Hamilton Jacobi Bellman Approach for some applied optimal control problems.

Mohamed Assellaou

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Mohamed ASSELLAOU

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Mme.	Marianne Akian	Présidente du Jury
M.	Olivier Bokanowski	Co-directeur de thèse
M.	Jean-Baptiste Caillau	Examinateur
M.	Kristian Debrabant	Rapporteur
M.	Roberto Ferretti	Examinateur
M.	Nicolas Forcadel	Examinateur
M.	Matthias Gerdts	Rapporteur
Mme	Hasnaa Zidani	Co-directrice de thèse

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Abbreviations

DPP	Dynamic Programming Principle.
HJ	Hamilton Jacobi.
HJB	Hamilton Jacobi Bellman.
USC	Upper semi continuous.
LSC	Lower semi continuous.

Notations

$\mathbb{R}^d$	Euclidean d-dimensional space.
$\mathbb{R}^{d\times n}$	Space of $(d \times m)$ real matrices.
$\mathcal{K}$	Closed subset of state constraints.
$\mathcal{C}$	Closed target set.
$\mathcal{E}pi_\Phi$	Epigraph of Φ.
$\mathcal{S}^d$	Space of symetric matrices in $\mathbb{R}^d \times d$.
Ω	Space of realizations ω.
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space.
$\{\mathcal{F}_t\}_{t\geq 0}$	Filtration.
$W(.)$	Brownian motion.
$B_r(x)$	Ball of center x and radius r.
$d_\mathcal{C}(.)$	Euclidean distance function to the set $\mathcal{C}$.
$1_\mathcal{C}(.)$	Indicator function for the set $\mathcal{C}$.
$\bar{\mathcal{J}}^{2,+}v, \bar{\mathcal{J}}^{2,-}v$	Closed semi jets of v.
$D\Phi$	Gradient of Φ.
$D^2\Phi$	Hessian of Φ.

Dédicace à ma famille

CHAPITRE I

GENERAL INTRODUCTION

The present thesis provides theoretical and numerical contributions to the Hamilton-Jacobi-Bellman approach for optimal control problems. The main objetive is to extend in practice this approach for problems with unusual forms.

Optimal control is a mathematical branch of the control theory used in many engineering disciplines such as mechanics, electrics, etc... Its aim is to find a control law for a controlled stochastic or ordinary dynamical system while minimizing or maximizing some criterion.

Deterministic optimal control has been closely linked to industrial applications since its inception in the 1950s, starting with the aerospace, among which was the problem of an optimal trajectory of an airplane. Stochastic optimal control will appear later in the 70s in finance. Merton in [84] studied the portfolio optimization and Black and Scholes [26] introduced the notion of financial model. The optimal control problems can be approched by the method of dynamic programming developed by Bellman in [22, 23, 21] which states that the value function associated with the optimal control problem satisfies a particular equality called a *dynamic programming principle* (DPP). This principle essentially allows to solve recursively an optimal control problem by its decomposition into a sequence of subproblems.

If the value function is smooth enough, then one can characterize it by an infinitesimal version of the dynamic programming principle, the *Hamilton Jacobi Bellman Partial differential equation* (HJB). For deterministic problems, the value function satisfies an HJB equation of first order while the for stochastic case, it satisfies a second order HJB equation. In general, due to the lack of regularity, the value function can not be characterized as a solution of a PDE in any classical sense. In [45, 46], Crandall and Lions propose a weak version for general solutions by introducing the notion of viscosity solution. The theory of viscosity solutions was extended to second order by Lions in [82, 81]. This theory provides a good context for proving existence, uniqueness and stability for a large class of non-linear partial differential equations including HJB equations (see [69, 70, 71, 73]). In this work, we will refer to Bardi and Capuzzo-Dolcetta [12] and Barles [13] for the first order case and Flemming and Soner [61] and Crandall, Ishii and Lions in [44] for the second order case.

The viscosity theory gives a suitable framework for dealing with numerical solutions of nonlinear HJB equations. Many numerical approaches were adapted to compute the weak solutions of PDE in the viscosity sense. In particular, Finite Difference (see [47, 97]) and Semi Lagrangian (see [55, 58]) methods were developed for first order

HJB equations. For the second order case, the first convergence result of numerical methods to viscosity solutions of second order HJB equations was given by Barles-Souganidis (see [17]). Since then, the field of error estimates for such numerical solutions has been of growing interest. Krylov [75] obtained error bounds for the second order HJB equation for constant diffusion coefficient. The case of diffusion coefficient depending on time, space and also on the control variable was developed in [14, 15, 16]. Several other extensions have been analysed in the literature, see [31] for stopping-game problems, [32] for impulsive control systems, [39, 24, 25] for integro partial differential HJB equations, and [34] for a general class of coupled HJB systems.

These early works concentrated on problems without constraints whereas in many applications, it is more crucial to take into account some state constraints. In the deterministic setting, the state-constrained optimal control problem has been discussed in the context of controllability assumptions and the associated value function has been characterized as the unique viscosity solution of a HJ equation. First, Soner introduced the so-called "Inward pointing" constraint qualification (IPQ) (see [98]- [99]). The IPQ condition allows to approximate each trajectory that reaches the boundary of the constraints set by a sequence of *viable trajectories*. Later, Frankowska et al formulated in [63]-[64], the "outward pointing" constraint condition (OPQ). The OPQ condition states that each point on the boundary of the set of constraints can be reached by a trajectory coming from the interior of this set. Since these conditions are not always satisfied, alternative ways using viability theory have been explored (see [11, 40, 41, 8]). In [3], Altarovici and coauthors propose to describe the value function of the state-constrained control problem by means of a Lipschitz continuous value function of an auxiliary control problem free of state constraints.

In the stochastic framework, the so-called chance constraints play an important role. In particular, the problem of the *safety region* defined as the set from which it is possible to reach a target set, with threshold probability is one of the most relevant applications in this field. For discrete time stochastic systems, this backward reachable set has been analysed and characterized by an adequate stochastic optimal control problem in [1] and [2]. In this case, the control problem is solved using the dynamic programming approach. In the context of financial mathematics, the problem of characterizing the backward reachable set with a given probability was first introduced by Föllmer and Leukert [62]. This problem was also studied and converted into the class of stochastic target problems by Touzi, Bouchard and Elie in [33].

The purpose of this thesis is to use the Hamilton Jacobi Bellman approach to study some optimal control problems from theoretical and computational points of view. First, we present in Chapter 2 the general background of HJB approach used in this thesis. In Chapter 3, a stochastic optimal control problem of the Mayer form with unbounded and discontinuous value function is considered. The “Krylov regularization” approach is extended here to obtain the error estimates using some refined consistency estimates. Then, the analysis of the reachability problem for stochastic systems is considered in Chapter 4 where we use essentially the level set approach and our results of error estimate for discontinuous value function. The Chapter 5 is devoted to the study of a state-constrained optimal control problem with a maximum cost. We introduce the auxiliary control problem to deal with the state contraints and we derive some HJB equations. Here, we are interested, in particular, in the analysis of the optimal feedback control and the associated trajectories for which we give some important results. The problem discussed in Chapter 6 is motivated by a real application : the abort landing in presence of windshear. In particular, we reconstruct the optimal trajectories and the associated optimal feedback controls using several algorithms.

Part 1 : Error estimate of second order HJB equations. Application :Probabilistic reachability analysis.

This part is detailed in Chapters 3 and 4. The first contribution of this part is to study the numerical approximations of unbounded and discontinuous value function associated with some stochastic control problems. We derive error estimates for monotone schemes based on a Semi-Lagrangian method (or more generally in the form of a Markov chain approximation). To achieve our goal, we follow two steps. The first one consists in determining the error estimates between our control problem with discontinuous function and a regularized control problem and the second step consists in considering the error estimate of the numerical scheme of the regularized problem.

More precisely, let $T > 0$ be a fixed final horizon. Consider a controlled process $X^u_{t,x}$ satisfying :

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \quad \forall s \in [t, T], \\ X(t) = x, \end{cases} \tag{1.1}$$

where the diffusion σ and drift b are two Lipschitz continuous functions, $W(\cdot)$ is the classical Brownian motion, and u is a control function that takes values in a compact subset U of $\mathbb{R}^m$ $(q \geq 1)$. Under suitable assumptions on b, σ and on $\mathcal{U}$, equation (1.1) admits a unique solution (see Chapter 3 for precise assumptions).

Consider the following control problem :

$$\vartheta(t, x) := \sup_{u \in \mathcal{U}} \mathbb{E}\big[\Phi(X^u_{t,x}(T))\big], \tag{1.2}$$

where $\Phi : \mathbb{R}^d \to \mathbb{R}$ is measurable, with linear growth. In this thesis, we are interested in error estimates of numerical approximations of ϑ.

Let us introduce a family of Lipschitz continuous functions $(\Phi_\varepsilon)_\varepsilon$ converging pointwisely to Φ. Then, the value function ϑ can be approximated by the value functions ϑ_ε defined as :

$$\vartheta_\varepsilon(t, x) := \sup_{u \in \mathcal{U}} \mathbb{E}\big[\Phi_\varepsilon(X^u_{t,x}(T))\big].$$

It is known that under quite general assumptions on the data and on Φ_ε, one can show that ϑ_ε converges pointwisely towards ϑ, when $\varepsilon \to 0$. However, the error estimate $\vartheta - \vartheta_\varepsilon$ is not a very classical result. Here, the error estimate of $\vartheta - \vartheta_\varepsilon$ depends on the measure of the set where the two functions Φ and Φ_ε differ. The result that will be studied here is obtained under an ellipticity condition of the diffusion matrix, i.e

- The diffusion σ depends only on (t, x) and there exists a real number $\Lambda \geq 1$, such that :

$$\forall (t,x) \in (0,T) \times \mathbb{R}^d, \quad \Lambda I_d \geq \sigma(t,x)\sigma(t,x)^T \geq \Lambda^{-1} I_d, \tag{1.3}$$

where I_d is the identity matrix and the inequalities (1.3) are in the sense of symmetric matrices : $\Lambda \|\xi\|^2 \geq \langle \xi, \sigma\sigma^T \xi\rangle \geq \Lambda^{-1}\|\xi\|^2$, $\forall \xi \in \mathbb{R}^d$.

Thanks to the bounds on the *density of probability* of the process $X^u_{t,x}(\cdot)$, for a given $(t,x) \in [0,T)$ and an admissible control $u \in \mathcal{U}$, the error estimate between ϑ and the approximated value function ϑ_ε can be achieved using some technical results. Thus, the following estimate holds, for every $0 \leq t < T$, $x \in \mathbb{R}^d$, and $0 < \varepsilon < \varepsilon_0$ ($\varepsilon_0 \in]0,1]$), there exists $C_0 > 0$, such that :

$$|\vartheta(t,x) - \vartheta_\varepsilon(t,x)| \leq C_0 \frac{1 + |x|^2 + |\log \varepsilon|}{(T-t)^{d/2}}\ \varepsilon.$$

The second step in the approximation of ϑ is to discretize the Hamilton-Jacobi-Bellman equation satisfied by ϑ_ε. Indeed the value function ϑ_ε is the unique Lipschitz continuous viscosity solution of :

$$\begin{aligned} -\partial_t v + \mathcal{H}(t,x,Dv,D^2v) = 0 \qquad & \text{in } (0,T) \times \mathbb{R}^d, \\ v(T,x) = \phi(x) \quad & \text{in } \mathbb{R}^d, \end{aligned}$$

where $\mathcal{H}(t,x,p,Q) := \sup_{a \in U} \left(-b(t,x,a)\cdot p - \mathrm{Tr}([\sigma\sigma^{\mathsf{T}}](t,x,a)Q)\right)$.

In the case where the drift b and the diffusion σ are bounded and where the value function ϑ_ε is itself bounded, error estimates of monotone schemes have been obtained first by Krylov [75] for the case where σ is a constant function. These results were developed further in [14, 15, 16] introducing new tools that allow to consider the case where σ can depend on time, space and also on the control variable. Several other extensions of the theory have been analysed in the literature. Let us mention some of these extensions for stopping-game problems [31], impulsive control systems [32], for integro partial differential HJB equations [39, 24, 25], and for a general class of coupled HJB systems [34]. Note also that the case of fully uniformly elliptic operators have been also studied by Cafarelli-Souganidis [37] using a different approach than the one introduced by Krylov.

Let us point out that the result given for the error estimates between the numerical scheme and the value function of the regularized value function is independent from the ellipticity condition and the diffusion may depend on the control. We aim to give new error estimates for Semi-Lagrangian schemes [38], in the case of unbounded Lipschitz continuous b and σ (as well as the solution v itself).

Let $h = dt > 0$ be a given time step. We define a semi-discrete scheme in its abstract form as (for $x \in \mathbb{R}^d$) :

$$V^N(x) = \phi(x)$$

and, for every $n = N, \dots, 1$,

$$V^{n-1}(x) = \mathcal{S}^h(t_n, x, V^n),$$

with, for any $t_n \in [0, T]$, $x \in \mathbb{R}^d$.

Under some assumptions, we state the following results on the error estimates of the semi discrete scheme, i.e, there exists $C \geq 0$, $\forall n \in [0, \dots, N]$,

$$|V^n(x) - v(t_n, x)| \leq C\,(1 + |x|)^{7/4}\, h^{1/4}. \tag{1.5}$$

The above result (1.5) is an extension to the error estimates known in the literature for bounded Hölder continuous value functions with bounded and Lipschitz continuous drift b and diffusion σ, see [14, 16, 49]. The proof given here is based on classical shaking and regularization techniques introduced by Krylov [75, 76] combined with a precise consistency estimate and an interpretation of the numerical scheme as value function of a discrete-time control problem.

Moreover, for a spatial discretization of $\mathbb{R}^d$, denote by V^Δ the solution of the fully discrete scheme. Then, under some regularity assumptions, we get the existence of $L > 0$ such that for every $R > 0$, we have :

$$\|v - V^\Delta\|_{L^\infty(\mathbb{B}_R)} \leq L\left(R^{7/4}h^{1/4} + \frac{|\Delta x|}{h}\right).$$

The study of the error estimates for unbounded and discontinuous value functions is motivated by several problems involving functions which are not necessary continuous and bounded. Among these problems, we have the characterization of the probabilistic backward reachable sets. In fact, the latter can be characterized using the level set and the HJB approaches making use of the results on the above error estimates.

The second contribution consists in studying the approximation of chance-constrained reachable sets. These sets will be characterized as level sets of a discontinuous value function associated with an adequate stochastic control problem.

More clearly, let $\mathcal{C}$ be a non-empty subset of $\mathbb{R}^d$ ("the target"). Let $\rho \in [0, 1[$ and $t \leq T$. Consider the backward reachable set under probability of success ρ, that is, the set of initial points x for which the probability that there exists a process $X^u_{t,x}$ solution of (1.1), associated with an admissible control $u \in \mathcal{U}$ and that reaches $\mathcal{C}$ at time T is higher than ρ :

$$\Omega^\rho_t = \left\{ x \in \mathbb{R}^d \,\middle|\, \exists u \in \mathcal{U},\ \mathbb{P}[X^u_{t,x}(T) \in \mathcal{C}] > \rho \right\}.$$

The sets Ω_t^ρ can be characterized by using the *level-set approach*. This approach has been introduced in [92] to model front propagation problems. Then, the method has attracted a big interest for studying backward reachable sets of continuous non-linear dynamical systems under general conditions, see [87, 28] and the references therein. The idea of using the level set approach in discrete time stochastic setting has been also considered in [1, 2, 6]. In this case, the value function is obtained by solving the dynamic programming principle.

It is important to observe that Ω_t^ρ is equivalent to :

$$\Omega_t^\rho = \left\{ x \in \mathbb{R}^d \;\middle|\; \exists u \in \mathcal{U},\; \mathbb{E}[\mathbb{1}_{\mathcal{C}}(X_{t,x}^u(T))] > \rho \right\}.$$

Moreover, by considering the control problem (1.2) with $\phi(x) := \mathbb{1}_{\mathcal{C}}(x)$, it is possible to show that for every $\rho > 0$ and every $t \in [0,T]$, the backward reachable set Ω_t^ρ is given by the *level-set* :

$$\Omega_t^\rho = \{x \in \mathbb{R}^d, \vartheta(t,x) > \rho\}.$$

In the present thesis, we are interested in the approximation of the probabilistic backward reachable sets for time-continuous stochastic processes. We analyse the approach and we provide error estimates between the exact sets and their numerical approximation. More precisely, we consider the following "regularized" control problem :

$$\vartheta^\varepsilon(t,x) := \sup_{u\in\mathcal{U}} \mathbb{E}[\Phi^\varepsilon(X_{t,x}^u(T))],$$

where Φ^ε is a "regularized" indicator function. Let us denote by $\vartheta^{\varepsilon,\Delta}$ a numerical approximation of ϑ^ε obtained by solving the fully discretized scheme. In the aim to obtain an error estimate of $\vartheta - \vartheta^{\varepsilon,\Delta}$, we obtain an approximation of Ω_t^ρ, i.e, for $0 \le t \le T - \delta$,

$$\left\{ x,\; \vartheta^{\varepsilon,\Delta}(x,t) > \rho + C\frac{R^{\frac{7}{4}}}{\delta^{\frac{d}{4}}}\Delta x^{\frac{1}{10}} \right\} \subset \Omega_t^\rho \cap \mathbb{B}_R \subset \left\{ x,\; \vartheta^{\varepsilon,\Delta}(x,t) > \rho - C\frac{R^{\frac{7}{4}}}{\delta^{\frac{d}{4}}}\Delta x^{\frac{1}{10}} \right\}.$$

Let us mention that other numerical methods for reachability analysis have been introduced and analysed in the literature. The most natural numerical algorithm consists in using Monte Carlo simulations to generate a set of trajectories starting from a given initial position $x \in \mathbb{R}^d$. Then the percentage of trajectories reaching the target gives an approximation of the probability of success (for reaching the target) when starting from this position x. On the other hand, for linear stochastic systems, a bound for the probability of hitting a target can be obtained by using the enclosing hulls of the probability density function for time intervals, see [5, 4], for instance. Note that these approaches are used to calculate the probabilities of success but do not allow to define the entire set of points that have the same given probability. In addition, Monte-Carlo-based methods often require a large number of simulations to obtain a good accuracy. We will use such simulations in Section 4.3 of chapter 4 to validate our level-set approach.

Part 2 : Feedback control analysis for the state-constrained control problem with maximum cost. Application : Abort landing in presence of windshear.

This part contains the results of the Chapters 5 and 6 and it is dedicated to the analysis of the state-constrained control problem with maximum cost. First, we show that it is possible to characterize the value function of such a problem as a level set of the value function of an auxiliary optimal control problem. Then, we prove that the auxiliary value function is Lipschitz continuous and is the unique viscosity solution of a HJB equation. Furthermore, with a special choice of the initial condition, the solution of the auxiliary HJB equation satisfies some boundary conditions out of a set that we can easily compute. In particular, this result is useful in the case when the dynamics function is Lipschitz continous only on a compact set. Next, we establish a link between the optimal trajectory associated with the control problem, the optimal trajectory associated with the auxiliary control problem and the optimal trajectories for some exit time problem associated with a *viable kernel set*. We prove that the optimal trajectories can be constructed as a limit of trajectories with piecewise constant controls. The convergence result is also extended to an approximate auxiliary value function. The same study will be done for the state-constrained control problem with a Bolza cost. Then, we focus on a concrete problem : abort landing problem of the airplane in presence of windshear. The problem consists in maximizing the lower altitude over a time horizon in order to avoid the crash on the ground. Many simulations are perfomed in this work including two models in order to validate our approach for real applications.

More precisely, let $T > 0$ be a given time horizon, consider the dynamical system

$$\dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u(s)}), \text{ a.e. } \mathbf{s} \in (\mathbf{0}, \mathbf{T}), \tag{1.6a}$$

$$\mathbf{y}(0) = y, \tag{1.6b}$$

where $f : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ is a continuous function (see Section 5.2 for the precise assumptions) and $\mathbf{u} : [0,T] \to U$ is a measurable function. Let us denote $\mathbf{y} = \mathbf{y}_y^{\mathbf{u}}$ the absolutely continuous solution of (1.6). Let $\mathcal{K} \in \mathbb{R}^d$ be a given non-empty closed set and consider the following value function : defined by,

$$\vartheta_1(t, y) := \min_{\mathbf{u} \in L^\infty((0,t),U)} \left\{ \max_{\theta \in [0,T]} \Phi_1(\mathbf{y}_y^{\mathbf{u}}(\theta)) \;\middle|\; \mathbf{y}_y^{\mathbf{u}}(\theta) \in \mathcal{K} \quad \forall \theta \in [0,t] \right\},$$

with the convention that $\inf \emptyset = +\infty$. The function $\Phi_1 : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function (see section 5.2).

In the unconstrained case ($\mathcal{K} = \mathbb{R}^d$), this control problem has already been studied by Barron and coauthors [18, 19] where they proved that the control problem with maximum cost can be approximated by the control problem with L_p-cost maximum-cost for which the standard theory of viscosity solutions can be applied. This problem has been also used to characterize the reachable sets in [28]. Quincampoix and Serea considered in [93] the control problem with l.s.c infimum cost where the Epigraph of the value function is characterized by a *viable Kernel* and the value function is the smallest l.s.c supersolution to a Hamilton Jacobi equation. In the context of differential games, the problem has been studied in [94] for a Lipschitz continuous infimum cost and [96] for maximum bounded cost.

Here, we want to analyse the general case where the set $\mathcal{K}$ is nonempty closed set. The paper [19] inspires us to work first on the control problem with L_p-cost that can be considered as an approximation of the maximum-cost. Here, we do not make any approximation between both problems. Even so, we will compare later the optimal trajectories of such problems for a concrete application (see [35]). So, we investigate the following state-constrained control problem with Bolza cost :

$$\vartheta_2(t,y) := \min_{\mathbf{u}\in L^\infty((0,t),U)} \left\{ \int_0^t \Phi_2(\mathbf{y}_y^{\mathbf{u}}(s))ds \;\middle|\; \mathbf{y}_y^{\mathbf{u}}(\theta) \in \mathcal{K} \quad \forall \theta \in [0,t] \right\},$$

where $\Phi_2(y) := r\Phi_1(y)^q$ for all $y \in \mathbb{R}^d$ and r and q are positive constants. For the unconstrained case ($\mathcal{K} = \mathbb{R}^d$), under classical assumptions on the dynamics and Φ_2, the value function ϑ_2 is the unique continuous viscosity solution of an HJ equation [13, 12]. This result has also been extended to the lower semicontinuous setting [20, 63].

For the constrained case (when $\mathcal{K} \neq \mathbb{R}^d$), the value function ϑ_2 may be discontinuous. Nevertheless, ϑ_2 satisfies a Dynamic programming principle. Moreover, a state-space constrained Hamilton-Jacobi-Bellman equation can be associated with ϑ_2 (see [98]- [99]) taking the following form :

$$-\partial_t\vartheta_2 + \mathcal{H}(t,y,\nabla\vartheta_2) = 0, \qquad \text{in } (0,T)\times\mathcal{K}, \tag{1.7a}$$
$$\vartheta_2(0,y) = 0, \qquad \text{in } \mathcal{K}, \tag{1.7b}$$

where $\mathcal{H}(t,y,p) := \max_{a\in U}(-f(t,y,a).p - \Phi_2(y))$. In Soner's formulation, a function ϑ_2 is a viscosity solution of (1.7) provided it is sub-solution in $(0,T)\times \overset{\circ}{\mathcal{K}}$ (where $\overset{\circ}{\mathcal{K}} := \mathcal{K}\backslash\partial\mathcal{K}$) and a super-solution on $(0,T)\times\mathcal{K}$. The uniqueness of the solution of the HJ equation (1.7) is more complicated to prove and it requires restrictive controllability assumptions on $\mathcal{K}$ and the dynamics.

The pointing qualification conditions are the most known controllability assumptions. The first classical one is called 'inward pointing condition (IPQ)' (see [98]-[99]). It states that each point of $\partial\mathcal{K}$, there exist a field of the system pointing into $\mathcal{K}$. In other words, this condition ensures that the trajectory hitting the $\partial\mathcal{K}$ can be approximated by a sequence of trajectories that remain in $\mathcal{K}\backslash\partial\mathcal{K}$. The uniqueness can then be established from the Lipschitz property of the value function obtained from this condition. The "outward pointing condition" (OPQ) is an other pointing condition (see [63] - [64]) ensuring that each point on the boundary of $\mathcal{K}$ can be reached by a trajectory coming from $\mathcal{K}\backslash\partial\mathcal{K}$. Under this condition, the value function is the unique lower semi continuous solution of a Hamilton Jacobi equation. Note that these pointing conditions are not satisfied for all control problems.

In the general case where the controllability assumptions do not necessary hold, it is more convenient to exploit the idea of the characterization of the value function by its Epigraph (see for instance [11, 40, 41, 8, 93])). Altarovici et al in [3] studied the case of Lipschitz continuous distributed cost Φ_2. Here, we show that this result can be extended to locally Lipschitz continuous functions Φ_2 with polynomial growth. Thus, the function ϑ_2 can be described by means of continuous value function of an auxiliary control problem. The new value function is free of constraints and it can be characterized as the unique continuous viscosity solution of a variational HJ equation. More precisely, let g be a Lipschitz continuous function satisfying $g(x) \leq 0 \iff x \in \mathcal{K}$, then the approach consists in introducing the following auxiliary control problem :

$$w_2(t, y, z) := \inf_{\mathbf{u} \in L^\infty((0,t),U)} \left(\int_0^t \Phi_2(\mathbf{y}_y^{\mathbf{u}}(\theta)) - z \right) \bigvee \max_{\theta \in (0,t)} g(\mathbf{y}_y^{\mathbf{u}}(\theta)),$$

where $a \bigvee b := \max(a, b)$. The function w_2 can be characterized as the unique continuous viscosity solution of a HJ equation of variational type,

$$\min\Big(\partial_t w_2 + H(y, \nabla_y w_2, \partial_z w_2), \quad w_2(t, y, z) - g(y)\Big) = 0, \text{ in } \quad [0, T) \times \mathbb{R}^d \times \mathbb{R}, \tag{1.8a}$$

$$w_2(0, y, z) = (-z) \vee g(y), \qquad \text{in } \quad \mathbb{R}^d \times \mathbb{R}. \tag{1.8b}$$

where $H(y, p_1, p_2) := \sup_{u \in U} \big(- f(y, u).p_1 + \Phi_2(y).p_2 \big)$, for all $p_1 \in \mathbb{R}^d, p_2 \in \mathbb{R}$. Moreover, under the classical assumptions on f, and Φ_2, the epigraph [1] of ϑ_2 satisfies $\mathcal{E}pi(\vartheta_2(t, \cdot)) = \{(y, z) \in \mathbb{R}^d \times \mathbb{R},\ w_2(t, y, z) \leq 0\}$. Thus, one can determine the value functions ϑ_2 in terms of level sets of w_2, i.e,

$$\vartheta_2(t, y) = \min\{z,\ w_2(t, y, z) \leq 0\}, \tag{1.9}$$

Let us emphasis that the variational Hamilton Jacobi equation (1.8) is defined in all domain. From a numerical point of view, this is not good since one has to choose the suitable boundary conditions for solving the problem on a compact set. Here, we prove that with a wise choice of the Lipschitz function g, one may obtain that the function ϑ_2 is the unique viscosity solution of a variational Hamilton Jacobi equation with a Dirichlet condition.

In the reality, using the same arguments, one can show that the function ϑ_1 can also be characterized by the levels sets of the value function of an auxiliary control problem that takes the following form :

$$w_1(t, y, z) := \inf_{\mathbf{u} \in L^\infty((0,t),U)} \max_{\theta \in [0,t]} \Psi_1(\mathbf{y}_y^{\mathbf{u}}(\theta), z),$$

where $\Psi_1(y, z) = (\Phi(y) - z) \vee g(y)$. In the same way, one can get the important property (1.9) for ϑ_1 and w_1. The function w_1 is characterized as the unique solution of a variational Hamilton Jacobi equation with initial data $\Psi_1(y, z) = \big(\Phi(y) - z\big) \bigvee g(y)$.

The central objective of this work is to study the optimal trajectories associated with the state-constrained optimal control problems discussed above. The auxiliary

1. The epigraph at time t is defined by $\mathcal{E}pi(\vartheta_2(t, \cdot)) := \{(y, z) \in \mathbb{R}^d \times \mathbb{R},\ \vartheta_2(t, y) \leq z\}$.

control problem has the advantage to handle the technical difficulties related to the resolution of the control problem under state constraints. Moreover, it provides a good framework for the study of the optimal trajectories and the associated optimal feedback control. Indeed, one can establish a link between the optimal trajectory associated with the control problem and the optimal trajectory associated with the auxiliary control problem.

In addition, we show that the control problem with maximum cost is linked to a *viability kernel* while the control problem with Bolza cost is linked to a *Capture basin*. We define an exit time function for the viability problem and a maximum time function for the reachability problem. Under some initial conditions, we show that the optimal trajectories constructed from the maximum time function are also optimal for the control problems whose value functions are respectively ϑ_1 and w_1. In the same manner, we make the link between the optimal trajectories constructed from the exit time function and the Bolza problems. An other important contribution is to extend the classical result for the convergence of optimal trajectories in [95] to the state-constrained control problems with maximum cost and with Bolza cost as well. The algorithm is based on an approximation of the set valued optimal feedback control map using the Dynamic Programming Principle for the auxiliary value functions. Moreover, we establish an equivalent result for the same algorithm using an approximated solution of the auxiliary control problem.

This work is motivated by a real application : abort landing problem in presence of windshear. We aim to determine the maximal altitude that the airplane can reach in order to prevent a crash on the ground. In the references [86], [85], the authors propose an optimal control with maximum cost for which an approximate solution for the problem is given with the associated optimal trajectories and feedback control. This solution was improved in [35] and [36] by considering the switching structure of the problem that has bang-bang subarcs and singular arcs. Here, we still study the same control problems but with different dynamics (see section 6.1 for more details).

The presence of constraints in the definition of our optimal control problem precludes the characterization of the value function -without any controllability assumption- as the unique solution of a Hamilton Jacobi equation. Therefore, we consider the Hamilton Jacobi approach developed for the maximum cost problem and the Bolza problem discussed in the theoretical part. Moreover, we introduce and compare some algorithms of reconstruction of optimal trajectories associated with both state-constrained control problems.

Conclusion

In this thesis, we study different control problems corresponding to some applications by making use of the so-called Hamilton Jacobi Bellman approach. The main aim of the study is to validate the behaviour of this approach when applied to concrete cases.

The first part is concerned by the error estimates for second order HJB equations with discontinuous and unbounded data. These estimates were obtained in two steps. First, we derive error estimate between a possible discontinous and unbounded value function of a stochastic optimal control problem and the regularized value function associated with a regularized control problem. Then, we focus on the error estimates for numerical solutions of second order HJB equation satisfied by the regularized value function with unbounded data. Here, these obtained errors are based on classical shaking and regularization techniques. An application of this study is the probabilistic reachablility problem. Indeed, the chance-constrainted backward reachable set is characterized by the level set of a discontinous value function. Then, the result of the theoretical study is applied to approximate this discontinous value function and the backward reachable set itself.

The second part considers some classes of state-constrained optimal control problems with maximum cost and with Bolza cost as well. For each control problem, we show that the Epigraph of the value function can be described in terms of a new value function associated with an auxiliary control problem. We show that the auxiliary value function enjoyes some regularities and it is the unique viscosity solution of a Hamilton Jacobi equation with a Dirichlet condition. We also prove a convergence result of some approximated trajectories to the continuous one. Next, we consider a concrete application : the problem of abort landing during low altitude windshears. Many algorithms of reconstruction of optimal trajectories and the associated feedback control are compared from a numerical and theoretical points of view.

Chapitre II

Background for Hamilton Jacobi Bellman approach

In this chapter we recall all the definitions and basic results on the control problems and the Hamilton-Jacobi Bellman approach, we will refer in the following chapters. The concept of optimal control can be described as the fact to influence the behavior of a dynamical system with the aim to optimize a payoff function depending on the control inputs to the system. A modern method to deal with this class of problems is the dynamic programming principle developed by [22, 23, 21] and considered as an important facet of the control theory.

This approach allows to reduce the core of the study to the analysis of the solutions to a nonlinear partial differential equation called Hamilton Jacobi Bellman equation (HJB). In general case, the value function is not smooth enough to be the unique solution of a HJB equation in a classical sense. The theory of viscosity solutions, first introduced in the early of 80s of the last century by M. G. Crandall and P.-L. Lions [45, 46], provides a convenient framework to study the HJB equations. We refer also to [12] and [44] for a detailed introduction of the theory of viscosity solution in the deterministic and stochastic setting as well.

The chapter is organized as follows. First, we introduce the deterministic unconstrained and state-constrained optimal control problems and the corresponding value functions. The HJ equation is then derived in both cases using the Dynamical programming approach. We also discuss the feedback control problems and the different ways to reconstruct trajectories. Some known numerical schemes are presented in section 2.1. In section 2.2, we give a formulation for the stochastic optimal control problems with finite or infinite time horizon and discuss the existence of solutions. The value function is then described as the unique solution of a Hamilton-Jacobi-Bellman equation. We also discuss some numerical approximate solutions of a second order HJB equation and the error estimates of the approximation. In section 2.3, the main notions and results of viscosity theory for a first and second order Hamilton-Jacobi Bellman equations are presented.

2.1 Deterministic control problems

Let $T > 0$ be a finite time. Consider the following nonlinear controlled dynamical system :

$$\dot{y}(s) = f(y(s), u(s)), \quad s \in (0, T], \tag{2.1a}$$
$$y(0) = x, \tag{2.1b}$$

where $x \in \mathbb{R}^d$ and $u(.) \in \mathcal{U}$ is the control function representing the decision or the policy of the controller and $\mathcal{U}$ is the set of all admissible control functions defined as,

$$\mathcal{U} := \{u(.) : [0, +\infty) \to U, \text{ Lebesgue measurable } \},$$

and U is a subset of $\mathbb{R}^k$ $(k \geq 1)$. Consider the following assumptions on U and the function f :

$$U \subset \mathbb{R}^k \text{ is a compact set ;} \tag{H_1}$$

$$\begin{cases} (i) & f(\cdot,\cdot) \text{ is continuous on } \mathbb{R}^d \times U; \\ (ii) & \exists L_f \geq 0 \text{ such that } \ x_1, x_2 \in \mathbb{R}^d, \ a \in U : \\ & \quad |f(x_1, a) - f(x_2, a)| \leq L_f |x_1 - x_2|; \end{cases} \tag{H_2}$$

Let us recall that if $f(., a)$ (for all $a \in U$) has continuous gradient f_x, then, the Lipschitz continuity property of f in (H_2) is equivalent to $|f_x(x, a)| \leq L_f$ for all $a \in U$.

It is known that the assumptions (H_1)-(H_2) ensure the existence and the uniqueness of the solution of the differential equation (2.1). For $u \in \mathcal{U}$ and $x \in \mathbb{R}^d$, we denote the solution $y_x^u(.)$. Note that the solution $y_x^u(.)$ (for $u \in \mathcal{U}$) is in the class of absolutely continuous functions $W^{1,1}([0, T])$ and it satisfies the following :

$$y_x^u(s) = x + \int_0^s f(y(\theta), u(\theta)) d\theta, \quad \forall s \in (0, T].$$

Furthermore, the following proposition holds (see [12, Chapter III])

Proposition 2.1.1 *Assume (H_1)-(H_2) hold. For each $x \in \mathbb{R}^d$, there exists constant C depending only on L_f, T such that, for all $u \in \mathcal{U}$, $t, s \in [0, T]$ and $x', x \in \mathbb{R}^d$,*

$$\begin{aligned} \left| y_x^u(t) - y_{x'}^u(t) \right| &\leq C|x - x'|, \\ \left| y_x^u(t) - y_x^u(s) \right| &\leq C(1 + |x|)|t - s|, \\ \left| y_x^u(t) \right| &\leq C(1 + |x|). \end{aligned}$$

Now, consider the set of all absolutely continuous functions solutions of (2.1) in $[0, t]$ $(t \leq T)$ starting from x :

$$S_{[0,t]}(x) := \left\{ y := y_x^u, \ y \text{ is absolute continuous solution to (2.1) in } [0, t] \text{ for } u \in \mathcal{U} \right\}.$$

Under the assumptions (H_1)-(H_2), the set $S_{[0,t]}(x)$ is compact in $W^{1,1}([0,t])$. Moreover, the set valued map $x \to S_{[0,t]}(x)$ is Lipschitz continuous from $\mathbb{R}^d$ in $C([0,t];\mathbb{R}^d)$ (see [9] for instance) with respect to the Hausdorff metric.

Now, let $\mathcal{K}$ be a given non-empty and closed subset of $\mathbb{R}^d$. Let us define the set of all the *admissible* solutions of (2.1) in $[0,t]$ starting from x by :

$$S^{\mathcal{K}}_{[0,t]}(x) := \Big\{ y := y^u_x,\ y \in S_{[0,t]}(x),\ \text{and}\ \ y^u_x(s) \in \mathcal{K}\ \forall s \in [0,t] \Big\}.$$

Let us recall that the assumptions (H_1)-(H_2) are not sufficient to prove the compactness of $S^{\mathcal{K}}_{[0,t]}(x)$ in $W^{1,1}([0,t])$. The compactness of $S^{\mathcal{K}}_{[0,tc]}(x)$ requires an additional hypothesis of the type,

$$f(x,U) \text{ is a convex set } \forall x \in \mathbb{R}^d; \tag{H_3}$$

In the following, our overall task will be to determine what is the best control for our system. For this, we will consider a criterion to optimize. Typically, we will minimize a cost or a payoff functional.

2.1.1 Unconstrained control problem

In this section, we present a class of optimal control problems of the bolza form for which there are classical results on the characterization of the value function with partial derivatives equations of HJB type (see [12, 60, 13, 80] and the references therein).

Finite time horizon

The bolza cost functional J associated to the trajectories of the system (2.1) in the finite horizon time is :

$$J(t,x,a) := \int_0^t \ell(y^a_x(s), a(s))ds + \varphi(y^a_x(t)).$$

We call $\ell : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}$ a running cost function and $\varphi : \mathbb{R}^d \to \mathbb{R}$ a terminal cost function. If $\varphi(x) = 0$ then the problem is said to be in *Lagrange form.* If $\ell(t,x,v) = 0$, the problem is in *Mayer form.*

The bolza and the Lagrange forms of the problem can be easily rewritten in a Mayer's type form ([100, Remark 3.2.(iii)]), by adding a state variable following the dynamics given by the running cost ℓ.

We will consider the following assumptions on the running cost function ℓ and the terminal cost function φ,

$$\begin{cases} (i) & \ell(\cdot,\cdot) \text{ is continuous on } \mathbb{R}^d \times U; \\ (ii) & \exists L_\ell \geq 0 \text{ such that } \forall x,y \in \mathbb{R}^d, u \in U : \\ & \qquad |\ell(x,u) - \ell(y,u)| \leq L_\ell |x-y|. \end{cases} \tag{H_4}$$

$$\varphi \text{ is a Lipschitz continuous function.} \qquad (H_5)$$

The problem is to find $u(.) \in \mathcal{U}$ which minimizes J. The dynamic value function associated to the cost functional J, is then defined by :

$$\vartheta(t,x) := \inf_{a \in \mathcal{U}} J(t,x,a). \qquad (2.3)$$

The first regularity result on the value function ϑ is the following Lipschitz property (see [12, Chapter III, proposition 3.1]).

Proposition 2.1.2 *Assume* (H_1)-(H_2) *and* (H_4)-(H_5) *hold. The value function* ϑ *is Lipschitz continuous, i.e* $\exists\, L_\vartheta > 0$ *depending only on* L_f, T, L_ℓ *such that for all* $T \geq s \geq t$ *and* $x, y \in \mathbb{R}^d$, *the following holds :*

$$|\vartheta(t,x) - \vartheta(s,y)| \leq L_\vartheta \Big(|x-y| + |t-s| \Big).$$

The idea of the dynamic programming is that the value function ϑ satisfies a functional equation called the Dynamic Programming Principle.

Proposition 2.1.3 *Assume* (H_1)-(H_2) *and* (H_4)-(H_5) *hold. Let* $t \in [0,T]$ *and* $x \in \mathbb{R}^d$ *be given. Then, for all* $s \in [0,t]$, *we have :*

$$\vartheta(t,x) = \inf_{u \in \mathcal{U}} \left\{ \int_0^s \ell(y_x^u(r), u(r))dr + \vartheta(t-s, y_x^u(s)) \right\}. \qquad (2.4)$$

In particular, the dynamic programming principle states that the function

$$s \to \int_0^s \ell(y_x^u(r), u(r))dr + \vartheta(t-s, y_x^u(s)), \qquad (2.5)$$

is non decreasing, for all control $u \in \mathcal{U}$. If there exists an optimal control $u^* \in \mathcal{U}$ for the problem (2.3) then (2.5) (with $u = u^*$) is constant.

The Dynamic Programming Principle (DPP) allows to split the trajectories so as to calculate the value function at the point (t,x) in terms of the value function at time $t-s$ starting with the position of the trajectory at time s.

If the function ϑ is differentiable, one can derive ϑ to get its differential version, the Hamilton-Jacobi-Bellman (HJB) equation. More precisely, define the Hamiltonian in the following way

$$H(x,p) := \sup_{a \in U} \big\{ -f(x,a).p - \ell(x,a) \big\}. \qquad (2.6)$$

Then, the value function ϑ is solution of

$$\begin{cases} \partial_t \vartheta(t,x) + H(x, \nabla\vartheta(t,x)) = 0, & \text{in } [0,T] \times \mathbb{R}^d, \\ \vartheta(0,x) = \varphi(x), & \text{in } \mathbb{R}^d, \end{cases} \qquad (2.7)$$

where $\nabla\vartheta(t,x)$ denotes the gradient of ϑ at the point x.

Usually the value function ϑ is not differential which means that ϑ does not satisfy (2.7). The theory of viscosity solution presented briefly in section 2.3 gives a good framework to characterize the value function ϑ.

Infinite time horizon

Let $T = +\infty$. Consider the cost functional J associated to the trajectories of the system (2.1) in the infinite horizon time :

$$J(x, u) := \int_0^{+\infty} \ell(y_x^u(s), u(s)) e^{-\lambda t} ds,$$

where $\lambda > 0$ is a discount factor and the distributed cost ℓ satisfies (H_4). The value function in the infinite time horizon is given by

$$\vartheta(x) = \inf_{u \in \mathcal{U}} J(x, u).$$

The value function ϑ satisfies the following regularity result (see [12, Chapter III, proposition 2.1])

Proposition 2.1.4 *Assume (H_1)-(H_2) and (H_4) hold. The value function ϑ is Hölder continuous, i.e $\exists\, L_\vartheta > 0$ depending only on L_f, T, L_ℓ such that for all $x, y \in \mathbb{R}^d$, we have :*

$$|\vartheta(x) - \vartheta(y)| \leq L_\vartheta |x - y|^\gamma,$$

where the exponent γ is depending on λ and L_f.

The value function ϑ satisfies a dynamic programming priniciple of the following form.

Proposition 2.1.5 *Assume (H_1)-(H_2) and (H_4) hold. For all $x \in \mathbb{R}^d$ and $t > 0$, the value function ϑ satisfies the following DPP :*

$$\vartheta(x) = \inf_{u \in \mathcal{U}} \left\{ \int_0^t \ell(y_x^u(r), u(r)) e^{-\lambda r} dr + \vartheta(y_x^u(t)) e^{-\lambda t} \right\}. \tag{2.8}$$

For differentiable function ϑ, the value function can be described by the following Hamilton-Jacobi-Bellman (HJB) equation :

$$\lambda \vartheta(x) + H(x, \nabla \vartheta(x)) = 0, \quad \text{in } \mathbb{R}^d, \tag{2.9}$$

where H is defined by (2.6).

In the general case, the value function ϑ is not necessary differentiable. In the section 2.3, we present a very relevant tool when dealing with non smooth value functions : viscosity solutions.
In many applications, the control problems are subject to constraints on the state variable and the control as well. In the following subsection, we will present some works on the state constrained optimal control problems.

2.1.2 State constrained control problem

For a given finite horizon $T > 0$, consider the dynamical system (2.1). Let $(t,x) \in [0,T] \times \mathbb{R}^d$, and $\mathcal{K}$ be a given non-empty and closed subset of $\mathbb{R}^d$. Consider a control problem whose value function is defined by

$$\vartheta(t,x) := \min \left\{ \int_0^t \ell(y_x^u(s), u(s))ds + \varphi(y_x^u(t)) \;\middle|\; y_x^u \in S_{[0,t]}^{\mathcal{K}}(x) \right\}, \tag{2.10}$$

where we consider the convention $\vartheta(t,x) = +\infty \quad \text{if } S_{[0,t]}^{\mathcal{K}}(x) = \emptyset$.

In the case when $\mathcal{K} = \mathbb{R}^d$ the problem turns out to be an unconstrained optimal control problem.

Here, the constraint set $\mathcal{K}$ is not equal to $\mathbb{R}^d$ and it is known that without assuming additional assumptions, the value functions ϑ may be discontinuous. However, if we assume that for any $x \in \mathbb{R}^d$ there exists a viable trajectory $(y_x(.) \in \mathcal{K})$, i.e,

$$\forall x \in \mathbb{R}^d, \quad S_{[0,t]}^{\mathcal{K}}(x) \neq \emptyset,$$

then, a state-constrained Hamilton-Jacobi equation is associated to the value function ϑ (see [98]- [99]) taking the following form,

$$\partial_t u + \mathcal{H}(x, \nabla_x u) = 0 \quad , \text{ in } (0,T) \times \mathcal{K}, \tag{2.11a}$$

$$u(0,x) = \varphi, \qquad \text{in } \mathcal{K}, \tag{2.11b}$$

where $\mathcal{H}$ is defined on (2.6). A function u is a viscosity solution of (2.11) in Soner's formulation, if it is sub-solution in $(0,T)\times \overset{\circ}{\mathcal{K}}$ (where $\overset{\circ}{\mathcal{K}} := \mathcal{K} \backslash \partial\mathcal{K}$) and a super-solution in $(0,T) \times \mathcal{K}$. The uniqueness of the solution of the HJ equation (2.11) is more complicate to prove. Indeed, without any contrability assumption on the behavior of the solution on the boundary, the state-space HJB equation may have several solutions (in the constrained viscosity sense), see [29, 72]. Actually, uniqueness requires restrictive controllability assumptions on $\mathcal{K}$ and on the dynamics.

The Pointing qualification conditions are the most known controllability assumptions. The first one was introduced by Soner (see [98]- [99]) and it is called the "Inward pointing" constraint qualification and it takes the following form,

$$\exists \beta > 0, \; \forall x \in \partial\mathcal{K}, \; \exists \alpha(x) \in U, \quad f(x, \alpha(x)).n(x) \leq -\beta, \tag{2.12}$$

where $n(x)$ is the exterior normal vector. The condition (2.12) is a strengthened viability condition and it means that from each point $x \in \partial\mathcal{K}$ a trajectory enters into $\overset{\circ}{\mathcal{K}}$, while the viability condition guarantees that a solution stays in $\mathcal{K}$ forever. Moreover, each trajectory hitting the boundary can be approximated by a sequence of trajectories staying in $\mathcal{K}$. The uniqueness of the solution (2.11) can then be established from the Lipschitz property of the value function obtained form this condition. For weaker inward pointing assumptions, we refer to [52, 88, 89].

In the papers [63] and [64], Frankowska et al, introduced another pointing condition called "outward pointing" constraint condition taking the form :

$$\forall x \in \partial\mathcal{K}, \; \exists \alpha(x) \in U, \quad f(x, \alpha(x)).n(x) > 0, \tag{2.13}$$

This condition (2.13) states that each point on the boundary of $\mathcal{K}$ can be reached by a trajectory coming from the interior of $\mathcal{K}$. The value function is not necessary continuous under (2.13), but is still characterized as the unique lower semi continuous solution of a Hamilton Jacobi equation. Note that these pointing conditions can not be satisfied for all control problems.

An alternative way for dealing with the constrained case is to consider the epigraph of value function. Using the viability theory [7], one can characterize the epigraph of the value function (see [11, 40, 41, 8]).

In [3], the authors introduced an alternative way to deal with the state constraints by showing that the epigraph of ϑ can be described by means of a Lipschitz continuous function, which will turn out to be the value function of an auxiliary control problem free of state constraints. More precisely, the approach consists of introducing the auxiliary control problem whose value function defined by

$$w(t,x,z) := \inf_{u\in\mathcal{U}} \left(\int_0^t \ell(y_x^u(s), u(s))ds + \varphi(y_x^u(t)) - z \right) \bigvee \max_{\theta\in(0,t)} g(y_x^u(\theta)), \tag{2.14}$$

where $a \bigvee b := \max(a,b)$, and g is a Lipschitz continuous function satisfying,

$$g(x) \leq 0 \Longleftrightarrow x \in \mathcal{K},$$

(one can take g to be the signed distance to $\mathcal{K}$ for instance).
The idea behind the use of an auxiliary control problem suggests the introduction of an additional state variable, but it presents the best tool for characterizing the epigraph without controllability assumptions. To see this, let us define the epigraph of ϑ at time t by

$$\mathcal{E}pi(\vartheta(t,\cdot)) := \{(x,z) \in \mathbb{R}^d \times \mathbb{R},\ \vartheta(t,x) \leq z\},$$

then, the following proposition holds

Proposition 2.1.6 *Assume (H_1)-(H_5) hold. Then for any $t \in [0,T]$ and $(x,z) \in \mathbb{R}^d \times \mathbb{R}$,*

$$\mathcal{E}pi(\vartheta(t,\cdot)) = \{(x,z) \in \mathbb{R}^d \times \mathbb{R},\ w(t,x,z) \leq 0\},$$

and $\vartheta(t,x) = \min\{z,\ w(t,x,z) \leq 0\}$.

The auxiliary problem is free of constraints and the value function w can be characterized as the unique continuous viscosity solution of the following variational HJ equation (obstacle type) :

$$\min\left(-\partial_t v + \max_{a\in\mathcal{U}}(-f(x,a)\nabla_x v + \ell(x,a)\partial_z v,\quad v - g(x) \right) = 0,$$
$$(t,x,z) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}, \tag{2.15a}$$
$$v(T,x,z) = (\varphi(x) - z) \vee g(x), \quad (x,z) \in \mathbb{R}^d \times \mathbb{R}. \tag{2.15b}$$

The control problems with L_∞-cost were studied in [18, 19]. The exact penalization of the state constraints (i.e $\max_{\theta\in(t,T)} g(y_{t,x}^{\alpha}(\theta))$) has also been applied in the reachability analysis of some state constrained nonlinear systems, (see [28] and [6]).

This approach can be applied to general optimal control problems with state constraints, namely, the cases of finite horizon control problems, infinite horizon control problems and differential games as well (see [28]).

Particular case of minimum time function

Let us consider the minimum time function $\mathcal{T}$ defined as the time needed by the system to reach a given closed target set $\mathcal{C} \subset \mathbb{R}^d$ with an admissible trajectory y_x^u (which remains in $\mathcal{K}$), that is,

$$\mathcal{T}(x) := \min\Big\{t \ \mid \exists u \in \mathcal{U}, \ \ y_x^u(t) \in \mathcal{C}, \ \ y_x^u(s) \in \mathcal{K}, \ \forall s \in [0,t]\Big\},$$

with the convention that $\mathcal{T}(x) := +\infty$ if $y_x^u(t) \notin \mathcal{C}, \forall t > 0$ and $u \in \mathcal{U}$ or $\exists s \in [0,t]$ s.t $y_x^u(s) \notin \mathcal{K}$.
When $\mathcal{K} = \mathbb{R}^d$ it is known that under some metric properties (see [12]), the value function $\mathcal{T}$ is the unique viscosity solution of a Hamilton Jacobi Bellman equation of the following form.

$$\begin{cases} \mathcal{T}(x) + \sup_{a\in U}\Big\{ -f(x,a).D\mathcal{T}(x)\Big\} = 1, & \text{on } \mathbb{R}\backslash\mathcal{C}, \\ \mathcal{T}(x) = 0, & \text{on } \partial\mathcal{C}. \end{cases} \tag{2.16}$$

Now, let $\mathcal{K} \neq \mathbb{R}^d$. Then, the minimum time function can not be characterized by an HJB equation. Actually, function $\mathcal{T}$ even does not satisfy the dynamic programming principle (see [27]). However, it can be characterized by level set of a Lipschitz value function ϑ of an auxiliar optimal control problem (see [28]). Let us consider the following Lipschitz value function $g_{\mathcal{C}}$ defined in $\mathbb{R}^d$ to $\mathbb{R}$ such that,

$$g_{\mathcal{C}}(x) \leq 0 \Leftrightarrow x \in \mathcal{C},$$

(one can take $g_{\mathcal{C}}$ to be the distance function to $\mathcal{C}$). Consider the following Mayer problem,

$$\vartheta(t,x) = \inf\big\{g_{\mathcal{C}}(y_x^u(t)) \mid \ u \in \mathcal{U}, \ \ y_x^u(\theta) \in \mathcal{K}, \ \ \forall\theta \in [0,t]\big\}.$$

Let us emphasise that ϑ is the value function of a state-constrained problem. The approach of the exact penalization will be used in order to eliminate the state constraints. Let us consider the following Lipschitz value function $g_{\mathcal{K}}$ defined in $\mathbb{R}^d$ to $\mathbb{R}$ such that,

$$g_{\mathcal{K}}(x) \leq 0 \Leftrightarrow x \in \mathcal{K}.$$

Then, we consider the following optimal control problem :

$$w(t,x) := \inf\bigg\{ \max\Big(g_{\mathcal{C}}(y_x^u(t)), \max_{s\in[0,t]} g_{\mathcal{K}}(y_x^u(s))\Big), \ y_x^u(.) \in S_{[0,t]}(x)\bigg\}. \tag{2.17}$$

It is important to remark here that the value function is free of constraints. Then, the following proposition provides a characterization of the minimum time function and the capture basin by the value function w (see [28] for the proof).

Proposition 2.1.7 *Assume (H_1)-(H_3) hold. The minimum time function $\mathcal{T}$ is related to w by the following relation :*

$$\mathcal{T}(x) = \inf\Big\{t \geq 0 \mid \vartheta(t,x) \leq 0\Big\} = \inf\Big\{t \geq 0 \mid w(t,x) \leq 0\Big\}.$$

The minimum time function (and the capture basin) can then be determinated by the level set of a Lipschitz value function ϑ (see [28]) without additional state variable.

In addition, the value function w is the unique viscosity solution of a Hamilton Jacobi equation. More precisely, consider the Hamiltonian defined in (2.6). Then, we have

Proposition 2.1.8 *Assume (H_1)-(H_3) hold. Then, the value function is the unique viscosity solution of the following obstacle problem,*

$$\begin{aligned} \max\Big(\partial_t w + H(x, Dw), w - g_{\mathcal{K}}(x)\Big) &= 0, \ \ t > 0, \ \ x \in \mathbb{R}^d, \\ w(0,x) = \max(g_{\mathcal{K}}(x), g_{\mathcal{C}}(x)), & \qquad\qquad\quad x \in \mathbb{R}^d. \end{aligned}$$

One of the issues discussed in this thesis is to extend the results reported above to state-constrained control problems with different optimality criterions.

2.1.3 Feedback control synthesis

The main goal of the computations in the control theory is to recover the approximate optimal controls and optimal trajectories. If the value function can be characterized as the unique solution of a Hamilton Jacobi equation, it is possible to reconstruct the feedback control by different procedures, in particular, by the use of Dynamic programming principle or the nonlinear partial differential equation or a combination of various techniques (see [56]).

Here, we recall the DPP procedure of the reconstruction of the trajectory from a Lipschitz value function ϑ solution to a Hamilton Jacobi equation (see [95]) by considering a piecewise constant control defined for N intervals of time representing the discretization of the time interval. Consider the following algorithm

Algorithm. 1 Let $(t_0 = 0, t_1 ... t_{n-1}, t_n = T)$ be a uniform partition and $h = \frac{T}{n}$ be the time step. Let $\{y^n(.)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$ ($i \geq 1$) and let $\{u^n(.)\}$ be the corresponding sequence of controls. Set $y^n(t_0) = x$.

Step 1 Compute the optimal control at t_k :

$$a_k \in \arg\min_u \Big\{\vartheta\big(t_{n-k}, y^n(t_k) + hf(y^n(t_k), u)\big)\Big\}. \tag{2.19}$$

Step 2 , Define the $u_k(t) := a_k$ be a constant control in $t \in (t_k, t_{k+1}]$ and $y^n(t)$ on $(t_k, t_{k+1}]$ be the solution to

$$\dot{y}(t) := f(y(t), u_k(t)) \text{ a.e } \; t \in (t_k, t_{k+1}]; \tag{2.20}$$

with intial condition $y(t_k)$.

The following proposition shows that the sequence of trajectories generated by the above algorithm converges uniformly to some optimal trajectory (see [95]).

Theorem 2.1.9 *Assume (H_1)-(H_5) hold. Let $\{y^n(.), u^n(.)\}$ be a sequence generated by algorithm 1 for $n \geq 1$. Then, the sequence of trajectories $\{y^n(.), u^n(.)\}$ has cluster points with respect to the uniform convergence topology. For any cluster point $\bar{y}(.)$ there exists a control low $\bar{u}(.)$ such that the pair $(\bar{y}(.), \bar{u}(.))$ is optimal for the problem associated to ϑ.*

It is known that the reconstruction of the optimal trajectories using the DPP provides good results. However, it is not expected in general to get the same results for the control law, in particular, for nonlinear problems. The regularization methods, namely, the regularization by variation of the control and the regularization of the post treatment allow to improve the behaviour of the control (see Chapter 6 for more details).

2.1.4 Numerical scheme

Numerical solutions of PDE obtained by applying the Dynamic programming principle to optimal control problems is a challenging topic that can solve many engineering issues. Here we are interested in the numerical solution of the Hamilton Jacobi equation (2.7). The most known methods of approximation of the solution of PDE are the finite difference method (see [47, 97]) and Semi lagrangian method (see [55, 58]). The Finite difference schemes are based on interpolations of discrete data using polynomials or other simple functions. The interpolations in second and high orders are necessarily oscillatory near a discontinuity. The Essentially Non-Oscillatory (ENO) proposed in [66] allows to obtain uniformly high order accurate non-oscillatory interpolation for piecewise smooth functions.

Another interesting method is called Semi Lagrangian and it was first introduced for conservation laws in [43]. The Semi Lagrangian was first used for stationary Hamilton Jacobi Bellman equations related to optimal control problems [53, 54, 57]. We refer to [59] for evolutive problems. A Semi-Lagrangian scheme is obtained by discretizing in time the dynamic programming principle and this provides an interesting interpretation of the schemes in terms of a discrete representation formula for the value function.

Finite difference method

Here, we propose a finite difference scheme to approximate the solution of the equation (2.7). Let us define the following grid,

$$\mathcal{G} :\equiv \left\{ (n\Delta t, I\Delta x), \quad n \in \mathbb{N}, I \in \mathbb{Z}^d \right\}.$$

where $\Delta x, \Delta t > 0$. Let V_I^n be the approximation of the solution V at the node (t_n, y_I).

Given a numerical Hamiltonian $\mathcal{H} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ (consistent with the Hamiltonian H) , the following scheme based on a Runge-Kutta method of first order for time discretization,

$$\begin{cases} V_I^{n+1} = V_I^n - \Delta t \mathcal{H}(x_I, D^+ V^n(x_I), D^- V^n(x_I)), \\ V_{I,j}^0 = \varphi_I, \end{cases} \tag{2.21}$$

where $\varphi_{I,j}$ is the appoximation of the initial data of the control problem at the node (x_I) and the discrete space gradient of the function V^n at the point (x_I),

$$D^{\pm} V^n(x_I) = (D^{\pm}_{y_1} V^n(x_I), .., D^{\pm}_{x_d} V^n(x_I)),$$

where the ENO scheme of second order can be used to approximate the derivatives $D^{\pm}_{y_i} W$.

Semi Lagrangian scheme

Let us consider the Semi Lagrangian scheme that approximates the solution of the equation (2.7). Let $\Delta x, \Delta t > 0$ be respectively the space and the time steps. Let V_I^n be the approximation of the solution V at the node (t_n, y_I). Consider the following approximation $-\nabla \vartheta(t_n, x_I).f(x_I, a) =$

$$\frac{\vartheta\big(t_n, x_I - f(x_I, a)\Delta t\big) - \vartheta(t_n, x_I)}{\Delta t} + O(\Delta t). \tag{2.22}$$

Replacing in (2.7), the term $\partial_t \vartheta$ by the forward finite differences and the directional derivatives by (2.22), we get the following explicit Semi Lagrangian scheme

$$\begin{cases} V_I^{n+1} = \min_{a \in U} \left[V_I^n\big(t_n, x_I - f(x_I, a)\Delta t\big) + \ l(x_I, a)\Delta t \right], \\ V_I^N = \varphi_I, \end{cases} \tag{2.23}$$

where φ_I is the appoximation of the initial data of the control problem at the node (y_I) and l is the approximation of ℓ at the node (y_I, a). The value of V on the right-hand side is calculated by an interpolation procedure based on the values on the nodes of the grid $\mathcal{G}$.

