|_{L^p}^p \leq f(t) + \tau^{-1+\alpha}g(t) + \int_0^t (\|b\|_{C^1} + \|\sigma\|_{C^1})^p \left\|\sup_{0\leq u\leq s} |\nabla^{k+1} X_u^\tau|\right\|_{L^p}^p ds \tag{4.85}$$

where f and g are bounded by bounds of the type $\exp(C(\|b\|_{C^{0,s}} + \|\sigma\|_{C^{0,s}})$. Now applying Gronwall's lemma leads to a majorization of the type:

$$\|\sup_{0\leq u\leq t} |\nabla^{k+1} X_u^\tau|\|_{L^p}^p \leq (1 + \tau^{(-1+\alpha)p} \exp(C(\|b\|_{C^{0,s}} + \|\sigma\|_{C^{0,s}}) \cdot t) \tag{4.86}$$

and if $0 < \alpha' < \alpha$:

$$\int_0^1 \tau^{-1+(1-\alpha')p}\|\sup_{0\leq u\leq t}|\nabla^{k+1}X_u^\tau|\|_{L^p}^p d\tau \leq A \cdot \exp(B(\|b\|_{C^{0,s}} + \|\sigma\|_{C^{0,s}})\cdot t) \quad (4.87)$$

Now we need to control $\partial_\tau \nabla^k X_t^\tau$. We note that, since $s \geq 1$, the properties of the Poisson kernel guarantee the differentiability of b_τ and σ_τ in τ together with a uniform control on these derivatives. Therefore the theory of stochastic flows (cf [40] or [38], the results generalize easily to processes with values in Hilbert spaces) proves the existence of $\partial_\tau\left(\nabla^k X_t^\tau\right)$ and that the computations below are valid.

Starting from (4.84) and replacing $k+1$ by k we find a stochastic differential equation verified by $\partial_\tau \nabla^k X_t^\tau$. First:

$$\partial_\tau \int_0^t \sigma_\tau^{(k)}(s,X_s^\tau)\nabla^{\otimes k}X_s^\tau dW_s = \int_0^t \left[\partial_\tau\sigma_\tau^{(k)}(s,X_s^\tau) + \sigma_\tau^{(k+1)}(s,X_s^\tau)\partial_\tau X_s^\tau\right]\nabla^{\otimes k}X_s^\tau dW_s$$
$$+ k\int_0^t \sigma_\tau^{(k)}(s,X_s^\tau)\nabla^{\otimes(k-1)}X_s^\tau \otimes \partial_\tau \nabla X_s^\tau dW_s \quad (4.88)$$

and

$$\partial_\tau \int_0^t \sigma_\tau'(s,X_s^\tau)\nabla^k X_s^\tau dW_s = \int_0^t \left[\partial_\tau\sigma_\tau'(s,X_s^\tau) + \sigma_\tau''(s,X_s^\tau)\partial_\tau X_s^\tau\right]\nabla^k X_s^\tau dW_s$$
$$+ \int_0^t \sigma_\tau'(s,X_s^\tau)\partial_\tau \nabla^k X_s^\tau dW_s \quad (4.89)$$

There also are similar Lebesgue integrals involving b. Finally, we obtain the form:

$$\partial_\tau X_t^\tau = \int_0^t \left[\sigma_\tau^{(k+1)}(s,X_s^\tau)\partial_\tau X_s^\tau\nabla^{\otimes k}X_s^\tau + \sigma_\tau'(s,X_s^\tau)\partial_\tau\nabla^k X_s^\tau\right]dW_s$$
$$+ \int_0^t \left[b_\tau^{(k+1)}(s,X_s^\tau)\partial_\tau X_s^\tau\nabla^{\otimes k}X_s^\tau + b_\tau'(s,X_s^\tau)\partial_\tau\nabla^k X_s^\tau\right]ds$$
$$+ \int_0^t P(s,X)dW_s + \int_0^t Q(s,X)ds + \int_0^{min(t,\cdot)} R(s,X)ds \quad (4.90)$$

where P, Q and R are polynomials involving terms of order at most $k-1$ in τ-derivatives end at most k in space and Malliavin derivatives, so once again by applying the Burkholder Davis Gundy inequality, theorem 4.4.1.2, Hölder's theorem and finally Gronwall's lemma we obtain:

$$\left\|\sup_{0\leq u\leq t}|\partial_\tau\nabla^k Xu^\tau|\right\|_{L^p}^p \leq (1 + \tau^{(-1+\alpha)p})\exp(C(\|\sigma\|_{C^{(0,s)}} + \|b\|_{C^{(0,s)}})) \quad (4.91)$$

and for $0 < \alpha' < \alpha$:

$$\int_0^1 \tau^{-1+(1-\alpha')p} \left\| \sup_{0 \le u \le t} |\partial_\tau \nabla^k X u^\tau| \right\|_{L^p}^p d\tau \le A \exp(B(\|b\|_{C^{0,s}} + \|\sigma\|_{C^{0,s}}) \cdot t) \quad (4.92)$$

By the definition of the real interpolation spaces, noting $s' = k + \alpha'$ this yields: $X_t \in \mathbb{D}_{p,s'}$ and:

$$\|X_t\|_{\mathbb{D}_{p,s'}} \le A \exp(B(\|b\|_{C^{0,s}} + \|\sigma\|_{C^{0,s}}) \cdot t) \quad (4.93)$$

This ends the proof. $\square$

Remark 4.4.1.2. *Here as opposed to theorem 4.4.1.2 we do not obtain a uniform majorization. This is because we could not differentiate the sup with regard to τ.*

4.4.2   A sufficient condition for the non-degeneracy of the solution of an SDE

We now state a condition for the non-degeneracy of the solution of the stochastic differential equation (4.77). We suppose that $\sigma \in C^{0,s}$ for some $s \ge 2$ and that $\beta \in C^{0,1}$. Then our stochastic differential solution has a unique strong solution. We note $k = [s]$. For $1 \le j \le d$ let us consider the vector fields on $\mathbb{R}^N$ which are defined by: $V_j = \sigma_{\cdot,j} \partial_{x_j}$. Let L_0 be the set containing all the V_j; define by recurrence L_{i+1} the set containing the $[V_j, Z]$ where $Z \in L_i$. Here $[\cdot, \cdot]$ is the commutator of two vector fields. The L_i are well defined for $i \le k$. Adapting the proof from [15] we state:

Theorem 4.4.2.1. *Suppose that for some $n \le k - 2$ there exists $c > 0$ such that for all $\xi \in \mathbb{S}^{N-1}$:*

$$\sum_{i=0}^n \sum_{Z \in L_k} \langle \xi, Z \rangle (0, x) > c \quad (4.94)$$

Suppose also that $\sigma \in C^{(\beta, n+2)}$ for some $\beta > 0$. Then, if Σ_t is the Malliavin matrix of X_t, estimates of the following type hold for $t > 0$:

$$\left\| \frac{1}{\Sigma_t} \right\|_{L^p} \le C \cdot \frac{1}{t^\nu} \cdot e^{Kt} \quad (4.95)$$

Here C, ν and K are constants which do not depend on t. In particular:

$$\int_\epsilon^T \left\| \frac{1}{\Sigma_t} \right\|_{L^p}^p dt < \infty \quad (4.96)$$

The proof is entirely contained in [15]. We simply accept to cope with a theorem which is more tedious to state as we pay more attention to the orders of differentiablity; in so doing we avoid to suppose that the volatility is $C^{0,\infty}$.

4.4.3 Weak Itô formula for the solution of an SDE

We now combine the results in the two previous paragraphs to obtain tractable sufficient conditions which let us apply the weak Itô formula to the solution of the stochastic differential equation (4.77).

Theorem 4.4.3.1. *Suppose that the coefficients of the stochastic differential equation (4.77) are in $C^{(0,s)}$ for some $s > 2$ and let $k = [s]$. Suppose that the Hormander condition (4.94) holds for some $n \leq k - 2$ and that $\sigma \in C^{(\beta,n+2)}$. Then if T is a path with bounded variations in $\mathcal{S}_{p,-(s-1)}$ the following Itô formula holds:*

$$
\begin{aligned}
T_t \circ X_t - T_\epsilon \circ X_\epsilon \;=\; & M_t^\epsilon + \sum_i \int_\epsilon^t b_i(u, X_u) \cdot (\partial_i T_u) \circ X_u du \\
& + \; \frac{1}{2} \sum_{i,j} \int_\epsilon^t a_{ij}(u, X_u) \cdot (\partial_{ij} T_u) \circ X_u du \\
& + \; \int_\epsilon^t T(du) \circ X_u
\end{aligned}
\tag{4.97}
$$

and in the sense of divergences:

$$
M_t^{(\epsilon)} = \sum_j \int_\epsilon^t \sum_i [b_i(u, X_u) \cdot (\partial_i T_u) \circ X_u] \, dW_u^j
\tag{4.98}
$$

Of course, theorem 4.1.1.2 also extends into the following:

Theorem 4.4.3.2. *Let X and T be as in the previous theorem. Consider two functions $b_Y, \sigma_Y \in C^{(0,s)}$ and let Y be the only strong solution of the stochastic differential equation:*

$$
\begin{aligned}
dY_t \;&=\; b(t, Y_t)dt + \sigma(t, Y_t)dW_t & (4.99) \\
Y_0 \;&=\; y & (4.100)
\end{aligned}
$$

Finally let $f \in BV(C^s)$. Then the Itô formula holds for:

$$
f(t, Y_t) \cdot T_t \circ X_t
\tag{4.101}
$$

Let us note that we do not need to make any assumption on the nondegeneracy of Y.

4.5   Comparison to other extensions of the Itô formula

4.5.1   Extensions of the Follmer-Protter-Shiryaev type

We start by showing that from our Itô formula, we can recover a formula of the type of the Follmer-Protter-Shiryaev formula given in [20]:

Theorem 4.5.1.1 (Föllmer-Protter-Shiryaev formula). *Let f be a differentiable function such that $Df \in L^2_{loc}\left(\mathbb{R}^N, \mathbb{R}^N\right)$ and let W be a N-dimensionnal Brownian motion. Then:*

$$f(W_t) = f(0) + \sum_{i=1}^N \partial_i f(W_s) dW_s + \frac{1}{2} \cdot [f'(W), W]_t \qquad (4.102)$$

where the quadratic covariaation $[f'(W), W]$ is defined as the limit of Riemann sums:

$$[f'(W), W]_t = \sum_{i=1}^N \lim_{n\to\infty} \sum_{k=1}^n \left(f'(W_{\frac{k}{n}}) - f'(W_{\frac{k-1}{n}}) \right) \cdot \left(W_{\frac{k}{n}} - W_{\frac{k-1}{n}} \right) \qquad (4.103)$$

Now, suppose that we are in the setting of theorem 4.4.3.1. Suppose, in addition, that T_t has values, say, in $\mathcal{S}_{p,1}$ or in $W^{p,1}$. Then, the T_t and its first order derivatives are proper functions with at least L^p regularity so the Itô formula reads:

$$
\begin{aligned}
T(t, X_t) - T(\epsilon, X_\epsilon) \;=\; & \sum_i \int_\epsilon^t b_i(u, X_u) \cdot \partial_i T(u, X_u) du \\
& + \sum_j \int_\epsilon^t \sum_i [b_i(u, X_u) \cdot \partial_i T(u, X_u)] \, dW_u^j \\
& + \int_\epsilon^t T(du, X_u) \\
& + \frac{1}{2} \sum_{i,j} \int_\epsilon^t a_{ij}(u, X_u) \cdot (\partial_{ij} T_u) \circ X_u du \qquad (4.104)
\end{aligned}
$$

In the equation above, only the terms in the last line are distributions on the Wiener space, while all the other are proper random variables. Therefore, one expects that it is possible to let ϵ tend to 0, and to recover a usual (functional) Itô formula in this case. Also, it should be possible to identify the distribution term above with the quadratic covariation term in the Föllmer-Protter-Shiryaev formula.

First, we prove that this holds indeed in the uniformly elliptic case, ie if the function a is uniformly bounded from below. Indeed, in [35] it is proved that under this condition, and if the coefficients b and σ are C_b^∞ uniformly in time, then the process X_t has a smooth density $p_{X_t}(y)$ which verifies:

$$c^{-1}t^{-N/2}\exp\left(-\frac{c\|y-x\|^2}{t}\right) \leq p_{X_t}(y) \leq C^{-1}t^{-N/2}\exp\left(-\frac{C\|y-x\|^2}{t}\right) \quad (4.105)$$

Here, c and C are two constants, and we recall that $X_0 = x$ a.s. Moreover, by taking a close look at Kohatsu-Higa's proof, one notices that the most irregular distributions playing a role are of the type $\delta_y' \circ X_t$. Therefore, the proof is still valid if b and σ are such that these distributions are well defined. Recall that the Dirac mass δ_y is in the space $\mathcal{S}_{1+\epsilon, -\frac{N\epsilon}{1+\epsilon}}$ for any ϵ. Recall also that the uniform ellipticity hypothesis implies uniform non-degeneracy of the proces X. Therefore, by following the proof by Kohatsu-Higa and using theorem 3.2.0.13 we extend his result to the following:

Theorem 4.5.1.2 (Density control under uniform ellipticity). *Let X_t be the strong solution of the SDE*

$$
\begin{aligned}
dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t & (4.106) \\
X_0 &= x & (4.107)
\end{aligned}
$$

where $b, \sigma \in C^{0,2+\epsilon}$ and a is uniformly bounded from below. Then X_t admits a continuous and bounded density p_{X_t} at each $t > 0$ and one has:

$$c^{-1}t^{-N/2}\exp\left(-\frac{c\|y-x\|^2}{t}\right) \leq p_{X_t}(y) \leq C^{-1}t^{-N/2}\exp\left(-\frac{C\|y-x\|^2}{t}\right) \quad (4.108)$$

for some constants c, C.

