# Non-commutative generalization of some probabilistic results from representation theory 

Pierre Tarrago

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Pierre Tarrago

17th November 2015

# Non-commutative generalization of some probabilistic results from representation theory 

Under the supervision of<br>Philippe Biane and Roland Speicher

A DISSERTATION PRESENTED
in fulfilment of the thesis requirement for the degree of
Docteur de l'Université Paris-Est
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and the degree of
Doktor der Naturwissenschaften
durch die Naturwissenschaftlich-Technische Fakultät I der Universität des Saarlandes


#### Abstract

The subject of this thesis is the non-commutative generalization of some probabilistic results that occur in representation theory. The results of the thesis are divided into three different parts. In the first part of the thesis, we classify all unitary easy quantum groups whose intertwiner spaces are described by non-crossing partitions, and develop the Weingarten calculus on these quantum groups. As an application of the previous work, we recover the results of Diaconis and Shahshahani on the unitary group and extend those results to the free unitary group. In the second part of the thesis, we study the free wreath product. First, we study the free wreath product with the free symmetric group by giving a description of the intertwiner spaces: several probabilistic results are deduced from this description. Then, we relate the intertwiner spaces of a free wreath product with the free product of planar algebras, an object which has been defined by Bisch and Jones in [46. This relation allows us to prove the conjecture of Banica and Bichon. In the last part of the thesis, we prove that the minimal and the Martin boundaries of a graph introduced by Gnedin and Olshanski are the same. In order to prove this, we give some precise estimates on the uniform standard filling of a large ribbon Young diagram. This yields a positive answer to the conjecture that Bender, Helton and Richmond gave in [18.


#### Abstract

Abstrakt

In dieser Dissertation widme ich mich der nicht-kommutativen Verallgemeinerung probabilistischer Ergebnisse aus der Darstellungstheorie. Die Dissertation besteht aus einer Einleitung und drei Teilen, die jeweils separate Veröffentlichungen darstellen. In dem ersten Teil der Dissertation wird der Begriff von easy Quantengruppe im unitären Fall untersucht. Es wird eine Klassifikation aller unitären easy Quantengruppen in dem klassischen und freien unitären Fall gegeben. Des weiteren werden die probabilistischen Ergebnisse von [14] auf den unitären Fall ausgedehnt. In dem zweiten Teil der Dissertation widme ich mich zunächst dem freien Kranzprodukt einer kompakten Quantengruppe mit der freien symmetrischen Gruppe. Die Darstellungstheorie solcher Kranzprodukte wird beschrieben, und verschiedene probabilistische Ergebnisse werden aus dieser Beschreibung gezogen. Dann wird eine Beziehung zwischen freien Kranzprodukten und planaren Algebren hergestellt, die zu dem Beweis einer Vermutung von Banica und Bichon führt. In dem dritten Teil dieser Dissertation wird der Graph $\mathcal{Z}$ der Multiplikation der fundamentalen quasi-symmetrischen Basis untergesucht. Der minimale Rand dieses Graphs wurde schon von Gnedin und Olshanski identifiziert [42]. Wir beweisen jedoch, dass der minimale Rand und der Martin-Rand gleich sind. Als Nebenprodukt des Beweises erhalten wir mehrere asymptotische kombinatorische Ergebnisse bezüglich großer Ribbon-Young-Tableaus.


## Résumé

Le sujet de cette thèse est la généralisation non-commutative de résultats probabilistes venant de la théorie des représentations. Les résultats obtenus se divisent en trois parties distinctes.
Dans la première partie de la thèse, le concept de groupe quantique easy est étendu au cas unitaire. Tout d'abord, nous donnons une classification de l'ensemble des groupes quantiques easy unitaires dans le cas libre et classique. Nous étendons ensuite les résultats probabilistes de [14] au cas unitaire.
La deuxième partie de la thèse est consacrée à une étude du produit en couronne libre. Dans un premier temps, nous décrivons les entrelaceurs des représentations dans le cas particulier d'un produit en couronne libre avec le groupe symétrique libre: cette description permet également d'obtenir plusieurs résultats probabilistes. Dans un deuxième temps, nous établissons un lien entre le produit en couronne libre et les algèbres planaires: ce lien mène à une preuve d'une conjecture de Banica et Bichon.
Dans la troisième partie de la thèse, nous étudions un analoque du graphe de Young qui encode la structure multiplicative des fonctions fondamentales quasi-symétriques. La frontière minimale de ce graphe a déjà été décrite par Gnedin et Olshanski [42]. Nous prouvons que la frontière minimale coïncide avec la frontière de Martin. Au cours de cette preuve, nous montrons plusieurs résultats combinatoires asymptotiques concernant les diagrammes de Young en ruban.

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## Thesis summary

The subject of this thesis is the non-commutative generalization of some probabilistic results that occur in representation theory. The results of the thesis are divided into three parts, which are summarized here.

Weingarten calculus and free easy quantum groups Easy quantum groups have been defined in [15] as a class of orthogonal compact quantum groups whose associated intertwiners are described by set partitions. This class of compact quantum groups contains important examples of quantum groups as the classical orthogonal and symmetric groups and their free analogs, the free orthogonal and free symmetric groups (see [95, 96]). In a second paper [14], it has been possible to systematically develop the Weingarten calculus on these compact quantum groups in order to get some probabilistic results: in particular, they recovered the convergence results of Diaconis and Shahshahani (see [33]) on the orthogonal and symmetric group, and extended them to the free case. The usual gaussian and Poisson laws are replaced in the free case by the semicircular and the Marchenko-Pastur laws, their free analogs in free probability theory.
The first part of the thesis is devoted to the generalization of this framework in the unitary case. Namely the compact quantum groups are not assumed to be orthogonal anymore, but their intertwiner spaces are still described by set partitions with colors. The classical example is given by the classical unitary group whose intertwiner spaces are described by permutations (which can be seen as two-colored pair partitions) through the Schur-Weyl duality. We classify all unitary easy quantum groups whose intertwiner spaces are described by non-crossing partitions, and develop the Weingarten calculus on these quantum groups. As an application of the previous work, we recover the results of Diaconis and Shahshahani on the unitary group and extend those results to the free unitary group.

Free wreath product The free wreath product is a non-commutative analog of the classical wreath product. The free wreath product is an algebraic construction that produces a new compact quantum group from a compact quantum group and a non-commutative permutation group. This construction arises naturally in the study of quantum symmetries of lexicographical products of graphs. In the classical case, the representation theory of a wreath product is well-know (see for example [60], Part 1, Annex B) and the Haar measure has a straightforward expression. It is for example easy to prove that the fundamental character of a wreath product with the symmetric group $S_{n}$ converges toward a compound Poisson law as $n$ goes to infinity. However in the free case, the Haar state doesn't have any straightforward expression. For instance Banica and Bichon conjectured in [10] that in some cases, the fundamental character of a free wreath product is distributed as the free multiplicative convolution of the law of the two initial fundamental characters.
In the second part of the thesis, we study the free wreath product. First, we study the free wreath product with the free symmetric group by giving a description of the intertwiner spaces: several probabilistic results are deduced from this description. Then, we relate the intertwiner spaces of a free wreath product with the free product of planar algebras, an object which has been defined by Bisch and Jones in 46]. This relation allows us to express the law of the character of a free wreath product as a free multiplicative convolution of the initial laws, which proves the conjecture of Banica and Bichon.

Martin boundary of the Zig-zag lattice The ring $Q S y m$ of quasi-symmetric functions is a refinement of the ring of symmetric functions, in the sense that any symmetric function has a decomposition in terms of quasi-symmetric ones. An important basis of this ring is called the fundamental basis, and its elements have a monomial expansion similar to the Schur basis of the ring of symmetric functions: this expansion is indexed by semi-standard filling of ribbon Young diagrams for the fundamental basis of $Q S y m$ and by semi-standard filling of Young diagram for the Schur basis of Sym. The multiplication structure of the Schur basis is encoded by an important graph which is called the Young graph and denoted by $\mathcal{Y}$. This graph has many applications in the representation theory of the infinite group $S_{\infty}$ and in the probabilistic behavior of some discrete processes. It has been intensively studied by Thoma, Vershik and Kerov in [85, 86, 47]. In particular they identified the minimal and Martin boundaries of $\mathcal{Y}$, and proved that the two coincide. The analog of $\mathcal{Y}$ for the fundamental basis of $Q S y m$ is the graph $\mathcal{Z}$ of Zigzag diagrams. This lattice has been deeply studied by Gnedin and Olshanski who identified in [42] its minimal boundary. They conjectured that the minimal and Martin boundaries also coincide on $\mathcal{Z}$.
In the last part of the thesis, we prove that the minimal and the Martin boundaries of $\mathcal{Z}$ are the same. In order to prove this, we give some precise estimates on the uniform standard filling of a large ribbon Young diagram: we prove that in a uniform filling, the fillings of distant cells become independent in a certain sense. This yields a positive answer to the conjecture that Bender, Helton and Richmond gave in [18.

## Résumé de la thèse

Le sujet de cette thèse est la généralisation non-commutative de résultats probabilistes venant de la théorie des représentations. Les résultats obtenus se divisent en trois parties qui sont résumées ici.

Groupes quantiques easy et calcul de Weingarten: La théorie des représentations de certains groupes et groupes quantiques orthogonaux compacts mettent en jeu un même objet combinatoire, les partitions d'ensembles finis. Ceci est le cas pour le groupe orthogonal et le groupe symétrique, ainsi que pour le groupe orthogonal libre et le groupe symétrique libre: ces deux derniers sont des groupes quantiques qui ont été introduits par Wang [95, 96] comme version non-commutative de leurs homologues classiques. Dans [15], Banica et Speicher ont généralisé ces exemples en définissant les groupes quantiques easy. Il y a dans cette classe deux situations extrêmes: celle où le groupe quantique est un groupe classique et celle où la théorie des représentations du groupe est décrite par des partitions non-croisées. Dans ce dernier cas, le groupe quantique est dit libre. La classification de tous les groupes quantiques easy dans le cas classique et libre a été initiée par Banica et Speicher, puis complétée par Weber [15, 97]. Dans un troisième temps, Raum et Weber [71] ont réussi à classifier l'ensemble des groupes quantiques easy.
Pour un tel groupe quantique, le calcul de Weingarten [28] donne un moyen efficace de calculer les intégrales par rapport à la mesure de Haar sur le groupe quantique. Avec l'aide du calcul de Weingarten, Banica, Curran et Speicher [14] ont pu obtenir plusieurs resultats probabilistes dans le cas des groupes quantiques easy libres ou classiques: par exemple, ils ont étendu à l'ensemble de ces groupes quantiques les théorèmes asymptotiques de Diaconis et Shahshahani 33] sur les traces des groupes orthogonaux et symétriques.
Dans la première partie de la thèse, le concept de groupe quantique easy est étendu au cas unitaire. Tout d'abord, nous donnons une classification de l'ensemble des groupes quantiques easy unitaires dans le cas libre et classique. Nous étendons ensuite les résultats probabilistes de [14] au cas unitaire.

Produit en couronne libre: Le produit en couronne libre est une construction algébrique dûe à Bichon [21] qui associe un groupe quantique compact à un sous-groupe quantique du groupe symétrique libre pour créer un nouveau groupe quantique, d'une manière analogue au produit en couronne classique. Alors que la mesure de Haar d'un produit en couronne classique a une expression simple en fonction des mesures de Haar des groupes initiaux, il n'y a dans le cas libre aucun moyen d'obtenir une formulation explicite de l'état de Haar. Banica et Bichon ont conjecturé dans [10] que la loi du caractère fondamental d'un produit en couronne libre est dans certains cas la convolution multiplicative libre des lois de caractère des groupes quantiques initiaux.
La deuxième partie de la thèse est consacrée à une étude plus approfondie du produit en couronne libre. Dans un premier temps, nous décrivons les entrelaceurs des représentations dans le cas particulier d'un produit en couronne libre avec le groupe symétrique libre: cette description permet également d'obtenir plusieurs résultats probabilistes. Dans un deuxième temps, nous établissons un lien entre le produit en couronne libre et les algèbres planaires: ce lien mène à une preuve de la conjecture de Banica et Bichon précitée.

Frontière de Martin du graph $\mathcal{Z}$ : Le graphe de Young est un graphe qui encode la structure multiplicative de l'anneau des fonctions symétriques dans la base de Schur [85, 86, 47]. Cet
anneau, également défini comme l'anneau commutatif universel engendré par un nombre infini et denombrable de variables, joue un rôle important dans la théorie des représentations du groupe symétrique et du groupe unitaire. En retirant la condition de commutativité dans cet anneau, on obtient un nouvel anneau non-commutatif qui a été introduit [4] comme l'anneau des fonctions symétriques non-commutatives. Un résultat fondamental est qu'on peut associer à cet anneau non-commutatif un anneau commutatif, l'anneau des fonctions quasi-symétriques, qui présente un structure combinatoire similaire à celle de l'anneau des fonctions symétriques. L'anneau des fonctions quasi-symétriques possède ainsi une base semblable à la base de Schur, la base des fonctions fondamentales quasi-symétriques.
Dans la troisième partie de la thèse, nous étudions un analoque du graphe de Young qui encode la structure multiplicative de la base des fonctions fondamentales. La frontière minimale de ce graphe a déjà été décrite par Gnedin et Olshanski [42]. Nous prouvons que la frontière minimale coïncide avec la frontière de Martin. Au cours de cette preuve, nous montrons plusieurs résultats combinatoires asymptotiques concernant les diagrammes de Young en ruban.

## Zusammenfassung der Dissertation

In dieser Dissertation widme ich mich der nicht-kommutativen Verallgemeinerung probabilistischer Ergebnisse aus der Darstellungstheorie. Die Dissertation besteht aus einer Einleitung und drei Teilen, die jeweils separate Veröffentlichungen darstellen.

Easy Quantengruppen und Weingarten-Kalkül: In mehreren Fällen besitzen orthogonale Gruppen und Quantengruppen eine ähnliche Darstellungstheorie, deren kombinatorische Struktur mit Hilfe von mengentheoretischen Partitionen beschrieben wird: dies gilt zum Beispiel für die symmetrische Gruppe und die orthogonale Gruppe sowie für die freie symmetrische Quantengruppe und die freie orthogonale Quantengruppe, wobei letztere als nicht-kommutative Verallgemeinerung von ersteren von Wang [95, 96] definiert wurden. In [15] wurden easy Quantengruppen von Banica und Speicher zur Systematisierung dieses Phänomens eingeführt. Im Rahmen der easy Quantengruppen gibt es zwei extreme Situationen: diejenige, in der die easy Quantengruppe eine klassische Gruppe ist und diejenige, in der die Darstellungstheorie der easy Quantengruppe mit Hilfe von nicht-kreuzenden Partitionen beschrieben wird. In letzterem Fall wird die easy Quantengruppe frei genannt. Die Klassifikation aller klassischen und aller freien Quantengruppen wurde von Banica, Speicher und Weber [15, 97] erreicht und später für alle easy Quantengruppen von Raum und Weber [71] vollendet.
Für eine easy Quantengruppe existiert eine effiziente Methode, die Weingarten-Kalkül genannt wird [28], um Integrale bezüglich des Haarmaßes zu berechen. Mit dem Weingarten-Kalkül konnten Banica, Curran und Speicher [14] mehrere probabilistische Ergebnisse im Rahmen der easy Quantengruppen erlangen: insbesondere wurde der Grenzwertsatz von Diaconis und Shahshahani 33 bezüglich der Verteilung des fundamentalen Charakters der symmetrischen und orthogonalen Gruppen auf alle klassischen und freien easy Quantengruppen ausgedehnt.
In dem ersten Teil der Dissertation wird der Begriff von easy Quantengruppe im unitären Fall untersucht. Es wird eine Klassifikation aller unitären easy Quantengruppen in dem klassischen und freien unitären Fall gegeben. Des weiteren werden die probabilistischen Ergebnisse von [14] auf den unitären Fall ausgedehnt.

Freies Kranzprodukt: Das freie Kranzprodukt ist eine von Bichon 21 eingeführte nichtkommutative Version des klassichen Kranzprodukts, mit Hilfe dessen eine neue Quantengruppe aus einer kompakten Quantengruppe und einer Untergruppe der freien symmetrischen Quantenruppe erzeugt wird. Während das Haarmaß für ein klassisches Kranzprodukt eine einfache Gestalt hat, gibt es für das Haarmaß eines freien Kranzprodukts keine explizite Formulierung. Banica und Bichon [10] stellten jedoch die Vermutung auf, dass die Verteilung des fundamentalen Charakters eines freien Kranzprodukts in vielen Fällen die multiplikative freie Faltung der Verteilungen der beiden originären Charaktere ist.
In dem zweiten Teil der Dissertation widme ich mich zunächst dem freien Kranzprodukt einer kompakten Quantengruppe mit der freien symmetrischen Gruppe. Die Darstellungstheorie solcher Kranzprodukte wird beschrieben, und verschiedene probabilistische Ergebnisse werden aus dieser Beschreibung gezogen. Dann wird eine Beziehung zwischen freien Kranzprodukten und planaren Algebren hergestellt, die zu dem Beweis der Vermutung von Banica und Bichon führt.

Martin-Rand des Graphs $\mathcal{Z}$ : Der Young-Graph $\mathcal{Y}$ beschreibt die multiplikative Struktur des Rings der symmetrischen Funktionen in der sogenannten Schur-Basis 85, 86, 47. Dieser

Ring ist der universelle kommutative Ring mit abzählbar unendlich vielen Variablen, der eine große Rolle in der Darstellungstheorie der symmetrischen und unitären Gruppen spielt. Wenn man die Kommutativität der Variablen wegfallen lässt, erhält man einen neuen Ring, der der Ring der nicht-kommutativen symmetrischen Funktionen genannt wird [4]. Der Punkt ist, dass man daraus trotzdem einen kommutativen Ring erzeugen kann, der ähnlich dem Ring der symmetrischen Funktionen ähnlich ist. Insbesondere gibt es in diesem neuen Ring ein Gegenstück der Schur-Basis, das die fundamentale quasi-symmetrische Basis genannt wird.
In dem dritten Teil dieser Dissertation wird der Graph $\mathcal{Z}$ der Multiplikation dieser fundamentalen quasi-symmetrischen Basis untergesucht. Der minimale Rand dieses Graphs wurde schon von Gnedin und Olshanski identifiziert [42]. Wir beweisen jedoch, dass der minimale Rand und der Martin-Rand gleich sind. Als Nebenprodukt des Beweises erhalten wir mehrere asymptotische kombinatorische Ergebnisse bezüglich großer Ribbon-Young-Tableaus.

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## Part I

## Introduction

## Chapter 1

## Partitions and free probability

This chapter is an introduction to set partitions, a class of objects that underlies the combinatorics of free probability. We set the notations, explain how set partitions are transformed into linear morphisms, and describe their role in free probability.

### 1.1 Set partitions

### 1.1.1 Definition and notations

## Definition of a set partition:

Definition 1.1. Let $n, r \geq 1$. A set partition of $n$ with $r$ parts is a set $p$ of subsets $B_{1}, \ldots, B_{r}$ of $\llbracket 1 ; n \rrbracket$ such that $\bigcup_{i=1}^{r} B_{i}=\{1, \ldots, n\}$ and for $1 \leq i<j \leq r, B_{i} \cap B_{j}=\emptyset$.

A set $B_{i}$ in the definition above is called a block of $p$. A block of cardinal one is called a singleton and a block of cardinal 2 is called a pair. When no confusion is possible, a set partition of $n$ with $r$ parts is simply called a partition of $n$. The set of all set partitions of $n$ is denoted by $P(n)$ and the number of blocks of a partition $p$ is denoted by $b(p) . P(0)$ denotes the empty set. We write $i \sim_{p} j$ if and only if $i$ and $j$ are in a same block of $p$. This is an equivalence relation on $\{1, \ldots, n\}$. Assigning to each equivalence relation the set of its equivalence classes yields a bijection between equivalence relations of $\{1, \ldots, n\}$ and set partitions of $n$.
A set partition is depicted by drawing the integers 1 to $n$ on a row, and the blocks as lines between them. Figure 1.1 is an example of such a drawing for $n=8$ and $p=\{\{1,3,4\},\{2,7\},\{5,8\},\{6\}\}$.


Figure 1.1: Partition $\{\{1,3,4\},\{2,7\},\{5,8\},\{6\}\}$ with 4 blocks.
We distinguish several subsets of $P_{n}$ :

- The set $P_{2}(n)$ of pair partitions: these are partitions such that all blocks are pairs.
- The set $N C(n)$ of non-crossing partitions: these are the partitions $p$ of $n$ such that if $1 \leq i<j<k<l \leq n$ and $i \sim_{p} k$ and $j \sim_{p} l$, then $j \sim_{p} k$. This means that we can draw $p$ such that the blocks do not cross each other. For example $\{\{1,2,5\},\{3,4\},\{6,8\},\{7\}\}$


Figure 1.2: Pair partition $\{\{1,5\},\{2,4\},\{6,8\},\{3,7\}\}$ with 4 blocks.
is a non-crossing partition:


Figure 1.3: Non-crossing partition $\{\{1,2,5\},\{3,4\},\{6,8\},\{7\}\}$ with 4 blocks

- The set $N C_{2}(n)$ of non-crossing pair partitions: $N C_{2}(n)$ is the set $N C(n) \cap P_{2}(n)$.


Figure 1.4: Non-crossing pair partition $\{\{1,6\},\{2,3\},\{4,5\},\{7,8\}\}$ with 4 blocks

The lattice of set partitions Let $p, q$ be two partitions of $n$, One says that $p$ refines $q$ (denoted by $p \leq q$ ) if any block of $p$ is contained in a block of $q$. $\leq$ yields a partial order on the set $P(n)$ (resp. $N C(n)$ ). One can check that for $p, q \in P(n)$ (resp. $N C(n)$ ), there always exist a unique supremum $p \vee q$ and infimum $p \wedge q$ of $p$ and $q$ in $P(n)$ (resp. $N C(n)$ ), yielding that $(P(n), \leq)$ and $(N C(n), \leq)$ are actually lattices.
Note that $N C(n)$ is a subset of $P(n)$ but not a sublattice, since two elements of $N C(n)$ may have a supremum in $P(n)$ that differs from the one in $N C(n)$. For example $\{\{1,3\},\{2\},\{4\}\}$ and $\{\{1\},\{3\},\{2,4\}\}$ are both in $N C(n)$; their supremum in $N C(n)$ is $\{\{1,2,3,4\}\}$ whereas their supremum in $P(n)$ is $\{\{1,3\},\{2,4\}\}$. However, for all $p, q \in N C(n), p \wedge q$ is again in $N C(n)$. To distinguish both lattice, we write $\wedge_{P}, \vee_{P}$ for the supremum in $P(n)$ and $\wedge_{N C}, \vee_{N C}$ for the one in $N C(n)$ (the subscripts are omitted when there is no confusion).

Two colored set partitions Let $S$ be a denumerable set. A $S$-coloring of $\llbracket 1, r \rrbracket$ is a map $c: \llbracket 1, r \rrbracket \rightarrow S$. A $S$-colored partition $p$ of $r$ is a partition $\tilde{p}$ of $P(r)$ together with a coloring $c$ of $\llbracket 1, r \rrbracket$. The partition $\tilde{p}$ is called the uncolored version of $p$. The set of $S$-colored partitions with a particular coloring is denoted by $P(c)$ or $P(c(1), \ldots, c(r))$. We replace $P$ by $P_{2}, N C$ or $N C_{2}$ to emphasize the shape of the partitions.
A two-colored partition $p$ of $n$ is a $S$-partition with $S=\{\circ, \bullet\}$, a set of cardinal 2 . The integer $n$ is thus fixed by the definition of $c$, and is also denoted by $|c|$. A two-colored partition corresponds to a coloring of the extreme points of the blocks of $\tilde{p}$ with elements of $\{\circ, \bullet\}$. We denote by $P^{\circ \bullet}(n)$ (resp. $\left.P_{2}^{\bullet \bullet}(n), N C^{\bullet \bullet}(n), N C_{2}^{\bullet \bullet}(n)\right)$ the set of two-colored partitions (resp. pair partitions, non-crossing partitions, non-crossing pair partitions). For each map $c: \llbracket 1 ; n \rrbracket \rightarrow\{0, \bullet\}$,
$P^{\circ \bullet}(c)$ denotes the set of two-colored partitions such that the coloring is given by the map $c$ (and the same for the three other kinds of partitions).

### 1.1.2 Two-level partitions

Definition 1.2. A two-level partition is a set partition $p$ of $n$ with a distinguished integer $l \in \llbracket 0 ; n \rrbracket$. The integers lower than $l$ are called the upper points and the integers greater than $l+1$ the lower points.

For $k, l \geq 0$, the set of two-level set partitions of $k+l$ with $k$ upper points and $l$ lower points is denoted by $P(k, l)$ (and by $P^{\bullet \bullet}(k, l)$ when the partition is colored). The subset of non-crossing two-level partitions (resp. two-level pairing, non-crossing two-level pairings) is denoted $N C(k, l)$ (resp. $\left.P_{2}(k, l), N C_{2}(k, l)\right)$ or $N C^{\bullet \bullet}(k, l)\left(\right.$ resp. $\left.P_{2}^{\circ \bullet}(k, l), N C_{2}^{\circ \bullet}(k, l)\right)$ depending on whether they are considered colored or not. When no confusion is possible, a two-level set partition is simply called a partition. For $c_{1}: \llbracket 1 ; k \rrbracket \rightarrow\{\circ, \bullet\}$ and $c_{2}: \llbracket k+1 ; k+l \rrbracket \rightarrow\{\circ, \bullet\}$, we denote by $P\left(c_{1}, c_{2}\right)$ (resp. $\left.N C\left(c_{1}, c_{2}\right)\right)$ the set of two-level partitions in $P(k, l)$ such that the coloring of the upper points is given by $c_{1}$ and the one of the lower points by $c_{2}$.
A two-level set partition is drawn with two rows of integers, in such a way that the numbering is cyclic:


Figure 1.5: Two-level partition in $P(3,5)$ with block structure $\{\{5,6,8\},\{2,7\},\{1,4\},\{3\}\}$
The integers are omitted when they do not play any role.
Note that the lattice structure on $P(n)$ (resp. $N C(n)$ ) extends to the case of two-level partitions $P(k, l)$ (resp. $N C(k, l)$ ). In the latter case, the lattice structure is the same as the one of $P(k+l)$, forgetting the role of lower and upper points. In the case of colored partitions, the same identification is made to also give a lattice structure to the set $P^{\circ \bullet}(c)$ for each $c: \llbracket 1, k+l \rrbracket \rightarrow$ $\{\circ, \bullet\}$.

Operations on two-level colored partitions Several operations can be performed on twolevel colored partitions. The easiest is to give a pictorial description of each of these operations.

- The tensor product of two partitions $p \in P^{\circ \bullet}(k, l)$ and $q \in P^{\circ \bullet}\left(k^{\prime}, l^{\prime}\right)$ is the partition $p \otimes q \in P^{\circ \bullet}\left(k+k^{\prime}, l+l^{\prime}\right)$ obtained by horizontal concatenation (writing $p$ and $q$ side by side). The first $k$ points of the $k+k^{\prime}$ upper points are connected by $p$ to the first $l$ of the $l+l^{\prime}$ lower points, and the remaining $k^{\prime}$ upper points are connected to the remaining $l^{\prime}$ lower points by $q$.
- The horizontal reflection of a partition $p \in P^{\circ \bullet}(k, l)$ is given by the reflection of $p$ through the horizontal axis. We also call it the involution of the partition $p$ and denote it by $p^{*}:=R_{h}(p)$.
- The vertical reflection of a partition $p \in P^{\bullet \bullet}(k, l)$ is given by the reflection $R_{v}(p) \in P^{\circ \bullet}(k, l)$ of $p$ through the vertical axis.


Figure 1.6: Tensor product of two partitions


Figure 1.7: Horizontal reflexion of a partition

- The composition of two partitions $q \in P^{\bullet \bullet}(k, l)$ and $p \in P^{\bullet \bullet}(l, m)$ is the partition $p q \in$ $P^{\bullet \bullet}(k, m)$ obtained by vertical concatenation (writing $p$ below $q$ ): First connect $k$ upper points by $q$ to $l$ middle points and then connect these middle points to $m$ lower points by $p$. This yields two kinds of objects : a partition, connecting $k$ upper points with $m$ lower points, and a certain number $r l(p, q)$ of blocks containing only middle points. The latter blocks and all the middle points $l$ are removed. Note that we can compose two partitions $q \in P^{\bullet \bullet}(k, l)$ and $p \in P^{\bullet \bullet}\left(l^{\prime}, m\right)$ only if
(i) the numbers $l$ and $l^{\prime}$ coincide,
(ii) the colorings match, i.e. the color of the $j$-th lower point of $q$ coincides with the color of the $j$-th upper point of $p$, for all $1 \leq j \leq l$.


Figure 1.8: Composition of two partitions

- The inversion of colors of a partition $p \in P^{\bullet \bullet}(k, l)$ is given by the partition $R_{c}(p) \in$ $P^{\circ \bullet}(k, l)$ with same uncolored partition as $p$, but with all the colors inverted.
- The verticolor reflection of a partition $p$ is given by $\tilde{p}:=R_{v} R_{c}(p)$.
- The rotation of a partition: Let $p \in P^{\bullet \bullet}(k, l)$ be a partition connecting $k$ upper points with $l$ lower points. Shifting the very left upper point to the left of the lower points and inverting its color gives rise to a partition in $P^{\circ \bullet}(k-1, l+1)$, a rotated version of $p$. Note that the point still belongs to the same block after rotation. We may also rotate the leftmost lower point to the very left of the upper line (again inverting its color), and we may as well rotate in the right hand side of the lines. In particular, for a partition


Figure 1.9: Verticolor reflection of a partition
$p \in P^{\circ \bullet}(0, l)$, we may rotate the very left point to the very right and vice versa. Such a rotation on one line does not change the colors of the points.

Here is a list of basic two-colored partitions that play an important role in Part II.

- The two identity partitions ${ }_{0}$, $\bullet \in P^{\bullet \bullet}(1,1)$ connects one upper point with one lower point of the same color. Note that ${ }_{0}$ and $\dot{C}$ are not identity partitions.
- The bicolored pair partitions $\quad$, $\quad \bullet \in P^{\bullet \bullet}(0,2)$ connect two lower points of different colors. We also have their horizontally reflected versions $\bullet \cup, 0 \bullet \in P^{\circ \bullet}(2,0)$. The unicolored pair partitions are $\quad \bullet, \quad \circ \in P^{\bullet \bullet}(0,2)$ and $\bullet \bullet$, 으 $\in P^{\bullet \bullet}(2,0)$.
- The singleton partitions $\uparrow, \uparrow \in P^{\circ \bullet}(0,1)$ consist of a single lower point respectively. Their reflected versions are denoted by $\downarrow, \downarrow \in P^{\circ \bullet}(1,0)$.
- We also have four block partitions like
- All preceding examples are partitions consisting of a single block. The crossing partition 8 $\in P^{\bullet \bullet}(2,2)$ however consists of two blocks. It connects a white upper left point to a white lower right point, as well as a black upper right point to a black lower left point; we also have other colorings like or od. These partitions are not in $N C^{\bullet \bullet}(2,2)$.

Category of partitions A collection $\mathcal{C}$ of subsets $\mathcal{C}\left(\varepsilon, \varepsilon^{\prime}\right) \subseteq P^{\circ \bullet}\left(\varepsilon, \varepsilon^{\prime}\right)$ (indexed by all the words $\varepsilon, \varepsilon^{\prime}$ in $\{0, \bullet\}$ ) is a category of partitions, if it is closed under the tensor product, the composition and the involution, and if it contains the bicolored pair partitions $\square_{\bullet}$ and $\square^{\circ}$ as well as the identity partitions ${ }_{0}$ and $\bullet$. We say that $\mathcal{C} \subseteq \mathcal{C}^{\prime}$ if for any pair of words $\varepsilon, \varepsilon^{\prime}$ in $\{0, \bullet\}, \mathcal{C}(\varepsilon) \subseteq \mathcal{C}\left(\varepsilon^{\prime}\right)$ in the set-theoretic sense.
An easy check yields that the set of all partitions $P^{\circ \bullet}$, the set of all pair partitions $P_{2}^{\circ \bullet}$ (i.e. all blocks have length two), the set of all non-crossing partitions $N C^{\bullet \bullet}$, and the set of all noncrossing pair partitions $N C_{2}^{\circ \bullet}$ form each of them a category of partition. Similarly let $P_{2, \text { altenating }}^{\circ \bullet}$ (resp. $N C_{2, \text { alternating }}^{\circ}$ ) be the set of pair partitions (resp. non-crossing pair partitions) with pairs having endpoints of opposite colors if these endpoints are on the same level and endpoints of same color if they are on different levels. Then $P_{2, \text { altenating }}^{\circ \bullet}$ and $N C_{2, \text { alternating }}^{\circ}$ are also categories of partitions. We have moreover the relation :

$$
\begin{array}{ccccc}
P_{2, \text { altenating }}^{\circ \bullet} & \subsetneq & P_{2}^{\circ \bullet} & \subsetneq & P^{\circ \bullet}  \tag{1.1.1}\\
N C_{2, \text { alternating }}^{\circ \bullet} & \subsetneq & \cup & \cup C_{2}^{\circ \bullet} & \subsetneq \\
\hline & N C^{\bullet \bullet}
\end{array}
$$

### 1.1.3 Contraction of tensor products

Kernel of a sequence of integers Let $\vec{i}=\left(i_{1}, \ldots, i_{n}\right)$ be a sequence of integers. This sequence defines an equivalence relation on $\llbracket 1, n \rrbracket$ by saying that $r \sim_{i} s$ if and only if $i_{r}=i_{s}$. The set partition associated to the relation $\sim_{\vec{i}}$ through the bijection given in Section 1.1.1 is denoted $\operatorname{ker}(\vec{i})$. If $\vec{i}=\left(i_{1}, \ldots, i_{k}\right), \vec{j}=\left(j_{1}, \ldots, j_{l}\right)$ are two sequences of integers of respective length $k$ and $l$, we can similarly define a set partition $\operatorname{ker}(\vec{i}, \vec{j})$ in $P(k, l)$ by the same construction: this is the twolevel partition $\left(\operatorname{ker}\left(j_{1}, \ldots, j_{l}, i_{k}, \ldots, j_{1}\right), l\right)$. Note that we reversed the order of the indices $i$ for convenience in later computations. If we specifiy a coloring $\varepsilon: \llbracket 1, k \rrbracket \rightarrow\{\circ, \bullet\}, \varepsilon^{\prime}: \llbracket 1, l \rrbracket \rightarrow\{0, \bullet\}$, we can assume that this partition is in $P\left(\varepsilon, \varepsilon^{\prime}\right)$. Here is an example of such a construction:


Figure 1.10: The partition $\operatorname{ker}((7,1,2),(3,1,3,3,7))$
Given a sequence of integers of length $n$ and a partition $p \in P(n)$, we set $\delta_{p}(\vec{i})=1$ if $p \leq \operatorname{ker}(\vec{i})$ and $\delta_{p}(\vec{i})=0$ otherwise. Similarly if $p \in P(k, l)$ and $\vec{i}, \vec{j}$ are sequences of integers of respective length $k$ and $l$, we set $\delta_{p}(\vec{i}, \vec{j})=1$ if $p \leq \operatorname{ker}(\vec{i}, \vec{j})$ and $\delta_{p}(\vec{i}, \vec{j})=0$ otherwise.