2.2 Stochastic control problems

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\}$ be a filtered probability space satisfying the usual condition $((\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\{\mathcal{F}_t\}_{t \geq 0}$ is a right continuous filtration contained in $\mathcal{F}$ i.e $\mathcal{F}_{t+} \equiv \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$, $\forall t \geq 0$ and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $W(.)$ be a given m-dimensional Brownian motion, and $T > 0$. We denote by $\mathcal{U}$ the set of progressively measurable processes valued in $U \subset \mathbb{R}^q$ $(q \geq 1)$. Let $(X^u_{t,x}(s))_{0 \leq s \leq T}$ be a controlled process valued in $\mathbb{R}^d$ solution of the following stochastic differential equation :

$$\begin{cases} dX(s) = f(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s) & s \in (t, T), \\ X(t) = x. \end{cases} \tag{2.24}$$

where the initial condition $x \in \mathbb{R}^d$. We consider the following assumptions on the *drift* $f : [0, T] \times \mathbb{R}^d \times U \times \mathbb{R}^d$, the *volatility* $\sigma : [0, T] \times \mathbb{R}^d \times U \times \mathbb{R}^{d \times p}$ and U :

$$U \subset \mathbb{R}^m \text{ is a compact set ;} \tag{A_1}$$

$$\begin{cases} (i) & f(\cdot,\cdot,\cdot) \text{ and } \sigma(\cdot,\cdot,\cdot) \text{ are continuous on } [0,T] \times \mathbb{R}^d \times U; \\ (ii) & \exists L_0 \geq 0 \text{ such that } \forall x, y \in \mathbb{R}^d, t \in [0,T], u \in U : \\ & \quad |f(t,x,u) - f(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \leq L_0|x-y|; \end{cases} \tag{A_2}$$

Note that the above uniform Lipschitz condition on f and σ and the compactness of U guarantee the existence of a controlled process on the time interval $[t, T]$ for each given initial data x, and for every admissible control $u \in \mathcal{U}$ (see [61] for more details).

A process $X^u_{t,x}$ solution of (2.24) associated to a control $u \in \mathcal{U}$ will be said *admissible*. Moreover, there exists K_0 depending only on L_0, T, d and m (see [101, page 42] or [61, Appendice D] or [100, Theorem 3.1])) : such that for any $u \in \mathcal{U}$, $0 \leq t \leq t' \leq T$ and $x, x' \in \mathbb{R}^d$

$$\mathbb{E}\left[\sup_{\theta \in [t,t']} \left|X^u_{t,x}(\theta) - X^u_{t,x'}(\theta)\right|^2\right] \leq K_0^2|x - x'|^2, \tag{2.25a}$$

$$\mathbb{E}\left[\sup_{\theta \in [t,t']} \left|X^u_{t,x}(\theta) - X^u_{t',x}(\theta)\right|^2\right] \leq K_0^2(1 + |x|^2)\ |t - t'|. \tag{2.25b}$$

Furthermore, for every $p \geq 1$, there exists $K_p > 0$ such that :

$$\mathbb{E}\left[\sup_{\theta \in [t,t']} \left|X^u_{t,x}(\theta) - x\right|^p\right] \leq K_p(1 + |x|^p)\ |t - t'|. \tag{2.25c}$$

The solution $X^u_{t,x}(\cdot)$ satisfies almost surely the following,

$$X^u_{t,x}(\cdot) = x + \int_t^{\cdot} f(s, X^u_{t,x}(s), u(s))ds + \int_t^{\cdot} \sigma(s, X^u_{t,x}(s), u(s))d\mathcal{W}(s).$$

We denote by $X^u_x(\cdot)$ the solution of (2.24) for autonomous systems ($t = 0$).

Roughly speaking, the control problem on a finite time interval is to minimize the following bolza functional :

$$J(t, x, u) = \mathbb{E}\left[\int_t^T \ell(s, X^u_{t,x}(s), u(s))ds + \varphi(X^u_{t,x}(T))\right]. \tag{2.26}$$

In the case of infinite time horizon, the control problem consists of minimizing :

$$J(x,u) = \mathbb{E}\bigg[\int_0^{+\infty} e^{-\lambda s}\ell(X_x^u(s),u(s))ds\bigg]. \tag{2.27}$$

Let us consider the following assumptions on the running cost ℓ and terminal cost φ

$$\ell \text{ is continuous in all its arguments and Lipschitz continuous with respect to } x, \tag{A_3}$$

$$\varphi \text{ is a Lipschitz continuous function.} \tag{A_4}$$

In a finite time horizon setting, the value function denoted by ϑ is the map that associates to any $t \in [0,T]$ and $x \in \mathbb{R}^d$ the optimal value in (2.26), that is,

$$\vartheta(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}\bigg[\int_t^T \ell(s,X_{t,x}^u(s),u(s))ds + \varphi(X_{t,x}^u(T))\bigg]. \tag{2.28}$$

If the time horizon is infinite, the value function ϑ is a function of x and it associates to x the optimal value in (2.27), that is,

$$\vartheta(x) = \inf_{u\in\mathcal{U}} \mathbb{E}\bigg[\int_0^{+\infty} e^{-\lambda s}\ell(X_x^u(s),u(s))ds\bigg].$$

Here, an obvious question is the existence of optimal control u^* such that,

$$\vartheta(t,x) = J(t,x,u^*),$$

for the case of finite case, and $\vartheta(x) = J(x,u^*)$ for the infinite case. The following discussions recall some classical results for the existence of optimal controls.

2.2.1 Existence of optimal controls

The existence of optimal controls is an important topic in optimal control theory. In a deterministic setting, the compactness of trajectories that follows from a convexity property of the dynamics, ensures the existence of an optimal control. In a stochastic setting, the problem is more complicated and it depends on formulations of the control problem, namely, strong and weak formulations.

If the Brownian basis $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P, W(.)\}$ is fixed at the beginning and the control is a progressively measurable process with respect to the filtration, then the optimal control problem has a strong formulation. In this case, there is a lack of a compact structure on the set of stochastic trajectories. However, in the particular case of linear function b and σ of the state and the control variable, one can get the existence of optimal solution. More precisely, let us consider the following stochastic linear controlled system

$$\begin{cases} dX(t) = (AX(t) + Bu(t))dt + (CX(t) + Du(t))dW(t), & t \in (0,T), \\ X(0) = x. \end{cases} \tag{2.29}$$

where A, B, C and D are matrices of suitable sizes and $W(\cdot)$ is a one-dimensional Brownian motion. The finite horizon Bolza problem (2.28) is considered (with autonomous systems ($t = 0$) and ℓ independent of t). Consider the following hypotheses,

$$\begin{cases} (i)\ \ell \text{ and } \varphi \text{ are convex and the set } U \text{ is convex and closed.} \\ (ii)\ \text{For some } k_0,\ k_1 > 0, \\ \qquad \ell(x,u) \geq k_0|u|^2 - k_1,\ \varphi(x) \geq -k_1,\ \forall (x,u) \in \mathbb{R}^d \times U. \end{cases} \tag{A_5}$$

$$U \text{ is convex and compact and } \varphi \text{ and } \ell \text{ are convex.} \tag{A_6}$$

The following proposition is an existence result in the case of linear dynamics (see [101, Chapter II Theorem 5.2]).

Proposition 2.2.1 *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a fixed filtered probability space. Under either (A_5) or (A_6), if (2.28) is finite, then it admits an optimal control.*

If the probability space and the filtration vary, then they represent *parts* of the so-called weak control. Here, even for general optimal control problems, there is a possibility to prove the existence of the control in a "weak formulation" (see [91, 78]). More precisely, let us give the definition of a weak admissible control.

Definition 2.2.2 *A 6-tuple $\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W(\cdot), u(\cdot))$ is called a weak admissible control and $(X(.), u(.))$ is a weak admissible pair if*

- *$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a filtred probability space satisfying the usual conditions ;*
- *$W(\cdot)$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$;*
- *$u(\cdot)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in U.*
- *$X(\cdot)$ is the unique solution of the equation (2.24) in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ under $u(\cdot)$.*

Denote by Π the set of all weak admissible controls. Under the weak formulation, the stochastic optimal control problem can be stated as follows :

$$\inf_{\pi\in\Pi} \mathbb{E}_\pi \left[\int_t^T \ell(s, X^u_{t,x}(s), u(s))ds + \varphi(X^u_{t,x}(T)) \right], \tag{2.30}$$

where $\mathbb{E}_\pi$ denotes the expectation with respect to the probability measure $\mathbb{P}$. Consider the following convexity property, i.e, for every $t \in [0, T], x \in \mathbb{R}^d$

$$\begin{aligned}(f, \sigma\sigma^T, \ell)(t, x, U) := \Big\{ &(b_i(t,x,u), (\sigma\sigma^T)_{ij}(t,x,u), \ell(t,x,u)), \\ &i = 1, \ldots, d, j = 1, \ldots, m, u \in U \Big\},\end{aligned} \tag{A_7}$$

is a convex set. Then, the following proposition is an existence result of a control in a weak formulation setting ([101, Chapter II, Theorem 5.3]) :

Proposition 2.2.3 *Assume* (A_1)-(A_4) *hold. Let* (A_7) *be also satisfied. If* (2.30) *is finite, then it admits an optimal control in* Π.

Remark 2.2.4 *The result still works for the case of unbounded controls (see [67] and [83]).*

In the following, we report some classical results on the characterization of the value functions in infinite and finite time horizons using the Dynamic programming principle. For differential functions, it is also possible to characterize the value function as the unique solution of a nonlinear partial differential equation called second order Hamilton Jacobi Bellman equation.

2.2.2 Finite time horizon

The main idea of the dynamic programming approach is that the value function ϑ satisfies a functional equation called the *Dynamic Programming Principle* (DPP).

Theorem 2.2.1 *Assume* (A_1)-(A_4). *Then, for any* $(t,x) \in [0,T) \times \mathbb{R}^d$ *and for any* $\{\mathcal{F}_t\}$*-stopping time* θ *with values in* $[t,T]$

$$\vartheta(t,x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[\vartheta(\theta, X^u_{t,x}(\theta)) + \int_t^\theta \ell(s, X^u_{t,x}(s), u(s))ds\right].$$

If ϑ is two times differentiable, by using the Itô formula it is possible to prove that v is a solution of the following equation :

$$\begin{cases} -\partial_t \vartheta(t,x) + H(t,x,D_x\vartheta, D^2_x\vartheta) = 0 & t \in [0,T), x \in \mathbb{R}^d, \\ \vartheta(T,x) = \varphi(x) & x \in \mathbb{R}^d. \end{cases} \tag{2.31}$$

where we denoted by $D\vartheta$ and $D^2\vartheta$ respectively the gradient and the Hessian matrix of ϑ. The function $H : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}^d \times \mathbb{R}$ (with $\mathcal{S}^d$ we denote the set of $d \times d$ symmetric matrices), namely the Hamiltonian of the system, is defined by

$$H(t,x,p,Q) := \sup_{u \in U}\left\{ - f(t,x,u) \cdot p - \frac{1}{2} Tr[\sigma\sigma^T(t,x,u)Q] - \ell(t,x,u)\right\},$$

for any $t \in [0,T], x, p \in \mathbb{R}^d$ and $Q \in \mathcal{S}^d$.

2.2.3 Infinite time horizon

For the value function in infinite time horizon, it is still possible to establish the *Dynamic programming principle* and derive from it the appropriate Hamilton Jacobi Bellman equation.

Theorem 2.2.2 *Assume* (A_1)- (A_4). *Then, for any* $x \in \mathbb{R}^d$ *and for any* $\{\mathcal{F}_t\}$*-stopping time* $\theta \geq 0$

$$\vartheta(x) = \inf_{u \in \mathcal{U}} \mathbb{E}\left[e^{-\lambda\theta}\vartheta(X^u_x(\theta)) + \int_0^\theta e^{-\lambda t}\ell(X^u_x(t), u(t))dt\right].$$

If ϑ is twice differentiable, by using the Itô formula it is possible to prove that v is a solution of the following equation :

$$\lambda\vartheta(x) + H(x, \nabla\vartheta, \nabla^2\vartheta) = 0, \qquad x \in \mathbb{R}^d, \tag{2.32}$$

For $\sigma \equiv 0$, equations (2.31) and (2.32) are reduced to the first order equations corresponding to the deterministic control problems.
It is clear that the value function of an optimal control problem may not be always twice differentiable. In such cases, it cannot be expected that the value function should solve the HJB equation in any classical sense. Even so, it will be seen that the value function is a viscosity solution of the HJB equation and even more it will be the unique viscosity solution (see section 2.3 for more details). This nice notion of solution is relevant for optimal control problems and it provides a good framework to deal with existence and uniqueness of solutions for generalized HJB equations.

Let us point out that the optimal control problems described in this chapter are associated to problems in finance, aerodynamic and many other domains. Then, it would be relevant to try to solve numerically these concrete problems. The next subsection proposes to discuss the numerical approximations of the solutions of second order Hamilton Jacobi Bellman equation.

2.2.4 Numerical approximations

The first convergence result of numerical methods to viscosity solutions was given by Barles-Souganidis (see [17]). Error bounds for the second order HJB equation were found by Krylov [75] for a case where σ is a constant function. These results were developed further in [14, 15, 16] by introducing new tools that allow to consider the case where σ can depend on time, space and also on the control variable. Several other extensions of the theory have been analysed in the literature, let us mention some of the extensions for stopping-game problems [31], for impulsive control systems [32], for integro partial differential HJB equations [39, 24, 25], and for a general class of coupled HJB systems [34]. Note also that the case of fully uniformly elliptic operators have been also studied by Cafarelli-Souganidis [37] using a different approach than the one introduced by Krylov.

Let us report some findings in the appoximations of solutions of Hamilton Jacobi Bellman equations, in particular, the error bounds for numerical solution for second order HJB equations (see [14, 15, 16]).

We shall consider only the evolutionary case (by analogy the results still hold for stationary case). For this, let $\mathcal{G}_h$ be a grid and consider the scheme denoted by S,

$$\begin{aligned} S(h, x, t, v_h(x,t), [v_h]_{t,x}) &= 0, && \text{in } \mathcal{G}_h^+ := \mathcal{G}_h \backslash \{t = 0\}, \\ v_h(0, x) &= \varphi_h(x), && \text{in } \mathcal{G}_h^0 := \mathcal{G}_h \cap \{t = 0\}. \end{aligned}$$

where $[v_h]$ is a function defined from v_h representing the value of v_h at other points than (t, x) (e.g interpolation).

The following discussion is about an important result on the error estimates for second order HJB equation on unbounded domain. We should mention that this result is not directly applicable to the bounded domain problem. Consider the following HJB equation :

$$\partial_t \vartheta + H(t, x, D_x \vartheta(t, x), D_x^2 \vartheta(t, x)) = 0, \quad \text{on } Q_T := (0, T) \times \mathbb{R}^d,$$
$$\vartheta = \varphi, \quad \text{on } \{T\} \times \mathbb{R}^d.$$

Assume that g, σ, f are bounded uniformly in α in the following norm :

$$|u|_1 := |u|_{L^\infty(\mathbb{R}\times\mathbb{R}^d)} + [u]_1, \text{ and } [u]_1 := \sup_{(t,x)\neq(s,y)} \frac{|u(t,x) - u(s,y)|}{|x-y| + \sqrt{|t-s|}}.$$

Consider that the following hypotheses are satisfied for the scheme :

[Monotonicity]. There exists $\lambda, \mu \geq 0$, $h_0 \geq 0$ such that if $|h| < h_0$, $u \leq v$ are function $C_b(\mathcal{G}_h)$, and $\phi(t) = e^{\mu t}(a + bt) + c$ for $a, b, c \geq 0$, then for any $r \in \mathbb{R}$,

$$S(h, x, t, r + \phi(t), [u + \phi]) \geq S(h, x, t, r, [v]) + \frac{b}{2} - \lambda c, \quad \text{in } \mathcal{G}_h^+.$$

[Regularity] For every h and $\phi \in C_b(\mathcal{G}_h)$, the function $(t, x) \to S(h, t, x, \phi(t, x), [\phi]_{t,x})$ is bounded and continuous in $\mathcal{G}_h^+$ and the function $r \to S(h, t, x, r, [\phi]_{t,x})$ is uniformly continuous for bounded r, uniformly in $(t, x) \in \mathcal{G}_h^+$.

[Sub-consistency]. There exists a function $E_1(K, h, \varepsilon)$ such that for any sequence $\{\phi_\varepsilon\}_{\varepsilon>0}$ of smooth functions satisfying

$$|\partial_t^{\beta_0} D^{\beta'} \phi_\varepsilon(x, t)| \leq K\varepsilon^{1-2\beta_0-|\beta'|}, \quad \text{in } Q_T, \quad \text{for any} \quad \beta_0 \in N, \beta' = (\beta_i')_i \in N^N,$$

where $|\beta'| = \sum_{i=1}^N \beta_i$, the following inequality holds :

$$S(h, t, x, \phi_\varepsilon(t, x), [\phi_\varepsilon]_{t,x}) \leq \partial_t \phi_\varepsilon + H(t, x, D\phi_\varepsilon, D^2\phi_\varepsilon) + E_1(K, h, \varepsilon), \quad \text{in } \mathcal{G}_h^+.$$

[Super-consistency]. There exists a function $E_2(K, h, \varepsilon)$ such that for any sequence $\{\phi_\varepsilon\}_{\varepsilon>0}$ of smooth functions satisfying

$$|\partial_t^{\beta_0} D^{\beta'} \phi_\varepsilon(x, t)| \leq K\varepsilon^{1-2\beta_0-|\beta'|}, \quad \text{in } Q_T, \quad \text{for any} \quad \beta_0 \in N, \beta' = (\beta_i')i \in N^N,$$

the following inequality holds :

$$S(h, t, x, \phi_\varepsilon(t, x), [\phi_\varepsilon]t, x) \geq \partial_t \phi_\varepsilon + H(t, x, D\phi_\varepsilon, D^2\phi_\varepsilon) + E_2(K, h, \varepsilon), \quad \text{in } \mathcal{G}_h^+,$$

Then, the following result on the error estimate of the approximate solution is obtained.

Theorem 2.2.3 *Assume the assumptions hold. If the scheme admits a unique solution $v_h \in C_b(\mathcal{G}_h)$, then for h sufficiently small, the following inequalities hold.*

Upper bound. *There exists C depending on $|\sigma|_1$, $|f|_1$, $|\varphi|_1$, $|\ell|_1$ and μ for all $(x,t) \in \mathcal{G}_h$,*

$$\vartheta - v_h \leq e^{\mu t}|(\varphi - \varphi_h(.,T))^+|_\infty + C\min_{\varepsilon>0}\Big(\varepsilon + E_1(|\vartheta|_1, h, \varepsilon)\Big).$$

Lower bound. *There exists C depending on $|\sigma|_1$, $|f|_1$, $|\varphi|_1$, $|\ell|_1$ and μ for all $(x,t) \in \mathcal{G}_h$,*

$$\vartheta - v_h \geq -e^{\mu t}|(\varphi - \varphi_h(.,T))^+|_\infty - C\min_{\varepsilon>0}\Big(\varepsilon^{\frac{1}{3}} + E_2(|\vartheta|_1, h, \varepsilon)\Big).$$

It is important to remark that the disymmetry in the consistency hypothesis allows to describe how the bound are obtained from the consistency requirements using the "Krylov regularization". In [49], the authors try to improve the lower bound for a large class of Semi Lagrangian schemes without using switching systems.

In the following chapter, we will extend the theory of error estimates to the unbounded Lipschitz setting using again the "Krylov regularization" and some regularization technics combined with the consistency estimate for a class of Semi Lagrangian schemes.

2.3 Viscosity solutions

The theory of viscosity solutions allows continuous functions to be solutions of nonlinear HJB equations of first and second order as well. Moreover, it provides very general existence and uniqueness theorems and gives precise formulations of general boundary conditions.

2.3.1 First order case

Here, we present some definitions and classical results of the theory of the continuous viscosity solutions of the second order Hamilton Jacobi equation :

$$F(x, \vartheta, D\vartheta) = 0, \qquad x \in \mathcal{D}, \tag{2.34}$$

where $\mathcal{D}$ is an open set. The case $\mathcal{D} = \mathbb{R}^d$ corresponds to (2.9) and

$$F(x, r, p) := \lambda r + H(x, p),$$

and $\mathcal{D} = (0,T) \times \mathbb{R}^d$ to the evolutionary case of equation (2.7) and

$$F((t,x), r, (p_1, p)) := -p_1 + H((t,x), p).$$

Definition 2.3.1 (Viscosity solutions) *(i) A function u is viscosity supersolution of (2.34) if u is lower semicontinuous and for any $\varphi \in C^1(\mathcal{D};\mathbb{R})$:*

$$F(x,\varphi,D\varphi) \geq 0 \qquad \textit{on} \quad \mathcal{D},$$

for all local minimum point $x \in \mathcal{D}$ of $u-\varphi$.
(ii) A function u is viscosity subsolution of (2.34) if u is upper semicontinuous and for any $\varphi \in C^1(\mathcal{D};\mathbb{R})$:

$$F(x,\varphi,D\varphi) \leq 0 \qquad \textit{on} \quad \mathcal{D},$$

for all local maximum point $x \in \mathcal{D}$ of $u-\varphi$.

The function u is a viscosity solution of (2.34) if and only if it is both a viscosity supersolution and subsolution and the final condition is satisfied as well.

Viscosity solutions were introduced by Crandall and Lions in [46] (see also [45] for earlier contribution). The notions of viscosity solutions can be defined in an alternative way using the semi differentials (see [12, 13]).

The value function ϑ is then characterized as follows :

Proposition 2.3.2 *Assume the assumptions (H_1)-(H_2) and (H_4)-(H_5) hold. Then, the value function ϑ is the unique viscosity solution of (2.34).*

The uniqueness represents the hard part of the proof of the proposition (2.3.2) and it requires a comparison result between viscosity supersolutions and viscosity subsolutions. The classical method for proving the comparison principle is based on doubling variable technique (see [12, 13]). The idea of the proof consists of considering the maximum point $(x_\varepsilon, y_\varepsilon)$ of the following

$$u_1(x) - u_2(y) - \frac{|x-y|^2}{\epsilon},$$

for $\mathcal{D} = \mathbb{R}^d$ and for the evolutionary case $\mathcal{D} = (0,T) \times \mathbb{R}^d$, the maximum point $((t_\varepsilon, x_\varepsilon), (s_\varepsilon, y_\varepsilon)$ of the following

$$u_1(t,x) - u_2(s,y) - \frac{|x-y|^2 + |t-s|^2}{\epsilon},$$

The information given by the following test functions allow to obtain the desired comparison principle, namely,

$$\begin{aligned}
\phi_1(x) &:= u_2(y_\varepsilon) + \frac{|x-y_\varepsilon|}{\varepsilon}, \\
\phi_2(y) &:= u_1(x_\varepsilon) + \frac{|x_\varepsilon - y|}{\varepsilon},
\end{aligned}$$

for $\mathcal{D} = \mathbb{R}^d$ and for time depend case, the following viscosity test functions

$$\begin{aligned}
\phi_1(t,x) &:= u_2(s_\varepsilon, y_\varepsilon) + \frac{|x-y_\varepsilon|^2 + |t-s_\varepsilon|^2}{\varepsilon}, \\
\phi_2(s,y) &:= u_1(t_\varepsilon, x_\varepsilon) + \frac{|x_\varepsilon - y|^2 + |t_\varepsilon - s|^2}{\varepsilon}.
\end{aligned}$$

2.3.2 Second order case

The viscosity theory is not restricted to the first order HJB equations and it can be extended to the second order HJB equations. This subsection is devoted to the definitions and classical results of the theory of the continuous viscosity solutions of the first order Hamilton Jacobi equation :

$$F(x, v, Dv, D^2v) = 0, \qquad x \in \mathcal{D}, \tag{2.37}$$

where $\mathcal{D}$ is an open set in $\mathbb{R}^d$ and Dv and D^2v respectively the gradient and the Hessian matrix of v in a viscosity sense. The equation (2.32) corresponds to the case $\mathcal{D} = \mathbb{R}^d$ and

$$F(x, r, p, Q) := \lambda r + H(x, p, Q),$$

and the equation (2.31) to $\mathcal{D} = (0, T) \times \mathbb{R}^d$ and

$$F((t, x), r, (p_1, p), Q) := -p_1 + H(t, x, p, (Q_{ij})_{i,j\geq 2}).$$

Definition 2.3.3 (Viscosity solutions) *(i) An usc function v on $\mathcal{D}$ is a* viscosity sub-solution *of (2.37), if for each function $\varphi \in C^2(\mathcal{D})$, at each maximum point x of $v - \varphi$ the following inequalities hold*

$$F(x, \varphi, D\varphi, D^2\varphi) \leq 0 \qquad \text{on} \quad \mathcal{D}.$$

(ii) A lsc function v on $\mathcal{D}$ is a super-solution *of (2.37), if for each function $\varphi \in C^2(\mathcal{D})$, minimum point x of $v - \varphi$ the following inequalities hold*

$$F(x, \varphi, D\varphi, D^2\varphi) \geq 0 \qquad \text{on} \quad \mathcal{D}.$$

A continuous function v is a viscosity solution of (2.37) if it is both a sub- and super-solution.

In [82, 81], Lions extends the notion of viscosity to the second order case (stochastic case) and characterizes the value function associated to optimal control problems for controlled diffusion as the unique continuous viscosity solution of a second order HJB equation.

It is well known that the uniqueness is obtained as a consequence of some comparison result between sub- and super-solutions. The classical method used for proving the comparison inequality for first order equations (deterministic setting) is not relevant here and the information given by the test functions on the second order derivatives is not sufficient to prove the result. In [74], Jensen gives the comparison principle for a family of equations (2.37) independent of x using the notions of sup and inf-convolution approximations. Since, many contribution have been added to this first work on uniqueness of the solution of second order HJB equations (see [69, 70, 71, 73]). Let us introduce another equivalent definition of viscosity solution involving the notion of semijets (see [101, Proposition 5.6] for the equivalence with definition 2.3.3).

Definition 2.3.4 *(i) An usc function v on $\mathcal{D}$ is a* viscosity sub-solution *of* (2.37) *if*

$$F(x, v, p, X) \leq 0 \qquad \text{for any } x \in D, (p, X) \in \overline{\mathcal{J}}^{2,+} v(x),$$

where the closed superjet *of v at some point x is defined by*

$$\begin{aligned} \overline{\mathcal{J}}^{2,+} v(x) :=& \big\{ (p, X) \in \mathbb{R}^d \times \mathcal{S}^d : \quad \exists x_n \to x, \exists (p_n, X_n), \quad s.t. \\ & v(x) \leq v(x_n) + p_n \cdot (x - x_n) + \frac{1}{2} X_n (x - x_n) \cdot (x - x_n) + o\|x_n - x\|^2 \\ & \ \ and \ \ (p_n, X_n) \to (p, X) \big\}. \end{aligned}$$

(ii) A lsc function v on $\mathcal{D}$ is a viscosity super-solution *of* (2.37) *if*

$$F(x, v, p, X) \geq 0 \qquad \text{for any } x \in D, (p, X) \in \overline{\mathcal{J}}^{2,-} v(x),$$

where the closed superjet *of v at some point x is defined by*

$$\begin{aligned} \overline{\mathcal{J}}^{2,-} v(x) :=& \big\{ (p, X) \in \mathbb{R}^d \times \mathcal{S}^d : \quad \exists x_n \to x, \exists (p_n, X_n), \quad s.t. \\ & v(x) \geq v(x_n) + p_n \cdot (x - x_n) + \frac{1}{2} X_n (x - x_n) \cdot (x - x_n) + o\|x_n - x\|^2 \\ & \ \ and \ \ (p_n, X_n) \to (p, X) \big\}. \end{aligned}$$

A continuous function v is a viscosity solution of (2.37) *if it is both a sub- and super-solution.*

Remark 2.3.5 *The notions of sub and super-jets allow to extend the notions of semi differentials (sub and super differential leading to a definition of Dv and $D^2 v$ in the nonsmooth case) to the second order case.*

The "Crandall-Ishii lemma" is a very useful tool associated to the semijet which allows to avoid the explicit regularization by convolution of the solutions (see [44, Theorem 3.2] for more details) :

Lemma 2.3.6 (Crandall-Ishii lemma) *Let D_i be a locally compact subset of $\mathbb{R}^{d_i}$ for $i = 1, \ldots, k$, $D := D_1 \times \ldots \times D_k \subseteq \mathbb{R}^d$ $(d = d_1 + \ldots + d_k)$, $v_i \in USC(D_i)$ and φ be a twice continuously differentiable function in a neighborhood of D. Set*

$$v(x) := v_1(x_1) + \ldots + v_k(x_k),$$

for $x \equiv (x_1, \ldots, x_k) \in D$ and suppose that $\hat{x} \in D$ is a local maximum point for $v - \varphi$. Then for any $\alpha > 0$ there exists $X_i \in \mathcal{S}^{d_i}$ such that

$$(D_{x_i} \varphi(\hat{x}), X_i) \in \overline{\mathcal{J}}^{2,+} v_i(\hat{x}_i), \qquad \text{for } i = 1, \ldots, k,$$

and the following matrix inequalities hold

$$-\left(\frac{1}{\alpha} + \|D^2 \varphi(\hat{x})\| \right) I_d \leq \begin{pmatrix} X_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & X_k \end{pmatrix} \leq D^2 \varphi(\hat{x}) + \alpha (D^2 \varphi(\hat{x}))^2,$$

(where for a matrix $A \in \mathcal{S}^d$ we define $\|A\| := \sup\{|A\xi \cdot \xi| : |\xi| \leq 1\}$).

We refer to [44] for the case of parabolic HJ equations making use of the concept of parabolic semijets.

CHAPITRE III

ERROR ESTIMATES FOR SECOND ORDER H-J-B EQUATIONS WITH POSSIBLY DISCONTINUOUS AND UNBOUNDED DATA

Publications of this chapter

M. Assellaou, O. Bokanowski and H. Zidani, *Error Estimates for Second Order Hamilton-Jacobi-Bellman Equations. Approximation of Probabilistic Reachable Sets*, DCDS- Serie A, vol. 35(9), pp. 3933 - 3964, 2015.

3.1 Introduction

Throughout this chapter, we denote by $T > 0$ a fixed final horizon. Consider a controlled process $X^u_{t,x}$ satisfying :

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \quad \forall s \in [t, T], \\ X(t) = x, \end{cases} \tag{3.1}$$

where the diffusion σ and drift b are two Lipschitz continuous functions, $W(\cdot)$ is the classical Brownian motion, and u is a control function that takes values in a compact subset U of $\mathbb{R}^q$ $(q \geq 1)$. Under suitable assumptions on b, σ and on $\mathcal{U}$, equation (3.1) admits a unique solution (see Section 3.2, for precise assumptions). Now, consider the following control problem

$$\vartheta(t, x) := \sup_{u \in \mathcal{U}} \mathbb{E}\big[\Phi(X^u_{t,x}(T))\big], \tag{3.2}$$

where $\Phi : \mathbb{R}^d \to \mathbb{R}$ is measurable, with linear growth. In this chapter, we are interested in error estimates of numerical approximations of ϑ.

The first approximation that will be considered here is a very classical one that consists of introducing a family of Lipschitz continuous functions $(\Phi_\varepsilon)_\varepsilon$ converging pointwisely to Φ. Then the value function ϑ can be itself approximated by the value functions ϑ_ε defined as :

$$\vartheta_\varepsilon(t,x) := \sup_{u\in\mathcal{U}} \mathbb{E}\big[\Phi_\varepsilon(X^u_{t,x}(T))\big].$$

It is known that under quite general assumptions on the data and on Φ_ε, one can show that ϑ_ε converges pointwisely towards ϑ, when $\varepsilon \to 0$. In this chapter, we are also interested in the error estimate of $\vartheta - \vartheta_\varepsilon$ depending on the measure of the set where the two functions Φ and Φ_ε differ. The result that will be studied here is obtained under an ellipticity condition of the diffusion matrix. An extension to more general degenerate matrices is still a challenging problem that is not covered in this chapter.

The second step in the approximation of ϑ is to discretize the Hamilton-Jacobi-Bellman equation satisfied by ϑ_ε. Indeed ϑ_ε will be shown to be the unique continuous viscosity solution of :

$$\begin{aligned} -\partial_t\vartheta_\varepsilon + \mathcal{H}(t,x,D\vartheta_\varepsilon,D^2\vartheta_\varepsilon) = 0, &\qquad \text{in } (0,T)\times\mathbb{R}^d,\\ \vartheta_\varepsilon(T,x) = \Phi_\varepsilon(x), &\qquad \text{in } \mathbb{R}^d, \end{aligned}$$

where $\mathcal{H}(t,x,p,Q) := \sup_{a\in U}\big(-b(t,x,a)\cdot p - \mathrm{Tr}([\sigma\sigma^{\mathsf{T}}](t,x,a)Q)\big)$. In the case when the drift b and the diffusion σ are bounded and when the value function ϑ_ε is itself bounded, the error estimates of monotone schemes have been obtained first by Krylov [75] for a case where σ is a constant function. These results were developed further in [14, 15, 16] by introducing new tools that allow to consider the case where σ can depend on time, space and also on the control variable. Several other extensions of these errors have been analysed in the literature. Let us mention some of them for stopping-game problems [31], for impulsive control systems [32], for integro partial differential HJB equations [39, 24, 25], and for a general class of coupled HJB systems [34]. Note also that the case of fully uniformly elliptic operators have been also studied by Cafarelli-Souganidis [37] using a different approach than the one introduced by Krylov.

Here, we extend the theory of error estimates to the unbounded Lipschitz setting. The proof is still based on "Krylov regularization" and on some refined consistency estimates. To the best of our knowledge, this is the first result in the case where b, σ and the solution to the HJB equation itself are unbounded with respect to the space variable (with linear growth).

The chapter is organized as follows : Section 3.2 introduces the notations and the setting of the control problems (3.2). In section 3.3, we derive an error estimate for the value functions when the payoff function Φ is approximated by smooth functions. In section 3.4, we analyse the error estimates for a semi-Lagrangian scheme for the approximation of the value function.

3.2 Setting of the problem. Basic assumptions

Throughout this chapter, $|\cdot|$ denotes the Euclidean norm for any $\mathbb{R}^N$ type space, and $\mathbb{B}_R$ is the closed ball centred at the origin and with radius R.

For a given set $\mathcal{S} \subset \mathbb{R}^N$, the indicator function is given by $\mathbb{1}_{\mathcal{S}}(x) = 1$ if $x \in \mathcal{S}$ and $\mathbb{1}_{\mathcal{S}}(x) = 0$ otherwise. The distance function to $\mathcal{S}$ is $d(x, \mathcal{S}) = \inf\{|x-y| : \; y \in \mathcal{S}\}$. We also denote by $\mu(\mathcal{S})$ the measure of $\mathcal{S}$ with respect to Lebesgue's measure.
For any real valued function $\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, we say that $\varphi \in C^{k,l}([0,T] \times \mathbb{R}^d)$ (for non-negative integers k, l) iff all the partial derivatives $\partial^i_{t^i}\partial^j_{x^j}\varphi$, for $0 \leq i \leq k$ and $0 \leq j \leq l$, exist and are continuous functions. Moreover, we denote by $\|\varphi\|_0$ the norm given by :

$$\|\varphi\|_0 := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\varphi(t,x)|,$$

and for the matrix $D^k\varphi$ (the k-th derivative with respect to the variable x) :

$$\|D^k\varphi\|_0 := \max_{\alpha_i \geq 0, \; \sum \alpha_i = k} \|\frac{\partial^k}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}\varphi\|_0.$$

Let $\{\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a filtered probability space, $W(.)$ be a given m-dimensional Brownian motion, and $T > 0$. We denote by $\mathcal{U}$ the set of progressively measurable processes valued in $U \subset \mathbb{R}^q$ $(q \geq 1)$ where U is a non empty compact set. Let $(X^u_{t,x}(s))_{0\leq s\leq T}$ be a controlled process valued in $\mathbb{R}^d$ solution of the following stochastic differential equation :

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), & \forall s \in [t,T] \\ X(t) = x, \end{cases} \tag{3.3}$$

where $\sigma : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times m}$ and $b : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ are two continuous functions satisfying the following standard assumption :

(H1a) there exists $L_0 > 0$ such that for any $(s,t,x,y,u) \in [0,T]\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times U$, we have :

$$|b(t,x,u) - b(s,y,u)| + |\sigma(t,x,u) - \sigma(s,y,u)| \leq L_0(|x-y| + |t-s|^{\frac{1}{2}}).$$

For convenience, we assume also that $|\sigma(0,0,u)| + |b(0,0,u)| \leq L_0$ for any $u \in U$.

Assumption (H1a) imposes that b and σ are Lipschitz continuous with respect to x and $\frac{1}{2}$- Hölder continuous with respect to t.

For a part of the results that will be presented in this chapter, we will also need an ellipticity condition that we state as follows :

(H1b) σ depends only on (t,x) and there exists a real number $\Lambda \geq 1$, such that :

$$\forall (t,x) \in (0,T) \times \mathbb{R}^d, \quad \Lambda I_d \geq \sigma(t,x)\sigma(t,x)^T \geq \Lambda^{-1} I_d, \tag{3.4}$$

where I_d is the identity matrix and the inequalities (3.4) are in the sense of symmetric matrices : $\Lambda\|\xi\|^2 \geq \langle \xi, \sigma\sigma^T\xi\rangle \geq \Lambda^{-1}\|\xi\|^2$, $\forall \xi \in \mathbb{R}^d$.

Assumption (H1b) will be used in section 3.3. It is very useful to derive Aronson type estimates [50] on the density of probability associated with the process $X^u_{t,x}$, precise statement is given in Lemma 3.3.3.

Remark 3.2.1 *Note that more generally, assumption (H1b) can be replaced by a weak Hörmander condition where the diffusion takes part only in some components and the noise propagates through a chain of differential equations. Similar Aronson type estimates can then be obtained, following [50],and the results of the chapter could be generalized to this context.*

Throughout the chapter, we denote by $\Phi : \mathbb{R}^d \to \mathbb{R}$ a given final cost function satisfying the assumption :

(H2) Φ is measurable, and with linear growth, i.e, there exists $M_0 > 0$ such that :

$$|\Phi(x)| \leq M_0(1 + |x|) \qquad \text{a.e. } x \in \mathbb{R}^d.$$

Now, consider the following optimal control problem :

$$\vartheta(t, x) := \sup_{u \in \mathcal{U}} \mathbb{E}\big[\Phi(X^u_{t,x}(T))\big]. \tag{3.5}$$

Under assumptions (H1a)-(H2), the value function ϑ is well defined but it may be discontinuous. Moreover, according to [65], if Φ is upper semi-continuous (u.s.c) and under some additional convexity assumptions on the drift and the diffusion coefficients, ϑ is u.s.c and satisfies the following HJB equation :

$$-\partial_t \vartheta + \mathcal{H}(t, x, D\vartheta, D^2\vartheta) = 0 \qquad \text{in } (0, T) \times \mathbb{R}^d, \tag{3.6a}$$

$$\vartheta(T, x) = \Phi(x) \quad \text{in } \mathbb{R}^d. \tag{3.6b}$$

In this chapter, we are interested in the error estimates theory for numerical approximations of the value function ϑ. Since ϑ is discontinuous, we shall first introduce a regularized problem with a controlled error with respect to the original problem, and on which further analysis and numerical approximation will be more convenient. For this aim, we consider a family of regularized functions $(\Phi_\varepsilon)_{\varepsilon>0}$, and denote by $\mathcal{D}_\varepsilon$ (for any $\varepsilon > 0$) the set where Φ_ε and Φ take different values :

$$\mathcal{D}_\varepsilon := \{x \in \mathbb{R}^d \mid \Phi_\varepsilon(x) \neq \Phi(x)\}.$$

Then we consider the following assumption :

(H3) (i) For every $\varepsilon \in]0, 1]$, $\Phi_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L_\varepsilon \geq 0$,
(ii) there exists a constant $M_0 > 0$ (independent of ε), such that

$$|\Phi_\varepsilon(x)| \leq M_0(1 + |x|), \qquad x \in \mathbb{R}^d,$$

(iii) there exists a constant $M_1 > 0$ (independent of ε), such that for any $A > 0$

$$\mu(\mathcal{D}_\varepsilon \cap \mathbb{B}_A) \leq M_1\, A\, \varepsilon, \qquad \varepsilon \in]0, 1].$$

(The constant M_0 in (H3)-(ii) can be chosen to be the same constant as in (H2) without loss of generality.)

Of course the existence of such approximated Lipschitz continuous functions implicitly imposes some more requirements on the function Φ itself. However, (H3) is not too restrictive and is satisfied in many cases. For instance, it is possible to construct a family Φ_ε satisfying (H3) when the function Φ is piecewise Lipschitz continuous function with discontinuities lying in a union of compact regular submanifolds of dimension $d-1$. See also remark 4.2.2 or Chapter 4 for construction of such approximations in some particular cases.

Notice also that if $\mathcal{D}_\varepsilon \subset \mathbb{B}_A$ for some given $A \geq 1$ and for all $\varepsilon \in]0,1]$, then (H3)-(iii) is simply equivalent to assume that there exists $M_1 > 0$ such that $\mu(\mathcal{D}_\varepsilon) \leq M_1 \varepsilon$ for every $\varepsilon \in]0,1]$.

Now, consider an approximation of ϑ given by the value function associated to the following control problem :

$$\vartheta_\varepsilon(t,x) := \sup_{u \in \mathcal{U}} \mathbb{E}\big[\Phi_\varepsilon(X^u_{t,x}(T))\big]. \tag{3.7}$$

Under $(H3)$ and using the ellipticity condition $(H1b)$, we shall derive an error estimate of $\vartheta - \vartheta_\varepsilon$.

The next step will then consist in the obtention of error estimates for numerical approximation of ϑ_ε Lipschitz continuous function with linear growth in x variable. For every $\varepsilon > 0$, this new value function can be characterized as unique Lipschitz viscosity solution of the HJB equation :

$$-\partial_t \vartheta_\varepsilon + \mathcal{H}(t,x,D\vartheta_\varepsilon, D^2\vartheta_\varepsilon) = 0, \qquad \text{in } (0,T) \times \mathbb{R}^d, \tag{3.8a}$$

$$\vartheta_\varepsilon(T,x) = \Phi_\varepsilon(x) \quad \text{in } \mathbb{R}^d. \tag{3.8b}$$

The Lipschitz regularity is suitable for deriving the error estimates when the HJB equation is approximated by a monotone scheme. However, error estimates for second order HJB equations have been studied so far only for Lipschitz *bounded solutions, as well as* bounded *coefficients* (b, σ)*.* Here, we are concerned by the case of unbounded coefficients with respect to the x variable (which is the case of many real applications such as call options in mathematical finance). We recall here that throughout all the chapter the control set U is bounded.

3.3 The regularized problem

Notation. Throughout this sections and the following ones, the constant C will denote a generic positive real number that may depend only on T, d, m, L_0, K_0, K_p.

3.3.1 Error estimate for the regularization procedure

Here we focus on the error estimate between ϑ and the approximated value function ϑ_ε.

Theorem 3.3.1 *Assume (H1a), (H1b), (H2), and (H3). Let ϑ and ϑ_ε be the value functions defined respectively by* (3.5) *and* (3.7)*.*

(i) There exist a constant $C_0 > 0$ (depending only on $T, L_0, \Lambda, M_0, M_1$) and $\varepsilon_0 \in]0,1]$, such that for every $0 < \varepsilon < \varepsilon_0$ the following estimate holds :

$$|\vartheta(t,x) - \vartheta_\varepsilon(t,x)| \leq C_0 \frac{1 + |x|^2 + |\log \varepsilon|}{(T-t)^{d/2}}\ \varepsilon, \tag{3.9}$$

for every $0 \leq t < T$ and $x \in \mathbb{R}^d$.
(ii)Furthermore, if there exists $A > 0$ such that $\mathcal{D}_\varepsilon \subset \mathbb{B}_A$ for every $\varepsilon \in]0,1]$, then there exist $C_1, C_2 > 0$ (depending only on T, M_0, M_1 and A) such that the following estimate holds :

$$|\vartheta(t,x) - \vartheta_\varepsilon(t,x)| \leq \frac{C_1}{(T-t)^{d/2}}\, e^{-C_2 \frac{d(x,\mathcal{D}_\varepsilon)^2}{(T-t)}}\ \varepsilon, \tag{3.10}$$

for every $\varepsilon \in]0,1]$, for every $x \in \mathbb{R}^d$ and every $0 \leq t < T$.

Remark 3.3.2 *In particular, if there exists $A > 0$ such that $\mathcal{D}_\varepsilon \subset \mathbb{B}_A$ for every $\varepsilon > 0$, then Theorem 3.3.1 leads directly to the following bound (since $dist(x, \mathcal{D}_\varepsilon) \geq 0$) :*

$$|\vartheta(t,x) - \vartheta_\varepsilon(t,x)| \leq \frac{C_1}{(T-t)^{d/2}} \varepsilon, \tag{3.11}$$

for every $0 \leq t < T$ and every $x \in \mathbb{R}^d$. Moreover, by using (3.10) and the fact that $e^{-r} \leq C/r^{d/2}$ for all $r > 0$ (for some constant $C \geq 0$), we conclude that there exists $C_1' \geq 0$ depending on T, M_0, M_1, A such that :

$$|\vartheta(t,x) - \vartheta_\varepsilon(t,x)| \leq \frac{C_1'}{[dist(x, \mathcal{D}_\varepsilon)]^d}\ \varepsilon, \tag{3.12}$$

for every $0 \leq t \leq T$ and any $x \in \mathbb{R}^d \setminus \mathcal{D}_\varepsilon$.

Before giving the proof of Theorem 3.3.1, we first recall some known results on the *density of probability* of the process $X^u_{t,x}(\cdot)$, for a given $(t,x) \in [0,T)$ and an admissible control $u \in \mathcal{U}$. We will denote by $y \longmapsto p^u(t,x;s,y)$ the density of probability function associated to the process $X^u_{t,x}(s)$ (for a given admissible control $u \in \mathcal{U}$).

Lemma 3.3.3 *Assume (H1a) and (H1b). There exist $c_1, c_2, c_3 > 0$ such that for any $(t,s,x,y) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^d$ such that $t < s$, and for any admissible control $u \in \mathcal{U}$, the following estimate holds :*

$$|p^u(t,x;s,y)| \leq \frac{c_1}{(s-t)^{\frac{d}{2}}}\, e^{-c_2 \frac{|x-y|^2}{2(s-t)}}\, e^{c_3 |x|^2}. \tag{3.13}$$

Proof. Let $u \in \mathcal{U}$. From [50], there exists $c_1, c_2 > 0$ such that for every $0 \leq t < s \leq T$ and every $x, y \in \mathbb{R}^d$, we have :

$$p^u(t,x;s,y) \leq \frac{c_1}{(s-t)^{\frac{d}{2}}} \exp\left(- c_2 \frac{|\theta^u_{t,x}(s) - y|^2}{s-t} \right), \tag{3.14}$$

where $\theta(s) := \theta^u_{t,x}(s)$ is the solution of the deterministic differential equation :

$$\frac{d}{ds}\theta(s) = b(s, \theta(s), u(s)), \quad s \geq t,$$
$$\theta(t) = x.$$

Note that by [50, Theorem 1.1], the constants c_1 and c_2 depend only on d, Λ and do not depend neither on θ^u nor on the control. Therefore, the estimate (3.14) is valid for any control function $u \in \mathcal{U}$. By assumption (H1a) and some classical estimates, we get $|\theta^u_{t,x}(s) - x| \leq L_0(1+|x|)(s-t)e^{L_0(s-t)}$. On the other hand, by straightforward calculation, we get : $|x - y| \leq |\theta^u_{t,x}(s) - y| + |\theta^u_{t,x}(s) - x|$. Hence

$$-\frac{|\theta^u_{t,x}(s) - y|^2}{s-t} \leq -\frac{|x-y|^2}{2(s-t)} + 2L_0^2(1+|x|^2)(s-t)e^{2L_0(s-t)}.$$

Therefore, we obtain

$$\begin{aligned} p^u(t, x; s, y) &\leq \frac{c_1}{(s-t)^{\frac{d}{2}}}\, e^{-c_2\frac{|x-y|^2}{2(s-t)}} e^{c_3(1+|x|^2)}, \\ &\leq \frac{c_1 e^{c_3}}{(s-t)^{\frac{d}{2}}}\, e^{-c_2\frac{|x-y|^2}{2(s-t)}} e^{c_3|x|^2}, \end{aligned}$$

with $c_3 = 2c_2L_0^2Te^{2L_0T}$, which gives the desired upper bound.

The lower bound can be derived in the same way. ■

Now we turn to the proof of theorem 3.3.1.

Proof of Theorem 3.3.1 Let $t \in [0, T]$ and $x \in \mathbb{R}^d$. We have

$$\begin{aligned} |\vartheta(t, x) - \vartheta_\varepsilon(t, x)| &= \left| \sup_{u\in\mathcal{U}} \mathbb{E}[\Phi(X^u_{t,x}(T))] - \sup_{u\in\mathcal{U}} \mathbb{E}[\Phi_\varepsilon(X^u_{t,x}(T))] \right|, \\ &\leq \sup_{u\in\mathcal{U}} \mathbb{E}[\left|\Phi(X^u_{t,x}(T)) - \Phi_\varepsilon(X^u_{t,x}(T))\right|], \\ &\leq \sup_{u\in\mathcal{U}} \int_{\mathbb{R}^d} |\Phi(y) - \Phi_\varepsilon(y)|\, p^u(t, x; T, y)dy, \end{aligned}$$

where $y \to p^u(t, x; T, y)$ is the density of probability associated to the process $X^u_{t,x}(T)$ associated to a control function $u \in \mathcal{U}$. Since $\mathrm{supp}(\Phi - \Phi_\varepsilon) \subset \mathcal{D}_\varepsilon$, it comes :

$$|\vartheta(t, x) - \vartheta^\varepsilon(t, x)| \leq \sup_{u\in\mathcal{U}} \int_{\mathcal{D}_\varepsilon} |\Phi(y) - \Phi_\varepsilon(y)|\, p^u(t, x; T, y)dy. \tag{3.15}$$

We first consider the proof of (ii). We assume that $\mathcal{D}_\varepsilon \subset \mathbb{B}_A$ for some $A > 0$ and for every $\varepsilon > 0$. Then by taking into account Lemma 3.3.3 (using the fact that for every $y \in \mathcal{D}_\varepsilon$, we have $|x - y| \geq \mathrm{dist}(x, \mathcal{D}_\varepsilon)$), and by assumption (H3) (which implies also that for any $y \in \mathbb{R}^d$, $|\Phi(y) - \Phi_\varepsilon(y)| \leq 2M_0(1 + |y|)$), we get :

$$\begin{aligned} &\int_{\mathcal{D}_\varepsilon} (\Phi(y) - \Phi_\varepsilon(y))p^u(t, x; T, y)dy, \\ &\quad \leq 2M_0(1 + A)c_1(T-t)^{-\frac{d}{2}}\, e^{-c_2\frac{d(x,\mathcal{D}_\varepsilon)^2}{T-t}} e^{c_3A^2}\mu(\mathcal{D}_\varepsilon \cap \mathbb{B}_A), \\ &\quad \leq 2M_0M_1c_1(T-t)^{-\frac{d}{2}}\, e^{-c_2\frac{d(x,\mathcal{D}_\varepsilon)^2}{T-t}} e^{c_3A^2}(1 + A)A\,\varepsilon, \end{aligned} \tag{3.16}$$

for every $u \in \mathcal{U}$, which concludes to the desired bound for (ii).

We come back to the general case (i). For $\varepsilon \in]0,1]$, let $X_\varepsilon \in \mathbb{R}$ be the unique parameter such that $X_\varepsilon \geq \frac{d-1}{2c_2}$ (where c_2 is introduced in Lemma 3.3) and

$$X_\varepsilon^{(d-1)/2} e^{-c_2 X_\varepsilon} = \varepsilon. \tag{3.17}$$

Then, as $\varepsilon \to 0$, it holds $X_\varepsilon \sim \frac{1}{c_2}|\log \varepsilon|$ and therefore, $X_\varepsilon = O(|\log(\varepsilon)|)$. Introduce also the positive constant $R_\varepsilon := \sqrt{(T-t)X_\varepsilon}$. Let $u \in \mathcal{U}$ be fixed. Using the estimate (3.14), we obtain a first bound as follows :

$$\begin{aligned}
\int_{\mathcal{D}_\varepsilon \cap (\theta^u_{t,x}(T)+\mathbb{B}_{R_\varepsilon})} & (\Phi(y) - \Phi_\varepsilon(y)) p^u(t,x;T,y)dy, \\
&\leq 2M_0(1 + |\theta^u_{t,x}(T)| + R_\varepsilon) \frac{c_1}{(T-t)^{\frac{d}{2}}} \mu\Big(\mathcal{D}_\varepsilon \cap (\theta^u_{t,x}(T) + \mathbb{B}_{R_\varepsilon})\Big), \\
&\leq 2M_0 M_1 (1 + |\theta^u_{t,x}(T)| + R_\varepsilon)^2 \frac{c_1}{(T-t)^{\frac{d}{2}}}\, \varepsilon, \\
&\leq 2M_0 M_1 C (1 + |x| + R_\varepsilon)^2 \frac{c_1}{(T-t)^{\frac{d}{2}}}\, \varepsilon,
\end{aligned} \tag{3.18}$$

where we have used that $|\theta^u_{t,x}(T)| \leq C(1+|x|)$ for some constant $C > 0$ that only depends on T and L_0 (and does not depend on u). On the other hand,

$$\begin{aligned}
\int_{\mathcal{D}_\varepsilon \cap\ \mathbb{R}^d \setminus (\theta^u_{t,x}(T)+\mathbb{B}_{R_\varepsilon})} & (\Phi(y) - \Phi_\varepsilon(y)) p^u(t,x;T,y)dy, \\
&\leq \int_{\mathbb{R}^d \setminus (\theta^u_{t,x}(T)+\mathbb{B}_{R_\varepsilon})} |\Phi(y) - \Phi_\varepsilon(y))| p^u(t,x;T,y)dy, \\
&\leq \int_{|y-\theta^u_{t,x}(T)| \geq R_\varepsilon} 2M_0(1+|y|) \frac{c_1}{(T-t)^{d/2}}\, e^{-c_2 \frac{|\theta^u_{t,x}(T)-y|^2}{T-t}}\, dy, \\
&= \int_{|y| \geq R_\varepsilon} 2M_0(1 + |y + \theta^u_{t,x}(T)|) \frac{c_1}{(T-t)^{d/2}}\, e^{-c_2 \frac{|y|^2}{T-t}}\, dy, \\
&\leq 2c_1 M_0 \int_{|z| \geq \frac{R_\varepsilon}{\sqrt{T-t}}} (1 + |\theta^u_{t,x}(T)| + \sqrt{T-t}|z|) e^{-c_2|z|^2}\, dz.
\end{aligned}$$

On the other hand, we have the following Lemma (see the proof in Appendix .B) :

Lemma 3.3.4 *For any $\alpha \geq 0$, there exists a constant $q_\alpha > 0$ (depending also on c_2 and d), such that*

$$\int_{|z| \geq a, z \in \mathbb{R}^d} |z|^\alpha e^{-c_2|z|^2} dz \leq q_\alpha a^{\alpha+d-1} e^{-c_2 a^2}, \ \text{as } |a| \geq 1.$$

Hence, with $a := R_\varepsilon / \sqrt{(T-t)} = \sqrt{X_\varepsilon}$, it comes :

$$\begin{aligned}
\int_{\mathcal{D}_\varepsilon \cap\ \mathbb{R}^d \setminus (\theta^u_{t,x}(T)+\mathbb{B}_{R_\varepsilon})} & (\Phi(y) - \Phi_\varepsilon(y)) p^u(t,x;T,y)dy, \\
&\leq 2c_1 M_0 \Big(q_0 (1 + C(1+|x|)) X_\varepsilon^{\frac{d-1}{2}} + q_1 \sqrt{T-t} X_\varepsilon^{\frac{d}{2}} \Big) e^{-c_2 X_\varepsilon}, \\
&\leq 2c_1 M_0 C (1 + |x| + R_\varepsilon) X_\varepsilon^{\frac{d-1}{2}} e^{-c_2 X_\varepsilon},
\end{aligned} \tag{3.19}$$

for some constant $C \geq 0$, under the condition that $X_\varepsilon \geq 1$ (which is satisfied whenever ε is small enough).

By combining (3.18) and (3.19), and taking into account that the two estimates do not depend on the control variable u, we get for every ε small enough :

$$|\vartheta(t,x) - \vartheta^\varepsilon(t,x)| \leq 2(c_1+M_1)M_0C(1+|x|+R_\varepsilon)^2\Big((T-t)^{-d/2}\,\varepsilon + X_\varepsilon^{\frac{d-1}{2}}\,e^{-c_2 X_\varepsilon}\Big). \quad (3.20)$$

By using the definition of X_ε and its properties, we get :

$$\begin{aligned}|\vartheta(t,x) - \vartheta^\varepsilon(t,x)| &\leq 2(c_1+M_1)M_0C(1+|x|+R_\varepsilon)^2(1+(T-t)^{-d/2})\varepsilon,\\ &\leq 2(c_1+M_1)M_0C(1+|x|+C|\log\varepsilon|^{1/2})^2(1+(T-t)^{-d/2})\varepsilon,\\ &\leq C_0\frac{1+|x|^2+|\log\varepsilon|}{(T-t)^{d/2}}\varepsilon,\end{aligned}$$

where the constant $C_0 > 0$ depends only on $T, L_0, M_0, M_1, c_1, c_2$, which concludes the proof of (i).

■

3.3.2 Some regularity results for ϑ_ε

In this subsection, we provide an upper bound of the Hölder constant of ϑ^ε. By using the fact that the function Φ_ε is L_ε-Lipschitz continuous, we obtain the following :

Lemma 3.3.5 *Assume (H1a), (H2) and (H3). There exists a constant $C > 0$ such that for every $\varepsilon > 0$, the value function ϑ_ε satisfies :*

$$|\vartheta_\varepsilon(t,x) - \vartheta_\varepsilon(t,y)| \leq CL_\varepsilon|x-y|,$$

for all $x, y \in \mathbb{R}^d$, $t \in [0,T]$. Moreover,

$$|\vartheta_\varepsilon(t,x) - \vartheta_\varepsilon(s,x)| \leq CL_\varepsilon(1+|x|)\,|t-s|^{\frac{1}{2}}, \quad (3.21)$$

for all $x \in \mathbb{R}^d$, $t, s \in [0,T]$

Proof. (i) By straightforward calculations, we obtain :

$$\begin{aligned}|\vartheta_\varepsilon(t,x) - \vartheta_\varepsilon(t,y)| &\leq \sup_{u\in\mathcal{U}} |\mathbb{E}[\Phi_\varepsilon(X^u_{t,x}(T))] - \mathbb{E}[\Phi_\varepsilon(X^u_{t,y}(T))]|,\\ &\leq \sup_{u\in\mathcal{U}} \mathbb{E}[|\Phi_\varepsilon(X^u_{t,x}(T)) - \Phi_\varepsilon(X^u_{t,y}(T))|].\end{aligned}$$

Then by using the Lipschitz regularity of Φ_ε, it comes that :

$$|\vartheta_\varepsilon(t,x) - \vartheta_\varepsilon(t,y)| \leq L_\varepsilon \sup_{u\in\mathcal{U}} \mathbb{E}\big[\big|X^u_{t,x}(T) - X^u_{t,y}(T)\big|\big].$$

By using (2.25), we get the inequality :

$$|\vartheta_\varepsilon(t,x) - \vartheta_\varepsilon(t,y)| \leq K_0L_\varepsilon\,|x-y|.$$

(ii) Without loss of generality, we assume that $s = t + h$ for some $h > 0$. By using the definition of ϑ, we have :

$$
\begin{aligned}
|\vartheta_\varepsilon(t+h,x) - \vartheta_\varepsilon(t,x)| &\leq \sup_{u\in\mathcal{U}} \left| \mathbb{E}\left[\Phi_\varepsilon(X^u_{t+h,x}(T))\right] - \mathbb{E}\left[\Phi_\varepsilon(X^u_{t,x}(T))\right] \right|, \\
&\leq \sup_{u\in\mathcal{U}} \mathbb{E}\left[\mathbb{E}\left[\left|\Phi_\varepsilon(X^u_{t+h,x}(T)) - \Phi_\varepsilon(X^u_{t+h,X^u_{t,x}(t+h)}(T))\right| \mid \mathcal{F}_{t+h}\right] | \mathcal{F}_t\right], \\
&\leq \sup_{u\in\mathcal{U}} L_\varepsilon\, \mathbb{E}\left[\mathbb{E}\left[\left|X^u_{t+h,x}(T) - X^u_{t+h,X^u_{t,x}(t+h)}(T)\right| \mid \mathcal{F}_{t+h}\right] | \mathcal{F}_t\right].
\end{aligned}
$$

Finally, taking into account (2.25), we deduce that :

$$|\vartheta_\varepsilon(t+h,x) - \vartheta_\varepsilon(t,x)| \leq K_0^2 L_\varepsilon (1+|x|)\, h^{1/2}.$$

Therefore, taking any $C \geq \max(K_0, K_0^2)$, the desired result follows. ■

It is also known that ϑ_ε satisfies the following dynamic programming principle and the HJB equation :

Proposition 3.3.6 *Assume (H1), (H2) and (H3).*
(i) Let $(t,x) \in [0,T] \times \mathbb{R}^d$, and denote $\mathcal{T}_{[t,T]}$ the set of $(\mathcal{F}_\theta)_{\theta\in[t,T]}$-adapted stopping times with values a.e. in $[t,T]$. Let $\{\tau^u; u \in \mathcal{U}\}$ be a subset of $\mathcal{T}_{[t,T]}$ (independent of $\mathcal{F}_t$). Then

$$\vartheta_\varepsilon(t,x) = \sup_{u\in\mathcal{U}} \mathbb{E}[\vartheta_\varepsilon(\tau^u, X^u_{t,x}(\tau^u))]. \tag{3.22}$$

(ii) The function ϑ_ε is the unique continous viscosity solution (see definition 3.3.7), with linear growth, of the following HJB equation :

$$-\partial_t\vartheta_\varepsilon + \mathcal{H}(t,x,D\vartheta_\varepsilon, D^2\vartheta_\varepsilon) = 0, \qquad \text{in } (0,T)\times\mathbb{R}^d, \tag{3.23a}$$

$$\vartheta_\varepsilon(T,x) = \Phi_\varepsilon(x) \quad \text{in } \mathbb{R}^d, \tag{3.23b}$$

where $\mathcal{H}$ denotes the Hamiltonian function defined by :

$$\mathcal{H}(t,x,p,Q) := \inf_{u\in U}\left\{-\frac{1}{2}Tr(\sigma(t,x,u)\sigma^T(t,x,u)\,Q) - b(t,x,u)\cdot p\right\}, \tag{3.24}$$

for every $t\in[0,T], x\in\mathbb{R}^d, p\in\mathbb{R}^d$ and for every symmetric $d\times d$-matrix Q.