Remark 4.5.1.1. *Because of the uniform ellipticity we do not need to assume that the coeficients are Hölder in time as we did in theorem 4.4.3.1 as this hypothesis was used only to prove the non-degeneracy. Also, if we suppose more space regularity on the coefficients, then the X_t have more Malliavin regularity according to theorem 4.4.1.3, and hence p_{X_t} has more regularity because of theorem 3.4.0.18*

Remark 4.5.1.2. *Under the conditions of theorem 4.5.1.2 for all $t > 0$ X_t has full support.*

We give another result of the same type . In [42], sections IV.11 and following, the following result is given:

Theorem 4.5.1.3 (Fundamental solution to a parabolic Cauchy problem). *Consider a second order parabolic operator $\mathcal{P}(t, x, \partial_t, \partial_x)$ on $]0, T[\times\mathbb{R}^N$. Suppose that, for some $0 < \alpha < 1$ the coefficients of $\mathcal{P}$ are jointly $\frac{\alpha}{2}$-Hölder in time and α-Hölder in space. Let $0 < \tau < t$; $\xi \in \mathbb{R}^N$. Then for $\tau \leq t \leq T$ there exists a fundamental solution Z to the equation:*

$$\mathcal{P}(t, x, \partial_t, \partial_x) Z_{\tau,\xi}(t, x) = \delta_\tau \otimes \delta_\xi \qquad (4.109)$$

which is continuous and bounded in x for any t. Moreover, there exisits a number $C > 0$ such that Z verifies:

$$|Z_{\tau,\xi}(t, x)| \leq C \cdot \frac{1}{(t - \tau)^{-\frac{N}{2}}} \cdot \exp\left(-C\frac{|x - \xi|^2}{t - \tau}\right) \qquad (4.110)$$

The same reference also provides estimates for small (fractional) order derivatives of Z. The unicity of the Cauchy problem is adressed to separately in the same book. Now, consider the SDE (4.106, 4.107) and suppose that its coefficients are Lipschitz in space and α-Hölder in time for some $\alpha > 0$. Then on one hand our SDE admits a unique strong solution, and on the other hand the parabolic operator associated to the SDE verifies the conditions of theorem 4.5.1.3. Therefore X_t has to have a continuous and bounded density at any $t > 0$ which verifies a majorization of the above type and we have:

Theorem 4.5.1.4 (Density existence and control without uniform ellipticity). *Let X_t be the strong solution of the SDE (4.106, 4.107) where b, σ are α-Hölder in time for some $\alpha > 0$ and Lipschitz in space. Then X_t admits a continuous and bounded density p_{X_t} at each $t > 0$ and one has:*

$$p_{X_t}(y) \leq C \cdot \frac{1}{(t - \tau)^{-\frac{N}{2}}} \cdot \exp\left(-C\frac{|x - \xi|^2}{t - \tau}\right) \qquad (4.111)$$

for some constant $C > 0$.

We note that although the second theorem seems easier to apply, it does not bound the density from below.

We now prove the following:

Theorem 4.5.1.5. *Suppose that T_t is a BV path taking values in $\mathcal{S}_{p,1}$ (or $W_{p,1}$) for some $p \geq 2$ and that X verifies the conditions of either theorem 4.5.1.3 or 4.5.1.2. Then it makes sense to let ϵ tend to 0 in formula (4.104)*

Proof. We only need to prove that every integral is well defined in L^p as ϵ tends to 0. More precisely, we prove the L^p convergence of every integral except the one

involving distributions on the Wiener space. Then, by elementary algebra, this integral has to converge, too.

The proof of the convergence of every term relies on ideas similar to those we developed in the proof of theorem 4.1.1.1. For example,

$$\left\| \int_0^t b(u, X_u) \partial_i T(u, X_u) du \right\|_{L^p} \leq \|b\|_\infty \int_0^t \|\partial_i T(u, X_u)\|_{L^p} du \qquad (4.112)$$

and

$$\|\partial_i T(u, X_u)\|_{L^p}^p = \int_{\mathbb{R}^N} |\partial_i T(u, x)|^p p_{X_u}(x) dx \qquad (4.113)$$

so finally

$$\left\| \int_0^t b(u, X_u) \partial_i T(u, X_u) du \right\|_{L^p} \leq \|b\|_\infty \int_0^t \|p_{X_u}\|_\infty^{1/p} \partial_i(u, \cdot)\|_{L^p} du \qquad (4.114)$$

Now $u \mapsto \|\partial_i(u, \cdot)\|_{L^p}$ is continuous because of the hypotheses on T and $\|p_{X_u}\|_\infty$ is controled by theorem 4.5.1.2, so finally this last integral is finite. Therefore we may apply the dominated convergence theorem to obtain our convergence.

The Brownian term is dealt with in a similar way, using the Burkholder Davies Gundy theorem. For the BV integral, notice that:

$$E\left[\left| \int_0^t T(du, X_u) \right| \right] \leq \int_0^t |T|(du, p_{X_u}) \qquad (4.115)$$

$\square$

Remark 4.5.1.3. *In [47] Moret and Nualart give a similar result for a general semimartingale. We only the specialization of their result to solutions of SDEs, as obtaining the existence and the regularity of a density for a general semimartingale is a more difficult problem than doing it just for the solution of an SDE.*

Remark 4.5.1.4. *In [20] the order 2 term is expressed as a quadradtic covariation rather than a $\mathbb{D}'$-valued integral, so we have the identification:*

$$\frac{1}{2} \sum_{i,j} \int_0^t a_{ij}(u, X_u) \cdot (\partial_{ij} T_u) \circ X_u du = \frac{1}{2} [(Df)(X), X]_t \qquad (4.116)$$

Remark 4.5.1.5. *We need that $p \geq 2$ because we apply the Burkholder Davies Gundy theorem to the Brownian integral. See the proof of theorem 4.1.1.1 for details.*

We may extend the above result a little bit in the specific case where $N = 1$ and the coefficients b and σ are time homogeneous. Indeed in [7] the following result is proved:

Theorem 4.5.1.6. *Let $b \in C_b^2(\mathbb{R})$ and $\sigma \in C^{n \vee 2}(\mathbb{R})$ for some $n \in \mathbb{N}$ and $x \in \mathbb{R}$ such that:*

$$\sigma(x) = \cdots = \sigma^{(n-1)}(x) = 0 \qquad (4.117)$$

and

$$b(x)\sigma^{(n)}(x) \neq 0 \qquad (4.118)$$

Let X_t be the solution of the SDE:

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \qquad (4.119)$$

Then at all $t > 0$, X_t admits a density $p_{X_t} \in C_b^0(\mathbb{R})$ such that:

$$p_{X_t}(x) \leq \frac{C}{t^{n+\frac{1}{2}}} \qquad (4.120)$$

Remark 4.5.1.6. *The conditions on the volatility exactly express hypoellipticity as in dimension 1 the Lie algebra generated by the volatility function and its derivatives is of dimension at most 1. The case where $n = 0$ corresponds to ellipticity. If $a = \sigma^2$ is bounded from below, one has unifrom ellipticity, which implies that one may take $n = 0$ i the above framework. Finally, the differentiability conditions on b and σ are slightly less demanding in this case than in theorem 4.4.3.1, and extra regularity may be proven for the density by using theorem 3.4.0.18.*

From this result and our weak Itô forula we recover the following result which is contained in [7]:

Theorem 4.5.1.7. *Let $\sigma, b \in C_b^2(\mathbb{R})$, let $x \in \mathbb{R}$ such that $b(x)\sigma(x) \neq 0$, and let X_t be the solution of:*

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \qquad (4.121)$$

Then, for any BV path T_t with values in $\mathcal{S}_{p,1}$ (resp $W^{p,1}$) one may let ϵ tend to zero in formula (4.104).

Proof. Similarly to above we start with our weak Itô formula and we only need to show that all the integrals starting at 0 do exist in L^p. We proceed as in the proof of theorem 4.5.1.5 and we write:

$$\left\| \int_0^t b(X_u)\partial_i T(u, X_u)du \right\|_{L^p} \leq \|b\|_\infty \int_0^t \|p_{X_u}\|_\infty \|\partial_i(u, \cdot)\|_{L^p}du \qquad (4.122)$$

In the above inequation, the density exists and the right hand side integral is finite because of theorem 4.5.1.6. The other terms are dealt with similarly. $\square$

Remark 4.5.1.7. *Once again we expressed the crochet in the Follmer-Protter-Shiryaev formula as an integral in $\mathbb{D}'$. $n = 0$ is needed to obtain the t-integrability of p_{X_t}; this is a slight improvement over the condition of uniform ellipticity required in the previous theorem.*

4.5.2 Around the local time

Results in dimension 1

Let (X_t) be as in theorem 4.4.3.1, with $N = 1$. If $L_t^K(X)$ is the local time of the process X at time t and point K, then the Tanaka formula reads:

$$
\begin{aligned}
&|X_t - K| - |x - K| \\
&= \int_0^t sgn(X_s - K)b(s, X_s)ds + \int_0^t sgn(X_s - K)\sigma(s, X_s)dW_s + \frac{1}{2}L_t^K(X)
\end{aligned}
$$
(4.123)

Moreover the function $x \mapsto |x - K|$ is in $\mathcal{S}_{p,-\alpha}$ for any $\alpha > 1 - \frac{1}{p}$ and the function $x \mapsto sgn(x - K)$ is in $\mathcal{S}_{p,-\alpha}$ for any $\alpha > -\frac{1}{p}$. We therefore may apply theorem 4.4.3.1 and we obtain:

$$
\begin{aligned}
&|X_t - K| - |X_\epsilon - K| \\
&= \int_\epsilon^t sgn(X_s - K)b(s, X_s)ds + \int_\epsilon^t sgn(X_s - K)\sigma(s, X_s)dW_s \\
&+ \frac{1}{2}\int_\epsilon^t \sigma(s, X_s)^2\delta_K \circ X_s ds
\end{aligned}
$$
(4.124)

As in the previous section, it is tempting to let ϵ tend to 0 in the above and to obtain the identification:

$$
L_t^K(X) = \int_0^t \sigma(s, X_s)^2\delta_K \circ X_s ds
$$
(4.125)

which is a time disintegration of the local time in a negative order Sobolev space, in the sense of Pettis integrals. The results in the previous section do not apply directly as the absolute value and the sign are not regular enough functions; we will however prove that similar (and more precise) results do hold. We start with the:

Lemma 4.5.2.1. *Consider a non-degenerate $X \in \mathbb{D}_{\infty^-,2}$ and $f \in C_b^1$. Then the quantities below are well defined and:*

$$
f(X) \cdot \delta_K \circ X = f(K) \cdot \delta_K \circ X
$$
(4.126)

Proof. First, since we work in dimension 1, we note that $\delta_K \in \mathcal{S}_{p,-(1-\frac{1}{p})-\epsilon}$ for any $1 < p < \infty$ and $\epsilon > 0$. In particular, let us choose ϵ such that $1 - \frac{1}{p}) - \epsilon < 1$. Then by theorem 3.2.0.15 $\delta_K \circ X$ is well defined in $\mathbb{D}_{p,-(1-\frac{1}{p})-\epsilon}$. Also by the chain

rule $f(X) \in \mathbb{D}_{p^*,1} \subset \mathbb{D}_{p^*,(1-\frac{1}{p})+\epsilon}$, so by Hölder's theorem the left hand side is well defined, at least in $\mathbb{D}_{1,-(1-\frac{1}{p})-\epsilon}$.