The maps $T_{p}$ 's: Let $V^{\circ}, V^{\bullet}$ be two Hilbert spaces of dimension $n$, and let $\left(e_{i}^{\circ}\right)_{1 \leq i \leq n},\left(e_{i}^{\bullet}\right)_{1 \leq i \leq n}$ be respectively an orthonormal basis of $V^{\circ}$ and $V^{\bullet}$. For any word $\varepsilon=\varepsilon_{1} \ldots \varepsilon_{r}$ in $\{0, \bullet\}$, the scalar product $\langle$,$\rangle on these Hilbert spaces is extended to a scalar product on the tensor product$ $V^{\varepsilon}=V^{\varepsilon_{1}} \otimes \cdots \otimes V^{\varepsilon_{r}}$ by saying that the basis $\left\{e_{i_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes e_{i_{r}}^{\varepsilon_{r}}\right\}_{1 \leq i_{1}, \ldots, i_{r} \leq n}$ is othonormal. For each partition $p \in P^{\circ \bullet}\left(\varepsilon, \varepsilon^{\prime}\right)$, one can define a map $T_{p}: V^{\varepsilon} \rightarrow V^{\varepsilon^{\prime}}$ by the relation

$$
\left\langle T_{p}\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right), e_{j_{1}} \otimes \cdots \otimes e_{j_{k^{\prime}}}\right\rangle=\delta_{p}(\vec{i}, \vec{j}) .
$$

If $p$ is considered without colors, the same definition holds by considering tensor products of a unique vector space $V$ of dimension $n$.
With this definition of the maps $T_{p}^{\prime} s$, the operations on two-level partitions defined in Paragraph 1.1 .2 transpose to the usual operations on linear maps between Hilbert spaces as follows:

- $T_{p} \otimes T_{q}=T_{p \otimes q}$.
- $T_{p} \circ T_{q}=n^{r l(p, q)} T_{p q}$.
- $T_{p}^{*}=T_{R_{h}(p)}$.

Some linear maps can be easily expressed by the maps $T_{p}^{\prime} s$. For example $T_{\rho}$ and $T$, are respectively the identity map on $V^{\circ}$ and $V^{\bullet}, T_{\text {天。 }}$ and $T_{\text {.. }}$ are the scalar products on $V^{\circ}$ and $V^{\boldsymbol{\bullet}}$.

### 1.2 Free independence and set partitions

### 1.2.1 Non-commutative probability spaces and freeness

The free independence is a phenomenon arising in the study of non-commutative random variables. The latter are a generalization of probabilty spaces in the framework of non-commutative algebras.

Definition 1.3. A non-commutative probability space $(A, \varphi)$ is a unital $*$-algebra $A$ with a linear functional $\varphi$, such that $\varphi\left(\mathbf{1}_{A}\right)=1$.
$\varphi$ is called the expectation on $A$ and is usually a trace (namely $\varphi(a b)=\varphi(b a)$ for $a, b \in A)$. The joint law of $a_{1}, \ldots, a_{r} \in A$ is defined as the expectation map

$$
\varphi_{a_{1}, \ldots, a_{r}}:\left\{\begin{array}{ccc}
\mathbb{C}<X_{1}, \ldots, X_{r}> & \longrightarrow & \mathbb{C} \\
X_{i_{1}} \ldots X_{i_{p}} & \mapsto & \varphi\left(a_{i_{1}} \ldots a_{i_{p}}\right.
\end{array}\right.
$$

Example 1.4. There are two basic examples of noncommutative probability spaces:

- If $(\Omega, \mathbb{P})$ is a classical probability space, then the algebra $L^{\infty-}(\Omega)=\bigcup_{p \geq 1} L^{p}(\Omega)$ of measurable functions having all moments finite is a noncommutative probability space, and the linear functional is given by the expectation $\mathbb{E}$ with respect to $\mathbb{P}$.
- Let $(\Omega, \mathbb{P})$ be a classical probability space and consider the algebra $M_{n} \otimes L^{\infty-}(\Omega)$. This algebra is again a noncommutative probability space with expectation given by the map $A \mapsto \mathbb{E}\left(\frac{1}{n} \operatorname{Tr}(A)\right)$.

When $A$ is a $C^{*}$-algebra and $a$ is a normal element of $A$ (i.e $a a^{*}=a^{*} a$ ), the spectral theorem yields that $a$ is an actual random variable on its spectrum, with moments given by $\left.\left\{\varphi\left(a^{k}\left(a^{*}\right)^{k^{\prime}}\right)\right\}_{k, k^{\prime} \geq 0}\right\}$. The law of $a$ is denoted by $\mu_{a}$.
If two commuting random variables $a, b$ are independent, the knowledge of the respective laws of $a$ and of $b$ suffices to compute the expectation of any polynomial in $a$ and $b$.
The concept of freeness is the analog of the independence of classical random variables in the setting of highly non-commutative variables. It has been introduced by Voiculescu around 1983 (see [90], see also [93] for an introduction to the subject).

Definition 1.5. Let $(A, \varphi)$ be a non-commutative probability space and $A_{1}, \ldots, A_{r}$ be subalgebras of $A . A_{1}, \ldots, A_{r}$ are called free (or freely independent) if for any sequence $\left(a_{1}, \ldots, a_{p}\right)$ with $a_{i} \in A_{k_{i}}, k_{i} \neq k_{i+1}$ for $1 \leq i \leq p-1$ and $\varphi\left(a_{i}\right)=0$ for $1 \leq i \leq p$, the relation

$$
\varphi\left(a_{1} \ldots a_{p}\right)=0
$$

holds. The variables $x_{1}, \ldots x_{r}$ are called freely independent if the algebras respectively generated by $x_{1}, \ldots, x_{r}$ are free.

In particular if $a_{1}, \ldots, a_{r}$ are free, the data $\left\{\varphi\left(a_{i}^{n}\right)\right\}_{1 \leq i \leq r, n \geq 1}$ suffices to characterize the joint law of $\left(a_{1}, \ldots, a_{r}\right)$.

Example 1.6. Originally introduced to study free products of $C^{*}$ - algebra and free group factors, free probability has drawn hudge interests when it has been discovered by Voiculescu in [92] that free probability encodes the limit law of large matrices with independent entries.

If $(A, \varphi)$ is $C^{*}$-algebra and $a_{1}, a_{2}$ are two free self-adjoint elements of $A$, then $a_{1}+a_{2}$ is again self-adjoint. We denote by $\mu_{a_{1}} \boxplus \mu_{a_{2}}$ the law of $a_{1}+a_{2}$, which depends only on $\mu_{a_{1}}$ and $\mu_{a_{2}}$ by the remark above. If $a_{2} \geq 0, a_{2}^{1 / 2} a_{1} a_{2}^{1 / 2}$ is again a self-adjoint element, and we denote by $\mu_{a_{1}} \boxtimes \mu_{a_{2}}$ the law of $a_{2}^{1 / 2} a_{1} a_{2}^{1 / 2}$. If $\varphi$ is tracial and $a_{1}$ is also positive, $\mu_{a_{1}} \boxtimes \mu_{a_{2}}$ is also equal to $a_{1}^{1 / 2} a_{2} a_{1}^{1 / 2}$.

### 1.2.2 Classical and free cumulants

In the classical case, the computation of the additive convolution of two independent random variables is greatly simplified by the use of the Fourier transform. In the free case, an anologuous method exists with the so-called R-transform introduced by Voiculescu in [91. However the R -transform is a complicated object, mainly because it involves using the inverse of analytic functions with respect to the composition.
In [77], Speicher introduced a combinatorial method to compute the sum of two free random variables. It is based on the notion of free cumulants, a non-commutative analog of cumulants in classical probability. We present both classical and free cumulants at the same time, since they will both be used in following chapters.
Let $\left\{f_{i}\right\}_{i \geq 1}$ be a family of multilinear functionals on $A$ such that $f_{i}$ is $i$-multilinear (namely $f_{i}: A^{\otimes i} \rightarrow \mathbb{C}$ ). For $\pi$ a partition of $r$ and $a_{1}, \ldots, a_{r}$ elements of $A, f_{\pi}$ denotes the $r$-multilinear map defined by

$$
f_{\pi}\left(a_{1}, \ldots, a_{r}\right)=\prod_{B=\left\{i_{1}, \ldots, i_{s}\right\} \in \pi} f_{s}\left(a_{i_{1}}, \ldots, a_{i_{s}}\right) .
$$

For $i \geq 1$ and $a \in A$, we denote by $f_{i}(a)$ the quantity $f_{i}(a, \ldots, a)$. The expectation $\varphi$ yields such a family of multilinear maps $\left\{m_{i}\right\}_{i \geq 1}$ with the relation $m_{i}\left(a_{1}, \ldots, a_{i}\right)=\varphi\left(a_{1} \ldots a_{i}\right)$; the $r-$ th moment of $a$ is $m_{r}(a)$.

Definition 1.7. The classical (resp. free) cumulants of $(A, \varphi)$ is the unique family of multilinear maps $\left\{c_{i}\right\}_{i \geq 1}$ (resp. $\left\{k_{i}\right\}_{i \geq 1}$ ) such that

$$
m_{r}=\sum_{\pi \in P(r)} c_{\pi},\left(r e s p . m_{r}=\sum_{\pi \in N C(r)} k_{\pi}\right) .
$$

$c_{r}(a)\left(r e s p . k_{r}(a)\right)$ is called the $r-$ th cumulant (resp. $r$-th free cumulant) of $a$.
The existence and unicity of such families is easily proved by recurrence on $r$. The relation in the definition is also known as the moment-cumulant formula. By this formula, the knowledge of $\left\{k_{i}(a)\right\}$ is equivalent to the knowledge of the law of $a$ in $(A, \varphi)$, and the same holds with classical cumulants.
Thanks to the poset structure on the set of partitions and the set of non-crossing partitions, there exists a direct formula to express the cumulants in terms of moments. In a finite poset $(G, \leq)$, the Moebius function $\mu_{G}: G \times G \rightarrow \mathbb{R}$ is defined as the unique function satisfying $\sum_{g^{\prime} \leq h \leq g} \mu_{G}\left(g^{\prime}, h\right)=\delta_{g, g^{\prime}}$ for $g^{\prime} \leq g$ and $\mu_{G}(h, g)=0$ if $h \not \leq g$. Let $\mu_{P}$ and $\mu_{N C}$ denote respectively the Moebius function on the poset of partitions and non-crossing partitions. The following result is due to Speicher ([77) in the non-commutative case.

Theorem 1.8. The cumulants and free cumulants have the following expression :

$$
c_{r}=\sum_{\pi \in P(r)} \mu_{P}\left(\pi, \mathbf{1}_{r}\right) m_{\pi}, k_{r}=\sum_{\pi \in N C(r)} \mu_{N C}\left(\pi, \mathbf{1}_{r}\right) m_{\pi} .
$$

In both case the Moebius function is given by an explicit formula. Let $\pi \leq \sigma$ in $P(r)$ (resp. $N C(r))$. One can easily show that the interval $[\pi, \sigma]$ is isomorphic as a poset to $\left.P\left(k_{1}\right) \times \cdots \times P_{k_{n}}\right)$ (resp. $\left.N C\left(k_{1}\right) \times \cdots \times N C\left(k_{n}\right)\right)$ for some positive integers $k_{1}, \ldots, k_{n}$. Then

$$
\mu_{P}(\pi, \sigma)=\prod_{1 \leq i \leq n}(-1)^{k_{i}-1}\left(k_{i}-1\right)!
$$

for $P(r)$, and

$$
\mu_{N C}(\pi, \sigma)=\prod_{1 \leq i \leq n}(-1)^{k_{i}-1} C_{k_{i}-1}
$$

in the non-crossing case. The formula in the non-crossing case has also been proved by Speicher in [77.
The important property of cumulants (resp. free cumulants) is that they characterize independence (resp. free independence). The free part of the following result comes from [77]:

Theorem 1.9. Let $a_{1}, \ldots, a_{r}$ be $r$ elements of $A . a_{1}, \ldots, a_{r}$ are independent (resp. free) if and only if $c_{n}\left(a_{i(1)}, \ldots, a_{i(n)}\right)$ (resp. $k_{n}\left(a\left(i_{1}\right), \ldots, a\left(i_{n}\right)\right)$ ) vanishes for any non-constant function $i: \llbracket 1, n \rrbracket \rightarrow \llbracket 1, r \rrbracket$.

Therefore if $a_{1}$ and $a_{2}$ are free then we have the simple relation $k_{r}\left(a_{1}+a_{2}\right)=k_{r}\left(a_{1}\right)+k_{r}\left(a_{2}\right)$ for all $r \geq 1$. A same formula involving free cumulants exists to compute the law $\mu_{a_{1}} \boxtimes \mu_{a_{2}}$ (see [66]).
In the classical setting several distributions arising as universal limit distributions have very simple expression in terms of cumulants:

- The standard Gaussian variable $N$, with density $d_{N}(x)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{x^{2}}{2}}$, has cumulants $c_{2}(N)=$ 1 and $c_{i}(N)=0$ for $i \neq 2$. This distribution appears as the limit distribution of $\frac{X_{1}+\cdots+X_{n}}{\sqrt{n}}$ when $n \rightarrow+\infty$ and $\left(X_{i}\right)_{i \geq 1}$ is a family of i.i.d centered random variables of variance 1 . A standard complex gaussian variable $Z$ is defined as $Z=\frac{1}{\sqrt{2}}(X+i Y)$, with $X$ and $Y$ two independent standard gaussian variables. All cumulants of $Z$ vanish except $c_{2}\left(Z, Z^{*}\right)$ and $c_{2}\left(Z^{*}, Z\right)$ which are equal to 1 .
- The Poisson variable $P$, with distribution $\mathbb{P}(P=n)=\frac{e^{-1}}{n!}$, has cumulants $c_{i}(P)=1$ for all $i \geq 1$. This distribution is the limit distribution of $Y_{1}^{n}+\cdots+Y_{n}^{n}$ as $n$ goes to infinity, where $\left(Y_{i}^{j}\right)_{1 \leq i \leq j}$ is a family of independent variables, $Y_{i}^{j}$ being a Bernoulli variable with law $\frac{n-1}{n} \delta_{0}+\frac{1}{n} \delta_{1}$.
- The compound Poisson variable $P_{\mu}$ with original probability measure $\mu$ is defined by the formula $P_{\mu}=\sum_{i=1}^{P} Z_{i}$, where $P$ is a Poisson variable and $\left(Z_{i}\right)_{i \geq 1}$ is a sequence of i.i.d $\mu$-distributed random variables (also independent from $P$ ). The cumulants of $P_{\mu}$ are $c_{i}\left(P_{\mu}\right)=m_{i}(\mu)$ for $i \geq 1$. This distribution is the limit distribution of $Z_{1} Y_{1}^{n}+\cdots+Z_{n} Y_{n}^{n}$ as $n$ goes to infinity, where $\left(Y_{i}^{j}\right)_{1 \leq i \leq j}$ is distributed as before and independent from $\left(Z_{i}\right)_{i \geq 1}$.

In the free case, the same phenomenon arises (see 66] for a detailed exposition of each case):

- The semi-circular variable $s$, having density $d_{s}(x)=\frac{1}{2 \pi} \mathbf{1}_{|x| \leq 2} \sqrt{4-x^{2}}$, has free cumulants $k_{2}(s)=1$ and $k_{i}(s)=0$ for $i \neq 2$. This distribution is the the limit distribution of $\frac{X_{1}+\cdots+X_{n}}{\sqrt{n}}$ when $n \rightarrow+\infty$ and $\left(X_{i}\right)_{i \geq 1}$ is a family of free identically distributed centered random variables of variance 1 .
A standard circular variable $c$ is defined by $c=\frac{1}{\sqrt{2}}\left(s_{1}+i s_{2}\right), s_{1}$ and $s_{2}$ being two free standard semi-circular variables. All the free cumulants of $c$ vanish except $k_{2}\left(c, c^{*}\right)$ and $k_{2}\left(c^{*}, c\right)$ that are equal to one.
- The free Poisson variable $p$, with density $d_{p}(x)=\frac{1}{\pi x} \mathbf{1}_{0 \leq x \leq 4} \sqrt{4-(2-x)^{2}}$ has free cumulants $k_{i}(p)=1$ for all $i \geq 1$. This distribution is the limit distribution of $Y_{1}^{n}+\cdots+Y_{n}^{n}$ as $n$ goes to infinity, where $\left(Y_{i}^{j}\right)_{1 \leq i \leq j}$ is a family of free independent variables, $Y_{i}^{j}$ being a Bernoulli variable with law $\frac{n-1}{n} \bar{\delta}_{0}+\frac{1}{n} \delta_{1}$.
- The free compound Poisson variable $p_{\mu}$ with original probability measure $\mu$ is defined as the limit distribution of $Z_{1} Y_{1}^{n}+\cdots+Z_{n} Y_{n}^{n}$ as $n$ goes to infinity, where $\left(Y_{i}^{j}\right)_{1 \leq i \leq j}$ is as before and $\left(Z_{i}\right)_{i \geq 1}$ is a free family of $\mu$-distributed random variables such that $Z_{i}$ is classicaly independent from $Y_{i}^{j}$ for all $j \geq i$. The $i$-th free cumulant of $p_{\mu}$ is the $i-$ th moment of $\mu$.

There is an obvious similarity between the three classical examples and the free ones. This correspondance has lead to a systematic bijection, the Bercovici-Pata bijection, between distributions arising as a limit of sums of independent variables and the ones arising as limits of sum of free variables (see [19]).
The cumulant description of the aforementioned distributions and Theorem 1.9 yield interesting combinatorial formulae for some joint moments of free variables. Let us state for example the following result that will be used in Chapter 5:

Proposition 1.10. Let $c_{1}, \ldots, c_{k}$ be $k$ free standard circular elements, and write $c_{-i}=c_{i}^{*}$. Then for $j_{1}, \ldots, j_{r} \in\{-k, \ldots,-1,1, \ldots k\}$,

$$
m_{r}\left(c_{j_{1}}, \ldots, c_{j_{r}}\right)=\#\left\{p \in N C_{2}\left(j_{1}, \ldots, j_{r}\right) \mid \forall\left\{b_{1}, b_{2}\right\} \in p, j_{b_{1}}=-j_{b_{2}}\right\} .
$$

Proof. By the moment cumulant formula,

$$
m_{r}\left(c_{j_{1}}, \ldots, c_{j_{r}}\right)=\sum_{\pi \in N C\left(j_{1}, \ldots, j_{r}\right)} k_{\pi}\left(c_{j_{1}}, \ldots, c_{j_{r}}\right) .
$$

Since $\left(c_{i}\right)_{1 \leq i \leq k}$ is a free family and for each $i \geq 1$, only $k_{2}\left(c_{i}, c_{i}^{*}\right)$ and $k_{2}\left(c_{i}^{*}, c_{i}\right)$ are non-zero, the result follows.

## Chapter 2

## Probabilistic aspects of representation theory: the unitary group

In this chapter we briefly review how the representation theory of the unitary group $U_{n}$ leads to interesting probabilistic results. This chapter is mainly intended for probabilists having no backgrounds on representation theory, and serves as a motivation for the non-commutative results in the following chapter. The first section presents the framework of compact groups and their associated probability space, and introduces the representation theory of the unitary group. The irreducible representations of $U_{n}$ are indexed by symmetric functions; the theory of these functions is quickly reviewed in the second section. The third section is devoted to the full description of the representations of the unitary group and to the description of the Schur-Weyl duality. The fourth section introduces the Weingarten calculus and the fifth section gives some applications of this method. Finally, generalization to other groups are discussed in Section 6.

### 2.1 Compact groups as probability spaces

This section follows the book [26], and the reader should refer to this reference for omitted proofs.

### 2.1.1 Compact group and Haar measure

Definition 2.1. A compact group $G$ is a group that is also a compact topological space, with the property that the maps $\left(g_{1}, g_{2}\right) \mapsto g_{1} g_{2}$ from $G \times G$ to $G$ and $g \mapsto g^{-1}$ from $G$ to $G$ are continuous maps.

In the definition above, $G \times G$ is considered with the product topology.
Example 2.2. The group $\mathbb{U}$ of complex numbers of modulus 1 with the topology inherited from $\mathbb{C}$ is a compact group.
More generally, the unitary group $U_{n}$ consisting of matrices $U$ in $M_{n}(\mathbb{C})$ which satisfy $U U^{*}=$ $I d_{n}$ is a compact group; the same is true for any closed subgroup of $U_{n}$.

The main feature of a compact group $G$ is the existence of an invariant probability measure on the topological space $G$ :

Theorem 2.3. Let $G$ be a compact group. There exists a unique regular probability measure $\int_{G}$ on $G$ such that for any measurable set $X \subseteq G$ and any $g \in G$,

$$
\int_{G} g \cdot X=\int_{G} X \cdot g=\int_{G} X
$$

This probability measure is called the Haar measure.
In the statement above, g.X (resp. X.g) denotes the set $\{g x, x \in X\}$ (resp. $\{x g, x \in X\}$ ). For $G=\mathbb{U}$, the Haar measure is simply the Lebesgue measure on the circle.

Remark 2.4. If $G$ is only assumed to be locally compact, there still exists a unique regular measure (up to a constant multiple) being invariant by left translation. However this measure is not necessarily invariant by right translation. Consider for example the group $G=\left\{\left.\left(\begin{array}{ll}y & x \\ 0 & 1\end{array}\right) \right\rvert\, x, y \in \mathbb{R}, y>0\right\}$, with the left invariant measure $\mu_{L}$ given by $d \mu_{L}=y^{-2} d x d y$ and the right invariant measure $\mu_{R}$ given by $d \mu_{R}=y^{-1} d x d y$. A locally compact group such that left and rignt invariant translations coincide is called unimodular.

Any compact group is thus a natural probability space with the Haar measure. Since it is also a topological space, it carries a canonical class of measurable functions with respect to $\int_{G}$, namely the algebra $C(G, \mathbb{C})$ of continuous complex functions on $G$. If $f \in C(G, \mathbb{C})$, we denote by $\bar{f}$ the complex conjugate of this function. With this conjugation, $C(G, \mathbb{C})$ is a commutative *-algebra.

### 2.1.2 Representations of a compact group

In most cases, the Haar measure doesn't have any straightforward expression and therefore the computation of the law of any element of $C(G, \mathbb{C})$ may become cumbersome. Fortunately, the representations theory helps to better understand these random variables.
Definition 2.5. A representation $(V, \rho)$ of $G$ is the data of a vector space $V$ and a continuous morphism of groups $\rho: G \rightarrow G l(V)$.
A subspace $W \subseteq V$ is called invariant if for all $g \in G, \rho(g)(W) \subseteq W$, and $v \in V$ is called a fixed point if $\rho(g) v=v$ for all $g \in V$. A representation $V$ is said irreductible if $V$ is finite dimensional and has no invariant subspace apart from $\{0\}$ and $V$.
An intertwiner operator from $(V, \rho)$ to $\left(V^{\prime}, \rho^{\prime}\right)$ is a linear map $T: V \rightarrow V^{\prime}$ such that, for any $g \in G, \rho^{\prime}(g) \circ T=T \circ \rho(g)$. The space of morphisms between $(V, \rho)$ and $\left(V^{\prime}, \rho^{\prime}\right)$ is denoted $\operatorname{Mor}_{G}\left(\rho, \rho^{\prime}\right)$.
Two representations $(V, \rho)$ and $\left(V^{\prime}, \rho^{\prime}\right)$ are isomorph if there is an invertible map $T$ in $\operatorname{Mor}_{G}\left(\rho, \rho^{\prime}\right)$.
From now on we will only consider finite dimensional representations and denote by $d(\rho)$ the dimension of the vector space $V$ of a representation $(V, \rho)$.
Given two representations $(V, \rho)$ and $\left(V^{\prime}, \rho^{\prime}\right)$ of $G$, we can construct the direct sum representation $\rho \oplus \rho^{\prime}$ on $V \oplus W$ (resp. the tensor product representation $\rho \otimes \rho^{\prime}$ on $V \otimes W$ ) by taking the direct sum of the maps $\rho$ and $\rho^{\prime}$ (resp. tensor product of the maps $\rho$ and $\rho^{\prime}$ ):

$$
\rho \oplus \rho^{\prime}(g)=\rho(g) \oplus \rho^{\prime}(g) \in G l(V \oplus W), \rho \otimes \rho^{\prime}(g)=\rho(g) \otimes \rho^{\prime}(g) \in G l(V \otimes W) .
$$

Finally if $(V, \rho)$ is a representation of $G$, we can define the dual representation $\rho^{*}$ on the dual $V^{*}$ of $V$ by

$$
\rho^{*}(f)(v)=f\left(\rho\left(g^{-1}\right) v\right),
$$

where $f \in V^{*}$ and $v$ is any vector in $V$.

Example 2.6. As an example of the previous constructions, let $(V, \rho)$ be a representation of $G$. It yields a representation $\tilde{\rho}$ of $G$ on $\operatorname{End}(V)$ defined by

$$
\tilde{\rho}(f)(v)=\rho(g)\left[f\left(\rho\left(g^{-1}\right) v\right)\right],
$$

with $f \in \operatorname{End}(V)$ and $v \in V$. It is possible to prove that, as a representations of $G, \operatorname{End}(V)$ is isomorphic to $V \otimes V^{*}$. Moreover the set of fixed points of $\operatorname{End}(V)$ under the action of $G$ is precisely the space of intertwiners $\operatorname{Mor}_{G}\left(\rho, \rho^{\prime}\right)$.

Let $(V, \rho)$ be a finite dimensional representation of $G$. Any scalar product $\langle\langle.,\rangle$.$\rangle on V$ defines an average scalar product $\langle., .\rangle_{G}$ on $V$ by the formula :

$$
\left\langle v_{1}, v_{2}\right\rangle_{\rho}=\int_{G}\left\langle\left\langle\rho(g) v_{1}, \rho(g) v_{2}\right\rangle\right\rangle d g,
$$

where $v_{1}, v_{2} \in V$.
The invariance of the Haar measure implies that $\langle., .\rangle_{\rho}$ is $G$-invariant, namely

$$
\left\langle\rho(g) v_{1}, \rho(g) v_{2}\right\rangle_{\rho}=\left\langle v_{1}, v_{2}\right\rangle_{\rho} .
$$

With the latter scalar product we can prove that the irreducible representations are the building block of the representation theory of $G$ :

Proposition 2.7. Let $(V, \rho)$ be a finite dimensional representation of $G$. Then $V$ is the direct sum of irreducible representations of $G$.

Noticing that the eigenspaces and the image of an intertwiner are invariant subspaces yields the description of the intertwiner space between irreducible representations:

Lemma 2.8 (Schur Lemma). Let $(V, \rho),\left(V^{\prime}, \rho\right)$ be two irreducible representations, and $T \in$ $\operatorname{Mor}\left(\rho, \rho^{\prime}\right)$. Then $T$ is either 0 or an isomorphism.
If $(V, \rho)$ is an irreducible representation and $f \in \operatorname{Mor}(\rho, \rho)$, then there exists a scalar $\lambda \in \mathbb{C}$ such that $f=\lambda I d$.

### 2.1.3 Matrix coefficients

The goal is now to construct a family of random variables which is dense in $C(G, \mathbb{C})$ and whose law with respect to the Haar measure could be theoretically computed. This family is a class of particular continuous functions based on the $G$-invariant scalar products $\langle., .\rangle_{\rho}$.

Definition 2.9. A matrix coefficient on $G$ is a function $\varphi$ on $G$ of the form

$$
\varphi(g)=\left\langle\rho(g) v_{1}, v_{2}\right\rangle_{\rho},
$$

with $(V, \rho)$ a representation of $G$ and $v_{1}, v_{2} \in V$.
The name of these functions is clear if we consider an orthonormal basis $\left(e_{1}, \ldots, e_{d}\right)$ of $V$ with respect to the scalar product $\langle., .\rangle_{G}$. With respect to this basis, the representation $\rho$ is

$$
\rho: g \mapsto\left(\begin{array}{ccc}
\rho_{11}(g) & \cdots & \rho_{1 d}(g) \\
\vdots & & \vdots \\
\rho_{d 1}(g) & \cdots & \rho_{d d}(g)
\end{array}\right)
$$

with $\rho_{i j}(g)=\left\langle\rho(g) e_{j}, e_{i}\right\rangle_{\rho}=\operatorname{Tr}\left(\rho(g) E_{j i}\right)$, where $E_{i j} e_{k}=\delta_{j k} e_{i}$. From now on, each representation $(V, \rho)$ is considered with a particular choice of orthonormal basis $\left(e_{i}\right)_{1 \leq i \leq d(\rho)}$ with respect to
an invariant scalar product on $V$ : in this way a canonical set of matrix coefficients $\left\{\rho_{i j}\right\}_{1 \leq i, j \leq d(\rho)}$ is associated to each representation $(V, \rho)$.
Since we have defined a representation $(V, \rho)$ as a continuous map $G \rightarrow G l(V)$, any matrix coefficient on $G$ is continuous.
Considering direct sums, dual and tensor products of representations shows that the sum, the conjugate and the product of matrix coefficients are again matrix coefficients. The trivial representation $g \mapsto 1$ yields the unit element of $C(G, \mathbb{C})$. Therefore the vector space $\mathcal{A}$ of matrix coefficients on $G$ is a unital $*$-subalgebra of $C(G, \mathbb{C})$. Since any representation of $G$ is a direct sum of irreducible representation, a basis of $\mathcal{A}$ is given by the set of matrix coefficients $\mathcal{I}=\left\{\rho_{i j}\right\}_{\rho}$ irreductible $1 \leq i, j \leq d(\rho)$.
It is possible to construct intertwiners from $(V, \rho)$ to $\left(V^{\prime}, \rho^{\prime}\right)$ by averaging on $G$ matrix coefficients coming from these two representations. A careful study of these intertwiners yields the first following important result:

Theorem 2.10 (Schur orthogonality relations, [26] Thm 2.3, Thm 2.4). Let ( $V, \rho$ ) and $\left(V^{\prime}, \rho^{\prime}\right)$ be two non-isomorphic irreducible representations. Then the matrix coefficients are orthogonal with respect to the Haar measure. Namely for $1 \leq i, j \leq d(\rho)$ and $1 \leq k, l \leq d\left(\rho^{\prime}\right)$,

$$
\int_{G} \rho_{i j}(g) \overline{\rho_{k l}^{\prime}(g)} d g=0
$$

If $(V, \rho)$ and $\left(V^{\prime}, \rho^{\prime}\right)$ are isomorphic irreducible representations, we can identify their basis and in this case

$$
\int_{G} \rho_{i j}(g) \overline{\rho_{k l}(g)} d g=\frac{1}{d(\rho)} \delta_{i k} \delta_{j l} .
$$

Therefore the matrix coefficients of irreducible representations yield an orthonormal basis of $\mathcal{A}$ with respect to the Haar measure on $G$ :

$$
\mathcal{A}=\bigoplus_{(V, \rho) \text { irred. }} \bigoplus_{1 \leq i, j \leq d(\rho)} \mathbb{C} \rho_{i j} .
$$

The subspace $\bigoplus_{1 \leq i, j \leq d(\rho)} \mathbb{C} \rho_{i j}$ is denoted by $W_{\rho}$.
The second important Theorem is that the algebra $\mathcal{A}$ is dense in $C(G, \mathbb{C})$ :
Theorem 2.11 (Peter-Weyl Theorem, Thm 4.1 in [26]). The matrix coefficients are dense in $C(G, \mathbb{C})$. In particular $\mathcal{I}$ is an orthonormal basis of $L^{2}(G)$, the space of square-integrable functions on $G$.

The proof of this Theorem is a bit evolved in the general case. However, if $G$ is already described as a subgroup of $G l_{n}(\mathbb{C})$, the proof of the density is a straightforward consequence of Stone-Weierstrass Theorem.

Example 2.12. Let us apply these results to the unit circle $\mathbb{U}$. In this case, since the group is commutative, an irreducible representation $(V, \rho)$ of $\mathbb{U}$ is one dimensional and thus it is just a group homomorphism $\rho: \mathbb{U} \rightarrow \mathbb{C}^{\times}$. Since $\rho$ has to be continuous, there exists $n \in \mathbb{Z}$ such that $\rho=e_{n}$, where $e_{n}(z)=z^{n}$ for all $z \in \mathbb{U}$. Reciprocally any function of this type is indeed an irreducible representation of $\mathbb{U}$.
From Example 2.2, the Haar measure on $\mathbb{U}$ is just the Lebesgue measure on the unit circle. Therefore by the content of this paragraph, the set of functions $\left\{e_{n}\right\}_{n \in \mathbb{Z}}$ is an othonormal basis of $L^{2}(\mathbb{U})$ with respect to the Lebesgue measure on the circle: the decomposition of any continuous
function in this basis is exactly the usual Fourier expansion.
Thus the constructions made in this paragraph are a generalization of the usual Fourier expansion on the circle to a general non-commutative compact group.

To sum up, any compact group $G$ comes naturally with a probability measure $\int_{G}$, and a particular set $\mathcal{I}$ of random variables that form an orthonormal basis of $L^{2}\left(G, \int_{G}\right)$. To fully describe this probability space, we need to know the joint law of these random variables. This is equivalent to knowing the expansion of products of matrix coefficients in the basis $\mathcal{I}$. A theoretical answer to this problem will be given in Section 2.4 with the Weingarten formula.
The concrete computations are hard to achieve. A smaller space of continuous functions, the space of class functions, is easier to handle and still give interesting informations on the compact group.

### 2.1.4 Characters and the unitary group

Definition 2.13. Let $(V, \rho)$ be a representation of $G$. The character of $\rho$ is the function

$$
\chi_{\rho}(g)=\operatorname{Tr}(\rho(g)) .
$$

The character is said irreducible if it is the character of an irreducible representation. A virtual character is a function of the form $\chi_{1}-\chi_{2}$ with $\chi_{1}$ and $\chi_{2}$ characters.

Since the direct sum and tensor product of representations are again representations, the set of characters is stable by addition and multiplication. Since the character of the trivial representation is the constant unit function, the set of virtual characters forms therefore a ring $C l(G)$.
Note moreover that since the trace is invariant by conjugation,

$$
\chi\left(h g h^{-1}\right)=\operatorname{Tr}\left(\rho\left(h g h^{-1}\right)\right)=\operatorname{Tr}\left(\rho(h) \rho(g) \rho(h)^{-1}\right)=\operatorname{Tr}(\rho(g))
$$

with $g, h \in G$. Thus by linear extension of this relation, $C l(G)$ is a ring of functions which are constant on conjugacy classes of $G$. A function $f$ on $G$ such that $f\left(h g h^{-1}\right)=f(g)$ for all $g, h \in G$ is called a class function. Actually the vector space $\mathbb{C} C l(G)$ spanned by the characters is dense in the space of class functions in $L^{2}(G)$ (see [26], Thm 2.6).
Theorem 2.10 yields the following straightforward result :
Proposition 2.14. Let $(V, \rho),\left(V^{\prime}, \rho\right)$ be two representations of $G$. Then

$$
\int_{G} \chi_{\rho}(g) \overline{\chi_{\rho^{\prime}}(g)} d g=\operatorname{dim} \operatorname{Mor}\left(\rho, \rho^{\prime}\right) .
$$

If $\rho, \rho^{\prime}$ are irreducible representations,

$$
\int_{G} \chi_{\rho}(g) \overline{\chi_{\rho^{\prime}}(g)} d g= \begin{cases}1 & \text { if }(V, \rho) \simeq\left(V^{\prime}, \rho^{\prime}\right) \\ 0 & \text { if }(V, \rho) \nsucceq\left(V^{\prime}, \rho^{\prime}\right) .\end{cases}
$$

In particular the set of irreducible characters is a basis of $\operatorname{Cl}(G)$.
Therefore the set of irreducible characters forms a basis of the $L^{2}$-space of class functions. To compute the moments of a character with respect to the Haar measure amounts to decompose tensor products of the representation of this character into irreducible ones.