Definition 3.3.7 *A usc function $\overline{\vartheta}$ (resp. lsc function $\underline{\vartheta}$) on $[0,T]\times\overline{\mathbb{R}}^d$ is a viscosity sub-solution (resp. super-solution) of (3.23), if for each function $\varphi \in C^{1,2}([0,T]\times\overline{\mathbb{R}}^d)$, at each maximum (resp. minimum) point (t,x) of $\overline{\vartheta}-\varphi$ (resp. $\underline{\vartheta}-\varphi$) the following inequalities hold*

$$\begin{cases} -\partial_t\varphi + \mathcal{H}(t,x,D_x\varphi,D_x^2\varphi) \leq 0 & \text{in } [0,T)\times\mathbb{R}^d, \\ \min\left(-\partial_t\varphi + \mathcal{H}(t,x,D_x\varphi,D_x^2\varphi), \overline{\vartheta}-\Phi_\varepsilon\right) \leq 0 & \text{on } \{T\}\times\mathbb{R}^d. \end{cases}$$

$$\Bigg(\textit{resp.}$$

$$\begin{cases} -\partial_t\varphi + \mathcal{H}(t,x,D_x\varphi,D_x^2\varphi) \geq 0 & \text{in } [0,T)\times\mathbb{R}^d, \\ \max\left(-\partial_t\varphi + \mathcal{H}(t,x,D_x\varphi,D_x^2\varphi), \overline{\vartheta}-\Phi_\varepsilon\right) \geq 0 & \text{on } \{T\}\times\mathbb{R}^d. \end{cases} \Bigg)$$

The proof of Proposition 3.3.6 can be found in [61, Chapter 5]. For the uniqueness of viscosity solution we will use in particular the following comparison principle that holds for unbounded solutions (see [48] for the proof).

Proposition 3.3.8 *Let $v_1, v_2 : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and assume that v_1 is u.s.c and v_2 is l.s.c, that there exists a constant $c \geq 0$ and $p \geq 1$ such that $v_1 \geq -c(1+|x|^p)$ and $v_2 \leq c(1+|x|^p)$ for all $x \in \mathbb{R}^d$, and that v_1 and v_2 are respectively viscosity subsolution and supersolution of (3.23). Then $v_1(t,x) \leq v_2(t,x), \forall (t,x) \in [0,T] \times \mathbb{R}^d$.*

3.4 Error estimate for numerical approximations by a Semi-Lagrangian scheme

3.4.1 Time semi-discrete scheme

We aim at approximating v, the unique continous viscosity solution, with linear growth, of the following HJB equation :

$$-\partial_t v + \mathcal{H}(t,x,Dv,D^2v) = 0, \qquad \text{in } (0,T) \times \mathbb{R}^d, \tag{3.25a}$$

$$v(T,x) = \phi(x) \quad \text{in } \mathbb{R}^d. \tag{3.25b}$$

where $\mathcal{H}$ is the same Hamiltonian function as in (3.24). This is the same as HJB equation (3.23) but with a general terminal data ϕ instead of Φ_ε. Throughout this section, we assume satisfied the assumption $(H1)$ on the drift and diffusion coefficients b, σ. Also ϕ is assumed to be a Lipschitz continuous function, with Lipschitz constant L_ϕ. The Hölder constant of v will be denoted by L_v.

We aim to give new error estimates for semi-Lagrangian schemes [38], in the case of Lipschitz continuous b and σ yet that can be unbounded (as well as the solution v itself).

For convenience, we will denote by σ_k the column vectors of the matrix σ :

$$\sigma(t,x,a) = \big[\sigma_1, \ldots, \sigma_m\big](t,x,a),$$

For $p > 0$ and a smooth function ψ defined on $[0,T] \times \mathbb{R}^d$, we have the following approximation of Camili and Falconi (see [38]),

$$\begin{aligned} &\frac{1}{2}Tr[(\sigma(t,x,a)\sigma^T(t,x,a)\ D^2\phi] + b(t,x,a)D\psi \\ :\simeq &\sum_{k=1}^{m} \frac{[\psi](t,x+\frac{p}{m}b(t,x,a)+\sqrt{p}\sigma_k(t,x,a)) - 2[\psi](t,x) + [\psi](t,x+\frac{p}{m}b(t,x,a)-\sqrt{p}\sigma_k(t,x,a))}{2p} \end{aligned}$$

where $[B]$ is the interpolation of B. Let $h := \frac{p}{m}$ be the time step. We obtain,

$$\begin{aligned} &\frac{1}{2}Tr[(\sigma\sigma^T)(t,x,a)\ D^2\phi] + b(t,x,a)D\psi \\ :\simeq &\sum_{k=1}^{m} \frac{[\psi](t,x+h\ b(t,x,a)+\sqrt{hm}\ \sigma_k(t,x,a)) - 2[\psi](t,x) + [\psi](t,x+h\ b(t,x,a)-\sqrt{hm}\ \sigma_k(t,x,a))}{2mh} \\ :\simeq &\sum_{k=1}^{2m} \frac{[\psi](t,x+h\ b(t,x,a)+\sqrt{h}\ \bar{\sigma}_k(t,x,a)) - [\psi](t,x)}{2mh}, \end{aligned}$$

where the vectors $(\bar{\sigma}_k)_{k=1,\ldots,2m}$ are defined by :

$$\bar{\sigma}_k(t,x,a) := (-1)^k \sqrt{m}\ \sigma_{\lfloor \frac{k-1}{2} \rfloor}(t,x,a), \tag{3.26}$$

(where $\lfloor p \rfloor$ denotes the integer part of $p \in \mathbb{R}$). Consider the following operator,

$$L_h^a[\psi](t,x) := \frac{1}{2m} \sum_{i=1}^{2m} \frac{\psi(t, x + h\ b(t,x,a) + \sqrt{h}\ \bar{\sigma}_k(t,x,a)) - \psi(t,x)}{h},$$

and the approximation of the time derivative of ψ, i.e,

$$\partial_t \psi(t,x) \simeq \frac{\psi(t+h,x) - \psi(t,x)}{h}.$$

It follows from the equation (3.25) and the above approximations that,

$$\psi(t+h,x) = \psi(t,x) + h \max_{a \in U} L_h^a[\psi](t,x)$$

Then, one can propose the following semi-discrete SL scheme (for $x \in \mathbb{R}^d$) :

$$V^N(x) = \phi(x), \tag{3.27a}$$

$$V^{n-1}(x) = \mathcal{S}^h(t_n, x, V^n), \text{ for every } n = N, \ldots, 1, \tag{3.27b}$$

with, for any $t \in [0,T]$, $x \in \mathbb{R}^d$, and any function $w : \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{S}^h(t,x,w) := \frac{1}{2m} \max_{a \in U} \left\{ \sum_{k=1}^{2m} w(x + hb(t,x,a) + \sqrt{h}\bar{\sigma}_k(t,x,a)) \right\}.$$

By V we will denote the linear interpolation of $V^0, \cdots, V^n$ on $t_0, \cdots, t_N$.

The main result of this section is the following :

Theorem 3.4.1 *Assume that (H1a) is satisfied and that ϕ is Lipschitz continuous function with Lipschitz constant L_ϕ. There exists $C \geq 0$, $\forall n \in [0, \ldots, N]$,*

$$|V^n(x) - v(t_n,x)| \leq C L_\phi \,(1 + |x|)^{7/4}\, h^{1/4}.$$

The above theorem is an extension to the error estimates known in the literature for bounded Hölder continuous value functions with bounded and Lipschitz continuous drift b and diffusion σ, see [14, 16, 49]. The proof given here is based on classical shaking and regularization techniques introduced by Krylov [75, 76] combined with a precise consistency estimate and an interpretation of the numerical scheme as value function of a discrete-time control problem.

Remark 3.4.1 *More precisely if we assume, for some constant $L_{0,0}, L_{0,1} \geq 0$:*

$$|b(t,x,a)| + |\sigma(t,x,a)| \leq L_{0,0} + L_{0,1}|x|, \quad |\phi(x)| \leq L_{0,0} + L_{0,1}|x|, \tag{3.28}$$

for every $(t,x,a) \in [0,T] \times \mathbb{R}^d \times U$, then there exists a constant $C \geq 0$, such that :

$$|V^n(x) - v(t_n,x)| \leq C L_\phi \,(1 + L_{0,0} + L_{0,1}|x|)^{7/4}\, h^{1/4}.$$

In particular if ϕ, b and σ are bounded functions then the previous estimates hold with $L_{0,1} := 0$ and we find the usual error estimate bounded by $h^{1/4}$ up to a universal constant (i.e. no growth term in $|x|^{7/4}$).

Properties of (3.27)

First, we derive the following consistency property :

Lemma 3.4.2 *For any regular function $\varphi \in C^{2,4}([0,T]\times\mathbb{R}^d)$, denoting $\varphi^n(x) = \varphi(t_n,x)$ and $\mathcal{E}^n_\varphi(x)$ as*

$$\mathcal{E}^n_\varphi(x) := -\partial_t\varphi(t_n,x) + \mathcal{H}(t_n,x,D\varphi,D^2\varphi) - \frac{\varphi^{n-1}(x) - \mathcal{S}^h(t_n,x,\varphi^n)}{h},$$

where $\mathcal{S}^h$ is defined in (3.27), it holds

$$|\mathcal{E}^n_\varphi(x)| \le C\,(\|\varphi_{tt}\|_0 + \sum_{k=2,3,4}\|D^k\varphi\|_0)\ \sup_{a\in U}\Big(|b(t_n,x,a)|^2 + |\sigma(t_n,x,a)|^4\Big)\ h,$$

where $C \ge 0$ is a constant independent of n, h and φ.

Proof. The result is straightforward by first using a Taylor expansion of fourth order of $\varphi(t-h, y+\sqrt{h}\bar{\sigma}_k(t,x,a))$ around $\varphi(t,y)$, where $y = x + b(t,x,a)h$, and then by using a second order Taylor expansion of the result around x. ■

In particular, by using the Lipschitz regularity of b and σ, and their linear growthness, it also holds that

$$|\mathcal{E}^n_\varphi(x)| \le C\,(\|\varphi_{tt}\|_0 + \sum_{k=2,3,4}\|D^k\varphi\|_0)\ (1+|x|^4)\ h.$$

Remark 3.4.3 *The factor $\sqrt{m}$ in (3.26) is in order that the consistency estimate of Lemma 3.4.2 holds.*

Remark 3.4.4 *More precisely if we assume (3.28) for some constant $L_{0,0}, L_{0,1} \ge 0$, then it also holds, for a constant $C \ge 0$:*

$$|\mathcal{E}^n_\varphi(x)| \le C\,(\|\varphi_{tt}\|_0 + \sum_{k=2,3,4}\|D^k\varphi\|_0)\ (1+L_{0,0}+L_{0,1}|x|)^4\ h. \tag{3.29}$$

Now, by considering $Q \in \{1,\dots,2m\}$ a random variable such that $\mathbb{P}[Q=k] = \frac{1}{2m}$, it follows that the scheme (3.27) is equivalent to :

$$V^{n-1}(x) = \max_{a\in U}\mathbb{E}\Big[V^n(x + hb(t_n,x,a) + \sqrt{h}\,\bar{\sigma}_Q(t_n,x,a))\Big].$$

For the sequel of the section, it will be useful to define recursively the Markov chain $Z^{k,a}_{n,x}$ as follows. For a given $x \in \mathbb{R}^d$, a given $k \ge n \ge 0$, a sequence of controls $a = (a_n,\dots,a_k,\dots)$ with $a_i \in U$, and a sequence $(Q_n, Q_{n+1},\dots,Q_k,\dots)$ of i.i.d. random variables with same law as Q,

- If $k = n$,

$$Z^{n,a}_{n,x} := x.$$

- If $k \geq n$,

$$Z_{n,x}^{k+1,a} := Z_{n,x}^{k,a} + hb(t_k, Z_{n,x}^{k,a}, a_k) + \sqrt{h}\bar{\sigma}_{Q_k}(t_k, Z_{n,x}^{k,a}, a_k).$$

Clearly, $Z_{n,x}^{k,a}$ will depend only of n, x, the first $k-n$ values of $(a_n, \dots, a_{k-1})$ and of $Q_n, \dots, Q_{k-1}$.

The scheme can then be written equivalently in the form

$$V^{n-1}(x) = \max_{a \in U} \mathbb{E}\Big[V^n(Z_{n,x}^{n+1,a_0})\Big]. \tag{3.30}$$

By standard estimates, there exists a constant $C > 0$ (depending on L_0, d, m and T) such that for all $x, y \in \mathbb{R}^d$, and for all $0 \leq n \leq k \leq N$ and $0 \leq m \leq N-k$, we have :

$$\max_{a_n} \mathbb{E}\Bigg[\max_{a_{n+1}} \mathbb{E}\Big[... \max_{a_{k-1}} \mathbb{E}\big[|Z_{n,x}^{k,a}|^4\big]...\Big]\Bigg] \leq C(1+|x|^4), \tag{3.31a}$$

$$\max_{a_n} \mathbb{E}\Bigg[\max_{a_{n+1}} \mathbb{E}\Big[... \max_{a_{k-1}} \mathbb{E}\big[|Z_{n,x}^{k,a} - Z_{n,y}^{k,a}|\big]...\Big]\Bigg] \leq C|x-y|, \tag{3.31b}$$

$$\max_{a_n,\dots,a_k} \mathbb{E}\Bigg[\max_{a_{n+1}} \mathbb{E}\Big[... \max_{a_{k+m-1}} \mathbb{E}\big[\big|Z_{n,x}^{k+m,a} - Z_{n,x}^{k,a}\big|\big] \leq C(1+|x|)(t_{k+m}-t_k)^{\frac{1}{2}}. \tag{3.31c}$$

For sake of completeness, a proof of the above estimates is given in Appendix C. Finally, we recall that the scheme is $\frac{1}{2}$-hölder in time and Lipschitz continuous in space :

Lemma 3.4.5 *There exists $C > 0$ (independent of h), for $0 \leq n \leq n+k \leq N$:*

$$|V^{n+k}(x) - V^n(y)| \leq CL_\phi\Big((1+|x|)(t_{n+k}-t_n)^{\frac{1}{2}} \; + \; |x-y|\Big),$$

for $x, y \in \mathbb{R}^d$.

Proof. By recursion we have

$$\begin{aligned} V^n(x) &= \max_{a_0} \mathbb{E}\Big[V^{n+1}(Z_{n,x}^{n+1,a_0})\Big], \\ &= \max_{a_0} \mathbb{E}\Big[\max_{a_1} \mathbb{E}\Big[\cdots \max_{a_{k-1}} \mathbb{E}\big[V^{n+k}(Z_{n,x}^{n+k,a})\big] \cdots\Big]\Big]. \end{aligned}$$

In particular, with $k = N-n$ and knowing that $V^N(x) = \phi(x)$, it comes

$$V^n(x) = \max_{a_0} \mathbb{E}\Big[\max_{a_1} \mathbb{E}\Big[\cdots \max_{a_{N-n-1}} \mathbb{E}\big[\phi(Z_{n,x}^{N,a})\big] \cdots\Big]\Big].$$

By using the property that $\max_{a_i} \mathbb{E}[f(a_i)] \leq \mathbb{E}[\max_{a_i} f(a_i)]$, we deduce that

$$|V^n(x) - V^n(y)| \leq L_\phi \mathbb{E}\Big[\max_a \big|Z_{n,x}^{N,a} - Z_{n,y}^{N,a}\big|\Big], \tag{3.32}$$

and in the same way

$$|V^{n+k}(x) - V^n(x)| \leq L_\phi \mathbb{E}\Big[\max_a \big|Z_{n+k,x}^{N,a} - Z_{n,x}^{N,a}\big|\Big]. \tag{3.33}$$

By combining the inequalities (3.32) and (3.33), and (3.31), we deduce the statement of Lemma 3.4.5. ∎

Upper bound

First, we consider a regular super-solution of (3.25), denoted w, and we aim to derive an upper bound for

$$e^n := V^n - w^n,$$

where $w^n(x) = w(t_n, x)$.

Lemma 3.4.6 *Let w be a regular super-solution of (3.25). For all $0 \le n \le N$ and every $x \in \mathbb{R}^d$, we have :*

$$\begin{aligned} e^n(x) \le & \max_{a_{n+1}} \mathbb{E}\Big[\max_{a_{n+2}} \mathbb{E}\Big[... \max_{a_N} \mathbb{E}\Big[e^N(Z^{N+1,a}_{n+1,x})\Big]...\Big]\Big], \\ & + h \sum_{n+1 \le k \le N} \max_{a_{n+1}} \mathbb{E}\Big[\max_{a_{n+2}} \mathbb{E}\Big[... \max_{a_{k-1}} \mathbb{E}\Big[\max_{a_{k-1}} \mathcal{E}^k_w(Z^{k,a}_{n+1,x})\Big]...\Big]\Big]. \end{aligned}$$

Proof. By definition of the scheme,

$$V^{n-1}(x) = \mathcal{S}^h(t_n, x, V^n) = \max_{a_n \in U} \mathbb{E}\Big[V^n(Z^{n+1,a_n}_{n,x})\Big],$$

and by the consistency estimate of Lemma 3.4.2 and the super-solution property, it comes that

$$w^{n-1}(x) \ge \mathcal{S}^h(t_n, x, w^n) - h\mathcal{E}^n_w(x) \; = \; \max_{a_n \in U} \mathbb{E}\Big[w^n(Z^{n+1,a_n}_{n,x})\Big] - h\mathcal{E}^n_w(x).$$

Therefore, for $e^n = V^n - w^n$ we get the estimate

$$e^{n-1}(x) \le \max_{a_n \in U} \mathbb{E}\Big[e^n(Z^{n+1,a_n}_{n,x})\Big] + h\mathcal{E}^n_w(x).$$

Hence

$$\begin{aligned} e^{n-1}(x) & \le \max_{a_n} \mathbb{E}\Big[\max_{a_{n+1}} \mathbb{E}\Big[e^{n+1}(Z^{n+2,a_{n+1}}_{n+1,Z^{n+1,a_n}_{n,x}})\Big] + h\mathcal{E}^{n+1}_w(Z^{n+1,a_n}_{n,x})\Big] + h\mathcal{E}^n_w(x), \\ & \le \max_{a_n} \mathbb{E}\Big[\max_{a_{n+1}} \mathbb{E}\Big[e^{n+1}(Z^{n+2,a}_{n,x})\Big]\Big] + h \max_{a_n} \mathbb{E}\Big[\mathcal{E}^{n+1}_w(Z^{n+1,a_n}_{n,x})\Big] + h\mathcal{E}^n_w(x), \\ & .. \\ & .. \\ & \le \max_{a_n} \mathbb{E}\Big[\max_{a_{n+1}} \mathbb{E}\Big[... \max_{a_{n+k}} \mathbb{E}\Big[e^{n+k}(Z^{n+k+1,a}_{n,x})\Big]...\Big]\Big], \\ & +h \sum_{0 \le j \le k} \max_{a_n} \mathbb{E}\Big[\max_{a_{n+1}} \mathbb{E}\Big[... \max_{a_{n+j-1}} \mathbb{E}\Big[\mathcal{E}^{n+j}_w(Z^{n+j,a_n}_{n,x})\Big]...\Big]\Big], \end{aligned}$$

(where we have denoted $\max\limits_{a_n \in U} \equiv \max\limits_{a_n}$, and the term j in the sum corresponds to $\mathbb{E}\big[\mathcal{E}_w^n(x)\big]$). We finally obtain :

$$
\begin{aligned}
e^{n-1}(x) \le \max_{a_n} \mathbb{E}\Big[\max_{a_{n+1}} \mathbb{E}\Big[\dots \max_{a_N} \mathbb{E}\Big[e^N(Z_{n,x}^{N+1,a}) \Big] \dots \Big]\Big], \\
+ h \sum_{n \le k \le N} \max_{a_n} \mathbb{E}\Big[\max_{a_n+1} \mathbb{E}\Big[\dots \max_{a_{k-1}} \mathcal{E}_w^k(Z_{n,x}^{k,a}) \Big] \dots \Big]\Big].
\end{aligned}
$$

The desired result follows by changing n into $n+1$. ■

The previous result holds for any smooth function w that is super-solution of (3.25). The viscosity solution v is just Hölder continuous. However it is possible to construct a regular function $w \equiv v_\eta$, *close* to v, and which is a classical supersolution of (3.25). More precisely, by using the shaking coefficients techniques introduced in [77] combined with a standard regularization by mollification, we have the following result.

Lemma 3.4.7 *Under assumption (H1a), for every $\eta > 0$ there exists a C^∞ function v_η such that v_η is a classical super-solution to* (3.25)*. Moreover, there exists $C > 0$ such that for every $\eta > 0$ the following estimates hold :*

$$
|v(t,x) - v_\eta(t,x)| \le C L_\phi (1 + |x|)\eta, \tag{3.34a}
$$

$$
\Big|\frac{\partial^k v_\eta}{dt^k}(t,x)\Big| \le \frac{C L_\phi}{\eta^{2k-1}}(1+|x|) \quad \textit{and} \quad \Big\|\frac{\partial^k v_\eta}{dx^k}\Big\|_0 \le \frac{C L_\phi}{\eta^{k-1}}, \tag{3.34b}
$$

for any $k \ge 1$, and for every $(t,x) \in [0,T] \times \mathbb{R}^d$.

The proof of this result can be found in [77] under the additional assumption that b, σ and ϕ are bounded functions. However the arguments used in [77] can be easily extended to the case when (H1a) is satisfied and ϕ is a Lipschitz function (not necessarily bounded). For convenience of the reader, the outline of the proof is given in Appendix A.
Now, we have all the ingredients to conclude the upper bound :

Proof of theorem 3.4.1 : upper bound of $V^n - v(t_n, \cdot)$
Let $\eta > 0$. By using Lemma 3.4.7, there exists $C \ge 0$ such that for every $x \in \mathbb{R}^d$, it holds :

$$
\begin{aligned}
&\mathcal{E}_{v_\eta}^n(x) \le C L_\phi (1 + |x|^5)\,\frac{h}{\eta^3}, \\
&\max_{a_n} \mathbb{E}\Big[\max_{a_{n+1}} \mathbb{E}\Big[\dots \max_{a_{N-1}} \mathbb{E}\Big[\big| V^N(Z_{n,x}^{N,a}) - v_\eta(T, Z_{n,x}^{N,a}) \big| \Big] \dots \Big]\Big] \le C L_\phi (1 + |x|)\eta.
\end{aligned}
$$

By applying the result of Lemma 3.4.6 with $w = v_\eta$, and taking into account estimates (3.31), we obtain :

$$
V^n(x) - v_\eta(t_n, x) \le C L_\phi (1 + |x|)\eta + T C L_\phi (1 + |x|^5)\,\frac{h}{\eta^3}.
$$

Therefore for $|x| \le R$, we can choose for η an optimal value of order $\eta \equiv Rh^{1/4}$ to derive the "upper" bound :

$$\|(V^n - v(t_n,\cdot))_+\|_{L^\infty(\mathbb{B}_R)} \le CL_\phi R^2 h^{1/4},$$

for any $n = 0, \cdots, N$, with C independent of h, ϕ and R. ■

Lower bound

Now, we aim at deriving the lower bound estimate for the semi-discrete scheme (3.27). For this, we will apply exactly the same techniques as used for the upper bound, reversing the role of the equation and the scheme. The key point is that the solution V of the semi-discrete scheme is also Hölder continuous. We first build a function V^η by considering a scheme with shaking coefficients :

$$V^\eta(t,x) = \max_{\substack{-\eta^2 \le e_1 \le 0 \\ |e_2| \le \eta \\ a \in U}} \mathbb{E}\Big[V^\eta(t+h, x + (hb + \sqrt{h}\bar{\sigma})(t+e_1, x+e_2, a))\Big], \qquad (3.35a)$$

$$\text{in } [-2\eta^2, T) \times \mathbb{R}^d,$$

$$V^\eta(T,x) = \phi(x), \quad \text{in } \mathbb{R}^d, \qquad (3.35b)$$

(where σ, b are extended in time interval $[-2\eta^2, T]$ in such way (H1a) is still valid). We define by convolution $V_\eta := V^\eta * \rho_\eta$ where ρ_η is a sequence of mollifiers defined by $\rho_\eta(t,x) := \frac{1}{\eta^{d+2}}\rho(\frac{t}{\eta^2}, \frac{x}{\eta})$ and with ρ such that $\{\rho_\eta\}_\eta$ is the sequence of mollifiers defined by

$$\rho_\eta = \frac{1}{\eta^{d+2}}\rho(\frac{t}{\eta^2}, \frac{x}{\eta}),$$

$$\rho \in C^\infty(\mathbb{R}^{d+1}), \ \rho \ge 0, \quad \text{supp } \rho \subset [0,1] \times B_1, \quad \int_{\mathbb{R}}\int_{\mathbb{R}^d} \rho(s,x)\, dxds = 1. \quad (3.36)$$

Then, by using the same arguments as in Appendix A, we get the following Lemma.

Lemma 3.4.8 *Under assumption (H1a), for every $\eta > 0$, V_η is a C^∞ function such that*

$$V_\eta(t,x) - \mathcal{S}^h(t+h, x, V_\eta(t+h,\cdot)) \ge 0 \qquad \forall (t,x) \in [0, T-h] \times \mathbb{R}^d.$$

Moreover, there exists $C > 0$ depending on T and L_0 such that for every $\eta > 0$ the following estimates hold for every $t \in [0,T]$ and every $x \in \mathbb{R}^d$:

$$|V(t,x) - V_\eta(t,x)| \le CL_\phi\ (1+|x|)\eta \qquad (3.37a)$$

$$|\frac{\partial^k V_\eta}{dt^k}(t,x)| \le \frac{CL_\phi}{\eta^{2k-1}}(1+|x|) \quad \textit{and} \quad \|\frac{\partial^k V_\eta}{dx^k}\|_0 \le \frac{CL_\phi}{\eta^{k-1}}, \qquad (3.37b)$$

for any $k \ge 1$.

By straithforward calculations, one can check that :

$$\mathcal{E}^n_{V_\eta}(x) \geq -\frac{CL_\phi}{\eta^3}(1+|x|^5)\,h.$$

By using the consistency estimate of Lemma 3.4.2, and the Hölder estimates on V_η (that can be inferred from the one of the scheme), we then deduce :

$$\begin{aligned}
-\partial_t V_\eta + \mathcal{H}(t_n,x,DV_\eta,D^2V_\eta) &= \frac{V^{n-1}_\eta(x) - \mathcal{S}^h(t_n,x,V^n_\eta)}{h} + \mathcal{E}^n_{V_\eta}(x),\\
&\geq \mathcal{E}^n_{V_\eta}(x),\\
&\geq -\frac{CL_\phi}{\eta^3}(1+|x|^5)\,h,
\end{aligned}$$

(for some constant $C \geq 0$). In the same way we establish the same estimate for any $t \in [t_{n-1}, t_n]$. Hence

$$-\partial_t V_\eta + \mathcal{H}(t,x,DV_\eta,D^2V_\eta) \geq -\frac{CL_\phi}{\eta^3}(1+|x|^5)\,h, \quad \forall (t,x) \in [0,T]\times\mathbb{R}^d. \tag{3.38}$$

Let ζ be the following function :

$$\zeta(t,x) := \bar{C}L_\phi e^{\lambda(T-t)}\Big((1+|x|^5)\frac{h}{\eta^3} + \sqrt{1+|x|^2}\eta\Big),$$

where $\lambda > 0$, $\bar{C} > 0$ will be fixed later on. The definitions of $\mathcal{H}$ and of ζ, and the linear growth of b and σ with respect to $|x|$, yield the following bounds :

$$\begin{aligned}
-\partial_t\zeta \;&=\; \lambda\zeta \geq \lambda\bar{C}e^{\lambda(T-t)}\Big((1+|x|^5)\frac{h}{\eta^3} + |x|\eta\Big),\\
\mathcal{H}(t,x,D\zeta,D^2\zeta) &\geq -K\bar{C}e^{\lambda(T-t)}\big(|x|^5\frac{h}{\eta^3} + |x|\eta\big),
\end{aligned}$$

for some constant $K \geq 0$ tht depends on L_0. In particular, choosing $\lambda := K+1$, it holds

$$-\partial_t\zeta + \mathcal{H}(t,x,D\zeta,D^2\zeta) \geq \bar{C}\Big((1+|x|^5)\frac{h}{\eta^3} + \sqrt{1+|x|^2}\eta\Big). \tag{3.39}$$

Combining (3.38) and (3.39), it comes for any $t \in [0,T]$ and $x \in \mathbb{R}^d$:

$$\begin{aligned}
&-\partial_t(V_\eta+\zeta) + \mathcal{H}(t,x,D(V_\eta+\zeta),D^2(V_\eta+\zeta)),\\
&\quad\geq -\partial_t V_\eta + \mathcal{H}(t,x,DV_\eta,D^2V_\eta) - \partial_t\zeta + \mathcal{H}(t,x,D\zeta,D^2\zeta),\\
&\quad\geq -CL_\phi(1+|x|^5)\frac{h}{\eta^3} + \bar{C}\Big((1+|x|^5)\frac{h}{\eta^3} + \sqrt{1+|x|^2}\eta\Big),\\
&\quad\geq 0,
\end{aligned} \tag{3.40}$$

for any $\bar{C} \geq CL_\phi$. On the other hand it also holds

$$\begin{aligned}
V_\eta(T,x) + \zeta(T,x) &\geq \phi(x) - CL_\phi(1+|x|)\eta + \bar{C}\sqrt{1+|x|^2}\eta,\\
&\geq 0,
\end{aligned} \tag{3.41}$$

for any $\bar{C} \geq \sqrt{2}CL_\phi$. Hence by choosing $\bar{C} := \sqrt{2}CL_\phi$, (3.40) and (3.41) hold and $V_\eta + \zeta$ is a viscosity super-solution of (3.25). Furthermore, $V_\eta + \zeta$ has a quadratic upper bound growth : $V_\eta(t,x) + \zeta(t,x) \leq c_a\ (1 + |x|^5)$ for some constant $c_a > 0$. The exact solution v is also a viscosity sub-solution of (3.25), with a linear growth (so a linear bound from below of the form $v(t,x) \geq -c_b(1+|x|$ for some constant $c_b > 0$). Therefore, according to the comparison principle stated in Lemma 3.3.8, it comes that $V_\eta + \zeta \geq v$ on $[0,T] \times \mathbb{R}^d$. By consequence,

$$v(t_n,x) - V^n(x) \leq \zeta(t_n,x) \leq \sqrt{2}CL_\phi e^{\lambda T}\big((1+|x|^5)\,\frac{h}{\eta^3} + \sqrt{1+|x|^2}\eta\big).$$

Finally, by choosing η such that $\eta^4 \equiv R^4 h$ and for $|x| \leq R$ we obtain the following reverse estimate, for some constant $C \geq 0$:

$$\|(v(t_n,.) - V^n)_+\|_{L^\infty(\mathbb{B}_R)} \leq CL_\phi\, R^2\, h^{1/4}.$$

This concludes the proof of theorem 3.4.1.

3.4.2 Fully discrete scheme

Now, Consider a spatial discretisation of $\mathbb{R}^d$ (which can be assumed uniform for simplicity) : for some given mesh steps $\Delta x_i > 0$, $x_i = i\Delta x \equiv (i_1\Delta x_1, \ldots, i_d\Delta x_d)$ with $i \in \mathbb{Z}^d$. We will denote $|\Delta x|$ the Euclidean norm of Δx, $\mathcal{G} := \{i\Delta x,\ i \in \mathbb{Z}^d\}$, and $\mathcal{G}_h := \{t_0, \ldots, t_N\} \times \mathcal{G}$.

Fully discrete scheme : for $n = N, \ldots, 1$, for all $x_i \in \mathcal{G}$:

$$V_i^{n-1} = V^{n-1}(x_i) = \frac{1}{2m} \max_{a\in U}\left\{\sum_{k=1}^{2m}[V^n](x_i + h\, b(t_n,x_i,a) + \sqrt{h}\,\bar{\sigma}_k(t_n,x_i,a))\right\}, \tag{3.42}$$

where $[V^n]$ denotes the bilinear interpolation of (V_i^n) on (x_i), and with

$$V_i^N = V^N(x_i) = \phi(x_i),\ \forall x_i \in \mathcal{G}. \tag{3.43}$$

Theorem 3.4.2 *Assume (H1a) and assume ϕ is a L_ϕ-Lipschitz continuous function. Let v be the continuous solution of (3.25) , and let V^Δ be the numerical solution satisfying the scheme (3.42), with $\Delta = (h, \Delta x)$ the time and space steps. There exists $C > 0$ depending only on T, L_0 such that for every $R > 0$, we have :*

$$\|v - V^\Delta\|_{L^\infty(\mathbb{B}_R)} \leq CL_\phi\left(R^2 h^{1/4} + \frac{|\Delta x|}{h}\right).$$

Proof. By theorem 3.4.1, we have already an error estimate between v and the solution V of the semi-discrete scheme (3.27). Now, the error between V and V^Δ is a classical result (see [49, 30] for details). ∎

3.4.3 Comments

The convergence and error estimates results of this section are still valid for more general schemes in the form of Markov chain approximations as (3.30) with general probability density (see also [79, 90]). Indeed, all the arguments developed in this section are mainly based on the formulation (3.30) and do not depend on the probability distribution of the random variable Q neither on the formulation as semi-Lagrangian scheme (3.27).

For the numerical simulations in Chapter 4, we will consider the scheme (related to Milstein's approximation) :

$$\mathcal{S}^h(t,x,w) := \frac{1}{2m} \max_{a\in U} \left\{ \sum_{k=1}^{2m} w(x + h\widetilde{b}(t,x,a) + \sqrt{h}\bar{\sigma}_k(t,x,a)) \right\} \tag{3.44}$$

where $\widetilde{b}(t,x,a) := \frac{1}{2}(b(t,x,a)+b(t,x+hb(t,x,a),a))$. This scheme is a little bit more precise for the approximation of the deterministic part of the processes when $b(\cdot,\cdot,a)$ is non-constant. The error estimates of the present section can be easily extended to such an approximation.

Appendix

Appendix A. Proof of Lemma 3.4.7

First, we notice that the functions σ and b are defined for times $t \in [0,T]$, but they can be extended to times $[-2\eta^2, T+2\eta^2]$ in such a way that assumption (H1a) still holds.

For any $\eta > 0$, let $\mathcal{E}$ be the set of progressively measurable processes (α,χ) valued in $[-\eta^2, 0] \times \mathbb{B}_\eta \subset \mathbb{R} \times \mathbb{R}^d$ that is,

$$\mathcal{E} := \left\{\text{prog. meas. process } (\alpha,\chi) \text{ valued in } [-\eta^2,0] \times \mathbb{B}_\eta\right\}.$$

Now, consider the function v^η associated to the perturbed control problem (with $\eta > 0$) :

$$v^\eta(t,x) := \inf_{\substack{u\in\mathcal{U},\\ (\alpha,\chi)\in\mathcal{E}}} \mathbb{E}\left[\phi\left(X_{t,x}^{u,(\alpha,\chi)}(T)\right)\right],$$

where $X_{t,x}^{u,(\alpha,\chi)}$ is the solution of the perturbed system of SDEs

$$\begin{cases} dX(s) = b(s+\alpha(s), X(s)+\chi(s), u(s))ds + \sigma(s+\alpha(s), X(s)+\chi(s), u(s))d\mathcal{B}(s), \\ X(t) = x. \end{cases}$$

By classical arguments, we can show that v^η is the unique viscosity solution of the perturbed HJB equation :

$$\begin{cases} -\partial_t v_t^\eta + \inf\limits_{-\eta^2\le s\le 0, |e|\le\eta} \mathcal{H}(t+s, x+e, Dv^\eta, D^2 v^\eta) = 0 \quad \text{in } Q_{\eta^2}, \\ v^\eta(T,x) = \phi(x) \quad \text{in } \mathbb{R}^d, \end{cases} \tag{3.45}$$

where $Q_{\eta^2} := (-\eta^2, T] \times \mathbb{R}^d$. Using similar argument to those used in Lemma 3.3.5, the function v^η satisfies the following relations :

$$|v^\eta(t,x) - v^\eta(t,y)| \leq CL_\phi|x-y|, \tag{3.46}$$

$$|v^\eta(t,x) - v^\eta(s,x)| \leq CL_\phi(1+|x|)|t-s|^{\frac{1}{2}}, \tag{3.47}$$

for all $x, y \in \mathbb{R}^d$, $t, s \in [0,T]$.

The bound on the difference between the pertubed function v^η and the value function v at every point (t,x) follows by the Lipschitz property of ϕ and (H1a)

$$\begin{aligned}
|v(t,x) - v^\eta(t,x)| &= \Big| \sup_{\substack{u\in\mathcal{U}\\(\alpha,e)\in\mathcal{E}}} \mathbb{E}\big[\phi(X^u_{t,x}(T))\big] - \sup_{\substack{u\in\mathcal{U}\\(\alpha,e)\in\mathcal{E}}} \mathbb{E}\big[\phi(X^{u,(\alpha,e)}_{t,x}(T))\big]\Big|,\\
&\leq \Big| \sup_{\substack{u\in\mathcal{U}\\(\alpha,e)\in\mathcal{E}}} \mathbb{E}\big[\phi(X^u_{t,x}(T)) - \phi(X^{u,(\alpha,e)}_{t,x}(T))\big]\Big|,\\
&\leq L_\phi \sup_{\substack{u\in\mathcal{U}\\(\alpha,e)\in\mathcal{E}}} \mathbb{E}\big[\big|X^u_{t,x}(T) - \phi(X^{u,(\alpha,e)}_{t,x}(T)\big|\big].
\end{aligned} \tag{3.48}$$

On the other hand, for every $\tau \in [t,T]$, we have :

$$\begin{aligned}
&\mathbb{E}\big[\big|X^u_{t,x}(\tau) - X^{u,(\alpha,\chi)}_{t,x}(\tau)\big|^2\big],\\
&\leq \mathbb{E}\Big[\Big| \int_t^\tau \big[b(s+\alpha(s), X^{u,(\alpha,\chi)}_{t,x}(s) + \chi(s), u(s)) - b(s, X(s), u(s))\big]ds,\\
&+ \int_t^\tau \big[\sigma(s+\alpha(s), X^{u,(\alpha,\chi)}_{t,x}(s) + \chi(s), u(s)) - \sigma(s, X(s), u(s))\big]dW(s)\Big|^2\Big].
\end{aligned}$$

With assumption (H1a), Cauchy-Schwartz inequality and Gronwall Lemma, we obtain that :

$$\mathbb{E}\big[\big|X^u_{t,x}(\tau) - X^{u,(\alpha,\chi)}_{t,x}(\tau)\big|^2\big] \leq C\eta^2. \tag{3.49}$$

By combining (3.48) and (3.49), we finally get :

$$|v(t,x) - v^\eta(t,x)| \leq L_\phi C\eta.$$

by a change of variables, we see that for $-\eta^2 \leq s \leq 0, |e| \leq \eta$, the function $v^\mu(\cdot - s, \cdot - e)$ is a supersolution of the following equation :

$$-\partial_t\varphi + \mathcal{H}(t,x,D\varphi,D^2\varphi) = 0 \quad \text{ in } (-\eta^2, T+s) \times \mathbb{R}^d. \tag{3.50}$$

In order to regularize v^η, we construct the following sequence $v_\eta = v^\eta * \rho_\eta$ where $\{\rho_\eta\}_\eta$ is the sequence of mollifiers defined by

$$\begin{aligned}
&\rho_\eta = \frac{1}{\eta^{d+2}}\rho(\frac{t}{\eta^2}, \frac{x}{\eta}),\\
&\rho \in C^\infty(\mathbb{R}^{d+1}),\ \rho \geq 0, \quad \text{supp}\,\rho \subset [0,1] \times B_1, \quad \int_{\mathbb{R}}\int_{\mathbb{R}^d} \rho(s,x)\,dxds = 1.
\end{aligned} \tag{3.51}$$

A Riemmann-sum approximation shows that $v_\eta(t,x)$ can be viewed as the limit of convex combinaisons of $v^\eta(t-s,x-e)$. By the stability result for viscosity supersolutions, and by using same arguments as in [14, Appendix A], we can conclude that v_η is itself a supersolution of (3.50).
Moreover, for a small $\eta << 1$ and using (3.46)-(3.47), we have

$$\begin{aligned}|v_\eta(t,x)-v^\eta(t,x)| &= \int_0^1\int_{\mathbb{B}_1}|v^\eta(t-\eta^2\tau,x-\eta z)-v^\eta(t,x)|\rho(\tau,z)\,dzd\tau,\\ &\leq CL_\phi\eta\int_0^1\int_{\mathbb{B}_1}\big((1+|x|+\eta|z|)\sqrt{\tau}+|z|\big)\rho(\tau,z)\,dzd\tau.\end{aligned}$$

Thus, we obtain for $\eta\in(0,1)$ that $|v_\eta(t,x)-v^\eta(t,x)|\leq CL_\phi(1+|x|)\eta$, which together with (3.48) yield to the desired estimate :

$$|v(t,x)-v_\eta(t,x)|\leq CL_\phi\eta(1+|x|)\quad \text{For all } t,x\in[0,T]\times\mathbb{R}^d,$$

for a positive constant $C>0$.

Bound estimates (3.34b) for the derivatives of v_η can be derived in straightforward way by using the definition of mollification and the Hölder estimates of v^η. see [32] for instance.

Appendix B. Proof of Lemma 3.3.4

By using spherical coordinates in $\mathbb{R}^d$ it first holds that,

$$I(a):=\int_{|z|\geq a,z\in\mathbb{R}^d}|z|^\alpha dz=|S_d|\int_a^\infty r^\beta e^{-c_2r^2}dr,\quad \text{with } \beta:=\alpha+d-1,$$

where $|S_d|$ denotes the surface of the unit sphere of $\mathbb{R}^d$. Then the following identity holds :

$$2(1+\frac{(1-\beta)r^2}{2c_2r^2})r^\beta e^{-c_2r^2}=\frac{-d}{c_2dr}(r^{\beta-1}e^{-c_2r^2}).$$

In this case when $a^2\geq 2\frac{\beta-1}{2c_2}=\frac{|\beta-1|}{c_2}$, and for $r\geq a$, we observe that $1\leq 2(1+\frac{(1-\beta)}{2c_2r^2})$, hence,

$$r^\beta e^{-c_2r^2}\leq\frac{-1}{c_2}\frac{d}{dr}(r^{\beta-1}e^{-c_2r^2}).$$

By integration over $r\in[a,\infty]$, we obtain $I(a)\leq\frac{|S_d|}{c_2}a^{\beta-1}e^{-c_2a^2}$. On the other hand, if $a\in J:=\big[1,\sqrt{\frac{|\beta-1|}{c_2}}\big]$, then $F(a):=I(a)/(a^{\beta-1}e^{-c_2a^2})$ is a continuous function on the interval J so it is bounded by some $q_\alpha>0$. We can furthermore choose $q_\alpha\geq\frac{|S_d|}{c_2}$. In all cases, for $a\geq 1$, it holds that $F(a)\leq q_\alpha$. Hence $I(a)\leq q_\alpha a^{\alpha+d-2}e^{-c_2a^2}$. Using that $a^{\alpha+d-2}\leq a^{\alpha+d-1}$ for $a\leq 1$, we obtain the desired result.

Appendix C. Proof of estimates (3.31)

Let us first prove, by recursion, the following estimate : there exists $C \geq 0$ such that, for any $p \in \{2, 4\}$, for $0 \leq n \leq k \leq N$ and $\forall x$ an $h \leq 1$:

$$\begin{aligned} & \max_{a_n} \mathbb{E}\left[\max_{a_{n+1}} \mathbb{E}\left[\dots \max_{a_{k-1}} \mathbb{E}\left[\left| Z^{k,a}_{n,x} \right| \right] \dots \right] \right], \\ & \leq e^{C(k-n)h} \left(\left| x \right|^p + C(k-n)h(1 + |x|^p) \right). \end{aligned} \tag{3.52}$$

For $p = 4$ and using that $(k - n)h \leq Nh = T$, this will give the desired estimate (3.31a).

We first start with the case $p = 2$. Using conditional expectations, let us first show that, for some constant $C_1, C_2 \geq 0$:

$$\max_{a_{k-1}} \mathbb{E}\big[|Z^{k,a}_{n,x}| \, \big| Z^{k-1,a}_{n,x} = y \big] \leq |y|^2 e^{C_1 h} + (1 + |y|^2) C_2 h. \tag{3.53}$$

Denoting $b(y) = b(t_k, y, a_k)$ as well as $\bar{\sigma}_q(y) = \bar{\sigma}_q(t_k, y, a_k)$, it holds,

$$\begin{aligned} \mathbb{E}\big[|Z^{k,a}_{n,x}| \, \big| Z^{k-1,a}_{n,x} = y \big] &= \mathbb{E}\big[\big| y + hb(y) + \sqrt{h} \bar{\sigma}_Q(y) \big|^2 \big] \\ &= \big| y + hb(y) \big|^2 + \mathbb{E} \big| \bar{\sigma}_Q(y) \big|^2, \end{aligned}$$

Where we have used that $\mathbb{E}[\bar{\sigma}_Q(y)] = 0$ by the definition of the random variable $\bar{\sigma}_Q$. Hence it holds, since $\bar{\sigma}_Q(y)$ has a linear growth in y :

$$\mathbb{E}\big[|Z^{k,a}_{n,x}| \, \big| Z^{k-1,a}_{n,x} = y \big] \leq \big| y + hb(y) \big|^2 + Ch(1 + |y|^2).$$

Notice that for $h \leq 1$ it holds, for the Euclidean norm, and for any vectors A and B of $\mathbb{R}^d$,

$$|A + hB|^2 \leq |A|^2(1 + h) + 2h|B|^2, \tag{3.54}$$

(using Cauchy-schwartz inequality and $h^2 \leq h$). Hence we obtain a bound of the form

$$\begin{aligned} & \max_{a_{k-1}} \mathbb{E}\left[\max_{a_{k-2}} \mathbb{E}\left[|Z^{k,a}_{n,x}|^2 \mid Z^{k-2,a}_{n,x} = y \right] \right], \\ & \leq \max_{a_{k-2}} \mathbb{E}\left[\left| Z^{k-1,a}_{n,x} \right|^2 e^{C_1 h} + C_2 h(1 + |y|^2) \right| Z^{k-2,a}_{n,x} = y \Big], \\ & \leq \left(|y|^2 e^{C_1 h} + C_2 h(1 + |y|^2) \right) e^{C_1 h} + C_2 h(1 + |y|^2), \\ & \leq |y|^2 e^{2C_1 h} + C_2 h(1 + |y|^2)(1 + e^{C_1 h}). \end{aligned} \tag{3.55}$$

By a recursion argument and since $Z^{n,a}_{n,x} = x$, it holds,

$$\begin{aligned} & \max_{a_n} \mathbb{E}\left[\max_{a_{n+1}} \mathbb{E}\left[\dots \max_{a_{k-1}} \mathbb{E}\left[\left| Z^{k,a}_{n,x} \right|^2 \dots \right] \right] \right], \\ & \leq e^{C_1(k-n)h} |x|^2 + C_2 h(1 + |x|^2) \sum_{j=0,\dots,k-n-1} e^{C_1 j h}, \\ & \leq e^{C_1(k-n)h} \left(|x|^2 + C_2(k-n)h(1 + |x|^2) \right). \end{aligned} \tag{3.56}$$

Now we turn to the case $p = 4$. Let us show that a similar estimate to (3.53) holds,

$$\max_{a_{k-1}} \mathbb{E}\left[\left| Z_{n,x}^{k,a} \right|^4 \mid Z_{n,x}^{k-1,a} = y \right] \leq |y|^4 e^{C_1 h} + (1 + |y|^4) C_2 h. \tag{3.57}$$

The rest of the proof of (3.52) then follows the same idea as for the case $p = 2$ and is left to reader.

To prove (3.57), assuming first that $d = 1$ to simplify the argument, denoting $A = y + hb(y)$ and $B = \bar{\sigma}_Q(y)$, we have

$$\begin{aligned} \mathbb{E}\left[\left| Z_{n,x}^{k,a} \right|^4 \mid Z_{n,x}^{k-1,a} = y \right] &= \mathbb{E}\left[\left| A + \sqrt{h} B^4 \right| \right], \\ &= |A|^4 + 16 h A^2 \mathbb{E}\big[B^2\big] + h^2 \mathbb{E}\big[B^4\big], \end{aligned}$$

where we have used that $\mathbb{E}[B] = \mathbb{E}[B^3] = 0$. Then, $\mathbb{E}[B^2] \leq C(1 + |y|^2)$ and $\mathbb{E}[B^4] \leq C(1 + |y|^4)$, it can be shown that $|y + hb(y)|^4 \leq |y|^4(1 + Ch) + Ch(1 + |y|^4)$ for some constant $C \geq 0$ (for instance by using twice (3.54)), and (3.57) is deduced from these estimates. The case $d \geq 1$ can be treated in a similar way.

The proof of (3.31b) can be obtained in a similar way as for the proof of (3.31a) for $p = 2$. It is first established that

$$\max_{a_n} \mathbb{E}\left[\max_{a_{n+1}} \mathbb{E}\left[\ldots \max_{a_{k-1}} \mathbb{E}\left[\left| Z_{n,x}^{k,a} - Z_{n,y}^{k,a} \right|^2 \right] \ldots \right] \right] \leq C \left| x - y \right|^2.$$

Then, using that $\mathbb{E}\big[|X|\big] \leq (\mathbb{E}\big[|X|^2\big])^{\frac{1}{2}}$.

Finally we consider the proof of (3.31c). In a complete manner as for the proof of (3.52), we can establish that for any given $x_0 \in \mathbb{R}^d$,

$$\begin{aligned} &\max_{a_n} \mathbb{E}\left[\max_{a_{n+1}} \mathbb{E}\left[\ldots \max_{a_{n+1}} \mathbb{E}\left[\left| Z_{n,x}^{k,a} - x_0 \right|^2 \right] \ldots \right] \right], \\ &\leq e^{C(k-n)h} \Big(|x - x_0|^2 + C(k - n) h (1 + |x|^2) \Big). \end{aligned}$$

In particular for $x_0 = x$, for some other constant $C \geq 0$, we obtain :

$$\begin{aligned} &\max_{a_n} \mathbb{E}\left[\max_{a_{n+1}} \mathbb{E}\left[\ldots \max_{a_{k-1}} \mathbb{E}\left[\left| Z_{n,x}^{k,a} - x \right|^2 \right] \ldots \right] \right], \\ &\leq C(1 + |x|^2)(k - n) h. \end{aligned} \tag{3.58}$$

By using (3.58) and the fact that for $y := Z_{n,x}^{k,a}$, $Z_{n,x}^{k+m,a} = Z_{k,y}^{m,a'}$ (with controls $a' = (a_k, a_{k+1}, \ldots, a_{k+m-1})$), we have

$$\max_{a_k} \mathbb{E}\left[\max_{a_{k+1}} \mathbb{E}\left[\ldots \max_{a_{k+m-1}} \mathbb{E}\left[\left| Z_{n,x}^{k+m,a} - Z_{n,x}^{k,a} \right|^2 \mid Z_{n,x}^{k,a} = y \right] \ldots \right] \right] \leq C(1 + |y|^2) m h.$$

Then

$$\max_{a_k} \mathbb{E}\left[\max_{a_{k+1}} \mathbb{E}\left[\ldots \max_{a_{k+m-1}} \mathbb{E}\left[\left| Z_{n,x}^{k+m,a} - Z_{n,x}^{k,a} \right|^2 \right] \ldots \right] \right] \leq C \mathbb{E}\big[1 + \left| Z_{n,x}^{k,a} \right|^2\big] m h,$$

By using (3.31c), the right hand side term is bounded by $C(1 + |x|^2) m h = C(1 + |x|^2)(t_{m+k} - t_k)$. Using again inequalities of the type $\mathbb{E}[|X|] \leq \mathbb{E}[|X|^2]^{\frac{1}{2}}$, we obtain the desired bound (3.31c).

CHAPITRE IV

PROBABILISTIC REACHABILITY ANALYSIS

Publications of this chapter

M. Assellaou, O. Bokanowski and H. Zidani, *Error Estimates for Second Order Hamilton-Jacobi-Bellman Equations. Approximation of Probabilistic Reachable Sets*, DCDS- Serie A, vol. 35(9), pp. 3933 - 3964, 2015.

M. Assellaou, O. Bokanowski and H. Zidani, *Probabilistic safety reachability analysis*, ICCOPT, Lisboa, July 2013.

4.1 Introduction

This chapter is devoted to reachability analysis for stochastic systems. The main objective is to approximate the reachable sets using the error estimates developed in the last chapter. More precisely, let $\mathcal{C}$ be a non-empty subset of $\mathbb{R}^d$ ("the target"). Let $\rho \in [0,1[$ and $t \leq T$. Consider the backward reachable set under probability of success ρ, that is, the set of initial points x for which the probability that there exists a process $X^u_{t,x}$ solution of (3.1), associated with an admissible control $u \in \mathcal{U}$ and that reaches $\mathcal{C}$ at time T is higher than ρ :

$$\Omega^\rho_t = \left\{ x \in \mathbb{R}^d \,\middle|\, \exists u \in \mathcal{U},\ \mathbb{P}[X^u_{t,x}(T) \in \mathcal{C}] > \rho \right\}. \tag{4.1}$$

The sets Ω^ρ_t can be characterized by using the *level-set approach*. Indeed, it is straightforward to see that Ω^ρ_t is equivalent to :

$$\Omega^\rho_t = \left\{ x \in \mathbb{R}^d \,\middle|\, \exists u \in \mathcal{U}, \mathbb{E}[\mathbb{1}_{\mathcal{C}}(X^u_{t,x}(T))] > \rho \right\}.$$

Moreover, by considering the control problem (3.2) with $\Phi(x) := \mathbb{1}_{\mathcal{C}}(x)$, it is possible to show that for every $\rho > 0$ and every $t \in [0,T]$, the backward reachable set Ω^ρ_t is given by the *level-set* :

$$\Omega^\rho_t = \{x \in \mathbb{R}^d, \vartheta(t,x) > \rho\}.$$

The level-set approach has been introduced in [92] to model front propagation problems. Then, the method has attracted a big interest for studying backward reachable sets of continuous non-linear dynamical systems under general conditions, see [87, 28] and the references therein. The idea of using the level set approach in discrete time stochastic setting has been also considered in [1, 2, 6]. In this case, the value function is obtained by solving the dynamic programming principle. In the present chapter, we are interested in the approximation of the probabilistic backward reachable sets for time-continuous stochastic processes. We analyse the approach and we provide error estimates between the exact sets and their numerical approximation.

Let us mention that other numerical methods for reachability analysis have been introduced and analysed in the literature. The most natural numerical algorithm consists in using Monte Carlo simulations to generate a set of trajectories starting from a given initial position $x \in \mathbb{R}^d$. Then the percentage of trajectories reaching the target gives an approximation of the probability of success (for reaching the target) when starting from this position x. On the other hand, for linear stochastic systems, a bound for the probability of hitting a target can be obtained by using the enclosing hulls of the probability density function for time intervals, see [5, 4], for instance. Note that these approaches are used to calculate the probabilities of success but do not allow to define the entire set of points that have the same given probability. In addition, Monte-Carlo-based methods often require a large number of simulations to obtain a good accuracy.

This chapter is organised as follows : section 4.2, we study the characterization and approximation of probabilistic backward reachable sets. Section 4.3 is devoted to some illustrative numerical examples.

4.2 Problem statement

Let $\mathcal{C}$ be a nonempty subset of $\mathbb{R}^d$ with non-zero measure ("the target"). Let $\rho \in [0, 1[$ and $t \leq T$. Consider Ω_t^ρ the backward reachable set under probability of success ρ, that is, the set of initial points x for which the probability that there exists trajectory $X_{t,x}^u$ solution of (3.3), associated with an admissible control $u \in \mathcal{U}$ and that reaches $\mathcal{C}$ at time T is at least ρ :

$$\Omega_t^\rho = \left\{ x \in \mathbb{R}^d \,\middle|\, \exists u \in \mathcal{U},\ \mathbb{P}[X_{t,x}^u(T) \in \mathcal{C}] > \rho \right\}. \tag{4.2}$$

Such backward reachable sets play an important role in many applications. For instance the set Ω_t^ρ can be interpreted as a "safety region" for reaching $\mathcal{C}$, with confidence ρ. For time discrete stochastic systems, stochastic backward reachable sets of the form of (4.2) have been analysed and characterized via an adequate stochastic optimal control problem in [1] and [2]. In this case, the control problem is solved via the dynamic programming approach. In the context of financial mathematics, the problem of characterizing the backward reachable set with a given probability was first introduced by Föllmer and Leukert [62]. This problem was also studied and converted into the class of stochastic target problems by Touzi, Bouchard and Elie in [33].

In order to characterize the domain Ω_t^ρ for different values of ρ, we consider the level-set approach and introduce the following optimal control problem :

$$\vartheta(t,x) := \sup_{u\in\mathcal{U}} \mathbb{E}\big[\mathbb{1}_{\mathcal{C}}(X_{t,x}^u(T))\big] \equiv \sup_{u\in\mathcal{U}} \mathbb{P}\big[X_{t,x}^u(T) \in \mathcal{C}\big]. \tag{4.3}$$

Therefore, it is straightforward to show the following :

Proposition 4.2.1 *Assume (H1a), and let ϑ defined in (4.3). Then, $\forall t \in [0,T]$:*

$$\Omega_t^\rho = \{x \in \mathbb{R}^d,\ \vartheta(t,x) > \rho\}. \tag{4.4}$$

Following the results of chapter 3, we first regularize the function $\mathbb{1}_{\mathcal{C}}(\cdot)$ by functions Φ^ε (for $\varepsilon > 0$), defined as follows :

$$\Phi^\varepsilon(x) = \min(1, \max(0, -\frac{1}{\varepsilon} d(x,\mathcal{C}))). \tag{4.5}$$

Notice that the Φ^ε is $\frac{1}{\varepsilon}$-Lipschitz continuous (see Figure 4.1).

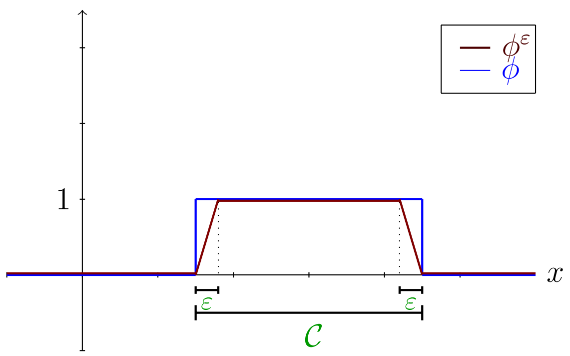

FIGURE 4.1 – Regularization Φ^ε of the indicator function $\mathbb{1}_{\mathcal{C}}$ for a given set $\mathcal{C}$.