Also, by equation (3.94) for any $\phi \in \mathbb{D}$, $p_{X,\phi}$ is well defined and there comes:

$$
\begin{aligned}
\langle f(X) \cdot \delta_K \circ X, \phi \rangle &= E\left[f(X)\phi | X = K\right] p_{X,\phi}(X) \\
&= E\left[f(K)\phi | X = K\right] p_{X,\phi}(X) \\
&= \langle f(K) \cdot \delta_K \circ X, \phi \rangle
\end{aligned}
\tag{4.127}
$$

so we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we will prove the following:

Theorem 4.5.2.1. *Let X be as in theorem 4.5.1.2. Then, for any K:*

$$
L_t^K(X) = \int_0^t \sigma(s,K)^2 \delta_K \circ X_s ds
\tag{4.128}
$$

and for any $r > 2$, $\alpha < \frac{1}{r}$:

$$
L_t^K(X) \in \mathbb{D}_{r,\alpha}
\tag{4.129}
$$

Proof. First we shall prove that under these conditions it is legitimate to let ϵ tend to 0 in equation (4.124). By theorem 4.4.1.3, for any $t > 0$, X_t is non-degenerate and $X_t \in \mathbb{D}_{\infty^-,2}$. Also, X_t admits a density p_{X_t} which is as in theorem 4.5.1.2. In particular, this density is continuous and bounded. We therefore may apply lemma 3.2.0.6 for $\gamma = 1$ and we obtain that $1_{X_t > K} \in \mathbb{D}_{p,\alpha}$, and for some ρ which does not depend on t and some constant $C(p,\alpha)$, $\alpha < \frac{1}{r}$:

$$
\|1_{X_t > K}\|_{\mathbb{D}_{p,\alpha}} \leq C \left(1 + \|\nabla X_t\|_{L^\rho}\right) \|p_{X_t}\|_\infty
\tag{4.130}
$$

Now since $p \geq 2$ by the Burkholder Davies Gundy theorem and Hölder's inequality:

$$
\left\| \int_0^t \sigma(s,X_s) sgn(X_s - K) dW_s \right\|_{\mathbb{D}_{p,\alpha}} \leq C \int_0^t \|\sigma(s,X_s)\|_{\mathbb{D}_{q,\alpha}} \|sgn(X_s - K)\|_{\mathbb{D}_{r,\alpha}}
\tag{4.131}
$$

with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Clearly:

$$
\|\sigma(s,X_s)\|_{\mathbb{D}_{q,\alpha}} \leq \|\sigma(s,X_s)\|_{\mathbb{D}_{q,1}} \leq \|\sigma\|_\infty \|X_s\|_{\mathbb{D}_{q,1}}
\tag{4.132}
$$

Also, writing the sign as the difference of two indicators:

$$
\|sgn(X_s - K)\|_{\mathbb{D}_{r,\alpha}} \leq C \left(1 + \|\nabla X_s\|_{L^\rho}\right) \|p_{X_s}\|_\infty
\tag{4.133}
$$

Now putting everything together and controlling the different terms through theorems 4.4.1.3 and 4.5.1.2 the integrability is clear. The other terms are dealt with similarly so we have obtained:

$$L_t^K(X) = \int_0^t \sigma(s, X_s)^2 \delta_K \circ X_s ds \in \mathbb{D}_{r,\alpha} \tag{4.134}$$

Now, on one hand the previous lemma applies, and on the other hand one can take r as close to 2 as one chooses, so we are done. □

Remark 4.5.2.1. *A more general result of that type was given in [3]; we simply want to show how the ideas in theorem 4.1.1.1 relate to the Tanaka formula. Let us especially note how the term:*

$$\int_{\mathbb{R}} L_t^x(X) T''(dx) \tag{4.135}$$

which appears in the Itô-Tanaka formula is related to the following term from our Itô formula:

$$\int_0^t \sigma(s, X_s)^2 \cdot T'' \circ X_s ds \tag{4.136}$$

These terms coincide when they are both defined but the second one may be defined for more general distributions than just Radon measures if X is regular enough.

We may also give another result related to the one in [3]. More precisely, we will prove:

Theorem 4.5.2.2. *Let X be as in theorem 4.5.1.2. Let $p > 2$ and $\alpha < \frac{1}{p}$. Then, for any t, $K \in \mathbb{R} \mapsto L_t^K(X) \in \mathbb{D}_{p,\alpha}$ is a continuous function.*

Proof. Consider $K_1 < K_2 \in \mathbb{R}$. One sees that $1_{]K_1, K_2[} \in W_{p,\alpha}$ for any $\alpha < \frac{1}{p}$. Indeed, $1_{]K_1, K_2[} \in L^p$ and $\nabla 1_{]K_1, K_2[} = \delta_{K_1} - \delta_{K_2} \in W_{p,\alpha-1}$. Therefore by lemma 3.2.0.5, for some q:

$$\begin{aligned}
\|1_{]K_1, K_2[}(X_t)\|_{\mathbb{D}_{p,\alpha}} &\leq C\|p_{X_t}\|_\infty (1 + \|\nabla X\|_{L^q}) \|1_{]K_1, K_2[}\|_{W^{p,\alpha}} \\
&\leq C\|p_{X_t}\|_\infty (1 + \|\nabla X\|_{L^q}) \\
&\quad \cdot \left(|K_1 - K_2| + \|(Id - \Delta)^{-\frac{1-\alpha}{2}} (\delta_{K_2} - \delta_{K_1})\|_{L^p} \right)
\end{aligned} \tag{4.137}$$

Now, for any $\beta > 1 - \frac{1}{p}$, $K \in \mathbb{R} \mapsto (Id - \Delta)^{-\beta} \delta_K \in L^p$ is continuous, so we may control the RHS. We obtain our result by writing $L_t^{K_1}(X) - L_t^{K_1}(X)$ with the Tanaka formula, mimicking the proof of theorem 4.5.2.1 and using the above estimate. □

Actually, the proof above gives an even more precise result, which means that Malliavin regularity may be traded off for trajectory regularity:

Theorem 4.5.2.3. *Let X be as in theorem 4.5.1.2. Let $p > 2$, $\alpha < \frac{1}{p}$ and $\gamma < \alpha$. Then, for any t, $K \in \mathbb{R} \mapsto L_t^K(X) \in \mathbb{D}_{p,\alpha-\gamma}$ is a γ-Hölder function.*

From this, we may recover an already well-known result:

Corollary 4.5.2.1. *Let X be as in theorem 4.5.1.2. Then, for any t, the (space) trajectories of $L_t(X)$ are almost surely γ-Hölder for any $\gamma < \frac{1}{2}$.*

We aready know that this corollary is optimal (cf, for example, the case of the Brownian motion), therefore there is little hope to extend theorem 4.5.2.1 to the case $p < 2$. Actually, in [3] it is shown that the Brownian motion provides a counter example. Let us also note that the case $p = 2$ has been dealt with by other methods (chaos expansion) for the Brownian motion only, see [51] and [50].

Here is another result of the same type:

Theorem 4.5.2.4. *Let X be as in theorem 4.5.1.2. Then $\frac{\partial}{\partial K} L_t^K(X)$ is well defined in $\mathbb{D}_{p,-(1-\frac{1}{p})-\epsilon}$ for any $\epsilon > 0$, and:*

$$\frac{\partial}{\partial K} L_t^K(X) = 2 \int_0^t \sigma(s,K) \frac{\partial}{\partial K} \sigma(s,K) \delta_K \circ X_s ds - \int_0^t \sigma(s,K)^2 \delta_K' \circ X_s ds \quad (4.138)$$

Finally, $K \in \mathbb{R} \mapsto L_t^K(X) \in \mathbb{D}_{p,-(1-\frac{1}{p})-\epsilon}$ is a continuous function.

It is interesting to compare this with the following result from [13]: if W is a brownian motion, then for every t, $x \mapsto L_t^x(W)$ is a semimartingale in its own filtration and there is the:

Theorem 4.5.2.5 (Bouleau-Yor formula). *Let f be a differentiable function on $\mathbb{R}$, with a locally bounded derivative and let W be a 1-dimensionnal Brownian motion. Then:*

$$f(W_t) = f(0) + \int_0^t f'(W_s) dW_s - \frac{1}{2} \cdot \int_{\mathbb{R}} f'(x) d_x L_t^x(W) \quad (4.139)$$

It is then possible to show that the term:

$$\int_{\mathbb{R}} f'(x) d_x L_t^x(X) \quad (4.140)$$

coincides with one of the type:

$$\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x} L_t^x(W) dx \quad (4.141)$$

where the latter is defined as a Pettis integral in some negative order Gross-Sobolev space. Also, such a quantity could be defined for a more general stochastic process than the Brownian motion, ie one which verifies the hypotheses in theorem 4.5.1.2.

Results in multiple dimensions

Some authors have shown how to define the local time of a multi-dimensionnal Brownian motion as a distribution on the Wiener space, see for example [27] and the references therein. In this paragraph we will show how the method we developped in the previous paragraph allows us to extend this idea to more general diffusions.

We start be recalling that, if w_n denotes the volume of the n-dimensionnal unit ball, the fundamental solution E_n of the Laplace operator is defined as follows:

$$E_2(x) = \frac{1}{w_2} \log(|x|) = \frac{1}{2\pi} \log(|x|) \tag{4.142}$$

and if $N > 2$:

$$E_N(x) = -\frac{1}{N(N-2)w_n} \cdot |x|^{2-N} \tag{4.143}$$

By design, $\Delta E_n = \delta_0$. Also, it can be checked that $E_n \in \mathcal{S}_{p,-N(1-\frac{1}{p})-\epsilon}$ for any $\epsilon > 0$. Now let X be the unique strong solution of the SDE:

$$dX_t^{(i)} = b_i(s, X_s)ds + \sigma_i(s, X_s)dW_s^{(i)} \tag{4.144}$$
$$X_0 = x \tag{4.145}$$

Suppose that the conditions of theorem 4.4.3.1 hold for some $s > N(1-\frac{1}{p})$ (if the coefficients are Hölder in time and $2 + \epsilon$ in space this is verified for small enough p). Then we may apply the weak Itô formula and we obtain:

$$E_N(X_t - K) - E_N(X_\epsilon - K)$$
$$= \int_\epsilon^t b_i(s, X_s)\partial_i E_N(X_s - K)ds + \int_\epsilon^t sgn(X_s - K)\sigma(s, X_s)dW_s$$
$$+ \frac{1}{2}\int_\epsilon^t |\sigma(s, X_s)|^2 \delta_K \circ X_s ds \tag{4.146}$$

Now we may apply the same method as in the dimension 1 case in order to prove that under these conditions:

Theorem 4.5.2.6. *Suppose that the conditions of theorem 4.4.3.1 hold for some $s > N(1-\frac{1}{p})$ and $p > 2$. Then for any $\epsilon > 0$ the local time of X is well defined in $\mathbb{D}_{p,1-N(1-\frac{1}{p})-\epsilon}$ as the following Bochner integral:*

$$L_t^K(X) := \int_0^t |\sigma(s, K)|^2 \delta_K \circ X_s ds \tag{4.147}$$

Let us remark that since there has to be $p > 2$ in the theorem above, unless $N = 1$ there always is:

$$1 - N \cdot \left(1 - \frac{1}{p}\right) < 0 \tag{4.148}$$

Therefore, the local time of a multi-dimensionnal process is not a proper random variable in general. This is the case even for very regular processes: for example it is well known that the brownian local time is not defined as a stochastic process in dimension $N > 2$.

We finish this section with the following result, which is obtained by applying the Itô formula to $\mathcal{K}^{-\delta} E_n$ rather than E_n:

Theorem 4.5.2.7. *Let X be as in the previous theorem, $p > 2$ and $t > 0$.*

$$\alpha < 1 - N \cdot \left(1 - \frac{1}{p}\right) < 0 \tag{4.149}$$

Then, for any $\gamma > 0$, the function:

$$K \in \mathbb{R}^N \mapsto L_t(K) \in \mathbb{D}_{p,\alpha-\gamma}\left(\mathbb{R}^N\right) \tag{4.150}$$

is γ-holderian.

This allows us to understand our weak Itô formula as a multi-dimensionnal, weak extension of the Bouleau-Yor formula.

Chapter 5

Applications to some problems in analysis

5.1 Variational inequations in a markovian frame-work

In this section, we still consider $X_t^u(x)$ the unique strong solution on $[u, T]$ of the stochastic differential equation (4.77) such that $X_u = x$. Here, we suppose that the assumptions of theorem 4.4.3.1 hold for some $s > 2$, $n \leq k - 2$ and $\beta > 0$. We will note $s = 1 + \delta$. We note A_t for the generator of X:

$$A_t = \sum a_{i,j}(t, x)\partial_i\partial_j + \sum b_i(t, x)\partial_i \tag{5.1}$$

and:

$$K_t = \partial_t + A_t \tag{5.2}$$

We note $Z_{t,T}$ the set of all stopping times τ such that $t \leq \tau \leq T$ a.s. Now we give ourselves some continuous, bounded and positive functions f and g and a $C^{(0,s)}$, positive function r and we consider the following problem:

$$
\begin{aligned}
K_t u - ru &\leq -g \tag{5.3} \\
u &\geq f \tag{5.4} \\
(K_t u - ru)(f - u) &= 0 \tag{5.5} \\
u(T, \cdot) &= f \tag{5.6}
\end{aligned}
$$

We will note $(\mathcal{A})$ for the set of assumptions we made on X, f, g, etc. and $(\mathcal{V})$ for the variational problem we consider. Now we prove:

Theorem 5.1.0.8. *Under* $(\mathcal{A})$, *let* $u \in BV(\mathcal{C}^0)$ *be a solution of the problem* $(\mathcal{V})$. *Then:*

$$u(t, x) = \sup_{\tau \in Z_{t,T}} E \left[f(X_\tau^t(x)) \exp\left(-\int_t^\tau r(u, X_u^t(x))du \right) \right.$$
$$\left. + \int_t^\tau g(s, X_s^t(x)) \exp\left(-\int_t^s r(u, X_u^t(x))du \right) ds \right] \quad (5.7)$$