Example 2.15. Let $(V, \rho)$ be a representation of $U_{n}$ and $U \in U_{n}$. Since $U$ is a unitary matrix, it is thus diagonalizable in an orthonormal basis and there exists $P \in U_{n}$ and $\Delta$ a diagonal matrix with modulus one coefficients such that $U=P \Delta P^{-1}$.
Since $\chi_{\rho}$ is a class function and $U$ is diagonalizable, $\chi_{\rho}(U)=\chi_{\rho}\left(P \Delta P^{-1}\right)=\chi_{\rho}(\Delta)$. Therefore the value of the character on $U$ only depends on the eigenvalues of $U$.
If $\sigma \in S_{n}$ is a permutation, the matrix $W(\sigma)=\left(\delta_{i \sigma(j)}\right)_{1 \leq i, j \leq n}$ is in $U_{n}$, and

$$
W(\sigma)\left(\begin{array}{ccc}
\lambda_{1} & & \\
& \ddots & \\
& & \lambda_{n}
\end{array}\right) W(\sigma)^{-1}=\left(\begin{array}{ccc}
\lambda_{\sigma^{-1}(1)} & & \\
& \ddots & \\
& & \lambda_{\sigma_{n}^{-1}}
\end{array}\right)
$$

Since $\chi_{\rho}\left(W(\sigma) \Delta W(\sigma)^{-1}\right)=\chi_{\rho}(\Delta)$, the value of the character on $U$ is a symmetric function of the eigenvalues of $U$.

The previous example shows that the theory of symmetric functions plays a role in the study of characters of the unitary group.

### 2.2 Symmetric functions

In this section we briefly review the basics of symmetric functions. Most of the results come from [60], and the reader should refer to this book for complete proofs.

### 2.2.1 Young diagrams

Definition 2.16. Let $n \geq 1$. A partition $\lambda$ of $n$, also written $\lambda \vdash n$, is a finite decreasing sequence of integers $\left(\lambda_{1} \geq \lambda_{2} \geq \lambda_{r}>0\right)$ such that $\sum \lambda_{i}=n$. The length of $\lambda$ is the length of the sequence of non-zero integers.

The set of partitions of $n$ is denoted $\mathcal{Y}_{n}$. For each partition $\lambda, m_{k}(\lambda)$ denotes the number of elements equal to $k$ in $\left(\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{r}\right)$. With this notation $\lambda$ is also written $\lambda=1^{m_{1}(\lambda)} \ldots n^{m_{n}(\lambda)}$.
A partition is pictorially represented by a Young diagram, which is an array of $n$ cells with $\lambda_{1}$ cells on the first row, $\lambda_{2}$ cels on the second and so on. The Young diagram of the partition $(7,4,2,1)$ is drawn in Figure 2.1.


Figure 2.1: Young diagram of $(7,4,2,1)$.

Definition 2.17. A partial order is defined on $\mathcal{Y}_{n}$ by saying that $\lambda \leq \mu$ if and only if $l(\lambda) \geq l(\mu)$ and

$$
\lambda_{1} \leq \mu_{1}, \lambda_{1}+\lambda_{2} \leq \mu_{1}+\mu_{2}, \ldots, \lambda_{1}+\cdots+\lambda_{l(\mu)} \leq \mu_{1}+\cdots+\mu_{l(\mu)}
$$

The transpose $\lambda^{t}$ of a partition $\lambda$ is defined as the partition corresponding to the symmetry of the Young diagram of $\lambda$ through the diagonal axis. For example the transpose of $(7,4,2,1)$ is the partition $(4,3,2,2,1,1,1)$, as suggests the following picture :


Figure 2.2: Young diagram of $(7,4,2,1)$ and its transpose.

A Young tableau $T$ is the assignment of a positive integer to each cell of a Young diagram $\lambda . \lambda$ is then refered as the shape of $T$. For $\left\{x_{i}\right\}_{i \geq 1}$ an infinite set of commutating variables, a monomial $x^{T}$ is assigned to each Young tableau $T$ with the formula $x^{T}=x_{1}^{\text {number of } 1 \text { in } T} x_{2}^{\text {number of } 2 \text { in } T} \ldots$ Since the Young diagram has a finite number of cells, the aforementioned product is finite. The next figure is an example of such correspondance:

Figure 2.3: Young tableau $T$ of shape $(7,4,2,1)$ giving $x^{T}=x_{1}^{2} x_{4}^{3} x_{5} x_{6}^{3} x_{7}^{2} x_{8} x_{10} x_{11}$.

### 2.2.2 Symmetric functions

Definition 2.18. A symmetric function $f$ is a polynomial in $n$ variables $x_{1}, \ldots, x_{n}$ such that for all permutation $\sigma \in S_{n}, f\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)=f\left(x_{1}, \ldots, x_{n}\right)$.
A rational symmetric function $g$ is a polynomial in $n$ variables $x_{1}, \ldots, x_{n}$ and their inverse $x_{1}^{-1}, \ldots, x_{n}^{-1}$ such that $g\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)=g\left(x_{1}, \ldots, x_{n}\right)$.

We denote by $\Lambda_{n}$ (resp. $\Lambda_{n}^{ \pm}$) the ring of symmetric functions (resp. rational symmetric functions) in $n$ variables with integer coefficients in the basis of monomials. This is a graded ring with the grading given by the degree of an homogeneous polynomial.
Let $e_{n}$ be the monomial $x_{1} \ldots x_{n}$. Since any rational symmetric function has a monomial of lowest degree, any rational symmetric function $g$ is equal to $\frac{1}{e_{m}^{m}} f$, with f a symmetric function. Therefore we will only consider symmetric functions in this subsection.
A straightforward basis of $\Lambda_{n}$ is the so-called monomial basis, whose elements are indexed by partition $\lambda$ with $l(\lambda) \leq n$. The monomial symmetric polynomial $m_{\lambda}$ is defined as the sum of $x_{1}^{\lambda_{1}} x_{2}^{\lambda_{2}} \ldots x_{n}^{\lambda_{n}}$ (where we set $\lambda_{r+1}=\cdots=\lambda_{n}=0$ ) and all the different monomials obtained from $x_{1}^{\lambda_{1}} x_{2}^{\lambda_{2}} \ldots x_{n}^{\lambda_{n}}$ by permuting the indices in the variables $\left\{x_{i}\right\}_{1 \leq i \leq n}$. For example

$$
m_{(3,1,1)}\left(x_{1}, x_{2}, x_{3}\right)=x_{1}^{3} x_{2} x_{3}+x_{1} x_{2}^{3} x_{3}+x_{1} x_{2} x_{3}^{2}
$$

Besides this basis, there exist three bases which can be constructed with Young tableaux:

- Let $R T(\lambda)$ denote the set of tableaux of shape $\lambda$ such that the integers are weakly increasing along the rows, and define

$$
h_{\lambda}\left(x_{1}, \ldots, x_{n}\right)=\sum_{T \in R T(\lambda)} x^{T}
$$

These functions are called homogeneous symmetric polynomials.

- Let $C T(\lambda)$ denote the set of tableaux of shape $\lambda$ such that the integers are strictly increasing along the columns, and define

$$
\tilde{e}_{\lambda}\left(x_{1}, \ldots, x_{n}\right)=\sum_{T \in C T(\lambda)} x^{T} .
$$

The functions $e_{\lambda}=\tilde{e}_{\lambda^{t}}$ are called the elementary symmetric polynomials.

- Let $\operatorname{SSY}(\lambda)$ denote the set of tableaux of shape $\lambda$ such that the integers are weakly increasing along the rows and stricly increasing along the colums; such tableau is called a semi-standard Young tableau. Define

$$
s_{\lambda}\left(x_{1}, \ldots, x_{n}\right)=\sum_{T \in S S Y T(\lambda)} x^{T} .
$$

These functions are called Schur polynomials. These are the most important polynomials in the study of the representations of $U_{n}$.
The sets $\left\{h_{\lambda}\right\}_{l(\lambda) \leq n},\left\{\tilde{e}_{\lambda}\right\}_{l(\lambda) \leq n},\left\{s_{\lambda}\right\}_{l(\lambda) \leq n}$ are all bases of $\Lambda_{n}$. In the case of the elementary and Schur polynomials this result is straightforward: indeed after ordering the bases with respect to the order $\leq$ on partitions (as defined in 2.17), the transition matrix between $\left\{m_{\lambda}\right\}$ and $\left\{\tilde{e}_{\lambda}\right\}$ (or $s_{\lambda}$ ) is upper-triangular with 1 on the diagonal. Complete proofs and complement can be found in [60], Part I, Ch 6.
From the list of bases above, $e_{\lambda}=e_{\lambda_{1}} \ldots e_{\lambda_{r}}$. Thus, since $\left\{e_{\lambda_{t}}\right\}_{l\left(\lambda^{t}\right) \leq n}$ is a basis of $\Lambda_{n}, \Lambda_{n}$ can be identified with the free commutative ring $\mathbb{Z}\left[e_{1}, \ldots, e_{n}\right]$, with $e_{r}$ being the polynomial

$$
e_{r}\left(x_{1}, \ldots, x_{n}\right)=\sum_{i_{1}<\cdots<i_{r}} x_{i_{1}} \ldots x_{i_{r}} .
$$

A fifth family of symmetric functions arises by considering power sums. Namely let $p_{k}=$ $\sum_{1 \leq i \leq n} x_{i}^{k}$ and $p_{\lambda}=p_{\lambda_{1}} \ldots p_{\lambda_{r}}$. Although the set $\left\{p_{\lambda}\right\}_{l(\lambda) \leq n}$ is not a basis of $\Lambda_{n}$, it is still a basis of $\mathbb{Q} \otimes \Lambda_{n}$.

### 2.2.3 Hall inner product

Projective limit The map $\Phi_{n}: \Lambda_{n} \rightarrow \Lambda_{n-1}$ defined by $\Phi_{n}\left(e_{i}\right)=\left\{\begin{array}{cl}e_{i} & \text { if } i \neq n \\ 0 & \text { if } i=n\end{array}\right.$ is a surjective homomorphism of graded algebra from $\Lambda_{n}$ to $\Lambda_{n-1}$. We can thus define the projective limit $\Lambda=\lim _{\leftarrow} \Lambda_{n}$.
$\Lambda$ can be seen as the algebra of symmetric polynomials in an infinite denumerable set of variables $\left\{x_{1}, \ldots, x_{n}, \ldots\right\}$, with the grading given by the degree of homogeneous polynomials. For example the monomial symmetric polynomial $m_{(3,1,1)}$ is defined as

$$
m_{(3,1,1)}\left(\left\{x_{1}, \ldots, x_{n}, \ldots\right\}\right)=\sum_{i_{1}, i_{2}, i_{3} \text { distinct }} x_{i_{1}}^{3} x_{i_{2}} x_{i_{3}} .
$$

The bases given for $\Lambda_{n}$ are also bases of $\Lambda$ if we drop the restriction $l(\lambda) \leq n$ on the partitions indexing elements of the bases.
An important result is that the coefficients of any expansion of an element in $\Lambda_{n}$ in one of the bases we gave before is constant for $n$ large enough. Therefore any algebraic result obtained in $\Lambda$ on a finite set of elements can be considered as also true in $\Lambda_{n}$ for $n$ large enough. For example, if we write $m_{\lambda} m_{\mu}=\sum_{l(\nu) \leq n} a_{\lambda \mu}^{\nu}(n)$ the expansion of $m_{\lambda} m_{\mu}$ in the basis $\left\{m_{\lambda}\right\}_{l(\lambda) \leq n}$ of $\Lambda_{n}$, the coefficients $a_{\lambda \mu}^{\nu}(n)$ are independent of $n$ as soon as $n \geq l(\lambda)+l(\mu)$.

Hall inner product We introduce here the Hall inner product in $\Lambda$. Note that the same construction and results exist for $\Lambda_{n}$.

Definition 2.19. The Hall inner product $\langle.,$.$\rangle is the bilinear form on \Lambda$ defined by its value on the basis $\left\{m_{\lambda}\right\}$ and $\left\{h_{\lambda}\right\}$ as

$$
\left\langle m_{\lambda}, h_{\mu}\right\rangle=\delta_{\lambda \mu} .
$$

This bilinear form is naturally extended to $\mathbb{Q} \otimes \Lambda$.
This bilinear form is actually an inner product, as we will see in Example 2.21 .
It is possible to characterize dual bases with respect to this product. If $\left\{x_{i}\right\},\left\{y_{i}\right\}$ are two denumerable infinite sets of variables, the basis $\left\{m_{\lambda}\right\}$ and $\left\{h_{\lambda}\right\}$ are related by the Cauchy formula

$$
\prod \frac{1}{1-x_{i} y_{j}}=\sum_{\lambda} m_{\lambda}\left(\left\{x_{i}\right\}\right) h_{\lambda}\left(\left\{y_{i}\right\}\right) .
$$

The Cauchy formula implies that if $f$ is a symmetric function, then $\left\langle\prod \frac{1}{1-x_{i} y_{i}}, f\left(\left\{x_{i}\right\}\right)\right\rangle=f\left(\left\{y_{i}\right\}\right)$. The latter equality yields a proof of the following fact:

Proposition 2.20. Two bases $\left\{f_{\lambda}\right\}$ and $\left\{g_{\lambda}\right\}$ consisting of homogeneous polynomials are dual with respect to $\langle.$, . $\rangle$ if and only if

$$
\sum_{\lambda} f_{\lambda}\left(\left\{x_{i}\right\}\right) g_{\lambda}\left(\left\{y_{i}\right\}\right)=\prod \frac{1}{1-x_{i} y_{j}} .
$$

Let us apply this proposition to the power sums basis.
Example 2.21. If we set $z_{\lambda}=\prod m_{k}!\prod k^{m_{k}}$ (recall that $m_{k}$ is the number of parts of $\lambda$ equal to $k$ ), then

$$
\begin{aligned}
\sum_{\lambda} \frac{1}{z_{\lambda}} p_{\lambda}\left(\left\{x_{i}\right\}\right) p_{\lambda}\left(\left\{y_{i}\right\}\right) & =\sum_{r} \sum_{\lambda, l(\lambda)=r} \frac{1}{\prod m_{k}(\lambda)!} \sum_{\substack{a_{1}, \ldots, a_{r} \\
b_{1}, \ldots, b_{r}}} \prod\left(\frac{\left(x_{a_{j}} y_{b_{j}}\right)^{\lambda_{j}}}{\lambda_{j}}\right) \\
& =\sum_{r} \frac{1}{r!}\left(-\sum_{i, j} \log \left(1-x_{i} y_{j}\right)\right)^{r} \\
& \left.=\exp \left(-\sum_{i, j} \log \left(1-x_{i} y_{j}\right)\right)\right)=\prod_{i, j} \frac{1}{1-x_{i} y_{j}}
\end{aligned}
$$

Therefore from the last proposition, the dual basis of $\left\{p_{\lambda}\right\}$ is $\left\{z_{\lambda} p_{\lambda}\right\}$. This implies that

$$
\left\langle p_{\lambda}, p_{\mu}\right\rangle=\delta_{\lambda, \mu} z_{\lambda} .
$$

In particular $\langle.,$.$\rangle is positive definite and symmetric.$
An important result is that the Schur basis is an orthonormal basis of $\Lambda$ : namely $\left\langle s_{\lambda}, s_{\mu}\right\rangle=$ $\delta_{\lambda \mu}$. The proof is done in [60], Part I, Ch. 4. By a linear algebraic argument, an inner product on $\Lambda$ has at most one orthonormal basis, up multiplication by $\pm 1$. The Schur basis is therefore the unique orthonormal basis of $\Lambda$, and the unique graded basis such that

$$
\prod_{i, j} \frac{1}{1-x_{i} y_{j}}=\sum_{\lambda} s_{\lambda}\left(\left\{x_{i}\right\}\right) s_{\lambda}\left(\left\{y_{i}\right\}\right) .
$$

### 2.3 Representation theory of the unitary group and probabilitistic applications

It is now possible to describe the ring of virtual characters $C l\left(U_{n}\right)$. This description gives a way to compute moments of characters with respect to the Haar measure: some important results obtained by Diaconis and Shahshahani in [33] are given in example of this machinery.

### 2.3.1 Irreducible representations of $U_{n}$

The content of this subsection comes from [26] Ch. 35 and 38.

Rational and polynomial representations The unitary group $U_{n}$ has a fundamental representation given by its canonical embedding in $G l_{n}(\mathbb{C})$. Denote by $\left(u_{i j}\right)_{1 \leq i, j \leq n}$ the matrix coefficients of this representation. A representation $(V, \rho)$ of $U_{n}$ is said rational (resp. polynomial) if the matrix coefficients of $\rho$ are rational (resp. polynomial) expressions of the $u_{i j}$. In the next paragraph we will classify all rational and polynomial representation of $U_{n}$.
In any case the character of a continuous representation of $U_{n}$ is a rational function of the eigenvalues. Indeed let $(V, \rho)$ be a continuous representation of $U_{n}$ and consider its restriction to the $n$-torus $T_{n}=\left\{\left(\begin{array}{ccc}e^{i \vartheta_{1}} & & \\ & \ddots & \\ & & e^{i \vartheta_{N}}\end{array}\right)\right\}$. This yields a continuous representation $(V, \tilde{\rho})$ of $T_{n}$. Since $T_{n}$ is the commutative product of $n$ different copies of $\mathbb{U}, V$ decomposes in $\operatorname{dim}(V)$ vectors $v_{i}$, each of them being a one-dimensional continuous representation of $T_{n}$. Each one-dimensional continuous representation of $T_{n}$ has the form $\tilde{\rho}\left(e^{i \vartheta_{1}}, \ldots, e^{i \vartheta_{n}}\right)=e^{i\left(k_{1} \vartheta_{1}+\cdots+k_{n} \vartheta_{n}\right)}$ with $k_{1}, \ldots, k_{n} \in \mathbb{Z}$; therefore the character $\chi_{\tilde{\rho}}\left(e^{i \vartheta_{1}}, \ldots, e^{i \vartheta_{n}}\right)$ is a rational function of $e^{i \vartheta_{1}}, \ldots, e^{i \vartheta_{n}}$ with non-negative integer coefficient in front of each monomial. But we have seen in Example 2.15 that for $U \in U_{n}$, $\chi_{\rho}(U)$ is equal to $\chi_{\rho}(\Delta)$, with $\Delta \in T_{n}$ being a diagonal matrix such that $U=P \Delta P^{-1}$. Thus if $(V, \rho)$ is a continuous representation of $U_{n}, \chi_{\rho}(U)$ is the evaluation of an element of $\Lambda_{n}^{ \pm}$on the eigenvalues of $U$. From now on the characters of $U_{n}$ are thus identified with elements of $\Lambda_{n}^{ \pm}$, and the ring of virtual characters with a subring of $\Lambda_{n}^{ \pm}$.

Examples of rational representations We review here some basic examples of rational representations, with the identification of the associated character as an element of $\Lambda_{n}^{ \pm}$. Recall that in order to identify the character, it suffices to consider the restriction of the representation to the $n$-torus $T_{n}$. In the sequel, $\left(v_{1}, \ldots, v_{n}\right)$ denotes the canonical basis of $\mathbb{C}^{n}$, and for each unitary matrix $U, u=\left\{u_{1}, \ldots, u_{n}\right\}$ denote its eigenvalues.

- The fundamental representation: this is the identity map on $U_{n}(\mathbb{C})$. Therefore the character is just the symmetric function $\sum u_{i}=m_{1}(u)=e_{1}(u)$.
- The determinant: the determinant det : $U_{n} \rightarrow \mathbb{C}$ is a group homomorphism, and thus a represenation. By the relation between determinant and eigenvalues of a matrix, the associated character is the elementary symmetric function $e_{n}\left(u_{1}, \ldots, u_{n}\right)=u_{1} \ldots u_{n}$.
- One can generalize the previous representation by considering powers of the determinant: for $m \in \mathbb{Z}$ the map $\operatorname{det}^{m}: U \mapsto(\operatorname{det}(U))^{m}$ is again a group homomorphism, and the associated character is the symmetric function $e_{n}^{m}$.
- The $m$-fold exterior representation: let $1 \leq m \leq n$ and let $\bigwedge^{m} \mathbb{C}^{n}$ be the $m$-fold exterior product of $\mathbb{C}^{n}$. The latter is the quotient of the tensor product $\left(\mathbb{C}^{n}\right)^{\otimes m}$ by the relations $v_{i_{1}} \otimes \cdots \otimes v_{i_{m}}=0$ if $i_{j}=i_{k}$ for some $j \neq k$. A basis of $\wedge^{m} \mathbb{C}^{n}$ is given by $\left\{v_{i_{1}} \wedge \cdots \wedge\right.$ $\left.v_{i_{m}}\right\}_{i_{1}<i_{2}<\cdots<i_{m}}$. The map $\left(U, v_{i_{1}} \wedge \cdots \wedge v_{i_{m}}\right) \mapsto U\left(v_{i_{1}}\right) \wedge \cdots \wedge U\left(v_{i_{m}}\right)$ gives a representation of $U_{n}$ on $\Lambda^{m} \mathbb{C}^{n}$. Considering the restriction to $T_{n}$ yields that the associated character is exactly the elementary symmetric function $e_{m}$.

Note that all these representations are polynomial, except for det ${ }^{m}$ with $m<0$.
Moreover, taking tensor products of the last example with different positive values of $m$ yields that any symmetric function of the form $e_{1}^{k_{1}} \ldots e_{n}^{k_{n}}$ with $k_{1}, \ldots, k_{r}>0$ are characters of some polynomial representations. Since $\Lambda_{n}=\mathbb{Z}\left[e_{1}, \ldots, e_{n}\right]$, taking direct sums of these representations shows that any element of $\Lambda_{n}$ corresponds to a virtual character.
From Section 2.2, any rational symmetric function is of the form $e_{n}^{-m} f$ with $f \in \Lambda_{n}$, and thus $C l\left(U_{n}\right) \simeq \Lambda_{n}^{ \pm}$.

Is there a continuous representation of $U_{n}$ which is not rational ? It seems not clear whether there exists a continuous representation that is not rational. For example $G l_{n}(\mathbb{C})$ has continuous non rational representations: consider for instance the group homomorphism $G=\left(g_{i j}\right)_{1 \leq i, j \leq n} \mapsto \bar{G}=\left(\bar{g}_{i j}\right)_{1 \leq i, j \leq n}$. However in the case of the unitary group the answer is negative: we have seen that the character of any continuous representation corresponds to an element of $\Lambda_{n}^{ \pm}$. But since, from the previous paragraph, $\Lambda_{n}^{ \pm}$is already spanned by characters of rational representations, any continuous representation is actually rational.

Irreducible characters of $U_{n}$ It remains to find which elements of $\Lambda_{n}^{ \pm}$correspond to irreducible characters.
Since the representations $\operatorname{det}^{m}$ are all one-dimensional, the functions $e_{n}^{m}$ correspond to irreducible characters. Therefore let us consider only the polynomial representations. The irreducible characters can be directly obtained thanks to the Hall inner product on $\Lambda_{n}$. Consider the graded algebra $\mathbb{C}\left[u_{i j}\right]=\bigoplus_{d \geq 0} \mathbb{C}\left[u_{i j}\right]_{d}$ of polynomials in the variables $u_{i j}$, with $\mathbb{C}\left[u_{i j}\right]_{d}$ being the subspace of homogeneous polynomials of degree $d$.
Since the set $\left\{u_{i j}\right\}_{1 \leq i, j \leq n}$ is a set of matrix coefficients of the fundamental representation, $\mathbb{C}\left[u_{i j}\right]$ is a subalgebra of the algebra $\mathcal{A}$ of matrix coefficients of $U_{n}$. Recall that

$$
\mathcal{A}=\bigoplus_{(V, \rho) \text { irred. }} W_{\rho},
$$

with $W_{\rho}=\bigoplus_{1 \leq i, j \leq d(\rho)} \mathbb{C} \rho_{i j}$. Since $\rho_{i j}(g)=\left\langle\rho(g) e_{i}, e_{j}\right\rangle, U_{n} \times U_{n}$ acts on $W_{\rho}$ as

$$
\left(\left(h, h^{\prime}\right) \cdot \rho_{i j}\right)(g)=\left\langle\rho(g)\left(\rho(h) e_{i}\right), \rho\left(h^{-1}\right) e_{j}\right\rangle
$$

This shows that as a representation of $U_{n} \times U_{n}, W_{\rho} \simeq V_{\rho} \otimes V_{\rho}^{*}$. It is an easy computation to check that $V_{\rho} \otimes V_{\rho}^{*}$ is an irreducible representation of $U_{n} \times U_{n}$. Therefore the $U_{n} \times U_{n}$ representation $\mathcal{A}$ has a decomposition into irreducible $U_{n} \times U_{n}$ representations

$$
\mathcal{A} \simeq \bigoplus_{(V, \rho) \text { irred. }} V_{\rho} \otimes V_{\rho}^{*} .
$$

Since $\mathbb{C}\left[u_{i j}\right]_{d}$ is invariant under the action of $U_{n} \times U_{n}$, it has a unique decomposition into $U_{n} \times U_{n}$ irreducible representations. The polynomial form of these representations yields that

$$
\mathbb{C}\left[u_{i j}\right]=\bigoplus_{(V, \rho) \text { poly. irred. }} W_{\rho} \simeq=\bigoplus_{(V, \rho) \text { poly. irred. }} V_{\rho} \otimes V_{\rho}^{*}
$$

Looking at the trace of the action of $\left(\left(\begin{array}{lll}x_{1} & & \\ & \ddots & \\ & & x_{n}\end{array}\right),\left(\begin{array}{lll}y_{1}^{-1} & & \\ & \ddots & \\ & & \\ & & \\ & & y_{n}^{-1}\end{array}\right)\right)$ on both sides of the equaliyy above yields

$$
\prod_{1 \leq i, j \leq n} \frac{1}{1-x_{i} y_{j}}=\sum_{\lambda, l(\lambda) \leq n} f_{\lambda}\left(x_{1}, \ldots, x_{n}\right) f_{\lambda}\left(y_{1}, \ldots, y_{n}\right)
$$

Remark 2.22. There is obviously a convergence problem here, since the $x_{i}, y_{i}$ have modulus one as eigenvalues of $U_{n}$. However this equality between functions should be considered as an equality between each homogeneous component, and the value of the whole serie doesn't play any role.

From Section 2.2 .3 we deduce that, with the appropriate labelling, $f_{\lambda}=s_{\lambda}$. This yields the following result:

Theorem 2.23. The ring of virtual characters of $U_{n}$ is isomorphic to $\Lambda_{n}^{ \pm}$, and the basis of irreducible characters is given by the set $\left\{e_{n}^{-m} s_{\lambda}\right\} \underset{l(\lambda) \leq n}{m \geq 0}$.
The algebra $\mathbb{C} C l_{p o l}\left(U_{n}\right)$ of polynomial characters is isomorphic to $\mathbb{C} \otimes \Lambda_{n}$, and the basis of irreducible polynomial characters is given by the Schur basis.
Through the isomorphism $\Phi: \mathbb{C} C l_{p o l}\left(U_{n}\right) \mapsto \mathbb{C} \otimes \Lambda_{n}$ the $L^{2}$-scalar product with respect to the Haar measure on $U_{n}$ yields the Hall inner product on $\Lambda_{n}$.

### 2.3.2 Probabilistic applications

The random variables $\chi_{\lambda}$ Let us label by $\lambda$ the irreducible representation whose character is given by $s_{\lambda}$ through the map $\Phi$ in Theorem 2.23 , and denote by $\chi_{\lambda}$ the associated character. Then for $n$ distinct partitions $\lambda^{1}, \ldots, \lambda^{r}$, the joint law of ( $\chi_{\lambda^{1}}, \ldots, \chi_{\lambda^{r}}$ ) can be explicitly computed. Indeed the product formula on the Schur basis in $\Lambda_{n}$ is given by the LittlewoodRichardson coefficients $\left\{c_{\lambda \mu}^{\nu}\right\}$ as

$$
\begin{equation*}
s_{\lambda} s_{\mu}=\sum_{l(\nu) \leq n} c_{\lambda \mu}^{\nu} s_{n u} \tag{2.3.1}
\end{equation*}
$$

These coefficients have a combinatorial nature (see [60], Ch.9), which allows to algorithmically compute them. Let us write $M_{\lambda}=\left(c_{\lambda \mu}^{\nu}\right)_{\mu \nu}$ the matrix of the multiplication by $s_{\lambda}$ in the basis $\left\{s_{\mu}\right\}_{l(\mu) \leq n}$. Then for $m_{1}, \ldots, m_{r}, n_{1}, \ldots, n_{r}>0$,

$$
\left.\int_{U_{n}}\left(\left(\chi_{\lambda_{1}}^{m_{1}} \ldots \chi_{\lambda_{r}}^{m_{r}}\right) \overline{\left(\chi_{\lambda_{1}}^{n_{1}} \ldots \chi_{\lambda^{r}}^{n_{r}}\right.}\right)\right)=\left(M_{\lambda^{1}}^{m_{1}} \ldots M_{\lambda_{r}}^{m_{r}}\left(M_{\lambda_{r}}^{t}\right)^{n_{r}} \ldots\left(M_{\lambda_{1}}^{t}\right)^{n_{1}}\right)_{(0)(0)}
$$

where (0) is the empty partition corresponding to the constant function 1. However, the Littlewood-Richardson coefficients are hard to compute and the previous formula is difficult to deal with.

Diaconis-Shahshahani results Other kinds of formulae can occur by expanding characters in different bases and evaluating the Hall inner product in these bases. Let us look at the power sum basis, and write $X_{i}=\operatorname{Tr}\left(U^{i}\right)$ for $i \geq 1$. Diagonalizing $U$ shows that the random variable $X_{k}$ correspond to the power sum $p_{k}$ through the map $\Phi$. Therefore if $a_{1}, \ldots, a_{r}, b_{1}, \ldots, b_{r} \geq 0$,

$$
\int_{U_{n}}\left(X_{1}^{a_{1}} \ldots X_{r}^{a_{r}}\right) \overline{\left(X_{1}^{b_{1}} \ldots X_{r}^{p_{r}}\right)}=\left\langle p_{\mu}, p_{\nu}\right\rangle_{\Lambda_{n}}
$$

with $\mu=1^{a_{1}} \ldots r^{a_{r}}$ and $\nu=1^{b_{1}} \ldots r^{b_{r}}$. For $n \geq\left(\sum a_{i} i\right) \vee\left(\sum b_{i} i\right)$, the expansion of $p_{\mu}, p_{\nu}$ in the Schur basis is independent of $n$, and thus the evaluation of the Hall inner product on $p_{\mu}$ and $p_{\nu}$ is the same in $\Lambda_{n}$ and $\Lambda$ for $n$ large enough. Therefore in this case, from the computation of the Hall inner product for the power sum basis in Section 2.2.3,

$$
\int_{U_{n}}\left(X_{1}^{a_{1}} \ldots X_{r}^{a_{r}}\right) \overline{\left(X_{1}^{b_{1}} \ldots X_{r}^{p_{r}}\right)}=\delta_{\mu \nu} z_{\mu} .
$$

Since $z_{\mu}=\prod i^{a_{i}} a_{i}!$, this yields for $n \geq\left(\sum i a_{i}\right) \vee\left(i \sum b_{i}\right)$

$$
\int_{U_{n}}\left(X_{1}^{a_{1}} \ldots X_{r}^{a_{r}}\right) \overline{\left(X_{1}^{b_{1}} \ldots X_{r}^{p_{r}}\right)}=\prod \int_{U_{n}} X_{i}^{a_{i}} \overline{X_{i}^{b_{i}}},
$$

with $\int_{U_{n}} X_{i}^{a_{i}} \overline{X_{i}^{b_{i}}}=\delta_{a_{i} b_{i}} i^{a_{i}} a_{i}!$.
The latter are exactly the moments of a symmetric Gaussian complex variable with mean 0 and variance $i$ (see 1.2.2). This is the content of the following Theorem of Diaconis and Shahshahani:

Theorem 2.24 ([33). As $n$ goes to $+\infty$, the random vector $\left(\operatorname{Tr}\left(U^{i}\right)\right)_{i \geq 1}$ converges in moments to a family of independent symmetric complex gaussian variables $\left(Z_{i}\right)_{i \geq 1}$, such that $Z_{i}$ has mean 0 and variance $i$.

By using the representation theory of the symmetric group $S_{n}$, Diaconis and Evans also computed in [32] the value of the variance of $\operatorname{Tr}\left(U^{i}\right)$ for all $n \geq 1$ and found that $\int_{U_{n}} \operatorname{Tr}\left(U^{i}\right) \overline{\operatorname{Tr}\left(U^{j}\right)}=$ $\delta_{i j}(i \wedge n)$. This allowed them to extend this convergence to all symmetric functions having certain Fourier expansions. They could also prove that the convergence of these random variables is stronger than the convergence in moments.

Concrete realization of the Hall inner product The Hall inner product, abstractly defined on the bases $\left\{m_{\lambda}, h_{\lambda}\right\}$ of $\Lambda_{n}$, can be concretely defined as the inner product of a $L^{2}$-space. Indeed, the Haar measure on $U_{n}$ yields a probability measure on the torus $T_{n}$ (identified with $\left.[0,2 \pi]^{n}\right)$ as the pushforwards measure through the map sending $U$ to its eigenvalues. Some care is needed because of the ordering of the eigenvalues, but eventually this yields the existence of a measure $d m$ on $T_{n}$, invariant under the action of $S_{n}$, such that for $\mu, \nu$ with $l(\mu), l(\nu) \leq n$,

$$
\int_{T_{n}} s_{\mu}\left(e^{i \vartheta_{1}}, \ldots, e^{i \vartheta_{n}}\right) \overline{s_{\nu}\left(e^{i \vartheta_{1}}, \ldots, e^{i \vartheta_{n}}\right)} d m\left(\vartheta_{1}, \ldots, \vartheta_{m}\right)=\delta_{\mu \nu} .
$$

The density of the measure $d m$ with respect to the Lebesgue measure can be explicitly computed with the Weyl integration formula (see [98]):

Theorem 2.25. The density of dm with respect to the Lebesgue measure is given by

$$
d m\left(\vartheta_{1}, \ldots, \vartheta_{n}\right)=\frac{1}{(2 \pi)^{n} n!} \prod_{j<k}\left|e^{i \vartheta_{k}} e^{i \vartheta_{j}}\right|^{2} .
$$

### 2.4 Weingarten calculus for $U_{n}$

In the last section we have used representation theory to obtained probabilistic results on class functions. The goal is to extend this approach to any element of $C\left(U_{n}, \mathbb{C}\right)$. This has been done by Collins in [28, and more generally by Collins and Sniady in [29]. The content of this section comes mainly from [29]. This section is particularly exhaustive, since the generalization of the Weingarten formula to quantum groups is the main motivation of the first part of this thesis.

