



**HAL**  
open science

# The Landau-Lifshitz-Gilbert equation driven by Gaussian noise

Antoine Hocquet

► **To cite this version:**

Antoine Hocquet. The Landau-Lifshitz-Gilbert equation driven by Gaussian noise. Analysis of PDEs [math.AP]. Ecole Polytechnique, 2015. English. NNT : . tel-01265433v2

**HAL Id: tel-01265433**

**<https://pastel.hal.science/tel-01265433v2>**

Submitted on 15 Feb 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

École doctorale n° 447 : Sciences et Technologies

# Doctorat THÈSE

pour obtenir le grade de docteur délivré par

## l'École Polytechnique

Spécialité “Mathématiques Appliquées”

*présentée et soutenue publiquement par*

**Antoine HOCQUET**

le 7 décembre 2015

## **The Landau-Lifshitz-Gilbert equation driven by Gaussian noise**

Directeur de thèse : **Anne DE BOUARD**

Co-encadrant de thèse : **François ALOUGES**

### **Jury**

|                                  |  |            |
|----------------------------------|--|------------|
| <b>M. Benjamin Goldys</b>        | Professeur, University of Sydney               | Rapporteur |
| <b>M. Massimiliano Gubinelli</b> | Professeur, Université Paris Dauphine          | Rapporteur |
| <b>M. Andreas Prohl</b>          | Professeur, Universität Tübingen               | Rapporteur |
| <b>M. Olivier Goubet</b>         | Professeur, Université de Picardie Jules Verne | Examineur  |
| <b>M. Lorenzo Zambotti</b>       | Professeur, Université Pierre et Marie curie   | Examineur  |
| <b>M. François Alouges</b>       | Professeur, École Polytechnique                | Directeur  |
| <b>Mme Anne De Bouard</b>        | Directrice de Recherches au CNRS               | Directrice |

**POLYTECHNIQUE ParisTech**

**Centre de Mathématiques Appliquées (CMAP)**

UMR CNRS 7641, Route de Saclay, 91128 Palaiseau, France



## *Remerciements*

Je souhaite tout naturellement remercier mes directeurs. Qu'ils trouvent en ces quelques mots l'expression de ma sincère gratitude.

Merci Anne, pour tes nombreux conseils, pour tes connaissances encyclopédiques, et pour ta rigueur mathématique. J'ai toujours enregistré ce que tu m'as dit, et il m'est arrivé de méditer tes réflexions pendant des heures. Merci pour ta volonté de toujours corriger mes défauts de rédaction, et surtout merci de m'avoir relu maintes et maintes fois. Grâce à toi je m'exprime plus clairement, et aussi je me rends compte à quel point c'était un effort (que dis-je : une torture) de me relire, du moins à mes débuts. Merci également de ne pas m'avoir cru quand j'ai dit des choses fausses. Merci également de m'avoir donné un bon coup de pied au  $\&\P\textcircled{C}$  lorsque j'en avais besoin.

Avec François, votre travail d'encadrement a été complémentaire, et j'ai eu la chance d'avoir à mes côtés deux chercheurs très doués, apportant chacun un regard différent sur le même sujet. Justement, puisqu'on parle de toi François, je voudrais te remercier pour tes conseils, ton expertise, pour ton intuition impressionnante, notamment lorsque que je parle avec toi d'un sujet qui n'est pas ton domaine de prédilection. Merci également pour ton optimisme, étant moi-même plutôt du genre opposé. J'admets qu'un minimum ne fait pas de mal, surtout dans la recherche.

Je remercie mes professeurs de Master 2 qui m'ont guidé dans mon choix de thèse, je pense particulièrement à Arnaud Debussche, qui m'a encouragé à aller au CMAP.

Je suis très reconnaissant envers mes rapporteurs Massimiliano Gubinelli, Benjamin Goldys et Andreas Prohl d'avoir accepté de relire ma thèse. Merci à Olivier Goubet d'avoir accepté de faire partie du jury de thèse. Merci particulièrement à Lorenzo Zambotti, pour la même raison, mais aussi pour les échanges très intéressants que nous avons eu à Saint-Flour. Je le remercie chaleureusement de m'avoir accueilli dans son groupe de travail pour les prochains mois. Cela s'adresse aussi à Cyril Labbé, avec qui j'ai pu parler cet été, et qui m'a également accordé du temps sur les structures de régularité, au lieu d'aller jouer au ping pong entre les séminaires.

Je veux remercier maintenant tous les membres du CMAP, sans distinction, à commencer par l'équipe administrative. Merci particulièrement à Nassera et Alexandra pour leur bonne humeur, et leurs nombreux sourires. Merci d'avoir aimé mon tee-shirt "Thor". Merci à Sylvain Ferrand d'avoir toujours su régler mes problèmes informatiques (notamment ce fameux jour maudit où je devais rendre ma thèse mais mon ordinateur a planté, avec bien entendu tous mes fichiers à l'intérieur), et aussi Thomas Aballe.

Merci à l'ensemble des chercheurs, et particulièrement à ceux avec qui j'ai pu échanger sur les mathématiques : à Carl Graham qui m'a expliqué le théorème de Yamada-Watanabe, à Vincent Bansaye qui m'a expliqué le théorème de Harris sur les chaînes de Markov, à Benoît

Merlet qui m'a apporté la solution d'un problème de compacité (merci aussi pour son humour), à Olivier Pantz qui m'a aidé à comprendre les subtilités de Freefem++, et grâce à qui j'ai pu faire de jolies animations (c'est toujours utile pour les présentations). Merci aussi à Igor Kortchemski et Clément Erignoux qui m'ont conseillé de faire Saint-Flour. Désolé Igor pour ma blague pas drôle sur les Bogdanov.

Merci également aux doctorants qui m'ont aidé que ce soit humainement parlant, ou mathématiquement. Je pense notamment à ceux qui étaient là en 2012 lorsque je suis arrivé, comme Camille Coron, Florent Barret, qui ont répondu à mes questions (angoissées) sur les semimartingales, en toute gratuité (je vous envoie bientôt les 50€, promis), ou encore Lætitia Giraldi qui m'a un peu considéré comme son petit frère. Je pense aussi à Aymeric Maury qui m'a filé un petit coup de pouce sur MATLAB (petit mais utile), et Étienne Adam en Probabilités. J'ai beaucoup aimé l'ambiance du CMAP, c'est un lieu d'échange avec beaucoup de doctorants étrangers : Italiens, Allemands, Chinois, Chiliens, Brésiliens, Argentins, Polonais, Russes, Néo-Zélandais, Auvergnats... Merci à tous ceux avec lesquels j'ai passé des bons moments (bien sûr ceux déjà cités en font partie) : Gwenaël Mercier, Hélène Leman (qui me fait la gentillesse de rire à mes bides), Massil Achab, Thibaut Jaisson, Charline Smadi, Romain Poncet, Faisal Wahid, Lucas Gérin et Chesnel, Xavier Dupuis, Laurent Pfeiffer, Antoine Hochart, Florine Bleuse, Gustaw Matulewicz, Vianney Bœuf, Jean-Léopold Vié, Justina Gianatti, Simona Schiavi, Mohamed Lakhal, Étienne Corman (embrassade spéciale pour l'aide au déménagement), Aline Marguet, Guillaume Mazanti, et aussi les quelques nouveaux Ludovic Sachelli, Tristan Roget, Raphaël Forien, Hadrien De March.

Il est facile d'oublier de remercier ceux qui nous ont fait aimer les mathématiques : mon professeur de terminale monsieur Lemaître, ainsi que mes professeurs de classe préparatoire Philippe Patte et MRB. Je pense également aux professeurs d'Orsay : Benjamin Graille qui m'a aidé pour l'oral de l'ENS, Raphaël Cerf qui m'a donné goût aux Probabilités.

Ici commencent les remerciements plus personnels. Merci à mes parents de m'avoir soutenu et encouragé à ne rien lâcher, vous m'avez permis de faire des études passionnantes, et ça c'est cool. Merci à Mamie Nenette, Tante Cécile, à ma Tante Véronique qui a tenu à venir à la soutenance, à toute ma famille, et merci à mes cousins de n'avoir pas fait de doctorat, comme ça je suis le premier dans un domaine, et ça c'est cool aussi. J'ai une pensée pour Mamie Marcelle qui malheureusement n'aura pas pu assister à ma soutenance, je tiens à lui dédier cette thèse. Merci à mes frères et sœurs, Nicolas, Estelle et Benoît, je suis content de vous avoir. Merci à ma belle-famille, Catherine et Jean qui m'ont toujours bien accueilli comme si j'étais leur fils, et aussi Céline, Jean-Yves, Sandrine et Andrea.

Enfin, pour finir, merci à toi, Anne-Sophie, pour ton amour, ton soutien quotidien, pour ta tolérance envers mes études longues. Merci pour ton enthousiasme et ta joie de vivre. Je suis heureux avec toi. J'espère que l'avenir nous réserve de belles choses.

## **Table des matières**

|          |  |           |
|----------|--|-----------|
| (i)      | L'équation de Landau-Lifshitz-Gilbert . . . . .  | 1         |
| (i-a)    | Le modèle dynamique du micromagnétisme . . . . .   | 1         |
| (i-b)    | Un parallèle important avec le flot des applications harmoniques. . . . .  | 4         |
| (i-c)    | Du phénomène de dissipation à celui des fluctuations thermiques. . . . .   | 6         |
| (i-d)    | Influence des fluctuations pour LLG fini-dimensionnelle . . . . .  | 7         |
| (i-e)    | Analyse mathématique de la dimension infinie. . . . .  | 8         |
| (ii)     | Revue des différents résultats obtenus. Comparaison avec le cas déterministe. . . . .  | 10        |
| (ii-a)   | Formule de l'énergie, existence de solutions faibles . . . . .   | 10        |
| (ii-b)   | Problèmes liés à l'existence et l'unicité locales de solutions pour (SLLG)<br>(pour un domaine $\mathcal{O}$ de dimension quelconque). . . . . | 13        |
| (ii-c)   | Problèmes spécifiques liés à SLLG en 2D, présentation des chapitres 2<br>et 4. . . . .   | 16        |
| (ii-d)   | Explosion en temps fini et unicité des solutions en 2D. . . . .  | 19        |
| (ii-e)   | Discrétisation de SLLG. . . . .  | 22        |
| <b>1</b> | <b>Local solvability</b> . . . . .   | <b>29</b> |
| 1        | Introduction . . . . .   | 30        |
| (1.a)    | Preliminaries . . . . .  | 30        |
| (1.b)    | Note on the infinite dimensional diffusion term. . . . .   | 31        |
| (1.c)    | Itô/Stratonovitch correction term, and main ansatz of the equation . . . . .   | 33        |
| (1.d)    | Results . . . . .  | 34        |
| 2        | Preliminaries . . . . .  | 36        |
| 3        | Proof of Theorem 1.1 . . . . .   | 39        |
| <b>2</b> | <b>Global solutions with finite energy</b> . . . . .   | <b>45</b> |
| 1        | Introduction . . . . .   | 46        |
| (1.a)    | Motivations . . . . .  | 46        |
| (1.b)    | Notion of solution and main Theorem . . . . .  | 48        |
| 2        | Proof of Theorem 2.2 . . . . .   | 51        |
| (2.a)    | Main estimates . . . . .   | 51        |
| (2.b)    | Consequences of Proposition 2.2 . . . . .  | 54        |
| (2.c)    | Skorohod space and the use of Gyöngy-Krylov Lemma . . . . .  | 57        |
| (2.d)    | Uniqueness and global well-posedness . . . . .   | 64        |
| A        | Appendix . . . . .   | 67        |
| (1)      | The energy formulae . . . . .  | 67        |

|          |  |            |
|----------|--|------------|
| (2)      | Some technical proofs . . . . .  | 70         |
| <b>3</b> | <b>A uniqueness criterion in dimension two</b>                           | <b>73</b>  |
| 1        | Motivations . . . . .  | 74         |
| 2        | Statement of the result . . . . .  | 76         |
| 3        | Proof of Theorem 3.3 . . . . .   | 78         |
| 4        | Proof of Lemma 3.1 . . . . .   | 80         |
| (4.a)    | Idea of the proof . . . . .  | 80         |
| (4.b)    | Use of Helmholtz decomposition . . . . .                                 | 81         |
| (4.c)    | Proof of Claim 3.1 : bounds on the gradient part . . . . .               | 83         |
| (4.d)    | Decomposition of “ $\nabla^\perp \beta_1 \cdot \nabla w$ ”. . . . .      | 84         |
| (4.e)    | Step 4 : Parabolic estimates and conclusion by Wentz’s Theorem . . . . . | 85         |
| <b>4</b> | <b>Finite time singularity of the stochastic harmonic map flow</b>       | <b>91</b>  |
| 1        | Introduction and main result . . . . .                                   | 92         |
| (1.a)    | Motivations . . . . .  | 92         |
| (1.b)    | Local existence and uniqueness . . . . .                                 | 95         |
| (1.c)    | Statement of the main result . . . . .                                   | 96         |
| 2        | Proof of Theorem 4.2 . . . . .   | 97         |
| 3        | Proof of Claim 4.1 . . . . .   | 99         |
| 4        | Proof of Proposition 4.1 . . . . .                                       | 106        |
| (4.a)    | An interpolation Lemma. . . . .  | 106        |
| (4.b)    | Proof when $\beta \in (4/3, 2]$ . . . . .                                | 108        |
| (4.c)    | Proof when $\beta \in (2, 4]$ . . . . .                                  | 109        |
| A        | Appendix: the comparison principle . . . . .                             | 111        |
| B        | Appendix: some technical proofs . . . . .                                | 114        |
| (4.B.1)  | Proof of Lemmata 4.1 . . . . .   | 114        |
| (4.B.2)  | Proof of Lemma 4.2 . . . . .   | 115        |
| <b>5</b> | <b>A new Semi Discrete Scheme for Stochastic LLG</b>                     | <b>117</b> |
| 1        | Introduction . . . . .   | 118        |
| 2        | Main result . . . . .  | 119        |
| 3        | Energy estimates . . . . .   | 122        |
| 4        | Tightness . . . . .  | 125        |
| 5        | Convergence of the martingale part . . . . .                             | 131        |
| 6        | Identification of the limit . . . . .                                    | 135        |
| <b>6</b> | <b>Numerical Studies</b>   | <b>143</b> |
| 1        | The numerical Method . . . . .   | 144        |
| (1.a)    | Preliminaries . . . . .  | 144        |
| (1.b)    | Approximations of white and space-regular noises . . . . .               | 145        |
| (1.c)    | Practical implementation of Algorithm 5.1 . . . . .                      | 146        |
| 2        | Influence of noise on blowing-up solutions . . . . .                     | 150        |

# Introduction

## (i) L'équation de Landau-Lifshitz-Gilbert

### (i-a) Le modèle dynamique du micromagnétisme

La plupart des données numériques sont stockées à l'aide des matériaux ferromagnétiques, qui par définition possèdent une aimantation spontanée. Afin d'en donner une description, Pierre Weiss a proposé en 1907 une théorie des *domains*, qui consiste à considérer l'aimant comme un assemblage de petites régions uniformément aimantées. À l'intérieur de ces domaines (les "domaines de Weiss"), les moments magnétiques sont tous alignés, la direction d'alignement semblant ainsi discontinue aux frontières, voir la figure 1. Cependant, il a été observé au début des années 1930 par les physiciens Bloch et Heisenberg que la largeur de ces frontières n'est pas nulle. En considérant des échelles plus petites, on voit qu'en réalité l'aimantation reste continue d'un domaine à l'autre, voir figure 2. Si le matériau est soumis à un champ magnétique extérieur  $H$ , les frontières des parois bougent, de sorte que les domaines dont les moments magnétiques  $M$  sont parallèles à  $H$  auront tendance à s'élargir. Les physiciens soviétiques Lev Davidovich Landau (1908–1968) et Eugenii Myhailovich Lifshitz (1915–1985), dans leur effort pour déterminer la distribution des moments magnétiques  $M$  à l'intérieur des parois, ont proposé en 1935 l'équation suivante

$$\frac{dM}{dt} = \gamma M \times H_{\text{eff}} - \frac{\lambda}{M_s} M \times (M \times H_{\text{eff}}) . \quad (\text{LL})$$

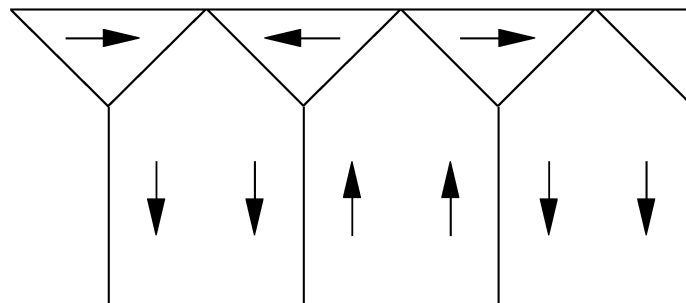


FIGURE 1 – Exemple réaliste de domaines magnétiques adjacents, suggéré par L.D. Landau et E.M. Lifshitz.



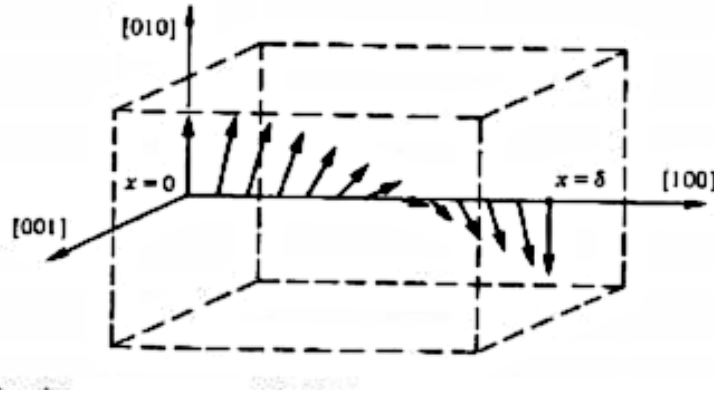


FIGURE 2 – Zoom à l’intérieur d’une paroi de Bloch : l’aimantation passe brusquement d’un alignement sur  $(0, 1, 0)$  à un alignement sur  $(0, -1, 0)$ .

Dans (LL), le premier terme de droite décrit la précession du champ effectif  $H_{\text{eff}}$ , autour duquel tourne le moment magnétique. Ce terme peut être obtenu dans le cadre d’une théorie phénoménologique générale, fondée sur l’observation que pour de faibles températures l’amplitude du moment magnétique est conservée. La théorie de Weiss prédit en effet qu’en dessous de la température dite “de Curie” :

$$|M| = M_s , \quad (1)$$

où “l’aimantation à saturation”  $M_s = M_s(T)$  ne dépend que de la température. La présence dans (LL) du double produit vectoriel dans le membre de droite est également proposée pour des raisons phénoménologiques. D’une part, la constance de l’amplitude (1) impose que chaque contribution à  $dM/dt$  soit orthogonale à  $M$ . D’autre part, le champ effectif est construit de sorte qu’à l’équilibre,  $M$  et  $H_{\text{eff}}$  sont alignés, ce qui rend nécessaire la présence d’un terme de vitesse dirigé de  $M$  vers  $H_{\text{eff}}$ . L.D. Landau et E.M. Lifshitz ont cependant souligné qu’en dehors du cas où sa valeur est très faible, le paramètre  $\lambda$  dans (LL) ne pouvait pas correspondre physiquement à une “constante d’amortissement”, des valeurs croissantes de  $\lambda$  ayant pour effet d’accélérer la relaxation vers l’état d’équilibre. De manière générale, l’amortissement d’un système physique en mouvement macroscopique s’accompagne nécessairement d’un transfert d’énergie cinétique et potentielle vers une énergie thermique (mouvement microscopique), et donc d’une perte de vitesse.

Dans le cas présent du mouvement d’une chaîne de spins, les détails concernant le phénomène de dissipation sont cependant trop complexes pour être pris en compte explicitement dans l’équation. Outre les pertes d’énergie liées aux courants de Foucault, les mécanismes par lesquels se produit la dissipation s’expliquent par des degrés de liberté microscopiques du système avec lequel  $M$  interagit, par exemple au travers des défauts du réseau cristallin du matériau, ou encore par ses vibrations. Cependant les mécanismes dominants ne semblent toujours pas encore avoir été identifiés aujourd’hui, voir par exemple à ce sujet [Gil04] ainsi que les références incluses.

Afin de remédier au problème soulevé par L.D. Landau et E.M. Lifshitz, le physicien Thomas L. Gilbert a proposé en 1955, à l’aide notamment du formalisme lagrangien, une autre façon d’introduire l’amortissement dans le système. L’équation proposée par Gilbert [Gil55], qui est

aujourd'hui la formulation admise d'un point de vue physique, s'écrit :

$$\frac{dM}{dt} = \gamma M \times H_{\text{eff}} - \frac{\alpha}{M_s} M \times \frac{dM}{dt} . \quad (2)$$

Le premier aspect commun à souligner, concernant les équations (LL) et (2), est que toute solution préserve la contrainte (1) : il suffit pour s'en rendre compte de multiplier scalairement ces équations par  $M$ , puis d'intégrer en temps. Dans l'équation (2), la constante  $\alpha$  est cette fois un paramètre d'amortissement, proportionnel à l'intensité de la dissipation du système. Cette équation est généralement dénommée "forme de Gilbert", faisant allusion au fait qu'il s'agit en réalité de l'équation (LL), mais avec des constantes physiques différentes. En effet, en multipliant vectoriellement (2) à gauche par  $M$ , puis en réutilisant la relation obtenue ainsi que l'égalité  $|M| = M_s$ , on voit par un calcul immédiat que les équations (LL) et (2) sont mathématiquement identiques. À redéfinition près de la constante de précession, on retrouve l'équation (LL), et la relation liant la dissipation à la constante de L.D. Landau et E.M. Lifshitz s'écrit

$$\lambda = \frac{\alpha}{1 + \alpha^2} . \quad (3)$$

Entre temps, W.F. Brown avait introduit dans les années 1940 le micromagnétisme [Bro63a], qui est une théorie des milieux continus consistant à décrire l'aimantation d'un matériau ferromagnétique  $\mathcal{O} \subseteq \mathbb{R}^3$  par un champ de vecteurs

$$u : \mathcal{O} \longrightarrow \mathbb{R}^3 ,$$

et où les configurations admissibles au repos sont celles qui minimisent une énergie  $\mathcal{E}$ , ce qui de manière équivalente se traduit par l'alignement ponctuel de  $u$  sur un champ effectif continu  $H_{\text{eff}}$ , défini comme étant l'opposé du gradient de l'énergie.

Le passage du modèle discret au modèle continu peut être dérivé heuristiquement de la manière suivante. Considérons un ensemble discret de spins  $M_i$ , situés dans différents points de l'espace  $x_i$ ,  $i = 1, 2, \dots, n$ , que l'on représente par une fonction d'aimantation continue  $u$  et renormalisée de sorte que  $u(x_i) = M_i/M_s$  pour tout  $i$ . Les équations du mouvement s'expriment  $dM_i/dt = \gamma M_i \times H_i - (\lambda/M_s) M_i \times (M_i \times H_i)$ ,  $i = 1, \dots, n$ , où le champ effectif  $H_i$  doit *a priori* prendre en compte les effets générés par tous les spins. Dans une situation simplifiée, l'interaction dominante localement en  $x_i$  est celle de  $M_i$  avec ses voisins immédiats. Une hypothèse simple est de supposer que les spins, à la manière d'une boussole qui s'aligne sur le champ terrestre, s'alignent sur le champ magnétique que créent les particules voisines. Ainsi, dans une structure de type cristalline, le vecteur  $M_i$  aura tendance à s'aligner sur la moyenne locale, obtenue sur l'ensemble des points  $x_i \pm \delta \vec{e}_j$  où  $\vec{e}_j$ ,  $j = 1, \dots, d$  décrit un ensemble de directions, et  $\delta$  est la distance séparant deux voisins. Cela donne un champ effectif local sous la forme  $H_{\text{eff}} = J \sum_{1 \leq j \leq d} u(x_i + \delta \vec{e}_j) + u(x_i - \delta \vec{e}_j)$ , où  $J$  est une constante d'échange qui classiquement est proportionnelle à l'inverse du carré de  $\delta$ . Posant pour simplifier  $J = \delta^{-2}$ , et prenant toutes constantes égales à 1, nous obtenons que  $u \times H_{\text{eff}} = u \times \Delta u + O(\delta^3)$ ,  $\Delta u$  désignant ici le Laplacien de  $u$  sur chaque composante. L'équation continue s'écrit donc

$$\frac{\partial u}{\partial t} = \gamma u \times \Delta u + u \times (u \times \Delta u) . \quad (\text{LLG})$$

Dans le cas où seule cette interaction est prise en compte, le champ effectif  $H_{\text{eff}} := \Delta u$  est égal au gradient de la fonctionnelle d'énergie d'échange  $E(u) := 1/2 \int_D |\nabla u|^2 dx$ . Une intégration

par parties montre en effet que l'on a  $\Delta u = -\frac{\partial E(u)}{\partial u}$ . Cette énergie pénalise les variations locales de  $u$  : la minimiser revient à aligner localement  $u$  sur sa moyenne.

En toute généralité, l'équation donnée par L.D. Landau et E.M. Lifshitz incorpore dans  $H_{\text{eff}}$  d'autres champs, générés par des phénomènes physiques de nature différente, bien que reliés. La théorie de W.F. Brown inclut ainsi plusieurs énergies. L'énergie extérieure correspond au même phénomène que le précédent, mais à plus grande échelle. Sa minimisation aligne  $u$  sur un champ extérieur donné, indépendant de  $u$ . L'énergie d'anisotropie facilite l'alignement de  $u$  sur des directions prescrites par la structure cristalline du matériau. Enfin, l'énergie démagnétisante est un terme qui rend compte du fait que la configuration de  $u$  est soumise au champ magnétique  $h_d$  qu'elle produit elle-même. Non local, il s'agit souvent du terme le plus difficile à traiter en pratique, dans les applications numériques. La somme de toutes ces contributions  $\mathcal{E}$  est appelée "énergie libre" (ou énergie de Brown), et le champ effectif se définit à l'aide de la relation  $H_{\text{eff}} = -\partial\mathcal{E}/\partial u$ . Il n'existe pas, en général, de minimiseur commun à ces différents termes, et on observe en général plusieurs configurations d'équilibre pour  $u$ , toutes solutions (locales) du problème de minimisation

$$\min_{u: |u|_{\mathbb{R}^3} \equiv 1} \mathcal{E}(u) . \quad (4)$$

Suivant l'esprit de la littérature mathématique [[SSB86](#), [AS92](#), [GH93](#), [BGJ13](#)], nous traitons dans ce manuscrit le cas d'un champ effectif dérivant de l'énergie d'échange seulement. Les résultats qualitatifs (existence, unicité, régularité, etc.) obtenus dans ce cas particulier peuvent en général s'adapter à des énergies/champs plus généraux. Des exemples de tels traitements sont par exemple effectués dans [[Vis85](#)], ou encore [[CF01a](#)]. À partir de maintenant, nous ferons systématiquement l'hypothèse que  $\mathcal{E} = E \equiv 1/2 \int_{\mathcal{O}} |\nabla u|^2$ .

### (i-b) Un parallèle important avec le flot des applications harmoniques.

Le problème de minimisation (4) a des liens profonds avec la géométrie différentielle, et notamment avec la notion d'application harmonique à valeurs dans une variété. Des problèmes variationnels de cette forme apparaissent naturellement aux géomètres, l'exemple le plus parlant étant le calcul des géodésiques d'une variété donnée  $\mathcal{N}$ , consistant à chercher un chemin  $u : [0, 1] \rightarrow \mathcal{N}$  qui minimise la distance parcourue entre deux points. Ici, la contrainte que l'on voit apparaître dans (4) s'exprime simplement comme l'appartenance de chaque vecteur image à la variété  $\mathbb{S}^2 = \{X \in \mathbb{R}^3, |X| = 1\}$ .

Étant données deux variétés riemanniennes  $\mathcal{M}$ ,  $\mathcal{N}$  et une application  $u : \mathcal{M} \rightarrow \mathcal{N}$ , on peut définir de manière intrinsèque l'intégrale

$$E(u) = \int_{\mathcal{M}} \frac{|du(x)|^2}{2} d\text{vol}_{\mathcal{M}} . \quad (5)$$

Les *applications harmoniques* sont par définition des fonctions régulières (au minimum de classe  $\mathcal{C}^2$ ) qui minimisent la forme quadratique (5), appelée intégrale de Dirichlet. Heuristiquement, une telle application  $u$  représente une manière optimale de "tendre" la variété de départ  $\mathcal{M}$  dans celle de d'arrivée – dans cette métaphore  $\mathcal{M}$  représenterait un objet idéalement élastique, voir l'illustration figure 3.

Il est également possible de définir des applications *faiblement* harmoniques en minimisant l'énergie de Dirichlet dans l'espace  $H^1(\mathcal{M}; \mathcal{N})$ , pendant de l'espace de Sobolev classique dans un contexte riemannien. Un calcul simple permet de voir que toute application  $u : \mathcal{O} \rightarrow \mathbb{S}^2$

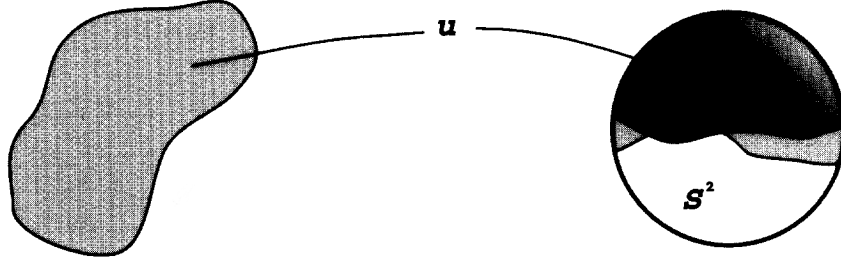


FIGURE 3 – Application harmonique à valeurs dans la sphère. La variété de départ est “tendue” dans  $\mathbb{S}^2$ .

d’énergie finie, minimisante pour  $E$ , est solution au sens des distributions de l’équation d’Euler-Lagrange

$$\Delta u + u|\nabla u|^2 = 0 \text{ dans } \mathcal{O} , \quad (6)$$

Ici, la fonction  $|\nabla u|^2 = \sum_{i,j} (\partial_j u^i)^2$  n’est autre que le multiplicateur de Lagrange associé à la contrainte ponctuelle imposée sur  $u$ . Si l’on munit les candidats  $u$  de conditions au bord, par exemple en imposant que  $u|_{\partial\mathcal{O}} = g$ , la méthode directe du calcul des variations montre qu’il existe une application faiblement harmonique dès lors que  $H^1(\mathcal{O}; \mathbb{S}^2) \cap \{u : u|_{\partial\mathcal{O}} = g\}$  n’est pas vide. En multipliant vectoriellement à gauche par  $u$  l’équation (6), puis en injectant la relation dans (LLG), il est immédiat de voir que la fonction  $f(t, x) = u(x)$  est une solution stationnaire pour (LLG).

En résumé, étant donnée une condition initiale  $u_0$  d’énergie finie, il existe au moins une configuration à l’équilibre pour l’énergie d’échange, valant  $u_0|_{\partial\mathcal{O}}$  au bord. En temps long, il est donc raisonnable d’espérer que les solutions de (LLG) (si elles existent) se stabiliseront “autour d’une application harmonique”, un point d’équilibre du système, minimiseur local de l’énergie d’échange.

Mentionnons également que l’équation (6) apparaît, lorsque  $\dim \mathcal{O} = 3$ , dans la théorie des cristaux liquides [CGH91]. Ces cristaux sont constitués de particules orientées décrites par un champ de vecteurs unitaires, et minimisant à l’équilibre l’énergie de Frank-Oseen.

Appliquant maintenant la formule du double produit vectoriel et utilisant le fait que les applications à valeurs dans la sphère satisfont  $0 = \Delta(|u|^2/2) = \Delta u \cdot u + |\nabla u|^2$ , l’équation (LLG) est, au moins formellement, équivalente à  $\partial_t u = \gamma u \times \Delta u + \Delta u + u|\nabla u|^2$ . Si le système est “suramorti” (i.e.  $\gamma = 0$ ), on retrouve l’équation du *flot des applications harmoniques à valeurs dans  $\mathbb{S}^2$* , i.e.

$$\partial_t u = \Delta u + u|\nabla u|^2 . \quad (\text{HMF})$$

Il existe une abondante littérature sur les propriétés du flot de (HMF). Pour un exposé complet sur le sujet, on peut entre autres consulter les références [EL78, EL88]. Les auteurs J. James Eells, J.H. Sampson ont en 1964 donné le point de départ dans un article fondateur [ES64]. Cette équation apparaît essentiellement pour une raison géométrique : étant donnée une application  $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ , on cherche à savoir s’il existe une application harmonique dans sa classe d’homotopie. Plus informellement, on aimerait savoir s’il existe une *déformation continue* de  $\varphi$  vers une application harmonique. L’existence d’une telle déformation peut être montrée par

une technique de flot. Supposons l'existence d'une solution  $u$  à (HMF) globale en temps, de condition initiale  $\varphi$ , telle que la limite  $\psi := \lim_{t \rightarrow \infty} u(t, \cdot)$  existe et que la convergence soit "suffisamment forte" (typiquement dans un espace de fonctions continues sur  $\mathcal{O}$ ). Il peut être montré, en utilisant la formule fondamentale de l'énergie, voir ci-dessous l'équation (7), qu'une telle application  $\psi$  est nécessairement une solution stationnaire de (HMF), et donc une application harmonique. Le flot  $t \mapsto u(t, \cdot)$  fournit ainsi la déformation désirée  $\varphi \rightsquigarrow \psi$ .

### (i-c) Du phénomène de dissipation à celui des fluctuations thermiques.

Du point de vue mathématique, toute solution (régulière) de (LLG) doit vérifier la relation de dissipation

$$E(t) - E(0) + \int_0^t \mathcal{D}(\sigma) d\sigma = 0, \quad (7)$$

où  $\mathcal{D}(\sigma)$  désigne la "tension"  $\int_{\mathcal{O}} |\partial_t u(\sigma, x)|^2 dx$ . Il suffit pour voir cela de multiplier scalairement l'équation par  $\Delta u$ , et d'intégrer en espace, puis en temps. Bien que fondamental, le fait que l'énergie décroisse au cours du temps n'est cependant pas étonnant, l'équation phénoménologique proposée par L.D. Landau et E.M. Lifshitz ayant été construite, en partie, dans ce but.

Comme déjà mentionné plus haut, le phénomène d'*amortissement* trouve ses origines dans des interactions microscopiques entre le champ  $u$  et son environnement. Moyennés, les effets dûs à ces degrés de liberté microscopiques provoquent une perte d'énergie du système, une dissipation. À mesure que l'on augmente la température du matériau, un autre effet important de ces interactions microscopiques se fait ressentir. Il s'agit de *fluctuations*, pour lesquelles il est désormais communément admis, dans la littérature physique, qu'elles sont l'effet d'interactions avec des particules : les phonons. Un cristal vibrant suivant un mode de fréquence  $\nu$  peut gagner ou céder de l'énergie, dans une quantité qui soit un multiple entier du quantum  $\hbar\nu$ , où  $\hbar$  désigne la constante de Planck. Dans le formalisme de la mécanique quantique, la dualité onde/corpuscule veut que les "paquets d'énergie" échangés, qui proviennent d'objets ondulatoires (ici les modes de vibration d'un cristal), correspondent également à des échanges de particules, les phonons.

La prise en compte des effets dûs à l'agitation thermique dans le micromagnétisme date de 1946 et des travaux de Louis Néel [Née46]. Elle a ensuite été formalisée en 1963 par W.F. Brown [Bro63b], et a depuis été développée par d'autres auteurs, voir par exemple [KMM<sup>+</sup>99]. Dans la théorie du micromagnétisme, le moyen le plus commun pour prendre en compte ces interactions est d'introduire un champ  $H_{\text{thm}}$  que l'on ajoute au champ effectif dans (LLG). Cela conduit à considérer l'équation de Landau-Lifshitz-Gilbert stochastique [Bro63b] :

$$\frac{dM}{dt} = -\gamma M \times (H_{\text{eff}} + H_{\text{thm}}) - \frac{\lambda}{M_s} M \times (M \times (H_{\text{eff}} + H_{\text{thm}})) . \quad (8)$$

À température ambiante, les phonons ont une "très faible" durée de vie, en comparaison du temps moyen de relaxation du système. Les forces générées par l'interaction de ces particules avec  $u$  ont des temps de corrélation de l'ordre de  $10^{-13}$  sec. Dans le cadre d'une "particule monodomaine" (i.e. un matériau constitué d'un seul domaine de Weiss, uniformément aimanté), le temps de relaxation lié à une perturbation instantanée est de l'ordre de  $10^{-10}$  sec., ce qui permet de considérer le spectre des forces d'agitation thermiques comme *blanc*, voir par exemple la discussion sur le sujet dans [Bro63b], ainsi que les références incluses.

Cette propriété subsiste dans le cas d'un nombre fini de spins  $M_1, \dots, M_n$ , situation dans laquelle, d'après les arguments développés dans [Ber07], les différentes composantes  $H_{\text{thm}}^1, \dots, H_{\text{thm}}^n$  peuvent également être considérées comme indépendantes. Mathématiquement parlant, cela revient à considérer que le champ  $H_{\text{thm}}$  satisfait aux propriétés statistiques suivantes

$$\mathbb{E} H_{\text{thm}}(t) = 0 \quad , \quad \mathbb{E} H_{\text{thm}}^i(t) \cdot H_{\text{thm}}^j(s) = 2D\delta_{ij}\delta(t-s) \quad , \quad (9)$$

où  $\delta_{ij}$  désigne le symbole de Kronecker, tandis que  $\delta(t)$  est la mesure de Dirac. La lettre  $D$  désigne ici une constante proportionnelle à la température du matériau via la relation [Ber07, p. 801] :

$$D = \frac{\lambda}{1 + \lambda^2} \frac{kT}{\gamma M_s V} \quad , \quad (10)$$

$k$  étant la constante de Boltzmann, et  $V$  le volume d'un monodomaine du matériau. De plus, en raison du grand nombre d'interactions entrant en jeu dans  $H_{\text{thm}}$ , et puisqu'elles ont toutes les mêmes propriétés statistiques, il est communément admis que le processus  $t \rightarrow H_{\text{thm}}(t)$  est *Gaussien*, ce qui peut se justifier mathématiquement par le Théorème de la Limite Centrale.

### (i-d) Influence des fluctuations pour LLG fini-dimensionnelle

Lorsque nous parlons de LLG "fini-dimensionnelle", nous faisons référence à l'équation portant sur un nombre fini de spins, c'est à dire au système d'équations différentielles stochastiques engendré par (8), avec  $M = M_i$ ,  $H = H_i$  et  $H_{\text{thm}} = H_{\text{thm}}^i$ , pour  $1 \leq i \leq n$ .

L'ajout de bruit dans l'équation de Landau-Lishitz-Gilbert est un problème mathématique dont l'analyse a débuté dans la dernière décennie. Cette analyse présente un intérêt industriel, les matériaux ferromagnétiques étant devenus d'usage courant. La nécessité grandissante d'optimiser le stockage magnétique de l'information constitue une des raisons principales de l'intérêt porté à l'analyse de (8). Les bits sont stockés dans les matériaux ferromagnétiques, au moyen d'une orientation particulière du moment magnétique. L'information contenue dans un disque dur correspond à une configuration du champ  $M$ , on peut par exemple imaginer que le vecteur  $M$  pointe vers le haut pour signifier "1", et inversement vers le bas pour coder un "0". Il faut s'assurer cependant que le vecteur  $M$  soit à l'équilibre, ce qui peut être obtenu en exerçant artificiellement un champ extérieur, nécessitant un apport d'énergie.

Le modèle déterministe atteint ses limites lorsque l'on cherche à stocker ces bits dans des espaces de plus en plus restreints. Cela est lié au phénomène de "switching" : malheureusement, en raison des fluctuations thermiques, l'aimantation en un point peut s'inverser spontanément. Pour provoquer un "switch", les fluctuations doivent au préalable apporter une quantité minimale d'énergie à la configuration. Ce quantum  $\Delta E$  est au mieux proportionnel au volume de l'échantillon. On voit donc que plus l'espace occupé par l'échantillon est restreint, plus la probabilité d'inversion spontanée est grande, ce qui pose un problème pratique dans le stockage de l'information. La compréhension de ce mécanisme, dont la non prise en compte dans (8) constituerait une impasse, est donc cruciale pour l'amélioration des capacités de stockage.

### (i-e) Analyse mathématique de la dimension infinie.

Il n'existe pas, à ce jour, de dérivation mathématique rigoureuse du modèle stochastique continu. L'hypothèse cependant retenue dans la littérature sur le sujet [BBNP13a, BGJ13, GLT13], par analogie avec le cas d'un nombre fini de spins, est que (i) les fluctuations thermiques doivent être incorporées dans (LLG) via l'ajout d'un terme  $H_{\text{thm}}$  au champ effectif ; (ii) ce terme est un *bruit blanc Gaussien en espace-temps*. En pratique, les auteurs cités supposent néanmoins l'existence d'une corrélation spatiale pour  $H_{\text{thm}}$ , le cas du bruit blanc espace-temps rendant hors de portée le traitement mathématique de l'équation.

Dans la suite  $\zeta$  désigne un bruit blanc espace-temps, à valeurs dans  $\mathbb{R}^3$ . Nous représentons un matériau ferromagnétique par un domaine  $\mathcal{O}$  de dimension deux ou trois. Après mise sans dimension de l'équation, nous sommes donc ramenés au problème suivant, pour des constantes  $\gamma \in \{0, 1\}$  et  $\epsilon$  données.

Trouver  $u = u(t, x)$ , tel que l'on ait :

$$\partial_t u = -u \times (u \times (\Delta u + \epsilon \zeta)) + \gamma u \times (\Delta u + \epsilon \zeta) , \quad \text{pour tout } (t, x) \text{ dans } \mathbb{R}_+ \times \mathcal{O} , \quad (11)$$

où  $u$  vérifie les conditions au bord de type

$$\begin{cases} \text{Dirichlet : } u(t, x) = u_0(x) , & \text{ou} \\ \text{Neumann homogènes : } \frac{\partial u}{\partial n}(t, x) = 0 , & \end{cases} \quad \text{pour tout } (t, x) \text{ dans } \mathbb{R}_+ \times \partial \mathcal{O} , \quad (12)$$

avec pour donnée initiale :

$$u(0, x) = u_0(x) , \quad \text{pour tout } x \text{ dans } \mathcal{O} , \quad (13)$$

et  $u$  est supposée vérifier la contrainte locale sur sa magnitude :

$$u(t, x) \in \mathbb{S}^2 = \{x \in \mathbb{R}^3, |x| = 1\} , \quad \text{pour tout } (t, x) \text{ dans } \mathbb{R}_+ \times \mathcal{O} . \quad (14)$$

Une force de type bruit blanc Gaussien est classiquement formalisée dans la littérature par la dérivée faible d'un processus de Wiener cylindrique. Un tel processus est généralement donné par une somme formelle

$$W(t) = \sum_{k \in \mathbb{N}} \beta_k(t) e_k , \quad (15)$$

où les coordonnées sur la base orthonormée  $\{e_k, k \geq 0\}$  de l'espace  $L^2(\mathcal{O})$  sont des mouvements browniens réels  $\beta_k$ , indépendants entre eux. Il s'agit du point de vue adopté dans l'ouvrage [DZ08]. On peut vérifier que le processus  $\zeta := dW/dt$  définit une mesure aléatoire sur les boréliens  $A$  de  $\mathbb{R}_+ \times \mathcal{O}$ , telle que : (i) pour tout  $A$ ,  $\zeta(A)$  est une variable aléatoire gaussienne, de variance égale à la mesure de Lebesgue de  $A$  ; (ii) si  $(A_i)$  désigne une famille disjointe de boréliens, alors les  $\zeta(A_i)$  correspondants forment une famille indépendante. Heuristiquement,

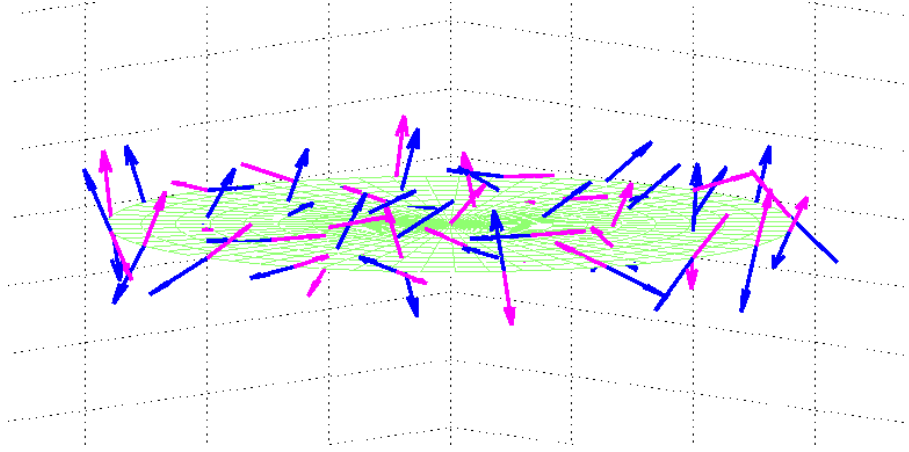


FIGURE 4 – Exemple de configuration sur le disque unité. En bleu :  $u$ , en violet :  $H_{\text{eff}} = \Delta u$ .

on peut voir  $\zeta(A)$  comme étant “la quantité de bruit contenue dans l’ensemble  $A$ ”, le bruit n’ayant, pour des raisons de régularité, pas de sens ponctuel.

En effet, le ratio  $(W(t + \Delta t) - W(t))/\Delta t$  diverge presque sûrement à la limite  $\Delta t \rightarrow 0$ . La manière correcte d’interpréter l’équation obtenue est de procéder par dualité, en définissant une intégrale stochastique en temps, par rapport à la mesure aléatoire  $dW$ . Cela ne peut malheureusement pas se faire de manière canonique : afin de définir précisément ces intégrales, il nous faut une “règle additionnelle” pour interpréter l’intégrale “ $\int_0^t \Phi dW$ ”, ou de manière équivalente, le produit “ $\Phi \frac{dW}{dt}$ ”.

L’interprétation de *Stratonovich* consiste à passer par une discrétisation via des sommes partielles du type  $\sum_{0 \leq i \leq n-1} \Phi(t_{i+1/2})(W(t_{i+1}) - W(t_i))$ , où l’on évalue l’intégrand au milieu de chaque intervalle de temps  $[t_i, t_{i+1})$ , puis à passer à la limite lorsque la taille de la discrétisation en temps tend vers 0. Si l’on choisit, à la place, d’évaluer l’intégrand par sa valeur à gauche de chaque intervalle, on définit alors l’intégrale d’Itô, préférée par les mathématiciens pour ses abondantes propriétés en théorie des probabilités, notamment celle de définir une martingale. À contrario, le calcul de Stratonovich est souvent préféré dans les applications physiques, car les résultats obtenus coïncident avec le passage formel à la limite  $\tau \rightarrow 0$ , où  $\tau$  désigne le temps de corrélation des fluctuations du système. Ces deux interprétations ne coïncident pas en général lorsque l’intégrand implique l’inconnue i.e. lorsque  $\Phi = \Phi(u)$ , voir par exemple [G<sup>+</sup>85, chap. 3].

Par ailleurs, l’intégrale d’Itô ne respecte pas la “règle de la chaîne” classique, à savoir que la différentielle de  $F(u)$  n’est pas égale en général à  $F'(u)du$ , quelle que soit la régularité de la fonctionnelle  $F$ . En particulier, le fait qu’ici  $du$  soit en tout point orthogonal à  $u$  n’implique pas pour autant que la magnitude locale soit préservée le long du flot : il est faux, au sens d’Itô, de dire que “ $d(|u|^2) = u \cdot du$ ”. En réalité, une interprétation Itô de (11) mènerait à une magnitude non constante, ce qui est proscrit, au regard du modèle. L’intégrale de Stratonovich, au contraire, nous assure que la contrainte (14) est préservée au cours du temps.

Les deux points de vue peuvent toutefois être réconciliés si l’on interprète une intégrale de Stratonovich comme une intégrale d’Itô plus un terme additionnel de drift. On a en effet la



relation (formelle)

$$\int_{[0,T]} \Phi \circ dW = \frac{1}{2} \underbrace{\int_{[0,T]} \sum_{k \geq 0} \Phi'(u) [\Phi e_k] e_k dt}_{:=F(u)} + \int_{[0,T]} \Phi dW, \quad (16)$$

où “ $\circ$ ” signifie que la règle de Stratonovitch est utilisée, l’intégrale de droite étant celle d’Itô, et  $(e_k)$  est comme dans (15). Pour LLG stochastique sous la forme (11), c’est l’intégrale de Stratonovitch qui est choisie dans [Bro63b]. En rajoutant le terme  $F(u)$  dans le membre de droite de (11), on peut vérifier formellement, en appliquant la formule d’Itô, que la contrainte sur la magnitude locale de  $u$  est préservée.

Dans la littérature sur (11) (voir section suivante), on omet en général d’ajouter  $\zeta$  à  $\Delta u$  dans le terme dissipatif, la raison souvent évoquée étant que dans les applications physiques, le paramètre d’amortissement est très faible devant la constante de précession  $\gamma$ . Un raisonnement plus mathématique permet cependant de mieux justifier la validité de cette hypothèse, puisqu’il a été remarqué par plusieurs auteurs [KRVE05, Rez04, GPL98, NP13] qu’en dimension finie i.e. pour l’équation différentielle stochastique associée à un nombre fini de spins, les deux formulations menaient à une seule et même équation de Fokker-Planck sur la loi des solutions, à redéfinition près de la constante  $\epsilon$ . Cela justifie ainsi l’égalité en loi pour les solutions de ces deux versions de SLLG *a priori* différentes. Il existe une raison géométrique simple expliquant cette propriété. Considérons un mouvement brownien tridimensionnel  $W(t)$ . Partant d’un même point  $B_0$  sur la sphère, les processus  $B, B'$  solutions de  $dB = B \times dW$ ,  $dB' = -B' \times (B' \times dW)$  définissent tous deux un mouvement brownien sur  $S^2$ . Le processus  $B$  correspond, dans un voisinage de  $t = 0$ , au même mouvement que  $B'$ , mais tourné de  $90^\circ$  selon le vecteur  $B_0$ , voir l’illustration fig. 5. Il en est de même, à constante multiplicative près, du processus  $\tilde{B}$  solution de  $d\tilde{B} = \tilde{B} \times dW - \tilde{B} \times (\tilde{B} \times dW)$ . Nous donnons dans le chapitre 1 une justification formelle de cette équivalence dans le cas de la dimension infinie. Cela nous permet de nous ramener au cas d’une diffusion dépendant linéairement de l’inconnue  $u$ , et non de manière quadratique.

## (ii) Revue des différents résultats obtenus. Comparaison avec le cas déterministe.

### (ii-a) Formule de l’énergie, existence de solutions faibles

Afin de préserver la contrainte sur la magnitude locale, mais aussi dans l’esprit de la littérature physique, l’interprétation choisie pour l’équation (11) est celle de Stratonovich. Le bruit dans l’équation (11) étant sous forme *multiplicative*, la correction définie dans (16) n’est pas nulle et vaut  $F_\epsilon(u) = \epsilon^2 \sum_{e \in \mathbb{B}} (u \times \phi e) \times \phi e$ ,  $\mathbb{B}$  désignant une base orthonormée de  $L^2(\mathcal{O}; \mathbb{R}^3)$ . Le formalisme développé dans la section précédente nous permet de nous ramener à l’équation stochastique en dimension infinie :

$$du = (\Delta u + u|\nabla u|^2 + \gamma u \times \Delta u + F_\epsilon(u))dt + \epsilon u \times dW. \quad (\text{SLLG})$$

où  $t \geq 0 \mapsto u(t)$ , est un processus à valeurs dans un espace fonctionnel de la variable d’espace. Il s’agit de manière équivalente d’une équation aux dérivées partielles stochastique, la différence

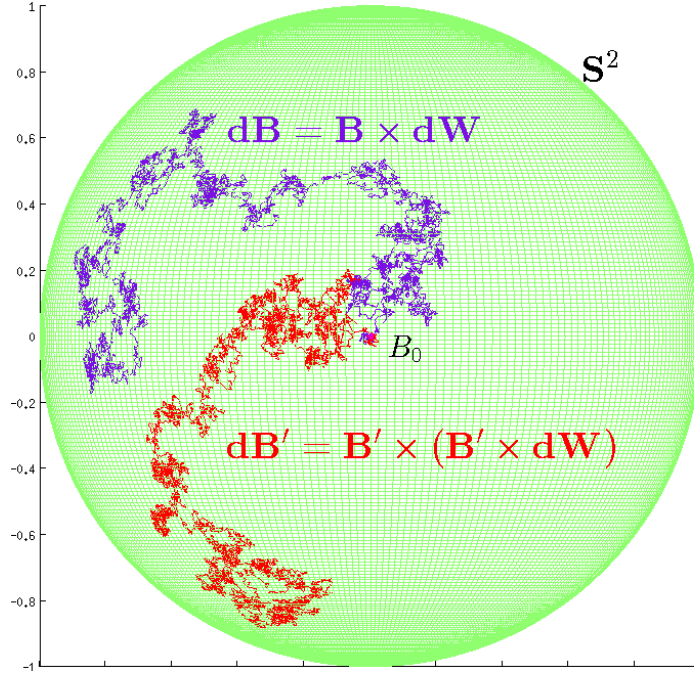


FIGURE 5 – Simulation numérique de deux mouvements browniens sur la sphère, partant d’un même point  $B_0$ . Pour un domaine constitué d’un spin  $M$ , les lois de  $\bar{M}$  et  $\bar{M}$  solutions de  $d\bar{M}/dt = \gamma\bar{M} \times (\bar{H} + \zeta) - \bar{M} \times (\bar{M} \times (\bar{H} + \zeta))$ , et  $d\bar{M}/dt = \gamma\bar{M} \times (\bar{H} + \zeta) - \bar{M} \times (\bar{M} \times \bar{H})$  sont identiques à condition de redéfinir  $\gamma$ .

résidant seulement dans le point de vue utilisé. Nous suivons en effet la présentation effectuée dans [DZ08], sachant que d’autres sont possibles, voir par exemple [Wal86].

L’analyse mathématique de (SLLG) a débuté très récemment. Elle est encore balbutiante aujourd’hui et peu de travaux existent sur le sujet, au contraire de l’équation déterministe (LLG) qui possède une littérature très fournie, voir par exemple les références [Vis85, SSB86, AS92, GH93, DW07, Mel05, CF97, CF01b, YSB98, GD08], ou plus spécifiquement concernant l’analyse numérique : [BP06, BKP05, BKP08, AJ06, Alo08, AKT12]. Dans le résultat suivant, le domaine  $\mathcal{O} \subseteq \mathbb{R}^3$  est soit borné, soit égal à l’espace entier.

**Théorème d’existence pour LLG** ([Vis85, AS92]). *Pour l’équation (SLLG) avec conditions au bord de type Neumann homogène, il existe une solution  $u : \mathbb{R}_+ \times \mathcal{O} \rightarrow \mathbb{S}^2$ , au sens des distributions, et telle que  $E(u(t)) \leq E(u_0)$  pour tout  $t \geq 0$ .*

La preuve de ce théorème repose essentiellement sur une méthode de compacité impliquant la formule (7). On a une estimation *a priori* dans l’espace des applications d’énergie finie pour tout temps, d’où découlent des propriétés de convergence pour des solutions approchées, soit par des approximations de type “schéma numérique”, soit par des solutions de l’équation projetée sur un espace fini-dimensionnel (méthode de Galerkin). Cette approche est celle utilisée par les auteurs de ce théorème. Elle permet d’obtenir des solutions “faibles”, vérifiant (LLG) au sens des distributions. Dans [AS92], on utilise de surcroît une méthode de pénalisation déjà mise en œuvre par Y. Chen pour l’existence de solutions faibles pour (HMF) [Che89], permettant de “relâcher” la contrainte  $|u|_{\mathbb{R}^3} \equiv 1$ .

Dans le cas de LLG stochastique (SLLG), une formule de dissipation existe et s'écrit formellement (voir le chapitre 2)

$$E(t) - E(0) + \int_0^t \mathcal{D}(\sigma) d\sigma = X(t) + C_0 t \quad (17)$$

où la constante  $C_0$  vaut  $+\infty$  si le processus de Wiener  $W$  est cylindrique, et cette fois la “tension”  $\mathcal{D}(\sigma)$  vaut  $\int_{\mathcal{O}} |u(\sigma, x) \times \Delta u(\sigma, x)|^2 dx$ , c'est à dire formellement “ $\int_{\mathcal{O}} |\partial_t u(\sigma, x) - u(\sigma, x) \times \dot{W}(\sigma, x)|^2 dx$ ”, au regard de l'équation (SLLG). Le processus  $X$  est une *martingale* : de moyenne nulle, il contient le terme de premier ordre lié aux fluctuations de l'énergie induites par le bruit. Sans hypothèse supplémentaire, il vaut également  $+\infty$  presque sûrement, quel que soit  $t$ .

Nous voyons au travers de la formule (17), que l'existence même d'une solution semble être compromise pour le cas du bruit blanc espace-temps, rendant impossible l'application de la formule de dissipation dans la quête d'estimation *a priori* pour  $u$ . En dimension trois d'espace, il semble donc illusoire, dans ce cas, de chercher à obtenir des solutions (même dans un sens “faible”), d'autant qu'il existe des exemples d'équations stochastiques “voisines” de (SLLG) qui sont mal posées en dimension d'espace  $> 1$ , y compris lorsque le terme de bruit est de type additif (ce qui serait un cas plus favorable), voir [HRW12]. Bien que ce ne soit pas le sujet de la thèse, mentionnons toutefois que la théorie des structures de régularité ne s'applique pas à (SLLG) en dimension trois, car l'hypothèse de *sous-criticalité locale* n'est pas satisfaite [Hai14, chap. 8]. Cette dernière n'est pas non plus vérifiée en dimension deux, où l'équation correspond au cas critique de la théorie, (SLLG) en 2D étant invariante par rescaling parabolique.

Le peu de littérature existant sur l'équation (SLLG) [BGJ13, BGJ12, BBNP13b, BBNP13a, GLT13, GP<sup>+</sup>09] traite le cas d'un bruit blanc en temps, mais régularisé en espace (à l'exception de [BBP13] où les auteurs discrétisent un bruit blanc à l'aide d'éléments finis). Cela se traduit, pour des fonctions test  $\varphi, \psi$  dans  $C^\infty(\mathcal{O})$ , par la propriété statistique  $\mathbb{E}[\langle \zeta(t), \varphi \rangle \langle \zeta(s), \psi \rangle] = \delta(t - s) \langle Q\varphi, \psi \rangle$ , où l'opérateur de covariance  $Q$  est supposé à trace finie dans l'espace  $L^2(\mathcal{O})$ . En décomposant ce dernier sous la forme  $Q = \phi\phi^*$  il s'agit de remplacer  $W$  par le processus de Wiener

$$W_\phi(t) = \sum_{k \in \mathbb{N}} \beta_k(t) \phi e_k, \quad (18)$$

où cette fois ci la somme converge presque sûrement dans  $L^2(\mathcal{O})$ . Dans ces conditions, en supposant que nous avons également remplacé  $F(u)$  par la correction adéquate  $F_\phi(u)$  (voir (16)), la formule (17) fait sens, dès lors que l'opérateur  $\phi : L^2(\mathcal{O}) \rightarrow H^1(\mathcal{O})$  est de classe *Hilbert-Schmidt*, et nous pouvons raisonnablement espérer appliquer une méthode de compacité tirée de la formule de dissipation, à la manière de celles développées par A. Visintin, F. Alouges et A. Soyeur dans la littérature déterministe (avec bien entendu une adaptation des arguments).

Le premier résultat notable pour l'équation (SLLG) a été obtenu par Z. Brzezniak, B. Goldys, et T. Jegaraj en 2012, et concerne l'existence de solutions martingales faibles. Il s'agit de l'article *Weak solutions of a stochastic Landau–Lifshitz–Gilbert equation : Applied Mathematics Research eXpress*, 2013(1) :1–33[BGJ13]. Le terme de “solution martingale faible” est utilisé en théorie des EDPS afin de distinguer la notion de solution faible au sens *probabiliste* du terme, de la notion de solution faible au sens des distributions. Nous pouvons définir ces solutions de la

manière suivante, pour un intervalle de temps fixé  $[0, T]$ . Définissons l'espace

$$\mathfrak{X} := L^2\left(0, T; H^1(\mathcal{O}; \mathbb{R}^3)\right) \cap \mathcal{C}\left([0, T]; L^2(\mathcal{O}; \mathbb{R}^3)\right) \\ \cap \left\{ v : [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^3, \int_{[0, T]} \mathcal{D}_v(t) dt < \infty \right\}, \quad (19)$$

où  $\mathcal{D}_v(t)$  désigne à nouveau la “tension”  $\int_{\mathcal{O}} |v(t, x) \times \Delta v(t, x)|^2 dx$ .

**Définition 1.** Étant donnée une condition initiale  $u_0$  d'énergie finie, une *solution martingale faible* de (SLLG), est une mesure de probabilité  $\mu$  sur l'espace de trajectoires  $\mathfrak{X}$  défini par (19) telle que si l'on note  $(t, \xi) \in [0, T] \times \mathfrak{X} \mapsto \xi(t)$  le processus canonique, alors

- 1)  $\mu$ -presque sûrement :  $\xi(0) = u_0$ , et  $\xi(t, x) \in \mathbb{S}^2$  pour presque tout  $t, x$  ;
- 2) le processus

$$M(t) = \xi(t) - \xi(0) - \int_0^t (\Delta \xi + \xi |\nabla \xi|^2 + \xi \times \Delta \xi + F_\phi(\xi)) dt$$

est une martingale de variation quadratique  $(\mathcal{Q}_t) f = - \int_0^t \phi^*(\xi(s) \times (\xi(s) \times \phi f)) ds$ , pour tout  $t \in [0, T]$  et  $f$  dans  $L^2(\mathcal{O}; \mathbb{R}^3)$ .

Notons qu'en raison de l'identité vectorielle  $|-v \times (v \times \Delta v) + v \times \Delta v|_{\mathbb{R}^3}^2 = |-v \times (v \times \Delta v)|_{\mathbb{R}^3}^2 + |v \times \Delta v|_{\mathbb{R}^3}^2 = 2|v \times \Delta v|_{\mathbb{R}^3}^2$ , valable dès que la contrainte sur la magnitude de  $v$  est satisfaite, la condition de sommabilité sur la tension  $\sigma \mapsto \mathcal{D}(\sigma)$  indique que l'intégrale dans 2) est convergente au sens de Bochner dans  $L_x^2$ . Dans [BGJ13], les auteurs montrent qu'il existe une solution martingale faible de (SLLG) en 3D, pour toute condition initiale d'énergie finie, et tout processus de Wiener  $W_\phi(t, x)$  donné par un produit  $h(x) \times W_1(t)$ , où  $W_1$  est un mouvement brownien réel et  $h$  est une fonction bornée (admettant des dérivées sommables). L'hypothèse que le processus de Wiener ait la forme d'un tel produit peut cependant être levée puisque l'article ultérieur [BBNP13a] montre la convergence, toujours vers une solution martingale faible, d'une discrétisation de (SLLG), où  $W_\phi$  est supposé avoir la forme plus générale (18), avec  $\phi : L^2(\mathcal{O}) \rightarrow H^2(\mathcal{O})$  Hilbert-Schmidt.

Le théorème d'existence montré par Z. Brzezniak, B. Goldys et T. Jegaraj, ne fournit pas d'information sur l'unicité, même locale, des solutions. Il n'y a cependant pas d'unicité à espérer en 3D, car un résultat obtenu par F. Alouges et A. Soyeur [AS92, Thm. 1.6] montre que l'on peut trouver une infinité de solutions faibles pour l'équation déterministe (LLG), appartenant à l'espace d'énergie qui est le pendant déterministe de  $\mathfrak{X}$ , voir (19). Il est donc peu probable, au meilleur de notre connaissance, d'espérer obtenir un critère d'unicité “naturel” en dimension trois, à moins que le bruit ait un effet inattendu sur l'unicité – voir par exemple à ce sujet les travaux de G. Da Prato, A. Debussche et C. Odasso [DD03a, DO06] sur l'équation de Navier-Stokes stochastique.

## (ii-b) Problèmes liés à l'existence et l'unicité locales de solutions pour (SLLG) (pour un domaine $\mathcal{O}$ de dimension quelconque).

Pour les équations aux dérivées partielles, la question de l'unicité soulève des problèmes liés à la régularité des solutions. Les différentes notions de solution (faible ou forte), lorsqu'elles ne

coïncident pas, peuvent entraîner un défaut d'unicité. L'exemple de l'équation stationnaire liée à (LLG) est instructif : nous avons vu que pour l'équation applications harmoniques à valeurs dans la sphère (6), il existe plusieurs notions de solution. En dehors des notions de solution classique et faible pour  $0 = \Delta u + u|\nabla u|^2$  (définies dans la section (i-a)), il existe également des applications dites *faiblement stationnaires*. Nous avons en fait la situation suivante :

$$\begin{aligned} & \{\text{applications harmoniques}\} \\ & \subseteq \{\text{applications faiblement stationnaires}\} \\ & \subseteq \{\text{applications faiblement harmoniques}\} . \end{aligned} \quad (20)$$

avec généralement des inclusions strictes, voir par exemple [Hé196] pour les définitions. Dans [AS92], les auteurs considèrent une application faiblement harmonique  $f$  qui n'est pas "faiblement stationnaire" et montrent qu'il existe une solution faible  $u$  de (LLG) partant à  $t = 0$  de  $f$ , qui n'est pas constante en temps. Or, pour  $t$  dans  $\mathbb{R}_+$ , la famille d'applications  $u^t$ , définie comme

$$u^t(s) := f \text{ , pour } s \leq t \text{ , et } u^t(s) := u(s - t) \text{ pour } s \geq t \text{ ,}$$

définit un continuum de solutions faibles pour (LLG), prouvant ainsi la non-unicité des solutions faibles. Cette preuve est en fait adaptée d'un argument utilisé pour montrer le même résultat sur le flot des applications harmoniques [Cor90].

Comme déjà mentionné plus haut, il existe un parallèle intéressant entre les deux équations équations (LLG) et (HMF), et on constate, dans la lignée de l'exemple précédent, et au regard de la littérature déterministe sur les deux sujets, que les résultats qui existent sur (HMF) peuvent souvent se généraliser à LLG. C'est une méthodologie que nous avons adoptée dans cette thèse, à savoir que nous avons en premier lieu étudié l'équation suramortie

$$du = (\Delta u + u|\nabla u|^2 + F_\phi(u))dt + u \times dW_\phi \text{ ,} \quad (\text{SHMF})$$

puis généralisé à (SLLG) les résultats obtenus pour (SHMF).

Outre les questions liées à l'unicité des solutions martingales faibles pour (SLLG), on peut se demander s'il existe des solutions *fortes locales* au sens probabiliste, c'est à dire qu'ayant fixé un processus de Wiener  $W_\phi$  sur un espace de probabilité  $(\Omega, \mathcal{F}, \mathbb{P})$ , on voudrait savoir s'il existe  $u$  satisfaisant dans  $L_x^2$  l'équation :

$$u(t) - u(0) = \int_0^t (\Delta u + u|\nabla u|^2 + \gamma u \times \Delta u + F_\phi(u))ds + \int_0^t u \times dW_\phi . \quad (21)$$

au moins jusqu'à un certain temps d'arrêt  $\tau > 0$ .

Dans le chapitre 1, nous montrons le résultat suivant pour le cas  $\gamma = 0$ .

**Théorème I : Résultat principal du premier chapitre, existence et unicité locales pour SHMF en dimension  $n \geq 2$ .** — *Supposons que la condition initiale  $u_0$  appartient à l'espace de Sobolev  $W^{1,p}(\mathcal{O})$  pour  $p > n$ , et qu'elle satisfait la contrainte (14). Il existe alors, pour tout  $\phi : L^2 \rightarrow W^{1,p}$  qui soit  $\gamma$ -radonifiant, une unique solution locale pour (SHMF).*

La preuve de ce résultat repose sur une méthode de point fixe avec troncature de la nonlinéarité  $u|\nabla u|^2$ , déjà développée par exemple par A. De Bouard et A. Debussche dans [DD99, DD03b], pour l'équation de Schrödinger stochastique. L'idée de départ est de définir une application  $v \mapsto u(v)$ , donnant la solution de

$$du - \Delta u dt = (v|\nabla v|^2 + F_\phi(v))dt + v \times dW_\phi \text{ , } u(0) = u_0 \text{ ,}$$

lorsque  $v$  est une variable aléatoire supportée dans l'espace  $X := \mathcal{C}([0, T]; W^{1,p})$ . À l'aide des effets régularisants du semigroupe de la chaleur  $t \rightarrow e^{t\Delta}$ , on peut estimer les quantités du type  $\mathbb{E}\|v\|_X$ ,  $\mathbb{E}\|u(v_1) - u(v_2)\|_X$ , permettant ainsi de trouver un point fixe pour l'application  $v \mapsto u(v)$ , ce qui donne le résultat voulu. En raison du terme stochastique dans l'équation, on ne peut pas traiter directement l'équation d'origine : il faut d'abord se ramener à une équation modifiée, dans laquelle la nonlinéarité est tronquée de sorte que pour  $R > 0$  donné, la solution  $u = u_R$  demeure dans une boule de  $X$ . On obtient ensuite des solutions de l'équation d'origine par un argument de localisation.

Malheureusement, cette méthode ne semble pas fonctionner dans le cas de (SLLG). Nous en expliquons ici brièvement les raisons. Dans le cas déterministe, l'existence de solutions régulières (au moins continues par rapport à  $x$ ) est une conséquence d'un théorème dû essentiellement à une série d'articles par H. Amann, voir [Ama84, Ama85, Ama86]. En substance, le résultat d'H. Amann établit que toute équation de la forme

$$\partial_t u + A(t, x, u, \nabla u)u = f(u, \nabla u) \quad (22)$$

où  $\{A(t), t \geq 0\}$  est une famille d'opérateurs d'ordre deux, *fortement elliptiques*, et dépendant de manière Höldérienne de la variable  $t$ , admet des solutions continues dès lors que les données  $f$  et  $u_0$  sont suffisamment régulières. Par ailleurs, la solution est donnée par la formule de "variation des constantes"

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(u(s), \nabla u(s))ds, \quad (23)$$

où la famille d'opérateurs  $U(t, s)$ ,  $s \leq t$  est l'analogie du semigroupe  $e^{(t-s)\Delta}$  dans le cas d'une famille d'opérateurs dépendant du temps. Tout comme le semigroupe de la chaleur, la famille  $U(t, s)$  a la propriété "agréable" de régulariser le second membre  $f$  et ce dans les mêmes échelles (typiquement un second membre dans  $L^2([0, T] \times \mathcal{O})$  entraîne que la solution  $u$  est dans  $L^2([0, T]; H^2(\mathcal{O}))$ ).

Un calcul immédiat montre que la condition d'ellipticité forte est satisfaite par la famille d'opérateurs  $A_v(t)u := -\Delta u - v \times \Delta u$ , pour  $v$  fixé dans  $X$ , voir la définition (1.21) donnée au chapitre 1. La preuve du résultat d'H. Amann se fait alors en deux temps. On travaille d'abord sur le problème linéaire associé à (22), en fixant une famille d'opérateurs non autonome  $A_v(t, x) := A(t, x, v, \nabla v)$  et en définissant les opérateurs d'évolution  $U_v(t, s)$ . On établit ensuite la formule de variation des constantes pour le problème avec second membre  $f(u, \nabla u)$ , utilisant l'analogie avec l'équation de la chaleur. Afin de faire fonctionner l'argument de point fixe sur l'application  $u \mapsto u(v)$ , on trouve dans un second temps des estimations sur la perturbation de la famille d'opérateurs du type " $\|(U_{v_1}(t, s) - U_{v_2}(t, s))f\|_X \leq C\|v_1 - v_2\|_X\|f\|_X$ ".

La principale obstruction dans le cas stochastique, lorsque  $\gamma \neq 0$ , est que l'intégrand dans " $\int_0^t U(t, s)v(s) \times dW(s)$ " n'est pas adapté. Il faut avoir recours à l'intégrale forward définie au moyen du calcul de Malliavin. Pour ce type d'intégrale, il n'existe pas à notre connaissance, d'estimations concluantes sur les quantités du type  $\mathbb{E}\|U_{v_1(\omega)}(t, s) - U_{v_2(\omega)}(t, s)\|_X$ , ce qui est un obstacle à la méthode utilisée ci-dessus.

Signalons qu'il existe d'autres preuves de la solvabilité locale de (HMF) [Ham75, ES64, p. 122, resp. p. 134] ainsi que de (LLG) [GD08, Thm. 4.2.5 p. 154], mais ces méthodes ne semblent pas s'appliquer, pour des raisons de régularité, au cas d'un second membre augmenté d'un bruit blanc en temps.

### (ii-c) Problèmes spécifiques liés à SLLG en 2D, présentation des chapitres 2 et 4.

Les équations (SLLG) et (SHMF) sont connues pour présenter des caractéristiques singulières en dimension deux d'espace. On peut citer, à ce sujet, la présentation faite dans [Hé196] pour le cas de (6)-(HMF).

Les propriétés algébriques de l'équation LLG (déterministe ou non) nous assurent que la formule (17) est vérifiée (ou son équivalent (7) pour l'équation non bruitée), par toute solution, dès lors que celle-ci ne présente pas de singularité. En plus de donner une estimation sur l'énergie d'échange  $E(u)$ , l'estimation (17) nous donne une borne sur l'intégrale de la tension orthogonale dans  $\mathbb{R}^3$  du laplacien de  $u$  sur  $(\text{Vect } u)^\perp$ . D'autre part, multipliant scalairement l'équation (LLG) par  $u$ , nous obtenons formellement que  $0 = \Delta u \cdot u + |\nabla u|^2$ . Écrivant ensuite que  $\Delta u = P_u^\perp \Delta u + u \cdot \Delta u$ , nous voyons qu'une estimation *a priori* pour les solutions de (SLLG) dans l'espace  $L^2(\mathcal{O}; H^2(\mathcal{O}))$  peut être obtenue à condition d'avoir une borne sur la quantité

$$\int_0^T \int_{\mathcal{O}} |\nabla u|^4 dx dt . \quad (24)$$

En 2D, si  $\mathcal{O}$  désigne une surface compacte sans bord, M. Struwe [Str85], a montré l'inégalité d'interpolation suivante pour toute application  $u$  :

$$\int_0^T \int_{\mathcal{O}} |\nabla u|^4 dy dt \leq c_0 \sup_{(t,x) \in [0,T] \times \mathcal{O}} \left( \int_{L^2(B_r(x))} |\nabla u(t,y)|^2 dy \right) \cdot \int_0^T \left( \int_{\mathcal{O}} |\nabla^2 u|^2 dx + r^{-2} \int_{\mathcal{O}} |\nabla u|^2 dx \right) dt , \quad (25)$$

où  $B_r(x)$  désigne la boule  $\{y \in \mathcal{O} : |x - y| \leq r\}$ . La propriété essentielle observée par M. Struwe, et découlant de (25), est qu'en 2D toute solution qui vérifie l'existence de  $r > 0$ , tel que la quantité

$$\epsilon(r, t) := \sup_{(s,x) \in [0,t] \times \mathcal{O}} \int_{B_r(x)} |\nabla u(s,y)|^2 dy , \quad (26)$$

soit plus petite qu'une donnée géométrique  $\epsilon_1$ , est une solution *classique* pour (HMF). On a en fait le théorème suivant.

**Théorème de Struwe ([Str85]).** *Pour toute donnée initiale d'énergie finie  $u_0$ , il existe une unique suite finie  $T_0 < T_1 < \dots < T_K$  avec  $T_i \in (0, \infty]$  pour tout  $1 \leq i \leq K$ , ainsi qu'une unique solution faible  $u : \mathbb{R}_+ \times \mathcal{O} \rightarrow \mathbb{S}^2$  de (HMF), appartenant pour tout  $T > 0$  à l'espace d'énergie  $\mathfrak{X} = \mathfrak{X}_T$  défini par (19), vérifiant*

1  $u|_{[T_{i-1}, T]}$  appartient à  $L^2([T_{i-1}, T]; H^2(\mathcal{O}))$  pour tout  $1 \leq i \leq K$  et  $T < T_i$  ;

2  $u$  est solution classique de (HMF) sur  $[0, \infty) \times \mathcal{O}$ , à l'exception d'un nombre fini de points singuliers  $(T_i, x_{i,l})$ ,  $(i, l) \in \llbracket 1, K \rrbracket \times \llbracket 0, L_i \rrbracket$  où  $u$  vérifie la propriété de bubbling  $\limsup_{t \uparrow T_i, r \rightarrow 0} \int_{B_r(x_i)} |\nabla u(t,y)|^2 dy \geq \epsilon_1$ , pour tout  $l$  dans  $\llbracket 0, L_i \rrbracket$ .