Proof. We will only prove the case where $t = 0$, the other cases being similar. We note $X_t = X_t^0(x)$. We also introduce:

$$l_t = \exp\left(-\int_0^t r(u, X_u)du \right) \quad (5.8)$$

We note that we have the injection $\mathcal{C}^0 \hookrightarrow \mathcal{S}_{p,-N/p-\epsilon}$ for any $\epsilon > 0$. Therefore, for p big enough and ϵ small enough so that $N/p + \epsilon < s - 1$, we may apply the generalized Itô formula in theorem 4.4.3.2 and for any $\epsilon > 0$ we obtain the following:

$$l_t u(t, X_t) - l_\epsilon u(\epsilon, X_\epsilon) - \int_\epsilon^t l_s \left((A_s u)(s, X_s) - (ru)(s, X_s) \right) ds - \int_\epsilon^t l_s u(ds, X_s) = M_t$$
$$(5.9)$$

for some weak martingale M. Also $K_t u - ru + g$ is a negative distribution, therefore the following is a positive distribution on the Wiener space:

$$D_t := M_t - l_t u(t, x_t) + l_\epsilon u(\epsilon, X_\epsilon) - \int_\epsilon^t l_s g(s, X_s)ds \quad (5.10)$$

Now for $\alpha > 0$ introduce the processes

$$v_t^\alpha = P_\alpha \left[l_t u(t, X_t) + \int_0^t l_s g(s, X_s)ds \right] \quad (5.11)$$

and

$$M_t^\alpha = P_\alpha M_t \quad (5.12)$$

These have moments of all orders. We note that the following is a positive random variable:

$$D_t^\alpha := P_\alpha D_t = M_t^\alpha - v_t^\alpha + P_\alpha \left[l_\epsilon u(\epsilon, X_\epsilon) + \int_0^\epsilon l_s g(s, X_s)ds \right] \quad (5.13)$$

Let τ be any stopping time; we evaluate at $t = \tau$ and take expectations. M^α is a uniformly integrable martingale starting from 0, so:

$$E[M_\tau^\alpha] = 0 \quad (5.14)$$

We know that the Ornstein-Uhlenbeck semigroup preserves the expectations, so:

$$E\left[P_\alpha\left[l_\epsilon u(\epsilon, X_\epsilon) + \int_0^\epsilon l_s g(s, X_s)ds\right]\right] = E\left[l_\epsilon u(\epsilon, X_\epsilon) + \int_0^\epsilon l_s g(s, X_s)ds\right] \quad (5.15)$$

Finally:

$$v_\tau^\alpha = P_\alpha\left[l_t u(t, X_t) + \int_0^t l_s g(s, X_s)ds\right]_{|t=\tau} \quad (5.16)$$

Since for any p $l_t u(t, X_t) + \int_0^t l_s g(s, X_s)ds$ is bounded in L^p (by $\|u\|_\infty + T\|g\|_\infty$), the process $l.u(\cdot, X.) + \int_0^\cdot l_s g(s, X_s)ds$ is uniformly integrable. Therefore, so is v_t^α as a bivariate process, and letting α tend to 0 we obtain:

$$E[v_\tau] = E\left[l_\tau u(\tau, X_\tau) + \int_0^\tau l_s g(s, X_s)ds\right] \quad (5.17)$$

Putting everything together one obtains:

$$E\left[l_\epsilon u(\epsilon, X_\epsilon) + \int_0^\epsilon l_s g(s, X_s)ds\right] \geq E\left[l_\tau u(\tau, X_\tau) + \int_0^\tau l_s g(s, X_s)ds\right] \quad (5.18)$$

Since this is for any stopping time τ and for any ϵ, letting ϵ tend to 0 there comes:

$$u(0, x) \geq \sup_{\tau \in Z_{0,T}} E[l_\tau u(\tau, X_\tau)] \quad (5.19)$$

Now to finish the proof we just need to find one stopping time τ_0 for which $u(0, x) = E[l_{\tau_0} u(\tau_0, X_{\tau_0})]$ stands. Consider the open set:

$$D = \{(t, y), u(t, y) \neq f(y)\} \quad (5.20)$$

and define a stopping time as:

$$\tau_x = \inf\{t, (t, X_t) \notin D\} \quad (5.21)$$

If $\tau_x = 0$ a.s. then $u(0, x) = f(x)$ and it is clear that τ_x realizes the supremum in this case. Otherwise, from the $0 - 1$ law for the Brownian filtration, $\tau_x > 0$ a.s. Now let $(u_m, m \in \mathbb{N})$ be a smooth approximation of u. By the usual Itô formula, for every m and for every $t > \epsilon > 0$:

$$l_{t \wedge \tau_x} u_m(t \wedge \tau_x, X_{t \wedge \tau_x}) - l_{\epsilon \wedge \tau_x} u_m(\epsilon \wedge \tau_x, X_{\epsilon \wedge \tau_x})$$
$$= \int_{\epsilon \wedge \tau_x}^{t \wedge \tau_x} l_s\left(\partial_s + A_s + r\right) u_m(s, X_s)ds + M_t^{(m,n,\epsilon)} \quad (5.22)$$

where $(M^{(m,n,\epsilon)}, t > \epsilon)$ is a martingale in the Brownian filtration starting at ϵ. Now let $\phi \in \mathbb{D}$. One has:

$$
\begin{aligned}
E\left[\phi \cdot \int_{\epsilon \wedge \tau_x}^{t \wedge \tau_x} l_s \left(\partial_s + A_s + r\right) u_m(s, X_s) ds\right] & \\
= \; & E\left[\phi \cdot \int_\epsilon^t l_s \left(\partial_s + A_s + r\right) u_m(s, X_s) 1_{(s,X_s) \in D_x}\right] ds \\
= \; & \int_\epsilon^t E\left[l_s \phi \left(\partial_s + A_s + r\right) u_m(s, X_s) 1_{(s,X_s) \in D_x} ds\right] \\
= \; & \int_\epsilon^t \int \left(\partial_s + A_s + r\right) u_m(s, y) 1_{(s,y) \in D_x} p_{X_s, l_s \phi}(y) dy ds \\
= \; & \int \int_{]\epsilon, t[\times \mathbb{R}^N \cap D} \left(\partial_s + A_s + r\right) u_m(s, y) p_{X_s, l_s \phi}(y) dy ds \qquad (5.23)
\end{aligned}
$$

Similarly:

$$
E\left[\phi \cdot \int_{\epsilon \wedge \tau_x}^{t \wedge \tau_x} l_s g(s, X_s) ds\right] = \int \int_{]\epsilon, t[\times \mathbb{R}^N \cap D} g(s, y) p_{X_s, l_s \phi}(y) dy ds \qquad (5.24)
$$

By theorem 3.4.0.18 we know that for $s > 0$:

$$
p_{X_s, l_s \phi} \in \mathcal{S}_{p, \delta} \qquad (5.25)
$$

Then, by writing the integration by parts:

$$
p_{X_s, l_s \phi}(x) = E\left[(Id + \mathcal{L})^{\delta/2} (l_s \cdot \phi) \cdot (Id + \mathcal{L})^{\delta/2} (\delta_x \circ x_s)\right] \qquad (5.26)
$$

one checks that the following path is continuous (for any value of p):

$$
s \in]\epsilon, T[\mapsto p_{X_s, l_s \phi} \in \mathcal{S}_{p, \delta} \qquad (5.27)
$$

Now we know that $\left(\partial_s + A_s + r\right) u_m - g$ converges to 0 in the sense of distributions on D; more precisely on D we have the equality:

$$
\Sigma \sigma_{ij} \partial_{ij} u = -\partial_t u - g + ru - \Sigma_i b_i \partial_i u \qquad (5.28)
$$

and $u \in \mathcal{S}_{p, -\frac{N}{p} - \epsilon}$ for any p so the convergence takes place in $\mathcal{S}_{p, -\frac{N}{p} - 1 - \epsilon}(D)$; in particular it takes place in $\mathcal{S}_{p, -\delta}(D)$ for small enough p. Choosing such a small p and p^* for the regularity of $p_{X_s, l_s \phi}$ one finally gets:

$$
\int_{\epsilon \wedge \tau_x}^{t \wedge \tau_x} l_s \left[\left(\partial_s + A_s + r\right) u_m(s, X_s) + g(s, X_s)\right] ds \to 0 \qquad (5.29)
$$

at least in $\mathbb{D}'$. We therefore have:

$$l_{t \wedge \tau_x} u(t \wedge \tau_x, X_{t \wedge \tau_x}) - l_{\epsilon \wedge \tau_x} u(\epsilon \wedge \tau_x, X_{\epsilon \wedge \tau_x}) + \int_{\epsilon \wedge \tau_x}^{t \wedge \tau_x} l_s g(s, X_s) ds = M_t^{(\epsilon)} \quad (5.30)$$

where $M^{(\epsilon)}$ is a weak martingale. Now $M^{(\epsilon)}$ is a proper, bounded in L^p process because the LHS is, so it is straightforward to check that $M^{(\epsilon)}$ is a martingale in the usual sense. In particular the expectations of the LHS and the RHS both are 0 at any time. Also, the LHS is bounded surely by $2\|u\|_\infty + T\|g\|_\infty$ so by taking expectations and letting ϵ tend to 0 and t tend to infinity by the dominated convergence theorem one gets:

$$E \left[l_{\tau_x} u(\tau_x, X_{\tau_x}) + \int_0^{\tau_x} l_s g(s, X_s) ds \right] = u(0, x) \quad (5.31)$$

so by the definition of τ_x:

$$E \left[l_{\tau_x} f(X_{\tau_x}) + \int_0^{\tau_x} l_s g(s, X_s) ds \right] = u(0, x) \quad (5.32)$$

and the proof is finished. □

Remark 5.1.0.2 (Improvements w.r.t the previous results)**.** *The coefficients b, σ, r need 2 or 3 regularities, not smoothness. f only needs to be continuous. There is no need for ellipticity, nor even for hypoellipticity: only Malliavin nondegeneracy conditions are required.*

Remark 5.1.0.3 (A simpler proof under stronger hypotheses:)**.** *Suppose K_t is hypoelliptic and r is smooth. Since $K_t u = ru$ in the sense of the distributions on D, u has to be a smooth function on D because r is smooth. Therefore the classical Itô formula may be applied, and if τ_n announces τ_x:*

$$l_{\tau_n} u(\tau_n, X_{\tau_n}) - u(0, x) = \int_0^{\tau_n} l_s ({}^t\sigma(s, X_s) \nabla u(s, X_s)) dW_s \quad (5.33)$$

Therefore the left hand side above is a (usual, strong) discrete time martingale convergeing a.s. to $l_{\tau_x} u(\tau_x, X_{\tau_x}) - u(0, x)$ and since τ_n announces τ_x our martingale is uniformly integrable, so the convergence takes place in L^1 and this yields:

$$u(0, x) = E \left[l_{\tau_x} u(\tau_x, X_{\tau_x}) \right] \quad (5.34)$$

which finishes our proof.

Remark 5.1.0.4. *Another method might have been the following: since*

$$\int_{\epsilon \wedge \tau_x}^{t \wedge \tau_x} l_s \left(\partial_s + A_s + r \right) u_m(s, X_s) ds \to 0 \tag{5.35}$$

in $\mathbb{D}'$, one may use the positive distribution methods to deduce from this that the convergence takes place in law. Then on may prove that the LHS is tense (perhaps need to restrict to increasing compacts) by the Aldous criteria. From this we deduce that our sequence of martingales is tense (one tense term + one term converging to 0); we also know that it converges in finite distributions. Therefore the limit has to be a local martingale (see [31] for the precise criterion) and since it is bounded a.s. by the Burkholder-Davies-Gundy theorem and the Novikov criterion it is a martingale. This is similar but more complicated; it was the first idea I had.

Remark 5.1.0.5 (On the condition: $u \in BV(C^0)$). *In our theorem we worked under the hypothesis that $u \in BV(C^0)$. In practice this might be hard to verify or one may only have that u is BV in time for every point in space and C^0 in space for every point in time but not have joint regularity. Then, one may still make the following observation: $u \in BV(\mathcal{S}')$ if and only if for every $\phi \in \mathcal{S}$, $t \mapsto \langle u(t, \cdot), \phi \rangle$ is a BV function. In our precise case, there is:*

$$\tilde{u}(t) := \langle u(t, \cdot), \phi \rangle = \int u(t, x) \cdot \phi(x) dx \tag{5.36}$$

and it is then easy to derive:

$$Var(\tilde{u}) \leq \int Var\left(u(\cdot, x) \right) \cdot \phi(x) dx \tag{5.37}$$

Then we see that if $x \mapsto Var u(\cdot, x)$ in L^p, by Hölder's theorem $u \in BV(L^p)$. More generally, if there exists some $\delta > 0$ such that:

$$x \mapsto \frac{Var\left(u(\cdot, x) \right)}{\left(1 + |x|^2 \right)^{\delta/2}} \in L^p \tag{5.38}$$

then $u \in BV(\mathcal{S}_{p,-\delta})$. Indeed we have already observed that:

$$\left\| x \mapsto \left(1 + |x|^2 \right)^{\delta/2} \cdot \phi(x) \right\|_{L^{p*}} \leq C \cdot \| \phi \|_{\mathcal{S}_{p*,\delta}} \tag{5.39}$$

We now prove the following reciprocal to the previous theorem:

Theorem 5.1.0.9. *Under the assumptions $(\mathcal{A})$, the function:*

$$u(t, x) = \sup_{\tau \in Z_{t,T}} E \left[f(X_\tau^t(x)) \exp \left(- \int_t^\tau r(u, X_u^t(x)) du \right) \right.$$

$$\left. + \int_t^\tau g(s, X_s^t(x)) \exp \left(- \int_t^s r(u, X_u^t(x)) du \right) ds \right] \tag{5.40}$$

is a solution to the variational inequation $(\mathcal{V})$.