We have seen that the algebra of polynomial matrix coefficients is exactly $\mathbb{C}\left[u_{i j}\right]$. From Section 2.3.1, rational representations of $U_{n}$ are tensor products of polynomial representations with a one dimensional representation of the form $\operatorname{det}^{-m}=(\overline{\operatorname{det}})^{m}$ for $m \geq 0$. Since det $\in \mathbb{C}\left[u_{i j}\right]$, the algebra of matrix coefficients is therefore $\mathbb{C}\left(\left[u_{i j}, \bar{u}_{i j}\right]\right.$.
The main goal is to compute the integral

$$
\begin{equation*}
I_{\vec{i}, \overrightarrow{i^{\prime}}, \vec{j}, \vec{j}^{\prime}}=\int_{U_{n}} u_{i_{1} j_{1}} \ldots u_{i_{d} j_{d}} \bar{u}_{i_{1}^{\prime} j_{1}^{\prime}} \ldots \bar{u}_{i_{d^{\prime}}^{\prime} j_{d^{\prime}}^{\prime}} \tag{2.4.1}
\end{equation*}
$$

for $\vec{i}, \vec{j} \in \llbracket 1 ; n \rrbracket^{d}$ and $\overrightarrow{i^{\prime}}, \overrightarrow{j^{\prime}} \in \llbracket 1 ; n \rrbracket^{d^{\prime}}$. Note that we can always assume that $d=d^{\prime}$, since otherwise by invariance of the Haar measure by a scalar rotation,

$$
\begin{aligned}
\int_{U_{n}} u_{i_{1} j_{1}} \ldots u_{i_{n} j_{n}} \bar{u}_{i_{1}^{\prime} j_{1}^{\prime}} \ldots \bar{u}_{i_{d^{\prime}}^{\prime} j_{d^{\prime}}^{\prime}} & =\int_{U_{n}}\left(z u_{i_{1} j_{1}}\right) \ldots\left(z u_{i_{n} j_{n}}\right)\left(\bar{z} \bar{u}_{i_{1}^{\prime} j_{1}^{\prime}}\right) \ldots\left(\bar{z} \bar{u}_{i_{d^{\prime}}^{\prime} j_{d^{\prime}}^{\prime}}\right) \\
& =z^{d-d^{\prime}} \int_{U_{n}} u_{i_{1} j_{1}} \ldots u_{i_{n} j_{n}} \bar{u}_{i_{1}^{\prime} j_{1}^{\prime}} \ldots \bar{u}_{i_{d^{\prime}}^{\prime} j_{d^{\prime}}^{\prime}}=0 .
\end{aligned}
$$

### 2.4.1 The method

Weingarten calculus is based on the following observation: if $G$ is a compact group and $(V, \rho)$ is a representation of $G$, then $p_{\rho}: v \mapsto \int_{G} \rho(g) v d g$ is the orthonal projection (with respect to the invariant scalar product) on the vector space of fixed points of $(V, \rho)$. The latter is a consequence of the invariance of the Haar measure by left multiplication and of Theorem 2.10.
The idea is thus to consider the integral $(2.4 .1)$ as the average of an endomorphism of $\left(\mathbb{C}^{n}\right)^{\otimes d}$ with respect to the action of $U_{n}$ on $\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$. From the previous phenomenon, this average is a fixed point of $\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$, and thus an intertwiner of the representation of $U_{n}$ on $\left(\mathbb{C}^{n}\right)^{\otimes d}$. Relating intertwiners of $\left(\mathbb{C}^{n}\right)^{\otimes d}$ with a particular action of the symmetric group $S_{d}$ gives then a combinatorial formula for the integral (2.4.1).

Expressing integrals as elements of $\operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$ Let us denote by $E_{i j}$ the matrix $\left(\delta_{r i} \delta_{s j}\right)_{1 \leq r, s \leq n}$ in $M_{n}(\mathbb{C})$ (the latter is identified with $\operatorname{End}\left(\mathbb{C}^{n}\right)$ through the action on the canonical basis). Then $E_{j_{1} j_{1}^{\prime}} \otimes \cdots \otimes E_{j_{d j} j_{d}^{\prime}} \in \operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right) . U_{n}$ acts on $\left(\mathbb{C}^{n}\right)^{\otimes d}$ by the $d$-fold tensor product of the fundamental representation; thus as in Example 2.6, $U_{n}$ acts on $\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$ by conjugation. For $U \in U_{n}$,

$$
U E_{j_{1} j_{1}^{\prime}} U^{*} \otimes \cdots \otimes U E_{j_{d j}^{\prime} d_{d}^{\prime}} U^{*}=M_{j_{1} j_{1}^{\prime}} \otimes \cdots \otimes M_{j_{d j}^{\prime} j_{d}^{\prime}}
$$

with $M_{j_{r} j_{r}^{\prime}}$ being the matrix

$$
\left(\begin{array}{ccc}
U_{1 j_{r}} \bar{U}_{1 j_{r}^{\prime}} & \ldots & U_{1 j_{r}} \bar{U}_{n j_{r}^{\prime}} \\
\vdots & \ddots & \\
U_{n j_{r}} \bar{U}_{1 j_{r}^{\prime}} & & U_{n j_{r}} \bar{U}_{n j_{r}^{\prime}}
\end{array}\right)
$$

Thus $\operatorname{Tr}\left(M_{j_{r} j_{r}^{\prime}} E_{i_{r}^{\prime} i_{r}}\right)=U_{i_{r} j_{r}} \bar{U}_{i_{r}^{\prime} j_{r}^{\prime}}$ and

$$
\operatorname{Tr}\left(\left(M_{j_{1} j_{1}^{\prime}} \otimes \cdots \otimes M_{j_{d} j_{d}^{\prime}}\right)\left(E_{i_{1}^{\prime} i_{1}} \otimes \cdots \otimes E_{i_{d}^{\prime} i_{d}}\right)\right)=U_{i_{1} j_{1}} \ldots U_{i_{d} j_{d}} \bar{U}_{i_{1}^{\prime} j_{1}^{\prime}} \ldots \bar{U}_{i_{d}^{\prime} j_{d}^{\prime}}
$$

Integrating with respect to the Haar measure yields

$$
I_{\vec{i}, \vec{i}^{\prime}, \vec{j}, \vec{j}^{\prime}}=\operatorname{Tr}\left(\left(\int U E_{j_{1} j_{1}^{\prime}} U^{*} \otimes \cdots \otimes U E_{j_{d j_{d}^{\prime}}^{\prime}} U^{*} d U\right)\left(E_{i_{1}^{\prime} i_{1}} \otimes \cdots \otimes E_{i_{d}^{\prime} i_{d}}\right)\right)
$$

To emphasize the geometric aspect of the right hand-side, this equality can be written as

$$
\left.I_{\vec{i}, \vec{i}^{\prime}, \vec{j}, \vec{j}^{\prime}}=\left\langle p_{\operatorname{End}\left(\left(\mathbb{C}^{n}\right) \otimes d\right)}\left(E_{j_{1} j_{1}^{\prime}} \otimes \cdots \otimes E_{j_{d} j_{d}^{\prime}}\right), E_{i_{1} i_{1}^{\prime}} \otimes \cdots \otimes E_{i_{d} i_{d}^{\prime}}\right)\right\rangle
$$

with $\langle$,$\rangle being the invariant scalar product (A, B) \mapsto \operatorname{Tr}\left(A B^{*}\right)$ in $\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$. Since $p_{\operatorname{End}\left(\left(\mathbb{C}^{n}\right) \otimes d\right)}$ is a projector, the latter quantity is the same as $\left\langle p_{\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d)}\right.}\left(E_{j_{1} j_{1}^{\prime}} \otimes \cdots \otimes E_{j_{d j}^{\prime} j_{d}^{\prime}}\right), p_{\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)}\left(E_{i_{1} i_{1}^{\prime}} \otimes\right.\right.$ $\left.\left.\cdots \otimes E_{i_{d} i_{d}}^{\prime}\right)\right\rangle$. From Example 2.6, the space of fixed points of $\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$ is exactly $\operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d},\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$. Therefore evaluating the above scalar product requires a good description of $\operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d},\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$.

### 2.4.2 Schur-Weyl duality

Let $S_{d}$ denote the symmetric group of order $d . S_{d}$ is a finite group with cardinal $d$ !. By a result of Young (see [60] Part I, Ch. 7), the irreducible representations of $S_{d}$ are indexed by the integer partitions of $d$. The irreducible representation corresponding to $\lambda$ is also called the Specht module of the partition $\lambda$ and denoted by $S_{\lambda}$.
The representation theory of $S_{d}$ has a very rich combinatorial structure. Looking at the representations of $S_{d}$ for several $d$ yields in particular a link between the theory of representation of $S_{d}$ and symmetric functions :

Theorem 2.26 (Frobenius character formula, [60] p.114). Let $\mu \vdash d$ and $\nu \vdash d$ be two partitions. The value of the irreducible character of $S_{\mu}$ on a permutation $\sigma$ with cycle decomposition $\nu$ is

$$
\begin{equation*}
\chi_{\mu}(\sigma)=\left\langle s_{\mu}, p_{\nu}\right\rangle \tag{2.4.2}
\end{equation*}
$$

where the scalar product on the right hand side is the Hall inner product on $\Lambda$.
$S_{d}$ acts also on $\left(\mathbb{C}^{n}\right)^{\otimes d}$ by permuting the entries of the tensor product. Namely for $\sigma \in S_{d}$, the representation $\left(\left(\mathbb{C}^{n}\right)^{\otimes d}, w\right)$ is defined by

$$
w(\sigma)\left(v_{1} \otimes \cdots \otimes v_{d}\right)=v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)} .
$$

This action commutes with the action of $U_{n}$ on each component of the tensor product and thus $w(\sigma) \in \operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d},\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$. Moreover

$$
\left\langle E_{j_{1} j_{1}^{\prime}} \otimes \cdots \otimes E_{j_{d j} j_{d}^{\prime}}, w(\sigma)\right\rangle=\delta_{j_{1} \sigma\left(j_{1}^{\prime}\right)} \cdots \delta_{j_{d} \sigma\left(j_{d}^{\prime}\right)} .
$$

Therefore we know a particular subset of $\operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d},\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$, namely the set $\{w(\sigma)\}_{\sigma \in S_{d}}$, for which the scalar product with $E_{j_{1} j_{1}^{\prime}} \otimes \cdots \otimes E_{j_{d j_{d}^{\prime}}^{\prime}}$ is particularly simple. The question is to know whether the knowledge of all these scalar products is enough to reconstruct $p_{\text {End }\left(\left(\mathbb{C}^{n}\right) \otimes d\right)}\left(E_{j_{1} j_{1}^{\prime}} \otimes\right.$ $\left.\cdots \otimes E_{j_{d} j_{d}^{\prime}}\right)$. The answer is positive if $\{w(\sigma)\}_{\sigma \in S_{d}}$ spans the vector space $\operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d},\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$. This is exactly the content of the Schur Weyl duality.

Theorem 2.27 (Schur-Weyl duality, [26], Ch.36). As a $U_{n} \times S_{d}$-representation, $\left(\mathbb{C}^{n}\right)^{\otimes d}$ decomposes as

$$
\left(\mathbb{C}^{n}\right)^{\otimes d}=\bigoplus_{\lambda, l(\lambda) \leq n} V_{\lambda} \otimes S_{\lambda} .
$$

In particular, the action $w$ of $S^{d}$ yields a surjective map

$$
w: \mathbb{C}\left[S_{d}\right] \rightarrow \operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d},\left(\mathbb{C}^{n}\right)^{\otimes d}\right)
$$

which restricts to an isomorphism

$$
\tilde{w}=\bigoplus_{\lambda, l(\lambda) \leq n} M_{d(\lambda)}(\mathbb{C}) \rightarrow \operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d},\left(\mathbb{C}^{n}\right)^{\otimes d}\right),
$$

where $\mathbb{C} S_{d}$ is identified with $\bigoplus_{\lambda \vdash d} M_{d(\lambda)}(\mathbb{C})$ as a semi-simple algebra.

The decomposition of $\mathbb{C} S_{d}$ as a direct sum of matrix algebras indexed by irreducible representations of $S_{d}$ is the content of Artin-Wedderburn's Theorem (see [74], Section 1.10).
We will give here a proof of Theorem 2.27 based on the Frobenius character formula. Note however that its apparent simplicity is misleading, since the Frobenius character formula is a nontrivial result.

Proof. The action of $S_{d}$ and $U_{n}$ commutes, yielding an action $\varphi$ of $S_{d} \times U_{n}$ on $\left(\mathbb{C}^{n}\right)^{\otimes d}$. We can thus decompose $\left(\mathbb{C}^{n}\right)^{\otimes d}=\bigoplus_{\rho \text { irred. of } U_{n}} V_{\rho} \otimes W_{\rho}$, where $V_{\rho}$ the irreducible representation of $U_{n}$ with character $s_{\rho}$ and $W_{\rho}$ a representation of $S_{d}$.
Moreover one can prove (see [57], Section 2) that if $\mu=\left(\mu_{1} \geq \cdots \geq \mu_{r}\right)$ is the cycle decomposition of $\sigma$ and $e^{i \vartheta_{1}}, \ldots, e^{i \vartheta_{n}}$ are the eigenvalues of $U \in U_{n}$, then

$$
\begin{equation*}
\chi_{\varphi}(\sigma, U)=\operatorname{Tr}\left(U^{\mu_{1}}\right) \operatorname{Tr}\left(U^{\mu_{2}}\right) \ldots \operatorname{Tr}\left(U^{\mu_{r}}\right)=p_{\mu}\left(e^{i \vartheta_{1}}, \ldots, e^{i \vartheta_{n}}\right) . \tag{2.4.3}
\end{equation*}
$$

For example if $\sigma$ is just the cycle $[1, \ldots, d]$,

$$
\begin{aligned}
\chi_{\varphi}(\sigma, U) & =\sum_{1 \leq i_{1}, \ldots, i_{d} \leq n}\left(U e_{i_{2}} \otimes \cdots \otimes U e_{i_{d}} \otimes U e_{i_{1}}, e_{i_{1}} \otimes \cdots \otimes e_{i_{d}}\right) \\
& =\sum_{1 \leq i_{1}, \ldots, i_{d} \leq n} U_{i_{1} i_{2}} \ldots U_{i_{n-1} i_{n}} U_{i_{n} i_{1}}=\operatorname{Tr}\left(U^{d}\right) .
\end{aligned}
$$

Since $\chi_{\varphi}(\sigma, U)=\sum_{(V, \rho) \text { irred of } U_{n}} s_{\rho}(U) \chi_{W_{\rho}}(\sigma)$ and the Schur functions $\left\{s_{\rho}\right\}_{l(\rho) \leq n}$ form an orthonormal basis of the class functions in $L^{2}\left(U_{n}, \int_{U_{n}}\right)$,

$$
\chi_{W_{\rho}}(\sigma)=\int_{U_{n}} \chi_{\varphi}(\sigma, U) \overline{s_{\rho}(U)} d U
$$

Therefore $\chi_{W_{\rho}}(\sigma)=\int_{U_{n}} p_{\mu}(U) \overline{s_{\rho}(U)} d U=\left\langle p_{\mu}, s_{\rho}\right\rangle_{\Lambda_{n}}$. Since $l(\rho) \leq n,\left\langle p_{\mu}, s_{\rho}\right\rangle_{\Lambda_{n}}=\left\langle p_{\mu}, s_{\rho}\right\rangle_{\Lambda}$. Thus by the Frobenius character formula (2.4.2), $\chi_{W_{\rho}}(\sigma)=\chi_{\rho}$ and $W_{\rho} \simeq S_{\rho}$, the Specht module of the partition $\rho$.
 with $A, B$ two elements of $\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$. To each element $A$, one can associate the function $f_{A}$ on $S_{d}$ defined by $f_{A}(\sigma)=\langle A, w(\sigma)\rangle$; by the Schur-Weyl duality, the intertwiner space of $\left(\mathbb{C}^{n}\right)^{\otimes d}$ is spanned by $\{w(\sigma)\}_{\sigma \in S_{d}}$ and thus the data of $f_{A}$ and $f_{B}$ is enough to compute $\left\langle p_{\operatorname{End}\left(\left(\mathbb{C}^{d}\right)^{\otimes d}\right.}(A), p_{\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)}(B)\right\rangle$. The matter is therefore to relate exactly $\left\langle p_{\operatorname{End}\left(\left(\mathbb{C}^{d}\right)^{\otimes d}\right.}(A), p_{\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)}(B)\right\rangle$ to $\left(f_{A}, f_{B}\right)_{L^{2}\left(S^{d}\right)}$.

### 2.4.3 Convolution algebra

Convolution algebra Let $G$ be a compact group. We have seen in Section 2 that $L^{2}\left(G, \int_{G}\right)=$ $\bigoplus_{(V, \rho \text { irred })} \bigoplus_{1 \leq i, j \leq d(\rho)} \rho_{i j}$.
Let $(V, \rho)$ be an irreducible representation. We identify $\operatorname{End}(V)$ with $M_{d(\rho)}(\mathbb{C})$ through the particular orthogonal basis $\left(e_{1}, \ldots, e_{d(\rho)}\right)$ chosen in Section 2.1.3. Thus there is a linear map $\Phi_{\rho}: M_{d(\rho)}(\mathbb{C}) \rightarrow L^{2}\left(G, \int_{G}\right)$ sending $E_{i j}$ to $d(\rho) \rho_{i j}$, and this linear map is an isomorphism onto the vector space $C_{\rho}$ of matrix coefficients of the irreducible representation $\rho$. $\Phi_{\rho}$ maps $A \in M_{d(\rho)}(\mathbb{C})$ to

$$
g \mapsto d(\rho) \operatorname{Tr}\left(\rho(g) A^{*}\right) .
$$

However $M_{d(\rho)}(\mathbb{C})$ has a richer structure given by the matrix multiplication, and by isomorphism this structure transposes to the vector space $\bigoplus_{1 \leq i, j \leq d(\rho)} \rho_{i j}$.

Definition 2.28. The convolution algebra on $G$ is the $*-\operatorname{algebra}(C(G, \mathbb{C}), *)$, with the product * given by

$$
f_{1} * f_{2}(g)=\int_{G} f_{1}(h) f_{2}\left(h^{-1} g\right) d h .
$$

The involution $f^{*}$ is given by

$$
f^{*}(g)=\bar{f}\left(g^{-1}\right) .
$$

There exists a state $\varepsilon$ on this algebra which is defined by the formula $f \mapsto f(e)$, where $e$ is the unit in $G$.

Note in particular that $\varepsilon\left(f_{1} * f_{2}^{*}\right)=\int_{G} f_{1}(h) \bar{f}_{2}(h) d h=\left\langle f_{1}, f_{2}\right\rangle_{L^{2}\left(G, \int_{G}\right)}$.
By Schur orthogonality's Theorem, if ( $e_{i}$ ) (resp. $\left(f_{i}\right)$ ) is the chosen basis of $(V, \rho)$ (resp. $\left(V^{\prime}, \rho^{\prime}\right)$ ),

$$
\begin{aligned}
\Phi_{\rho}\left(E_{i j}^{\rho}\right) \Phi_{\rho^{\prime}}\left(E_{k l}^{\rho^{\prime}}\right)=d(\rho) d\left(\rho^{\prime}\right) \rho_{i j} * \rho_{k l}^{\prime}(g) & =d(\rho) d\left(\rho^{\prime}\right) \int_{G}\left\langle\rho(h) e_{j}, e_{i}\right\rangle\left\langle\rho^{\prime}\left(h^{-1} g\right) f_{l}, f_{k}\right\rangle d h \\
& =d(\rho) d\left(\rho^{\prime}\right) \int_{G}\left\langle\rho(h) e_{j}, e_{i}\right\rangle \overline{\left\langle\rho^{\prime}(h) f_{k}, \rho(g) f_{l}\right\rangle} d h \\
& =d(\rho) \delta_{\rho \rho^{\prime}} \delta_{k l} \rho_{i l}=\Phi_{\rho}\left(E_{i j}^{\rho} E_{k l}^{\rho}\right) .
\end{aligned}
$$

Since $\Phi_{\rho}\left(E_{i j}^{*}\right)=\Phi_{\rho}\left(E_{j i}\right)=d(\rho) \rho_{j i}$ and $\rho_{j i}=\left\langle\rho(g) e_{i}, e_{j}\right\rangle=\overline{\left\langle\rho\left(g^{-1}\right) e_{j}, e_{i}\right\rangle}$ yields also $\Phi_{\rho}\left(E_{i j}^{*}\right)=$ $\Phi_{\rho}\left(E_{i j}\right)^{*}, \Phi_{\rho}$ is a $*-$ algebra isomorphism. With this isomorphism, the scalar product given by the trace on $M_{d(\rho)}(\mathbb{C})$ gives the scalar product $\frac{1}{d(\rho)}\langle.,$.$\rangle on C_{\rho}$.

Fourier transform of a representation Let $(W, \vartheta)$ be a finite dimensional representation of $G$, and let $\mathcal{A}_{\vartheta}$ be the matrix algebra generated by $\{\vartheta(g)\}_{g \in G}$. Since $W$ is finite dimensional, $W=\bigoplus_{(V, \rho) \text { irred }} V^{\oplus r_{\rho}} \simeq \bigoplus_{(V, \rho)} \mathbb{C}^{r_{\rho}} \otimes V$, with $\sum r_{\rho}<\infty$. Since $G$ doesn't act on the left of each tensor product, $A \in \mathcal{A}_{\vartheta}$ has the form

$$
\begin{equation*}
A=\bigoplus_{(V, \rho) \text { irred }} I d_{\mathbb{C}^{r_{\rho}}} \otimes A_{\rho}, \tag{2.4.4}
\end{equation*}
$$

with $A_{\rho} \in M_{d(\rho)}(\mathbb{C})$.

Definition 2.29. The Fourier transform of $A \in \operatorname{End}(W)$ is the function $f_{A} \in C(G, \mathbb{C})$ defined by

$$
f_{A}(g)=\operatorname{Tr}\left(\vartheta(g) A^{*}\right) .
$$

Since on $W \simeq \bigoplus_{(V, \rho) \text { irred. }} \mathbb{C}^{r_{\rho}} \otimes V, \vartheta(g)$ has the form $\vartheta(g)=\bigoplus_{(V, \rho) \text { irred. }} I d_{r_{\rho}} \otimes \rho(g)$, for $A \in \mathcal{A}_{\vartheta}$

$$
f_{A}(g)=\sum_{(V, \rho) \text { irred. }} r_{\rho} \operatorname{Tr}\left(A_{\rho}^{*} \rho(g)\right)=\sum_{(V, \rho)} \frac{r_{\rho}}{d(\rho)} \Phi\left(A_{\rho}\right),
$$

where $A$ has been decomposed as in (2.4.4.
Let $B \in \mathcal{A}_{\vartheta}$ be another operator with the expansion $B=\bigoplus_{(V, \rho) \text { irred }} I d_{\mathbb{C}^{r} \rho} \otimes B_{\rho}$. Then on one hand,

$$
\operatorname{Tr}\left(A B^{*}\right)=\sum_{(V, \rho)} r_{\rho} \operatorname{Tr}\left(A_{\rho} B_{\rho}^{*}\right)=\sum_{(V, \rho)} \frac{r_{\rho}}{d(\rho)}\left\langle\Phi\left(A_{\rho}\right), \Phi\left(B_{\rho}\right)\right\rangle,
$$

and on the other hand in $C(G, \mathbb{C})$,

$$
\left\langle f_{A}, f_{B}\right\rangle_{L^{2}(G)}=\sum_{(V, \rho)}\left(\frac{r_{\rho}}{d(\rho)}\right)^{2}\left\langle\Phi\left(A_{\rho}\right), \Phi\left(B_{\rho}\right)\right\rangle .
$$

Thus if we denote by $P_{\vartheta}$ the operator multiplying each element of $C_{\rho}$ by $\frac{d(\rho)}{r(\rho)}$, then

$$
\operatorname{Tr}\left(A B^{*}\right)=\left\langle P_{\vartheta} f_{A}, f_{B}\right\rangle=\left(P_{\vartheta} * f_{A} * f_{B}^{*}\right)(e) .
$$

Characters in the convolution algebra Since $P_{\vartheta}$ is diagonal on each space $C_{\rho}$, it lies in the center of $(C(G, \mathbb{C}), *)$. Note that for $g, g^{\prime} \in G$;

$$
f\left(g g^{\prime}\right)=\int_{G} \delta_{g^{-1}}(h) f\left(h^{-1} g^{\prime}\right) d h=\left(\delta_{g^{-1}} * f\right)\left(g^{\prime}\right)
$$

and

$$
\left(f * \delta_{g^{-1}}\right)\left(g^{\prime}\right)=\int_{G} f(h) \delta_{g^{-1}}\left(h^{-1} g^{\prime}\right) d h=f\left(g^{\prime} g\right) .
$$

Thus a function $f$ lies in the center $\mathcal{Z}$ of $(C(G, \mathbb{C}), *)$ if and only if for all $g, g^{\prime} \in G, f\left(g g^{\prime}\right)=$ $f\left(g^{\prime} g\right)$. This means that the center of the convolution algebra coincides with the space of class functions, and has a basis consisting in the irreducible characters $\left\{\chi_{\rho}\right\}_{\rho \text { irred }}$.
Moreover if $\rho$ is an irreductible representation, then by the Schur orthogonality's Theorem,

$$
\chi_{\rho} * \rho_{i j}^{\prime}(g)=\frac{1}{d(\rho)} \delta_{\rho, \rho^{\prime}} \rho_{i j}^{\prime} .
$$

Thanks to this formula, we can express the operator $P_{\vartheta}$ above as :

$$
P_{\vartheta}=\sum_{(V, \rho) \text { irred }} \frac{d(\rho)^{2}}{r_{\rho}} \chi_{\rho}
$$

Weingarten Calculus for $U_{n}$ Let $W g$ denotes the function $P_{\left(\mathbb{C}^{n}\right) \otimes d}$. Applying the result of the last paragraph to the representations of $S_{d}$ on $\left(\mathbb{C}^{n}\right)^{\otimes d}$ in order to compute $\left\langle p_{\text {End }\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)}\left(E_{j_{11} j_{1}^{\prime}} \otimes\right.\right.$ $\left.\left.\left.\cdots \otimes E_{j_{d} j_{d}^{\prime}}\right), E_{i_{1}^{\prime} i_{1}} \otimes \cdots \otimes E_{i_{d}^{\prime} i_{d}}\right)\right\rangle$ yields

$$
\begin{aligned}
I_{\vec{i}, \vec{i}^{\prime}, \vec{j}, \vec{j}^{\prime}} & \left.=\left\langle p_{\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)}\left(E_{j_{1} j_{1}^{\prime}} \otimes \cdots \otimes E_{j_{d j_{d}^{\prime}}}\right), p_{\operatorname{End}\left(\left(\mathbb{C}^{n}\right)^{\otimes d}\right)}\left(E_{i_{1} i_{1}^{\prime}} \otimes \cdots \otimes E_{i_{d} i_{d}^{\prime}}\right)\right)\right\rangle \\
& =\left(W g * f_{E_{j_{1} j_{1}^{\prime}}} \otimes \cdots \otimes E_{j_{d j_{d}^{\prime}}} * f_{E_{i_{1} i_{1}^{\prime}}^{*}}^{*} \otimes \otimes E_{i_{d} i_{d}^{\prime}}\right)(e)
\end{aligned}
$$

Computing the last product in the convolution algebra gives the Weingarten formula obtained by Collins in [28]:

Theorem 2.30.

$$
I_{\vec{i}, \vec{i}^{\prime}, \vec{j}, \vec{j}^{\prime}}=\sum_{\sigma, \tau \in S_{d}} \delta_{j_{1} \sigma\left(j_{1}^{\prime}\right)} \ldots \delta_{j_{d} \sigma\left(j_{d}^{\prime}\right)} \delta_{i_{1} \tau\left(i_{1}^{\prime}\right)} \ldots \delta_{i_{d} \tau\left(j_{d}^{\prime}\right)} W g\left(\sigma \tau^{-1}\right) .
$$

with $W g(\sigma)=\frac{1}{(d!)^{2}} \sum_{\lambda \vdash d, l(\lambda) \leq n} \frac{d(\lambda)^{2}}{\operatorname{dim} V_{\lambda}} \chi_{\lambda}(\sigma)$.

### 2.5 Application of the Weingarten calculus

We review here some applications of the Weingarten calculus. The main motivation of [28] for developing the Weingarten calculus was to compute the coefficients of the so-called ItzyksonZuber integrals $(z, X, Y) \mapsto \int_{U_{n}} \exp \left(n z \operatorname{Tr}\left(X U Y U^{*}\right) d U\right.$. However we will only give results concerning asymptotic freeness and second order freeness, since the latter involve free probability. One should refer to [28] for more details on the asymptotic expansion of the Itzykson-Zuber integrals.

### 2.5.1 Asymptotic of the Weingarten function and pair partitions

All the results of this section rely on the asymptotic value of the Weingarten function. Suppose from now on that $n \geq d$. In this case, $W g(\sigma)=\sum_{\lambda \vdash d} \frac{d(\lambda)^{2}}{\operatorname{dim} V_{\lambda}} \chi_{\lambda}(\sigma)$ and $W g$ is precisely the inverse of the character $\chi_{\left(\mathbb{C}^{n}\right) \otimes d}$ in the convolution algebra $S_{d}$. Applying expression 2.4.3) to $\sigma \times I d$ yields

$$
\chi_{\left(\mathbb{C}^{n}\right)^{\otimes d}}=n^{c(\sigma)},
$$

where $c(\sigma)$ is the number of cycles of $\sigma$. Since $\chi_{\left(\mathbb{C}^{n}\right)^{\otimes d}}$ is polynomial in $n$, it is expected that $W g$ is rational in $n$. Actually, Collins proved in [28] that

$$
W g(\sigma)=\sum_{k \geq 0} a_{k}(\sigma) n^{-k-(2 d-c(\sigma))},
$$

and that

$$
a_{0}(\sigma)=(d!)^{2} \prod_{\mu_{i}} l_{\mu_{i}-1}(-1)^{\mu_{i}-1},
$$

where $\mu$ is the partition coming from the cycle decomposition of $\sigma$ and $l_{k}=\frac{(2 k)!}{k!(k+1)!}$ is the $k$-th Catalan number. There exists also a combinatorial description of the other coefficients $a_{k}$.
A simpler proof for the expression of $a_{0}$, based on Biane's algebra (see [20]), has been given by Collins and Sniady in [29]. Note in particular that the expression of $a_{0}$ is a particular value of the Moebius function of the lattice of non-crossing partitions (see Section 1.2.2).
The higher order term in the expansion is given by the value of $W g$ on the identity, with $W g(\sigma)=(d!)^{2} n^{-d} \delta_{e \sigma}\left(1+O\left(n^{-1}\right)\right)$. An independent proof of this first order expansion can be given using a scalar product on $\mathbb{C} S_{d}$.
Let (.,.) be the scalar product defined on $\mathbb{C} S_{d}$ by the formula $(\sigma, \tau)=n^{c\left(\sigma^{-1} \tau\right)}$ and let $G_{n d}$ be the scalar product matrix $((\sigma, \tau))_{\sigma, \tau \in S_{d}}$. With these notations $\left(G_{n d}\right)_{\sigma \tau}=\chi_{\left(\mathbb{C}^{n}\right)^{\otimes d}}\left(\sigma^{-1} \tau\right)$. Thus if we set $G_{n d}^{-1}=\left(a_{\sigma}(\tau)\right)_{\sigma, \tau \in S_{d}}$, then the functions $a_{\tau}$ have to satisfy the relation

$$
\sum_{\mu} a_{\sigma}(\mu) \chi_{\left(\mathbb{C}^{n}\right)^{\otimes d}}\left(\mu^{-1} \tau\right)=\delta_{\sigma \tau}
$$

In the convolution algebra of $S_{d}$, this means that $a_{\sigma} * \chi_{\left(\mathbb{C}^{n}\right) \otimes d}=\frac{1}{d!} \delta_{\sigma}$. Therefore $a_{\sigma}=\frac{1}{d!} \delta_{\sigma} * W g$ and

$$
\left(G_{n d}^{-1}\right)_{\sigma \tau}=a_{\sigma}(\tau)=\frac{1}{d!} \sum_{h} \frac{1}{d!} \delta_{\sigma}(h) W g\left(h^{-1} \tau\right)=\frac{1}{(d!)^{2}} W g\left(\sigma^{-1} \tau\right)
$$

Thus $\left(G_{n d}^{-1}\right)_{\sigma \tau}=\frac{1}{(d!)^{2}} W g\left(\sigma^{-1} \tau\right)$. On the other hand since $c\left(\sigma \tau^{-1}\right)<d$ for $\sigma \tau^{-1} \neq e, G_{n d}=$ $n^{d}\left(I d+o\left(n^{-1}\right)\right)$. Inverting $G_{n d}$ in the latter first order expansion yields $\left(G_{n d}^{-1}\right)=n^{-d}(I d+$ $o\left(n^{-1}\right)$ ), which gives the first order expansion of the Weingarten formula.
The method using the Gram-Schmidt matrix of the scalar product on the intertwiner spaces of $U_{n}$ will be generalized to a large class of quantum groups in Chapter 5.

### 2.5.2 Asymptotic freeness of unitary invariant random matrices

Second order freeness A second-order probability space intends to capture both expectations and fluctuations of non-commutative random variables. Second-order probability spaces and second-order freeness have been introduced by Mingo and Speicher in [64] to express the flucutations of large random matrices. Unless specified otherwise, all the content of this subsection comes from [64].

Definition 2.31. A second-order probability space is the data of a probability space $(A, \varphi)$ with a bilinear functional $\tilde{\varphi}: A \times A \rightarrow \mathbb{C}$ which is tracial in both arguments and such that $\tilde{\varphi}\left(., \mathbf{1}_{A}\right)=$ $\tilde{\varphi}\left(\mathbf{1}_{A},.\right)=0$.