Then, we consider the following "regularized" control problem :

$$\vartheta^\varepsilon(t,x) := \sup_{u\in\mathcal{U}} \mathbb{E}[\Phi^\varepsilon(X_{t,x}^u(T))],$$

and we denote by $\vartheta^{\varepsilon,\Delta}$ a numerical approximation of ϑ^ε obtained by solving the fully discretized scheme (3.42). In order to obtain an error estimates of $\vartheta - \vartheta^{\varepsilon,\Delta}$, we shall need to assume the following hypothesis on the target set $\mathcal{C}$:

(H4) $\mathcal{C}$ is a non-empty Borelean subset of $\mathbb{R}^d$. Moreover, if we denote by $\mathcal{C}_\varepsilon$ the set defined by :

$$\mathcal{C}_\varepsilon := \{x \in \mathcal{C},\ d_{\partial\mathcal{C}}(x) \le -\varepsilon\},$$

where $d_{\partial\mathcal{C}}$ is the signed distance to $\mathcal{C}$, then, there exists a constant $M_1 > 0$ such that, for every $A > 0$,

$$\mu((\mathcal{C} \setminus \mathcal{C}_\varepsilon) \cap \mathbb{B}_A) \le M_1 A\, \varepsilon.$$

Remark 4.2.2 *The above assumption is satisfied in many cases, for example when $\mathcal{C}$ is a half space or when $\mathcal{C}$ is a finite union of bounded, convex polytopes $\mathcal{O}_i \subset \mathbb{R}^d$ with non empty interiors :*

$$\mathcal{C} := \bigcup_{i=1,\ldots,p} \mathcal{O}_i.$$

Theorem 4.2.3 *Assume that (H1a), (H1b) and (H4) are satisfied.*
(i) There exists $C > 0$ such that for every $\varepsilon > 0$ and every $\Delta = (h, \Delta x)$ mesh steps, for every $\forall t \in [0, T)$, $\forall x \in \mathbb{R}^d$, the following holds :

$$|\vartheta^{\varepsilon,\Delta}(t,x) - \vartheta(t,x)| \le C\frac{1 + |x|^2 + |\log \varepsilon|}{(T-t)^{\frac{d}{2}}}\,\varepsilon + \frac{C}{\varepsilon}\left(|x|^{\frac{7}{4}} h^{\frac{1}{4}} + \frac{\Delta x}{h}\right) \tag{4.6}$$

(ii) If there exists $A > 0$ such that $\mathcal{C} \setminus \mathcal{C}_\varepsilon \subset \mathbb{B}_A$ for every $\varepsilon \in]0, 1]$, then there exists $C > 0$ such that

$$|\vartheta^{\varepsilon,\Delta}(t,x) - \vartheta(t,x)| \le C\frac{1}{(T-t)^{\frac{d}{2}}} e^{-C_2\frac{d(x,\mathcal{C}\setminus\mathcal{C}_\varepsilon)^2}{T-t}}\,\varepsilon + \frac{C}{\varepsilon}\left(|x|^{\frac{7}{4}} h^{\frac{1}{4}} + \frac{\Delta x}{h}\right) \tag{4.7}$$

for every $\forall t \in [0, T)$, $\forall x \in \mathbb{R}^d$, $\forall \varepsilon \in]0, 1]$.

Proof. Under assumption (H4), all the requirements of assumptions (H2)-(H3) are satisfied for $\Phi = \mathbb{1}_{\mathcal{C}}$ and the regularized function Φ_ε defined in (4.5). Thus the result of theorems 3.3.1 and 3.4.2 can be applied and leads to the result. ∎

To get the optimal rates in (4.6) and (4.7), one can choose ε, h and Δx in such way to minimize the error in the right hand side of the estimates. For instance in case there exists $A > 0$ such that $\mathcal{C} \setminus \mathcal{C}_\varepsilon \subset \mathbb{B}_A$, and for every $0 \le T - \delta$ (with $\delta > 0$), and for every $x \in \mathbb{B}_R$ with $R > 1$, the error estimate in (4.7) becomes :

$$|\vartheta^{\varepsilon,\Delta}(t,x) - \vartheta(t,x)| \le C\frac{\varepsilon}{\delta^{\frac{d}{2}}} + \frac{C}{\varepsilon} R^{\frac{7}{4}}\left(h^{\frac{1}{4}} + \frac{\Delta x}{h}\right).$$

The optimal estimate is then obtained by choosing ε, $\frac{h^{1/4}}{\varepsilon}$ and $\frac{1}{\varepsilon}\frac{\Delta x}{h}$ to be of the same order. This leads to $\varepsilon \sim h^{1/8} \sim \Delta x^{1/10}$ and to the following estimate :

$$|\vartheta^{\varepsilon,\Delta}(t,x) - \vartheta(t,x)| \le C\frac{R^2}{\delta^{\frac{d}{4}}}\Delta x^{1/10}. \tag{4.8}$$

Therefore, we obtain the following approximation of Ω_t^ρ, for $0 \le t \le T - \delta$:

$$\left\{x,\ \vartheta^{\varepsilon,\Delta}(x,t) > \rho + C\frac{R^2}{\delta^{\frac{d}{4}}}\Delta x^{\frac{1}{10}}\right\} \subset \Omega_t^\rho \cap \mathbb{B}_R \subset \left\{x,\ \vartheta^{\varepsilon,\Delta}(x,t) > \rho - C\frac{R^2}{\delta^{\frac{d}{4}}}\Delta x^{\frac{1}{10}}\right\} \tag{4.9}$$

Hence we can approximate the region Ω_t^ρ by level sets of the numerical approximation of $\vartheta^{\varepsilon,\Delta}$. The above approximation, of order $O(\Delta x^{1/10})$, is rough. However, in practice, we have observed numerically that it is sufficient to take $h \equiv \Delta x$ and, in that case, the error behaves like $O(\Delta x)$, so the errors in (4.8) or (4.9) are also of the order of Δx (see section 4.3 for more details).

To conclude, we have given a simple numerical approximation procedure for the characterisation of probabilistic backward reachable sets and how to control rigorously the error made in the approximation.

4.3 Numerical simulations.

In all this section the numerical scheme considered is the fully discrete Semi-Lagrangian scheme (3.44) where the maximization operation is performed on a subset of control values $\{a_1, ..., a_{N_u}\}$ that represents a discretization of U with a mesh size Δu. In all the simulations, the regularization parameter ε will be chosen as $\varepsilon = \frac{1}{\Delta x}$.

Example 1. We consider the following stochastic differential equation with no drift term and no control :

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \sigma \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} \tag{4.10}$$

where $c = 0.2$ and $\sigma = 0.2$. The time horizon is $T = 1.0$. The target $\mathcal{C}$ is the diamond of summits $(-0.8, 1), (1.2, 1), (0.8, 1), (-1.2, 1)$ (see Figure 4.2(up-left)). If we consider the initial data

$$\phi(x, y) = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

then the exact solution to the HJB equation is known and the level-set function is given by,

$$\vartheta(t, x, y) = v_1(t, x + cy)\, v_2(t, y)$$

where $v_1(t, r) = v_2(t, r) := \frac{1}{\sqrt{2\pi\ \sigma^2\ t}} \int_{-1-r}^{1-r} e^{-\frac{s^2}{2\sigma^2\ t}} ds$.

First, in Figure 4.2 (up-right) shows the backward reachable set Ω_t^ρ for $\rho = 0.05$ at time $t = 0$. Fo this simulation, we have considered the computational domain $D = [-4, 4]^2$ with a uniform grid and zero condition outside the domain D :

$$\vartheta(t, x, y) = 0, \quad \forall t \in [0, T],\ \forall (x, y) \notin D \tag{4.12}$$

(which amounts to take homogenous Dirichlet boundary condition on ∂D). One can observe a good matching between the numerical front (computed using the scheme approximation), and the exact front (computed by using the exact value function).

In Figure 4.2(down), we have also plotted different sets corresponding to different level set values (when using $\Delta x = \Delta y = 0.016$). This corresponds to different level of confidence for reaching the target.

Remark 4.3.1 *The reason behind the use of a diamond as a target set is to validate the behaviour of the numerical scheme for non standard target shape.*

We consider also a case with smaller target, and set $\sigma = 0.5$ (σ is taken large in order to see the impact of the diffusion). Similar simulations as in before are performed in this case and the results are given in Figure 4.3, where we can observe again a good approximation of the reachable set.

In Table 4.1, we summarize the error estimates between the exact solution and the numerical approximation, for $\sigma = 0.25$, showing L^∞, L^1 and L^2 errors. We have chosen h and Δx of the same order ($h \equiv \Delta x$), and we observe roughly a convergence of first order.

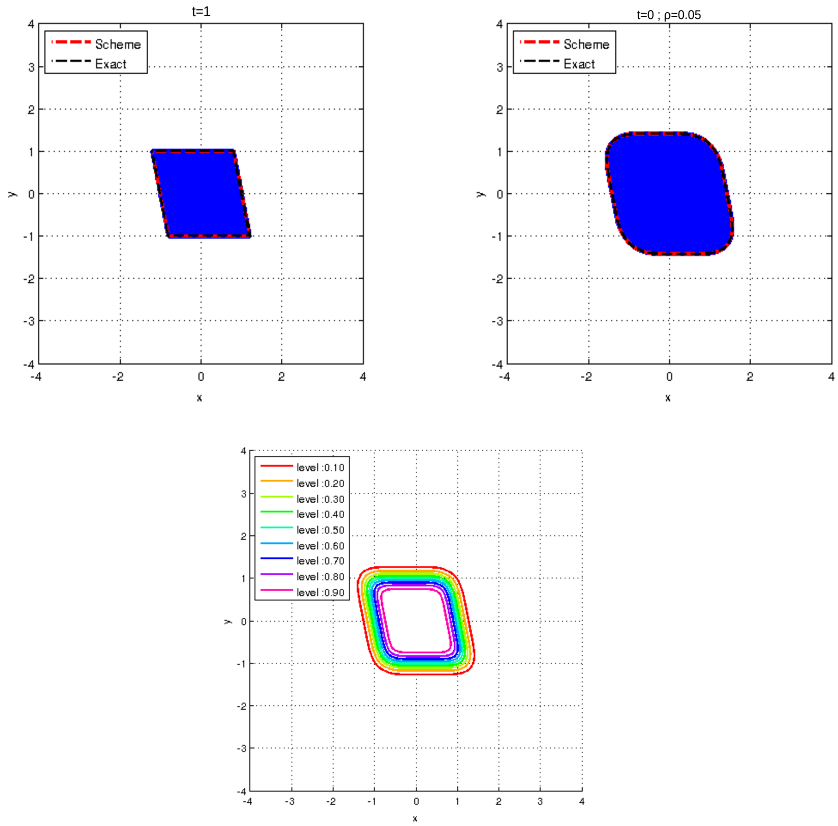

FIGURE 4.2 – Example 1 with $\sigma = 0.2$: Target (up-left) ; backward reachable set Ω_0^ρ for $\rho = 0.05$ (up-right) ; backward reachable sets Ω_0^ρ for different values of ρ (down).

TABLE 4.1 – Example 1 with $\sigma = 0.25$: Error estimates $\vartheta - V$ at time $t = 0$, using $h = T/N$ and $\Delta x = h$.

N	error L^∞	error L^1	error L^2	CPU time (s)
20	3.31 e-2	1.02 e-1	3.88 e-2	3.33 $\times 10^{-1}$
40	1.56 e-2	4.65 e-2	1.82 e-2	2.43 $\times 10^{0}$
80	6.97 e-3	1.99 e-2	8.01 e-3	2.05 $\times 10^{1}$
160	3.78 e-3	1.14 e-2	4.34 e-3	1.55 $\times 10^{2}$
320	2.01 e-3	6.13 e-3	2.36 e-3	1.27 $\times 10^{3}$

Remark 4.3.2 *In this simple case, we can observe numerically that the error estimate decreases with order 1 wich is better than what one can prove theoretically (order of $1/4$). Also, the choice $h = \Delta x$ seems to give better numerical approximations than what we would get if we choose the optimal ratio between h and Δx established in Chapter 3. The first-order numerical behavior can be justified in this example by the fact that the exact solution is very smooth on $[0, T) \times \mathbb{R}^2$.*

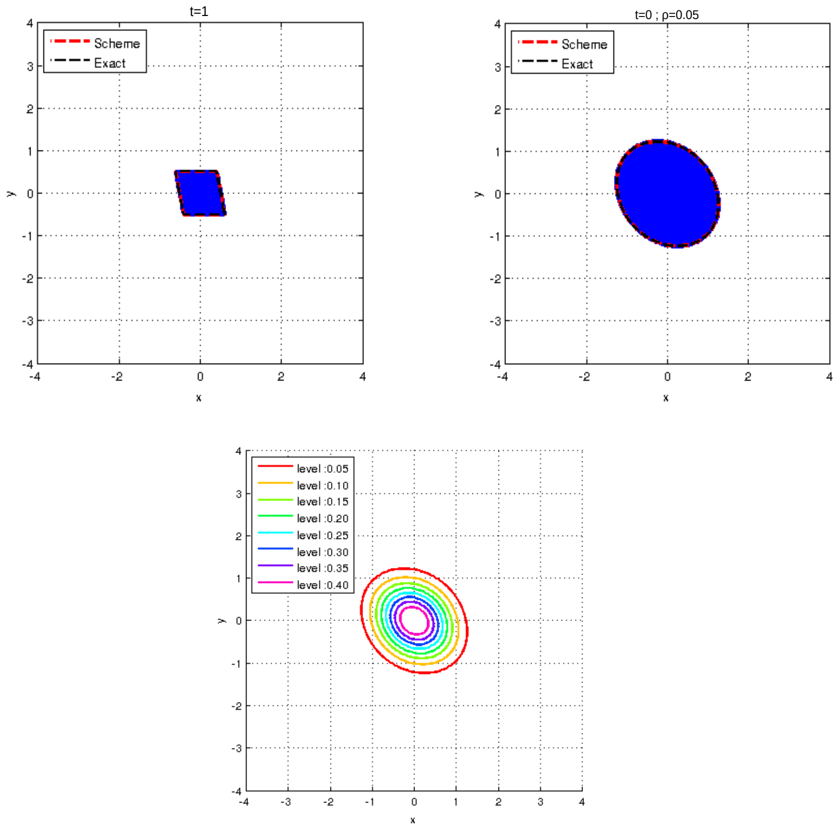

FIGURE 4.3 – (Example 1) initial data (up-left), capture basin Ω_0^ρ for $\rho = 0.05$ (up-right) or different ρ values (center), for $\underline{\sigma = 0.5}$.

Example 2. Now we deal with the following controlled stochastic system :

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = u\sigma \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} \tag{4.13}$$

where u is a control taking values in $[0, 1]$, $c = 0.2$, $\sigma = 0.25$ and $T = 1.0$. The initial data and the boundary conditions are the same as the ones used in the first example. For this kind of problem the exact solution is not known. The solution obtained with $N = 160$ is taken as the reference solution. The error estimates computed at time $t = 0$ are summarized in Table 4.2. As for Example 1, we observe again a convergence of order 1.

TABLE 4.2 – (Example 2) Error table, using $\Delta x = h = T/N$.

N	error L^∞	error L^1	error L^2	CPU time (s)
10	1.68 e-1	1.16 e-3	8.41 e-3	6.42×10^{-1}
20	8.80 e-2	7.49 e-4	4.34 e-3	5.06×10^{0}
40	4.43 e-2	2.55 e-4	1.79 e-3	4.01×10^{1}
80	1.62 e-2	1.12 e-4	7.13 e-4	3.20×10^{2}

Example 3. In this example, we consider a controlled stochastic system with a drift :

$$dx(t) = \begin{pmatrix} -1 & -4 \\ 4 & -1 \end{pmatrix} x(t)dt + u(t)dt + \begin{pmatrix} 0.7 & 0 \\ 0 & 0.7 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} \tag{4.14}$$

where $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ and $u_i \in [-0.1, 0.1]$, for $i = 1, 2$.

The linear system (4.14) is used in [5] to illustrate an appoximation of the probability of reaching a target by using enclosing hulls of probability density functions. Here, we set $T = 1.75$ and consider a target set represented by the green square in Figure 4.4. We compute for different times $t \in \{0.75; 0.25; 0\}$, the set Ω_t^ρ for $\rho = 0.4$ (see Figure 4.4).

The numerical simulation is performed on a computational domain $D = [-8, 8]^2$ with a uniform grid and boundary conditions as (4.12). Once the numerical approximation V of the value function and the backward reachable set $\Omega_{t_n}^\rho$ are computed, and in order to validate the numerical simulations, we generate different trajectories starting from the backward reachable set using the algorithm described below. Let $\bar{x}$ be a given initial position, the following algorithm aims to reconstruct a trajectory on starting at time t_n from the position $\bar{x}$:

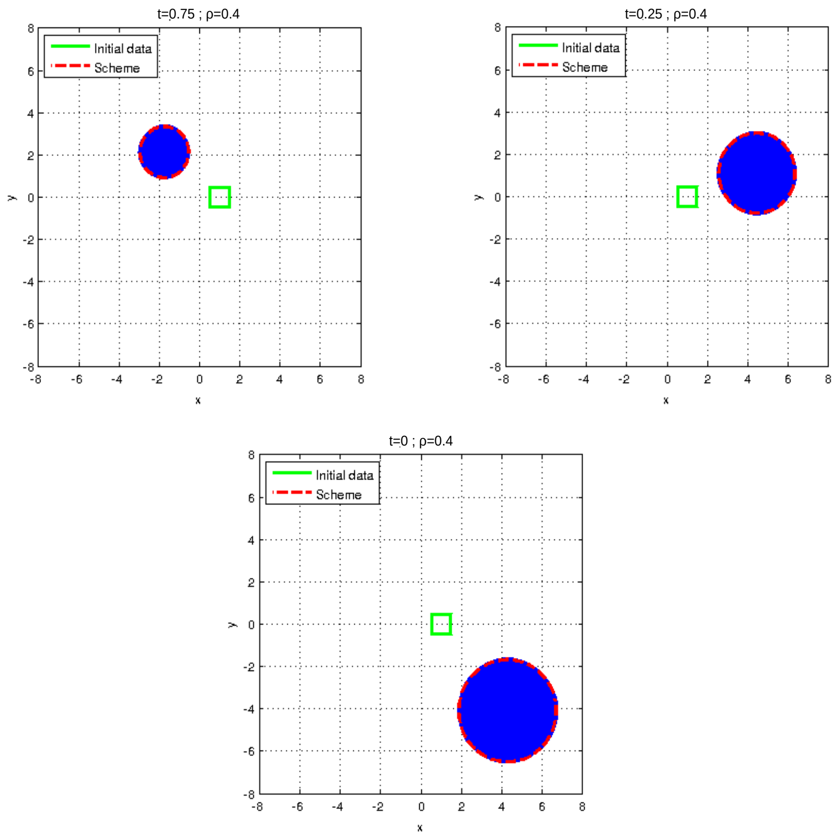

FIGURE 4.4 – (Example 3) Backward reachable sets at different times $t \in \{0.75,\ 0.25,\ 0\}$ for a time horizon $T = 1.75$. The target set is represented by the green square.

In order to validate our results, we then generate different trajectories starting from the capture basin using the following algorithm :

Algorithm (trajectory reconstruction) Initialization : Set $X_n = \bar{x}$.

For $k = 0$ to $N-1$ by step 1 :

Step 1 Compute optimal control at $t = t_k$:

$$u_k = \arg\max_{a \in \{a_1, \ldots, a_{N_u}\}} \mathbb{E}[V(X^a_{k+1}, t_{k+1})] \tag{4.15}$$

Step 2 Compute the next point at iteration k :

$$X_{k+1} := X_k + b(t_k, X_k, u_k)dt + \sigma(t_k, X_k, u_k)\sqrt{h}B \tag{4.16}$$

where B is a random variable with a normal law $\mathcal{N}(0,1)$.

Figure 4.7 shows some controlled processes issued from a starting point located in the backward reachable sets Ω^ρ_t for $t \in \{0.75, 0.25, 0\}$.

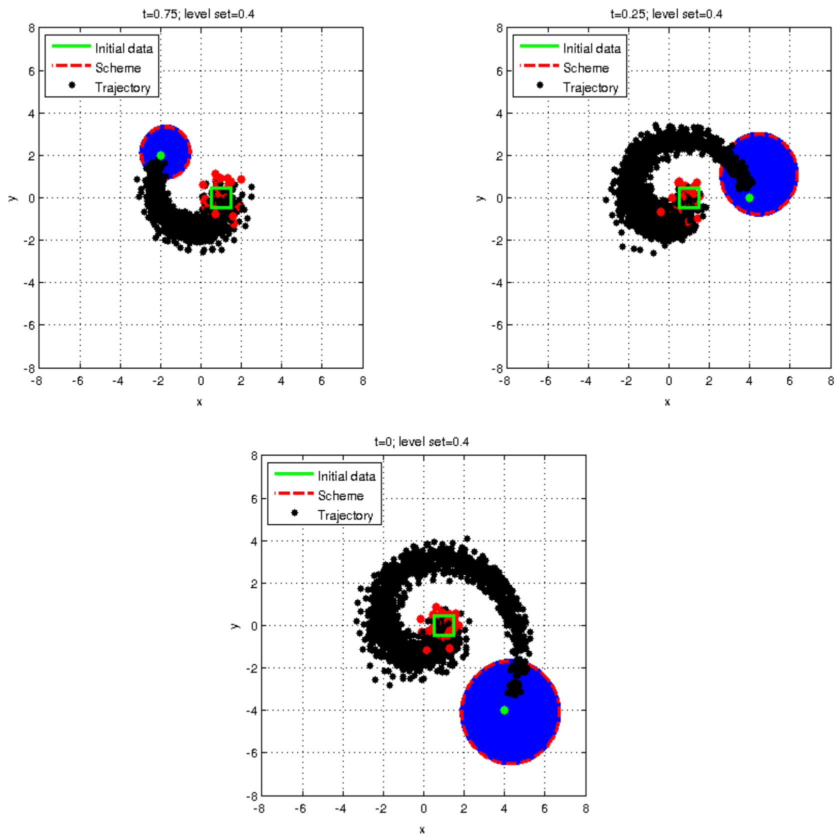

FIGURE 4.5 – (Example 3) Behaviour of controlled processes starting from the backward reachable sets at times $t \in \{0.75,\ 0.25,\ 0\}$ for a final time horizon $T = 1.75$.

Now consider $\bar{x} := (-1.0, 2.0)^T$ and set $t_n = 0.75$. We compute an approximation $V(t_n, \bar{x})$ of the level-set function (by numerically solving the corresponding HJB equation). On a grid with the discretization parameters $\Delta u = \Delta = h = \frac{T}{N}$, we obtain the values given in the second column, of Table 4.3. The third column of this table gives the differences between two values. The last value $V(t_n, \bar{x})$ is approximately 0.503 and the difference between the last two computed values is of order 0.014. Hence we can extrapolate numerically to say that a process starting from $\bar{x}$ reaches the target set with a probability 0.503 ± 0.007.

TABLE 4.3 – (Example 3) The value of $V(t_n, \bar{x})$ for different mesh parameters, and the difference between two successive values (here $t_n = 0.75$ and $\bar{x} = (-1.0, 2.0)^T$).

N	$v(t,x)$ estimate	differences
10	0.41768	-
20	0.46359	0.046
40	0.48926	0.026
80	0.50326	0.014

Now, let $N = 20$ and call the trajectory reconstruction algorithm described above to generate some trajectories starting from $\bar{x}$ (by Monte Carlo simulations). The results are reported in Table 4.4 with M is the number of the simulated trajectories, p is the percentage of trajectories reaching the target set and C.I denotes the confidence intervall at 95%. The results of Table 4.4 show that the value $V(t_n, \bar{x})$ at point $\bar{x}$ is inside the confidence interval.

TABLE 4.4 – (Example 3) Percentage of p of simulated trajectories that reach the target set, corresponding confidence interval (C.I.), and a Monte Carlo error estimate (MC-error)

M	p	C.I.	MC-error
3000	0.51233	(0.4944, 0.5302)	0.0179
6000	0.51317	(0.5005, 0.5258)	0.0127
12000	0.51575	(0.5068, 0.5247)	0.0090
25000	0.50912	(0.5029, 0.5153)	0.0062
50000	0.50876	(0.5044, 0.5131)	0.0044
100000	0.50969	(0.5066, 0.5128)	0.0031

Example 4. In this example, we consider a controlled stochastic system with a drift :

$$dx(t) = \begin{pmatrix} -1 & -\pi \\ \pi & -1 \end{pmatrix} x(t)dt + \begin{pmatrix} u_1(t) + 0.5 & 0 \\ 0 & u_2(t) + 0.5 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} \tag{4.17}$$

where $u_i \in [-0.1, 0.1]$, for $i = 1, 2$.

Let us emphasise that the case of the linear system (4.17) is not considered in the theoretical study in chapter 3 since the control appear in the components of the diffusion. Here, we set $T = 1.5$ and consider a target set represented by the green square in Figure 4.4. We compute for different times $t \in \{0.0; 0.25; 0.5; 0.75\}$, the set Ω_t^ρ for $\rho = 0.4$ (see Figure 4.6).

The numerical simulation is performed on a computational domain $D = [-8, 8]^2$ with a uniform grid and boundary conditions as (4.12). Once the numerical approximation V of the value function and the backward reachable set $\Omega_{t_n}^\rho$ are computed, and in order to validate the numerical simulations, we generate different trajectories starting from the backward reachable set using the algorithm described above. Let $\bar{x}$ be a given initial position, the following algorithm aims to reconstruct a trajectory on starting at time t_n from the position $\bar{x}$:

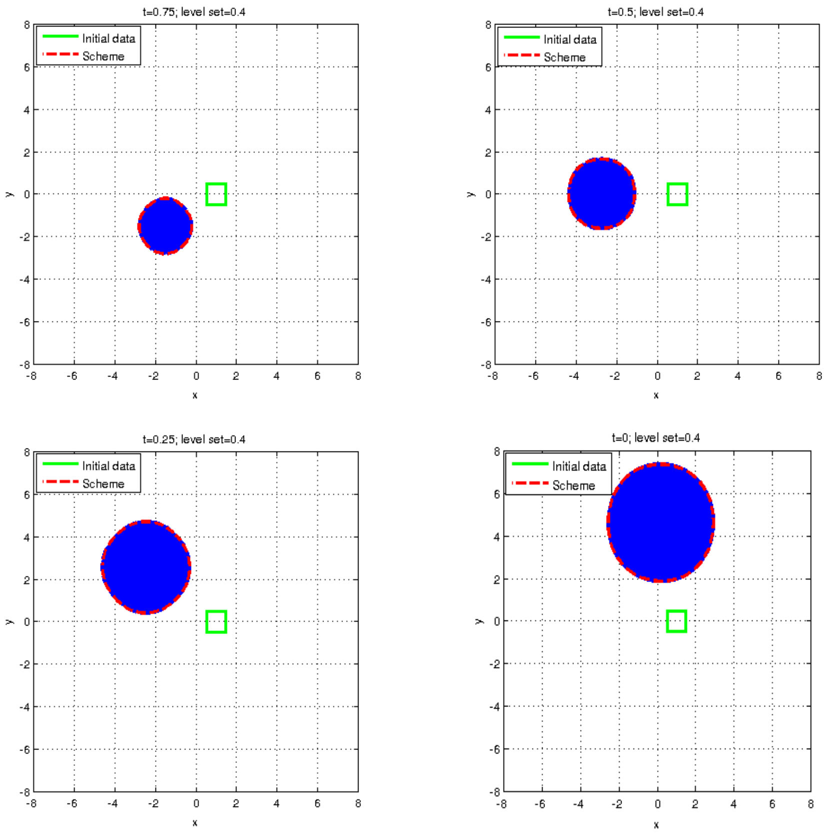

FIGURE 4.6 – (Example 3) Reachable sets at different times $t \in \{0.0,\ 0.25, 0.5,\ 0.75\}$ for a time horizon $T = 1.25$. The target set is represented by the green square.

Next, we will use the algorithm described in the last example to generate different trajectories starting from the capture basin . Figure 4.7 shows some controlled processes issued from a starting point located in the backward reachable sets Ω_t^ρ for $t \in \{0.0, 0.25, 0.5, 0.75\}$.

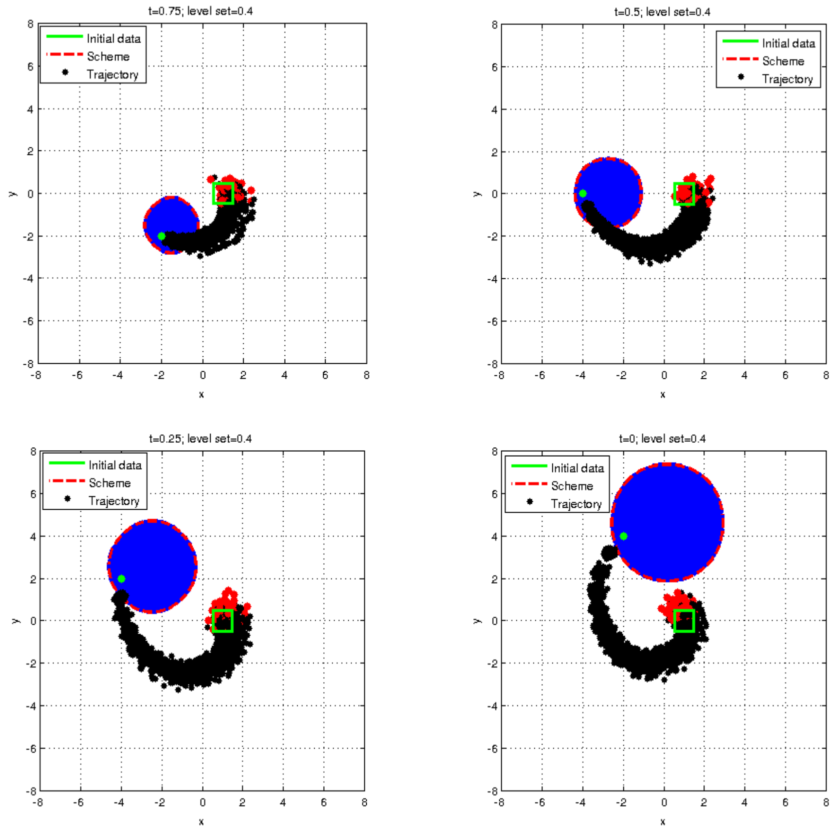

FIGURE 4.7 – (Example 3) Behaviour of controlled processes starting from the backward reachable sets at times $t \in \{0.0,\ 0.25,\ 0.5, 0.75\}$ for a final time horizon $T = 1.5$.

Let us consider $\bar{x} := (-2.0, 0.0)^T$ and set $t_n = 0.75$ and compute an approximation $V(t_n, \bar{x})$ of the level-set function (by numerically solving the corresponding HJB equation). On a grid with the discretization parameters $\Delta u = \Delta = h = \frac{T}{N}$, we obtain the values given in the second column, of Table 4.5. Hence, we can conclude that approximately $V(t_n, \bar{x}) = 0.786 \pm 0.009$.

TABLE 4.5 – (Example 4) The value of $V(t_n, \bar{x})$ for different mesh parameters, and the difference between two successive values (here $t_n = 1.0$ and $\bar{x} = (-2.0, 0.0)^T$).

N	$v(t,x)$ estimate	differences
10	0.64208	-
20	0.72736	0.08528
40	0.76679	0.03943
80	0.78655	0.01976

Now, let $N = 20$ and call the trajectory reconstruction algorithm described above to generate some trajectories starting from $\bar{x}$ (by Monte Carlo simulations). The results are reported in Table 4.6 with M is the number of the simulated trajectories, p is the percentage of trajectories reaching the target set and C.I denotes the confidence intervall at 95%. The results of Table 4.6 show that the value $V(t_n, \bar{x})$ at point $\bar{x}$ is slightly closed to the confidence interval but not inside.

Table 4.6 – (Example 3) Percentage of p of simulated trajectories that reach the target set, corresponding confidence interval (C.I.), and a Monte Carlo error estimate (MC-error)

M	p	C.I.	MC-error
3000	0.77133	(0.7563, 0.7864)	0.0150
6000	0.77083	(0.7602, 0.7815)	0.0106
12000	0.77517	(0.7677, 0.7826)	0.0074
25000	0.77488	(0.7697, 0.7801)	0.0052
50000	0.77384	(0.7702, 0.7775)	0.0036
100000	0.77361	(0.7710, 0.7762)	0.0026

CHAPITRE V

HJB APPROACH FOR STATE CONSTRAINED CONTROL PROBLEM WITH MAXIMUM COST

Publications of this chapter

M. Assellaou, O. Bokanowski, A. Desilles and H. Zidani, *Feedback control analysis for state constrained control problem with maximum cost*, in preparation.

5.1 Introduction

The purpose of this chapter is to study a Hamilton Jacobi Bellman approach for state-constrained control problems with maximum cost and Bolza cost. In particular, we are interested by the characterization of the value functions of such problems and the analysis of the associated optimal trajectories. Let $T > 0$ be a finite time horizon and consider the following dynamical system :

$$\dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)), \text{ a.e. } s \in (0, T), \tag{5.1a}$$

$$\mathbf{y}(0) = y, \tag{5.1b}$$

where $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is a Lipschitz continuous function, U is a compact set, and $\mathbf{u} : [0, T] \to U$ is a measurable function. Denote $\mathbf{y} = \mathbf{y}_y^{\mathbf{u}}$ the absolutely continuous solution of (5.1). Let $\mathcal{K} \subset \mathbb{R}^d$ be a given non-empty closed set and consider the following value function :

$$\vartheta_1(t, y) := \min_{\mathbf{u} \in L^\infty((0,t),U)} \left\{ \max_{\theta \in [0,T]} \Phi_1(\mathbf{y}_y^{\mathbf{u}}(\theta)) \;\middle|\; \mathbf{y}_y^{\mathbf{u}}(\theta) \in \mathcal{K} \quad \forall \theta \in [0, t] \right\},$$

with the convention that $\inf \emptyset = +\infty$. The function $\Phi_1 : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous function.

In the case when $\mathcal{K} = \mathbb{R}^d$, this control problem has been studied by Barron and coauthors [18, 19] where the control problem with maximum cost is approximated by a sequence of control problems with L_p-cost. In [93], the control problem with l.s.c infimum cost has been considered from a viability point of view where the Epigraph of the value function is characterized by a *viability Kernel*. In the context of differential games, the problem has been studied in [94] for a Lipschitz continuous infimum cost and [96] for maximum bounded cost.

In the general case where the set of state constraints $\mathcal{K}$ is nonempty closed set of $\mathbb{R}^d$, we shall follow an approach closely related to [3]. The idea is first to look for the characterization of the value function by its Epigraph (see for instance [11, 40, 41, 8, 93]). Indeed, the function ϑ_1 can be described by a Lipschitz continuous value function of an auxiliary control problem free of state constraints. Moreover, the auxiliary value function can be characterized as the unique Lipschitz continuous viscosity solution of a variational HJ equation. More precisely, let g be a Lipschitz continuous function satisfying $g(x) \leq 0 \iff x \in \mathcal{K}$ and introduce the following auxiliary control problem :

$$w_1(t, y, z) := \inf_{\mathbf{u} \in L^\infty((0,t),U)} \left(\max_{\theta \in [0,t]} \Psi_1(\mathbf{y}_y^{\mathbf{u}}(\theta), z) \right),$$

where

$$\Psi_1(y, z) := \Big(\Phi_1(y) - z \Big) \vee g(y)$$

and $a \bigvee b := \max(a, b)$. The new function w_1 can be characterized as the unique Lipschitz continuous viscosity solution of a HJ equation of variational type :

$$\min \Big(\partial_t w_1 + H(y, \nabla_y w_1), \quad w_1(t, y, z) - \Psi_1(y, z) \Big) = 0, \text{ in } [0, T) \times \mathbb{R}^d \times \mathbb{R}, \quad (5.2a)$$

$$w_1(0, y, z) = \Psi_1(y, z), \qquad \text{in } \quad \mathbb{R}^d \times \mathbb{R}, \qquad (5.2b)$$

where $H(y, p) := \sup_{u \in U} \big(-f(y, u).p \big)$. Moreover, under the classical assumptions on f, and Φ_1, the epigraph [1] of ϑ_1 satisfies $\mathcal{E}pi(\vartheta_1(t, \cdot)) = \{(y, z) \in \mathbb{R}^d \times \mathbb{R},\ w_1(t, y, z) \leq 0\}$. Thus, one can determine the value functions ϑ_1 in terms of level sets of w_1, i.e,

$$\vartheta_1(t, y) = \min\{z,\ w_1(t, y, z) \leq 0\},$$

The new value function is defined in all domain. In order to reduce our search of the solution only on a closed set, we show that with a wise choice of the Lipschitz continuous function g as well as a slightly modified function Ψ_1, we can obtain a function $\widetilde{w_1}$ characterized as the unique Lipschitz continuous viscosity solution of a variational Hamilton Jacobi equation with specific Dirichlet boundary conditions (this will be detailed in section 5.3.1).

We then focus on the analysis of the optimal trajectories associated with the state constrained optimal control problem with maximum cost. Indeed, the characterization of the state-constrained control problem using auxiliary control problem provides a tool to deal with optimal trajectories. In this framework, we show that under some initial condition, the optimal trajectory of the auxiliary control problem is also optimal for the control problem whose value function is ϑ_1. Thanks to this result, we propose algorithms to generate optimal trajectories corresponding to the auxiliary optimal control problem. On the other hand, we show that the control

1. The epigraph at time t is defined by $\mathcal{E}pi(\vartheta_1(t, \cdot)) := \{(y, z) \in \mathbb{R}^d \times \mathbb{R},\ \vartheta_1(t, y) \leq z\}$.

problems discussed above are linked to some *viability kernel* associated with an *exit time function*.

In addition, a convergence result of the approximated optimal trajectory to the continuous optimal trajectory is included in this Chapter. More precisely, we extend the result of [95] to the optimal control problem with maximum criterion. The main idea of this approach is to define an approximation of the set-valued map of the optimal feedback control using the dynamic programming principle.

Another contribution of this Chapter is to work also on the control problem with Bolza cost that can be considered as an approximation of the maximum-cost. So, we investigate the following state constrained control problem with Bolza cost :

$$\vartheta_2(t,y) := \min_{\mathbf{u}\in L^\infty((0,t),U)} \left\{ \int_0^t \Phi_2(\mathbf{y}_y^{\mathbf{u}}(s))ds \;\middle|\; \mathbf{y}_y^{\mathbf{u}}(\theta) \in \mathcal{K} \quad \forall \theta \in [0,t] \right\},$$

with the convention that $\inf \emptyset = +\infty$. The function $\Phi_2 : \mathbb{R}^d \to \mathbb{R}$ is a locally Lipschitz continuous function with polynomial growth (for instance $\Phi_2(y) := r\Phi_1(y)^q$ where r and q are positive constants). Note that the value function ϑ_2 can be discontinuous. Nevertheless, ϑ_2 satisfies a Dynamic programming principle. Moreover, a state-space constrained Hamilton-Jacobi-Bellman equation can be associated with ϑ_2 (see [98]-[99]) taking the following form :

$$-\partial_t \vartheta_2 + \mathcal{H}(t,y,\nabla\vartheta_2) = 0, \quad \text{in } (0,T)\times\mathcal{K}, \tag{5.3a}$$

$$\vartheta_2(0,y) = 0, \quad \text{in } \mathcal{K}, \tag{5.3b}$$

where $\mathcal{H}(t,y,p) := \max_{a\in U}(-f(t,y,a).p - \Phi_2(y))$. In Soner's formulation, a function ϑ_2 is a viscosity solution of (5.3) provided it is sub-solution in $(0,T)\times \overset{\circ}{\mathcal{K}}$ (where $\overset{\circ}{\mathcal{K}} := \mathcal{K}\backslash\partial\mathcal{K}$) and a super-solution on $(0,T)\times\mathcal{K}$. The uniqueness of the solution of the HJ equation (5.3) is more complicated to prove and it requires restrictive controllability assumptions on $\mathcal{K}$ and the dynamics (see [98]- [99] for the IPQ condition and [63] - [64] for the OPQ condition).

The viability tools [7, 10] and non-smooth analysis allow to characterize the value function or more precisely its epigraph, see [11, 40, 41, 8] and the references therein.

Altarovici et al in [3] studied the case where Φ_2 is Lipschitz continuous without any controllability assumption. In this Chapter, we show that this result can be extended for locally Lipschitz continuous distributed cost with polynomial growth. Thus, the function ϑ_2 can be described by means of a continuous value function free of state constraints associated with an auxiliary control problem. More precisely, let us consider the following auxiliary control problem :

$$w_2(t,y,z) := \inf_{\mathbf{u}\in L^\infty((0,t),U)} \left(\int_0^t \Phi_2(\mathbf{y}_y^{\mathbf{u}}(\theta)) - z \right) \bigvee \max_{\theta\in(0,t)} g(\mathbf{y}_y^{\mathbf{u}}(\theta)),$$

where $a \bigvee b := \max(a,b)$. In the same manner, the function w_2 can be characterized as the unique continuous viscosity solution of a HJ equation of variational type and the epigraph of $\vartheta_2(t,.)$ is described in terms of the level sets of w_2. Furthermore, we study the optimal trajectories associated with the state constrained optimal control problem of Bolza type. We show again that under some initial condition, the optimal trajectory of the auxiliary control problem is also optimal for the control problem whose value function is ϑ_2. On the other hand, we established a link between the control problems whose value function are ϑ_2 and w_2 and some *Backward reachable set* associated with a maximum time function.

This Chapter is organized as follows. Section 5.2 contains the setting and the assumptions of the problem. The main results of the state constrained control problem with a maximum cost including the auxiliary control problems and the analysis of the optimal feedback control is presented in section 5.3. Section 5.4 is devoted to the state constrained control problem with Bolza cost and the anaylsis of the corresponding optimal feedback controls.

5.2 Problem Formulation

For a given non-empty compact subset U of $\mathbb{R}^k$ and a finite time $T>0$, define the set of admissible control to be,

$$\mathcal{U} := \left\{ \mathbf{u} : (0,T) \to \mathbb{R}^k, \text{ measurable, } \mathbf{u}(t) \in U \quad \text{a.e} \right\}.$$

Consider the following control system :

$$\begin{cases} \dot{\mathbf{y}}(s) := f(\mathbf{y}(s), \mathbf{u}(s)), & \text{a.e} \quad s \in [0,T], \\ \mathbf{y}(0) := y, \end{cases} \tag{5.4}$$

where $\mathbf{u} \in \mathcal{U}$ and the function f is defined and continuous on $\mathbb{R}^d \times U$ and that it is Lipshitz continuous in the variable y, i.e,

$$\begin{cases} (i)\ f : \mathbb{R}^d \times U \to \mathbb{R}^d \text{ is continuous,} \\ (ii)\ \exists L > 0 \text{ s.t. } \forall (y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d,\ \forall u \in U, \\ |f(y_1,u) - f(y_2,u)| \leq L(|y_1 - y_2|). \end{cases} \tag{$\mathbf{H_1}$}$$

Denote $\mathbf{y}_y^{\mathbf{u}}(.)$ the trajectory corresponding to the control $\mathbf{u}$ and the initial point y (if it exists). Under the assumption ($\mathbf{H_1}$), the solutions of the above differential equation (5.4) belong to the class of absolutely continuous functions $W^{1,1}([0,T])$. Moreover, the set of all absolutely continuous solutions of (5.4) on $[0,t] \subseteq [0,T]$ starting from a given state vector y, i.e :

$$S_{[0,t]}(y) := \{\mathbf{y}_y^{\mathbf{u}}, \quad \mathbf{y} \text{ absolutely continuous solution of (5.4) for some } \mathbf{u} \in \mathcal{U}\}.$$

is a compact subset of $W^{1,1}([0,t])$ for the topology of $C([0,t];\mathbb{R}^d)$. Moreover, under the assumptions ($\mathbf{H_1}$), the set valued map $y \rightsquigarrow S_{[0,t]}(y)$ is Lipschitz continuous from $\mathbb{R}^d$ in $C([0,t];\mathbb{R}^d)$ (see [9]) with respect to the Hausdorff metric.

Let $\mathcal{K} \subset \mathbb{R}^d$ be a compact set of state constraints. We will say that a solution $\mathbf{y}_y^{\mathbf{u}} \in S_{[0,t]}(y)$ is admissible for $\mathcal{K}$ on $[0,t]$ if

$$\forall\, \theta \in [0,t],\ \mathbf{y}_y^{\mathbf{u}}(\theta) \in \mathcal{K}.$$

Let us denote

$$S_{[0,t]}^{\mathcal{K}}(y) := \{\mathbf{y}_y^{\mathbf{u}} \in S_{[0,t]}(y), \ \ \text{s.t.} \ \forall\, \theta \in [0,t],\ \mathbf{y}_y^{\mathbf{u}}(\theta) \in \mathcal{K}\},$$

the set, eventually empty, of all admissible trajectories starting from a given initial vector y.

Remark 5.2.1 *Note that in some cases the function f may not be defined or Lipshitz continuous on the whole state space $\mathbb{R}^d$. For instance, this is the case in the application studied in the Chapter 6. In such situations we will assume that at least the function f is Lipshitz continuous on some compact set containing $\mathcal{K}$. More precisely, assume that there exists $\widetilde{\mathcal{K}}$ such that $\mathcal{K} \subset \widetilde{\mathcal{K}}$ and :*

$$\begin{cases} (i)\ f : \widetilde{\mathcal{K}} \times U \to \ \mathbb{R}^d \text{ is continuous,} \\ (ii)\ \exists L > 0 \text{ s.t. } \forall (y_1, y_2) \in \widetilde{\mathcal{K}} \times \widetilde{\mathcal{K}},\ \forall u \in U, \\ |f(y_1,u) - f(y_2,u)| \le L(|y_1 - y_2|). \end{cases} \qquad (\mathbf{H_1}\ \mathbf{local})$$

Then, there exists a Lipshitz continuous extension of f out of $\widetilde{\mathcal{K}}$, due to the lemma 5.2.2 below. In what follows we will assume that such extension exists and we will use the same notation f for it. So, we assume that the hypothesis ($\mathbf{H_1}$) and all others defined below hold for f or, if necessary, for its extension.

Lemma 5.2.2 *(McShane-Whitney extension theorem). Let $f : \mathcal{K}\times U \to R$, $\mathcal{K} \subset \mathbb{R}^d$, be a continuous function such that,*

$$|f(x,u) - f(y,u)| \le L(|x-y|), \qquad \forall (x,y,u) \in \mathbb{R}^d \times \mathbb{R}^d \times U.$$

Then, there exists a continuous function $f_e : \mathbb{R}^d \times U \to R$ such that $f_e\big|_{\mathcal{K}\times U} = f$, and

$$|f_e(y_1,u) - f_e(y_2,u)| \le L(|y_1-y_2|), \qquad \forall (y_1,y_2,u) \in \mathbb{R}^d \times \mathbb{R}^d \times U.$$

Proof. For $u \in U$, define the function $f_e(x,u) = \inf_{y\in\mathcal{K}} \big\{f(y,u) + L|x-y|\big\}$. It is obvious that the functions

$$(x,u) \to f(y,u) + L|x-y|, \ \ \forall y \in \mathcal{K},$$

are L-Lipschitz continuous in x. It is also known that the infimum of Lipschitz continuous functions is Lipschitz continuous, that is, the function f_e is L-Lipschitz continuous in x.

By definition of f_e we have $f_e(x,u) \le f(x,u)$ for each $x \in \mathcal{K}$ and $u \in U$. On the other hand, from the Lipschitz property of f, we have, for each $x \in \mathcal{K}$, $u \in U$,

$$f(x,u) \le f(y,u) + L|y-x|, \ \ \forall y \in \mathcal{K}$$

Taking the infimum over $y \in \mathcal{K}$ yields that $f(x,u) \le f_e(x,u)$. It follows that $f_e(x,u) = f(x,u)$ for each $x \in \mathcal{K}$ and $u \in U$.

∎

Let us define the maximum cost J_1 and the Bolza cost J_2,

$$J_1(T, y, \mathbf{u}) := \max_{\theta \in [0,T]} \Phi_1(\mathbf{y}_{y_0}^{\mathbf{u}}(\theta)),$$
$$J_2(T, y, \mathbf{u}) := \int_0^T \Phi_2(\mathbf{y}_{y_0}^{\mathbf{u}}(\theta)) d\theta,$$

where the cost function $\Phi_1(y)$ is assumed to satisfy the following :

$$\begin{cases} (i)\ \Phi_1 : \mathbb{R}^d \to\ \mathbb{R} \text{ is continuous.} \\ (ii)\ \exists L_1 > 0 \text{ s.t. } \forall (y, y') \in \mathbb{R}^d \times \mathbb{R}^d, \\ |\Phi_1(y) - \Phi_1(y')| \leq L_1(|y - y'|). \end{cases} \tag{$\mathbf{H_2}$}$$

and the distributed cost $\Phi_2(y)$ is assumed to satisfy the following

$$\begin{cases} (i)\ \Phi_2 : \mathbb{R}^d \to\ \mathbb{R} \text{ is continuous.} \\ (ii)\ \ \text{For each } y \in \mathbb{R}^d,\ \exists L_y, \delta_0 > 0\ \text{ s.t} \\ |y - y'| \leq \delta_0 \Rightarrow |\Phi_2(y) - \Phi_2(y')| \leq L_y(|y - y'|). \\ (iii)\ \exists C > 0,\ \forall y \in \mathbb{R}^d,\ \ |\Phi_2(y)| \leq C(1 + |y|^q) \text{ where } q > 0\ . \end{cases} \tag{$\mathbf{H_3}$}$$

In this chapter, two optimal control problems are studied related to the functionals J_1 and J_2 :

$$\inf\left\{ J_1(T, y, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U},\ \ \mathbf{y}_y^{\mathbf{u}}(s) \in \mathcal{K},\ s \in [0, T] \right\}, \tag{5.5}$$

and

$$\inf\left\{ J_2(T, y, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U},\ \ \mathbf{y}_y^{\mathbf{u}}(s) \in \mathcal{K},\ s \in [0, T] \right\}. \tag{5.6}$$

The existence of the minimum of the control which minimizes (5.5) and (5.6) is not obvious and it depends on the compactness of the set of admissible trajectories. Indeed, we will assume that the function f satisfies the convexity hypothesis :

$$\forall y \in \mathbb{R}^d,\ f(y, U) = \{f(y, u),\ u \in U\} \ \text{ is a convex set.} \tag{$\mathbf{H_4}$}$$

Under this last assumption, the set $S_{[0,T]}^{\mathcal{K}}(y)$ is compact. Moreover, the set valued map $x \to S_{[0,T]}^{\mathcal{K}}(x)$ is Lipschitz continuous from $\mathbb{R}^d$ in $C([0, T]; \mathbb{R}^d)$ (see [9]) with respect to the Hausdorff metric.

The optimal control problems discussed above are subject to value functions defined below. One can associate with the maximum cost functional J_1 the following value function :

$$\vartheta_1(t, y) = \inf\left\{ J_1(t, y, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U},\ \ \mathbf{y}_y^{\mathbf{u}}(s) \in \mathcal{K},\ \forall s \in [0, t] \right\}, \tag{5.7}$$

with the convention that $\inf\{\emptyset\} := +\infty$. For the Bolza cost functional J_2, the associated value function is denoted by ϑ_2,

$$\vartheta_2(t,y) = \inf\left\{ J_2(t,y,\mathbf{u}) \;\middle|\; \mathbf{u} \in \mathcal{U}, \quad \mathbf{y}_y^{\mathbf{u}}(s) \in \mathcal{K}, \ \forall s \in [0,t] \right\}, \tag{5.8}$$

with the convention that $\inf\{\emptyset\} := +\infty$. Note that, in this constrained case (when $\mathcal{K} \neq \mathbb{R}^d$), the value functions ϑ_1 and ϑ_2 may be discontinuous. Moreover, there is no classical result to be applied to the value function ϑ_1. However, the function ϑ_2 can be characterized, under some controllability assumption of (IPQ) type, as the unique Lipschitz continuous viscosity solution of a state-constrained Hamilton-Jacobi equation (see [98]- [99]) and under some controllability assumption of (OPQ) type, as the unique lower semi continuous solution of a Hamilton Jacobi equation (see [63]- [64]). Since these assumptions are not satisfied for a wide class of problems, one has to look for other methods to characterize these value functions.

On the hand, the viability theory (see [7, 10]) and non-smooth analysis allows to characterize the epigraph of the value function, see [11, 40, 41, 8].

In this work, we consider the viability point of view and we look at the characterization of the epigraphs of the value functions ϑ_1 and ϑ_2 (see [3]) using the level-set and the Hamilton Jacobi approaches. More precisely, these epigraphs are linked to some auxiliary optimal control problems free of constraints whose value functions are continuous. For each control problem, the new auxiliary value function is the unique viscosity solution of a Hamilton-Jacobi equation.

5.3 State constrained control problem with maximum cost

In order to characterize the epigraph of the value function ϑ_1 in terms of a value function free of constraints associated with an auxiliary optimal control problem, we introduce the following augmented dynamics $\widehat{f}$ for $u \in U$ and $\widehat{y} := (y,z) \in \mathbb{R}^d \times \mathbb{R}$:

$$\widehat{f}((y,z),u) = \begin{pmatrix} f(y,u) \\ 0 \end{pmatrix}.$$

Let $\widehat{\mathbf{y}}(s) := \widehat{\mathbf{y}}_{\{y,z\}}(s) := (\mathbf{y}_y^{\mathbf{u}}(s), \mathbf{z}_{y,z}^{\mathbf{u}}(s))$ (where $\mathbf{z}_{y,z}^{\mathbf{u}}(.) := z$) be the associated augmented solution of :

$$\dot{\widehat{\mathbf{y}}}(s) = \widehat{f}(\widehat{\mathbf{y}}(s), \mathbf{u}(s)), \quad s \in (0,T), \tag{5.9a}$$

$$\widehat{\mathbf{y}}(0) = (y,z)^T. \tag{5.9b}$$

Define the corresponding set of feasible trajectories,

$$\widehat{S}_{[0,T]}(\widehat{y}) := \{\widehat{\mathbf{y}} = (\mathbf{y}_y^{\mathbf{u}}, \mathbf{z}_{y,z}^{\mathbf{u}}), \quad \widehat{\mathbf{y}} \text{ satisfies (5.9) for some } \mathbf{u} \in \mathcal{U}\}, \tag{5.10}$$

for $\widehat{y} = (y,z) \in \mathbb{R}^d \times \mathbb{R}$.

Remark 5.3.1 *Under the assumption* ($\mathbf{H_4}$)*, for every $\hat{y} \in \mathbb{R}^d \times \mathbb{R}$, the augmented dynamics $\widehat{f}(\widehat{y}, U)$ is convex. Therefore, $\widehat{S}_{[0,T]}(\widehat{y})$ is a compact subset of $W^{1,1}([0,T])$ for the topology of $C([0,T];\mathbb{R}^{d+1})$ (see [9]).*

In what follows, we will define an auxiliary optimal control problem whose value function is Lipschitz continuous. The auxiliary value function is characterized by a HJ equation.

5.3.1 Auxiliary control problem for the L_∞-running cost case

Hamilton-Jacobi equation

Let g be a Lipschitz continuous function characterizing the constraints set $\mathcal{K}$ in the following way :

$$\forall y \in \mathcal{K},\ \ g(y) \leq 0 \Leftrightarrow y \in \mathcal{K}. \tag{5.11}$$

Denote by $L_g > 0$ the Lipschitz constant of g. Note that the function g exists because of the closeness of $\mathcal{K}$ (for instance the signed distance to $\mathcal{K}$ satisfies the condition (5.11)). Therefore, for $u \in \mathcal{U}$, we have,

$$\mathbf{y}_y^{\mathbf{u}}(s) \in \mathcal{K}, \forall s \in [0,t] \ \ \Leftrightarrow \max_{s\in[0,t]} g(\mathbf{y}_y^{\mathbf{u}}(s)) \leq 0.$$

Consider the following auxiliary control problem defined below by its value function :

$$w_1(t,y,z) := \inf_{\widehat{\mathbf{y}}\in\widehat{S}_{[0,t]}(\widehat{y})} \max_{\theta\in[0,t]} \Psi_1(\mathbf{y}(\theta), z), \tag{5.12}$$

where

$$\Psi_1(y,z) := (\Phi_1(y) - z) \vee g(y), \tag{5.13}$$

(with $a \vee b = \max(a,b)$). The new control problem is free of any additional assumption on the set $\mathcal{K}$ neither on the dynamics $\widehat{f}$. The following proposition shows that the level sets of this new value function w_1 characterize the epigraph of ϑ_1.

Proposition 5.3.2 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_4}$) *hold and let $(t,y,z) \in [0,T] \times \mathcal{K} \times \mathbb{R}$. The value functions w_1 is related to ϑ_1 by the following relations :*

$$\begin{aligned} &(i) \quad \vartheta_1(t,y) - z \leq 0 \Leftrightarrow w_1(t,y,z) \leq 0, \\ &(ii) \quad \vartheta_1(t,y) = \min\Big\{ z \in \mathbb{R}\ ,\ \ w_1(t,y,z) \leq 0 \Big\}. \end{aligned}$$

Proof. (i) Assume $\vartheta_1(t,y) \leq z$. There exists a minimizing sequence $\{\mathbf{y_n}\}_\mathbf{n} \subset S_{[0,T]}(y)$ of admissible trajectories such that,

$$\lim_{\mathbf{n}\to\infty} \left[\max_{0\leq\theta\leq t} \Phi_1(\mathbf{y_n}(\theta)) - z \right] = \vartheta_1(t,y) - z \leq 0.$$

Since all the trajectories $\mathbf{y_n}$ are admissible, we have $\max\limits_{0\le\theta\le t} g(\mathbf{y_n}(\theta)) \le 0$. Hence

$$w_1(t,y,z) \le \liminf_{\mathbf{n}\to\infty} \left[\max_{0\le\theta\le t} \big(\Phi_1(\mathbf{y_n}(\theta)) - z\big) \bigvee \max_{0\le\theta\le t} g(\mathbf{y_n}(\theta)) \right] \le 0.$$

Conversely, assume $w_1(t,y,z) \le 0$. By remark 5.3.1, there exists a trajectory $\widehat{\mathbf{y}} \in \widehat{S}_{[0,T]}(y,z)$ starting from $\widehat{y} = (y,z)$ such that,

$$0 \ge w_1(t,y,z) = \max_{0\le\theta\le t} \Psi_1(\mathbf{y}(\theta), z).$$

The last inequality is equivalent to,

$$\max_{0\le\theta\le t} \big(\Phi_1(\mathbf{y}(\theta)) - z\big) \le 0 \quad \text{and} \quad \max_{0\le\theta\le t} g(\mathbf{y}(\theta)) \le 0.$$

It follows that $\vartheta_1(t,y) \le z$ and (i) is proved. The proof of (ii) follows from (i). ■

Remark 5.3.3 *Note that the value function $\vartheta_1(t,.)$ is l.s.c. and then its epigraph is a closed set. Moreover, from the previous proposition, we have*

$$Epi\big(\vartheta_1(t,.)\big) = \Big\{ \widehat{y} = (y,z) \in \mathcal{K}\times\mathbb{R} \mid w_1(t,\widehat{y}) \le 0 \Big\}.$$

The value function w_1 enjoys more regularity properties and recall that it satisfies the dynamic programming principle. Let us define the following Hamiltonian, for all $y, p \in \mathbb{R}^d$:

$$H(y,p) := \sup_{u\in U} \big(- f(y,u)\cdot p \big).$$

Proposition 5.3.4 *Assume ($\mathbf{H_1}$)-($\mathbf{H_2}$) hold.*

(i) The value function w_1 is Lipschitz continuous in $\mathbb{R}^d\times\mathbb{R}$, i.e, $\forall \hat{y}, \hat{y'} \in \mathbb{R}^d\times\mathbb{R}$, $t,t' \in [0,T]$,

$$|w_1(t,\hat{y}) - w_1(t,\hat{y'})| \le \max(L_g, L_1)e^{Lt}|\hat{y} - \hat{y'}|$$
$$|w_1(t,\hat{y}) - w_1(t',\hat{y})| \le \max(L, C_f)e^{LT}|t - t'|$$

where $C_f := \max_{a\in U} |f(0,a)|$.

(ii) For any $t \in [0,T]$, $h \ge 0$, such that $t+h \le T$,

$$w_1(t+h,y,z) = \inf_{\widehat{\mathbf{y}}:=(\mathbf{y},\mathbf{z})\in\widehat{S}_{[0,h]}(y,z)} \Big\{ w_1(t,\mathbf{y}(h),z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}(\theta),z) \Big\}.$$

(iii) Furthermore, the function w_1 is the unique Lipschitz continuous viscosity solution of the following HJ equation :

$$\min\Big(\partial_t w_1(t,y,z) + H(y,\nabla_y w_1),\ w_1(t,y,z) - \Psi_1(y,z) \Big) = 0, \qquad \text{in }]0,T]\times\mathbb{R}^d\times\mathbb{R}, \tag{5.14a}$$
$$w_1(0,y,z) = \Psi_1(y,z), \qquad \text{in } \mathbb{R}^d\times\mathbb{R}. \tag{5.14b}$$

Proof. The proof of the Lipschitz property is given in the Appendix A. The DPP is classical and its proof can be found in [19]. For the proof of the (iii) see for instance [3]. ■

Hamilton-Jacobi equation with a wise choice of the obstacle function

The reason behind the introduction of an auxiliary control problem is to deal with a problem free of state constraints. However, from a control problem defined only on a set $\mathcal{K}$, we deal with an auxiliary control problem whose value function is characterized with a Hamilton Jacobi equation defined in all domain. In addition, if the dynamics function f is Lipshitz continuous only on a closed set $\widetilde{\mathcal{K}}$ containing the constraints set $\mathcal{K}$, then it is extended out of $\widetilde{\mathcal{K}}$. In such a case, it can be difficult to solve the equation (5.14) out of $\widetilde{\mathcal{K}}$. In order to overcome this problem , we want to restrict our search of the solution on the set $\widetilde{\mathcal{K}}$. In fact, we show that with a wise choice of the function g, we will have to deal with an HJ equation with a Dirichlet condition. Let $\widetilde{c} > 0$ and define the following extended set $\widetilde{\mathcal{K}}$:

$$\widetilde{\mathcal{K}} :\equiv \mathcal{K} + \tilde{c}B.$$

where B is the unit ball in $\mathbb{R}^d$ centred in the origin. Let $g(y) := (d_{\mathcal{K}}(y) \wedge \widetilde{c}), \forall y \in \mathbb{R}^d$ where $d_{\mathcal{K}}$ the signed distance to $\mathcal{K}$. In particular, we have

$$g(y) < \widetilde{c}, \quad \forall y \in \overset{\circ}{\widetilde{\mathcal{K}}} \qquad \text{and} \qquad g(y) = \widetilde{c}, \quad \forall y \notin \overset{\circ}{\widetilde{\mathcal{K}}} \,. \tag{5.15}$$

Define

$$\widetilde{\Psi}_1(y,z) = \Big((\Phi_1(y) - z) \vee g(y) \Big) \wedge \widetilde{c},$$

where Ψ_1 is defined in (5.13). Let $\widetilde{w}_1$ be defined by :

$$\widetilde{w}_1(t,y,z) := \inf_{\widehat{\mathbf{y}}=(\mathbf{y},z)\in \widehat{S}_{[0,T]}(\widehat{\mathbf{y}})} \max_{\theta\in[0,t]} \widetilde{\Psi}_1(\mathbf{y}(\theta),z).$$

It is important to emphasise that the function $\widetilde{w_1}$ can be used to characterize the epigraph of the value function $\vartheta_1(t,.)$ in the same manner that it is done by the function w_1. In fact, it can be easily shown (using the definitions w_1 and $\widetilde{w_1}$) and the fact that $\widetilde{c} > 0$) that

$$\widetilde{w}_1(t,y,z) \le 0 \;\Leftrightarrow\; w_1(t,y,z) \le 0.$$

The following theorem provides the characterization of the value function $\widetilde{w_1}$ as the unique Lipschitz continuous viscosity solution of the same Hamilton Jacobi equation (5.14) but with a Dirichlet boundary condition. This allows to reduce the solution of the problem to a closed set. Apart from this set, the solution remains constant.