Le résultat a ensuite été étendu à (LLG) en 2D [GH93], suivant essentiellement le même argument. Cela a notamment pour conséquence le fait que toute solution  $u$  d'énergie finie est une solution globale en temps, vérifiant l'équation (HMF) au sens classique, à moins qu'en temps fini la condition  $\epsilon(r, t) \leq \epsilon_1$  ne soit violée pour toute valeur de  $r > 0$ . Par la suite, K-C. Chang a montré un résultat similaire [KC89], pour (HMF) en 2D avec  $\partial\mathcal{O} \neq \emptyset$ .

Pour l'équation stochastique (SLLG), nous avons montré dans le chapitre 2 le résultat suivant. Ici les solutions sont comprises dans un sens probabiliste "fort", i.e. nous avons construit des solutions sur un espace de probabilité et pour un processus de Wiener  $W_\phi$  prescrits  $(\Omega, \mathcal{F}, \mathbb{P}; W_\phi)$ . L'intervalle de temps  $[0, T]$  est fixé, et nous traitons le cas du tore bidimensionnel  $\mathcal{O} = \mathbb{T}^2$ .

**Théorème II : Résultat principal du second chapitre, existence et unicité en 2D de solutions localement régulières, globales en temps.** — Pour toute donnée initiale  $u_0$  d'énergie finie, et tout  $\phi : L^2(\mathcal{O}) \rightarrow H^1(\mathcal{O})$  Hilbert-Schmidt, il existe une unique suite de temps d'arrêts  $\vartheta^1, \dots, \vartheta^J$  ainsi qu'un unique processus  $u : \Omega \rightarrow \mathfrak{X}$  progressivement mesurable (où  $\mathfrak{X}$  est encore défini comme en (19)), tels que

i) presque sûrement  $\vartheta^J = T$ ;  $u$  est solution de (SLLG) sur chaque intervalle de temps  $[\vartheta^j, \vartheta^{j+1})$ , pour  $0 \leq j \leq J - 1$ ;

ii) si pour  $0 \leq j \leq J - 1$ , la suite de temps d'arrêt  $\vartheta^{j,k}$  tend presque sûrement vers  $\vartheta^{j+1}$  par valeurs croissantes, alors la solution  $u$  appartient localement pour tout  $k \geq 0$  à l'espace  $\mathfrak{X} \cap L^2(\vartheta^j, \vartheta^{j,k}; H^2(\mathcal{O}))$ , presque sûrement;

iii) pour tout  $j$ , le temps  $\vartheta^{j+1}$  est caractérisé par le fait que  $\epsilon(r, \vartheta^{j+1}) > \epsilon_1$  quelle que soit la valeur de  $r > 0$ .

Pour démontrer ce résultat, nous utilisons de manière cruciale l'identité de l'énergie (7), couplée avec l'inégalité d'interpolation (25). Nous supposons cependant acquise l'existence de solutions fortes locales pour (SLLG), presque sûrement continues à valeurs dans l'espace  $H^2(\mathbb{T}^2)$ , ce qui n'a malheureusement pas pu être démontré pour le moment, en raison des difficultés soulevées par une généralisation, au cas  $\gamma \neq 0$ , du théorème d'existence locale du chapitre 1. Toutefois, si l'existence de telles solutions régulières venait à être contredite, le résultat demeure dans le cas du flot stochastique des applications harmoniques en 2D.

La propriété (iii) caractérise les temps d'explosions. À l'aide d'un calcul simple, on peut voir que cette dernière impose à la solution de perdre un "quantum" d'énergie valant un multiple entier de  $\epsilon_1$ . Ce type de singularité est dénommé, dans la littérature déterministe, phénomène de "bubbling", faisant allusion au fait qu'un ou plusieurs point(s)  $(\vartheta^j, x)$  concentre une quantité d'énergie non nulle. De manière peu rigoureuse, on peut dire le point  $(\vartheta^j, x)$  a pour image par l'application  $u$  l'ensemble de la sphère  $\mathbb{S}^2$ , voir l'illustration numérique fig. 6.



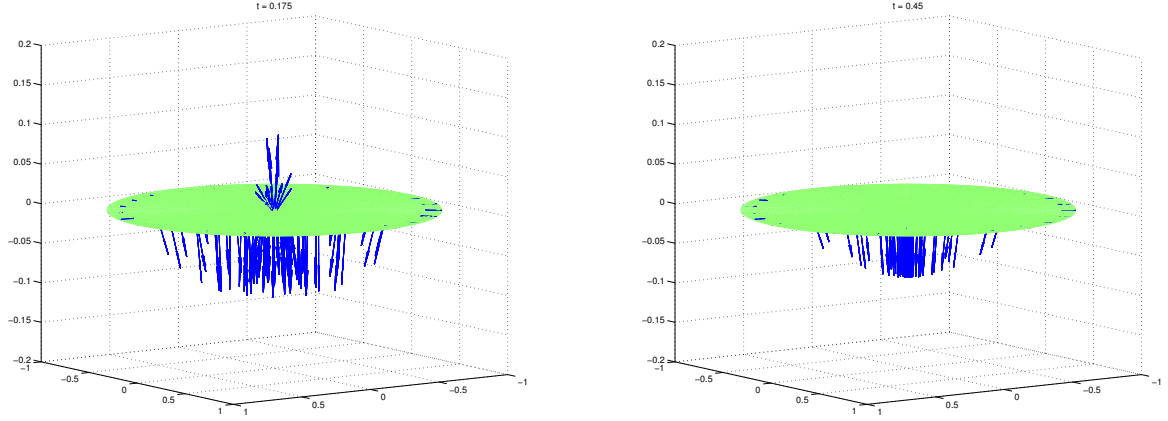


FIGURE 6 – En 2D : Exemple de bubbling en  $x_0 = (0, 0)$ , dans le cas où  $\mathcal{O} = \mathbb{D}$ , le disque unité ; le champ  $u : \mathbb{R}_+ \times \mathbb{D} \rightarrow \mathbb{S}^2$  est représenté par les flèches bleues. On a prolongé  $u$  après le bubbling en lui attribuant la valeur  $(0, 0, -1)$  au centre du disque. À gauche :  $u$  avant le temps d’explosion  $\vartheta$  ; à droite : prolongation de la solution après  $\vartheta$  (solution de Struwe).

Notre théorème ne dit cependant rien sur l’existence effective de solutions explosives en temps fini, i.e. pour lesquelles il existe un temps d’arrêt  $\vartheta < \infty$  tel que (iii) soit observé. Dans le cas déterministe, lorsque  $\dim \mathcal{O} = n \geq 3$  et que la variété d’arrivée est la sphère  $\mathbb{S}^n$ , l’explosion en temps fini du flot des applications harmoniques est un fait connu, démontré en 1989 par J.M. Coron et J.M. Ghidaglia [CG89]. La méthode utilisée par ces auteurs est d’injecter des solutions à symétrie sphérique dans l’équation (HMF), s’écrivant sous la forme

$$u(t, \vec{x}) = \left( \frac{\vec{x}}{|\vec{x}|_{\mathbb{R}^n}} \sin h(t, |\vec{x}|); \cos h(t, |\vec{x}|) \right) , \quad \vec{x} \in \mathbb{R}^n , \quad (27)$$

puis d’écrire l’équation correspondante sur la fonction scalaire  $h = h(t, r)$ . Celle-ci s’écrit :

$$\partial_t h = \partial_{rr} h + \frac{n-1}{r} \partial_r h + \frac{n-1}{2r^2} \sin 2h , \quad \text{pour } t, r > 0 . \quad (28)$$

L’intérêt principal d’une telle approche est de pouvoir énoncer un théorème de comparaison pour l’équation parabolique (28). On construit ensuite des sous solutions dont la dérivée explose en temps fini  $t^*$  au point  $(t^*, 0)$  (la symétrie impose dans ce cas que l’explosion ne peut avoir lieu qu’au point  $x = 0_{\mathbb{R}^n}$ ), entraînant par comparaison l’explosion de  $h$  également. La preuve donnée repose cependant sur l’hypothèse que l’application  $u$  soit à valeurs dans  $\mathbb{S}^n$  pour  $n \geq 3$ .

On peut, en dimension deux d’espace, considérer des solutions sous la forme (27) dites “équivariantes”, qui soient par conséquent à valeurs dans la sphère unité  $\mathbb{S}^2$ . Cet aspect propre à la dimension deux, a permis de montrer l’explosion de (HMF), plus tard en 1992, dans le papier [CDY<sup>+</sup>92]. Les auteurs de cette note considèrent également des solutions sous la forme (27), i.e. des solutions “équivariantes”, et utilisent également un théorème de comparaison pour l’équation (28), montrant qu’il existe des conditions initiales  $h_0$  qui génèrent une singularité en temps fini. Dans un article antérieur [CD91], il avait été remarqué que toute solution  $h$  de (28) en 2D est nécessairement globale si la condition “ $\sup_{r \in [0,1]} |h_0(r)| < \pi$ ” est remplie. La condition initiale, dans le cas déterministe, détermine si oui ou non la solution  $h$  explose en temps fini.

Dans le cadre du flot stochastique des applications harmoniques en 2D, nous avons montré dans le chapitre 4 le résultat suivant. Ici  $W(t)$  désigne un processus de Wiener régulier en espace, tel que presque sûrement, on ait  $W(t, x) \in \mathbb{R}$ , et la valeur de  $W$  en  $(t, x)$  ne dépend que de  $(t, |x|)$ . Nous considérons l'équation sur le disque unité  $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ .

**Théorème III : Résultat principal du chapitre quatre, explosion en temps fini pour SHMF en 2D.** — *Considérons une condition initiale  $u_0$  sous la forme équivariante (27), avec  $h_0(1) = h_0(1) = 0$ , et telle que  $h_0$  soit de classe  $\mathcal{C}^1(0, 1)$ . Notons pour tout  $t \geq 0$*

$$u^\perp(t, x) = \left( \frac{\vec{x}}{|x|_{\mathbb{R}^2}} \cos h(t, |x|); -\sin h(t, |x|) \right), \quad (29)$$

et supposons que  $u$  soit solution de l'équation stochastique

$$du = (\Delta u + u|\nabla u|^2 + F_\phi(u)) + u^\perp dW(t, |x|), \quad u(0) = u_0. \quad (30)$$

avec données constantes au bord.

Quelle que soit  $u_0$  comme ci-dessus, et quel que soit  $t^* > 0$ , la probabilité pour que  $u$  explose avant  $t^*$  est positive, c'est à dire que  $\mathbb{P}(\sup_{t \leq t^*} |\nabla u|_{L^\infty} = \infty) > 0$ .

Ce résultat contraste avec le cas déterministe, où seules certaines classes de données initiales peuvent conduire au bubbling. Pour des raisons qui tiennent à la nécessité de préserver l'équivariance des solutions, nous ne pouvons pas considérer le flot stochastique des applications harmoniques dans sa forme générale (SHMF). Néanmoins, l'explosion en temps fini pour (SHMF) est observée numériquement au chapitre 6. On observera également au chapitre 6 que ce phénomène de bubbling est stable à  $u_0$  fixé pour des “petites” valeurs du paramètre gyromagnétique  $\gamma$ . Si l'on choisit  $u_0$  tel que le flot de (HMF) provoque un bubbling sur  $u$ , alors le bubbling avec  $\mathbb{P} > 0$  se produit non seulement pour (SHMF), mais également pour (SLLG), dans la mesure où le paramètre de précession  $\gamma$  est plus faible qu'une certaine constante  $\gamma_*$ .

Quoi qu'il en soit, un résultat théorique d'explosion pour (SLLG) en 2D paraît hors de portée, le problème étant toujours ouvert lorsqu'il n'y a pas de bruit.

## (ii-d) Explosion en temps fini et unicité des solutions en 2D.

Dans l'esprit de ce qui a été développé plus haut pour la dimension trois, voir (20), la “souplesse” de la notion de solution faible permet de définir des exemples de non-unicité en 2D pour (1.17). Cette observation a été faite indépendamment par Bertsch, Dal Passo, Van Der Hout [BDvdH02], et P. Topping dans [Top02]. Après bubbling, une solution  $u$  de (HMF) peut être prolongée continûment en temps dans l'espace  $L^2(\mathcal{O})$  (voir fig. 6) il suffit pour voir cela d'utiliser la construction de M. Struwe. La solution  $\tilde{u}$  ainsi prolongée repart de la limite faible de  $u(t)$  dans  $H^1(\mathcal{O})$  lorsque  $t \rightarrow t^*$ , avec perte d'énergie quantifiée  $k \in_1, k \geq 1$ .

Lorsque le domaine spatial est le disque  $\mathbb{D}$ , les auteurs du premier papier ont montré le résultat suivant.

**Théorème de non unicité des solutions faibles de HMF en 2D ([BDvdH02]).** *Il existe une donnée initiale  $u_0$  d'énergie finie et valant  $\vec{k} := (0, 0, 1)$  au bord du disque  $\mathbb{D}$ , telle que (HMF) associée à  $u|_{\partial D} = \vec{k}$ ,  $u(0) = u_0$ , admet une infinité de solutions  $(u^\tau)_{\tau > \bar{\tau}}$ .*

Dans ce théorème, toutes les solutions considérées sont équivariantes. La notation

$$u \leftrightarrow h \text{ ,}$$

signifie ici que  $u$  et  $h$  sont liées via la relation (27), (dans les coordonnées sphériques  $h : [0, 1] \rightarrow \mathbb{R}$  est en fait la colatitude de  $u$ ).

Le point de départ de la preuve est de considérer une solution explosive en temps fini  $t_* > 0$   $u : [0, t_*) \times [0, 1] \rightarrow \mathbb{S}^2$ , classique sur  $[0, t_*)$ , donnée par  $u \leftrightarrow h$ . Remarquons que la solution faible obtenue par M. Struwe, que nous appellerons par la suite “solution de Struwe”, conserve son caractère équivariant avant et après l’explosion en  $t = t_*$ , au sens où si l’on note  $\tilde{u}$  la solution globale en temps donnée par le théorème de M. Struwe, alors on a  $\tilde{u} \leftrightarrow \tilde{h}$ , pour une certaine  $\tilde{h} : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ . Il suffit pour voir cela de prolonger  $h|_{[0, t_*)}$  par la valeur  $\pi$  en l’origine (les autres points du disque ne présentent pas de singularité), et de voir que la solution redémarrant de cette donnée est bien dans la “classe de Struwe”, d’où le fait qu’elle soit égale à  $\tilde{u}$ .

Si l’on définit le premier temps  $> t^*$  tel que la fonction  $r \in [0, 1] \mapsto \tilde{h}(\tau, r)$  vérifie l’existence d’une constante  $C > 0$  telle que

$$h(t, r) \leq \pi - Cr \text{ , pour tous } t > \bar{\tau} \text{ , et } r \text{ dans } [0, 1] \text{ .} \quad (31)$$

Alors, pour tout  $\tau > 0$ , via une approximation de  $\tilde{h}(\tau, \cdot)$  par une suite de données  $h_n$  astucieusement choisies, les auteurs montrent que l’on peut définir une application  $\bar{h}^\tau$  telle que  $u^\tau \leftrightarrow h^\tau$  a les propriétés suivantes :

- (a)  $u^\tau$  est solution faible de (HMF) et coïncide avec  $\tilde{u}$  sur  $[0, \tau)$  ;
- (b) l’énergie de  $u^\tau$  présente une discontinuité à droite au temps  $t = \tau$ , sous la forme du gain instantané du quantum  $\epsilon_1$  défini par M. Struwe, et précédemment perdu au temps  $t_*$  : nous avons la relation

$$\lim_{t \downarrow \tau} E(u^\tau(t)) = E(\tilde{u}(\tau)) + \epsilon_1 \text{ .}$$

Comme pour le cas 3D et le résultat obtenu par F. Alouges et A. Soyeur pour (LLG), les auteurs obtiennent un continuum de solutions  $(u^\tau)_{\tau > \bar{\tau}}$  en 2D, solutions faibles du même problème. La propriété (b) montrent que ces applications présentent une seconde singularité, dénommée dans la littérature “backward bubbling”.

Un célèbre résultat de A. Freire [Fre95] montre que si l’on impose à une solution faible  $u$  de (HMF) d’avoir une énergie *décroissante* au cours du temps, alors  $u$  est nécessairement égale à la solution de Struwe. En d’autres termes ce résultat énonce que la condition

$$E(u(t)) \leq E(u(s)) \text{ , pour tout } 0 \leq s \leq t \leq T \text{ ,} \quad (32)$$

est un critère d’unicité pour les solutions faibles de (HMF). Ce résultat a été adapté pour (LLG) dans la référence [YSB98] où les auteurs fournissent cependant une preuve incorrecte, voir à ce sujet l’article [Har04] qui explique comment adapter correctement le résultat pour (LLG). La décroissance de l’énergie  $E$  le long du flot de (HMF) proscrit en particulier le phénomène (b), induit par “backward bubbling”. P. Topping a construit des solutions faibles de (HMF) en 2D pour lesquelles on observe au en  $t^*$  un phénomène de bubbling suivi *instantanément* d’un backward bubbling. Une telle solution  $v$  ne peut être égale à la solution de Struwe  $u$ , car cette dernière doit perdre de l’énergie pour être prolongée, voir la figure 7. Cela paraît contradictoire

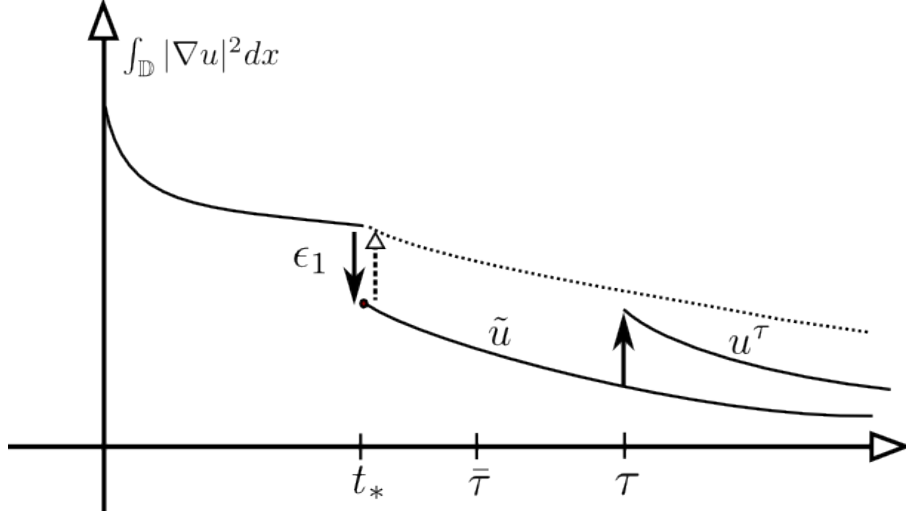


FIGURE 7 – Non unicité des solutions faibles en 2D. Tracé de l'énergie le long du temps. En pointillés : la solution de P. Topping.

avec le résultat de A. Freire, en effet l'énergie  $E_v$  de la solution construite par P. Topping est décroissante pour *presque* tout  $t$ , mais ce n'est qu'une apparence : la subtilité réside dans le fait que la relation de décroissance (32) doit être vérifiée pour *tout*  $t \geq s$ ,  $E(u(t))$  signifiant l'énergie de l'application donnée par la *trace* de  $u$  sur  $\{t\} \times \mathbb{D}$ .

Dans le cas de l'équation bruitée (SLLG), nous avons cherché à savoir, dans le chapitre 3, s'il existait pour les solutions faibles en 2D, un critère d'unicité trajectorielle portant sur la fonctionnelle d'énergie. Du fait des fortes fluctuations de la martingale  $X(t)$  dans (17), la fonctionnelle  $E$  ne peut toutefois pas décroître en temps presque sûrement. Cependant, une étude approfondie de la preuve donnée par A. Freire dans le cas déterministe nous a permis d'obtenir le résultat suivant. Ici, nous notons toujours par  $C_0$  la constante liée à la corrélation spatiale du bruit dans l'égalité (17). On rappelle que cette constante est finie dès que  $\phi : L^2(\mathcal{O}) \rightarrow H^1(\mathcal{O})$  est Hilbert-Schmidt.

**Théorème IV : Résultat principal du chapitre trois, critère d'unicité trajectorielle pour les solutions de SLLG en 2D.** — Soient deux solutions martingales faibles  $u_j$   $j = 1, 2$ , de (SLLG), définies sur un même espace de probabilité, un même intervalle de temps  $[0, T]$ , et telles que  $\phi : L^2(\mathcal{O}) \rightarrow H^1(\mathcal{O})$  soit Hilbert-Schmidt. On définit pour  $j = 1, 2$  la fonctionnelle d'énergie renormalisée :

$$\mathcal{G}_j(t) = \int_{\mathcal{O}} |\nabla u_j(t, x)|^2 dx - C_0 t . \quad (33)$$

Alors, l'identité de sur-martingale

$$\mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t)] \leq \mathcal{G}(s) \text{ presque sûrement pour tout } 0 \leq s \leq t \leq T \text{ et } j = 1, 2 , \quad (34)$$

entraîne que  $u_1 = u_2$ .

Le théorème se montre à l'aide de l'inégalité de Wentz. Cette inégalité entraîne qu'en 2D, le terme non linéaire  $u|\nabla u|^2$  dans l'équation (SLLG) est légèrement plus régulier que ce à quoi l'on pourrait s'attendre. En effet, une analyse rapide donne que pour  $u$  dans l'espace d'énergie  $\mathfrak{X}$

et vérifiant la contrainte (14), la quantité  $u|\nabla u|^2$  est un élément de l'espace de Lebesgue  $L^1(\mathcal{O})$ . En dimension deux  $L^1(\mathcal{O})$  n'est pas inclus dans  $H^{-1}(\mathcal{O})$ . Il s'agit en effet du cas "limite dans l'injection de Sobolev" en 2D : l'assertion " $H_x^1 \hookrightarrow L_x^\infty$ " est fautive, bien que  $H_x^1 \hookrightarrow L_x^p$  soit vrai pour tout  $1 \leq p < \infty$ .

Afin d'obtenir un critère d'unicité, nous voudrions voir à quelle condition la propriété " $u(t)|\nabla u(t)|^2 \in L^1(\mathcal{O})$ " peut être améliorée pour

$$"u(t)|\nabla u(t)|^2 \in H^{-1}(\mathcal{O})" \quad (35)$$

localement en temps. Cette volonté que  $u|\nabla u|^2$  soit un élément du dual de  $H_0^1$  s'explique par le fait que l'équation de la chaleur  $\partial_t f - \Delta f = g$ , avec une source  $g$  appartenant à cet espace, localement en temps, est *bien posée*. Pour l'équation stationnaire, le "coup de pouce" permettant d'obtenir un supplément inattendu de régularité a été découvert par F. Hélein, et sa mise en œuvre a permis de prouver la régularité des applications faiblement harmoniques en dimension deux dans un célèbre article [Hé190].

Du fait de sa structure algébrique particulière, le terme  $u|\nabla u|^2$  peut cependant s'écrire comme un produit  $a \cdot \nabla u$ , où  $a$  est un terme en lien avec l'énergie  $E$ . Si l'application  $u$  est faiblement harmonique, alors  $a$  est un terme de *divergence nulle*. Pour ce genre de produit  $a \cdot \nabla u$ , l'inégalité de Wente, qui est en quelque sorte le pendant du "Lemme div – rot" de F. Murat et J.L. Tartar [Mur81, Tar83] pour la dimension deux d'espace, nous dit qu'en réalité  $u|\nabla u|^2 \in H_x^{-1}$ .

Dans la démonstration du résultat de A. Freire, le point crucial est que l'hypothèse (32) engendre de la régularité à droite en temps pour l'application  $t \geq 0 \mapsto a(t)$ . Si l'énergie renormalisée  $\mathcal{G}$  est une sur-martingale, nous avons montré dans le chapitre 3 que cette propriété de régularité à droite est préservée. Cela permet de montrer que pour des temps courts, localement en temps, le processus  $a$  peut être considéré comme un terme de divergence nulle plus un reste tendant vers 0.

## (ii-e) Discrétisation de SLLG.

Déjà au niveau déterministe, la discrétisation de (LLG) pose le problème crucial du *respect de la contrainte*

$$|u(t, x)| = 1 \quad , \quad \text{pour tout } (t, x) \in \mathbb{R}_+ \times \mathcal{O} \quad .$$

Un "bon" schéma numérique pour (LLG) doit également être compatible avec la structure de Lyapunov (i.e. dissipative) du système. Dans la littérature déterministe, on distingue entre autres deux approches (précisons que ce n'est pas exhaustif : pour un aperçu global du sujet on pourra consulter [KP06]). Ici on considère une semi-discrétisation en temps

$$u^0 = u_0, \quad u^1 \sim u(\Delta t), \dots, \quad u^n \sim u(n\Delta t), \quad \dots$$

Une possibilité est d'utiliser un algorithme de projection.

La méthode de projection est un algorithme en deux temps dans lequel on cherche d'abord, pour chaque pas de temps  $n$ , une inconnue  $u_*^n = u^n + v^n$ , sans tenir compte de la contrainte géométrique. La valeur intermédiaire  $u_*^n$  est ensuite projetée ponctuellement sur la sphère, définissant ainsi  $u^{n+1}$ . On définit pour chaque  $u : \mathcal{O} \rightarrow \mathbb{R}^3$ , "l'espace tangent"

$$K(u) := \{v \in H^1(\mathcal{O}; \mathbb{R}^3), u(x) \cdot v(x) = 0 \text{ pour tout } x \text{ dans } \mathcal{O}\} \quad (36)$$

dans lequel on énonce une formulation variationnelle à chaque pas de temps, afin de trouver l'incrément intermédiaire  $v^n$ . L'avantage décisif d'une telle formulation réside dans le fait que la non linéarité

$$u^n |\nabla u^n|^2 ,$$

étant proportionnelle à  $u^n$ , disparaît lorsque l'on prend son produit avec des fonctions test dans  $K(u^n)$ . Cela permet de se ramener, pour chaque pas de temps, à l'inversion d'un système *linéaire*, ce qui est un gain considérable de temps de calcul, par rapport aux autres algorithmes existant dans la littérature. Il est cependant nécessaire, au préalable, de comprendre les éventuels effets induit par l'étape de renormalisation. La forme de **(LLG)** utilisée dans ce premier algorithme est celle de Gilbert  $\partial_t u - u \times \partial_t u = \Delta u + u |\nabla u|^2$ . Pour un paramètre  $\theta \in (1/2, 1]$  donné, cela donne l'algorithme suivant, pour lequel la convergence a été prouvée par F. Alouges et P. Jaisson dans [AJ06] : pour  $n = 0, 1, \dots$ ,

$$\left[ \begin{array}{l} \text{Trouver } v^n \in K(u^n) \text{ tel que pour tout } \varphi \in K(u^n) , \\ \int_{\mathcal{O}} v^n \cdot \varphi - \int_{\mathcal{O}} u^n \times v^n \cdot \varphi = -2 \int_{\mathcal{O}} \nabla(u^n + \theta \Delta t v^n) \cdot \nabla \varphi . \\ \text{Définir pour tout } x \text{ dans } \mathcal{O} : u^{n+1}(x) = \frac{u^n(x) + v^n(x)}{|u^n(x) + v^n(x)|} , \text{ et itérer.} \end{array} \right.$$

L'autre approche que nous détaillons ici est celle du "point milieu", proposée par A. Prohl et S. Bartels dans [BP06]. Notant  $u^{n+1/2} := (1/2)(u^{n+1} + u^n)$ , il s'agit de remplacer, dans l'algorithme de projection ci dessus, les deux étapes de resolution puis projection par l'unique étape suivante :

$$\left[ \begin{array}{l} \text{Trouver } u^{n+1} \text{ telle que pour tout } \varphi : \mathcal{O} \rightarrow \mathbb{R}^3 , \\ \int_{\mathcal{O}} (u^{n+1} - u^n) \cdot \varphi - \int_{\mathcal{O}} u^n \times (u^{n+1} - u^n) \cdot \varphi = -2 \int_{\mathcal{O}} (u^{n+1/2} \times \Delta u^{n+1/2}) \cdot \varphi \end{array} \right. \quad (37)$$

où ici on utilise, à la place de la forme de Gilbert, l'équivalence formelle de **(LLG)** avec l'équation  $\partial_t u + u \times \partial_t u = u \times \Delta u$ . La contrainte sur la magnitude est ici respectée, comme on peut le voir en considérant  $\varphi := u^{n+1/2}$  dans (37), observant que l'on a l'identité  $(u^{n+1} - u^n) \cdot u^{n+1/2} = 1/2(|u^{n+1}|^2 - |u^n|^2)$ .

Les deux approches mènent à des résultats concluants de convergence, dans les cas semi-discrétisé en temps [AJ06], et totalement discrétisé [BKP05, BP06]. Les preuves se basent sur un pendant discret de la formule de dissipation (7).

L'adaptation de ces résultats pose cependant des problèmes pratiques pour l'équation stochastique, liés notamment au défaut de régularité du terme de bruit. Rappelons que la "forme de Gilbert" est l'équation obtenue par multiplication vectorielle à gauche de **(LLG)** par  $(\text{id}_{\mathbb{R}^3} - u \times)$ . Une telle multiplication donne formellement dans le cas stochastique l'équation

$$du - u \times du = 2(\Delta u + u |\nabla u|^2 + G_\phi(u))dt + (\text{id} - u \times)u \times dW_\phi ,$$

où l'on note  $G_\phi(u) := 1/2(\text{id} - u \times)F_\phi(u)$ . Dans [BBNP13a], les auteurs prouvent la convergence d'une version stochastique de l'algorithme de point milieu donné par A. Prohl et S. Bartels, et la limite est une solution martingale faible. Ce schéma consiste essentiellement à ajouter un terme  $\langle u^{n+1/2} \times \Delta W^n, \varphi \rangle_{L^2_x}$  dans le problème variationnel (37). Notons qu'il n'y a pas de terme de type "correction d'Itô" dans l'algorithme, car le choix du point milieu permet de

retrouver, à la limite d'un pas de discrétisation  $\Delta t \rightarrow 0$ , l'intégrale de Stratonovich. Rappelons que l'équation est à l'origine formulée dans cette convention. Signalons qu'un autre exemple de discrétisation d'une équation de Stratonovitch a également été traité dans [DD04], ou est également utilisé un schéma de type "point milieu" mais semi-discret en temps.

Dans le chapitre 5 nous avons adapté l'algorithme de projection au cas de (SLLG), ce qui a donné lieu à la publication d'un article [ADH14]. Notons que l'article [GLT13] propose déjà une adaptation de cet algorithme, mais en ayant recours à une technique ne fonctionnant que pour un bruit monodimensionnel. Pour  $\theta \in (1/2, 1]$ ,  $T > 0$ ,  $\Delta t = T/N$ , et notant toujours  $K(u)$  l'espace tangent (36), ainsi que  $G_\phi = (\text{id} - u \times) F_\phi$ , l'algorithme s'énonce comme suit.

ALGORITHME DE PROJECTION POUR SLLG. — Pour tout  $n = 0, 1, \dots, N - 1$ , notant  $\bar{u}^{n+\theta} := (1 - \theta)u^n + \theta v^n$ ,

$$\left[ \begin{array}{l} \text{trouver } v^n(\omega, \cdot) \in K_n \text{ tel que pour tout } \varphi \in K(u^n) : \\ \quad \langle v^n - u^n \times v^n, \varphi \rangle = -2\Delta t \langle \nabla \bar{u}^{n+\theta}, \nabla \varphi \rangle + \langle G(u^n), \varphi \rangle + \langle (\text{id} - u \times) u^n \times \Delta W^n, \varphi \rangle ; \\ \text{projeter ponctuellement la fonction } u^n + v^n \text{ sur la sphère via} \\ \quad u^{n+1}(\omega, x) = \frac{u^n(\omega, x) + v^n(\omega, x)}{|u^n(\omega, x) + v^n(\omega, x)|_{\mathbb{R}^3}} , \text{ et répéter jusqu'à ce que } n = N - 1. \end{array} \right.$$

Si l'on considère en plus du temps, une discrétisation en espace sous forme d'éléments finis, alors l'étape de projection sur la sphère s'écrit au moyen de l'opérateur d'interpolation nodale  $\mathcal{I}_h$  :

$$u^{n+1} := \mathcal{I}_h \left( \frac{u^n + v^n}{|u^n + v^n|_{\mathbb{R}^3}} \right) \quad (38)$$

Il a été montré dans [ACDP04], que la condition :

$$\int_{\mathbb{D}} \nabla \psi_i \cdot \nabla \psi_j \leq 0 , \quad (39)$$

où les  $\psi_i$  désignent les fonctions de base de la triangulation, entraîne que le procédé de renormalisation est dissipatif au sens :

$$\int_{\mathbb{D}} \left| \nabla \mathcal{I}_h \left( \frac{\varphi(x)}{|\varphi(x)|_{\mathbb{R}^3}} \right) \right|_{\mathbb{R}^{3 \times 2}}^2 dx \leq \int_{\mathbb{D}} |\nabla \varphi(x)|^2 dx , \quad (40)$$

pour tout élément fini  $\varphi$ , dès que  $|\varphi|_{\mathbb{R}^3}$  est plus grand que 1 en les noeuds. Un tel résultat relie les propriétés d'un maillage à une propriété analytique. Il est prouvé que la condition (39) est satisfaite dans les deux cas suivants :

- en 2D, dans le cas d'une triangulation de Delaunay ;
- en 3D, lorsque tous les angles dièdres des tétraèdres sont inférieurs à  $\pi/2$ ,

voir par exemple les références incluses dans [Alo08].

Cette relation permet, au niveau discret, de récupérer une formule de dissipation du type de (17). Nous avons obtenu les résultats suivants aux chapitres 5 et 6.

**Théorème V : Résultats principaux des des chapitres cinq et six, convergence de l’algorithme de projection.** — Définissons la solution approchée, pour  $t \in [0, T]$

$$u_N(t, \cdot) := u^n \text{ pour } n \text{ tel que } t \in [n\Delta t, (n+1)\Delta t) .$$

Alors la loi  $\mathcal{L}(u_N)$  converge, lorsque  $N \rightarrow \infty$  et à sous-suite près, vers une solution martingale faible de (SLLG).

Il en est de même si on considère une discrétisation  $u_{h,k}$  en temps et en espace, à l’aide d’éléments finis de type  $P_1$  obtenus par une triangulation respectant (39). L’algorithme considéré est alors le même que le précédent, en prenant soin toutefois de remplacer le produit scalaire  $L^2$  par une version discrétisée  $\langle \cdot, \cdot \rangle_{V_h}$ .

Nous complétons notre étude, dans le chapitre 6, par des exemples numériques. Nous utilisons le logiciel gratuit Freefem++, pour lequel toute dimension  $1 \leq d \leq 3$  d’espace peut être traitée. Néanmoins nous nous focalisons ici sur la dimension deux. Nous observons numériquement l’explosion des solutions en dimension deux, au sens “bubbling” dans le cas de (SLLG).

Pour les applications numériques, nous utilisons notamment une approximation  $\chi^n \in V_h$  des incréments Gaussiens  $\Delta W^n$  tirée de [BBP13], pour le bruit blanc espace temps :

$$\chi^n(x) := \sum_{\substack{\psi_i \text{ dans la} \\ \text{base de } V_h}} \frac{(\gamma_1^{n,i}, \gamma_2^{n,i}, \gamma_3^{n,i})}{\sqrt{\int_{\mathbb{D}} (\psi_i(y))^2 dy}} \psi_i(x) , \text{ avec } \gamma^{n,i} \text{ suite i.i.d. } \sim \mathcal{N}(0, \text{id}_{\mathbb{R}^3}) .$$

Le bruit régularisé en espace peut être incorporé dans l’algorithme de discrétisation espace temps via le remplacement  $\chi^n \leftarrow (-\Delta_h)^{-1} \chi^n$ ,  $\Delta_h$  étant le Laplacien discrétisé sur  $V_h$ . Cela n’est pas beaucoup plus coûteux en temps de calcul, car le Laplacien doit de toute façon être inversé pour donner la solution  $v^n$ .





## General notation

**Basic notation.** For an unknown  $u = (u^1, u^2, u^3) : \mathcal{O} \rightarrow \mathbb{R}^3$  the symbol “ $\Delta u$ ” means Laplace operator on each coordinate, namely  $\Delta u = \sum_{j=1}^n (\partial_{jj} u^1, \partial_{jj} u^2, \partial_{jj} u^3)$ , whereas  $\nabla u$  denotes the gradient, namely  $\nabla u_j^i = \partial_j u^i$  for  $(i, j) \in \llbracket 1, 3 \rrbracket \times \llbracket 1, n \rrbracket$ . The notation  $|\nabla u|^2$  or  $|\nabla u|_{\mathbb{R}^{2 \times 3}}^2$  means  $\sum_{i,j} (\partial_j u^i)^2$ . We will denote by  $a \cdot b$ , where  $a, b \in E$  the standard scalar product in Euclidean spaces. The symbol  $\times$  denotes the cross product, namely if  $u, v \in \mathbb{R}^3$ ,

$$u \times v = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) .$$

For a given filtered probability space  $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t), \bar{\mathbb{P}})$ , and a random variable  $X$ , the notation  $\bar{\mathbb{E}}[X]$  means the expectation of  $X$ , namely  $\bar{\mathbb{E}}[X] = \int_{\bar{\Omega}} X(\omega) \bar{\mathbb{P}}(d\omega)$ , whereas  $\bar{\mathbb{E}}_{\mathcal{F}_s}[X]$ , or  $\mathbb{E}[\cdot | \mathcal{F}_s]$  means the conditional expectation with respect to  $\mathcal{F}_s$ . Recall that, given a Hilbert space  $H$ , a *stochastic basis*

$$\mathfrak{B} = (\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \in [0, T]}, W) , \quad (0.1)$$

means a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  together with a filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  and a  $(\mathcal{F}_t)$ ,  $H$ -valued Wiener process  $W$ .

**Functional spaces and norms.** If  $\mathcal{O} \subseteq \mathbb{R}^n$ ,  $n \in \mathbb{N}$ , standard Lebesgue spaces of  $p$ -summable classes of functions from  $\mathcal{O}$  into a Banach space  $X$  will be denoted as  $L^p(\mathcal{O}; X)$  or in abbreviate form  $L_x^p X$ , if  $\mathcal{O}$  refers without ambiguity to the values taken by the space variable  $x$ . If  $X$  is a finite dimensional space, we shall repeatedly use the abbreviation  $L_x^p$  instead of  $L_x^p X$ . Similarly, when  $T > 0$  refers to a bound for the time variable  $t$ ,  $L^p([0, T]; X)$  can be abbreviate  $L_T^p X$ , or  $L_T^p$  if  $X$  is finite dimensional.

The space of continuous functions over a domain  $D \subseteq \mathbb{R}^n$  with values in some Banach space  $X$  will be denoted  $\mathcal{C}(D; X)$ . If the domain is a time interval  $[0, T]$  where  $T > 0$ , we shall use the abbreviation  $\mathcal{C}_T X$ , instead of  $\mathcal{C}([0, T]; X)$ . In a similar way, if  $\alpha \in (0, 1)$ , then  $\mathcal{C}^\alpha(I; X)$  or  $\mathcal{C}_T^\alpha X$  if  $I = [0, T]$  will denote the space of  $\alpha$ -Hölder maps from  $I$  into  $X$ . Classical Sobolev spaces of  $X$ -valued functions will be denoted by  $W^{\alpha, p}(D; X)$ ,  $\alpha \in \mathbb{R}$ , or  $H^\alpha(D; X)$  when  $p = 2$ , (see e.g. [AF03]). To be more explicit, in the whole manuscript we adopt the convention that when  $D = [0, T]$ , and  $X$  denotes a functional space for the variable  $x$ , then

$$\|f\|_{W^{\alpha, p}([0, T]; X)} := \left( \iint_{[0, T]^2} \frac{|f(s) - f(t)|_X^p}{|t - s|^{1 + \alpha p}} ds dt \right)^{\frac{1}{p}} , \quad (0.2)$$

for each measurable  $f : [0, T] \rightarrow X$ ,  $\alpha \in (0, 1)$ , and  $q \in [1, \infty)$ . As above, if there is no ambiguity, we will abbreviate these spaces by  $W_T^{\alpha, p} X$ ,  $W_x^{\alpha, p}$ ,  $H_T^\alpha$  etc.

We use simple bars  $|\cdot|_X$  for norms of functions of the space variable, and double bars  $\|\cdot\|_E$  for space-time functions. For instance if  $f : \mathcal{O} \rightarrow \mathbb{R}$ ,  $|f|_{L_x^2}$  means  $(\int_{\mathcal{O}} f(x)^2 dx)^{1/2}$ , whereas for  $g : [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ ,  $\|g\|_{L_T^2 L_x^2}$  means  $(\int_{[0, T] \times \mathcal{O}} g(t, x)^2 dt dx)^{1/2}$ .

If  $E$  is finite dimensional, the inner product in the space  $L^2(\mathcal{O}; E)$  will be denoted by  $\langle \cdot, \cdot \rangle_{L_x^2}$ , namely

$$\forall f, g \in L^2(D; E), \quad \langle f, g \rangle_{L_x^2} := \int_D f(x) \cdot g(x) dx ,$$

and where there is no ambiguity, we simply denote by  $\langle f, g \rangle$  this quantity. The brackets will be also used for the dual pairing between a Banach space  $X$  and its dual  $X'$ , and we will precise the pair by the notation  $\langle \cdot, \cdot \rangle_{X', X}$ . To avoid any ambiguity, we never use these brackets for finite-dimensional scalar product, for which the notation  $a \cdot b$  is systematically used.

**Notation on operators.** Throughout the manuscript, if  $X, Y$  denotes Banach spaces,  $\mathcal{L}(X, Y)$  is the space of all continuous linear operators from  $X$  into  $Y$ , endowed with the standard operator norm. If  $X = Y$ , we will abbreviate  $\mathcal{L}(X, X)$  by  $\mathcal{L}(X)$ .

If  $H, K$  denote Hilbert spaces,  $H$  being separable, we denote by  $\text{tr}_{H, K}$  (or simply  $\text{tr}$  when  $H = K = L_x^2$ ) the trace of operators from  $H$  into  $K$ , namely

$$\text{tr } \Phi = \sum_{\varepsilon \in \mathcal{B}} \langle \Phi \varepsilon, \varepsilon \rangle_K , \quad \Phi \in \mathcal{L}(H; K) . \quad (0.3)$$

where  $\mathcal{B}$  denotes an orthonormal basis of  $H$ . The notation  $\mathbb{L}_2(H; K)$  is used for the space of Hilbert-Schmidt operators from  $H$  into  $K$ , i.e. operators  $\phi \in \mathcal{L}(H, K)$  such that

$$|\phi|_{\mathbb{L}_2(H, K)}^2 := \text{tr}_{H, K} \phi \phi^* < \infty , \quad (0.4)$$

In abbreviate form, when  $H = L_x^2$  and  $K = H_x^s$ ,  $s \in \mathbb{R}$ , it is denoted by  $\mathbb{L}_2^{0, s}$ . The particular case of  $\mathbb{L}_2(L_x^2; L_x^2) = \mathbb{L}_2^{0, 0}$  is denoted by  $\mathbb{L}_2$ .

We will also denote by  $R(H, X)$  the space of  $\gamma$ -radonifying operators from a separable Hilbert space  $H$  into a Banach space  $X$ , namely the space of elements  $\phi \in \mathcal{L}(H, X)$ , such that for some orthonormal basis  $(e_k)_{k \in \mathbb{N}}$  of  $H$  :

$$|\phi|_{R(H, X)}^2 := \int_{\tilde{\Omega}} \left| \sum_{k \in \mathbb{N}} \gamma_k(\tilde{\omega}) \phi e_k \right|_X^2 \tilde{\mathbb{P}}(d\tilde{\omega}) < \infty , \quad (0.5)$$

where  $(\gamma_k)_{k \in \mathbb{N}}$  denotes an i.i.d. sequence of real-valued normal gaussian r.v. on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We recall that this definition does not depend on the particular choice of the sequence  $(e_k)$  – see [Brz97]-[BP99] and references therein.

**Constants.** The symbols  $c, c', c''$ , etc. will designate absolute constants unless we explicitly mention the dependence with respect to peculiar elements  $\phi, \psi$  by  $c(\phi), c(\phi, \psi)$  etc.

**Abbreviation.** The initials “ONB” mean “orthonormal basis”.

## CHAPTER 1.

# *Local solvability*

---

The first chapter is devoted to the question of local existence and uniqueness of solutions of SLLG. We begin by describing the different possible ways of introducing a Gaussian noise in the equation, and we recall the underlying functional framework. We present the different ansatz of the equation studied in the manuscript and we then define a precise notion of mild solution in the case when there is no gyromagnetic term  $u \times \Delta u$ , i.e. in the “overdamped” case. For that specific version of (SLLG), we interpret the equation as a fixed point problem in some functional space  $L^q(\Omega; X_T)$ , where  $\Omega$  is a probability space on which the noise is defined, and the choice of  $X_T$  is motivated by the nonlinear term  $u|\nabla u|^2$ . After a few reminders on stochastic convolutions and the so-called “maximal inequality”, we precise how the noise term must be regularized in space through its correlation operator. For a truncated equation, where the nonlinear term is modified, existence and uniqueness of solutions are obtained by using the regularizing effects of the associated linear equation. We then show how to eliminate the truncation, obtaining finally the solutions of our equation, up to a maximal time  $\tau > 0$ .

---

# 1 Introduction

## (1.a) Preliminaries

Let  $\mathcal{O} \subseteq \mathbb{R}^n$  be a bounded domain, where  $n \in \mathbb{N}$ , and let  $\Omega$  be the sample set. In the context given previously, solving SLLG means finding  $u = u(\omega, t, x)$  such that

$$u(\omega, t, x) \in \mathbb{S}^2 = \{X \in \mathbb{R}^3, |X|_{\mathbb{R}^3} = 1\}, \quad \forall (\omega, t, x) \in \Omega \times \mathbb{R}^+ \times \mathcal{O} \text{ a.e.} \quad (1.1)$$

and fulfilling

$$\begin{cases} \partial_t u = -u \times \left( u \times (\Delta u + \zeta) \right) + u \times (\Delta u + \zeta), & \text{for } (\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathcal{O}, \\ Bu = Bu_0(x), & \text{for } (\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \partial\mathcal{O}, \\ u|_{t=0} = u_0(x), & \text{for } (\omega, x) \in \Omega \times \mathcal{O}, \end{cases} \quad (1.2)$$

where  $\zeta = \zeta(\omega, t, x)$  denotes the noisy perturbations of the effective field, and  $B$  denotes either Neumann or trace operator. As already mentioned in the preceding chapter, the term  $\zeta$  should ideally refer to a *Gaussian space-time white noise* (see the review article [Ber07] and references therein). It should be uncorrelated in space in the sense:

$$\mathbb{E} [\zeta(t, x)\zeta(t', x')] = \delta(x - x')\delta(t - t'), \quad (1.3)$$

where  $\delta$  means the Dirac distribution, and the equality holds in the sense of distributions. However, due to the lack of regularity of the space time white noise, equation (1.2) is not expected to possess a well defined solution in this case, and we regularize the noise in space, letting

$$\mathbb{E} [\zeta(t, x)\zeta(t', x')] = c(x, x')\delta(t - t'), \quad (1.4)$$

where the correlation map  $c(x, x'), (x, x') \in \mathcal{O} \times \mathcal{O}$  is defined by a kernel-type formula  $c(x, x') = \int_{\mathcal{O}} k(x, z)k(z, x')dz$  with some  $k \in L^2(\mathcal{O} \times \mathcal{O}; \mathbb{R})$ . Denoting by “ $\dot{\cdot}$ ” the time derivative, one has classically (see [DZ08]) the formal relation

$$\zeta(t, x) = \dot{W}_\phi(t, x),$$

where  $t \mapsto W_\phi(t), t \geq 0$ , is some  $L^2(\mathcal{O}; \mathbb{R}^3)$ -valued Wiener process, with a certain correlation operator  $Q = \phi\phi^* : L_x^2 \rightarrow L_x^2$  of trace class, or equivalently,  $\phi$  is a Hilbert-Schmidt class operator from  $L_x^2$  into itself – see (0.4). If  $(\beta_k)_{k \in \mathbb{N}}$ , stands for a sequence of real valued and independent brownian motions, and if  $(e_k)_{k \in \mathbb{N}}$  is an orthonormal basis of  $L^2(\mathcal{O}, \mathbb{R}^3)$ , a Wiener process  $W$  is formally given by the formula:

$$W_\phi(t) = \sum_{k \in \mathbb{N}} \beta_k(t)\phi e_k, \quad t \geq 0,$$

see [DZ08, chap. 4].

Define for each  $X \in \mathbb{S}^2$  the linear map  $\sigma_0(X)$  by:

$$\sigma_0(X)\zeta := X \times (\zeta - X \times \zeta), \quad \forall \zeta \in \mathbb{R}^3, \quad (1.5)$$

so that (1.2) formally rewrites:

$$\begin{cases} \partial_t u = \Delta u + u|\nabla u|^2 + u \times \Delta u + \sigma_0(u)\dot{W}_\phi , & \text{in } \Omega \times \mathbb{R}_+ \times \mathcal{O} , \\ Bu = Bu_0 , & \text{in } \Omega \times \mathbb{R}_+ \times \partial\mathcal{O} , \\ u|_{t=0} = u_0 , & \text{in } \Omega \times \mathcal{O} . \end{cases} \quad (1.6)$$

As long as we have not specified the choice of the stochastic integral, (1.6) may have different meanings [Van81]. For the reasons mentioned in the previous chapter (see also [Rez04]), in order to satisfy the geometrical constraint (1.1), the product  $\sigma_0(u(t, x))\zeta(t, x)$  must be understood in the Stratonovich sense  $\sigma_0(u(t, x)) \circ \zeta(t, x)$ .

### (1.b) Note on the infinite dimensional diffusion term.

Let  $B$  be a Brownian Motion in  $\mathbb{R}^3$ , and  $X$  a process on  $\mathbb{R}^3$  with  $|X_0|_{\mathbb{R}^3} = 1$ . Denote by  $P^\perp(X_t)$  the orthogonal projection on  $(\text{Vect } X_t)^\perp$  in  $\mathbb{R}^3$ , and by “ $\circ$ ” the Stratonovich product. Then, a Brownian motion on the manifold  $\mathbb{S}^2$  is classically modeled by the SDE

$$dX = P^\perp(X) \circ dB , \quad (1.7)$$

see e.g. the book [Elw82]. It was noticed in [PW83] that the alternative SDE

$$dX = X \times \circ dB , \quad (1.8)$$

where  $\times$  denotes cross product, also generates a Brownian Motion in the sphere, in the sense that for any  $\varphi \in \mathcal{C}^2(\mathbb{S}^2)$ , the map  $v(t, x) = \mathbb{E}[\varphi(X_t)]$ , where  $t \geq 0$  and we set  $X_0 = x \in \mathbb{S}^2$ , solves the initial value problem

$$\begin{cases} \partial_t v(t, x) = \frac{1}{2} \Delta_{\mathbb{S}^2} v(t, x) , & t > 0 , \quad x \in \mathbb{S}^2 , \\ v(0, x) = \varphi(x) , & x \in \mathbb{S}^2 , \end{cases}$$

where  $\Delta_{\mathbb{S}^2}$  is the Laplace-Beltrami operator. We note that the SDE (1.8) has the advantage of being linear, whereas (1.7) is quadratic.

It is now tempting to solve LLG in the form (1.6) but considering the cross product with  $u(t, x)$ , instead of the linear map  $\sigma_0(u(t, x))$ , see (1.5). By analogy with the finite dimensional case, we can guess that this should not cause any change in the law of a solution  $u$ . Define for  $X \in \mathbb{S}^2$  the linear map  $\sigma_1(X)$  by:

$$\sigma_1(X)\zeta = X \times \zeta , \quad \forall \zeta \in \mathbb{R}^3 . \quad (1.9)$$

Disregarding the drift terms, which are equal in both formulations, we focus on the infinite dimensional Stratonovich equations:

$$\begin{cases} dz^j = \sigma_j(z^j(t, x)) \circ dW_\phi , & (t, x) \in \mathbb{R}_+ \times \mathcal{O} , \\ z^j(0) = \xi \in L^2(\mathcal{O}; \mathbb{S}^2) , \end{cases} \quad (1.10)$$

for  $j = 0, 1, 2$ ,  $z^j$  being the unknown, where we define in addition:

$$\sigma_2(X)\zeta = P^\perp(X)\zeta , \quad \forall (X, \zeta) \in \mathbb{S}^2 \times \mathbb{R}^3 . \quad (1.11)$$

Apart from the matter of solvability, the question is to know whether associated solutions  $z^0, z^1, z^2$  have equal laws. If  $\varphi$  is continuous and bounded on  $L_x^2$ , and if for  $j = 0, 1, 2$ ,  $z^j(\cdot, \xi)$ , denotes a solution of (1.10), then the Kolmogorov equation on  $v^j(t, \xi) = \mathbb{E}[\varphi(z^j(t, \xi))]$  formally writes:

$$\begin{cases} \partial_t v^j = \frac{1}{2} \operatorname{tr}[D_{\xi\xi}^2 v^j(t, \xi) \sigma_j(\xi) \phi \phi^* \sigma_j(\xi)^*] , & \text{for } t > 0 \text{ and } \xi \in L^2(\mathcal{O}; \mathbb{S}^2) , \\ v^j(0, \xi) = \varphi(\xi) , & \text{for } \xi \in L^2(\mathcal{O}; \mathbb{S}^2) . \end{cases} \quad (1.12)$$

Considering the case of space-time white noise, namely when the correlation operator is the identity  $\operatorname{id} : L_x^2 \rightarrow L_x^2$ , we observe that, at a formal level, the three choices (1.5)-(1.9)-(1.11) lead to the same diffusion operator  $\mathcal{A}$  on  $\mathcal{H} = C_b^1(L^2(\mathcal{O}; \mathbb{R}^3))$ . Indeed, observe that for any unit vector  $X \in \mathbb{R}^3$ :

$$-X \times (X \times \cdot) = P^\perp(X) ,$$

where as above  $\zeta \in \mathbb{R}^3 \mapsto P^\perp(X)\zeta$  denotes the projection map on  $T_X \mathbb{S}^2$ . By the additional fact that  $(X \times)^* = -X \times$ ,  $X \in \mathbb{R}^3$ , we have

$$\sigma_1(\xi)(\sigma_1(\xi))^* = P^\perp(\xi) = P^\perp(\xi)P^\perp(\xi)^* ,$$

almost everywhere for  $x \in \mathcal{O}$  provided  $\xi(x) \in \mathbb{S}^2$ . Hence  $v^0, v^1$  solve the same Kolmogorov equation.

The situation is similar for  $j = 0$ , noticing first that  $\sigma_0 = \sigma_1 + \sigma_2$ . Moreover, denoting by “ $\dagger$ ” the adjoint in  $\mathcal{L}(\mathbb{R}^3)$ , there holds the identity  $X \times \dagger (P^\perp(X)) = -P^\perp(X) \dagger (X \times)$ , for any  $X \in \mathbb{S}^2$ . Therefore, taking any  $\xi \in L^2(\mathcal{O}; \mathbb{S}^2)$ , we have

$$\sigma_0 \sigma_0(\xi)^* = \sigma_1(\xi) \sigma_1(\xi)^* + \sigma_2(\xi) \sigma_2(\xi)^* , \quad \text{a.e. on } \mathcal{O} .$$

Whence, denoting  $\mathcal{A}(\xi)$  the abstract operator  $1/2 \operatorname{tr}[D_{\xi\xi}^2 \cdot \sigma_1(\sigma_1)^*]$ , then we see that  $v^0$  solves  $\partial_t v^0 = 2\mathcal{A}(\xi)v$ , so that replacing  $\sigma_0$  by  $\sigma_0/\sqrt{2}$  in the equation fulfilled by  $z^0$  gives the same Kolmogorov equation as that of  $z^j$  for  $j = 0, 1$ . This justifies the assumption made in the literature (see e.g the presentation made in [BGJ13]) that one can consider the following stochastic Landau-Lifshitz-Gilbert equation instead of (1.6):

$$du = (\Delta u + u|\nabla u|^2 + u \times \Delta u)dt + u \times \circ dW_\phi . \quad (1.13)$$

*Remark 1.1.* It is intuitively clear that the three noise terms corresponding to  $\sigma_i$  with  $0 \leq i \leq 2$ , namely

$$\begin{aligned} & (u(t, x) \times \cdot + P_{u(t,x)}^\perp) \circ (\dot{W}_1(t, x), \dot{W}_2(t, x), \dot{W}_3(t, x)) , \\ \text{resp. } & u(t, x) \times \circ (\dot{W}_1(t, x), \dot{W}_2(t, x), \dot{W}_3(t, x)) , \\ \text{resp. } & P_{u(t,x)}^\perp \circ (\dot{W}_1(t, x), \dot{W}_2(t, x), \dot{W}_3(t, x)) , \end{aligned}$$

are “degenerate”, in the sense that

$$\ker_{\mathbb{R}^3}(\sigma_i(u(t, x))) \neq \{0\} ,$$

their range being the two dimensional tangent space  $T_{u(t,x)} \mathbb{S}^2 = (\operatorname{Vect}\{u(t, x)\})^\perp$ . The presence of three coordinates in  $\dot{W}$  therefore appears to be redundant, and one may ask if there is a “more

intrinsic” way of incorporating noise in (1.6), i.e. involving only two coordinates  $\dot{W}_1, \dot{W}_2$ . In particular, for any  $X \in \mathbb{S}^2$ , we are looking for a linear map  $\sigma_3(X) \in \mathcal{L}(\mathbb{R}^2; \mathbb{R}^3)$  that would give the same generator as that of (1.5)-(1.9)-(1.9) when considering the SPDE (1.10).

If we work in spherical coordinates  $u(t, x) = (\sin \varphi(t, x) \cos \theta(t, x), \sin \varphi(t, x) \sin \theta(t, x), \cos \varphi(t, x))$ , and if  $(\vec{e}_\rho(t, x) \equiv u(t, x), \vec{e}_\theta(t, x), \vec{e}_\varphi(t, x))$  denotes the classical mobile frame associated to  $u(t, x)$ , it can be formally checked that then equation (1.10) with  $W := (W_1, W_2)$  and

$$\sigma_3(u(t, x)) \circ \dot{W} := \dot{W}_1(t, x) \circ \frac{\vec{e}_\theta(t, x)}{\sin \varphi(t, x)} + \dot{W}_2(t, x) \circ \vec{e}_\varphi(t, x) , \quad (1.14)$$

leads to a similar Kolmogorov equation (for a detailed calculus in finite dimension, we refer the reader to [Rez04]). This confirms the intuition that the term  $u(t, x) \times \circ \dot{W}(t, x)$  is essentially two dimensional.

### (1.c) Itô/Stratonovitch correction term, and main ansatz of the equation

In order to work with a non-anticipative integral, we change (1.13) to its Itô form. Using the formal relation between the Stratonovich and Itô differentials, a corresponding formulation of (1.13) is obtained by adding a correction term to the drift of (1.2). In this sense if  $\Phi$  denotes the map  $u \mapsto \sigma(u)(\phi \cdot)$ , the noise term can be rewritten as follows (see e.g. [BBNP13b, BBNP13a, BGJ13]) :

$$\begin{aligned} u \times \circ dW_\phi &= u \times dW_\phi + \frac{1}{2} \sum_{e \in \mathcal{B}^3} [\Phi'(u) \cdot (\Phi(u)e)] e dt \\ &= u \times dW_\phi + \frac{1}{2} \sum_{\substack{\alpha \in \{1,2,3\} \\ l \in \mathbb{N}}} (u \times \phi e_l^\alpha) \times \phi e_l^\alpha dt. \end{aligned}$$

where here we have again denoted by “ $\circ d$ ” the Stratonovich differential, and by “ $d$ ” the Itô differential, and  $\mathcal{B}^3$  means an adapted basis  $e_l^1 := (\varepsilon_l, 0, 0)$ ,  $e_l^2 := (0, \varepsilon_l, 0)$ ,  $e_l^3 := (0, 0, \varepsilon_l), \dots$  etc.,  $(\varepsilon_l)_{l \geq 0}$  being an ONB of  $L^2(\mathcal{O}; \mathbb{R})$ . The term

$$F_\phi(u) = \frac{1}{2} \sum_{\substack{\alpha \in \{1,2,3\} \\ l \in \mathbb{N}}} (u \times \phi e_l^\alpha) \times \phi e_l^\alpha \quad (1.15)$$

is what we may call in the sequel “the Itô correction”. Note that this defines an element of  $L^1(\mathcal{O}; \mathbb{R}^3)$  as soon as  $\phi$  is Hilbert-Schmidt from  $L^2(\mathcal{O}; \mathbb{R}^3)$  into itself, which will be a a systematic assumption in the sequel of the manuscript.

Let  $(\beta_k)_{k \in \mathbb{N}}$  denote a sequence of independent real-valued standard Brownian motion, let  $\phi : L^2(\mathcal{O}; \mathbb{R}^3) \rightarrow L^2(\mathcal{O}; \mathbb{R}^3)$  an Hilbert-Schmidt linear operator, and denote by  $(e_k)_{k \in \mathbb{N}}$  an ONB of  $L^2(\mathcal{O}; \mathbb{R}^3)$ . The considerations above lead us to the following Itô formulation, which is the



main ansatz treated in the manuscript:

$$\left\{ \begin{array}{l} du = \left( \Delta u + \gamma u \times \Delta u + u |\nabla u|^2 + F_\phi(u) \right) dt + u \times dW_\phi , \quad \text{in } \Omega \times \mathbb{R}^+ \times \mathcal{O} , \\ \text{with boundary conditions:} \\ Bu(t) = Bu_0 , \quad \text{in } \Omega \times \mathbb{R}^+ \times \partial\mathcal{O} , \\ u|_{t=0} = u_0 , \quad \text{in } \Omega \times \mathcal{O} , \end{array} \right. \quad (\text{SLLG})$$

where  $u$  must satisfy the local constraint on the magnitude

$$|u(\omega, t, x)|_{\mathbb{R}^3} = 1 , \quad \text{for } d\mathbb{P} \otimes dt \otimes dx \text{ a.e. } (\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O} , \quad (\text{C})$$

and here we use the following notations:

$$\left[ \begin{array}{l} \gamma \in \{0, 1\} , \\ W_\phi(\omega, t) = \sum_{k \in \mathbb{N}} \beta_k(\omega, t) \phi e_k , \\ B = \text{Neumann or trace operator} , \end{array} \right. \quad (1.16)$$

## (1.d) Results

In the sequel we denote by  $n$  the dimension of the domain  $\mathcal{O}$ , and, in order to simplify the presentation, we assume that  $n \geq 2$ . For technical reasons that are developed above, we are only able to give the local existence and uniqueness in the overdamped case  $\gamma = 0$ , i.e. for the stochastic Flow of Harmonic Maps:

$$\left\{ \begin{array}{l} du = (\Delta u + u |\nabla u|^2 + F_\phi(u)) dt + u \times dW_\phi , \quad \text{on } \Omega \times \mathbb{R}_+ \times \mathcal{O} , \\ Bu = Bu_0 , \quad \text{on } \Omega \times \mathbb{R}_+ \times \mathcal{O} , \\ u(0) = u_0 , \quad \text{on } \Omega \times \mathcal{O} , \end{array} \right. \quad (1.17)$$

where the initial data  $u_0$  is compatible with (C).

**Theorem 1.1** (Local solvability in the overdamped case). *Let  $B$  denote either Neumann or Dirichlet operator. Let  $W_\phi$  be a  $\phi\phi^*$ -Wiener process on the space  $L^2(\mathcal{O}; \mathbb{R}^3)$ , and take  $p \in (n, \infty)$ . Assume that  $\phi$  is  $\gamma$ -radonifying from  $L_x^2$  onto  $W_x^{1-n/p, p}(D, \mathbb{R}^3) \cap \ker B$ . Then, for every  $u_0 \in W_x^{1, p}$  such that (C) holds, there exists a unique local mild solution to (1.17) which is supported locally in the spaces  $\mathcal{C}_T W_x^{1, p}$  for  $T > 0$ , in the sense that there exists a stopping time  $\tau = \tau(u_0)$ , such that a.s.  $u(\omega) \in \mathcal{C}([0, \tau]; W_x^{1, p})$ , and for  $0 \leq t < \tau$ :*

$$u(t) = S(t)u_0 + \int_0^t S(t-s) (u |\nabla u|^2 + F_\phi(u)) (s) ds + \int_0^t S(t-s) u(s) \times dW_\phi(s) , \quad (1.18)$$

where  $S(\cdot)$  is the Heat Semigroup  $e^{\Delta}$ , and  $\{\tau < \infty\} \subseteq \{\limsup_{t \uparrow \tau} |u(\omega, t, \cdot)|_{W_x^{1, p}} = \infty\}$ . Moreover  $u$  satisfies (C) up to  $\tau$ .

Furthermore, if more regularity is assumed on the data, namely  $u_0 \in W_x^{\beta-n/p,p}$  and  $\phi \in R(L_x^2; W_x^{\beta,p} \cap \ker B \cap \ker B^2)$  for a certain  $\beta$  with  $2 \leq \beta < 4$ , then there exists a positive stopping time  $\tau^\beta(u_0) \leq \tau$ , such that the solution  $u$  obtained above is supported, locally up to  $\tau^\beta(u_0)$ , in the spaces  $\mathcal{C}_T W_x^{\beta,p}$ .

**Remark 1.2.** The constraint (C) on  $u_0$  plays actually no crucial role in the proof of the existence and uniqueness part in Theorem 1.1. If the local constraint is not assumed for the initial data, then the result above stays true, except that instead of being with values in  $\mathbb{S}^2$ , the solution  $u(t, \cdot)$  will satisfy  $|u(\omega, t, x)|_{\mathbb{R}^3} = |u_0(x)|_{\mathbb{R}^3}$  a.e. , see the last part of the proof below.

**Idea of the proof.** The basic idea of this proof is rather simple. We interpret equation (1.17) as an Itô evolution equation in the Banach space  $L_x^p$ ,  $p > n$ , and we associate to any adapted process  $v \in L^2(\Omega; \mathcal{C}([0, T]; W_x^{1,p}))$  the unique solution  $u(v)$  of the linear evolution equation

$$\dot{u} - \Delta u = v|\nabla v|^2 + F_\phi(v) + v \times \dot{W}_\phi, \quad 0 < t \leq T, \quad u(0) = u_0 . \quad (1.19)$$

Then, we try to find a fixed point for the map  $v \mapsto u(v)$ , which gives a solution to our problem. Formally, the solution of (1.22) is given by the “variation of constants” formula:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)(v(s)|\nabla v(s)|^2 + F_\phi(v(s)))ds + \int_0^t S(t-s)v(s) \times dW_\phi(s) , \\ \text{a.s. for } 0 \leq t \leq T . \quad (1.20)$$

The reason why this method works is twofold:

- the choice of the space  $W_x^{1,p}$  with  $p > n$  gives enough regularity to control the terms due to the nonlinearity  $u|\nabla u|^2$ , since on the one hand  $W_x^{1,p} \hookrightarrow L_x^\infty$ , and the convolution with  $S(t) = e^{t\Delta}$  maps continuously  $L_x^{p/2}$  onto  $W_x^{1,p}$ .
- the stochastic integrals  $\int \Phi dW_\phi$  are well defined objects in  $L_x^p$  spaces, which are UMD and type 2 for  $p \geq 2$ . For instance, if  $\Phi : \Omega \times [0, T] \rightarrow L_x^p$  is a predictable  $\gamma$ -radonifying operator-valued process, then the stochastic convolution

$$Z^\Phi(t) = \int_0^t S(t-s)\Phi(s)dW_\phi(s)$$

is a.s.  $L_x^p$ -valued, and we can find bounds for the quantity  $\mathbb{E}[\sup_{0 \leq \tau \leq t} |Z^\Phi(\tau)|_{W_x^{1,p}}^2]$ , expressed in terms of the norm of the stochastic process  $v$ , see section 2.

Unfortunately, this method fails when dealing with the case  $\gamma = 1$ , where the higher order term contains a nonlinearity. This equation can however be treated, *in the deterministic case*, where the local existence and uniqueness have been obtained by H. Amann in a series of papers [Ama85, Ama86]. The author gives local solutions for a general class of equations of the form

$$\frac{\partial u}{\partial t} + A(t, x, u(t, x), \nabla u(t, x))u = F(t, x, u, \nabla u)$$

with  $A(t, x, u, \nabla u) = \sum_{\alpha, \beta} a_{\alpha, \beta}(t, x, u(t, x), \nabla u(t, x)) \partial_\alpha \partial_\beta$ , each map  $a_{\alpha, \beta}$  being  $\mathcal{L}(\mathbb{R}^3)$ -valued. This class is defined by the fact that the family  $A(\cdot)$  fulfills the *strong parabolicity property*, namely

$$\operatorname{Re} \left\{ \sum_{r, s=1}^3 \sum_{\alpha, \beta=1}^2 a_{\alpha, \beta}^{r, s}(t, x, u(t, x)) \xi^\alpha \xi^\beta \lambda_r \bar{\lambda}_s \right\} > 0, \quad \forall (\xi, \lambda) \in \mathbb{R}^n \setminus 0 \times \mathbb{C}^3 \setminus 0, \quad (1.21)$$

which is satisfied by the family associated to **(SLLG)** i.e.  $u \mapsto A(u)u = \Delta u + u \times \Delta u$ . The method presented here for **(1.17)** could be generalized as follows: associate to any  $v : [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^3$  the unique solution  $u(v)$  of the Itô equation

$$\dot{u} + A_v(t, x)u = v(\nabla v)^2 + F_\phi(v) + v \times \dot{W}_\phi, \quad s < t \leq T, \quad u(0) = u_0, \quad (1.22)$$

and try to find a fixed point for the map  $v \mapsto u(v)$ . Formally, the solution of **(1.22)** is given by **[Sob61, Tan60]**:

$$u(t) = U_v(t, 0)u_0 + \int_0^t U_v(t, s)(b(v(s)) + F_\phi(v(s)))ds + \int_0^t U_v(t, s)v(s) \times dW_\phi(s), \quad (1.23)$$

where for  $0 \leq s \leq t \leq T$ , and  $f \in L_x^p$ , the term  $U_v(t, s)f$  denotes the value taken at time  $t$  by the solution  $X$  of:

$$\frac{\partial X}{\partial t} + A_v(t, x)X = 0 \quad \text{in } [s, T] \times \mathcal{O}, \quad X(s) = f.$$

However, we the fixed point is done in a functional space for the variables  $\omega, t, x$ , and if  $v$  denotes a predictable process, giving a correct sense to **(1.23)** is not obvious. Indeed, for each fixed  $t_0 > 0$ , the processes  $\omega, s \mapsto U_{v(\omega)}(t_0, s)$  are not adapted in general, so that the sense of the anticipative integral

$$\text{“ } \int_0^t U_v(t, s)u \times dW_\phi \text{ ”},$$

is not clear. Note that we can still define the Skorohod integral, which is an extension of the Itô integral for anticipating integrands, but in that case the map obtained is no longer a solution of **(1.22)**, see e.g. **[Le690]**. It is however possible to obtain a solution using the notion of *forward integral*, see **[LN98]** and references therein, but this notion requires heavier hypotheses on  $u$ . In addition, estimates of the type

$$\mathbb{E} \|u(v_1) - u(v_2)\|^q \leq \mathbb{E} \|v_1 - v_2\|^q,$$

seem untreated in the existing literature when the underlying operator  $U$  depends on the element  $v$ .

## 2 Preliminaries

We now focus on the linear stochastic equation in the space  $L^p(\mathcal{O}; \mathbb{R}^3)$ ,  $p \in [2, \infty)$ :

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \Phi(\omega, t)dW_\phi, & \text{in } \Omega \times \mathbb{R}_+ \times \mathcal{O}, \\ Bu = Bu_0, & \text{on } \Omega \times \mathbb{R}_+ \times \partial\mathcal{O}, \\ u(0, \cdot) = u_0, & \text{in } \Omega \times \mathcal{O}, \end{cases} \quad (1.24)$$

where we assume that  $\omega, t \mapsto \Phi(\omega, t)$  is a predictable, operator-valued process. When  $p = 2$ , in the case where each  $\Phi(\omega, t) : L_x^2 \rightarrow L_x^2$  is of Hilbert-Schmidt class and satisfies

$$\|\Phi\|_{T, L_x^2}^2 := \mathbb{E} \int_0^T |\Phi(\omega, t)|_{\mathbb{L}_2}^2 dt < \infty ,$$

then solutions of (1.24) are classically given by the formula

$$Z^\Phi(t) = \int_0^t S(t-s)\Phi(s)dW_\phi(s) , \quad t \in [0, T] . \quad (1.25)$$

In case  $S$  is analytic, we have with full probability (see [DZ08, thm. 6.10 & thm. 6.12]):

$$Z^\Phi \in \mathcal{C}([0, T]; L_x^2) \cap L^2(0, T; H_x^1) ,$$

whose norm is bounded, in expectation, by  $\|\Phi\|_{T, L_x^2}$ . These optimal regularity results, obtained by Da Prato and Zabczyk in [DZ92], were later generalized in a Banach instead of Hilbert space framework, using the formalism of  $\gamma$ -radonifying operators, see (0.5). For instance, the following proposition is a consequence of Theorem 3.2 in [Brz97].

**Proposition 1.1** (Maximal inequality). *Let  $p \geq 2$ . Assume that we are given real numbers  $q \geq 1$  and  $\nu \geq 0$  with*

$$\frac{2}{q} < \nu .$$

*Then for any measurable adapted process  $\Phi : \Omega \times [0, T] \rightarrow R(L_x^2; W_x^{\nu, p})$  with*

$$\mathbb{E} \left[ \left( \int_0^T |\Phi(t)|_{R(L_x^2; W_x^{\nu, p})}^2 dt \right)^{\frac{q}{2}} \right] < \infty ,$$

*the process  $Z^\Phi$  – see (1.25) – is well defined and possesses a modification in  $L^q(\Omega; \mathcal{C}([0, T]; W_x^{1, p}))$ . Moreover*

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |Z^\Phi(t)|_{W_x^{1, p}}^q \right] \leq C(\nu, q) \mathbb{E} \left[ \left( \int_0^T |\Phi(t)|_{R(L_x^2; W_x^{\nu, p})}^2 dt \right)^{\frac{q}{2}} \right] ,$$

*for some constant  $C(\nu, q) > 0$ , independent of  $\Phi$ .*

To deal with the loss of regularity due to the product in the term “ $u \times \dot{W}_\phi$ ”, we need the following.

**Lemma 1.1.** *Fix numbers  $\nu \geq 0$ ,  $0 < \theta < 1$ , and  $p > n$ . Let  $v \in W_x^{\nu+\theta, p}$ , and  $\phi \in R(L_x^2; W_x^{\nu, \frac{n}{\theta}})$ . The operator  $f \mapsto v \times \phi f$  is well defined as an element of  $\mathcal{L}(L_x^2; W_x^{\nu, p})$ , and belongs to  $R(L_x^2; W_x^{\nu, p})$ . Moreover there exists a constant  $c > 0$  independent of  $v$ , such that:*

$$|v \times \phi \cdot|_{R(L_x^2; W_x^{\nu, p})} \leq c |v|_{W_x^{\nu+\theta, p}} |\phi|_{R(L_x^2; W_x^{\nu, \frac{n}{\theta}})} . \quad (1.26)$$

*Proof.* Let  $\theta \in (0, 1)$ . For  $\nu = 0$ , if  $\mathcal{B}^3$  is any ONB of  $L^2(\mathcal{O}; \mathbb{R}^3)$ , we have for any  $v \in W^{\theta,p}(\mathcal{O}; \mathbb{R}^3)$ :

$$|v \times \phi|_{R(L_x^2; L_x^p)}^2 = \tilde{\mathbb{E}} \left| \sum_{e \in \mathcal{B}^3} \gamma_e(\tilde{\omega}) v \times \phi e \right|_{L_x^p}^2$$

where  $(\gamma_e)_{e \in \mathcal{B}^3}$  is an i.i.d. sequence of real Gaussian random variables. Hölder inequality then yields for all  $q \in [1, \infty)$ , and  $e \in \mathcal{B}^3$ :  $|v \times \phi e|_{L_x^p} \leq |v|_{L^q} |\phi|_{R(L_x^2; L^{q'})}$ , where  $1/q + 1/q' = 1/p$ . By the classical Sobolev embedding  $W_x^{\theta,p} \hookrightarrow L_x^{np/(n-\theta p)}$ , the particular choice of  $q = np/(n - \theta p)$  yields

$$|v \times \phi|_{R(L_x^2; L_x^p)} \leq |v|_{W_x^{\theta,p}} |\phi|_{R(L_x^2; L_x^{\frac{n}{\theta}})} .$$

Assume now that  $v \in W_x^{2+\theta}$  and  $\phi \in R(L_x^2; W_x^{2,n/\theta})$ . With the same notations as above, there holds:

$$\begin{aligned} |v \times \phi|_{R(L_x^2; W_x^{2,p})} &\leq c \left( |v \times \phi|_{R(L_x^2; L_x^p)}^2 + \tilde{\mathbb{E}} \left| \sum_{e \in \mathcal{B}^3} \gamma_e(\tilde{\omega}) \Delta(v \times \phi e) \right|_{L_x^p}^2 \right)^{\frac{1}{2}} \\ &\leq c \left( |v|_{W_x^{\theta,p}}^2 |\phi|_{R(L_x^2; L_x^{\frac{n}{\theta}})}^2 + 3|\Delta v|_{W_x^{\theta,p}}^2 \tilde{\mathbb{E}} \left| \sum_{e \in \mathcal{B}^3} \gamma_e(\tilde{\omega}) \phi e \right|_{L_x^{\frac{n}{\theta}}}^2 \right. \\ &\quad \left. + 3|\nabla v|_{W_x^{\theta,p}}^2 \tilde{\mathbb{E}} \left| \sum_{e \in \mathcal{B}^3} \gamma_e(\tilde{\omega}) \nabla \phi e \right|_{L_x^{\frac{n}{\theta}}}^2 + 3|v|_{W_x^{\theta,p}}^2 \tilde{\mathbb{E}} \left| \sum_{e \in \mathcal{B}^3} \gamma_e(\tilde{\omega}) \Delta \phi e \right|_{L_x^{\frac{n}{\theta}}}^2 \right)^{\frac{1}{2}} \\ &\leq c' |v|_{W_x^{2+\theta,p}} |\phi|_{R(L_x^2; W_x^{2,\frac{n}{\theta}})} , \end{aligned}$$

which yields the desired bound.