Proof. That $u(t, x) \geq f(x)$ and $u(T, x) = f(x)$ are obvious. Also, it is known (cf [33]) that u is jointly continuous. Finally it is decreasing in time because so is the set $Z_{t,T}$, hence u is BV in time and:

$$Var\left(u(\cdot, x)\right) = u(0, x) - u(T, x) \tag{5.41}$$

Therefore:

$$\sup_x Var\left(u(\cdot, x)\right) \leq 2 \cdot \|f\|_\infty \tag{5.42}$$

so by the same arguments as in remark 5.1.0.5 one gets $u \in BV\left(\mathcal{S}_{p, -N/p-\epsilon}\right)$. We now consider the following adapted process:

$$S_t := l_t u(t, X_t) + \int_0^t l_s g(s, X_s) ds \tag{5.43}$$

We will prove it is supermartingale. Indeed, from the expression of u and the strong Markov property X has as the strong solution of an SDE, one has:

$$u(t, X_t) = \sup_{t \leq \tau \leq T} E\left[\frac{l_\tau}{l_t} f(X_\tau) + \int_t^\tau \frac{l_s}{l_t} g(s, X_s) ds | \mathcal{F}_t\right] \tag{5.44}$$

and since $l_\cdot$ and $\int_0^\cdot l_s g(s, X_s) ds$ are adapted processes:

$$S_t = \sup_{t \leq \tau \leq T} E\left[l_\tau f(X_\tau) + \int_0^\tau l_s g(s, X_s) ds | \mathcal{F}_t\right] \tag{5.45}$$

One sees that this is the Snell envelope for the cost function

$$Y_u := l_u f(X_u) + \int_0^u l_s g(s, X_s) ds \tag{5.46}$$

therefore by the results in [33] we know that it is a supermartingale. Also, S is (almost) surely bounded:

$$|S_t| \leq \|u\|_\infty + (T - t)\|g\|_\infty \leq \|f\|_\infty + T\|g\|_\infty \tag{5.47}$$

In particular, S is a class D supermartingale, therefore it admits a Doob-Meyer decomposition, cf [17]:

$$S_t - u(0, x) = M_t + D_t \tag{5.48}$$

where M is a martingale and D is a decreasing process, and $M_0 = D_0 = 0$. Now since $u \in BV\left(\mathcal{S}_{p, -N/p-\epsilon}\right)$, as in the proof of the previous theorem, one may write the weak Itô formula:

$$l_t u(t, X_t) - l_\epsilon u(t, X_\epsilon) = \int_\epsilon^t l_s \left(\partial_s + A_s + r\right) u(s, X_s) ds + \int_\epsilon^t l_s \sigma(s, X_s) \partial_x \sigma(s, X_s) dW_s \tag{5.49}$$

and therefore:

$$S_t - S_\epsilon = \int_\epsilon^t l_s \left(\left(\partial_s + A_s + r \right) u(s, X_s) + g(s, X_s) \right) ds + \int_\epsilon^t l_s \sigma(s, X_s) \partial_x \sigma(s, X_s) dW_s$$
(5.50)

The first term of the RHS is a weak BV process, and it second term is a weak martingale. Now apply the operator $(Id + \mathcal{L})^{-\frac{k}{2}}$ for big enough k so they become a BV process and a martingale, respectively. Also by (5.48):

$$l_t u(t, X_t) - u(\epsilon, X_\epsilon) = (M_t - M_\epsilon) + (D_t - D_\epsilon)$$
(5.51)

so by the unicity of the decomposition of a semimartingale as a local martingale plus a BV process one may identify:

$$(Id + \mathcal{L})^{-\frac{k}{2}} \int_\epsilon^t l_s \sigma(s, X_s) \partial_x \sigma(s, X_s) dW_s \;\; = \;\; (Id + \mathcal{L})^{-\frac{k}{2}} (M_t - M_\epsilon)$$
(5.52)

$$(Id + \mathcal{L})^{-\frac{k}{2}} \int_\epsilon^t \left(l_s \left(\partial_s + A_s + r \right) u(s, X_s) + g(s, X_s) \right) ds \;\; = \;\; (Id + \mathcal{L})^{-\frac{k}{2}} (D_t - D_\epsilon)$$
(5.53)

so in particular:

$$\int_\epsilon^t l_s \left(\left(\partial_s + A_s + r \right) u(s, X_s) + g(s, X_s) \right) ds = D_t - D_\epsilon$$
(5.54)

therefore this weakly BV process is a proper process, and it is decrasing. Now let ϕ be smooth and a.s. positive. For any $0 < \epsilon < t < u$:

$$E \left[\phi \int_\epsilon^u l_s \left(\left(\partial_s + A_s + r \right) u(s, X_s) + g(s, X_s) \right) ds \right]$$
$$\leq \;\; E \left[\phi \int_\epsilon^t \left(\left(\partial_s + A_s + r \right) u(s, X_s) + g(s, X_s) \right) ds \right]$$
(5.55)

so:

$$E \left[\phi \int_t^u l_s \left(\left(\partial_s + A_s + r \right) u(s, X_s) + g(s, X_s) \right) ds \right]$$
$$= \;\; \int_t^u E \left[\phi l_s \left(\left(\partial_s + A_s + r \right) u(s, X_s) + g(s, X_s) \right) \right] ds$$
$$\leq \;\; 0$$
(5.56)

The function under the integral is:

$$s \;\; \mapsto \;\; E \left[\phi l_s \left(\left(\partial_s + A_s + r \right) u(s, X_s) + g(s, X_s) \right) \right]$$
$$= \;\; \int \left(\left(\partial_s + A_s + r \right) u(s, y) + g(s, y) \right) p_{X_s, l_s \phi}(y) dy$$
(5.57)

which is bounded on $]\epsilon, T[$, therefore it is licit to take limits in the Newton ratio and to differentiate at least in the absolutely continuous sense to obtain, for all $s > \epsilon$:

$$E\left[\phi l_s\left((\partial_s + A_s + r)u(s, X_s) + g(s, X_s)\right)\right] \leq 0 \tag{5.58}$$

Apply this to $\phi = h(X_s)$ for some regular enough, positive function h. Then there comes:

$$\langle(\partial_s + A_s + r)u(s, \cdot) + g, p_{X_s, l_s}h\rangle \leq 0 \tag{5.59}$$

Also by proposition 3.4.0.5 $supp(p_{X_s}) = supp(p_{X_s, l_s})$ because l_s is a.s. strictly positive. From this we deduce that $(\partial_s + A_s + r)u + g \leq 0$ in the sense of the distributions on $supp(p_{X_s})$. To finish the proof, we only need to verify that $(\partial_s + A_s + r)u + g = 0$ in the sense of the distributions on the open set $supp(p_{X_s}) \cap \{u(s, \cdot) \neq f\}$. As in the proof of the previous theorem, we introduce:

$$D = \{(t, y), u(t, y) \neq f(y)\} \tag{5.60}$$

and define a stopping time as:

$$\tau_x = \inf\{t, (t, X_t) \notin D\} \tag{5.61}$$

We note that $\tau_x < \infty$ a.s. because we consider a finite horizon problem; also τ_x is the smallest optimal stopping time for our optimal stopping problem. By the Doob stopping theorem and the definition of τ_x one sees that:

$$E[l_{\tau_x}f(X_{\tau_x}) + \int_0^{\tau_x} l_s g(s, X_s)ds] = E[S_{\tau_x}] = u(0, x) + E[D_{\tau_x}] \tag{5.62}$$

and therefore $E[D_{\tau_x}] = 0$. Now let (u_m) be a smooth approximation of u. By methods similar to thoses developed in the proof of the previous theorem, there comes:

$$E\left[\int_{\epsilon \wedge \tau_x}^{t \wedge \tau_x} l_s\left((\partial_s + A_s + r)u_m(s, X_s) + g(s, X_s)\right)ds\right]$$
$$= \int \int_{]\epsilon, t[\times \mathbb{R}^N \cap U} \left((\partial_s + A_s + r)u_m(s, y) + g(s, y)\right)p_{X_s, l_s}(y)dyds \tag{5.63}$$

so by taking limits and comparing with $E[B_{\tau_x}]$ one sees that for any s:

$$\langle(\partial_s + A_s + r)u + g, p_{X_s, l_s}\rangle_U = 0 \tag{5.64}$$

and since we already know that $(\partial_s + A_s + r)u + g$ is a negative distribution we are done. $\qquad\square$

Remark 5.1.0.6. *[52] and some references therein provide results about the support of p_{X_t}. In particular, within the hypotheses we made on b and σ, the results in [52] imply that $\{p_{X_t} > 0\}$ is an open, connected set, so we may work with the distributions on $\mathrm{supp}(p_{X_t})$. Also, in many cases, for example when A is elliptic, we have lower bounds for p_{X_t}, so p_{X_t} has full support.*

Remark 5.1.0.7. *By using results on positive distributions it should be possible to get rid of the necessity that the discount factor l_t is regular. Then the results would hold simply with a continuous interest rate.*

Remark 5.1.0.8. *To do: extend the results to the case where the horizon is not finite and to the case where the domain is not $\mathbb{R}^N$.*

5.2   On a quasi variational inequation

Conclusion

Appendix A

Some results in topology

In this appendix we give some results on the theory of topological vector spaces which are required for our study. Especially, we will make use of the structure of $\mathcal{S}$ and $\mathcal{S}'$ as nuclear spaces, barreled spaces and Montel spaces, so we recall the results about those spaces which are relevant to our purpose. We will mostly follow [69], [24], [64], [36] and [57].

Let us consider a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ which we equip with its archimedean topology. Let E be some vector space on $\mathbb{K}$; we equip E with a topology Θ. We recall the:

Definition A.0.0.1 (Toplogical vector space). *(E, Θ) is a topological vector space, TVS for short, if the applications $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda \cdot x$ are continuous on $E \times E$ and $\mathbb{K} \times E$ respectively.*

We recall that the topology of a TVS is fully defined by a basis of neighbourhoods of 0.

In the sequel of this appendix, $E, F, \dots$ will denote topological vector spaces (we omit to note their respective topologies).

A.1 Locally convex topological vector spaces

A.1.1 First definitions

Definition A.1.1.1 (Locally convex topological vector space). *A TVS E is said to be locally convex if $0 \in E$ has a basis of neighbourhoods consisting in convex sets.*

A crucial notion will be that of:

Definition A.1.1.2 (Barrel). *A part $A \subset E$ is a barrel if it verifies:*

- *A is absorbing: for every $x \in E$ there exists $c > 0$ s.t. $|\lambda| < c \Rightarrow \lambda \cdot x \in A$;*

- *A is balanced: for every $x \in A$, for every λ s.t. $|\lambda| < 1$, $\lambda \cdot x \in A$;*

- *A is closed;*

- *A is convex.*

We have the:

Proposition A.1.1.1. *In a TVS, 0 has a basis of neighbourhoods consisting in barrels.*

A.1.2 Characterization by seminorms

Definition A.1.2.1 (Seminorm). *$p : E \mapsto \mathbb{R}_+$ is a seminorm on E if it verifies:*

- *For every $x, y \in E$ $p(x + y) \leq p(x) + p(y)$;*

- *For every $\lambda \in \mathbb{K}$, $x \in E$: $p(\lambda \cdot x) = |\lambda| \cdot p(x)$*

We note that if p is a seminorm there has to be $p(0) = 0$. The continuity of seminorms is easily characterized:

Proposition A.1.2.1. *Let p be a seminorm on E. Then the three following are equivalent:*

- *The open unit ball $\{x | p(x) < 1\}$ is an open set;*

- *p is continuous at 0;*

- *p is continuous.*

Also, in that case, the open unit ball of p is a barrel.

Conversely there is the:

Proposition A.1.2.2. *Let T be a barrel in E. Then there exists a unique seminorm p such that T is the open unit ball of p and p is continuous if and only if T is a neighbourhood of 0. p is the gauge of the barrel T:*

$$p(x) = \inf \{\lambda \geq 0 | x \in \lambda \cdot T\} \tag{A.1}$$

These two results combine into the:

Theorem A.1.2.1. *In a locally convex topological vector space, the open unit balls of the continuous seminorms form a basis of neighbourhoods of 0.*

Now if we introduce the:

Definition A.1.2.2 (Basis of continuous seminorms)**.** *A family $\mathcal{P}$ of continuous seminorms on a locally convex TVS is a basis of continuous seminorms if for every $p \in \mathcal{P}$, there exists $q \in \mathcal{P}$ such that $p = O(q)$.*

the two notions combine in the:

Theorem A.1.2.2. *In a locally convex TVS:*

- *Let $\mathcal{P}$ be a basis of continuous seminorms; if $p \in \mathcal{P}$ we note U_p for its open unit ball. Then the λU_p, $\lambda > 0$, $p \in \mathcal{P}$ are a basis of neighbourhoods of 0.*

- *Conversely if one gives oneself a basis of neighbourhoods of 0 consisting in barrels, then the associated seminorms are a basis of continuous seminorms.*

The result we will use in practice is the following:

Corollary A.1.2.1. *Let E, F be two locally convex TVS. Then a linear mapping $f : E \mapsto F$ is continuous if and only if for every continuous seminorm q on F there exists a continuous seminorm p on E such that for every $x \in E$:*

$$q(f(x)) \le p(x) \tag{A.2}$$

Especially a linear form on E is continuous if and only if it is continuous for at least one continuous seminorm on E.