Theorem 5.3.5 *Assume* ($\mathbf{H_1}$)-($\mathbf{H_2}$) *hold. The function w_1 is the unique Lipschitz continuous viscosity solution of the following Hamilton Jacobi equation :*

$$\min\left(\partial_t \widetilde{w_1}(t,y,z) + H(y, \nabla_y \widetilde{w_1}), \widetilde{w_1}(t,y,z) - \widetilde{\Psi_1}(y,z) \right) = 0,$$

$$\textit{in } \;]0,T] \times \overset{\circ}{\widetilde{\mathcal{K}}} \times \mathbb{R}, \tag{5.16a}$$

$$\widetilde{w_1}(0,y,z) = \widetilde{\Psi_1}(y,z), \qquad \textit{in } \; \overset{\circ}{\widetilde{\mathcal{K}}} \times \mathbb{R}. \tag{5.16b}$$

$$\widetilde{w}_1(t,y,z) = \widetilde{c}, \qquad \textit{for all } t \in [0,T], \; y \notin \overset{\circ}{\widetilde{\mathcal{K}}} \;\; \textit{and} \;\; z \in \mathbb{R}, \tag{5.16c}$$

Proof. The first part (i) follows from (iii) of the proposition 5.3.4. Let prove the assertion (ii). For $(y,z) \notin \widetilde{\mathcal{K}} \times \mathbb{R}$, we have

$$g(y) \geq \widetilde{c}.$$

It follows that $\big(\Phi_1(y) - z\big) \vee g(y) \geq \widetilde{c}$. Then, by definition of $\widetilde{\Psi_1}$, we have

$$\widetilde{\Psi_1}(y,z) = \widetilde{c}.$$

Now, by using the fact that $w_1(t,y,z) \geq \widetilde{\Psi_1}(y,z)$ which follows from (5.14), we get

$$w_1(t,y,z) \geq \widetilde{c}, \quad \forall y \notin \widetilde{\mathcal{K}} \ \text{ and } \ z \in \mathbb{R}.$$

By definition of w_1, we have $w_1(t,y,z) \leq \widetilde{c}$. This ends the proof of (ii) and then the proof of the theorem 5.3.5. ∎

Remark 5.3.6 *If the function Φ_1 satisfies the following :*

$$\Phi_1(y) \in [\underline{m}, \overline{M}], \quad \forall y \in \widetilde{\mathcal{K}},$$

Then, it suffices to consider the variable $z \in]\underline{\underline{m}}, \overline{\overline{M}}[$ such that $\underline{\underline{m}} < \underline{m}$ and $\overline{\overline{M}} > \overline{M}$. Indeed, in this case, we still have the important relation between ϑ_1 and w_1, i.e,

$$\vartheta_1(t,y) = \inf\Big\{ z \in]\underline{\underline{m}}, \overline{\overline{M}}[\mid \widetilde{w_1}(t,y,z) \leq 0 \Big\}$$

In addition, the function w_1 is the unique Lipschitz continuous viscosity solution of the following HJ equation :

$$\min\Big(\partial_t \widetilde{w_1}(t,y,z) + H(y, \nabla_y \widetilde{w_1}), \widetilde{w_1}(t,y,z) - \widetilde{\Psi_1}(y,z) \Big) = 0,$$

$$\text{in } \ ]0,T] \times \overset{\circ}{\widetilde{\mathcal{K}}} \times]\underline{\underline{m}}, \overline{\overline{M}}[, \tag{5.17a}$$

$$\widetilde{w_1}(0,y,z) = \widetilde{\Psi_1}(y,z), \qquad \text{in } \ \overset{\circ}{\widetilde{\mathcal{K}}} \times]\underline{\underline{m}}, \overline{\overline{M}}[. \tag{5.17b}$$

$$\widetilde{w_1}(t,y,z) = \widetilde{c}, \quad \forall\, t \in [0,T], \ y \notin \overset{\circ}{\widetilde{\mathcal{K}}} \ \text{ and } z \in]\underline{\underline{m}}, \overline{\overline{M}}[, \tag{5.17c}$$

We don't need any boundary condition on the axis of z because the dynamics is zero $\dot{\mathbf{z}}(t) = 0$.

The control problems discussed in this Chapter can be described by a viability kernel associated with the epigraph of the function Φ_1. Moreover, an exit time function corresponding to the viability kernel will be defined and used later for a reconstruction of the optimal trajectories.

5.3.2 Link with the exit time function and viability problems

The aim of this subsection is to make a link between the control problems discussed in the previous subsection and a viability problem we will define in the following. First, let us define the following set :

$$\mathcal{D}_1 := \Big\{ \widehat{y} = (y,z) \in \mathbb{R}^{d+1} \mid y \in \mathcal{K} \quad \text{and} \quad \widehat{y} \in \mathcal{E}pi(\Phi_1) \Big\}.$$

Let us define also the exit time function $\mathcal{T}_1 : \mathbb{R}^{d+1} \to [0,T]$, which associates with each starting point $(y,z) \in \mathbb{R}^{d+1}$, the maximum time to remain inside the epigraph of the function Φ_1 with an *admissible* trajectory $\widehat{\mathbf{y}}^u_{\widehat{y}}(.)$ solution of (5.9) associated with an admissible control $u \in \mathcal{U}$, i.e,

$$\mathcal{T}_1(y,z) := \sup \left\{ t \in [0,T] \;\middle|\; \exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \widehat{\mathbf{y}}^u_{\widehat{y}}(\theta) \in \mathcal{D}_1, \; \forall \theta \in [0,t] \right\}. \tag{5.18}$$

Now, let us recall the definition of *the viability kernel* with time horizon t associated with $\mathcal{D}_1$, i.e, the set of starting points for which it is possible to remain in $\mathcal{D}_1$ for the time interval $[0,t]$,

$$\mathcal{V}iab_1(t) := \left\{ \widehat{y} \in \mathbb{R}^{d+1} \;\middle|\; \exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \widehat{\mathbf{y}}^u_{\widehat{y}}(\theta) \in \mathcal{D}_1, \; \forall \theta \in [0,t] \right\}.$$

The following proposition gives a link between the value functions w_1, v_1 and the *viability kernel* at time $t \in [0,T]$.

Proposition 5.3.7 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_4}$) *hold. Then, for* $t \in [0,T]$ *the* viability kernel $\mathcal{V}iab_1(t)$ *is related to the value functions* w_1 *and* ϑ_1 *by the following relations :*

$$\mathcal{V}iab_1(t) = \big\{ \widehat{y} = (y,z) \in \mathbb{R}^{d+1} \;\big|\; \vartheta_1(t,y) \le z \;\big\} = \big\{ (y,z) \in \mathbb{R}^{d+1} \;\big|\; w_1(t,y,z) \le 0 \;\big\}.$$

Proof. Let $\mathcal{V}_1(t) := \big\{ (y,z) \in \mathbb{R}^{d+1} \;\big|\; w_1(t,y,z) \le 0 \;\big\}$. The fact that $\mathcal{V}_1(t) = \big\{ \widehat{y} = (y,z) \in \mathbb{R}^{d+1} \;\big|\; \vartheta_1(t,y) \le z \;\big\}$ is a consequence of proposition 5.3.2. It remains to prove that $\mathcal{V}iab_1(t) = \mathcal{V}_1(t)$,

Let $\widehat{y} = (y,z) \in \mathcal{V}iab_1(t)$. It follows from the definition of $\mathcal{V}iab_1(t)$ that

$$\exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \widehat{\mathbf{y}}^u_{\widehat{y}}(\theta) \in \mathcal{D}_1, \; \forall \theta \in [0,t].$$

By definition of $\mathcal{D}_1$, the following equivalence holds :

$$\exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \Psi_1\big(\widehat{\mathbf{y}}^u_{\widehat{y}}(\theta)\big) \le 0, \; \forall \theta \in [0,t] \Longleftrightarrow \exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \max_{\theta \in [0,t]} \Psi_1\big(\widehat{\mathbf{y}}^u_{\widehat{y}}(\theta)\big) \le 0. \tag{5.19}$$

Taking the infimum over the set of trajectories, yields that $w_1(t,y,z) \le 0$.

Conversely, assume that $(y,z) \in \mathcal{V}_1(t)$. Then, by definition of $\mathcal{V}_1(t)$ we have $w_1(t,y,z) \le 0$. By Remark 5.3.1, there exists a trajectory $\widehat{\mathbf{y}}^{\mathbf{u}}_{\widehat{y}} \in \widehat{S}_{[0,T]}(y,z)$ starting from $\widehat{y} = (y,z)$ depending on the control $\mathbf{u} \in \mathcal{U}$ such that,

$$0 \ge w_1(t,y,z) = \max_{0 \le \theta \le t} \Psi_1(\mathbf{y}^{\mathbf{u}}_y(\theta), z).$$

By using (5.19), we get $(y,z) \in \mathcal{V}iab_1(t)$ and the proof is completed. ∎

The following proposition gives a tool to bypass the regularity issues of $\mathcal{T}_1$. Here, we focus on the characterization of *the viability kernel* $\mathcal{V}iab_1(t)$. Moreover, the value function ϑ_1 is written in terms of the exit time function.

Theorem 5.3.8 *Assume* $(\mathbf{H_1})$*,* $(\mathbf{H_2})$ *and* $(\mathbf{H_4})$ *hold. Then the exit time function* $\mathcal{T}_1$ *satisfies the following relations :*

$$
\begin{array}{ll}
(i) & \mathcal{T}_1(y,z) = \sup\left\{t \in [0,T] \;\middle|\; (y,z) \in \mathcal{V}iab_1(t)\;\right\} = \sup\left\{t \in [0,T] \;\middle|\; w_1(t,y,z) \leq 0\;\right\},\\
(ii) & \mathcal{T}_1(y,z) = t \Rightarrow\; w_1(t,y,z) = 0,\\
(iii) & \vartheta_1(t,y) = \inf\left\{z \;\middle|\; \mathcal{T}_1(y,z) \geq t\right\}.
\end{array}
$$

Proof. Let $\widetilde{\mathcal{T}}_1(y,z) := \sup\left\{t \in [0,T] \;\middle|\; (y,z) \in \mathcal{V}iab_1(t)\;\right\}$. The fact that $\widetilde{\mathcal{T}}_1(y,z) := \sup\left\{t \in [0,T] \;\middle|\; w_1(t,y,z) \leq 0\;\right\}$ is a consequence of proposition (5.3.7). It remains to prove that $\mathcal{T}_1(y,z) = \widetilde{\mathcal{T}}_1(y,z)$. Let $t := \mathcal{T}_1(y,z)$ and assume that $t < \infty$. By remark 5.3.1, there exists an admissible trajectory $\widehat{\mathbf{y}}^{\mathbf{u}}_{\widehat{y}}(.)$ such that

$$
\widehat{\mathbf{y}}^{\mathbf{u}}_{\widehat{y}}(\theta) \in \mathcal{D}_1,\;\; \forall\theta \in [0,t].
$$

This implies that $\widehat{y} \in \mathcal{V}iab_1(t)$. Hence $\widetilde{\mathcal{T}}_1(y,z) \geq t$ by definition of $\widetilde{\mathcal{T}}_1(y,z)$.

Now, let $\widetilde{t} := \widetilde{\mathcal{T}}_1(y,z)$. For any $n \geq 1$, there exists a sequence $(t_n)_{n\geq 1}$ such that $t_n \leq \widetilde{t}$ such that $t_n \to \widetilde{t}$ as $n \to \infty$ and $w_1(t_n,y,z) \leq 0$, $\forall n \geq 1$. By remark 5.3.1, there exists an admissible trajectory $\widehat{\mathbf{y}}_n := \widehat{\mathbf{y}}^{\mathbf{u_n}}_{\widehat{y}}$ such that $\widehat{\mathbf{y}}_n(\theta) \in \mathcal{D}_1,\;\; \forall\theta \in [0,t_n]$. Since $\mathcal{D}_1$ is closed and using the compactness arguments of trajectories, it is possible to extract a subsequence $\widehat{\mathbf{y}}_n$ converging uniformly on $[0,\widetilde{t}]$ to $\widehat{\mathbf{y}}$ (with $t_n \to \widetilde{t}$) such that $\widehat{\mathbf{y}}(\theta) \in \mathcal{D}_1,\;\; \forall\theta \in [0,\widetilde{t}]$. Thus, by definition of $\mathcal{T}_1(y,z)$, we obtain $\mathcal{T}_1(y,z) \geq \widetilde{t}$, which concludes the proof of (i).

Now, let us prove (ii). Let $t := \mathcal{T}_1(y,z)$. As a consequence of (i), we have that for all $n \geq 1$, $\exists t_n \in [t - \frac{1}{n}, t[$ s.t $w_1(t_n,y,z) \leq 0$. Since w_1 is continuous, one obtain $w_1(t,y,z) \leq 0$. On the other hand, $\forall t_1 \geq t$, $w_1(t_1,y,z) > 0$, then $w_1(t,y,z) \geq 0$ and the proof (ii) is completed.

For the proof of (iii), one can deduce directly from (i) and proposition 5.3.7 that, for all (t,y,z),

$$
\mathcal{T}_1(y,z) \geq t \Longleftrightarrow \vartheta_1(t,y) \leq z. \tag{5.20}
$$

The statement of (iii) follows from (5.36). ∎

In the following, a link is established between the optimal trajectory associated with the auxiliary control problem whose value function is w_1, the optimal trajectory of the control problem whose value function is ϑ_1, and the optimal trajectories associated with the exit time function.

Proposition 5.3.9 *Assume* $(\mathbf{H_1})$*,* $(\mathbf{H_2})$ *and* $(\mathbf{H_4})$ *hold. Let* $y \in \mathcal{K}$ *such that* $\vartheta_1(T,y) < \infty$*. Define* $z := \vartheta_1(T,y)$*.*

(i) Let $\widehat{\mathbf{y}}^* = (\mathbf{y}^*, \mathbf{z}^*)$ be the optimal trajectory for the auxiliary control problem (5.12) associated with the initial point $(y, z) \in \mathcal{K} \times \mathbb{R}$. Then, the trajectory $\mathbf{y}^*$ is optimal for the control problem (5.7).

(ii) Let $\widehat{\mathbf{y}}^* = (\mathbf{y}^*, \mathbf{z}^*)$ be an optimal trajectory for the exit time problem (5.18) associated with the initial point $(y, z) \in \mathcal{K} \times \mathbb{R}$. Then, $\widehat{\mathbf{y}}^*$ is also optimal for the auxiliary control problem (5.12).

Proof. Let $(y, z) \in \mathcal{K} \times \mathbb{R}$ such that $\vartheta_1(T, y) = z$. First, let us prove (i). Let $\widehat{\mathbf{y}}^* = (\mathbf{y}^*, \mathbf{z}^*)$ be the optimal trajectory for the auxiliary control problem (5.12) associated with the initial point $(y, z) \in \mathcal{K} \times \mathbb{R}$.

Using the proposition 5.3.2, we have that

$$\vartheta_1(T, y) = z \Rightarrow w_1(T, y, z) \leq 0.$$

Since $\widehat{\mathbf{y}}^*$ is an optimal trajectory of the problem (5.12) associated with (y, z), it follows that

$$w_1(T, y, z) = \max_{s \in [0,T]} \Psi_1(\mathbf{y}^*(s), z) \leq 0.$$

Using the definition of Ψ_1, we get,

$$\max_{s \in [0,T]} \Phi_1(\mathbf{y}^*(s)) \leq z, \qquad \text{and} \qquad \max_{s \in [0,T]} g(\mathbf{y}^*(s)) \leq 0.$$

Since $\vartheta_1(T, y) = z$, it follows from that,

$$\max_{s \in [0,T]} \Phi_1(\mathbf{y}^*(s)) \leq \vartheta_1(T, x) \quad \text{and} \quad \mathbf{y}^*(s) \in \mathcal{K}, \ \forall s \in [0, T].$$

By definition of ϑ_1 one can conclude that

$$\vartheta_1(T, x) = \max_{s \in [0,T]} \Phi_1(\mathbf{y}^*(s)) \quad \text{and} \quad \mathbf{y}^*(s) \in \mathcal{K}, \ \forall s \in [0, T].$$

Therefore, $\mathbf{y}^*$ is an optimal trajectory for (5.7) associated with (y, z) and the proof is achieved.

Now, let us prove (ii). Let $\widehat{\mathbf{y}}^* = (\mathbf{y}^*, \mathbf{z}^*)$ be an optimal trajectory for the exit time problem (5.18) associated with the initial point $(y, z) \in \mathcal{K} \times \mathbb{R}$ such that $\vartheta_1(T, y) = z$

Let $\tau := \mathcal{T}_1(y, z)$. Since $\widehat{\mathbf{y}}^*$ is an optimal trajectory of the problem (5.18), it follows from the definition of $\mathcal{T}_1$ that,

$$\widehat{\mathbf{y}}^*(\theta) := (\mathbf{y}^*(\theta), \mathbf{z}^*(\theta)) \in \mathcal{D}_1, \ \forall \theta \in [0, \tau].$$

Then, we have,

$$\max_{s \in [0,\tau]} \Phi_1(\mathbf{y}^*(s)) \leq z, \qquad \text{and} \qquad \max_{s \in [0,\tau]} g(\mathbf{y}^*(s)) \leq 0.$$

Since $\vartheta_1(T,y) = z$, we obtain that

$$\max_{s\in[0,\tau]} \Phi_1(\mathbf{y}^*(s)) \le \vartheta_1(T,x) \quad \text{and} \quad \mathbf{y}^*(s) \in \mathcal{K},\ \forall s \in [0,\tau].$$

If $\tau = T$, then from the definition of ϑ_1, one can conclude that $\mathbf{y}^*$ is an optimal trajectory for (5.7) associated with (y,z).

Assume $\tau < T$. Then, from the fact that $\vartheta_1(T,y) = z$, there exists a minimizing sequence $\{\mathbf{y_n}\}_n \subset S_{[0,T]}(y)$ of admissible trajectories such that,

$$\lim_{\mathbf{n}\to\infty} \left[\max_{0\le\theta\le T} \Phi_1(\mathbf{y_n}(\theta)) - z \right] = \vartheta_1(T,y) - z = 0.$$

Since all the trajectories $\mathbf{y_n}$ are admissible, we have $\max\limits_{0\le\theta\le T} g(\mathbf{y_n}(\theta)) \le 0$. It follows that,

$$\lim_{\mathbf{n}\to\infty} \left[\max_{0\le\theta\le T} \Psi_1(\mathbf{y_n}(\theta), z) \right] \le 0. \tag{5.21}$$

From (5.21) and by definition of the exit time function $\mathcal{T}_1$ we have that $\mathcal{T}_1(y,z) := \tau \ge T$ and this is a contradiction and the proof of the proposition is completed. ■

Remark 5.3.10 *The previous links we have just established are very important in the sense that they allow to consider several algorithms for the reconstruction of optimal trajectories. On one hand, one can consider the algorithm of reconstruction of trajectories using the value function w_1 of the auxiliary control problem defined in $[0,T] \times \mathbb{R}^d \times \mathbb{R}$. However, for high dimensions, this kind of algorithms may be very slow. On the other hand, the reconstruction using the exit time function may be an alternative way since $\mathcal{T}_1$ is defined only on $\mathbb{R}^d \times \mathbb{R}$.*

5.3.3 Reconstruction of optimal trajectories

The purpose of this section is to present some ways to reconstruct optimal trajectories of the control problem whose value function is ϑ_1. It is important here to recall the context of our analysis. The value function ϑ_1 is in general discontinuous in presence of constraints. In the previous section we have shown that the epigraph of ϑ_1 can be characterized by a level set of a Lipschitz continuous value function w_1 of an auxiliary optimal control problem. This procedure has a dual importance. First, we handle the technical difficulties related to the direct computation of ϑ_1 and we focus on a Lipschitz value function characterized as the unique Lipschitz continuous viscosity solution of a Hamilton Jacobi equation. The second importance is the possibility to generate optimal trajectories using the auxiliary optimal control problem for which the value function is Lipschitz continuous.

Reconstruction by minimizing the value function

Here, we want to extend the classical result of [95] for the unconstrained optimal control problem to the state constrained control problem with maximum criterion. The main idea of this approach is to define an approximation of the set-valued optimal feedback map using the dynamic programming principle.

Algorithm A. For a given $n \in \mathbb{N}$, $n \geq 1$, let us consider $(t_0 = 0, t_1, ..., t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with the time step $h = \frac{T}{n}$. Let $\{\mathbf{y}^n(\cdot), \mathbf{z}^n(\cdot)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$ $(i \geq 1\ )$. Set $\mathbf{z}^n(\cdot) := z$ and $\mathbf{y}^n(t_0) = y$ (where $\vartheta_1(t_n, y) = z$).

Step 1 Knowing the state $\mathbf{y}^n(t_k)$, choose the optimal control at time t_k such that :

$$u_k^n \in \arg\min_{u\in U} \Big(w_1(t_{n-k-1}, \mathbf{y}^n(t_k) + hf(\mathbf{y}^n(t_k), u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}^n(t_k) + \theta f(\mathbf{y}^n(t_k), u), z) \Big).$$

Step 2 Define $\mathbf{u}^n(t) := u_k^n$ a constant control on the interval $t \in (t_k, t_{k+1}]$ and $\mathbf{y}^n(t)$ on $(t_k, t_{k+1}]$ as the solution of

$$\dot{\mathbf{y}}(t) := f(\mathbf{y}(t), \mathbf{u}^n(t)) \text{ a.e } \ t \in (t_k, t_{k+1}],$$

with initial condition $\mathbf{y}^n(t_k)$ at t_k and $\mathbf{z}^n(\cdot) := z$.

An important result of this section is the following theorem showing that the algorithm A will converge to an optimal trajectory.

Theorem 5.3.11 *Assume* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_4})$ *hold. Let* w_1 *be the unique Lipshitz continuous viscosity solution of the equation (5.14)) for* $n \geq 1$ *and* $h = \frac{T}{n}$. *Let* $\{\mathbf{y}^n(\cdot), \mathbf{z}^n(\cdot), \mathbf{u}^n(\cdot)\}$ *be a sequence generated by algorithm A for* $n \geq 1$. *Then, the sequence of trajectories* $\{\mathbf{y}^n(\cdot)\}_n$ *has cluster points with respect to the uniform convergence topology. For any cluster point* $\bar{\mathbf{y}}(\cdot)$ *there exists a control law* $\bar{\mathbf{u}}(\cdot)$ *such that* $(\bar{\mathbf{y}}(\cdot), \bar{\mathbf{z}}(\cdot), \bar{\mathbf{u}}(\cdot))$ *is optimal for the control problem (5.12).*

In order to prove the theorem 5.3.11, we first establish the following property :

Proposition 5.3.12 *Let* $\Omega \subset \mathbb{R}^d$ *be a compact set. Assume* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_4})$. *Then for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $h \in [0, \delta]$, *for all* $\xi \in \Omega$ *and* $t \in [0, T]$

$$\left| \frac{1}{h} \left(\min_{u\in U} \left\{ w_1(t-h, \xi + hf(\xi, u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\xi + \theta f(\xi, u)) \right\} - w_1(t, \xi, z) \right) \right| < \varepsilon.$$

Proof. of Proposition 5.3.12. Let us introduce a set-valued map $R(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ defining for a $\xi \in \mathbb{R}^d$ and $\tau \in [0, T]$ the reachable set of the dynamic system f for the trajectories starting from ξ and at time τ

$$R(\tau, \xi) = \{\mathbf{y}(\tau),\ \mathbf{y} \in \mathbf{S}_{[\mathbf{0},\tau]}(\xi)\}. \tag{5.22}$$

Let $L_{\Psi_1} > 0$ and $L_w > 0$ be the Lipshitz constants of Ψ_1 and w_1 respectively. Due to the hypothesis (H_1) there exists a constant $r > 0$ such that for all $\xi \in \Omega,\ \tau \in [0, T]$ $R(\tau, \xi) \subset rB$ (with B the unit ball centered at zero).

Fix a $\varepsilon > 0$. Using the properties of the reachable set (5.22) (see proposition 5.3 in [95]), let us choose $\delta > 0$ such that for all $h \in (0,\delta) \cap (0,T)$ and for all $\xi \in \Omega$

$$\max(L_{\Psi_1}, L_w) \cdot \mathrm{dist}_H(\frac{1}{h}(R(h,\xi) - \xi), f(\xi,U)) < \frac{\varepsilon}{2}.$$

where dist_H is the Hausdorf distance. Now, let take $\xi \in \Omega$ and $t \in [0,T]$. Recall that, due to the dynamic programming principle (5.14a) we have :

$$w_1(t,\xi,z) = \inf_{\mathbf{y} \in S_{[0,h]}(\xi)} \left(w_1(t-h, \mathbf{y}(h), z) \bigvee \max_{\theta \in [0,h]} \Psi_1(\mathbf{y}(\theta)) \right).$$

Then

$$\begin{aligned}
&\left| \min_{u \in U} \left\{ w_1(t-h, \xi + hf(\xi,u), z) \bigvee \max_{\theta \in [0,h]} \Psi_1(\xi + \theta f(\xi,u)) \right\} - w_1(t,\xi,z) \right| \\
\leq &\left| \min_{u \in U} \left\{ w_1(t-h, \xi + hf(\xi,u), z) \bigvee \max_{\theta \in [0,h]} \Psi_1(\xi + \theta f(\xi,u)) \right\} - \inf_{\mathbf{y} \in S_{[0,h]}(\xi)} \left(w_1(t-h, \mathbf{y}(h), z) \bigvee \max_{\theta \in [0,h]} \Psi_1(\mathbf{y}(\theta)) \right) \right| \\
\leq &\left| \sup_{u \in U, \mathbf{y} \in S_{[0,h]}(\xi)} \left\{ w_1(t-h, \xi + hf(\xi,u), z) \bigvee \max_{\theta \in [0,h]} \Psi_1(\xi + \theta f(\xi,u)) - w_1(t-h, \mathbf{y}(h), z) \bigvee \max_{\theta \in [0,h]} \Psi_1(\mathbf{y}(\theta)) \right\} \right|.
\end{aligned}$$

Let $u \in U$ and $\mathbf{y} \in S_{[0,h]}(\xi)$. Then, there exists $\mathbf{u} \in \mathcal{U}$ such that $\mathbf{y} = \mathbf{y}_\xi^{\mathbf{u}}$ on $[0,h]$. Let us show that

$$\left| \max_{\theta \in [0,h]} \Psi_1(\xi + \theta f(\xi,u)) - \max_{\theta \in [0,h]} \Psi_1(\mathbf{y}(\theta)) \right| < h \cdot \frac{\varepsilon}{2}. \tag{5.23}$$

Indeed, for all $\theta \in [0,h]$,

$$\begin{aligned}
|\Psi_1(\xi + \theta f(\xi,u)) - \Psi_1(\mathbf{y}(\theta))| &\leq L_{\Psi_1} |\xi + \theta f(\xi,u) - \mathbf{y}(\theta)| \\
&\leq L_{\Psi_1} \mathrm{dist}_H(R(\theta,\xi), \xi + \theta f(\xi,U)) = L_{\Psi_1} \mathrm{dist}_H(R(\theta,\xi) - \xi, \theta f(\xi,U)) \\
&= L_{\Psi_1} \theta \mathrm{dist}_H(\theta^{-1}(R(\theta,\xi) - \xi), f(\xi,U)) < \theta\varepsilon \leq h\frac{\varepsilon}{2}.
\end{aligned}$$

Then the inequality (5.23) holds. With the similar arguments, one can show also that

$$\begin{aligned}
|w_1(t-h, \xi + hf(\xi,u), z) - w_1(t-h, \mathbf{y}(h), z)| &\leq L_w |\xi + \theta f(\xi,u) - \mathbf{y}(\theta)| \\
&\leq L_w \cdot \mathrm{dist}_H(R(h,\xi) - \xi, hf(\xi,U)) \\
&= L_W \cdot h \cdot \mathrm{dist}_H(h^{-1}(R(h,\xi) - \xi), f(\xi,U)) \\
&< h \cdot \frac{\varepsilon}{2}.
\end{aligned}$$

Recall that for all reals $(a,b,c) \in \mathbb{R}^3$,

$$|\max(a,c) - \max(b,c)| \leq |a-b|. \tag{5.24}$$

So, for all $u \in U$ and all $\mathbf{y} \in S_{[0,h]}(\xi)$, we have

$$
\begin{aligned}
&\left| w_1(t-h, \xi + hf(\xi,u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\xi + \theta f(\xi,u)) - w_1(t-h, \mathbf{y}(h), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}(\theta)) \right| \\
&\leq \left| w_1(t-h, \xi + hf(\xi,u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\xi + \theta f(\xi,u)) - w_1(t-h, \xi + hf(\xi,u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}(\theta)) \right| \\
&+ \left| w_1(t-h, \xi + hf(\xi,u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}(\theta)) - w_1(t-h, \mathbf{y}(h), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}(\theta)) \right| \\
&\leq \left| \max_{\theta\in[0,h]} \Psi_1(\xi + \theta f(\xi,u)) - \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}(\theta)) \right| + |w_1(t-h, \xi + hf(\xi,u), z) - w_1(t-h, \mathbf{y}(h), z)| \\
&< h\varepsilon.
\end{aligned}
$$

and the the proof of the proposition 5.3.12 is achieved. ∎

Proof. of Theorem 5.3.11. The proof is given in two steps :

Step 1. For all $n \geq 1$, the function $(\mathbf{y}^n(.), \mathbf{u}^n(.))$ satisfies

$$\dot{\mathbf{y}}^n(t) := f(\mathbf{y}^n(t), \mathbf{u}^n(t)) \text{ a.e } \ t \in [0,T].$$

Recall that the function f verifies the linear growth property (that follows from the Lipschitz property). Then, using the Gronwall's lemma, one can show that for all $t \in [0,T]$,

$$
\begin{aligned}
|\mathbf{y}^n(t)| &= \left| y + \int_0^t f(\mathbf{y}^n(s), \mathbf{u}^n(s)) ds \right| \\
&\leq |y| + L \int_0^t \big(1 + |\mathbf{y}^n(s)|\big) ds \leq (|y| + Lt)e^{Lt} \leq c_T, \qquad (5.25)
\end{aligned}
$$

where L is the Lipschitz constant of f and c_T depends only on T and $|y|$. For all $(t,s) \in [0,T]^2$, due to the linear growth property of f we obtain the inequality

$$
\begin{aligned}
|\mathbf{y}^n(t) - \mathbf{y}^n(s)| &= \left| \int_s^t f(\mathbf{y}^n(r), u) dr \right| \\
&\leq L \int_s^t (1 + c_T)) dr \leq L(1 + c_T)|t - s|, \qquad (5.26)
\end{aligned}
$$

So, the sequence $(\mathbf{y}^n(.), \mathbf{u}^n(.))$ is bounded and equicontinuous. By Arzela-Ascoli theorem, there exists a uniformly convergent subsequence $\{\mathbf{y}^{n_i}(.)\}$ that converges to a continuous solution $\bar{\mathbf{y}}(.)$ as $n_i \to \infty$. The Dunford Pettis criterion allows to extract a subsequence of $\dot{\mathbf{y}}^i$ converging weakly to a limit $l(.)$, i.e

$$\dot{\mathbf{y}}^i \rightharpoonup l, \qquad \text{in } L^1(\mathbb{R}^d, [0,T]).$$

Using $\mathbf{y}^i(t) = \mathbf{y}^i(s) + \int_s^t \dot{\mathbf{y}}^i(r) dr$ and the dominated convergence theorem, we obtain

$$\bar{\mathbf{y}}(t) = y + \int_0^t l(s) ds.$$

Hence, $\bar{\mathbf{y}}$ is an absolutely continuous function on $[0,T]$ and $\dot{\bar{\mathbf{y}}}(.) = l(.)$ a.e.

Set $L(t,y,p) = \max_{q\in f(y,U)} < p,q >$. Let $\{A_i\}$ be a sequence of measurable subsets of $[0,T]$ such that measure $(A_i) \to T$ and 1_i be the indicator function of A_i. Let V be any measurable set in $[0,T]$, then we have,

$$\begin{aligned}
&0 \leq \limsup_{i\to\infty} \int_{V\cap A_i} \left[L(t,\mathbf{y}^i(t),p) - < p, \dot{\mathbf{y}}^i(t) > \right] dt \\
&0 \leq \int_V \limsup_{i\to\infty} 1_i L(t,\mathbf{y}^i(t),p)dt + \limsup_{i\to\infty} \int_{V\cap A_i} < -p, \dot{\mathbf{y}}^i(t) > dt \\
&0 \leq \int_V L(t,\bar{\mathbf{y}}(t),p)dt + \limsup_{i\to\infty} \int_V < -p, \dot{\mathbf{y}}^i(t) > dt \\
&\quad + \limsup_{i\to\infty} \int_{V\cap A_i^c} < p, \dot{\mathbf{y}}^i(t) > dt \\
&0 \leq \int_V \left[L(t,\bar{\mathbf{y}}(t),p) - < p, \dot{\bar{\mathbf{y}}}(t) > \right] dt.
\end{aligned}$$

It follows from the arbitrariness of V that $L(t,\bar{\mathbf{y}}(t),p) \geq < p, \dot{\bar{\mathbf{y}}}(t) >$. Since L is continuous in p, it follows (see proposition 2.1.4 of [42]) that

$$\dot{\bar{\mathbf{y}}}(t) \in f(\bar{\mathbf{y}}(t),U).$$

By Fillipov's selection theorem, we get a control function $\bar{\mathbf{u}}(.)$ such that $(\bar{\mathbf{y}}(.),\bar{\mathbf{u}}(.))$ is an admissible solution for the optimal control problem.

Step 2. The optimality of $(\bar{\mathbf{y}}(.),\bar{\mathbf{u}}(.))$ will follow if,

$$\max_{\theta\in[0,T]} \Psi_1(\mathbf{y}^n(\theta),z) \to w_1(T,y,z), \quad \text{as} \quad n\to\infty.$$

Let L_w the Lipschitz constant of w_1.

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \max_{\theta\in[0,T]} \Psi_1(\mathbf{y}^n(\theta),z) - w_1(T,y,z)) \right| \\
&\leq \limsup_{n\to\infty} \sum_{k=0}^{n-1} \left| w_1(t_{n-k-1},\mathbf{y}^n(t_{k+1}),z) \bigvee \max_{\theta\in[t_0,t_{k+1}]} \Psi_1(\mathbf{y}^n(\theta),z) - w_1(t_{n-k},\mathbf{y}^n(t_k),z) \bigvee \max_{\theta\in[t_0,t_k]} \Psi_1(\mathbf{y}^n(\theta),z) \right|.
\end{aligned}$$

By using (5.24), one can simplify the last expression so that we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \max_{\theta\in[0,T]} \Psi_1(\mathbf{y}^n(\theta),z) - w_1(T,y,z)) \right| \\
&\leq \limsup_{n\to\infty} \sum_{k=0}^{n} \left| w_1(t_{n-k-1},\mathbf{y}^n(t_{k+1}),z) \bigvee \max_{\theta\in[t_k,t_{k+1}]} \Psi_1(\mathbf{y}^n(\theta),z) - w_1(t_{n-k},\mathbf{y}^n(t_k),z) \right| \\
&\leq \limsup_{n\to\infty} \sum_{k=0}^{n} \left| w_1(t_{n-k-1},\mathbf{y}^n(t_k) + hf(\mathbf{y}^n(t_k),u_k^n),z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}^n(t_k) + \theta f(\mathbf{y}^n(t_k),u_k^n),z) - w_1(t_{n-k},\mathbf{y}^n(t_k),z) \right| \\
&+ \limsup_{n\to\infty} \sum_{k=0}^{n} \left| w_1(t_{n-k-1},\mathbf{y}^n(t_{k+1}),z) \bigvee \max_{\theta\in[t_k,t_{k+1}]} \Psi_1(\mathbf{y}^n(\theta),z) \right. \qquad (5.27) \\
&\left. - w_1(t_{n-k-1},\mathbf{y}^n(t_k) + hf(\mathbf{y}^n(t_k),u_k^n),z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\mathbf{y}^n(t_k) + \theta f(\mathbf{y}^n(t_k),u_k^n),z) \right|.
\end{aligned}$$

.
Since u_{k-1} is chosen to achieve the minimum, the first term in the right hand side of the last inequality is zero (due to the proposition 5.3.12). For the second term we use the same arguments that in the proof of the proposition 5.3.12 to show that for all $k = 0, \dots, n-1$

$$\begin{aligned}&\Big| w_1(t_{n-k-1}, \mathbf{y}^n(t_{k+1}), z) - w_1(t_{n-k-1}, \mathbf{y}^n(t_k) + hf(\mathbf{y}^n(t_k), u_k^n), z) \Big| \\ &\leq L_{w_1} h \mathrm{dist}_H \Big(h^{-1}\big(R(h, \mathbf{y}^n(t_k)) - \mathbf{y}^n(t_k)\big), f(\mathbf{y}^n(t_k), U) \Big) < h\varepsilon,\end{aligned}$$

and

$$\begin{aligned}&\Big| \max_{\theta \in [t_k, t_{k+1}]} \Psi_1(\mathbf{y}^n(\theta), z) - \max_{\theta \in [0,h]} \Psi_1(\mathbf{y}^n(t_k) + \theta f(\mathbf{y}^n(t_k), u_k^n), z) \Big| \\ &\leq L_{\Psi_1} h \mathrm{dist}_H \Big(h^{-1}\big(R(h, \mathbf{y}^n(t_k)) - \mathbf{y}^n(t_k)\big), f(\mathbf{y}^n(t_k), U) \Big) < h\varepsilon.\end{aligned}$$

Then the second member of right hand side of the inequality (5.27) is also zero and this achieves the proof. ∎

Remark 5.3.13 *Let us point out that the strong convergence of the control requires restrictive assumptions and it leads, in general, to second order optimality conditions. These assumptions are not considered for our problems. This is why we make the choice to deal only with the convergence of the trajectories and we don't study the convergence of the control. However, in the framework of relaxed control, the weak convergence of the control has been studied in [51, Theorem 5.1] for infinite horizon optimal control problem and this result is easily extended to our case.*

Reconstruction using an approximate value function

Let W_1 be a numerical approximate solution of the problem (5.12) such that the error estimates between the exact solution and the approximate one is,

$$|W_1(t, y, z) - w_1(t, y, z)| \leq E_1(\Delta t, \Delta y), \tag{5.28}$$

where E_1 depends on Δt and Δy such that if $W_1(.,.) \to w_1(.,.)$ as $\Delta t, \Delta y \to 0$. For instance, one can consider the approximate solution of the equation (5.14) constructed by the Finite Difference scheme or the Semi Lagrangian scheme.

The following result is an extension of the convergence of the trajectories for piecewise constant control with the algorithm A using an approximate solution of the control problem.

Theorem 5.3.14 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_4}$) *hold. Let w_1 be the unique Lipschitz continuous viscosity solution of the equation* (5.14)*. Let W_h be an approximation of w_1 satisfying* (5.28)*. Let $\{\mathbf{y}^n(.), \mathbf{u}_1^n(.)\}$ and $\{\mathbf{Y}^n(.), \mathbf{u}_2^n(.)\}$ be the sequences generated by the algorithm A for w_1 and W_1 respectively. Then, for all $\varepsilon > 0$ we have the following estimates :*

$$\Big| \max_{\theta \in [0,T]} \Psi_1(\mathbf{Y}^n(\theta), z) - \max_{\theta \in [0,T]} \Psi_1(\mathbf{y}^n(\theta), z) \Big| \leq 3T\varepsilon + (2n+3)E_1(\Delta t, \Delta y).$$

The proof of the theorem is based on the following property :

Proposition 5.3.15 *Let $\Omega \subset \mathbb{R}^d$ be a compact set. Assume* ($\mathbf{H_1}$)*,* ($\mathbf{H_2}$) *and* ($\mathbf{H_4}$)*. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in [0, \delta]$, for all $\xi \in \Omega$ and $t \in [0, T]$*

$$\left|\frac{1}{h}\left(\min_{u\in U}\left\{W_1(t-h, \xi + hf(\xi, u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\xi + \theta f(\xi, u))\right\} - W_1(t, \xi, z)\right)\right| < \varepsilon + 2\frac{E_1(\Delta t, \Delta y)}{h}.$$

Proof.
Let us denote by

$$A_u(W_1) = \left\{W_1(t-h, \xi + hf(\xi, u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\xi + \theta f(\xi, u))\right\},$$
$$A_u(w_1) = \left\{w_1(t-h, \xi + hf(\xi, u), z) \bigvee \max_{\theta\in[0,h]} \Psi_1(\xi + \theta f(\xi, u))\right\}.$$

We have

$$\begin{aligned}
&\left|\min_{u\in U} A_u(W_1) - W_1(t, y, z)\right| \\
&\leq \left|\min_{u\in U} A_u(W_1) - \min_{u\in U} A_u(w_1)\right| + \left|\min_{u\in U} A_u(w_1) - w_1(t, \xi, z)\right| + |w_1(t, \xi, z) - W_1(t, \xi, z)| \\
&\leq \left|\min_{u\in U} A_u(W_1) - \min_{u\in U} A_u(w_1)\right| + h\varepsilon + E_1(h, \Delta y) \\
&\leq \left|\sup_{u\in U}\left\{A_u(W_1) - A_u(w_1)\right\}\right| + E_1(h, \Delta y) + h\varepsilon.
\end{aligned}$$

By using (5.24), we have

$$\begin{aligned}
\left|\sup_{u\in U}\left\{A_u(W_1) - A_u(w_1)\right\}\right| &\leq \left|\sup_{u\in U}\left\{W_1(t-h, \xi + hf(\xi, u), z) - w_1(t-h, \xi + hf(\xi, u), z)\right\}\right| \\
&\leq E_1(h, \Delta y).
\end{aligned}$$

■

Proof. of theorem 5.3.14. By using the theorem 5.3.11 and (5.28), we obtain that,

$$\begin{aligned}
&\left|\max_{\theta\in[0,T]} \Psi_1(\mathbf{Y}^n(\theta), z) - \max_{\theta\in[0,T]} \Psi_1(\mathbf{y}^n(\theta), z))\right| \\
&\leq \left|\max_{\theta\in[0,T]} \Psi_1(\mathbf{Y}^n(\theta), z) - W_1(T, y, z)\right| + \left|w_1(T, y, z) - W_1(T, y, z)\right| + \left|w_1(T, y, z) - \max_{\theta\in[0,T]} \Psi_1(\mathbf{y}^n(\theta), z))\right| \\
&\leq \left|\max_{\theta\in[0,T]} \Psi_1(\mathbf{Y}^n(\theta), z) - W_1(T, y, z)\right| + E_1(\Delta t, \Delta y) + T\varepsilon.
\end{aligned}$$

By using again (5.24), one can simplify the last expression so that we have

$$\begin{aligned}
&\left|\max_{\theta\in[0,T]}\Psi_1(\mathbf{Y}^n(\theta),z)-W_1(T,y,z)\right|\\
&\leq\sum_{k=0}^{n-1}\left|W_1(t_{n-k-1},\mathbf{Y}^n(t_{k+1}),z)\bigvee\max_{\theta\in[t_0,t_{k+1}]}\Psi_1(\mathbf{Y}^n(\theta),z)-W_1(t_{n-k},\mathbf{Y}^n(t_k),z)\bigvee\max_{\theta\in[t_0,t_k]}\Psi_1(\mathbf{Y}^n(\theta),z)\right|\\
&+\left|W_1(0,\mathbf{Y}^n(T),z)\bigvee w_1(0,\mathbf{Y}^n(T),z)-W_1(0,\mathbf{Y}^n(T),z)\right|+\left|W_1(T,y,z)\bigvee w_1(T,y,z)-W_1(T,y,z)\right|\\
&\leq\sum_{k=0}^{n-1}\left|W_1(t_{n-k-1},\mathbf{Y}^n(t_{k+1}),z)\bigvee\max_{\theta\in[t_0,t_{k+1}]}\Psi_1(\mathbf{Y}^n(\theta),z)-W_1(t_{n-k},\mathbf{Y}^n(t_k),z)\right|+2E_1(\Delta t,\Delta y).
\end{aligned}$$

It follows from above that

$$\begin{aligned}
&\left|\max_{\theta\in[0,T]}\Psi_1(\mathbf{Y}^n(\theta),z)-\Psi_1(\mathbf{y}^n(\theta),z)\right|\\
&\leq\sum_{k=0}^{n-1}\left|W_1(t_{n-k-1},\mathbf{Y}^n(t_{k+1}),z)\bigvee\max_{\theta\in[t_0,t_{k+1}]}\Psi_1(\mathbf{Y}^n(\theta),z)-W_1(t_{n-k},\mathbf{Y}^n(t_k),z)\right|+3E_1(\Delta t,\Delta y)+T\varepsilon\\
&\leq\sum_{k=0}^{n-1}\left|W_1(t_{n-k-1},\mathbf{Y}^n(t_k)+hf(\mathbf{Y}^n(t_k),u_k^n),z)\bigvee\max_{\theta\in[0,h]}\Psi_1(\mathbf{Y}^n(t_k)+\theta f(\mathbf{Y}^n(t_k),u_k^n),z)-W_1(t_{n-k},\mathbf{Y}^n(t_k),z)\right|\\
&+\sum_{k=0}^{n-1}\left|W_1(t_{n-k-1},\mathbf{Y}^n(t_{k+1}),z)\bigvee\max_{\theta\in[t_k,t_{k+1}]}\Psi_1(\mathbf{Y}^n(\theta),z)\right.\\
&\quad\left.-W_1(t_{n-k-1},\mathbf{Y}^n(t_k)+hf(\mathbf{Y}^n(t_k),u_k^n),z)\bigvee\max_{\theta\in[0,h]}\Psi_1(\mathbf{Y}^n(t_k)+\theta f(\mathbf{Y}^n(t_k),u_k^n),z)\right|+3E_1(\Delta t,\Delta y)+T\varepsilon.
\end{aligned}$$

Since u_k is chosen to achieve the minimum, if follows from the proposition that,

$$\begin{aligned}
&\left|\max_{\theta\in[0,T]}\Psi_1(\mathbf{Y}^n(\theta),z)-\Psi_1(\mathbf{y}^n(\theta),z)\right|\\
&\leq 2T\varepsilon+(2n+3)E_1(\Delta t,\Delta y)+\sum_{k=0}^{n-1}\left|W_1(t_{n-k-1},\mathbf{Y}^n(t_{k+1}),z)\bigvee\max_{\theta\in[t_k,t_{k+1}]}\Psi_1(\mathbf{Y}^n(\theta),z)\right. \quad (5.29)\\
&\quad\left.-W_1(t_{n-k-1},\mathbf{Y}^n(t_k)+hf(\mathbf{Y}^n(t_k),u_k^n),z)\bigvee\max_{\theta\in[0,h]}\Psi_1(\mathbf{Y}^n(t_k)+\theta f(\mathbf{Y}^n(t_k),u_k^n),z)\right|.
\end{aligned}$$

Now, let L_{W_1} be the Lipschitz constant of W_1. By using again the arguments that in the proof of the proposition 5.3.12, one can show that for all $k=0,\ldots,n-1$

$$\begin{aligned}
&\left|W_1(t_{n-k-1},\mathbf{Y}^n(t_{k+1}),z)-W_1(t_{n-k-1},\mathbf{Y}^n(t_k)+hf(\mathbf{Y}^n(t_k),u_k^n),z)\right|\\
&\leq L_{W_1}h\,\mathrm{dist}_H\Big(h^{-1}\big(R(h,\mathbf{Y}^n(t_k))-\mathbf{Y}^n(t_k)\big),f(\mathbf{Y}^n(t_k),U)\Big)<h\frac{\varepsilon}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\max_{\theta\in[t_k,t_{k+1}]}\Psi_1(\mathbf{Y}^n(\theta),z)-\max_{\theta\in[0,h]}\Psi_1(\mathbf{Y}^n(t_k)+\theta f(\mathbf{Y}^n(t_k),u_k^n),z)\right|\\
&\leq L_{W_1}h\,\mathrm{dist}_H\Big(h^{-1}\big(R(h,\mathbf{Y}^n(t_k))-\mathbf{Y}^n(t_k)\big),f(\mathbf{Y}^n(t_k),U)\Big)<h\frac{\varepsilon}{2}.
\end{aligned}$$

Then, from (5.29), we obtain the following result :

$$\left|\max_{\theta\in[0,T]}\Psi_1(\mathbf{Y}^n(\theta),z)-\Psi_1(\mathbf{y}^n(\theta),z)\right|\leq 3T\varepsilon+(2n+3)E_1(\Delta t,\Delta y).$$

■

Corollary 5.3.16 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_4}$) *hold. Let* w_1 *be the unique lipschitz continuous viscosity solution of the equation* (5.14)*. Let* W_h *be an approximation of* w_1 *given by the Finite Difference scheme or the Semi Lagrangian scheme. Let* $\{\mathbf{Y}^n(.), \mathbf{u}_2^n(.)\}$ *be the sequence generated by the algorithm A for* W_1*. Then, if we have the following estimates :*

$$\max_{\theta\in[0,T]} \Psi_1(\mathbf{Y}^n(\theta), z) \to \max_{\theta\in[0,T]} \Psi_1(\bar{\mathbf{y}}(\theta), z) = w_1(T, y, z),$$

where $\bar{\mathbf{y}}$ *is the optimal trajectory for the problem* (5.12)*.*

Proof. Let $\{\mathbf{y}^n(.), \mathbf{u}_1^n(.)\}$ and $\{\mathbf{Y}^n(.), \mathbf{u}_2^n(.)\}$ be the sequences generated by the algorithm A for w_1 and W_1 respectively. By theorem 5.3.14, we have, for all $\varepsilon > 0$,

$$\left| \max_{\theta\in[0,T]} \Psi_1(\mathbf{Y}^n(\theta), z) - \max_{\theta\in[0,T]} \Psi_1(\mathbf{y}^n(\theta), z) \right| \leq 3T\varepsilon + (2n+3)E_1(\Delta t, \Delta y).$$

Since the error estimate of the Finite Difference scheme and the Semi Lagrangian scheme are the same and it takes the following form :

$$\Big| W_1(t, y, z) - w_1(t, y, z) \Big| \leq k(\Delta t^{\frac{1}{2}} + \Delta y).$$

By taking $(\Delta t^{\frac{1}{2}} + \Delta y) \simeq \circ(h)$ and letting $h \to 0$, we obtain the result. ∎

Reconstruction using the exit time function

An alternative way to deal with the reconstruction of the optimal trajectories for (5.7), is to use the exit time function $\mathcal{T}_1$. Unfortunately, there is no convergence result for such algorithm. However, we have seen that the optimal trajectories constructed from $\mathcal{T}_1$ are also optimal for (5.12) and (5.7). Let us formulate the Dynamic Programming Principle for the exit time problem.

Proposition 5.3.17 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_4}$) *hold. Then, for all* $(y, z) \in \mathbb{R}^{d+1}$*,*

$$\mathcal{T}_1(y, z) := \sup_{\alpha\in\mathcal{U}} \left\{ \mathcal{T}_1(\mathbf{y}^\alpha(t), z) + t \right\} \bigwedge T, \quad \forall t \in \big[0, \mathcal{T}_1(y, z)\big].$$

Proof. Let $\theta := \sup_{\alpha\in\mathcal{U}} \left\{ \mathcal{T}_1(\mathbf{y}^\alpha(t), z) + t \right\}$. By definition of $\mathcal{T}_1$, there exists $\alpha \in \mathcal{U}$ such that,

$$\widetilde{\mathbf{y}}_{\widetilde{y}}^{\alpha}(\tau) \in \mathcal{D}_1, \ \forall \tau \in [0, \mathcal{T}_1(y, z)].$$

Define $\bar{\alpha}$ by $\bar{\alpha}(s) := \alpha(s+t)$ for all $s \in [0, \mathcal{T}_1(y, z) - t]$. Then, we have

$$\widetilde{\mathbf{y}}_{\widetilde{\mathbf{y}}(t)}^{\bar{\alpha}}(s) \in \mathcal{D}_1, \ \forall s \in [0, \mathcal{T}_1(y, z) - t].$$

It follows from the definition of $\mathcal{T}_1(\mathbf{y}^{\bar{\alpha}}(t), z)$ that

$$\mathcal{T}_1(\mathbf{y}^{\bar{\alpha}}(t), z) \geq \mathcal{T}_1(y, z) - t.$$

Taking the supremum over the set of controls $\mathcal{U}$, we obtain that $\mathcal{T}_1(y, z) \leq \theta$. Conversely, let us prove that $\mathcal{T}_1(y, z) \geq \theta$ and consider $a \in \mathcal{U}$ such that,

$$\widetilde{\mathbf{y}}^{a}_{\widetilde{y}}(s) \in \mathcal{D}_1, \ \forall s \in [0, t].$$

From the definition of $\mathcal{T}_1(\mathbf{y}^{a}(t), z)$, we have that there exists $\bar{a} \in \mathcal{U}$ such that,

$$\widetilde{\mathbf{y}}^{\bar{a}}_{\widetilde{\mathbf{y}}^{a}(t)}(s) \in \mathcal{D}_1, \ \forall s \in [0, \mathcal{T}_1(\mathbf{y}^{a}(t), z)].$$

Now, define the following control :

$$\alpha(s) = a(s) 1_{\left[0,t\right]} + \bar{a}(s-t) 1_{\left[t, \mathcal{T}_1(\mathbf{y}^{a}(t), z)\right]}.$$

Note that $\alpha \in \mathcal{U}$. Then, we have that,

$$\widetilde{\mathbf{y}}^{\alpha}_{y}(s) \in \mathcal{D}_1, \ \forall s \in [0, \mathcal{T}_1(\mathbf{y}^{a}(t), z) + t].$$

By definition of $\mathcal{T}_1(y, z)$, we have

$$\mathcal{T}_1(y, z) \geq \mathcal{T}_1(\mathbf{y}^{a}(t), z) + t.$$

We get the result by taking the supremum over the set of controls. From the fact that $\mathcal{T}_1(y, z) \leq T$, we conclude the Dynamic Programming Principle. ■

Remark 5.3.18 *The value function ϑ_1 is upper semi contiunous and the exit time function is lower semi continuous. Under pointing controllability assumptions of inward type, one can obtain that both of ϑ_1 and $\mathcal{T}_1$ are Lipschitz continuous and the convergence of the trajectories in the previous section can be extended to the algorithms using ϑ_1 and $\mathcal{T}_1$.*

The following formal algorithm uses the exit time function to reconstruct optimal trajectories for (5.7),

Algorithm B. For a given $n \in \mathbb{N}$, let $(t_0 = 0, t_1 ... t_{n-1}, t_n = T)$ be an uniform partition of $[0, T]$ with the time step $h = \frac{T}{n}$. Let $\{\mathbf{y}^n(\cdot), \mathbf{z}^n(\cdot)\}$ be a trajectory defined recursively at t_i ($i \geq 1$) with $\mathbf{z}^n(\cdot) := z$. Set $\mathbf{y}^n(t_0) = y$ and $\vartheta_1(T, y) := z$.

Step 1 Knowing the state $\mathbf{y}^n(t_k)$ choose the optimal control at t_k :

$$u^n_k \in \arg\max_{u \in U} \left\{ \left(\left[\mathcal{T}_1\right]\left(\mathbf{y}^n(t_k) + hf\left(\mathbf{y}^n(t_k), u\right), z\right) + h \right) \bigwedge \ T \right\},$$

where $\left[A\right]$ denotes the linear interpolation of A.

Step 2 The next point of the state corresponding to the minimizing value $\mathbf{u}(t_k) = u_k^n$ is :

$$\mathbf{y}^n(t_{k+1}) := \mathbf{y}^n(t_k) + hf\big(\mathbf{y}^n(t_k), \mathbf{u}(t_k)\big).$$

and $\mathbf{z}^n(\cdot) := z$.

Remark 5.3.19 *For the numerical point of view, one can stress that the choice of the algorithm using the exit time function allows to reduce the dimension of the matrix that will be saved for the reconstruction and thus making the economy of the CPU time and the available memory. Moreover, with the algorithm using the exit time function, one can save the matrix just at the last time. For high dimensions, it cannot be possible to use the algorithm of reconstruction by minimizing the value function because of CPU time and the memory required for this operation. The algorithm of reconstruction using the exit time function can be seen as the best alternative for reconstruction algorithms in high dimensions.*

5.4 State constrained control problem with Bolza cost

The main objective of this section is to prove similar results to that obtained in the last section, for the present state-constrained control problem with Bolza cost. Let us mention that this problem has been studied in [3] for Lipschitz continuous cost. Here, we deal with the case of locally Lipschitz continuous cost with polynomial growth. First of all, we want to characterize the epigraph of the value function in terms of a locally Lipschitz continuous value function function associated to some auxiliary control problem. More precisely, introduce the following augmented dynamics $\bar{f}$ for $u \in U$ and $\bar{y} := (y, z) \in \mathbb{R}^d \times \mathbb{R}$,

$$\bar{f}((y,z),u) = \begin{pmatrix} f(y,u) \\ -\Phi_2(y) \end{pmatrix}.$$

Let the associated augmented trajectory be denoted $\bar{\mathbf{y}}(s) := \bar{\mathbf{y}}_{\{y,z\}}(s) := (\mathbf{y}_y^{\mathbf{u}}(s), \zeta_{y,z}^{\mathbf{u}}(s))$ (where $\zeta_{z,y}^{\mathbf{u}}(s) := z - \int_0^s \Phi_2(\mathbf{y}_y^{\mathbf{u}}(\theta))d\theta$) solution of :

$$\dot{\bar{\mathbf{y}}}(\mathbf{s}) = \bar{f}\big(\bar{\mathbf{y}}(s), \mathbf{u}(s)\big), \quad s \in (0,T), \tag{5.30a}$$

$$\bar{\mathbf{y}}(0) = (y,z)^T. \tag{5.30b}$$

Define the associated set of feasible trajectories,

$$\bar{S}_{[0,T]}(\bar{y}) := \{\bar{\mathbf{y}} = (\mathbf{y}_y^{\mathbf{u}}, \zeta_{y,z}^{\mathbf{u}}), \quad \bar{\mathbf{y}} \text{ satisfies } (5.30) \text{ for some } \mathbf{u} \in \mathcal{U}\},$$

for $\bar{y} = (y,z) \in \mathbb{R}^d \times \mathbb{R}$.

Remark 5.4.1 *Since $f(y,U)$ is convex, the augmented dynamics $\bar{f}(\bar{y},U)$ is convex (since $\Phi_2(\cdot)$ is not depending on the control u) and therefore $\bar{S}_{[0,T]}(\bar{y})$ is compact for the topology of $C([0,T];\mathbb{R}^{d+1})$ (see [9]). Moreover, the function ϑ_2 is lsc in $[0,T]\times\mathbb{R}^d$.*

In the same manner, we will consider some auxiliary optimal control problems for which the continuous value functions (which will not be globally Lipschitz continuous) are characterized by different HJB equations.

5.4.1 Auxiliary optimal control with Bolza cost

Hamilton-Jacobi equation

Let us consider the new auxiliary control problem associated with a bolza problem and defined in the following way :

$$w_2(t,y,z) := \inf_{\bar{\mathbf{y}}=(\mathbf{y},z)\in \bar{S}_{[0,t]}(\bar{\mathbf{y}})} \left\{ \Big(-\zeta^u_{y,z}(t)\Big) \bigvee \Big(\max_{\theta\in[0,t]} g(\mathbf{y}^u_y(\theta))\Big) \right\}, \tag{5.31}$$

The new control problem is free of any additional assumption of the set $\mathcal{K}$ neither on the dynamics $\bar{f}$. Then, the following equivalences between w_2 and ϑ_2 holds.