The case  $\nu \in [0, 2]$  is obtained by interpolation, and higher orders work similarly.  $\blacksquare$

We now give a basic result that justifies the choice of the functional space  $W_x^{1,p}$  with  $p > n$  is the following.

**Lemma 1.2** (Hypercontractivity property of  $S$ ). *Consider  $1 \leq q \leq p \leq \infty$ . Let  $S(t) : L_x^2 \rightarrow L_x^2$ ,  $t \geq 0$ , denote the Heat Semigroup. For all  $t > 0$ ,  $S(t)$  has a continuous extension from  $L_x^q$  into  $W_x^{1,p}$ , and there exists  $K > 0$ , independent of  $t > 0$ , such that the following estimate holds:*

$$|S(t)|_{\mathcal{L}(L_x^q, W_x^{1,p})} \leq \frac{K}{t^{\frac{1}{2} + \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}} , \quad (1.27)$$

*Proof.* The proof can be found in [Rot84], where a more precise statement is made on the value of the constant  $K$ . We indicate the proof for the sake of completeness. Set  $\alpha = (n/2)(1/q - 1/p) \in (0, 1)$ . Recall that the operator  $(\Delta, W_x^{2,p})$  is sectorial and therefore generates an analytic semigroup  $S$  such that  $|\Delta S(t)|_{\mathcal{L}(L_x^q)} \leq c_0/t$ , for some constant  $c_0 > 0$  independent of  $t > 0$  (see e.g. [Paz83, thm. 5.2]). By Gagliardo-Nirenberg inequality we have

$$\begin{aligned} |S(t)y|_{L_x^p} &\leq c |S(t)y|_{W_x^{2,q}}^\alpha |S(t)y|_{L_x^q}^{1-\alpha} \\ &\leq c' (|\Delta S(t)y|_{L_x^q} + |S(t)y|_{L_x^q})^\alpha |S(t)y|_{L_x^q}^{1-\alpha} \\ &\leq c'' \left( \frac{|y|_{L_x^q}}{t} + |S(t)y|_{L_x^q} \right)^\alpha |S(t)y|_{L_x^q}^{1-\alpha} , \end{aligned}$$

and the inequality follows, observing that  $|S(t)|_{\mathcal{L}(L_x^p)} \leq 1$ .  $\blacksquare$

### 3 Proof of Theorem 1.1

The proof follows essentially the same approach as that of [DD02]. Fix  $p > n$ , and consider  $u_0 \in W_x^{1,p}$ , and  $\phi \in R(L_x^2; W_x^{1-n/p,p} \cap \ker B)$ . We apply Picard's fixed point Theorem on the complete space  $L^q(\Omega; \mathcal{C}_T W_x^{1,p})$ , for  $T$  depending on the maximal size of  $|v(t)|_{W_x^{1,p}}$ , and  $q \geq 1$  depending on  $p, n$ . Since the noise term cannot be estimated pathwise, we cannot operate a fixed point in  $L^2(\Omega; B_R)$ , where  $B_R$  is some ball in  $\mathcal{C}_T W_x^{1,p}$ . This naturally leads us to truncate the nonlinearities by the use of a cut-off function  $\theta : \mathbb{R}^+ \rightarrow [0, 1]$ , which has the following properties:

$$\theta \in C_0^\infty(0, 2) \ , \quad \theta(x) = 1, \text{ for all } 0 \leq x \leq 1 \ . \quad (1.28)$$

For  $R > 0$ , and  $x \in \mathbb{R}^+$ , we denote  $\theta_R(x) = \theta(\frac{x}{R})$ . We first consider the case of a modified version of (1.17) where the nonlinear term has been truncated, and prove existence and uniqueness for this equation. We therefore consider the fixed point problem  $u(t) = \psi^R(u)$ , where for a fixed  $R > 0$ , we define the map  $\psi^R$  on  $L^2(\Omega; \mathcal{C}_T W_x^{1,p})$  by the formula:

$$\begin{aligned} \psi^R(v) = & S(t)u_0 + \int_0^t S(t-s) [\theta_R(|v(t)|_{W_x^{1,p}})v(s)|\nabla v(s)|^2] ds \\ & + \int_0^t S(t-s)[F_\phi(v(s))]ds + \int_0^t S(t-s)[v(s) \times dW_\phi(s)] \ , \ t \in [0, T] \ . \end{aligned} \quad (1.29)$$

Note that we can without difficulty replace  $u_0$  by a random element  $u_0 \in L^2(\Omega; W_x^{1,p})$  which is  $\mathcal{F}_0$ -measurable.

**Treatment of the truncated equation.** We are first looking for a fixed point of the map  $\psi^R = \psi^{R, u_0, T, \phi}$  i.e. we solve in the mild sense the following Itô equation:

$$du - \Delta u dt = (\theta_R(|u|_{W_x^{1,p}})u|\nabla u|^2 + F_\phi(u))dt + u \times dW_\phi \quad (E_R)$$

with  $u|_{t=0} = u_0$ . We show that provided  $T$  is sufficiently small, depending on  $R$  and  $u_0$ , then  $\psi^R$  defines a contraction in  $L^q(\Omega; \mathcal{C}_T W_x^{1,p})$ .

**Claim 1.1.** For  $q > 2p/(p-n)$ ,  $\psi^R$  maps  $L^q(\Omega; \mathcal{C}_T W_x^{1,p})$  onto itself.

**Claim 1.2.** For  $q > 2p/(p-n)$ , if  $T_*(u_0, R) > 0$  is chosen sufficiently small, then the map

$$\psi^R : L^q(\Omega; \mathcal{C}_T W_x^{1,p}) \rightarrow L^q(\Omega; \mathcal{C}_T W_x^{1,p})$$

is a contraction.

*Proof of Claim 1.2.* We begin our proof by the following remark: if  $v \in L^q(\Omega; \mathcal{C}_T W_x^{1,p})$ , then for  $t \in [0, T]$ , on the event  $\{\sup_{0 \leq s \leq t} |v(s)|_{W_x^{1,p}} \leq R\}$ , we have:

$$\begin{aligned} \left| \int_0^t S(t-s)v(s)|\nabla v(s)|^2 ds \right|_{W_x^{1,p}} & \leq \int_0^t \frac{K}{(t-s)^{\frac{1}{2} + \frac{n}{2}(\frac{2}{p} - \frac{1}{p})}} |v|\nabla v|^2|_{L_x^{p/2}} \\ & \leq \int_0^t \frac{K}{(t-s)^{\frac{1}{2} + \frac{n}{2p}}} |v|_{L_x^\infty} |\nabla v|_{L_x^p}^2 \\ & \leq \int_0^t \frac{c(p)K}{(t-s)^{\frac{1}{2} + \frac{n}{2p}}} |v|_{W_x^{1,p}} |\nabla v|_{L_x^p}^2 \\ & \leq c'(p)R^3 T^{\frac{1}{2} - \frac{n}{2p}} \ , \end{aligned} \quad (1.30)$$

where we have used respectively Lemma 1.2, and the Sobolev embedding  $W_x^{1,p} \hookrightarrow L_x^\infty$ , valid since  $p > n$ .

We now denote by  $b(v) = v|\nabla v|^2$ , and for  $j = 1, 2$ ,  $\theta_{R,j}(s) := \theta_R(|v_j(s)|)$ . We write:

$$\begin{aligned} |\psi(v_1) - \psi(v_2)|_{W_x^{1,p}} &= |S \star (\theta_{R,1}b(v_1) - \theta_{R,2}b(v_2))|_{W_x^{1,p}} + |S \star (F_\phi(v_1) - F_\phi(v_2))|_{W_x^{1,p}} \\ &\quad + |S \diamond ((v_1 \times \cdot) - (v_2 \times \cdot))|_{W_x^{1,p}} \\ &= I + II + III , \end{aligned}$$

where  $\star$ , resp.  $\diamond$  denotes the convolution, resp. stochastic convolution. Define for  $j = 1, 2$ :

$$t_j^R = \inf \{t \in [0, T], |v_j(t)|_{W_x^{1,p}} \geq 2R\} ,$$

with the convention that  $t_j^R = T$  if the set is empty, and assume without loss of generality that  $t_1^R \leq t_2^R$ . Writing  $[0, T] = [0, t_1^R] \cup [t_1^R, t_2^R] \cup [t_2^R, T]$ , we have

$$\begin{aligned} I &\leq \int_0^{t_1^R} |\theta_{R,1}(s) - \theta_{R,2}(s)| |S(t-s)b(v_1)|_{W_x^{1,p}} ds \\ &\quad + \int_0^{t_1^R} \theta_{R,2}(s) |S(t-s)[b(v_1) - b(v_2)]|_{W_x^{1,p}} ds \\ &\quad + \int_{t_1^R}^{t_2^R} \theta_{R,2}(s) |S(t-s)b(v_2)|_{W_x^{1,p}} ds \\ &= I_1 + I_2 + I_3 . \end{aligned}$$

Using  $|\theta'_R|_{L^\infty(\mathbb{R})} \leq c/R$ , and (1.30), the first term gives:

$$\begin{aligned} I_1 &\leq \int_0^{t_1^R} (\theta_{R,1}(s) - \theta_{R,2}(s)) \frac{|b(v_1)|_{L_x^{p/2}}}{(t-s)^{1/2+n/(2p)}} ds \\ &\leq \frac{c}{R} \|v_1 - v_2\|_{C_T W_x^{1,p}} T^{1/2-n/(2p)} R^3 \quad \text{a.s.} \end{aligned}$$

Writing  $b(v_1) - b(v_2) = (v_1 - v_2)|\nabla v_1|^2 + v_2(|\nabla v_1|^2 - |\nabla v_2|^2)$ , by the same steps as that of (1.30), we have:

$$I_2 \leq cT^{1/2-n/(2p)} R^2 \|v_1 - v_2\|_{C_T W_x^{1,p}} \quad \text{a.s.} ,$$

and, since  $\theta_{R,1}(s) \equiv 0$  for  $s \in [t_1^R, t_2^R]$ , then

$$\begin{aligned} I_3 &= \int_{t_1^R}^{t_2^R} (\theta_{R,2} - \theta_{R,1}) |S(t-s)b(v_2)|_{W_x^{1,p}} ds \\ &\leq c(\theta) R^2 T^{1/2-n/(2p)} \|v_1 - v_2\|_{C_T W_x^{1,p}} \quad \text{a.s.} \end{aligned}$$

Using (1.15), the hypotheses on  $\phi$ , and denoting by  $\mathcal{B}^3$  an orthonormal basis of  $L_x^2$  the second term is estimated by

$$\begin{aligned} |F_\phi(v_1) - F_\phi(v_2)|_{L_x^{p/2}} &\leq \frac{1}{2} \sum_{e \in \mathcal{B}^3} |(v_1 - v_2) \times \phi e| \times \phi e|_{L_x^{p/2}} \\ &\leq c|v_1 - v_2|_{L_x^\infty} |\phi|_{R(L_x^2; L_x^p)}^2 . \end{aligned} \tag{1.31}$$

Therefore,

$$II \leq cT^{1/2-n/(2p)} |\phi|_{R(L_x^2; W_x^{1,p})}^2 \|v_1 - v_2\|_{\mathcal{C}_T W_x^{1,p}} \quad \text{a.s.}$$

Now, since  $p > n$ , fix a constant  $0 < \theta < 1$  such that  $p = n/\theta$ . Using successively Proposition 1.1 with  $\nu = 1 - \theta$ , and then Lemma 1.1, we obtain:

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t S(t-s)(v_1 - v_2) \times dW_\phi \right|_{W_x^{1,p}}^q \right] \\ \leq c(q) \mathbb{E} \left[ \left( \int_0^T |(v_1 - v_2) \times \phi|_{R(L_x^2; W_x^{1,p})}^2 dt \right)^{\frac{q}{2}} \right] \\ \leq c(q) \mathbb{E} \left[ \left( \int_0^T |v_1 - v_2|_{W_x^{1,p}}^2 |\phi|_{R(L_x^2; W_x^{1-n/p, p})}^2 dt \right)^{\frac{q}{2}} \right] \end{aligned}$$

provided  $q > 2/\nu$ . Thus, we have the bound:

$$III \leq cT^{\frac{q}{2}} \mathbb{E} \left[ \|v_1 - v_2\|_{\mathcal{C}_T W_x^{1,p}}^q \right] |\phi|_{R(L_x^2; W_x^{1,p})}^q .$$

Summing these inequalities, we obtain

$$\begin{aligned} \mathbb{E} \left[ \|\psi^R(v_1) - \psi^R(v_2)\|_{\mathcal{C}_T W_x^{1,p}}^q \right] \leq c \left( R, |\phi|_{R(L_x^2; W_x^{1-n/p, p})}, q \right) \max \left\{ T^{\frac{q}{2} - \frac{qn}{2p}}, T^{q/2} \right\} \\ \cdot \mathbb{E} \left[ \|v_1 - v_2\|_{\mathcal{C}_T W_x^{1,p}}^q \right] . \quad (1.32) \end{aligned}$$

Claim 1.2 is now proved, since if we take

$$T \leq \frac{1}{2} \min \left\{ c \left( R, |\phi|_{R(L^2, W_x^{1-n/p, p})} \right)^{-\frac{1}{\frac{q}{2} - \frac{qn}{2p}}}, c \left( R, |\phi|_{R(L^2, W_x^{1-n/p, p})} \right)^{-\frac{2}{q}} \right\} ,$$

then the map  $\psi^R$  is a contraction in  $L^q(\Omega; \mathcal{C}_T W_x^{1,p})$ .  $\blacksquare$

Claim 1.1 follows immediately by the same arguments as that of Claim 2, using Lemma 1.2, Lemma 1.1, and Proposition 1.1. Hence, for  $T \leq T_*(R)$ ,  $\psi^R$  is a strict contraction, and Picard's Theorem ensures the existence of a unique fixed point  $u_R \in \mathcal{C}_T W_x^{1,p}$ , which is a mild solution of  $(E_R)$ .

**Eliminating the cut-off.** For  $m \in \mathbb{N}$ , denote by  $u_m$  the unique solution of  $(E_R)$  with  $R = m$  given by the previous paragraph. We define:

$$\tau_m = \inf \{ t \in [0, T] \mid |u_m|_{W_x^{1,p}} \geq m \} . \quad (1.33)$$

**Claim 1.3.** The sequence  $(\tau_m)$  is non-decreasing and  $u_{m+1}(t) = u_m(t)$  for  $t \in [0, \tau_m \wedge \tau_{m+1}]$  a.s.

*Proof.* Fix  $m \in \mathbb{N}$ ,  $t \in [0, \tau]$ , where we define  $\tau = \tau_m \wedge \tau_{m+1}$ . Write:

$$\begin{aligned} u_{m+1}(t) - u_m(t) = \int_0^t S(t-s)(b(u_{m+1}) - b(u_m)) ds + \int_0^t S(t-s)(F_\phi(u_{m+1}) - F_\phi(u_m)) ds \\ + \int_0^t S(t-s)(u_{m+1} - u_m) \times dW_\phi . \end{aligned}$$



For  $j = m, m + 1$ , define the map  $y_j$  as the solution of the linear equation:

$$\begin{cases} dy_j - \Delta y_j dt = F_\phi(y_j) dt + y_j \times dW_\phi, & t \in [\tau, T], \\ y_j(\tau) = u_j(\tau), \end{cases}$$

where the existence and uniqueness of  $y_j$  follow by standard arguments for linear SPDES. Then, extend the solutions  $u_j$  on the whole interval  $[0, T]$  by defining for  $j = m, m + 1$ :

$$\tilde{u}_j(t) = \begin{cases} u_j(t) & \text{if } t \in [0, \tau], \\ y_j(t) & \text{if } t \in [\tau, T]. \end{cases}$$

By definition of  $\tilde{u}_m, \tilde{u}_{m+1}$  on  $[0, T]$  we have for  $t \in [0, T]$ :

$$\begin{aligned} \tilde{u}_{m+1}(t) - \tilde{u}_m(t) &= \int_0^{t \wedge \tau} S(t-s)(b(\tilde{u}_{m+1}) - b(\tilde{u}_m)) ds \\ &\quad + \int_0^t S(t-s)(F_\phi(\tilde{u}_{m+1}) - F_\phi(\tilde{u}_m)) ds \\ &\quad + \int_0^t S(t-s)(\tilde{u}_{m+1} - \tilde{u}_m) \times dW_\phi \\ &= X + Y + Z. \end{aligned}$$

Using the same arguments as that of (1.30):

$$\begin{aligned} \|X\|_{C_T W_x^{1,p}} &\leq m^3 c(p) T^{1/2-n/(2p)} \|\tilde{u}_{m+1} - \tilde{u}_m\|_{C_T W_x^{1,p}}, \quad \text{a.s.}, \\ \|Y\|_{C_T W_x^{1,p}} &\leq c(p) T^{1/2-n/(2p)} |\phi|_{R(L_x^2; L_x^p)}^2 \|\tilde{u}_{m+1} - \tilde{u}_m\|_{C_T W_x^{1,p}}, \quad \text{a.s.}, \end{aligned}$$

and

$$\mathbb{E} \left[ \|Z\|_{C_T W_x^{1,p}}^q \right] \leq c(q) T^{\frac{q}{2}} |\phi|_{R(L_x^2; W_x^{1-n/p, p})} \mathbb{E} \left[ \|\tilde{u}_{m+1} - \tilde{u}_m\|_{C_T W_x^{1,p}}^q \right],$$

provided  $q > 1/\nu$ .

We finally obtain the estimate

$$\mathbb{E} \left[ \|\tilde{u}_{m+1} - \tilde{u}_m\|_{C_T W_x^{1,p}}^q \right] \leq \tilde{m}^2 c(q) \max\{T^{\frac{q}{2}}, T^{\frac{q}{2} - \frac{qn}{2p}}\} |\phi|_{R(L_x^2; W_x^{1,p})} \mathbb{E} \left[ \|\tilde{u}_{m+1} - \tilde{u}_m\|_{C_T W_x^{1,p}}^q \right],$$

and this proves that  $\tilde{u}_{m+1} - \tilde{u}_m \equiv 0$  on  $[0, T]$  provided  $T$  is small enough. The conclusion follows by a reiteration procedure, which proves Claim 1.3.  $\blacksquare$

We define now the stopping time

$$\tau^* = \lim_{m \rightarrow \infty} \tau_m,$$

and the uniqueness part shows that we can define without ambiguity  $u$  on  $[0, \tau^*]$  with

$$u(t) = u_m(t) \text{ if } \exists m \in \mathbb{N}, t \leq \tau_m,$$

without ambiguity. Finally,  $(u, \tau^*)$  is a mild local solution of (1.17). This finishes the proof of the existence and uniqueness in  $C_T W_x^{1,p}$ .

**Further regularity.** To simplify the presentation, we show the result for  $\beta = 3$ . Other values work by interpolation.

Consider  $u_0 \in W_x^{3,p}$  and  $\phi \in R(L_x^2; W_x^{3,p})$ . If  $q > 2p/(p-n)$ , we define  $u \in L^q(\Omega; \mathcal{C}([0, T]); W_0^{1,p})$  as the process obtained by the argument above. We then apply  $(-\Delta)$  to

$$u(t) = S(t)u_0 + \int_0^t S(t-s)u|\nabla u|^2 ds + \int_0^t S(t-s)F_\phi(u) ds + \int_0^t u \times dW_\phi, \quad t \in [0, \tau],$$

and take the norm in  $W_x^{1,p}$ . Straightforward computations show that for  $u \in W_x^{3,p}$ ,

$$\Delta(u|\nabla u|_{\mathbb{R}^{3 \times n}}^2) = \Delta u|\nabla u|_{\mathbb{R}^{3 \times n}}^2 + 2 \operatorname{tr}(\nabla u)\nabla^2 u \nabla u + u(\nabla \Delta u) \cdot (\nabla u) + u|\nabla^2 u|_{\mathbb{R}^{3 \times n \times n}}^2,$$

so that using Sobolev embeddings, we obtain:  $|\Delta(u|\nabla u|^2)|_{L_x^{p/2}} \leq c|u|_{W_x^{3,p}}^3$ . Besides, there holds a.s. for  $t < \tau$ :

$$|\Delta(F_\phi(u(t)))|_{\mathcal{C}_T W_x^{1,p}} \leq c \left( |\phi|_{R(L_x^2; W_x^{3-n/p,p})} \right) |u(t)|_{W_x^{3,p}}.$$

The bound on the drift part of  $u(t)$  follows now the same steps as that of (1.30): we have

$$\begin{aligned} |S \star b(u)(t)|_{W_x^{3,p}} + |S \star F_\phi(u)(t)|_{W_x^{3,p}} &\leq c(p, |\phi|_{R(L_x^2; W_x^{2,p})}) t^{1/2-n/(2p)} \\ &\cdot \sup_{s \in [0, t]} \left( |u(s)|_{W_x^{3,p}}^3 + |u(s)|_{W_x^{3,p}} \right), \end{aligned} \quad (1.34)$$

a.s. for  $t < \tau$ .

Lastly, using again Proposition 1.1 yields:

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| (-\Delta) \int_0^{\vartheta \wedge t} S(t-s)u \times dW_\phi(s) \right|_{W_x^{1,p}}^q \right] \\ \leq \mathbb{E} \left[ \left( \int_0^T |\Delta(u(s) \times \phi)|_{R(L_x^2; W_x^{1-n/p,p})}^2 ds \right)^{q/2} \right] \\ \leq c(q) |\phi|_{R(L_x^2; W_x^{3-n/p,p})} T^{\frac{q}{2}} \mathbb{E} \left[ \|u\|_{\mathcal{C}_T W_x^{3,p}}^q \right], \end{aligned}$$

for every stopping time  $\vartheta < \tau$ .

Gathering these inequalities, we see by a standard generalization of Grönwall's Lemma that there exists a positive stopping time  $\bar{\tau} \leq \tau$  such that  $\omega, t \mapsto u(t \wedge \bar{\tau}(\omega))$  belongs pathwise to  $\mathcal{C}_T W_x^{3,p}$ .

**The pointwise constraint.** We adapt the proof of the deterministic case given in [GH93]. We treat the more general case  $\gamma = 1$ , assuming that we are given a local solution of SLLG that is supported locally in  $\mathcal{C}_T W_x^{1,p}$ .

*Remark 1.3.* Note that formally

$$du = (-u \times (u \times \Delta u) + u \times \Delta u) dt + u \times \circ dW_\phi$$

whence in some sense  $u \perp du$  and  $d(|u|^2) = 0$ . However the fact that any solution  $u \in L^q(\Omega; \mathcal{C}_T W_x^{1,p})$  satisfies (C) is not completely obvious, because the identity

$$\Delta u + u|\nabla u|^2 = -u \times (u \times \Delta u), \quad (1.35)$$

which ensures the orthogonality of the right hand side with respect to  $u$ , comes precisely from  $|u|^2 \equiv 1$ .

Define the functional

$$\psi(u) = \frac{1}{4} \| |u|^2 - 1 \|_{L^2_x}^2, \quad \text{for } u \in L^2(\mathcal{O}; \mathbb{R}^3),$$

and notice that for  $h, k \in L^2(\mathcal{O}; \mathbb{R}^3)$  we have

$$\begin{cases} D\psi(u) \cdot h = \langle u \cdot h, |u|^2 - 1 \rangle \\ D^2\psi(u) \cdot (h, k) = \langle h \cdot k, |u|^2 - 1 \rangle + 2\langle u \cdot h, u \cdot k \rangle. \end{cases}$$

Consider now, up to the stopping time  $\tau > 0$ , a regular solution  $u$  of

$$du = (\Delta u + |\nabla u|^2 + u \times \Delta u + F_\phi(u)) dt + u \times dW_\phi,$$

$$\text{with: } \begin{cases} \text{either } u = u_0 \text{ on } \{0\} \times \mathcal{O} \cup [0, \tau) \times \partial\mathcal{O}, \\ \text{either } u = u_0 \text{ on } \{0\} \times \mathcal{O} \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \cup [0, \tau) \times \partial\mathcal{O}. \end{cases}$$

By ‘‘regular’’ we mean that  $u$  is supported in  $\mathcal{C}([0, \tau); W^{2,p})$ , with  $p > n$ .

On the one hand,  $\int_0^t \langle u \cdot (u \times dW_\phi), |u|^2 - 1 \rangle = 0$  a.s. for every  $t \in [0, \tau)$ , whereas on the other hand the terms  $\int_0^t \langle u \cdot F_\phi(u), |u|^2 - 1 \rangle dt$  and  $-\sum_{e \in \mathcal{B}^3} \int_0^t \{ \langle \phi e \cdot \phi e, |u|^2 - 1 \rangle + 2\langle (u \cdot \phi e), (u \cdot \phi e) \rangle \} dt$  are equal for each ONB  $\mathcal{B}^3$  of  $L^2(\mathcal{O}; \mathbb{R}^3)$ . Noticing furthermore that  $\Delta(|u|^2/2) = u \cdot \Delta u + |\nabla u|^2$ , Itô Formula now yields:

$$\begin{aligned} \psi(u(t)) - \psi(u_0) &= \int_0^t \langle u \cdot (\Delta u + u|\nabla u|^2 + u \times \Delta u), |u|^2 - 1 \rangle dt \\ &= \int_0^t \int_{\mathcal{O}} \left( \frac{\Delta(|u|^2 - 1)}{2} + (|u|^2 - 1)|\nabla u|^2 \right) (|u|^2 - 1) dx dt. \end{aligned}$$

Integrating by parts, there holds for a.e.  $t \in [0, \tau)$ :

$$\int_{\mathcal{O}} \frac{\Delta(|u|^2 - 1)}{2} (|u|^2 - 1) dx = -\frac{1}{2} \int_{\mathcal{O}} |\nabla(|u|^2 - 1)|^2 dx + \int_{\partial\mathcal{O}} \partial u / \partial n \cdot u (|u|^2 - 1) dx,$$

and the boundary term equals 0, no matter if we choose Dirichlet or Neumann homogeneous boundary conditions. This implies finally  $\psi(u(t)) \leq \sup_{s \in [0, t]} |\nabla u(s)|_{L_x^\infty} \int_0^t \psi(s) ds$  for every  $t < \tau$ , which ensures by Grönwall’s Lemma, and the Sobolev embedding  $W_x^{2,p} \hookrightarrow W_x^{1,\infty}$ , that  $\psi(u) \equiv 0$  on  $[0, \tau)$ .

By an argument of density, the assumption that  $u \in \mathcal{C}([0, \tau); W_x^{2,p})$  can be removed. Indeed, take any  $u(0) \in W_x^{1,p}$  fulfilling the local constraint. There exists a sequence  $u_n(0) \rightarrow u_0$  for the  $W^{1,p}$  topology, with  $u_{0,n} \in W^{2,p}$  for each  $n$ . and the corresponding solutions  $u_n$  satisfy (C) locally in time. But the energy inequality is fulfilled for each  $n$ — see the Appendix of Chapter 2 below — so that  $\mathbb{E} |\nabla u_n(t \wedge \tau')|_{L_x^2}^2 \leq c$ , with a constant independent of  $n$  and the stopping time  $\tau' < \tau$ , provided each solution  $u_n$  is associated to a covariance operator  $\phi \in \mathbb{L}_2(L_x^2; H_x^1)$ . We can now assume the convergence, up to a subsequence, of  $u_n(t \wedge \tau') \rightarrow u(t \wedge \tau')$  weakly in  $L^2(\Omega \times \mathcal{O})$ , and a.e., so that  $|u|^2 = 1$  a.e. Theorem 1.1 is now proved.  $\blacksquare$

## CHAPTER 2.

# *Global solutions with finite energy*

---

In this chapter, we give a new result on the global well-posedness of (SLLG) on the two dimensional torus, for a space of trajectories that satisfy a rather “natural” energy inequality. This generalizes a result of M. Struwe for the deterministic heat flow of harmonic maps from surfaces  $M$  with  $\partial M = \emptyset$ . Notice that the result has been adapted for the deterministic (LLG) in dimension two by B. Guo and M-C. Hong, and also for the boundary value problem by Chang K-C.

Using a compactness method that relies on the energy inequality, the proof gives essentially a unique possible limit point. A general argument first developed by I. Gyöngy and N. Krylov implies that the solution obtained is strong in the probabilistic sense. This is somehow related to a famous result of T. Yamada and S. Watanabe giving the existence of strong solution by the fact that existence in law and pathwise uniqueness hold. As a consequence of our result, we show, as in the deterministic case, that singularities can occur in the form of a “bubbling phenomena” only, meaning that the energy density “concentrates” at some finite set of points  $(t_l, x_l)$ . We explain then how the solution can restart from a singularity, loosing a “quantum” of energy  $k\epsilon_1 > 0, k \geq 1$ , and we show that the solution obtained by this method is global in time.

---

# 1 Introduction

## (1.a) Motivations

The Landau-Lifshitz-Gilbert equation and the harmonic map flow have specific aspects in two dimensional spatial domains, the main reason being that the “natural energy space” namely  $H^1(\mathcal{O})$  is “almost” a functional space on which the Cauchy problem is well-posed (see also the discussion made in the book [Hél96] for the stationary problem). In that direction, we saw in chapter 1 that for  $p > 2 = \dim \mathcal{O}$ , then the stochastic harmonic map flow is well-posed for functions with  $p$ -summable derivatives and under some conditions on  $\phi$ , the squareroot of the spatial correlation operator of the noise.

Consider the following version of (SLLG), where the space domain is the two-dimensional torus  $\mathbb{T}^2$ :

$$\begin{cases} du = (\Delta u + u|\nabla u|^2 + u \times \Delta u + F_\phi(u))dt + u \times dW_\phi , & \text{on } \Omega \times [0, T] \times \mathbb{T}^2 \\ u|_{t=0} = u_0 , & \text{on } \Omega \times \mathbb{T}^2 , \end{cases} \quad (2.1)$$

with the same notations as before, see (1.15)-(1.16). For the case of space-time white noise, namely when  $\phi = \text{id}_{L_x^2}$ , then the very notion of solution for (2.1) seems unclear, despite the fact that recent theoretical work has been made to treat such equations in [Hai14]. Although we shall not use these results in this manuscript, it is worth to be mentioned that here the assumption of *local subcriticality* – which is necessary to apply the theory of regularity structures – is not fulfilled. Indeed, the equation (2.1) has the somehow remarkable property that it remains invariant as we operate the parabolic rescaling

$$\theta_\lambda : (t, x) \mapsto \left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right) , \quad (2.2)$$

for  $\lambda > 0$ . For a Gaussian space-time white noise  $\zeta$  in a two-dimensional domain of space, if  $\lambda > 0$ , if  $(\tau, y)$  are the new variables  $(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ , and if  $v = u \circ \theta_\lambda^{-1}$ , then the equation

$$\partial_t u = \Delta_x u + u|\nabla_x u|^2 + u \times \Delta_x u + u \times \zeta ,$$

reads

$$\partial_\tau v = \Delta_y v + v|\nabla_y v|^2 + v \times \Delta v + v \times \eta ,$$

where  $\eta := \lambda^2 \zeta \circ \theta_\lambda^{-1}$  remains a Gaussian space-time white noise, due to self-similarity properties of  $\zeta$  in two-dimensional space domains.

On the other hand, the time increments of the functional

$$E(t) = \frac{1}{2} \int_{\mathbb{T}^2} |\nabla u(t, x)|^2 dx , \quad (2.3)$$

which is the “natural energy” associated to (2.1), contain a term which becomes infinite as the correlation approaches the delta function—see (2.A.2). The numerical experiments of Chapter 6 seem to confirm that  $E$  does not remain uniformly bounded as the size of the space discretization tends to 0. For these reasons, we are led to consider a more regular noise in space, namely with a correlation operator  $Q = \phi\phi^*$  of trace class, at least from  $L_x^2$  into  $H_x^1$ . We saw in Chapter 1 that in the overdamped case, i.e. for the stochastic harmonic map flow, the equation is well

posed on the torus, under the assumptions that (i)  $u_0$  belongs to  $W_x^{1,p}$  for some  $p > 2$ ; (ii) the operator  $\phi : L_x^2 \rightarrow W_x^{1,p}$  is  $\gamma$  radonifying. However, at a formal level, the increments of the energy remain bounded if:

1.  $u_0 \in H_x^1 \cap \{u : u(x) \in \mathbb{S}^2 \text{ a.e.}\}$ ;
2.  $\phi \in \mathbb{L}_2(L_x^2; H_x^1)$ .

With these weakened assumptions, it turns out that strong solutions, in a sense precised below, exist globally. This is the main result of that chapter.

Several authors have noticed (see e.g. [GH93]) the existing parallel between the Landau-Lifshitz-Gilbert equation, and the so-called *Harmonic Maps Flow*, namely

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = u|\nabla u|^2, & t \in \mathbb{R}_+, \\ u|_{t=0} = u_0, \end{cases} \quad (2.4)$$

which is obtained by neglecting gyromagnetic effects and taking into account the exchange energy only. This equation was first studied by J. Eells Jr. and J.H. Sampson in their seminal paper [ES64]. Their aim was to construct a harmonic map between two manifolds  $u : M \rightarrow N$ , lying in the homotopy class of a given map  $u_0$ . By using the heat flow (2.4) they managed to find such  $u$ , but under the restriction that the sectional curvature of  $N$  is nonpositive. Later, M. Struwe improved their result, showing in particular the following result of existence and uniqueness.

**Theorem 2.1** ([Str85]). *Let  $N$  be a manifold, and  $M$  be a Riemannian surface without boundary. For any initial value  $u_0 \in H^1(M, N)$ , there exists a unique solution  $u$  of (2.4) on  $M \times [0, \infty)$  which belongs locally to the spaces*

$$V^T = \left\{ u : (0, T] \times M \rightarrow N, \right. \\ \left. \text{ess sup}_{0 \leq t \leq T} \int_M |\nabla u(t, x)|^2 dM + \int_0^T \int_M |\nabla^2 u|^2 + |\partial_t u|^2 dM dt < \infty \right\}. \quad (2.5)$$

*The solution is regular with the exception of at most finitely many points  $(t_l, x_l)$ ,  $1 \leq l \leq L$ , characterized by the condition*

$$\limsup_{t \rightarrow t_l^-} \int_{B(x_l, r)} |\nabla u(t, x)|^2 dM \geq \epsilon_1, \quad \forall r \in (0, r_0],$$

*for a certain parameter  $\epsilon_1 = \epsilon_1(M)$ .*

Later the boundary value problem for (2.4) was treated by K-C. Chang [KC89], and led to similar conclusions on the characterization of the blow-up. A similar theorem has been shown in [GH93], concerning this time the (deterministic) LLG equation, essentially by applying the same reasoning as that of M. Struwe.

In this work we construct global solutions of (2.1), using the machinery of Stochastic Integration of Hilbert space valued processes (see e.g. [DZ08]). Adapting the arguments used in [Str85] is not obvious, and we are led to use new ones, e.g. to control energy locally, see Proposition 2.3.

*Additional Notation.* We will repeatedly use the abbreviation

$$B_r(x) = \{y \in \mathbb{T}^2, |y - x| \leq r\} ,$$

Whenever  $x \in \mathbb{T}^2$ ,  $r > 0$ , and  $f : \mathbb{T}^2 \rightarrow \mathbb{R}$  (or  $\mathbb{R}^d$ ), we denote by

$$\|f\|_{L^2(B_r(x))} = \left( \int_{B_r(x)} |f(y)|^2 dy \right)^{\frac{1}{2}}$$

If  $\tilde{\phi}$  is a linear operator on  $L^2(\mathcal{O}; \mathbb{R})$ , we shall denote by

$$I_3 \otimes \tilde{\phi} \tag{2.6}$$

the operator acting on  $L^2(\mathcal{O}; \mathbb{R}^3)$  by the formula:

$$I_3 \otimes \phi(f_1, f_2, f_3) = (\tilde{\phi}f_1, \tilde{\phi}f_2, \tilde{\phi}f_3). \tag{2.7}$$

When  $0 \leq \tau_1 \leq \tau_2 \leq T$  denote two stopping times, define

$$L^2_{\text{loc}}([\tau_1, \tau_2]; H_x^2) := \left\{ f : [\tau_1, \tau_2] \rightarrow H_x^2, \exists \zeta^k \uparrow \tau_2, \int_{[\tau_1, \zeta^k]} |f(s)|_{H_x^2}^2 ds < \infty, \forall k \geq 1 \right\} . \tag{2.8}$$

## (1.b) Notion of solution and main Theorem

In this chapter, we fix a Stochastic basis  $\mathfrak{F} = (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), \hat{W})$ , see (0.1), and we assume that  $\hat{W}$  is a  $(\mathcal{F}_t)$  cylindrical Wiener process formally given, when  $\mathcal{B}^3$  denotes an ONB of  $L^2(\mathbb{T}^2; \mathbb{R}^3)$  and  $(\beta_e)_{e \in \mathcal{B}^3}$  a family of independent real standard brownian motions, by the relation

$$\hat{W}(t) = \sum_{e \in \mathcal{B}^3} \beta_e(t) e , \tag{2.9}$$

As in Chapter 1, when  $\phi : L_x^2 \rightarrow L_x^2$  is a Hilbert-Schmidt operator,  $W_\phi$  denotes the  $\phi\phi^*$ -Wiener process

$$W_\phi = \phi \hat{W} , \tag{2.10}$$

which is an abbreviation for  $\sum_{e \in \mathcal{B}^3} \beta_e(t) \phi e$ . We consider the following notion of a local solution for (2.1).

**Definition 2.1.** Assume that a stochastic basis  $\mathfrak{F}$  is given,  $\tau_1, \tau_2$  are stopping times, and  $u_1 \in L^2(\Omega; H_x^1)$ . Given a progressively measurable process  $u : \Omega \times [0, T] \rightarrow H_x^1$ , we say that  $(u, \tau_1, \tau_2)$ , or simply  $(u, \tau_2)$  if  $\tau_1 = 0$ , is a *local strong solution* of (2.1) on  $[\tau_1, \tau_2]$ , with initial data  $u_1$  if the following conditions are fulfilled:

- (i)  $\tau_1 < \tau_2$ ,  $\mathbb{P}$  - a.s.
- (ii) the process  $u$  has paths in  $\mathcal{C}([\tau_1, \tau_2]; H_x^1)$  and  $\int_{\tau_1}^{\tau_2} |u \times \Delta u + \Delta u + u|\nabla u|^2 + F_\phi(u)|_{L_x^2} dt < \infty$ ,  $\mathbb{P}$  - a.s.;

(iii)  $\mathbb{P}$  – a.s. , for  $t \in [\tau_1, \tau_2)$  :

$$u(t) - u_1 = \int_{\tau_1}^t (u \times \Delta u + \Delta u + u|\nabla u|^2 + F_\phi(u))dt + \int_{\tau_1}^t u \times dW_\phi ,$$

in the sense of Bochner, respectively Itô integral in  $L_x^2$ ;

(iv) for  $\mathbb{P} \otimes dt \otimes dx$  a.e.  $(\omega, t, x)$  with  $\tau_1(\omega) \leq t < \tau_2(\omega)$ , there holds

$$|u(\omega, t, x)|_{\mathbb{R}^3} = 1 . \quad (2.11)$$

Our main result reads as follows.

**Theorem 2.2.** *Let  $W_\phi$  be as in (2.9)-(2.10), with  $\phi = I_3 \otimes \tilde{\phi} \in \mathbb{L}_2^{0,1}$ , see (2.6). For all  $T > 0$ , and  $u_0 \in H^1(\mathbb{T}^2; \mathbb{R}^3) \cap \{v : v(x) \in \mathbb{S}^2 \text{ a.e.}\}$ , there exists a solution  $u \in L^2(\Omega \times [0, T]; H^1(\mathbb{T}^2; \mathbb{R}^3))$  to (2.1) which is global in the following sense: there exists an increasing sequence of stopping times  $\vartheta^0 = 0 < \vartheta^1 < \vartheta^2 < \dots < \vartheta^{J(\omega)}$ , such that*

$$\mathbb{P}(\vartheta^J = T) = 1 ,$$

and for each  $j \in \llbracket 0, J-1 \rrbracket$ ,  $(u|_{[\vartheta^j, \vartheta^{j+1}]}, \vartheta^j, \vartheta^{j+1})$  is a local strong solution of (2.1) (in the sense of Definition 2.1), with the initial data  $U^j$  at time  $\vartheta^j$ , uniquely determined, when  $1 \leq j \leq J$ , by

$$u(\zeta_k) \rightharpoonup U^j , \text{ weakly in } H^1(\mathbb{T}^2; \mathbb{R}^3) , \text{ d}\mathbb{P} - \text{a.s.}$$

for all sequence of stopping times  $\zeta_k \uparrow \vartheta^j$ .

Moreover, for all  $j \geq 1$ , we have with full probability:

$$u|_{[\vartheta^j, \vartheta^{j+1}]} \in \mathcal{C}([\vartheta^j, \vartheta^{j+1}]; H_x^1) \cap L_{\text{loc}}^2([\vartheta^j, \vartheta^{j+1}]; H_x^2) ,$$

see definition (2.8), and the stopping times  $\vartheta^j$  are characterized by

$$\mathbb{P}\left(\forall k \in \mathbb{N}^*, \exists(\zeta_k, x) \in [\vartheta^j, \vartheta^{j+1}] \times \mathbb{T}^2, |\nabla u(\zeta_k)|_{L^2(B_{1/k}(x))}^2 \geq \epsilon_1\right) = 1 ,$$

where  $\epsilon_1 > 0$  is a parameter that does not depend on  $u_0, \phi, T$ .

Besides, uniqueness holds in the following sense: if  $(v, \sigma)$  denotes a local strong solution with  $v(0) = u(0)$ , such that for all  $\epsilon > 0$ ,

$$\text{d}\mathbb{P} - \text{a.s.}, \exists \zeta > 0, \sup_{(t,x) \in [0, \zeta] \times \mathbb{T}^2} |\nabla v(t)|_{L^2(B_R(x))}^2 \leq \epsilon ,$$

then  $u|_{[0, \sigma]} = v$ .

**Outline of the proof.** Consider an initial data  $u_0 \in H_x^1 \cap \{v : v(x) \in \mathbb{S}^2 \text{ a.e.}\}$ , together with a correlation  $\phi \in \mathbb{L}_2^{0,1}$  for the Wiener process  $W_\phi$ . Our strategy is to approximate the couple  $(u_0, \phi)$  by a sequence of “more regular ones”, namely  $(v_n(0), \phi_n) \in H_x^2 \times \mathbb{L}_2^{0,2}$ , and to derive some uniform estimates on the corresponding strong solutions  $(v_n, \tau_n)$ , where  $\tau_n$  denotes the maximal existence time in  $H_x^2$ .

To establish energy estimates on the associated sequence  $v_n$  of solutions, we follow the deterministic tools used by Struwe for the harmonic maps flow, together with probabilistic



techniques. It was shown in [Str85] that uniform bounds in the space  $V^T$  – see (2.5) – can be obtained, provided the  $H_x^1$  norm does not “concentrate too much”, i.e. under the condition that

$$\inf_{r>0} \sup_{(t,x) \in [0,T] \times \mathbb{T}^2} |\nabla v_n(t)|_{L^2(B_r(x))}^2 = 0 . \quad (2.12)$$

The main differences with the deterministic case can be summarized as follows.

- Since the existence time of  $v_n$  depends on  $\omega \in \Omega$  it is natural to define stopping times  $\zeta_{n,k}(\omega)$  corresponding to the case when a certain threshold of energy over a ball  $B_{1/k}(x)$  is attained by  $v_n$ . This will permit to fix a common interval  $[0, T]$  of existence for the “stopped” solutions  $u_{n,k}(t)$  defined below.
- Compactness methods for SPDEs obtained by uniform bounds in spaces  $L^p(\Omega)$  give in general the convergence of the laws  $\mathcal{L}(v_n)$ , which is not easy to handle. Since we need to take the limit in the equation satisfied by  $v_n$ , we will first argue on the so-called Skorohod space to have better convergence properties.

The first part (2.a) is devoted to the “Main estimates” which are essentially obtained as consequences of the energy formula (2.A.2) and Remark 2.2. The second step is the so-called “tightness argument”, which is summarized in the part (2.b). It basically describes a way to circumvent the fact that energy bounds imply convergence in law only. This is a technical tool which is also used in Chapter 5, as well as in [BGJ13, BBNP13a]. Once this work is done, we show that the limit obtained is “essentially unique”, implying, by a basic result observed by I. Gyöngy and N. Krylov (see also [YW+71] for a celebrated result in the same flavour), that the equation on  $v_n$  passes to the limit  $n \rightarrow \infty$ . This gives a local solution  $u$  to our problem. To show that  $u$  necessarily reaches the time  $T > 0$  “after a finite number of bubbling(s)”, we use the fact that any concentration of energy at a point  $(t_0, x_0)$  is accompanied with a constant loss of energy  $\epsilon_1 > 0$ , so that roughly speaking, it cannot happen too rapidly (unless the energy  $E$  would become negative).

We recall now a particular case of Burkholder-Davies-Gundy inequality that will be used below.

**Proposition 2.1 (BDG).** *Let  $W$  denote a  $\phi\phi^*$ -Wiener process in  $L_x^2$ , where  $\phi \in \mathbb{L}_2^{0,1}$ . Let  $\xi : \Omega \times [0, T] \rightarrow \mathbb{L}_2(H_x^1; L_x^2)$ ,  $Z : \Omega \times [0, T] \rightarrow H_x^1$  be two progressively measurable processes, such that*

$$\mathbb{P} \left( \int_0^T |(\xi\phi)^* Z(s)|_{L_x^2}^2 dt < \infty \right) = 1 .$$

*Then, the process defined for  $t \in [0, T]$  by*

$$X(t) = \int_0^t \langle Z(s), \xi(s) dW \rangle ,$$

*has a continuous modification, and*

$$\mathbb{E} \left[ \sup_{t \in [0, T]} X(t) \right] \leq C_0 \mathbb{E} \left[ \int_0^T |(\xi\phi)^* Z(s)|_{L_x^2}^2 ds \right] ds$$

*where the constant  $C_0$  does not depend on the individual elements  $Z, \xi$ .*

*Proof.* Notice that the quadratic variation process of  $X$  writes

$$\ll X \gg_t = \int_0^t |(\xi(s)\phi)^* Z(s)|_{L_x^2}^2 ds, \quad t \in [0, T]. \quad (2.13)$$

Applying Theorems 3.15 and 4.27 in [DZ08] to the Hilbert space  $H = L_x^2$ , the conclusion follows.  $\blacksquare$

*Remark 2.1.* In order to prove Theorem 2.2, we need to consider, for each  $n \geq 0$ , a sequence of mild solutions  $v_n$  to (2.1) associated with an  $H_x^2$ -valued Wiener process, and with support in the space  $\mathcal{C}([0, T]; H_x^2)$ , see (A4) below. Although the existence of such a sequence is a consequence of Theorem 1.1 in the overdamped case  $\gamma = 0$  (with slight modifications), we still ignore if such solutions exist for the case of SLLG. We however admit the local existence of these solutions. If it turns out that this statement is false, Theorem 2.2 still holds true for the stochastic flow of harmonic maps on  $\mathbb{T}^2$ , with obvious changes in the statement.

## 2 Proof of Theorem 2.2

### (2.a) Main estimates

We consider a sequence of  $\phi_n \phi_n^*$  Wiener processes in  $L_x^2$  formally defined by

$$W_n := \phi_n \hat{W}, \quad \text{namely } \sum_{e \in \mathcal{B}^3} \beta_e(\cdot) \phi_n e,$$

for  $\mathcal{B}^3$  ONB of  $L_x^2$ , where  $\phi_n \in \mathbb{L}_2(L_x^2; H_x^2)$ . Then, we make the following assumptions:

$$\forall n \geq 0, \quad v_n \in \mathcal{C}([0, \tau_n]; H_x^2) \text{ a.s.}, \quad \text{and } v_n \text{ fulfills (2.11);} \quad (\text{A1})$$

$$v_n(0) \rightarrow u_0 \text{ in } H_x^1, \quad \frac{1}{2} |\nabla v_n|_{L_x^2}^2 \leq cE_0 \equiv \frac{c}{2} |u_0|_{L_x^2}^2, \quad (\text{A2})$$

$$\phi_n \rightarrow \phi \text{ in } \mathbb{L}_2(L_x^2; H_x^1). \quad (\text{A3})$$

$$\left[ \begin{array}{l} \text{The processes } (v_n, \tau_n) \text{ are local strong solutions for the} \\ \text{equation } (E_n): \\ dv_n = (\Delta v_n + v_n \times \Delta v_n + v_n |\nabla v_n|^2 + F_{\phi_n}(v_n)) dt + v_n \times dW_n. \end{array} \right. \quad (\text{A4})$$

Moreover, in order to simplify the presentation, we shall only consider ‘‘isotropic correlations’’, namely such that there exist  $\tilde{\phi}_n, \tilde{\phi} : L^2(\mathbb{T}^2; \mathbb{R}) \rightarrow L^2(\mathbb{T}^2; \mathbb{R})$ , with

$$\phi_n = I_3 \otimes \tilde{\phi}_n, \quad \phi = I_3 \otimes \tilde{\phi}, \quad (\text{A5})$$

see notation (2.6).

Let  $\epsilon_1 > 0$ , and define the following stopping times for  $n \in \mathbb{N}$ :  $\zeta_{n,0} = 0$  and for  $k \geq 1$ ,

$$\zeta_{n,k} = \inf \left\{ 0 \leq t < \tau_n \wedge T, \quad \sup_{x \in \mathbb{T}^2} |\nabla v_n(t)|_{L^2(B_{1/k}(x))} \geq \epsilon_1 \right\}. \quad (2.14)$$

For now, we do not fix the parameter  $\epsilon_1$ , although in the proof of the next proposition, we shall see that it has an ‘‘optimal’’ value that only depends on geometrical quantities through inequality (2.21). In particular, it is independent of  $T, n, k$ , and the sample  $\omega \in \Omega$ .

**Proposition 2.2.** Denote by  $u_{n,k}$ ,  $k \in \mathbb{N}$ , the “stopped” process:

$$u_{n,k}(t) = \begin{cases} v_n(t) & \text{if } 0 \leq t \leq \zeta_{n,k} , \\ e^{-(t-\zeta_{n,k})(\Delta)^2} v_n(\zeta_{n,k}) & \text{if } \zeta_{n,k} < t \leq T . \end{cases} \quad (2.15)$$

(i) For all  $n \in \mathbb{N}$ ,  $t \in [0, T]$ :

$$\mathbb{E} \left[ \frac{|\nabla u_{n,k}(t)|_{L_x^2}^2}{2} - \frac{|\nabla u_{n,k}(0)|_{L_x^2}^2}{2} + \int_0^{\zeta_{n,k}} |u_{n,k}(s) \times \Delta u_{n,k}(s)|_{L_x^2}^2 ds \right] = |\nabla \tilde{\phi}_n|_{\mathbb{L}_2}^2 \mathbb{E}[\zeta_{n,k}] . \quad (2.16)$$

(ii) For all  $\rho \in [1, \infty)$ ,  $n \in \mathbb{N}$ :

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\nabla u_{n,k}(t)|_{L_x^2}^{2\rho} \right] \leq c(\rho, E_0, T, \phi) , \quad (2.17)$$

$$\mathbb{E} \left[ \left( \int_0^T |u_{n,k} \times \Delta u_{n,k}|_{L_x^2}^2 dt \right)^\rho \right] \leq c(\rho, E_0, T, \phi) , \quad (2.18)$$

where  $c(\rho, E_0, T, \phi) > 0$  does not depend on  $n \in \mathbb{N}$ , neither on  $k \in \mathbb{N}$ .

(iii) There exists a sufficiently small  $\epsilon_1 > 0$  in the definition (2.14) of  $\zeta_{n,k}$ , such that for any  $k \in \mathbb{N}$ ,  $n \in \mathbb{N}$ , and  $\rho \in [1, \infty)$ :

$$\mathbb{E} \left[ \left( \int_0^T |\Delta u_{n,k}|_{L_x^2}^2 dt \right)^\rho \right] \leq c(k, \rho, E_0, T, \phi) , \quad (2.19)$$

$$\mathbb{E} \left[ \left( \int_0^T |\nabla u_{n,k}|_{L_x^4}^4 dt \right)^\rho \right] \leq c(k, \rho, E_0, T, \phi) , \quad (2.20)$$

where the constant  $c(k, \rho, E_0, T, \phi) > 0$  does not depend on  $n \in \mathbb{N}$ .

*Remark 2.2.* In the next paragraphs we shall use the following inequality whose proof can be found in [Str85]: for any  $T > 0$ , there exists a constant  $c_0 > 0$ , such that for all  $v \in \mathcal{C}_T H_x^2$ , for all  $r > 0$ :

$$\int_0^T |\nabla v|_{L_x^4}^4 dt \leq c_0 \sup_{(t,x) \in [0,T] \times \mathbb{T}^2} |\nabla v(t)|_{L^2(B_r(x))}^2 \left( \int_0^T |\nabla^2 v|_{L_x^2}^2 dt + r^{-2} \int_0^T |\nabla v|_{L_x^2}^2 dt \right) . \quad (2.21)$$

Therefore, the definition (2.15) implies

$$\int_0^T |\nabla u_{n,k}|_{L_x^4}^4 dt \leq c_1 \epsilon_1 \left( \int_0^T |\Delta u_{n,k}|_{L_x^2}^2 dt + k^2 \int_0^T |\nabla u_{n,k}|_{L_x^2}^2 dt \right) , \quad (2.22)$$

for all  $k, n \in \mathbb{N}$ , where  $c_1 = c_0 \times c$ ,  $c = c(\mathbb{T}^2)$  being chosen so that  $|\nabla^2 u|_{L_x^2}^2 \leq c |\Delta u|_{L_x^2}^2$ ,  $\forall u \in H_x^2$ .

*Proof of Proposition 2.2.*

*Proof of (i)* – First note that  $v_n$  and  $u_{n,k}$  coincide on  $[0, \zeta_{n,k}]$ , therefore by (A1)-(A4),  $(u_{n,k}, \zeta_{n,k})$  is a also strong solution of the problem  $(E_n)$ , and its trajectories belong a.s. to  $\mathcal{C}_T H_x^2$ , we have the energy formula (2.A.2), namely

$$\frac{1}{2} |\nabla u_{n,k}(t)|_{L_x^2}^2 - \frac{1}{2} |\nabla u_{n,k}(0)|_{L_x^2}^2 + \int_0^t |u_{n,k} \times \Delta u_{n,k}|_{L_x^2}^2 ds = t |\nabla \phi_n|_{\mathbb{L}^2}^2 + X_{n,k}(t) , \quad (2.23)$$

a.s. for all  $t \in [0, \zeta_{n,k})$  where  $X_{n,k}$  denotes the martingale  $\int_0^t \langle \nabla u_{n,k}, u_{n,k} \times \nabla dW_n \rangle$ . Estimate (2.16) now follows by taking the expectation.

*Proof of (ii)* – Notice that in (2.23) the martingale term writes

$$X_{n,k}(t) = \int_0^t \langle Z(s), \xi(s) dW_\phi \rangle ,$$

where the processes  $Z(s) := \nabla u_{n,k}(s)$  and  $\xi(s) := u_{n,k}(s) \times \nabla \cdot$  satisfy the hypotheses of Proposition 2.1, so that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} X_{n,k} \right] \leq C_0 \int_0^T |\phi_n^* \operatorname{div}(u_{n,k} \times \nabla u_{n,k})|_{L_x^2}^2 ds . \quad (2.24)$$

Since for each  $n$ ,  $\phi_n : L_x^2 \rightarrow H_x^1$  is a Hilbert Schmidt operator, then it is also continuous, and we have the diagram:

$$\begin{array}{ccc} L_x^2 & \xrightarrow{\phi_n} & H_x^1 \xrightarrow{\nabla} L_x^2 , \\ L_x^2 & \xleftarrow{\phi_n^*} & H_x^{-1} \xleftarrow{\operatorname{div}} L_x^2 , \end{array} \quad (2.25)$$

each arrow meaning that the map is continuous. Therefore, using (2.24) with (2.11) gives

$$\mathbb{E} \left[ \sup_{t \in [0, T]} X_{n,k} \right] \leq c(\phi_n) \mathbb{E} \int_0^T |\nabla u_{n,k}|_{L_x^2}^2 ds . \quad (2.26)$$

By (A3), and the classical inequality  $|T|_{\mathcal{L}(H;K)} \leq |T|_{\mathbb{L}_2(H;K)}$ , we have

$$|(\phi_n - \phi)^* \circ \operatorname{div}|_{\mathcal{L}(L_x^2; L_x^2)} \rightarrow 0 ,$$

so that the constant  $c(\phi_n)$  in (2.26) is uniform in  $n$  and depends on  $|\phi|_{\mathbb{L}_2^{0,1}}$  only.

Taking the power  $\rho \in [1, \infty)$  in (2.23), the supremum in time, the expectation, and using (2.26), we obtain the bound:

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\nabla u_{n,k}(s)|_{L_x^2}^{2\rho} \right] \leq c \left( E_0, \rho, |\phi|_{\mathbb{L}_2^{0,1}}, T \right) \int_0^t \mathbb{E} \left[ \sup_{0 \leq \sigma \leq s} |\nabla u_{n,k}(\sigma)|_{L_x^2}^{2\rho} \right] ds , \quad t \in [0, T] .$$

And (2.17) is obtained by applying Grönwall Lemma to the map  $t \mapsto \mathbb{E}[\sup_{0 \leq s \leq t} |\nabla u(s)|^{2\rho}]$ .

The bound (2.18) follows, using again (2.23), (2.26), and (2.17).

*Proof of (iii)* – Notice that by (A1), we have pointwisely:

$$\begin{aligned} |u_{n,k} \times \Delta u_{n,k}|_{\mathbb{R}^3} &= | - u_{n,k} \times (u_{n,k} \times \Delta u_{n,k}) |_{\mathbb{R}^3} \\ &= |P_{u_{n,k}}^\perp(\Delta u_{n,k})|_{\mathbb{R}^3} \\ &= |\Delta u_{n,k} + u_{n,k} |\nabla u_{n,k}|^2|_{\mathbb{R}^3} , \end{aligned}$$

where for  $\vec{u} \in \mathbb{S}^2$ ,  $P_{\vec{u}^\perp}$  denotes the pointwise orthogonal projection on  $(\text{Vect}\{\vec{u}\})^\perp$  in  $\mathbb{R}^3$ . Hence, for  $t \in [0, \zeta_{n,k}]$  we have a.s.

$$\begin{aligned} \int_0^t |u_{n,k} \times \Delta u_{n,k}|_{L_x^2}^2 ds &= \int_0^t \langle P_{u_{n,k}^\perp}(\Delta u_{n,k}), P_{u_{n,k}^\perp}(\Delta u_{n,k}) \rangle ds \\ &= \int_0^t \langle \Delta u_{n,k}, \Delta u_{n,k} + u_{n,k} |\nabla u_{n,k}|^2 \rangle ds, \end{aligned}$$

so that expanding this term in (2.23) gives a.s. for  $t \in [0, \zeta_{n,k}]$ :

$$\begin{aligned} \frac{1}{2} |\nabla u_{n,k}(t)|_{L_x^2}^2 - \frac{1}{2} |\nabla u_{n,k}(0)|_{L_x^2}^2 + \int_0^t |\Delta u_{n,k}|_{L_x^2}^2 ds - X_{n,k}(t) - t |\nabla \phi_n|_{\mathbb{L}_2}^2 \\ \leq \int_0^t \langle -\Delta u_{n,k}, u_{n,k} |\nabla u_{n,k}|^2 \rangle ds \\ \leq \frac{1}{2} \int_0^t |\Delta u_{n,k} \cdot u_{n,k}|_{L_x^2}^2 ds + \frac{1}{2} \int_0^t |\nabla u_{n,k}|_{L_x^4}^4 ds. \end{aligned}$$

Applying then inequality (2.22) to  $\int_0^t |\nabla u_{n,k}|_{L_x^4}^4 ds$ , and using (2.11), we obtain the bound:

$$\begin{aligned} \frac{1}{2} \int_0^t |\Delta u_{n,k}|_{L_x^2}^2 ds \leq \frac{c_1 \epsilon_1}{2} \left( \int_0^t |\Delta u_{n,k}|_{L_x^2}^2 ds + k^2 \int_0^t |\nabla u_{n,k}|_{L_x^2}^2 ds \right) \\ + \frac{1}{2} |\nabla u_{n,k}(0)|_{L_x^2}^2 - \frac{1}{2} |\nabla u_{n,k}(t)|_{L_x^2}^2 + X_{n,k}(t) + t |\nabla \phi_n|_{\mathbb{L}_2}^2, \quad (2.27) \end{aligned}$$

a.s. for  $t \in [0, \zeta_{n,k}]$ . Taking the power  $\rho \in [1, \infty)$  and the expectation in (2.27), we obtain the uniform bound on  $\mathbb{E}[\int_0^{\zeta_{n,k}} |\Delta u_{n,k}|_{L_x^2}^2 dt]$ , provided  $\epsilon_1$  is chosen to be sufficiently small, namely  $< c_1^{-1}$ , see (2.22).

*Bounds on the time interval  $[\zeta_{n,k}, T]$ .* Observe that we have the classical inequality

$$|\Delta e^{-t(\Delta)^2} f|_{L_x^2}^2 \leq \frac{c |\nabla f|_{L_x^2}^2}{t^{\frac{1}{2}}}, \quad \text{for } t > 0, \quad \text{and } f \in H_x^1,$$

(the proof is identical as that of (4.44), for the sectorial operator  $A$  defined in Chap. 4). Therefore, by the definition (2.15) we have

$$\mathbb{E} \int_{\zeta_{n,k}}^T |\Delta u_{n,k}(t)|_{L_x^2}^2 dt \leq \mathbb{E} \int_{\zeta_{n,k}}^T \frac{c |\nabla u(\zeta_{n,k})|_{L_x^2}^2}{(t - \zeta_{n,k})^{\frac{1}{2}}} dt,$$

which is bounded by a constant  $c(E_0, T, \phi)$ , using part (ii).

Lastly, we obtain the uniform bound (2.20) by applying again (2.22), (2.17) and (2.19).  $\blacksquare$

## (2.b) Consequences of Proposition 2.2

By a direct application of Proposition 2.2, we have the following

**Corollary 2.1.** Define for  $n \in \mathbb{N}$ :

$$\mathbf{u}_n := (u_{n,k})_{k \in \mathbb{N}} .$$

Then, for any pair of extractions  $(n_l, m_l)_{l \in \mathbb{N}}$ , and every  $\beta \in (0, 1/2)$ , the family of laws

$$\mathcal{L}((\mathbf{u}_{n_l}, W_{n_l}), (\mathbf{u}_{m_l}, W_{m_l})) , \quad l \in \mathbb{N} ,$$

is tight in the space  $\left( \prod_{k \in \mathbb{N}} (L_T^2 H_x^1 \cap C_T L_x^2) \times C_T^\beta L_x^2 \right)^2$ .

**Remark on notation.** When a pair of extractions  $(n_l, m_l)_{l \in \mathbb{N}}$  is fixed, we shall denote by:

$$\begin{cases} \phi_l^1 := \phi_{n_l} , \quad \phi_l^2 := \phi_{m_l} \quad \text{and} , \\ F_l^j := F_{\phi_l^j} \quad \text{for } j = 1, 2 . \end{cases} \quad (2.28)$$

**Corollary 2.2.** There exists a probability space  $(\Omega', \mathcal{F}', \mathbb{P}')$ , carrying processes,  $\mathfrak{z}^j = (z_k^j)_{k \in \mathbb{N}}$ ,  $\mathfrak{z}_l^j = (z_{l,k}^j)_{k \in \mathbb{N}}$ ,  $l \in \mathbb{N}$ ,  $j = 1, 2$ , taking values in  $\prod_{k \in \mathbb{N}} C_T L_x^2 \cap L_T^2 H_x^1$ , and  $L_x^2$ -Wiener processes  $W_l^j$  (resp.  $W_\phi^j$ ) with covariance  $\phi_l^j \phi_l^{j*}$  (resp.  $\phi \phi^*$ ) such that:

(a) for each  $l \in \mathbb{N}$ :  $\mathcal{L}((\mathbf{u}_{n_l}, W_{n_l}), (\mathbf{u}_{m_l}, W_{m_l})) = \mathcal{L}((\mathfrak{z}_l^1, W_l^1), (\mathfrak{z}_l^2, W_l^2))$

(b) for  $j = 1, 2$ ,  $\mathbb{P}'$ -a.s.

$$\begin{aligned} z_{l,k}^j &\xrightarrow[l \rightarrow \infty]{} z_k^j \quad \text{in } L_T^2 H_x^1 \cap C_T L_x^2, \quad \forall k \in \mathbb{N}, \\ W_l^j &\xrightarrow[l \rightarrow \infty]{} W_\phi^j \quad \text{in } C_T^\beta L_x^2 . \end{aligned} \quad (2.29)$$

(c) For  $j = 1, 2$ ,  $k, l \in \mathbb{N}$ , we define as in (2.14):

$$\kappa_{l,k}^j = \inf \left\{ 0 \leq t \leq T, \quad \sup_{x \in \mathbb{T}^2} |\nabla z_{l,k}^j(t)|_{L^2(B_{1/k}(x))} \geq \epsilon_1 \right\} . \quad (2.30)$$

Then, a.s. for  $t \in [0, \kappa_{l,k}^1]$ ,

$$z_{l,k}^j(t) - z_{l,k}^j(0) - \int_0^t (\Delta z_{l,k}^j + z_{l,k}^j |\nabla z_{l,k}^j|^2 + F_l^j(z_{l,k}^j)) dt = \int_0^t z_{l,k}^j \times dW_l^j , \quad (2.31)$$

in the sense of Bochner, resp. Itô integral in  $L_x^2$ .

(d) uniformly in  $t \in [0, T]$ , for  $j = 1, 2$ ,

$$\int_0^t z_{l,k}^j(s) \times dW_l^j(s) \xrightarrow[l \rightarrow \infty]{} \int_0^t z_k^j(s) \times dW_\phi^j(s) ,$$

strongly  $L_x^2$ , in probability .

(e) The processes  $(z_k^j, \kappa_k^j)$ ,  $j = 1, 2$ , are local solutions of (2.1) on  $\mathfrak{A}^j$ , up to the stopping times

$$\kappa_k^j = \text{ess inf} \left\{ t \geq 0, \quad |\nabla z_k^j|_{L^2(B_{1/k}(x))} \geq \epsilon_1 \right\} . \quad (2.32)$$

*Proof of Corollary 2.1.* The proof is rather similar than that of [BGJ13, Lemma 4.2]. It uses the previous a priori estimates, together with the following well-known compactness Lemma (see [FG95, Theorem 2.1]): Note that it will be also needed in Chapter 5, for the proof of Proposition 5.2.

**Lemma 2.1.** *If  $B_0 \subseteq B \subseteq B_1$  are Banach spaces, such that  $B_0, B_1$  are reflexive, and the embedding of  $B_0$  in  $B$  is compact, and if  $(\beta, p, q) \in (0, 1) \times (1, \infty) \times (1, \infty)$  with  $\beta p > 1$  then*

$$L^q(0, T; B_0) \cap W^{\beta, p}(0, T; B_1) \hookrightarrow L^q(0, T; B) , \quad (2.33)$$

$$\mathcal{C}(0, T; B_0) \cap W^{\beta, p}(0, T; B_1) \hookrightarrow \mathcal{C}(0, T; B) , \quad (2.34)$$

and the embeddings are compact.

Applying Proposition 2.2, we have for all  $l \in \mathbb{N}$ , and all  $k \in \mathbb{N}$ :

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\nabla u_{n_l, k}|_{L_x^2}^2 ds \right] \leq c(E_0, T, \phi) \mathbb{E} \int_0^T |\Delta u_{n_l, k}|_{L_x^2}^2 ds \leq c(k, E_0, T, \phi) , \quad (2.35)$$

with constants that do not depend on  $l$ , and the same estimates holds with  $m_l$  instead of  $n_l$ . To obtain compactness, we need however additional uniform estimates in some Sobolev space  $W^{\beta, p}(0, T; B_1)$ , where  $B_1$  can be any reflexive Banach space containing  $L_x^2$ , and  $\beta p > 1$ .

As in the proof of Lemma 4.1 in [BGJ13], we can write, using the equation on  $v_n$ :

$$\begin{aligned} u_{n_l, k}(t) - u_{n_l, k}(0) &= \int_0^t (u_{n_l, k} \times \Delta u_{n_l, k} + \Delta u_{n_l, k} + u_{n_l, k} |\nabla u_{n_l, k}|^2) ds \\ &\quad + \int_0^t F_{\phi_{n_l}}(u_{n_l, k}) ds + \int_0^t u_{n_l, k} \times dW_{n_l}(s) \\ &= J_{l, k}^1(t) + J_{l, k}^2(t) + J_{l, k}^3(t) , \end{aligned}$$

for all  $l \in \mathbb{N}$ , a.s. for  $t \in [0, \zeta_{n_l, k})$ . By (A1), this equation holds in the sense of Bochner, and Itô integrals in  $L_x^2$ . Besides, (2.11) implies that

$$\begin{aligned} &\int_0^t |u_{n_l, k} \times \Delta u_{n_l, k} + \Delta u_{n_l, k} + u_{n_l, k} |\nabla u_{n_l, k}|^2|_{L_x^2}^2 ds \\ &= \int_0^t |u_{n_l, k} \times \Delta u_{n_l, k} - u_{n_l, k} \times (u_{n_l, k} \times \Delta u_{n_l, k})|_{L_x^2}^2 ds \\ &= 2 \int_0^t |u_{n_l, k} \times \Delta u_{n_l, k}|_{L_x^2}^2 ds \end{aligned}$$

so that the bound

$$\mathbb{E} [\|J_{l, k}^1\|_{W^{1, 2}(0, T; L_x^2)}^2] \leq c(|\phi|_{\mathbb{L}_2^{0, 1}}, T) . \quad (2.36)$$

is obtained as a consequence of inequality (2.18).

Moreover, by (1.15), (2.11) and (A3), we obtain

$$\mathbb{E} [\|J_{l, k}^2\|_{W^{1, 2}(0, T; L_x^2)}^2] \leq c(|\phi|_{\mathbb{L}_2}, T) , \quad (2.37)$$

independently of  $l$ .

Lastly, using Lemma 2.1 from [FG95], for any  $\beta \in (0, \frac{1}{2})$ ,  $\infty > p \geq 2$  there exists a constant depending only on  $\beta, p, |\phi|_{\mathbb{L}_2}, T$  such that:

$$\mathbb{E}[\|J_{l,k}^3\|_{W^{\beta,p}(0,T;L_x^2)}^p] \leq c(\beta, p, |\phi|_{\mathbb{L}_2}, T) , \quad (2.38)$$

The tightness follows from standard arguments, using the same inequalities for  $m_l$ . It suffices to apply (2.34) to  $B_0 = H^1, B = B_1 = L^2$ , (2.33) with  $q = 2, B_0 = H_x^2, B = H_x^1, B_1 = L_x^2$ , so that the embedding

$$\mathcal{C}([0, T]; H_x^1) \cap W^{\beta,p}([0, T]; L_x^2) \cap L^2([0, T]; H_x^2) \hookrightarrow L^2([0, T]; H_x^1)$$

is compact. The tightness is now a consequence of estimates (2.35)-(2.36)-(2.37)-(2.38), together with Tychonov Theorem, Markov inequality, and Prokhorov Theorem. However, we refer the reader to Chapter 5, Proposition 5.2 for a more detailed proof of the same type of statement. ■

*Sketch of the proof of Corollary 2.2.* The proof is standard. We however summarize the main ideas. All the following arguments are directly adapted from [Sko14], where a more detailed discussion is made.

(a)-(b). These properties are a consequence of Skorohod embedding Theorem (see [WI81]), Corollary 2.1 and classical properties of Wiener processes.

Statements (c) and (d) are consequences of the following identities

$$\mathbb{E} \left[ (M_{l,k}^j(t) - M_{l,k}^j(s)) \varphi(W_l^j|_{[0,s]}) \right] = 0 , \quad (2.39)$$

$$\mathbb{E} \left[ \left( \langle M_{l,k}^j(t), a \rangle - \langle M_{l,k}^j(s), b \rangle - \int_s^t \langle z_{l,k}^j \times \phi_l^j a, z_{l,k}^j \times \phi_l^j b \rangle ds \right) \varphi(W_l^j|_{[0,s]}) \right] = 0 , \quad (2.40)$$

for any  $a, b \in L^2$ , where  $M_{l,k}^j(t) = z_{l,k}^j(t) - z_{l,k}^j(0) - \int_0^t ((\text{id} + z_{l,k}^j \times) \Delta z_{l,k}^j + z_{l,k}^j |\nabla z_{l,k}^j|^2 + F_l^j(z_{l,k}^j)) ds$ . These relations are due to the equality of the laws. Moreover we can take the limits in (2.39) and (2.40) to obtain that the processes  $M_k^j(t) = z_k^j(t) - z_k^j(0) - \int_0^t ((\text{id} + z_k^j \times) \Delta z_k^j + z_k^j |\nabla z_k^j|^2 + F_\phi(z_k^j)) ds$ , for  $j = 1, 2$  satisfy identities of the type above with  $\phi$ , resp.  $W_\phi^j$ , instead of  $\phi_l^j$ , resp.  $W_l$ . Indeed thanks to Proposition 2.2, all the terms are uniformly integrable in  $\omega \in \Omega'$  and converge  $\mathbb{P}$ -a.s. By the representation theorem for square integrable martingales, we can always enlarge  $\mathfrak{P}'$ , so that each  $M_{l,k}^j$  is a stochastic integral with respect to some Wiener process  $\tilde{W}_l$ . However the use of a classical regularization procedure (see [Ben95]) shows that these processes are actual stochastic integrals with respect to the Wiener processes  $W_\phi^l, W_l$ ,  $l \in \mathbb{N}$ , of  $\mathfrak{P}'$ . The statement (e) follows. ■

## (2.c) Skorohod space and the use of Gyöngy-Krylov Lemma

To obtain the convergence of the full sequence  $(u^n)$  towards a strong probabilistic solution (i.e. in the original space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), W)$ ), we make use of the following lemma, whose proof is immediate, but which turns out to have deep implications. It was first used by I. Gyöngy and N. Krylov in [GK96].

**Lemma 2.2.** *Let  $(Z_n)_{n \geq 0}$  denote a sequence of random elements in a Polish space  $E$  equipped with its Borel  $\sigma$ -algebra. Then the following properties are equivalent:*



- (i) the sequence  $(Z_n)_{n \geq 0}$  converges in probability to a random variable  $Z$ ;
- (ii) for every pair of extractions  $m_l, n_l, l \in \mathbb{N}$ , there exists a subsequence of  $\mathcal{L}(Z_{m_l}, Z_{n_l})_{l \in \mathbb{N}}$  which converges weakly to a law supported on the diagonal  $\{(x, y) \in E \times E, x = y\}$ .

In order to apply Lemma 2.2, we show the following.