A.1.3 Metrizable locally convex TVS

First we recall the general:

Definition A.1.3.1 (Metrizable TVS)**.** *The TVS E is said to be metrizable if it is Haussdorf and if $0 \in E$ admits a countable basis of neighbourhoods.*

and:

Theorem A.1.3.1. *A TVS is metrizable if and only if its topology is defined by a translation-invariant distance.*

In the case of locally convex TVS, we have the more precise:

Theorem A.1.3.2. *Let E be a metrizable, locally convex TVS; there exists a countable basis of continuous seminorms $(p_j)_{j \in \mathbb{N}}$ on E. Let also $(a_j) \in \mathbb{R}_+$ be such that $\sum a_j < \infty$ and define:*

$$d(x,y) = \sum_{j=0}^{\infty} a_j \cdot \frac{p_j(x-y)}{1 + p_j(x-y)} \tag{A.3}$$

Then d is a translation-invariant distance on E which defines the toplogy on E.

We also introduce the:

Definition A.1.3.2 (Fréchet space). *A Fréchet space is a metrizable, locally convex TVS which is complete.*

A.2 Duality

Let E be a TVS (E is not necessarily locally convex); we note E' for its topological dual, ie the vector space of the continuous linear forms on E. We recall that a subset A of a general TVS E is called bounded if for every neighbourhood U of 0 there exists a number λ such that $U \subset \lambda \cdot A$ and we introduce the following:

Definition A.2.0.3 (Polar of a subset). *Let $A \subset E$. We define the polar set of A as:*

$$A^\circ = \left\{ x' \in E' \;\middle|\; \sup_{x \in A} |x'(x)| < 1 \right\} \tag{A.4}$$

We note that A° is a convex and balanced set and that there is:

$$(A \cup B)^\circ = A^\circ \cap B^\circ \tag{A.5}$$

and

$$(\lambda \cdot A)^\circ = (1/\lambda) \cdot A^\circ \tag{A.6}$$

Moreover if A is bounded then A° is absorbing. One may therefore prove the:

Theorem A.2.0.3. *Let $\mathfrak{S}$ be a nonempty family of bounded sets in A such that:*

- *For every $A, B \in \mathfrak{S}$ there exists $C \in \mathfrak{S}$ such that $A \cup B \subset C$;*

- *For every $A \in \mathfrak{S}$ and $\lambda \in \mathbb{K}$ there exists $B \in \mathfrak{S}$ such that $\lambda \cdot A \subset B$.*

Then, $\mathfrak{S}^\circ := \{A^\circ, A \in \mathfrak{S}\}$ defines a basis of neighbourhoods of 0 for a certain topology on E'.

Relevant examples are:

- The weak topology, obtained by taking the finite subsets of E for $\mathfrak{S}$;

- The convex-compact topology, obtained by taking the convex and compact subsets of E for $\mathfrak{S}$;

- The compact topology, obtained by taking the compact subsets of E for $\mathfrak{S}$;

- The strong topology, obtained by taking all bounded subsets of E for $\mathfrak{S}$.

A.3 Barreled spaces and Montel spaces

A.3.1 Barreled spaces

Definition A.3.1.1 (Barreled space). *A topological vector space E is said to be barreled if every barrel in E is a neighbourhood of 0 in E.*

It may be proved that any Baire space is barreled; we will only need the following:

Theorem A.3.1.1. *Any Fréchet space is barreled.*

Barreled spaces have many properties which are useful in analysis. The one we will use is the following:

Theorem A.3.1.2 (Closed graph theorem). *Let E and F be two TVS; suppose that E is barreled and that F is complete. Then, if $u : E \to F$ is a linear map with closed graph, then u is continuous.*

A.3.2 Montel spaces

Definition A.3.2.1 (Montel space). *A TVS E is said to be a Montel space if it verifies the following properties:*

- *E is Haussdorf;*

- *E is locally convex;*

- *E is barreled;*

- *Every closed and bounded space in E is compact.*

We are interested in the following property of Montel spaces:

Theorem A.3.2.1. *In the dual of a Montel space, every weakly convergent sequence is strongly convergent.*

Let us also mention:

Theorem A.3.2.2. *The strong dual of a Montel space is a Montel space.*

A.4 Nuclear spaces

A.4.1 Topological tensor products

Let E and F be two TVS and let $\mathcal{B}(E, F)$ be the vector space of continuous bilinear forms on $E \times F$. For $(x, y) \in E \times F$ we consider the evaluation mapping:

$$u_{x,y} = f \in \mathcal{B}(E, F) \mapsto f(x, y) \in \mathbb{K} \tag{A.7}$$

Clearly, $u_{x,y} \in \mathcal{B}(E, F)'$, the dual space of $\mathcal{B}(E, F)$. Now let us consider:

$$\chi = (x, y) \in E \times F \mapsto u_{x,y} \in \mathcal{B}(E, F)' \tag{A.8}$$

We note $E \otimes F$ for the linear hull of $\chi(E \times F)$ in $\mathcal{B}(E, F)'$; it will sometimes make intuitive sense to note $x \otimes y$ for $u_{x,y}$.

Definition A.4.1.1 (Tensor product)**.** *$E \otimes F$ is the topological tensor product of E and F.*

Definition A.4.1.2 (Canonical bilinear map)**.** *χ is the canonical bilinear map of $E \times F$ into $E \otimes F$.*

We have the:

Theorem A.4.1.1. *Let E, F and G be three topological vector spaces. Then the following application:*

$$u \in \mathcal{L}(E \otimes F, G) \mapsto u \circ \chi \in \mathcal{B}(E, F; G) \tag{A.9}$$

is an isomorphism. Especially, there is:

$$(E \otimes F)' \approx \mathcal{B}(E, F) \tag{A.10}$$

We may characterize the topology on $E \otimes F$ through the following:

Theorem A.4.1.2. *Let p and q be two seminorms on E and F respectively, such that p and q are the gauges of the barrels T_p and T_q respectively. Then, the following defines a seminorm on $E \otimes F$:*

$$r(z) = \inf \left\{ \sum p(x_i) \cdot q(y_i) \mid u = \sum x_i \otimes y_i \right\} \tag{A.11}$$

Moreover there is:

$$r(x \otimes y) = p(x) \cdot q(y) \tag{A.12}$$

and r is the gauge of the smallest barrel containing $\chi(T_1 \times T_2)$. We note:

$$r = p \otimes q \tag{A.13}$$

If E and F are metrizable locally convex TVS with respective countable basis of seminorms (p_i) and (q_j) we will usually equip $E \otimes F$ with the metrizable locally convex TVS topology with basis of seminorms $(p_i \otimes q_j)$. In the case where E and F are Banach spaces, we simply obtain a Banach space structure on $E \otimes F$, with norm $\| \cdot \|_E \otimes \| \cdot \|_F$. This special case is relevant to the construction of the spaces $\mathbb{D}_{p,k}$ in Malliavin calculus.

We conclude this paragraph with the:

Theorem A.4.1.3. *Let E and F be two metrizable, locally convex TVS. Then any element of $E \otimes F$ may be written in the form:*

$$\sum_{i=1}^{\infty} \lambda_i \cdot x_i \otimes y_i \tag{A.14}$$

where $\sum |\lambda_i| < \infty$.

A.4.2 Nuclear mappings in Banach spaces

Let E and F be two Banach spaces; we equip the strong dual E' of E with its natural Banach space structure. If:

$$v \in E' \otimes F = \sum_{i=1}^{r} f_i \otimes y_i \tag{A.15}$$

we define an element $u \in \mathcal{L}(E, F)$ as:

$$u(x) = \sum_{i=1}^{r} f_i(x) \cdot y_i \tag{A.16}$$

Then the linear mapping:

$$v \in E' \otimes F \mapsto u \in \mathcal{L}(E, F) \tag{A.17}$$

is an (algebraic) isomorphism; let us note τ for the extension of this isomorphism to the completion of $E' \otimes F$. Then the nuclear linear mappings in $\mathcal{L}(E, F)$ are those linear applications which are in the image of τ.

A.4.3 Nuclear mappings in a locally convex TVS

We start with some notation.

Let T be a barrel in the vector E. Then, the familly $(n^{-1} \cdot T, n \in \mathbb{N} - \{0\})$ is a neighbourhood basis of 0 for some metrizable locally convex TVS topology

on E. Let also p be the gauge of T; then a norm is defined on the vector space $E_T := E/p^{-1}(0)$ by:

$$\|\hat{x}\| = p(x) \tag{A.18}$$

for any $x \in \hat{x}$. We note $\tilde{E}_T$ the completion of E_T for this norm and we note that the canonical injection:

$$\phi_T : E \to \tilde{E}_T \tag{A.19}$$

is continuous.

Now let B be a bounded and non-empty barrel in E; we define a subspace of E as:

$$E_1 := \bigcup_{n=1}^{\infty} n \cdot B \tag{A.20}$$

The gauge p_B of B is a norm on E_1 and we note E_B for the normed space (E_1, p_B). If B is complete E_B is a Banach space. We introduce the following notation for the canonical embedding:

$$\psi_B = E_B \hookrightarrow E \tag{A.21}$$

and we note that ψ_B is continuous. Finally, we note that if T and B coincide then the spaces E_T and E_B coincide as well.

We introduce the following:

Definition A.4.3.1 (Bounded linear map). *Let E and F be two locally convex TVS. $u \in \mathcal{L}(E, F)$ is said to be a bounded linear map if there exists a neighbourhood U of $0 \in E$ such that $u(U)$ is a bounded subset of F.*

Let us note that if u is a bounded mapping, one may take U to be a barrel; then $u(U)$ is included in a bounded barrel $B \subset F$ and one may write:

$$u = \psi_B \circ u_0 \circ \phi_U \tag{A.22}$$

where $u_0 \in \mathcal{L}(E_U, F_B)$. Also, if F_B is a Banach space, u_0 has a continuous extension to $\mathcal{L}(\tilde{E}_U, F_B)$ which we still denote u_0. This leads us to the following:

Definition A.4.3.2 (Nuclear mapping). *A mapping between two locally convex TVS E and F is said to be nuclear if it may be written as in equation (A.22), with U a barrel, B a bounded barrel such that F_B is a Banach space and u_0 a nuclear mapping (in the sense of Banach spaces).*

It is interesting to note that for every $\mathfrak{S}$-topology on $\mathcal{L}(E, F)$, the nuclear maps are contained in the closure of $E' \otimes F$ (viewed as a subspace of $\mathcal{L}(E, F)$). Therefore from theorem A.4.1.3 one obtains the:

Theorem A.4.3.1 (Characterization of nuclear maps)**.** *Let E and F be two locally convex TVS. A linear map $u \in \mathcal{L}(E, F)$ is nuclear if and only if it may be written:*

$$u(x) = \sum_{n=1}^{\infty} \lambda_n \cdot f_n(x) \cdot y_n \tag{A.23}$$

where $(\lambda_n) \in l^1(\mathbb{K})$, (f_n) is an equicontinuous sequence in E' and (y_n) is a sequence included in some bounded barrel $B \subset F$ for which F_B is complete.

Sometimes we will abusively note:

$$u = \sum_{n=1}^{\infty} \lambda_n \cdot f_n \otimes y_n \tag{A.24}$$

A.4.4 Nuclear spaces

Having defined the notion of nuclear mapping we are able to introduce the:

Definition A.4.4.1 (Nuclear space)**.** *A locally convex TVS is said to be nuclear if it admits a base $\mathfrak{B}$ of neighbourhoods of 0 such that:*

- *Every $V \in \mathfrak{B}$ is a barrel;*

- *For every $V \in \mathfrak{B}$, the canonical mapping $E \to \tilde{E}_B$ is nuclear.*

The following characterizaztion is more tractable:

Theorem A.4.4.1. *Let E be a locally convex TVS; the three following are equivalent:*

- *E is nuclear;*

- *Any continuous linear map from E into any Banach space B is nuclear;*

- *Let $\mathfrak{B}$ be a base of neighbourhoods of 0 which are all barrels; then for every $T_1 \in \mathfrak{B}$ there exists $T_1 \in \mathfrak{B}$ such that $T_2 \subset T_1$ and the canonical map $\tilde{E_{T_2}} \to \tilde{E_{T_1}}$ is nuclear.*

For Fréchet spaces, the following is even simpler:

Theorem A.4.4.2. *A Fréchet E space is nuclear if and only if it is the projective limit of a sequence of Hilbert spaces with nuclear injections, ie there exists a decreasing sequence of Hlibert spaces (H_n) such that:*

$$E = \bigcap_{n=0}^{\infty} H_n \tag{A.25}$$

and for every $m < n$:

$$H_n \hookrightarrow H_m \tag{A.26}$$

is nuclear.