Proposition 5.4.2 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *hold and let* $(t,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$. *The value function* w_2 *is related to* ϑ_2 *by the following relations :*

$$\begin{aligned} &(i) \quad \vartheta_2(t,y) - z \le 0 \Leftrightarrow w_2(t,y,z)\le 0,\\ &(ii) \quad \vartheta_2(t,y) = \min\Big\{ z\in\mathbb{R}\ ,\ w_2(t,y,z)\le 0\Big\}. \end{aligned}$$

Proof. The proof is similar to that given in the proposition 5.3.2. ■

In this case, the value function of the auxiliary optimal control problem w_2 is not globally Lipschitz continuous, but it satisfies a polynomial growth property and a Dynamic Programming Principle. Define the Hamiltonian :

$$\mathcal{H}(y,p_1,p_2) := \sup_{u\in U}\big(-f(y,u).p_1 + \Phi_2(y).p_2\big) \text{ for } y,p_1\in\mathbb{R}^d, p_2\in\mathbb{R}$$

Proposition 5.4.3 *Assume* ($\mathbf{H_1}$) *and* ($\mathbf{H_3}$) *hold.*

(i) The function w_2 *is locally Lipschitz continuous. The function* w_2 *has a polynomial growth property, i.e, there exists* $C>0$, *such that,*

$$|w_2(t,y,z)| \le C(1+|y|^q+|z|).$$

(ii) For any $t\in[0,T]$, $h\ge 0$, *such that* $t+h\le T$,

$$w_2(t+h,y,z) = \inf_{\bar{\mathbf{y}}:=(\mathbf{y},\zeta)\in\bar{S}_{[0,t]}(y,z)} \Big\{ w_2(t,\mathbf{y}(h),\zeta(h)) \bigvee \max_{\theta\in[0,h]} g(\mathbf{y}(\theta))\Big\}.$$

(iii) The function w_2 *is the unique continuous viscosity solution of the following HJ equation :*

$$\min\Big(\partial_t w_2(t,y,z) + \mathcal{H}(y,\nabla_y w_2,\partial_z w_2), w_2(t,y,z)-g(y)\Big) = 0, \qquad \textit{in } \ ]0,T]\times\mathbb{R}^d\times\mathbb{R} \tag{5.32a}$$

$$w_2(0,y,z) = g(y)\vee(-z), \qquad \textit{in } \mathbb{R}^d\times\mathbb{R}. \tag{5.32b}$$

Proof. The proof of the polynomial growth and the Lipschitz property is given in Appendix B. The proof of (ii) is classical and its proof can be found also in [19].

The proof of w_2 is a solution of (5.32) is similar to the proof of w_1 is solution of (5.14). For the uniqueness of the solution, let us first consider the following lemma :

Lemma 5.4.4 *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$*. Then* $\mathcal{H}$ *is continuous and it satisfies,*

(i) *Let* λ *be a modulus of continuity of* Φ_2*, i.e,*

$$|\Phi_2(y_1) - \Phi_2(y_2)| \leq \lambda(|y_1 - y_2|, R), \quad \forall R > 0 \quad \text{and} \quad y_1, y_2 \in B(\circ, R),$$

such that $\lambda(., R)$ *is continuous, nondecreasing and* $\lambda(0, R) = 0$*. Then, for all* $R > 0$*,* $p \in \mathbb{R}^d$*,* $m \in \mathbb{R}$*,* $y_1, y_2 \in B_R$*,*

$$|\mathcal{H}(y_1, p, m) - \mathcal{H}(y_2, p, m)| \leq L|p|.|y_1 - y_2| + |m|.\lambda(|y_1 - y_2|, R).$$

(ii) *For all* $p_1, p_2, y \in \mathbb{R}^d$*,* $m_1, m_2 \in \mathbb{R}$*,*

$$|\mathcal{H}(y, p_1, m_1) - \mathcal{H}(y, p_2, m_2)| \leq L(|y|^q + 1)(|p_1 - p_2| + |m_1 - m_2|).$$

Proof. By using the Lipschitz continuity of f and the continuity of Φ_2, we obtain for all $R > 0$, $p \in \mathbb{R}^d$, $m \in \mathbb{R}$, $y_1, y_2 \in B_R$ and $a \in \mathcal{U}$,

$$\begin{aligned} |\mathcal{H}(y_1, p, m) - \mathcal{H}(y_2, p, m)| &\leq \sup_{u \in \mathcal{U}} |(f(y_2, u) - f(y_1, u)).p| + |m|.|\Phi_2(y_1) - \Phi_2(y_2)| \\ &\leq |f(y_2, a) - f(y_1, a)|.|p| + |m|.|\Phi_2(y_1) - \Phi_2(y_2)| + \varepsilon \\ &\leq L|p|.|y_2 - y_1| + |m|.\lambda(|y_1 - y_2|, R) + \varepsilon; \end{aligned}$$

where $\lambda(s, R) \to 0$ as s goes to 0 for all $R > 0$. Then the first inequality of lemma 5.4.4 follows from the arbitrariness of ε. Let us prove the second inequality of lemma 5.4.4. For all $y, p_1, p_2 \in \mathbb{R}^d$, $m_1, m_2 \in \mathbb{R}$ and $a \in \mathcal{U}$,

$$\begin{aligned} |\mathcal{H}(y, p_1, m_1) - \mathcal{H}(y, p_2, m_2)| &\leq |(f(y, a).(p_2 - p_1)| + |\Phi_2(y).(m_1 - m_2)| + \varepsilon \\ &\leq |f(y, a)|.|p_1 - p_2| + |\Phi_2(y)|.|m_1 - m_2| + \varepsilon \\ &\leq L(1 + |y|^q).(|p_1 - p_2| + |m_1 - m_2|) + \varepsilon. \end{aligned}$$

Since ε is arbitrary the proof of lemma 5.4.4 is complete. ∎

The proof of uniqueness of solutions of the equation 5.32 follows from the comparison principle (see Appendix D) using lemma 5.4.4. ∎

Hamilton-Jacobi equation with a wise choice of the obstacle function

By using the same idea as for $\widetilde{w_1}$, we want to show that if the function g satisfies some conditions, then the solution of (5.32) is constant out of some closed set.

For this, let us assume that the function g is defined by (5.15). Let $\widetilde{w_2}$ be defined by :

$$\widetilde{w_2}(t, y, z) := \inf_{\bar{\mathbf{y}} = (\mathbf{y}, z) \in \bar{S}_{[0,t]}(\bar{\mathbf{y}})} \left\{ \left(\left(-\zeta^u_{y,z}(t) \right) \vee \max_{\theta \in [0,t]} g(\mathbf{y}^u_y(\theta)) \right) \wedge \widetilde{c} \right\}$$

Then, the following theorem allows to characterize the value function $\widetilde{w_2}$ as the unique viscosity solution of (5.32). Moreover, the solution is constant outside some unbouded closed set.

Theorem 5.4.5 *Assume* ($\mathbf{H_1}$) *and* ($\mathbf{H_3}$) *hold. The function* $\widetilde{w_2}$ *is the unique continuous viscosity solution of the following HJ equation :*

$$\min\Big(\partial_t\widetilde{w_2}(t,y,z)+\mathcal{H}(y,\nabla_y\widetilde{w_2},\partial_z\widetilde{w_2}),\widetilde{w_2}(t,y,z)-g(y)\wedge\widetilde{c}\Big)=0,$$

$$\textit{in}\ \ ]0,T]\times\overset{\circ}{\widetilde{\mathcal{K}}}\times\mathbb{R} \tag{5.33a}$$

$$\widetilde{w_2}(0,y,z)=\Big(g(y)\vee(-z)\Big)\wedge\widetilde{c},\qquad \textit{in}\ \ \overset{\circ}{\widetilde{\mathcal{K}}}\times\mathbb{R}. \tag{5.33b}$$

$$\widetilde{w_2}(t,y,z)=\widetilde{c},\quad\forall\ t\in[0,T],\ y\notin\overset{\circ}{\widetilde{\mathcal{K}}}\quad \textit{and}\ z\in\mathbb{R}$$

Proof. The first part (i) follows from (iii) of the proposition 5.4.3. Let us prove the assertion (ii). For $y\notin\overset{\circ}{\widetilde{\mathcal{K}}}$ and $z\in\mathbb{R}$, we have

$$g(y)\wedge\widetilde{c}=\widetilde{c}.$$

It follows that :

$$\Psi_2(y,z)=\widetilde{c}.$$

From (5.32), we have $w_1(t,y,z)\geq\Psi_2(y,z)$, which implies that,

$$\widetilde{w_2}(t,y,z)\geq\widetilde{c},\quad\forall\ t\in[0,T],\ y\notin\overset{\circ}{\widetilde{\mathcal{K}}}\ \text{ and }z\in\mathbb{R}.$$

By definition of $\widetilde{w_2}$, we have $\widetilde{w_2}(t,y,z)\leq\widetilde{c}$. This ends the proof of (ii) and then the proof of the theorem 5.4.5. ■

Remark 5.4.6 *Assume that the function* Φ_2 *satisfies the following :*

$$\Phi_2(y)\in[\underline{m},\overline{M}],\quad\forall y\in\mathcal{K},$$

and $z\in]T\underline{m}-\widetilde{c},T\overline{M}+\widetilde{c}[$*. Then, we still have the characterization of* ϑ_2 *by* w_2*, i.e,*

$$\vartheta_2(t,y)=\inf\Big\{z\in]T\underline{m}-\widetilde{c},T\overline{M}+\widetilde{c}[\ \mid\ \widetilde{w_2}(t,y,z)\leq 0\Big\}$$

In addition, the function w_2 *is the unique continuous viscosity solution of the following HJ equation :*

$$\min\Big(\partial_t\widetilde{w_2}(t,y,z)+H(y,\nabla_y\widetilde{w_2},\partial_z\widetilde{w_2}),\widetilde{w_2}(t,y,z)-g(y)\wedge\widetilde{c}\Big)=0,$$

$$\textit{in}\ \ ]0,T]\times\overset{\circ}{\widetilde{\mathcal{K}}}\times]T\underline{m}-\widetilde{c},T\overline{M}+\widetilde{c}[,$$

$$\widetilde{w_2}(0,y,z)=\Big((-z)\vee g(y)\Big)\wedge\widetilde{c},\qquad \textit{in}\ \ \overset{\circ}{\widetilde{\mathcal{K}}}\times]T\underline{m}-\widetilde{c},T\overline{M}+\widetilde{c}[.$$

$$\widetilde{w}_2(t,y,z)=\widetilde{c},\quad\forall\ t\in[0,T],\ y\notin\overset{\circ}{\widetilde{\mathcal{K}}}\quad\textit{and}\quad z\in[T\underline{m}-\widetilde{c},T\overline{M}+\widetilde{c}],$$

For the boundary condition, it is easy to check that

$$\widetilde{w_2}(t, y, T\underline{m} - \widetilde{c}) = \widetilde{c}, \quad \forall t \in [0, T], \quad y \in \overset{\circ}{\widetilde{\mathcal{K}}} .$$

However, for the boundary condition on $z = T\overline{M} + \widetilde{c}$, one can make a slight change on the definition of $\widetilde{w_2}$ so as to obtain $\widetilde{w_2}(t, y, z) := \widetilde{c}$, $\forall t \in [0, T]$ and $y \in \overset{\circ}{\widetilde{\mathcal{K}}}$. (see the Chapter 6 for more details).

In the following, we will establish a link between the value functions w_2 and ϑ_2 and a *Backward reachable set*. The latter is associated to some maximum time function.

5.4.2 Link with the maximum time function and reachability problems

Define the maximum time function $\mathcal{T}_2 : \mathbb{R}^{d+1} \to [0, T]$, which associates to each point $\bar{y} := (y, z) \in \mathbb{R}^{d+1}$, the maximum time to reach the target set $\mathcal{C} := \mathcal{K} \times \mathbb{R}_+$ with an admissible trajectory $\bar{\mathbf{y}}^u_{\bar{y}}(\cdot) := (\mathbf{y}(\cdot), \zeta(\cdot))$ solution of (5.9) associated with an admissible control $u \in \mathcal{U}$, i.e,

$$\mathcal{T}_2(y, z) := \sup \left\{ t \in [0, T] \;\middle|\; \exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \mathbf{y}^u_y(\theta) \in \mathcal{K}, \; \forall \theta \in [0, t], \;\; \text{and} \; \zeta^u_{y,z}(t) \geq 0 \right\}. \tag{5.35}$$

Now, define *the Capture basin* with time horizon t associated to $\mathcal{C}$, i.e, set of starting points for which it is possible to reach the target set $\mathcal{C}$ at time t,

$$Cap_{\mathcal{C}}(t) := \left\{ (y, z) \in \mathbb{R}^{d+1} \;\middle|\; \exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \mathbf{y}^u_y(\theta) \in \mathcal{K}, \; \forall \theta \in [0, t], \;\; \text{and} \; \zeta^u_{y,z}(t) \geq 0 \right\}.$$

In the following proposition, we give a link between the value functions w_2, v_2 and the set $Cap_{\mathcal{C}}(t)$.

Proposition 5.4.7 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *hold. Then, for $t \in [0, T]$ the* Capture basin *$Cap_{\mathcal{C}}(t)$ is related to the value functions w_2 and ϑ_2 by the following relations :*

$$Cap_{\mathcal{C}}(t) = \left\{ (y, z) \in \mathbb{R}^{d+1} \;\middle|\; \vartheta_2(t, y) \leq z \right\} = \left\{ (y, z) \in \mathbb{R}^{d+1} \;\middle|\; w_2(t, y, z) \leq 0 \right\}.$$

Proof. Let $\mathcal{V}_2(t) := \left\{ (y, z) \in \mathbb{R}^{d+1} \;\middle|\; w_2(t, y, z) \leq 0 \right\}$. The fact that $\mathcal{V}_2(t) = \left\{ \widehat{y} = (y, z) \in \mathbb{R}^{d+1} \;\middle|\; \vartheta_2(t, y) \leq z \right\}$ is a consequence of proposition 5.4.2. It remains to prove that $Cap_{\mathcal{C}}(t) = \mathcal{V}_2(t)$,

Let $\widehat{y} = (y, z) \in Cap_{\mathcal{C}}(t)$, it follows from the definition of $Cap_{\mathcal{C}}(t)$ that

$$\exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \zeta^u_{y,z}(t) \geq 0, \;\; \text{and} \; \mathbf{y}^u_y(\theta) \in \mathcal{K}, \; \forall \theta \in [0, t],$$

which is equivalent to,

$$\exists u \in \mathcal{U}, \;\; \text{s.t} \;\; \big(-\zeta^u_{y,z}(t) \big) \bigvee \max_{\theta \in [0,t]} g\big(\mathbf{y}^u_y(\theta) \big) \leq 0.$$

Taking the infimum over the set of trajectories, yields that $w_2(t,x,y) \leq 0$.

Conversely, assume that $(y,z) \in \mathcal{V}_2(t)$. Then by definition of $\mathcal{V}_t$ we have $w_2(t,y,z) \leq 0$. By Remark 5.4.1, there exists a trajectory $\bar{\mathbf{y}}^{\mathbf{u}}_{\bar{y}} \in \bar{S}_{[0,T]}(y,z)$ starting from $\bar{y} = (y,z)$ depending on the control $\mathbf{u} \in \mathcal{U}$ such that,

$$0 \geq w_2(t,y,z) = \big(-\zeta^{u}_{\bar{y}}(t)\big) \bigvee \max_{\theta \in [0,t]} g\big(\mathbf{y}^{u}_{y}(\theta)\big) \leq 0$$

It follows that $\bar{y} = (y,z) \in Capp_{\mathcal{C}}(t)$ and the proof is completed. ■

The following theorem presents the obtained relations between the maximum time function $\mathcal{T}_2$, the Capture basin $Capp_{\mathcal{C}}(t)$ and the value functions w_2 and ϑ_2.

Theorem 5.4.8 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *hold. Then the maximum time function $\mathcal{T}_2$ satisfies the following relations : :*

$$\begin{array}{ll} (i) & \mathcal{T}_2(y,z) = \sup\big\{t \in [0,T] \;\big|\; (y,z) \in Capp_{\mathcal{C}}(t) \;\big\} = \sup\big\{t \in [0,T] \;\big|\; w_2(t,y,z) \leq 0 \;\big\}, \\ (ii) & \mathcal{T}_2(y,z) = t \Rightarrow \; w_2(t,y,z) = 0, \\ (iii) & \vartheta_2(t,y) = \inf\big\{z \;\big|\; \mathcal{T}_2(y,z) \geq t\big\}. \end{array}$$

Proof. Let $\widetilde{\mathcal{T}_2}(y,z) := \sup\big\{t \in [0,T] \;\big|\; (y,z) \in Capp_{\mathcal{C}}(t) \;\big\}$. The fact that $\widetilde{\mathcal{T}_2}(y,z) := \sup\big\{t \in [0,T] \;\big|\; w_2(t,y,z) \leq 0 \;\big\}$ is a consequence of proposition (5.4.7). It remains to prove that $\mathcal{T}_2(y,z) = \widetilde{\mathcal{T}_2}(y,z)$. Let $t := \mathcal{T}_2(y,z)$ and assume that $t < \infty$. There exists an admissible trajectory $\bar{\mathbf{y}}^{\mathbf{u}}_{\bar{y}}(.) = (\mathbf{y}^{\mathbf{u}}_{y}(.), \zeta^{\mathbf{u}}_{y,z}(.))$ such that

$$\mathbf{y}^{\mathbf{u}}_{y}(\theta) \in \mathcal{K}, \; \forall \theta \in [0,t], \text{ and } \zeta^{\mathbf{u}}_{y,z}(t) \geq 0$$

This implies that $\bar{y} = (y,z) \in Capp_{\mathcal{C}}(t)$. Hence $\widetilde{\mathcal{T}_2}(y,z) \geq t$ by definition of $\widetilde{\mathcal{T}_2}(y,z)$.

Now, let $\tilde{t} := \widetilde{\mathcal{T}_2}(y,z)$. For any $n \geq 1$, there exists a sequence $(t_n)_{n \geq 1}$ such that $t_n \leq \tilde{t}$ such that $t_n \to \tilde{t}$ as $n \to \infty$ and $w_2(t_n,y,z) \leq 0$, $\forall n \geq 1$. There exists an admissible trajectory $\bar{\mathbf{y}}_n := \bar{\mathbf{y}}^{\mathbf{u_n}}_{\bar{y}}$ such that $\mathbf{y}_n(\theta) \in \mathcal{K}$, $\forall \theta \in [0,t_n]$ and $\zeta_n(t_n) \geq 0$. Since $\mathcal{C}$ and $\mathcal{K} \in \mathbb{R}$ are closed and using the compactness arguments of trajectories, it is possible to extract a subsequence $\bar{\mathbf{y}}_n$ converging uniformly on $[0,\tilde{t}]$ to $\bar{\mathbf{y}}$ (with $t_n \to \tilde{t}$) such that $\zeta(\tilde{t}) \geq 0$ and, $\mathbf{y}(\tilde{t}) \in \mathcal{K}$ $\forall \theta \in [0,\tilde{t}]$. Thus, by definition of $\mathcal{T}_2(y,z)$, we obtain $\mathcal{T}_2(y,z) \geq \tilde{t}$, which concludes the proof of (i).

Now, let us prove (ii). Let $t := \mathcal{T}_2(y,z)$. As a consequence of (i), we have that for all $n \geq 1$, $\exists t_n \in [t - \frac{1}{n}, t[$ s.t $w_2(t_n,y,z) \leq 0$. Since w_2 is continuous, one obtain $w_2(t,y,z) \leq 0$. On the other hand, $\forall t_1 \geq t$, $w_2(t_1,y,z) > 0$, then $w_1(t,y,z) \geq 0$ and the proof (ii) is completed.

For the proof of (iii), one can deduce directly from (i) and proposition 5.3.7 that, for all (t,y,z),

$$\mathcal{T}_2(y,z) \geq t \Longleftrightarrow \vartheta_2(t,y) \leq z. \tag{5.36}$$

The statement of (iii) follows from (5.36). ■

We show also that, under some initial condition, the optimal trajectories of the control problems (5.8) and (5.31) can be linked to the optimal trajectories associated with the maximum time function $\mathcal{T}_2$.

Proposition 5.4.9 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *hold. Let* $y \in \mathcal{K}$ *such that* $\vartheta_2(T, y) < \infty$*. Define* $z := \vartheta_2(T, y)$*.*

(i) Let $\bar{\mathbf{y}}^* = (\mathbf{y}^*, \zeta^*)$ *be the optimal trajectory for the auxiliary control problem* (5.31) *associated with the initial point* $(y, z) \in \mathcal{K} \times \mathbb{R}$*. Then, the trajectory* $\mathbf{y}^*$ *is optimal for the control problem* (5.8)*.*

(ii) Let $\bar{\mathbf{y}}^* = (\mathbf{y}^*, \zeta^*)$ *be an optimal trajectory for the maximum time problem* (5.35) *associated with the initial point* $(y, z) \in \mathcal{K} \times \mathbb{R}$*. Then,* $\bar{\mathbf{y}}^*$ *is also optimal for the auxiliary control problem* (5.31)*.*

Proof. Let $(y, z) \in \mathcal{K} \times \mathbb{R}$ such that $\vartheta_2(T, y) = z$. First, let us prove that the optimal trajectory $\bar{\mathbf{y}}^* = (\mathbf{y}^*, \zeta^*)$ for the auxiliary control problem (5.31) associated with the initial point $(y, z) \in \mathcal{K} \times \mathbb{R}$ ensures that $\mathbf{y}^*$ is an optimal trajectory of the control problem (5.8).

Using the proposition 5.4.2, we have :

$$\vartheta_2(T, y) = z \Rightarrow w_2(T, y, z) \leq 0.$$

Since $\bar{\mathbf{y}}^* := (\mathbf{y}^*, \zeta^*)$ is an optimal trajectory of the problem (5.31) associated with (y, z), it follows that,

$$w_2(T, y, z) = -\zeta^*(T) \vee \max_{s \in [0,T]} g(\mathbf{y}^*(s)) \leq 0.$$

It follows that,

$$-\zeta^*(T) = \int_0^T \Phi_2(\mathbf{y}^*(\theta))d\theta - z \leq 0, \qquad \text{and} \qquad \max_{s \in [0,T]} g(\mathbf{y}^*(s)) \leq 0. \tag{5.37}$$

Since $\vartheta_2(T, y) = z$, it follows from (5.37), that

$$\int_0^T \Phi_2(\mathbf{y}^*(\theta))d\theta \leq \vartheta_2(T, x) \quad \text{and} \quad \mathbf{y}^*(s) \in \mathcal{K}, \ \forall s \in [0, T].$$

By definition of ϑ_1 one can conclude that,

$$\vartheta_2(T, x) = \int_0^T \Phi_2(\mathbf{y}^*(\theta))d\theta \quad \text{and} \quad \mathbf{y}^*(s) \in \mathcal{K}, \ \forall s \in [0, T].$$

Therefore, $\mathbf{y}^*$ is an optimal trajectory for (5.7) associated with (y, z) and the proof is completed.

Now, let us prove that the optimal trajectory $\widehat{\mathbf{y}}^* = (\mathbf{y}^*, \zeta^*)$ for the maximum time problem (5.35) associated with the initial point $(y, z) \in \mathcal{K} \times \mathbb{R}$ such that $\vartheta_1(T, y) = z$ ensures that $\mathbf{y}^*$ is the optimal trajectory of the problem (5.8).

Let $\tau := \mathcal{T}_2(y,z)$. Since $\widehat{\mathbf{y}}^*(\cdot) = (\mathbf{y}^*, \zeta^*)$ is an optimal trajectory of the problem (5.35), it follows from the definition of $\mathcal{T}_2$ that,

$$-\zeta^*(\tau) \le 0, \qquad \text{and} \quad \max_{s\in[0,\tau]} g(\mathbf{y}^*(s)) \le 0.$$

Since $\vartheta_2(T,y) = z$, we obtain

$$\int_0^\tau \Phi_2(\mathbf{y}^*(\theta))d\theta \le \vartheta_2(T,x) \quad \text{and} \quad \mathbf{y}^*(s) \in \mathcal{K},\ \forall s \in [0,\tau].$$

If $\tau = T$, then from the definition of ϑ_2 one can conclude that $\mathbf{y}^*$ is an optimal trajectory for (5.8) associated with (y,z).

Assume $\tau < T$. Since $\vartheta_2(T,y) = z$, then there exists a minimizing sequence $\mathbf{y_n}$ of admissible trajectories of $S_{[0,T]}(y)$ such that,

$$\lim_{\mathbf{n}\to\infty} \left[\int_0^T \Phi_2(\mathbf{y_n}(s))ds - z \right] = \vartheta_2(T,y) - z = 0.$$

Since all the trajectories are admissible, we have $\max_{0\le\theta\le t} g(\mathbf{y_n}(\theta)) \le 0$. It follows that,

$$\lim_{\mathbf{n}\to\infty} \left[\left(\int_0^T \Phi_2(\mathbf{y_n}(s))ds - z \right) \vee \max_{0\le\theta\le T} g(\mathbf{y_n}(\theta)) \right] \le 0. \tag{5.38}$$

From (5.38) and the definition of the maximum time function $\mathcal{T}_2$, we obtain $\mathcal{T}_2(y,z) := \tau \ge T$ and this is a contradiction. ■

5.4.3 Reconstruction of optimal trajectories

This section is devoted to the reconstruction of optimal trajectories associated with the control problem for which the value function is ϑ_2. We have seen that the epigraph of ϑ_2 can be characterized by a level set of a Lipschitz continuous value function w_2 of an auxiliary optimal control problem. The Lipschitz property of w_2 allows to generate optimal trajectories.

Reconstruction by minimizing the value function

Consider the following version of the algorithm dedicated to the control problem (5.31) :

Algorithm C. For a given $n \in \mathbb{N}$ let us consider $(t_0 = 0, t_1 ... t_{n-1}, t_n = T)$ a uniform partition of $[0,T]$ with the time step $h = \frac{T}{n}$. Let $\{\mathbf{Y}^n(.)\} := \{\mathbf{y}^n(.), \zeta^n(.)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$ ($i \ge 1$). Set $\mathbf{Y}^n(t_0) = (y,z)$ (where $\vartheta_2(t_n,y) = z$).

Step 1 Knowing the state $\mathbf{Y}^n(t_k)$, compute the optimal control at t_k :

$$a_k \in \arg\min_u \left\{ w_2\Big(t_{n-k-1}, \mathbf{Y}^n(t_k) + h\bar{f}(\mathbf{Y}^n(t_k),u)\Big) \bigvee \max_{\theta\in[0,h]} g(\mathbf{y}^n(t_k) + \theta f(\mathbf{y}^n(t_k),u)) \right\}.$$

Step 2 , Define $\mathbf{u}_k(t) := a_k$ be a constant control in $t \in (t_k, t_{k+1}]$ and $\mathbf{Y}^n(t) = (\mathbf{y}^n(t), \zeta^n(t))$ on $(t_k, t_{k+1}]$ be the solution to

$$\dot{\mathbf{Y}}(t) := \bar{f}(\mathbf{Y}(t), \mathbf{u}_k(t)) \quad \text{a.e} \quad t \in (t_k, t_{k+1}].$$

with the initial condition $\mathbf{Y}(t_k) = (\mathbf{y}(t_k), \zeta(t_k))$.

Theorem 5.4.10 *Assume that the hypothesis* ($\mathbf{H_1}$), ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *hold. Let* w_2 *be the unique continuous viscosity solution of the equation* (5.32) *and let* $\{\mathbf{Y}^n(.), \mathbf{u}^n(.)\}$ *be a sequence generated by algorithm C for* $n \geq 1$*. Then, the sequence of trajectories* $\{\mathbf{Y}^n(.)\}_n$ *has cluster points with respect to the uniform convergence topology. For any cluster point* $\bar{\mathbf{Y}}(.)$ *there exists a control low* $\bar{\mathbf{u}}(.)$ *such that the pair* $(\bar{\mathbf{Y}}(.), \bar{\mathbf{u}}(.))$ *is optimal for the problem* (5.31)*.*

Proof. The proof is similar to that given for theorem 5.3.11 and it is given in Appendix C. ∎

Reconstruction using an approximate value function

Let W_2 be the solution given by a numerical scheme approximating the value function w_2 such that the error estimates takes the following form :

$$|W_2(t, y, z) - w_2(t, y, z)| \leq E_2(\Delta t, \Delta y, \Delta z), \tag{5.39}$$

where E_2 depends on Δt, Δy and Δz such that if $W_2(., .) \to w_2(., .)$ as $\Delta t, \Delta y, \Delta z \to 0$.

Let $\mathbf{Y}$ be the trajectory associated with W_2 given by the algorithm C. Then the following holds :

Theorem 5.4.11 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *hold. Let* w_2 *be the unique continuous viscosity solution of the equation* (5.14)*. Let* W_2 *be an approximation of* w_2 *satisfying* (5.39)*. Let* $\{\mathbf{Y}_1^n(.), \mathbf{u}_1^n(.)\}$ *and* $\{\mathbf{Y}_2^n(.), \mathbf{u}_2^n(.)\}$ *be the sequences respectively generated by the algorithm C for* w_2 *and* W_2 *respectively. Then, for all* $\varepsilon > 0$ *we have the following estimates :*

$$\left| \Big(-\zeta_1^n(T) \vee \max_{\theta \in [0,T]} g(\mathbf{y}_1^n(\theta)) \Big) - \Big(-\zeta_2^n(T) \vee \max_{\theta \in [0,T]} g(\mathbf{y}_2^n(\theta)) \Big) \right| \leq 3T\varepsilon + (2n+3)E_2(\Delta t, \Delta y, \Delta z).$$

where $\mathbf{Y}_1 := (\mathbf{y}_1, \zeta_1)$ *and* $\mathbf{Y}_2 := (\mathbf{y}_2, \zeta_2)$*.*

Proof. The proof of the theorem 5.4.11 is similar to that given for the theorem 5.3.14. ∎

Reconstruction using the maximum time function

The reconstruction of optimal trajectories corresponding to (5.8) can be done using the maximum time function $\mathcal{T}_2$ since it satisfies a Dynamic Programming Principle.

Proposition 5.4.12 *Assume* ($\mathbf{H_1}$), ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *hold. Then, for all* $(y,z) \in \mathbb{R}^{d+1}$,

$$\mathcal{T}_2(y,z) := \sup_{\alpha\in\mathcal{U}} \left\{ \mathcal{T}_2(\mathbf{y}^\alpha(t), \zeta^\alpha(t)) + t \right\} \bigwedge T, \quad \forall t \in \big[0, \mathcal{T}_2(y,z)\big].$$

Proof. The same proof is given for proposition 5.3.17. ■

The following formal algorithm uses the maximum time function to reconstruct optimal trajectories for (5.7),

One can derive also a similar algorithm for the maximum time function $\mathcal{T}_2$ in the following manner :

Algorithm D. For a given $n \in \mathbb{N}$ let us consider $(t_0 = 0, t_1 ... t_{n-1}, t_n = T)$ a uniform partition of $[0,T]$ with the time step $h = \frac{T}{n}$. Let $\mathbf{Y}^n(.) := \{\mathbf{y}^n(.), \zeta^n(.)\}$ be a trajectory defined recursively at t_i ($i \geq 1$). Set $\mathbf{y}^n(t_0) = y$ and $\zeta^n(t_0) = z$.

Step 1 Knowing the state $\mathbf{y}^n(t_k)$ choose the optimal control at t_k :

$$u_k^n \in \arg\max_{u\in U} \left\{ \left(\big[\mathcal{T}_2\big] \big(\mathbf{Y}^n(t_k) + h\bar{f}(\mathbf{Y}^n(t_k), u)\big) + h \right) \bigwedge \ T \ \right\}, \tag{5.40}$$

where $\big[A\big]$ denotes the linear interpolation of A.

Step 2 The next point of the state corresponding to the minimizing value $\mathbf{u}(t_k) = u_k^n$ is :

$$\mathbf{Y}^n(t_{k+1}) := \mathbf{Y}^n(t_k) + h\bar{f}\big(\mathbf{Y}^n(t_k), \mathbf{u}(t_k)\big). \tag{5.41}$$

In the next Chapter, we propose to work on a real application in order to see the behaviour of our approach for state constrained control problems with maximum cost problem and Bolza cost. In particular, we generate optimal trajectories using the algorithms described above and we propose other formal algorithms.

Appendix

Appendix A. Proof of the Lipschitz property of w_1

Consider $\widehat{y} = (y,z), \widehat{y}' = (y',z') \in \mathbb{R}^d \times \mathbb{R}$, $t \in [0,T]$, and $w_1(0,\widehat{y}) \equiv \max(\varphi(y) - z, g(y)) \wedge \widetilde{c}$. By using the definition of w_1 and the elementary inequalities

$$\max(A,B) - \max(C,D) \leq \max(A-C, B-D), \quad \text{and} \quad \inf A_\alpha - \inf B_\alpha \leq \sup(A_\alpha - B_\alpha), \tag{5.42}$$

we obtain :

$$|w_1(t,\widehat{y}) - w_1(t,\widehat{y}')| \le \sup_{\alpha\in\mathcal{U}} \max_{\theta\in(0,T)} \left(\left| \Psi_1(\mathbf{y}^\alpha_y(\theta), z) - \Psi_1(\mathbf{y}^\alpha_{y'}(\theta), z) \right| \right), \qquad (5.43)$$
$$\le \sup_{\alpha\in\mathcal{U}} \left(L_{\Psi_1} \max_{\theta\in(t,T)} \left|\mathbf{y}^\alpha_y(\theta) - \mathbf{y}^\alpha_{y'}(\theta)\right| \right)$$

where L_{Ψ_1} denote the Lipschitz constant of Ψ_1. By using the Lipschitz property of the function $\widetilde{f}$, for any $\theta \in (0,T)$, $\left|\widehat{\mathbf{y}}^\alpha_{\widehat{y}}(\theta) - \widehat{\mathbf{y}}^\alpha_{\widehat{y}'}(\theta)\right| \le e^{\widehat{L}(\theta)}|\widehat{y} - \widehat{y}'| \le e^{\widehat{L}T}|\widehat{y} - \widehat{y}'|$ (here $\widehat{L}$ is the Lipschitz constant of $\widehat{f}$), we can conclude that

$$|w_1(t,\widehat{y}) - w_1(t,\widehat{y}')| \le L_{\Psi_1} e^{\widehat{L}T}|\widehat{y} - \widehat{y}'|. \qquad (5.44)$$

Alternatively, let $\widehat{y} = (y,z)$ be in $\mathbb{R}^d \times \mathbb{R}$, and let $t \ge 0$, $h \ge 0$. Using the fact that $w_1(t,\widehat{y}) \ge \Psi_1(\widehat{y})$, we can deduce from the dynamic programming principle for w_1 that

$$\begin{aligned}
&|w_1(t+h,\widehat{y}) - w_1(t,\widehat{y})| \\
&\le \sup_\alpha \max \left(\left|w_1(t, \widehat{\mathbf{y}}^\alpha_{\widehat{y}}(h)) - w_1(t,\widehat{y}))\right|, \max_{\theta\in(0,h)} \left|\Psi_1(\mathbf{y}^\alpha_y(\theta), z) - \Psi_1(y,z)\right| \right) \\
&\le \max \left(L_{\Psi_1} e^{\widehat{L}T} |\widehat{\mathbf{y}}^\alpha_{\widehat{y}}(h) - \widehat{y}|,\ L_{\Psi_1} \max_{\theta\in(0,h)} |\mathbf{y}^\alpha_y(\theta) - y| \right)
\end{aligned}$$

where we have used (5.44).
Furthermore, denoting $C_f := \max_{a\in\mathcal{U}} |f(0,a)| < \infty$, we have $|f(y,a)| \le C_f + L|y|$. Hence, by a Gronwall estimate, we have $|\mathbf{y}^\alpha_y(\theta) - y| \le (C_f + L|y|)e^{Lh}h \le (C_f + L|y|)e^{LT}h$ for $\theta \in (0,h)$. We can obtain in the same fashion the estimate $|\widehat{\mathbf{y}}^\alpha_{\widehat{\mathbf{y}}}(\theta) - \widehat{y}| \le (\widehat{C}_f + \widehat{L}|\widehat{y}|)e^{\widehat{L}T}h$ for every $\theta \in (0,h)$.
Therefore, we conclude that $|w_1(t',\widehat{y}) - w_1(t,\widehat{y})| \le C(1+|\widehat{y}|)|t'-t|$ for some constant $C > 0$. Combining the inequalities above we get

$$|w_1(t',\widehat{y}') - w_1(t,\widehat{y})| \le C(1+|\widehat{y}|)\,(|t'-t| + |\widehat{y}' - \widehat{y}|),$$

for some constant $C \ge 0$. In particular, we obtain the linear growth condition $|w_1(t,\widehat{y})| \le C(1+|\widehat{y}|)$.

Appendix B. Proof of properties of w_2

Proof. Locally Lipschitz property. Consider $\bar{y} = (y,z) \in \mathbb{R}^d \times \mathbb{R}$, $t \in [0,T]$. By using the elementary inequalities (5.42), there exists $u \in \mathcal{U}$, let $y' \in V(y)$ (where $V(y)$ denotes a neighborhood of y),

$$\begin{aligned}
&\left|w_2(t,y,z) - w_2(t,y',z)\right| \\
&\le \max_{\theta\in(0,t)} \left|\Psi_2(\bar{\mathbf{y}}^u_{y,z}(\theta) - \Psi_2(\bar{\mathbf{y}}^u_{y',z}(\theta)\right|, \\
&\le \max_{\theta\in(0,t)} \left| \int_0^\theta \left(\Phi_2(\mathbf{y}^u_y(s)) - \Phi_2(\mathbf{y}^u_{y'}(s))\right) ds \bigvee \left(g(\mathbf{y}^u_y(\theta)) - g(\mathbf{y}^u_{y'}(\theta))\right) \right|
\end{aligned}$$

On the other hand, by using the Lipschitz property of the function f, for any $\theta \in (0,T)$, $\left|\mathbf{y}_y^\alpha(\theta) - \mathbf{y}_{y'}^\alpha(\theta)\right| \le e^{L(\theta)}|y-y'| \le e^{LT}|y-y'|$ (here L is the Lipschitz constant of f)

Then, the set $V(y)$ can be defined such that $\mathbf{y}_{y'}^u(\theta) \in V(\mathbf{y}_y^u(\theta))$ for all $\theta \in [0,T]$. It follows that,

$$\left|w_2(t,y,z) - w_2(t,y',z)\right| \le L_V \max_{\theta\in(0,t)} \left|\mathbf{y}_y^u(\theta) - \mathbf{y}_{y'}^u(\theta)\right|,$$

we can conclude that

$$|w_1(t,y,z) - w_1(t,y',z)| \le L_V e^{LT}|y-y'|.$$

The Lipschitz property in z is obvious and one can get for all $y' \in V(y)$, and $z, z' \in \mathbb{R}$,

$$|w_1(t,y,z) - w_1(t,y',z)| \le L_V e^{LT}|y-y'| + |z-z'|.$$

The Lipschitz property in t follows in the same way from the Dynamic programming principle and classical argument as in the previous proof of Lipschitz property for w_1. ■

Proof. Polynomial growth property. Consider $\bar{y} = (y,z) \in \mathbb{R}^d \times \mathbb{R}$, $t \in [0,T]$. By using the elementary inequalities (5.42), there exists $u \in \mathcal{U}$,

$$\begin{aligned}
|w_2(t,\bar{y})| &\le \max_{\theta\in(0,t)} \left|\Psi_2(\bar{\mathbf{y}}_{\bar{y}}^\alpha(\theta)\right|, && (5.45)\\
&\le \max_{\theta\in(0,t)} \left| -\zeta_{\bar{y}}^\alpha(\theta) \bigvee g(\mathbf{y}_y^{\mathbf{u}}(\theta))\right| && (5.46)\\
&\le \max_{\theta\in(0,t)} \left|\Big(\int_0^\theta \Phi_2(\mathbf{y}_y^{\mathbf{u}}(s))ds - z\Big) \bigvee \Big(g(0) + L_g|\mathbf{y}_y^{\mathbf{u}}(\theta)|\Big)\right| && (5.47)\\
&\le \max_{\theta\in(0,t)} \Big(c_1 \int_0^\theta (1+|\mathbf{y}_y^{\mathbf{u}}(s)|^q)ds + |z|\Big) \bigvee \Big(g(0) + L_g|\mathbf{y}_y^{\mathbf{u}}(\theta)|\Big)
\end{aligned}$$

where L_g is Lipschitz constant of g. By Lipschitz assumption of f, for any $\theta \in (0,T)$, $\left|\mathbf{y}_y^{\mathbf{u}}(\theta)\right| \le L(|y| + \sqrt{K\theta})e^{K\theta}$ (where $K = L + \sup\{|f(0,a) \mid a \in U\}$ and L is the Lipschitz constant of f). If follows that,

$$\begin{aligned}
|w_2(t,\bar{y})| &\le \left|\Big(c_1 \int_0^t (1+|y|^q + c_2)ds - z\Big) \bigvee L_g|y| + c_3\right|\\
&\le \left|\Big(Tc_1(1+|y|^q + c_2) - z\Big) \bigvee L_g|y| + c_3\Big)\right.\\
&\le \left|\Big(Tc_1(1+|y|^q + c_2) + |z| + L_g|y| + c_3\Big)\right.\\
&\le c_T\Big(1+|y|^q + |z| + |y|\Big)\\
&\le c_T\Big(1+|\bar{y}|^q\Big)
\end{aligned}$$

where c_T is a positive constant depending on T, L, L_g, and L_1. We conclude that the value function w_2 has a polynomial growth. ■

Appendix C. Proof of theorem 5.4.10

The proof of the theorem is based on the following property :

Proposition 5.4.13 *Let $\Omega \subset \mathbb{R}^{d+1}$ be a compact set. Assume (H_1)-(H_3). Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in [0, \delta]$, for all $\xi := (y, z) \in \Omega$ and $t \in [0, T]$*

$$\left| \frac{1}{h} \left(\min_{u \in U} \left\{ w_2(t-h, \xi + h\bar{f}(\xi, u)) \bigvee \max_{\theta \in [0,h]} g(y + \theta f(y,u)) \right\} - w_2(t, \xi) \right) \right| < \varepsilon$$

Proof.

Let us introduce a set-valued map $\bar{R}(\cdot,\cdot) : [0,T] \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ defining for a $\xi := (y,z) \in \mathbb{R}^{d+1}$ and $\tau \in [0,T]$ the reachable set of the dynamic system $\bar{f}$ for the trajectories starting from ξ and at time τ

$$\bar{R}(\tau, \xi) = \{\mathbf{Y}(\tau),\ \mathbf{Y} = (\mathbf{y}, \zeta) \in \bar{\mathbf{S}}_{[\mathbf{0},\tau]}(\xi)\} \tag{5.48}$$

Let $L_g > 0$ and $L_{w_2} > 0$ be the Lipshitz constants of g and w_2 respectively. Due to the hypothesis (H_1) there exists a constant $r > 0$ such that for all $\xi \in \Omega,\ \tau \in [0,T]$ $\bar{R}(\tau, \xi) \subset rB$ (with B the unit ball centered at zero).

Fix a $\varepsilon > 0$. Using the properties of the reachable set (5.48) (see proposition 5.3 in [95]), let us choose $\delta > 0$ such that for all $h \in (0, \delta) \cap (0, T)$ and for all $\xi \in \Omega$

$$\max(L_g, L_{w_2}) \cdot \mathrm{dist}_H(\frac{1}{h}(\bar{R}(h,\xi) - \xi), \bar{f}(\xi, U)) < \frac{\varepsilon}{2}$$

where dist_H is the Hausdorf distance. Now, let us take $\xi := (y,z) \in \Omega$ and $t \in [0,T]$.

Recall that, due to the dynamic programming principle (5.14a) we have :

$$w_2(t,\xi) = \inf_{\mathbf{Y} := (\mathbf{y},\zeta) \in \bar{S}_{[0,h]}(\xi)} \left(w_2(t-h, \mathbf{Y}(h)) \bigvee \max_{\theta \in [0,h]} g(\mathbf{y}(\theta)) \right)$$

Then

$$\begin{aligned}
&\left| \min_{u \in U} \left\{ w_2(t-h, \xi + h\bar{f}(\xi,u)) \bigvee \max_{\theta \in [0,h]} g(y + \theta f(y,u)) \right\} - w_2(t,\xi) \right| \\
\le &\left| \min_{u \in U} \left\{ w_2(t-h, \xi + h\bar{f}(\xi,u)) \bigvee \max_{\theta \in [0,h]} g(y + \theta f(y,u)) \right\} - \inf_{\mathbf{Y} \in \bar{S}_{[0,h]}(\xi)} \left(w_2(t-h, \mathbf{Y}(h)) \bigvee \max_{\theta \in [0,h]} g(\mathbf{y}(\theta)) \right) \right| \\
\le &\left| \sup_{u \in U, \mathbf{Y} \in \bar{S}_{[0,h]}(\xi)} \left\{ w_2(t-h, \xi + h\bar{f}(\xi,u)) \bigvee \max_{\theta \in [0,h]} g(y + \theta f(y,u)) - w_2(t-h, \mathbf{Y}(h)) \bigvee \max_{\theta \in [0,h]} g(\mathbf{y}(\theta)) \right\} \right|
\end{aligned}$$

Let $u \in U$ and $\mathbf{y} \in S_{[0,h]}(\xi)$. Then there exists $\mathbf{u} \in \mathcal{U}$ such that $\mathbf{y} = \mathbf{y}_\xi^{\mathbf{u}}$ on $[0,h]$. Let us show that

$$\left| \max_{\theta \in [0,h]} g(\xi + \theta f(\xi, u)) - \max_{\theta \in [0,h]} g(\mathbf{y}(\theta)) \right| < h \cdot \frac{\varepsilon}{2} \tag{5.49}$$

Indeed, for all $\theta \in [0,h]$

$$\begin{aligned}
|g(y + \theta f(y,u)) - g(\mathbf{y}(\theta))| &\le L_g \left| y + \theta f(y,u) - \mathbf{y}(\theta) \right| \\
&\le L_g \mathrm{dist}_H(R(\theta, y), y + \theta f(y,U)) = L_g \mathrm{dist}_H(R(\theta,y) - y, \theta f(\xi, U)) \\
&= L_g \theta \mathrm{dist}_H(\theta^{-1}(R(\theta,y) - y), f(y,U)) < \theta\varepsilon \le h\frac{\varepsilon}{2}
\end{aligned} \tag{5.50}$$

Then the inequality (5.49) holds. With the similar arguments one can show also that

$$\begin{aligned}
\left|w_2(t-h,\xi+h\bar f(\xi,u))-w_2(t-h,\mathbf{Y}(h))\right| &\le L_{w_2}\left|\xi+\theta f(\xi,u)-\mathbf{y}(\theta)\right| \\
&\le L_{w_2}\cdot \mathrm{dist}_H(\bar R(h,\xi)-\xi,hf(\xi,U)) \\
&= L_{w_2}\cdot h\cdot \mathrm{dist}_H(h^{-1}(\bar R(h,\xi)-\xi),f(\xi,U)) \\
&< h\cdot\frac{\varepsilon}{2}
\end{aligned}$$

By recalling again (5.24), we have, for all $u\in U$ and all $\mathbf{Y}:=(\mathbf{y},\zeta)\in S_{[0,h]}(\xi)$,

$$\begin{aligned}
&\left|w_2(t-h,\xi+h\bar f(\xi,u))\bigvee \max_{\theta\in[0,h]} g(y+\theta f(y,u))-w_2(t-h,\mathbf{Y}(h))\bigvee \max_{\theta\in[0,h]} g(\mathbf{y}(\theta))\right| \\
\le&\left|w_2(t-h,\xi+h\bar f(\xi,u))\bigvee \max_{\theta\in[0,h]} g(y+\theta f(y,u))-w_2(t-h,\xi+h\bar f(\xi,u))\bigvee \max_{\theta\in[0,h]} g(\mathbf{y}(\theta))\right| \\
+&\left|w_2(t-h,\xi+h\bar f(\xi,u))\bigvee \max_{\theta\in[0,h]} g(\mathbf{y}(\theta))-w_2(t-h,\mathbf{Y}(h))\bigvee \max_{\theta\in[0,h]} g(\mathbf{y}(\theta))\right| \\
\le&\left|\max_{\theta\in[0,h]} g(y+\theta f(y,u))-\max_{\theta\in[0,h]} g(\mathbf{y}(\theta))\right|+\left|w_2(t-h,\xi+hf(\xi,u))-w_2(t-h,\mathbf{Y}(h))\right| \\
<&\, h\varepsilon
\end{aligned}$$

and the proof of the proposition is done. ∎

Proof.
Step 1. For all $n\ge 1$, the function $(\mathbf{Y}^n(.),\mathbf{u}^n(.))$ satisfies

$$\dot{\mathbf{Y}}^n(t):=\bar f(\mathbf{Y}^n(t),\mathbf{u}^n(t))\quad \text{a.e}\quad t\in[0,T]$$

Recall the definition of the augmented dynamics in this case,

$$\bar f(t):=\begin{pmatrix} f_e(\mathbf{y}^n(t),\mathbf{u}^n(t)) \\ -\Phi_2(\mathbf{y}^n(t))\end{pmatrix}\quad \text{a.e}\quad t\in[0,T]$$

Let us denote $\mathbf{Y}^n(.):=(\mathbf{y}^n(.),\zeta^n(.))$. The boundedness and the equicontinuity of $\mathbf{y}^n(.)$ follows respectively from (5.25) and (5.26). Let us prove the boundedness and the equicontinuity of $\zeta^n(.)$. Indeed, the boundedness of $\zeta^n(.)$ follows from the boundedness of $\mathbf{y}^n(.)$ and the following polynomial growth property of function $y\to\Phi_2(y)$ (that follows from the definition of Φ_2), i.e there exists $c_{\Phi_2}>0$ s.t,

$$\Phi_2(y)\le c_\Phi(1+|y|^q)$$

we obtain that,

$$|\zeta^n(t)|\le |z|+rc_T^q t\le c_4$$

where c_T is given by (5.25) and the boundedness of the sequence $\mathbf{Y}^n(.)$ follows. Moreover, the equicontinuity of $\{\mathbf{Y}^n(.)\}$ holds using similar arguments.

By Arzela-Ascoli theorem we know that there exists a uniformly convergent subsequence $\{\mathbf{Y}^{n_i}(.)\}$ that converges to the continuous solution $\bar{\mathbf{Y}}(.)$ as $n_i\to\infty$. The

Dunford Pettis criterion allows to extract a subsequence converging weakly to a trajectory limit $l(.)$, i.e

$$\dot{\mathbf{Y}}^i \rightharpoonup l, \qquad \text{in } L^1(\mathbb{R}^{d+1}, [0,T])$$

Using the dominated convergence theorem, we obtain that $\bar{\mathbf{Y}}$ is an absolutely continuous function on $[0,T]$ and $\dot{\bar{\mathbf{Y}}}(.) = l(.)$ a.e.

Set $L(t,Y,p) = \max_{q \in \bar{f}(Y,U)} < p, q >$. Let $\{A_i\}$ be a sequence of measurable subsets of $[0,T]$ such that measure $(A_i) \to T$ and 1_i be the indicator function of A_i. Let V be any measurable set in $[0,T]$, then we have,

$$\begin{aligned}
&0 \leq \limsup_{i\to\infty} \int_{V\cap A_i} \Big[L\Big(t, \mathbf{Y}^i(t), p\Big) - \langle p, \dot{\mathbf{Y}}^i(t)\rangle \Big] dt \\
&0 \leq \int_V \limsup_{i\to\infty} 1_i L\Big(t, \mathbf{Y}^i(t), p\Big) dt + \limsup_{i\to\infty} \int_{V\cap A_i} \langle p, \dot{\mathbf{Y}}^i(t)\rangle dt \\
&0 \leq \int_V L\Big(t, \bar{Y}(t), p\Big) dt + \limsup_{i\to\infty} \int_V \langle p, \dot{\mathbf{Y}}^i(t)\rangle dt \\
&\quad + \limsup_{i\to\infty} \int_{V\cap A_i^c} \langle p, \dot{\mathbf{Y}}^i(t)\rangle dt \\
&0 \leq \int_V \Big[L\Big(t, \bar{\mathbf{Y}}(t), p\Big) - \langle p, \dot{\bar{\mathbf{Y}}}(t)\rangle \Big] dt
\end{aligned}$$

It follows from the arbitrariness of V that $L(t, \bar{\mathbf{Y}}(t), p) \geq < p, \dot{\bar{\mathbf{Y}}}(t) >$. Since L is continuous in p, it follows (see proposition 2.1.4 of [42]) that

$$\dot{\bar{\mathbf{Y}}}(t) \in \bar{f}(\bar{\mathbf{Y}}(t), U)$$

By Fillipov's Lemma, we get a control function $\bar{\mathbf{u}}(.)$ such that $(\bar{\mathbf{Y}}(.), \bar{\mathbf{u}}(.))$ is an admissible solution for the optimal control problem.

Step 2. The optimality of $(\bar{\mathbf{Y}}(.), \bar{\mathbf{u}}(.))$ will follow if,

$$\Big(-\zeta(T)\Big) \bigvee \Big(\max_{\theta\in[0,T]} g(\mathbf{y}^n(\theta))\Big) \to w_2(T,(y,z)) \quad \text{as} \quad n\to\infty$$

Let L_w be the Lipschitz constant of w_1.

$$\begin{aligned}
&\limsup_{n\to\infty} \Big| \Big(-\zeta(T)\Big) \bigvee \Big(\max_{\theta\in[0,T]} g(\mathbf{y}^n(\theta))\Big) - w_2(T,(y,z)) \Big| \\
&\leq \limsup_{n\to\infty} \sum_{k=0}^{n-1} \Big| w_2(t_{n-k-1}, \mathbf{Y}^n(t_{k+1})) \bigvee \max_{\theta\in[t_0,t_{k+1}]} g(\mathbf{y}^n(\theta)) - w_2(t_{n-k}, \mathbf{Y}^n(t_k)) \bigvee \max_{\theta\in[t_0,t_k]} g(\mathbf{y}^n(\theta)) \Big|
\end{aligned}$$

By recalling again (5.24), one can simplify the last expression so that we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\left|\Big(-\zeta(T)\Big)\bigvee\Big(\max_{\theta\in[0,T]} g(\mathbf{y}^n(\theta))\Big) - w_2(T,(y,z))\right| \\
&\leq \limsup_{n\to\infty}\sum_{k=0}^{n}\left|w_2(t_{n-k-1},\mathbf{Y}^n(t_{k+1}))\bigvee\max_{\theta\in[t_k,t_{k+1}]} g(\mathbf{y}^n(\theta)) - w_2(t_{n-k},\mathbf{Y}^n(t_k))\right| \\
&\leq \limsup_{n\to\infty}\sum_{k=0}^{n}\left|w_2(t_{n-k-1},\mathbf{Y}^n(t_k)+h\bar{f}(\mathbf{y}^n(t_k),u_k^n))\bigvee\max_{\theta\in[0,h]} g(\mathbf{y}^n(t_k)+\theta f(\mathbf{y}^n(t_k),u_k^n)) - w_2(t_{n-k},\mathbf{Y}^n(t_k))\right| \\
&+ \limsup_{n\to\infty}\sum_{k=0}^{n}\Big|w_2(t_{n-k-1},\mathbf{Y}^n(t_{k+1}))\bigvee\max_{\theta\in[t_k,t_{k+1}]} g(\mathbf{y}^n(\theta)) \qquad (5.51)\\
&\quad - w_2(t_{n-k-1},\mathbf{Y}^n(t_k)+hf(\mathbf{Y}^n(t_k),u_k^n))\bigvee\max_{\theta\in[0,h]} g(\mathbf{y}^n(t_k)+\theta f(\mathbf{y}^n(t_k),u_k^n))\Big|
\end{aligned}
$$

Since u_k is chosen to achieve the minimum, the first term in the right hand side of the last inequality is zero (due to the proposition 5.4.13. For the second term we use the same arguments that in the proof of the proposition 5.4.13 to show that for all $k = 0, \dots, n-1$

$$
\begin{aligned}
&\left|w_2(t_{n-k-1},\mathbf{Y}^n(t_{k+1})) - w_2(t_{n-k-1},\mathbf{Y}^n(t_k)+h\bar{f}(\mathbf{Y}^n(t_k),u_k^n))\right| \\
&\leq L_{w_2} h \,\mathrm{dist}_H\Big(h^{-1}\big(\bar{R}(h,\mathbf{Y}^n(t_k)) - \mathbf{Y}^n(t_k)\big), \bar{f}(\mathbf{Y}^n(t_k),U)\Big) < h\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\max_{\theta\in[t_k,t_{k+1}]} g(\mathbf{y}^n(\theta),z) - \max_{\theta\in[0,h]} g(\mathbf{y}^n(t_k)+\theta f(\mathbf{y}^n(t_k),u_k^n))\right| \\
&\leq L_g h \,\mathrm{dist}_H\Big(h^{-1}\big(R(h,\mathbf{y}^n(t_k)) - \mathbf{y}^n(t_k)\big), f(\mathbf{y}^n(t_k),U)\Big) < h\varepsilon
\end{aligned}
$$

Then the second member of right hand side of the inequality (5.51) is also zero and this achieves the proof.

■

Appendix D. Comparison principle for HJ equations with obstacle terms

The aim of this section is to prove a comparison principle for the following HJ equation in the presence of an obstacle term

$$
\begin{aligned}
&\min(u_t(t,x) + H(x,\nabla u), u(t,x) - u_0(x)) = 0 \quad \text{on } (0,T)\times\mathbb{R}^d, && (5.52a)\\
&u(0,x) = u_0(x), \quad x\in\mathbb{R}^d, && (5.52b)
\end{aligned}
$$

where $T > 0$ and $u_0 \in C(\mathbb{R}^d)$, and $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous and assumed to satisfy the following there exists $C \geq 0$ such that

$$
\begin{cases}
\text{for all } x \text{ in } \mathbb{R}^d, p_1, p_2 \in \mathbb{R}^d,\\
|H(x,p_1) - H(x,p_2)| \leq C(|x|^q + 1)|p_1 - p_2|,
\end{cases} \qquad (\mathbf{A_1})
$$

$$\begin{cases}\text{for any } R>0, \text{ there exists a function } w_R:[0,\infty)\to[0,\infty),\ \lim_{r\to 0+} w_R(r)=0, \text{ s.t,}\\ |H(x,p)-H(y,p)|\le C|p|.w_R(|x-y|).\end{cases} \tag{$\mathbf{A_2}$}$$

For every $p \in \mathbb{R}^d$, $x, y \in B_R$, where B_R denotes the open ball centered at 0 and of radius R

Theorem 5.4.14 *Let u, v be two functions of $C([0,T]\times\mathbb{R}^d)$, and let g, h be in $C(\mathbb{R}^d)$. We assume that u (resp. v) is a sub-solution (resp. super-solution) of (5.52a) in $(0,T)\times\mathbb{R}^d$:*

$$\min(u_t(t,x)+H(x,\nabla u(t,x)), u(t,x)-g(x)) \le 0 \quad \textit{in } (0,T)\times\mathbb{R}^d, \tag{5.53a}$$
$$\min(v_t(t,x)+H(x,\nabla v(t,x)), v(t,x)-h(x)) \ge 0 \quad \textit{in } (0,T)\times\mathbb{R}^d. \tag{5.53b}$$

We denote $g(x) := u(0,x)$ and $h(x) := v(0,x)$. Then for all $t \in [0,T]$,

$$\sup_{\mathbb{R}^d}(u(t,.)-v(t,.)) \le \sup_{\mathbb{R}^d}(g-h). \tag{5.54}$$

Proof. The result without the obstacle term can be found in Ishii [68]. It suffices to prove the result for $T>0$ small enough, the result for any $T>0$ can then be deduced by immediate recursion.

Assuming that $C>0$, we take $T = 1/(2qC)$ and $L = L(x_0) := 2qC(|x_0|^q+1)$, and we define the set

$$\mathcal{O}_{x_0} := \left\{(t,x)\in(0,T)\times\mathbb{R}^d,\ |x-x_0| < (Lt)^{\frac{1}{q}}\right\}.$$

(the case $C=0$ is trivial). We claim that for every $t_0 \in (0,T)$

$$u(t_0,x_0)-v(t_0,x_0) \le \sup_{B_r(x_0)}(g-h), \tag{5.55}$$

(where $B_r(x_0)$ is the ball of radius $r = (L(T-t_0))^{\frac{1}{q}}$ centered in x_0) which concludes (5.54). Let us consider $t_0 \in (0,T)$ and prove the claim. First, notice that for any $(t,x)\in\mathcal{O}_{x_0}$, there holds $C(|x|^q+1) \le qC(|x_0|^q+1)+qC|x-x_0|^q \le \frac{L}{2}+qCLT \le \frac{L}{2}+\frac{1}{2}L = L$, and thus

$$|H(x,p_1)-H(x,p_2)| \le L|p_1-p_2|, \qquad x\in B_s(x_0).$$

where $s = (LT)^{\frac{1}{q}}$. We also define for any $(\bar t,\bar x)\in\mathcal{O}_{x_0}$ and $\tau\in(\bar t,T)$ the set

$$\mathcal{O}_{\bar t,\bar x,\tau} := \left\{(t,x)\in(\bar t,\tau)\times\mathbb{R}^d,\ |x-\bar x| < (L(t-\bar t))^{\frac{1}{q}}\right\}.$$

According to Crandall-Lions [46], and Ishii [68], the following Lemma holds :

Lemma 5.4.15 *If u, v belongs to $C(\overline{\mathcal{O}}_{\bar t,\bar x,t})$ with $t\in(\bar t,T)$, and are respectively viscosity solutions of*

$$-u_t+H(x,\nabla u)\le 0 \quad \textit{in } \mathcal{O}_{\bar t,\bar x,t}, \tag{5.56a}$$
$$-v_t+H(x,\nabla v)\ge 0 \quad \textit{in } \mathcal{O}_{\bar t,\bar x,t}, \tag{5.56b}$$

then

$$u(\bar{t},\bar{x}) - v(\bar{t},\bar{x}) \leq \sup_{B_{\bar{r}}(\bar{x})} (u(t,.) - v(t,.)).$$

where $\bar{r} = (L(t-\bar{t}))^{\frac{1}{q}}$.