**Proposition 2.3.** Define  $\kappa_{l,k}^j$  by (2.30).

- (i) For all  $k \in \mathbb{N}$ , with full probability:  $z_k^1 = z_k^2$ .
- (ii) Provided  $k \in \mathbb{N}$  is sufficiently large, then for any limit point  $\kappa_k^*$  of the sequence  $\{\kappa_{l,k}^1 \wedge \kappa_{l,k}, l \in \mathbb{N}\}$  for the convergence in probability, we have

$$\mathbb{P}'(\kappa_k^* > 0) = 1 .$$

*Remark 2.3.* Using the equality of the laws, the processes  $z_{l,k}^j, l, k \in \mathbb{N}, j = 1, 2$ , verify immediate counterparts of estimates of Proposition 2.2. Combining the uniform estimates on moments of  $\int_0^T |\nabla z_{l,k}^j|_{L_x^2}^2 dt$  with Corollary 2.2 gives

$$z_{l,k}^j \xrightarrow[l \rightarrow \infty]{} z_k^j \text{ strongly in } L^2(\Omega' \times [0, T]; H_x^1) . \quad (2.41)$$

*Proof of Proposition 2.3.*

*Proof of (i)* – We first prove that there exists  $k_0 \in \mathbb{N}$  such that for all  $l, k \in \mathbb{N}$  with  $k \geq k_0$ , then  $\mathbb{P}'(\kappa_{l,k} > 0) = 1$ . Fix  $l, k \in \mathbb{N}$ . We observe that:

$$\begin{aligned} \mathbb{P}'(\kappa_{l,k} > 0) &= 1 - \mathbb{P}'(\kappa_{l,k} = 0) \\ &= 1 - \lim_{N \rightarrow \infty} \mathbb{P}'(\kappa_{l,k} \leq 1/N) . \end{aligned} \quad (2.42)$$

To show that  $\mathbb{P}'(\kappa_{l,k} \leq 1/N) \rightarrow 0$  as  $N \rightarrow \infty$ , we use the following covering argument: there exists a constant  $c(\mathbb{T}^2) > 0$ , such that for all  $k \in \mathbb{N}^*$ , there exists a finite set of points  $\{P_1, P_2, \dots, P_{N_k}\} \subseteq \mathbb{T}^2$  with the properties:

$$\begin{cases} \forall x \in \mathbb{T}^2, \exists i \in \llbracket 1, N_k \rrbracket, B_{1/k}(x) \subseteq B_{3/k}(P_i) , \\ N_k \leq c(\mathbb{T}^2)k^2 , \end{cases} \quad (2.43)$$

see Figure 2.1. Now, for each  $k \in \mathbb{N}^*$ , and each  $P_i$ , consider  $\eta = \eta_{3/k,i} \in \mathcal{C}_0^\infty(B_{4/k}(P_i))$  with

$$0 \leq \eta \leq 1 , \quad \sup_{x \in B_{4/k}(P_i)} |\nabla \eta(x)| \leq c_1 k , \quad (2.44)$$

for some  $c_1 > 0$  independent of  $i, k$ . To lighten the notations, denote by

$$\text{cov}_k = \{\eta = \eta_{3/k,i}, 1 \leq i \leq N_k\}$$

where the functions  $\eta_{3/k,i}$  are as above, so that in particular  $\#\text{cov}_k = N_k$  is finite. Using the bound on the local dissipation, namely (2.A.6), we have for all  $\eta \in \text{cov}_k$ :

$$\frac{1}{2} (|\eta \nabla z_{l,k}^j(t)|_{L^2}^2 - |\eta \nabla z_{l,k}^j(t)|_{L^2}^2) \leq V_{l,k}^\eta(t) , \quad (2.45)$$

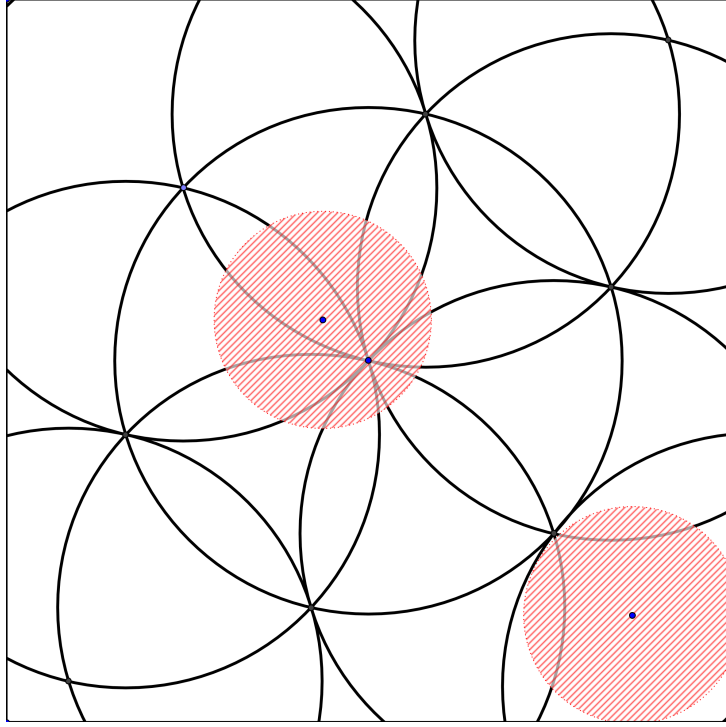


Figure 2.1 – There exists a prescribed set of points  $\{P_i, 1 \leq i \leq N_k\}$ , such that every ball of radius  $1/k$  (in red) lies at least in one ball  $B_{3/k}(P_{i_0})$ .

where we denote for  $t \in [0, T]$ :

$$V_{l,k}^\eta(t) := t|\eta \nabla \tilde{\phi}_l^j|_{\mathbb{L}_2}^2 + c_1 k^2 \int_0^t |\nabla z_{l,k}^j|_{L_x^2}^2 ds + \int_0^t \langle \eta \nabla z_{l,k}^j, \eta z_{l,k}^j \times \nabla dW_l^j(s) \rangle, \quad (2.46)$$

making the slight abuse of notation  $|\eta \nabla \phi_l^j|_{\mathbb{L}_2}^2 = \sum_{\varepsilon \in \mathcal{B}} \int_{\mathbb{T}^2} \eta(x)^2 |\nabla \tilde{\phi}_\varepsilon(x)|_{\mathbb{R}^2}^2 dx$ . By strong convergence of  $z_{l,k}^j(0)$  towards  $z_k^j(0)$  in  $H_x^1$ , and  $\text{Supp}(\eta) \subseteq B_{3/k}(P_i)$  for some  $i \in \llbracket 1, N_k \rrbracket$ , we may find a sufficiently large  $k_0 \in \mathbb{N}^*$  such that

$$\forall k \geq k_0, \quad \max_{1 \leq i \leq N_k} |\eta_{3/k,i} \nabla z_{l,k}^j(0)|_{L_x^2}^2 \leq \epsilon_1/2, \quad \text{uniformly in } l \in \mathbb{N}, \text{ and } j = 1, 2. \quad (2.47)$$

Moreover, Proposition 2.1 applied on the martingale part  $\int_0^t \langle \eta \nabla z_{l,k}^j, \eta z_{l,k}^j \times \nabla dW_l^j(s) \rangle$  of  $V_{l,k}$  gives, similarly to the proof of Proposition 2.2,

$$\mathbb{E} \left[ \sup_{0 \leq t \leq 1/N} V_{l,k}^\eta(t) \right] \leq \frac{|\eta \nabla \tilde{\phi}_l^1|_{\mathbb{L}_2}^2}{N} + c_1 k^2 \mathbb{E} \int_0^{\frac{1}{N}} |\nabla z_{l,k}^j(s)|_{L_x^2}^2 ds + c(|\phi|_{\mathbb{L}_2^{0,1}}) \mathbb{E} \int_0^{\frac{1}{N}} |\eta \nabla z_{l,k}^j(s)|_{L_x^2}^2 ds \quad (2.48)$$

By (2.45), (2.47), and (2.48), we see that

$$\left\{ |\eta \nabla z_{l,k}^j|_{L_x^2}^2 \geq \epsilon_1 \right\} \subseteq \left\{ 2V_{l,k}^\eta \geq \frac{\epsilon_1}{2} \right\}$$

so that using on the other hand, (2.43), and Markov inequality, we obtain

$$\begin{aligned}
\mathbb{P}'(\kappa_{l,k} \leq 1/N) &= \mathbb{P}' \left( \sup_{t \in [0, 1/N]} \sup_{x \in \mathbb{T}^2} |\nabla z_{l,k}^j|_{L^2(B_{1/k}(x))}^2 \geq \epsilon_1 \right) \\
&\leq \mathbb{P}' \left( \bigcup_{1 \leq i \leq N_k} \left\{ \sup_{0 \leq t \leq 1/N} |\eta_{3/k,i} \nabla z_{l,k}^j(t)|_{L_x^2}^2 \geq \epsilon_1 \right\} \right) \\
&\leq \sum_{\eta \in \text{cov}_k} \mathbb{P}' \left( \sup_{t \in [0, 1/N]} V_{l,k}^\eta(t) \geq \frac{\epsilon_1}{2} \right) \\
&\leq \frac{4}{\epsilon_1} \sum_{\eta \in \text{cov}_k} \left( \frac{|\eta \nabla \tilde{\phi}_l^1|_{\mathbb{L}_2}^2}{N} + c_1 k^2 \mathbb{E} \int_0^{\frac{1}{N}} |\nabla z_{l,k}^j(s)|_{L_x^2}^2 ds \right. \\
&\quad \left. + c(|\phi|_{\mathbb{L}_2^{0,1}}) \mathbb{E} \int_0^{\frac{1}{N}} |\eta \nabla z_{l,k}^j(s)|_{L_x^2}^2 ds \right).
\end{aligned} \tag{2.49}$$

By Remark 2.3, we see that the right hand side of (2.49) converges to 0 as  $N \rightarrow \infty$ , and that the convergence holds uniformly in  $l \in \mathbb{N}$  and  $j = 1, 2$ .

Lastly, when  $k \geq 0$ , denote by  $\kappa_k^*$  any limit point, in the sense of convergence in probability, of the sequence  $\{\kappa_{l,k}, l \in \mathbb{N}\}$ . Using the triangle inequality, write

$$\left\{ \kappa_k^* \leq \frac{1}{N} \right\} \subseteq \left\{ |\kappa_{l,k} - |\kappa_{l,k} - \kappa_k^*|| \leq \frac{1}{N} \right\},$$

so that:

$$\mathbb{P}' \left( \kappa_k^* \leq \frac{1}{N} \right) \leq \mathbb{P}' \left( \kappa_{l,k} \leq \frac{2}{N} \right) + \mathbb{P}' \left( |\kappa_{l,k} - \kappa_k^*| \geq \frac{1}{N} \right).$$

The conclusion follows by  $|\kappa_{l,k} - \kappa_k^*| \xrightarrow{\mathbb{P}'} 0$  as  $l \rightarrow \infty$  (up to extraction), and the uniform convergence of  $\mathbb{P}'(\kappa_{l,k} \leq 1/N)$  as  $N \rightarrow \infty$ .

*Proof of (ii)* – Fix  $l \in \mathbb{N}$ ,  $k \in \mathbb{N}$ , and set  $y_{l,k} = z_{l,k}^1 - z_{l,k}^2$ . There holds a.s. for  $t \in [0, \kappa_{l,k}^1 \wedge \kappa_{l,k}^2]$ , in differential notation:

$$\begin{aligned}
dy_{l,k} &= \left( \Delta y_{l,k} + z_{l,k}^2 \times \Delta y_{l,k} + y_{l,k} \times \Delta z_{l,k}^1 \right. \\
&\quad \left. + F_l^1(z_{l,k}^1) - F_l^2(z_{l,k}^2) + y_{l,k} |\nabla z_{l,k}^1|^2 + z_{l,k}^2 (|\nabla z_{l,k}^1|^2 - |\nabla z_{l,k}^2|^2) \right) dt \\
&\quad + y_{l,k} \times dW_l^1 + z_{l,k}^2 \times d(W_l^1 - W_l^2).
\end{aligned}$$

Applying Itô formula to the process  $\frac{1}{2}|y_{l,k}(t)|_{L_x^2}^2$ ,  $t \in [0, T]$ , and writing for  $t \in [0, T]$ :

$$\begin{aligned}
A_{l,k}(t) &:= |\nabla z_{l,k}^1(t)|_{L_x^4}^4 + |\nabla z_{l,k}^2(t)|_{L_x^4}^4 \\
Y_{l,k}(t) &:= 1_{\{0 \leq t \leq \kappa_{l,k}^1 \wedge \kappa_{l,k}^2\}} |y_{l,k}(t)|_{L_x^2}^2,
\end{aligned} \tag{2.50}$$

a similar calculus as that of the proof of uniqueness in [GH93] (the calculus being identical as that of the computations done in chapter 3 below, in the proof of Theorem 3.3, but in case  $R_{l,k} = 0$  and  $y_{l,k}(0) = 0$ ) gives the inequality:

$$Y_{l,k}(t) \leq Y_{l,k}(0) + T_{l,k} + c_0 \int_0^t A_{l,k}(s) Y_{l,k}(s) ds, \quad \text{a.s. for } t \in [0, T], \tag{2.51}$$

where  $c_0 > 0$  denotes a universal constant, and we put all the martingale plus correction terms in

$$\begin{aligned} T_{l,k} &= \sup_{t \in [0, T]} \left| \int_0^t \langle y_{l,k}, z_{l,k}^2 \times d(W_l^1 - W_l^2) \rangle \right| + \int_0^T \langle y_{l,k}, F_l^1(z_{l,k}^j) - F_l^2(z_{l,k}^j) \rangle dt \\ &+ \left| \int_0^T \frac{1}{2} D^2(|y_{l,k}|_{L_x^2}^2/2) \cdot d \ll \int_0^T y_{l,k} \times dW_l^1 + \int_0^T z_{l,k}^2 \times d(W_l^1 - W_l^2) \gg_t \right|. \end{aligned} \quad (2.52)$$

Note that  $Y_{l,k}(0) \rightarrow 0$  as  $l \rightarrow \infty$ , and it is a tedious, but easy proof to show that  $\mathbb{E}[T_{l,k}] \rightarrow 0$  as  $l \rightarrow \infty$  (we refer the reader to the appendix). For  $\lambda > 0$ , we write  $\tau_{\lambda,l} = \inf\{0 \leq t \leq T, A_{l,k}(t) > \lambda c_0^{-1}\}$ . Applying pathwise Grönwall Lemma in (2.51), we see that for  $t \in [0, T]$ ,

$$1_{\{t \leq \tau_{\lambda,l}\}} Y_{l,k}(t) \leq (Y_{l,k}(0) + T_{l,k}) e^{\lambda T}$$

Therefore, integrating  $|y_{l,k}(t)|_{L_x^2}^2$  over  $t \in [0, T] = [0, \tau_{\lambda,l}] \cup [\tau_{\lambda,l}, T]$ , and taking the expectation gives:

$$\begin{aligned} \mathbb{E} \int_0^T Y_{l,k}(t) dt &\leq T(Y_{l,k}(0) + \mathbb{E}[T_{l,k}]) e^{\lambda T} + \mathbb{E} \int_0^T Y_{l,k}(t) 1_{[\tau_{\lambda,l}, T]}(t) dt \\ &\leq T(Y_{l,k}(0) + \mathbb{E}[T_{l,k}]) e^{\lambda T} \\ &\quad + \mathbb{E} \left[ \left( \int_0^T Y_{l,k}(t)^2 dt \right)^{1/2} \left( \int_0^T \mathbb{P}'(A_{l,k}(t) > \lambda) dt \right)^{1/2} \right], \end{aligned}$$

by Cauchy-Schwarz inequality, and thus by  $|Y_{l,k}| \leq 2$  a.e., and Markov inequality, we obtain:

$$\begin{aligned} \mathbb{E} \int_0^T Y_{l,k}(t) dt &\leq T(Y_{l,k}(0) + \mathbb{E}[T_{l,k}]) e^{\lambda T} + 4T^{1/2} \left( \int_0^T \mathbb{P}'(A_{l,k}(t) > \lambda) dt \right)^{1/2} \\ &\leq T(Y_{l,k}(0) + \mathbb{E}[T_{l,k}]) e^{\lambda T} + \frac{4T^{1/2}}{\lambda^{1/2}} \left( \int_0^T \mathbb{E}[A_{l,k}(t)] dt \right)^{1/2}. \end{aligned}$$

Now, since  $Y_{l,k}(0) + T_{l,k} \rightarrow 0$  as  $l \rightarrow \infty$ , fix  $\epsilon > 0$  and use the uniform estimates on  $\mathbb{E}[\int_0^T |\nabla z_{l,k}^j|_{L_x^4}^4 ds]$ ,  $j = 1, 2$  – see Remark 2.3 – to choose  $\lambda > 0$ , depending only on  $E_0, |\phi|_{\mathbb{L}^{0,1}}$  and  $k \in \mathbb{N}$  such that

$$\frac{4T^{1/2}}{\lambda^{1/2}} \left( \mathbb{E} \int_0^T A_{l,k}(s) ds \right)^{1/2} \leq \epsilon/2 \text{ for all } l \in \mathbb{N}.$$

Then, choose  $l$  sufficiently large so that  $T(Y_{l,k}(0) + T_{l,k}) e^{\lambda} \leq \epsilon/2$ . This shows that  $\lim_{l \rightarrow \infty} \mathbb{E} \int_0^T Y_{l,k}(t) dt = 0$ , and Fatou Lemma implies that

$$|y_k(t)|_{L_x^2} = 0 \text{ a.s. for } t \in [0, \kappa_k^1 \wedge \kappa_k^2],$$

Lastly, one can write

$$|y_{l,k}(t)|_{L_x^2} = 1_{\{0 \leq t \leq \kappa_{l,k}^1 \wedge \kappa_{l,k}^2\}} |y_{l,k}(t)|_{L_x^2} + 1_{\{\kappa_{l,k}^1 \wedge \kappa_{l,k}^2 < t \leq T\}} |y_{l,k}(t)|_{L_x^2}, \text{ a.s.}$$

Taking the limit as  $l \rightarrow \infty$ , and using  $Y_{l,k} \equiv 0$  gives for any possible limit point  $\kappa_k^*$  of  $\kappa_{l,k}^1 \wedge \kappa_{l,k}^2$

$$\begin{aligned} |y_k(t)|_{L_x^2} &= 1_{\{\kappa_k^* < t \leq T\}} |y_k(t)|_{L_x^2}, \text{ a.s.} \\ &= |y_k(\kappa_k^*)|_{L_x^2} \\ &= 0, \end{aligned}$$

On the other hand, using (2.49), we have

$$\begin{aligned}
\mathbb{P}'(\kappa_k^* > 0) &= \lim_{N \rightarrow \infty} \mathbb{P}'(\kappa_k^* \geq \frac{1}{N}) \\
&= \lim_{N \rightarrow \infty} \lim_{l \rightarrow \infty} \mathbb{P}'(\kappa_{l,k} \geq \frac{1}{N}) . \\
&\geq \lim_{N \rightarrow \infty} \lim_{l \rightarrow \infty} 1 - \max_{j=1,2} \frac{4}{\epsilon_1} \sum_{\eta \in \text{cov}_k} \left( \frac{|\eta \nabla \tilde{\phi}_l^1|_{L^2}^2}{N} + c_1 k^2 \mathbb{E} \int_0^{\frac{1}{N}} |\nabla z_{l,k}^j(s)|_{L_x^2}^2 ds \right. \\
&\qquad \qquad \qquad \left. + c(|\phi|_{L^2}) \mathbb{E} \int_0^{\frac{1}{N}} |\eta \nabla z_{l,k}^j(s)|_{L_x^2}^2 ds \right) \\
&= 1 ,
\end{aligned}$$

since the convergence as  $l \rightarrow \infty$  holds uniformly in  $N \geq 1$ . Therefore we can iterate the argument above over the whole interval  $[0, T]$ , which completes the proof of Proposition 2.3. ■

By Lemma 2.2 we can now return to our initial stochastic basis i.e.  $\mathfrak{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \hat{W})$  on which we have now limits  $u_k$  in probability for the processes  $u_{n,k}$ . The following result shows that the convergence obtained is actually better than expected.

**Corollary 2.3** (Existence of a strong solution). *Let  $k \in \mathbb{N}$ .*

(i) *There exists  $k_0 \in \mathbb{N}$  such that for each  $k \geq k_0$ , the sequence  $\{\zeta_{n,k}, n \in \mathbb{N}\}$  defined in (2.14) converges in probability to a stopping time  $\zeta_k$ . Moreover*

$$\mathbb{P}(\zeta_k > 0) = 1 , \quad \forall k \geq k_0 . \quad (2.53)$$

(ii) *As  $n \rightarrow \infty$ , the processes  $u_{n,k}$  converges in  $L^2(\Omega; \mathcal{C}_T H_x^1 \cap L_T^2 H_x^2)$  to a local strong solution  $(u_k, \zeta_k)$  of (2.1), in the sense of Definition 2.1.*

*Proof of Corollary 2.3.* The previous proposition shows in particular that  $\mathcal{L}(u_{n_l}, u_{m_l})$  tends to a law that is supported on the diagonal of  $E = \prod_{k \in \mathbb{N}} (\mathcal{C}_T L_x^2 \cap L_T^2 H_x^1)$ . By Lemma 2.2, there exists a random variable  $u = (u_k) \in E$  such that  $u_n = (u_{n,k})$  converges in probability to  $u$  i.e. for all  $\epsilon > 0, k \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \sup_{t \in [0, T]} |u_k(t) - u_{n,k}(t)|_{L_x^2}^2 + \int_0^T |\nabla u_k - \nabla u_{n,k}|_{L_x^2}^2 ds > \epsilon \right) = 0 . \quad (2.54)$$

*Proof of (i)* – Since the couples  $(\zeta_{n_l, k}, \zeta_{m_l, k})$  and  $(\kappa_{l,k}^1, \kappa_{l,k}^2)$  share the same law – see the definitions (2.14) and (2.30) – another application of Lemma 2.2 to the sequence  $\{\zeta_n\} = \{(\zeta_{n,k})_{k \in \mathbb{N}}, n \geq 0\}$ , implies the existence, for all  $k \geq 0$ , of the *unique* limit  $\zeta_k$  of  $\zeta_{n,k}$  as  $n \rightarrow \infty$ , and the convergence holds in probability. The fact that  $\mathbb{P}(\zeta_k > 0) = 1$  for  $k$  large enough follows by Proposition 2.3, and  $\mathcal{L}(\zeta_{n_l, k}, \zeta_{m_l, k}) = \mathcal{L}(\kappa_{l,k}^1, \kappa_{l,k}^2)$ .

*Proof of (ii).a*–Convergence in  $L^2(\Omega; \mathcal{C}_T L_x^2 \cap L_T^2 H_x^1)$ . Uniform estimates of Proposition 2.2-(ii), together with (2.54) imply by a standard argument that the convergence holds in the spaces  $L^\rho(\Omega; \mathcal{C}_T L_x^2 \cap L_T^2 H_x^1)$ , for all  $\rho \in [1, \infty)$ . Using again the same regularization procedure as that of [Ben95], observe that a.s.

$$u_k(t) - u_k(0) = \int_0^{t \wedge \zeta_k} (\Delta u_k + u_k \times \Delta u_k + F_\phi(u_k) + u_k |\nabla u_k|^2) + \int_0^{t \wedge \zeta_k} u_k \times dW_\phi ,$$

for  $t \in [0, T]$ , in the sense of Bochner, resp. Itô integrals in  $L_x^2$ . By the equality of the laws, the first integral above is well-defined and (2.11) is fulfilled for  $u_k$  since for  $j = 1, 2$ ,

$$\mathbb{P}' \left( \int_0^T |z^k \times \Delta z_k^j|_{L_x^2} dt < \infty, \int_0^T \left| |z_k^j|_{\mathbb{R}^3}^2 - 1 \right|_{L_x^1} dt = 0 \right) = 1$$

The stochastic integral  $\int_0^{t \wedge \zeta_k} u_k \times dW_\phi$  is also well defined since  $\phi \in \mathbb{L}_2^{0,1}$ .

*Proof of (ii).b–Convergence in  $L^2(\Omega; \mathcal{C}_T H_x^1 \cap L_T^2 H_x^2)$ .* Fix  $k \in \mathbb{N}$ , and denote  $\delta_{n,k} = u_{n,k} - u_k$ . We have a.s. on  $[0, \zeta_k \wedge \zeta_{k,l}]$ :

$$\begin{aligned} d\delta_{n,k} = & \left( \Delta\delta_{n,k} + u_k \times \Delta\delta_{n,k} + \delta_{n,k} \times \Delta u_{n,k} \right. \\ & \left. + F_n(u_{n,k}) - F_\phi(u_k) + \delta_{n,k} |\nabla z_{l,k}^1|^2 + u_k (|\nabla u_{n,k}|^2 - |\nabla u_k|^2) \right) dt \\ & + \delta_{n,k} \times dW_l^1 + u_k \times d(W_n - W_\phi) . \end{aligned}$$

Adapting the energy inequality (2.A.2) for  $\frac{1}{2} |\nabla \delta_{n,k}|_{L_x^2}^2$  (the proof involves similar computations based on Itô formula), integrating by parts, and using  $ab \leq a^2/2 + b^2/2$ , gives a.s. for  $t \in [0, \zeta_{n,k}]$ :

$$\begin{aligned} \frac{1}{2} |\nabla \delta_{n,k}(t)|_{L_x^2}^2 - \frac{1}{2} |\nabla \delta_{n,k}(0)|_{L_x^2}^2 + \int_0^t |\Delta \delta_{n,k}|_{L_x^2}^2 ds \\ \leq R_{n,k} + \int_0^t \langle \Delta \delta_{n,k}, \delta_{n,k} |\nabla u_{n,k}|^2 \\ + u_k \nabla(u_{n,k} + u_k) \cdot \nabla \delta_{n,k} + \delta_{n,k} \times \Delta u_k \rangle ds \quad (2.55) \end{aligned}$$

where we put the martingale plus correctional terms in

$$\begin{aligned} R_{n,k} = & \sup_{t \in [0, T]} \left| \int_0^t \frac{1}{2} D^2(|\nabla \delta_{n,k}|_{L_x^2}^2 / 2) \cdot d \ll \int_0^t \delta_{n,k} \times dW_n - \int_0^t u_k \times d(W_n - W) \gg_t \right| \\ & + \sup_{t \in [0, T]} \left| \int_0^t \langle \nabla \delta_{n,k}, \delta_{n,k} \times \nabla dW_n \rangle + \langle \nabla \delta_{n,k}, u_k \times \nabla d(W_n - W) \rangle \right| \\ & + \int_0^T \langle \nabla \delta_{n,k}, \nabla (F_n(u_{n,k}) - F_\phi(u_k)) ds \rangle \quad (2.56) \end{aligned}$$

Taking the supremum in (2.55), the expectation, and using  $ab \leq a^2/2 + b^2/2$ , we obtain

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq \zeta_{n,k}} |\nabla \delta_{n,k}(s)|_{L_x^2}^2 + \int_0^{\zeta_{n,k}} |\Delta \delta_{n,k}|_{L_x^2}^2 ds \right] \\ \leq |\nabla \delta_{n,k}(0)|_{L_x^2}^2 + \mathbb{E}[R_{n,k}] + \mathbb{E} \int_0^{\zeta_{n,k}} |\delta_{n,k} \times \Delta u_k|_{L_x^2}^2 dt \\ + c \mathbb{E} \int_0^{\zeta_{n,k}} \left( |\delta_{n,k} |\nabla u_{n,k}|^2|_{L_x^2}^2 + |\nabla(u_{n,k} + u_k)|_{L_x^4}^2 |\nabla \delta_{n,k}|_{L_x^4}^2 \right) dt . \quad (2.57) \end{aligned}$$

We refer the reader to Appendix A for the proof that  $\mathbb{E}[R_{n,k}] \rightarrow 0$ , as  $n \rightarrow \infty$ . For  $t \in [0, T]$  denote by  $f_n(t) := \mathbb{E}[\mathbb{1}_{[0, \zeta_{n,k}]}(t) (|\delta_{n,k} \times \Delta u_k|_{L_x^2}^2(t) + |\delta_{n,k} |\nabla u_{n,k}|^2|_{L_x^2}^2(t) + |\nabla(u_{n,k} + u_k)|_{L_x^4}^2 |\nabla \delta_{n,k}|_{L_x^4}^2(t))]$ . We recall that by Vitali convergence Theorem, to obtain  $\int_0^T f_n(t) dt \rightarrow 0$ , it suffices to check the two conditions

1.  $\{f_n\}$  is uniformly integrable;
2.  $f_n(t) \rightarrow 0$  for a.a.  $t \in [0, T]$ .

Integrability of  $t \mapsto \mathbb{E}[|\nabla u_{n,k}(t)|_{L_x^4}^4]$  and  $t \mapsto \mathbb{E}[|u_{n,k} \times \Delta u_{n,k}(t)|_{L_x^2}^2]dt$ , uniformly with respect to  $n \in \mathbb{N}$ , is obtained in Proposition 2.2. The fact that  $f_n(t) \rightarrow 0$  follows by  $\delta_{n,k} \rightarrow 0$   $d\mathbb{P} \otimes dt \otimes dx$  a.e. (up to extraction). This proves Corollary 2.3.  $\blacksquare$

## (2.d) Uniqueness and global well-posedness

By the definition (2.14), we have

$$\zeta_{n,k} \leq \zeta_{n,k+1} \quad , \quad d\mathbb{P} - \text{a.s.} \quad (2.58)$$

for each  $k, n \in \mathbb{N}$ , and thus denoting by  $\zeta_k := \lim_{n \rightarrow \infty} \zeta_{n,k}$ , recalling that  $v_n$  is the sequence defined in (A1)-(A2), using then the equality  $u_{n,k}|_{[0, \zeta_{n,k}]} = u_{n,k+1}|_{[0, \zeta_{n,k}]}$  we see that for each  $k$  the following equality holds a.s. :

$$\lim_{n \rightarrow \infty} \mathbb{1}_{[0, \zeta_k]}(t) v_n(t) = \begin{cases} u_k(t) & \text{if } t \in [0, \zeta_k) \\ 0 & \text{otherwise.} \end{cases}$$

In particular,  $u_k|_{[0, \zeta_k]} = u_{k+1}|_{[0, \zeta_k]}$ , so that when  $t \in [0, T]$ , the following notation is not ambiguous

$$\hat{u}(t) = \begin{cases} u_k(t) & \text{if } t \in [0, \zeta_k) \text{ for some } k \geq 1 \\ 0 & \text{otherwise,} \end{cases} \quad (2.59)$$

and defines a local strong solution on each  $[0, \zeta_k]$ ,  $k \geq 0$ .

We define now the stopping time:

$$\vartheta^1 = \sup_{k \in \mathbb{N}} \zeta_k \quad , \quad (2.60)$$

By the previous results  $(\hat{u}, \hat{\vartheta})$  solves locally (2.1) with the pathwise regularity:

$$\hat{u}(\omega) \in \mathcal{C}([0, \hat{\vartheta}); H_x^1) \cap L_{\text{loc}}^2([0, \hat{\vartheta}); H_x^2) \quad , \quad \text{for } d\mathbb{P} - \text{a.e. } \omega \in \Omega \quad , \quad (2.61)$$

see (2.8).

*Remark 2.4 (Uniqueness).* Let  $r > 0$ , and assume that we are given a local solution  $(v, \sigma)$  to (2.1), such that:  $v(0) = u(0)$ , and

$$d\mathbb{P} - \text{a.s.} \quad , \quad \forall t \in [0, \sigma) \quad , \quad \sup_{x \in \mathbb{T}^2} |\nabla u(t)|_{L^2(B_r(x))}^2 \geq \epsilon_1 \quad .$$

Then, using a standard regularization procedure, one can see that  $v$  satisfies all the estimates of Proposition 2.2, with  $r^{-1}$  instead of  $k$ , and  $(v, \sigma)$  instead of  $(u_{n,k}, \zeta_{n,k})$ . Moreover, adapting the proof of Proposition 2.3, there holds:

$$v|_{[0, \sigma \wedge \zeta_k]} = u_k|_{[0, \sigma \wedge \zeta_k]}$$

provided  $k \geq k_0$  and  $r \leq r_0$ , depending on  $u(0)$  only, and by iteration we obtain that:

$$\hat{u}|_{[0, \sigma]} = v \quad , \quad d\mathbb{P} - \text{a.s.}$$

**Lemma 2.3.** (i) the sequence  $\{U_k := u_k(\zeta_k), k \geq 0\}$  has a unique weak limit  $U$  in  $L^2(\Omega; H_x^1)$ .

(ii) the random set

$$\text{Sing}(\vartheta^1) = \left\{ x \in \mathbb{T}^2, \liminf_{k \rightarrow \infty} |\nabla \hat{u}(\zeta_k)|_{L^2(B_{1/k}(x))}^2 > 0 \right\}$$

is finite,  $d\mathbb{P}$ - a.s. , and if  $L$  denotes its cardinal, then a.s.

$$|\nabla U|_{\mathbb{T}^2}^2 \leq \lim_{s \rightarrow \vartheta^1_-} |\nabla u(s)|_{L_x^2}^2 - L\epsilon_1 .$$

(iii) For  $m \in \mathbb{N}^*$ , define a measurable process  $u : \Omega \times [0, T] \rightarrow H_x^1$ , and a stopping time  $\vartheta^m$  recursively by  $(u(\cdot)|_{[0, \vartheta^1]}, \vartheta^1) = (\hat{u}, \hat{\vartheta})$  the solution defined by (2.59) and  $d\mathbb{P}$ -a.s. :

$$\left[ \begin{array}{l} u(\vartheta^m) = \text{weak } \lim_{t \uparrow \vartheta^m} u^{(m)}(t) \text{ in } L^2(\Omega; H_x^1) , \text{ and} \\ \vartheta^{m+1} = \hat{\vartheta} + \vartheta^m , \\ u|_{[\vartheta^m, \vartheta^{m+1})}(\cdot - \vartheta^m) = \hat{u}(\cdot) , \end{array} \right. \left\{ \begin{array}{l} \text{where } (\hat{u}, \hat{\vartheta}) \text{ is the strong solution starting} \\ \text{from } u(\vartheta^m) \text{ at } t = 0, \text{ obtained} \\ \text{through Corollary 2.3.} \end{array} \right. \quad (2.62)$$

Then,

$$\mathbb{P}(\exists j \geq 0, \vartheta^j = T) = 1 . \quad (2.63)$$

*Proof.*

*Proof of (i)* – Let  $(U_k)_{k \in \mathbb{N}} = (u_k(\zeta_k))_{k \in \mathbb{N}}$ . The existence of an actual random variable  $U$  supported in  $H_x^1$  such that, up to a subsequence,  $U_k \rightharpoonup U$  weakly in  $L^2(\Omega; H_x^1)$ , is a consequence of the Banach Alaoglu Theorem, together with the energy estimate (i) of Proposition 2.2.

To show that this limit is unique, write for  $k, p \in \mathbb{N}$ :

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} |u_{k+p}(t) - u_k(t)|_{H_x^{-1}}^2 \right] &= \mathbb{E} \left[ \sup_{t \in [\zeta_k, \zeta_{k+p}]} |u_{k+p}(t) - u_k(t)|_{H_x^{-1}}^2 \right] \\ &\leq c \mathbb{E} \int_{\zeta_k}^{\zeta_{k+p}} \left| \Delta u_{k+p} + u_{k+p} \times \Delta u_{k+p} \right. \\ &\quad \left. + u_{k+p} |\nabla u_{k+p}|^2 + F_\phi(u_{k+p}) \right|_{H_x^{-1}}^2 dt \\ &\quad + c(|\phi|_{\mathbb{L}_2^{0,1}}) \mathbb{E} \int_{\zeta_k}^{\zeta_{k+p}} |u_{k+p}|_{H_x^{-1}}^2 dt \end{aligned}$$

Using the bounds of Proposition 2.2, together with  $\limsup_{k \rightarrow \infty} |\zeta_{k+l} - \zeta_k| = 0$ , a.s. , (this holds by growth and boundedness of  $\{\zeta_k\}$ ), we obtain that  $(u_k)_{k \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\Omega; \mathcal{C}_T H_x^{-1})$ . This implies that the process  $\hat{u}$  defined in (2.59) is supported in  $\mathcal{C}_T H_x^{-1}$ , In particular, the limit  $U$  is unique and equal to  $u(\vartheta^1)$ .



*Proof of (ii)* – The fact that  $\#\text{Sing}(U)$  is finite follows by additiveness of the energy. More precisely if  $\vartheta^1 \leq T$ , take an arbitrary  $\epsilon > 0$ . For  $k \in \mathbb{N}$ , there exists  $X_k \in \mathbb{T}^2$  such that  $|\nabla u_k(\zeta_k)|_{L^2(B_{1/k}(X_k))}^2 \geq \epsilon_1 - \epsilon$ . Using semicontinuity of the  $L_x^2$  norm, we obtain

$$\begin{aligned} |\nabla U|_{L^2(\mathbb{T}^2)}^2 &\leq \liminf_{k \rightarrow \infty} |\nabla U_k|_{L^2(\mathbb{T}^2 \setminus B_{1/k}(X_k))}^2 \\ &\leq \liminf_{k \rightarrow \infty} |\nabla U_k|_{L^2(\mathbb{T}^2)}^2 - \epsilon_1 + \epsilon . \end{aligned} \quad (2.64)$$

The loss of energy at  $t = \vartheta^1$  is at least equal to the optimal geometrical parameter  $\epsilon_1 > 0$ , which does not depend on the individual elements  $u, T$ . This shows the finiteness of  $\text{Sing}(u(\vartheta^1))$ .

*Proof of (iii)* – The fact that one can reiterate the procedure follows by  $U = u^1(\vartheta^1) \in L^2(\Omega; H_x^1)$ : one can always approximate  $U$  by a sequence  $U_n \in L^2(\Omega; H_x^2)$ . These are not deterministic elements as  $u(0), v_n(0)$ , but note that there is no difficulty to adapt the argument above, using for instance the conditional expectation with respect to the  $\sigma$ -algebra  $\mathcal{F}_{\vartheta^1}$ , instead of  $\mathbb{E}$ .

We now turn to the proof of (2.63). Use the notations (2.62), and denote by

$$\vartheta^\infty = \text{a. s. -} \lim_{m \rightarrow \infty} \vartheta^m \in [0, T] .$$

Define the  $\mathbb{N} \cup \{\infty\}$ -valued discrete process

$$N_m = \# \left\{ x \in \mathbb{T}^2, \liminf_{\epsilon \rightarrow 0^+} |\nabla u(\vartheta^m - \epsilon)|_{L^2(B_{1/k}(x))}^2 > 0 \right\}, \quad \text{a.s. for } m \in \mathbb{N} . \quad (2.65)$$

By (2.64), if  $N_\infty$  denotes  $\lim_{m \rightarrow \infty} N_m \in \mathbb{N} \cup \{\infty\}$ , then

$$\begin{aligned} \mathbb{P}(\forall m \in \mathbb{N}, \vartheta^m < T) &\leq \mathbb{P}(\forall L \in \mathbb{N}, N_\infty \geq L) \\ &\leq \lim_{L \rightarrow \infty} \mathbb{P}(N_\infty \geq L) \\ &\leq \lim_{L \rightarrow \infty} \frac{\mathbb{E}[N_\infty]}{L} . \end{aligned} \quad (2.66)$$

By (2.64), and Proposition 2.2 we see that

$$\begin{aligned} \mathbb{E}|\nabla u(\vartheta)^1|_{L_x^2}^2 &\leq \lim_{k \rightarrow \infty} |\nabla u(\zeta_k)|^2 - \epsilon_1 \mathbb{E}[N_1] \\ &\leq \mathbb{E}|\nabla u(0)|_{L_x^2}^2 + c(\phi) \mathbb{E}[\vartheta] - \epsilon_1 \mathbb{E}[N_1] , \end{aligned}$$

and a straightforward induction implies

$$\mathbb{E}|\nabla u(\vartheta^m)|^2 \leq |\nabla u(0)|^2 c(\phi) T - \epsilon_1 \mathbb{E}[N_m] ,$$

which gives the bound

$$\mathbb{E}[N_m] \leq (\epsilon_1)^{-1} (|\nabla u(0)|^2 + c(\phi) T) . \quad (2.67)$$

The conclusion now follows from (2.66) and (2.67): we have  $\mathbb{P}(\forall m \in \mathbb{N}, \vartheta^m < T) = 0$ , and thus  $\mathbb{P}(\exists m \in \mathbb{N}, \vartheta^m = T) = 1$ .

This finishes the proof the Lemma, and Theorem 2.2. ■

# A Appendix

## (1) The energy formulae

Let  $(u, \tau)$  be a local solution of (SLLG) on some (regular) domain  $\mathcal{O} \subseteq \mathbb{R}^d$ ,  $d \in \mathbb{N}^*$ . Assume that  $u$  has trajectories in  $\mathcal{C}_T H_x^2$ , and that  $W$  is a  $\phi\phi^*$ -Wiener process in  $L^2(D; \mathbb{R}^3)$  such that

$$\begin{cases} \phi = I_3 \otimes \tilde{\phi} \text{ , with} \\ \tilde{\phi} \in \mathbb{L}_2(L^2(\mathcal{O}; \mathbb{R}); H^1(\mathcal{O}; \mathbb{R})) \text{ ,} \end{cases}$$

see (2.6). Assume that  $u$  verifies

- (a) either constant in time Dirichlet boundary conditions  $u|_{\partial\mathcal{O}} \equiv \gamma \in H^{3/2}(\partial\mathcal{O})$ ;
- (b) either homogeneous Neumann boundary conditions  $\partial u / \partial n \equiv 0$ ;
- (c) either periodic boundary conditions (in case  $\mathcal{O}$  is a square).

Denote

$$E(t, \omega) = \frac{1}{2} \int_{\mathcal{O}} |\nabla u(\omega, t, x)|_{\mathbb{R}^{3 \times 2}}^2 dx \text{ , } (\omega, t) \in \Omega \times [0, \tau(\omega)) \text{ .} \quad (2.A.1)$$

Let  $\mathcal{B}$  denote an ONB of  $L^2(\mathcal{O}; \mathbb{R})$ .

**Increments of the Dirichlet Energy.** *Almost surely, for  $t \leq \tau(\omega)$ :*

$$E(t) - E(0) + \int_0^t |u \times \Delta u|_{L_x^2}^2 ds = t \sum_{\varepsilon \in \mathcal{B}} |\nabla \tilde{\phi} \varepsilon|_{L_x^2}^2 + \int_0^t \langle \nabla u, u \times d\nabla W \rangle \text{ .} \quad (2.A.2)$$

*Proof.* For  $a, b \in H^1(\mathcal{O}, \mathbb{R}^3)$ ,  $u \in H_x^2$ , the first two differentials of  $E$  in  $H_x^1$  are

$$\begin{cases} DE(u) \cdot a = \int_{\mathcal{O}} \nabla u(x) \cdot \nabla a(x) dx \text{ ,} \\ \quad \quad \quad = -\langle \Delta u, a \rangle + \int_{\partial\mathcal{O}} \nabla u(\sigma) \cdot \vec{n}(\sigma) a(\sigma) d\sigma \text{ ,} \\ D^2 E(u) \cdot (a, b) = \int_{\mathcal{O}} \nabla a(x) \cdot \nabla b(x) dx \text{ .} \end{cases}$$

Note that in the three cases (a)-(b)-(c),  $DE(u) \cdot a = \langle -\Delta u, a \rangle$ . These are bounded maps on trajectories  $t \mapsto u(t, \omega)$ ,  $t \in [0, \tau(\omega))$ , for a.e.  $\omega \in \Omega$ , and therefore, a slight modified version of [DZ08, Thm. 4.32, p. 106] (but the proof is identical) gives the relation for  $t < \tau$ :

$$\begin{aligned} E(t) - E(0) &= \int_0^t \langle -\Delta u, \Delta u + u \times \Delta u + F_\phi(u) \rangle + \int_0^t \langle \nabla u, \nabla(u \times dW_\phi) \rangle \\ &\quad + \frac{1}{2} \int_0^t \sum_{e \in \mathcal{B}^3} |\nabla(u \times \phi e)|_{L_x^2} \text{ , a.s. ,} \end{aligned}$$

where  $\mathcal{B}^3$  is the natural adapted basis of  $L^2(\mathcal{O}; \mathbb{R}^3)$

$$e_1^1 = (\varepsilon_1, 0, 0), \quad e_1^2 = (0, \varepsilon_1, 0), \quad e_1^3 = (0, 0, \varepsilon_1), \quad e_2^1 = (\varepsilon_2, 0, 0), \quad e_2^2 = (0, \varepsilon_2, 0) \dots$$

Due to orthogonality, immediate computations imply:

$$\int_0^t \langle \nabla u, \nabla(u \times dW_\phi) \rangle = \int_0^t \langle \nabla u, u \times d\nabla W \rangle .$$

Disregarding the time variable for clarity, we now expand

$$\begin{aligned} \frac{1}{2} \sum_{e \in \mathcal{B}^3} |\nabla(u \times \phi e)|_{L_x^2}^2 &= \frac{1}{2} \sum_{\varepsilon \in \mathcal{B}} \left| \nabla \left( \tilde{\phi} \varepsilon \begin{bmatrix} 0 \\ u^3 \\ -u^2 \end{bmatrix} \right) \right|_{L_x^2}^2 + \left| \nabla \left( \tilde{\phi} \varepsilon \begin{bmatrix} -u^3 \\ 0 \\ u^1 \end{bmatrix} \right) \right|_{L_x^2}^2 \\ &\quad + \left| \nabla \left( \tilde{\phi} \varepsilon \begin{bmatrix} u^2 \\ -u^1 \\ 0 \end{bmatrix} \right) \right|_{L_x^2}^2 \\ &= \frac{1}{2} \sum_{\varepsilon \in \mathcal{B}} \int_{\mathcal{O}} \sum_{j=1}^d \left\{ 2(\partial_j \tilde{\phi} \varepsilon(x))^2 ((u^1)^2 + (u^2)^2 + (u^3)^2)(x) \right. \\ &\quad \left. + 2(\tilde{\phi} \varepsilon(x))^2 ((\partial_j u^1)^2 + (\partial_j u^2)^2 + (\partial_j u^3)^2)(x) \right. \\ &\quad \left. + 2(\tilde{\phi} \varepsilon(x))(\partial_j \tilde{\phi} \varepsilon(x))(u^1 \partial_j u^1 + u^2 \partial_j u^2 + u^3 \partial_j u^3)(x) \right\} dx \\ &= \sum_{\varepsilon \in \mathcal{B}} |\nabla \circ \tilde{\phi} \varepsilon|_{L_x^2}^2 + |(\tilde{\phi} \varepsilon) \nabla u|_{L_x^2}^2 , \end{aligned} \tag{2.A.3}$$

where for the last equality we used the pointwise constraint on the magnitude **(C)**, in both forms  $\sum_{i=1}^3 (u^i)^2 \equiv 1$  and  $\nabla u \cdot u \equiv 0$ .

On the other hand, going back to the definition of  $F_\phi(u)$  – see **(1.15)** – we have:

$$\begin{aligned} \langle \nabla u, \nabla(F_\phi(u)) \rangle &= \frac{1}{2} \sum_{e \in \mathcal{B}^3} \langle \nabla u, \nabla((u \times \phi e) \times \phi e) \rangle \\ &= \frac{1}{2} \sum_{l \in \mathbb{N}} \left\{ \sum_{\alpha=1}^3 \left( \langle \nabla u, (\nabla u \times \phi e_l^\alpha) \times \phi e_l^\alpha \rangle \right. \right. \\ &\quad \left. \left. + \langle \nabla u, (u \times \nabla \phi e_l^\alpha) \times \phi e_l^\alpha \rangle + \langle \nabla u, (u \times \phi e_l^\alpha) \times \nabla \phi e_l^\alpha \rangle \right) \right\} \\ &= \frac{1}{2} \sum_{\varepsilon \in \mathcal{B}} (-2|(\tilde{\phi} \varepsilon) \nabla u|_{L_x^2}^2 + 4\langle \nabla u, u(\nabla \phi \varepsilon)(\phi \varepsilon) \rangle) \\ &= - \sum_{\varepsilon \in \mathcal{B}} |(\phi \varepsilon) \nabla u|_{L_x^2}^2 , \end{aligned} \tag{2.A.4}$$

where we have used again the pointwise constraint **(C)** through the properties

$$\left[ \begin{aligned} \sum_{i=1}^3 (u^i(\omega, t, x))^2 &= 1 , \text{ and } \nabla u \cdot u(\omega, t, x) = 0 , \text{ for a.e. } (\omega, t, x) , \\ \sum_{\vec{a} \in \vec{B}} (\vec{v} \times \vec{a}) \times \vec{a} &= -2|\vec{v}|_{\mathbb{R}^3}^2 , \forall \vec{v} \in \mathbb{R}^3 , \text{ with } \vec{B} = \text{ONB of } \mathbb{R}^3 . \end{aligned} \right. \tag{2.A.5}$$

Summing all these contributions, we obtain

$$E(t) - E(0) = \int_0^t \langle -\Delta u, \Delta u + u|\nabla u|^2 + u \times \Delta u \rangle ds \\ + t \sum_{\varepsilon \in \mathcal{B}} |\nabla \tilde{\phi} \varepsilon|_{L_x^2}^2 + \int_0^t \langle \nabla u, u \times d\nabla W \rangle, \text{ a.s. for } 0 \leq t < \tau .$$

Since for  $v \in H_x^2$ , by (C), and by skew-symmetry of  $u(t, x) \times \cdot$ :

$$\int_0^t \langle \Delta u, \Delta u + u|\nabla u|^2 + u \times \Delta u \rangle ds = \int_0^t \langle \Delta u, -u \times (u \times \Delta u) + u \times \Delta u \rangle ds \\ = - \int_0^t |u \times \Delta u|_{L_x^2}^2 ds ,$$

which gives the formula (2.A.2). ■

**Local dissipation.** Let  $\eta \in C_0^\infty(\mathcal{O}; \mathbb{R})$ ,  $r > 0$ , and  $x_0 \in \mathcal{O}$ , such that:

- (1)  $\text{Supp}(\eta) \subseteq B_r(x_0)$  (i.e. the ball of radius  $r$  centered at  $x_0$ );
- (2)  $\sup_{x \in \mathcal{O}} |\eta'(x)| \leq c/r$ .

Then:

$$\frac{|\eta \nabla u(t)|_{L^2}^2}{2} - \frac{|\eta \nabla u(0)|_{L^2}^2}{2} \leq t |\eta \nabla \tilde{\phi}|_{\mathbb{L}^2}^2 + \frac{c_1}{r^2} \int_0^t |\nabla u|_{L_x^2}^2 ds + \int_0^t \langle \eta \nabla u, \eta u \times \nabla dW_\phi \rangle, \quad (2.A.6)$$

where we denote

$$|\eta \nabla \tilde{\phi}|_{\mathbb{L}^2}^2 = \sum_{\varepsilon \in \mathcal{B}} \int_{\mathcal{O}} \eta(x)^2 |\nabla \tilde{\phi} \varepsilon(x)|_{\mathbb{R}^2}^2 dx .$$

*Proof.* Similarly to the proof of (2.A.2), Itô Formula writes for  $(1/2)|\eta \nabla u|_{L_x^2}^2$ :

$$\frac{|\eta \nabla u(t)|^2}{2} - \frac{|\eta \nabla u(0)|_{L_x^2}^2}{2} - \int_0^t \left( \langle \eta^2 \nabla u, \nabla(F_\phi(u)) \rangle + \frac{1}{2} \sum_{e \in \mathcal{B}^3} |\eta \nabla(u \times \phi e)|_{L_x^2}^2 \right) ds \\ = \int_0^t \langle \eta^2 \nabla u, \nabla(u \times \Delta u - u \times (u \times \Delta u)) \rangle ds \\ + \int_0^t \langle \eta^2 \nabla u, \nabla(u \times dW_\phi) \rangle \text{ a.s. ,}$$

Moreover, the same computations as that of the proof of (2.A.2) lead to the identity

$$\int_0^t \left( \langle \eta^2 \nabla u, \nabla(F_\phi(u)) \rangle + \frac{1}{2} \sum_{e \in \mathcal{B}^3} |\eta \nabla(u \times \phi e)|_{L_x^2}^2 \right) ds \\ = \int_0^t \left( \sum_{\varepsilon \in \mathcal{B}} \int_{\mathcal{O}} (\eta(x))^2 |\nabla \tilde{\phi} \varepsilon(x)|_{\mathbb{R}^2}^2 dx \right) ds . \quad (2.A.7)$$

Indeed, notice that the computations in (2.A.3)-(2.A.4) remain identical when replacing  $\langle \cdot, \cdot \rangle$  by  $\langle \eta \cdot, \eta \cdot \rangle$ . Therefore, we obtain a.s.

$$\begin{aligned}
& \frac{|\eta \nabla u(t)|^2}{2} - \frac{|\eta \nabla u(0)|_{L_x^2}^2}{2} - \int_0^t |\eta \nabla \tilde{\phi}|_{\mathbb{L}_2}^2 ds \\
&= \int_0^t \langle \eta^2 \nabla u, \nabla(u \times \Delta u - u \times (u \times \Delta u)) \rangle ds + \int_0^t \langle \eta^2 \nabla u, \nabla(u \times dW_\phi) \rangle \\
&= \int_0^t \left( - \langle 2\eta \nabla \eta \nabla u + \eta^2 \Delta u, u \times \Delta u - u \times (u \times \Delta u) \rangle \right) ds + \int_0^t \langle \eta^2 \nabla u, u \times \nabla dW_\phi \rangle \\
&= \int_0^t \left( - 2 \langle \eta \nabla \eta \nabla u, u \times \Delta u - u \times (u \times \Delta u) \rangle - \langle \eta^2 u \times \Delta u, u \times \Delta u \rangle \right) ds \\
& \quad + \int_0^t \langle \eta^2 \nabla u, u \times \nabla dW_\phi \rangle \\
&\leq \int_0^t \left( - |\eta u \times \Delta u|_{L_x^2}^2 + |\eta u \times \Delta u|_{L_x^2}^2 + \frac{c}{r^2} |\nabla u|_{L_x^2}^2 \right) ds + \int_0^t \langle \eta^2 \nabla u, u \times \nabla dW_\phi \rangle ,
\end{aligned}$$

where we have used respectively (2.A.7),  $\Delta u + u|\nabla u|^2 = -u \times (u \times \Delta u)$ , the skew symmetry of  $u \times \cdot$  and  $ab \leq a^2/2 + b^2/2$ . This proves (2.A.6).  $\blacksquare$

## (2) Some technical proofs

**Claim 2.1.** With the notations of the proof of Proposition 2.3, define for  $n \in \mathbb{N}^2$ :

$$\begin{aligned}
T_{l,k} &= \sup_{t \in [0, T]} \left| \int_0^t \langle y_{l,k}, z_{l,k}^2 \times d(W_l^1 - W_l^2) \rangle \right| + \int_0^T \langle y_{l,k}, F_l^1(z_{l,k}^j) - F_l^2(z_{l,k}^j) \rangle dt \\
&+ \left| \int_0^T \frac{1}{2} D^2(|y_{l,k}|_{L_x^2}^2/2) \cdot d \ll \int_0^T y_{l,k} \times dW_l^1 + \int_0^T z_{l,k}^2 \times d(W_l^1 - W_l^2) \gg_t \right| . \quad (2.A.8)
\end{aligned}$$

Then, for all  $k \in \mathbb{N}$ ,  $\mathbb{E}[T_{l,k}] \rightarrow 0$  as  $l \rightarrow \infty$ .

*Proof.* Since it does not play any role in this proof, we omit the index  $k$  for more clarity. We write  $\mathbb{E}[T_l] = I + II + III$ , and treat each term separately. First,

$$\begin{aligned}
I &= \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \langle y_l, z_l^2 \times d(W_l^1 - W_l^2) \rangle \right| \\
&\leq \mathbb{E} \left[ \left( \int_0^T \langle y_l, z_l^2 \times (\phi_l^1 - \phi_l^2)(\phi_l^1 - \phi_l^2)^* (-z_l^2) \times y_l \rangle dt \right)^{1/2} \right] \\
&\leq c \mathbb{E} \left[ \|y_l\|_{L_T^2 L_x^2}^{1/2} \|(\phi_l^1 - \phi_l^2)(\phi_l^1 - \phi_l^2)^* y_l\|_{L_T^2 L_x^2}^{1/2} \right] \\
&\leq c' |(\phi_l^1 - \phi_l^2)|_{\mathbb{L}_2^{0,0}} \mathbb{E}[\|y_l\|_{L_T^2 L_x^2}] \rightarrow 0 \quad \text{as } l \rightarrow \infty ,
\end{aligned}$$

where we have used respectively Proposition 2.1, (2.11), and  $|\phi_l^1 - \phi_l^2|_{\mathcal{L}(L_x^2, L_x^2)} \leq c|\phi_l^1 - \phi_l^2|_{\mathbb{L}_2}$ . Secondly:

$$\begin{aligned}
II &= \mathbb{E} \int_0^T \sum_{e \in \mathcal{B}^3} \left( \langle y_l, (z_l^1 \times \phi_l^1 e) \times \phi_l^1 e \rangle - \langle y_l, (z_l^2 \times \phi_l^2 e) \times \phi_l^2 e \rangle \right. \\
& \quad \left. + |y_l \times \phi_l^1 e|_{L_x^2}^2 + |z_l^2 \times (\phi_l^1 - \phi_l^2) e|_{L_x^2}^2 \right) dt .
\end{aligned}$$

Since  $|z_l|_{L_x^\infty}$  is bounded, and  $|\phi_n - \phi|_{\mathbb{L}_2(L^2; H_x^1)} \rightarrow 0$ , using  $H_x^1 \hookrightarrow L_x^4$  we obtain for each  $j = 1, 2$ ,

$$\sum_{e \in \mathcal{B}^3} (z_l^j \times \phi_l^j e) \times \phi_l^j e \rightarrow \sum_{e \in \mathcal{B}^3} (z_l^j \times \phi e) \times \phi e \text{ in } L_x^2 .$$

and so  $II \rightarrow 0$ . The second term is treated by the same argument.

The term  $III$  is handled in the same way, expanding the quadratic variation as

$$\begin{aligned} \mathbb{E} \left| \int_0^T \frac{1}{2} D^2(|y_l|_{L_x^2}^2/2) \cdot d \ll \int_0^T y_l \times dW_l^1 + \int_0^T z_l^2 \times d(W_l^1 - W_l^2) \gg_t \right| \\ \leq 2 \mathbb{E} \left( \int_0^T \sum_{e \in \mathcal{B}^3} |y_l \times \phi_l^1 e|_{L_x^2}^2 dt + \int_0^T \sum_{e \in \mathcal{B}^3} |z_l \times (\phi_l^1 - \phi_l^2) e|_{L_x^2}^2 dt \right) \rightarrow 0 . \quad \blacksquare \end{aligned}$$

**Claim 2.2.** With the notations defined in the proof of Corollary 2.3, Define for  $n, k \in \mathbb{N}$ :

$$\begin{aligned} R_{n,k} = \sup_{t \in [0, T]} \left| \int_0^t \frac{1}{2} D^2(|\nabla \delta_{n,k}|_{L_x^2}^2/2) \cdot d \ll \int_0^t \delta_{n,k} \times dW_n - \int_0^t u_k \times d(W_n - W) \gg_t \right| \\ + \sup_{t \in [0, T]} \left| \int_0^t \langle \nabla \delta_{n,k}, \delta_{n,k} \times \nabla dW_n \rangle + \langle \nabla \delta_{n,k}, u_k \times \nabla d(W_n - W) \rangle \right| \\ + \int_0^T \langle \nabla \delta_{n,k}, \nabla (F_n(u_{n,k}) - F_\phi(u_k)) ds \rangle \quad (2.A.9) \end{aligned}$$

For all  $k \in \mathbb{N}$ ,  $R_{n,k} \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* We again omit the index  $k \in \mathbb{N}$ . Proposition 2.1 gives

$$\begin{aligned} I = \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \langle \nabla \delta_n, \delta_n \times \nabla dW_n \rangle + \langle \nabla \delta_n, u \times \nabla d(W_n - W) \rangle \right| \\ \leq \left( \mathbb{E} \int_0^T \langle \nabla \delta_n, \delta_n \times \nabla \phi_n \phi_n^* \nabla^* (-\delta_n) \times \nabla \delta_n \rangle dt \right)^{1/2} \\ + \left( \mathbb{E} \int_0^T \langle \nabla \delta_n, u \times \nabla (\phi_n - \phi) (\phi_n - \phi)^* \nabla^* (-u) \times \nabla \delta_n \rangle dt \right)^{1/2} \\ = I_1 + I_2 . \end{aligned}$$

By  $|\delta_n|_{\mathbb{R}^3} \leq 2$  we have:

$$I_1 \leq 2 \left( \mathbb{E} \int_0^T |\nabla \delta_n|_{L_x^2}^2 |\nabla \phi_n|_{\mathcal{L}(L_x^2)}^2 dt \right)^{1/2} ,$$

and  $I_1 \rightarrow 0$  since  $\delta_n \rightarrow 0$  in  $L^2(\Omega \times [0, T]; H_x^1)$  and  $|\nabla \phi_n| \leq c(|\phi|_{\mathbb{L}_2^{0,1}})$ . Similarly:

$$I_2 \leq 2 \left( \mathbb{E} \int_0^T |\nabla \delta_n|_{L_x^2}^2 |\nabla (\phi_n - \phi)|_{\mathcal{L}(L_x^2)}^2 dt \right)^{1/2} \rightarrow 0 ,$$

because of  $\phi_n \rightarrow \phi$  in  $\mathbb{L}_2^{0,1}$ .

Using Hölder inequality, we obtain for the second term:

$$\begin{aligned}
II \leq \mathbb{E} \int_0^T |\nabla \delta_n|_{L_x^4} \sum_{e \in \mathcal{B}^3} & \left( |\nabla \delta_n|_{L_x^4} |\phi e|_{L_x^2}^2 + |\delta_n|_{L_x^\infty} |\nabla \phi_n e|_{L_x^2} |\phi_n e|_{L_x^4} \right. \\
& + 2 |\nabla u|_{L_x^4} |\phi_n e - \phi e|_{L_x^4} |\phi_n e + \phi e|_{L_x^4} \\
& \left. + 2 |u|_{L_x^\infty} |\nabla(\phi_n - \phi) e|_{L_x^2} |\phi_n e + \phi e|_{L_x^4} \right) dt
\end{aligned}$$

and  $II \rightarrow 0$  follows by  $\mathbb{E} \|\nabla(u_n - u)\|_{L_T^2 L_x^4} \rightarrow 0$ ,  $|\phi_n - \phi|_{\mathbb{L}_2^{0,1}} \rightarrow 0$ , and  $H_x^1 \hookrightarrow L_x^4$ .

Lastly,

$$III \leq 2\mathbb{E} \int_0^T \left( |\nabla(\delta_n \times \phi_n e)|_{L^2}^2 + |\nabla(u \times \phi_n e - \phi e)|_{L^2}^2 \right) dt,$$

and  $III \rightarrow 0$  by the same justifications as that of  $II$ . ■

## CHAPTER 3.

### *A uniqueness criterion in dimension two*

---

For the Harmonic Map Flow in dimension 2, there is a famous result by A. Freire that there exists a unique energy decreasing weak solution. For the stochastic perturbation, there is no hope for solutions of the equation to be energy decreasing. We however adapt Freire's criteria of uniqueness. Indeed, we prove that under the assumption that the energy functional satisfies a supermartingale-type property, which turns out to be the stochastic counterpart of assuming that the energy decreases, then pathwise uniqueness holds.

---



# 1 Motivations

Let  $\mathcal{O}$  denote a bounded (regular) surface. For  $T > 0$ ,  $u_0 \in H^1(\mathcal{O}; \mathbb{R}^3)$ , we are concerned with giving a criteria of uniqueness for solutions of the boundary value problem:

$$\begin{cases} du - \Delta u dt = (\gamma u \times \Delta u + u|\nabla u|^2 + F_\phi(u)) + u \times dW_\phi, & \text{on } \Omega \times [0, T] \times \mathcal{O}, \\ u|_{\partial\mathcal{O}} = u_0|_{\partial\mathcal{O}}, & \text{on } \Omega \times [0, T], \\ u(0) = u_0, & \text{on } \Omega. \end{cases} \quad (3.1)$$

where we keep the notations (2.10)-(1.15),  $\gamma \in \{0, 1\}$ , and  $u_0 \in H_x^1 \cap \{v : |v(x)| = 1 \text{ a.e.}\}$  is given. When  $W_\phi \equiv 0$ , a *weak solution* for LLG can be defined as a map  $u \in H^1([0, T] \times \mathcal{O})$  fulfilling:

(i)  $\nabla u \in L^\infty([0, T]; L_x^2)$ ;  $\partial_t u \in L^2([0, T] \times \mathcal{O})$ ;  $|u|_{\mathbb{R}^3} = 1$  a.e.;

(ii) for all  $\varphi \in \mathcal{C}_0^\infty((0, T) \times \overset{\circ}{\mathcal{O}})$ :

$$\gamma \int_{[0, T] \times \mathcal{O}} \partial_t u \cdot \varphi dx dt + \int_{[0, T] \times \mathcal{O}} (u \times \partial_t u) \cdot \varphi dx dt = \sum_{i=1,2} (u \times \partial_i u) \cdot \partial_j \varphi dx dt; \quad (3.2)$$

(iii)  $u|_{\{0\} \times \mathcal{O}} = u_0$ ,  $u|_{[0, T] \times \partial\mathcal{O}} = u_0|_{\partial\mathcal{O}}$  in the sense of traces;

(iv) for all  $t \in [0, T]$ ,

$$\frac{|\nabla u(t, \cdot)|_{L_x^2}^2}{2} + \frac{1}{1 + \gamma} \int_0^t |\partial_t u(s, \cdot)|_{L_x^2}^2 ds \leq \frac{|\nabla u_0|_{L_x^2}^2}{2}.$$

Struwe's Theorem (Thm. 2.1) and others in related papers (see [KC89, GH93, Str96]) prove in particular that weak solutions exist globally time, for  $\gamma = 0$  or 1. Their existence is also proved in dimensions higher than 2, see e.g. [Che89] for  $\gamma = 0$ , [Vis85, AS92] for  $\gamma = 1$ . For  $\dim \mathcal{O} \geq 3$ , examples of nonuniqueness are given in [Cor90, AS92], for  $\gamma = 0$  and 1.

In dimension two, nonuniqueness has remained an open problem until M. Bertsh, R. Dal Passo, R. Van der Hout [BDvdH02], and independently P. Topping [Top02], finally found some way of constructing examples for  $\gamma = 0$ . The main reason for the difficulty to find two different weak solutions for

$$\partial_t u - \Delta u = u|\nabla u|^2, \quad u(0) = u_0,$$

is that for each  $t > 0$ , roughly speaking, the nonlinearity  $u(t, \cdot)|\nabla u(t, \cdot)|^2$  "almost belongs to  $H_x^{-1}$ ", and the Laplace equation with a right hand side in  $H_x^{-1}$  is well-posed in  $H_x^1$ . On the other hand, we have the following uniqueness result.

**Theorem 3.1** ([Fre95]). *Set  $\gamma = 0$ . Let  $\mathcal{O}$  be a smooth surface with possibly empty boundary. If  $u_1, u_2 \in H^1([0, T] \times \mathcal{O})$  satisfy (i)-(ii)-(iii), and if moreover the following criterion is fulfilled for  $j = 1, 2$ :*

(iv')  $|\nabla u_j(t)|_{L_x^2}^2 \leq |\nabla u_j(s)|_{L_x^2}^2$  for every  $s < t$ ,

where  $\nabla u(t)$  denotes the trace of  $\nabla u$  onto  $\{t\} \times \mathcal{O}$ , then  $u_1 = u_2$ .

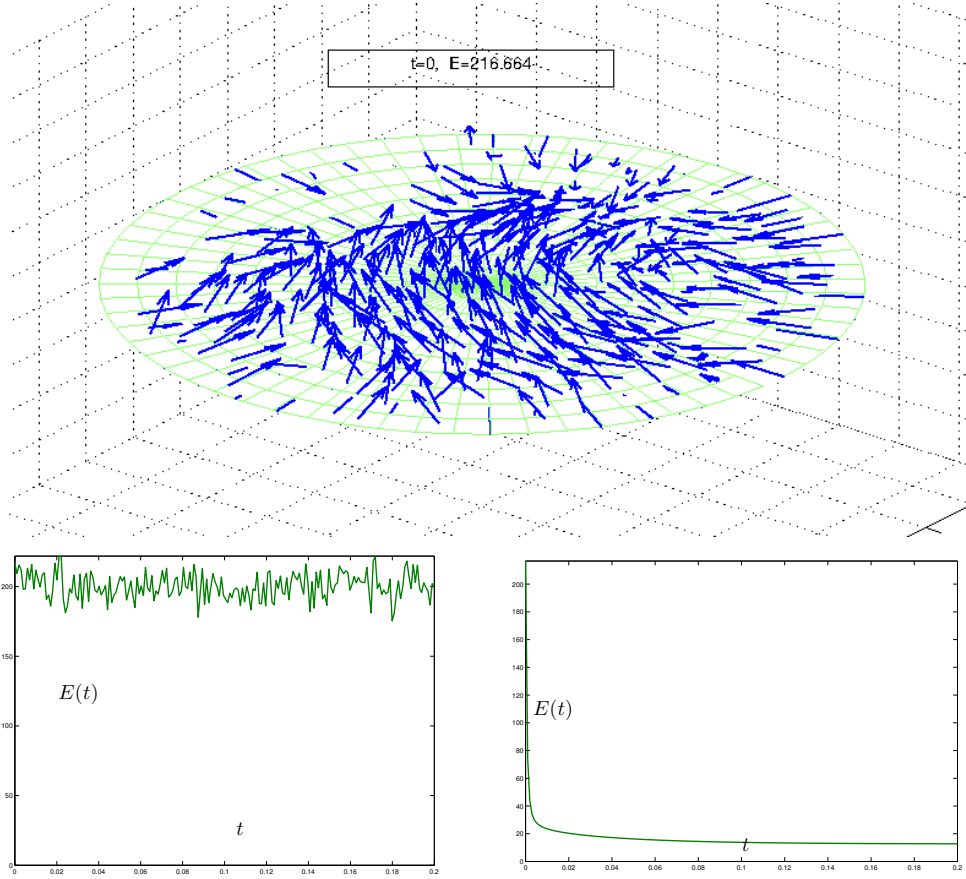


Figure 3.1 – Evolution of the energy, with noise (left), and without (right). The common initial data is displayed above. We use the discretization detailed in Chapter 6. The increments of the noise are regularized in space so that  $\dot{W}(t, \cdot) \in H_x^1$ , a.s. as the size of the triangles  $h_{\min} \rightarrow 0$ .

When  $\gamma = 1$ , weak solutions for the deterministic problem are not expected to be unique, but the problem still remains open. However, the result of A. Freire stays true, see [YSB98]. We mention however that the authors give an incorrect proof, since they seem to use the “density of  $C^\infty([0, T] \times \mathcal{O})$  into  $L^\infty([0, T]; \mathcal{C}^1(\mathcal{O}))$ ”, which is false. In [Har04] the author gives a counterexample to this density statement, but explains how the uniqueness result for LLG can be correctly obtained.

Set now  $\gamma = 1$ , and consider the stochastic case, with a correlation  $\phi\phi^*$  such that  $\phi = I_3 \otimes \tilde{\phi} \in \mathbb{L}_2(L_x^2; H_x^1)$ , see (2.6). We recall the energy formula for solutions  $u$  supported in  $\mathcal{C}([0, \tau]; H_x^2)$  for some stopping time  $\tau > 0$ : if  $t \in [0, \tau)$ , then

$$\frac{|\nabla u(t)|_{L_x^2}^2}{2} + \int_0^t |u \times \Delta u|_{L_x^2}^2 ds = \frac{|\nabla u_0|_{L_x^2}^2}{2} + t|\nabla \tilde{\phi}|_{\mathbb{L}_2}^2 + X_t, \quad \text{a.s.}, \quad (3.3)$$

where  $(X_t)$  is some (local) martingale (see (2.A.2)). At this stage, when considering noise in (3.2), we see that given a trajectory  $\omega \in \Omega$ , the energy  $t \mapsto E(\omega, t)$  has no actual reason to decrease, see Figure 3.1. Due to the additional drift term  $|\nabla \phi|_{\mathbb{L}_2}^2$ , this may not be true even if we only consider the mean value  $\mathbb{E}|\nabla u(t)|_{L_x^2}^2$  instead of the pathwise energy. Therefore one may doubt on the fact that Theorem 3.1 can be adapted in this setting.

Nevertheless, provided every term makes sense in (3.3), one sees that the Dirichlet energy minus some linear growth  $c(\phi)t$  is actually a *supermartingale*. If we denote by  $\mathcal{G} : \Omega \times [0, T]$  this quantity, namely

$$\mathcal{G}(t) := \frac{|\nabla u(t, \cdot)|_{L_x^2}^2}{2} - |\nabla \tilde{\phi}|^2 t, \quad \text{for } t \in [0, T], \quad (3.4)$$

then the following supermartingale-type property

$$\mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t)] - \mathcal{G}(s) = -\mathbb{E}_{\mathcal{F}_s} \int_0^t |u \times \Delta u|_{L_x^2}^2 ds, \quad 0 \leq s \leq t \leq T, \quad (3.5)$$

is, roughly speaking, an analogue of

$$\frac{|\nabla f(t, \cdot)|_{L_x^2}^2}{2} - \frac{|\nabla f(s, \cdot)|_{L_x^2}^2}{2} = -\frac{1}{2} \int_s^t |\partial_t f(\sigma, \cdot)|_{L_x^2}^2 d\sigma, \quad (3.6)$$

for a classical solution  $f$  of the deterministic problem, see e.g. [KC89, Prop. 2.3]. It is therefore natural to ask whether, requiring the property  $\mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t)] - \mathcal{G}(s) \leq 0$  instead of  $E(t) - E(s) \leq 0$ ,  $0 \leq s \leq t \leq T$ , we can recover pathwise uniqueness of the solutions of (3.1). It turns out that the answer is positive.

## 2 Statement of the result

We now give a notion of weak solution for (3.1) which will be also used in Chapter 5. Since the word “weak” may lead to some confusions, we rather employ the terminology “martingale solution” used e.g. in [DZ08]. This means that we are looking for solutions to the associated martingale problem. In the sequel we assume that the Wiener process  $W_\phi$  is such that every term in (3.3) makes sense, namely we have

$$\tilde{\phi} \in \mathbb{L}_2(L^2(\mathcal{O}; \mathbb{R}); H^1(\mathcal{O}; \mathbb{R})) , \quad (3.7)$$

and  $\phi = I_3 \otimes \tilde{\phi}$ , see (2.6).

*Remark 3.1* (On the weak sense of “ $u \times \Delta u$ ”). As noticed by Chen in [Che89], one can interpret the term  $u \times \Delta u$  in the weak form  $\operatorname{div}(u \times \nabla u)$ , namely if  $u \in H_x^1$ , then by  $u \times \Delta u$  we mean an element of  $W_x^{-1,4/3}$ , such that the identity

$$\langle (u \times \Delta u), \varphi \rangle_{W_x^{-1,4/3}, W_x^{1,4}} = \sum_{j=1,2} \langle \partial_j u, u \times \partial_j \varphi \rangle_{L_x^2},$$

holds true for every  $\varphi \in W_x^{1,4}$  with  $\varphi|_{\partial\mathcal{O}} = 0$ .

**Definition 3.1** (Martingale solution). Given  $T > 0$ , a martingale solution on  $[0, T]$  of (5.1) is given by a stochastic basis  $\mathfrak{F} = (\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \in [0, T]}, W)$ , where  $W$  has covariance  $\phi\phi^*$ , and  $u : \Omega \times [0, T] \rightarrow H_x^1$ , a progressively measurable process satisfies the following assumptions:

(1) for  $\mathbb{P}$  – a.e.  $\omega \in \Omega$ ,

$$u(\omega, \cdot) \in \mathcal{C}([0, T]; L_x^2), \quad \text{and } |u(\omega, t, x)| = 1 \text{ for a.e. } (t, x) \in [0, T] \times \mathcal{O}; \quad (3.8)$$

$$(2) \mathbb{E} \left[ \text{ess sup}_{t \in [0, T]} |\nabla u(t)|_{L_x^2}^2 + \int_0^T |u \times \Delta u|_{L_x^2}^2 dt \right] < \infty ;$$

(3)  $u$  satisfies (5.1) in the sense

$$u(t) = u_0 + \int_0^t \left( \Delta u(s) + u(s) |\nabla u(s)|^2 + u(s) \times \Delta u(s) + F_\phi(u)(s) \right) ds \\ + \int_0^t u(s) \times dW(s), \quad \forall t \in [0, T], \mathbb{P} - \text{a.s.},$$

where the first integral is the Bochner integral, and the second is the Itô integral, in the space  $L_x^2$ .

This notion of a solution is motivated by the following result obtained by Z. Brzezniak, B. Goldys and T. Jegaraj, for the associated Neumann problem in three dimensions:

$$\begin{cases} du - \Delta u dt = (u \times \Delta u + u |\nabla u|^2 + F_\phi(u)) + u \times dW_\phi, & \text{in } \Omega \times [0, T] \times \mathcal{O}, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Omega \times [0, T] \times \partial \mathcal{O}, \\ u(0) = u_0, & \text{in } \Omega \times \mathcal{O}. \end{cases} \quad (3.9)$$

**Theorem 3.2** ([BGJ13]). *Let  $\mathcal{O}$  be a bounded, three dimensional domain. Let  $T > 0$ ,  $u_0 \in H^1(\mathcal{O}; \mathbb{R}^3)$  with  $|u(x)| = 1$ ,  $dx$ -a.e. and  $\frac{\partial u_0}{\partial n} = 0$ . Consider a one-dimensional noise*

$$\phi \dot{W}(t, x) = \dot{W}_1(t) h(x),$$

where  $h \in W^{1, \infty}(\mathcal{O}; \mathbb{R}^3)$ , and  $W_1$  is a real valued Wiener process.

*There exists a martingale solution  $(\mathfrak{P}, u)$  to (3.9), in the sense of Definition 3.1.*

The proof relies on Faedo-Galerkin approximations, for which the authors give uniform estimates based on a finite dimensional analogue of (3.3). However, no result on uniqueness as that of Theorem 3.1 is expected, since nonuniqueness for deterministic solutions is a known fact when  $\dim \mathcal{O} = 3$ , see Section 1.