We mention a couple more results about nuclear spaces:

Theorem A.4.4.3. *A Fréchet space is nuclear if and only if its strong dual is nuclear.*

Theorem A.4.4.4. *A complete barreled space which is nuclear is a Montel space.*

Appendix B

The Bochner and Pettis integrals

For this appendix our setting will be a Banach space X and a measured space $(\Omega, \mathcal{F}, \nu)$; we will also note X' for the strong topological dual space of X. The aim is to define and study the measurability and the integrability of X-valued borelian functions on Ω. We stick to the case of a positive measure for simplicity. Our reference is [18], where all the proofs may be found.

B.1 Measurability

Definition B.1.0.2 (Simple function). *$f : \Omega \to X$ is called a simple function if one may write a finite linear combination:*

$$f = \sum_i 1_{E_i} \alpha_i \tag{B.1}$$

where $E_i \in \mathcal{F}$ and $\alpha_i \in X$.

Definition B.1.0.3 (Strong measurability). *$f : \Omega \to X$ is (strongly) measurable if there exists a sequence (f_n) of simple functions such that the following holds in the ν-a.s. sense:*

$$\lim_{n \to \infty} \|f - f_n\|_X = 0 \tag{B.2}$$

Definition B.1.0.4 (Weak measurability). *$f : \Omega \to X$ is weakly measurable if for every $x^* \in X'$ the real-valued function $\langle x^*, f \rangle$ is measurable in the usual (finite-dimensionnal) sense.*

One may verify that in the case where X is of finite dimension, these two notions of measurability coincide with the usual one. Also there is the:

Theorem B.1.0.5. *$f : \Omega \to X$ is ν-measurable if and only if the two following conditions hold:*

- *f is ν-weakly measurable;*

- *There exists a separable subspace Y of X such that f is ν-a.s. Y-valued.*

In particular, if X is a separable Banach space, as will be the case in our practical applications, the notions of weak and strong measurability coincide.

B.2   The Bochner integral

If f is a simple function, $f = \sum_i 1_{E_i}\alpha_i$, we define its (Bochner) integral as:

$$\int_\Omega f d\nu = \sum_i \nu(E_i)\alpha_i \tag{B.3}$$

Then let us introduce the:

Definition B.2.0.5 (Bochner integrability). *We say that a (strongly) measurable function $f : \Omega \to X$ is ν-Bochner integrable if there exists a sequence of simple functions such that:*

$$\lim_{n\to\infty} \int_\Omega \|f - f_n\|_X d\nu = 0 \tag{B.4}$$

and then we set:

$$\int_\Omega f d\nu = \lim \int_\Omega f_n d\nu \tag{B.5}$$

Now we state the more convenient characterization which we will use in practice:

Theorem B.2.0.6. *$f : \Omega \to X$ is ν-Bochner integrable if and only if:*

$$\int_\Omega \|f\|_X d\nu < \infty \tag{B.6}$$

We will also make use of the following results:

Theorem B.2.0.7 (Dominated convergence theorem). *Let (f_n) be a sequence of Bochner-integrable functions. We suppose that:*

- *$\lim_{n\to\infty} \nu\left(\{\|f - f_n\|_X > \epsilon\}\right) = 0$ for every $\epsilon > 0$;*

- *There exists a ν-integrable, $\mathbb{R}_+$-valued function g such that $\|f_n\| \le g$ for every n.*

Then f is (strongly measurable and) Bochner integrable and:

$$\lim \int_\Omega f_n d\nu = \int_\Omega f d\nu \tag{B.7}$$

Theorem B.2.0.8 (Absolute continuity of the Bochner integral). *Let f be a Bochner integrable function; then:*

$$\lim_{\nu(E)\to 0} \int_E f d\nu = 0 \tag{B.8}$$

where the integrals above are well defined as:

$$\int_E f d\nu = \int_\Omega 1_E f d\nu \tag{B.9}$$

Theorem B.2.0.9 (Triangle inequality). *If f is a Bochner integrable function then:*

$$\left\| \int_\Omega f d\nu \right\|_X \le \int_\Omega \|f\|_X d\nu \tag{B.10}$$

Theorem B.2.0.10 (Differentiation). *If f and g are two Bochner integrable functions such that for every $E \in \Sigma$:*

$$\int_E f d\nu = \int_E g d\nu \tag{B.11}$$

then $f = g$ ν-a.s.

Theorem B.2.0.11 (Hille). *Let X and Y be two separable Banach spaces and $T \in \mathcal{L}_c(X, Y)$. If both f and $T \circ f$ are Bochner-integrable then:*

$$T \left(\int_\Omega f d\nu \right) = \int_\Omega T \circ f d\nu \tag{B.12}$$

B.3   The Pettis integral

Lemma B.3.0.1. *Consider a weakly measurable $f : \Omega \to X$ such that for every $x^* \in X'$:*

$$\langle x^*, f \rangle \in L^1(\nu) \tag{B.13}$$

Then for every $E \in \Sigma$ there exists a unique element $I_E \in X^{**}$ such that for every $x^* \in X'$:

$$I_E(x^*) = \int_E \langle x^*, f \rangle \, d\nu \tag{B.14}$$

I_E is the Dunford integral of f on the measurable set E. If X is reflexive, which will be the case in our applications, we may consider that $I_E \in X$ and we call it a Pettis integral. Then the Pettis integral coincides with the Bochner integral for strongly measurable functions.

Appendix C

Some results in semigroup theory

In this section we give a brief introduction to the theory of semigroups and generators. We mostly follow [84] and [14]; [60] provides an introduction to how the theory applies to probability while [1], [70] or [71] relates it to the construction and the properties of spaces which are of common use in analysis.

C.1 Definition

Let X be some Banach space. We start by introducing the notion with which this appendix deals:

Definition C.1.0.6 (Semigroup on X). $(T_t)_{t \geq 0}$, or T_t for short, is said to be a semigroup on X if it verifies the three following properties:

- $\forall t \geq 0$: $T_t \in \mathcal{L}_c(X)$

- $\forall t_1, t_2 \geq 0$: $T_{t_1+t_2} = T_{t_1} \circ T_{t_2}$

- $T_0 = Id$

Definition C.1.0.7 (Semigroup of class C^0). A semigroup T_t is said to be of class C^0 if for every $x \in X$ the following limit holds in the strong sense:

$$\lim_{t \to 0} T_t x = x \tag{C.1}$$

Definition C.1.0.8 (Contraction semigroup). If a semigroup T_t is such that for every $t \geq 0$, $\|T_t\|_{\mathcal{L}_c(X)} \leq 1$, (resp. < 1) we call T_t a contraction semigroup (resp. a strict contraction semigroup).

C.2 The infinitesimal generator

Let T_t be some semigroup of class C^0 on X. We introduce the following strong limit, whenever it exists:

$$Ax = \lim_{t \to 0} \frac{T_t x - x}{t} \tag{C.2}$$

and we introduce the:

Definition C.2.0.9 (Infinitesimal generator of a semigroup)**.** *A is the infinitesimal generator of T_t.*

Definition C.2.0.10 (Domain of the infinitesimal generator)**.** *The set $\mathcal{D}(A)$ of those $x \in X$ such that Ax is well defined is called the domain of A.*

The properties of an infinitesimal generator that we will use are summarized in the:

Theorem C.2.0.12. *One has:*

- *$\mathcal{D}(A)$ is a subspace of X and $A \in \mathcal{L}\left(\mathcal{D}(A), X\right)$*

- *$\mathcal{D}(A)$ is dense in X (for the topology of $\| \cdot \|_X$)*

- *$\|x\|_A := \|x\|_X + \|Ax\|_X$ defines a norm on $\mathcal{D}(A)$ and $\mathcal{D}(A)$ equipped with that norm is a Banach space.*

- *If $x \in X$ and $t > 0$ then $T_t x \in \mathcal{D}(A)$ and the following relations hold:*

$$AT_t x = \frac{d}{dt}\left(T_t x\right) \tag{C.3}$$

$$T_t x - x = \int_0^t \left(T_s Ax\right) ds \tag{C.4}$$

where the integral is defined in the Bochner sense, see appendix B.

- *If $x \in \mathcal{D}(A)$:*

$$AT_t x = T_t Ax \tag{C.5}$$

Similarly, one may define powers A^r for $r \in \mathbb{N}$, $\mathcal{D}(A^{r+1})$ being a subspace of $\mathcal{D}(A^r)$. Here are the properties we will use about the powers of A:

Theorem C.2.0.13. *One has:*

- *$\bigcap_{r \in \mathbb{N}} \mathcal{D}(A^r)$ is dense in X and contains all the $T_t x$, $x \in X$.*

- If $x \in \mathcal{D}(A^r)$ *the following hold:*

$$\frac{d^r}{dt^r} (T_t x) = A^r T_t x = T_t A^r x \tag{C.6}$$

  and as a Bochner integral:

$$T_t x - \sum_{k=0}^{r-1} \frac{t^k}{k!} A^k x = \frac{1}{(r-1)!} \cdot \int_0^t (t-s)^{r-1} \cdot T_s A^r x \, ds \tag{C.7}$$

- *For every* $x \in X$:

$$(T_t - Id)^r x = \int_0^t \ldots \int_0^t T_{s_1 + \cdots + s_r} x \, ds_1 \ldots ds_r \tag{C.8}$$

Equation (C.7) justifies that we will sometimes abusively note:

$$T_t = e^{t \cdot A} \tag{C.9}$$

C.3 Resolvent of a semigroup

We start with the:

Theorem C.3.0.14 (Hille).

$$w_0 := \sup_{t \in \mathbb{R}_+} \frac{1}{t} \log \|T_t\|_{\mathcal{L}_c(X)} < \infty \tag{C.10}$$

Then, for $\lambda > w_0$, one may define the following integral in the sense of Bochner:

$$R_\lambda(A)x := \int_0^\infty e^{-\lambda t} T_t x \, dt \tag{C.11}$$

Definition C.3.0.11 (Resolvent of a semigroup). $R_\lambda(A)$ *is the resolvent of the semigroup* T_t.

This is interesting because of the:

Proposition C.3.0.1. *For* $\lambda > w_0$, *for every* $x \in X$, $R_\lambda(A)x \in \mathcal{D}(A)$ *and:*

$$R_\lambda A \left(\lambda \cdot Id - A \right) x = x \tag{C.12}$$

Therefore we will sometimes think of $R_\lambda(A)$ as the inverse of $\lambda \cdot Id - A$.

C.4 Is an operator the generator of a semigroup?

C.5 Fractional powers of a contraction semigroup and subordination

Suppose that T_t is a strict contraction semigroup. Then, using the notation of theorem C.3.0.14, one has $w_0 < 0$ so for $0 < \beta < 1$ it is possible to define, in the sense of Bochner:

$$(-A)^\beta x = \frac{\sin(\beta\pi)}{\pi} \cdot \int_0^\infty \lambda^{\beta-1} \left(\lambda \cdot Id - A\right)^{-1} (-Ax) d\lambda \qquad (C.13)$$

$$= \frac{1}{\Gamma(\beta)} \cdot \int_0^\infty \lambda^{-\beta-1} \left(T_\lambda x - x\right) d\lambda \qquad (C.14)$$

We call these fractional powers of A; this makes intuitive sense because within a hilbertian setting one may diagonalize A and there is:

$$x^\beta = x \cdot \frac{\sin(\beta\pi)}{\pi} \cdot \int_0^\infty \frac{\lambda}{\lambda + x} d\lambda \qquad (C.15)$$

$$= \frac{1}{\Gamma(\beta)} \cdot \int_0^\infty \frac{e^{-\lambda x} - 1}{\lambda^{\beta+1}} \qquad (C.16)$$

On the other hand, it is possible to construct the family T_t^β of the so-called subordinated semigroups from T_t; indeed, in the sense of Bochner, define:

$$T_t^\beta x = \int_0^\infty t_s \lambda_t^\beta(ds) \qquad (C.17)$$

where the λ_t^β are defined through their Laplace transforms as:

$$\int_0^\infty e^{-\gamma s} \lambda_t^\beta(ds) = e^{-\gamma^\beta t} \qquad (C.18)$$

Then, the generator of T_t^β is $-(-A)^\beta$.

Combining the notions of resolvent and of fractional powers, one also may define $(\lambda \cdot Id - A)^{-s}$ for a general $s > 0$. Indeed, in the sense of Bochner we set:

$$(\lambda \cdot Id - A)^{-s} x = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} T_t x \, dt \qquad (C.19)$$

Appendix D

Interpolation theory

In this appendix we give a brief review of the real and complex methods of interpolation for Banach spaces. In our applications, we will mostly refer to the case where the spaces we interpolate are defined as the domain of the infinitesimal generator of some semigroup, so we also give more specific results for the interpolation of semigroups. We mainly follow [10] and [14]; classic examples (Sobolev and Besov spaces, Hölder spaces, etc.) are explained in detail in [1], [67], [70] and [71] for example. We give no proofs and refer to these references instead.