The natural construction of a second order probability space is made by considering an algebra $A$ together with a linear map $\vartheta: A \rightarrow L^{\infty-}(\Omega)$ sending $\mathbf{1}_{A}$ to the constant function $\mathbf{1}_{\Omega}$, and such that $\vartheta\left(a^{*}\right)=\overline{\vartheta(a)}$. Then $(A, \vartheta)$ yields a second-order probability space with the maps $\varphi, \tilde{\varphi}$ defined by

$$
\varphi(a)=\mathbb{E}(\vartheta(a)), \tilde{\varphi}(a, b)=\operatorname{Cov}(\vartheta(a), \vartheta(b))
$$

Note that $\vartheta$ does not need to be an algebra homomorphism. Actually if $\vartheta$ is an algebra homomorphism, the map $\tilde{\varphi}$ doesn't give further information than $\varphi$, since in this case

$$
\operatorname{Cov}(\vartheta(a), \vartheta(b))=\mathbb{E}(\vartheta(a b))-\mathbb{E}(\vartheta(a)) \mathbb{E}(\vartheta(b))=\varphi(a b)-\varphi(a) \varphi(b)
$$

Example 2.32. Let $A=M_{n} \otimes L^{\infty-}(\Omega)$. The trace maps any random matrix to a random variable, and therefore from the discussion above, $\left(A, \mathbb{E}\left(\frac{1}{n} \operatorname{Tr}().\right), \operatorname{Cov}\left(\frac{1}{n} \operatorname{Tr}(),. \frac{1}{n} \operatorname{Tr}().\right)\right)$ is a secondorder probability space.

Definition 2.33. Let $(A, \varphi, \tilde{\varphi})$ be a second-order probability space and $A_{1}, \ldots, A_{r}$ be subalgebras of $A . A_{1}, \ldots, A_{r}$ are called second-order free if they are free and if for all centered elements $a_{1}, \ldots, a_{p}, b_{1}, \ldots b_{p^{\prime}}$ with $a_{i} \in A_{k_{i}}, b_{j} \in A_{k_{j}^{\prime}}, k_{i} \neq k_{i+1}, k_{j}^{\prime} \neq k_{j+1}^{\prime}, k_{p} \neq k_{1}$ and $k_{p^{\prime}}^{\prime} \neq k_{1}^{\prime}$,

- if $p=p^{\prime}=1$ and $k_{1} \neq k_{1}^{\prime}, \tilde{\varphi}\left(a_{1}, b_{1}\right)=0$.
- otherwise

$$
\tilde{\varphi}\left(a_{1} \ldots a_{p}, b_{p^{\prime}} \ldots b_{1}\right)=\delta_{p p^{\prime}} \sum_{i=0}^{p-1} \varphi\left(a_{1} b_{1+i}\right) \ldots \varphi\left(a_{p} b_{p+i}\right)
$$

where the indices are understood modulo $p$.
As for freeness, second order freeness allows to recover $\tilde{\varphi}$ from the value of $\varphi$ and $\tilde{\varphi}$ on each subalgebra $A_{i}$.

Second-order limit distribution Let $\left\{A_{n}\right\}_{n \geq 1}$ be a family of unital $*$-algebras and let $\left\{\vartheta_{n}\right\}_{n \geq 1}$ be a family of linear maps $\vartheta_{n}: A_{n} \rightarrow L^{\infty-}(\Omega, \mathbb{C})$ with $\vartheta_{n}\left(a^{*}\right)=\overline{\vartheta_{n}(a)}$ for any $a \in A_{n}$. A sequence $\left(\left(a_{1}^{n}, \ldots, a_{p}^{n}\right)\right)_{n \geq 1}$ of $p$-tuples $\left(a_{1}^{n}, \ldots, a_{p}^{n}\right)$ in $A_{n}$ has a second-order limit distribution $(\varphi, \tilde{\varphi})$ if and only if

- the family $\left(\vartheta_{n}\left(P\left(a_{1}^{n}, \ldots, a_{p}^{n}\right)\right)\right)_{P \in \mathbb{C}<X_{i}>_{1 \leq i \leq p}}$ converges in moment to a family of complex gaussian variables $(\vartheta(P))_{P \in \mathbb{C}<X_{i}>_{1 \leq i \leq p}}$. The expectation of $\vartheta(P)$ is given by a functional $\varphi: \mathbb{C}<X_{i}>\rightarrow \mathbb{C}$ and covariances are given by a bilinear functional $\tilde{\varphi}: \mathbb{C}<X_{i}>\times \mathbb{C}<$ $X_{i}>\longrightarrow \mathbb{C}$.
- the space $\mathbb{C}<X_{1}, \ldots, X_{p}>$ with the functional $\varphi$ and $\tilde{\varphi}$ is a second-order probability space.

Note that $\mathbb{C}<X_{1}, \ldots, X_{p}>$ denotes here the $*$-algebra of noncommutative polynomials in $X_{i}, X_{i}^{*}$. We will write $X_{i}^{*}=X_{i}^{-1}$ in the sequel.

Example 2.34. To illustrate the meaning of a second-order limit distribution, let us consider the convergence result of Diaconis and Shahshahani in Theorem 2.24. Let $\mathbb{C}\left[U_{n}\right]$ be the unital algebra of polynomials in $U(n)$, with $U(n)$ being a unitary matrix chosen randomly according to the Haar
measure on $U_{n}$, and let $\vartheta_{n}: \mathbb{C}\left[U_{n}\right] \rightarrow L^{\infty-}\left(U_{n}, \int_{U_{n}}\right)$ define by $\vartheta_{n}(B)=\operatorname{Tr}(B)-\frac{n-1}{n} \mathbb{E}(\operatorname{Tr}(B))$. Then from Theorem 2.24, $U(n)$ has a second-order limit distribution $(\mathbb{C}[X], \varphi, \tilde{\varphi})$ with $\varphi\left(X^{i}\right)=0$ and $\tilde{\varphi}\left(X^{i}, X^{j}\right)=\delta_{i(-j)^{i}}$.

Definition 2.35. Let $\left\{A_{n}\right\}_{n \geq 1}$ be a family of unital algebras and let $\left\{\vartheta_{n}\right\}$ be a family of linear maps $\vartheta_{n}: A_{n} \mapsto L^{\infty-}(\Omega, \mathbb{C})$. A couple of sequences $\left(\left(a_{1}^{n}, \ldots, a_{p}^{n}\right),\left(b_{1}, \ldots, b_{q}^{n}\right)\right)_{n \geq 1}$ is asymptotically second order free if and only if the sequence of $p+q$ elements $\left(\left(a_{1}^{n}, \ldots, a_{p}^{n}, b_{1}, \ldots, b_{q}^{n}\right)\right)_{n \geq 1}$ has a second order limit distribution $\left(\mathbb{C}<X_{1}, \ldots, X_{p}, Y_{1}, \ldots, Y_{q}>, \varphi, \tilde{\varphi}\right)$ such that $\mathbb{C}<X_{i}>$ and $\mathbb{C}<Y_{i}>$ are second order free in $\left(\mathbb{C}<X_{1}, \ldots, X_{p}, Y_{1}, \ldots, Y_{q}>, \varphi, \tilde{\varphi}\right)$.

Second order freeness for random matrices The two following theorems are two striking applications of the asymptotic computation of the Weingarten formula. The first one is directly based on the evalutation of integrals of type (2.4.1) with the Weingarten calculus :

Theorem 2.36. [[63], [62]] For each $n \geq 1$ let $U_{n}(1), \ldots, U_{n}(p)$ be p independent Haar-distributed unitary matrices of dimension $n$. Let $A_{n}=\mathbb{C}\left[U_{n}(1), \bar{U}_{n}(1), \ldots, U_{n}(p), \bar{U}_{n}(p)\right]$ with $\vartheta_{n}=\operatorname{Tr}()-$. $\frac{n-1}{n} \int \operatorname{Tr}($.$) .$
Then the family $\left(U_{n}(1), \bar{U}_{n}(1), \ldots, U_{n}(p), \bar{U}_{n}(p)\right)$ is asymptotically second order free and the second order distribution $\left(\mathbb{C}\left[X_{i}\right], \varphi, \tilde{\varphi}\right)$ (resp. $\left.\mathbb{C}\left[Y_{i}\right]\right)$ of each $U_{n}(i)$ (resp. $\bar{U}_{n}(i)$ ) is given by

$$
\varphi\left(X_{i}^{k}\right)=0, \tilde{\varphi}\left(X_{i}^{k}, X_{i}^{k^{\prime}}\right)=\delta_{k,-k^{\prime}} k
$$

A generalization of this result is given for the free unitary group in Chapter 5.
The second theorem should be seen as a generalization of the asymptotic freeness result of Voiculescu on independent random matrices in [92]. We consider here $\left(A_{n}, \vartheta_{n}\right)$ as the $*-$ algebra of random matrices with the $*-$ linear map $\vartheta_{n}=\operatorname{Tr}()-.\frac{n-1}{n} \mathbb{E}(\operatorname{Tr}()$.$) . An n$-dimensional random matrix $A$ is said unitarily invariant if the law of $A$ is the same as the law of $U A U^{*}$ for all $U \in U_{n}$.

Theorem 2.37. [[63]]] Let $\left(\left[A_{n}^{1}, \ldots, A_{n}^{p}\right]\right)_{n \geq 1}$ and $\left(\left[B_{n}^{1}, \ldots, B_{n}^{q}\right]\right)_{n \geq 1}$ be two sequences of random matrices, each of them having a second order limit distribution. Suppose moreover that the entries of $\left[A_{n}^{1}, \ldots, A_{n}^{p}\right]$ are independent to the ones of $\left[B_{n}^{1}, \ldots, B_{n}^{q}\right]$, and that the law of each $B_{n}^{i}$ is unitarily invariant.
Then $\left(\left[A_{n}^{1}, \ldots, A_{n}^{p}\right]\right)_{n \geq 1}$ and $\left(\left[B_{n}^{1}, \ldots, B_{n}^{q}\right]\right)_{n \geq 1}$ are asymptotically second order free.
The Weingarten calculs is the cornerstone of the proof. Indeed, since the law of each $B_{n}^{i}$ is unitarily invariant, any expectation of products of $B_{n}^{i}$ results in elements of $\operatorname{Mor}_{U_{n}}\left(\left(\mathbb{C}^{n}\right)^{\otimes d},\left(\mathbb{C}^{n}\right)^{\otimes d}\right)$. Therefore computing expectations of traces of products of these matrices with the ones of $\left[A_{n}^{1}, \ldots, A_{n}^{p}\right]$ yields projections on the space of intertwiners, and thus the use of the machinery of Section 5 .

### 2.6 Generalization to other groups: the Tannaka-Krein duality

We have seen that the computation of an integral with respect to the Haar measure was done through the following procedure :

1. Relate the integral to the scalar product of the projection of two operators on the space of intertwiners of the group.
2. Find a spanning set of the space of intertwiners, on which orthogonal projections have a straightforward expression.
3. Use the properties of the space of intertwiners to recover the scalar product from the orthogonal projections on each element of the spanning set.

Although the first step is a straightforward, the second and the third ones depend heavily on the group and may not be possible. Fortunately there exist some groups for which the second and third steps are still feasible. This will lead to define a category of groups whose intertwiners have a nice combinatorial description (see Section 3.2 and [15]). This combinatorial description is greatly simplified by the rich structure underlying the intertwiner spaces. This structure is given by the Tannaka-Krein duality.

### 2.6.1 Tannaka-Krein duality for compact matrix group

The Tannaka-Krein duality describes the operations which exist on intertwiner spaces, and establish a bijection between compact groups and collections of spaces stable under these operations. We will only give the result in the case of a matrix compact group but a similar result exists for general compact groups.
Let $G \subseteq U_{n}$ be a compact subgroup of $U_{n}$. There exists a natural family or representations of $G$, indexed by words in $\{\circ, \bullet\}$. Namely let $\left(V^{\circ}, \rho^{\circ}\right)$ be the fundamental representation of $G$ given by the identity morphism $\rho^{\circ}: G \rightarrow U_{n}$ and let $\left(V^{\bullet}, \rho^{\bullet}\right)$ be its dual representation given by the morphism $\rho^{\bullet}\left(\left(g_{i j}\right)_{1 \leq i, j \leq n}\right)=\left(\bar{g}_{i j}\right)_{\leq i, j \leq n}$. Note that as vector spaces, $V^{\circ} \simeq V^{\bullet} \simeq \mathbb{C}^{n}$. Let $\left(e_{i}\right)_{1 \leq i \leq n}$ be a basis of $V^{\circ}$ which is orthogonal with respect to the invariant scalar product. The dual basis in $V^{\boldsymbol{\bullet}}$ is denoted by $\left(\bar{e}_{i}\right)_{1 \leq i \leq n}$ and the pairing between both bases is denoted by $\langle.,$.$\rangle .$ Note that $\langle.,$.$\rangle is an intertwiner from V^{\circ} \otimes V^{\bullet}$ to the trivial representation $V^{\emptyset} \simeq \mathbb{C}$.
Taking tensor products of these two representations yields the existence, for any finite word $\varepsilon=\varepsilon_{1} \ldots, \varepsilon_{r}$ in $\{\circ, \bullet\}$, of a representation $\left(V^{\varepsilon}, \rho^{\varepsilon}\right)$ with

$$
V^{\varepsilon}=V^{\varepsilon_{1}} \otimes \cdots \otimes V^{\varepsilon_{r}}, \rho^{\varepsilon}=\rho^{\varepsilon_{1}} \otimes \cdots \otimes \rho^{\varepsilon_{r}} .
$$

Let us denote by $\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)$ the vector space of intertwiners from $\left(V^{\varepsilon}, \rho^{\varepsilon}\right)$ to $\left(V^{\varepsilon^{\prime}}, \rho^{\varepsilon^{\prime}}\right)$. Thus for any $G \subseteq U_{n}, \operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)$ is a vector subspace of $\mathcal{L}\left(V^{\varepsilon}, V^{\varepsilon^{\prime}}\right)$, the space of linear maps from $V^{\varepsilon}$ to $V^{\varepsilon^{\prime}}$.

Remark 2.38. The collection of vector spaces $\left\{\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$ satisfies several properties :
$\bullet I d_{V^{\circ}} \in \operatorname{Mor}(\circ, \circ), I d_{V} \bullet \in \operatorname{Mor}(\bullet, \bullet),\langle.,.\rangle \in \operatorname{Mor}(\bullet \bullet, \emptyset),\langle.,.\rangle \in \operatorname{Mor}(\bullet \circ, \emptyset)$.

- If $T_{1} \in \operatorname{Mor}\left(\varepsilon_{1}, \varepsilon_{2}\right), T_{2} \in \operatorname{Mor}\left(\varepsilon_{2}, \varepsilon_{3}\right)$, then $T_{2} \circ T_{1} \in \operatorname{Mor}\left(\varepsilon_{1}, \varepsilon_{3}\right)$.
- If $T_{1} \in \operatorname{Mor}\left(\varepsilon_{1}, \varepsilon_{3}\right), T_{2} \in \operatorname{Mor}\left(\varepsilon_{2}, \varepsilon_{4}\right)$, then $T_{1} \otimes T_{2} \in \operatorname{Mor}\left(\varepsilon_{1} \varepsilon_{2}, \varepsilon_{3} \varepsilon_{4}\right)$.
- If $T \in \operatorname{Mor}\left(\varepsilon_{1}, \varepsilon_{2}\right), T^{*} \in \operatorname{Mor}\left(\varepsilon_{2}, \varepsilon_{1}\right)$.

Of course, since the functions on a compact group form a commutative algebra,

$$
\begin{equation*}
T: e_{i} \otimes e_{j} \mapsto e_{j} \otimes e_{i} \text { is in } \operatorname{Mor}(\circ \circ, \infty) . \tag{2.6.1}
\end{equation*}
$$

All these properties are straightforward deductions of Section 1.2.
Example 2.39. For $U_{n}$, the space of intertwiners $\operatorname{Mor}_{U_{n}}\left(\varepsilon_{1}, \varepsilon_{2}\right)$ is exactly the vector space spanned by all $T_{p}$ for $p \in P_{2, \text { alternating }}\left(\varepsilon_{1}, \varepsilon_{2}\right)$ (as defined in Section 1.1.3). If $\varepsilon=\varepsilon^{\prime}=\circ^{d}$. Each allowed pair partition is encoding a permutation of $S_{d}$. In particular the scalar product on permutations considered in Section 5.1 is the scalar product between the maps $T_{p}$ for $p$ in $P_{2, \text { alternating }}\left(\circ^{d}, \circ^{d}\right)$.

The striking fact is that the collection $\left\{\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)\right\}_{\varepsilon, \varepsilon^{\prime}}$ completely characterizes $G$ : this is the content of Tannaka's Theorem. Note that the actual statement of the following Theorem is a bit more evolved (refer to [81] for an exact statement):

Theorem 2.40 (81]). $G$ can be reconstructed from the data of $\left\{\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)\right\}_{\varepsilon, \varepsilon^{\prime}}$. In particular $\left\{\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)\right\}_{\varepsilon, \varepsilon^{\prime}}$ completely characterizes $G$.

Krein's Theorem gives an answer to the dual question: to find all collections of vector spaces $\left\{H\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$ being the collection of intertwiners of any compact subgroup of $U_{n}$. Of course such a collection has to satisfy the conditions (2.38). Krein's Theorem asserts that these conditions are enough:

Theorem 2.41 (51). Let $\left\{H\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$ be a collection of vector spaces such that $H\left(\varepsilon, \varepsilon^{\prime}\right) \subseteq$ $\mathcal{L}\left(V^{\varepsilon}, V^{\varepsilon^{\prime}}\right)$. If $\left\{H\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$ fulfills the four conditions (2.38) and the commutativity relation 2.6.1], then there exists a compact subgroup $G$ of $U_{n}$ such that for all $\varepsilon, \varepsilon^{\prime}$,

$$
H\left(\varepsilon, \varepsilon^{\prime}\right)=\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)
$$

By Tannaka's Theorem, the compact group $G$ coming from Krein's Theorem is uniquely determined by $\left\{H\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$.

### 2.6.2 Other groups with intertwiners described by set partition

In this thesis we are mainly interested in groups (and later quantum groups) whose associated intertwiner spaces are spanned by maps $T_{p}$ 's as in the case of $U_{n}$.

Compact classical groups The orthogonal group $O_{n}$ is the group of matrices $O \in G l_{n}(\mathbb{R})$ such that $O O^{t}=I d$, and the symplectic group $S p_{n}$ is the group of matrices $T \in U_{2 n}$ such that $T J T^{t}=J$, with $J=\left(\begin{array}{cc}0 & I d_{n} \\ -I d_{n} & 0\end{array}\right)$.
In both cases, the fundamental representation and the dual ones are isomorphic: the isomorphisms are given by the map $I d$ in the orthogonal case and by the map $J$ in the symplectic case. Therefore it is enough to specify the description of the intertwiner spaces $\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)$ in the case $\varepsilon=o^{k}$ and $\varepsilon^{\prime}=o^{k^{\prime}}$. Let us simply denote these vector spaces by $\operatorname{Mor}_{G}\left(k, k^{\prime}\right)$.
The intertwiners of the compact classical groups $O_{n}$ and $S p_{n}$ are also described by pair partitions. Refer to Section 1.1.3 for the definition of the maps $T_{p}^{\prime} s$ for a given set partition and a Hilbert space $V$. In the case of $O_{n}$,

$$
\operatorname{Mor}_{O_{n}}\left(k, k^{\prime}\right)=\left\langle T_{p}\right\rangle_{p \in P_{2}\left(k, k^{\prime}\right)} .
$$

Moreover $\left.\left\{T_{p}\right\}_{p \in P_{2}\left(k, k^{\prime}\right)}\right\}$ is a basis of $\operatorname{Mor}_{O_{n}}\left(k, k^{\prime}\right)$ for $n \geq k+k^{\prime}$. A same result holds for $S p_{n}$, but it is necessary to adapt the maps $T_{p}$ 's to the non-degenerate bilinear form given by $J$. Using this description of the interwiners, it has been shown by Diaconis and Shahshahani in [33] that the random vectors $\left(\operatorname{Tr}\left(O_{n}^{k}\right)\right)_{k \geq 1}$ and $\left(\operatorname{Tr}\left(T_{n}^{k}\right)\right)_{k \geq 1}$ converge in moments respectively to a gaussian vector $\left(o_{k}\right)_{k \geq 1}$ and $\left(t_{k}\right)_{k \geq 1}$, with covariance matrices $\mathbb{E}\left(o_{k} o_{k^{\prime}}\right)=\mathbb{E}\left(t_{k} t_{k^{\prime}}\right)=\delta_{k k^{\prime}} k$ and expectations $\mathbb{E}\left(o_{k}\right)=\delta_{k}$ even and $\mathbb{E}\left(t_{k}\right)=-\delta_{k}$ even.
In [29], Collins and Sniady used the Weingarten calculus to compute the Haar integral of arbitrary polynomials in the coefficients of the fundamental and dual representations of these two groups. In particular the same results as in Section 5 exist in the orthogonal and symplectic case.

Symmetric group A permutation $\sigma_{n}$ of $S_{n}$ can be embedded into $O_{n}$ by considering its permutation action on the vector space $\mathbb{C}^{n}$ : this action simply permutes the elements of the canonical basis $\left(e_{i}\right)_{1 \leq i \leq n}$. This yields a representation $\left(v(\sigma)_{i j}\right)_{1 \leq i, j \leq n}$ of $S_{n}$ called the fundamental representation. Thus $S_{n}$ can be seen as a compact matrix group through this fundamental representation $v$.
As for the orthogonal group, the fundamental representation is isomorphic to the dual representation, and therefore it is enough to describe the interwiner spaces for $\varepsilon=\circ^{k}, \varepsilon^{\prime}=o^{k^{\prime}}$. The description is once again achieved by using the maps $\left\{T_{p}\right\}$ for general partitions. Indeed the spaces of intertwiners are $\operatorname{Mor}_{S_{n}}\left(k, k^{\prime}\right)=\left\langle T_{p}\right\rangle_{p \in P\left(k, k^{\prime}\right)}$, and the set $\left\{T_{p}\right\}_{p \in P\left(k, k^{\prime}\right)}$ is a basis of $\operatorname{Mor}_{S_{n}}\left(k, k^{\prime}\right)$ for $n \geq k+k^{\prime}$ (see [44]). If $n \leq k+k^{\prime}$, a basis is given by restricting to the set $\left\{T_{p}\right\}$ where $p$ is a partition having less than $n$ blocks.
Diaconis ans Shahshahani proved in [33] the convergence in moments of the random vector $\left(\operatorname{Tr}\left(\sigma_{n}^{k}\right)\right)_{k \geq 1}$ toward a vector of independent random variables $\left(s_{k}\right)_{k \geq 1}$, $s_{k}$ having a Poisson distribution with parameter $\frac{1}{k}$. Their proof of the result doesn't use the description of the intertwiner spaces. A proof involving this description has been done by Banica, Curran and Speicher in (14.)
As it was already said in Section 4.2, symmetric functions play also an important role in the representation theory of $S_{n}$. Indeed irreducible representations $S_{\nu}$ of $S_{n}$ are indexed by Young diagrams $\nu$ with $n$ cells. Note first that there is a natural inclusion $S_{l} \times S_{m} \subseteq S_{n}$ for $l+m=n$. Therefore an irreducible representation $\rho$ of $S_{n}$ is not necessarily an irreducible representation of $S_{l} \times S_{m}$ and the decomposition of $\rho$ into irreducible representations of $S_{l} \times S_{m}$ is given by the multiplicative structure of the ring of symmetric functions in the Schur basis: namely if $\lambda \vdash l, \mu \vdash m$ and $\nu \vdash n$, there is a decomposition

$$
S_{\nu}=\bigoplus_{\substack{\lambda \vdash-l \\ \mu \vdash m}}\left(S_{\lambda} \otimes S_{\mu}\right)^{\oplus c_{\lambda \mu}^{\nu}},
$$

where $c_{\lambda \mu}^{\nu}$ are the Littlewood-Richardson coefficients (see 2.3.1) and 60, Part I, Ch. 9 for their precise definition).
The Weingarten calculus for the symmetric group is not as much developed as the one for the classical Lie groups. We have seen in Section 5 that the precision of the Weingarten calculus is given by the ability to invert a Gram-Schmidt matrix. In the unitary case, this was greatly simplified by the Schur-Weyl duality and the well-known representation theory of the different symmetric groups $S_{k}$, for $k \geq 1$. The same method applies also to the orthogonal and symplectic case. However in the symmetric case, the Schur-Weyl theory involves another family of algebras, namely the partition algebras $P_{k}(n)$ for $n, k \geq 1$. The understanding of the algebraic properties of this family is recent (see the work of Halverson and Ram in [44 for example), and therefore the Weingarten calculus has still not been fully achieved in this setting.

Wreath product with $S_{n}$ Let $G$ be a classical group and $n \geq 1$. Then $S_{n}$ acts on $G^{n}$ by the automorphisms

$$
\begin{equation*}
s: \sigma \in S_{n} \mapsto s(\sigma) \cdot\left(g_{1}, \ldots, g_{n}\right)=\left(g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}\right) . \tag{*}
\end{equation*}
$$

Definition 2.42. The wreath product between $G$ and $S_{n}$, denoted $G \imath S_{n}$, is the semi-direct product of $G^{n}$ and $S_{n}$, where $S_{n}$ acts on $G^{n}$ by the action (*). In other words,

$$
G \imath S_{n}=\left\{\left(\left(g_{1}, \ldots, g_{n}\right), \sigma\right), g_{i} \in G, \sigma \in S_{n}\right\},
$$

with the product

$$
\left(\left(g_{1}, \ldots, g_{n}\right), \sigma\right) \cdot\left(\left(g_{1}^{\prime}, \ldots, g_{n}^{\prime}\right), \mu\right)=\left(\left(g_{1} g_{\sigma^{-1}(1)}^{\prime}, \ldots, g_{n} g_{\sigma^{-1}(n)}^{\prime}\right), \sigma \mu\right) .
$$

If $G$ is a matrix compact group, there is an equivalent way to define $G$ 亿 $S_{n}$. Namely let $\left(u_{k l}(g)\right)_{1 \leq k, l \leq p}$ be the fundamental representation of $G$. Then $G \imath S_{n}$ can be defined as the subgroup of $U_{n \times p}$ consisting of the matrices $\left\{\left(v_{i j}(\sigma) u_{k l}\left(g_{i}\right)\right)_{\substack{1 \leq i, j \leq n \\ 1 \leq k, l \leq p}}\right\}$ for $\sigma \in S_{n}$ and $g_{1}, \ldots, g_{n} \in G$. If $G$ is a compact group, $G \imath S_{n}$ is compact as well and thus there exists a Haar measure on $G \imath S_{n}$. It is easy to see that $G \imath S_{n}$ is isomorphic to $G \times \cdots \times G \times S_{n}$ as a measure space and that the Haar measure on $G \imath S_{n}$ is given by $d \lambda_{G l S_{n}}=\bigotimes_{i} d g_{i} \otimes d \sigma$, where $d_{g}$ denotes the Haar measure on $G$ and $d \sigma$ the normalized counting measure on $S_{n}$.
Let $G \subseteq U_{m}$. In this case $G \imath S_{n} \subseteq U_{n} \otimes U_{m}$, and by the Tannaka-Krein duality, the description of $G\} S_{n}$ is completely given by the data of $\operatorname{Mor}_{G l S_{n}}\left(\varepsilon, \varepsilon^{\prime}\right)$ for all words $\varepsilon, \varepsilon^{\prime}$ in $\{0, \bullet\}$. Actually it is a straightforward computation to express $\operatorname{Mor}_{G i S_{n}}\left(\varepsilon, \varepsilon^{\prime}\right)$ in terms of $\operatorname{Mor}_{S_{n}}\left(\varepsilon, \varepsilon^{\prime}\right)$ and $\left\{\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right\}_{\varepsilon, \varepsilon^{\prime}}\right.$ : an element of $\operatorname{Mor}_{G l S_{n}}\left(\varepsilon, \varepsilon^{\prime}\right)$ is given by a partition $p \in P\left(\varepsilon, \varepsilon^{\prime}\right)$ together with an element of $\operatorname{Mor}_{G}\left(\varepsilon_{B}, \varepsilon_{B}^{\prime}\right)$ for each block $B$ of $p, \varepsilon_{B}$ and $\varepsilon_{B}^{\prime}$ being respectively the restriction of $\varepsilon$ and $\varepsilon^{\prime}$ to the elements in $B$ (see Chapter 6 for more details on the subject).
As consequence, we get the convergence in law of $\operatorname{Tr}\left(u_{G l S_{n}}\right)$ toward a compound Poisson distribution with initial law $\operatorname{Tr}\left(u_{G}\right)$.
Note that the irreducible representations of $G \imath S_{n}$ are described by generalization of Schur functions. Refer to [60], Part I, Appendix B for an exposition in the case of a wreath product $G \imath S_{n}$ with $G$ a finite group.

Remark 2.43. For two sets $X, Y$ denote by $\mathcal{F}(X, Y)$ the set of maps from $X$ to $Y$. The wreath product is a more general construction than the one presented here. Let $G$ be a group. For any set $X$ and group $F$ acting on $X$, the wreath product $G 2_{X} F$ is the set $\mathcal{F}(X, G) \times F$ with the product $\star$ defined as follows: for $h, h^{\prime} \in \mathcal{F}(X, G)$ and $f, f^{\prime} \in F,(h, f) \star\left(h^{\prime}, f^{\prime}\right)=\left(\tilde{h}, f f^{\prime}\right)$, where for $x \in X$,

$$
\tilde{h}(x)=h(x) h^{\prime}\left(f^{-1}(x)\right),
$$

with the product on the right hand side being done in $G$.
In Chapter 6 , we will study this more general wreath product for $G$ a compact group, $X$ a finite set and $F$ a permutation group of $X$. In this case the construction is exactly the same as in the case of the symmetric group, and $G \sum_{X} F$ is compact.

## Chapter 3

## Compact quantum group

In this chapter we introduce the notion of compact matrix quantum group and give the noncommutative version of the results of last chapter. This lead to an overview of the results obtained in the thesis.

### 3.1 Noncommutative spaces and quantum groups

### 3.1.1 What is a compact quantum group ?

Non-commutative spaces The notion of quantum groups fits into the more general framework of noncommutative spaces. The starting idea is that most properties of a classical object, like a topological or a measurable space, can be seen through the algebra of functions on this object. Thus by a considering noncommutative generalization of these algebras, it is possible to define noncommutative analogs to the classical objects.

Example 3.1 (Historical example). The most trivial example is the one of complex functions on a unique point. Classically this space is just $\mathbb{C}$, with multiplication given by the canonical one on $\mathbb{C}$. The noncommutative generalization is obtained by replacing $\mathbb{C}$ by the algebra $M_{n}(\mathbb{C})$ of $n$-dimensional matrices. This is exactly what Heisenberg, Born and Jordan (see (24]) did when replacing the orbital position $x$ and momentum $p$ of an electron by two matrices $X$ and $P$ (which were infinite dimensional in this case).
This example can be transposes to the case of $\mathbb{C}$-valued functions on $r$ points. In this case the algebra $\mathbb{C}^{r}$ with the pointwise multiplication turns into a matrix algebra $\bigoplus_{i=1}^{r} M_{n_{i}}(\mathbb{C})$. Note that the classical algebra $\mathbb{C}^{r}$ coincides with the center of $\bigoplus_{i=1}^{r} M_{n_{i}}(\mathbb{C})$.

In the previous example there is no particular interest in defining topological or measurable noncommutative spaces, since the classical space is a finite set. The correct approach to the definition of functions on a noncommutative topological space is the one of $C^{*}$-algebras:

Definition 3.2. $A C^{*}$-algebra $A$ is a $*-$ algebra over $\mathbb{C}$ with a norm $\|$.$\| such that$

- $A$ is complete with respect to $\|$.$\| .$
- for all $x, y \in A,(x y)^{*}=y^{*} x^{*}$.
- for all $x, y \in A,\|x y\| \leq\|x\|\|y\|$ and $\left\|x^{*} x\right\|=\|x\|^{2}$.

This definition is the most natural one for two reasons. First, the algebra of complex functions on a locally compact Hausdorff space, considered with the $\|\cdot\|_{\infty}$-norm, is a commutative $C^{*}$-algebra. Actually any commutative $C^{*}$-algebra is of this form:
Theorem 3.3 (Gelfand's Theorem). Let A be a commutative $C^{*}$-algebra. There exists a locally compact Hausdorff space $X$ such that $A$ is isomorphic to $C^{0}(X)$, the algebra of complex functions on $X$ vanishing at infinity.
If $A$ is unital, then $X$ is compact.
Secondly, if $A$ is finite dimensional, we recover the construction made in Example 3.1.
Theorem 3.4 (Artin-Wedderdurn's Theorem). Let $A$ be a finite dimensional $C^{*}$-algebra. Then there exist $r>0$ and $n_{1}, \ldots, n_{r}>0$ such that

$$
A \simeq \bigoplus_{i=1}^{r} M_{n_{i}}(\mathbb{C})
$$

If $X, Y$ are compact spaces, a continuous map $\varphi$ from $X$ to $Y$ yields a $C^{*}$-morphism $\Phi$ : $C(Y) \rightarrow C(X)$ defined by $\Phi(f)(x)=f(\varphi(x))$. If $\varphi$ is an injective map (resp. surjective, bijective), then $\Phi$ is a surjective map (resp. injective, resp. invertible). Therefore $C^{*}-$ morphisms encode continuous maps between non-commutative topological spaces.
Remark 3.5 (Where are the points in a noncommutative space ?). Even if the right way to see a noncommutative space is to consider the functions defined on it, it is still possible to recover a topological space from a general $C^{*}$-algebra. If $A$ is commutative, we have seen that $A$ isomorphic to $C^{0}(X)$ for a locally compact Hausdorff space $X$. In this case one can show that any irreducible (continuous) representation of $A$ is of the form $e v_{x}: a \mapsto a(x)$ for an element $x \in X$.
Similarly if $A$ is a general $C^{*}$-algebra, we define the $\operatorname{spectrum} \operatorname{Spec}(A)$ as the set of equivalence classes of continuous representations of the $C^{*}$-algebra. It is possible to define a topology on $\operatorname{Spec}(A)$ such that in the commutative case, $\operatorname{Spec}\left(C^{0}(X)\right) \simeq X$. In Example 3.1, this yields as expected that $\operatorname{Spec}(A)$ is a discrete space with $r$ elements. Therefore formally, evaluating $a \in A$ on $x \in \operatorname{Spec}(A)$ is taking the image of a in the irreducible representation $x$.
This point of view is however often limited, since in many cases, $\operatorname{Spec}(A)$ is just a point.
It is also possible to define noncommutative measurable spaces. This yields the notion of von Neumann algebra, which won't be explained here (refer to [80]).