Consider the set

$$\Sigma := \{(t,x) \in \overline{\mathcal{O}}_{x_0},\ u(t,x) \leq g(x)\},$$

and its complement in $\mathcal{O}_{x_0}$

$$\Omega := \mathcal{O}_{x_0} \backslash \Sigma.$$

Since u is a sub-solution of (5.53a), u is also a sub-solution of $u_t + H(x, \nabla u) = 0$ on the open set Ω. Furthermore, v being a super-solution of (5.53b), it is also a super-solution of $v_t + H(x, \nabla v) = 0$ on the open set Ω.
On the other hand, from (5.53b), it follows that $v(t,x) \geq g(x)$ everywhere. Hence

$$\forall (t,x) \in \Sigma, \quad u(t,x) - v(t,x) \leq g(x) - h(x) \leq \sup_{B_r(x_0)} (g-h). \tag{5.57}$$

Now, assume that

$$u(t_0,x_0) - v(t_0,x_0) > M := \sup_{B_r(x_0)} (g-h), \tag{5.58}$$

and define, for $\tau \in [0,T]$ (with $\tau > t_0$), the set

$$\mathcal{O}_{t_0,x_0,\tau} := \left\{(t,x) \in (t_0,\tau) \times \mathbb{R}^d,\ |x-x_0| < (L(t-t_0))^{\frac{1}{q}}\right\}.$$

Using the continuity of $u-v$ in (t_0,x_0), there exists a neighborhood Γ of (t_0,x_0) in $\mathcal{O}_{x_0}$ satisfying :

$$u(t,x) - v(t,x) > M \geq \sup_{B_r(x_0)} (g-h) \quad \forall (x,t) \in \Gamma.$$

Taking into account (5.57), it follows that Γ is necessarily included in Ω. Hence, there exists a $\tau > t_0$ such that the set $\mathcal{O}_{t_0,x_0,\tau}$ is also included in Ω. Set

$$t_1 := \sup\left\{\tau \in (t_0,T],\ \mathcal{O}_{t_0,x_0,\tau} \bigcap \Gamma = \emptyset\right\}.$$

($\mathcal{O}_{t_0,x_0,t_1}$ is the greatest set $\mathcal{O}_{t_0,x_0,\tau}$ such that $\mathcal{O}_{t_0,x_0,\tau} \subset \Omega$.) Applying Lemma 5.4.15 to the set $\mathcal{O}_{t_0,x_0,t_1}$, we obtain

$$u(t_0,x_0) - v(t_0,x_0) \leq \sup_{B_p(x_0)} (u(t_1,.) - v(t_1,.)).$$

where $p = L(t_1-t_0)^{\frac{1}{q}}$. If $t_1 = T$, then $u(t_0,x_0) - v(t_0,x_0) \leq \sup_{B_s(x_0)}(u_T - v_T) = M$, which contradicts (5.58). Hence $t_1 < T$. We consider a point x_1 of the ball $\overline{B}_p(x_0)$ at which a maximum of $u(t_1,\cdot) - v(t_1,\cdot)$ is attained. We get

$$M < u(t_0,x_0) - v(t_0,x_0) \leq u(t_1,x_1) - v(t_1,x_1).$$

We re-iterate the previous argument to obtain the existence of a t_2 in (t_1, T) corresponding to the greatest set of the form $\mathcal{O}_{t_1,x_1,t_2}$ and satisfying $\mathcal{O}_{t_1,x_1,t_2} \subset \Omega$, and then the existence of a point x_2 in $\overline{B}_{(L(t_2-t_1))^{\frac{1}{q}}}(x_1) \subset \overline{B}_{(L(t_2-t_0))^{\frac{1}{q}}}(x_0)$ such that

$$M < u(t_1, x_1) - v(t_1, x_1) \leq u(t_2, x_2) - v(t_2, x_2),$$

and so on. Therefore we construct an increasing sequence of times (t_k), and a sequence of points (x_k) such that the points (t_k, x_k) belong to the set $\mathcal{O}_{x_0}$. Because $\overline{\mathcal{O}}_{x_0}$ is a compact set, we can extract a convergent subsequence limiting to (t^*, x^*) in $\overline{\mathcal{O}}_{x_0}$. Moreover,

$$M < u(t^*, x^*) - v(t^*, x^*).$$

If $t^* = T$, we obtain a contradiction. Hence $t^* < T$. Now by continuity, we must have $u(t, x) - v(t, x) > M$ in a neighborhood of (t^*, x^*), for instance in the tube $\mathcal{C}$ centered at (t^*, x^*) defined by

$$\mathcal{C} \ := \ ]t^* - \tau_0, t^* + \tau_0[\ \times (B_n(x^*)),$$

where $n = (L\tau_0)^{\frac{1}{q}}$, for $\tau_0 > 0$ sufficiently small. On the other hand, as soon as $t^* - t_k < \tau_0$, we have $\mathcal{O}_{t_k,x_k,t_{k+1}} \subset \mathcal{O}_{t_k,x_k,t^*} \subset \mathcal{C} \subset \Omega$. This contradicts the fact that t_{k+1} is the maximum time τ such that $\mathcal{O}_{t_k,x_k,\tau} \subset \Omega$. This concludes the proof of (5.55). ■

Chapitre VI

Optimal feedback control for the abort landing problem in presence of windshear

Publications of this chapter

M. Assellaou, O. Bokanowski, A. Desilles and H. Zidani, *Optimal feedback control for the abort landing problem in presence of windshear* , in preparation.

6.1 Introduction

The problem of the control of the aircraft flying through the wind shear is one of the most important issues in the aerodynamics of the flight, in particular, in a landing framework. Indeed, several aircraft accidents have been attributed to wind shear [35]. This meteorological phenomenon is defined as the change on speed and direction of the wind over a small distance. This change of the wind affects the aircraft motion relative to the ground and it has more significant effects during the landing case.

As the aircraft passes through the wind shear level, the aircraft suffers a loss of the lift force and the airspeed. The pilot encounters a headwind with transition to the tailwind coupled with a descending air which spreads horizontally near the ground. This generates a significant threat of the resulting inertia wind shear force.

The penetration landing in presence of wind shear is unsafe in a high altitude. The abort landing problem is the best strategy to avoid the failed landing. This procedure consists in steering the aircraft to the maximum altitude that can reach in order to prevent a crash on the ground. In the references [86], [85], the authors propose a Chebyshev-type optimal control for which an approximate solution for the problem is given with the associated feedback control. This solution was improved in [35] and [36] by considering the switching structure of the problem that has bang-bang subarcs and singular arcs.

The present chapter is concerned by a similar optimal problem as in [86], [85], [35], [36]. The Hamilton Jacobi approach is used in order to calculate the corresponding value function. The presence of constraints on the definition of the optimal control problem precludes the characterization of the value function -without any controllability assumption- as the unique solution of a Hamilton Jacobi equation. Here, we will characterize the epigraph of the value function, as it is done in the theoretical part of this chapter, using some auxiliary optimal control problems.

In this chapter, we will introduce two optimal control problems with maximum and Bolza types associated with different models of equations of motion. Then, we will consider the Finite Difference and the Semi Lagrangian schemes for which we will study the stability issues. Next, we will reconstruct the associated optimal trajectories and feedback control using different algorithms of reconstruction discussed in the last section. Furtheremore, we introduce additional algorithms and focus on the regularity issue of the feedback control. In this framework, we discuss the regularization approach to improve the history of the feedback control. Many simulations will be included in this chapter involving data of a Boeing 727 aircraft model.

This chapter will be organized as follows. Section 6.2 contains the background of the aerodynamic of flight. The formulation of the control problem is given in section 6.3. In sections 6.5-6.6, we give the numerical schemes and the study of the stability. Then, we reconstruct the trajectories using different algorithms proposed in sections 6.7 and 6.8.

6.2 Background for the flight aerodynamic

6.2.1 Equation of motion

Consider the flight of an airplane in a vertical plane over a flat earth where the thrust, the aerodynamic and weight are forces acting on the center of gravity of the airplane and they lie in the plane of symmetry. Let $\mathbf{V}$ be the velocity vector of the airplane relative to the atmosphere. In order to clarify the derivation of equations of motion, let us define the following coordinate systems (see Figure 6.1) :

$*$ The ground axes system $Ex_ey_ez_e$ fixed to the surface of earth at mean sea level.
$*$ The wind axes system denoted by $Ox_wy_wz_w$ moving with the airplane and the x_w axis coincides with the velocity vector.

The path angle γ is defining the wind axes orientation with respect to the ground horizon axes. Let $\mathbf{E_0}$ be the position vector of center of the gravity relative to the ground.

The derivation of the position vector and the Newton's second law with the external force acting on the airplane yield the following equations,

$$\mathbf{V_0} = \frac{\mathrm{d}\,\mathbf{E_0}}{\mathrm{dt}}, \tag{6.1a}$$

$$\mathbf{F} = m\frac{d\mathbf{V}_0}{dt}, \tag{6.1b}$$

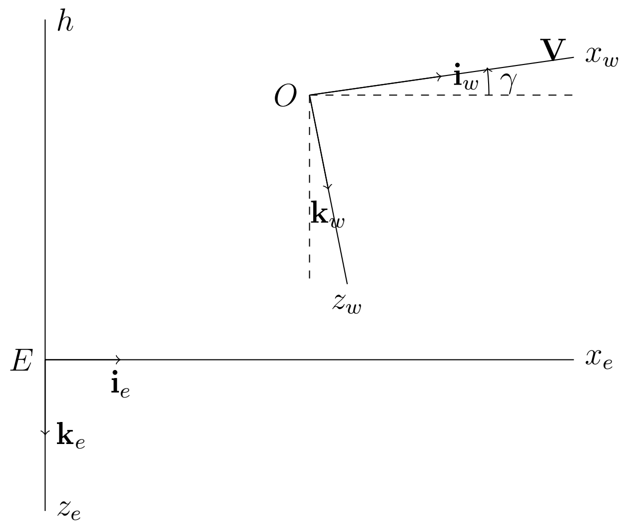

FIGURE 6.1 – Coordinate Systems for Flight in a Vertical Plane.

where $\mathbf{V_0} = \mathbf{V} + \mathbf{w}$ is the resultant velocity of the airplane relative to the ground axes system ($\mathbf{w}$ is the velocity of the atmosphere relative to the ground axis system) and $\mathbf{F}$ is the resultant force vector acting on the center of gravity of the aircraft. Figure 6.2 shows the following forces :

- The vector of the thrust force $\mathbf{F_T}$. The modulus of the thrust force is, in general, of the form $F_T := F_T(V, \beta)$ where (V is the modulus of the velocity $\mathbf{V}$ and β is the power setting of the engine which identifies the ratio of the actual revolution per minute (rpm, measure of rotation's frequency) per the maximum allowable rpm.)

- The vectors of the lift and drag forces $\mathbf{F_L}$ and $\mathbf{F_D}$.The norms of the lift and the drag forces are supposed to satisfy the following relations :

$$F_L = \frac{1}{2}\rho V^2 S c_l, \quad F_D = \frac{1}{2}\rho V^2 S c_d, \tag{6.2}$$

where ρ is the air density on altitude and S is the wing area. The coefficients c_d and c_l may depend on the angle of attack and the nature of the airplane.

- The vector of the weight force $\mathbf{F_P}$. the modulus of the weight obeys the known relation $F_p = mg$ (m ,g are respectively the aircraft mass and gravitational force per unit mass).

Let the velocity vector of the atmosphere be written in the ground axis system :

$$\mathbf{w} = w_x \mathbf{i}_e - w_h \mathbf{k}_e,$$

where w_x and w_h are respectively the horizontal component and the vertical component of the wind velocity. The vector $\mathbf{V_0}$ in the ground axis system takes the following form,

$$\mathbf{V_0} = (V\cos\gamma + w_x)\mathbf{i}_e - (V\sin\gamma + w_y)\mathbf{k}_e. \tag{6.3}$$

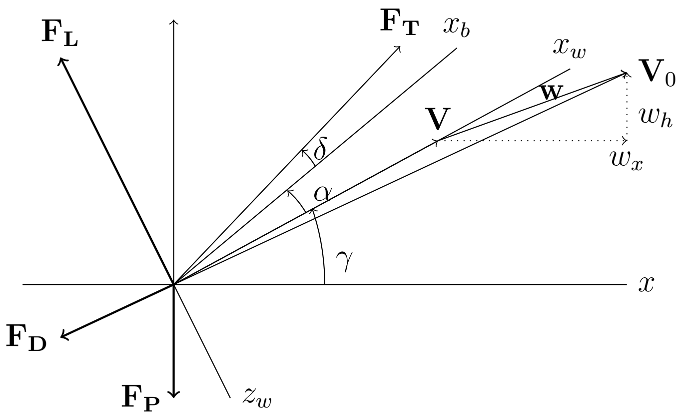

FIGURE 6.2 – Forces acting on the aircraft in flight in a moving atmosphere.

Then, by using (6.1a) with (6.3), the following scalar system is obtained :

$$\begin{cases} \dot{x} = V\cos\gamma + w_x, \\ \dot{h} = V\sin\gamma + w_h, \end{cases} \tag{6.4}$$

In the wind coordinate system, the forces acting on the aircraft can be written in the following form :

$$\begin{aligned} &\mathbf{F_T} = F_T\cos(\alpha+\delta)\mathbf{i}_w - F_T\sin(\alpha+\delta)\mathbf{k}_w, \qquad &&\mathbf{F_D} = -F_D\mathbf{i}_w, \\ &\mathbf{F_P} = -F_P\sin\gamma\mathbf{i}_w + F_P\cos\gamma\mathbf{k}_w, &&\mathbf{F_L} = -F_L\mathbf{k}_w, \end{aligned}$$

The resultant external force becomes in this coordinate system :

$$\mathbf{F} = (F_T\cos(\alpha+\delta) - F_D - F_P\sin\gamma)\mathbf{i}_w - (F_T\sin(\alpha+\delta) + F_L - F_P\cos\gamma)\mathbf{k}_w. \tag{6.5}$$

The acceleration of the airplane with respect to the ground can be written in the wind axes :

$$\begin{aligned} \frac{\mathrm{d}\mathbf{V}_0}{\mathrm{dt}} &= \dot{V}\mathbf{i}_w + V\frac{\mathrm{d}\mathbf{i}_w}{\mathrm{dt}} + \dot{w}_x\mathbf{i}_e - \dot{w}_h\mathbf{k}_e \\ &= \dot{V}\mathbf{i}_w - V\dot{\gamma}\mathbf{k}_w + \dot{w}_x(\cos\gamma\mathbf{i}_w + \sin\gamma\mathbf{k_w}) \\ &- \dot{w}_h(-\sin\gamma\mathbf{i}_w + \cos\gamma\mathbf{k}_w) \\ &= (\dot{V} + \dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma)\mathbf{i}_w - (V\dot{\gamma} - \dot{w}_x\sin\gamma + \dot{w}_h\cos\gamma)\mathbf{k}_w. \end{aligned} \tag{6.6}$$

where the time derivatives of the components of the wind are given by,

$$\begin{aligned} \dot{w}_x &= \frac{\partial w_x}{\partial x}(V\cos\gamma + w_x) + \frac{\partial w_x}{\partial h}(V\sin\gamma + w_h). \\ \dot{w}_h &= \frac{\partial w_h}{\partial x}(V\cos\gamma + w_x) + \frac{\partial w_h}{\partial h}(V\sin\gamma + w_h). \end{aligned}$$

Replacing (6.5) and (6.6) in the Newton's law (6.1b), the balance of forces gives the following scalar equations :

$$\begin{cases} \dot{V} = \frac{F_T}{m}\cos(\alpha+\delta) - \frac{F_D}{m} - \frac{F_P}{m}\sin\gamma - (\dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma) \\ \dot{\gamma} = \frac{1}{V}(\frac{F_T}{m}\sin(\alpha+\delta) + \frac{F_L}{m} - \frac{F_P}{m}\cos\gamma + (\dot{w}_x\sin\gamma - \dot{w}_h\cos\gamma)). \end{cases} \tag{6.7}$$

By adding (6.4) to (6.7), we obtain the equations of motion in a vertical plane over a flat plane :

$$\begin{cases} \dot{x} = V\cos\gamma + w_x \\ \dot{h} = V\sin\gamma + w_h \\ \dot{V} = \frac{F_T}{m}\cos(\alpha+\delta) - \frac{F_D}{m} - g\sin\gamma - (\dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma) \\ \dot{\gamma} = \frac{1}{V}(\frac{F_T}{m}\sin(\alpha+\delta) + \frac{F_L}{m} - g\cos\gamma + (\dot{w}_x\sin\gamma - \dot{w}_h\cos\gamma)). \end{cases} \tag{6.8}$$

6.2.2 Different models

In flight, the aircraft movements cause a change of the forces balance and affects the value of the angle of attack α. Then, α can be a control of the dynamics (6.8). The airplane can also be controlled by moving a device used for piloting some fixed wing aircraft called the throttle. Moving the throttle leads to the change of the flow rate which causes a change of the revolution per minute (rpm). The power setting β used to control the relative rpm of the engine, can also be a control corresponding to the throttle.

Instead of considering the control variables, one can think to control its derivatives and take into consideration an augmented state variables vector. When the control is non smooth enough, this last procedure presents a best regularization approach of the control variable. Indeed, in [35], the control is $u = \dot{\alpha}$ and the state variables are increased by one.

Here, we present two models of the equation of motion depending on the choice of the controls variables and the control dependence of the aerodynamic forces. Let d be the dimension of any model described below and T a fixed time horizon and y_0 the initial point, the equations of motion are supposed to be in the following form,

$$\dot{\mathbf{y}}(s) = f(\mathbf{y}(s), \mathbf{u}(s)), \quad s \in [0, T], \tag{6.9a}$$

$$\mathbf{y}(0) = y_0, \tag{6.9b}$$

where $\mathbf{y}(s) \in \mathbb{R}^d$ is the vector of state variables at time $s \in [0, T]$ and f the dynamic depending on the model. The vector of controls is $\mathbf{u}$. Denote $\mathbf{y}_{y_0}^{\mathbf{u}}(.)$ the trajectory corresponding to the control u and the initial point y_0.

M1- model : The first model of the equations of motion is a four dimensional differential equations given by (6.8) (the function f is given by the right hand side of (6.8)), where $\mathbf{y}(.) = (x(.), h(.), V(.), \gamma(.))$ is the state variables. The vector control here is $\mathbf{u}(.) = (\alpha(.), \beta(.))$ (where α is the angle of attack and β is the power setting). The aerodynamic forces are only functions of velocity i.e

$$F_D = F_D(V), \quad F_L = F_L(V),$$

whereas the thrust function is assumed to be a function of the power setting β and the velocity V, i.e, $F_T := \beta F_T(V)$.

M2- model : The second model considered here is a five dimensional differential equation of the following form,

$$\begin{cases} \dot{x} = V\cos\gamma + w_x, \\ \dot{h} = V\sin\gamma + w_h, \\ \dot{V} = \frac{F_T}{m}\cos(\alpha+\delta) - \frac{F_D}{m} - g\sin\gamma - (\dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma), \\ \dot{\gamma} = \frac{1}{V}(\frac{F_T}{m}\sin(\alpha+\delta) + \frac{F_L}{m} - g\cos\gamma + (\dot{w}_x\sin\gamma - \dot{w}_h\cos\gamma)), \\ \dot{\alpha} = \mathbf{u}. \end{cases} \tag{6.10}$$

The equations of motion involve the five state variables whose derivatives appear in the equations of motion, that is, $\mathbf{y}(.) = (x(.), h(.), V(.), \gamma(.), \alpha(.))$. The control $\mathbf{u}$ in this model is the angular speed of the angle of attack α.The aerodynamic forces are depending on the velocity V and the control α and the thrust function is assumed to be a function of velocity, that is

$$F_D = F_D(V,\alpha), \quad F_L(V,\alpha), \quad F_T = \beta F_T(V). \tag{6.11}$$

where the power setting β is equal to 1.

Remark 6.2.1 *The first model M1 is based on some approximation of the aerodynamic forces. Nevertheless, it will help us to explore some numerical aspects for this real application, namely, the stability issue of the considered schemes. The 5D-model M2 has already been studied in [86], [85], [35], [36] for a power setting β in the from of an affine function for the first three seconds and equal to 1 for the following. Here, we justify our choice of β by the fact that the pilot has to achieve the maximal thrust when he realizes that the windshear phenomenon occurs. The model M2 is more realistic since the aerodynamic forces are depending on the velocity and the angle of attack as well. This is a good example to validate our approach developed in the Chapter 5 for real applications. Both models satisfy the assumptions considered in this Chapter 5 (see Appendix A). In the next section, we will present the control problem we want to solve and the associated constraints.*

6.3 Statement of the problem

6.3.1 State and control constraints

This subsection presents different physical types of control and state constraints that prevent the resolution of the problems in all domain. These constraints are characterizing the structure of the airplane and the sensitivity of the trajectories in the abort landing problem. Denote by $\mathcal{K}$ the set of state constraints that will contain all the constraints corresponding to the state variables. The following constraints are characterizing the aircraft :

$*$ The velocity norm V can not be negative. The velocity is zero means that the aircraft is not moving which is not compatible with our problem. Moreover, the velocity is characterizing the aircraft and it satisfies $V(\cdot) \leq V_{\max}$ (where $V_{\max}$ is given by the aircraft constructor). The airplane can not fly with a velocity lower than a threshold $V_{\min}$ ($V_{\min} \leq V \leq V_{\max}$).

$*$ The inclination angle γ and the angle of attack α are obviously belonging to $[-\pi, \pi]$. In addition, these angles can not exceed some upper and lower bounds given by the constructor that is $\alpha(.) \in [\alpha_{\min}, \alpha_{\max}]$ and $\gamma(\cdot) \in [\gamma_{\min}, \gamma_{\max}] \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}]$. The power setting β belongs to $[0, 1]$.

Besides the constraints of the aircraft structure, we have the constraints corresponding to the abort landing problem :

$*$ The altitude h is strictly positive. Otherwise, it is zero, then the aircraft can touch the ground and this is what we want to avoid for this abort landing problem. Moreover, the altitude is supposed to satisfy the following,

$$h(\cdot) \in [h_{\min}, h_{\max}],$$

where $h_{\min}$ defines the lower altitude below which the abort landing is very difficult and $h_{\max}$ is a reference altitude (the cruise altitude for instance).

$*$ There is no special assumption that can be made for the coordinate x of the horizontal axis. However, the time horizon T is fixed which implies that $x(.) \in [x_{\min}, x_{\max}]$.

6.3.2 Optimality criterion

In the case of the wind-shear, the Airport Traffic Control Tower has to choose between two options. The first one is to penetrate inside the wind shear area and try to make a successful landing. If the altitude is high enough, it is more safety to choose another option : the abort landing, so as to avoid any unexpected unstability of the aircraft. In this thesis, only the second option is considered.

Starting from an initial point $y_0 \in \mathbb{R}^d$, the optimality criterion is defined as the lower altitude over a time interval , that is,

$$\min_{\theta \in [0,T]} h(\theta),$$

where $h(\theta)$ is the altitude at time θ corresponding to the second component of the vector $\mathbf{y}(\theta) := \mathbf{y}_y^{\mathbf{u}}(\theta)$ solution of (6.9) at time θ starting from y.

Instead of considering the lower altitude as a cost function, one can choose to optimize the peak value over a time interval of the difference between the reference altitude H_r and the instantaneous altitude. Let us define the function Φ by the following,

$$\Phi(y) = H_r - h,$$

where h is the second component of vector y. The state constrained control problem with a maximum cost related to Φ is given by the following form :

$$(\mathbf{P_1}) \qquad := \max_{\theta\in[0,T]} \left\{ \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta)) \;\middle|\; \mathbf{u}\in\mathcal{U}, \;\; \mathbf{y}_y^{\mathbf{u}}(s)\in\mathcal{K}, \; s\in[0,T] \right\}$$

In the case when $H_r - h(t) \geq 0, \;\forall t\in[0,T]$, the Supremum norm functional can be approximated by the Bolza functional using the relation between the Holder norm and the supremum norm , i.e,

$$\lim_{k\to\infty} \left(\int_0^T (H_r - h(t))^{2k} \right)^{\frac{1}{2k}} = \sup_{0\leq t\leq T} (H_r - h(t)).$$

Let P_2 be the state constained control problem with Bolza cost :

$$(\mathbf{P_2}) \qquad := \left\{ r\int_0^T \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta))^q d\theta \;\middle|\; \mathbf{u}\in\mathcal{U}, \;\; \mathbf{y}_y^{\mathbf{u}}(s)\in\mathcal{K}, \; s\in[0,T] \right\}$$

where $r = \frac{1}{\max_{\mathcal{K}} \Phi(y)^q}$, q are positive constants. Both control problems (P_1) and (P_2) have been studied in the previous Chapter 5 for closed set $\mathcal{K}$.

6.4 Hamilton Jacobi Bellman approach

Let us consider the value functions corresponding to the state constrained control problem discussed above. One can associate with (P_1), the following value function,

$$\vartheta_1(t,y) := \min_{\mathbf{u}\in\mathcal{U}} \left\{ \max_{\theta\in[0,t]} \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta)) \;\middle|\; \mathbf{u}\in\mathcal{U}, \;\; \mathbf{y}_y^{\mathbf{u}}(s)\in\mathcal{K}, \; s\in[0,t] \right\}$$

For the Bolza problem (P_2), one can consider the value function denoted by ϑ_2,

$$\vartheta_2(t,y) := \min_{\mathbf{u}\in\mathcal{U}} \left\{ r\int_0^t \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta))^q d\theta \;\middle|\; \mathbf{u}\in\mathcal{U}, \;\; \mathbf{y}_y^{\mathbf{u}}(s)\in\mathcal{K}, \; s\in[0,t] \right\}$$

The Hamilton Jacobi Bellman approach developed in the Chapter 5 is applied here in order to compute the value function ϑ_2 and ϑ_1.

6.4.1 Problem (P_1)

Let g be the signed distance function to $\mathcal{K}$ and let $\widetilde{c} > 0$ and define the following extended set $\widetilde{\mathcal{K}}$,

$$\widetilde{\mathcal{K}} :\equiv \mathcal{K} + \tilde{c}B.$$

where B is the unit ball in $\mathbb{R}^d$ centred in origin. Define

$$\widetilde{\Psi}_1(y,z) = \Big((\Phi_1(y) - z) \vee g(y) \Big) \wedge \widetilde{c},$$

where Ψ_1 is defined in (5.13). Let $\widetilde{w}_1$ be defined by :

$$\widetilde{w}_1(t,y,z) := \inf_{\widehat{\mathbf{y}}=(\mathbf{y},z)\in\widehat{S}_{[0,t]}(y,z)} \max_{\theta\in[0,t]} \widetilde{\Psi}_1(\mathbf{y}(\theta), z).$$

Define the Hamiltonian :

$$H(y,p) := \sup_{u\in U}\big(- f(y,u).p\big).$$

Since $\mathcal{K}$ is compact, then the function Φ_1 satisfies the following :

$$\Phi_1(y) \in [\underline{m}, \overline{M}], \quad \forall y \in \mathcal{K},$$

Then, the value function $\widetilde{w_1}$ describes the epigraph of ϑ_1.

$$\vartheta_1(t,y) = \inf\Big\{ z \in]\underline{m}-1, \overline{M}+1[, \ w_1(t,y,z) \le 0 \Big\}.$$

(where 1 means a small constant). Moreover, the value function w_1 is the unique Lipschitz continuous viscosity solution of the following Hamilton Jacobi equation,

$$\min\Big(\partial_t \widetilde{w_1}(t,y,z) + H(y, \nabla_y \widetilde{w_1}), \widetilde{w_1}(t,y,z) - \widetilde{\Psi_1}(y,z) \Big) = 0,$$

$$\text{in } [0,T]\times \overset{\circ}{\widetilde{\mathcal{K}}} \times]\underline{m}-1, \overline{M}+1[, \qquad (6.12a)$$

$$\widetilde{w_1}(0,y,z) = \widetilde{\Psi_1}(y,z), \qquad \text{in } \overset{\circ}{\widetilde{\mathcal{K}}} \times]\underline{m}-1, \overline{M}+1[. \qquad (6.12b)$$

$$\widetilde{w}_1(t,y,z) = \widetilde{c}, \quad \text{for all } t\in[0,T],\ y \notin \overset{\circ}{\widetilde{\mathcal{K}}} \text{ and } \quad z \in]\underline{m}-1, \overline{M}+1[\quad (6.12c)$$

We don't need any boundary condition on the boundary of z because the dynamics is zero $\dot{\mathbf{z}}(t) = 0$.

6.4.2 Problem (P_2)

Since $\mathcal{K}$ is compact then, the function $\Phi_2(\cdot) := r\Phi(\cdot)^q$ satisfies the following,

$$\Phi_2(y) \in [\underline{m}, \overline{M}], \quad \forall y \in \widetilde{\mathcal{K}},$$

Let g be the signed distance to $\mathcal{K}$ and let $\widetilde{w_2}$ be the value function defined by :

$$\widetilde{w_2}(t,y,z) := \inf_{\bar{\mathbf{y}}=(\mathbf{y},z)\in \bar{S}_{[0,t]}(y,z)} \max_{\theta\in[0,t]} \widetilde{\Psi_2}(\mathbf{y}(\theta), \zeta(\theta))$$

where $\zeta(t) := z - \int_0^t r\Phi(\mathbf{y}(s))^q ds$ and $\widetilde{\Psi_2}(y,z) := \Big((-z) \vee g(y) \vee (z - T\overline{M}) \Big) \wedge \widetilde{c}$. Then, the level sets of the value function $\widetilde{w_2}$ describe the epigraph of ϑ_1, i.e,

$$\vartheta_2(t,y) = \inf\Big\{ z \in]T\underline{m} - \widetilde{c}, T\overline{M} + \widetilde{c}[\mid \widetilde{w_2}(t,y,z) \le 0 \Big\}$$

In addition, the function $\widetilde{w_2}$ is the unique continuous viscosity solution of the following HJ equation. More precisely, define the Hamiltonian :

$$\mathcal{H}(y,p_1,p_2) := \sup_{u\in U}\big(- f(y,u).p_1 + \Phi_2(y).p_2\big)$$

$$\min\Big(\partial_t\widetilde{w_2}(t,y,z)+\mathcal{H}(y,\nabla_y\widetilde{w_2},\partial_z\widetilde{w_2}),\widetilde{w_2}(t,y,z)-g(y)\wedge\widetilde{c}\Big)=0,$$

$$\text{in }\ [0,T]\times\overset{\circ}{\widetilde{\mathcal{K}}}\times]T\underline{m}-\widetilde{c},T\overline{M}+\widetilde{c}[,\tag{6.13a}$$

$$\widetilde{w_2}(0,y,z)=\Big((-z)\vee g(y)\Big)\wedge\widetilde{c},\qquad\text{in }\ \overset{\circ}{\widetilde{\mathcal{K}}}\times]T\underline{m}-\widetilde{c},T\overline{M}+\widetilde{c}[,\tag{6.13b}$$

$$\widetilde{w}_2(t,y,z)=\widetilde{c},\quad\forall\, t\in[0,T],\ y\notin\overset{\circ}{\widetilde{\mathcal{K}}}\ \text{ and }\ \ z\in]T\underline{m}-\widetilde{c},T\overline{M}+\widetilde{c}[,\tag{6.13c}$$

$$w_2(t,y,T\underline{m}-\widetilde{c})=\widetilde{c},\quad\forall t\in[0,T],\quad y\in\overset{\circ}{\widetilde{\mathcal{K}}},\tag{6.13d}$$

$$w_2(t,y,T\overline{M}+\widetilde{c})=\widetilde{c},\quad\forall t\in[0,T],\quad y\in\overset{\circ}{\widetilde{\mathcal{K}}}\,.\tag{6.13e}$$

Now, it remains to solve numerically these problems and in particular to focus on the numerical solutions of (6.12) and (6.13).

6.5 Numerical Schemes

In this section, we are interested in the numerical solution of the Hamilton Jacobi equation (5.14) and (5.32). The most known methods of approximation of the solution of PDE are the finite difference method (see [47, 97]) and Semi lagrangian method (see [55, 58]).

6.5.1 Finite Difference scheme

The Finite difference schemes are based on interpolations of discrete data using polynomials or other simple functions. Let us propose a FD scheme to approximate the solution of the equation (5.14). Define the following grid,

$$\mathcal{G}:=\big\{(n\Delta t,I\Delta y,j\Delta z),\quad n\in\mathbb{Z},I\in\mathbb{Z}^d,\ j\in\mathbb{Z}\big\},$$

where $\Delta y,\Delta z,\Delta t>0$. Let $W^n_{I,j}$ be the approximation of the solution w_1 at the node (t_n,y_I,z_j).

Given a numerical Hamiltonian $\mathcal{H}:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ (consistent with the Hamiltonian H) , the following scheme based on a Runge-Kutta method of first order for time discretization,

$$\begin{cases}W^{n+1}_{I,j}=\max\Big(W^n_{I,j}+\Delta t\mathcal{H}(y_I,D^+W^n(y_I,z_j),D^-W^n(y_I,z_j)),\varphi_{I,j}\Big)\\ W^N_{I,j}=\varphi_{I,j},\end{cases}$$

where $\varphi_{I,j}$ is the appoximation of the initial data of the control problem at the node (y_I,z_i) and the discrete space gradient of the function W^n at the point (y_I,z_j),

$$D^{\pm}W^n(y_I,z_j)=(D^{\pm}_{y_1}W^n(y_I,z_j),..,D^{\pm}_{y_d}W^n(y_I,z_j)),$$

where we have used the ENO scheme of second order to approximate the derivatives $D^{\pm}_{y_i}W$.
Note that to the Essentially Non-Oscillatory ENO scheme introduced by [66] allows to obtain uniformly second order accurate non-oscillatory interpolations.

Remark 6.5.1 *If the numerical Hamiltonian is Lipschitz continuous on all its arguments and monotone (i.e $\frac{\partial \mathcal{H}}{\partial P_i^+}(y_I, P^+, P^-) \leq 0, \quad \frac{\partial \mathcal{H}}{\partial P_i^-}(y_I, P^+, P^-) \geq 0.$) and if the following CFL condition is satisfied*

$$\frac{\Delta t}{\Delta y}\sum_{i=1}^{d}\left\{\left|\frac{\partial \mathcal{H}}{\partial P_i^+}(y_I, P^+, P^-)\right| + \left|\frac{\partial \mathcal{H}}{\partial P_i^-}(y_I, P^+, P^-)\right|\right\} \leq 1,$$

Then, the scheme is consistent and it converges to the continuous solutions. Moreover, the discrete continuous error estimate takes the following form :

$$\sup_{(t_n, z_I, y_j)\in\mathcal{G}} |w_1(t_n, y_I, z_j) - W^n_{I,j}| \leq K_T \Delta t^{\frac{1}{2}}$$

where K_T is depending only on T (see [28] for more details).

6.5.2 Semi Lagrangian scheme

Another famous method is the Semi Lagrangian and it was first introduced in [43]. The Semi Lagrangian was first used for stationary Hamilton Jacobi Bellman equations related to optimal control problems [53, 54, 57]. We refer to [59] for evolutive problems. A Semi-Lagrangian scheme is obtained by discretizing in time the dynamic programming principle.

Let us consider the Semi Lagrangian scheme that approximates the solution of the equation (5.14). We have the following approximation $-\nabla_y W(t_n, x_I, z_j).f(x_I, a) =$

$$-\frac{W\big(t_n, y_I + f(y_I, a)\Delta t, z_j\big) - W(t_n, y_I, z_j)}{\Delta t} + O(\Delta t) \tag{6.14}$$

Replacing (6.14) in (5.14), we get the following Semi Lagrangian scheme,

$$\begin{cases} W^{n+1}_{I,j} = \min_{a\in U}\Big(W\big(t_n, y_I + f(y_I, a)\Delta t, z_j\big)\Big) \bigvee \varphi_{I,j} \\ W^N_{I,j} = \varphi_{I,j} \end{cases}$$

where $\varphi_{I,j}$ is the appoximation of the initial data of the control problem at the node (y_I, z_j). The value of V on the right-hand side is calculated by an interpolation procedure based on the values on the nodes of $\mathcal{G}$.

The Semi Lagrangian scheme is independent from any CFL condition and the discrete continuous error is the same as that given by the FD scheme. Later we will compare the numerical convergence of the Semi Lagrangian scheme with the Finite Difference scheme.

6.5.3 Analytical forms of the Hamiltonian

In this subsection, we give the analytical forms of the Hamiltonian for models M1. and M2.

Model M1

Since the control is acting only in the third and forth dimension, one can have an explicit form of the Hamiltonian using the definition of the hamiltonian given by :

$$H(y,P) := \max_{(\alpha,\beta)\in U} -f(y,\alpha,\beta).P. \tag{6.15}$$

It follows from the dynamics that,

$$\begin{aligned}
-f(y,\alpha,\beta).P &= -\frac{\beta\mathcal{T}}{m}\cos(\alpha+\delta)P_3 - \frac{\beta\mathcal{T}}{mV}\sin(\alpha+\delta)P_4 + C_1(y,P_1,P_2)\\
&= -\frac{\beta\mathcal{T}}{m}\Big(\cos(\alpha+\delta)P_3 + \frac{P_4}{V}\sin(\alpha+\delta)\Big) + C_1(y,P_1,P_2)\\
&= -\frac{\beta\mathcal{T}}{m}\Big(\sqrt{P_3^2+\frac{P_4^2}{V^2}}\cos(\alpha+\delta+\theta)\Big) + C_1(y,P_1,P_2).
\end{aligned}$$

where $\theta = \arctan\big(-\frac{P_4}{P_3 V}\big)$ (P_i are the components of the vector P) and $C_1(y,P_1,P_2)$ is the part of the Hamiltonian independing form the controls. Then, the value of the controls (α^*,β^*) which maximizes $H(\alpha,\beta,\,x,P)$ is minimizing $\beta\cos(\alpha+\delta+\theta)$.

Assume $\alpha \in [\alpha_{\min},\alpha_{\max}] \in [0,\pi]$ and $\beta \in [\beta_{\min},\beta_{\max}] \in \mathbb{R}^+$. The value α^* is depending on θ, P_3 and P_4 :

1- For $P_4 = 0$ and $P_3 = 0$, any control can be taken for the maximum in (6.15).

2- For $P_4 \neq 0$ or $P_3 \neq 0$, we have four cases,

$$\begin{cases}
\alpha^* = \alpha_{\min}, & \theta \in [-\pi, -\big(\frac{\alpha_{\max}+\alpha_{\min}}{2}+\delta\big)],\\
\alpha^* = \alpha_{\max}, & \theta \in [-\big(\frac{\alpha_{\max}+\alpha_{\min}}{2}+\delta\big), \pi-\delta-\alpha_{\max}],\\
\alpha^* = \pi-\delta-\theta, & \theta \in [\pi-\delta-\alpha_{\max}, \pi-\delta-\alpha_{\min}],\\
\alpha^* = \alpha_{\min}, & \theta \in [\pi-\delta-\alpha_{\min}, \pi].
\end{cases}$$

On the other hand, it is obvious that the value of the control β^* is taking this form,

$$\begin{cases}
\beta^* = \beta_{\min}, & \cos(\alpha+\theta+\delta) > 0,\\
\beta^* = \beta_{\max}, & \text{otherwise}.
\end{cases}$$

Model M2

Taking into account that the control appears only in the fifth component of the dynamical system f, one can have the following Hamiltonian expression :

$$H(x,P) := \sum_{i=1}^{4}\big(-f_i P_i\big) - \big(P_5 u_{\max} \vee P_5 u_{\min}\big).$$

where $\{f_i\}_{1\le i\le 5}$ denote the five components of (6.10) and P_i is the components of P.

Remark 6.5.2 *The value of the optimal control obtained above is an approximation of the optimal control of a numerical Hamiltonian of a basic Finite Difference scheme taking this form,*

$$\mathcal{H}_{FD}(x, P^+, P^-) := \max_{(\alpha,\beta)\in U} \left(\sum_{i=1}^{5} \max(0, f_i(x,\alpha,\beta))P_i^- + \min(0, f_i(x,\alpha,\beta))P_i^+ \right)$$

where $\{f_i\}_i$ *are the components of* (6.8) *or* (6.10).

6.5.4 Numerical data

The model data of a Boeing B 727 aircraft is considered. The wind velocity components relative to the windshear model are satisfying the following relations :

$$w_x(x) = kA(x), \quad w_y(x,h) = k\frac{h}{h_*}B(x), \tag{6.16}$$

where $A(x)$ and $B(x)$ are functions depending only on the x axis given by,

$$A(x) = \begin{cases} -50 + ax^3 + bx^4, & 0 \le x \le 500, \\ \frac{1}{40}(x - 2300) & 500 \le x \le 4100, \\ 50 - a(4600 - x)^3 - b(4600 - x)^4, & 4100 \le x \le 4600, \\ 50, & 4600 \le x, \end{cases}$$

$$B(x) = \begin{cases} dx^3 + ex^4, & 0 \le x \le 500, \\ -51\exp(-c(x - 2300)^4), & 500 \le x \le 4100, \\ d(4600 - x)^3 + e(4600 - x)^4, & 4100 \le x \le 4600, \\ 0, & 4600 \le x, \end{cases} \tag{6.17}$$

where all the constants appearing in the relations of the forces and the wind are given in tables 6.1.

TABLE 6.1 – Boeing 727 aircraft model data.

Eqs (6.8)-(6.2)		Eqs (6.18)	
$\rho =$	$2.203\ e^{-3}$ Ib sec^2 ft^{-4}	$A_0 =$	$4.456\ e^4$ Ib
$S =$	$1.56\ e^3$ ft^2	$A_1 =$	-23.98 Ib sec ft^{-1}
$g =$	32.172 ft sec^{-2}	$A_2 =$	$1.442e^{-2}$ Ib sec^{-2} ft^{-2}
$mg =$	$1.5\ e^5$ Ib		
$\delta =$	$3.49\ e^{-2}$ rad		

Eqs (6.20)		Eqs (6.16)-(6.17)	
$B_0 =$	0.1552	$k \in$	$[0,1]$
$B_1 =$	0.12369 rad^{-1}	$h_* =$	1000 ft
$B_2 =$	2.4203 rad^{-2}	$a =$	$6\ e^{-8}$ sec^{-1} ft^{-2}
$C_0 =$	0.7125	$b =$	$-4\ e^{-11}$ sec^{-1} ft^{-3}
$C_1 =$	6.0877 rad^{-1}	$c =$	$-ln(25/30.6)\ e^{-12}$ ft^{-4}
$C_2 =$	-9.0277 rad^{-2}	$d =$	$-8.02881\ e^{-8}$ sec^{-1} ft^{-2}
$\alpha_* =$	0.2094 rad	$e =$	$6.28083\ e^{-11}$ sec^{-1} ft^{-3}

Depending on the model of the dynamics, the thrust, the drag and the lift forces are taking differents forms.

M1-model

The thrust force is supposed to have a polynomial dependance on the velocity and linear dependence on the power setting β,

$$F_T := F_T(\beta, V) = \beta(A_0 + A_1\ V + A_2\ V^2). \tag{6.18}$$

The drag and lift forces are the components of the aerodynamic force and they take the following forms :

$$F_L := F_L(V) = \frac{1}{2}\rho V^2 S c_l,$$
$$F_D := F_D(V) = \frac{1}{2}\rho V^2 S c_d,$$

where $c_l = C_0 + C_1\alpha_*$ and $c_d = B_0 + B_1\alpha_*$ are constansts.

The controls belong to the following interval :

$$\alpha(.) \in \big[5\,\text{deg}, 17.2\,\text{deg}\big], \quad \beta(.) \in \big[0, 1\big].$$

Define the set state constraints $\mathcal{K}$,

$$\mathcal{K} = \big[x_{\min}, x_{\max}\big] \times \big[h_{\min}, h_{\max}\big] \times \big[v_{\min}, v_{\max}\big] \times \big[\gamma_{\min}, \gamma_{\max}\big]. \tag{6.19}$$

where the constants $x_{\min}$, $x_{\max}$, $h_{\min}$, $h_{\max}$, $v_{\min}$, $v_{\max}$, $\gamma_{\min}$ and $\gamma_{\max}$ are given in table 6.2.

M2-model

The thrust force is supposed to have a polynomial dependance on the velocity,

$$F_T := F_T(V) = A_0 + A_1\ V + A_2\ V^2.$$

The drag and lift forces take the same relations. The drag coefficient has a polynomial dependence on α and the lift coefficient is linearly depending on α until a swiching point where the dependence becomes polynomial, i.e :

$$c_d = B_0 + B_1\alpha + B_2\alpha^2, \quad c_l(\alpha) = \begin{cases} C_0 + C_1\alpha, & \alpha \leq \alpha_*, \\ C_0 + C_1\alpha + C_2(\alpha - \alpha_*)^2 & \alpha_* \leq \alpha \leq \alpha_{\max}, \end{cases} \tag{6.20}$$

Table 6.2 – Coefficients defining the set of constraints

Eq (6.19) and Eq (6.21)	
$x_{\min} = -500.0$	$x_{\max} = 9500.0$
$h_{\min} = 500.0$	$h_{\max} = 1000$
$v_{\min} = 170.0$	$v_{\max} = 260.0$
$\gamma_{\min} = -4.0\text{deg}$	$\gamma_{\max} = 14.0\,\text{deg}$
$\alpha_{\min} = 0.0$	$\alpha_{\max} = 17.2\,\text{deg}$

The control belongs to the following interval $\omega(.) \in \big[-3.0\,\text{deg}, 3.0\,\text{deg}\big]$ and the set of state constraints $\mathcal{K}$ is given by

$$\mathcal{K} = \big[x_{\min}, x_{\max}\big] \times \big[h_{\min}, h_{\max}\big] \times \big[v_{\min}, v_{\max}\big] \times \big[\gamma_{\min}, \gamma_{\max}\big] \times \big[\alpha_{\min}, \alpha_{\max}\big] \quad (6.21)$$

where the constants are given in table 6.2.

The computational domain $\tilde{\mathcal{K}} := \mathcal{K} + \tilde{c}B$ (where B is the unit ball centered in the origin) will be ajusted according to the set $\mathcal{K}$.

Remark 6.5.3 *The following change of variable is made in order to obtain an homogeneous grid,*

$$\mathbf{Y}(.) := \phi(\mathbf{X}(.)) := \frac{\mathbf{X}(\cdot) - \mathbf{X}_{\min}(\cdot)}{\mathbf{X}_{\max}(\cdot) - \mathbf{X}_{\min}(\cdot)}.$$

where $X_{\min}$ and $X_{\max}$ are the vectors containing respectively the lower and upper bounds on the computational domain $\tilde{\mathcal{K}}$.

6.6 Stability analysis

The aim of this subsection is to be convinced that the choices we will make of the method to approximate the solutions and the setting of the scheme are the best. In the last section, we discussed the methods often used in the literature, namely, the Finite difference and the Semi Lagrangian schemes. In the following, we study the stability issue for these two schemes corresponding to the solutions of the auxiliary control problems.

Here, these schemes are based on the Runge-Kutta method of second order for time discretization. The 5D model M2 requires a big memory and the associated CPU time is very large for a big number of the grid points. Let us emphasise that for a relevant analysis of the stability for the schemes, we can not consider a small number of points of grid points. On the hand, the 4D model M1 allows to choose a more important setting for the space steps.

For these problems, the exact solutions are not known. Nevertheless, we will calculate a reference solution for each control problem and we will study the error against this solution. More precisely we compute the following absolute L^{∞}-, L^1- and L^2-errors against the reference solution W_{ref} :

$$
\begin{aligned}
e_{L_a^{\infty}}(W(T,.)) &:= \max_{x\in\mathcal{G}} |W_{ref}(T,x) - W(T,x)|, \\
e_{L_a^{1}}(W(T,.)) &:= \left(\frac{1}{\Pi N_i}\sum_{x\in\mathcal{G}} |W_{ref}(T,x) - W(T,x)|\right), \\
e_{L_a^{2}}(W(T,.)) &:= \left(\frac{1}{\Pi N_i}\sum_{x\in\mathcal{G}} |W_{ref}(T,x) - W(T,x)|^2\right)^{\frac{1}{2}},
\end{aligned}
$$

where N_i defines the number of points per the axis i. Furthermore, for a complete view of the errors obtained, we will compute also the following relative errors,

$$
\begin{aligned}
e_{L_r^{\infty}}(W(T,.)) &:= \frac{e_{L_a^{\infty}}(W(T,.))}{||W_{ref}(T,.)||_{L_\infty}}, \\
e_{L_r^{1}}(W(T,.)) &:= \frac{e_{L_a^{1}}(W(T,.))}{||W_{ref}(T,.)||_{L_1}}, \\
e_{L_r^{2}}(W(T,.)) &:= \frac{e_{L_a^{2}}(W(T,.))}{||W_{ref}(T,.)||_{L_2}}.
\end{aligned}
$$

6.6.1 Auxiliary control problem for the L_∞ running cost associated with M.1

Let us recall first that for the case of the auxiliary control problem with maximum cost, the augmented dynamics is zero for the additional variable z. Let z to be a constant ($N_z = 1$ for the number of points per the axis z).

For each method of approximation, we consider a reference solution. The reference solution is obtained with $N_{ref} = 160^4$ grid points and $CFL = 0.5$ for the Finite Difference scheme. The analytical form of the Hamiltonian allows to compute directly the approximated solution using the FD scheme without discretization of the control.

On the other hand, the Semi Lagrangian scheme needs to discretize the control. We have used $N_c = 10^2$ discrete points for the control values. The time step is $\Delta t_{ref} = 0.1$.

We have denoted Δt the time step and $N_{\mathcal{G}}$ the total number of grid points. In table 6.3, the errors and the CPU times are given for the Finite Difference scheme associated with the maximum cost problem at time $t = T$. From this table, we observe roughly a convergence of first order. Note that the computations are performed using 30 threads.

TABLE 6.3 – Error table for FD scheme of the control problem with maximum cost associated with the model M1

$N_{\mathcal{G}}$	Δt	error L^2		error L^1		error L^∞		CPU time (s)
		$e_{L^2_a}$	$e_{L^2_r}$	$e_{L^1_a}$	$e_{L^1_r}$	$e_{L^\infty_a}$	$e_{L^\infty_r}$	
20^4	1.69 e-2	3.85 e-2	7.42 e-2	1.68 e-2	3.02 e-2	3.12 e-1	6.32 e-1	2.16 e+2
40^4	8.47 e-3	1.97 e-2	4.02 e-2	7.80 e-3	1.57 e-2	2.02 e-1	4.03 e-1	7.31 e+2
80^4	4.24 e-3	7.40 e-3	1.56 e-2	2.63 e-3	5.59 e-3	8.01 e-2	1.16 e-1	8.97 e+3
$40^3 \times 20$	1.38 e-2	2.15 e-2	4.34 e-2	8.41 e-3	1.64 e-2	2.23 e-1	4.47 e-1	4.53 e+2

The next table 6.4 shows the absolute and the relative errors associated with the Semi Lagrangian scheme against the reference solution at time $t = T$. The number of discrete points for the control values ($N_c = 10^2$) remains unchanged. From table 6.4, we observe a convergence of first order. The convergence of the FD scheme is slightly better than that obtained with SL scheme. Note that the fact that CPU times for SL scheme are lower than those obtained for the FD scheme is due to the time step which is considered very lower than the time step for the FD scheme.

TABLE 6.4 – Error table for SL scheme of the control problem with maximum cost associated with the model M1

$N_{\mathcal{G}}$	Δt	error L^2		error L^1		error L^∞		CPU time (s)
		$e_{L^2_a}$	$e_{L^2_r}$	$e_{L^1_a}$	$e_{L^1_r}$	$e_{L^\infty_a}$	$e_{L^\infty_r}$	
20^4	8.00 e-1	5.53 e-2	1.04 e-1	2.36 e-2	4.32 e-2	4.47 e-1	8.94 e-1	9.71
40^4	4.00 e-1	2.91 e-2	6.01 e-2	1.24 e-2	2.52 e-2	2.59 e-1	5.19 e-1	2.67 e+2
80^4	2.00 e-1	1.18 e-2	2.51 e-2	4.78 e-3	1.03 e-2	1.08 e-1	2.16 e-1	7.72 e+3
$40^3 \times 20$	4.00 e-1	2.86 e-2	5.83 e-2	1.23 e-2	2.43 e-2	2.83 e-1	5.67 e-1	1.38 e+2

Let us mention that the errors of both schemes for the setting $N_{\mathcal{G}} = 40^3 \times 20$ are very close to the errors given by $N_{\mathcal{G}} = 40^4$. This can be justified by the fact that the inclination angle belongs to a small interval. This setting will be used in the following section to reconstruct trajectories.

In order to determine the sentivity of the errors for refined grid of the interval of controls, we report in the table 6.5 the errors and the CPU times for different numbers of points per control against the reference solution obtained with $N_{\mathcal{G}} = 40^3 \times 20$ grid points and $N_c = 80^2$ discrete points for the control values. From this table, one can see that the solutions are not improved when we increase the number of points per control. Therefore, a refined discretization of the control intervals is not relevant. On the other hand, the FD scheme doesn't need to discretize the control since the solution is directly computed using the analytical form of the Hamiltonian.

TABLE 6.5 – Error table for SL scheme of the control problem with maximum cost associated with M1 using the reference solution obtained for $N_{\mathcal{G}} = 40^4 \times 20$, $Dt = 0.4$ and $N_c = 80^2$.

N_c	error L^2		error L^1		error L^∞		CPU time (s)
	$e_{L^2_a}$	$e_{L^2_r}$	$e_{L^1_a}$	$e_{L^1_r}$	$e_{L^\infty_a}$	$e_{L^\infty_r}$	
10^2	4.14 e-5	8.55 e-5	1.49 e-5	3.01 e-5	1.41 e-3	2.83 e-3	1.38 e+2
20^2	2.23 e-5	4.59 e-5	7.56 e-6	1.52 e-5	4.49 e-4	8.98 e-4	7.09 e+2
40^2	6.87 e-6	1.41 e-5	2.24 e-6	4.53 e-6	1.80 e-4	3.61 e-4	2.69 e+3

6.6.2 Auxiliary control problem for the Bolza cost associated with M.1

Let us also check the behaviour of the errors for the control problem with Bolza cost. Let the solution obtained with $N_{ref} = 80^4 \times 10$ grid points (where $N = 80$ is the number of points per the first forth components of the state and $N_z = 10$ for the additional variable z) and $CFL = 0.5$ be a reference solution at time $t = T$ for the Finite Difference scheme. For the Semi Lagrangian scheme, we take the same number of grid points $N_{ref} = 80^4 \times 10$, and we consider $N_c = 10^2$ for the reference solution at time $t = T$ with the time step $\Delta t_{ref} = 0.1$.

The next table 6.6 shows the absolute and the relative errors corresponding to the FD scheme of the control problem associated with the Bolza cost against the reference solution at time $t = T$. The number of points per the variable z ($N_z = 10$) remains unchanged.

TABLE 6.6 – Error table for FD scheme of the control problem with Bolza cost associated with the model M1

$N_{\mathcal{G}}$	Δt	error L^2		error L^1		error L^∞		CPU time (s)
		$e_{L^2_a}$	$e_{L^2_r}$	$e_{L^1_a}$	$e_{L^1_r}$	$e_{L^\infty_a}$	$e_{L^\infty_r}$	
10^5	3.06 e-2	9.79 e-2	1.86 e-1	8.02 e-2	1.32 e-1	3.20 e-1	6.40 e-1	2.73 e+1
$20^4 \times 10$	1.53 e-2	5.36 e-2	1.14 e-1	3.61 e-2	7.47 e-2	2.40 e-1	4.81 e-1	3.16 e+2
$40^4 \times 10$	7.71 e-3	2.26 e-2	5.14 e-2	1.29 e-2	2.99 e-2	1.30 e-1	2.60 e-1	6.12 e+3
$40^3 \times 20 \times 10$	1.25 e-1	3.09 e-2	6.91 e-2	1.77 e-2	3.99 e-2	2.11 e-1	4.23 e-1	2.05 e+3

We report in the next table 6.7 as we did for the maximum cost problem, the errors and CPU times associated with the Semi Lagrangian scheme of the control problem with Bolza cost. From table 6.7, we observe a convergence of first order. Again, one can observe that the convergence of the FD scheme is better than that obtained with SL scheme.

TABLE 6.7 – Error table for SL scheme of the control problem with Bolza cost associated with the model M1

$N_{\mathcal{G}}$	Δt	error L^2		error L^1		error L^∞		CPU time (s)
		$e_{L^2_a}$	$e_{L^2_r}$	$e_{L^1_a}$	$e_{L^1_r}$	$e_{L^\infty_a}$	$e_{L^\infty_r}$	
10^5	8.00 e-1	1.48 e-1	2.81 e-1	1.19 e-1	1.98 e-1	4.89 e-1	9.79 e-1	1.81 e+1
$20^4 \times 10$	4.00 e-1	8.12 e-2	1.70 e-1	5.54 e-2	1.19 e-1	2.92 e-1	5.84 e-1	3.91 e+2
$40^4 \times 10$	2.00 e-1	3.97 e-2	9.09 e-2	2.28 e-2	5.24 e-2	1.59 e-1	3.10 e-1	8.44 e+3
$40^3 \times 20 \times 10$	4.00 e-1	4.59 e-2	1.03 e-1	2.66 e-2	6.01 e-2	2.23 e-1	4.47 e-1	5.56 e+3

In order to see how the errors behave for refined grid of the interval of controls, we consider the reference solution obtained with $N_{\mathcal{G}} = 20^4 \times 10$ grid points and $N_c = 40^2$ discrete points for the control values. From this table 6.8, one can see again that the errors of the solutions given by the SL scheme for $N_c = 40^2$ are not very important if we compare with that obtained with $N_c = 10^2$.

TABLE 6.8 – Error table for SL scheme of the control problem with maximum cost associated with M1 using the reference solution obtained for $N_{\mathcal{G}} = 20^4 \times 10$, $Dt = 0.4$ and $N_c = 40^2$.

N_c	error L^2		error L^1		error L^∞		CPU time (s)
	$e_{L^2_a}$	$e_{L^2_r}$	$e_{L^1_a}$	$e_{L^1_r}$	$e_{L^\infty_a}$	$e_{L^\infty_r}$	
10^2	7.11 e-4	1.35 e-3	3.72 e-4	6.45 e-4	2.70 e-2	5.41 e-2	3.91 e+2
20^2	2.02 e-4	3.86 e-4	1.08 e-4	1.88 e-4	1.10 e-2	2.19 e-2	1.51 e+3

In this section, we did a study of the stability of the FD and SL schemes for both control problems discussed in the theoretical part of this chapter using the model M1. This provides some information of the advantages and disadvantages of the each scheme. In the following section, we will reconstruct the optimal trajectories using some algorithms of reconstruction of trajectories for the control problems with both schemes.

6.7 Analysis of the simplified model M1

This section is devoted to the analysis of the model M1. This model is based on some approximations of the aerodynamic forces and it allows to verify some aspects of the reconstruction of trajectories. Indeed, we will focus on the reconstruction by minimizing an approximate solution of the value function and the simulations are supposed to reinforce our choice of the method of appoximation. Then, we will justify the choice we will make of the model to be used later for differents methods of reconstruction of trajectories.

The study of the stability provided some tools to compare the FD and SL schemes. Indeed, the FD scheme seems to have significant convergence to the reference solution and the solutions are computed using an analytical form of the Hamiltonian. On the other hand, the solution given by the SL scheme can be performed using a relatively greater time step (there is no restriction or CFL condition to be respected while choosing the time step). Unfortunately, the solution is not improved for a refined grid of the control intervals and this doesn't encourage the choice of the SL scheme. Here, we will compare again these two schemes in order to justify our choice of the approximation method.

Let us we compare the trajectories given by the FD and the SL schemes starting from differents altitudes using a moderate wind coefficient $k = 0.6$ in order to see which method will be used after. Let $CFL = 0.5$ for the FD scheme and $Dt = 0.4$ for the SL scheme. Consider the following initial points :

$$\begin{aligned} y_0 &= (0.0, 600.0, 239.7, -2.249°) \\ y_1 &= (0.0, 650.0, 239.7, -2.249°) \\ y_2 &= (0.0, 700.0, 239.7, -2.249°) \\ y_3 &= (0.0, 750.0, 239.7, -2.249°) \end{aligned}$$

Throughout this section, let $N_{\mathcal{G}} = 40^3 \times 20$ be the grid points (where 40 is the number of points per axis for the first three components, namely, x, h and v and 20 is the number of points per the axis of the inclination angle) and $N_c = 50 \times 10$ be the number of discrete points for the control values. In the table 6.9, we report the value of the optimality criterion J_1 at time T for each scheme starting from the initial points y_i $(i = \{1, .., 4\})$. Note that the value of the optimality criterion is computed using the following :

$$\inf \left\{ z \;\middle|\widetilde{W_1}(T, y, z) \leq \eta \right\},$$

where $\widetilde{W_1}$ is the scaled approximated solution and $\eta = 0.02$ is expected to be a small error due to the interpolation of the scheme. From this table, one can observe that the SL scheme is suboptimal compared to the FD scheme. Indeed, the lower altitude for the FD scheme is slightly greater than the one obtained by the SL method.