D.1 General definitions

In this chapter we will consider two normed spaces A_0 and A_1; we will also note:

$$\bar{A} = (A_0, A_1) \tag{D.1}$$

Definition D.1.0.12 (Compatible normed spaces)**.** *The normed vector spaces A_0 and A_1 are said to be compatible if there exists a Haussdorf space $\mathcal{A}$ such that the following (continuous) injections hold:*

$$A_0, A_1 \hookrightarrow \mathcal{A} \tag{D.2}$$

Then it is possible to define the vector spaces $A_0 \cap A_1$ and $A_0 + A_1$ and to equip them with the norms:

$$\|a\|_{A_0 \cap A_1} = \max\left(\|a\|_{A_0}, \|a\|_{A_1}\right) \tag{D.3}$$

$$\|a\|_{A_0 + A_1} = \inf_{a_0 + a_1 = a}\left(\|a\|_{A_0} + \|a\|_{A_1}\right) \tag{D.4}$$

We have the easy:

Proposition D.1.0.2. *If A_0 and A_1 are compatible Banach spaces, then $A_0 \cap A_1$ and $A_0 + A_1$ are Banach spaces.*

In practice we will mostly consider the case where $A_0 \subset A_1 = \mathcal{A}$ but we still state the general results as this does not involve any extra difficulty. We introduce the:

Definition D.1.0.13 (Intermediate space). *A is said to be an intermediate space in $\bar{A}$ if the following (continuous) injections hold:*

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1 \tag{D.5}$$

Definition D.1.0.14 (Interpolation space). *The intermediate space A in $\bar{A}$ is said to be an interpolation space in $\bar{A}$ if for every $T \in \mathcal{L}_c(\bar{A})$ one also has $T \in \mathcal{L}_c(A)$.*

The key result is the:

Theorem D.1.0.15. *If A is an interpolation space in $\bar{A}$ there exists a number $C > 0$ and a number $\theta \in [0, 1]$ such that for every $T \in \mathcal{L}_c(\bar{A})$:*

$$\|T\|_{\mathcal{L}_c(\bar{A})} \leq C \cdot \|T\|_{\mathcal{L}_c(A_0)}^{1-\theta} \cdot \|T\|_{\mathcal{L}_c(A_1)}^{\theta} \tag{D.6}$$

Definition D.1.0.15. *In the framework of the above theorem, A is said to be an interpolation space of exponent θ.*

D.2   The real method of interpolation

We start with some notation. Let $a \in A_0 + A_1$; we define a borelian function on $\mathbb{R}_+$ by the equation:

$$K(t, a) = \inf_{a = a_0 + a_1} \left(\|a_0\|_{A_0} + t \cdot \|a_1\|_{A_1} \right) \tag{D.7}$$

If $a \in A_0 \cap A_1$ we define a borelian function on $\mathbb{R}_+$ by the equation:

$$J(t, a) = \max \left(\|a\|_{A_0}, t \cdot \|a\|_{A_1} \right) \tag{D.8}$$

Finally if ϕ is a positive borelian function on $\mathbb{R}_+$, for $0 \leq \theta \leq 1$ and $q \in [1, \infty]$ we introduce the quantity (which is possibly $+\infty$):

$$\Phi_{\theta, q}(\phi) = \left(\int_0^\infty \left(t^{-\theta} \phi(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \tag{D.9}$$

On one hand we define a space through K by setting:

$$\|a\|_{\theta, q, K} = \Phi_{\theta, q} \left(K(\cdot, a) \right) \tag{D.10}$$

and:

$$\bar{A}_{\theta,q,K} = \left\{ a \in A_0 + A_1 \middle| \|a\|_{\theta,q,K} < \infty \right\} \tag{D.11}$$

On the other hand we define a set $\bar{A}_{\theta,q,J}$ based on J as the set of those $a \in A_0 \cap A_1$ which may be represented as:

$$a = \int_0^\infty u(t) \frac{dt}{t} \tag{D.12}$$

for some borelian $u : \mathbb{R}_+ \mapsto A_0 \cap A_1$ such that $\Phi\left(J(\cdot, u(\cdot))\right) < \infty$ and we introduce:

$$\|a\|_{\theta,q,J} = \inf_u \Phi\left(J(\cdot, u(\cdot))\right) \tag{D.13}$$

Now we may state the main result in this paragraph:

Theorem D.2.0.16 (Real interpolation). *One has:*

- $\left(\bar{A}_{\theta,q,K}, \|a\|_{\theta,q,K}\right)$ *and* $\left(\bar{A}_{\theta,q,J}, \|a\|_{\theta,q,J}\right)$ *are two normed vector spaces, to which werefer as the K-method (resp. the J-method) real interpolation spaces.*

- *These spaces are interpolation spaces of exponent θ.*

- *If $0 < \theta < 1$ and $q \in [1, \infty]$ the K and J spaces coincide with equivalence of norms. Then we simply note $\bar{A}_{\theta,q}$ or equivalently $(A_0, A_1)_{\theta,q}$ for that space.*

- *If A_0 and A_1 are Banach spaces then so is $\bar{A}_{\theta,q}$.*

- *If $q < \infty$ then $A_0 \cap A_1$ is dense in $\bar{A}_{\theta,q}$.*

We wil also need the two following results:

Theorem D.2.0.17 (Reiteration theorem). *If A_0, A_1 are two compatible Banach spaces and if $0 < \theta, \theta_0, \theta_1, \eta < 1$ verify:*

$$\theta = (1 - \eta) \cdot \theta_0 + \eta \cdot \theta_1 \tag{D.14}$$

then the following holds:

$$\left(\bar{A}_{\theta_0,q}, \bar{A}_{\theta_1,q}\right)_{\eta,q} = \bar{A}_{\theta,q} \tag{D.15}$$

Theorem D.2.0.18 (Duality theorem). *If A_0 and A_1 are two compatible Banach spaces such that $A_0 \cap A_1$ is dense in both A_0 and A_1 then:*

$$\bar{A}'_{\theta,q} = (A'_0, A'_1)_{\theta,q^*} \tag{D.16}$$

D.3   The complex method of interpolation

We introduce:

$$S := \left\{ z \in \mathbb{C} \,\Big|\, 0 \leq \Re z \leq 1 \right\} \tag{D.17}$$

Then if A_0 and A_1 are two compaible Banach spaces, we define a vector space $\mathcal{F}(\bar{A})$ as the set of those functions f such that:

- f is C_b^0 on S;

- f is holomorphic on S°;

- For $j \in \{0, 1\}$: $t \in \mathbb{R}_+ \mapsto f(j + it) \in A_j$ is continuous;

- For $j \in \{0, 1\}$: $\lim_{|t| \to \infty} f(j + it) = 0$

and we state the:

Theorem D.3.0.19. *$\mathcal{F}(\bar{A})$ is a Banach space when equipped with the norm:*

$$\|f\|_{\mathcal{F}(\bar{A})} = \max \left(\sup_t \|f(it)\|_{A_0}, \sup_t \|f(1 + it)\|_{A_1} \right) \tag{D.18}$$

Now for $0 \leq \theta \leq 1$ we introduce the set:

$$[\bar{A}]_\theta = \left\{ a \in A_0 + A_1 \,\Big|\, \exists f \in \mathcal{F}(\bar{A}), \ f(\theta) = a \right\} \tag{D.19}$$

and we state:

Theorem D.3.0.20 (Complex interpolation). *$[\bar{A}]_\theta$ is a vector space; it also is a Banach space if equipped with the norm:*

$$\|a\|_\theta = \inf \left\{ \|f\|_{\mathcal{F}(\bar{A})} \,\Big|\, f(\theta) = a \right\} \tag{D.20}$$

Moreover $[\bar{A}]_\theta$ is an interpolation space of order θ and $A_0 \cap A_1$ is dense in $[\bar{A}]_\theta$.

Remark D.3.0.1. *[10] introduces another complex interpolation method. However, it is shown that the two methods are equivalents when at least one of the spaces A_0 or A_1 is reflexive, which will always be the case in our applications. Also, we will not make any direct use of the other method, therefore we do not develop it here.*

We will also make use of the following results:

Theorem D.3.0.21 (Duality theorem)**.** *If at least one of the Banach spaces A_0 or A_1 is reflexive, then:*

$$[\bar{A}]'_\theta = (A'_0, A'_1)_{[\theta]} \tag{D.21}$$

Theorem D.3.0.22 (Reiteration theorem)**.** *If A_0, A_1 are two compatible Banach spaces and if $0 < \theta, \theta_0, \theta_1, \eta < 1$ verify:*

$$\theta = (1 - \eta) \cdot \theta_0 + \eta \cdot \theta_1 \tag{D.22}$$

then the following holds:

$$\left([\bar{A}]_{\theta_1}, [\bar{A}]_{\theta_0}\right)_\eta = [\bar{A}]_\theta \tag{D.23}$$

Theorem D.3.0.23 (Relationship between the complex and real interpolation methods)**.** *One has:*

- $\bar{A}_{\theta,1} \subset [\bar{A}]_\theta \subset \bar{A}_{\theta,\infty}$

- $\left([\bar{A}]_{\theta_0}, [\bar{A}]_{\theta_1}\right)_{\eta,p} = \bar{A}_{\theta,p}$

- $\left(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1}\right)_\eta = \bar{A}_{\theta,p}$

where $\theta = (1 - \eta) \cdot \theta_0 + \eta \cdot \theta_1$ and $\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$.

D.4   Semigroup interpolation

If T_t is a semigroup of class C^0 with generator G on some Banach space X, it is interesting to consider the Favard class $X_{\alpha,r,q}$, which is defined, for $r \in \mathbb{N}$, $0 < \alpha < r$ and $q \in [1, \infty]$ as the following real interpolation space:

$$X_{\alpha,r,q} = (X, \mathcal{D}(A^r))_{\frac{\alpha}{r}, q} \tag{D.24}$$

Then we have the:

Theorem D.4.0.24. *On $X_{\alpha,r,q}$, the following quantity:*

$$\|x\| := \|x\|_X + \left(\int_0^\infty \|(T_t - Id)^r x\|_X^q \, t^{-\alpha q} \frac{dt}{t}\right)^{\frac{1}{q}} \tag{D.25}$$

defines a norm which is equivalent to the usual real interpolation norms.

One may also prove:

Theorem D.4.0.25. *Let us write $[\alpha] = k$ and $\{\alpha\} = \beta$. Then, $x \in X_{\alpha,r,q}$ if and only if $x \in \mathcal{D}(A^k)$ and $A^k x \in X_{\beta,1,q}$. Also, the following defines a norm on $X_{\alpha,r,q}$ which is equivalent to the previous one:*

$$\|x\| = \|x\|_{A^k} + \left(\int_0^\infty \|(T_t - Id)x\|_{A^k}^q \, t^{-\alpha q} \frac{dt}{t}\right)^{\frac{1}{q}} \tag{D.26}$$

We recall that $\| \cdot \|_{A^k}$ has been defined in appendix C.

Appendix E

A result from PDE theory

We make use of the following results which we copy from [70]. We start with the:

Definition E.0.0.16 (Good weight function). *ρ is said to be a good weight function on $\mathbb{R}^N$ if it is a positive, C^∞ function such that:*

- *for any multi-index γ, $|D^\gamma \rho| = O\left(\rho^{1+|\gamma|}\right)$ holds;*

- *for some $a > 0$, $\rho^{-a} \in L^1$.*

Now, being given a good weight function ρ, a non-zero integer m, a real number μ and a real number ν such that $\nu > \mu + 2m$ and $\nu \geq 0$, we may introduce:

Definition E.0.0.17 (Family $\mathcal{A}_{\mu,\nu}^m\left(\mathbb{R}^N\right)$ of differential operators). *These are the differential operators*

$$Au = \sum_{l=0}^m \sum_{|\alpha|=2l} \rho^{\chi_{2l}} b_\alpha D^\alpha u + \sum_{|\beta|<2m} a_\beta D^\beta u \tag{E.1}$$

where the b_α and the a_β are C^∞ functions and:

- *$\chi_l = \frac{1}{2m}\left(\nu\left(2m-l\right)+\mu l\right)$*

- *$b_0 \geq C > 0$*

- *$(-1)^m \sum_{|\alpha|=2m} b_\alpha(x)\xi^\alpha \geq C|\xi|^{2m}$*

- *for $1 \leq l \leq m-1$: $(-1)^l \sum_{|\alpha|=2l} b_\alpha(x)\xi^\alpha \geq 0$*

- *for all multi-indexes β and γ: $D^\gamma a_\beta = o\left(\rho^{\chi_{|\beta|}+|\gamma|}\right)$*

In the above framework, [70] provides the following:

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Theorem E.0.0.26 (Estimates for an operator in $\mathcal{A}^m_{\mu,\nu}\left(\mathbb{R}^N\right)$)**.** *Let $A \in \mathcal{A}^m_{\mu,\nu}\left(\mathbb{R}^N\right)$, $\chi \in \mathbb{R}$ and $1 < p < \infty$. Then there exists a real number C_1 and two positive numbers $C_{2,3}$ depending only on A, p and χ such that, for any complex number λ such that $Re\lambda \leq C_1$, the following holds:*

$$C_2 \|u\|_{W_p^{2m}(\rho^{\chi+p\mu},\rho^{\chi+p\nu})} \geq \|Au - \lambda u\|_{L^p(\rho^\chi)} \geq C_3 \|u\|_{W_p^{2m}(\rho^{\chi+p\mu},\rho^{\chi+p\nu})} \qquad \text{(E.2)}$$

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