We recall the notion of pathwise uniqueness, see e.g. [WI81].

**Definition 3.2** (Pathwise Uniqueness). We say that pathwise uniqueness holds for (3.1) if, whenever  $(u, W)$  and  $(u', W')$  are two solutions of (3.1) defined on a same probability space, then  $u_0 = u'_0$  and  $W = W'$  implies  $u = u'$ .

To avoid unuseful complications, we choose to work on the unit disk  $\mathbb{D} = \{x \in \mathbb{R}^2, |x| \leq 1\}$ , but the result stays true in the Riemannian framework given in Theorem 3.1. We therefore consider the equation (3.1) with  $\mathcal{O} = \mathbb{D}$ .

**Theorem 3.3** (Conditional pathwise uniqueness). *Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), W_\phi)$  be a stochastic basis with right continuous filtration, and  $T > 0$ . Let  $u_j$ ,  $j = 1, 2$  denote solutions of (3.1) on  $\Omega \times [0, T] \times \mathbb{D}$ , in the sense given in definition 3.1, with an associated  $\phi \in \mathbb{L}_2(L_x^2; H_x^1)$  of the form  $I_3 \otimes \tilde{\phi}$ , see (2.6). Assume moreover that for  $j = 1, 2$ ,  $u_j$  verifies the criterion that the renormalized energy functional  $\mathcal{G}_j$  given by*

$$\mathcal{G}_j : (\omega, t) \in \Omega \times [0, T] \longmapsto \frac{1}{2} |\nabla u_j(\omega, t)|_{L_x^2}^2 - |\nabla \tilde{\phi}|_{\mathbb{L}_2}^2 t$$

is a supermartingale with respect to  $(\mathcal{F}_t)$ , where  $\nabla u(\omega, t)$  denotes the trace of the map  $\nabla u(\omega)$  onto  $\{t\} \times \mathcal{O}$ , and we recall that  $|\nabla \phi|_{\mathbb{L}_2}^2$  means  $\sum_{\varepsilon \in \mathcal{B}} |\nabla \tilde{\phi}_\varepsilon|_{L_x^2}^2$  for an ONB  $\mathcal{B}$  of  $L^2(\mathbb{D}; \mathbb{R})$ . Then we have

$$\mathbb{P}(u_1 = u_2) = 1 .$$

Synthetically speaking, pathwise uniqueness holds conditionnally to the fact that  $\mathcal{G}$  is a supermartingale.

*Remark 3.2.* In Chapter 2, we have shown that when the spatial domain is the two-dimensional torus  $\mathbb{T}^2$ , there exist a stochastic process  $v$  on  $\Omega$ , a nondecreasing sequence of stopping times  $\vartheta^0 = 0 < \vartheta^1 \vartheta^2, \dots, \vartheta^j$ , such that  $\mathbb{P}$ -a.s.

$$\exists J \in \mathbb{N}, \quad \vartheta^J = T \tag{3.10}$$

with for all  $j \in \llbracket 0, J-1 \rrbracket$ ,

$$\begin{cases} v|_{[\vartheta^j, \vartheta^{j+1})} \in \mathcal{C}([\vartheta^j, \vartheta^{j+1}); H_x^1) \text{ and} \\ v|_{[\vartheta^j, \vartheta^{j+1})} \in L_{\text{loc}}^2(\vartheta^j, \vartheta^{j+1}; H_x^2) ; \end{cases} \tag{3.11}$$

and  $v$  is a solution of (3.1). Moreover there holds

$$v|_{[\vartheta^j, \vartheta^{j+1})} \in L_{\text{loc}}^4(\vartheta^j, \vartheta^{j+1}; W_x^{1,4}) , \tag{3.12}$$

indeed, it suffices to apply the following Gagliardo-Nirenberg type inequality: (see for instance [LSU68]): there exists  $c > 0$  such that for all  $f \in H_x^1$ ,

$$|f|_{L_x^4}^2 \leq c |f|_{H_x^1} |f|_{L_x^2} . \tag{3.13}$$

We believe however that the assumption  $\mathcal{O} = \mathbb{T}^2$  could be relaxed, using e.g. the arguments presented in [KC89] for the deterministic Dirichlet boundary value problem, when  $u(0)|_{\partial\mathcal{O}} \in H^{3/2}(\partial\mathcal{O}; \mathbb{R}^3)$ . Adapting the proof of Theorem 2.2 when  $\partial\mathcal{O} \neq \emptyset$  would certainly lead to similar conclusions. Therefore, we admit the existence of a solution  $v$  to (3.1), satisfying (3.10)-(3.11). However, the proof we give here remains true when  $\mathbb{D}$  is replaced by  $\mathbb{T}^2$ .

### 3 Proof of Theorem 3.3

Let  $T > 0$ , and denote by  $u_j, j = 1, 2$  two solutions of (3.1), starting from the same initial data  $u(0) \in H_x^1$ , and with the same boundary condition  $u(0)|_{\partial\mathcal{O}}$ . Assume without loss of generality that  $u_2$  is the strong solution given by Remark 3.2. Set  $w = u_1 - u_2$ , denote by  $b(u_j)$  the nonlinearity  $u_j |\nabla u_j|^2$  and note that  $w$  satisfies for  $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{D}$ :

$$\begin{cases} dw = (\Delta w + b(u_1) - b(u_2) + w \times \Delta u_1 + F_\phi(w) + u_2 \times \Delta w) dt + w \times dW_\phi , \\ w(t, \cdot)|_{\partial\mathbb{D}} = 0 , \\ w(0, \cdot) = 0 . \end{cases} \tag{3.14}$$

Observing that  $\langle w, F_\phi(w) \rangle = -1/2 \sum_{e \in \mathcal{B}^3} \langle w \times \phi e, w \times \phi e \rangle$  – see (1.15) – then Itô formula on  $1/2|w|_{L_x^2}^2$  gives us formally

$$\begin{aligned}
d \left( \frac{|w|_{L_x^2}^2}{2} \right) &= \langle w, dw \rangle + \frac{1}{2} \sum_{e \in \mathcal{B}^3} |w \times \phi e|_{L_x^2}^2 dt \\
&= \langle w, \Delta w + w \times \Delta u_1 + F_\phi(w) + u_2 \times \Delta w + b(u_1) - b(u_2) \rangle dt \\
&\quad + \langle w, w \times dW_\phi \rangle - \langle w, F_\phi(w) \rangle dt \\
&= \left( -|\nabla w|_{L_x^2}^2 + \langle w, u_2 \times \Delta w \rangle + \langle w, b(u_1) - b(u_2) \rangle \right) dt \\
&= \left( -|\nabla w|_{L_x^2}^2 - \langle \nabla w, u_2 \times \nabla w \rangle - \langle w, \nabla u_2 \cdot \nabla w \rangle \right. \\
&\quad \left. + \langle w, b(u_1) - b(u_2) \rangle \right) dt, \quad \text{a.s.},
\end{aligned}$$

with  $\mathcal{B}^3$  ONB of  $L_x^2$ , and  $\nabla u_2 \cdot \nabla w$  denotes the  $\mathbb{R}^3$ -valued map  $(\sum_{k=1,2} \partial_k u_2^i \cdot \partial_k w^i)_{1 \leq i \leq 3}$ . Note that no noise appears since it is pointwise orthogonal to the gradient of  $|w|_{L_x^2}^2$  with respect to  $w$ , and the term  $\langle \nabla w, u_2 \times \nabla w \rangle$  vanishes by skew-symmetry. Notice that this formula can be rigorously derived by a standard regularization procedure. Writing now  $b(u_1) - b(u_2) = w|\nabla u_1|^2 + u_2(\nabla u_1 + \nabla u_2) \cdot \nabla w$ , using respectively Hölder Inequality and the Young Inequality  $ab \leq a^2 + (1/4)b^2$ , we have the following formal computations:

$$\begin{aligned}
I &:= \int_0^t \langle w, b(u_1) - b(u_2) \rangle ds \leq \int_0^t (|w|_{L_x^4}^2 |\nabla u_1|_{L_x^4}^2 + |w|_{L_x^4} |\nabla u_1 + \nabla u_2|_{L_x^4} |\nabla w|_{L_x^2}) ds \\
&\leq c_1 \int_0^t (|\nabla u_1|_{L_x^4}^2 + |\nabla u_2|_{L_x^4}^2) |w|_{L_x^4}^2 ds + \frac{1}{4} \int_0^t |\nabla w|_{L_x^2}^2 ds,
\end{aligned}$$

so that (3.13) applied to  $f = w$ , and again Young inequality yields

$$\begin{aligned}
I &\leq c_2 \int_0^t (|\nabla u_1|_{L_x^4}^2 + |\nabla u_2|_{L_x^4}^2) |\nabla w|_{L_x^2} |w|_{L_x^2} ds + \frac{1}{4} \int_0^t |\nabla w|_{L_x^2}^2 ds \\
&\leq c_3 \int_0^t (|\nabla u_1|_{L_x^4}^4 + |\nabla u_2|_{L_x^4}^4) |w|_{L_x^2}^2 ds + \frac{1}{2} \int_0^t |\nabla w|_{L_x^2}^2 ds.
\end{aligned}$$

On the other hand, we have similarly

$$\begin{aligned}
II &:= \int_0^t -\langle w, \nabla u_2 \cdot \nabla w \rangle ds \leq \int_0^t |w|_{L_x^4} |\nabla u_2|_{L_x^4} |\nabla w|_{L_x^2} ds \\
&\leq c_4 \int_0^t |w|_{L_x^4}^2 |\nabla u_2|_{L_x^4}^2 ds + \frac{1}{4} \int_0^t |\nabla w|_{L_x^2}^2 ds,
\end{aligned}$$

and using again (3.13) with  $f = w$ , and Young Inequality, gives:

$$\begin{aligned}
II &\leq c_5 \int_0^t |w|_{L_x^2} |\nabla w|_{L_x^2} |\nabla u_2|_{L_x^4}^2 ds + \frac{1}{4} \int_0^t |\nabla w|_{L_x^2}^2 ds \\
&\leq c_6 \int_0^t |w|_{L_x^2}^2 |\nabla u_2|_{L_x^4}^4 ds + \frac{1}{2} \int_0^t |\nabla w|_{L_x^2}^2 ds.
\end{aligned}$$

We obtain therefore the following relation

$$\frac{1}{2}|w(t)|_{L_x^2}^2 \leq c \int_0^t (|\nabla u_1|_{L_x^4}^4 + |\nabla u_2|_{L_x^4}^4) |w(s)|_{L_x^2}^2 ds \quad , \quad \text{for a.e. } (\omega, t) \in \Omega \times [0, T] \quad , \quad (3.15)$$

so that using Grönwall Lemma gives a.s.

$$u_1|_{[0, \tau]} = u_2|_{[0, \tau]} \quad ,$$

provided  $\tau$  is a stopping time such that the map  $|\nabla u_1|_{L_x^4}^4 + |\nabla u_2|_{L_x^4}^4$  belongs pathwise to  $L^1(0, \tau)$  – note also that in this case all the computations above make sense.

By Remark 3.2, there exists a stopping time  $\tau_1 \in (0, T]$ , such that

$$\text{a.s.}, \quad u_2|_{[0, \tau_1]} \in L^4([\tau_0, \tau_1]; W_x^{1,4}) \quad .$$

In the sequel, our aim is to show the following Lemma, whose statement implies Theorem 3.3.

**Lemma 3.1.** *There exists a stopping time  $\tau \in (0, T]$  such that  $u_1|_{[0, \tau]} \in L^4(0, \tau; W_x^{1,4})$ .*

*Proof that Lemma 3.1 implies Theorem 3.3.* If the Lemma holds true, then by (3.15) we have  $\mathbb{P}$  – a.s.:  $u_1|_{[0, \tau]} = u_2|_{[0, \tau]}$  and by reiteration of the argument, we see that  $u_1$  and  $u_2$  coincide on  $[0, \vartheta^1)$  – recall that  $u_2 \in L_{\text{loc}}^2(0, \vartheta^1; H_x^2)$ , see the definition 2.8. The value at time  $\vartheta^1$  is now imposed by (3.8), so that Theorem 3.3 follows by induction on each  $[\vartheta^j, \vartheta^{j+1})$ ,  $j \in \llbracket 1, J-1 \rrbracket$ . ■

*Remark 3.3.* In the following, we can therefore assume without loss of generality that  $T$  is a stopping time such that:

$$T < \vartheta^1 \quad , \quad (3.16)$$

for in the case  $T \geq \vartheta^1$ , the argument above shows how to reiterate the procedure.

*Additional Notation.* In the sequel, we denote by  $\nabla^\perp$  the “orthogonal gradient of a map  $f \in H^1(\mathbb{D}; \mathbb{R})$ ”, namely the vector field

$$\nabla^\perp f = (\partial_2 f, -\partial_1 f) \quad , \quad (3.17)$$

whereas  $\text{div}^\perp$  denotes its formal adjoint, i.e. for every  $F = (F^1, F^2) \in H^1(\mathbb{D}; \mathbb{R}^2)$ ,

$$\text{div}^\perp F = -\partial_2 F^1 + \partial_1 F^2 \quad . \quad (3.18)$$

## 4 Proof of Lemma 3.1

### (4.a) Idea of the proof

In two dimensions, we are in the critical situation where

$$H_x^1 \hookrightarrow L_x^p \quad , \quad \forall p \in [1, \infty) \quad ,$$

and yet the assertion “ $H_x^1 \hookrightarrow L_x^\infty$ ” is false, implying in particular that  $L_x^1$  is not a subspace of  $H_x^{-1}$ . For the reason explained in Section 1, in order to prove Lemma 3.1, we aim to show that the difference of the solutions  $w = u_1 - u_2$  satisfies locally a heat equation in which the right hand side belongs at each time to  $H_x^{-1}$  and not  $L_x^1$  only. Recall Wentz’s inequality.

**Theorem 3.4** ([Wen69]). For  $\alpha, \beta \in H^1(\mathbb{D}; \mathbb{R})$ , let  $\varphi$  be solution of

$$\begin{cases} \Delta\varphi = \{\alpha, \beta\} \text{ on } \mathbb{D} , \\ \varphi|_{\partial\mathbb{D}} = 0 , \end{cases}$$

where  $\{\alpha, \beta\}$  denotes the Poisson bracket  $\partial_1\alpha\partial_2\beta - \partial_2\alpha\partial_1\beta$ . Then  $\varphi \in C(\mathbb{D}; \mathbb{R}) \cap H^1(\mathbb{D}; \mathbb{R})$ , and

$$|\varphi|_{L^\infty} + |\nabla\varphi|_{L_x^2} \leq c|\nabla a|_{L_x^2} |\nabla b|_{L_x^2} , \quad (3.19)$$

for a constant independent of  $\varphi$ .

Let  $u$  denote a weak solution in the sense given by Definition 3.1. Since  $u(t)$  is supported in  $H_x^1 \cap L_x^\infty$  for a.e.  $t \in [0, T]$ , we know a priori that  $u(t)|\nabla u(t)|^2$  belongs to  $L_x^1$ . Using  $|u(\omega, t, x)|_{\mathbb{R}^3} \equiv 1$  a.e., and Helein's "trick" [Hél90], we can write for  $i = 1, 2, 3$ :

$$\begin{aligned} u^i |\nabla u|^2 &= \sum_{j,k} (u^i \partial_k u^j - u^j \partial_k u^i) \partial_k u^j \\ &= \sum_{j,k} a^{i,j,k} \partial_k u^j \\ &= (a \cdot \nabla u)^i . \end{aligned} \quad (3.20)$$

If we assume moreover that  $u$  is stationary and harmonic, namely if  $\Delta u + u|\nabla u|^2 = 0$ , then the fact that a.e.  $u(x) \propto \Delta u(x)$  implies  $\operatorname{div}(u \times \nabla u) = 0$ , which means exactly

$$\operatorname{div} a = 0 .$$

Hence, by Helmholtz decomposition Theorem (Thm. 3.5 below) there exists  $\beta \in H^1(\mathbb{D}; \mathbb{R})$  such that  $a = \nabla^\perp \beta$ , and the identity (3.20) implies that the term  $u|\nabla u|^2$  is the Poisson bracket given by  $\nabla^\perp \beta \cdot \nabla u$ , that is  $\{\beta, u\}$ . Theorem 3.4 now applies, so that

$$u|\nabla u|^2 \in H_x^{-1} .$$

In the present situation, the map  $a(t)$  associated with  $u(t)$  has however nonzero divergence. To prove Theorem 3.3, we write  $a$  as a sum of two terms, one of them being an orthogonal gradient, whose divergence is zero. The other part is treated in (4.c).

## (4.b) Use of Helmholtz decomposition

The following theorem can be found in [DL12], as a consequence of Prop. 1 p. 215, and Prop. 3 p. 222.

**Theorem 3.5** (Helmholtz). If  $\mathbb{D}$  denotes the unit disk of  $\mathbb{R}^2$ , then we have the following orthogonal decomposition:

$$L^2(\mathbb{D}; \mathbb{R}^2) = \nabla H^1(\mathbb{D}, \mathbb{R}) \oplus (\nabla^\perp H^1(\mathbb{D}; \mathbb{R}) \cap \{u \in H^1(\mathbb{D}; \mathbb{R}^2), u \cdot n = 0 \text{ on } \partial\mathbb{D}\}) ,$$

see the notation (3.17), i.e. for all  $a \in L^2(\mathbb{D}; \mathbb{R}^2)$ , there exist  $\alpha \in H^1(\mathbb{D}; \mathbb{R})$  and  $\beta \in H^1(\mathbb{D}; \mathbb{R})$  with  $\nabla^\perp \beta \cdot n = 0$  a.e. on  $\partial\mathcal{O}$ , such that:

$$a^k = \nabla \alpha^k + \nabla^\perp \beta^k , \quad \forall 1 \leq k \leq 2 ,$$



and the corresponding projections are continuous in the sense that there exists a universal constant  $C > 0$ , not depending on  $a$  such that

$$|\alpha|_{H_x^1} + |\beta|_{H_x^1} \leq C|a|_{L_x^2} .$$

Fix  $1 \leq i, j \leq 3$ . Applying Theorem 3.5, we write for each  $t \in [0, T]$ :

$$a^{i,j}(t) = \nabla \alpha^{i,j}(t) + \nabla^\perp \beta^{i,j}(t) ,$$

where  $a$  is defined by (3.20) with  $u = u_1$ . Taking the divergence, we obtain

$$\operatorname{div}(a^{i,j}) = u_1^i \Delta u_1^j - u_1^j \Delta u_1^i , \quad (3.21)$$

implying  $\|\operatorname{div}(a^{i,j})\|_{L_T^2 L_x^2} \leq c \|u_1 \times \Delta u_1\|_{L_T^2 L_x^2}$ . On the other hand since  $\operatorname{div}(a^{i,j}) = \operatorname{div}(\nabla \alpha^{i,j})$ , by Calderon-Zygmund inequality together with the estimate

$$\|a^{i,j}\|_{L_T^\infty L_x^2} \leq c \|\nabla u_1^{i,j}\|_{L_T^\infty L_x^2} , \quad (3.22)$$

we have for an absolute constant

$$\|\nabla \alpha\|_{L_T^2 H_x^1} \leq c \|u_1 \times \Delta u_1\|_{L_T^2 L_x^2} , \quad \mathbb{P} - \text{a.s.} , \quad (3.23)$$

and the right hand side is finite, see Definition 3.1-(2). Using again the Helmholtz decomposition, but for the strong solution  $u_2$ , we write  $u_2 |\nabla u_2|^2 = a_2 \cdot \nabla u_2$ , where  $a_2 = \nabla \alpha_2 + \nabla^\perp \beta_2$ , and also denote by  $a_1 = a$ ,  $\alpha_1 = \alpha$ ,  $\beta_1 = \beta$ . This gives us the following equation on  $w$

$$\begin{cases} dw - \Delta w dt = \left( f + u_1 \times \Delta u_1 - u_2 \times \Delta u_2 + F_\phi(w) + \nabla^\perp \beta_1 \cdot \nabla w \right) dt + w \times dW_\phi , \\ \quad \text{on } \Omega \times [0, T] \times \mathbb{D} , \\ w(t, \cdot) = 0 , \quad \text{on } \Omega \times \partial \mathbb{D} , \\ w(0, \cdot) = 0 , \quad \text{in } \Omega \times \mathbb{D} , \end{cases} \quad (3.24)$$

where we define on  $\Omega \times [0, T] \times \mathbb{D}$ , the map:

$$f = \nabla \alpha_1 \cdot \nabla w + \nabla^\perp (\beta_1 - \beta_2) \cdot \nabla u_2 + \nabla (\alpha_1 - \alpha_2) \cdot \nabla u_2 . \quad (3.25)$$

We proceed as in [Fre95], by showing the following

**Claim 3.1.** With probability one,

- (i)  $f$  defined by (3.25), belongs to  $L_T^4 L_x^{4/3}$ ;
- (ii)  $u_1 \times \Delta u_1 - u_2 \times \Delta u_2$  belongs to  $L_T^4 W_x^{-1,4}$ ;
- (iii)  $F_\phi(w)$  belongs to  $L_T^4 W_x^{-1,4}$ .

#### (4.c) Proof of Claim 3.1 : bounds on the gradient part

First, observe that as a consequence of inequality (3.12), we have for all  $f \in L_T^2 H_x^1 \cap L_T^\infty L_x^2$ :

$$\begin{aligned} \|f\|_{L_T^4 L_x^4} &\leq c \left( \int_0^T |f|_{H_x^1}^2 |f|_{L_x^2}^2 dt \right)^{1/4} \\ &\leq \|f\|_{L_T^2 H_x^1}^{1/2} \|f\|_{L_T^\infty L_x^2}^{1/2} . \end{aligned} \quad (3.26)$$

We shall use in this paragraph the following notation

$$E_T := \operatorname{ess\,sup}_{0 \leq t \leq T} (|\nabla u_1(t)|_{L_x^2}^2 + |\nabla u_2(t)|_{L_x^2}^2) . \quad (3.27)$$

*Proof of (i).* Using respectively Hölder inequality,  $\|w\|_{L_T^\infty L_x^2}^2 \leq 2E_T$  and then (3.26), we obtain:

$$\begin{aligned} \|\nabla \alpha_1 \cdot \nabla w\|_{L_T^4 L_x^{4/3}} &\leq \|\nabla \alpha_1\|_{L_T^4 L_x^4} \|\nabla w\|_{L_T^\infty L_x^2} \\ &\leq c \|\nabla \alpha_1\|_{L_T^2 H_x^1}^{1/2} \|\nabla \alpha_1\|_{L_T^2 L_x^2}^{1/2} E_T^{1/2} \end{aligned}$$

so that by (3.23) and (3.22),

$$\begin{aligned} \|\nabla \alpha_1 \cdot \nabla w\|_{L_T^4 L_x^{4/3}} &\leq c \|u_1 \times \Delta u_1\|_{L_T^2 L_x^2}^{1/2} \|\nabla \alpha_1\|_{L_T^2 L_x^2}^{1/2} E_T^{1/2} \\ &\leq c \|u_1 \times \Delta u_1\|_{L_T^2 L_x^2}^{1/2} E_T^{3/4} . \end{aligned}$$

Using again Hölder inequality, the continuity of the map  $a \mapsto \beta$ ,  $L_x^2 \rightarrow H_x^1$ , (3.22) and (3.26):

$$\begin{aligned} \|\nabla^\perp(\beta_1 - \beta_2) \cdot \nabla u_2\|_{L_T^4 L_x^{4/3}} &\leq c \|\nabla^\perp(\beta_1 - \beta_2)\|_{L_T^\infty L_x^2} \|\nabla u_2\|_{L_T^4 L_x^4} \\ &\leq c E_T^{1/2} \|u_2\|_{L_T^2 H_x^2}^{1/2} \|\nabla u_2\|_{L_T^2 L_x^2}^{1/2} \\ &\leq c E_T^{3/4} \|u_2\|_{L_T^2 H_x^2} , \end{aligned}$$

which is finite a.s. since we assumed  $T < \vartheta^1$ , see Remark 3.3.

Similarly, we have by Hölder inequality:

$$\begin{aligned} \|\nabla(\alpha_1 - \alpha_2) \cdot \nabla u_2\|_{L_T^4 L_x^{4/3}} &\leq \|\nabla(\alpha_1 - \alpha_2)\|_{L_T^\infty L_x^2} \|\nabla u_2\|_{L_T^4 L_x^4} \\ &\leq E_T^{1/2} \|\nabla u_2\|_{L_T^4 L_x^4} \end{aligned}$$

which finishes the proof of part (i), since by Remark 3.2 and  $T < \vartheta$ , we have  $u_2 \in L_T^4 W_x^{1,4}$ ,  $\mathbb{P}$ - a.s.

*Proof of (ii).* Using Remark 3.1, we can write for  $q = 1, 2$

$$u_q \times \Delta u_q = \operatorname{div}(a_q) .$$

Therefore by the decomposition  $a_q = \nabla \alpha_q + \nabla^\perp \beta_q$ , we have:

$$\begin{aligned} \|u_1 \times \Delta u_1 - u_2 \times \Delta u_2\|_{L_T^4 W_x^{-1,4}} &= \|\operatorname{div}(\nabla(\alpha_1 - \alpha_2))\|_{L_T^4 W_x^{-1,4}} \\ &\leq c(\mathbb{D}) \|\alpha_1 - \alpha_2\|_{L_T^4 W_x^{1,4}} \\ &\leq c'(\mathbb{D}) \|\nabla(\alpha_1 - \alpha_2)\|_{L_T^4 L_x^4} \end{aligned}$$

whence using (3.26) we obtain

$$\begin{aligned} \|u_1 \times \Delta u_1 - u_2 \times \Delta u_2\|_{L_T^4 W_x^{-1,4}} &\leq c \sum_{q=1,2} \|\nabla \alpha_q\|_{L_T^2 H_x^1}^{1/2} \|\nabla \alpha_q\|_{L_T^\infty L_x^2}^{1/2} \\ &\leq c \sum_{q=1,2} \|u_q \times \Delta u_q\|_{L_T^2 L_x^2}^{1/2} (E_T)^{1/4}, \quad (\text{by (3.23)}), \end{aligned}$$

which proves part (ii) of the claim.

*Proof of (iii).* By (1.15), there holds:

$$\int_0^T |F_\phi(w(t, \cdot))|_{L_x^2}^4 dt \leq \int_0^T |w(t, \cdot)|_{L_x^\infty}^4 \left( \sum_{e \in \mathcal{B}^3} |\phi e|_{L_x^4}^2 \right)^2 dt,$$

which, by the embedding  $H_x^1 \hookrightarrow L_x^4$ , and  $|w(\omega, t, x)|_{\mathbb{R}^3} \leq 2$  a.e., is bounded by  $cT|\phi|_{\mathbb{L}_2^{0,1}}$ . This proves (iii).  $\blacksquare$

#### (4.d) Decomposition of “ $\nabla^\perp \beta_1 \cdot \nabla w$ ”.

The following lemma is essential. Its deterministic counterpart (with a similar statement, but under the assumption that the energy is nonincreasing) is a key ingredient to show Theorem 3.1, Although no proof is explicitly given in [Fre95], a justification was made later in [Fre96]. It arises, when  $W_\phi \equiv 0$ , as a consequence of the criterion (iv’), but we show here that for the stochastic equation, the assumption that  $\mathcal{G}_1$  is a supermartingale suffices to show similar conclusions. To lighten the notations, in this subsection we denote by  $a = a_1$ ,  $\alpha = \alpha_1$ ,  $\beta = \beta_1$ , and  $\mathcal{G} = \mathcal{G}_1$ .

**Lemma 3.2.** *Assume that the process  $\mathcal{G}$  defined in (3.4) is a supermartingale. Given  $\epsilon > 0$ , there exists a stopping time  $0 < \tau(\epsilon) \leq T$  and  $\beta_\epsilon \in L_\tau^\infty H_x^1$ ,  $\beta'_\epsilon \in \mathcal{C}^\infty([0, T] \times \mathbb{D})$  such that:*

$$\beta|_{[0, \tau]} = \beta_\epsilon + \beta'_\epsilon \quad \text{and} \quad \|\beta_\epsilon\|_{L_\tau^\infty H_x^1} \leq \epsilon.$$

*Proof.* Fix  $1 \leq i, j \leq 3$ . Write the Helmholtz decomposition at time  $t \geq 0$ , and with the notation (3.18), notice that:

$$\begin{aligned} \operatorname{div}^\perp(\nabla^\perp \beta^{i,j}(t)) &= \operatorname{div}^\perp(a^{i,j}(t)) \\ &= 2(\partial_1 u^j(t) \partial_2 u^i(t) - \partial_1 u^i(t) \partial_2 u^j(t)), \end{aligned} \quad (3.28)$$

The lemma follows from the following

**Claim 3.2.** With full probability,  $t \in [0, T] \mapsto |\nabla u(t)|_{L_x^2}^2$  is right continuous.

If claim 2 is true, we see that  $\nabla^\perp \beta(t) \rightarrow \nabla^\perp \beta(s)$  in  $W_x^{1,1}$  as  $t \rightarrow s+$ , and thus also in  $L_x^2$  by the embedding  $W_x^{1,1} \hookrightarrow L_x^2$ . Therefore  $\nabla^\perp \beta$  is also right continuous in  $L_x^2$ , and the conclusion of the lemma follows by density of  $\mathcal{C}_{t,x}^\infty$  in  $\mathcal{C}_T H_x^1$ .

*Proof of Claim 3.2.* Let  $s \in [0, T]$ . Define for  $p, n \in \mathbb{N}$ ,

$$\mathcal{U}(p, n) = \left\{ \omega \in \Omega : \exists t_n(\omega) \in \left[ s, s + \frac{1}{n+1} \right], |\mathcal{G}(t_n) - \mathcal{G}(s)| > \frac{1}{p+1} \right\}.$$

We have the equality

$$\{\omega \in \Omega : t \mapsto u(t) \in H_x^1 \text{ is not right continuous at } s\} = \bigcup_{p \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \mathcal{U}(p, n) .$$

Now, reasoning by contradiction, assume that there exists  $p \in \mathbb{N}$  such that  $\Omega_p = \bigcap_{n \in \mathbb{N}} \mathcal{U}(p, n)$  has positive probability. The use of the Optional Sampling Theorem (see e.g. [WI81, Chap. I §6], implies

$$\mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t_n) - \mathcal{G}(s)] \leq 0 \text{ a.s.} \quad (3.29)$$

Moreover, classical facts on supermartingales imply that  $\hat{\mathcal{G}}(s) = \text{a.s. -} \lim_{n \rightarrow \infty} \mathcal{G}(t_n)$  exists. Note that by the right continuity assumption on  $(\mathcal{F}_t)$ , the set  $\Omega_p$  is  $\mathcal{F}_s$  measurable. On the one's hand, there holds

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_s}[(\hat{\mathcal{G}}(s) - \mathcal{G}(s))\mathbf{1}_{\Omega_p}] &= \mathbb{E}[\mathbb{E}_{\mathcal{F}_s}[(\hat{\mathcal{G}}(s) - \mathcal{G}(s))\mathbf{1}_{\Omega_p}]] \\ &\leq \mathbb{E}[\mathbf{1}_{\Omega_p} \liminf_{n \rightarrow \infty} \mathbb{E}_{\mathcal{F}_s}[\mathcal{G}(t_n) - \mathcal{G}(s)]] \\ &\leq 0 , \quad \text{by (3.29).} \end{aligned} \quad (3.30)$$

On the other hand, since  $\mathbb{P}(u \in \mathcal{C}_T L_x^2) = 1$ ,  $\nabla u(t_n)$  converges a.s. towards  $\nabla u(s)$  in  $H_x^{-1}$ , and by  $\nabla u(s) \in L_x^2$ , a.s., we have

$$\text{a.s. , } \nabla u(t_n) \rightharpoonup \nabla u(s) \text{ weakly in } L_x^2 . \quad (3.31)$$

Using the lower semicontinuity of the map  $P \mapsto |P|_{L_x^2}^2$ ,  $L_x^2 \rightarrow \mathbb{R}$  with respect to the weak topology, we have  $\mathcal{G}(s) \leq \hat{\mathcal{G}}(s) = \lim \mathcal{G}(t_n)$  a.s., and therefore since on  $\Omega_p$ , we have  $|\mathcal{G}(t_n) - \mathcal{G}(s)| > 1/(p+1)$  for all  $n \geq 0$ , it follows that

$$\mathbf{1}_{\Omega_p} \left( \hat{\mathcal{G}}(s) - \mathcal{G}(s) \right) = \mathbf{1}_{\Omega_p} |\hat{\mathcal{G}}(s) - \mathcal{G}(s)| \geq \frac{\mathbf{1}_{\Omega_p}}{p+1} ,$$

which together with (3.30) and  $\mathbb{P}(\Omega_p) > 0$ , leads to a contradiction. ■

#### (4.e) Step 4 : Parabolic estimates and conclusion by Wentz's Theorem

Consider the deterministic linear parabolic equation

$$\begin{cases} \partial_t \varphi - \Delta \varphi = f(t, x) , & \text{in } (0, \tau) \times \mathbb{D} , \\ \varphi(x, \cdot) = 0 , & \text{on } \partial \mathbb{D} , \\ \varphi(\cdot, 0) = 0 , & \text{in } \mathbb{D} , \end{cases} \quad (3.32)$$

for a given  $f$ , and a stochastic version with multiplicative noise:

$$\begin{cases} dZ - \Delta Z dt = v(t, x) \times dW_\phi , & \text{in } \Omega \times (0, \tau) \times \mathbb{D} , \\ Z(x, \cdot) = 0 , & \text{on } \Omega \times \partial \mathbb{D} , \\ Z(\cdot, 0) = 0 , & \text{in } \Omega \times \mathbb{D} , \end{cases} \quad (3.33)$$

where  $v : \Omega \times [0, \tau] \rightarrow H_0^1(\mathbb{D}; \mathbb{R}^3)$  is a predictable process. We have the following result of regularity for (3.32)-(3.33).

**Proposition 3.1** (Parabolic estimates). (i) If the right hand side  $f$  belongs to  $L_T^2 H_x^{-1}$ , then there exists a unique solution  $\mathcal{U} f$  of (3.33) in the Hilbert space  $\mathbb{H} := \{\Psi \in L_T^2 H_x^1, \partial_t \Psi \in L_T^2 H_x^{-1}\}$  where (3.32) is understood in the sense of distributions. Moreover the map

$$\mathcal{U} : L_T^2 H_x^{-1} \rightarrow \mathbb{H} ,$$

is a bounded isomorphism.

(ii) Similarly, if  $f$  belongs to the Banach space  $L_T^4 W_x^{-1,4}$  then there exists a unique  $\mathcal{V} f$  in the space  $\mathbb{B} := \{\Psi \in L_T^4 W_x^{1,4}, \partial_t \Psi \in L_T^4 W_x^{-1,4}\}$ , solving (3.32) in the sense of distributions, and the map

$$\mathcal{V} : L_T^4 W_x^{1,4} \rightarrow \mathbb{B} ,$$

is a bounded isomorphism.

(iii) Assume that the predictable process  $v$  fulfills

$$v \in L^\infty(\Omega \times [0, \tau] \times \mathbb{D}) . \quad (3.34)$$

If the  $\phi\phi^*$ -Wiener process  $W_\phi$  is such that  $\phi \in R(L_x^2; L_x^4)$ , then there exists a unique solution  $Z$  of (3.33) supported in  $L_T^2 H_0^1(\mathbb{D}; \mathbb{R}^3)$ , in the sense that for all  $\zeta \in H^1(\mathbb{D}; \mathbb{R}^3)$ , with probability one, we have

$$\langle Z(t), \zeta \rangle_{L_x^2} = \int_0^t \langle Z(s), \Delta \zeta \rangle_{H_x^1, H_x^{-1}} ds + \int_0^t \langle v(s) \times dW_\phi(s), \zeta \rangle_{L_x^2} , \text{ for all } t \in [0, T] . \quad (3.35)$$

Moreover,  $Z$  is given by the Itô integral

$$Z(t) = \int_0^t S(t-s)v(s) \times dW_\phi , \quad t \in [0, T] , \quad (3.36)$$

which is well defined in the space  $L_x^2$ , and we have:

$$\mathbb{E} \|Z\|_{L_T^4 W_x^{1,4}}^4 < \infty . \quad (3.37)$$

For the reader's convenience, we recall a particular case of the Burkholder-Davies-Gundy inequality, whose general statement and proof can be seen in [Brz97, Thm. 2.4] (see also [BP99] and references therein).

Assume that  $p, q \in [2, \infty)$ . There exists a constant  $c_1 = c_1(T, p, q) > 0$ , such that for every predictable process  $\xi : \Omega \times [0, T] \rightarrow R(L_x^2; L_x^p)$  with  $\mathbb{E}[(\int_0^T |\xi(t) \circ \phi|_{R(L_x^2; L_x^p)}^2)^{q/2}] < \infty$ , there holds the inequality

$$\mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \xi(s) dW_\phi(s) \right|_{L_x^p}^q \leq c_1 \mathbb{E} \left[ \left( \int_0^T |\xi(s) \circ \phi|_{R(L_x^2; L_x^p)}^2 ds \right)^{q/2} \right] . \quad (3.38)$$

*Proof of Proposition 3.1.*

(i) and (ii). The proof of (i) can be found in [LM68]. The case of a right hand side  $f$  in the Banach space  $L_T^p L_x^q$  is treated in [Gri69], where the continuity of  $f \mapsto \varphi$ ,  $L^p L^q \rightarrow L^p W^{2,q}$  is shown, provided  $1 < q, p < \infty$ . The case where the right hand side belongs to  $L^p W_x^{-1,q}$  is obtained by considering  $(-\Delta)^{-1/2} f$  instead of  $f$ .

*Proof of (iii).* Denote by  $Z = \int_0^\cdot S(\cdot - s)u(s) \times dW_\phi(s)$ . The fact that  $Z$  is a well defined process with values in  $L_x^2$  follows from the bound

$$\mathbb{E} \int_0^T |Z(t)|_{L_x^2}^2 dt \leq c(S) \mathbb{E} \int_0^T \int_0^t |v(s) \times \phi \cdot|_{\mathbb{L}_2}^2 ds dt ,$$

the assumption that  $v \in L_{\omega,t,x}^\infty$ ,  $\phi \in R(L_x^2; L_x^4)$ , and  $R(L_x^2; L_x^4) \subseteq \mathbb{L}_2$ . Following [DZ08, Thm. 5.4] (where the assumptions are stronger but the proof is identical), in order to prove that  $Z$  satisfies (3.35), it suffices to show that  $\mathbb{E} \int_0^T |Z(t)|_{H_x^1}^4 dt < \infty$ , which, by the embedding  $L_T^4 W^{1,4} \hookrightarrow L_T^2 H_x^1$ , will follow from (3.37).

Recall the basic inequality:

$$|LK|_{R(L^2; W_x^{1,4})} \leq |L|_{\mathcal{L}(L_x^4; W_x^{1,4})} |K|_{R(L_x^2; L_x^4)} , \quad \text{for all } (L, K) \in \mathcal{L}(L^4; W_x^{1,4}) \times R(L_x^2; L_x^4) , \quad (3.39)$$

see e.g. [DD99, Lem. 2.1]. Therefore, we can write, using successively (3.38), (3.39):

$$\begin{aligned} \mathbb{E} \|Z(t)\|_{L_T^4 W_x^{1,4}}^4 &\leq c \int_0^T \mathbb{E} \left| \nabla \int_0^t S(t-s)v(s) \times dW_\phi \right|_{L_x^4}^4 dt \\ &\leq c' \int_0^T \mathbb{E} \left( \int_0^t |\nabla S(t-s)v(s) \times \phi \cdot|_{R(L_x^2; L_x^4)}^2 ds \right)^2 dt \\ &\leq c'' \int_0^T \mathbb{E} \left( \int_0^t |S(t-s)|_{\mathcal{L}(L_x^4; W_x^{1,4})} |v(s) \times \phi \cdot|_{R(L_x^2; L_x^4)}^2 ds \right)^2 dt . \end{aligned} \quad (3.40)$$

By hypercontractivity of the heat semigroup (see Lemma 1.2), for a constant  $c_0 > 0$  depending on  $S$  only, we have

$$|S(t)|_{\mathcal{L}(L_x^4; W_x^{1,4})} \leq \frac{c_0}{t^{1/2}} , \quad \text{for } t > 0 ,$$

so that (3.40) implies together with  $v \in L_{\omega,t,x}^\infty$ , that

$$\begin{aligned} \mathbb{E} \|Z(t)\|_{L_T^4 W_x^{1,4}}^4 &\leq c \int_0^T \left( \int_0^t \frac{1}{t-s} |\phi|_{R(L_x^2; L_x^4)}^2 ds \right)^2 dt \\ &= c |\phi|_{R(L_x^2; L_x^4)}^4 \int_0^T \left( \frac{1}{s} \star \mathbb{1}_{[0,t]} \right)^2 dt . \end{aligned} \quad (3.41)$$

Using the continuity of the so-called *Hilbert transform*  $f \mapsto (\frac{1}{s} \star f)$  from  $L^2([0, T])$  into itself, see e.g. [Ste70, Chap. II], we see that right hand side of (3.41) is finite, despite of the singular integral. This proves (3.37), and (iii).  $\blacksquare$

*End of the proof of Lemma 3.1.* Choose  $\epsilon, \tau > 0$  as in Lemma 3.2, so that

$$\|\nabla^\perp u_1\|_{L_\tau^\infty L_x^2} \leq \epsilon . \quad (3.42)$$

We then set for convenience:

$$f_\epsilon := \left( f + \nabla^\perp \beta'_\epsilon \cdot \nabla w + F_\phi(w) + u_1 \times \Delta u_1 - u_2 \times \Delta u_2 \right) \Big|_{[0, \tau]} , \quad (3.43)$$

so that by Claim 3.1 and the continuous embedding  $L_x^{4/3} \hookrightarrow W_x^{-1,4}$ , the map  $f_\epsilon$  actually defines an element of  $L_\tau^4 W_x^{-1,4}$ .

Considering the right hand side in (3.24) as given maps, equation (3.24) can be formally rewritten

$$w - (\partial_t - \Delta)^{-1}(\nabla^\perp \beta_\epsilon \cdot \nabla w) = (\partial_t - \Delta)^{-1} f_\epsilon + Z , \quad (3.44)$$

where  $(\partial_t - \Delta)^{-1}$  is understood with zero boundary value, and  $Z$  is the solution of (3.33), associated with  $v(t) = u_1(t) - u_2(t)$ . Note that  $v$  and  $\phi$  fulfill the hypotheses of Proposition 3.1-(iii), since  $\|u_1 - u_2\|_{L^\infty_{\omega,t,x}} \leq 2$  and  $\phi \in \mathbb{L}_2^{0,1} \subseteq R(L_x^2; L_x^4)$ , and therefore  $Z$  is given by the formula (3.36).

Theorem 3.4 now implies that the Poisson bracket  $\nabla^\perp \beta \cdot \nabla \Phi$  belongs to  $L_\tau^2 H_x^{-1}$ , so that using the notation of Proposition 3.1-(i), the parabolic estimate writes

$$\|\mathcal{U}(\nabla^\perp \beta_\epsilon \cdot \nabla \Psi)\|_{L_\tau^2 H_x^1} \leq c \|\nabla^\perp \beta_\epsilon \cdot \nabla \Psi\|_{L_\tau^2 H_x^{-1}} \quad \text{for all } \Psi \in L_\tau^2 H_x^1 . \quad (3.45)$$

Denoting by  $T_\epsilon \Psi := \mathcal{U} \nabla^\perp \beta_\epsilon \cdot \nabla \Psi$ , using successively Wente's inequality (see (3.19)) and (3.42), we obtain

$$\begin{aligned} \|T_\epsilon \Psi\|_{L_\tau^2 H_x^1} &\leq c \|\nabla^\perp \beta_\epsilon\|_{L_\tau^\infty L_x^2} \|\Psi\|_{L_\tau^2 H_x^1} , \\ &\leq c(T) \epsilon \|\Psi\|_{L_\tau^2 H_x^1} , \end{aligned}$$

with a constant depending on  $T > 0$  but not on  $\tau$  (observe that the norm  $\|\mathcal{U}\|_{\mathcal{L}(L^2([0,t]; H_x^{-1}); \mathbb{H})}$  is nondecreasing with  $t$ ). Therefore, we may take  $\epsilon < c(T)^{-1}$  so that

$$\|T_\epsilon\|_{(L_\tau^2 H_x^1)} < 1 . \quad (3.46)$$

The equation (3.44) rewrites in the form

$$\Psi - T_\epsilon \Psi = \mathcal{U} f_\epsilon + Z|_{[0,\tau]} \quad (3.47)$$

where, using Proposition 3.1-(i), (3.43) and Claim 3.1, we see that the term  $\mathcal{U} f_\epsilon$  is well defined as an element of  $\mathbb{H}$  since  $L_T^4 W_x^{-1,4}$  is continuously embedded in  $L_\tau^2 H_x^{-1}$ . We can also assume without restriction, using the part (iii), that  $Z|_{[0,\tau]} \in L_\tau^2 H_x^1$ .

By (3.46), the Neumann Series  $\sum_{n \in \mathbb{N}} (T_\epsilon)^n$  converges in  $\mathcal{L}(L_\tau^2 H_x^1)$  towards  $(\text{id} - T_\epsilon)^{-1}$ , and we obtain the uniqueness of  $\Psi_0$  solving (3.44), in the space  $L_\tau^2 H_x^1$ . Since moreover  $u_1$  and  $u_2$  belong to  $L_T^2 H_x^1$ , we obtain that necessarily

$$\Psi_0 = (u_1 - u_2)|_{[0,\tau]} . \quad (3.48)$$

Consider again (3.44) in the form

$$\Phi - \mathcal{V}(\nabla^\perp \beta_\epsilon \cdot \nabla \Phi) = \mathcal{V} f_\epsilon + Z .$$

where we have taken the notation of Proposition 3.1-(ii). Using the parabolic estimate (ii), and the continuous embedding  $W_x^{1,4/3} \hookrightarrow L_x^4$ , there comes

$$\begin{aligned} \|\mathcal{V}(\nabla^\perp \beta_\epsilon \cdot \nabla \Phi)\|_{L_\tau^4 W_x^{1,4}} &\leq c \|\nabla^\perp \beta_\epsilon \cdot \nabla \Phi\|_{L_\tau^4 W_x^{-1,4}} \\ &\leq c \|\nabla^\perp \beta_\epsilon \cdot \nabla \Phi\|_{L_\tau^4 L_x^{4/3}} , \end{aligned} \quad (3.49)$$

which, using Hölder Inequality, is bounded by  $c(T) \|\nabla^\perp \beta_\epsilon\|_{L_\tau^\infty L_x^2} \|\nabla \Phi\|_{L_\tau^4 L_x^4}$  and thus we have

$$\|\tilde{T}_\epsilon\|_{\mathcal{L}(L_\tau^4 W_x^{1,4})} \leq c'(T) \epsilon ,$$

where  $\tilde{T}_\epsilon$  denotes the linear operator  $T_\epsilon \Phi := \nabla^\perp \beta_\epsilon \cdot \nabla \Phi$ . Assuming that  $\epsilon < c'(T)$ , we obtain by the same argument as above the existence (and uniqueness) of  $\Phi_0$  solving (3.47), in the space  $L_\tau^4 W_x^{1,4}$ . Since  $L_\tau^4 W_x^{1,4} \subseteq L_\tau^2 H_x^1$ , the uniqueness of  $\Psi_0$  implies

$$\begin{aligned} \Phi_0 &= \Psi_0 \\ &= u_1|_{[0,\tau]} - u_2|_{[0,\tau]} \ , \end{aligned}$$

which shows in particular that  $u_2|_{[0,\tau]} \in L_\tau^4 W_x^{1,4}$ . This finishes the proof of Lemma 3.1, and Theorem 3.3. ■





## *Finite time singularity of the stochastic harmonic map flow*

---

Struwe’s construction [Str85] gives a necessary condition under which blow-up phenomena occurs for the harmonic map flow in two dimensions, and says nothing on the existence of such maps. Since an explicit construction has been given by K-C. Chang, W.Y. Ding and R. Ye in [CDY+92], it is known however that certain classes of solutions blow-up in finite time. The main argument used in this article is a comparison principle, together with a class of self-similar, blowing-up subsolutions. To construct such subsolutions, it is crucial to satisfy some symmetry assumption, in order to reduce the equation to a scalar problem, so that comparison principles can apply. The particular symmetry that was used in [CDY+92] is the so-called “equivariance” assumption.

When considering (SLLG) in dimension 2, it is not known however, how to prove the existence of blowing-up solutions, although there is numerical evidence that the result above remains true in that case (see Chapter 6 and references therein). The existence of such solutions is still an open problem in the deterministic case. Considering a stochastic version of the overdamped case, where some assumptions are made on the noise in order to preserve equivariance, we show in the present chapter that blowing-up solutions exist for the “stochastic heat flow”. In addition, we show that the set of initial data leading possibly to the bubblings (namely the “pre-blow-up set”) is reachable from *any* equivariant map  $u_0$ , implying that every equivariant initial data generates blow with positive probability.

The method to show the existence of a preblow-up set can be described (loosely) as follows. We first show, by means of a comparison principle for a perturbed equation, that given a trajectory of the related Ornstein-Uhlenbeck process  $Z$ , there exists an initial data  $u_0(z)$  that generates blow-up in finite time. Using then a topological argument, it is possible to show that there exist initial data generating blow-up for a “sufficiently large number of trajectories  $z$ ”, which gives a pre-blow-up set (we “reverse the quantifiers”).

---

# 1 Introduction and main result

## (1.a) Motivations

Consider the stochastic harmonic map flow from the unit disk  $\mathbb{D} = \{x \in \mathbb{R}^2, |x|_{\mathbb{R}^2} \leq 1\}$  into the sphere  $\mathbb{S}^2$ , namely:

$$\begin{cases} du = (\Delta u + u|\nabla u|^2)dt + \sigma(u) \circ dW , & \text{in } \Omega \times \mathbb{R}_+ \times \mathbb{D} , \\ u(t)|_{\partial\mathbb{D}} = u_0|_{\partial\mathbb{D}} , & \text{on } \Omega \times [0, T] \times \partial\mathbb{D} , \\ u(x, 0) = u_0(x) , & \text{in } \Omega \times \mathbb{D} , \end{cases} \quad (4.1)$$

where for  $X \in \mathbb{S}^2$ ,  $\sigma(X)$  is *a priori* defined as the linear map  $X \times \cdot$ ,  $W = (W_1, W_2, W_3)$  is a given Wiener process in  $L^2(\mathbb{D}; \mathbb{R}^3)$ , and unless otherwise stated, the boundary value is the fixed vertical unit vector, i.e. on  $\partial\mathbb{D}$ ,

$$u_0|_{\partial\mathbb{D}} = \vec{k} := (0, 0, 1) . \quad (4.2)$$

Blow-up phenomena for equation (4.1) with  $\sigma \equiv 0$  is a well-known fact. In the case of equivariant initial data, i.e. maps that have the following form for  $x \in \mathbb{D}$ :

$$u_0(x) = \left( \frac{x}{|x|} \sin h_0(|x|); \cos h_0(|x|) \right) , \quad (4.3)$$

where  $h_0 \in \mathcal{C}^1([0, 1])$  with  $h_0(0) = \alpha \in \mathbb{R}$  and  $h_0(1) = \beta \in \mathbb{R}$ , it was shown, that the corresponding solution  $u(t, x)$  of (4.1) with  $W \equiv 0$  writes also

$$u(t, x) = \left( \frac{x}{|x|} \sin h(t, |x|); \cos h(t, |x|) \right) , \quad (4.4)$$

with  $h$  satisfying an equation of the form:

$$\begin{cases} \frac{\partial h}{\partial t} = \partial_{rr}h + \frac{1}{r}\partial_r h - \frac{\sin 2h}{2r^2} , & \text{for } (t, r) \in \mathbb{R}^+ \times (0, 1) , \\ h(t, 0) = \alpha , \quad h(t, 1) = \beta , & \text{for } t \in \mathbb{R}^+ , \\ h(0, r) = h_0(r) , & \text{for } r \in (0, 1) . \end{cases} \quad (4.5)$$

and the solution  $u$  is global provided  $|h_0|_{\mathcal{C}([0,1])} \leq \pi$  – see [CD91]. Note that the number  $\beta$  is the angle formed by  $(\vec{k}, u_0|_{\partial\mathbb{D}})$ , so that in case (4.2) we have  $\beta = 0$ . In [CDY<sup>+</sup>92], the authors give explicit maps  $h_0$  which generates blow-up in the sense (4.6) but for the case where  $h(t, 1) \equiv \beta > \pi$ . In [BDvdH02] the authors show that similar conclusions hold in the case where  $\alpha = \beta = 0$  i.e. when  $u|_{\partial\mathbb{D}} \equiv \vec{k}$  and the degree of the application  $u$  is equal to zero. In this case the existence of actual  $h$  leading to (4.6) is obtained through slight modifications of the proof given in [CDY<sup>+</sup>92], their important result being longer how (4.6) leads to nonuniqueness of weak solutions. Note that for solutions of the form (4.4), one can always assume that the value at the origin is  $\alpha = 0$ . With this assumption, every regular equivariant map must satisfy  $(\partial_r)^{2m}h(t, 0) = 0$ , for all  $m \in \mathbb{N}$  – see [BDvdH02]. Summarizing the results above, we have the following theorem, for the case (4.2).

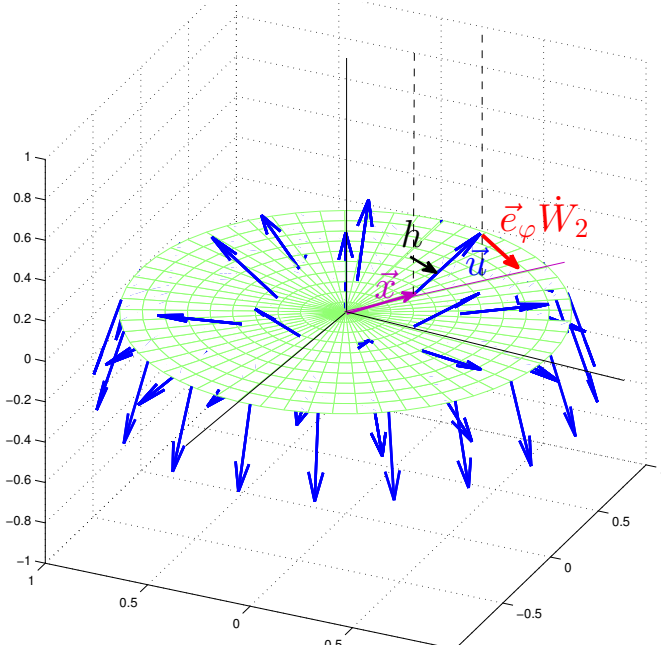


Figure 4.1 – “Equivariant noise”, represented in red.

**Theorem 4.1** ([CDY+92, BDvdH02]). *Assume  $\sigma = 0$ . There exist  $\bar{h}_0 \in C^1([0, 1])$  with  $\bar{h}_0(0) = \bar{h}_0(1) = 0$ , such that every solution  $u \in C^1(\mathbb{D}; \mathbb{S}^2)$  of the form (4.4) with  $h_0 \geq \bar{h}_0$  a.e., blows-up in finite time in the sense that*

$$\lim_{t \rightarrow T} |\nabla u(t, \cdot)|_{L^\infty(\mathbb{D}; \mathbb{R}^{2 \times 3})} = \infty, \quad (4.6)$$

for some  $T = T(u_0) > 0$ .

As mentioned in Chapter 1, there are several ways of modifying the linear map  $\sigma(u(t, x)) \in \mathcal{L}(\mathbb{R}^3)$ , so that the Kolmogorov equation associated to (4.1) remains (formally) invariant, replacing for instance  $\sigma(u(t, x)) = u(t, x) \times \cdot$ , by

$$\sigma_3(u(t, x)) \circ \dot{W} = \dot{W}_1(t, x) \circ \frac{\vec{e}_\theta(t, x)}{\sin \varphi(t, x)} + \dot{W}_2(t, x) \circ \vec{e}_\varphi(t, x),$$

where  $(\vec{e}_\rho, \vec{e}_\theta, \vec{e}_\varphi)$  denotes the standard mobile frame associated to the spherical coordinates of  $u(t, x)$ , see Rem. 1.1. Under the equivariant setting, we have  $\rho \equiv 1$ , and the colatitude  $\varphi(t, x)$  equals  $h(t, |x|)$ . There is no hope however to preserve equivariance along the flow if  $W_1 \neq 0$ , for in that case there would be a part of  $du$  “acting in the direction  $(\text{Vect}\{\vec{x}, \vec{k}\})^\perp$ ”. To guarantee the symmetry of  $u$ , we replace  $\dot{W}_1$  by 0, and also assume that  $\dot{W} := \dot{W}_2$  is radial, see Figure 4.1. This leads to the linear map  $\sigma = \sigma_4$  defined as

$$\begin{aligned} \sigma_4(u) \circ \dot{W}(t, |x|) &= \dot{W}(t, |x|) \circ \vec{e}_\varphi(u(t, |x|)) \\ &= \dot{W}(t, |x|) \circ \left( \frac{x}{|x|} \cos h(t, |x|); -\sin h(t, |x|) \right). \end{aligned} \quad (4.7)$$

A formal application of the Itô formula leads to the following equation with *additive noise*

$$\begin{cases} dh = \left( \partial_{rr}h + \frac{1}{r}\partial_r h - \frac{\sin 2h}{2r^2} \right) dt + dW , & \text{in } (t, r) \in \mathbb{R}^+ \times (0, 1) , \\ h(t, 0) = h(t, 1) = 0 , & \text{for } t \in \mathbb{R}^+ , \\ h(0, r) = h_0(r) , & \text{on } r \in (0, 1) . \end{cases} \quad (4.8)$$

This is a stochastic partial differential equation of the parabolic type, whose unknown is real valued, and which is related to the so-called *perturbed equation*:

$$\begin{cases} \partial_t v = \partial_{rr}v + \frac{1}{r}\partial_r v + \frac{2z - \sin 2(v+z)}{2r^2} , & \text{in } (t, r) \in \mathbb{R}^+ \times (0, 1) , \\ v(t, 0) = v(t, 1) = 0 , & \text{for } t \in \mathbb{R}^+ , \\ v(0, r) = h_0(r) , & \text{on } r \in (0, 1) . \end{cases} \quad (4.9)$$

where  $z = z(t, r)$  can be any trajectory in the support of the solution of the stochastic linear equation  $dZ = (\partial_{rr}Z + \partial_r Z/r - Z/r^2)dt + dW$ ,  $Z|_{t=0} = 0$ ,  $Z(\cdot, 0) = Z(\cdot, 1) = 0$ , see Remark 4.3 below.

The proof of Theorem 4.1 relies mainly on a comparison result for the parabolic equation (4.5), and in this work we shall employ a similar method. Several authors have studied comparison theorems for SPDE's (see for instance [BGP94]) but here the additiveness of the noise allows to appeal to deterministic theory only. A few changes in the proof of the first comparison principle for (4.5), given e.g. in [BDvdH02], gives the

**Comparison principle for (4.9).** Fix  $r_1 > 0$ , and denote  $J' = [0, r_1]$ . Let  $\kappa > 0$ ,  $z(t, r)$ ,  $(t, r) \in [0, \kappa] \times J$ , measurable map with  $z(\cdot, r=0) = 0$ , and such that there exists  $\kappa = \kappa(z) > 0$ , and regular maps  $f, g : [0, \kappa] \times J' \rightarrow \mathbb{R}$  with  $f(\cdot, r=0) = g(\cdot, r=0) = 0$ , fulfilling:

$$\begin{cases} \partial_t f \leq \partial_{rr}f + \frac{\partial_r f}{r} + \frac{2z - \sin 2(f+z)}{2r^2} , \\ \text{and} \\ \partial_t g \geq \partial_{rr}g + \frac{\partial_r g}{r} + \frac{2z - \sin 2(g+z)}{2r^2} , \end{cases} \quad \text{on } [0, \kappa] \times J' ,$$

$$f \leq g , \quad \text{on } \{t=0\} \times J' \cup [0, \kappa] \times \{r_1\} .$$

Then

$$f \leq g , \quad \text{on } [0, \kappa] \times J' .$$

A more precise statement of this and a proof are given in Appendix A.

*Additional Notation.* In the sequel we denote by  $J$  the compact interval  $[0, 1]$ . For  $1 \leq p < \infty$ , the notation  $L^p_{rdr}$  will be used to designate the Banach space of real valued measurable maps  $r \mapsto f(r)$ ,  $r \in J$ , such that

$$\|f\|_{L^p(J, rdr)} := \left( \int_0^1 |f(r)|^p r dr \right)^{\frac{1}{p}} < \infty .$$

The special case  $H = L^2_{rdr}$  defines a Hilbert space for the inner product

$$f, g \in H \mapsto \langle f, g \rangle = \int_0^1 f(r)g(r)r dr .$$

Norms in the Hilbert spaces  $V_\beta$  (see the definition (4.11)), when  $\beta \in \mathbb{R}$ , will be denoted by  $|\cdot|_\beta$ . For the sake of clarity, norms in spaces  $\mathcal{C}_T V_\beta$  will be denoted by  $\|\cdot\|_{T,\beta}$ , i.e.  $\|z\|_{T,\beta} = \sup_{0 \leq t \leq T} |z(t)|_\beta$ .

In the whole chapter, we consider a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$  satisfying the usual conditions, together with an adapted  $H$ -valued Wiener process regularized in space, that is:

$$W_\phi(t) = \sum_{k \in \mathbb{N}} \beta_k(t) \phi e_k , \quad (4.10)$$

where  $(\beta_k)_{k \in \mathbb{N}}$  stands for a sequence of real independent brownian motions in time,  $(e_k)_{k \in \mathbb{N}}$  is an ONB of  $H$ , and  $\phi : H \rightarrow H$  is a linear operator.

### (1.b) Local existence and uniqueness

We first need to introduce some functional spaces. Let  $A$  be the self-adjoint operator on  $H$  given, for  $\varphi$  in

$$D(A) = \left\{ f \in H : \partial_{rr} f \in H, \left( \frac{\partial_r f}{r} - \frac{f}{r^2} \right) \in H, f(0) = f(1) = 0 \right\} , \quad (4.11)$$

by

$$A\varphi = \partial_{rr}\varphi + \left( \frac{\partial_r}{r} - \frac{1}{r^2} \right) \varphi . \quad (4.12)$$

This operator has eigenpairs  $\{(e_k, \lambda_k), k \geq 1\}$  with  $(e_k)$  forming an ONB of  $H$ , while the values  $\lambda_k$  are negative and asymptotically quadratic in  $k$  – see Remark 4.5 below for a justification. Therefore, we can define, when  $\beta \in \mathbb{R}$ , the fractional power  $(-A)^{\beta/2}$  through

$$(-A)^{\beta/2} h := \sum_{k \in \mathbb{N}} (-\lambda_k)^{\beta/2} \langle h, e_k \rangle e_k ,$$

$$\text{for every } h \in V_\beta := \left\{ h \in H, \sum_{k \in \mathbb{N}} k^{2\beta} \langle h, e_k \rangle^2 < \infty \right\} . \quad (4.13)$$

The domains  $V_\beta$  endowed with the norm  $|h|_\beta := |(-A)^{\beta/2} h|_H$  are separable Hilbert spaces, and thus by the classical theory of SPDE's [DZ08], the process  $t, \omega \mapsto W_\phi(t, \omega)$  introduced in (4.10) has continuous paths in the space  $V_\beta$ , with full probability, provided  $\phi : H \rightarrow V_\beta$  is Hilbert-Schmidt.

Now, given  $\beta \geq 0$  and  $h_0 \in V_\beta$  equation (4.8) can be written as an infinite dimensional SDE in the space  $V_\beta$ :

$$\begin{cases} dh = (Ah + b(h))dt + dW_\phi , & \text{for } t \in \mathbb{R}^+ , \\ h(0) = h_0 , \end{cases} \quad (4.14)$$

where  $d$  denotes Itô differential, whereas the term  $b(h)$  denotes the nonlinearity

$$b(h)(r) = \frac{2h(r) - \sin 2h(r)}{2r^2} , \quad r \in J \setminus \{0\} . \quad (4.15)$$

As we will see in the proof of Proposition 4.1 below, the term  $b(h)$  has finite norm in  $H$ , provided  $h \in V_\beta$ , for sufficiently large  $\beta \geq 0$ .

**Proposition 4.1** (Existence, uniqueness and regularity of strong solutions). *Let  $4 \geq \beta > 4/3$ , and take  $\phi \in \mathbb{L}_2(H, V_\beta)$ . Then, for  $h_0 \in V_\beta$ , there exist a stopping time  $\tau^\beta(h_0)$ , and a unique  $h$  with paths in  $\mathcal{C}([0, \tau^\beta]; V_\beta)$ , a.s., mild solution of (4.14) in the sense that*

$$h(t) = S(t)h_0 + \int_0^t S(t-s)(b(h)(s))ds + \int_0^t S(t-s)dW_\phi(s) \quad , \quad \text{for } t \in [0, \tau^\beta) \quad , \quad \text{a.s.} \quad , \quad (4.16)$$

*$S(\cdot)$  being the semigroup  $e^{\cdot A}$ . The stopping time  $\tau^\beta$  is maximal in the sense given in (4.18). Moreover, the regularity propagates in the sense that if  $h_0 \in V_\beta$ , and if  $\phi \in \mathbb{L}_2(H, V_\beta)$ , with  $4 \geq \beta > 2$ , if  $\tau^\beta < \infty$ , setting  $\beta_* = (3 - \beta/2) \wedge (2 - \beta/4)$ , then a.s.  $\tau^\beta = \tau^{\beta_*}$ . In particular:*

$$\limsup_{t \rightarrow \tau^\beta(h_0)} |h(t)|_{\beta_*} = \infty \quad .$$

*Remark 4.1.* If  $f \in H$ , using the polar coordinates on the disk, define a map  $F \in L^2(\mathbb{D}; \mathbb{R}^2)$  by the expression  $F(x) = (f(r) \cos \theta, f(r) \sin \theta)$ , where  $x = (r \cos \theta, r \sin \theta)$ ,  $r \in J$ ,  $\theta \in [0, 2\pi]$ . Then  $|f|_H = c |F|_{L^2(\mathbb{D}; \mathbb{R}^2)}$ , and if  $f \in V_2$ , then  $F$  is in the domain of the Laplace operator with homogeneous Dirichlet boundary data on  $\partial\mathbb{D}$ , and  $\Delta F = (Af(r) \cos \theta, Af(r) \sin \theta)$ . The reader may also check that if we plugg the ansatz  $F$  in the expression  $\nabla^2 F$ , there holds

$$\int_{\mathbb{D}} |\nabla^2 F|^2 dx = 2\pi \int_0^1 (\partial_{rr} f)^2 r dr + 4\pi \int_0^1 \left( \frac{\partial_r f}{r} - \frac{f}{r^2} \right)^2 r dr \quad .$$

By a classical inequality, this justifies that the norms

$$|\partial_{rr} f|_H + \left| \left( \frac{\partial_r}{r} - \frac{1}{r^2} \right) f \right|_H \quad , \quad |f|_2 = |Af|_H \quad ,$$

are in fact equivalent on  $V_2$ .

*Remark 4.2.* For  $p \in [1, \infty)$ ,  $\beta \in \mathbb{R}$ ,  $f \in V_\beta$ , if  $\beta < 1$  and if

$$1 \leq p \leq p^* = \frac{2}{1 - \beta} \quad ,$$

the classical Sobolev Embedding Theorem in dimension 2 (see [AF03]) implies that  $|F|_{L^p(\mathbb{D})} \leq c |F|_{W_0^{\beta, 2}(\mathbb{D})} \simeq |(-\Delta)^{\beta/2} F|_{L^2(\mathbb{D})}$ . Thus, using the notations of Remark 4.1, since  $|F|_{L^p(\mathbb{D})} = (2\pi)^{\frac{1}{p}} |f|_{L_{rdr}^p}$ , and  $|(-\Delta)^{\beta/2} F|_{L^2(\mathbb{D})} = |f|_\beta$ , it is straightforward that we have the continuous embedding:

$$V_\beta \hookrightarrow L_{rdr}^p \quad .$$

Similarly if  $\beta > 1$ , then  $V_\beta \hookrightarrow \mathcal{C}(J; \mathbb{R})$ . In addition, by the formula  $|\nabla F|^2 = (\partial_r f)^2 + f^2/r^2$ , for any  $\beta > 2$ , there exists a constant  $c_\beta > 0$  such that for all  $f \in V_\beta$ ,  $|\partial_r f|_{L_{rdr}^\infty} \leq c_\beta |f|_\beta$ .

### (1.c) Statement of the main result

Our main result states as follows.

**Theorem 4.2.** *Let  $\phi \in \mathbb{L}_2(H, V_\beta)$  with  $\beta > 2$ , and assume that  $W_\phi$  is a  $\phi\phi^*$  - Wiener process. For any  $h_0 \in V_\beta$ , there exist a maximal local mild solution  $(h, \tau^\beta(h_0))$  of (4.8), which has paths in  $\mathcal{C}([0, \tau^\beta(h_0)]; V_\beta)$ , a.s. Moreover, if  $\ker\phi^* = \{0\}$ , then any solution blows up before a given time  $t_* > 0$  with non zero probability, namely for any  $h_0 \in V_\beta$  and  $t_* > 0$ , we have:*

$$\mathbb{P}(\tau^\beta(h_0) \leq t_*) > 0 \quad .$$

Moreover, if we set  $\beta_* = (3 - \beta/2) \wedge (2 - \beta/4)$ , one has a.s. on the event  $\{\tau(h_0) \leq t_*\}$ :

$$\limsup_{t \rightarrow \tau^\beta(h_0)} |h(t)|_{\beta_*} = \infty \quad .$$

**Outline of the proof.** The proof of Theorem 4.2 is given in the next section. It is based on the existence of a “pre-blow-up set”  $\mathfrak{H}$  in the state space of the process  $t, \omega \mapsto h(\omega, t)$ . In section 2, we give the proof of Theorem 4.2, given Claim 4.1, and two technical lemmas. This proposition basically states that there exists an open set of initial data  $h_0$  such that the associated solutions blow up in finite time, with positive probability. In section 3 we prove Claim 4.1, the main ingredient being Lemma 4.3, together with a topological argument. The proof of Lemma 4.3 relies on the comparison of the solution  $h$  with a certain class of subsolutions  $\psi$  given in Lemma 4.4. Section 4 is devoted to the proof of the local existence of a solution, namely Proposition 4.1. The proof relies on a classical fixed point argument. Appendix A is devoted to the proof of the comparison principle, whereas in Appendix B we prove the two technical lemmas of Section 2.

## 2 Proof of Theorem 4.2

The main argument in the proof of Theorem 4.2 is Claim 4.1, whose proof will be given in section 3.

*Remark 4.3.* Since the noise here is additive, solutions depend continuously – see Lemma 4.1 below – on the so-called Ornstein-Uhlenbeck process:

$$Z(t, \omega) = \int_0^t S(t-s) dW_\phi(s) \quad , \quad t \geq 0 \quad , \quad \omega \in \Omega \quad . \quad (4.17)$$

It will be then convenient, when  $z \in \mathcal{C}([0, \infty); V_\beta)$ , to write

$$\left( h_z(h_0), \tau_z(h_0) \right) \quad , \quad (4.18)$$

in order to designate the mild solution of (4.14) on  $\{Z = z\}$ , starting at  $h_0$ , together with its maximal time of existence (in some space  $V_\beta$ ), namely

$$\left[ \begin{array}{l} h_z(h_0) = v + z \quad , \\ \text{where the map } v = v(t, r) \text{ solves in the mild sense:} \\ \left\{ \begin{array}{l} \partial_t v = Av + b(v + z) \quad \text{on } [0, \tau_z(h_0)) \times J \quad , \\ v|_{t=0} = h_0 \quad , \end{array} \right. \end{array} \right. \quad (4.19)$$

with the understanding that the map  $t \mapsto h_z(h_0, t)$  cannot be extended continuously after  $\tau_z(h_0)$  i.e.  $|h_z(h_0, t)|_{V_\beta}$  becomes arbitrarily large as  $t \rightarrow \tau_z(h_0)$ .



**Claim 4.1.** Let  $\beta > 2$ , and  $\phi \in \mathbb{L}_2(H, V_\beta)$ . Then, for any  $t_* > 0$ , there exist two subsets  $\mathfrak{Z}$  of  $\mathcal{C}_{t_*} V_\beta$ , and  $\mathfrak{H}$  of  $V_\beta$ , with *nonempty interiors*, such that for all  $(z, h_0) \in \mathfrak{Z} \times \mathfrak{H}$ ,

$$\tau_z^\beta(h_0) \leq t_* .$$

We need to complete this statement with two other lemmas, whose proof are postponed to Appendix B.

**Lemma 4.1** (Continuous dependence). *Let  $T > 0$ ,  $\tilde{z} \in \mathcal{C}_T V_\beta$ , and  $\tilde{h}_0 \in V_\beta$  such that  $h_{\tilde{z}}(\tilde{h}_0, \cdot)$  exists on  $[0, T]$ . There exist open neighbourhoods  $\mathcal{V}$  of  $\tilde{z}$  in  $\mathcal{C}_T V_\beta$  and  $\mathcal{W}$  of  $\tilde{h}_0$  in  $V_\beta$  such that: for all  $(z, h_0) \in \mathcal{V} \times \mathcal{W}$ , a solution  $h_z(h_0, \cdot)$  of (4.16) exists on  $[0, T]$  and is unique in  $\mathcal{C}_T V_\beta$ . Moreover, using the notation defined in (4.18), the mapping*

$$(z, h_0) \in \mathcal{V} \times \mathcal{W} \longmapsto h_z(h_0, \cdot)|_{[0, T]} \in \mathcal{C}_T V_\beta ,$$

*is continuous.*

**Lemma 4.2** (controlability). *Take  $\beta > 0$ ,  $T_1 > 0$  and  $h_0, h_1 \in V_\beta$ . There exists a control  $z_1 \in \mathcal{C}_{T_1} V_\beta$  with  $z_1(0) = 0$ , such that:  $h_{z_1}(h_0, \cdot)$  exists on  $[0, T_1]$ , and*

$$h_{z_1}(h_0, T_1) = h_1 .$$

*Proof of Theorem 4.2.* It is sufficient to show the result with  $2t_*$  instead of  $t_*$ . Let  $t_* > 0$ , and take  $\mathfrak{Z}, \mathfrak{H}$  as in Claim 4.1. Since it is nonempty, we may consider an element  $h_1$  in the interior of  $\mathfrak{H}$ . By Lemma 4.2, there exists  $z_1 \in \mathcal{C}(0, t_*; V_\beta)$  such that  $h_{z_1}(h_0, \cdot)$  exists on  $[0, t_*]$  and  $h_{z_1}(h_0, t_*) = h_1$ . Using in addition Lemma 4.1, we see that there exists a neighbourhood  $\mathcal{V}_1$  of  $z_1$  in  $\mathcal{C}(0, t_*; V_\beta)$ , such that

$$\forall z \in \mathcal{V}_1 , h_z(h_0, t_*) \in \mathfrak{H} .$$

Since  $\ker \phi^* = \{0\}$ , then  $\phi$  has dense range in  $V_\beta$  and the process  $Z(t) = \int_0^t S(t-s) dW_\phi(s)$ ,  $t \geq 0$ , is non degenerate. Therefore,

$$\mathbb{P} \circ Z|_{[0, t_*]}^{-1}(\mathcal{V}_1) > 0 , \quad (4.20)$$

and similarly

$$\mathbb{P} \circ Z|_{[0, t_*]}^{-1}(\mathfrak{Z}) > 0 . \quad (4.21)$$

The result now follows by the fact that  $h$  fulfills the Markov property. More precisely, we define the extended state space  $V'_\beta = V_\beta \cup \{\Delta\}$  where the terminal state  $\Delta$  is an isolated point, and extend the stochastic flow  $h(h_0, t)$  as a family of processes with values in  $V'_\beta$ , where the value  $\Delta$  is reached for  $t \geq \tau^\beta(h_0)$ . By standard arguments (see e.g. [Rom11] and references therein), one can show that the processes obtained (which are still denoted by  $h(h_0, \cdot)$ ) are Markov. Denoting by  $P_t$  the operator acting on bounded borelian maps  $\varphi : V_\beta \rightarrow \mathbb{R}$ , by the relation  $P_t \varphi(h_0) = \mathbb{E}[\varphi(h(h_0, t))]$ , and by  $P(h_0, t; \Gamma) := P_t \mathbf{1}_\Gamma(h_0)$ , where  $\Gamma$  is borelian, and using the fact that on  $A := \{\tau(h_0) > t_*\}$  we have  $\tau(h_0) = t_* + \tau(h(h_0, t_*))$ , then there holds the relation:

$$\mathbb{P}(A \cap \{\tau(h_0) \leq 2t_*\}) = \int_{V_\beta} (P_{t_*} \mathbf{1}_{\{\tau(t_*, \cdot) \leq t_*\}})(h_1) P(h_0, t_*; dh_1) .$$

This implies in particular  $\mathbb{P}(\tau(h_0) \leq 2t_*) \geq \int_{\mathfrak{H}} (P_{t_*} \mathbf{1}_{\{\tau(t_*, \cdot) \leq t_*\}})(h_1) P(h_0, t_*; dh_1)$ , whence  $\mathbb{P}(\tau(h_0) \leq 2t_*) \geq \int_{\mathfrak{H}} \mathbb{P} \circ Z|_{[0, t_*]}^{-1}(\mathfrak{Z}) P(h_0, t_*; dh_1)$ , which is positive by (4.20) and (4.21). This proves Theorem 4.2.  $\blacksquare$

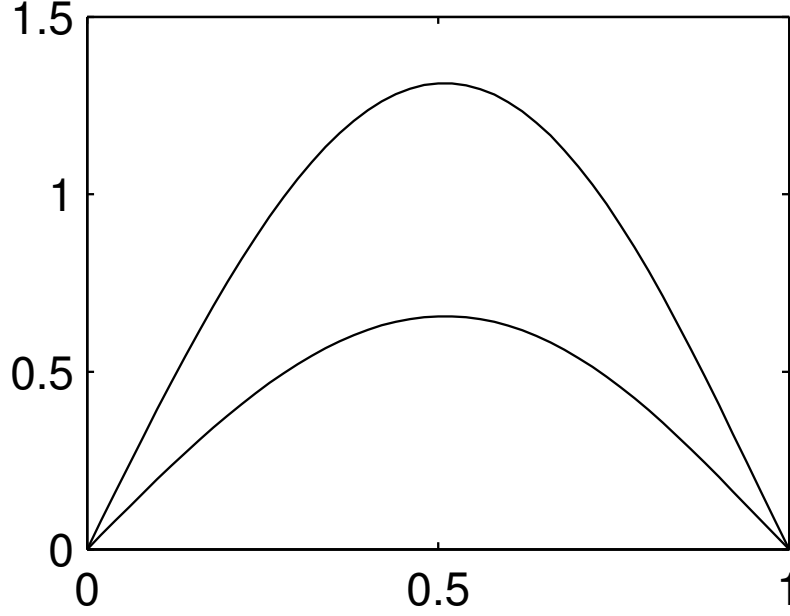


Figure 4.2 – Plots of  $\chi_1, \chi_2$  on  $J$ .