TABLE 6.9 – The value of the optimality criterion deduced from the value function at time $t = T$ for each scheme starting from different initial points.

Scheme	$J_1^*(T, y_0)$	$J_1^*(T, y_1)$	$J_1^*(T, y_2)$	$J_1^*(T, y_3)$
FD	4.01 e+2	3.80 e+2	3.72 e+2	3.47 e+2
SL	4.03 e+2	3.96 e+2	3.85 e+2	3.65 e+2

The figure 6.3 presents the history of state for the control problem with maximum-cost associated with the model M1 starting from the initial point y_3 and using the SL and FD schemes .

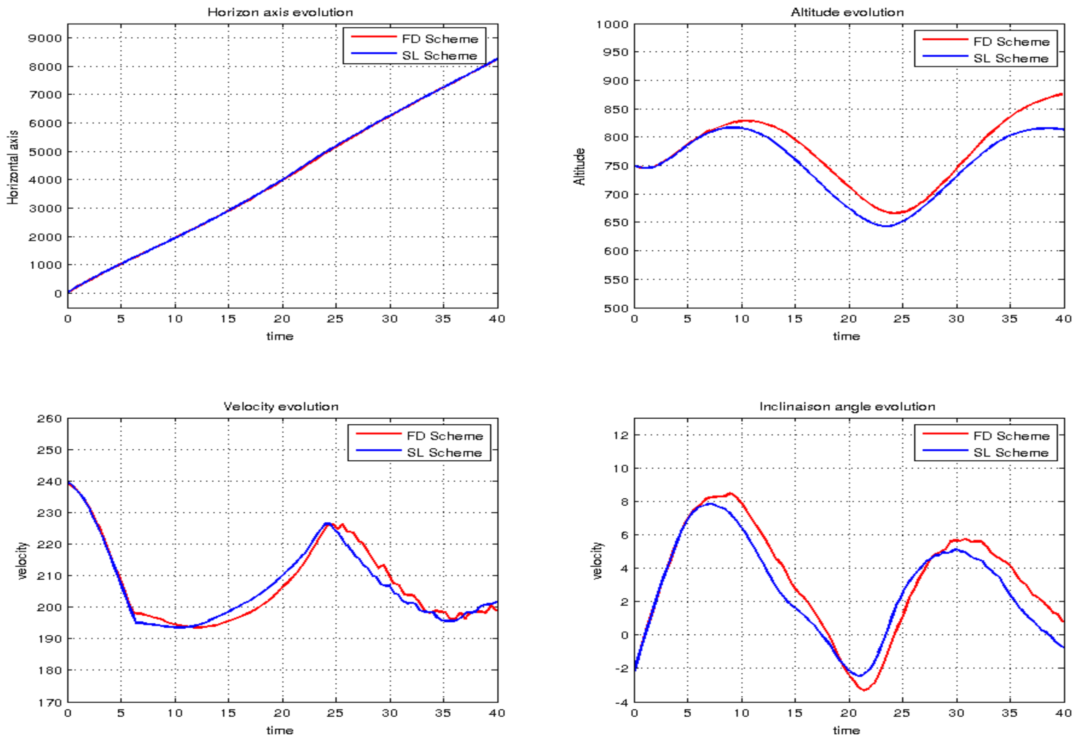

FIGURE 6.3 – History of the state components for the maximum-cost problem with model M1 using SL and FD schemes.

Note that the value of the optimality criterion is the difference between the reference altitude and the minimal altitude over the time interval. Then, from the reconstruction of the altitude over the time interval, one can compute the value of the optimality criterion. Table 6.10 gives the value of the optimality criterion deduced from the reconstruction of the altitude. We observe that these values are not the same values reported in the Table 6.9. Indeed, we expected that we will have this result because of the interpolation errors of the scheme.

TABLE 6.10 – The value of the optimality criterion at time $t = T$ obtained from the reconstruction of trajectories for each scheme starting from different initial points.

Scheme	$J_1^*(T, y_0)$	$J_1^*(T, y_1)$	$J_1^*(T, y_2)$	$J_1^*(T, y_3)$
FD	4.04 e+2	3.63 e+2	3.51 e+2	3.34 e+2
SL	4.07 e+2	3.80 e+2	3.60 e+2	3.56 e+2

Now, let us make a choice for the model to use in order to compare different algorithms of reconstruction of optimal trajectories. First, let us recall that the 4D model M1 is based on some approximations of the aerodynamic forces and the coefficient of the wind $k \in [0, 1]$ is an important parameter to take into account. The wind forces increase with respect to the parameter k. In Table 6.11, we observe that these approximations are not suitable with a coefficient k greater than 0.8 starting from y_1 (see also Figure 6.4 for the reconstruction of the altitude starting from y_1).

TABLE 6.11 – The optimality criterion at time $t = T$ obtained with FD scheme and starting from $y_0 = (0.0, 650.0, 239.7, -2.249.0\,\text{deg})$.

k	$J_1^*(T, y_1)$	Minimal altitude
0.5	3.54 e+2	6.46 e+2
0.6	3.63 e+2	6.37 e+2
0.7	4.77 e+2	5.23 e+2
0.8	$+\infty$	-

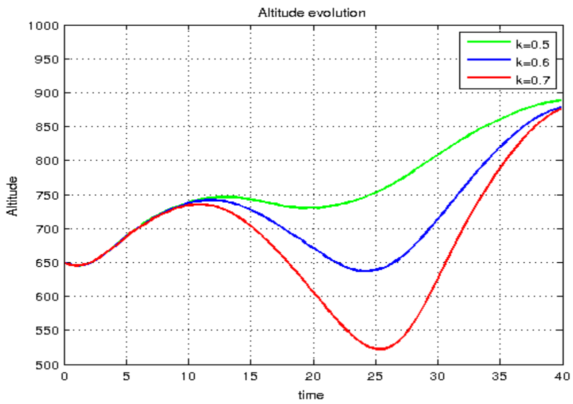

FIGURE 6.4 – History of the altitude for different coefficients of the wind starting from $y_1 = (0.0, 650.0, 239.7, -2.249.0\,\text{deg})$.

On the other hand, we know that the model M2 has already been used in [86], [85], [35], [36] and it is suitable with the data of the Boeing 727 aircraft model. For this reason, in the following, we will focus on the five dimensional model M2 and we will compare the results obtained for differents algorithm of reconstruction of optimal feedback controls.

6.8 Different methods of reconstruction of trajectories

There are several versions of the algorithm of reconstruction of trajectories starting from a numerical solution of an optimal control problem. In the previous Chapter 5, we discussed the algorithm of reconstruction by minimizing the value function and the algorithm of reconstruction involving the exit time function. Here, we analyse numerically these algorithms and we introduce other alternatives. Indeed, we compare the reconstructed optimal trajectory based on the value function and the trajectory constructed using the exit time function. Furtheremore, we introduce an algorithm based on the value function coupled with a penalization of the control variation. Such algorithm is supposed to provide a significant improvement in the quality of the optimal feedback control which can be nonsmooth due to the nonlinear structure of the problem.

Throughout this Section, we consider the following setting for the discretization of the space : we let $N_{\mathcal{G}} = 30^3 \times 20^2 \times 10$ (where 30 is the number of points per axis for the first three components, namely, x, h and v, 20 is the number of the points for the angles γ and α an 10 is the number of points for the additional variable z). Let $N_c = 50^2$ be the number of discrete points for the control values. Since the model is more realistic and the aerodynamic forces are close to the real ones, one can consider the strong wind whose coefficient is $k = 1$. Moreover, we start from an altitude close to the lower boundary of the constraints in the altitude, i.e, the initial point considered is,

$$y_0 = (0.0, 600.0, 239.7, -2.249\,\text{deg}, 7.373\,\text{deg}).$$

The procedure of reconstruction by minimizing the value function has been validated theoretically by proving the convergence of the sequence of trajectories to an optimal trajectory. For the algorithm using the exit time function, we don't have any theoretical result on the convergence but it is a good alternative in high dimension since it does not need to save the value at each iteration. In the next subsection, we want to observe numerically and compare the result with the solution given in [35]- [36].

6.8.1 Comparison with [36] using the exit time algorithm

In the Chapter 5 of this thesis, we established the link between the optimal trajectories corresponding to the controls problems and those constructed from an exit time function associated with a viability kernel set. We also gave a formal algorithm of the reconstruction of optimal trajectories using the exit time function. Here, we want to make a comparison between the optimal trajectories corresponding to the control problems with maximum cost and with Bolza cost. We use the exit time function and we consider again the same setting of the space and control steps and we let $q = 6.0$ and $r = \frac{1}{\max_{\mathcal{K}} \Phi(y)^q}$. Figure 6.5 shows the history of the state components for both control problems.

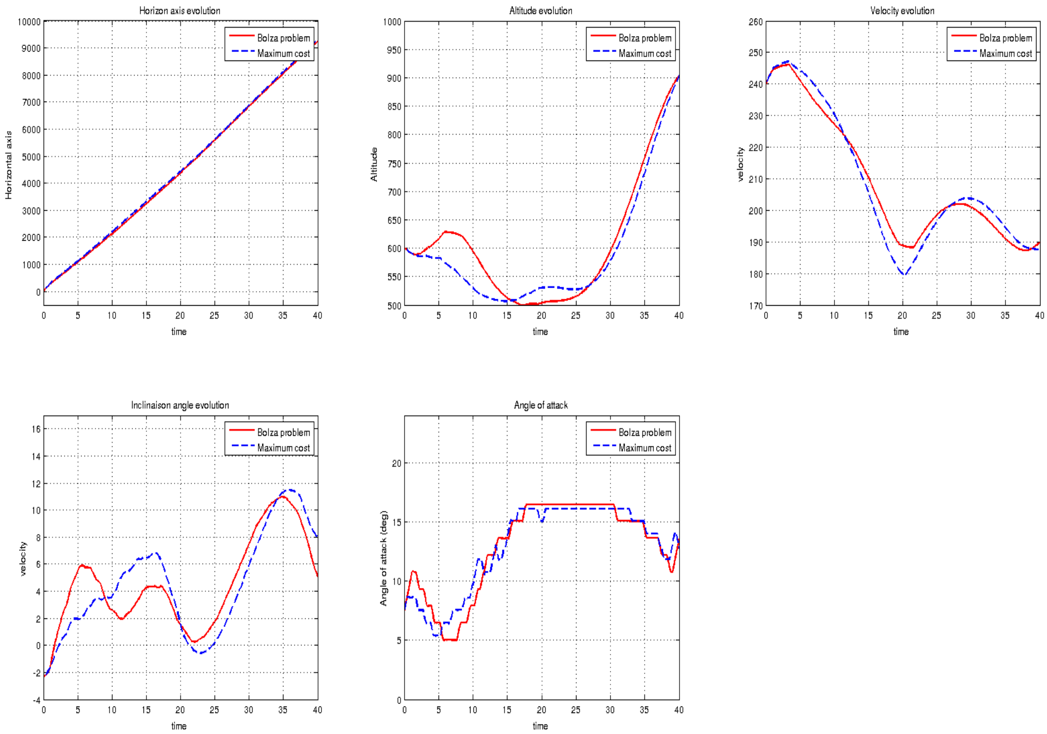

FIGURE 6.5 – History of the states for the control problems with Bolza cost and maximum cost using the exit time function.

Let us point out that the model M2 can be compared to the model used is [35]- [36]. The only difference between both models is that the thrust force depends affinely on time in [35]- [36] just for the first three seconds. Then, it takes the same formula as that used in this work. Figures 6.6-6.7 show that the result obtained here is close to that obtained in [36].

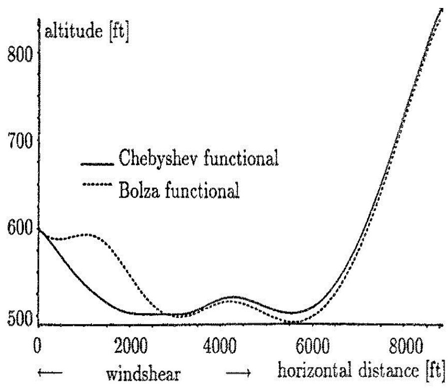

FIGURE 6.6 – Image copied from [36] : History of the altitude for both the Bolza problem and L_∞-cost problem.

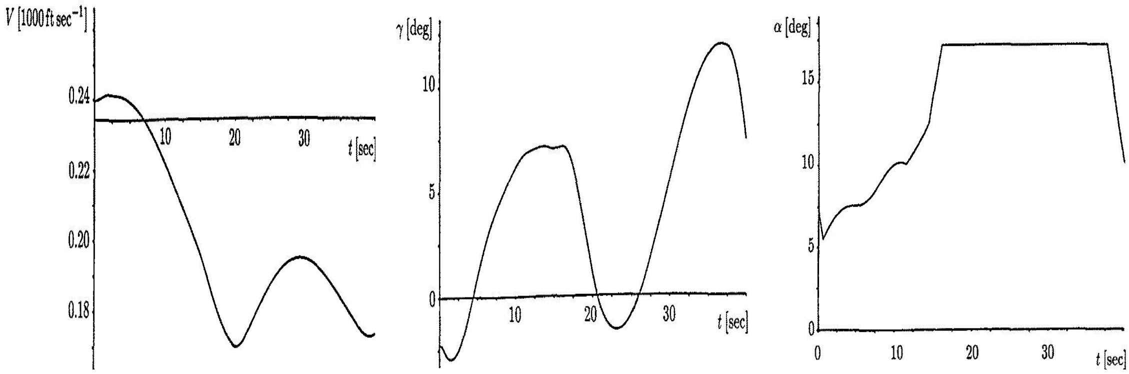

FIGURE 6.7 – Image copied from [36] : History of the velocity, the angular velocity and the angle of attack for the L_∞-cost problem.

Table 6.12 shows that the optimality criterion in our case is less than that deduced from the Figure 6.5 for both control problems.

TABLE 6.12 – The optimality criterion at time $t = T$ obtained with FD scheme compared with the Figure .

Control problem	$J_1^*(T, y_0)$	$J_2^*(T, y_0)$
Paper [36]	4.98 e+2	5.10 e+2
Exit time algorithm	4.94 e+2	5.01 e+2

Let us emphasise that there is no theoretical result related to the exit time algorithm. On the other hand, we have proved that the sequence of trajectories generated by the algorithm of the value function is convergent. In what follows, we concentrate on the control problem with maximum cost and we compare several algorithms of reconstruction involving the value function.

6.8.2 Reconstruction using the value function

The reconstruction by an approximate value function is based on a space discretized version of the algorithm of reconstruction based on the value function and it investigates if the control is minimizing the value function. This method has been analysed in the Chapter 5 and a convergence result has been proved.

Figures 6.8 and 6.9 show respectively the history of the state components and the control along the interval starting from the same initial point y_0. The results are close that obtained in the last experiences.

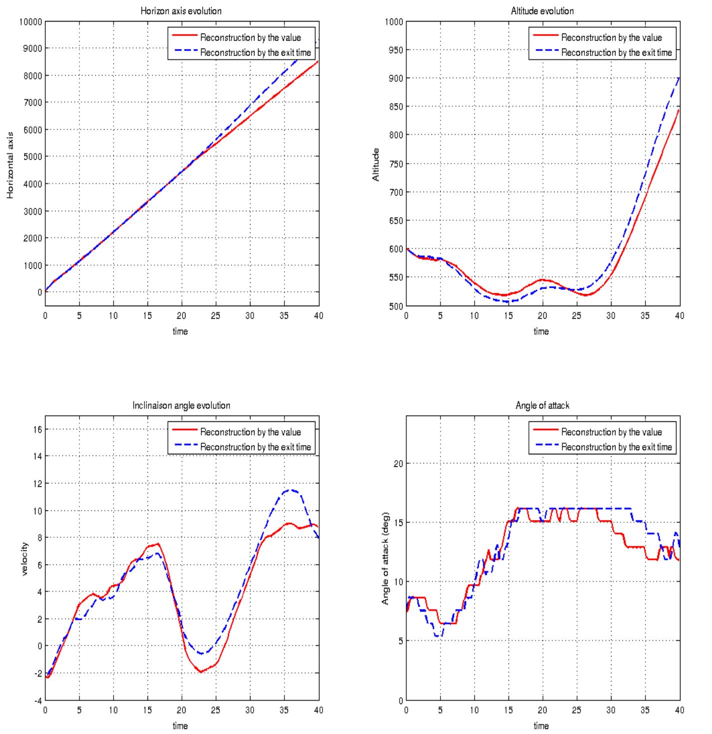

FIGURE 6.8 – History of the state components for the maximum-cost problem associated with the model M2 using the reconstruction by exit time function.

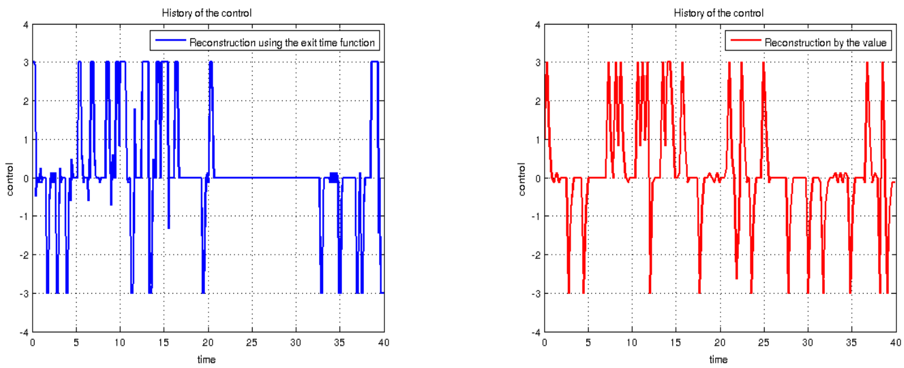

FIGURE 6.9 – History of the control for the maximum-cost problem associated with the model M2 using the reconstruction by exit time function.

Figure 6.9 show that the quality optimal control using the exit time is slightly better than the one obtained by the algorithm based on the value specially after t=20s. The same remark can be done for the evolution of the angle of attack in the Figure 6.8.

The last experiences require a huge memory and the CPU times are sometimes very big due to the matrices to be saved during the time interval. For high dimension and refined grid, it is very expensive in time and memory to consider the algorithm based on the value function. On the other hand, the algorithm of reconstruction based on the exit time function doesn't need to save at each iteration and the only matrix to be saved is at the end. The latter can then be considered as a good alternative to make the economy on memory but also in CPU time in high dimensions. In the following, we give other variants of algorithms that can be used for the reconstruction of the optimal trajectories.

6.8.3 Other variants for the algorithms of reconstruction

Reconstruction using the value and the penalization of control variation

The penalization by control variation aims to provide an improvement on the performance of the algorithm of reconstruction by the value. In particular, if many values of the control realize the minimization of the value function, then, this method allows to select the control for which the value is close to the last value of the control.

Algorithm. E For a given $n \in \mathbb{N}$, let us consider $(t_0 = 0, t_1 ... t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with the time step $h = \frac{T}{n}$. Let $\{\mathbf{y}^n(.)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$ $(i \geq 1$). Set $\mathbf{y}^n(t_0) = y$ (where $\vartheta_1(t_n, y) = z$). For a given $n \in \mathbb{N}$ let us consider $(t_0 = 0, t_1 ... t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with the time step $h = \frac{T}{n}$. Let $\{\mathbf{y}^n(.)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$ $(i \geq 1$). Set $\mathbf{y}^n(t_0) = y$ (where $\vartheta_1(t_n, y) = z$).

Step 1 Let λ be a positive constant. Knowing the state $\mathbf{y}^n(t_k)$, choose the optimal control at t_k :

$$u_k^n = \arg\min_{u \in U} \Big(w_1(t_{n-k-1}, \mathbf{y}^n(t_k) + hf(\mathbf{y}^n(t_k), u), z) \bigvee \max_{\theta \in [0,h]} \Psi_1(\mathbf{y}^n(t_k) + \theta f(\mathbf{y}^n(t_k), u), z) + \lambda |u - u_{k-1}^n| \Big).$$

Step 2 Define $\mathbf{u}(t_k) = c_k$. Then, the next point is : $\mathbf{y}^n(t_{k+1}) := \mathbf{y}^n(t_k) + hf(\mathbf{y}^n(t_k), \mathbf{u}(t_k).)$

From Figure 6.10, one can observe that the quality of the optimal feedback control with penalization of control variation $(\lambda > 0)$ is quite better than the one obtained without any regularization $(\lambda = 0)$. Moreover, the choice of $\lambda = 2.0$ is better than $\lambda = 1.0$. This affects also the quality of the angle of attack (see figure 6.11). The other state components seem to be the same.

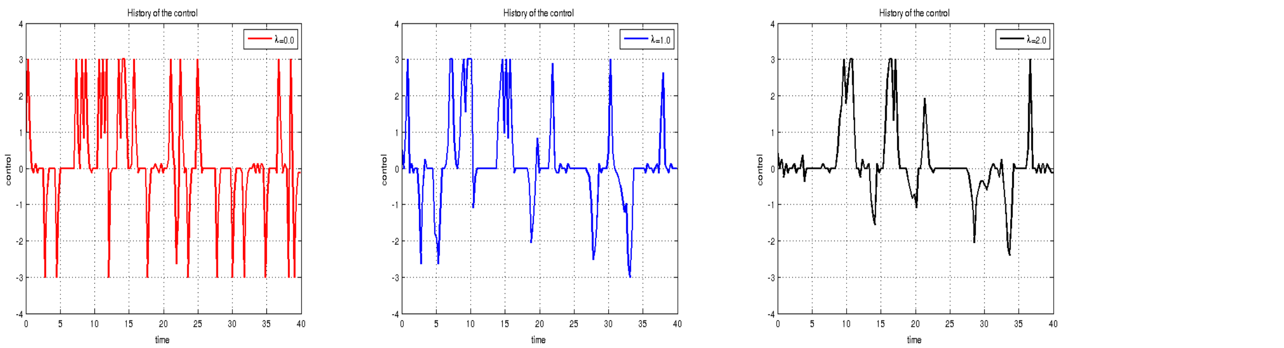

FIGURE 6.10 – Reconstruction of the optimal feedback control (speed of the angle of attack) for the control problem with maximum cost without with three values of $\lambda = 0.0,\ 1.0$ and 2.0.

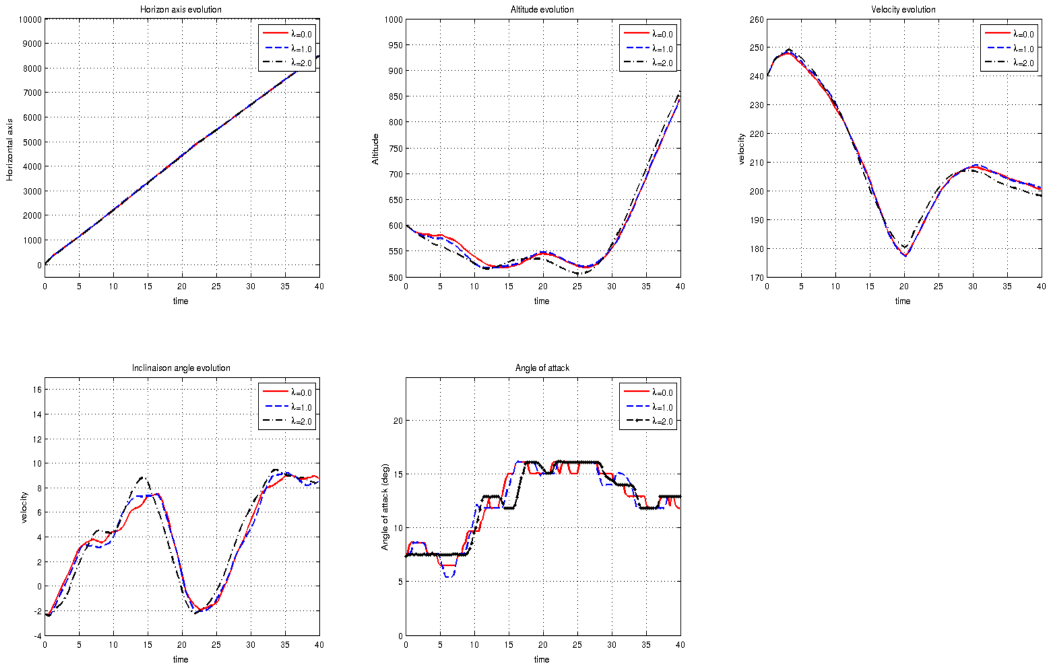

FIGURE 6.11 – Reconstruction of the state components for the control problem with maximum cost with three values of $\lambda = 0.0,\ 1.0$ and 2.0.

Besides the algorithms supposed to provide a regularization of the feedback control during the search of the optimal control, one may consider other algorithms that produce a posterior regularization of the control in order to improve the result. These algorithm are simple to implement and are based on the post treatment of the control graph.

Reconstruction using the Hamiltonian

Since the value function is the unique solution of (6.12), one can reconstruct the optimal trajectories from the Hamiltonian. Indeed, at each iteration, the control whose value minimizes the numerical Hamiltonian is the optimal one. The algorithm start by computing the approximation of the gradient of the value function and select the optimal control from the set of controls. Thanks to the analytical form of the Hamiltonian, we don't need to discretize the set of control and the only task to be achieved is to obtain the best approximation of the gradient. More precisely, the algorithm of reconstruction using the Hamiltonian takes this form,

Algorithm. F For a given $n \in \mathbb{N}$, let us consider $(t_0 = 0, t_1 ... t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with the time step $h = \frac{T}{n}$. Let $\{\mathbf{y}^n(.)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$ $(i \geq 1$). Set $\mathbf{y}^n(t_0) = y$ (where $\vartheta_1(t_n, y) = z$).

Step 1 For $1 \leq k \leq n$, let $y_k := \mathbf{y}^n(t_k)$. Calculate the space gradient of the function W at the point (t_k, y_k, z),

$$D^{\pm}W(t_k, y_k, z) = (D^{\pm}_{y_{k,1}}W(t_k, y_k, z), .., D^{\pm}_{y_{k,d}}W(t_k, y_k, z)),$$

where $y_k = (y_{k,1}, ...y_{k,d})$ and for a general function b,

$$D^{\pm}_{y_{k,i}} b(y_k, z) = \left[\pm \frac{b(y_{k^{i,\pm}}, z) - b(y_k, z)}{\Delta x} \right]$$

where $[A]$ is the linear interpolation of A and $k^{j,\pm} = (i_1, .., i_{j-1}, i_j \pm 1, i_{j+1}, ..i_d)$.

Compute the optimal control at t_k :

$$a_k = \arg\min_u H_{num}(u, y_k, D^+W(t_k, y_k, z), D^-W(t_k, y_k, z)) \quad (6.22)$$

where H_{num} is the numerical Hamiltonian.

Step 2 Define $\mathbf{u}(t_k) = a_k$. Then, the next point is :

$$\mathbf{y}^n(t_{k+1}) := \mathbf{y}^n(t_k) + hf\big(\mathbf{y}^n(t_k), \mathbf{u}(t_k)\big)$$

Let us emphasise that if the gradient is close to zero, all values of the control are solutions to (6.22). Moreover, the choice of the scheme of the numerical Hamiltonian can have a big importance. A scheme of the numerical Hamiltonian using a centered approximation of gradient of the value function may lead to more smooth feedback controls. Let us compare this procedure to the algorithm using the value function.

From the figure 6.12, one can observe that the evolution of the state is almost the same for both algorithms, but the history of the angle of attack by the method of the value seems to be more smooth than the one obtained by using the Hamiltonian.

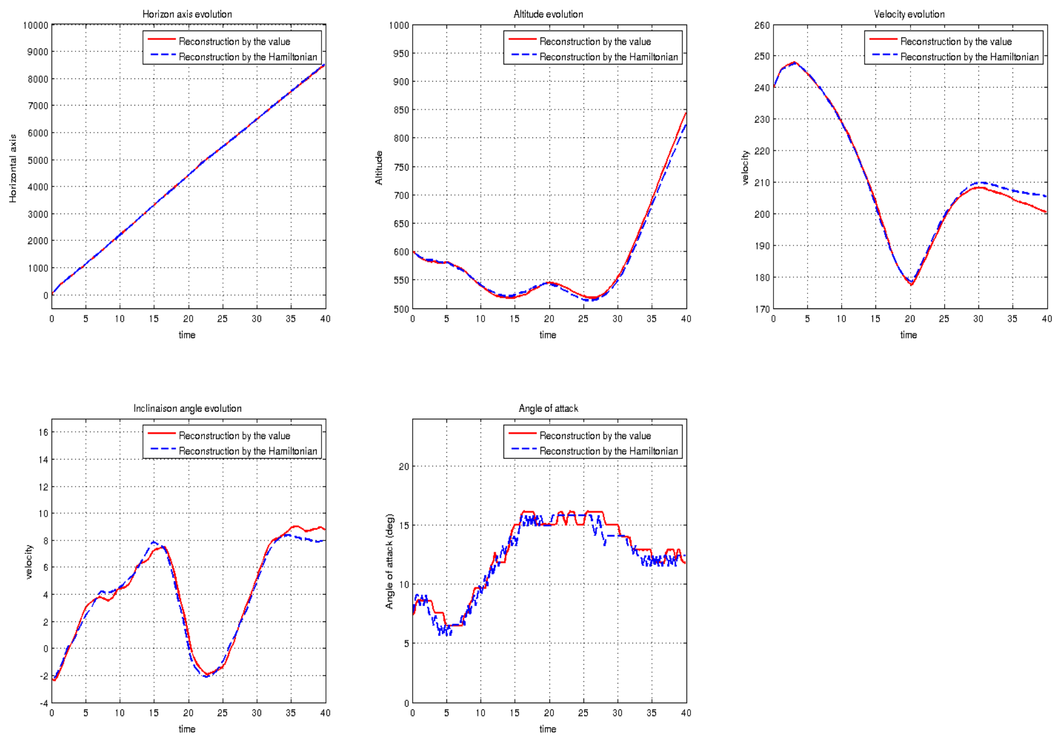

FIGURE 6.12 – Reconstruction of the state for the control problem with maximum cost using the method based on the value function and the method using the Hamiltonian.

In the figure 6.13, we can see that the quality of the control and the angle of attack using the algorithm based on the minimization of the value is slightly better than the result obtained with the algorithm using the Hamiltonian. This involves the nonsmoothness of the angle of attack that has a very unstable behaviour (see Figure 6.12).

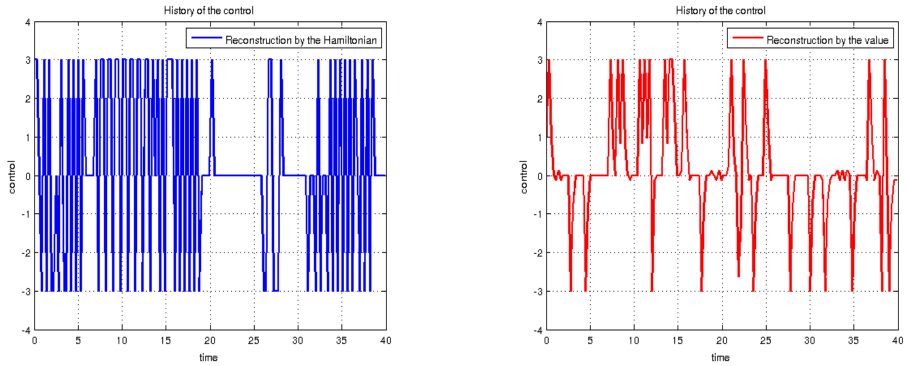

FIGURE 6.13 – Reconstruction of the optimal control for the control problem with maximum cost using the value function (right) and the Hamiltonian (left).

In order to make a suitable comparison between all the methods of reconstruction, we superimpose all reconstructed graphs of the altitude in a same Figure (see Figure 6.14). The optimality criterion obtained with each method is reported in the table 6.13. The reconstruction by minimizing the value function provides the best performance since the optimality criterion is maximal using this algorithm.

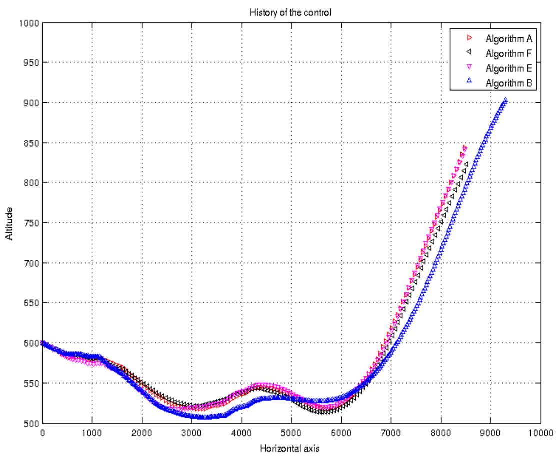

FIGURE 6.14 – History of the trajectory in the plan oxh for the control problem with maximum cost using several methods of reconstruction.

Table 6.13 – The optimality criterion at time $t = T$ obtained with several algorithms.

Algorithm	$J_1^*(T, y_0)$	Minimal altitude
Algorithm A	4.821 e+2	5.178 e+2
Algorithm B	4.936 e+2	5.063 e+2
Algorithm E	4.828 e+2	5.171 e+2
Algorithm F	4.865 e+2	5.134 e+2

On the other hand, the quality of the angle of attack and the control are not good using the value function and the hamiltonian. The regularization allows to improve partially the performance. From figures 6.15 6.16, one can conclude that the algorithm of reconstruction by the exit time function allows to obtain smooth behaviour of the control and the angle of attack.

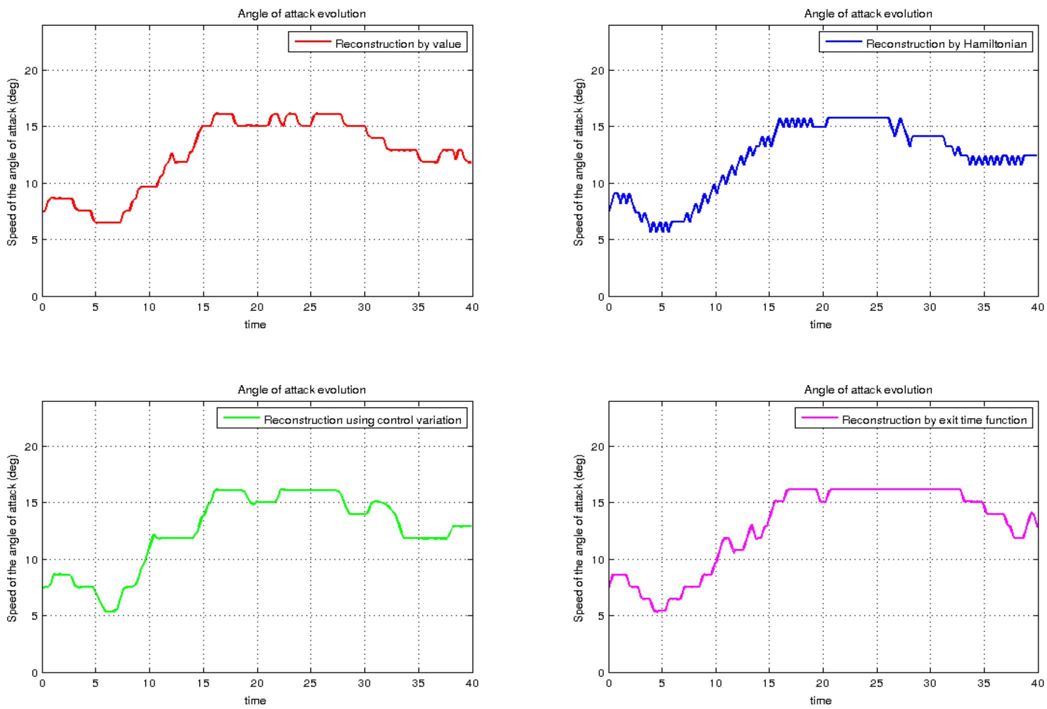

Figure 6.15 – History of the angle of attack for the control problem with maximum-cost using several methods of reconstruction.

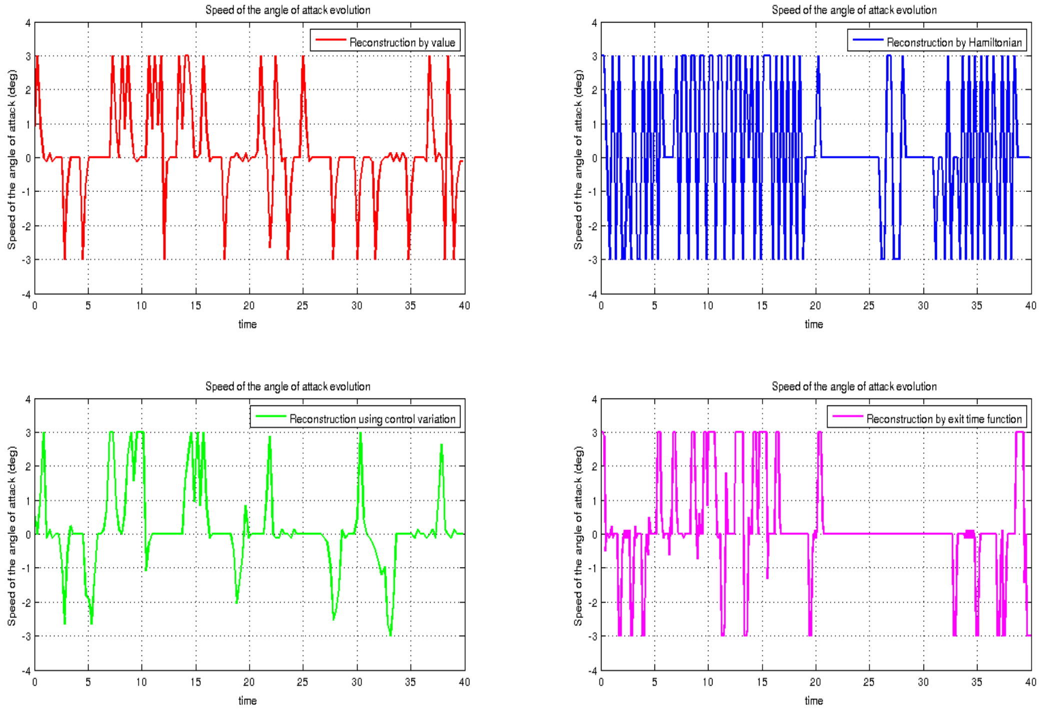

FIGURE 6.16 – History of the control for the control problem with maximum-cost using different methods of reconstruction.

Appendix A. Analysis of the properties of the dynamical systems for the landing abort problem

This subsection is concerned by the analysis of the properties of different models proposed for the study of the landing problem. First, in all the models the function $f : \mathcal{K} \times U \to \mathbb{R}$ is continuous in both variables (the forces F_D, F_L and F_P and the wind are supposed to be continuous functions in all its arguments).

Lipshitz continuity hypothesis

One can easily observe that, for all the models, the function f is defined on the constraints set $\mathcal{K}$ and it is possible to define more large set $\widetilde{\mathcal{K}}$ such that the function f has all partial derivatives in all state variables and that the Jacobian of f is bounded on $\widetilde{\mathcal{K}}$. In particular, on can define this set as follows

$$\widetilde{\mathcal{K}} := \mathcal{K} + \widetilde{c}B$$

where B is the ball of radius 1 centered in the origin and $\widetilde{c} < v_{\min}$. That means that the function f is Lipschitz continuous in y on the set $\widetilde{\mathcal{K}}$. Then one can define a Lipshitz continuous extension of f out of $\widetilde{\mathcal{K}}$.

Convexity hypothesis

Now, let verify if the models are satisfying the convexity property ($\mathbf{H_4}$) or not. In fact, this property may not be satisfied for all models considered in this paper.

Indeed, in general, the model **M1-model** don't satisfy ($\mathbf{H_4}$) and it will depend on the formulas of the aerodynamic forces.

Fortunately, the convexity property is satisfied for the other models. For **M1.2-model**, one can observe that the controls are appearing only in the third and the forth components of (6.8) or (6.10). Then, one can only focus on the convexity of the following set,

$$\mathcal{A} := \left\{ \left(\beta F_T^*(V)\cos(\alpha+\delta), \beta \frac{F_T^*(V)}{V}\sin(\alpha+\delta) \right) \,\middle|\, u = (\alpha, \beta) \in \mathcal{U} \right\}$$

The set $\mathcal{A}$ turns out to be a subset of the set of the ellipses $\mathcal{E}$,

$$\mathcal{E} := \left\{ (x, z) \in \mathbb{R}^2 \mid \frac{x^2}{V^2} + z^2 \leq \frac{F_T^*(V)^2}{V^2} \right\}$$

Since $\alpha \in [0, \alpha_{\max}] \subset [-\delta, 2\pi - \delta]$, then $\mathcal{A}$ represents the set of all arcs of $\mathcal{E}$ corresponding to $\alpha_{\max}$. The set $\mathcal{A}$ is convex and the convexity of $f(z, U)$ follows.

For **M2- model**, the convexity property ($\mathbf{H_4}$) is obvious and it follows from the linear dependence of f on the control ω.

Conclusion

In this thesis, we study some control problems with non standard forms motivated by applications using the so-called Hamilton Jacobi Bellman approach. The main goal is to validate in practice the behaviour of this approach while dealing with applications.

First, we consider a discontinuous and unbounded value function associated with some stochastic optimal control problems . We introduce a regularized value function and compute the error estimates between both functions. The regularized value function is the unique viscosity solution of a second order HJB equation with unbounded data. We derive error estimates for monotone schemes based on a Semi-Lagrangian method (or more generally in the form of a Markov chain approximation). These errors are based on classical shaking and regularization techniques.

The reachablility under chance constraints is one of the current issues in the control theory. Here, we characterize the starting points from which it is possible to reach a target with a probability greater than a threshold. We make use of the level set approach and the result of the first study to approximate the value function. It follows that the probabilistic reachable set is contained between two sets that we can compute.

Then, we study on a class of deterministic state-constrained optimal control problem with maximum cost. We want to deal with the constraints in the general case where no controllability assumption is made. We describe the epigraph of the value function by the auxiliary optimal control problem whose value function is Lipschitz continuous. We prove that our new Lipschitz continuous value function is the unique Lipschitz continuous viscosity solution of a Hamilton Jacobi equation with a Dirichlet condition. We prove a convergence result of a sequence of approximated optimal trajectories. In addition, we show that the auxiliary value function is linked to a *Viability Kernel* and the corresponding exit time function. In the same manner, we study the state-constrained control problem with Bolza cost and emphasise differences.

Finally, we consider the concrete problem of the abort landing during low altitude wind-shears. Many algorithms of reconstruction involving the value function and the exit time function are analysed from theoretical and computational points of view to generate optimal trajectories and the associated optimal feedback controls.

PERSPECTIVES

In the Chapter 6, we discussed the abort landing in presence of windshear. This real application is modeled using a deterministic dynamics. Whereas, in the reality, the wind does not have only a determistic term corresponding to the meterological predictions (see Chapter 6) but also a stochastic term representing inaccuracy and uncertainty in these predictions. Let $W(\cdot) = (W^1(\cdot), W^2(\cdot))$ be a 2-dimensional Browian motion and consider the following stochastic equation of motion :

$$\begin{cases} dx(t) = (V\cos\gamma + w_x)dt + \sigma_1 dW_t^1, \\ dh(t) = (V\sin\gamma + w_h)dt + \sigma_2 dW_t^2, \\ dV(t) = (\frac{F_T}{m}\cos(\alpha+\delta) - \frac{F_D}{m} - g\sin\gamma - (\dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma))dt \\ \qquad -\sigma_1\Big(\frac{\partial w_x}{\partial x}\cos\gamma + \frac{\partial w_h}{\partial x}\sin\gamma\Big)dW_t^1 - \sigma_2\Big(\frac{\partial w_x}{\partial h}\cos\gamma + \frac{\partial w_h}{\partial h}\sin\gamma\Big)dW_t^2, \\ d\gamma(t) = (\frac{1}{V}(\frac{F_T}{m}\sin(\alpha+\delta) + \frac{F_L}{m} - g\cos\gamma + (\dot{w}_x\sin\gamma - \dot{w}_h\cos\gamma)))dt \\ \qquad +\frac{\sigma_1}{V}\Big(\frac{\partial w_x}{\partial x}\sin\gamma - \frac{\partial w_h}{\partial x}\cos\gamma\Big)dW_t^1 + \frac{\sigma_2}{V}\Big(\frac{\partial w_x}{\partial h}\sin\gamma - \frac{\partial w_h}{\partial h}\cos\gamma\Big)dW_t^2, \\ d\alpha(t) = \mathbf{u}(t). \end{cases} \tag{6.23}$$

where the time derivatives of the components of the wind are given by,

$$\dot{w}_x := \frac{\partial w_x}{\partial x}(V\cos\gamma + w_x) + \frac{\partial w_x}{\partial h}(V\sin\gamma + w_h),$$
$$\dot{w}_h := \frac{\partial w_h}{\partial x}(V\cos\gamma + w_x) + \frac{\partial w_h}{\partial h}(V\sin\gamma + w_h),$$

(see Chapter 6 for other definitions). The above dynamics can be rewriten in the abstract form,

$$\begin{cases} d\mathbf{y}(t) := f(\mathbf{y}(t), \mathbf{u}(t))dt + \sigma(\mathbf{y}(t))dW(t) \\ \mathbf{y}(0) := y \end{cases}$$

where y is the initial point and f and σ are given by :

$$\sigma(y) = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 & 0 \\ -\sigma_1\Big(\frac{\partial w_x}{\partial x}\cos\gamma + \frac{\partial w_h}{\partial x}\sin\gamma\Big) & -\sigma_2\Big(\frac{\partial w_x}{\partial h}\cos\gamma + \frac{\partial w_h}{\partial h}\sin\gamma\Big) \\ \frac{\sigma_1}{V}\Big(\frac{\partial w_x}{\partial x}\sin\gamma - \frac{\partial w_h}{\partial x}\cos\gamma\Big) & \frac{\sigma_2}{V}\Big(\frac{\partial w_x}{\partial h}\sin\gamma - \frac{\partial w_h}{\partial h}\cos\gamma\Big) \\ 0 & 0 \end{pmatrix}$$

$$f(y,u) = \begin{pmatrix} V\cos\gamma + w_x \\ V\sin\gamma + w_h \\ \frac{F_T}{m}\cos(\alpha+\delta) - \frac{F_D}{m} - g\sin\gamma - (\dot{w}_x\cos\gamma + \dot{w}_h\sin\gamma) \\ \frac{1}{V}(\frac{F_T}{m}\sin(\alpha+\delta) + \frac{F_L}{m} - g\cos\gamma + (\dot{w}_x\sin\gamma - \dot{w}_h\cos\gamma)) \\ \mathbf{u}(t) \end{pmatrix}$$

Remark 6.8.1 *Notice that in this case the function f and σ are not Lipschitz continuous on the whole state space $\mathbb{R}^d$. Denote by $\mathcal{K}$ the set on which the functions f and σ are Lipschitz continuous. Then, there exist Lipshitz continuous extensions of f and σ out of $\mathcal{K}$. In what follows we consider that these extensions have the same notations f and σ.*

Note that the uniform Lipschitz properties of f and σ and the compactness of U guarantee the existence of a strong solution $\mathbf{y}_z^{\mathbf{u}}(\cdot) := \mathbf{x}_z^{\mathbf{u}}(\cdot), \mathbf{h}_z^{\mathbf{u}}(\cdot), \mathbf{V}_z^{\mathbf{u}}(\cdot), \gamma_z^{\mathbf{u}}(\cdot), \alpha_z^{\mathbf{u}}(\cdot))$ for each initial data y_0, and for every admissible control $u \in \mathcal{U}$ (see [61] for more details). A process $\mathbf{y}_y^u$ solution of (6.23) associated to a control $u \in \mathcal{U}$ will be said *admissible*. Moreover, there exists K_0 depending only on T, d (see [101, page 42] or [61, Appendice D]) : such that for any $u \in \mathcal{U}$, $0 \le t \le T$ and $y, y' \in \mathbb{R}^d$,

$$\mathbb{E}\left[\sup_{\theta\in[0,t]} \left|\mathbf{y}_y^u(\theta) - \mathbf{y}_{y'}^u(\theta)\right|^2\right] \le K_0^2|y-y'|^2, \tag{6.24a}$$

Furthermore, for every $p \ge 1$, there exists $K_p > 0$ depending only on T such that :

$$\mathbb{E}\left[\sup_{\theta\in[0,t]} \left|\mathbf{y}_y^u(\theta) - y\right|^p\right] \le K_p(1+|y|^p). \tag{6.24b}$$

Stochastic optimal control problem

In the deterministic setting, the problem of the abort landing consists to maximize the lower altitude in order to avoid the crash on the ground. Here, one can formulate the problem of the abort landing in presence of a wind with stochastic form as a reachability problem. More precisely, one can be interested by the characterization of the starting points from which the probability that the altitude of the airplane remains above a threshold $\underline{h}$ is greater than ρ, i.e, consider the following probabilistic backward reachable set,

$$\Omega^\rho := \left\{ y \in \mathbb{R}^d \;\mid \mathbf{u} \in \mathcal{U},\;\; \mathbb{P}\Big[\min_{\theta\in[0,T]} \mathbf{h}_y^{\mathbf{u}}(t) \ge \underline{h}\Big]) > \rho \right\}$$

Instead of considering the lower altitude one can consider the peak value of the difference between the reference altitude h_R and the instantaneous altitude.

$$\Omega^\rho := \left\{ z \in \mathbb{R}^d \;\mid \mathbf{u} \in \mathcal{U},\;\; \mathbb{P}\Big[\max_{\theta\in[0,T]} \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta)) \le h_R - \underline{h}\Big]) > \rho \right\}$$

where $\Phi(\mathbf{y}_y^{\mathbf{u}}(.)) = h_R - \mathbf{h}_y^{\mathbf{u}}(.)$. If the function Φ is positive (which is the case here), then we know that the maximum cost can be approximated by an L_p-cost, i.e,

$$\max_{\theta\in[0,T]} \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta)) = \lim_{q\to\infty} \Big(\int_0^T \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta))^q d\theta\Big)^{\frac{1}{q}}$$

Here, we want to deal with the characterization of the following backward reachable set :

$$\Gamma_q^\rho := \Big\{ y \in \mathbb{R}^d \ \mid \mathbf{u} \in \mathcal{U}, \ \ \mathbb{P}\Big[\int_0^T \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta))^q d\theta \leq \Big(h_R - \underline{h} \Big)^q \Big] > \rho \Big\}$$

The analyse of the comparison or convergence between both sets Γ_q^ρ and Ω^ρ is not discussed in this work. The backward reachable set Γ_q^ρ can be characterized using the value function of a suitable stochastic control problem.

Let us denote $\zeta_{y,z}(\cdot) := z + \int_0^\cdot \Phi(\mathbf{y}_y^{\mathbf{u}}(\theta))^q d\theta$ and $\lambda := \Big(h_R - \underline{h} \Big)^q$ and define the following stochastic optimal control problem

$$\vartheta(t,y,z) := \sup_{\mathbf{u}\in\mathcal{U}} \mathbb{P}\Big[\zeta_{y,z}^{\mathbf{u}}(t) - z \leq \lambda \Big] \tag{6.25}$$

Then, using the level set approach, it is easy to check that,

$$\Gamma^\rho = \Big\{ y \in \mathbb{R}^d \ \mid \ \vartheta(T,y,0) > \rho \Big\}$$

Regularized control problem

Let us first regularize the function $\mathbb{1}_{[0,\lambda]}(\cdot)$ by functions Ψ_ε (for $\varepsilon > 0$), defined as follows :

$$\Psi_\varepsilon(z) = \min(1, \max(0, -\frac{1}{\varepsilon} \max(z-\lambda, -z))). \tag{6.26}$$

Notice that the Ψ_ε is $\frac{1}{\varepsilon}$-Lipschitz continuous. Then, we consider the following "regularized" control problem :

$$\vartheta_\varepsilon(t,y,z) := \sup_{\mathbf{u}\in\mathcal{U}} \mathbb{E}\Big[\Psi_\varepsilon(\zeta_{y,z}^{\mathbf{u}}(t)) \Big] \tag{6.27}$$

Remark 6.8.2 *In chapter 5, we proved that, under an additional ellipticity condition on the diffusion σ, we obtain an error estimates of order $\circ(\varepsilon)$ between both value functions ϑ and ϑ_ε. Here, this condition is not necessary satisfied for the extended function σ. Moreover, the arguments used in the proof for the error estimates between the value functions ϑ and ϑ_ε in the Chapter 4, can not be readapted in this case since the dynamics in z can not be splitted in deterministic and stochastic terms.*

In the following we do not study the error estimates between the value function ϑ and ϑ_ε. The value function ϑ_ε enjoyes some regularity properties. Using the fact that the function Φ_ε is L_ε-Lipschitz continuous, we obtain the following :

Lemma 6.8.3 *Assume (A_1). There exists a constant $C > 0$ such that for every $\varepsilon > 0$, the value function ϑ_ε satisfies :*

$$|\vartheta_\varepsilon(t, y, z)| \le CL_\varepsilon(1 + |y|^q + |z|),$$

for all $y \in \mathbb{R}^d$, $z \in \mathbb{R}$, $t \in [0, T]$.

Proof. (i) By straightforward calculations, we obtain :

$$\begin{aligned} |\vartheta_\varepsilon(t, y, z)| &\le \sup_{u\in\mathcal{U}} |\mathbb{E}[\Psi_\varepsilon(\zeta^u_{\tilde{y}}(t))]|, \\ &\le \sup_{u\in\mathcal{U}} L_\varepsilon \mathbb{E}[1 + |\zeta^u_{\tilde{y}}(t)|]. \end{aligned}$$

Then by using the Lipschitz regularity of Φ, it comes that :

$$|\zeta^u_{\tilde{y}}(t)| \le L_\varepsilon \sup_{u\in\mathcal{U}} \mathbb{E}\big[|z| + L^q_\Phi\big|\int_0^t (1 + |\mathbf{y}^u_y(s)|)^q ds\big|\big].$$

By using (6.24), we get the inequality :

$$|\vartheta_\varepsilon(t, y, z)| \le L_\varepsilon\Big(|z| + C_T L^q_\Phi(1 + |y|^q)\Big).$$

■

It is also known that ϑ_ε satisfies the following dynamic programming principle and the HJB equation :

Proposition 6.8.4 *Assume (A_1).*

(i) Let $(t, x) \in [0, T] \times \mathbb{R}^d$, and denote $\mathcal{T}_{[0,t]}$ the set of $(\mathcal{F}_\theta)_{\theta\in[0,t]}$-adapted stopping times with values a.e. in $[0, t]$. Let $\{\tau^u; u \in \mathcal{U}\}$ be a subset of $\mathcal{T}_{[0,t]}$ (independant of $\mathcal{F}_0$). Then

$$\vartheta_\varepsilon(t + \tau^u, y, z) = \sup_{u\in\mathcal{U}} \mathbb{E}[\vartheta_\varepsilon(t, \mathbf{y}^u_y(\tau^u), \zeta^u_{y,z}(\tau^u))]. \tag{6.28}$$

(ii) The function ϑ_ε is a continous viscosity solution, with polynomial growth of the following HJB equation :

$$\partial_t\vartheta_\varepsilon + \mathcal{H}(y, D_y\vartheta_\varepsilon, \partial_z\vartheta_\varepsilon, D^2_y\vartheta_\varepsilon) = 0, \qquad \text{in } (0, T) \times \mathbb{R}^{d+1}, \tag{6.29a}$$

$$\vartheta_\varepsilon(0, y, z) = \Psi_\varepsilon(z) \qquad \text{in } \mathbb{R}^{d+1}, \tag{6.29b}$$

where $\mathcal{H}$ denotes the Hamiltonian function defined by :

$$\mathcal{H}(y, p_1, p_2, Q) := \inf_{u\in U}\Big\{-f(y, u)\cdot p_1\Big\} - \frac{1}{2}Tr(\sigma(y)\sigma^T(y)\,Q) - \Phi(y)^q \cdot p_2, \tag{6.30}$$

for every $y \in \mathbb{R}^d, p_1 \in \mathbb{R}^d, p_2 \in \mathbb{R}$ and for every symmetric $d \times d$-matrix Q.

Proof. The proof of Proposition 6.8.4 can be found in [61, Chapter 5]. For the uniqueness of unbounded solutions see [48]. ■

Time semi-discrete SL scheme

We aim at approximating v, the unique continous viscosity solution, with linear growth, of the following HJB equation :

$$\partial_t v + \mathcal{H}(y, D_y v, \partial_z v, D^2 v) = 0, \qquad \text{in } (0,T)\times\mathbb{R}^{d+1}, \tag{6.31a}$$
$$v(0,y,z) = \psi(z) \quad \text{in } \mathbb{R}^d. \tag{6.31b}$$

where $\mathcal{H}$ is the same Hamiltonian function as in (6.30). This is the same as HJB equation (6.29) but with a general Lipschitz continuous terminal data ψ instead of Ψ_ε, with Lipschitz constant L_ψ.

We aim to give new error estimates for semi-Lagrangian schemes [38], in the case of Lipschitz continuous b and σ yet that can be unbounded and locally Lipschitz continuous solution v with polynomial growth.

For convenience, we will denote by σ_k the column vectors of the matrix σ :

$$\sigma(y) = \big[\sigma_1, \dots, \sigma_m\big](y),$$

and let us denote the vectors $(\bar\sigma_k)_{k=1,\dots,2m}$ as follows

$$\bar\sigma_k(y) := (-1)^k \sqrt{m}\ \sigma_{\lfloor\frac{k-1}{2}\rfloor}(y), \tag{6.32}$$

(where $\lfloor p\rfloor$ denotes the integer part of $p\in\mathbb{R}$).

Let $h = h > 0$ denote a given time step, and consider a semi-discrete scheme defined as (for $y\in\mathbb{R}^d$) :

$$V^0(y,z) = \psi(z), \tag{6.33a}$$

and, for every $n = N, \dots, 1$,

$$V^{n+1}(y,z) = \mathcal{S}^h(t_n, y, z, V^n),$$

with, for any $t\in[0,T]$, $y\in\mathbb{R}^d$, $z\in\mathbb{R}$, and any function $w:\mathbb{R}^d\to\mathbb{R}$,

$$\mathcal{S}^h(t,y,z,w) := \frac{1}{2m}\max_{a\in U}\left\{\sum_{k=1}^{2m} w(y + hf(y,a) + \sqrt{h}\bar\sigma_k(y), z + h\Phi^q(y))\right\}.$$

By V we will denote the linear interpolation of $V^0, \cdots, V^n$ on $t_0, \cdots, t_N$.

Remark 6.8.5 *An important perspective in the following is to extend the result of the errors estimates for Lipschitz continuous initial data to locally Lipschitz continuous initial data with polynomial growth.*

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Abstract. The main objective of this thesis is to analyze the Hamilton Jacobi Bellman approach for some control problems of unusual forms. The first work is devoted to the numerical approximations of unbounded and discontinuous value functions associated with some stochastic control problems. We derive error estimates for monotone schemes based on a Semi-Lagrangian method (or more generally in the form of a Markov chain approximation). The proof is based on classical shaking and regularization techniques. The second contribution concerns the probabilistic reachablility analysis. In particular, we characterize the chance-constrained backward reachable set by a level set of a discontinuous value function and we use the first theoretical results to derive the corresponding error estimates. In the second part of this thesis, we study a class of state constrained optimal control problem with maximum cost. We first describe the epigraph of the value function by an auxiliary optimal control problem whose value function is Lipschitz continuous. We show that the new value function is the unique Lipschitz continuous viscosity solution of a Hamilton Jacobi equation with a Dirichlet condition. Here, we give a review of the optimal trajectories and the associated feedback control for such control problems. In particular, we prove the convergence of a sequence of approximated optimal trajectories to the continuous one. We establish a link between the control problem and a *viability kernel* associated with an exit time function. The obtained results for the state constrained control problem with maximum cost are then extended to the state constrained control problem with Bolza cost. The study is motivated by a real application : the abort landing during low altitude wind-shears. Many algorithms of reconstruction of optimal feedback trajectories are studied and compared from numerical and theoretical points of view.

Résumé. Le principal objectif de cette thèse est d'analyser l'approche Hamilton Jacobi Bellman appliquée à certains problèmes de contrôle optimal de formes non usuelles. La première étude concerne les estimations d'erreur des schémas monotones basés sur la méthode Semi-Lagrangian (ou plus généralement sous la forme d'une chaîne de Markov approchée). La preuve est basée sur des techniques classiques de secousse et de régularisation. Ensuite, on analyse un problème d'atteignabilité probabiliste. En particulier, on caractérise l'ensemble des points initiaux tel qu'on arrive à la cible avec une probabilité supérieure à un seuil. Ce problème est relié à une courbe de niveau d'une fonction valeur discontinue à laquelle on applique les premiers résultats théoriques pour obtenir les estimations d'erreur associées. La deuxième partie de la thèse traite d'un problème de contrôle optimal avec un coût maximum sous des contraintes d'état. Tout d'abord, on va décrire l'épigraphe de la fonction valeur par une fonction valeur Lipschitzienne d'un problème de contrôle optimal sans contraintes sur l'état. Cette nouvelle fonction est caractérizée comme l'unique solution Lipschitzienne de viscosité d'une équation Hamilton-Jacobi avec une condition de Dirichlet. Un résultat de convergence des trajectoires optimales approchées vers la trajectoire optimale continue est inclu dans ce travail. Cette étude est motivée par un problème d'atterrissage annulé avec un vent à basse altitude. De nombreux algorithmes de reconstruction de trajectoires optimales sont étudiés et comparés de points de vue théorique et numérique.