Compact quantum group Following the dual approach to the study of spaces, we want to translate the axioms of a compact group $G$ at the level of the continous functions on $G$, in order to construct noncommutative analogs.
If $(X, \bullet)$ is a compact Hausdorff space with a continuous semigroup structure $\bullet: X \times X \rightarrow X$, the algebra of continuous functions on $X$ inherits an additionnal structure. Namely it is possible to define the map

$$
\Delta:\left\{\begin{array}{ccc}
C(X) & \longrightarrow & C(X \times X) \\
f & \mapsto & \left(x, x^{\prime}\right) \mapsto f\left(x \bullet x^{\prime}\right) .
\end{array}\right.
$$

By Arzela-Ascoli Theorem, $C(X \times X) \simeq C(X) \otimes C(X)$ (where $C(X) \otimes C(X)$ is the norm completion of the algebraic tensor product). Since $(f g)\left(x \bullet x^{\prime}\right)=f\left(x \bullet x^{\prime}\right) g\left(x \bullet x^{\prime}\right)$ and $\bar{f}\left(x \bullet x^{\prime}\right)=$ $\overline{f\left(x \bullet x^{\prime}\right)}, \Delta$ is a $*$-homomorphism from $C(X)$ to $C(X) \otimes C(X)$. Moreover the associativity of the product on $X$ yields the relation:

$$
\begin{array}{r}
\left.(\Delta \otimes I d) \circ \Delta(f))\left(x_{1}, x_{2}, x_{3}\right)=\Delta(f)\left(\left(x_{1} \bullet x_{2}\right), x_{3}\right)=f\left(\left(x_{1} \bullet x_{2}\right) \bullet x_{3}\right)\right) \\
=f\left(x_{1} \bullet\left(x_{2} \bullet x_{3}\right)\right)=(I d \otimes \Delta) \circ \Delta(f)\left(x_{1}, x_{2}, x_{3}\right) .
\end{array}
$$

A map $\Delta$ satisfying the relation $(I d \otimes \Delta) \circ \Delta=(\Delta \otimes I d) \circ \Delta$ is called coassociative.
Let us consider the maps $\varphi:\left(x, x^{\prime}\right) \mapsto\left(x \bullet x^{\prime}, x^{\prime}\right)$ and $\varphi^{\prime}:\left(x, x^{\prime}\right) \mapsto\left(x, x \bullet x^{\prime}\right)$. If $X$ is a group, these maps are homeomorphisms of topological spaces. By duality $\varphi$ and $\varphi^{\prime}$ yield on $C(X) \otimes$ $C(X)$ the maps $\Phi(f, g)=\Delta(f)(1 \otimes g)$ and $\Phi^{\prime}(f, g)=(f \otimes 1) \Delta(g)$. Since $\varphi$ and $\varphi^{\prime}$ are injective, $\Phi$ and $\Phi^{\prime}$ are surjective maps, and therefore $\{(f \otimes 1) \Delta(g)\}_{f, g \in C(X)}$ and $\{\Delta(f)(1 \otimes g)\}_{f, g \in C(X)}$ are dense in $C(X) \otimes C(X)$.
Reciprocally, if these sets are dense, this means that the maps $\Phi, \Phi^{\prime}$ are surjective, and thus the maps $\varphi, \varphi^{\prime}$ are injective. But this is equivalent to the left and right cancellation property for the compact semigroup $X$, and therefore $X$ is actually a group.
The $C^{*}$-algebra $C(G)$ of functions on a compact group $G$ is therefore a commutative unital $C^{*}$-algebra with an associative coproduct $\Delta: C(G) \rightarrow C(G) \otimes C(G)$, and such that the sets $\{(f \otimes 1) \Delta(g)\}_{f, g \in C(G)}$ and $\{\Delta(f)(1 \otimes g)\}_{f, g \in C(G)}$ are dense in $C(G) \otimes C(G)$. This motivates the following definition, which has been introduced by Woronowicz:

Definition 3.6 (Woronowicz, (99). A compact quantum group is a unital $C^{*}$-algebra $A$ with a coassociative $C^{*}$-morphism $\Delta: A \rightarrow A \otimes A$ such that $\Delta(A)(1 \otimes A)=(A \otimes 1) \Delta(A)=A \otimes A$.

The $C^{*}$-algebra is often denoted $C(G)$ to emphasize its quantum group nature, even if there is no concrete underlying space $G$.
$\left(C(H), \Delta^{\prime}\right)$ is a quantum subgroup of $(C(G), \Delta)$ if there is a sujective $C^{*}$-morphism $\Phi: C(G) \rightarrow$ $C(H)$ such that $(\Phi \otimes \Phi) \Delta=\Delta^{\prime} \Phi$. If $\Phi$ is an isomorphism, then $H$ and $G$ are called isomorphic. As for $C^{*}$-algebras, a commutative compact quantum group is a classical group in the following sense :

Proposition 3.7. Let $A$ be a compact quantum group. If $A$ is commutative, then there exists a compact group $G$ such that $A \simeq C(G)$.

### 3.1.2 Representation theory

In this subsection we will introduce the representation theory of a compact quantum group for finite dimensional representations. The content of this subsection comes from [99].

Haar state Since the purpose is to extend probabilistic results from the classical group to the quantum case, one need a natural probability space on compact quantum groups. In the classical setting, this probability space was given by the Haar measure $\int_{G}$. This probability measure is the unique to satisfy the relations $\int_{G} f(g h) d g=\int_{G} f(h g) d g=\int_{G} f(g) d g$ for all continuous functions $f$ on $G$; equivalently, for any regular probability measure $\mu$ on $G$ and any function $f \in C(G, \mathbb{C}), \int_{G \times G} f(g h) d g d \mu(h)=\int_{G \times G} f(h g) d g d \mu(h)=\int_{G} f(g) d g$.
By the Riesz representation theorem, there is a bijection between regular signed finite measures on $G$ and bounded linear functionals on $C(G, \mathbb{C})$. This bijection restricts to a bijection between regular probability measures $\mu$ on $G$ and positive linear functionals $l$ on $C(G, \mathbb{C})$ such that $l(\mathbf{1})=1$. Positiveness means that $l(f) \geq 0$ is $f \geq 0$ on $G$; such positive linear functional $l$ with $l(\mathbf{1})=1$ is called a state on $C(G, \mathbb{C})$. If we use the dual approach of the last subsection, the Haar measure corresponds to the unique state $\int_{G}$ on $C(G, \mathbb{C})$ satisfying the relations :

$$
\begin{equation*}
(h \otimes l) \Delta=(l \otimes h) \Delta=h, \tag{3.1.1}
\end{equation*}
$$

for any other state $l$ on $C(G, \mathbb{C})$.
In the quantum framework, we don't have access to the space but only to the functions defined on it. Therefore we can not define measures, but only states: a state $\omega$ on a unital $C^{*}$-algebra $A$ is a linear functional which is positive, in the sense that $\omega\left(a a^{*}\right) \geq 0$ for any $a \in A$, and such
that $\omega(\mathbf{1})=1$.
One of the major results deduced from the axioms of a compact quantum group $C(G)$ is the existence of a state on $C(G)$ satisfying the relations 3.1.1.

Theorem 3.8 ( 99$]$ ). Let $C(G)$ be a compact quantum group. There exists a unique state $h$ on $C(G)$ such that for any bounded linear functional $\varphi$ on $C(G)$,

$$
(h \otimes \varphi) \Delta=(\varphi \otimes h) \Delta=h
$$

Therefore as in the classical case, a compact quantum group becomes naturally a noncommutative probability space with the Haar state $h$. We will mainly be interested in the behavior of elements of $C(G)$ with respect to this Haar state. The example of $U_{n}$ showed us that the representation theory of the group plays an important role in the computation of expectations with respect to this Haar state. Fortunately it is also possible to build a representation theory of a compact quantum group, and this representation theory is approximately the same as in the classical case.

Finite dimensional representations A finite-dimensional representation of a classical compact group is a finite-dimensional vector space $V$ together with a continuous map $\rho: G \mapsto$ $\operatorname{End}(V)$, such that $\rho\left(g g^{\prime}\right)=\rho(g) \rho\left(g^{\prime}\right)$ for all $g, g^{\prime} \in G$ and $\rho(e)=I d_{V}$. If the dimension of $V$ is $n$, the space of functions from $G$ to $\operatorname{End}(V)$ is isomorphic to the space $\operatorname{End}(V) \otimes C(G, \mathbb{C})$; thus the previous definition is equivalent to the data of a vector space $V$ together with an element $\rho$ in $\operatorname{End}(V) \otimes C(G, \mathbb{C})$ satisfying

$$
(I d \otimes \Delta) \circ \rho=\rho_{12} \rho_{13}
$$

where $(a \otimes b)_{12}=a \otimes b \otimes \mathbf{1}_{C(G)}$ and $(a \otimes b)_{13}=a \otimes \mathbf{1}_{C(G)} \otimes b$.
Applying $\rho$ to a vector $v \in V$ yields an element in $V \otimes C(G, \mathbb{C})$, and the image of a subspace $W$ of $V$ is a subspace of $V \otimes C(G, \mathbb{C})$.
This yields the following definition in the quantum case :
Definition 3.9. Let $(C(G), \Delta)$ be a compact quantum group. A finite dimensional representation of $C(G)$ is a finite-dimensional vector space $V$ with an element $\alpha \in \operatorname{End}(V) \otimes C(G)$ such that

$$
(\alpha \otimes I d) \circ \alpha=(I d \otimes \Delta) \circ \alpha
$$

as maps from $V$ to $V \otimes C(G) \otimes C(G)$.
An intertwiner from $(V, \alpha)$ to $\left(V^{\prime}, \alpha^{\prime}\right)$ is a linear map $T: V \rightarrow V^{\prime}$ such that

$$
\alpha^{\prime} \circ T=(T \otimes I d) \circ \alpha
$$

The vector space of intertwiners from $(V, \alpha)$ to $\left(V^{\prime}, \alpha^{\prime}\right)$ is denoted $\operatorname{Mor}_{G}\left(\alpha, \alpha^{\prime}\right)$.
Since $V$ is finite dimensional, we can express $\alpha$ in a basis $\left(e_{i}\right)_{1 \leq i \leq n}$ of $V$. This yields a matrix $\left(u_{i j}\right)_{1 \leq i, j \leq}$ in $M_{n}(C(G))$ such that

$$
\Delta\left(u_{i j}\right)=\sum_{k=1}^{n} u_{i k} \otimes u_{k j}
$$

Reciprocally any matrix in $M_{n}(C(G))$ satisfiying the above relations yields a finite dimensional representation of $C(G)$.

We can take tensor products, direct sums and dual of finite dimensional representations $\left(u_{i j}\right)_{1 \leq i, j \leq n}$ and $\left(v_{k l}\right)_{1 \leq k, l \leq m}$ by considering the following usual operations on $M_{n}(C(G)), M_{m}(C(G))$ :

$$
u \otimes v=\left(u_{i j} v_{k l}\right)_{\substack{1 \leq i, j \leq n \\
1 \leq k, l \leq m}} \in M_{m n}(C(G)), u \oplus v=\left(\begin{array}{cc}
u & 0 \\
0 & v
\end{array}\right) \in M_{n+m}(C(G))
$$

and $\bar{u}=\left(u_{i j}^{*}\right)_{1 \leq i, j \leq n}$.
The representation is called non-degenerate if $\left(u_{i j}\right)_{1 \leq i, j \leq n}$ is invertible in $M_{n}(C(G))$, and two representations are said equivalent if there is an invertible intertwiner from one to the other.
As in the classical case, an invariant subspace of $(V, \alpha)$ is a subspace $W \subseteq V$ such that $\alpha(W) \subseteq$ $W \otimes C(G)$ and a fixed vector is a an element $v$ of $V$ such that $\alpha(v)=v \otimes \mathbf{1}_{C(G)}$. A representation $(V, \alpha)$ is called irreducible if there is no invariant subspace except $\{0\}$ and $V$.
Thanks to the Haar state, if $(V, \alpha)$ is non-degenerate, it is still possible to defined a scalar product (.,.) on $V$ which is invariant with respect to $\alpha$ : namely (.,.) satisfies

$$
\left(\alpha\left(e_{i}\right), \alpha\left(e_{k}\right)\right)=\sum\left(e_{j}, e_{l}\right) \otimes u_{j i} u_{l k}^{*}=\left(e_{i}, e_{k}\right) \otimes \mathbf{1}
$$

To obtain this scalar product it suffices to take any scalar product $\langle.,$.$\rangle on V$, and to average $\langle.,$.$\rangle with respect to the Haar state:$

$$
\left(e_{i}, e_{k}\right)=\sum\left\langle e_{j}, e_{l}\right\rangle \otimes h\left(u_{j i} u_{k l}^{*}\right) .
$$

Therefore any non-degenerate representation $(V, \alpha)$ has a basis $\mathcal{B}$ such that the matrix $u$ of $\alpha$ in $\mathcal{B}$ verifies $u u^{*}=u^{*} u=I d$, where $\left(u^{*}\right)_{i j}=u_{j i}^{*}$. Such a matrix is called unitary. The main difference with the classical case is that the conjugate matrix defined by $\bar{u}_{i j}=u_{i j}^{*}$ is not necessarily unitary as well. However we can show that the representation associated to $\bar{u}$ has good properties, namely: it is non-degenerate (resp. irreducible) if $u$ is non-degenerate (resp.irreducible) and the representation associated to $\bar{u}$ doesn't depend, up to equivalence of representations, to the choice of a matrix $u$ for $(V, \alpha)$. A compact quantum group such that for any unitary representation $u, \bar{u}$ is also unitary, is called a compact quantum group of Kac type.

Matrix quantum group We have seen in Chapter 2 that the situation is much simpler when the group is already described as a subgroup of $U_{n}$ for some integer $n \geq 1$. In particular the Peter-Weyl Theorem, which is a deep result in the general case, has a much simpler proof in this case.

Definition 3.10. A compact matrix quantum group is a triple $\left(A,\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)$ such that :

- $A$ is a $C^{*}$-algebra.
- The $*-$ algebra generated by $\left\{u_{i j}\right\}_{1 \leq i, j \leq n}$ is dense in $A$.
- The map $\Phi: u_{i j} \mapsto \sum u_{i k} \otimes u_{i j}$ extends to a $C^{*}$-homorphism from $A$ to $A \otimes A$.
- The matrices $u=\left(u_{i j}\right)_{1 \leq i, j \leq n}$ and $\bar{u}=\left(u_{i j}^{*}\right)_{1 \leq i, j \leq n}$ are invertible in $M_{n}(A)$.
$u$ can always be chosen unitary, up to equivalence of representations. Since $\bar{u}$ is nondegenerate, there exists a matrix $F \in G l_{n}(\mathbb{C})$ such that $F \bar{u} F^{-1}=\left(u^{t}\right)^{-1}(F$ is the matrix encoding the invariant scalar product on the representation of $\bar{u}$ ).
One can prove that a matrix compact quantum group is actually a compact quantum group.

Therefore a compact matrix quantum group is just a compact quantum group $(C(G), \Delta)$ with a particular representation $u=\left(u_{i, j}\right)_{1 \leq i, j \leq n}$ whose coeffficients generate all the $C^{*}$-algera $C(G)$. A compact matrix quantum group is of Kac type if $F$ can be chosen equal to the identity. In this case we have

$$
u^{*} u=u u^{*}=\bar{u} u^{t}=u^{t} \bar{u}=I d .
$$

Remark 3.11. This definition is very convenient to define new compact matrix quantum group. Namely it suffices to specify relations among abstract variables $\left\{u_{i j}, u_{i j}^{*}\right\}_{1 \leq i, j \leq n}$ that are compatible with the coproduct defined above, and then to construct the universal $C^{*}$-algebra having these relations.
An important example is given by the free unitary quantum group $U_{n}^{+}$, introduced by Wang in [95]. The $C^{*}$-algebra is the universal $C^{*}$ - algebra generated by $n^{2}$ elements $u=\left(u_{i j}\right)_{1 \leq i, j \leq n}$ satisfying the relations $u^{*} u=u u^{*}=\bar{u} u^{t}=u^{t} \bar{u}=I d$. Since any other compact matrix quantum group of Kac type has also to fulfill these relations, $U_{n}^{+}$can be seen as the biggest compact matrix quantum group of Kac type of dimension $n$.
If we add the commutation relations $u_{i j} u_{k l}=u_{k l} u_{i j}$ for all $1 \leq i, j, k, l \leq n$, the resulting $C^{*}$-algebra is commutative and corresponds therefore to a compact matrix group. Actually this group is $U_{n}$, the unitary group of dimension $n$.

### 3.1.3 Tannaka-Krein Duality

We will present here the Tannaka-Krein duality in the framework of compact quantum groups. This duality extends the Tannaka-Krein duality of Chapter 2 to compact quantum groups.

Peter-Weyl Theorem and Schur's orthogonality Theorem The similarity in the representation theory of the classical and the quantum groups extend to the two majors Theorem of the first section of Chapter 2. Let $C(G)$ be a compact quantum group. A matrix coefficient of $C(G)$ is an element $u_{i j} \in C(G)$ coming from a finite-dimensional representation of $G$. The vector subspace of matrix coefficients of $C(G)$ is a $*$-algebra for the same reasons as in the classical case. This $*$-algebra is denoted by $C(G)_{0}$.

Theorem 3.12. Let $C(G)$ be a compact quantum group. The $*$-algebra $C(G)_{0}$ is dense in $C(G)$.

The Schur's orthogonality Theorem is in the quantum case is analogous to the one in the classical case. However one needs to modify a bit the orthogonality relations, because the dual of a unitary representation is not necessarily also unitary.
Let $\left\{\left(u_{i j}^{\alpha}\right)_{1 \leq i, j \leq d(\alpha)}\right\}_{\alpha \text { irred }}$ be the set of equivalence classes of irreducible representations (written in an orthonormal basis with respect to the invariant scalar product). We have seen that for each irreducible representation $u^{\alpha}$, the dual representation $\bar{u}^{\alpha}$ is not necessarily unitary but always nondegenerate, and therefore there exists $F^{\alpha} \in M_{d(\alpha)}(\mathbb{C})$ such that $F^{\alpha}\left(u^{\alpha}\right)^{t}\left(F^{\alpha}\right)^{-1} \bar{u}^{\alpha}=$ $I d_{M n(C(G))}$.
Let $L^{2}(C(G), h)$ be the completion of $C(G)$ with respect to the scalar product $(a, b) \mapsto h\left(a b^{*}\right)$.
Theorem 3.13 (Schur's orthogonality Theorem). The set $\left\{\left(u_{i j}^{\alpha}\right)_{1 \leq i, j \leq d(\alpha)}\right\}_{\alpha \text { irred }}$ forms a basis of $L^{2}(C(G), h)$, and for $\alpha, \beta$ irreducible representations, $1 \leq i, j \leq d(\alpha)$ and $1 \leq k, l \leq d(\beta)$,

$$
h\left(\left(u_{i p}^{\beta}\right)^{*} u_{j q}^{\alpha}\right)=\delta_{\alpha \beta} \delta_{p q} \frac{\left(Q_{\alpha}\right)_{j i}}{D_{\alpha}}
$$

with $Q_{\alpha}, D_{\alpha}$ only depending on $F^{\alpha}$. Similarly

$$
h\left(u_{j q}^{\beta}\left(u_{i p}^{\alpha}\right)^{*}\right)=\delta_{\alpha \beta} \delta_{i j} \frac{\left(Q_{\alpha}^{-1}\right)_{p q}}{D_{\alpha}} .
$$

If $G$ is of Kac type, $h\left(\left(u_{i p}^{\beta}\right)^{*} u_{j q}^{\alpha}\right)=\delta_{\alpha \beta} \delta_{p q} \delta_{i j} \frac{1}{d(\alpha)}$ and the Haar state is a trace:

$$
h(x y)=h(y x) \text { for all } x, y \in C(G, h) .
$$

In particular in the case of a compact quantum group of Kac type, the situation is very close to the one of classical compact groups.

Tannaka-Krein duality In the last part of the previous chapter, it has been shown that a classical compact group is essentially the same as a collection of vector spaces of linear maps stable under certain operations and that most of the properties of the group could be seen on this collection of spaces. The similarity between classical compact groups and compact quantum groups continues here, since the same kind of alternative description exists for a compact quantum group.
The natural framework to describe the representation theory of a compact quantum group is the one of concrete monoidal $W^{*}$-category with dual. We won't introduce the basics of category theory here and an interested reader should refer to [59 to get precise statements and theoretical explanation of the formalism introduced here. The definition is given from an abstract point of view, but keeping in mind the representations of a compact quantum group makes it more concrete.

Definition 3.14 ( 100 ). A concrete monoidal $W^{*}$-category (or $C M W^{*}$-category) $\mathcal{C}$ is a monoid $R$ together with a family of finite dimensional Hilbert spaces $\left\{H_{r}\right\}_{r \in R}$ such that $H_{r} \otimes H_{s}=$ $H_{r s}\left(\left(H_{r} \otimes H_{s}\right) \otimes H_{t}\right.$ is canonically identified with $\left.H_{r} \otimes\left(H_{s} \otimes H_{t}\right)\right)$, and a family of vector spaces $\operatorname{Mor}(r, s) \subseteq \mathcal{L}\left(H_{r}, H_{s}\right)$ with the following properties :

- $I d_{H_{r}} \in \operatorname{Mor}(r, r)$
- If $T \in \operatorname{Mor}\left(r, r^{\prime}\right)$ and $T^{\prime} \in \operatorname{Mor}\left(r^{\prime}, r^{\prime \prime}\right)$ then $T^{\prime} T \in \operatorname{Mor}\left(r, r^{\prime \prime}\right)$.
- If $T \in \operatorname{Mor}\left(r, r^{\prime}\right)$ then $T^{*} \in \operatorname{Mor}\left(r^{\prime}, r\right)$.
- If $T \in \operatorname{Mor}\left(r, r^{\prime \prime}\right)$ and $T^{\prime} \in \operatorname{Mor}\left(r^{\prime}, r^{(3)}\right)$ then $T \otimes T^{\prime} \in \operatorname{Mor}\left(r r^{\prime}, r^{\prime \prime} r^{(3)}\right)$.
- $H_{e}=\mathbb{C}$
$\mathcal{C}$ is called complete if moreover
- For any $r \in R$ and Hilbert space $H$ such that there exists a unitary operator $V: H_{r} \rightarrow H$, $H=H_{s}$ for some $s \in R$, and $V \in \operatorname{Mor}(r, s)$.
- For any projector $p \in \operatorname{Mor}(r, r)$, there exists $s \in R$ such that $H_{s}=p H_{r}$ and the embedding $i: H_{s} \rightarrow H_{r}$ is in $\operatorname{Mor}(s, r)$.
- For any $r, r^{\prime}$, there exists $s \in R$ such that $H_{r} \oplus H_{r^{\prime}}=H_{s}$ and the canonical inclusion $H_{r} \rightarrow H_{s}$ and $H_{r^{\prime}} \rightarrow H_{s}$ are respectively in $\operatorname{Mor}(r, s)$ and $\operatorname{Mor}\left(r^{\prime}, s\right)$.

Example 3.15 (100). The set of finite dimensional representations of a compact quantum group $(C(G), \Delta)$ together with the spaces of intertwiners between them is a complete $C M W^{*}$-category denoted $\operatorname{Rep} G$. Moreover $H$ is a quantum subgroup of $G$ if and only if $\operatorname{Rep} G$ is a subcategory of Rep $H$.

Note that we didn't formalize the fact that the dual of a finite dimensional representation is equivalent to a unitary representation. In the context of $C M W^{*}$-category, this is equivalent to the following definition :

Definition 3.16. Let $r \in R$ and $H_{r}$ the associated Hilbert space with basis $\left\langle e_{i}\right\rangle_{1 \leq i \leq n}$. $r$ has a complex conjugate $\bar{r} \in R$ if there is an invertible antilinear map $j: H_{r} \rightarrow \bar{H}_{\bar{r}}$ such that $\sum e_{i} \otimes j\left(e_{i}\right) \in \operatorname{Mor}(e, r \bar{r})$ and $\sum j^{-1}\left(e_{i}\right) \otimes e_{i} \in \operatorname{Mor}(e, \bar{r} r)$.
A CMW *-category such that any $r$ has a complex conjugate $\bar{r}$ is called a CMW ${ }^{*}$-category with conjugates.

For example, a finite dimensional representation $u^{\alpha}$ of a compact quantum group has a dual $\bar{u}^{\alpha}$, and the map $j$ is given by the matrix $F^{\alpha}$ as constructed in Paragraph 3.1.3.
In the classical case, it was easy to compare the categories of representations of two compact groups $G$ and $G^{\prime}$ since we only considered matrix compact groups of fixed dimension. Thus we just had to compare the intertwiner spaces $\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)$ and $\operatorname{Mor}_{G^{\prime}}\left(\varepsilon, \varepsilon^{\prime}\right)$ for all $\varepsilon, \varepsilon^{\prime}$. In the broader case of $C M W^{*}$-category, we still have to be able to compare these objects. This leads to the following definition:
Definition 3.17. Let $\mathcal{C}=\left(R,\left\{H_{r}\right\}_{r \in R},\left\{\operatorname{Mor}\left(r, r^{\prime}\right)\right\}_{r, r^{\prime} \in R}\right)$ and $\mathcal{C}^{\prime}=\left(S,\left\{K_{s}\right\}_{s \in S},\left\{\operatorname{Mor}\left(s, s^{\prime}\right)\right\}_{s, s^{\prime} \in S}\right)$ be two $C M W^{*}$-categories.
$\mathcal{C}$ and $\mathcal{C}^{\prime}$ are unitarily monoidally equivalent if there is a monoid morphism $\mathcal{F}: R \mapsto S$ and for each $r, r^{\prime}$ a vector space isomorphism $\mathcal{F}: \operatorname{Mor}\left(r, r^{\prime}\right) \rightarrow \operatorname{Mor}\left(\mathcal{F}(r), \mathcal{F}\left(r^{\prime}\right)\right)$ such that :

- for all $s \in S$, there exists $r \in R$ such that $\operatorname{Mor}(\mathcal{F}(r), s)$ contains a unitary operator.
- $\mathcal{F}$ respects the operations on $\mathcal{C}$ and $\mathcal{C}^{\prime}$ : namely if $T \in \operatorname{Mor}\left(r, r^{\prime}\right), T^{\prime} \in \operatorname{Mor}\left(r^{\prime}, r^{\prime \prime}\right), T^{\prime \prime} \in$ $\operatorname{Mor}\left(r^{\prime \prime}, r^{(3)}\right)$, then $\mathcal{F}\left(T^{*}\right)=\mathcal{F}(T)^{*}, \mathcal{F}\left(T^{\prime} T\right)=\mathcal{F}\left(T^{\prime}\right) \mathcal{F}(T)$ and $\mathcal{F}\left(T \otimes T^{\prime \prime}\right)=\mathcal{F}(T) \otimes \mathcal{F}\left(T^{\prime}\right)$.
$\mathcal{C}$ and $\mathcal{C}^{\prime}$ are unitarily isomorphic if they are unitarily monoidally equivalent, and moreover there exists for each $r \in R$ a unitary operator $F_{r}: H_{r} \rightarrow K_{\mathcal{F}(r)}$ such that:
- $F\left(r r^{\prime}\right)=F(r) \otimes F\left(r^{\prime}\right)$.
- If $T \in \operatorname{Mor}\left(r, r^{\prime}\right), F\left(r^{\prime}\right) \circ T=\mathcal{F}(T) \circ F(r)$.

This means that two $C M W^{*}$-categories are unitarily monoidally equivalent if they have the same structure (composition, tensor products, decomposition into simple pieces,...), but the concrete realization on Hilbert spaces are different; they are unitarily isomorphic if even the realization of these structure on Hilbert spaces is the same. In the latter case the two $C M W^{*}$-categories should be considered as being the same object.
We can now state the Tannaka-Krein duality in the compact quantum group case. This duality has been discovered and proved by Woronowicz in [100].

Theorem 3.18 (Tannaka-Krein's Duality). Let $\mathcal{C}$ be a complete CMW*-category with conjugate. There exists a compact quantum group $(C(G), \Delta)$ such that $\mathcal{C}$ is unitarily isomorphic to $\operatorname{Rep} G$. Moreover if $\left(C(H), \Delta^{\prime}\right)$ is another compact quantum group such that $\mathcal{C}$ is unitarily isomorphic to $\operatorname{Rep} H$, then $H$ is isomorphic to $G$.

There exist compact quantum group whose categories of representations are unitarily monoidally equivalent but not unitarily isomorphic. In the latter case the representations are still similar, and for example two compact quantum groups with unitarily monoidally equivalent categories of representations have isomorphic fusion rings.
In the case of a matrix compact quantum group of Kac type, the situtation is much simpler. Let $n \geq 1$. We are using here the notations of Section 2.6.1.

Theorem 3.19 (Tannaka-Krein's duality in the matrix case). Let $\left\{H\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$ be a collection of vector spaces such that $H\left(\varepsilon, \varepsilon^{\prime}\right) \subseteq \mathcal{L}\left(V^{\varepsilon}, V^{\varepsilon^{\prime}}\right)$. If $\left\{H\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$ fulfills the four conditions (2.38), then there exists a compact subgroup $(C(G), \Delta)$ of $U_{n}^{+}$such that for all $\varepsilon, \varepsilon^{\prime}$,

$$
H\left(\varepsilon, \varepsilon^{\prime}\right)=\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)
$$

Moreover the matrix compact quantum group is uniquely determined by the data of $\left\{H\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$.
Example 3.20. By the stability results 1.1 .3 of Chapter 1 , for each category of partition $\mathcal{C}$ and $n \geq 1$, the maps $T_{p}$ 's give a collection of vector spaces $\left\{\mathcal{C}_{n}\left(\varepsilon, \varepsilon^{\prime}\right)\right\}$ that fulfills the four conditions (2.38). Therefore for each catergory of partition $\mathcal{C}$ and $n \geq 1$, there is a quantum subgroup of $U_{n}^{+}$whose representation theory is encoded by $\mathcal{C}$.

### 3.2 Unitary easy quantum groups

### 3.2.1 Easy quantum groups

Free versions of the classical groups and their associated categories of representation We have seen in Chapter 2, Section 6 that we could associate categories of partition to certain classical groups. This correspondance is summarized by the following list :

| $U_{n}$ | $P_{2, \text { alternating }}^{\circ \bullet}$ |
| :---: | :---: |
| $O_{n}$ | $P_{2}^{\bullet \bullet}$ |
| $S_{n}$ | $P^{\circ}$ |

Figure 3.1: Correspondance between classical groups and categories of partitions through the $\operatorname{map} p \mapsto T_{p}$

Note that the size $n$ of the group is reflected through the map $p \mapsto T_{p}$ by the choice of the dimension of $V^{\circ}$ and $V^{\bullet}$ (which is theoretically speaking a concrete realization of the corresponding category of partition).
The relation $S_{n} \subseteq O_{n} \subseteq U_{n}$ corresponds to the inverse relation $P_{2, \text { alternating }}^{\circ \bullet} \subseteq P_{2}^{\circ \bullet} \subseteq P^{\circ \bullet}$, as it was predicted in Example 3.15. In Chapter 1, we have seen the existence of non-crossing analogs $N C^{\circ \bullet}, N C_{2}^{\circ \bullet}, N C_{2, \text { alternating }}^{\circ \bullet}$ of these categories of partitions. By Example 3.20, each category of partition $\mathcal{C}$ yields a subgroup of $U_{n}^{+}$for each $n \geq 1$. Actually the quantum groups corresponding to the three categories of non-crossing partitions aforementioned have already been introduced by Wang in [95] and [96], and the correspondance with categories of partitions has been proved by Banica in [3, [5] and [4. In each case the construction is done by using Remark 3.11;

- $N C_{2, \text { alternating }}^{0 \boldsymbol{}}$ corresponds to the free unitary group itself $U_{n}^{+}$.
- $N C_{2}^{o \bullet}$ corresponds to the free orthogonal group $O_{n}^{+}$. This quantum group is the quantum subgroup of $U_{n}^{+}$defined by the relation $u_{i j}^{*}=u_{i j}$. This is the biggest compact matrix quantum group $\left(C(G),\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)$ such that all the $u_{i j}^{\prime} \mathrm{s}$ are self-adjoint.
- $N C^{\circ \bullet}$ corresponds to the free symmetric group $S_{n}^{+}$. This quantum group is the quantum subgroup of $O_{n}^{+}$defined by the relation $\sum_{j} u_{i j}=\sum_{i} u_{i j}=\mathbf{1}$ and $u_{i j} u_{i k}=\delta_{j k} u_{i j}$ for all $1 \leq i, j, k \leq n$.
If we add the relations $u_{i j} u_{k l}=u_{k l} u_{i j}$ for all $1 \leq i, j, k, l \leq n, U_{n}^{+}, O_{n}^{+}$and $S_{n}^{+}$become respectively $U_{n}, O_{n}, S_{n}$. Thus we have the following inclusion relations:

$$
\begin{array}{lllll}
S_{n}^{+} & \subsetneq & O_{n}^{+} & \subsetneq & U_{n}^{+} \\
\cup & & \cup & & \cup \\
S_{n} & \subsetneq & O_{n} & \subsetneq & U_{n}
\end{array}
$$

Definition of easy quantum groups The diagram above is the starting point to define the class of unitary easy quantum groups. This class has been first introduced in the orthogonal case by Speicher and Banica in [15], and then extended to the unitary case in an unpublished article of Banica, Curran and Speicher ( 83$]$ ).
Definition 3.21. A compact matrix quantum group $G$ with $S_{n} \subseteq G \subseteq U_{n}^{+}$is called easy, if there is a category of partitions $\mathcal{C} \subseteq P^{\circ \bullet}$ such that for every words $\varepsilon$ and $\varepsilon^{\prime}{ }^{n}$ in $\{0, \bullet\}$, the space of intertwiners $\operatorname{Mor}_{G}\left(\varepsilon, \varepsilon^{\prime}\right)$ is spanned by all linear maps $T_{p}$ where $p$ is in $\mathcal{C}^{\circ \bullet}\left(\varepsilon, \varepsilon^{\prime}\right)$. An easy quantum group $G$ is called orthogonal easy quantum group, if $G \subseteq O_{n}^{+}$.

Refer to Chapter 1, Section 1 for the definition of the map $T_{p}$ for $p$ a two-colored partition. As we will see in Chapter 4, an easy quantum group is orthogonal if and only if $\circ$ ○ belongs to the associated category of partitions.

Example 3.22. Let us consider the set $\mathcal{H}$ of all partition $p \in P^{\bullet \bullet}$ such that each block of $p$ has an even number of elements. $\mathcal{H}$ is a category of partitions. This category corresponds to the hyperoctahedral group $H_{n}=Z_{2} \imath S_{n}$. This is a subgroup of $O_{n}$, and $C\left(H_{n}\right)$ is defined by taking the quotient of $C\left(O_{n}\right)$ by the relations $u_{i j} u_{i k}=u_{j i} u_{k i}=0$ for all $1 \leq i, j, k \leq n$ with $j \neq k$.
Once again, we can define the same with non-crossing partitions, yielding the free hyperoctahedral quantum group introduced by [11]. This quantum group is the subgroup of $O_{n}^{+}$defined by imposing the same relations as above.

The natural question is to find all the compact matrix quantum groups that are easy; this question is equivalent to the classification of all categories of partitions. From a probabilistic point of view, an answer to this question is interesting because for such quantum groups, we expect that the Weingarten calculus may have simpler combinatorial expressions, as this is the case in Chapter 2 for the classical groups. There exists also a method to study the representation theory of easy quantum groups, see 40.
The classification of all orthogonal easy quantum groups has been done in a serie of papers [15, [97, [71]. In this classification, there are two particularly simple cases: the case where the easy quantum group is a classical group and the case where the category of partitions associated to the quantum group is a category of non-crossing partitions. The classification in these both cases has been done in [15, 97]. In-between the situation is much harder to handle with, since there is an uncountable set of such easy quantum groups (see [71); we should stress nonetheless that the situation becomes simple again when restricting to easy quantum groups between $O_{n}$ and $O_{n}^{+}$. In the latter case there is only one such quantum group, namely the half-liberated orthogonal group $O_{n}^{*}$ (see [15]).