### 3 Proof of Claim 4.1

The first step is to show that given  $\beta > 2$ , and a fixed  $z \in \mathcal{C}_{2t_*} V_\beta$  with  $z(0) = 0$ , there exists a map  $\chi \in V_\beta$  (depending on  $z$ ) such that for every  $h_0$  lying over  $\chi$ , the associated solution  $h_z(h_0, \cdot)$  blows up before  $t_*$ . That will be stated in Lemma 4.3.

In the sequel, the following notation will we used for  $c > 0$ , and  $r \in J$ :

$$\begin{aligned} \chi_c(r) &:= 2cr - 3cr^3 + cr^5 \\ &= cr(1 - r^2)(2 - r^2) . \end{aligned} \tag{4.22}$$

see Fig. 4.2. The choice of this map is motivated to avoid matters of regularity in the spaces  $V_\beta$ ,  $\beta > 0$ . More precisely, on  $J \setminus \{0\}$ :  $A^2 \chi_c(r) = 48cr$ ,  $r \in J \setminus \{0\}$ , which belongs to  $V_{1/2-\epsilon}$  for any  $\epsilon > 0$  (this fact is left to the reader). Furthermore, we also have  $\chi_c(\partial J) \equiv 0$ , and  $A\chi_c(\partial J) \equiv 0$ , which ensures that the maps  $\chi_c$  belong to  $V_{9/2-\epsilon}$  for all  $\epsilon > 0$  and  $c > 0$ . Note that properly speaking, we do not consider maps but equivalence classes of maps. Nevertheless, since in the sequel  $\beta > 1$ , we restrict our attention to maps that have a continuous version (see Remark 4.2), we will say that an inequality  $h_0 \leq h_1$  is true for  $h_0, h_1 \in V_\beta$  as soon as it is true almost everywhere.

**Lemma 4.3 (Main Lemma).** *Let  $\beta > 2$ , and fix  $t_* > 0$ . There exists  $\bar{\eta} > 0$ , such that for all  $z \in \mathcal{C}_{2t_*} V_\beta$  with  $\|z\|_{2t_*, \beta} \leq \bar{\eta}$ , there exists a parabola  $\chi_* = \chi_*(z)$  belonging to the family (4.22), and satisfying the property that: if  $h_0 \in V_\beta$  with  $h_0 \geq \chi_*$ , then*

$$\tau_z^\beta(h_0) \leq t_* .$$

Moreover, the pre-blow-up set  $\mathfrak{S} = \{h_0 \in V_\beta, h_0 \geq \chi_*\}$ , has nonempty interior in  $V_\beta$ .

The main ingredient of the proof of Lemma 4.3 is the use of the comparison principle for the perturbed equation (4.9) with a certain class of subsolutions. The following lemma gives an explicit family of maps  $\{\psi_{\epsilon,\mu,\lambda_0,x_0}\}$  satisfying the differential inequality  $\partial_t \psi \leq A\psi + b(\psi + z)$ , up to a certain time  $t_+$ .

**Lemma 4.4.** *When  $\lambda_0, \epsilon, \delta > 0$  are real numbers, define  $\lambda = \lambda_{\epsilon,\delta,\lambda_0} : t \in [0, T_{\lambda_0}) \mapsto \lambda(t)$  as the solution of the ODE :*

$$\begin{cases} \lambda' = -\delta\lambda^\epsilon, & 0 \leq t \leq T_{\lambda_0} := \sup\{s \geq 0, \lambda(s) > 0\} = \frac{\lambda_0^{1-\epsilon}}{(1-\epsilon)\delta}, \\ \lambda(0) = \lambda_0. \end{cases} \quad (4.23)$$

Assume that there exists  $x_0 \in V_\beta$ ,  $z \in \mathcal{C}([0, T_{\lambda_0}]; V_\beta)$  with  $z(0) = 0$ , and  $t_+ = t_+(x_0, z) > 0$ , such that the process

$$x(t) = S(t)x_0 + z(t), \quad t \in [0, T_{\lambda_0}]$$

takes nonnegative values on  $[0, t_+ \wedge T_{\lambda_0}] \times J'$ , where  $J'$  is any compact subinterval of  $J$ , and  $S$  is the semigroup associated with  $A$ , see (4.12).

If we fix  $0 < \epsilon < 1$ , then there exist  $\bar{\mu} = \bar{\mu}(\epsilon)$ ,  $\bar{\delta} = \bar{\delta}(\epsilon)$ , such that for all  $\mu, \delta > 0$ , with  $\mu \geq \bar{\mu}$  and  $\delta \leq \bar{\delta}$ , for all  $\lambda_0 > 0$ , and  $\lambda = \lambda_{\epsilon,\delta,\lambda_0}$  as in (4.23), the map given by

$$\psi(r, t) = \arccos\left(\frac{\lambda(t)^2 - r^2}{\lambda(t)^2 + r^2}\right) + \arccos\left(\frac{\mu^2 - r^{2+2\epsilon}}{\mu^2 + r^{2+2\epsilon}}\right) + S(t)x_0(r),$$

satisfies the differential inequality

$$\partial_t \psi \leq A\psi + b(\psi + z) \text{ for } (t, r) \text{ in } [0, t_+ \wedge T_{\lambda_0}] \times J'. \quad (4.24)$$

*Proof of Lemma 4.4.* Let  $0 < \epsilon < 1$ . As in [CDY<sup>+</sup>92], we set for  $(\lambda, r) \in \mathbb{R}_+^* \times J'$ :

$$\varphi_\lambda(r) := \arccos\left(\frac{\lambda^2 - r^2}{\lambda^2 + r^2}\right), \quad \theta_{\epsilon,\mu}(r) := \arccos\left(\frac{\mu^2 - r^{2+2\epsilon}}{\mu^2 + r^{2+2\epsilon}}\right). \quad (4.25)$$

Recall that for any fixed triplet  $\lambda > 0, \epsilon > 0, \mu > 0$ , the maps  $\varphi_\lambda, \theta_{\epsilon,\mu}$  satisfy for  $r \in J'$  (see [CDY<sup>+</sup>92]):

$$A\varphi_\lambda(r) = \frac{\sin 2\varphi_\lambda(r) - 2\varphi_\lambda(r)}{2r^2} \quad (4.26)$$

$$A\theta_{\epsilon,\mu}(r) = \frac{(1+\epsilon)^2 \sin 2\theta_{\epsilon,\mu}(r) - 2\theta_{\epsilon,\mu}(r)}{2r^2} \quad (4.27)$$

Now, since  $\theta_{\epsilon,\mu}(r) \rightarrow 0$  as  $\mu \rightarrow \infty$ , it is possible to choose a parameter  $\bar{\mu}(\epsilon)$ , such that for all  $r \in J'$ ,

$$\cos \theta_{\epsilon,\mu}(r) \geq \frac{1}{1+\epsilon}. \quad (4.28)$$

We take  $\mu \geq \bar{\mu}$ , and denote abusively  $\theta = \theta_{\epsilon,\mu}(\cdot)$ ,  $\varphi = \varphi_{\lambda(\cdot)}(\cdot)$ , and  $Sx_0 = t \in \mathbb{R}_+ \mapsto S(t)x_0$ . Moreover, let  $z \in \mathcal{C}_{t_+} V_\beta$  being such that  $x = Sx_0 + z$  takes nonnegative values on  $[0, t_+] \times J'$ . For  $t \in [0, T_{\lambda_0})$ ,  $r \in J'$ , define:

$$\psi(t, r) = \varphi_{\lambda(t)}(r) + \theta(r) + S(t)x_0(r).$$

On the one side, using (4.26), (4.27), with classical trigonometric formulae, there comes pointwise for  $(t, r) \in [0, t_+] \times J'$ :

$$\begin{aligned}
A(\psi) + b(\psi + z) &= A(\varphi + \theta + Sx_0) + b(\varphi + \theta + x) \\
&= (2r)^{-2} \left[ (1 + \epsilon)^2 \underbrace{\sin 2\theta}_{=2 \cos \theta \sin \theta} + \overbrace{\sin 2\varphi - \sin 2(\varphi + \theta)}^{=-2 \sin \theta \cos(2\varphi + \theta)} \right. \\
&\quad \left. + 2x + \sin 2(\varphi + \theta) - \sin 2(\varphi + \theta + x) \right] + ASx_0 \\
&= (2r)^{-2} \left[ 2(1 + \epsilon)^2 \sin \theta \cos \theta - 2 \sin \theta \cos(2\varphi + \theta) + F_{\varphi, \theta}(x) \right] + ASx_0 \\
&\geq (2r)^{-2} \left[ 2 \sin \theta ((1 + \epsilon) - \cos(2\varphi + \theta)) + F_{\varphi, \theta}(x) \right] + ASx_0 \\
&\geq (2r)^{-2} [2\epsilon \sin \theta + F_{\varphi, \theta}(x)] + ASx_0 ,
\end{aligned}$$

where at the last line we used (4.28) to minorate  $\cos \theta \sin \theta$  by  $(1 + \epsilon)^{-1} \sin \theta$ , and we defined the family of maps

$$F_{\varphi, \theta}(x) = 2x - (\sin 2(\varphi + \theta + x) - \sin 2(\varphi + \theta)) , \quad x \in \mathbb{R} ,$$

for  $\varphi, \theta \in \mathbb{R}$ . Now, regardless of the values taken by the parameters  $\theta, \varphi$ , the map  $F_{\varphi, \theta}$  vanishes at the origin, and has nonnegative derivative on  $\mathbb{R}_+$ . We deduce that since the map  $x$  takes nonnegative values on  $[0, t_+] \times J'$ , then so is  $F_{\varphi, \theta}(x)$ . Moreover, simple computations show that for  $r \in J'$ :

$$\partial_t \psi(t, r) = \frac{2\delta \lambda(t)^\epsilon r}{\lambda(t)^2 + r^2} + AS(t)x_0(r) ,$$

and for  $(t, r) \in [0, t_+] \times J'$ ,

$$\begin{aligned}
A\psi + b(\psi + z)(t, r) &\geq \frac{\epsilon \sin \theta}{r^2} + AS(t)x_0(r) \\
&= \frac{2\epsilon \mu r^{\epsilon-1}}{\mu^2 + r^{2(1+\epsilon)}} + AS(t)x_0(r) \\
&\geq \frac{2\epsilon \mu r^{\epsilon-1}}{\mu^2 + 1} + AS(t)x_0(r) .
\end{aligned}$$

Thus, if for every  $(t, r)$  in  $[0, t_+] \times J'$ ,

$$\frac{2\epsilon \mu r^{\epsilon-1}}{\mu^2 + 1} \geq \frac{2\delta \lambda(t)^\epsilon r}{\lambda(t)^2 + r^2} ,$$

then (4.24) holds. Setting  $s = r/\lambda(t)$ , it is however sufficient to verify that:

$$\sup_{s \in \mathbb{R}} \frac{s^{2-\epsilon}}{1 + s^2} \leq \frac{\mu \epsilon}{\delta(\mu^2 + 1)} , \quad (4.29)$$

which is true if  $\delta \geq \bar{\delta}$  for a certain  $\bar{\delta} = \bar{\delta}(\epsilon, \bar{\mu})$ , where  $\bar{\mu} > 0$  is fixed by (4.28). This proves the lemma. ■

*Remark 4.4.* As in [CDY<sup>+</sup>92], observe that the subsolution  $\psi = \psi_{\epsilon, \mu, \lambda_0, x_0}$  constructed above blows up at the time  $t = T_{\lambda_0}$  where

$$T_{\lambda_0} = \frac{\lambda_0^{1-\epsilon}}{(1-\epsilon)\delta} . \quad (4.30)$$

Indeed, since  $x_0 \in V_\beta$  with  $\beta > 2$ , then  $|\partial_r S(t)x_0|_{L_{rdr}^\infty}$  is bounded for  $t \in [0, T_{\lambda_0})$  (see Remark 4.2), and:

$$\begin{aligned} \partial_r \psi(t, \cdot)|_{r=0} &= \partial_r \varphi(r, t)|_{r=0} + \partial_r \theta(r)|_{r=0} + \partial_r S(t)x_0(r)|_{r=0} \\ &= \frac{2}{\lambda(t)} + \partial_r S(t)x_0(0) \rightarrow \infty \text{ as } t \rightarrow T_{\lambda_0} . \end{aligned}$$

Let  $(h, \tau^\beta) = (h_Z(h_0, \cdot), \tau_Z^\beta(h_0))$  denote a strong solution of (4.14), where  $h_0 \in V_\beta$ , and  $Z$  is as in (4.17). Take  $x_0, z, t_+$  as in Lemma 4.4, and assume that  $\tau_z(h_0) \geq T_{\lambda_0}$ . If one can show that the subsolution  $f = \psi$  satisfies  $f \leq g$  on  $\{0\} \times J' \cup [0, t_+] \times \partial J'$ , where  $g = h - z$ , then by the comparison principle for (4.9),  $f \leq g$  on  $[0, t_+ \wedge T_{\lambda_0}) \times J'$ . Since the maps  $f, g$  vanish at the origin regardless of the time variable, it follows that:

$$\partial_r f(t, 0) \leq \partial_r g(t, 0), \quad \forall t \in [0, t_+ \wedge T_{\lambda_0}) .$$

In particular if we assume that  $T_{\lambda_0} \leq t_+$ , then  $|\partial_r h|_{L_{rdr}^\infty} \rightarrow \infty$  as  $t \rightarrow T_{\lambda_0}$ , which by Remark 4.2 implies blow-up also in the sense  $\limsup_{t \rightarrow T_{\lambda_0}} |h(t)|_\beta = \infty$ .

We can now turn to the proof of the main lemma.

*Proof of Lemma 4.3.* For our purpose, it is sufficient to assume that  $h_0 \in V_\beta$ , with  $\beta \in (2, 4]$ . By Proposition 4.1 we know that the solution  $h$  of (4.14) has a.s. continuous paths in  $V_\beta$ . Take  $z \in \mathcal{C}_{2t_*} V_\beta$ ,  $x_0 \in V_\beta$ , and define for  $t \leq 2t_*$ :  $x(t) = S(t)x_0 + z(t)$ . In this proof we denote by  $J'$  the compact interval  $[0, 1/2]$ .

*Step 1: nonnegativeness of  $x$  up to a positive time.* Assume that on  $J'$ ,  $x_0 \geq \chi_1$ , where  $\chi_1$  is the parabola defined by (4.22). We show that if the perturbation  $z$  is not too large in  $\mathcal{C}_{2t_*} V_\beta$ , then the map  $x$  defined above stays nonnegative for almost every  $r \in J'$ . We claim that there exists a constant  $\eta > 0$ , independent of  $x_0 \geq \chi_1$ , such that for all  $y \in V_\beta$ ,

$$|x_0 - y|_\beta \leq 2\eta \Rightarrow |y|_{J'} \geq 0 . \quad (4.31)$$

Indeed, since  $\beta > 2$ , then there exists  $c_\beta > 0$ , such that for all  $y \in V_\beta$ , (see Remark 4.2),

$$|\partial_r x_0 - \partial_r y|_{L_{rdr}^\infty(J')} \leq c_\beta |x_0 - y|_\beta .$$

Choose  $\eta = c/(2c_\beta)$ , where  $c$  is such that  $\chi_1(r) - cr \geq 0$  for  $r \in J'$  (note that  $c$  and so  $\eta$  do not depend on  $x_0 \in V_\beta$ ), so that  $|y - x_0|_\beta \leq \eta$  will imply  $|\partial_r y - \partial_r x_0|_{L_{rdr}^\infty} \leq c/2$ . We conclude by the Mean Value Theorem, observing first that both maps equal zero at the origin: if  $|x_0 - y|_\beta \leq \eta$ , then  $\forall r \in J'$ ,  $y(r) \geq x_0(r) - cr \geq \chi_1(r) - cr$  and thus  $y(r) \geq 0$ , which proves the claim.

Furthermore, for a fixed  $x_0 \in V_\beta$  with  $x_0 \geq \chi_1$ , since  $S$  is a strongly continuous semigroup, there exist a time  $t_+(x_0)$  such that

$$\text{for all } t \in [0, t_+(x_0)] , \quad |S(t)x_0 - x_0|_\beta \leq \eta ,$$

and thus for  $0 \leq t \leq t_+(x_0)$ ,  $z \in B(0, \eta)$ , if  $x = S(\cdot)x_0 + z$  is defined as above, there holds

$$\begin{aligned} |x(t) - x_0|_\beta &\leq |S(t)x_0 - x_0|_\beta + |z(t)|_\beta \\ &\leq 2\eta . \end{aligned}$$

We have to get rid of the dependence of  $t_+(x_0)$  with respect to  $x_0$ . But if  $x_0 \in V_\beta$  with  $x_0 \geq \chi_1$  on  $J$ , apply the linear comparison principle (see Rem. 4.6) on the whole interval  $J$  to  $f = S(\cdot)\chi_1$ ,  $g = S(\cdot)x_0$ ,  $\tau = 2t_*$  (note that we have  $f \leq g$  on  $\{0\} \times J \cup [0, 2t_*] \times \partial J$ ). Then we immediately have  $t_+(x_0) \geq t_+(\chi_1)$ . Now define  $t_+ = t_+(\chi_1)$ . We have proved that there exists  $\eta > 0$  such that for all  $t \in [0, t_+ \wedge 2t_*]$ , for all  $x_0 \in V_\beta$  with  $x_0 \geq \chi_1$  on  $J$ , for all  $z \in \mathcal{C}_{2t_*} V_\beta$  with  $\|z\|_{2t_*, \beta} \leq \eta$ , then  $x(t) \geq 0$  on  $J'$ , where  $x = S(\cdot)x_0 + z$ .

*Step 2. Construction of a pre-blow-up set for a fixed  $z$ .* Consider  $x_0 \in V_\beta$  with  $x_0 \geq \chi_1$  on  $J$ . It is sufficient to prove the proposition with  $t_* \wedge t_+$  instead of  $t_*$ , thus we can assume in the sequel that

$$t_+ = t_* .$$

By the previous step, for any  $z \in \mathcal{C}_{2t_+} V_\beta$ ,  $x(t)|_{J'} \geq 0$  for  $t \in [0, t_+]$ . Fix  $z \in \mathcal{C}_{2t_+} V_\beta$  with  $\|z\|_{2t_+, \beta} \leq \eta$ . In order to lighten the notations we denote by  $\tau = \tau_z^\beta$ , and  $h = h_z$ . Take any  $0 < \epsilon < 1$ , and fix  $\mu \geq \bar{\mu}(\epsilon)$ ,  $\delta \leq \bar{\delta}(\epsilon)$ ,  $\lambda = \lambda_{\epsilon, \delta, \lambda_0}$  – see (4.23)-(4.25)-(4.28)-(4.29) – where  $\lambda_0 > 0$  is chosen so that

$$T_{\lambda_0} = \frac{\lambda_0^{1-\epsilon}}{\delta(1-\epsilon)} \leq t_+ ,$$

so that we know by Lemma 4.4 that the map  $\psi_0 := \psi_{\epsilon, \mu, \lambda_0, x_0}$  defined as

$$\begin{aligned} \psi_0(t, r) &= \arccos\left(\frac{\lambda(t)^2 - r^2}{\lambda(t)^2 + r^2}\right) + \arccos\left(\frac{\mu^2 - r^{2+2\epsilon}}{\mu^2 + r^{2+2\epsilon}}\right) + S(t)x_0(r) , \\ &\text{for } (t, r) \in [0, T_{\lambda_0}] \times J , \end{aligned} \quad (4.32)$$

with  $\lambda(\cdot) = \lambda(\lambda_0)(\cdot)$  as in (4.23), satisfies

$$\partial_t \psi_0 \leq A\psi_0 + b(\psi_0 + z) \text{ on } [0, T_{\lambda_0}] ,$$

with a blow-up at  $t = T_{\lambda_0}$ . We aim to compare solutions  $h(h_0)$ ,  $h_0 \in V_\beta$ , with subsolutions of the ansatz (4.32), and conclude by Remark 4.4 that blow-up of  $h$  happens before  $t_+$ . For that purpose, it remains to ensure that

$$\psi_0 \leq h - z \text{ on } \Sigma := \{t = 0\} \times J' \cup [0, t_+] \times \partial J' . \quad (4.33)$$

If for instance  $h_0$  is such that for all  $t \in [0, t_+]$ ,

$$h\left(h_0, t; r = \frac{1}{2}\right) - z\left(t; r = \frac{1}{2}\right) > \sup_{(r, \lambda) \in J' \times \mathbb{R}_+^*} \left(\varphi_\lambda(r) + \theta(r) + S(t)x_0(r)\right) ,$$

then  $h(h_0, t; r = 1/2) - z(t; r = 1/2)$  lies over all possible value  $\psi_{\epsilon, \mu, \lambda, x_0}(t, 1/2)$ , with  $\psi$  as in (4.32), regardless of the values taken by the parameter  $\lambda_0 > 0$  and by  $0 \leq t \leq t_+$ . In particular, this will imply the bound needed on  $[0, t_+] \times \partial J'$ . Moreover, note that  $\pi$  is an upper bound for the family of maps  $(\varphi_\lambda(\cdot))_{\lambda > 0}$  (see the illustration in figure 4.3). This motivates the following definition: set

$$\gamma := \pi + |\theta_{\epsilon, \mu}|_{L_{rdr}^\infty} + \sup_{t \geq 0} |S(t)x_0|_{L_{rdr}^\infty} , \quad (4.34)$$

and define

$$t_\Sigma(h_0) = \inf \left\{ 0 \leq t \leq \tau(h_0), \quad h\left(h_0, t; r = \frac{1}{2}\right) \leq \gamma + z\left(t; r = \frac{1}{2}\right) \right\}, \quad (4.35)$$

with the understanding that  $t_\Sigma(h_0) = \tau(h_0)$  if the set is empty.

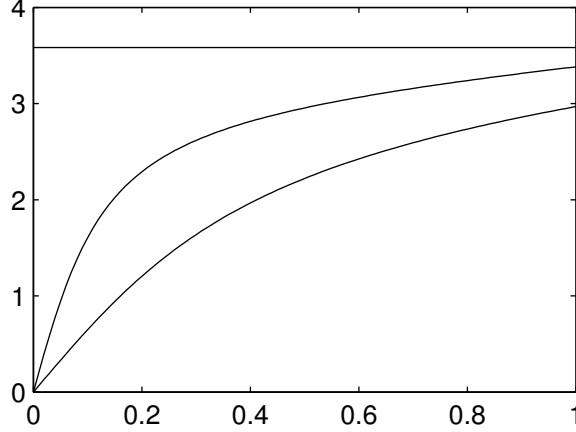


Figure 4.3 – Plots of  $\gamma, \psi_{\lambda_0}$  for  $\lambda_0 = 0.1, 0.01$

Note that  $\gamma$  is well-defined. Indeed for any  $u = \sum_k u_k e_k \in V_\beta$ , by Remark 4.2, since  $\beta > 1$ , the mapping  $t \mapsto |S(t)u|_{L_{rdr}^\infty} = |\sum_k u_k e^{-t(x_k)^2} e_k|_{L_{rdr}^\infty}$ ,  $t \geq 0$ , is bounded – see Section 4.

We claim now that there exists an integer  $k = k(z) \geq 1$  such that for all  $h_0 \in V_\beta$ , if  $h_0 \geq \chi_k$ , then

$$\tau(h_0) \leq t_+. \quad (4.36)$$

Indeed, let  $h_0 \in V_\beta$  with  $h_0|_{J'} \geq \psi_0|_{\{t=0\} \times J'}$  and assume that  $\tau(h_0) > t_+$ . Note that in particular the assertion

$$\text{“ } h\left(h_0, \cdot; r = \frac{1}{2}\right) \geq \gamma \text{ on } [0, t_+] \text{ ”}$$

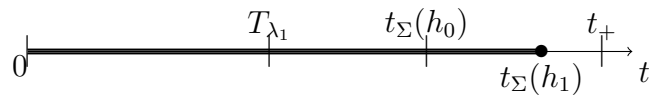
is false, otherwise by comparison with  $\psi_0$  plus Remark 4.4, we would have blow-up for  $h(h_0)$  before  $t_+$ . So we have  $t_\Sigma(h_0) \leq t_+$ .

Now, choose any  $\lambda_1 > 0$  with  $T_{\lambda_1} = \lambda_1^{1-\epsilon}/(\delta(1-\epsilon)) \leq t_\Sigma(h_0)$ , and define  $\psi_1 := \psi_{\epsilon, \mu, \lambda_1, x_0}$  by the formula (4.32) with  $\lambda_1$  instead of  $\lambda_0$ . Since the map  $\psi_1$  is Lipschitz, we can always find  $k \geq 1$  such that

$$\chi_k|_{J'} \geq \psi_1|_{\{t=0\} \times J'}$$

where  $\chi_k$  is as in (4.22). Consider any  $h_1 \in V_\beta$  with  $h_1 \geq \chi_k$  on  $J$ . One has the following alternative :

- (i) either  $t_\Sigma(h_1) \geq t_\Sigma(h_0)$ .

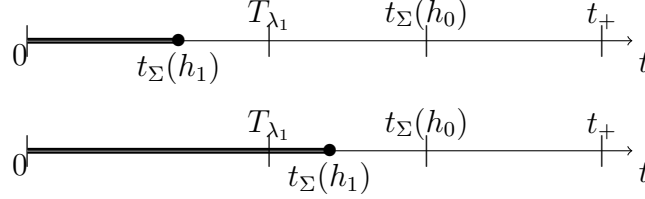


In this case, we have:

$$T_{\lambda_1} \leq t_{\Sigma}(h_1) \leq \tau(h_1) \text{ , and } h_1 \geq \chi_k \geq \psi_1 \text{ on } \{t = 0\} \times J' \cup [0, T_{\lambda_1}] \times \partial J' \text{ , (4.37)}$$

and the comparison principle for (4.9) can be applied on the thick part of the above line segment, in particular with  $\kappa = T_{\lambda_1}$ , and  $f = \psi_1$  and  $g = h(h_1) - z$ . By Remark 4.4 we obtain that  $h$  blows-up before  $T_{\lambda_1}$ , whence  $T_{\lambda_1} = \tau(h_1) \leq t_+$ .

(ii) Either  $t_{\Sigma}(h_1) < t_{\Sigma}(h_0)$ .



In this case, apply the comparison principle for (4.9) on the whole interval  $J$  with  $\kappa = \tau(h_0) \wedge \tau(h_1)$ ,  $f = h(h_0) - z$ , and  $g = h(h_1) - z$ , so that in particular:

$$\text{on } [0, \tau(h_0) \wedge \tau(h_1)] \text{ , there holds } f\left(\cdot, \frac{1}{2}\right) \leq g\left(\cdot, \frac{1}{2}\right) \text{ .}$$

Therefore, in this case one has necessarily  $\tau(h_1) = t_{\Sigma}(h_1)$ , otherwise we would have

$$g\left(t, \frac{1}{2}\right) \leq \gamma < f\left(t, \frac{1}{2}\right) \text{ for } t \in [t_{\Sigma}(h_1), t_{\Sigma}(h_0) \wedge \tau(h_1)] \text{ ,}$$

leading to a contradiction. Moreover, one has  $t_{\Sigma}(h_1) \leq t_+$ , and thus  $\tau(h_1) \leq t_+$ .

We see that in both cases (4.36) is true, and the claim implies that

$$\mathfrak{H} := \{h_1 \in V_{\beta} \text{ , } h_1 \geq \chi_{k(z)}\}$$

defines a pre-blow-up set for the individual element  $z$ .

*Step 3. Nonemptiness of  $\mathfrak{H}$ .* It suffices to show the result when  $k = 1$ , namely that the set  $\mathfrak{H} = \{h_1 \in V_{\beta} \text{ , } h_1 \geq \chi_1\}$  has nonempty interior for the topology of  $V_{\beta}$ .

Set  $h_0 = \chi_2 \in \mathfrak{H}$ , so that  $h_0 \in \mathfrak{H}$ . By Remark 4.2, since  $\beta > 2$ , there exists a sufficiently small radius  $R > 0$  such that if  $h_1 \in V_{\beta}$  with  $|h_1 - h_0|_{\beta}$ , then  $|\partial_r h_1 - \partial_r h_0|_{L_{rdr}^{\infty}} \leq 1/2$ . By the Mean Value Theorem, since  $h_0$  and  $h_1$  vanish for  $r \in \{0, 1\}$ , for all  $r \in [0, 1/2]$ :  $|h_1(r) - h_0(r)| \leq (1/2)r \leq r(1 - r)$ , and the same inequality holds when  $r \in [1/2, 1]$ . Thus for all  $r \in J$ :

$$|h_1(r) - h_0| \leq r(1 - r) \text{ .}$$

The reader may also check that

$$\forall r \in J, \chi_1(r) = r(1 - r^2)(2 - r^2) \geq r(1 - r) \text{ .}$$

Thus, for all  $h_1$  belonging to an open ball centered at  $h_0 = \chi_2$ , and for all  $r \in J$ :  $h_1(r) \geq \chi_2(r) - cr(1 - r) \geq \chi_1(r)$ , which means that  $h_1 \in \mathfrak{H}$ . This finishes the proof of Lemma 4.3. ■



We now turn to the proof of Claim 4.1.

*Proof of Claim 4.1.* Let  $4 \geq \beta > 2$ . Let  $\bar{\eta} > 0$  taken as in Lemma 4.3. So far, we have shown in Lemma 4.3 that given a trajectory  $z$  in the ball  $\mathbb{B} \subseteq \mathcal{C}_{2t^*}V_\beta$ , centered at zero and of radius  $\bar{\eta}$ , there exist an integer  $k(z)$ , such that for all  $h_0 \in V_\beta$ ,  $h_0 \geq \chi_{k(z)}$ , then  $\tau_z^\beta(h_0) \leq t_*$ . To conclude we need some globalization technique to reverse the quantifiers. Define

$$\mathfrak{F}_k(t_*) = \{z \in \mathbb{B}, \forall h_0 \in V_\beta \text{ with } h_0 \geq \chi_k \text{ on } J, \tau_z^\beta(h_0) \leq t_*\}.$$

We claim that  $\mathfrak{F}_k(t_*)$  is a closed subset of  $\mathbb{B}$ . Indeed, by definition: if  $z \in \mathbb{B} \setminus \mathfrak{F}_k(t_*)$ , there exists  $h_0 \in V_\beta$  with  $h_0 \geq \chi_k$  on  $J$  and  $\tau_z(h_0) > t_*$ . Let  $z^n \in \mathbb{B} \rightarrow z$  in  $\mathbb{B}$ , as  $n \rightarrow \infty$ . Let  $\epsilon > 0$  such that  $h_z(h_0, \cdot)$  is defined on  $[0, t_* + \epsilon]$ . By Lemma 4.1,  $h_{z^n}(h_0, \cdot)$  will be defined up to  $t_* + \epsilon$ , provided  $n$  is large enough. And thus  $(\mathfrak{F}_k(t_*))^c$  is an open set of  $\mathbb{B}$ , which proves the claim.

By Lemma 4.3, if  $z \in \mathbb{B}$ , then there exists  $k$  such that  $z \in \mathfrak{F}_k(t_*)$ , thus

$$\mathbb{B} = \bigcup_{k \in \mathbb{N}} \mathfrak{F}_k(t_*).$$

Hence, by Baire's Theorem, there exists at least one  $k^*$  such that  $\mathfrak{F}_{k^*}(t_*)$  has non-empty interior. Thus we can set  $\mathfrak{Z} = \mathfrak{F}_{k^*}(t_*)$ . If we define  $\mathfrak{H} = \{h_0 \in V_\beta, h_0 \geq \chi_{k^*}\}$ , then for all  $(z, h_0) \in \mathfrak{Z} \times \mathfrak{H}$ , there holds  $\tau_z(h_0) \leq t_*$ . This proves the proposition.  $\blacksquare$

## 4 Proof of Proposition 4.1

### (4.a) An interpolation Lemma.

In order to obtain estimates on the nonlinear term  $b(h) = (2h - \sin 2h)/(2r^2)$ , we use the following Lemma, which is based on expansion of elements of  $H$  in terms of the so-called *Fourier-Bessel series* – see [Wat95, chap. 18].

**Lemma 4.5.** *Let  $\beta > 0$ .*

- (i) *For any  $p \in [1, \infty]$ , take  $2/p - 1/2 < \nu < 2/p + 1$ , and define the operator  $T : D(T) \subseteq V_\beta \rightarrow L_{rdr}^p$  by  $Tf = \varphi/r^\nu$  for  $f \in D(T) = \{\varphi \in V_\beta, |\varphi/r^\nu|_{L_{rdr}^p} < \infty\}$ . Then,  $T$  can be extended to a continuous linear operator  $T : V_\beta \rightarrow L_{rdr}^p$ , as soon as  $\beta > 1 + \nu - 2/p$ .*
- (ii) *If  $p > 4$ , then the derivation  $\partial_r : D(\partial_r) \subseteq V_\beta \rightarrow L_{rdr}^p$ , where  $D(\partial_r) = \{\varphi \in V_\beta, |\partial_r \varphi|_{L_{rdr}^p} < \infty\}$ , can be extended to a continuous linear operator  $\partial_r : V_\beta \rightarrow L_{rdr}^p$ , provided  $\beta > 2 - \frac{2}{p}$ .*

*Remark 4.5.* The eigenvectors of  $(A, D(A))$  – see (4.11)-(4.12) – derive from the so called *Bessel functions of the first kind*. Recall that the order one Bessel function of the first kind, which is generally denoted by  $J_1(y)$ ,  $y \in \mathbb{R}^+$ , is determined by the ODE:

$$\begin{cases} y^2 \frac{dJ_1}{dy^2} + y \frac{dJ_1}{dy} + (y^2 - 1)J_1 = 0, & \text{for } y \geq 0, \\ J_1(0) = 0. \end{cases}$$

The zeros of  $J_1$  form a countable subset  $(x_k)_{k \geq 1}$  of  $\mathbb{R}_+^*$ , and it is a well known fact that, if we arrange them in ascending order (we will do this assumption in the sequel), then the  $x_k$ 's are asymptotically linear in  $k \in \mathbb{N}^*$ . For  $k \in \mathbb{N}^*$ , the mappings

$$e_k := \left( r \mapsto \frac{1}{|J_1(x_k \cdot)|_H} J_1(x_k r) , r \in J \right) , \quad (4.38)$$

define a family  $(e_k)_{k \in \mathbb{N}^*}$  of eigenvectors of  $A$ , with associated eigenvalues  $-(x_k)^2$ ,  $k \in \mathbb{N}^*$ . It forms an orthonormal basis of  $H$ . It follows that  $A$  generates an *analytical* semigroup  $t \mapsto S(t) = e^{tA}$ ,  $t \geq 0$ , on the hilbert space  $H$  (see e.g. [Paz83]). This provides also a precise definition of the fractional powers of  $-A$ , through

$$(-A)^{\beta/2} f := \sum_{k \in \mathbb{N}} (x_k)^\beta \langle f, e_k \rangle e_k , \text{ for } f \text{ in } V_\beta , \quad (4.39)$$

the series being convergent in  $H$ , see (4.13).

*Proof of Lemma 4.5.*

*Proof of (i).* Let  $p \in [1, \infty)$ . Using the orthonormal basis defined in (4.38), for  $k \geq 1$ , and setting  $c_k := |J_1(x_k \cdot)|_H^{-1}$ , one has by definition

$$\begin{aligned} \left| \frac{1}{r^\nu} e_k \right|_{L_{rdr}^p}^2 &= (c_k)^2 \left| \frac{1}{r^\nu} J_1(x_k \cdot) \right|_{L_{rdr}^p}^2 \\ &= (c_k)^2 (x_k)^{2\nu-4/p} \left( \int_0^{x_k} \frac{J_1(y)^p}{y^{p\nu}} y dy \right)^{2/p} , \end{aligned}$$

where we have done the change of variable  $y = x_k r$ . Then, using the well known properties concerning Bessel functions, see [Wat95, chap. 7], there exist constants  $c, c' > 0$  such that

$$J_1(y) \leq cy , \quad \forall y \in J \text{ and } |J_1(y)| \leq c' y^{-\frac{1}{2}} , \quad \forall y \in [1, \infty) ,$$

together with the fact that  $x_k$  is asymptotically linear in  $k \geq 1$  ([Wat95, p. 503-510]), we easily obtain that for  $\nu \geq 0$ ,

$$\int_0^{x_k} \frac{J_1(y)^p}{y^{p\nu}} y dy \leq c'' \left( \int_0^1 u^{p-p\nu+1} dy + \int_1^{x_k} y^{-p/2-p\nu+1} dy \right) . \quad (4.40)$$

If  $p, \nu$  satisfy the sufficient condition  $2/p - 1/2 < \nu < 2/p + 1$ , then each term in the right hand side of (4.40) is finite as  $k \rightarrow \infty$ . Using moreover the asymptotics  $(c_k)^2 = |J_1(x_k \cdot)|_H^{-2} = \mathcal{O}(k)$  (this fact is left to the reader), we have

$$\left| \frac{1}{r^\nu} e_k \right|_{L_{rdr}^p}^2 = \mathcal{O}(k^{1+2\nu-4/p}) .$$

Take  $\beta > 1 + \nu - 2/p$ . Using now triangle and Cauchy-Schwarz inequalities on the Fourier-Bessel series of  $f \in V_\beta$ , there holds the relation

$$\begin{aligned} |Tf|_{L_{rdr}^p} &\leq \sum_{k \geq 1} |\langle f, e_k \rangle| |Te_k|_{L_{rdr}^p} \\ &\leq |f|_\beta \left( \sum_{k \geq 1} (x_k)^{-2\beta} |Te_k|_{L_{rdr}^p}^2 \right)^{1/2} , \end{aligned} \quad (4.41)$$

and thus we have a continuous extension  $T : V_\beta \rightarrow L_{rdr}^p$ . The proof when  $p = \infty$  uses similar arguments, and is left to the reader.

*Proof of (ii).* The proof of the second assertion is rather similar, using the fact that  $|\partial_r e_k|_{L_{rdr}^p} = c_k x_k |J_1'(x_k \cdot)|_{L_{rdr}^p}$ , together with the derivative identity [Wat95, p. 17-19]

$$J_1'(y) = J_0(y) - \frac{J_1(y)}{y}, \quad y \geq 0, \quad (4.42)$$

which defines an element of  $L_{rdr}^p$  near the origin, for any  $p \geq 1$ . There exists a constant  $c > 0$  such that

$$|\partial_r e_k|_{L_{rdr}^p} \leq c k^{3/2-2/p} \left( c_p + \int_1^{x_k} (J_0(y) - J_1(y)/y)^p y dy \right)^{1/p}.$$

As for  $J_1$ , there exists  $c' > 0$ , such that:  $J_0 \leq c'y^{-1/2}$ , the other term  $J_1/y$  being smaller at infinity. Then, the integral in the right hand side is bounded, regardless of  $k \in \mathbb{N}^*$ , provided  $p > 4$ . We conclude by the inequality (4.41) with  $T$  replaced by  $\partial_r$ , observing that if  $\beta > 2 - 2/p$ , then  $\sum_k (x_k)^{-2\beta} k^{3-4/p}$  is summable.  $\blacksquare$

We are now able to prove the proposition.

#### (4.b) Proof when $\beta \in (4/3, 2]$

Consider parameters  $\beta > 0, T > 0$  that will be fixed later. Take  $h_0 \in V_\beta$ . As in (4.17), for  $\omega \in \Omega$ , set  $Z(\omega, t) := \int_0^t S(t-s) dW_\phi(s)$ . It is standard that since  $\phi \in \mathbb{L}_2(H, V_\beta)$ , the process  $Z$  is a random variable supported in the space  $\mathcal{C}_T V_\beta$  – see [DZ08]. Therefore we can take  $z \in \mathcal{C}_T V_\beta$ , and argue pathwise, considering the translated equation (4.19) with unknown  $v$ . If a solution  $v$  exists up to  $\tau = \tau(z) > 0$ , it is standard that  $h := v + z$  gives a solution of (4.14) on  $\{Z|_{[0,\tau]} = z|_{[0,\tau]}\}$ .

For each  $z \in \mathcal{C}_T V_\beta$ , we aim to find a fixed point  $v$  for the map  $\Gamma_{z,h_0}^T$ , defined as

$$\Gamma_{z,h_0}^T(v)(t) := S(t)h_0 + \int_0^t S(t-s)b(v(s) + z(s))ds, \quad \text{for } t \in [0, T]. \quad (4.43)$$

We show that if  $T_* > 0$  is sufficiently small, depending only on  $\|z\|_{T,\beta}$  and  $|h_0|_\beta$ , then the mapping  $\Gamma_{z,h_0}^{T_*}$  is a contraction of a certain ball of  $\mathcal{C}_{T_*} V_\beta$ . It relies mainly on the three following properties:

$$|(-A)^{\gamma/2} S(t)|_{\mathcal{L}(V_\beta)} \leq ct^{-\gamma/2}, \quad \text{for all } t > 0 \text{ and all } \gamma \in \mathbb{R}, \quad (4.44)$$

and when  $4/3 < \beta < 2$ ,

$$|b(v)|_H \leq c'|v|_\beta^3, \quad \text{for all } v \in V_\beta, \quad (4.45)$$

$$|b(u) - b(v)|_H \leq c''|u - v|_\beta(|u|_\beta^2 + |v|_\beta^2) \quad \text{for all } u, v \in V_\beta, \quad (4.46)$$

with constants depending on  $\gamma, \beta$  and  $A$  only.

The property (4.44) comes from  $|(-A)^{\gamma/2} h|_\beta^2 = \sum_k (x_k)^{2(\beta+\gamma)} e^{-2t(x_k)^2} \langle h, e_k \rangle^2$  together with

$$\sup_{x \geq 0} x^{2\gamma} e^{-2tx} \leq (\gamma/t)^{2\gamma} e^{-2\gamma}.$$

On the other hand, denoting by  $F : x \mapsto x - \sin 2x/2$ ,  $x \in \mathbb{R}$ , and using the inequality  $|F(x)| \leq c|x|^3$ ,  $x \in \mathbb{R}$ , for a certain  $c > 0$ , we have by an application of Lemma 4.5-(i) with  $\nu = 2/3$ ,  $p = 6$ :

$$\begin{aligned} |b(v)|_H &= \left| \frac{F(v)}{r^2} \right|_H \leq c \left| \frac{v}{r^{2/3}} \right|_{L_{rdr}^6}^3 \\ &\leq c' |v|_\beta^3, \end{aligned}$$

as soon as  $\beta > 4/3$ .

The third property is shown similarly, using the fact that

$$|F(x) - F(y)| \leq c|x - y|(x^2 + y^2),$$

for all  $x, y \in \mathbb{R}$ , and Hölder's inequality:

$$\begin{aligned} |b(u) - b(v)|_H &\leq c \left| \frac{u - v}{r^{2/3}} \right| \left( \left( \frac{u}{r^{2/3}} \right)^2 + \left( \frac{v}{r^{2/3}} \right)^2 \right) \Big|_H \\ &\leq c \left| \frac{u - v}{r^{2/3}} \right|_{L_{rdr}^6} \left( \left| \frac{u}{r^{2/3}} \right|_{L_{rdr}^6}^2 + \left| \frac{v}{r^{2/3}} \right|_{L_{rdr}^6}^2 \right), \end{aligned}$$

where we have used Hölder inequality. An application of Lemma 4.5-(i) with the same parameters as above leads to (4.46).

Now, fix  $4/3 < \beta < 2$ , and consider any  $z \in \mathcal{C}_T V_\beta$ , and  $h_0 \in V_\beta$ . If  $v \in \mathcal{C}_T V_\beta$ , taking the  $V_\beta$ -norm in (4.43) and using (4.44) and (4.45) gives:

$$\|\Gamma_{z, h_0}(v)\|_{T, \beta} \leq |h_0|_\beta + cT^{1-\beta/2} (\|v\|_{T, \beta}^3 + \|z\|_{T, \beta}^3). \quad (4.47)$$

Then, still using the expression (4.43), for  $u, v \in \mathcal{C}_T V_\beta$ , take the norm of  $\Gamma_{z, h_0}(u) - \Gamma_{z, h_0}(v)$ . By (4.44) and (4.46), one has:

$$\|\Gamma_{z, h_0}(u) - \Gamma_{z, h_0}(v)\|_{T, \beta} \leq cT^{1-\beta/2} (\|v\|_{T, \beta}^2 + 2\|z\|_{T, \beta}^2) \|u - v\|_{T, \beta}. \quad (4.48)$$

Take  $R > |h_0|_\beta$ . With these inequalities, it is clear that there exists  $T_* = T_*(|h_0|_\beta, \|z\|_{T, \beta}, R) \leq T$ , such that the application  $\Gamma_{z, h_0}$  maps the ball of radius  $R$  into itself, and is a contraction. Since the space  $\mathcal{C}_{T_*} V_\beta$  is complete, by Picard Theorem there exists a unique fixed point  $v$  which gives a mild solution to (4.19), up to  $t = T_*$ . The maximal solution is obtained by reiteration of the argument.

### (4.c) Proof when $\beta \in (2, 4]$

Consider  $2 < \beta \leq 4$ . Take  $h_0 \in V_\beta$  and assume that  $\phi \in \mathbb{L}^{0, \beta}$ . By the same argument as above, we can fix  $Z = z$ , where  $z$  is any map of  $\mathcal{C}_{T_*} V_\beta$ , and argue pathwise. Denote by  $(h, \tau_z^{\beta-2})$ , the maximal solution obtained by the method above, which therefore belongs to  $\mathcal{C}([0, \tau_z^{\beta-2}); V_\beta)$ . We aim to find an *a priori* bound for  $h \in \mathcal{C}_T V_{\beta-2}$  in the space  $\mathcal{C}_T V_\beta$  that guarantees its existence during a positive time. Write for  $0 \leq t < \tau_z^{\beta-2}$ :

$$h(t) = S(t)h_0 + \int_0^t (-A)^{(\beta-2)/2} S(t-s) [(-A)^{-1-(\beta-2)/2} (-Ab(h))(s)] ds + z(t). \quad (4.49)$$

and still using (4.44), the following inequality holds

$$|h(t)|_\beta \leq |S(t)h_0|_\beta + \int_0^t (t-s)^{-(\beta-2)/2} |Ab(h)(s)|_H ds + |z(t)|,$$

provided all quantities in the right hand side are finite. Therefore, there remains to evaluate the term  $|Ab(v)|_H$ . Using the expression  $A = \partial_{rr} + \frac{\partial_r}{r} - \frac{1}{r^2}$ , then easy computations lead to the formula

$$\begin{aligned} Ab(h) &= \frac{1 - \cos 2h}{r^2} \partial_{rr} h \\ &\quad + \frac{1 - \cos 2h}{r^3} \partial_r h - (3/2) \frac{2h - \sin 2h}{r^4} \\ &\quad + \frac{2 \sin 2h}{r^2} (\partial_r h)^2 + 3 \frac{2h - \sin 2h}{r^4} - 4 \frac{1 - \cos 2h}{r^3} \partial_r h , \end{aligned}$$

where, due to compensations, each line of the right hand side must be treated separately. Then, using the triangle inequality, we write for  $h \in V_\beta$ ,  $|Ab(h)|_H \leq I + II + III$ , and deal with each term. To avoid a profusion of constants  $c$ , from now until the end of the proof, we shall use the notation  $T_1(h) \lesssim T_2(h)$  if two terms involving  $h \in V_\beta$  are comparable up to a multiplicative constant that does not depend on  $h$ .

In the sequel, we fix an arbitrary  $\epsilon > 0$ . Using the bound  $|G(x)| \leq c|x|^2$ ,  $x \in \mathbb{R}$ , where  $G : x \in \mathbb{R} \mapsto 1 - \cos(2x)$ , Remark 4.1, and Lemma 4.5-(i) in the case  $\nu = 1$ ,  $p = \infty$ , the first term is majorized as follows:

$$\begin{aligned} I &\lesssim \left| \frac{h}{r} \right|_{L_{rd}^\infty}^2 |\partial_{rr} h|_H \\ &\lesssim |h|_{2+\epsilon}^2 |h|_2 , \end{aligned}$$

whereas for the second term we have:

$$\begin{aligned} II &\lesssim \left| \frac{h^2}{r^2} \left( \frac{\partial_r h}{r} - \frac{h}{r^2} \right) \right|_H + \left| \frac{1 - \cos 2h - 2h^2}{r^3} \partial_r h - (3/2) \frac{2h - \sin 2h - (4/3)h^3}{r^4} \right|_H \\ &= II_1 + II_2 . \end{aligned}$$

Then, using Lemma 4.5-(i) with  $\nu = 1$ ,  $p = \infty$ , and Remark 4.1, one has

$$\begin{aligned} II_1 &\lesssim \left| \frac{h}{r} \right|_{L_{rd}^\infty}^2 \left| \frac{\partial_r h}{r} - \frac{h}{r^2} \right|_H \\ &\lesssim |h|_{2+\epsilon}^2 |h|_2 . \end{aligned}$$

Moreover, using that for  $x \in \mathbb{R}$ ,

$$|1 - \cos 2x - 2x^2| \leq cx^4 , \quad \text{and} \quad |2x - \sin 2x - (4/3)x^3| \leq c|x|^5 ,$$

for an absolute constant  $c > 0$ , using Hölder's inequality, and applying Lemma 4.5-(i) with  $(\nu, p) = (3/4, 40/3)$ , and -(ii) with  $p = 5$ , (resp. -(i) with  $(\nu, p) = (4/5, 10)$ ), the following bound is obtained:

$$\begin{aligned} II_2 &\lesssim \left| \frac{h^4}{r^3} \partial_r h \right|_H + \left| \frac{h^5}{r^4} \right|_H \\ &\lesssim \left| \frac{h}{r^{3/4}} \right|_{L_{rd}^{40/3}}^4 |\partial_r h|_{L_{rd}^5} + \left| \frac{h}{r^{4/5}} \right|_{L^{10}}^5 \\ &\lesssim |h|_{8/5+\epsilon}^5 . \end{aligned}$$

The bound on  $III$  is obtained in a similar way. we write that  $III \leq III_1 + III_2$ , with

$$\begin{aligned} III_2 &= \left| \frac{2 \sin 2h - 4h}{r^2} (\partial_r h)^2 + 3 \frac{2h - \sin 2h - (4/3)h^3}{r^4} - 4 \frac{1 - \cos 2h - 2h^2}{r^3} \partial_r h \right|_H \\ &\lesssim \left| \frac{h}{r^{2/3}} \right|_{L_{rd}^{30}}^3 \left| \partial_r h \right|_{L_{rd}^5}^2 + \left| \frac{h}{r^{4/5}} \right|_{L_{rd}^{10}}^5 + \left| \frac{h}{r^{3/4}} \right|_{L_{rd}^{40/3}}^4 \left| \partial_r h \right|_{L_{rd}^5} \\ &\lesssim |h|_{8/5+\epsilon}^5 . \end{aligned}$$

The main term  $III_1$  has to be handled cautiously, since it involves typical compensations due to the expression of the domain  $D(A)$  – see Remark 4.1. One has

$$\begin{aligned} III_1 &= \left| \frac{h}{r^2} (\partial_r h)^2 + \frac{h^3}{r^4} - 2 \frac{h^2}{r^3} \partial_r h \right|_H \\ &= \left| \frac{h}{r} \partial_r h \left( \frac{\partial_r h}{r} - \frac{h}{r^2} \right) + \frac{h^2}{r^2} \left( \frac{h}{r^2} - \frac{\partial_r h}{r} \right) \right|_H \\ &\lesssim \left| \frac{h}{r} \right|_{L_{rd}^\infty} \left| \frac{\partial_r h}{r} - \frac{h}{r^2} \right|_H \left( \left| \partial_r h \right|_{L_{rd}^\infty} + \left| \frac{h}{r} \right|_{L_{rd}^\infty} \right) \\ &\lesssim |h|_2 |h|_{2+\epsilon}^2 , \end{aligned}$$

by Remark 4.1, and Lemma 4.5-(i) with  $(\nu, p) = (1, \infty)$ .

Now, going back to (4.49), we have for all  $t \in [0, \tau_z^{\beta-2})$ :

$$|h(t)|_\beta \leq c|h_0|_\beta + c_\epsilon \int_0^t (t-s)^{-\beta/2} Q(h(s)) ds + \|z\|_{\tau_z^{\beta-2}, \beta} \quad (4.50)$$

where, gathering all the terms above, we set:  $Q(h) = |h|_{8/5+\epsilon}^5 + |h|_{2+\epsilon}^2 |h|_2$ , for an arbitrary  $\epsilon > 0$ , and a constant  $c_\epsilon > 0$ .

Such a differential inequality proves the existence in  $V_\beta$ , at least for a positive time.

**Propagation of regularity.** Writing  $h_k = \langle h, e_k \rangle$ , if  $h \in V_\beta$ , an application of Hölder's inequality with  $(p, p') = (3/2, 3)$  gives us, for an arbitrary  $\epsilon > 0$ :

$$|h|_{2+\epsilon}^2 = \sum_{k \geq 1} (x_k)^{4-2/3\beta} (h_k)^{4/3} (x_k)^{2/3\beta} (h_k)^{2/3} \leq |h|_{3-\beta/2+3\epsilon/2}^{4/3} |h|_\beta^{2/3} .$$

Similarly, applying Hölder's inequality with  $(p, p') = (5/4, 5)$ :

$$|h|_{8/5+\epsilon}^2 = \sum_{k \geq 1} (x_k)^{(16-2\beta)/5+2\epsilon} (h_k)^{8/5} (x_k)^{2\beta/5} (h_k)^{2/5} \leq |h|_{2-\beta/4+4\epsilon/5}^{8/5} |h|_\beta^{2/5} ,$$

Thus, setting  $\beta_* = (3 - \beta/2) \wedge (2 - \beta/4)$ , we see that  $Q \leq |h|_{\beta_*}^4 |h|_\beta + |h|_{\beta_*}^2 |h|_\beta$ . All terms in the right hand side of (4.50) are linear in  $|h|_\beta$ . Applying Gronwall's Lemma, we see that  $|h|_\beta$  must be finite up to  $\tau_z^{\beta_*}$ , which ends the proof of Proposition 4.1. ■

## A Appendix: the comparison principle

Using a similar argument than that of Remark 4.1, one can show that

$$V_1 = \left\{ f \in H , \frac{f}{r} \in H , \partial_r f \in H \right\} .$$

Moreover, if  $J' \subseteq J$  denotes any compact interval, if  $f \in V_2$ , and  $g \in V_1$  with zero trace on  $\partial J'$ , one has the integration by parts formula:

$$\langle -Af, g \rangle_{J'} = \langle \partial_r f, \partial_r g \rangle_{J'} + \left\langle \frac{f}{r}, \frac{g}{r} \right\rangle_{J'} ,$$

where we have used the notation

$$\langle f, g \rangle_{J'} = \langle \mathbf{1}_{J'} f, \mathbf{1}_{J'} g \rangle , \quad f, g \in H . \quad (4.A.1)$$

The comparison principle states as follows.

**Proposition 4.2** (Comparison principle). *Fix a bound  $r_1 \in J$ , and denote by  $J'$  the subinterval  $[0, r_1]$ . Let  $\beta > 1$ . Let  $\kappa > 0$  and  $z \in \mathcal{C}_\kappa V_\beta$  with  $z(0) = 0$ . Using the notation (4.A.1), assume that we are given two maps  $f, g \in \mathcal{C}^1([0, \kappa]; V_\beta)$ , such that*

(i) *for all nonnegative  $\zeta \in \mathcal{C}_\kappa V_1$  such that  $\zeta(t, r) \equiv 0$  for  $(t, r) \in \{0\} \times J' \cup [0, \kappa] \times \partial J'$ , we have:*

$$\int_0^\kappa \langle \partial_t f, \zeta \rangle_{J'} dt \leq \int_0^\kappa \left( -\langle \partial_r f, \partial_r \zeta \rangle_{J'} - \left\langle \frac{f}{r}, \frac{\zeta}{r} \right\rangle_{J'} + \langle b(f+z), \zeta \rangle_{J'} \right) dt , \quad (4.A.2)$$

and

$$\int_0^\kappa \langle \partial_t g, \zeta \rangle_{J'} dt \geq \int_0^\kappa \left( -\langle \partial_r g, \partial_r \zeta \rangle_{J'} - \left\langle \frac{g}{r}, \frac{\zeta}{r} \right\rangle_{J'} + \langle b(g+z), \zeta \rangle_{J'} \right) dt ; \quad (4.A.3)$$

(ii) *On  $\{t = 0\} \times J' \cup [0, \kappa] \times \partial J'$ , there holds  $f \leq g$ .*

Then,

$$\forall (t, r) \in [0, \kappa] \times J' \text{ a.e. } , \quad f(t, r) \leq g(t, r) .$$

*Proof.* Take  $0 < T < \kappa$ . Let  $t \mapsto \zeta(t, \cdot) \in \mathcal{C}([0, T] \times [0, r_1])$  be a non negative map such that  $\zeta(t, r)$  vanishes for  $(t, r) \in \Sigma_T$ . Using (4.A.2) and (4.A.3):

$$\begin{aligned} \int_0^t \langle \zeta, \partial_t(f-g) \rangle_{J'} ds &\leq - \int_0^t \langle \partial_r \zeta, \partial_r(f-g) \rangle_{J'} ds \\ &\quad + \int_0^t \left\langle \zeta, \frac{\sin 2(g+z) - \sin 2(f+z)}{2r^2} \right\rangle_{J'} ds . \end{aligned} \quad (4.A.4)$$

Denote by  $[x]_+$ ,  $x \in \mathbb{R}$  the positive part of  $x$  and observe that we have for  $t \in [0, T]$ :  $\partial_t |[f-g]_+|_H^2 = 2 \langle [f-g]_+, \partial_t(f-g) \rangle_{J'}$ . Therefore, using (4.A.4) with the test map  $\zeta(t, \cdot) := [f-g]_+(t, \cdot)$  (the reader may check that it satisfies all the conditions required above), gives us

for  $t \in [0, T]$ :

$$\begin{aligned}
\frac{1}{2} |[f - g]_+(t)|_H^2 &= \int_0^t \partial_s |[f - g]_+|^2 ds \\
&\leq - \int_0^t \langle \partial_r ([f - g]_+), \partial_r (f - g) \rangle_{J'} ds \\
&\quad + \int_0^t \left\langle [f - g]_+, \frac{\sin 2(g + z) - \sin 2(f + z)}{2r^2} \right\rangle_{J'} ds \\
&= - \int_0^t \langle \mathbb{1}_{\mathbb{R}_+}(f - g), (\partial_r (f - g))^2 \rangle_{J'} ds \\
&\quad + \int_0^t \left\langle [f - g]_+, \frac{\sin 2(g + z) - \sin 2(f + z)}{2r^2} \right\rangle_{J'} ds \\
&\leq \int_0^t \left\langle [f - g]_+, \frac{\sin 2(g + z) - \sin 2(f + z)}{2r^2} \right\rangle_{J'} ds ,
\end{aligned} \tag{4.A.5}$$

where we have used the fact that the weak derivative of  $x \mapsto [x]_+$ ,  $x \in \mathbb{R}$  is the map  $x \mapsto \mathbb{1}_{\mathbb{R}_+}(x)$ ,  $x \in \mathbb{R}$ , and the nonpositiveness of the first term in the right hand side of (4.A.5).

Now, since  $\beta > 1$ , and  $T < \kappa$ , observe by Remark 4.2 that the maps  $f, g, z$  are uniformly continuous in  $t, r \in [0, T] \times J$ . Recall that  $V_\beta = D((-A)^{\beta/2})$  where  $D(-A)$  contains homogeneous Dirichlet data (see (4.11)). Hence, by the fact that  $\beta \geq 1/2$ , they all vanish on  $[0, T] \times \{r = 0\}$ . Using that for  $\epsilon > 0$  small enough,

$$\sin'([-\epsilon, \epsilon]) \subseteq \mathbb{R}_+ \setminus \{0\} ,$$

then there exist a constant  $\bar{r} = \bar{r}(T) > 0$  such that for every  $(t, r) \in [0, T] \times [0, \bar{r}]$ ,

$$[f - g]_+(\sin 2(g + z) - \sin 2(f + z)) \leq 0 .$$

Finally we write for all  $t \in [0, T]$ :

$$\begin{aligned}
|[f - g]_+(t)|_H^2 &\leq \int_0^t \int_0^{\bar{r}} \frac{\sin 2(g + z) - \sin 2(f + z)}{2r^2} [f - g]_+ r dr ds \\
&\quad + \frac{1}{2\bar{r}^2} \int_0^t \int_{\bar{r}}^1 (\sin 2(g + z) - \sin 2(f + z)) r dr ds \\
&\leq \frac{c}{\bar{r}^2} \int_0^t |[f - g]_+|_H^2 ds ,
\end{aligned}$$

and thus:

$$|[f - g]_+|_H^2(t) \leq \frac{c}{\bar{r}^2} \int_0^t |[f - g]_+|_H^2(s) ds , \text{ for } t \in [0, T]. \tag{4.A.6}$$

The conclusion follows, applying Gronwall's Lemma: on every subinterval  $[0, T] \subseteq [0, \kappa)$ , one has  $f \leq g$ , i.e.  $f \leq g$  on  $[0, \kappa)$ .  $\blacksquare$

*Remark 4.6.* If we are given  $\beta > 1$ ,  $\kappa > 0$ , and two maps  $f, g \in \mathcal{C}^1([0, \kappa); V_\beta)$  with  $f(0, \cdot) \leq g(0, \cdot)$  on  $J$ , satisfying moreover the linear inequations for all  $\zeta$  as above:

$$\int_0^\kappa \langle \partial_t f, \zeta \rangle_{J'} dt \leq - \int_0^\kappa \left( \langle \partial_r f, \partial_r \zeta \rangle_{J'} + \left\langle \frac{f}{r}, \frac{\zeta}{r} \right\rangle_{J'} \right) dt ,$$



and

$$\int_0^\kappa \langle \partial_t g, \zeta \rangle_{J'} dt \geq - \int_0^\kappa \left( \langle \partial_r g, \partial_r \zeta \rangle_{J'} + \left\langle \frac{g}{r}, \frac{\zeta}{r} \right\rangle_{J'} \right) dt .$$

Then, taking back the previous proof and replacing  $b$  by zero gives us the same conclusion. Indeed, in this case, we have instead of (4.A.5):

$$\frac{1}{2} \| [f - g]_+(t) \|_H^2 \leq \int_0^t \left\langle [f - g]_+, \frac{g - f}{r^2} \right\rangle_{J'} ds ,$$

and the end of the proof is similar.

## B Appendix: some technical proofs

### (4.B.1) Proof of Lemmata 4.1

The proof of Lemma 4.1 is directly adapted from other backgrounds – see for instance, in the context of the nonlinear Schrödinger equation, the references [DD02, DD03b]. We still prove the result for the sake of completeness. In the sequel, for  $R, T > 0$ , we denote by  $\mathbb{B}_T^R$  (resp.  $B^R$ ) the ball of radius  $R$  of center 0 in  $\mathcal{C}_T V_\beta$  (resp.  $V_\beta$ ).

*Proof of Lemma 4.1.* We prove the result when  $2 \geq \beta > 4/3$ . The reader may check that the case  $4 \geq \beta > 2$  works the same way. If  $(z, h_0) \in \mathcal{C}_T V_\beta \times V_\beta$ , on the time interval  $[0, \tau_z(h_0))$ , we define  $v(z, h_0, \cdot) = h_z(h_0, \cdot) - z$ . For our purpose, it is sufficient to show the continuity of  $v$ . Recall that for  $(T, z, h_0)$  we denote by  $\Gamma_{T,z,h_0}$  the map given for  $u \in \mathcal{C}_T V_\beta$  by

$$\Gamma_{T,z,h_0}(u)(t) = S(t)h_0 + \int_0^t b(u(s) + z(s)) ds , \quad t \in [0, T] . \quad (4.B.1)$$

**Continuity of the unique fixed point.** Take any  $\tilde{z} \in \mathcal{C}([0, \infty); V_\beta)$ ,  $\tilde{h}_0 \in V_\beta$ , and take  $R > |\tilde{h}_0|_\beta$ . We saw in section 4 that there exist constants  $c, c' > 0$  such that if

$$|\tilde{h}_0|_\beta + cT^{1-\beta/2}(R^3 + 8\|\tilde{z}\|_{T,\beta}^3) < R , \quad \text{and} \quad c'T^{1-\beta/2}(R^2 + 4\|\tilde{z}\|_{T,\beta}^2) < 1 , \quad (4.B.2)$$

then the map  $\Gamma_{T,z,h_0}$  is a contraction of the ball  $\mathbb{B}_T^R$ . Take  $T_* = T_*(R, |h_0|_\beta, \|z\|_{T,\beta})$  satisfying (4.B.2). It is immediate that inequalities (4.B.2) hold with any  $(h_0, z)$ , provided it lies in some neighbourhood  $\mathcal{V} \times \mathcal{W}$  of  $(\tilde{h}_0, \tilde{z})$ . Moreover, for such  $(h_0, z)$ , using (4.B.1) and (4.45) (see section 4), we have

$$\begin{aligned} \|v(z, h_0) - v(\tilde{z}, \tilde{h}_0)\|_{T_*,\beta} &\leq |h_0 - \tilde{h}_0|_\beta \\ &\quad + c'T_*^{1-\beta/2}(R^2 + \|z + \tilde{z}\|_{T_*,\beta}^2) \left( \|v(z, h_0) \right. \\ &\quad \left. - v(\tilde{z}, \tilde{h}_0)\|_{T_*,\beta} + \|z - \tilde{z}\|_{T_*,\beta} \right) , \end{aligned}$$

and since  $c'T_*^{1-\beta/2}(R^2 + \|z + \tilde{z}\|_{T_*,\beta}^2) \leq c'T_*^{1-\beta/2}(R^2 + 2\max(\|z\|_{T_*,\beta}, \|\tilde{z}\|_{T_*,\beta})^2) < 1$ , the inequality above implies the convergence of  $v(z, h_0)$  to  $v(\tilde{z}, \tilde{h}_0)$  as  $\|z - \tilde{z}\|_{T_*,\beta} + |h_0 - \tilde{h}_0|_{T_*,\beta} \rightarrow 0$ .

**Continuity of the flow.** Keeping the same notations, let  $T_* = T_*(R, |\tilde{h}_0|_\beta, \|\tilde{z}\|_{T,\beta})$  such that (4.B.2) holds. Define  $N = \lfloor T/T_* \rfloor$ . We shall prove the result by induction. For each  $k \in \{1, \dots, N\}$  denote by  $(H_k)$  the sentence: “there exists  $\delta_k > 0$ , such that if  $(z, h_0) \in \mathbb{B}_T^{\delta_k} \times B^{\delta_k}$ , then:  $\tau_z(h_0) > kT_*$ , and the map  $(z, h_0) \in \mathbb{B}_T^{\delta_k} \times B^{\delta_k} \rightarrow v(z, h_0, kT_*)$  is continuous.”

*Proof of  $(H_1)$ .* The case  $k = 1$  follows by the previous paragraph.

*Proof that  $(H_k) \Rightarrow (H_{k+1})$ .* Let  $k \geq 1$ , assume  $(H_k)$ , and chose  $\delta_{k+1} = \delta \wedge \min_{0 \leq i \leq k} \delta_i$ , where  $\delta > 0$  is such that  $|v(z, h_0, kT_*)|_\beta < R$  for  $(z, h_0) \in \mathbb{B}^{\delta_k} \times B^{\delta_k}$ . By the case  $k = 1$ , this proves that for such  $z, h_0$ , then  $v(z, h_0, \cdot)$  is defined up to  $(k+1)T_*$ , and by uniqueness of the solution, using the notation  $y(t) = z(t + kT_*) - S(t)z(kT_*)$ ,  $t \in [0, T_*]$ , we have:

$$v(z, h_0, (k+1)T_*) = v(y, v(z, h_0, kT_*), T_*) .$$

Still by the case  $k = 1$ , this expression defines a continuous map relatively to  $(z, h_0) \in \mathbb{B}_T^{\delta_k} \times B^{\delta_k}$ . This proves  $(H_{k+1})$ .

In particular, the case  $(H_N)$  is true, which implies the proposition. ■

## (4.B.2) Proof of Lemma 4.2

The proof is very similar from that of [DD02], although simpler since in our context the semigroup  $S$  has better regularizing effects – recall that  $S$  is analytical, see Rem.4.5. For  $t \in [0, T_1]$ , we set

$$\tilde{h}(t) := \frac{T_1 - t}{T_1} h_0 + \frac{t}{T_1} h_1 .$$

We define the maps

$$\begin{aligned} \mathbf{u}(t) &:= (\tilde{h}(t) - h_0) - \int_0^t \left( A\tilde{h}(s) + b(\tilde{h})(s) \right) ds , \\ \tilde{z}(t) &:= \int_0^t S(t-s) \frac{d\mathbf{u}}{ds} ds . \end{aligned}$$

We see in particular that the regularity of  $t \mapsto \mathbf{u}(t)$  is at least  $\mathcal{C}^1([0, T_1]; V_{\beta-2})$ . Due to their definition, the maps  $\tilde{h}, \mathbf{u}$  satisfy the forced equation

$$\frac{d\tilde{h}}{dt} = A\tilde{h} + b(\tilde{h}) + \frac{d\mathbf{u}}{dt} , \quad \text{for } t \in [0, T_1] .$$

Moreover  $\tilde{z}$  satisfies the heat type equation

$$\begin{cases} \frac{d\tilde{z}}{dt} = A\tilde{z} + \frac{d\mathbf{u}}{dt} , \\ \tilde{z}(0) = 0 . \end{cases}$$

By classical theory of parabolic equations, we have at least  $\tilde{z} \in \mathcal{C}_{T_1} V_\beta$ . Moreover, the map  $\tilde{v} := \tilde{h} - \tilde{z}$  satisfies the translated equation (4.19) with  $z = \tilde{z}$ . The conclusion of the lemma follows, since by uniqueness:  $\tilde{h} = h_{\tilde{z}}(h_0, \cdot)|_{[0, T_1]}$ , where here  $h$  denotes the functional defined in (4.18). ■



## A new Semi Discrete Scheme for Stochastic LLG

---

This chapter is based on a common work with Alouges, F. and De Bouard, A. published in the review *Stochastic Partial Differential Equations: Analysis and Computations*, see [ADH14].

For a given ferromagnetic domain, noise due to thermal fluctuations can induce phase transition between different equilibrium states. This has been the main topic of several papers during the last decade see e.g. [KRVE05, KORVE07], where the authors call for a treatment of nonuniform magnetization. The present work arises then from the growing necessity to develop numerical methods for the infinite dimensional stochastic LLG.

We propose a new convergent time semi-discrete scheme for (SLLG) when the domain  $\mathcal{O}$  has dimension 2 or 3. The main idea is to extend the projection algorithm of [AJ06] in the stochastic case. As in the deterministic case, our scheme has the advantage that despite that (SLLG) contains a nonlinear term, it is only linearly implicit at each time step. Indeed, the algorithm used here iterates a variational formulation defined on the “tangent space” of  $u^n \sim u(n\Delta t)$ , namely the linear subspace of  $H^1$ , composed with vector fields pointwisely orthogonal to  $u^n$ . This method permits to make the nonlinearity vanish, when tested against these specific maps. Besides, it involves a projection step, guaranteeing that the local constraint on the magnitude is automatically satisfied for a solution of the algorithm.

To prove the convergence, we use here a martingale approach. The starting point is to use the variational formulation (5.8) to derive some energy estimates for the approximate process  $u^{(\Delta t)}$ . We proceed by applying compactness results as that of [FG95], and passing to the limit on a discrete equation. Taking advantage of the projection step, we show that it permits to recover the additional drift that is along the solution. Finally, we obtain the convergence up to a subsequence, of  $u^{(\Delta t)}$  towards a weak martingale solution for SLLG.

---

# 1 Introduction

Developing numerical schemes for the simulation of LLG plays a prominent role in the modeling of ferromagnetic materials. We refer the reader to [Cim05, Cim07, CS04] for an overview of the literature on the subject. However, reliable schemes for the simulation of (SLLG) remain very few. Probably the first scheme for which convergence can be proved is given in [BBNP13a] and is based on a Crank-Nicolson-type time-marching evolution which relies on a nonlinear iteration solved by a fixed point method. On the other hand, there has been in the past recent years an intensive development of a new class of numerical methods for LLG, based on a linear iteration, and for which unconditional convergence and stability can be shown [Alo08, AJ06, BKP08, KVBP<sup>+</sup>14]. Our purpose in this chapter is to extend the ideas developed there. We aim to generalize the scheme in order to take into account the stochastic term in the following Landau-Lifshitz-Gilbert equation

$$\begin{cases} du = (\Delta u + u|\nabla u|^2 + u \times \Delta u + F_\phi(u))dt + u \times dW_\phi, & \text{in } \Omega \times \mathbb{R}^+ \times \mathcal{O}, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Omega \times \mathbb{R}^+ \times \partial\mathcal{O}, \\ u|_{t=0} = u_0, & \text{in } \Omega \times \mathcal{O}, \end{cases} \quad (5.1)$$

where the  $n$ -dimensional domain  $\mathcal{O}$ , for  $n = 2, 3$ , is bounded, and we still use the notations defined in (2.10), (1.15).

An important characteristic of the algorithm is that it contains a projection step, guaranteeing that the local constraint on the magnitude is satisfied exactly. Note that this approach has already been used in [GLT13] where a fully discrete scheme for (5.1) but with a one-dimensional noise is studied. The method used in that paper is based on the so-called Doss-Sussmann technique [Dos77, Sus77], which, thanks to a geometric transformation, allows to replace the stochastic PDE by an equivalent PDE with random coefficients. However, it is well known that this method only works with a one dimensional noise, and cannot be generalized to our setting. Instead, in the present paper, we apply the projection scheme directly on the original stochastic equation. This allows us to consider a more general noise, but requires a specific treatment of the corresponding term. We think that the methodology that we develop can be generalized to a stochastic differential or partial differential equations with a geometrical constraint. It is definitely different from – though related to – the approach of [LLVE08] (see Remark 5.1 below).

We only consider a time semi-discrete approximation of (5.1) for which we show the unconditional convergence when the time step tends to 0. Proving the convergence of the fully discrete approximation, using e.g. a finite element method in space, does not cause any major difficulty – see Theorem 6.2 of Chapter 6 below.

*Additional Notation.* Throughout this chapter, we assume that  $T > 0$  is a given constant and  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]}, (W_t)_{t \in [0, T]})$  is a stochastic basis, see (0.1).

For a given  $N \in \mathbb{N}^*$ , the notation  $\Delta t$  will always refer to  $\frac{T}{N}$ , and for  $0 \leq n \leq N - 1$ ,  $\Delta W^n$  is defined as the  $n^{\text{th}}$  increment of the Wiener process, namely  $W((n + 1)\Delta t) - W(n\Delta t)$ . Therefore,  $\Delta W^n$  is a gaussian random variable on  $L_x^2$  with covariance operator  $(\Delta t)\phi\phi^*$ . For lighter computations, we will denote in the sequel

$$\phi_i(x) := [\phi e_i](x), \quad (5.2)$$

and we also abbreviate the inner product  $\langle \cdot, \cdot \rangle_{L_x^2}$  as  $\langle \cdot, \cdot \rangle$ .

The pointwise orthogonal projection on  $u(t, x, \omega)$ , resp. on  $\langle u(t, x, \omega) \rangle^\perp$ , in the euclidean space  $\mathbb{R}^3$  will be noted

$$P_{u(t,x,\omega)} \quad , \quad \text{resp. } P_{u(t,x,\omega)^\perp} \quad , \quad (5.3)$$

or simply  $P_u$ , resp.  $P_{u^\perp}$ .

## 2 Main result

Unlike the approach used in [BBNP13a], we use the so-called *Gilbert Form* (GF) of (5.1), that is, the equation formally obtained by applying  $(\text{id} - u \times \cdot)$  to (5.1), namely:

$$du - u \times du = [2(\Delta u + u|\nabla u|^2) + (\text{id} - u \times)F_\phi(u)]dt + (\text{id} - u \times)(u \times dW) \quad , \quad (5.4)$$

where the notation “ $u \times du$ ” means Itô integral in the sense of semimartingales. Equivalence between (5.1) and (5.4) is not clearly stated in the literature and we therefore establish it in Remark 5.2.

This particular form allows us to overcome the difficulty of solving a nonlinear system at each step of the algorithm. Let us briefly describe why. Consider a uniform discretization in time  $0, \Delta t, \dots, n\Delta t, \dots, N\Delta t = T$ , and fix some “level of implicitness”  $\theta \in (0, 1)$ . Given a time step  $n \in \llbracket 0, N - 1 \rrbracket$  and  $u^n \sim u(n\Delta t)$ , an approximation  $v^n$  of  $\Delta t \partial u / \partial t(n\Delta t)$  can be found simply by solving a *linear* system of the form

$$\frac{(v^n - u^n \times v^n)}{\Delta t} = 2\Delta \bar{u}^{n+\theta} + (\text{id} - u^n \times)(F_\phi(u^n) + u^n \times \Delta W^n) \quad ,$$

where we denote  $\bar{u}^{n+\theta} = (1 - \theta)u^n + \theta v^n$ . Indeed, following the same idea as that of the deterministic case (see the references of the previous section), one may seek the unknown  $v^n$  in the subspace of  $H_x^1$  whose elements are pointwise orthogonal to  $u^n(x)$ , so that the nonlinear term  $u^n |\nabla u^n|^2$  vanishes when tested against functions that also satisfy this constraint. Roughly speaking, the test functions in the following formulation (5.8) “only see” the part of  $u^{n+1} - u^n$  which is orthogonal to  $u^n$ . We then pointwisely project  $u^n + v^n$  on  $\mathbb{S}^2$  by setting  $u^{n+1}(x) = (u^n(x) + v^n(x)) / |u^n(x) + v^n(x)|_{\mathbb{R}^3}$ , so that the local constraint on the magnitude is automatically satisfied. Apart from the effects related to the renormalization step, it seems intuitively clear that any limit point  $u$  of the càdlàg process defined by  $u_N(t) = u^n$  for  $t \in [n\Delta t, (n+1)\Delta t)$  should satisfy in some sense an (Itô) equation of the form:

$$\frac{\partial u}{\partial t} = u \times \Delta u + P_{u(t,x)^\perp} \left( \Delta u + F_\phi(u) \right) + u \times \dot{W}_\phi \quad .$$

Heuristically speaking, there are two main reasons why the terms  $u|\nabla u|^2$  and  $P_{u(t,x)^\perp} F_\phi(u)$  are actually recovered:

- because the limit satisfies  $|u|_{\mathbb{R}^3} \equiv 1$ , the term  $u|\nabla u|^2$  is implicitly contained in the algorithm: it appears using specific test functions of the form  $u \times (u \times \varphi)$ , see the proof of Proposition 5.6 ;
- since for each  $n \in \llbracket 1, N \rrbracket$  we have  $u^n \perp v^n$ , then  $|u^n + v^n|_{\mathbb{R}^3}^2 = 1 + |v^n|_{\mathbb{R}^3}^2$ . The renormalization step is then asymptotically equivalent to define the value  $u^{n+1} \leftarrow (u^n + v^n) / |u^n + v^n|_{\mathbb{R}^3}$ .

$v^n)(1 - (1/2)|v^n|_{\mathbb{R}^3}^2)$  which, according to the observation that the first order term in  $v^n$  is  $\Delta W^n$ , approximately equals  $u^n + v^n - (1/2)u^n|\Delta W^n|_{\mathbb{R}^2}^2$ . It turns out that the sum  $\sum_{n \leq N-1} -(1/2)u^n|\Delta W^n|^2$  of the additional terms that are due to the renormalization step approximates in some sense the integral  $\int_0^T P_{u^n(x)} F_\phi(u^n) dt$ , see Proposition 5.7.

We give now a rigorous description of the scheme. Fix a parameter

$$\theta \in \left(\frac{1}{2}, 1\right], \quad (5.5)$$

and assume that the operator  $\phi : L_x^2 \rightarrow H_x^2$  satisfies

$$|\phi|_{\mathbb{L}_2^{0,2}}^2 = \sum_{i \in \mathbb{N}} |\phi_i|_{H_x^2}^2 < \infty. \quad (5.6)$$

Our algorithm reads as follows, for a given  $N > 0$ :

ALGORITHM 5.1 : *PROJECTION ALGORITHM.* — *Fix*

$$u^0 := u_0 \in H_x^1 \cap \{u \in L_x^2, u(x) \in \mathbb{S}^2, \text{ a.e. } x \in \mathcal{O}\}, \quad (5.7)$$

and for any  $n \in \{0, \dots, N-1\}$ , suppose that the random variable  $u^n(\omega, \cdot) \in H_x^1$  is known. Let  $v^n(\omega, \cdot)$  be the unique solution in the space

$$\mathbb{W}_{N,n}(\omega) := \left\{ \psi \in H_x^1, \forall x \in \mathcal{O}, \psi(x) \perp u^n(\omega, x) \right\},$$

of the variational problem:  $\forall \varphi \in \mathbb{W}_{N,n}(\omega)$ ,

$$\begin{aligned} \langle v^n - u^n \times v^n, \varphi \rangle + 2\theta \Delta t \langle \nabla v^n, \nabla \varphi \rangle \\ = -2\Delta t \langle \nabla u^n, \nabla \varphi \rangle + \Delta t \langle (\text{id} - u^n \times) F_\phi(u^n), \varphi \rangle \\ + \langle (\text{id} - u^n \times)(u^n \times \Delta W^n), \varphi \rangle \end{aligned} \quad (5.8)$$

Then, we set, for all  $(\omega, x) \in \Omega \times \mathcal{O}$ ,

$$u^{n+1}(\omega, x) = \frac{u^n(\omega, x) + v^n(\omega, x)}{|u^n(\omega, x) + v^n(\omega, x)|_{\mathbb{R}^3}}. \quad (5.9)$$

Note that the formulation (5.8) is a  $\theta$ -scheme applied to the variational formulation of equation (5.4) (see [AJ06]). One has  $u^n \in H_x^1$  a.s., for any  $n \in \{0, \dots, N\}$  and  $(u^n)_{0 \leq n \leq N}$  is adapted to the filtration  $(\mathbb{F}_N^n)_{0 \leq n \leq N}$  defined by

$$\mathbb{F}_N^n := \sigma\{W(k\Delta t), 0 \leq k \leq n\}. \quad (5.10)$$

Indeed, it is not difficult to prove that under the above assumptions, problem (5.8) admits a unique solution  $v^n(\omega, \cdot) \in \mathbb{W}_{N,n}(\omega)$  (see [AJ06] for a proof in the deterministic case). The noise and correction terms do not alter the hypotheses of the Lax-Milgram theorem. Moreover, this solution depends continuously in  $H_x^1$  on the two arguments  $(u^n, \Delta W^n)$ , for the  $H_x^1 \times L_x^2$  topology. It implies in particular that the law of  $v^n$  on  $H_x^1$  only depends on the law of  $(u^n, \Delta W^n)$  on  $H_x^1 \times L_x^2$ .

*Remark 5.1.* As mentioned before, the approach here is different from the one in [LLVE08], where the approximation of solutions of some Stratonovich stochastic differential equation

$$dX = F(X_t)dt + \sigma(X_t) \circ dW \quad , \quad X_t \in M \quad , \quad t \geq 0 \quad ,$$

with values in a manifold  $M$ , is considered. Indeed, in [LLVE08], the scheme consists in using the explicit Euler scheme (which approximates the Itô equation  $dX = F(X_t)dt + \sigma(X_t)dW$ ) on one time step, and then projecting the solution on the manifold. Here, we do not approximate the Itô equation, since part of the Itô correction, namely  $P_{u^\perp}F_\phi(u)$ , is put in the increment. We will see that the projection on the manifold (the sphere here) brings the remaining part  $P_uF_\phi$ .

The notion of martingale solutions we use here is that of Definition 3.1. It is similar to the one used in [BBNP13a, BGJ13]. Our main result is given by the following theorem, and says that, up to a subsequence, the discrete solution  $u_N$  of the algorithm (PA) converges in law to a martingale solution of equation (5.1).

**Theorem 5.1** (Convergence of the algorithm). *For every  $N \in \mathbb{N}^*$ , we define the progressively measurable  $H_x^1$ -valued process  $u_N$  by:*

$$u_N(t) := u^n \quad \text{if } t \in [n\Delta t, (n+1)\Delta t) \quad , \quad n \in \{0, \dots, N-1\} \quad .$$

*In the sense given by Definition 3.1, there exists a martingale solution of (5.1)  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\bar{\mathcal{F}}_t)_{t \in [0, T]}, (\bar{W}_t)_{t \in [0, T]}, \bar{u})$ , and a sequence  $(\bar{u}_N)_{N \in \mathbb{N}^*}$  of random processes defined on  $\bar{\Omega}$ , with the same law as  $u_N$ , so that up to a subsequence, the following convergence holds:*

$$\bar{u}_N \xrightarrow[N \rightarrow \infty]{} \bar{u}, \quad \text{in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3).$$

**Outline of the Proof.** We proceed in several steps. In Section 3, we use (5.8) with appropriate test functions to establish uniform estimates for several processes related to  $(u^n)_{0 \leq n \leq N}$ . We first obtain a uniform bound on the Dirichlet energy of  $u_N$  which is a discrete counterpart of taking the expectation in the Energy Formula (2.A.2), for a continuous solution. Section 4 is devoted to the proof of the tightness of the sequence  $(u_N)$  on the space  $L^2([0, T] \times \mathcal{O}; \mathbb{R}^3)$ . After a change of probability space, we can assume that there exists an almost sure limit  $u$  of  $(u_N)$ , that is

$$u_N \xrightarrow[N \rightarrow \infty]{} u \quad \text{a.s. in } L^2([0, T] \times \mathcal{O}; \mathbb{R}^3).$$

Then, setting

$$u_N(t) = u_0 + F_N(t) + X_N(t), \tag{5.11}$$

where  $(X_N(n\Delta t))_{0 \leq n \leq N-1}$  defines an  $L_x^2$ -valued discrete parameter martingale, with respect to the filtration  $(\mathbb{F}_N^n)_{0 \leq n \leq N-1}$  (see (5.10)), and  $F_N(t)$  is, for each  $N$ , a deterministic function of  $u_N|_{[0, t]}$ , we use (5.8) and the previous energy estimates to identify  $F_N(t)$  and its limit up to a subsequence. In section 5, we show that, still up to a subsequence,  $X_N(t)$  converges to a limit  $X(t)$  which is a square-integrable continuous martingale with an explicit quadratic variation. Then, the martingale representation theorem implies the existence of a new filtered probability space for which the limit of the martingale part is a stochastic integral with respect to a Wiener process  $\bar{W}$  with covariance operator  $\phi\phi^*$ . Finally we use the limit of (5.11) and this latter stochastic integral in order to identify the equation satisfied by  $u(t)$ . The explicit form of the limit  $F(t)$  of  $F_N(t)$  as  $N \rightarrow \infty$  is the Bochner integral of the  $L_x^2$ -valued process  $t' \mapsto \Delta u(t') + u(t')|\nabla u(t')|^2 + u(t') \times \Delta u(t') + F_\phi(u)(t')$  on the time interval  $[0, t]$ , which allows us to conclude.



*Remark 5.2.* Note that if  $\bar{u}$  is a solution of (5.1), then we can rewrite the stochastic integral of the predictable process  $s \mapsto \bar{u}(s) \times$  with respect to the semimartingale  $\bar{u}$  as

$$\int_0^t \bar{u}(s) \times d\bar{u}(s) = \int_0^t \bar{u}(s) \times d \left( \int_0^s I(\sigma) d\sigma \right) + \int_0^t \bar{u}(s) \times d \left( \int_0^s \bar{u}(\sigma) \times d\bar{W}(\sigma) \right)$$

where  $I$  is given by  $\forall s \in [0, T]$ :

$$I(s) = \Delta \bar{u}(s) + \bar{u}(s) |\nabla \bar{u}(s)|^2 + \bar{u}(s) \times \Delta \bar{u}(s) + F_\phi(\bar{u})(s).$$

It then follows from classical properties of stochastic integrals with respect to semimartingales (see e.g. [Din00]) that

$$\int_0^t (\text{id} - \bar{u}(s) \times) d\bar{u}(s) = \int_0^t (\text{id} - \bar{u}(s) \times) I(s) ds + \int_0^t (\text{id} - \bar{u}(s) \times) d\bar{W}(\sigma),$$

and since for almost all  $\omega, t, x$ ,  $|\bar{u}(\omega, t, x)| = 1$ ,  $\bar{u}$  is also a solution to (5.4). Thus (5.1) and (5.4) are in fact equivalent.

### 3 Energy estimates

Fix  $N > 0$ , and set  $u^0 = u_0$ . Let  $(u^n)_{0 \leq n \leq N}$  and  $(v^n)_{0 \leq n \leq N}$  be given by the algorithm (PA). In all what follows, we write

$$A^n := u^n \times \Delta W^n. \quad (5.12)$$

This term corresponds to the noise term which is added at each step of the algorithm. Thanks to the Gaussianness of  $\Delta W^n$ , the fact that  $|u^n|_{L_x^\infty} \leq 1$ , and the Sobolev embeddings, we have the following obvious, but useful estimates: for all  $n \in \{0, \dots, N\}$ ,

$$\mathbb{E}[|A^n|_{L_x^2}^2] \leq \Delta t |\phi|_{\mathbb{L}_2}^2, \quad (5.13)$$

and

$$\mathbb{E}[|A^n|_{L^4}^4] \leq c(\Delta t)^2 |\phi|_{\mathbb{L}_{2,0,1}}^4. \quad (5.14)$$

**Proposition 5.1.** *There exists a constant  $c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}) > 0$ , independent of  $N \in \mathbb{N}^*$  and  $0 \leq n \leq N - 1$ , such that:*

$$\max_{0 \leq n \leq N} \mathbb{E} \left[ |\nabla u^n|_{L_x^2}^2 \right] \leq c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}), \quad (5.15)$$

$$\mathbb{E} \sum_{n=0}^{N-1} |v^n - A^n|_{L_x^2}^2 \leq c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}) \Delta t, \quad (5.16)$$

$$\mathbb{E} \sum_{n=0}^{N-1} |v^n|_{L_x^2}^2 \leq c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}), \quad (5.17)$$

$$\mathbb{E} \sum_{n=0}^{N-1} |\nabla v^n|_{L_x^2}^2 \leq c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}). \quad (5.18)$$

The proof of Proposition 5.1 uses the following remark, together with the estimate of Lemma 5.1 below, whose proof is postponed to the end of section 3.

*Remark 5.3.* The renormalization stage decreases the Dirichlet energy. Indeed, it was shown in [Alo97] that for any map  $\psi \in H_x^1$ , such that a.e. in  $\mathcal{O}$ ,  $|\psi(x)| \geq 1$ , one has

$$\int_{\mathcal{O}} \left| \nabla \left( \frac{\psi(x)}{|\psi(x)|} \right) \right|^2 dx \leq \int_{\mathcal{O}} |\nabla(\psi(x))|^2 dx. \quad (5.19)$$

**Lemma 5.1.** *For all  $\epsilon$  with  $0 < \epsilon < 2\theta - 1$ , there exists  $c = c(\epsilon, |\phi|_{\mathbb{L}_2^{0,2}}, T) > 0$  such that for all  $N \in \mathbb{N}^*$ , and  $n = 0, \dots, N - 1$ :*

$$\begin{aligned} \mathbb{E}[|\nabla u^{n+1}|_{L_x^2}^2] + \frac{(1-\epsilon)}{\Delta t} \mathbb{E}[|v^n - A^n|_{L_x^2}^2] + (2\theta - 1 - \epsilon) \mathbb{E}[|\nabla v^n|_{L_x^2}^2] \\ \leq (1 + c\Delta t) \mathbb{E}[|\nabla u^n|_{L_x^2}^2] + c\Delta t. \end{aligned} \quad (5.20)$$

We now prove Proposition 5.1 with the help of Lemma 5.1.

*Proof of Proposition 5.1.* In the sequel, we fix  $\epsilon \in (0, 2\theta - 1)$ . We first prove (5.15). We deduce from (5.20) that for all  $n = 0, \dots, N - 1$

$$\mathbb{E}[|\nabla u^{n+1}|_{L_x^2}^2] \leq (1 + c\Delta t) \mathbb{E}[|\nabla u^n|_{L_x^2}^2] + c\Delta t.$$

We then apply the discrete Gronwall lemma. There exists  $c = c(|\phi|_{\mathbb{L}_2^{0,2}}, T) > 0$  such that for all  $n = 0, \dots, N$

$$\mathbb{E}[|\nabla u^n|_{L_x^2}^2] \leq c(1 + \mathbb{E}[|\nabla u_0|_{L_x^2}^2]),$$

and (5.15) is proved.

We now turn to the proof of (5.16)-(5.18). We note that (5.20) implies in particular

$$\begin{aligned} \mathbb{E}[|\nabla u^{n+1}|_{L_x^2}^2] - \mathbb{E}[|\nabla u^n|_{L_x^2}^2] + \frac{(1-\epsilon)}{\Delta t} \mathbb{E}[|v^n - A^n|_{L_x^2}^2] \\ + (2\theta - 1 - \epsilon) \mathbb{E}[|\nabla v^n|_{L_x^2}^2] \leq c\Delta t \mathbb{E}[|\nabla u^n|_{L_x^2}^2] + c\Delta t. \end{aligned}$$

Summing these inequalities for  $n = 0 \dots N - 1$ , and using (5.15), we obtain

$$\begin{aligned} \mathbb{E}[|\nabla u^N|_{L_x^2}^2] - \mathbb{E}[|\nabla u^0|_{L_x^2}^2] + \frac{(1-\epsilon)}{\Delta t} \sum_{n=0}^{N-1} \mathbb{E}[|v^n - A^n|_{L_x^2}^2] + (2\theta - 1 - \epsilon) \sum_{n=0}^{N-1} \mathbb{E}[|\nabla v^n|_{L_x^2}^2] \\ \leq \sum_{n=0}^{N-1} c\Delta t \mathbb{E}[|\nabla u^n|_{L_x^2}^2] + \sum_{n=0}^{N-1} c\Delta t \\ \leq c(|\phi|_{\mathbb{L}_2^{0,2}}, T, u_0). \end{aligned}$$

This implies that:

$$\begin{aligned} \frac{(1-\epsilon)}{\Delta t} \sum_{n=0}^{N-1} \mathbb{E}[|v^n - A^n|_{L_x^2}^2] + (2\theta - 1 - \epsilon) \sum_{n=0}^{N-1} \mathbb{E}[|\nabla v^n|_{L_x^2}^2] \\ \leq c(|\phi|_{\mathbb{L}_2^{0,2}}, T, u_0) - \mathbb{E}[|\nabla u^N|_{L_x^2}^2] + \mathbb{E}[|\nabla u^0|_{L_x^2}^2] \\ \leq c'(|\phi|_{\mathbb{L}_2^{0,2}}, T, u_0). \end{aligned}$$

Thus, (5.16) and (5.18) follow. Finally, we may deduce (5.17) from (5.16) and (5.13). This ends the proof of Proposition 5.1.  $\blacksquare$

We now turn to the proof of the lemma.

*Proof of Lemma 5.1.* Let  $0 \leq n \leq N - 1$ . Since, by definition of the variational problem (5.8),  $u^n(x) \cdot v^n(x) = 0$ , almost everywhere, and almost surely, it follows that for a.e.  $x \in \mathcal{O}$ , a.s.

$$|u^n(x) + v^n(x)| = \sqrt{1 + |v^n(x)|^2} \geq 1, \quad ,$$

and by Remark 5.3, one has a.s.

$$\begin{aligned} |\nabla u^{n+1}|_{L_x^2}^2 &= \int_{\mathcal{O}} \left| \nabla \left( \frac{u^n + v^n}{|u^n + v^n|} \right) \right|^2 dx \\ &\leq \int_{\mathcal{O}} |\nabla u^n + \nabla v^n|^2 dx. \end{aligned}$$

Then, by expanding the right hand side of this inequality:

$$|\nabla u^{n+1}|_{L_x^2}^2 \leq |\nabla u^n|_{L_x^2}^2 + 2\langle \nabla u^n, \nabla v^n \rangle + |\nabla v^n|_{L_x^2}^2. \quad (5.21)$$

To find an expression of  $2\langle \nabla u^n, \nabla v^n \rangle$ , we use (5.8) with the test function

$$\varphi := v^n - A^n \in \mathbb{W}_{N,n}.$$

Then, observing that we have pointwise for a.e.  $x \in \mathcal{O}$

$$\begin{aligned} (v^n - u^n \times v^n) \cdot (v^n - A^n) - (\text{id} - u^n \times)(u^n \times \Delta W^n) \cdot (v^n - A^n) \\ = ((\text{id} - u^n \times)(v^n - A^n)) \cdot (v^n - A^n) \\ = |v^n - A^n|_{\mathbb{R}^3}^2, \end{aligned}$$

one has

$$\begin{aligned} 2\langle \nabla u^n, \nabla v^n \rangle &= -\frac{1}{\Delta t} |v^n - A^n|_{L_x^2}^2 - 2\theta |\nabla v^n|_{L_x^2}^2 + 2\theta \langle \nabla v^n, \nabla A^n \rangle \\ &\quad + 2\langle \nabla u^n, \nabla A^n \rangle + \langle (\text{id} - u^n \times) F_\phi(u^n), v^n - A^n \rangle, \end{aligned} \quad (5.22)$$

which, using (5.21) and taking the expectation yields to

$$\begin{aligned} \mathbb{E} [|\nabla u^{n+1}|_{L_x^2}^2] &+ \frac{1}{\Delta t} \mathbb{E} [|v^n - A^n|_{L_x^2}^2] + (2\theta - 1) \mathbb{E} [|\nabla v^n|_{L_x^2}^2] \\ &\leq \mathbb{E} [|\nabla u^n|_{L_x^2}^2] + 2\theta \mathbb{E} [\langle \nabla v^n, \nabla A^n \rangle] + 2\mathbb{E} [\langle \nabla u^n, \nabla A^n \rangle] \\ &\quad + \mathbb{E} [\langle (\text{id} - u^n \times) F_\phi(u^n), v^n - A^n \rangle], \end{aligned} \quad (5.23)$$

all terms on the left hand side being non negative, due to  $\theta \in (\frac{1}{2}, 1]$ .

Now, since  $u^n$  and  $\Delta W^n$  are independent, and  $\mathbb{E}[\Delta W^n] = 0$ , we have

$$\mathbb{E} [\langle \nabla u^n, \nabla A^n \rangle] = 0. \quad (5.24)$$

Moreover, by (5.12) and the Sobolev embedding  $H_x^2 \hookrightarrow L_x^\infty$ ,

$$\begin{aligned} \mathbb{E} [|\nabla A^n|_{L_x^2}^2] &\leq 2 \left( \mathbb{E} |\nabla u^n|_{L_x^2}^2 \mathbb{E} [|\Delta W^n|_{L_x^\infty}^2] + \mathbb{E} [|\nabla \Delta W^n|_{L_x^2}^2] \right) \\ &\leq c\Delta t |\phi|_{\mathbb{L}^2}^2 (\mathbb{E} [|\nabla u^n|_{L_x^2}^2] + 1). \end{aligned} \quad (5.25)$$

Therefore

$$\mathbb{E} [\langle \nabla v^n, \nabla A^n \rangle] \leq \frac{\epsilon}{2} \mathbb{E} [|\nabla v^n|_{L_x^2}^2] + \frac{c\Delta t}{2\epsilon} |\phi|_{\mathbb{L}_2^{0,2}}^2 (\mathbb{E} [|\nabla u^n|_{L_x^2}^2] + 1). \quad (5.26)$$

Similarly, one has

$$\mathbb{E} [\langle (\text{id} - u^n) \times F_\phi(u^n), v^n - A^n \rangle] \leq \frac{\Delta t}{2\epsilon} |\phi|_{\mathbb{L}_2^{0,2}}^4 + \frac{2\epsilon}{\Delta t} \mathbb{E} [v^n - A^n|_{L_x^2}^2]. \quad (5.27)$$

Using (5.24), (5.26) and (5.27) in (5.23) gives

$$\begin{aligned} \mathbb{E} [|\nabla u^{n+1}|_{L_x^2}^2] + \frac{1-\epsilon}{\Delta t} \mathbb{E} [v^n - A^n|_{L_x^2}^2] + (2\theta - 1 - \theta\epsilon) \mathbb{E} [|\nabla v^n|_{L_x^2}^2] \\ \leq \mathbb{E} [|\nabla u^n|_{L_x^2}^2] + \frac{c\theta\Delta t}{\epsilon} |\phi|_{\mathbb{L}_2^{0,2}}^2 (\mathbb{E} [|\nabla u^n|_{L_x^2}^2] + 1) + \frac{\Delta t}{4\epsilon} |\phi|_{\mathbb{L}_2^{0,2}}^4. \end{aligned} \quad (5.28)$$

Since  $\theta \leq 1$ , this proves the lemma.  $\blacksquare$

Fix  $N \in \mathbb{N}^*$ . Let

$$w_N^n = \frac{1}{\Delta t} (v^n - A^n) \quad \text{for } n = 0, \dots, N-1. \quad (5.29)$$

In the following, we will also denote by  $v_N$ ,  $w_N$ , the piecewise constant processes (indexed by the time interval  $[0, T]$ ), whose values on  $[n\Delta t, (n+1)\Delta t)$  are (respectively)  $v^n$ ,  $w^n$ . The previous energy estimates can now be written in the form:

$$\text{ess sup}_{t \in [0, T]} \mathbb{E} [|\nabla u_N(t, \cdot)|_{L_x^2}^2] \leq c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}), \quad (5.30)$$

$$\mathbb{E} \int_0^T |w_N(t, \cdot)|_{L_x^2}^2 dt \leq c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}), \quad (5.31)$$

$$\mathbb{E} \int_0^T |v_N(t, \cdot)|_{L_x^2}^2 dt \leq c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}) \Delta t, \quad (5.32)$$

and

$$\mathbb{E} \int_0^T |\nabla v_N(t, \cdot)|_{L_x^2}^2 dt \leq c(T, u_0, |\phi|_{\mathbb{L}_2^{0,2}}) \Delta t. \quad (5.33)$$

Note in addition that, since  $|u_N(t, x)| = 1$  for a.e.  $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$ , one has

$$\mathbb{E} \int_0^T |u_N(t, \cdot)|_{L_x^2}^2 dt \leq c(T). \quad (5.34)$$

## 4 Tightness

The aim of this section is to show that the sequence  $(u_N)_{N \in \mathbb{N}^*}$  is tight. Applying then the classical Prokhorov and Skorohod theorems (see for instance [DZ08]), we first get the relative compactness of the sequence of laws, and secondly we can assume the almost sure convergence

to a certain limit  $\bar{u}$ , up to a change of probability space. In the sequel, we use the following notations: for all  $t \in [0, T]$ , we set

$$X_N(t) := \sum_{0 \leq (n+1)\Delta t \leq t} A^n = \sum_{0 \leq (n+1)\Delta t \leq t} u^n \times \Delta W^n, \quad (5.35)$$

and

$$X^n := X_N(n\Delta t) = \sum_{0 \leq k \leq n-1} u^k \times \Delta W^k. \quad (5.36)$$

The process  $t \mapsto X_N(t)$  is the martingale part of the semi-martingale  $u_N$ . It is a martingale with respect to a natural piecewise constant filtration, and corresponds to the noise induced fluctuations of the process  $t \mapsto u_N(t)$ . In order to get an almost sure convergence for the martingale part  $X_N$ , we consider the triplet  $(u_N, X_N, W)_{N \in \mathbb{N}^*}$ , and show that it forms a tight sequence on a suitable space. This classical technique is used essentially to retrieve the noise term in this new probability space. This has the drawback that the new Wiener process depends on the integer  $N \in \mathbb{N}^*$ .

**Proposition 5.2.** *The sequence*

$$(u_N, X_N, W)_{N \in \mathbb{N}^*}$$

*is tight in the space*

$$L^2([0, T]; L_x^2) \times L^2([0, T]; L_x^2) \times \mathcal{C}(0, T; L_x^2).$$

Recall that by Lemma 2.1, the embedding

$$L^2(0, T; B_0) \cap H^\alpha(0, T; B) \hookrightarrow L^2(0, T; B)$$

is compact, for  $\alpha > 0$ , as soon as  $B_0$  is compactly embedded in  $B$  (where  $B$  and  $B_0$  denote two reflexive Banach spaces). We aim to apply this with  $B = L_x^2$ , and  $B_0 = H_x^1$ . Therefore, in order to deduce the tightness, we need uniform  $H^\alpha(0, T; L_x^2)$  estimates on  $u_N$  and  $X_N$  for some  $\alpha > 0$ . These estimates are stated in the following proposition.

**Proposition 5.3.** *For any  $\alpha \in (0, \frac{1}{2})$ , there exists a constant  $c = c(|\phi|_{\mathbb{L}^{0,2}}, T, \alpha)$  such that*

$$\mathbb{E} \left[ \|u_N\|_{H^\alpha(0, T; L_x^2)}^2 \right] \leq c, \quad (5.37)$$

$$\mathbb{E} \left[ \|X_N\|_{H^\alpha(0, T; L_x^2)}^2 \right] \leq c. \quad (5.38)$$

*Proof of Proposition 5.3.* We have to evaluate the following quantities for  $\alpha \in (0, \frac{1}{2})$ :

$$\iint_{[0, T]^2} \frac{\mathbb{E}[|u_N(t) - u_N(s)|_{L_x^2}^2]}{|t - s|^{1+2\alpha}} dt ds,$$

and

$$\iint_{[0, T]^2} \frac{\mathbb{E}[|X_N(t) - X_N(s)|_{L_x^2}^2]}{|t - s|^{1+2\alpha}} dt ds.$$

Notice that these integrals measure the regularity in time of the two processes  $t \mapsto u_N(t)$  and  $t \mapsto X_N(t)$ . Since  $X_N$  is the martingale part in the decomposition of the semi-martingale  $u_N$ , one expects  $u_N$  to be at least as regular as  $X_N$ . In the sequel, we take  $t, s \in [0, T]$ , and assume without loss of generality that  $t > s$ . Thus, we first evaluate the following quantity:

$$\mathbb{E}\left[|X_N(t) - X_N(s)|_{L_x^2}^2\right] = \mathbb{E}\left[\sum_{s < (n+1)\Delta t \leq t} |A^n|_{L_x^2}^2\right]. \quad (5.39)$$

Observe that for  $n \neq m$ , the random variables  $\Delta W^n, \Delta W^m$  are independent, with zero mean. Using also the fact that  $u^n$  is independent of  $\Delta W^k$  for  $1 \leq n \leq k \leq N-1$ , Fubini's theorem, and the identity  $a \cdot (b \times c) = a \times b \cdot c$ , for  $a, b, c \in \mathbb{R}^3$ , one has for  $m > n$ ,

$$\begin{aligned} \mathbb{E}[\langle A^n, A^m \rangle] &= \int_{\mathcal{O}} \mathbb{E}[(u^n(x) \times \Delta W^n(x)) \cdot (u^m(x) \times \Delta W^m(x))] dx \\ &= \int_{\mathcal{O}} \mathbb{E}[(u^n(x) \times \Delta W^n(x)) \times u^m(x)] \cdot \mathbb{E}[\Delta W^m(x)] dx \\ &= 0. \end{aligned} \quad (5.40)$$

Developing the sum (5.39) and using (5.13), one has

$$\begin{aligned} \mathbb{E}[|X_N(t) - X_N(s)|_{L_x^2}^2] &= \mathbb{E} \sum_{s < (n+1)\Delta t \leq t} |A^n|_{L_x^2}^2 + 2\mathbb{E} \sum_{\substack{s < (n+1)\Delta t \leq t \\ s < (m+1)\Delta t \leq t \\ n < m}} \langle A^n, A^m \rangle \\ &= \mathbb{E} \sum_{s < (n+1)\Delta t \leq t} |A^n|_{L_x^2}^2 \\ &\leq c|\phi|_{\mathbb{L}_2}^2 \left( \sum_{s < (n+1)\Delta t \leq t} \Delta t \right). \end{aligned}$$

We observe that the number of terms in the sum above is bounded by  $\frac{t-s}{\Delta t} + 1$ , and deduce that for all  $0 \leq s \leq t \leq T$ ,

$$\mathbb{E}\left[|X_N(t) - X_N(s)|_{L_x^2}^2\right] \leq c|\phi|_{\mathbb{L}_2}^2 (|t-s| + \Delta t). \quad (5.41)$$

Now, remark that (5.41) implies the uniform estimate of the  $H^\alpha$  norm. Indeed, since  $X_N$  is a piecewise constant function, the integrand  $\mathbb{E}[|X_N(t, \cdot) - X_N(s, \cdot)|_{L_x^2}^2]$  vanishes for  $(t, s) \in [n\Delta t, (n+1)\Delta t]^2$ , for  $0 \leq n \leq N-1$ . Using moreover (5.41), there exists a constant  $c = c(|\phi|_{\mathbb{L}_2}, T)$  such that

$$\begin{aligned} &\iint_{[0, T]^2} \frac{\mathbb{E}[|X_N(t) - X_N(s)|_{L_x^2}^2]}{|t-s|^{1+2\alpha}} dt ds \\ &\leq c \iint_{[0, T]^2} \frac{dt ds}{|t-s|^{2\alpha}} + c\Delta t \sum_{\substack{0 \leq m, n \leq N-1 \\ |n-m| \geq 1}} \int_{n\Delta t}^{(n+1)\Delta t} \int_{m\Delta t}^{(m+1)\Delta t} \frac{dt ds}{|t-s|^{1+2\alpha}} \\ &= A + B. \end{aligned}$$

Since the set

$$\bigcup_{\substack{0 \leq n, m \leq N-1 \\ |n-m| \geq 2}} [n\Delta t, (n+1)\Delta t] \times [m\Delta t, (m+1)\Delta t]$$

is contained in

$$\{(t, s) \in [0, T]^2, |t - s| > \Delta t\},$$

we remark that

$$B \leq \sum_{\substack{0 \leq m, n \leq N-1 \\ |n-m|=1}} \int_{n\Delta t}^{(n+1)\Delta t} \int_{m\Delta t}^{(m+1)\Delta t} \frac{c\Delta t \, dt \, ds}{|t-s|^{1+2\alpha}} + \iint_{\substack{[0, T]^2 \\ |t-s| > \Delta t}} \frac{c\Delta t \, dt \, ds}{|t-s|^{1+2\alpha}}.$$

Finally, we get

$$\begin{aligned} & \iint_{[0, T]^2} \frac{\mathbb{E}[|X_N(t) - X_N(s)|_{L_x^2}^2]}{|t-s|^{1+2\alpha}} \, dt \, ds \\ & \leq 2c \iint_{[0, T]^2} \frac{dt \, ds}{|t-s|^{2\alpha}} + 2c\Delta t \sum_{n=0}^{N-2} \int_{n\Delta t}^{(n+1)\Delta t} \int_{(n+1)\Delta t}^{(n+2)\Delta t} \frac{dt \, ds}{|t-s|^{1+2\alpha}}. \end{aligned} \quad (5.42)$$

The first term of the right hand side of (5.42) is bounded because  $\alpha \in (0, \frac{1}{2})$ . Then, it is easy to show that

$$\sum_{n=0}^{N-2} \int_{n\Delta t}^{(n+1)\Delta t} \int_{(n+1)\Delta t}^{(n+2)\Delta t} \frac{dt \, ds}{|t-s|^{1+2\alpha}} = O(\Delta t^{-2\alpha}), \quad (5.43)$$

and (5.38) is proved.

We now turn to (5.37). Note that since we have already estimated  $X_N$ , it remains only to consider  $u_N - X_N$ . Using the definition of  $u_N$  and  $X_N$  together with (5.9), we write:

$$\begin{aligned} & (u_N(t) - X_N(t)) - (u_N(s) - X_N(s)) \\ & = \sum_{s < (n+1)\Delta t \leq t} \left( u^{n+1} - u^n - A^n \right) \\ & = \sum_{s < (n+1)\Delta t \leq t} \left( \frac{u^n + A^n}{|u^n + A^n|} - u^n - A^n \right) + \sum_{s < (n+1)\Delta t \leq t} \left( \frac{u^n + v^n}{|u^n + v^n|} - \frac{u^n + A^n}{|u^n + A^n|} \right). \end{aligned}$$

Then, taking the  $L_x^2$  norm, and the expectation, we get

$$\begin{aligned} & \mathbb{E}[|u_N(t) - X_N(t) - (u_N(s) - X_N(s))|_{L_x^2}^2] \\ & \leq 2\mathbb{E}\left[ \left| \sum_{s < (n+1)\Delta t \leq t} \frac{u^n + A^n}{|u^n + A^n|} - u^n - A^n \right|_{L_x^2}^2 \right] \\ & \quad + 2\mathbb{E}\left[ \left| \sum_{s < (n+1)\Delta t \leq t} \frac{u^n + v^n}{|u^n + v^n|} - \frac{u^n + A^n}{|u^n + A^n|} \right|_{L_x^2}^2 \right]. \end{aligned} \quad (5.44)$$

For the first term in the right hand side of (5.44), observe that for any  $u, V \in \mathbb{R}^3$ , s.t.  $V \perp u$  and  $|u| = 1$ , one has:

$$\left| \frac{u+V}{|u+V|} - u - V \right| \leq \sqrt{1+|V|^2} - 1 \leq \frac{1}{2}|V|^2. \quad (5.45)$$

Using Cauchy-Schwarz inequality, and (5.45) on each term of the sum (recall that  $A^n \perp u^n$ , see (5.12)), one has:

$$\begin{aligned} \mathbb{E} \left[ \left| \sum_{s < (n+1)\Delta t \leq t} \frac{u^n + A^n}{|u^n + A^n|} - u^n - A^n \right|_{L_x^2}^2 \right] \\ \leq \left( \frac{t-s}{\Delta t} + 1 \right) \sum_{s < (n+1)\Delta t \leq t} \mathbb{E} \left[ \left| \frac{u^n + A^n}{|u^n + A^n|} - u^n - A^n \right|_{L_x^2}^2 \right] \\ \leq \frac{1}{4} \left( \frac{t-s}{\Delta t} + 1 \right) \sum_{s < (n+1)\Delta t \leq t} \mathbb{E} \left[ |A^n|_{L_x^4}^4 \right]. \end{aligned}$$

Then we have by (5.14):

$$\mathbb{E} \left[ \left| \sum_{s < (n+1)\Delta t \leq t} \frac{u^n + A^n}{|u^n + A^n|} - u^n - A^n \right|_{L_x^2}^2 \right] \leq c(|\phi|_{\mathbb{L}_2^{0,1}}, T)(|t-s| + \Delta t)^2. \quad (5.46)$$

Similarly, for the second term in (5.46), we use the fact that the map  $x \mapsto \frac{x}{|x|}$ , is 1-Lipschitz for  $|x| \geq 1$ , together with Cauchy-Schwarz inequality and (5.16). Then

$$\begin{aligned} \mathbb{E} \left[ \left| \sum_{s < (n+1)\Delta t \leq t} \frac{u^n + v^n}{|u^n + v^n|} - \frac{u^n + A^n}{|u^n + A^n|} \right|_{L_x^2}^2 \right] &\leq \left( \frac{t-s}{\Delta t} + 1 \right) \mathbb{E} \left[ \sum_{s < (n+1)\Delta t \leq t} |v^n - A^n|_{L_x^2}^2 \right] \\ &\leq c(|\phi|_{\mathbb{L}_2^{0,2}}, T)(|t-s| + \Delta t). \end{aligned} \quad (5.47)$$

Using (5.46) and (5.47) in (5.44), together with (5.38), we conclude that there exists a constant  $c = c(|\phi|_{\mathbb{L}_2^{0,2}}, T)$ , independent of  $N \in \mathbb{N}^*$ , such that

$$\mathbb{E} \left[ |u_N(t, \cdot) - u_N(s, \cdot)|_{L_x^2}^2 \right] \leq c(|t-s| + \Delta t).$$

We have proved the same inequality as for the process  $X_N$  (see (5.41)), thus the conclusion follows in the same way as before.  $\blacksquare$

We now turn to the proof of Proposition 5.2.

*Proof of Proposition 5.2.* Notice that by (5.34) and (5.30),  $(u_N)_{N \in \mathbb{N}^*}$  is bounded in  $L^2(\Omega \times [0, T]; H_x^1)$ . A similar bound holds for the process  $X_N$ , writing

$$\begin{aligned} \mathbb{E} \left[ \|X_N\|_{L^2(0, T; L_x^2)}^2 \right] &= \mathbb{E} \left[ \sum_{n=0}^{N-1} |X^n|_{L_x^2}^2 \Delta t \right] \\ &= \sum_{n=0}^{N-1} \Delta t \mathbb{E} \left[ \left| \sum_{k \leq n} A^k \right|_{L_x^2}^2 \right] \\ &= \sum_{n=0}^{N-1} \Delta t \left( \mathbb{E} \left[ \sum_{k \leq n} |A^k|_{L_x^2}^2 \right] + 2\mathbb{E} \left[ \sum_{0 \leq k < l \leq n} \langle A^k, A^l \rangle \right] \right). \end{aligned}$$

As before, the second term vanishes (see (5.40)), while using (5.13), the first term is bounded by  $c(|\phi|_{\mathbb{L}_2}, T)$ .



Similarly, for  $k \neq l$ , we have  $\mathbb{E}[\langle \nabla A^k, \nabla A^l \rangle] = 0$ . Moreover, by (5.25), we get

$$\begin{aligned} \mathbb{E}[|\nabla X_N|_{L_x^2}^2] &= \sum_{n=0}^{N-1} \Delta t \sum_{k \leq n} \mathbb{E} \left[ |\nabla A^k|_{L_x^2}^2 \right] \\ &\leq c |\phi|_{\mathbb{L}_2^{0,2}}^2 \sum_{n=0}^{N-1} \Delta t \sum_{k \leq n} \Delta t \\ &\leq c' (|\phi|_{\mathbb{L}_2^{0,2}}, T). \end{aligned}$$

Therefore, there exists a constant  $c = c(|\phi|_{\mathbb{L}_2^{0,2}}, T) > 0$  such that

$$\mathbb{E} \left[ |X_N|_{L^2(0,T;H_x^1)}^2 \right] \leq c.$$

The tightness of the sequence  $(u_N, X_N, W)$  is now obtained in a classical way. Let  $R > 0$ , and fix  $\alpha \in (0, \frac{1}{2})$ . We consider the space

$$E := L^2(0, T; L_x^2) \times L^2(0, T; L_x^2) \times \mathcal{C}([0, T]; L_x^2),$$

endowed with the product norm. By lemma 2.1, and a standard Ascoli compactness theorem, the space

$$\begin{aligned} F := L^2(0, T; H_x^1) \times L^2(0, T; H_x^1) \times \mathcal{C}([0, T]; H_x^1) \\ \cap H^\alpha(0, T; L_x^2) \times H^\alpha(0, T; L_x^2) \times \mathcal{C}^\alpha([0, T]; L_x^2) \end{aligned}$$

is compactly embedded in  $E$ . Using Markov inequality, one has

$$\begin{aligned} \mathbb{P} \left( (u_N, X_N, W) \notin B_F(0, R) \right) \\ \leq \frac{1}{R^2} \left( \mathbb{E} \left[ \|u_N\|_{L^2(0,T;H_x^1)}^2 \right] + \mathbb{E} \left[ \|u_N\|_{H^\alpha([0,T];L_x^2)}^2 \right] + \mathbb{E} \left[ \|X_N\|_{L^2(0,T;H_x^1)}^2 \right] \right. \\ \left. + \mathbb{E} \left[ \|X_N\|_{H^\alpha([0,T];L_x^2)}^2 \right] + \mathbb{E} \left[ \|W\|_{\mathcal{C}^\alpha([0,T];H_x^1)}^2 \right] \right). \end{aligned} \quad (5.48)$$

Then, using the bounds (5.37), and (5.38), (5.6), the classical properties of a  $\phi\phi^*$ -Wiener process and also (5.15), the right hand side of (5.48) tends to 0 as  $R \rightarrow \infty$  uniformly in  $N \in \mathbb{N}^*$ . Since the sets  $B_F(0, R)$  are precompacts in  $E$ , the sequence  $(u_N, X_N, W)_{N \in \mathbb{N}^*}$  is tight in  $E$ , and the proposition is proved.  $\blacksquare$

A simple application of Prokhorov and Skorohod theorem leads to the following corollary:

**Corollary 5.1.** *There exists a new probability space  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ , a sequence of random variables on this space  $(\bar{u}_N, \bar{X}_N, \bar{W}_N)_{N \in \mathbb{N}^*}$  taking its values in the space  $L^2(0, T; L_x^2) \times L^2(0, T; L_x^2) \times \mathcal{C}(0, T; L_x^2)$ , with the same laws, for each  $N \in \mathbb{N}^*$ , as  $(u_N, X_N, W)$ , and a triplet  $(\bar{u}, \bar{X}, \bar{W})$  of r.v. in  $L^2(0, T; L_x^2) \times L^2(0, T; L_x^2) \times \mathcal{C}(0, T; L_x^2)$ , so that up to a subsequence,*

$$\bar{u}_N \xrightarrow{N \rightarrow \infty} \bar{u} \quad \text{a.s. in } L^2([0, T]; L_x^2),$$

$$\bar{X}_N \xrightarrow{N \rightarrow \infty} \bar{X} \quad \text{a.s. in } L^2([0, T]; L_x^2),$$

$$\bar{W}_N \xrightarrow{N \rightarrow \infty} \bar{W} \quad \text{a.s. in } \mathcal{C}(0, T; L_x^2).$$

Since  $u_N, X_N$  are piecewise constant processes, the same is also true for their counterparts in the new probability space  $\bar{\Omega}$ . Emphasizing on their dependence with respect to  $N \in \mathbb{N}^*$ , we define the following discrete parameter processes for  $0 \leq n \leq N$ :

$$\begin{aligned}\bar{u}_N^n &:= \bar{u}_N(n\Delta t) \in L_x^2, \\ \bar{X}_N^n &:= \bar{X}_N(n\Delta t) \in L_x^2,\end{aligned}$$

and also

$$\begin{aligned}\Delta \bar{W}_N^n &:= \bar{W}_N((n+1)\Delta t) - \bar{W}_N(n\Delta t), \\ \bar{A}_N^n &:= \bar{u}_N^n \times \Delta \bar{W}_N^n,\end{aligned}$$

and  $\bar{v}_N^n$  as the unique solution of (5.8) associated to the data  $(\bar{u}_N^n, \Delta \bar{W}_N^n)$ , i.e. for all  $0 \leq n \leq N-1$ , and all  $\varphi \in \bar{\mathbb{W}}_{N,n}$ ,

$$\begin{aligned}\langle \bar{v}_N^n - \bar{u}_N^n \times \bar{v}_N^n, \varphi \rangle + 2\theta\Delta t \langle \nabla \bar{v}_N^n, \nabla \varphi \rangle \\ = -2\Delta t \langle \nabla \bar{u}_N^n, \nabla \varphi \rangle + \langle (\text{id} - \bar{u}_N^n \times)(\bar{u}_N^n \times \Delta \bar{W}_N^n), \varphi \rangle \\ + \Delta t \langle (\text{id} - \bar{u}_N^n \times)F_\phi(\bar{u}_N^n), \varphi \rangle,\end{aligned}\quad (5.49)$$

where  $\bar{\mathbb{W}}_{N,n}(\omega) := \{\psi \in H_x^1, \forall x \in \mathcal{O}, \psi(x) \perp \bar{u}_N^n(\omega, x)\}$ . These random variables have the same laws as their counterparts in  $\Omega$  that is (respectively)  $u^n, X^n, \Delta W_N^n$  and  $A^n := u^n \times \Delta W_N^n$ . We already noticed that  $\bar{v}_N^n$  depends continuously on the couple  $(\bar{u}_N^n, \Delta \bar{W}_N^n)$  through (5.49), and thus the law of  $\bar{v}_N^n$  is the same as the law of  $v^n$ . It also follows that we have the identity

$$\bar{u}_N^{n+1} = \frac{\bar{u}_N^n + \bar{v}_N^n}{|\bar{u}_N^n + \bar{v}_N^n|} \quad \text{a.s.} \quad (5.50)$$

We still need to define the following processes on  $\bar{\Omega}$ :  $\forall t \in [0, T]$ ,

$$\bar{v}_N(t) := \bar{v}_N^n, \quad \text{if } t \in [n\Delta t, (n+1)\Delta t), \quad (5.51)$$

and

$$\bar{w}_N(t) := \frac{\bar{v}_N^n - \bar{A}_N^n}{\Delta t}, \quad \text{if } t \in [n\Delta t, (n+1)\Delta t). \quad (5.52)$$

*Remark 5.4.* By (5.30) and a classical compactness argument, we may assume that up a subsequence the following convergence holds

$$\nabla \bar{u}_N \xrightarrow{N \rightarrow \infty} \nabla \bar{u} \quad \text{weakly in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^{3 \times 3}). \quad (5.53)$$

## 5 Convergence of the martingale part

In section 4 we proved that the process  $\bar{X}_N$  converges almost surely in  $L^2([0, T] \times \mathcal{O}; \mathbb{R}^3)$  to  $\bar{X}$ . Here we show that  $\bar{X}$  defines a square integrable continuous martingale with values in  $L_x^2$ . We define the filtration  $(\bar{\mathcal{F}}_t)_{t \in [0, T]}$  as

$$\bar{\mathcal{F}}_t = \sigma \{ \bar{W}(s), s \leq t \}. \quad (5.54)$$

**Proposition 5.4.** *The process  $t \in [0, T] \mapsto \bar{X}(t, \omega) \in L_x^2$  is a square integrable continuous martingale with respect to the filtration  $(\bar{\mathcal{F}}_t)$ , with quadratic variation characterized by the following relation holding for all  $a, b \in L_x^2$ :*

$$\langle \ll \bar{X} \gg_t a, b \rangle = \int_0^t \langle \bar{u} \times (\phi a), \bar{u} \times (\phi b) \rangle ds.$$

The proof needs an additional martingale-type uniform estimate on  $X_N$ .

**Proposition 5.5.** *For all  $q \in \mathbb{N}$ , there exists a constant  $c = c(|\phi|_{\mathbb{L}^2}, T, q) > 0$  independent of  $N \in \mathbb{N}^*$ , such that*

$$\mathbb{E} \left[ \max_{0 \leq n \leq N} |\bar{X}_N^n|_{L_x^2}^{2q} \right] \leq c.$$

To prove proposition 5.5, we state a discrete version of the Burkholder-Davis-Gundy inequality with values in a Hilbert space. The following result is a particular case of Proposition 2 of [Ass75], and we therefore omit the proof.

**Lemma 5.2.** *For a given discrete parameter martingale  $(M^n)_{0 \leq n \leq N}$  with values in a Hilbert space  $H$ , for any  $q \in \mathbb{N}^*$ , there exist  $c = c(q) > 0$  such that the following inequality holds:*

$$\mathbb{E} \left[ \max_{0 \leq n \leq N} |M^n|_H^{2q} \right] \leq c \mathbb{E} \left[ \left( \sum_{n=0}^{N-1} |M^{n+1} - M^n|_H^2 \right)^q \right].$$

*Remark 5.5.* Since for all  $N \in \mathbb{N}^*$ , the laws of  $X_N$  and  $\bar{X}_N$  are equal, note that for all  $t \in [0, T]$ , and almost surely,

$$\bar{X}_N(t) = \sum_{0 \leq (n+1)\Delta t \leq t} \bar{u}_N^n \times \Delta \bar{W}_N^n. \quad (5.55)$$

It is easily seen, using (5.49) and (5.50) that  $(\bar{u}_N^n)_{0 \leq n \leq N}$  is adapted to

$$\bar{\mathbb{F}}_N^n = \sigma \{ \bar{W}_N(k\Delta t); k \in \mathbb{N}^*, k \leq n \}, \quad (5.56)$$

and the process  $\bar{X}_N^n$  defines a martingale with respect to this filtration. In particular, we have the following identity: for all  $0 \leq n \leq n' \leq N$ , and any bounded continuous function  $\varphi$  on  $(L_x^2)^n$ ,

$$\mathbb{E} \left[ (\bar{X}_N^{n'} - \bar{X}_N^n) \varphi(\bar{W}_N(\Delta t), \dots, \bar{W}_N(n\Delta t)) \right] = 0. \quad (5.57)$$

The reader may also check that for any  $n, n', \varphi$  as above, and for all  $a, b \in L_x^2$ ,

$$\begin{aligned} & \mathbb{E} \left[ \left( \langle \bar{X}_N^{n'}, a \rangle \langle \bar{X}_N^{n'}, b \rangle - \langle \bar{X}_N^n, a \rangle \langle \bar{X}_N^n, b \rangle \right. \right. \\ & \quad \left. \left. - \sum_{n \leq k \leq n'-1} \Delta t \langle \bar{u}_N^k \times (\phi a), \bar{u}_N^k \times (\phi b) \rangle \right) \varphi(\bar{W}_N(\Delta t), \dots, \bar{W}_N(n\Delta t)) \right] = 0. \end{aligned} \quad (5.58)$$

Equation (5.58) gives us the quadratic variation of  $(\bar{X}_N^n)_{0 \leq n \leq N}$ .

*Proof of Proposition 5.5.* Assume that  $N \in \mathbb{N}^*$  is given. We apply Lemma 5.2 to the discrete parameter martingale  $(\bar{X}_N^n)_{0 \leq n \leq N}$ , which takes values in the Hilbert space  $H = L_x^2$ . By (5.55), and Hölder's inequality, one has

$$\begin{aligned} \mathbb{E} \left[ \left( \sum_{n=0}^{N-1} |\bar{X}_N^{n+1} - \bar{X}_N^n|_{L_x^2}^2 \right)^q \right] &= \mathbb{E} \left[ \left( \sum_{n=0}^{N-1} |\bar{u}_N^n \times \Delta \bar{W}_N^n|_{L_x^2}^2 \right)^q \right] \\ &\leq N^{q-1} \sum_{n=0}^{N-1} \mathbb{E} \left[ |\bar{u}_N^n \times \Delta \bar{W}_N^n|_{L_x^2}^{2q} \right]. \end{aligned}$$

It is known (see for instance [DZ08], corollary 2.17) that since  $\Delta \bar{W}_N^n$  is a gaussian random variable with covariance  $\Delta t \phi \phi^*$ , there exists a constant  $c(|\phi|_{\mathbb{L}_2}, q) > 0$  (independent of  $n$  and  $N$ ) such that:

$$\mathbb{E} \left[ |\Delta \bar{W}_N^n|_{L_x^2}^{2q} \right] \leq c(|\phi|_{\mathbb{L}_2}, q) \Delta t^q. \quad (5.59)$$

Therefore, recalling that  $|\bar{u}_N^n| = 1$  a.e., one has

$$\mathbb{E} \left[ \left( \sum_{n=0}^{N-1} |\bar{X}_N^{n+1} - \bar{X}_N^n|_{L_x^2}^2 \right)^q \right] \leq N (c(|\phi|_{\mathbb{L}_2}, q) \Delta t^q) N^{q-1} \leq c'(|\phi|_{\mathbb{L}_2}, T, q).$$

This proves proposition 5.5. ■

We now turn to the proof of Proposition 5.4.

*Proof of Proposition 5.4.*

$\bar{X}$  is a martingale. We use the identities (5.57) and (5.58). We have to show that for any bounded continuous function  $\varphi$  defined on the space  $(L_x^2)^K$ , any  $a, b \in L_x^2$  the following relations hold for almost all  $0 \leq s \leq t \leq T$ , all  $K \in \mathbb{N}^*$ , and  $t_1 \leq \dots \leq t_K < s$ :

$$\mathbb{E} \left[ (\bar{X}(t) - \bar{X}(s)) \varphi(\bar{W}(t_1), \dots, \bar{W}(t_K)) \right] = 0. \quad (5.60)$$

and

$$\begin{aligned} \mathbb{E} \left[ \left( \langle \bar{X}(t), a \rangle \langle \bar{X}(t), b \rangle - \langle \bar{X}(s), a \rangle \langle \bar{X}(s), b \rangle \right. \right. \\ \left. \left. - \int_s^t \langle \bar{u}(\sigma) \times (\phi a), \bar{u}(\sigma) \times (\phi b) \rangle d\sigma \right) \varphi(\bar{W}(t_1), \dots, \bar{W}(t_K)) \right] = 0. \quad (5.61) \end{aligned}$$

First, observe that as a consequence of Proposition 5.5, and Egorov's Theorem,

$$\bar{X}_N \xrightarrow{N \rightarrow \infty} \bar{X} \text{ in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3). \quad (5.62)$$

Hence, up to a subsequence, one has for almost all  $t, s \in [0, T]$ ,

$$\bar{X}_N(t) - \bar{X}_N(s) \xrightarrow{N \rightarrow \infty} \bar{X}(t) - \bar{X}(s), \text{ in } L^2(\bar{\Omega} \times \mathcal{O}; \mathbb{R}^3).$$

For all  $0 \leq k \leq K$ , if  $\lfloor \frac{t_k}{\Delta t} \rfloor$  denotes the floor of  $\frac{t_k}{\Delta t}$ , then  $\lfloor \frac{t_k}{\Delta t} \rfloor \Delta t$  tends to  $t_k$  as  $N \rightarrow \infty$ . Taking into account the almost sure continuity of the limit process  $\bar{W}$ , and the fact that the process  $\bar{W}_N$  converges almost surely to  $\bar{W}$  in  $\mathcal{C}([0, T]; L_x^2)$  as  $N$  tends to  $\infty$ , one has

$$(\bar{W}_N(\lfloor \frac{t_1}{\Delta t} \rfloor \Delta t), \dots, \bar{W}_N(\lfloor \frac{t_K}{\Delta t} \rfloor \Delta t)) \xrightarrow{N \rightarrow \infty} (\bar{W}(t_1), \dots, \bar{W}(t_K)) \text{ in } (L_x^2)^K.$$

The application  $\varphi$  being continuous, we conclude that

$$\mathbb{E} \left[ (\bar{X}_N(t) - \bar{X}_N(s)) \varphi \left( \bar{W}_N \left( \lfloor \frac{t_1}{\Delta t} \rfloor \Delta t \right), \dots, \bar{W}_N \left( \lfloor \frac{t_K}{\Delta t} \rfloor \Delta t \right) \right) \right] \\ \xrightarrow{N \rightarrow \infty} \mathbb{E} \left[ (\bar{X}(t) - \bar{X}(s)) \varphi(\bar{W}(t_1), \dots, \bar{W}(t_K)) \right].$$

On the other hand, by (5.57)

$$\mathbb{E} \left[ (\bar{X}_N(t) - \bar{X}_N(s)) \varphi \left( \bar{W}_N \left( \lfloor \frac{t_1}{\Delta t} \rfloor \Delta t \right), \dots, \bar{W}_N \left( \lfloor \frac{t_K}{\Delta t} \rfloor \Delta t \right) \right) \right] = 0,$$

and (5.60) is proved.

If  $a, b \in L^2([0, T] \times \mathcal{O}; \mathbb{R}^3)$ , then (5.58) implies that:

$$\mathbb{E} \left[ \left( \langle \bar{X}_N(t), a \rangle \langle \bar{X}_N(t), b \rangle - \langle \bar{X}_N(s), a \rangle \langle \bar{X}_N(s), b \rangle \right. \right. \\ \left. \left. - \sum_{s < (n+1)\Delta t \leq t} \Delta t \langle \bar{u}_N^n \times (\phi a), \bar{u}_N^n \times (\phi b) \rangle \right) \right. \\ \left. \cdot \varphi \left( \bar{W}_N \left( \lfloor \frac{t_1}{\Delta t} \rfloor \Delta t \right), \dots, \bar{W}_N \left( \lfloor \frac{t_K}{\Delta t} \rfloor \Delta t \right) \right) \right] = 0.$$

Moreover, the term

$$\langle \bar{X}_N(t), a \rangle \langle \bar{X}_N(t), b \rangle - \langle \bar{X}_N(s), a \rangle \langle \bar{X}_N(s), b \rangle$$

converges to

$$\langle \bar{X}(t), a \rangle \langle \bar{X}(t), b \rangle - \langle \bar{X}(s), a \rangle \langle \bar{X}(s), b \rangle$$

in  $L^1(\bar{\Omega})$ , while the term

$$\sum_{s < (n+1)\Delta t \leq t} \Delta t \langle \bar{u}_N^n \times (\phi a), \bar{u}_N^n \times (\phi b) \rangle$$

tends to

$$\int_s^t \langle \bar{u}(\sigma) \times \phi a, \bar{u}(\sigma) \times \phi b \rangle d\sigma,$$

in  $L^1(\bar{\Omega})$ . This proves (5.61). It remains to prove that  $\bar{X}$  has continuous trajectories.

*Proof of the continuity.* We prove that the limit  $\bar{X}$  satisfies the assumptions of Kolmogorov's test (see e.g. [DZ08], theorem 3.3). More precisely, we show that for any  $q \in \mathbb{N}^*$ , there exists  $c_q > 0$ , such that for almost every  $(t, s) \in [0, T]^2$ ,

$$\mathbb{E} \left[ |\bar{X}(t) - \bar{X}(s)|_{L_x^{2q}}^{2q} \right] \leq c_q |t - s|^q. \quad (5.63)$$

Let  $T \geq t > s \geq 0$ , and  $n, n' \in \mathbb{N}$ , the unique integers such that  $t \in [n'\Delta t, (n'+1)\Delta t)$  and  $s \in [n\Delta t, (n+1)\Delta t[$ . One has  $|t - n'\Delta t| \leq \Delta t$  and  $|s - n\Delta t| \leq \Delta t$ . We consider the discrete parameter martingale which starts at  $n\Delta t$ , and whose increments are the same as  $(\bar{X}_N^k)_{k \in \{0, \dots, N\}}$ . More precisely, let  $(M_N^l)_{0 \leq l \leq n' - n}$  be the discrete parameter process defined by

$$\bar{M}_N^l = \bar{X}_N^{n+l} - \bar{X}_N^n = \sum_{k=n+1}^{n+l} \bar{A}_N^k, \quad \text{for } 0 \leq l \leq n' - n.$$

The process  $(\bar{M}_N^l)_{0 \leq l \leq n' - n}$  defines a martingale for the discrete filtration  $(\bar{\mathbb{F}}_{(l+n)\Delta t})_{0 \leq l \leq n' - n}$ , (see (5.10)). Using similar arguments as for the proof of Proposition 5.5, and in particular Lemma 5.2,

$$\begin{aligned} \mathbb{E} \left[ |\bar{X}_N(t) - \bar{X}_N(s)|_{L_x^2}^{2q} \right] &\leq \mathbb{E} \left[ \max_{l=0, \dots, n' - n} |\bar{M}_N^l|_{L_x^2}^{2q} \right] \\ &\leq c \sum_{k=n+1}^{n'} \mathbb{E} [ |\bar{A}_N^k|_{L_x^2}^{2q} ] (n' - n)^{q-1} \\ &\leq c(|\phi|_{\mathbb{L}_2}, q) (n' \Delta t - n \Delta t)^q \\ &\leq c(|\phi|_{\mathbb{L}_2}, q) (|t - s| + \Delta t)^q. \end{aligned}$$

Then, (5.63) follows from (5.62) and Fatou's Lemma. Thus,  $\bar{X}$  defines a continuous martingale with respect to  $(\bar{\mathcal{F}}_t)_{t \in [0, T]}$  (see (5.54)). As we saw in the proof of Proposition 5.2 the processes  $\bar{X}_N$ , for  $N \in \mathbb{N}^*$  are square-integrable, uniformly in  $N$ , thus the almost sure limit  $\bar{X}$  is square-integrable. This proves Proposition 5.4.  $\blacksquare$

We are now ready to apply the continuous martingale representation theorem for Hilbert space-valued Wiener processes. We have shown that the limit process  $\bar{X}$  satisfies its hypotheses. The quadratic variation of  $\bar{X}$  is given, for any  $a, b \in L_x^2$ , by:

$$\langle \llbracket \bar{X} \rrbracket_t a, b \rangle = \int_0^t \langle \bar{u}(s) \times (\phi a), \bar{u}(s) \times (\phi b) \rangle ds, \quad t \in [0, T].$$

There exists an enlarged filtered probability space  $\tilde{\mathfrak{P}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{\mathbb{P}})$ , with  $\bar{\Omega} \subseteq \tilde{\Omega}$ , and a  $L_x^2$ -valued Wiener process  $\tilde{W}$  defined on  $\tilde{\mathfrak{P}}$ , with covariance operator  $\phi\phi^*$ , such that  $\bar{X}$ ,  $\bar{u}$  can be extended to random variables on this space, and

$$\bar{X}(t, \tilde{\omega}) = \int_0^t \bar{u}(s, \tilde{\omega}) \times d\tilde{W}(s, \tilde{\omega}), \quad \text{for all } t \in [0, T], \quad d\tilde{\mathbb{P}} - \text{a.s.} \quad (5.64)$$

## 6 Identification of the limit

In this section, the purpose is to find a relation between  $\bar{X}$  and the limit  $\bar{u}$ . Noticing that  $\sum_{0 \leq (n+1)\Delta t \leq t} (\bar{u}_N^{n+1} - \bar{u}_N^n) = \bar{u}_N(t) - u_0$ , and by definition of  $\bar{w}_N^n$  in (5.52), one may write  $\bar{X}_N(t)$  as:

$$\bar{X}_N(t) = \bar{u}_N(t) - u_0 - \sum_{0 \leq (n+1)\Delta t \leq t} \Delta t \bar{w}_N^n - \sum_{0 \leq (n+1)\Delta t \leq t} (\bar{u}_N^{n+1} - \bar{u}_N^n - \bar{v}_N^n) \quad (5.65)$$

for any  $t \in [0, T]$ .

**Proposition 5.6.** *Up to a subsequence:*

$$\bar{w}_N \xrightarrow{N \rightarrow \infty} \bar{w} \quad \text{weakly in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3),$$

with

$$\bar{w} = \Delta \bar{u} + \bar{u}(\nabla \bar{u})^2 + \bar{u} \times \Delta \bar{u} + P_{\bar{u}^\perp} [F_\phi(\bar{u})].$$

**Corollary 5.2.** *Up to a subsequence, for any  $t \in [0, T]$ ,  $\sum_{0 \leq (n+1)\Delta t \leq t} \Delta t \bar{w}_N^n$  converges weakly in  $L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3)$  to*

$$\int_0^t \left( \Delta \bar{u}(s) + \bar{u}(s) |\nabla \bar{u}(s)|^2 + \bar{u}(s) \times \Delta \bar{u}(s) + F_\phi(\bar{u})(s) \right) ds.$$

*Proof of Proposition 5.6.* Thanks to (5.31), the equality of the laws of  $w_N$  and  $\bar{w}_N$ , and Alaoglu theorem, we can assume that up to a subsequence,  $\bar{w}_N$  converges weakly to a limit  $\bar{w}$  in  $L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3)$ . Because of the strong convergence of  $\bar{u}_N$  to  $\bar{u}$  in  $L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3)$ , one has also:

$$(\text{id} - \bar{u}_N \times) \bar{w}_N \xrightarrow{N \rightarrow \infty} (\text{id} - \bar{u} \times) \bar{w} \quad \text{weakly in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3). \quad (5.66)$$

We shall first identify the limit of  $(\text{id} - \bar{u}_N \times) \bar{w}_N$ .

*Step 1: proof that a.s.  $(\text{id} - \bar{u} \times) \bar{w}(t, x) \perp \bar{u}(t, x)$ .* By definition of  $\bar{w}_N$ , almost surely, and for almost every  $(t, x) \in [0, T] \times \mathcal{O}$ , one has  $\bar{w}_N(\omega, t, x) \cdot \bar{u}_N(\omega, t, x) = 0$ . Thus for any  $\mathbb{R}$ -valued test function  $\varphi \in L^\infty(\bar{\Omega} \times [0, T] \times \mathcal{O})$ , one has

$$\mathbb{E} \int_0^T \int_{\mathcal{O}} (\bar{w}_N \cdot \bar{u}_N) \varphi dx dt = 0.$$

On the other hand, by weak convergence of  $\bar{w}_N$  and strong convergence of  $\bar{u}_N \varphi$ ,

$$\mathbb{E} \int_0^T \int_{\mathcal{O}} (\bar{w}_N \cdot \bar{u}_N) \varphi dx dt \xrightarrow{N \rightarrow \infty} \mathbb{E} \int_0^T \int_{\mathcal{O}} (\bar{w} \cdot \bar{u}) \varphi dx dt.$$

Thus,  $\bar{u}(\omega, t, x) \cdot \bar{w}(\omega, t, x) = 0$ , for almost all  $(\omega, t, x)$ , and  $(\text{id} - \bar{u} \times) \bar{w} \perp \bar{u}$ .

*Step 2: identification of the limit for specific test functions.* We use the definition of  $\bar{w}_N$  (5.52), and (5.8). Take

$$\Phi \in \mathcal{C}([0, T]; L^\infty(\bar{\Omega}; W_x^{1, \infty})),$$

and consider a test function of the form

$$\bar{u}(\omega, t, x) \times \Phi(\omega, t, x), \quad \omega \in \bar{\Omega}, t \in [0, T], x \in \mathcal{O}. \quad (5.67)$$

We approximate this test function by the sequence of piecewise constant functions  $(\bar{u}_N \times \Phi_N)$ , where we set for all  $N \in \mathbb{N}^*$ , all  $0 \leq n \leq N - 1$ , and for  $t \in [n\Delta t, (n+1)\Delta t)$ ,

$$\Phi_N(\omega, t, x) = \Phi_N^n(\omega, x) := \Phi(\omega, n\Delta t, x).$$

On the one hand, using the strong convergence of  $\bar{u}_N$  to  $\bar{u}$  in  $L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3)$ , and (5.30) we have:

$$\bar{u}_N \times \Phi_N \xrightarrow{N \rightarrow \infty} \bar{u} \times \Phi \quad \text{strongly in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3), \quad (5.68)$$

$$\nabla(\bar{u}_N \times \Phi_N) \text{ is bounded in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^{3 \times 3}) \text{ uniformly in } N, \quad (5.69)$$

and for any  $k \in \{1, 2, 3\}$ ,

$$\bar{u}_N \times \partial_{x_k} \Phi_N \xrightarrow{N \rightarrow \infty} \bar{u} \times \partial_{x_k} \Phi \quad \text{strongly in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3). \quad (5.70)$$

On the other hand, almost surely,  $(\bar{u}_N \times \Phi_N) \in \bar{\mathbb{W}}_{N,n}$ , and is therefore a suitable test function in the variational formulation (5.49). Using then (5.52) and the definition of  $\bar{A}_N^n$ , and summing over  $n \in \{0, \dots, N-1\}$ , one obtains:

$$\begin{aligned} \mathbb{E} \int_0^T \langle (\text{id} - \bar{u}_N \times) \bar{w}_N, \bar{u}_N \times \Phi_N \rangle dt &= -2\theta \mathbb{E} \int_0^T \langle \nabla \bar{v}_N, \nabla(\bar{u}_N \times \Phi_N) \rangle dt \\ &\quad - 2\mathbb{E} \int_0^T \langle \nabla \bar{u}_N, \nabla(\bar{u}_N \times \Phi_N) \rangle dt \\ &\quad + \mathbb{E} \int_0^T \langle (\text{id} - \bar{u}_N \times) F_\phi(\bar{u}_N), \bar{u}_N \times \Phi_N \rangle dt. \end{aligned} \quad (5.71)$$

The first term in the right hand side above converges to zero, because of (5.32), and (5.69). For the second term, we observe that since for all  $k = 1, 2, 3$ ,

$$\partial_{x_k} \bar{u}_N^n \cdot (\partial_{x_k} \bar{u}_N^n \times \Phi_N^n) = 0,$$

then

$$2\mathbb{E} \int_0^T \langle \nabla \bar{u}_N, \nabla(\bar{u}_N \times \Phi_N) \rangle dt = 2\mathbb{E} \int_0^T \left\langle \sum_{k=1,2,3} \partial_{x_k} \bar{u}_N, \bar{u}_N \times \partial_{x_k} \Phi_N \right\rangle dt.$$

By (5.70), and the weak convergence of  $\nabla \bar{u}_N$  to  $\nabla \bar{u}$  in  $L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3)$  (see Remark 5.4), this tends to  $2\mathbb{E} \int_0^T \sum_{k=1,2,3} \langle \partial_{x_k} \bar{u}, \bar{u} \times \partial_{x_k} \Phi \rangle dt$  as  $N \rightarrow \infty$ .

Eventually, it easily follows from assumption (5.6), the Sobolev embedding  $H_x^2 \subseteq L_x^\infty$ , and the boundedness of the sequence  $(\bar{u}_N)_N$  in  $L^\infty(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3)$  and (5.68) that the third term of the right hand side of (5.71) converges to

$$\mathbb{E} \int_0^T (\text{id} - \bar{u} \times) F_\phi(\bar{u}) dt = \frac{1}{2} \sum_{i \in \mathbb{N}} \mathbb{E} \int_0^T \langle (\text{id} - \bar{u} \times) ((\bar{u} \times \phi_i) \times \phi_i), \bar{u} \times \Phi \rangle dt.$$

Identifying all the limits in the right hand side of (5.71), we get:

$$\begin{aligned} \mathbb{E} \int_0^T \langle (\text{id} - \bar{u} \times) \bar{w}, \bar{u} \times \Phi \rangle dt &= -2\mathbb{E} \int_0^T \langle \nabla \bar{u}, \nabla(\bar{u} \times \Phi) \rangle dt \\ &\quad + \mathbb{E} \int_0^T \langle (\text{id} - \bar{u} \times) F_\phi(\bar{u}), \bar{u} \times \Phi \rangle dt. \end{aligned} \quad (5.72)$$

By a density argument, (5.72) remains true for any  $\Phi \in L^2(\bar{\Omega} \times [0, T]; H_x^1)$ .

*Step 3: identification of the limit for any test function.* We are going to use (5.72) with

$$\Phi := \bar{u} \times \Xi,$$

where  $\Xi \in L^2(\bar{\Omega} \times [0, T]; W_x^{1,\infty})$  and thus  $\Phi \in L^2(\bar{\Omega} \times [0, T]; H_x^1)$ . First, observe that for any unit vector  $V \in \mathbb{S}^2$ , one has

$$V \times (V \times \cdot) = -P_{V^\perp}, \quad (5.73)$$

where  $P_{V^\perp}$  denotes the orthogonal projection on  $V^\perp$ , hence from Step 1,

$$((\text{id} - \bar{u} \times) \bar{w}) \cdot \bar{u} \times (\bar{u} \times \Xi) = -((\text{id} - \bar{u} \times) \bar{w}) \cdot \Xi. \quad (5.74)$$



Moreover, for any  $1 \leq k \leq 3$ , since  $\bar{u} \cdot \partial_k \bar{u} = 0$ , one has

$$\begin{aligned} (\partial_{x_k} \bar{u} \times \bar{u}) \cdot \partial_{x_k} (\bar{u} \times \Xi) &= (\partial_{x_k} \bar{u} \times \bar{u}) \cdot ((\partial_{x_k} \bar{u} \times \Xi) + (\bar{u} \times \partial_{x_k} \Xi)) \\ &= |\partial_{x_k} \bar{u}|^2 \bar{u} \cdot \Xi - \partial_{x_k} \bar{u} \cdot \partial_{x_k} \Xi. \end{aligned} \quad (5.75)$$

Using (5.74) and (5.75) in (5.72) with  $\Phi := \bar{u} \times \Xi$ , we obtain:

$$\begin{aligned} -\mathbb{E} \int_0^T \langle (\text{id} - \bar{u} \times) \bar{w}, \Xi \rangle dt &= 2\mathbb{E} \int_0^T \langle \nabla \bar{u}, \nabla \Xi \rangle dt - 2\mathbb{E} \int_0^T \langle (\nabla \bar{u})^2 \bar{u}, \Xi \rangle dt \\ &\quad - \frac{1}{2} \mathbb{E} \int_0^T \langle P_{\bar{u}^\perp} (\text{id} - \bar{u} \times) F_\phi(\bar{u}), \Xi \rangle dt, \end{aligned} \quad (5.76)$$

from which we deduce that

$$(\text{id} - \bar{u} \times) \bar{w} = 2(\Delta \bar{u} + (\nabla \bar{u})^2 \bar{u}) + P_{\bar{u}^\perp} \left[ (\text{id} - \bar{u} \times) F_\phi(\bar{u}) \right],$$

in  $L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3)$ .