### 3.2.2 Free easy quantum groups

Following the last comment, we focus particularly on the easy quantum groups described by non-crossing partitions:

Definition 3.23. $A$ free easy quantum group is an easy quantum group $G_{n}$ such that the corresponding category of partitions $\mathcal{C}$ is a subcategory of $N C^{\circ \bullet}$ (or equivalently $S_{n}^{+} \subseteq G_{n}$ ).

The terminology goes back to Wang's papers [95, 96]; see also [11] or [39]. For example the three quantum groups $S_{n}^{+}, O_{n}^{+}$and $U_{n}^{+}$are all free easy quantum groups. Several other free easy quantum groups have already been discovered.

The orthogonal case As we said before the classification of all free easy quantum groups has been done in the orthogonal case by Banica, Speicher and Weber in [15, 97].

Proposition 3.24. Let $G_{n}$ be an orthogonal free easy quantum group. Then $G_{n}$ coincides with one of the following quantum groups.

- $O_{n}^{+}: u_{i j}=u_{i j}^{*}$, $u$ orthogonal, i.e. $\sum_{k} u_{i k} u_{j k}=\sum_{k} u_{k i} u_{k j}=\delta_{i j}$.
- $H_{n}^{+}: u_{i j}=u_{i j}^{*}$, $u$ orthogonal, $u_{i k} u_{j k}=u_{k i} u_{k j}=0$ if $i \neq j$.
- $S_{n}^{\prime+}: u_{i j}=u_{i j}^{*}$, u orthogonal, $u_{i k} u_{j k}=u_{k i} u_{k j}=0$ if $i \neq j, \sum_{k} u_{i k}=\sum_{k} u_{k j}$ for all $i, j$.
- $S_{n}^{+}: u_{i j}=u_{i j}^{*}=u_{i j}^{2}$, u orthogonal, $\sum_{k} u_{i k}=\sum_{k} u_{k j}=1$ for all $i, j$.
- $B_{n}^{\#+}: u_{i j}=u_{i j}^{*}$, u orthogonal, $\sum_{k} u_{i k}=\sum_{k} u_{k j}$ for all $i, j$.
- $B_{n}^{\prime+}: u_{i j}=u_{i j}^{*}$, u orthogonal, $r:=\sum_{k} u_{i k}=\sum_{k} u_{k j}$ for all $i, j, u_{i j} r=r u_{i j}$.
- $B_{n}^{+}: u_{i j}=u_{i j}^{*}$, u orthogonal, $\sum_{k} u_{i k}=\sum_{k} u_{k j}=1$ for all $i, j$.

Note that the quantum groups $B_{n}^{+}, B_{n}^{\#+}$ (with a different notation) and $S_{n}^{\prime+}$ appeared first in [15], and $B_{n}^{\prime+}$ was discovered in 97].
Moreover the category of partitions of each of these quantum groups can be explicitly described. For example $\mathcal{B}^{\#}$, the category of partitions associated to $B_{n}^{\#+}$, is the category of non-crossing partitions whose blocks are only pairs and singleton, and such that there is an even number of singletons between two elements of a same pair. We won't describe all of these categories, since a more general result will be given in Chapter 4.

Banica and Vergnioux's quantum reflection groups $H_{n}^{s+}$ The quantum reflection groups $H_{n}^{s+}$ were first defined by Banica and Vergnioux in [16] and studied by Banica, Belinschi, Capitaine and Collins in [9.

Definition 3.25. Given $n, s \in \mathbb{N}$, the quantum reflection group $H_{n}^{s+}$ is given by the universal $C^{*}$-algebra generated by elements $u_{i j}, 1 \leq i, j \leq n$ subject to the conditions:

- $u=\left(u_{i j}\right)$ and $\bar{u}=\left(u_{i j}^{*}\right)$ are unitaries
- all $u_{i j}$ are partial isometries (i.e. $u_{i j} u_{i j}^{*} u_{i j}=u_{i j}$ ) and the projections $u_{i j}^{*} u_{i j}$ and $u_{i j} u_{i j}^{*}$ coincide
- $u_{i j}^{s}=u_{i j} u_{i j}^{*}$

We define $H_{n}^{\infty+}$ by omitting the third of the above conditions.
Note that $H_{n}^{1+}=S_{n}^{+}$and $H_{n}^{2+}=H_{n}^{+}$. Furthermore, $H_{n}^{s+}=\widehat{\mathbb{Z}_{s}} \iota_{*} S_{n}^{+}$where $\iota_{*}$ denotes Bichon's free wreath product [21 and $\mathbb{Z}_{s}$ is shorthand for the cyclic group $\mathbb{Z} / s \mathbb{Z}$. Moreover, the quotient of the above $C^{*}$-algebras by the commutator ideal yields $C\left(H_{n}^{s}\right)$, where $H_{n}^{s}=\mathbb{Z}_{s} \imath S_{n}$. The quantum reflection groups have also been studied in [16], [9] and [55].

Proposition 3.26. Let $s \in \mathbb{N} \cup\{\infty\}$. The quantum reflection group $H_{n}^{s+}$ is easy and the corresponding category of partition is $\mathcal{H}^{s, s}$, the category of non-crossing partitions such that each block has the same number of black and white points modulo $s$.

This serie of quantum groups will be further studied in Chapter 5 and 6 , since they are also free wreath products.

Banica, Bichon, Capitaine and Collins's $H^{\#,+}$ The quantum group $H^{\#,+}$ has been defined in [9] as an auxiliary object.

Definition 3.27. The quantum group $H^{\#,+}$ is given by the universal $C^{*}$-algebra generated by $u_{i j}$ such that $u$ and $\bar{u}$ are unitaries and:

$$
u_{i k} u_{j k}^{*}=u_{k i} u_{k j}^{*}=u_{i k}^{*} u_{j k}=u_{k i}^{*} u_{k j}=0, \quad \text { whenever } i \neq j .
$$

It has been proven in [8] that

$$
H^{\#,+}
$$

is free easy with category $\mathcal{H}^{\#}$, the category of non-crossing partitions with blocks having an even number of elements with alternating colors.

### 3.2.3 Overview of the results

Let us review the results that are obtained in Part II. The results of Part II are from a joint work with Moritz Weber.

Classification of free easy quantum groups The first result is a classification of all free easy quantum groups. This classification is done in two steps. The first step is the classification of all the categories of non-crossing two-colored partitions, which can be summarized as follows :

Theorem 3.28 (Ch.3, Th. 4.41 and Th 4.42 ). There exist five denumerable families of categories of two-colored non-crossing two-colored partitions :

- $\mathcal{U}$
- $\mathcal{O}^{k}$ for $d \in 2 \mathbb{N}$
- $\mathcal{H}^{\#}$ and $\mathcal{H}^{d, k}$ for $d \mid k, k \geq 2$.
- $\mathcal{B}^{r, d, k}$ for $d \mid k, k \geq 1$ and $r \in\{*, 0, d / 2\}$ ( $r=d / 2$ is possible only if $d$ is even).
- $\mathcal{S}^{d, k}$ for $d \mid k, k \geq 1$.

In each case these categories have a combinatorial description.
In a second step we identify the compact quantum groups that corresponds to each of these categories of partitions. This identification is greatly simplified by the fact that a lot of free easy quantum groups have already been identified. The essential tool of the remaining part of this identification is the introduction of two algebraic operations, the tensor and the free complexifications denoted respectively $\tilde{x}$ and $\tilde{*})$ by $Z_{d}$. The tensor complexification has been already considered in an unpublished draft [83, and the free complexification with $\mathbb{Z}$ has been first introduced in [8]. This yields the following classification :

Theorem 3.29 (Ch.3, Th. 4.57). For each $n \geq 1$, the following correspondance holds between categories of partitions and unitary easy quantum groups:

1. the category $\mathcal{U}$ corresponds to the free unitary quantum group $U_{n}^{+}$. the category $\mathcal{O}^{k}$ corresponds to $O_{n}^{+} \tilde{\times} Z_{k}$.
2. the category $\mathcal{H}^{k, d}$ corresponds to $\left(Z_{d} \imath_{*} S_{n}^{+}\right) \tilde{\times} Z_{k} . Z_{d} \imath_{*} S_{n}^{+}$is also denoted $H_{n}^{d,+}$ and has been introduced by Banica and Vergnioux in [16].
3. the category $\mathcal{H}^{\#}$ corresponds to $H_{n}^{+\#}=H_{n}^{+} \tilde{*} Z_{2}$.
4. the category $\mathcal{B}^{k, d, *}\left(\right.$ resp $\left.\mathcal{B}^{k, d, 0}\right)$, corresponds to $\left.\left(C_{n}^{+} \tilde{*} Z_{d}\right) \tilde{\times} Z_{k}\right)$ (reps. $\left.\left(B_{n}^{+} \tilde{*} Z_{d}\right) \tilde{\times} Z_{k}\right)$ ).
5. the category $\mathcal{B}^{k, d, d / 2}$ corresponds to $\tilde{C}_{n}^{+, d} \tilde{\times} Z_{k}$.
6. the category $\mathcal{S}^{k, d}$ corresponds to $\left.\left(S_{n}^{+} \tilde{*} Z_{d}\right) \tilde{\times} Z_{k}\right)$.

As a corollary, we obtain also all the unitary easy quantum groups that are classical groups (see Ch.3, Th 4.61).

Weingarten calculus on free easy quantum groups We have seen in Chapter 2, Section 3 that for $u_{n}$ the fundamental representation of $U_{n}$, the family $\left(\operatorname{Tr}\left(u_{n}^{k}\right)_{k \geq 1}\right)$ converges in law toward a family of independent complex gaussian variables $\left(u_{k}\right)_{k \geq 1}$ such that $u_{k}$ has variance $k$; since this result holds also for $O_{n}$ and $S_{n}$ with different limit distributions, it is expected that the result can be generalized to all easy quantum groups. In [14], Banica, Curran and Speicher proved that the same phenomenon holds for orthogonal free easy quantum groups, with limit distributions involving free semicircular and free Poisson distributions.
In Chapter 5, we will extend this result to all free easy quantum groups. The main tool is the Weingarten formula, which takes a simpler expression for easy quantum groups. In the second part of Chapter 5, we prove that the second-order freeness for the unitary group (see Chapter 2, Th.2.36) has a natural analog in the free case: namely the family of traces of arbitrary reduced products of $u, u^{t}, \bar{u}, u^{*}$ converges in distribution to a family of circular variables.

### 3.3 Noncommutative permutations and free wreath product

The free wreath product is an algebraic construction that generalizes the usual wreath product between permutation groups and compact groups. It has been introduced by Bichon in [21] as a way to encode the quantum symmetries of a finite product of graphs.

### 3.3.1 Free wreath product

In the classical case, a permutation group is a subgroup of $S_{n}$ for some $n \geq 1$. The natural extension in the noncommutative case yields the following definition:

Definition 3.30. A non-commutative permutation group $F=\left(C(F),\left(v_{i j}\right)_{1 \leq i, j \leq n}\right)$ is a quantum subgroup of $S_{n}^{+}$. The non-commutative permutation group $F$ is said irreducible if $\operatorname{dim} \operatorname{Mor}_{F}(0,1)=$ 1.

From an algebraic point of view, this means that $F$ is a compact matrix quantum group whose fundamental representation matrix $v$ satisfies at least the following relations:

$$
v_{i j}^{*}=v_{i j}, \sum_{j} v_{i j}=\sum_{i} v_{i j}=\mathbf{1}, v_{i j} v_{i k}=\delta_{j k} v_{i j}
$$

for all $1 \leq i, j, k \leq n$. From a representation theoretic point of view, $\operatorname{Rep}\left(S_{n}^{+}\right)$is a subcategory of $\operatorname{Rep}(F)$, and therefore all the maps $T_{p}$ with $p \in N C^{\circ \bullet}$ are also intertwiners of $F$.
The free wreath product is the generalisation of the construction Chapter 2, Section 6.2.
Definition 3.31 (Bichon). ([21, Definition 2.2]) Let $G=(C(G), \Delta)$ be a compact quantum group and $F=\left(C(F),\left(v_{i j}\right)_{1 \leq i, j \leq n}\right)$ be a non-commutative permutation group. Let $\nu_{i}: C(G) \rightarrow$ $C(G)^{* n}$ be the canonical inclusion of $C(G)$ as the $i-t h$ copie in the free product $C(G)^{* n}, i=$
$1, \ldots, n$.
The free wreath product of $G$ by $F$ is the quotient of the $C^{*}$-algebra $C(G)^{* N} * C(F)$ by the two-sided ideal generated by the elements

$$
\nu_{k}(a) v_{k i}-v_{k i} \nu_{k}(a), \quad 1 \leq i, k \leq n, \quad a \in C(G) .
$$

It is denoted by $C(G) *_{w} C(F)$.
It has been proved in 21] that there exists a coproduct $\Delta$ on $C(G) *_{w} C(F)$ such that $\left(C(G) *_{w} C(F), \Delta\right)$ is a compact quantum group. This coproduct is defined as

$$
\Delta\left(v_{i j}\right)=\sum_{k=1}^{n} v_{i k} \otimes v_{k j}, \forall i, j \in\{1, \ldots, n\}
$$

and

$$
\Delta\left(\nu_{i}(a)\right)=\sum_{k=1}^{n} \nu_{i}\left(a_{(1)}\right) v_{i k} \otimes \nu_{k}\left(a_{(2)}\right)
$$

where $\Delta_{G}(a)=\sum a^{(1)} \otimes a^{(2)}$ is the value of the coproduct $\Delta_{G}$ on $a$ with the Seedler notations. We denote by $G \imath_{*} F$ the quantum group $\left(C(G) *_{w} C(F), \Delta\right)$.
Suppose that $G$ is a compact matrix quantum group with fundamental representation $\left(u_{k l}\right)_{1 \leq k, l \leq m}$. In this case $G z_{*} F$ is also a compact matrix quantum group with a fundamental representation $\left.\left(w_{i j, k l}\right)\right)_{\substack{1 \leq i, j \leq n \\ 1 \leq k, l \leq m}}$ defined by $w_{i j, k l}=v_{i j} u_{k l}^{i}$, where $u_{k l}^{i}$ denotes the element $\nu_{i}\left(u_{k l}\right)$.

Quantum symmetries of a graph The free wreath product construction is a natural construction when considering quantum symmetries of graphs. Let $\mathcal{G}$ be a finite graph with $n$ vertices $\{1, \ldots, n\}$ and adjacency matrix $d_{\mathcal{G}}=\left(d_{i j}\right)_{1 \leq i, j \leq n}$. We suppose that $\mathcal{G}$ doesn't have any loop. A symmetry of $\mathcal{G}$ is a permutation $\sigma$ of $\left\{v_{1}, \ldots, v_{n}\right\}$ such that $d_{\sigma(i) \sigma(j)}=d_{i j}$ for all $1 \leq i, j \leq n$. The set of symmetries of $\mathcal{G}$ forms a subgroup of $S_{n}$ called the symmetry group of $\mathcal{G}$ and denoted by $S(\mathcal{G})$. From a dual point of view, $C(S(\mathcal{G}))=C\left(S_{n}\right) /\left\langle v d_{\mathcal{G}}=d_{\mathcal{G}} v\right\rangle$, where $v$ is the fundamental matrix $\left(v_{i j}\right)_{1 \leq i, j \leq n}$ of $S_{n}$ defined in 2.6.2.
Definition 3.32 ([10]). The quantum symmetry group of $\mathcal{G}$ is the matrix quantum subgroup $\left(A(\mathcal{G}),\left(v_{i j}\right)_{1 \leq i, j \leq n}\right)$ of $\left(C\left(S_{n}^{+},\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)\right.$ defined as follows: $A(\mathcal{G})$ is the quotient of $C\left(S_{n}^{+}\right)$by the relation $u d=d u$ and $v_{i j}$ is the image of $u_{i j}$ in this quotient.

There is a natural operation on graphs yielding free wreath products on the level of the quantum symmetry group.
Definition 3.33. Let $\mathcal{F}, \mathcal{G}$ be two graphs without loop, with vertices indexed respectively by $\llbracket 1 ; n \rrbracket$ and $\llbracket 1, m \rrbracket$, and respective adjacency matrices $c$ and $d$. The lexicographical product $\mathcal{G} \circ \mathcal{F}$ is the graph with vertices indexed by $\llbracket 1, m \rrbracket \times \llbracket 1, n \rrbracket$, and adjacendy matrix

$$
D_{i j, k l}=\left\{\begin{array}{c}
d_{k l} \\
c_{i j} \\
\text { if } i \neq j
\end{array} \quad \text { if } i=j\right.
$$

Figure 3.2 is an example of such construction, with the lexicographical product of a segment with a square.
By a result of [27], $A(\mathcal{G}) \imath_{*} A(\mathcal{F}) \subseteq A(\mathcal{G} \circ \mathcal{F})$, with equality if and only if $(\mathcal{G}, \mathcal{F})$ respects the conditions of Sabidussi (see [73] for a description of this conditions). Note that the same result holds also in the classical case : namely $S(\mathcal{G}) 乙 S(\mathcal{F}) \subseteq S(\mathcal{G} \circ \mathcal{F})$, with equality if and only if $(\mathcal{G}, \mathcal{F})$ respects the conditions of Sabidussi.


Figure 3.2: Lexicographical product $\mathcal{G} \circ \mathcal{F}$, with $\mathcal{G}$ a segment and $\mathcal{F}$ a square.

Free product formulae An important invariant of a quantum permutation group $F$ is the law of the character $\chi_{F}$ of the fundamental representation $\left(v_{i j}\right)_{1 \leq i, j \leq n}$ under the Haar measure. Since $F$ is a quantum permutation group, $F$ is a subgroup of $O_{n}^{+}$and therefore $\chi$ is a self-adjoint element. Thus, although $C(F)$ is a non-commutative $C^{*}$-algebra, $\chi$ is a well-defined random variable on $\mathbb{R}$, whose associated measure is denoted by $\mu(F)$. From the representation theory of $F$, the $k$-th moment $c_{k}$ of $F$ is exactly $\operatorname{dim}\left(\operatorname{Fix}_{F}(k)\right)$, where $\operatorname{Fix}_{F}(k)$ denotes the vector space of invariant vectors of the $k$-th tensor power representation $v^{\otimes k}$.
By considering several examples of lexicographical products of graph and extended versions of this constructions, Banica and Bichon have been lead to the following conjecture in [10]:

Conjecture 3.34. Let $\mathcal{F}$ and $\mathcal{G}$ be two graphs such that $S(\mathcal{F})$ and $S(\mathcal{G})$ are respectively transitive on $\mathcal{F}$ and $\mathcal{G}$. Then

$$
\mu\left(A(\mathcal{G}) \imath_{*} A(\mathcal{F})=\mu(A(\mathcal{G})) \boxtimes \mu(A(\mathcal{F})) .\right.
$$

Recall that Q is the free multiplicative convolution as defined in Section 1.2.1. The condition that $S(\mathcal{F})$ is transitive on $\mathcal{F}$ means that for all $i, j \in \mathcal{F}$, there exists $\sigma \in S(\mathcal{F})$ such that $\sigma(i)=j$. Equivalently this means that $\operatorname{dim}\left(\operatorname{Fix}_{S(\mathcal{F})}(1)\right)=\operatorname{dim}\left(\operatorname{Fix}_{A(\mathcal{F})}(1)\right)=1$.
This conjecture comes from the theory of planar algebras and a similar result exists by taking a free product of certain planar algebras.

### 3.3.2 Planar algebra

A planar algebra is a collection of vector spaces which is stable under a set of transformations indexed by planar diagrams. They have been introduced by Jones in [46] in order to give a diagrammatic approach to the study of subfactors of finite factors.

Planar tangles A planar tangle $P$ of degree $k \geq 0$ consists of the following objects:

- A disk $D_{0}$ of $\mathbb{R}^{2}$, called the outer disk.
- Some disjoint disks $D_{1}, \ldots, D_{n}$ in the interior of $D_{0}$ which are called the inner disks.
- For each $0 \leq i \leq n$, a finite subset $S_{i} \in \partial D_{i}$ of cardinal $2 k_{i}$ (such that $k_{0}=k$ ) with a particular element $i_{*} \in S_{i}$. The elements of $S_{i}$ are called the distinguished points of $D_{i}$ and numbered counterclockwise starting from $i_{*} . k_{i}$ is called the degree of the inner disk $D_{i}$.
- A finite set of disjoint smooth curves $\left\{\gamma_{j}\right\}_{1 \leq j \leq r}$ such that each $\dot{\gamma}_{j}$ lies in the interior of $D_{0} \backslash \bigcup_{i \geq 1} D_{i}$ and such that $\bigcup_{1 \leq j \leq r} \partial \gamma_{j}=\bigcup_{0 \leq i \leq n} S_{i}$; it is also required that each curve meets a disk boundary orthogonally, and that its endpoints have opposite (resp. same) parity if they both belong to inner disks or both belong to the outer disk (resp. one belongs to an inner disk and the other one to the outer disk).
- A region of $P$ is a connected component of $D_{0} \backslash\left(\bigcup_{i \geq 1} D_{i} \cup\left(\bigcup \gamma_{j}\right)\right)$. Give a chessboard shading on the regions of $P$ in such a way that the interval components of type ( $2 i+1,2 i$ ) are boundaries of shaded regions.

In the above description, closed curves are allowed among the set $\left\{\gamma_{i}\right\}$. On each disk, the labelling of the distinguished points is fixed by the choice of a particular distinguished point $i_{*}$ : this choice is pictorially represented by adding a mark $*$ on the interval component directly preceeding this point. An example of planar tangle is given in Figure 3.3.


Figure 3.3: Planar tangle of degree 4

Planar tangles can be composed in the following way: suppose that $T_{1}$ and $T_{2}$ are tangles of respective degree $k$ and $k^{\prime}$, and that $T_{1}$ has an interior disk $D$ of degree $k^{\prime}$. Plugging $T_{2}$ inside $D$ in such way that the marked interval of the exterior disk of $T_{2}$ coincides with the marked interval of $D$, and then erasing the boundary of the exterior disk of $T_{2}$ (except the distinguished points of the exterior disk of $T_{2}$, which become usual points in the resulting picture) yields a new planar tangle $T_{1} \circ_{D} T_{2}$. An example of such a gluing is given in Figure 3.4.

Planar algebra A planar algebra is a family of finite-dimensional vector spaces $\left\{V_{k}\right\}_{k \in \mathbb{Z}_{>0} \cup\{+,-\}}$ together with an action of the planar tangles. Namely, each planar tangle $T$ of degree $k$ yields a linear maps $Z(T): \bigotimes_{\substack{D \text { internal } \\ \text { disk of } T}} V_{k_{D}} \rightarrow V_{k}$ with the compatibility condition

$$
Z(T)\left(v_{D_{1}} \otimes \cdots \otimes Z(S)\left(v_{D_{j}}\right) \otimes \cdots \otimes v_{D_{r}}\right)=Z\left(T \circ_{D_{j}} S\right)\left(v_{D_{1}} \otimes \cdots \otimes v_{D_{j}} \otimes \cdots \otimes v_{D_{r}}\right),
$$



Figure 3.4: Composition of planar tangles
for any planar tangles $T, S$ such that the degree of $S$ is the degree of the internal disc $D_{j}$ of $T$ and for all vectors $v_{D_{i}} \in V_{k_{i}}$ for $i \neq j$ and $v_{D_{j}} \in \bigotimes_{\substack{D_{\text {isternal }}^{\text {disk of } S}}} V_{k_{D}}$.
The planar algebra structure yields several natural operations on $\left\{V_{k}\right\}$. In particular each vector space $V_{k}$ becomes an algebra with the action of the planar tangle of Figure 3.5.


Figure 3.5: : Multiplication tangle of degree $\mathcal{P}_{6}$.
Under certain assumptions, each vector space $V_{k}$ is a semi-simple $*$-algebra and $\bigcup V_{k}$ is a tower of algebras with a common trace: such planar algebra is called a subfactor planar algebra (see [46]). One of the major results on planar algebras relates the dimension of each vector space $V_{k}$ of a subfactor planar algebra with the cardinality of some paths on a graph:

Theorem (Jones, 46]). Let $\mathcal{P}$ be a subfactor planar algebra. There exists a bipartite graph $G_{\mathcal{P}}$ with root vertex $*$ such that:

$$
\operatorname{dim} \mathcal{P}_{k}=\#\left\{\text { walk of length } 2 k \text { on } G_{\mathcal{P}} \text { starting and ending at } *\right\} .
$$

The random variable $\mu_{P}$ whose $k-$ th moment is $\operatorname{dim} \mathcal{P}_{k}$ is called the spectral measure of the planar algebra $\mathcal{P}$.

### 3.3.3 Overview of the results

The main motivation is the proof of Conjecture 3.34 Aiming this proof, we obtained several results on the intertwiner spaces of a free wreath product.

Free wreath product with $S_{n}^{+}$(joint work with François Lemeux) In Chapter 6, we study the free wreath product of a compact quantum group with the free symmetric group $S_{n}^{+}$. In particular we describe completely the intertwiner spaces in this case, which yields some useful expressions for the Weingarten matrix of a free wreath product with $S_{n}^{+}$. As a byproduct of these results, an asymptotic formula has been found for the law of the characters of $\mathbb{G} 2_{*} S_{n}^{+}$, where $\mathbb{G}$ is a fixed compact matrix quantum group and $n$ goes to $+\infty$.
Let $t \in(0,1]$ and let $\mathbb{G}$ be a matrix compact quantum group of Kac type and dimension $r$. Denote by $\chi_{\mathbb{G}}$ the law of the character of its fundamental representation. Let $\left.(\mathbb{G}\rangle_{*} S_{n}^{+},\left(w_{i j, k l}\right)_{1 \leq i, j \leq r, 1 \leq k, l \leq n}\right)$ be the matrix quantum group $\mathbb{G} 2_{*} S_{n}^{+}$with its fundamental representation $w$. For $1 \leq s \leq n$, denote by $\chi_{w}(s)$ the truncated character $\chi_{w}(s)=\sum_{\substack{1 \leq i \leq r \\ 1 \leq k \leq s}} w_{i i, k k}$.

Theorem 3.35. With respect to the Haar measure, if $s \sim t n$ for $n$ going to infinity,

$$
\chi_{\mathbb{G} \imath_{*} S_{n}^{+}}(s) \rightarrow \mathcal{P}_{t}\left(\chi_{\mathbb{G}}\right),
$$

where $\mathcal{P}_{t}\left(\chi_{\mathbb{G}}\right)$ is the free compound Poisson with parameter $t$ and original law $\chi_{\mathbb{G}}$.

Free wreath product and planar algebras (joint work with Jonas Wahl) In Chapter 7 , we establish a link between the free wreath product $G\rangle_{*} F$ of two irreducible non-commutative permutation groups and the free product of two planar algebras. The free product of two planar algebras is a construction which has been done by Bisch and Jones in [23], in order to deal with chains of inclusions of subfactors. In [6], Banica proved that the intertwiner spaces of any irreducible non-commutative permutation group $F$ is a planar algebra $\mathcal{P}(V)$.
Using the study of the case $G \imath_{*} S_{n}^{+}$, we prove that the intertwiner spaces of a free wreath product $G 2_{*} F$ is the free product of the planar algebras $\mathcal{P}(F)$ and $\mathcal{P}(G)$. Using the results of [23], we deduce a positive (and more general) answer to Conjecture 3.34 .
Theorem 3.36 ([10], Conj 3.1). Let $F$ and $G$ be two non-commutative permutation groups such that $\operatorname{dim}_{F} \operatorname{Mor}(0,1)=\operatorname{dim}_{G} \operatorname{Mor}(0,1)=1$. Then

$$
\mu\left(F \imath_{*} G\right)=\mu(F) \boxtimes \mu(G),
$$

where $\mu(\mathbb{G})$ denotes the law (with respect to the Haar measure) of the character of the fundamental representation of a matrix compact quantum group $\mathbb{G}$.

Using a result of Landau from [54] and the link between Boolean and free independence (see [17]), we also give a combinatorial proof to the fact that the spectral measure of a free product of irreducible planar algebras is the free multiplicative convolution of the spectral measures of the initial planar algebras (a result which is already proven in [23]).

### 3.4 Free fusion rings and non-commutative symmetric functions

We have seen in Chapter 2 that symmetric functions play an important role in the representation theory of classical easy groups : they encode the fusion rules of $U_{n}$ and $O_{n}$ by expressing the characters in terms of the eigenvalues of the matrices (Chapter 2, Section 3), and they describe the induction and restriction operations on the irreducible respresentations of $S_{n}$ (see Chapter 2, Section 6.2).
For free easy guantum groups, the analog of the symmetric functions has not been found yet. On the other hand, a non-commutative analog of the ring of symmetric functions called NSym has been introduced in [41. It appears that the fusion rules of free easy quantum groups share a common pattern which the multiplicative structure of the ring NSym in a particular basis.

### 3.4.1 Free fusion ring

The notion of free fusion ring has been introduced by Banica and Vergnioux in [16] to describe the fusion rules of free hyperoctahedral groups.
Let $R$ be a set with an involution $r \mapsto \bar{r}$ and a product $*: R \times R \rightarrow R \cup\{\emptyset\}$. The involution and the product are extended to the set $F(R)$ of words in $R$ with the formulae

$$
\overline{r_{1} \ldots r_{k}}=\overline{r_{k}} \ldots \overline{r_{1}},
$$

and

$$
\left(r_{1} \ldots r_{k}\right) *\left(s_{1} \ldots s_{l}\right)=r_{1} \ldots r_{k-1}\left(r_{k} * s_{l}\right) s_{2} \ldots s_{l}
$$

with the conventrion that $r_{k} \star s_{1}=\emptyset$ implies that $\left(r_{1} \ldots r_{k}\right) *\left(s_{1} \ldots s_{l}\right)=\emptyset$.
Definition 3.37. For $(R, *,-)$ has above, the free fusion ring with underlying set $(R, *,-)$ is the algebra whose basis is the words in $R$ with the product

$$
\begin{equation*}
x \otimes y=\sum_{x=v z, y=\bar{z} w} v w+v * w . \tag{3.4.1}
\end{equation*}
$$

This algebra is denoted $F(R, *,-)$.
Most of the free easy quantum groups have fusion rules following a free fusion ring:

- For $U_{n}^{+}$, the ring of characters is a free fusion ring with $R=\mathbb{Z}_{2}, \bar{r}=1-r$ and $r * s=\emptyset$.
- For $O_{n}^{+}$, the ring of characters is a free fusion ring with $R=\{1\}, \overline{1}=1$ and $1 * 1=\emptyset$.
- For $H_{n}^{s,+}$, the ring of characters is a free fusion ring with $R=\mathbb{Z}_{s}, \bar{r}=-r$ and $r * s=r+s$.

Freslon proved in [38] that the fusion ring of any free easy quantum group whose category of partitions is stable by removing block is a free fusion ring.
An alternative formula can be used to describe the multiplication in a free fusion ring. Indeed (3.4.1) is equivalent to the recurrence formula

$$
\begin{array}{r}
r_{1} \ldots r_{n} \otimes s_{1} \ldots s_{m}=r_{1} \ldots r_{n} s_{1} \ldots s_{m}+r_{1} \ldots r_{n-1}\left(r_{n} * s_{1}\right) s_{2} \ldots s_{m} \\
+ \\
\delta_{s_{1}=\overline{r_{n}}} r_{1} \ldots r_{n-1} \otimes s_{2} \ldots s_{m}
\end{array}
$$

The latter formula will be generalized in the last section of Chapter 6 to describe the fusion rules of general free wreath products with $S_{n}^{+}$. With this formula, Lemeux proved in [55] that a free fusion ring is isomorphic to the free ring $\mathbb{Z}\langle R\rangle$.

### 3.4.2 Noncommutative symmetric functions

The content of this subsection comes mainly from 84].

The Hopf algebra of symmetric functions In Chapter 2, the ring of symmetric functions $\Lambda$ has been defined as the algebra of symmetric polynomials in an infinite number of commuting variables $\left(X_{1}, \ldots, X_{n}, \ldots\right)$. The fundamental theorem of symmetric polynomials yields that $\Lambda$ is isomorphic, as a graded algebra, to the graded algebra of polynomials $\mathbb{Z}\left[e_{1}, \ldots, e_{n}, \ldots\right]$ with $\operatorname{deg}\left(e_{i}\right)=i$.
It turns out that $\Lambda$ has the structure of a Hopf algebra. Indeed let $\left(Y_{1}, \ldots, Y_{n}, \ldots\right)$ be another infinite family of variables and let $P \in \Lambda$. Then $P\left(\left\{X_{i}\right\},\left\{Y_{i}\right\}\right)$ has a decomposition

$$
P\left(\left\{X_{i}\right\},\left\{Y_{i}\right)=\sum_{i=1}^{r} P_{i}\left(\left\{X_{i}\right\}\right) \tilde{P}_{i}\left(\left\{Y_{i}\right\}\right),\right.
$$

for some $r \geq 1$ and $P_{i}, \tilde{P}_{i} \in \Lambda$. The formula $\Delta(P)=\sum P_{i} \otimes \tilde{P}_{i}$ defines a coproduct $\Delta$ on $\Lambda$. Since $(P Q)\left(\left\{X_{i}\right\},\left\{Y_{i}\right\}\right)=P\left(\left\{X_{i}\right\},\left\{Y_{i}\right\}\right) Q\left(\left\{X_{i}\right\},\left\{Y_{i}\right\}\right)$, this coproduct is an algebra morphism, turning $\Lambda$ into an Hopf algebra. The counit $\varepsilon$ of $\Lambda$ is given by $\varepsilon\left(e_{i}\right)=0, \varepsilon(1)=1$ and the antipode is given by the involutive automorphism $\omega\left(e_{i}\right)=(-1)^{i} h_{i}$.
The coproduct takes simple expressions on the basis of elementary functions and power sums, as we have

$$
\Delta\left(e_{n}\right)=\sum_{i=0}^{n} e_{i} \otimes e_{n-i}, \Delta\left(p_{n}\right)=1 \otimes p_{n}+p_{n} \otimes 1
$$

The Hall inner product defined in Chapter 2, Section 2 plays a particular role in this framework, since $\Lambda$ is a self-dual Hopf algebra with respect to this inner product. Namely

$$
\begin{equation*}
\langle P Q, R\rangle=\langle P \otimes Q, \Delta R\rangle \tag{3.4.2}
\end{equation*}
$$

for any $P, Q, R \in \Lambda$, and where $\langle.,$.$\rangle is canonically extended to the algebraic tensor product \Lambda \otimes \Lambda$.

Noncommutative symmetric function The ring of non-commutative symmetric functions NSym is the non-commutative version of $\Lambda$ obtained by removing the commutation relation on the $e_{i}^{\prime}$ s.

Definition 3.38. NSym is the graded free ring $\mathbb{Z}\left\langle S_{1}, \ldots, S_{n}, \ldots\right\rangle$ with the grading $\operatorname{deg}\left(S_{i}\right)=i$ and the comultiplication $\Delta\left(S_{n}\right)=\sum S_{i} \otimes S_{n-i}$.