*Step 4: end of the proof.* Note that if  $V \cdot \bar{u} = 0$ , then

$$(\text{id} - \bar{u} \times)^{-1} V = \frac{1}{2} (\text{id} + \bar{u} \times) V,$$

whence

$$\bar{w} = \Delta \bar{u} + \bar{u} (\nabla \bar{u})^2 + \bar{u} \times \Delta \bar{u} + P_{\bar{u}^\perp} [F_\phi(\bar{u})],$$

and Proposition 5.6 is proved. ■

*Proof of Corollary 5.2.* It is an immediate consequence of Proposition 5.6, and the fact that

$$\mathbb{E} \int_{\lfloor \frac{t}{\Delta t} \rfloor \Delta t}^t \langle \bar{w}_N(s), \Phi \rangle ds \text{ converges to } 0 \text{ as } N \rightarrow \infty,$$

for any  $\Phi \in L^2(\bar{\Omega} \times \mathcal{O}; \mathbb{R}^3)$ , by (5.31). ■

**Proposition 5.7.** *For almost every  $t \in [0, T]$ ,*

$$\sum_{0 \leq (n+1)\Delta t \leq t} (\bar{u}_N^{n+1} - \bar{u}_N^n - \bar{v}_N^n)$$

*converges strongly in  $L^1(\bar{\Omega} \times \mathcal{O}; \mathbb{R}^3)$  to the part of the Itô correction which is along  $\bar{u}$ , namely*

$$\int_0^t P_{\bar{u}} [F_\phi(\bar{u})] ds.$$

**Corollary 5.3.** *For almost every  $t \in [0, T]$ ,*

$$\bar{X}(t) = \bar{u}(t) - u_0 - \int_0^t (\Delta \bar{u} + \bar{u} (\nabla \bar{u})^2 + \bar{u} \times \Delta \bar{u} + F_\phi(\bar{u})) ds, \quad (5.77)$$

*and  $\bar{u} \in \mathcal{C}([0, T]; L_x^2)$ .*

*Proof of Proposition 5.7.* We set for each  $0 \leq n \leq N - 1$ :

$$R_N^n := \bar{u}_N^{n+1} - \bar{u}_N^n - \bar{v}_N^n + \frac{1}{2} |\bar{A}_N^n|^2 \bar{u}_N^n. \quad (5.78)$$

It suffices to prove the following two facts:

$$\sum_{0 \leq (n+1)\Delta t \leq t} R_N^n \xrightarrow[N \rightarrow \infty]{} 0 \quad (5.79)$$

strongly in  $L^1(\bar{\Omega} \times \mathcal{O}; \mathbb{R}^3)$ , and

$$\sum_{0 \leq (n+1)\Delta t \leq t} |\bar{u}_N^n \times \Delta \bar{W}_N^n|^2 \bar{u}_N^n \xrightarrow[N \rightarrow \infty]{} - \int_0^t P_{\bar{u}} [F_{\phi}(\bar{u})] ds, \quad (5.80)$$

strongly in  $L^2(\bar{\Omega} \times \mathcal{O}; \mathbb{R}^3)$ .

*Proof of (5.79).* We decompose  $R_N^n$  into four terms:

$$R_N^n = \frac{\bar{u}_N^n + \bar{v}_N^n}{\sqrt{1 + |\bar{v}_N^n|^2}} - \bar{u}_N^n - \bar{v}_N^n + \frac{1}{2} |\bar{A}_N^n|^2 \bar{u}_N^n = I + II + III + IV,$$

with

$$\begin{aligned} I &:= \bar{u}_N^n \left( \frac{1}{\sqrt{1 + |\bar{A}_N^n|^2}} - 1 + \frac{1}{2} |\bar{A}_N^n|^2 \right), & II &:= \bar{u}_N^n \left( \frac{1}{\sqrt{1 + |\bar{v}_N^n|^2}} - \frac{1}{\sqrt{1 + |\bar{A}_N^n|^2}} \right), \\ III &:= \bar{v}_N^n \left( \frac{1}{\sqrt{1 + |\bar{A}_N^n|^2}} - 1 \right), & \text{and} & & IV &:= \bar{v}_N^n \left( \frac{1}{\sqrt{1 + |\bar{v}_N^n|^2}} - \frac{1}{\sqrt{1 + |\bar{A}_N^n|^2}} \right), \end{aligned}$$

and treat each of them separately.

Convergence of  $I$ : Using

$$\left| \frac{1}{\sqrt{1 + x^2}} - \left(1 - \frac{1}{2}x^2\right) \right| \leq cx^4, \text{ for all } x \in \mathbb{R},$$

we get

$$\begin{aligned} \mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{u}_N^n \left( \frac{1}{\sqrt{1 + |\bar{A}_N^n|^2}} - 1 + \frac{1}{2} |\bar{A}_N^n|^2 \right) \right|_{L_x^1} &\leq \frac{1}{2} \mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} |\bar{A}_N^n|_{L_x^4}^4 \\ &\leq c |\phi|_{\mathbb{L}_2^{0,1}}^4 \sum_{0 \leq (n+1)\Delta t \leq t} \Delta t^2, \end{aligned}$$

which tends to zero as  $N$  tends to infinity.

Convergence of  $II$ : we have by Cauchy-Schwarz inequality and a Hölder-type inequality:

$$\begin{aligned}
\mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{u}_N^n \left( \frac{1}{\sqrt{1+|\bar{v}_N^n|^2}} - \frac{1}{\sqrt{1+|\bar{A}_N^n|^2}} \right) \right|_{L_x^1} & \\
& \leq c \mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| |\bar{v}_N^n|^2 - |\bar{A}_N^n|^2 \right|_{L_x^1} \\
& \leq c \mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{v}_N^n + \bar{A}_N^n \right|_{L_x^2} \left| \bar{v}_N^n - \bar{A}_N^n \right|_{L_x^2} \\
& \leq \frac{c}{2\sqrt{\Delta t}} \mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{v}_N^n - \bar{A}_N^n \right|_{L_x^2}^2 \\
& \quad + \frac{c\sqrt{\Delta t}}{2} \mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{v}_N^n + \bar{A}_N^n \right|_{L_x^2}^2.
\end{aligned}$$

We then use (5.13), (5.16) and (5.17) to conclude that  $II$  tends to 0 as  $N$  tends to infinity.  
Convergence of  $III$ : Working as above, one has

$$\begin{aligned}
\mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{v}_N^n \left( \frac{1}{\sqrt{1+|\bar{A}_N^n|^2}} - 1 \right) \right|_{L_x^1} & \\
& \leq \frac{\sqrt{\Delta t}}{2} \mathbb{E} \left( \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{v}_N^n \right|_{L_x^2}^2 \right) + \frac{c}{2\sqrt{\Delta t}} \mathbb{E} \left( \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{A}_N^n \right|_{L_x^4}^4 \right).
\end{aligned}$$

Then, using again (5.17), and (5.14), the above quantity tends to 0 as  $N \rightarrow \infty$ .

Convergence of  $IV$ : Cauchy-Schwarz inequality implies:

$$\begin{aligned}
\mathbb{E} \left[ \left| \bar{v}_N^n \left( \frac{1}{\sqrt{1+|\bar{v}_N^n|^2}} - \frac{1}{\sqrt{1+|\bar{A}_N^n|^2}} \right) \right|_{L_x^1} \right] & \\
& \leq \left( \mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{v}_N^n \right|_{L_x^2}^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \sum_{0 \leq (n+1)\Delta t \leq t} \left| \bar{v}_N^n - \bar{A}_N^n \right|_{L_x^2}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Using (5.16) and (5.17), we conclude that  $IV$  tends to 0 as  $N \rightarrow \infty$ . Finally (5.79) is proved.

*Proof of (5.80).* For each  $N \in \mathbb{N}^*$ , and  $0 \leq n \leq N-1$ , observe that if we denote by  $\mathbb{E}[\cdot | \bar{\mathbb{F}}_N^n]$  the conditional expectation with respect to  $\bar{\mathbb{F}}_N^n$  (see (5.56)), we have

$$\mathbb{E} \left[ |\bar{u}_N^n \times \Delta \bar{W}_N^n|^2 | \bar{\mathbb{F}}_N^n \right] = \Delta t \sum_{i \in \mathbb{N}} |\bar{u}_N^n \times \phi_i|^2. \quad (5.81)$$

We set

$$\sum_{0 \leq (n+1)\Delta t \leq t} \bar{u}_N^n |\bar{u}_N^n \times \Delta \bar{W}_N^n|^2 - \int_0^t \sum_{i \in \mathbb{N}} \bar{u} |\bar{u} \times \phi_i|^2 ds = I + II,$$

where

$$I = \sum_{0 \leq (n+1)\Delta t \leq t} \bar{u}_N^n \left( |\bar{u}_N^n \times \Delta \bar{W}_N^n|^2 - \sum_{i \in \mathbb{N}} |\bar{u}_N^n \times \phi_i|^2 \Delta t \right),$$

and

$$II = \sum_{0 \leq (n+1)\Delta t \leq t} \sum_{i \in \mathbb{N}} \bar{u}_N^n |\bar{u}_N^n \times \phi_i|^2 \Delta t - \int_0^t \sum_{i \in \mathbb{N}} \bar{u} |\bar{u} \times \phi_i|^2 ds.$$

Developing the square of the  $L_x^2$  norm of  $I$ , we get a sum over two indices  $k \leq n$ , which after taking the expectation contains the terms:

$$\mathbb{E} \left[ \left\langle \bar{u}_N^n (|\bar{u}_N^n \times \Delta \bar{W}_N^n|^2 - \sum_{i \in \mathbb{N}} |\bar{u}_N^n \times \phi_i|^2 \Delta t), \bar{u}_N^k (|\bar{u}_N^k \times \Delta \bar{W}_N^k|^2 - \sum_{i \in \mathbb{N}} |\bar{u}_N^k \times \phi_i|^2 \Delta t) \right\rangle \right].$$

When  $k < n$ , this is equal to

$$\mathbb{E} \left[ \left\langle \bar{u}_N^n \mathbb{E} [|\bar{u}_N^n \times \Delta \bar{W}_N^n|^2 - \sum_{i \in \mathbb{N}} |\bar{u}_N^n \times \phi_i|^2 \Delta t | \bar{\mathbb{F}}_N^n], \bar{u}_N^k (|\bar{u}_N^k \times \Delta \bar{W}_N^k|^2 - \sum_{i \in \mathbb{N}} |\bar{u}_N^k \times \phi_i|^2 \Delta t) \right\rangle \right].$$

Thus, using (5.81), these terms vanish. It follows, using again (5.81), that

$$\begin{aligned} \mathbb{E}[|I|_{L_x^2}^2] &= \sum_{0 \leq (n+1)\Delta t \leq t} \mathbb{E} \left[ \left| |\bar{u}_N^n \times \Delta \bar{W}_N^n|^2 - \sum_{i \in \mathbb{N}} |\bar{u}_N^n \times \phi_i|^2 \Delta t \right|_{L_x^2}^2 \right] \\ &= \sum_{0 \leq (n+1)\Delta t \leq t} \mathbb{E} \left[ \int_{\mathcal{O}} \left( |\bar{u}_N^n \times \Delta \bar{W}_N^n|^4 - \left( \sum_{i \in \mathbb{N}} |\bar{u}_N^n \times \phi_i|^2 \right)^2 (\Delta t)^2 \right) dx \right]. \end{aligned}$$

Both terms on the right hand side are bounded by  $c(T) |\phi|_{\mathbb{L}_2^{0,1}}^4 \Delta t$  (see (5.14)), hence tend to zero as  $N$  tends to infinity.

For II, we write

$$\sum_{0 \leq (n+1)\Delta t \leq t} \sum_{i \in \mathbb{N}} \bar{u}_N^n |\bar{u}_N^n \times \phi_i|^2 \Delta t = \int_0^{\lfloor \frac{t}{\Delta t} \rfloor \Delta t} \sum_{i \in \mathbb{N}} \bar{u}_N |\bar{u}_N \times \phi_i|^2 ds,$$

and note that  $\int_{\lfloor \frac{t}{\Delta t} \rfloor \Delta t}^t \sum_{i \in \mathbb{N}} \bar{u}_N |\bar{u}_N \times \phi_i|^2 ds$  tends to zero in  $L^2(\bar{\Omega} \times \mathcal{O}; \mathbb{R}^3)$ . Moreover, we have

$$\mathbb{E} \left[ \left| \int_0^t \sum_{i \in \mathbb{N}} \bar{u}_N |\bar{u}_N \times \phi_i|^2 ds - \int_0^t \sum_{i \in \mathbb{N}} \bar{u} |\bar{u} \times \phi_i|^2 ds \right|_{L_x^2} \right] \leq c |\phi|_{\mathbb{L}_2^{0,2}}^2 \mathbb{E} \int_0^T |\bar{u}_N - \bar{u}|_{L_x^2} ds,$$

which tends to 0 as  $N \rightarrow \infty$ . We conclude that

$$\sum_{0 \leq (n+1)\Delta t \leq t} \sum_{i \in \mathbb{N}} \bar{u}_N^n |\bar{u}_N^n \times \phi_i|^2 \Delta t \xrightarrow{N \rightarrow \infty} \int_0^t \sum_{i \in \mathbb{N}} \bar{u} |\bar{u} \times \phi_i|^2 ds$$

strongly in  $L^1(\bar{\Omega}; L_x^2)$ . Eventually, (5.80) follows from

$$\begin{aligned} \int_0^t \sum_{i \in \mathbb{N}} \bar{u} |\bar{u} \times \phi_i|^2 ds &= - \int_0^t P_{\bar{u}} \left[ \sum_{i \in \mathbb{N}} ((\bar{u} \times \phi_i) \times \phi_i) \right] ds \\ &= - \int_0^t P_{\bar{u}} [F_{\phi}(\bar{u})] ds. \end{aligned}$$

■

*Proof of Corollary 5.3.* We recall that we have for almost every  $t \in [0, T]$  and for all  $N \in \mathbb{N}^*$

$$\bar{X}_N(t) = \bar{u}_N(t) - u_0 - \sum_{0 \leq (n+1)\Delta t \leq t} \Delta t \bar{w}_N^n - \sum_{0 \leq (n+1)\Delta t \leq t} (\bar{u}_N^{n+1} - \bar{u}_N^n - \bar{v}_N^n)$$

and that, up to the extraction of a subsequence,

$$\bar{u}_N \longrightarrow \bar{u} \quad \text{strongly in } L^2(\bar{\Omega} \times [0, T] \times \mathcal{O}; \mathbb{R}^3),$$

and thus without loss of generality, we can assume that for almost any  $t \in [0, T]$ ,

$$\bar{u}_N(t) \longrightarrow \bar{u}(t) \quad L^2(\bar{\Omega} \times \mathcal{O}; \mathbb{R}^3).$$

Then, by Corollary 5.2 and Proposition 5.7,

$$- \sum_{0 \leq (n+1)\Delta t \leq t} \Delta t \bar{w}_N^n - \sum_{0 \leq (n+1)\Delta t \leq t} (\bar{u}_N^{n+1} - \bar{u}_N^n - \bar{v}_N^n)$$

converges weakly in  $L^2(\bar{\Omega} \times \mathcal{O}; \mathbb{R}^3)$  to

$$- \int_0^t (\Delta \bar{u} + \bar{u}(\nabla \bar{u})^2 + \bar{u} \times \Delta \bar{u} + P_{\bar{u}^\perp} [F_\phi(\bar{u})]) ds - \int_0^t P_{\bar{u}} [F_\phi(\bar{u})] ds .$$

The continuity of  $\bar{u}$  follows from (5.77), the continuity of  $\bar{X}$  (Proposition 5.4), and the fact that  $\bar{w} \in L^1(0, T; L_x^2)$  a.s. (see Proposition 5.6). ■

*End of the proof of Theorem 5.1.* The conclusion follows by (5.77), together with (5.64): there exists a martingale solution  $(\bar{\Omega}, \bar{\mathbb{P}}, \bar{\mathcal{F}}_{t \in [0, T]}, \bar{W}, \bar{u})$  of (5.1), i.e. for any  $t \in [0, T]$ ,

$$\bar{u}(t) = u_0 + \int_0^t (\Delta \bar{u} + \bar{u}(\nabla \bar{u})^2 + \bar{u} \times \Delta \bar{u} + F_\phi(\bar{u})) ds + \int_0^t \bar{u} \times d\bar{W}(s) . \quad \blacksquare$$

## CHAPTER 6.

# *Numerical Studies*

---

In this chapter, we numerically investigate the influence of the noise on LLG. We implement the new numerical scheme proposed in Chapter 5, and we show how to use it together with a finite element discretization in space. After giving a brief proof of the counterpart of the main theorem of Chapter 5, namely the convergence of the discretization of SLLG in space and time towards a martingale solution, we evidence pathwise blow-up behaviour with jumps of the energy functional, and we show smooth evolution of the expectation.

---

# 1 The numerical Method

## (1.a) Preliminaries

In this chapter we describe our numerical method related to (SLLG) with a precession parameter  $\gamma > 0$ , and dimensionless temperature  $\epsilon > 0$ . namely

$$du = (\gamma u \times \Delta u + \epsilon F_\phi(u) + \Delta u + u|\nabla u|^2)dt + \sqrt{\epsilon}u \times dW_\phi$$

in  $\Omega \times \mathbb{R}^+ \times \mathbb{D}$ ,  $u(0) = u_0$  (plus boundary conditions which will be precised in the examples below). We focus especially on a two-dimensional domain which may be the unit disk  $\mathbb{D} = \{x \in \mathbb{R}^2, |x|_{\mathbb{R}^2} \leq 1\}$ , or the torus  $\mathbb{T}^2$ . Recall that the scheme proposed in Chapter 5 is only semi-discrete in time, therefore a space discretization is needed. We choose finite element approximation, which is well suited for numerical investigation on multidimensional domains and was already studied for deterministic LLG, see e.g. [AJ06, AKT12, BKP08, BS05].

Let  $(\mathcal{T}_h)_{h>0}$  denote a regular family of conformal triangulations of the domain  $\mathbb{D} \subseteq \mathbb{R}^2$  parametrized by the space step  $h > 0$ . For  $h > 0$  we denote by  $L = L(h) \geq 1$  the number of vertices, by  $I$  the indices  $\llbracket 1, L \rrbracket$ , and by  $(x_i)_{i \in I}$  the vertices of  $\mathcal{T}_h$ . The standard basis of the so-called  $\mathcal{P}_1$  discretization, is denoted by  $(\psi_i)_{i \in I}$ , namely the functions  $(\psi_i)$  are continuous and affine on each triangle, and satisfy  $\psi_i(x_j) = \delta_{i,j}$ , the Kronecker symbol, for all  $i, j \in I$ . We denote by  $V_h$  the finite element space:

$$V_h = \left\{ u = \sum_{i \in I} u_i \psi_i, \text{ such that } u_i \in \mathbb{R}^3 \text{ for each } i \in I \right\}, \quad (6.1)$$

and if  $u_h \in V_h$ , then we define the so-called ‘‘tangent space’’ (see [AKT12]):

$$K(u_h) = \left\{ v = \sum_{i \in I} v_i \psi_i, \forall i \in I, u_i \cdot v_i = 0 \right\}. \quad (6.2)$$

We also denote by  $\mathcal{I}_h$  the standard interpolation operator, namely:

$$\begin{aligned} \mathcal{I}_h : \mathcal{C}(\mathbb{D}; \mathbb{R}^3) &\longrightarrow V_h \\ u &\longmapsto \sum_{i \in I} u(x_i) \psi_i. \end{aligned} \quad (6.3)$$

We denote by  $\langle \cdot, \cdot \rangle_h$  the bilinear form

$$\langle \varphi, \psi \rangle_h := \int_{\mathcal{O}} \mathcal{I}_h(\varphi(x) \cdot \psi(x)) dx, \text{ for all } \varphi, \psi \in \mathcal{C}(\mathcal{O}; \mathbb{R}^3). \quad (6.4)$$

Assume that we are given a stochastic basis  $\mathfrak{B} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \hat{W})$ , where  $\hat{W}$  is a cylindrical process formally defined as

$$\hat{W}(\omega, t) = \sum_{e \in \mathcal{B}^3} \beta_e(\omega, t) e, \quad (\omega, t) \in \Omega \times [0, T], \quad (6.5)$$

where  $\{\beta_e\}$  is an i.i.d. sequence of independent  $\mathbb{R}^3$ -valued Brownian motions, and  $\mathcal{B}^3$  is an ONB of  $L^2(\mathbb{D}; \mathbb{R}^3)$ . As in the preceding chapters, we consider the  $\phi\phi^*$  Wiener process  $W_\phi$  defined by applying formally the operator  $\phi$  to  $\hat{W}$ , namely  $W_\phi := \sum_{e \in \mathcal{B}^3} \beta_e \phi e$ .

## (1.b) Approximations of white and space-regular noises

Fix  $T > 0$ , take an integer  $N \in \mathbb{Z}$ , and define the time step  $k := T/N$ . For each  $n \in \llbracket 0, N-1 \rrbracket$ , we denote by

$$\Delta \hat{W}^n = \hat{W}((n+1)k) - W(nk) \sim \mathcal{N}(0, k \text{id}_{L^2(\mathbb{D}; \mathbb{R}^3)}) . \quad (6.6)$$

and  $\Delta W_\phi^n := \phi \Delta \hat{W}$ . Since for the numerical applications we cannot work directly with these “real increments”, we shall define first some approximations  $\chi^n \approx \Delta \hat{W}^n$  and  $\xi^n \approx \Delta W_\phi^n$  in  $V_h$ , and then use it for the variational formulations (6.16)-(6.17) below.

*Approximation 1* (white noise). We approximate the increments  $\Delta \hat{W}^n$  by  $\chi^n$  which for each  $0 \leq n \leq N-1$  is the element of  $V_h$  defined by:

$$\chi^n(x) := \sum_{i \in I} \frac{\sqrt{\epsilon k} (\gamma_1^{n,i}, \gamma_2^{n,i}, \gamma_3^{n,i})}{\sqrt{\int_{\mathbb{D}} (\psi_i(x))^2 dx}} \psi_i(x) , \quad \text{for } x \text{ in } \mathbb{D} , \quad (6.7)$$

where  $\{\gamma_j^{n,i} : 0 \leq n \leq N-1, 1 \leq i \leq L, 1 \leq j \leq 3\}$  denotes an i.i.d. family of real random variables with law  $\mathcal{N}(0, 1)$  (see also [BBP13] for an approximation of this form).

Now, recall that for  $g \in H^{-1}(\mathbb{D}; \mathbb{R})$ , the boundary value problem

$$\begin{cases} -\Delta f + f = g , & \text{in } \mathbb{D} , \\ f|_{t=0} = 0 , & \text{on } \mathbb{D} , \end{cases} \quad (6.8)$$

admits a unique solution  $f = \phi g$ , such that  $\phi : H_x^{-1} \rightarrow H^1$  is bounded. Furthermore, it is a classical fact that if  $\mathcal{B}$  denotes the orthonormal basis of  $L_x^2$  associated to the eigenvectors of  $-\Delta + I$  (with the same boundary conditions as above), then

$$\sum_{e \in \mathcal{B}} |\phi e|_{H_x^1}^2 < \infty ,$$

which means that  $\phi$  is Hilbert-Schmidt from  $L_x^2$  onto  $H_0^1$ . Therefore, if  $\hat{W}$  denotes a cylindrical Wiener process, the process formally defined as  $W = \phi \hat{W}$  is  $H_0^1$ -valued, and it guarantees the boundedness of the energy functional for the corresponding solution  $u$ , see Chapter 2.

For the problem (6.8), an approximation of  $f = \phi g$  in  $V_h$  is given by finding the unique solution  $f_h = \phi_h g \in V_h$  of

$$\langle \nabla f_h, \nabla \mu \rangle_h + \langle f_h, \mu \rangle_h = \langle g, \mu \rangle_{H^{-1}, H_0^1} , \quad \forall \mu \in V_h . \quad (6.9)$$

The Lax-Milgram Lemma also applies, and therefore

$$\phi_h : H^{-1}(\mathbb{D}; \mathbb{R}) \rightarrow V_h , \quad (6.10)$$

is correctly defined by (6.9), bounded from  $H^{-1}(\mathbb{D}; \mathbb{R})$  into  $H^1(\mathbb{D}; \mathbb{R})$ , uniformly in  $h > 0$ .

*Approximation 2.* (regular noise in space) For  $n = 0, 1, \dots, N$ , an approximation  $\xi^n$  of the increments  $\Delta W_\phi^n = W_\phi((n+1)k) - W_\phi(nk)$ , will be computed as follows.

- (1) First define approximations  $\chi^n$  of  $\Delta \hat{W}^n$  as in (6.7).



(2) Then, “regularize”  $\chi^n$  by defining the element  $\xi^n$  of  $V_h$  as:

$$\xi^n = \phi_h \chi^n , \quad (6.11)$$

for each  $0 \leq n \leq N - 1$ .

*Remark 6.1.* For Approximation 2, recall that if  $V_h$  consists of the continuous piecewise polynomials of degree  $\leq 1$  with respect to  $\mathcal{T}_h$ , then we have the standard error estimate for all  $g \in H^{-1}(\mathbb{D}; \mathbb{R}^3)$  (see e.g. [Yan05]):

$$|(\phi_h - \phi)g|_{L_x^2} \leq Ch|\phi g|_{H_x^1} = Ch|g|_{H_x^{-1}} , \quad (6.12)$$

where the constant is independent of  $g$ . Hence, for all  $n \in \llbracket 1, N - 1 \rrbracket$ , we have

$$|\xi^n - \phi \chi^n|_{L_x^2} \leq Ch|\chi^n|_{H^{-1}} ,$$

so that by (6.7) we have  $\mathbb{E}[|\xi^n - \phi \chi^n|_{L_x^2}^2] \leq C(\chi^n)kh$ . Notice however that in dimension two, the “real increments”  $\Delta \hat{W}^n$  belong to  $H_x^{-1-\epsilon}$  for all  $\epsilon > 0$ , but not to  $H_x^{-1}$ , so that the constant  $C(\chi^n)$  diverges as  $h \rightarrow 0$ .

### (1.c) Practical implementation of Algorithm 5.1

In what follows, we define the operators  $\phi$  and  $\phi_h$  as in the previous subsection. For these operators, there exist  $\tilde{\phi}_h$  and  $\tilde{\phi}$  acting on scalar functions, such that  $\phi f = (\tilde{\phi} f_1, \tilde{\phi} f_2, \tilde{\phi} f_3)$ , and  $\phi_h f = (\tilde{\phi}_h f_1, \tilde{\phi}_h f_2, \tilde{\phi}_h f_3)$ , for all  $f = (f_1, f_2, f_3)$  in  $L^2(\mathbb{D}; \mathbb{R}^3)$ .

We begin this paragraph by two important remarks on the practical implementation of Algorithms 6.1-6.2 below.

*Remark 6.2.* The term  $(\text{id} - u^n \times) F_\phi(u^n)$  has vanished in the algorithms below: *there is no Itô correction*. It is due to the fact that when the regularizing operator  $\phi$  is chosen “isotropic”, namely there exists  $\tilde{\phi} : L_x^2 \rightarrow L_x^2$  Hilbert-Schmidt, such that  $\phi f = (\tilde{\phi} f_1, \tilde{\phi} f_2, \tilde{\phi} f_3)$ ,  $\forall f \in L^2(\mathbb{D}; \mathbb{R}^3)$ , then for any orthonormal basis  $\mathcal{B}$  of  $L_x^2$ , and for all  $x \in \mathbb{D}$ , a.e. :

$$F_\phi(u)(x) = - \sum_{\varepsilon \in \mathcal{B}} (\tilde{\phi} \varepsilon(x))^2 u(x) . \quad (6.13)$$

This comes from the vector identity  $\sum_{\vec{a} \in \vec{\mathcal{B}}} (\vec{u} \times \vec{a}) \times \vec{a} = -2\vec{u}$ , for all  $\vec{u} \in \mathbb{R}^3$ , and  $\vec{\mathcal{B}}$  ONB of  $\mathbb{R}^3$ . Therefore in the semi-discrete case – note that in particular  $(\text{id}_{\mathbb{R}^3} - u^n \times) F_\phi(u^n) = F_\phi(u^n)$  – the term  $\langle (\text{id}_{\mathbb{R}^3} - u^n \times) F_\phi(u^n), \varphi \rangle_{L_x^2}$  vanishes when tested against a function  $\varphi$  of the tangent space  $K(u^n)$ . In the “completely discrete” case, it is reasonable to approximate, for  $u_h = \sum_i u_i \phi_i \in V_h$ , the term  $F_\phi(u_h)$  by the finite sum

$$F_h(u_h) := \frac{1}{2} \sum_{i \in I, a \in \vec{\mathcal{B}}} \mathcal{I}_h \left( \frac{(u_h \times (\tilde{\phi}_h \psi_i) \vec{a}) \times (\tilde{\phi}_h \psi_i) \vec{a}}{\int_{\mathbb{D}} (\psi_i(x))^2 dx} \right) = - \sum_{i \in I} ((\tilde{\phi}_h \psi_i)(x_i))^2 u_i \psi_i , \quad (6.14)$$

which also vanishes when tested against elements of  $K(u^n)$ .

However, if one aim to use more general spatial correlation, e.g. considering different regularities for different directions, then  $F_\phi(u)$  may have a part which is orthogonal to  $u(t, x)$

for each  $(t, x) \in \mathbb{R}_+ \times \mathbb{D}$ . In the semi-discrete case, the variational formulation of the Algorithm 5.1 of Chapter 5 writes for all  $0 \leq n \leq N - 1$ :

$$\begin{aligned} \langle v^n - u^n \times v^n, \varphi \rangle_{L_x^2} + 2\theta \Delta t \langle \nabla v^n, \nabla \varphi \rangle_{L_x^2} \\ = -2\Delta t \langle \nabla u^n, \nabla \varphi \rangle_{L_x^2} + \Delta t \langle (\text{id} - u^n \times) F_\phi(u^n)^\perp, \varphi \rangle_{L_x^2} \\ + \langle (\text{id} - u^n \times) u^n \times \Delta W_\phi^n, \varphi \rangle_{L_x^2}, \quad \forall \varphi \in K(u^n). \end{aligned}$$

where here we have just replaced the term  $F_\phi(u^n)$  by its pointwise projection  $F_\phi(u^n)^\perp := P_u^\perp F_\phi(u)$ .

In the completely discrete case below, this corresponds to adding the term  $\langle (\text{id} - u_{h,k} \times) F_h(u_{h,k})^\perp, \varphi \rangle_h$  in the variational formulation (6.17), where  $F_h^\perp(u_{h,k})$  denotes the element obtained by applying the pointwise orthogonal projection on  $\text{Vect}\{u_{h,k}\}^\perp$  to the second term of (6.14). As mentioned in Chapter 5, due to the renormalization step, the term of the Itô correction that is along  $u$ , namely  $(F_\phi(u) \cdot u)u$ , is recovered at the limit  $\Delta t \rightarrow 0$ .

*Remark 6.3.* Since our variational formulation comes from the so-called Gilbert Form, namely

$$du - u \times du = (\Delta u + u|\nabla u|^2 + F_\phi(u))dt + (\text{id}_{\mathbb{R}^3} - u \times)u \times dW_\phi,$$

a correct transcription of Algorithm 5.1 should be given at each time step  $1 \leq n \leq N - 1$  by:

$$\left[ \begin{array}{l} \text{Find } v^n \in K(u^n), \text{ such that for all } \varphi \in K(u^n) : \\ \langle v^n - u^n \times v^n, \varphi \rangle_h + 2\theta k \langle \nabla v^n, \nabla \varphi \rangle_h = -2k \langle \nabla u^n, \nabla \varphi \rangle_h + \langle (\text{id} - u^n \times)(u^n \times \chi^n), \varphi \rangle_h, \end{array} \right.$$

In Algorithm 6.1 below, we have instead replaced the term  $\langle (\text{id} - u^n \times)(u^n \times \chi^n), \varphi \rangle$  by the simpler one  $\langle u^n \times \chi^n, \varphi \rangle$ , which, roughly speaking, rather leads to approximate the equation  $du = (\Delta u + u|\nabla u|^2 + u \times \Delta u)dt + (\text{id}_{\mathbb{R}^3} - u \times)^{-1}(u \times \circ dW)$ , namely the equation obtained by formal multiplication of  $du - u \times du = 2(\Delta u + u|\nabla u|^2)dt + u \times \circ dW$  by  $(\text{id} - u \times)^{-1}$ . However, the vectorial identity  $(\text{id} - u^n \times \cdot)^{-1} = \frac{1}{2}(\text{id}_{\mathbb{R}^3} + u^n \wedge (u^n + u^n \times \cdot))$ , and Chapter 1 - Section (1.b) justifies that both equations

$$\begin{aligned} du &= (\Delta u + u|\nabla u|^2 + u \times \Delta u)dt + (\text{id}_{\mathbb{R}^3} - u \times)^{-1}(u \times \circ dW) \\ du &= (\Delta u + u|\nabla u|^2 + u \times \Delta u)dt + u \times \circ dW \end{aligned}$$

lead formally to identical Kolmogorov equations for their respective laws.

**ALGORITHM 6.1 : SPACE-TIME WHITE NOISE.** — Take  $\theta \in (\frac{1}{2}, 1]$ . Take a final time  $T > 0$ , an integer  $N > 0$ , and define the time step  $k = T/N > 0$ . Set  $n = 0$ . Start with an initial data

$$u^0 = \sum_{i \in I} u_i^0 \psi_i \in V_h \cap \left\{ u = \sum_{i \in I} u_i \psi_i, \quad u_i \in \mathbb{R}^3, \quad |u_i|_{\mathbb{R}^3}^2 = 1, \quad \forall i \in I \right\}. \quad (6.15)$$

Then, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ :

$$\left[ \begin{array}{l} \text{find } v^n = \sum_{i \in I} v_i^n \psi_i \in K(u_h^n) \text{ such that:} \\ \left\{ \begin{array}{l} \langle v^n - u^n \times v^n, \varphi \rangle_h + 2\theta k \langle \nabla v^n, \nabla \varphi \rangle_h = -2k \langle \nabla u^n, \nabla \varphi \rangle_h + \langle u^n \times \chi^n, \varphi \rangle_h, \\ \text{for all } \varphi \in K(u^n). \end{array} \right. \\ \text{Set a.s. in } V_h : u^{n+1} := \sum_{i \in I} \frac{u_i^n + v_i^n}{|u_i^n + v_i^n|_{\mathbb{R}^3}} \psi_i. \\ \text{Reiterate until } n = N. \end{array} \right. \quad (6.16)$$

For the case of a regular noise, namely when  $W$  is a  $\phi\phi^*$ -Wiener process in  $L_x^2$ , with  $\phi$  defined by (6.8) we need a supplementary step.

ALGORITHM 6.2 : REGULAR NOISE IN SPACE. — Compute Algorithm 6.1, replacing (6.16) by the following.

$$\left[ \begin{array}{l} \text{Find } \xi^n \in V_h, v^n \in K(u^n), \\ \left\{ \begin{array}{l} \langle \nabla \xi^n, \nabla \mu \rangle_h = \langle \chi, \mu \rangle_h, \text{ for all } \mu \in V_h, \text{ and} \\ \langle v^n - u^n \times v^n, \varphi \rangle_h + 2\theta k \langle \nabla v^n, \nabla \varphi \rangle_h \\ \quad = -2k \langle \nabla u^n, \nabla \varphi \rangle_h + \langle u^n \times \xi^n, \varphi \rangle_h, \text{ for all } \varphi \in K(u^n). \end{array} \right. \\ \text{Set a.s. in } V_h : u^{n+1} := \sum_{i \in I} \frac{u_i^n + v_i^n}{|u_i^n + v_i^n|_{\mathbb{R}^3}} \psi_i. \\ \text{Reiterate until } n = N. \end{array} \right. \quad (6.17)$$

Our aim is now to state an adaptation of Theorem 5.1. We however need some uniformity assumption on the shape of the triangles to ensure that the renormalization step is compatible with the energy estimates of Chapter 5. For a given triangulation  $\mathcal{T}_h$ , we make the analytical assumption on the basis elements  $(\psi_i)$ :

$$\int_{\mathbb{D}} \nabla \psi_i \cdot \nabla \psi_j \leq 0, \text{ for all } (i, j) \in I^2, i \neq j. \quad (6.18)$$

We have the following result [ACDP04, Bar05].

**Theorem 6.1.** For meshes that satisfy the property (6.18) for the  $P_1$  approximation, we have the following finite elements counterpart of inequality (5.19): if  $v = \sum_i v_i \psi_i \in V_h$  is such that

$$\forall i \in I, |v_i|_{\mathbb{R}^3} \geq 1,$$

then

$$\int_{\mathbb{D}} \left| \nabla \mathcal{I}_h \left( \frac{v(x)}{|v(x)|_{\mathbb{R}^3}} \right) \right|_{\mathbb{R}^{3 \times 2}}^2 dx \leq \int_{\mathbb{D}} |\nabla v(x)|^2 dx. \quad (6.19)$$

Notice that the condition (6.18) is fulfilled in the two important cases:

- (i) in dimension two, when the triangulation is of Delaunay type;
- (ii) in dimension three, when all dihedral angles of the tetraedra are smaller than  $\pi/2$ .

We refer to [Alo08] and references therein for a discussion on this topic.

**Theorem 6.2** (Convergence of Algorithm 6.1). *Assume that the  $P_1$  approximation satisfies (6.18), and define  $u^n$ , the sequence solving Algorithm (6.2) with  $\xi^n := \Delta W_\phi^n$  for each  $0 \leq n \leq N - 1$ . Let  $u_{h,k}$  be the progressively measurable  $V_h$ -valued process:*

$$u_{h,k}(t) := u^n \quad \text{if } t \in [nk, (n+1)k) \quad , \quad \text{for some } n \in \{0, \dots, N-1\} \quad .$$

where  $(u^n)$  is obtained by Algorithm 6.2 with  $\xi^n$  defined as  $W_\phi((n+1)k) - W_\phi(nk)$  for each  $n \in \llbracket 0, N-1 \rrbracket$ . Then, up to a subsequence,

$$\bar{u}_{h,k} \xrightarrow{h,k \rightarrow 0} \bar{u}, \quad \text{in } L^2(\bar{\Omega} \times [0, T] \times \mathbb{D}; \mathbb{R}^3).$$

*Proof.* The proof of Theorem 6.2 follows then the same steps as that of Theorem 5.1: using the peculiar test function

$$\varphi := v^n - u^n \times \chi^n \quad ,$$

we obtain the uniform estimates:

$$\begin{aligned} \operatorname{ess\,sup}_{t \in [0, T]} \mathbb{E} \left[ |\nabla u_{h,k}(t, \cdot)|_{L_x^2}^2 \right] &\leq C \quad , \quad \mathbb{E} \left[ \int_0^T |w_{h,k}(t, \cdot)|_{L_x^2}^2 dt \right] \leq C \quad , \\ \mathbb{E} \left[ \int_0^T |v_{h,k}(t, \cdot)|_{L_x^2}^2 dt \right] &\leq Ck \quad , \quad \mathbb{E} \left[ \int_0^T |\nabla v_{h,k}(t, \cdot)|_{L_x^2}^2 dt \right] \leq Ck \quad , \end{aligned}$$

where  $v_{h,k} := v^n$ , and  $w_{h,k} := (v^n - u^n \times \chi^n)$  on  $[nk, (n+1)k)$ , with a constant  $C$  depending on  $\phi, T, u^0, \theta$ , but not on  $h, k > 0$ . The first bound on the Dirichlet energy is a consequence of the assumption (6.18), using then the same proof as that of Lemma 5.1.

By the same compactness arguments as that of Chap. 5 - sec. 4, we obtain the convergence  $\mathcal{L}(u_{h,k}) \rightarrow \mathcal{L}(\bar{u})$  up to an extraction.

Set for  $t \in [0, T]$ ,  $X_{h,k}(t) = \sum_{0 \leq (n+1)k \leq t} \mathcal{I}_h(u^n \times \xi^n)$ . Then, the identities:

$$\mathbb{E} \left[ (X_{h,k}(t) - X_{h,k}(s)) \varphi(W_\phi(k), \dots, W_\phi(nk)) \right] = 0, \quad (6.20)$$

$$\begin{aligned} \mathbb{E} \left[ \left( \langle X_{h,k}(t), a \rangle_{L_x^2} \langle X_{h,k}(t), b \rangle_{L_x^2} - \langle X_{h,k}(s), a \rangle_{L_x^2} \langle X_{h,k}(s), b \rangle_{L_x^2} \right. \right. \\ \left. \left. - \sum_{n \leq l \leq n'-1} k \langle u_{h,k}^l \times (\phi_h a), u_{h,k}^l \times (\phi_h b) \rangle_{L_x^2} \right) \varphi(W_\phi(k), \dots, W_\phi(nk)) \right] = 0 \quad , \quad (6.21) \end{aligned}$$

for  $n, n'$  with  $s \in [nk, (n+1)k)$ ,  $t \in [n'k, (n'+1)k)$ ,  $a, b \in L_x^2$ , and  $\varphi$  bounded continuous on  $L_x^2$  (and therefore on  $V_h$ ) and all  $a, b \in (V_h)^2$ , are used to show convergence of the martingale part of  $u_{h,k}$  to a stochastic integral  $\int \bar{u}(s, \bar{\omega}) \times d\bar{W}_\phi(s, \bar{\omega})$ , which, with an adaptation of Propositions 5.6-5.7, gives us the desired equation on  $\bar{u}$ .  $\blacksquare$



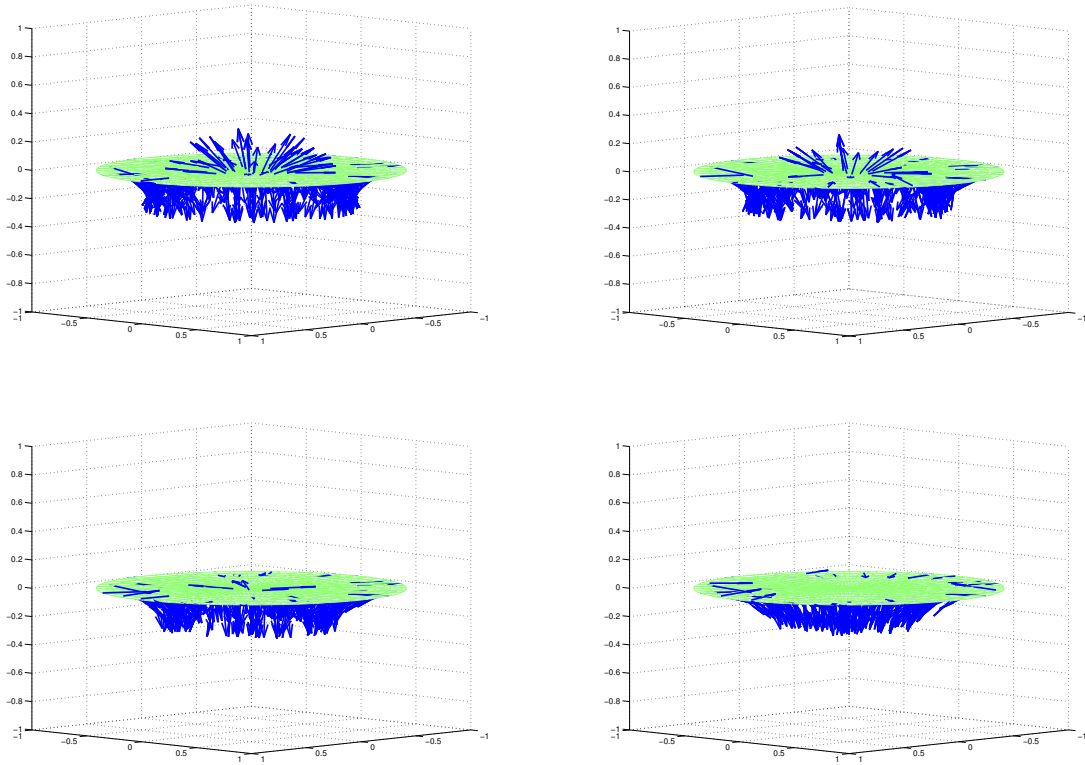


Figure 6.1 – For the overdamped case, blow-up still occurs even when for a noise of the form  $u \times (W_1(t, x), W_2(t, x), W_3(t, x))$ , instead of  $e_\varphi W_1(t, |x|)$ , see (4.7). Here we have plot the solution at times  $t = 0, t = 0.015, t = 0.05, t = 0.06$ . The parameters are  $\gamma = 0, k = 0.001, h_{\min} = 0.050518$ .

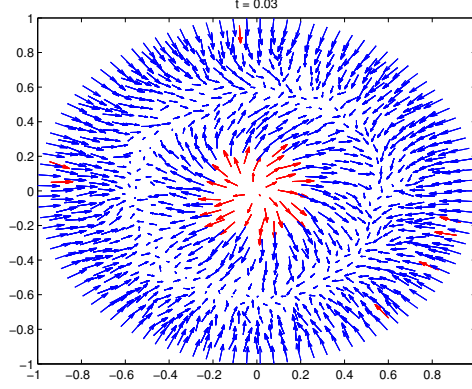


Figure 6.2 – Equivariant initial data taken for the computations of Figures 6.3 and 6.1.

In the case  $\gamma > 0, \epsilon = 0$ , we still observe blow-up. For this particular initial data, the gyromagnetic term has a regularizing effect: for all values  $\gamma \in [0, 0.425]$  we observe blow-up, whereas for  $\gamma \geq 0.43$  no such phenomena happens (even after  $T_f$ ). This motivates our peculiar choice of  $\gamma$  in (6.23). This type of regularizing effect was already observed, also for an equivariant initial data, but for another scheme, see [PV02], even though, according to [BKP08], this could be due to the particular choice of initial data, (in this work the authors choose a non equivariant initial data).

In figure 6.3, we see that for three different samples  $\omega_1, \omega_2, \omega_3$ , any scenario can happen. For  $\omega_1$  and  $\omega_3$  blow-up still occurs when  $\epsilon = 1$ . We observe that in finite time every vector is pointing downwards, evidencing the switching mechanism during which  $|\nabla u(t, \cdot)|_{L_x^\infty} \rightarrow \infty$ . For  $\omega_2$ , the noise seem to regularize the solution, no singularity forms.

We have put the plots of the energy (and energy density) in Figure 6.4.

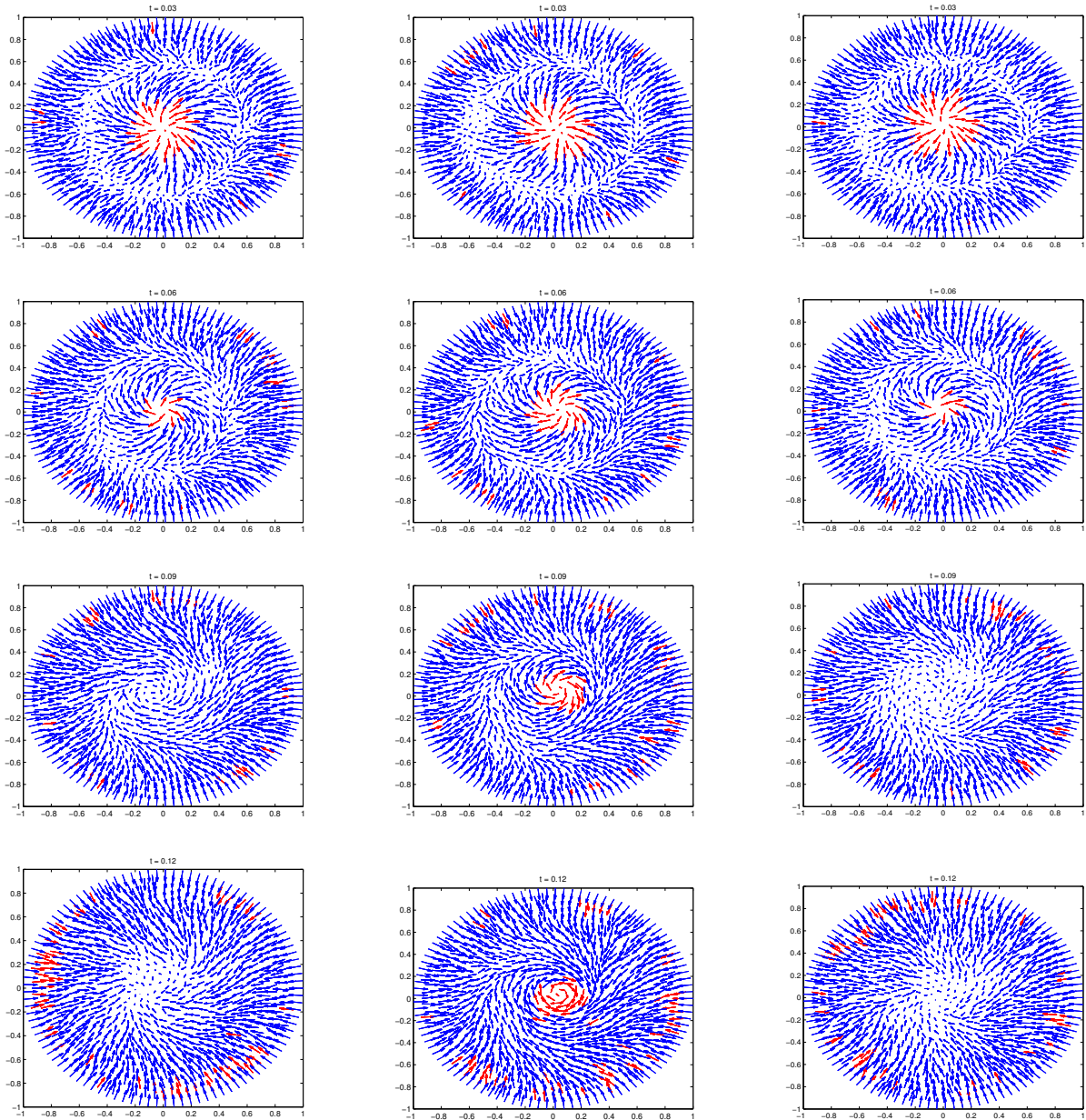


Figure 6.3 – View from above of 3 trajectories (the three columns) solution at times  $t = 0$ ,  $t = 0.015$ ,  $t = 0.05$ ,  $t = 0.06$  (rows). The color red means that  $u^3(\omega, t, x) > 0$ , whereas blue means  $u^3(\omega, t, x) \leq 0$ . The parameters are  $\gamma = 0$ ,  $k = 0.001$ ,  $h_{\min} = 0.050518$ . All solutions start with the same initial data, see figure 6.2.



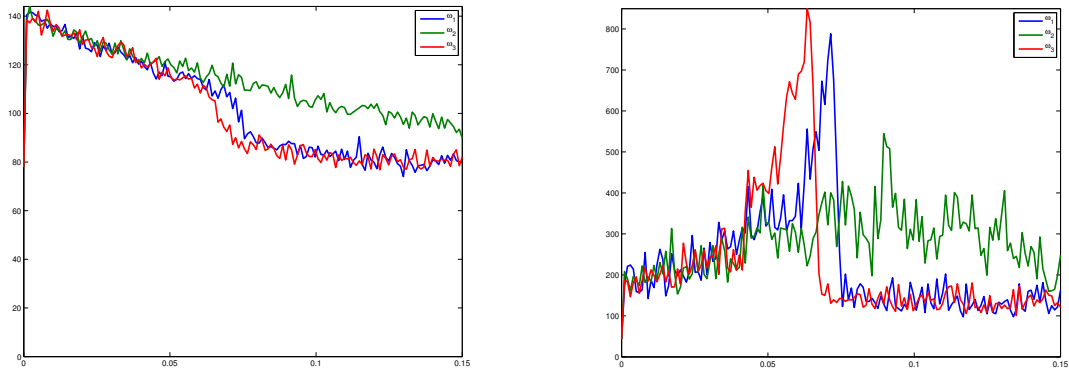


Figure 6.4 – Evolution of  $|\nabla u(t)|_{L^2_x}^2$  (left),  $|\nabla u(t)|_{L^\infty_x}^2$  (right) for the three trajectories of Example 1.

## **Bibliography**

- [ACDP04] François Alouges, Sergio Conti, Antonio DeSimone, and Yvo Pokern. Energetics and switching of quasi-uniform states in small ferromagnetic particles. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, **38**(2):235–248, 2004.
- [ADH14] François Alouges, Anne De Bouard, and Antoine Hocquet. A semi-discrete scheme for the stochastic Landau–Lifshitz equation. *Stochastic Partial Differential Equations: Analysis and Computations*, **2**(3):281–315, 2014.
- [AF03] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume **140** of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [AJ06] François Alouges and Pascal Jaisson. Convergence of a finite element discretization for the Landau–Lifshitz equations in micromagnetism. *Mathematical Models and Methods in Applied Sciences*, **16**(02):299–316, 2006.
- [AKT12] François Alouges, Evaggelos Kritsikis, and Jean-Christophe Toussaint. A convergent finite element approximation for Landau–Lifshitz–Gilbert equation. *Physica B: Condensed Matter*, **407**(9):1345–1349, 2012.
- [Alo97] François Alouges. A new algorithm for computing liquid crystal stable configurations: the harmonic mapping case. *SIAM Journal on Numerical Analysis*, **34**(5):1708–1726, 1997.
- [Alo08] François Alouges. A new finite element scheme for Landau–Lifshitz equations. *Discrete Contin. Dyn. Syst. Ser. S*, **1**(2):187–196, 2008.
- [Ama84] Herbert Amann. Existence and regularity for semilinear parabolic evolution equations. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, **11**(4):593–676, 1984.
- [Ama85] Herbert Amann. Global existence for semilinear parabolic systems. *J. reine angew. Math*, **360**:47–83, 1985.
- [Ama86] Herbert Amann. Quasilinear evolution equations and parabolic systems. *Transactions of the American Mathematical Society*, **293**(1):191–227, 1986.

- [AS92] François Alouges and Alain Soyeur. On global weak solutions for Landau–Lifshitz equations: existence and nonuniqueness. *Nonlinear Analysis: Theory, Methods & Applications*, **18**(11):1071–1084, 1992.
- [Ass75] Patrick Assouad. Espaces  $p$ -lisses et  $q$ -convexes. Inégalités de Bürkholder. *Séminaire d'Analyse fonctionnelle*, pages 1–7, 1975.
- [Bar05] Sören Bartels. Stability and convergence of finite-element approximation schemes for harmonic maps. *SIAM journal on numerical analysis*, **43**(1):220–238, 2005.
- [BBNP13a] L'ubomír Bañas, Zdzisław Brzeźniak, Mikhail Neklyudov, and Andreas Prohl. A convergent finite-element-based discretization of the stochastic Landau–Lifshitz–Gilbert equation. *IMA Journal of Numerical Analysis*, 2013.
- [BBNP13b] L'ubomír Bañas, Zdzisław Brzeźniak, Mikhail Neklyudov, and Andreas Prohl. *Stochastic Ferromagnetism–Analysis and Numerics*. De Gruyter, 2013.
- [BBP13] L'ubomír Bañas, Zdzisław Brzeźniak, and Andreas Prohl. Computational Studies for the Stochastic Landau–Lifshitz–Gilbert Equation. *SIAM Journal on Scientific Computing*, **35**(1):B62–B81, 2013.
- [BDvdH02] Michiel Bertsch, Roberta Dal Passo, and Rein van der Hout. Nonuniqueness for the Heat Flow of Harmonic Maps on the Disk. *Archive for rational mechanics and analysis*, **161**(2):93–112, 2002.
- [Ben95] Alain Bensoussan. Stochastic Navier-Stokes equations. *Acta Applicandae Mathematica*, **38**(3):267–304, 1995.
- [Ber07] Dmitri Vladimirovitch Berkov. Magnetization dynamics including thermal fluctuations. In Helmut Kronmüller and Stuart Parkin, editors, *Handbook of Magnetism and Advanced Magnetic Materials*, volume **2**, pages 795–823. Wiley Online Library, 2007.
- [BGJ12] Zdzisław Brzeźniak, Benjamin Goldys, and Terence Jegaraj. Large deviations for a stochastic Landau–Lifshitz equation, extended version. *arXiv preprint arXiv:1202.0370*, 2012.
- [BGJ13] Zdzisław Brzeźniak, Benjamin Goldys, and Terence Jegaraj. Weak solutions of a stochastic Landau–Lifshitz–Gilbert equation. *Applied Mathematics Research eXpress*, **2013**(1):1–33, 2013.
- [BGP94] Vlad Bally, István Gyöngy, and Étienne Pardoux. White noise driven parabolic SPDEs with measurable drift. *Journal of Functional Analysis*, **120**(2):484–510, 1994.
- [BKP05] Sören Bartels, Joy Ko, and Andreas Prohl. Numerical approximation of the Landau–Lifshitz–Gilbert equation and finite time blow-up of weak solutions. *preprint*, 2005.

- [BKP08] Sören Bartels, Joy Ko, and Andreas Prohl. Numerical analysis of an explicit approximation scheme for the Landau–Lifshitz–Gilbert equation. *Mathematics of Computation*, **77**(262):773–788, 2008.
- [BP99] Zdzisław Brzeźniak and Szymon Peszat. Space-time continuous solutions to SPDE’s driven by a homogeneous Wiener process. *Studia Mathematica*, **137**(3):261–299, 1999.
- [BP06] Sören Bartels and Andreas Prohl. Convergence of an implicit finite element method for the Landau–Lifshitz–Gilbert equation. *SIAM journal on numerical analysis*, **44**(4):1405–1419, 2006.
- [Bro63a] William Fuller Brown. *Micromagnetics*. Interscience, New York, 1963.
- [Bro63b] William Fuller Brown. Thermal fluctuations of a single-domain particle. *Physical Review*, **130**(5):1677, 1963.
- [Brz97] Zdzisław Brzeźniak. On stochastic convolution in Banach spaces and applications. *Stochastics: An International Journal of Probability and Stochastic Processes*, **61**(3-4):245–295, 1997.
- [BS05] L’ubomír Bañas and Marián Slodička. Space discretization for the Landau–Lifshitz–Gilbert equation with magnetostriction. *Computer methods in applied mechanics and engineering*, **194**(2-5):467–477, 2005.
- [CD91] Kung-Ching Chang and Wei-Yue Ding. A Result on the Global Existence for Heat Flows of Harmonic Maps from  $\mathbb{D}^2$  into  $\mathbb{S}^2$ . In *Nematics*, pages 37–47. Springer, 1991.
- [CDY<sup>+</sup>92] Kung-Ching Chang, Wei Yue Ding, Rugang Ye, et al. Finite-time blow-up of the heat flow of harmonic maps from surfaces. *Journal of Differential Geometry*, **36**(2):507–515, 1992.
- [CF97] Gilles Carbou and Pierre Fabrie. Comportement asymptotique des solutions faibles des équations de Landau–Lifshitz. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, **325**(7):717–720, 1997.
- [CF01a] Gilles Carbou and Pierre Fabrie. Regular solutions for Landau–Lifshitz equation in a bounded domain. *Differential and integral equations*, **14**(2):213–229, 2001.
- [CF01b] Gilles Carbou and Pierre Fabrie. Regular solutions for Landau–Lifshitz equation in  $\mathbb{R}^3$ . *Communications in Applied Analysis*, **5**(1):17–30, 2001.
- [CG89] Jean-Michel Coron and Jean-Michel Ghidaglia. Equations aux dérivées partielles. Explosion en temps fini pour le flot des applications harmoniques. *Comptes rendus de l’Académie des sciences. Série 1, Mathématique*, **308**(12):339–344, 1989.
- [CGH91] Jean-Michel Coron, Jean-Michel Ghidaglia, and Frédéric Hélein. *Nematics*, 1991.
- [Che89] Yunmei Chen. The weak solutions to the evolution problems of harmonic maps. *Mathematische Zeitschrift*, **201**(1):69–74, 1989.

- [Cim05] Ivan Cimrák. Error estimates for a semi-implicit numerical scheme solving the Landau–Lifshitz equation with an exchange field. *IMA journal of numerical analysis*, **25**(3):611–634, 2005.
- [Cim07] Ivan Cimrák. A survey on the numerics and computations for the Landau–Lifshitz equation of micromagnetism. *Archives of Computational Methods in Engineering*, **15**(3):1–37, 2007.
- [Cor90] Jean-Michel Coron. Nonuniqueness for the heat flow of harmonic maps. In *Annales de l’IHP Analyse non linéaire*, volume **7**, pages 335–344, 1990.
- [CS04] Ivan Cimrák and Marián Slodička. An iterative approximation scheme for the Landau–Lifshitz–Gilbert equation. *Journal of computational and applied mathematics*, **169**(1):17–32, 2004.
- [DD99] Anne De Bouard and Arnaud Debussche. A Stochastic Nonlinear Schrödinger Equation with Multiplicative Noise. *Communications in mathematical physics*, **205**(1):161–181, 1999.
- [DD02] Anne De Bouard and Arnaud Debussche. On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation. *Probability theory and related fields*, **123**(1):76–96, 2002.
- [DD03a] Giuseppe Da Prato and Arnaud Debussche. Ergodicity for the 3D stochastic Navier–Stokes equations. *Journal de mathématiques pures et appliquées*, **82**(8):877–947, 2003.
- [DD03b] Anne De Bouard and Arnaud Debussche. The stochastic nonlinear Schrödinger equation in  $H^1$ . 2003.
- [DD04] Anne De Bouard and Arnaud Debussche. A semi-discrete scheme for the stochastic nonlinear Schrödinger equation. *Numerische Mathematik*, **96**(4):733–770, 2004.
- [Din00] Nicolae Dinculeanu. *Vector Integration and Stochastic Integration in Banach Spaces*, volume **48**. John Wiley & Sons, 2000.
- [DL12] Robert Dautray and Jacques-Louis Lions. *Mathematical Analysis and Numerical Methods for Science and Technology: Volume 3 Spectral Theory and Applications*. Springer Science & Business Media, 2012.
- [DO06] Arnaud Debussche and Cyril Odasso. Markov solutions for the 3D stochastic Navier–Stokes equations with state dependent noise. *Journal of Evolution Equations*, **6**(2):305–324, 2006.
- [Dos77] Halim Doss. Liens entre equations différentielles stochastiques et ordinaires. *Ann. Inst. H. Poincaré Setc. B (N.S.)*, **13**(2):99–125, 1977.
- [DW07] Shijin Ding and Changyou Wang. Finite Time singularity of the Landau–Lifshitz–Gillbert Equation. *IMRN-International Mathematics Research Notices*, **2007**(12):rnm012, 2007.

- [DZ92] Giuseppe Da Prato and Jerzy Zabczyk. A note on stochastic convolution. *Stochastic Analysis and Applications*, **10**(2):143–153, 1992.
- [DZ08] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge University Press, 2008.
- [EL78] James Eells and Luc Lemaire. A report on harmonic maps. *Bulletin of the London mathematical society*, **10**(1):1–68, 1978.
- [EL88] James Eells and Luc Lemaire. Another report on harmonic maps. *Bulletin of the London Mathematical Society*, **20**(5):385–524, 1988.
- [Elw82] Kenneth David Elworthy. *Stochastic differential equations on manifolds*, volume **70**. Cambridge University Press, 1982.
- [ES64] James Eells and Joseph H. Sampson. Harmonic mappings of Riemannian manifolds. *American Journal of Mathematics*, pages 109–160, 1964.
- [FG95] Franco Flandoli and Dariusz Gatarek. Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probability Theory and Related Fields*, **102**(3):367–391, 1995.
- [Fre95] Alexandre Freire. Uniqueness for the harmonic map flow in two dimensions. *Calculus of Variations and Partial Differential Equations*, **3**(1):95–105, 1995.
- [Fre96] Alexandre Freire. Correction to “uniqueness for the harmonic map flow from surfaces to general targets”. *Commentarii Mathematici Helvetici*, **71**(1):330–337, 1996.
- [G<sup>+</sup>85] Crispin W. Gardiner et al. *Handbook of stochastic methods*, volume **4**. Springer Berlin, 1985.
- [GD08] Boling Guo and Shijin Ding. *Landau–Lifshitz Equations*. World Scientific Singapore, 2008.
- [GH93] Boling Guo and Min-Chun Hong. The Landau–Lifshitz equation of the ferromagnetic spin chain and harmonic maps. *Calculus of Variations and Partial Differential Equations*, **1**(3):311–334, 1993.
- [Gil55] Thomas L. Gilbert. A Lagrangian formulation of the gyromagnetic equation of the magnetization field. *Phys. Rev.*, **100**:1243, 1955.
- [Gil04] Thomas L. Gilbert. A phenomenological theory of damping in ferromagnetic materials. *Magnetics, IEEE Transactions on*, **40**(6):3443–3449, 2004.
- [GK96] István Gyöngy and Nicolai Krylov. Existence of strong solutions for Itô’s stochastic equations via approximations. *Probability theory and related fields*, **105**(2):143–158, 1996.
- [GLT13] Benjamin Goldys, Kim-Ngan Le, and Thanh Tran. A finite element approximation for the stochastic Landau–Lifshitz–Gilbert equation. 2013.

- [GP<sup>+</sup>09] Boling Guo, Xueke Pu, et al. Stochastic Landau–Lifshitz equation. *Differential and Integral Equations*, **22**(3/4):251–274, 2009.
- [GPL98] José Luis García-Palacios and Francisco J Lázaro. Langevin-dynamics study of the dynamical properties of small magnetic particles. *Physical Review B*, **58**(22):14937, 1998.
- [Gri69] Pierre Grisvard. Équations différentielles abstraites. *Ann. Sci. École Norm. Sup.*(4), **2**(3):311–395, 1969.
- [Hai14] Martin Hairer. A theory of regularity structures. *Inventiones mathematicae*, **198**(2):269–504, 2014.
- [Ham75] Richard S. Hamilton. *Harmonic maps of manifolds with boundary*, volume **471**. Springer, 1975.
- [Har04] Paul Harpes. Uniqueness and bubbling of the 2-dimensional Landau–Lifshitz flow. *Calculus of Variations and Partial Differential Equations*, **20**(2):213–229, 2004.
- [Hél90] Frédéric Hélein. Régularité des applications faiblement harmoniques entre une surface et une sphere. *CR Acad. Sci. Paris Sér. I Math*, **311**(9):519–524, 1990.
- [Hél96] Frédéric Hélein. *Applications harmoniques, lois de conservation et repères mobiles*. Diderot, 1996.
- [HRW12] Martin Hairer, Marc D. Ryser, and Hendrik Weber. Triviality of the 2D stochastic Allen-Cahn equation. *Electron. J. Probab*, **17**(39):1–14, 2012.
- [KC89] Chang Kung-Ching. Heat flow and boundary value problem for harmonic maps. In *Annales de l’IHP Analyse non linéaire*, volume **6**, pages 363–395, 1989.
- [KMM<sup>+</sup>99] Till Kamppeter, Franz G. Mertens, Esteban Moro, Angel Sánchez, and Alan R. Bishop. Stochastic vortex dynamics in two-dimensional easy-plane ferromagnets: Multiplicative versus additive noise. *Physical Review B*, **59**(17):11349, 1999.
- [KORVE07] Robert V. Kohn, Felix Otto, Maria G. Reznikoff, and Eric Vanden-Eijnden. Action minimization and sharp-interface limits for the stochastic Allen-Cahn equation. *Communications on pure and applied mathematics*, **60**(3):393–438, 2007.
- [KP06] Martin Kružík and Andreas Prohl. Recent developments in the modeling, analysis, and numerics of ferromagnetism. *SIAM review*, **48**(3):439–483, 2006.
- [KRVE05] Robert V. Kohn, Maria G. Reznikoff, and Eric Vanden-Eijnden. Magnetic elements at finite temperature and large deviation theory. *Journal of nonlinear science*, **15**(4):223–253, 2005.
- [KVB<sup>+</sup>14] Evaggelos Kritsikis, Adrien Vaysset, Liliana D. Buda-Prejbeanu, François Alouges, and Jean-Christophe Toussaint. Beyond first-order finite element schemes in micromagnetics. *J. Comput. Phys.*, **256**:357–366, 2014.

- [Le690] Jorge A. León. On equivalence of solution to stochastic differential equation with antipating evolution system. *Stochastic Analysis and Applications*, **8**(3):363–387, 1990.
- [LLVE08] Tony Lelièvre, Claude Le Bris, and Eric Vanden-Eijnden. Analyse de certains schémas de discrétisation pour des équations différentielles stochastiques contraintes. *Comptes Rendus Mathématique*, **346**(7):471–476, 2008.
- [LM68] Jacques-Louis Lions and Enrico Magenes. *Problèmes aux limites non homogènes: et applications. 2*. Dunod, 1968.
- [LN98] Jorge A. León and David Nualart. Stochastic evolution equations with random generators. *Annals of probability*, pages 149–186, 1998.
- [LSU68] Olga Ladyzhenskaya, Vsevolod Solonnikov, and Nina Uraltseva. Linear and quasilinear parabolic equations of second order. *Translation of Mathematical Monographs, AMS, Rhode Island*, 1968.
- [Mel05] Christof Melcher. Existence of Partially Regular Solutions for Landau–Lifshitz Equations in  $\mathbb{R}^3$ . *Communications in Partial Differential Equations*, **30**(4):567–587, 2005.
- [Mur81] François Murat. Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, **8**(1):69–102, 1981.
- [Née46] Louis Néel. Bases d’une nouvelle théorie générale du champ coercitif. In *Annales de l’Universite de Grenoble*, volume **22**, pages 299–343, 1946.
- [NP13] Mikhail Neklyudov and Andreas Prohl. The role of noise in finite ensembles of nanomagnetic particles. *Archive for Rational Mechanics and Analysis*, **210**(2):499–534, 2013.
- [Paz83] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983.
- [PH] Olivier Pironneau and Frédéric Hecht. FreeFEM (software file). *On the web* <http://www-rocq.inria.fr>.
- [PV02] Francesca Pistella and Vanda Valente. Numerical study of the appearance of singularities in ferromagnets. *Advances in Mathematical Sciences and Applications*, **12**(2):803–816, 2002.
- [PW83] Gareth C. Price and David Williams. Rolling with “slipping”: I. *Séminaire de probabilités de Strasbourg*, **17**:194–197, 1983.
- [Rez04] Maria G. Reznikoff. *Rare events in finite and infinite dimensions*. PhD thesis, New York University, 2004.
- [Rom11] Marco Romito. Uniqueness and blow-up for the noisy viscous dyadic model. *arXiv preprint arXiv:1111.0536*, 2011.



- [Rot84] Franz Rothe. *Global solutions of reaction-diffusion systems*, volume **1072**. Springer-Verlag Berlin, 1984.
- [Sko14] Anatoliy Volodymyrovych Skorokhod. *Studies in the theory of random processes*, volume **7021**. Courier Corporation, 2014.
- [Sob61] Pavel Evseyevich Sobolevskii. Equations of parabolic type in a Banach space. *Trudy Moskovskogo Matematicheskogo Obshchestva*, **10**:297–350, 1961.
- [SSB86] Pierre-Louis Sulem, Catherine Sulem, and Claude Bardos. On the continuous limit for a system of classical spins. *Communications in Mathematical Physics*, **107**(3):431–454, 1986.
- [Ste70] Elias M. Stein. *Singular integrals and differentiability properties of functions*, volume **2**. Princeton university press, 1970.
- [Str85] Michael Struwe. On the evolution of harmonic mappings of Riemannian surfaces. *Commentarii Mathematici Helvetici*, **60**(1):558–581, 1985.
- [Str96] Michael Struwe. Geometric evolution problems. *IAS/Park City Mathematics series*, **2**:1, 1996.
- [Sus77] Hector J. Sussmann. An interpretation of stochastic differential equations as ordinary differential equations which depend on the sample point. *Bull Am. Math. Soc.*, **83**(2):296–298, 1977.
- [Tan60] Hiroki Tanabe. On the equations of evolution in a Banach space. 1960.
- [Tar83] Luc Tartar. The compensated compactness method applied to systems of conservation laws. In *Systems of nonlinear partial differential equations*, pages 263–285. Springer, 1983.
- [Top02] Peter Topping. Reverse bubbling and nonuniqueness in the harmonic map flow. *International Mathematics Research Notices*, **2002**(10):505–520, 2002.
- [Van81] Nico G. Van Kampen. Itô versus Stratonovich. *Journal of Statistical Physics*, **24**(1):175–187, 1981.
- [Vis85] Augusto Visintin. On Landau–Lifshitz equations for ferromagnetism. *Japan journal of applied mathematics*, **2**(1):69–84, 1985.
- [Wal86] John B. Walsh. *An introduction to stochastic partial differential equations*. Springer, 1986.
- [Wat95] George Neville Watson. *A treatise on the theory of Bessel functions*. Cambridge university press, 1995.
- [Wen69] Henry Christian Wente. An existence theorem for surfaces of constant mean curvature. *Journal of Mathematical Analysis and Applications*, **26**(2):318–344, 1969.

- [WI81] Shinzo Watanabe and Nobuyuki Ikeda. *Stochastic differential equations and diffusion processes*. Elsevier, 1981.
- [Yan05] Yubin Yan. Galerkin finite element methods for stochastic parabolic partial differential equations. *SIAM journal on numerical analysis*, **43**(4):1363–1384, 2005.
- [YSB98] Chen Yunmei, Ding Shijin, and Guo Boling. Partial regularity for two dimensional Landau–Lifshitz equations. *Acta Mathematica Sinica*, **14**(3):423–432, 1998.
- [YW<sup>+</sup>71] Toshio Yamada, Shinzo Watanabe, et al. On the uniqueness of solutions of stochastic differential equations. *Journal of Mathematics of Kyoto University*, **11**(1):155–167, 1971.





**Résumé.** Cette thèse porte sur l'influence d'un bruit Gaussien dans l'équation de Landau-Lifshitz-Gilbert Stochastique (SLLG). Il s'agit d'une équation aux dérivées partielles stochastique non linéaire, avec une contrainte non convexe sur le module des solutions. Le chapitre 1 se consacre à la solvabilité locale de SLLG. Utilisant les propriétés classiques de l'intégration stochastique dans un espace de Banach, nous proposons une formulation *mild*, et donnons l'existence et l'unicité d'une solution locale en dimension  $n \geq 2$ , pour un bruit Gaussien régulier en espace, dans le cas suramorti. Par la suite, nous effectuons une étude spécifique de la dimension deux d'espace. Le chapitre 2 porte sur l'existence d'une solution forte, au sens probabiliste, en 2D. Une méthode de compacité nous permet d'obtenir une solution globale en temps, de manière unique. Le chapitre 3 s'intéresse à l'unicité des solutions faibles en 2D. Un résultat déterministe donnait l'unicité en supposant l'énergie décroissante au cours du temps, hypothèse impossible dans le cas bruité. Néanmoins, nous montrons l'unicité des solutions faibles dont l'énergie vérifie une propriété de *sur-martingale*. Le chapitre 4 donne l'existence, sous l'hypothèse de suramortissement, de solutions explosives en temps fini. Contrairement au cas déterministe, une singularité peut apparaître avec probabilité positive, *quelle que soit la donnée initiale choisie*. Revenant ensuite au cas d'une dimension quelconque, nous proposons au chapitre 5 un nouveau schéma numérique semi-discrétisé en temps, préservant de manière *exacte* le module des solutions, et nous montrons sa convergence en loi. Ce travail a fait l'objet d'un article, écrit en collaboration avec F. Alouges et A. De Bouard. Ce schéma est ensuite implémenté dans le chapitre 6 à l'aide d'éléments finis. Nous donnons une méthode pratique pour approcher un bruit régulier en espace. Le phénomène d'explosion est ensuite observé numériquement, ce malgré la présence d'un terme gyromagnétique, et d'un bruit plus général qu'au chapitre 4.

**Abstract.** This thesis is devoted to the influence of a noise term in the stochastic Landau-Lifshitz-Gilbert Equation (SLLG). It is a nonlinear stochastic partial differential equation with a non-convex constraint on the modulus of the solutions. We study in chapter 1 the question of local solvability. Using classical properties of stochastic integration with Banach space-valued processes, we propose a *mild* formulation, and give the existence and uniqueness of a local solution in any dimension, for a space-regular Gaussian noise. We focus then on the specific study of a two-dimensional space domain. Chapter 2 deals with the existence of a strong solution, in the probabilistic sense. A compactness method allows to obtain a global solution, in a unique way. Chapter 3 gives uniqueness of weak solutions, provided that the energy satisfies a *supermartingale* property. This is the stochastic counterpart of a known deterministic result giving the uniqueness of weak solutions, knowing that the energy decreases. Chapter 4 gives the existence, in the so-called “overdamped case”, of solutions that blow-up in finite time. We prove that, unlike the deterministic case, a singularity may appear with positive probability, *regardless of the initial data chosen*. Then we return to the case of general dimension of space, providing in chapter 5 a new time semi-discrete scheme for SLLG which preserves *exactly* the local constraint on the magnitude, and we show its convergence in law. This chapter is based on an article in collaboration with F. Alouges and A. De Bouard. In Chapter 6, we show how to implement it with a finite element discretization in space, and we give a practical method for approaching a regular noise in this framework. We also evidence numerical blow-up of the solutions, despite the presence of a gyromagnetic term, and of a more general noise than that of Chapter 4.