It is still possible to define an involution $\omega$ in such a way that NSym is a Hopf-algebra. Since the elements $S_{i}$ don't commute anymore, the basis of NSym is not indexed by integers partitions but by sequences of integers: a sequence $I$ of integers $\left(i_{1}, \ldots, i_{r}\right)$ such that $\sum i_{j}=n$ is called a composition of $n$ of length $r$ and denoted by $I \vdash n$. The set of all composition of $n$ is denoted by $\operatorname{Comp}(n)$. For example $\left\{S_{I}\right\}_{I \vdash n}$ is a basis of the degree $n$ subspace of NSym. The multiplication structure on $S_{I}$ is given by the simple expression $S_{I} S_{J}=S_{I . J}$, where $I . J$ is the concatenation of the sequences $I$ and $J$.
An other important basis is given by the ribbon Schur functions $\left\{R_{I}\right\}$. Define a partial order $\leq$ on $\operatorname{Comp}(n)$ by the relation $I \leq J$ if and only if $J=\left(j_{1}, \ldots, j_{r}\right)$ and $I=\left(j_{1}+\cdots+j_{s_{1}}, j_{s_{1}+1}+\right.$ $\left.\cdots+j_{s_{1}+s_{2}}, \ldots, j_{r-s_{t}}+\cdots+j_{r}\right)$, where $s \vdash r$. The Moebius function on $(\operatorname{Comp}(n), \leq)$ is the function $\mu(J, I)=\delta_{J \leq I}(-1)^{l(I)-l(J)}$, where $l(I)$ is the length of the composition $I$. The ribbon Schur function $R_{I}$ is defined in the same way as the free cumulant in Chapter 1:

$$
R_{I}=\sum_{J \leq I} \mu(J, I) S_{J} .
$$

Equivalently $S_{I}=\sum_{J \leq I} R_{J}$. The multiplication in the basis $R_{I}$ is given by the formula $R_{I} R_{J}=R_{I . J}+R_{I \triangleright J}$, where $\left(i_{1}, \ldots, i_{r}\right) \triangleright\left(j_{1}, \ldots, j_{s}\right)=\left(i_{1}, \ldots, i_{r-1}, i_{r}+j_{1}, j_{2}, \ldots, j_{s}\right)$. Note in particular that this formula yields that NSym is a free fusion ring without involution.
The commutative quotient of NSym is isomorphic to $\Lambda$ through the map $S_{i} \mapsto e_{i}$. By this map, $R_{I}$ is mapped to the ribbon Schur function $r_{I}$, which is a particular skew Schur function (see [60], Part I, Ch.5).

Quasi-symmetric functions Unlike the Hopf-algebra of symmetric functions, NSym is not self-dual and therefore there is no inner product such that the relation 3.4.2 holds. However it is still possible to construct the dual Hopf-algebra of NSym. For each composition $I=\left(i_{1}, \ldots, i_{r}\right)$ define the polynomial $M_{I}\left(\left\{X_{i}\right\}\right)$ in the commuting variables $\left\{X_{i}\right\}_{i \geq 1}$ as

$$
M_{I}(X)=\sum_{j_{1}<j_{2}<\cdots<j_{r}} x_{j_{1}}^{i_{1}} x_{j_{2}}^{i_{2}} \ldots x_{j_{r}}^{i_{r}} .
$$

Definition 3.39. The ring generated by the polynomials $M_{I}$ over $\mathbb{Z}$ is called the ring of quasisymmetric functions. This ring is denoted by QSym.

The product of $M_{J}$ and $M_{K}$ has a decomposition in $\left\{M_{I}\right\}$ with integers coefficients and thus $\left\{M_{I}\right\}$ is a basis of $Q S y m$. As for the ring of symmetric functions, a coproduct $\gamma$ is defined on $Q$ Sym by decomposing a quasi-symmetric function in the variables $\left\{X_{i}\right\} \cup\left\{Y_{i}\right\}$ into a product of quasi-symmetric functions respectively in the variables $\left\{X_{i}\right\}$ and in $\left\{Y_{i}\right\}$. This coproduct turns $Q S y m$ into a Hopf algebra which is the dual Hopf algebra of NSym. With the pairing $\langle\rangle:, Q S y m \times \mathbf{N S y m} \rightarrow \mathbb{R}$ defined by the formula $\left\langle M_{I}, S_{J}\right\rangle=\delta_{I J}$, the following equalities hold:

$$
\begin{aligned}
\langle f \otimes g, \Delta(P)\rangle & =\langle f g, P\rangle, \\
\langle\gamma(f), P \otimes Q\rangle & =\langle f, P Q\rangle
\end{aligned}
$$

for all $f, g \in Q S y m$ and $P, Q \in$ NSym. The definition of the basis $\left\{R_{I}\right\}$ of NSym in terms of $\left\{S_{I}\right\}$ yields that the dual basis of $\left\{R_{I}\right\}$ is the set of quasi-symmetric function $F_{I}$ with

$$
F_{I}=\sum_{J \geq I} M_{J} .
$$

The basis $\left\{F_{I}\right\}$ is called the basis of fundamental quasi-symmetric functions. Each $F_{I}$ has a decomposition in terms of monomials similar to the decomposition of classical Schur functions with the semi-standard filling of Young diagram. In the case of $F_{I}$, the decomposition is described by the semi-standard filling of ribbon Young diagram :


Figure 3.6: Skew Young tableau associated to the composition $I=(3,2,4,1)$.

### 3.4.3 Overview of the results

The results obtained in Part $I V$ are somehow independent of Part $I I$ and Part $I I I$, apart from the relation with free fusion rings.

NSym as a probability space At the end of Chapter 6, the ring NSym is embedded in the ring of characters of the free wreath product $H_{n}^{+, \infty}$. Through this embedding the ribbon Schur functions are characters of certain irreducible representations of $H_{n}^{+, \infty}$. Therefore the Haar measure on $H_{n}^{+, \infty}$ turns the ribbon Schur functions into noncommutative random variables exactly as in the case of $U_{n}$ with the ring of symmetric functions. The law of the ribbon Schur functions $\left\{R_{(n)}\right\}_{n \geq 1}$ can be described thanks to the fusion rules of $H_{n}^{+, \infty}$ found by Banica and Vergnioux in [16]. This leads to the following result :

Theorem 3.40. There exists a injective algebra morphism $\Phi: \mathbf{N S y m} \rightarrow C l\left(H_{n}^{+, \infty}\right)$. For any ribbon Schur function $R_{I}, \Phi\left(R_{I}\right)$ is an irreducible character.
The family of random variables $\left(\Phi\left(R_{(n)}\right)\right)_{n \geq 1}$ with respect to the Haar measure is distributed as $\left(s z^{n} s\right)_{n \geq 1}$, where $s$ is a semi-circular variable of variance 1 and $z$ is a uniform measure on the the unit circle and free from $s$.

The Martin boundary of Zigzag diagrams Since $\left\{F_{I}\right\}$ is a basis of $Q S y m$, for any composition $J$ the product $F_{J} F_{(1)}$ has a decomposition in the basis $\left\{F_{I}\right\}$. A combinatorial argument shows that the coefficients in this decomposition are 0 or 1 .
Therefore we can construct a graph whose vertices are the compositions, and such that there is an edge between $I$ and $J$ if and only if the coefficient of $F_{J}$ in the product $F_{I} F_{(1)}$ is equal to 1 . This graph is called the graph of Zigzag diagrams, and denoted by $\mathcal{Z}$. It has been deeply studied by Olshanski and Gnedin, who have identified in 42 the minimal boundary of the graph (see Chapter 9 for a detailed exposition of the different boundaries of a graph): this boundary is a measured space which encodes the behavior of directed random walks on the graph.
An analogous graph exists by considering the Schur basis of Sym. This graph, denoted by $\mathcal{Y}$, has vertices indexed by Young diagrams and edges between $s_{\lambda}$ and $s_{\mu}$ if and only if the coefficient of $s_{\mu}$ in $s_{(1)} s_{\lambda}$ is non-zero. This graph has played an important role in the study of certain irreducible representations of $S_{\infty}$ (see [86]), the group of permutations of $\mathbb{N}$ with finite
support. Through its study it has been shown in [86] that the Poisson boundary of $\mathcal{Y}$ coincides with its Martin boundary. The Martin boundary is a geometrical boundary which comes from a compactification of the graph $\mathcal{Y}$.
In Chapter 9 we prove an analogous result for $\mathcal{Z}$ :
Theorem 3.41. The Martin boundary of $\mathcal{Z}$ coincides with its Poisson boundary.
Note that in general the Martin boundary is a subset of the Poisson boundary. The result of Theorem 3.41 had been conjectured by Olshanski and Gnedin in 42]. In order to prove this fact, we obtain some estimates on the filling of large ribbon Young diagrams in Chapter 8.
Finally in Chapter 9, Section 7, we establish a link between paths on $\mathcal{Y}$ and paths on $\mathcal{Z}$, and we give a central limit theorem for the shape of the descent pattern of a large permutation.

## Part II

## Unitary easy quantum groups

## Chapter 4

## Classification of categories of non-crossing colored partitions

We have seen in Chapter 3 that categories of two-colored partitions yield matrix compact quantum groups whose intertwiners have a simple description : the use of the Weingarten formula is thus greatly simplified on these particular quantum groups. It is therefore interesting to find all the possible categories of two colored partitions. This task seems very difficult in the general case; however, if we assume that the partitions have to be non-crossing, the situation is much easier.
This chapter is devoted to the classification of all categories of non-crossing two-colored partitions and to the construction of their associated unitary easy quantum group. The full classification is given by Theorem 4.41 and 4.42 in Section 7. The list of associated easy quantum group is given in Theorem 4.57. As a corollary, we also obtain in Section 9 the classification of all classical easy groups.

### 4.1 Categories of two-colored partitions and first results

Recall that from Chapter 1, a category of partitions is a collection $\mathcal{C}$ of subsets $\mathcal{C}(k, l) \subseteq P^{\circ} \bullet(k, l)$ (for all $k, l \in \mathbb{N}_{0}$ ) is a category of partitions, if it is closed under the tensor product, the composition and the involution, and if it contains the bicolored pair partitions $\square_{\bullet}$ and $\square^{\circ}$ as well as the identity partitions 0 and

If $\mathcal{C}$ is the smallest category of partitions containing the partitions $p_{1}, \ldots, p_{n}$, we write $\mathcal{C}=\left\langle p_{1}, \ldots, p_{n}\right\rangle$ and say that $\mathcal{C}$ is generated by $p_{1}, \ldots, p_{n}$.

Lemma 4.1. Let $\mathcal{C} \subseteq P^{\circ}$ • be a category of partitions.
(a) $\mathcal{C}$ is closed under rotation and verticolor reflection.
(b) If $p \in P^{\bullet \bullet}(k, l)$ is a partition in $\mathcal{C}$, we can erase two neighbouring points of $p$ if they have different (!) colors, i.e. if the $j$-th and the ( $j+1$ )-th of the lower points have inverse colors, then the partition $p^{\prime} \in P^{\bullet \bullet}(k, l-2)$ is in $\mathcal{C}$ which is obtained from $p$ by first connecting the blocks to which the $j$-th and the $(j+1)$-th lower points belong respectively, and then erasing these two points. We may also erase neighbouring points of inverse colors on the upper line.
(c) Let $p_{1} \otimes p_{2} \in \mathcal{C}$. Then $p_{1} \otimes \tilde{p}_{1} \in \mathcal{C}$ and $p_{2} \otimes \tilde{p}_{2} \in \mathcal{C}$. Note that we do not have $p_{1} \in \mathcal{C}$ or $p_{2} \in \mathcal{C}$ in general.

Proof. (a) Firstly, let $p \in P^{\circ \bullet}(k, l)$ be a partition and let the first of the $k$ upper points be black. Let $r \in P^{\circ}(k-1, k-1)$ be the tensor product of the identity partitions and a with the same color pattern as the latter $k-1$ upper points of $p$. The composition ( $\circ \otimes p)(\square \bullet r)$ yields a partition $p^{\prime} \in P^{\circ}(k-1, l+1)$ which conincides with the partition obtained from $p$ when rotating the first upper points to the row of lower points. See also [15, Lem 2.7]. If now $p \in \mathcal{C}$, then also $p^{\prime} \in \mathcal{C}$ since all partitions we used are in the category. Similarly we prove the other cases of rotation.

Secondly, if $p \in P^{\circ \bullet}(k, l)$ is in $\mathcal{C}$, then also $p^{*} \in P^{\circ \bullet}(l, k)$ is in $\mathcal{C}$ by the definition of a category. Rotating the $k$ upper points to below and the lower points to the upper line yields $\tilde{p}$, which is in $\mathcal{C}$.
(b) Compose $p$ with $r_{1} \otimes \stackrel{\ominus}{\square} \otimes r_{2}$ or $r_{1} \otimes \stackrel{\ominus}{\square} \otimes r_{2}$, where $r_{1}$ and $r_{2}$ are suitable tensor products of the identity partitions of and ..
(c) By (a), we have $\tilde{p}_{2} \otimes \tilde{p}_{1} \in \mathcal{C}$ and thus $p_{1} \otimes p_{2} \otimes \tilde{p}_{2} \otimes \tilde{p}_{1} \in \mathcal{C}$. Using (b), we infer $p_{1} \otimes \tilde{p}_{1} \in \mathcal{C}$ and likewise $p_{2} \otimes \tilde{p}_{2} \in \mathcal{C}$ (using rotation).

Tensor product, composition, involution, and the operations of the preceding lemma are called the category operations.

### 4.1.1 Special operations on partitions

The category operations may be performed in any category of partitions. Other procedures are allowed if and only if certain key partitions are contained in the category.

Lemma 4.2. Let $\mathcal{C}$ be a category of partitions and let $p \in P^{\circ \bullet}(0, l)$ be a partition without upper points.
(a) If $\sqcap \otimes \square \in \mathcal{C}$, then $\mathcal{C}$ is closed under permutation of colors, i.e. if $p \in \mathcal{C}$, then $p^{\prime} \in \mathcal{C}$, where $p^{\prime}$ is obtained from $p$ by some permutation of the colors of the points (without changing the strings connecting the points).
(b) If $\uparrow \otimes \uparrow \in \mathcal{C}$, then $\mathcal{C}$ is closed under permutation of colors of neighbouring singletons. Furthermore, we may disconnect any point from a block and turn it into a singleton.
(c) If $\sqcap \bullet \bullet \in \mathcal{C}$, then $\mathcal{C}$ is closed under permutation of colors of neighbouring points belonging to the same block. Furthermore, we may connect neighbouring blocks.
(d) If $\sqcap \sqcap \in \mathcal{C}$, we have no permutation of colors in general, and we may only connect neighbouring blocks if they meet at two points with inverse colors.
(e) If $\hat{\hat{\imath} \bullet} \in \mathcal{C}$, then $\mathcal{C}$ is closed under arbitrary positioning of singletons, i.e. if $p \in \mathcal{C}$, then $p^{\prime} \in \mathcal{C}$, where $p^{\prime}$ is obtained from $p$ by shifting blocks of length one to other positions.
(f) If $\uparrow \widehat{\substack{\imath}} \in \mathcal{C}$, then we may swap a singleton with a neighbour point of inverse color. This procedure inverts both colors. In other words, if $p=X a b Y \in \mathcal{C}$ where $b$ is a singleton and $a$ is a point of color inverse to $b$, then $p^{\prime}=X b^{-1} a^{-1} Y \in \mathcal{C}$.

Proof. (a) By rotation, the partitions $\bullet \otimes \varrho_{0}$ and $\bullet \otimes!$ are in $\mathcal{C}$. Note that the partitions $\bullet$ and - themselves are not necessarily contained in $\mathcal{C}$. Composing $p$ with partitions $r_{1} \otimes \emptyset_{0} \otimes \bullet \otimes r_{2}$
where the $r_{i}$ are given by suitable tensor products of the identity partitions and © yields a transposition of the colors of the points of $p$.
(b) Using rotation, we infer $\hat{\delta} \otimes \uparrow \otimes \hat{\jmath} \otimes \hat{\bullet} \in \mathcal{C}$. Analoguous to (a), we see that we may permute the colors of neighbouring singletons. Furthermore, rotations of $\uparrow \otimes \uparrow$ yield partitions p
$\delta$ and in $P^{\circ \bullet}(1,1)$ consisting of two disconnected points of the same color. Composing a partition with suitable tensor products of the identity partitions $\quad \circ, \bullet$ and the partitions $\quad \stackrel{\ominus}{\circ}$, $\bullet$ yields a partition where some points are disconnected from their blocks (without changing the color).
(c) Again, similar to (a), we infer that $\mathcal{C}$ is closed under permutation of colors of neighbouring points belonging to the same block, using $\square_{\circ \bullet \bullet} \in \mathcal{C}$. Since we then also have $\square \bullet \bullet \in \mathcal{C}$, the partitions are all in $\mathcal{C}$ by rotation. Composing with them effects that some neighbouring blocks are connected.
(d) We argue as in (c), but we may only use and
 reflection.
(f) Use $\uparrow \uparrow \downarrow \downarrow$ or $\uparrow \bullet \stackrel{\downarrow}{\circ}$ etc.

We formulated the above lemma only for partitions having no upper points, but the statements may be extended to arbitrary partitions $p \in P^{\circ \bullet}(k, l)$. We then have to take into account that the colors are inverted whenever they are rotated from the upper line to the lower line or the converse.

### 4.1.2 The non- (or one-) colored case

Let us end this section with a comparison to the case of categories of non-colored partitions, which were studied in [15] and in other articles and which were completely classified in [71]. For the classification in the noncrossing case, see [15] and [97]. Recall that there are exactly seven categories, given by:

| $\langle\uparrow\rangle$ | 2 | $\langle, \Pi\rangle$ | $\bigcirc$ | $\langle\uparrow \otimes \uparrow\rangle$ | $\supseteq$ | $\langle\emptyset\rangle=N C_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $1 \cap$ |  | in |  |  |  | $1 \cap$ |
| $\langle\uparrow, \Pi \Pi\rangle=N C$ | $\supseteq$ | $\langle\uparrow \otimes \uparrow, \Pi \square\rangle$ |  | $\supseteq$ |  | $\langle\Pi \square\rangle$ |

By $P(k, l)$ we denote the set of non-colored partitions where all points have no color. Likewise we use the notations $P$ for all non-colored partitions and $N C$ for all non-colored noncrossing partitions. Categories of non-colored partitions are defined like categories of two-colored partitions when forgetting all colors, see for instance [15] or [71]. The key link between non-colored categories and two-colored categories is given by the partition $\square_{0}$ as may be seen in the next proposition. Note that and are rotated versions of $\square_{0}$. Composing a partition $p$ with these partitions, we can change the colors of the points of $p$ to every possible color pattern. Hence, categories containing $\square_{0}$ are non-colored categories, in this sense.

Let $\Psi: P^{\circ \bullet} \rightarrow P$ be the map given by forgetting the colors of a two-colored partition. For a set $\mathcal{C} \subseteq P$, we denote by $\Psi^{-1}(\mathcal{C}) \subseteq P^{\bullet \bullet}$ its preimage under $\Psi$.

Proposition 4.3. (a) Let $\mathcal{C} \subseteq P$ be a category of non-colored partitions. Then $\Psi^{-1}(\mathcal{C}) \subseteq P^{\circ \bullet}$ is a category of two-colored partitions containing the unicolored pair partition $\sqcap$ (or equivalently ••).
(b) Let $\mathcal{C} \subseteq P^{0 \bullet}$ be a category of two-colored partitions containing the unicolored pair partition $\Gamma_{\circ}$ (or equivalently $\left.\sqcap\right)$ ). Then $\Psi(\mathcal{C}) \subseteq P$ is a category of non-colored partitions and $\Psi^{-1}(\Psi(\mathcal{C}))=\mathcal{C}$.
Hence, there is a one-to-one correspondence between categories of non-colored partitions and categories of two-colored partitions containing $\circ$.

Proof. (a) It is easy to see from the definition that $\Psi^{-1}(\mathcal{C})$ is a category of partitions. Furthermore, $\Gamma_{\circ} \in \Psi^{-1}(\mathcal{C})$ since $\Psi\left(\Gamma_{\circ}\right)=\sqcap \in \mathcal{C}$.
(b) It is easy to see that $\Psi(\mathcal{C})$ is closed under tensor product and involution and that it contains the pair partition $\Pi$ and the identity partition $\mid$. The composition is a bit more subtle. If $p, q \in \Psi(\mathcal{C})$, their composition is in $\Psi(\mathcal{C})$ only if we can lift $p$ and $q$ to partitions in $\mathcal{C}$ whose color patterns allow the composition in $P^{\circ}$. But since and $\emptyset_{0}^{\bullet}$ are in $\mathcal{C}$ (by rotation), we can do so: If $p \in \Psi(\mathcal{C})$, there is a partition $p_{0} \in \mathcal{C}$ such that $\Psi\left(p_{0}\right)=p$. Composing it wih $!$ and - , we may assume that all points of $p_{0}$ are white. Now, $\Psi(\mathcal{C})$ is closed under composition since $\mathcal{C}$ is. Similarly, we prove $\Psi^{-1}(\Psi(\mathcal{C})) \subseteq \mathcal{C}$ using $\varrho_{0}$ and.

### 4.2 Dividing the categories into cases

The classification of categories of noncrossing partitions is given by a detailed case study which we will now prepare.

### 4.2.1 The cases $\mathcal{O}, \mathcal{H}, \mathcal{S}$ and $\mathcal{B}$

The first division into cases is given by the sizes of blocks. The next lemma is formulated for arbitrary categories of partitions (not necessarily noncrossing ones).

Lemma 4.4. Let $\mathcal{C} \subseteq P^{\circ \bullet}$ be a category of partitions.
(a) If $\hat{\jmath} \otimes \uparrow \notin \mathcal{C}$, then all blocks of partitions $p \in \mathcal{C}$ have length at least two.
(b) If $\bullet \bullet \in \mathcal{C}$, then all blocks of partitions $p \in \mathcal{C}$ have length at most two.

Proof. (a) Let $p \in \mathcal{C}$ be a partition containing a block of size one. By rotation and possibly verticolor reflection, it is of the form $\uparrow \otimes q$, with no upper points. By Lemma 4.1(b) we have $\uparrow \otimes \uparrow \in \mathcal{C}$.
(b) Let $p \in \mathcal{C}$ be a partition containing a block of size at least three. By rotation, it is of the form $p=a^{\varepsilon_{1}} X_{1} a^{\varepsilon_{2}} X_{2} a^{\varepsilon_{3}} X_{3}$ with no upper points, where the points $a^{\varepsilon_{i}}$ belong to the same block, and $\varepsilon_{i} \in\{1,-1\}$ depending on the color. The subwords $X_{1}, X_{2}$ and $X_{3}$ are possibly connected to the block on the $a^{\varepsilon_{i}}$. By verticolor reflection, we infer that the following partition is in $\mathcal{C}$.

$$
p \otimes \tilde{p}=a^{\varepsilon_{1}} X_{1} a^{\varepsilon_{2}} X_{2} a^{\varepsilon_{3}} X_{3} \otimes \tilde{X}_{3} b^{-\varepsilon_{3}} \tilde{X}_{2} b^{-\varepsilon_{2}} \tilde{X}_{1} b^{-\varepsilon_{1}}
$$

By 4.1(b), we obtain that a partition $q:=a^{\varepsilon_{1}} X_{1}^{\prime} a^{\varepsilon_{2}} a^{-\varepsilon_{2}} \tilde{X}_{1}^{\prime} a^{-\varepsilon_{1}}$ is in $\mathcal{C}$. Note that while the blocks on $a^{\varepsilon_{i}}$ and $b^{-\varepsilon_{i}}$ are not connected in $p \otimes \tilde{p}$, the points $a^{\varepsilon_{i}}$ and $a^{-\varepsilon_{i}}$ are connected in $q$
due to the procedure as described in Lemma 4.1(b). Using rotation, we infer that the partion $a^{-\varepsilon_{1}} a^{\varepsilon_{1}} X_{1}^{\prime} a^{\varepsilon_{2}} a^{-\varepsilon_{2}} \tilde{X}_{1}^{\prime}$ is in $\mathcal{C}$. Again, tensoring it with its verticolor reflected version and using Lemma 4.1(b), we obtain $a^{-1} a a^{-1} a \in \mathcal{C}$ which implies $\because \bullet \bullet \in \mathcal{C}$.

Note that unlike in the non-colored case, $\hat{\delta} \otimes \hat{\emptyset} \notin \mathcal{C}$ does not imply that all blocks have even size. Consider for instance $\langle\stackrel{\Gamma}{\circ}\rangle$ 〉.
Definition 4.5. Let $\mathcal{C} \subseteq P^{\circ \bullet}$ be a category of partitions. We say that:

- $\mathcal{C}$ is in case $\mathcal{O}$, if $\hat{\circ} \otimes \hat{\bullet} \notin \mathcal{C}$ and $\circ \bullet \circ \notin \mathcal{C}$.
- $\mathcal{C}$ is in case $\mathcal{B}$, if $\uparrow \otimes \uparrow \in \mathcal{C}$ and $\circ \bullet \bullet \notin \mathcal{C}$.
- $\mathcal{C}$ is in case $\mathcal{H}$, if $\uparrow \otimes \uparrow \notin \mathcal{C}$ and $\sqcap \bullet \circ \in \mathcal{C}$.
- $\mathcal{C}$ is in case $\mathcal{S}$, if $\uparrow \otimes \uparrow \in \mathcal{C}$ and $\sqcap \bullet \bullet \in \mathcal{C}$.


### 4.2.2 Global and local colorization

It is convenient to study categories $\mathcal{C} \subseteq N C^{\circ \bullet}$ case by case according to the above definition. According to Lemma 4.2 (a), we divide each of these cases into two subcases: Those categories $\mathcal{C}$ containing $\circ \circ \otimes \curvearrowleft$ behave very differently from those not containing this partition.

Definition 4.6. A category of partitions $\mathcal{C} \subseteq P^{\bullet \bullet}$ is

- globally colorized, if $\circ \circ \bullet \bullet \in \mathcal{C}$
- and locally colorized if $\neg \otimes .\llcorner\notin \mathcal{C}$.

By Lemma 4.2, we may permute the colors of the points of partitions in globally colorized categories. Hence the coloring of partitions turns out to be of a global nature - the difference between the number of white and black points is the only number that matters for the coloring of a partition in such categories.

### 4.2.3 The global parameter $k(\mathcal{C})$

Studying categories of noncrossing partitions case by case, we will use certain global and local (color) parameters.

Definition 4.7. Let $p \in P^{\circ \bullet}$.

- Denote by $c_{\circ}(p)$ the sum of the number of white points on the lower line of $p$ and the black points on the upper line.
- Denote by $c_{\bullet}(p)$ the sum of the number of black points on the lower line of $p$ and the white points on the upper line.
- Define $c: P^{\circ \bullet} \rightarrow \mathbb{Z}$ by $c(p):=c_{\circ}(p)-c_{\bullet}(p)$.

We will mainly consider partitions $p \in P^{\circ \bullet}(0, l)$ with no upper points. In this case $c_{\circ}$ is counting the white points whereas $c_{\bullet}$ is counting the black points of a partition. Note that rotating black points from the upper line to the lower line turns them into white points. In this sense, $c_{\circ}$ counts black points on the upper line as white points on the lower line.

Definition 4.8. Let $\mathcal{C}$ be a category of partitions. We set $k(\mathcal{C})$ as the minimum of all numbers $c(p)$ such that $c(p)>0$ and $p \in \mathcal{C}$, if such a partition exists in $\mathcal{C}$. Otherwise $k(\mathcal{C}):=0$. The parameter $k(\mathcal{C})$ is called the degree of reflection of $\mathcal{C}$. It is the global parameter of $\mathcal{C}$.

Note that we always find a partition $p$ in $\mathcal{C}$ such that $c(p)=0$; take for instance $p=\square_{\bullet}$. In the next lemma, we show that the map $c: P^{0 \bullet} \rightarrow \mathbb{Z}$ behaves well with respect to the category operations. In particular, if there exists a partition $p \in \mathcal{C}$ such that $c(p)<0$, then $\tilde{p} \in \mathcal{C}$ and $c(\tilde{p})=-c(p)>0$.

Lemma 4.9. For the map $c: P^{\circ \bullet} \rightarrow \mathbb{Z}$ the following holds true.

$$
\begin{aligned}
& \text { (a) } c(p \otimes q)=c(p)+c(q) \\
& \text { (b) } c(p q)=c(p)+c(q) \\
& \text { (c) } c\left(p^{*}\right)=-c(p) \\
& \text { (d) } c\left(p^{\prime}\right)=c(p) \text {, if } p^{\prime} \text { is obtained from } p \text { by rotation. } \\
& \text { (e) } c(\tilde{p})=-c(p)
\end{aligned}
$$

Proof. From the definition it is clear that (a), (c), (d) and (e) hold. To see the invariance under composition, let $w_{1}$ be the number of upper white points of $q$, and $b_{1}$ be the number of upper black points. Let $w_{2}$ be the number of lower white points of $q$ and likewise $b_{2}$ for the black points. Since $p$ and $q$ are composable, the numbers $w_{2}$ and $b_{2}$ also count the number of upper white and upper black points of $p$, respectively. Finally, let $w_{3}$ and $b_{3}$ be the number of lower white and black points of $p$ respectively. We thus have:

$$
\begin{aligned}
c(q) & =c_{\circ}(q)-c_{\bullet}(q)=\left(w_{2}+b_{1}\right)-\left(w_{1}+b_{2}\right) \\
c(p) & =\left(w_{3}+b_{2}\right)-\left(w_{2}+b_{3}\right) \\
c(p q) & =\left(w_{3}+b_{1}\right)-\left(w_{1}+b_{3}\right)
\end{aligned}
$$

This implies $c(p q)=c(p)+c(q)$.
The global parameter $k(\mathcal{C})$ gives rise to a complete description of all possible numbers $c(p)$ of a category $\mathcal{C}$.

Proposition 4.10. Let $\mathcal{C} \subseteq P^{\circ \bullet}$ be a category of partitions and let $k:=k(\mathcal{C}) \in \mathbb{N}_{0}$. Then $c(p) \in k \mathbb{Z}$ for all partitions $p \in \mathcal{C}$.

Proof. The statement is obvious for $k=0$ by definition and Lemma 4.9 (e), thus we may assume $k>0$. Let $p \in \mathcal{C}$, such that $c(p) \neq 0$. By Lemma 4.9(e), we may restrict to $c(p)>0$. Now, assume that there is a number $m \in \mathbb{N}_{0}$ such that $k m<c(p)<k(m+1)$. By the definition of $k(\mathcal{C})$, there is a partition $q \in \mathcal{C}$ such that $c(q)=k$. Put $r:=\tilde{q}^{\otimes m} \otimes p$. Then $r \in \mathcal{C}$ and $c(r)=-m c(q)+c(p)$. Hence $0<c(r)<k$ which contradicts the definition of $k(\mathcal{C})$.

### 4.2.4 The local parameters $d(\mathcal{C})$ and $K^{\mathcal{C}}\left(\begin{array}{ll}〔 & \text { • }) ~\end{array}\right.$

We also have some local parameters. The idea is to determine possible numbers $c\left(p_{1}\right)$ of subpartitions $p_{1}$ between two legs of a block of a partition $p \in \mathcal{C}$. The situation when these two legs have the same color behaves quite differently from the case of equally colored legs.


By rotation，we can reduce it to the following situation．
Definition 4．11．－Let $p \in N C^{\bullet \bullet}(0, l)$ be a partition with no upper points．Assume that $p$ can be decomposed as $p=p_{1} \otimes p_{2}$ ．If the first and the last point of $p_{2}$ belong to the same block and if $p_{2}$ has length at least two，we say that $p=p_{1} \otimes p_{2}$ is in nest decomposed form．
－Let $\operatorname{NDF}\left(\begin{array}{ll}\ulcorner & 〕\end{array}\right)$ be the set of all partitions $p=p_{1} \otimes p_{2}$ in nest decomposed form such that the first point of $p_{2}$ is white and the last one is black．By $N D F^{\mathcal{C}}\left(\begin{array}{ll}\Gamma \\ 0 & \text { 〕 }\end{array}\right)$ we denote


－By $K^{\mathcal{C}}\left(\ulcorner\right.$ ．$)$ we denote the set of all numbers $c\left(p_{1}\right) \in \mathbb{Z}$ such that $p=p_{1} \otimes p_{2} \in$

－Let $\mathcal{C} \subseteq N C^{\circ}$ be a category of noncrossing partitions．We define the following local parameter $d(\mathcal{C})$ ．If $K^{\mathcal{C}}\left(\begin{array}{l}\ulcorner \\ 0 \\ \bullet\end{array}\right)$ contains a number $t>0$ ，we put $d(\mathcal{C})$ as the minimum of those numbers．Otherwise $d(\mathcal{C}):=0$ ．

Example 4．12．（a）The partition $\Gamma_{\circ} \otimes . \bullet$ is in $\operatorname{NDF}(\complement)$ with $c\left(p_{1}\right)=2$ ，whereas $\llbracket \bullet \otimes$ ○○ is in $N D F\left(\begin{array}{ll}\ulcorner & 7\end{array}\right)$ with $c\left(p_{1}\right)=-2$ ．
（b）The partition $p:=\Gamma_{\circ} \otimes \square \otimes \Gamma_{\circ}$ is in nest decomposed form，where $p_{1}=\Gamma_{0} \otimes \Gamma_{\circ}$
 in $N D F\binom{〔}{\circ}$ with the decomposition $q_{1}:=\sqcap \otimes .$. and $q_{2}:=\sqcap$ ．
When working on the classfication of noncrossing categories，we will be interested in the
 subsection，we shall prove that we always have $K^{\mathcal{C}}\binom{\complement}{\bigcirc}=K^{\mathcal{C}}\binom{$ ■ }{$\bullet}=d \mathbb{Z}$ for $d=d(\mathcal{C})$ and

 As for $K^{\mathcal{C}}\left(\begin{array}{l}\text { ■ }\end{array}\right)$ the situation is a bit more complicated－for instance，it could be empty．

Proposition 4．13．Let $\mathcal{C} \subseteq N C^{\bullet \bullet}$ be a category of noncrossing partitions．Then，$K^{\mathcal{C}}\binom{\ulcorner }{\bullet}=d \mathbb{Z}$ for $d=d(\mathcal{C})$ ．Furthermore，if $k(\mathcal{C}) \neq 0$ ，then $d(\mathcal{C}) \neq 0$ and $d(\mathcal{C})$ is a divisor of $k(\mathcal{C})$ ．
 $p=p_{1} \otimes p_{2}$ and $q=q_{1} \otimes q_{2}$ be two partitions in $N D F^{\mathcal{C}}\binom{〔}{\bullet}$ ．Firstly，we have $p_{1} \otimes q_{1} \otimes q_{2} \otimes p_{2} \in \mathcal{C}$ using rotation．Using the pair partition $\cup \cup$ and composition，we can connect the last point of $q_{2}$ with the first point of $p_{2}$ ．This erases these two points and we obtain a partition $p_{1} \otimes q_{1} \otimes r \in \mathcal{C}$ in nest decomposed form such that the first point of $r$ is white and the last one is black．Thus，
 Now，this is a partition in nest decomposed form，such that the first point of $\tilde{p}_{2}$ is white and the last one is black．Thus，$-c\left(p_{1}\right)=c\left(\tilde{p}_{1}\right) \in K^{\mathcal{C}}\binom{$ ऽ }{ 〕 } ．

We deduce $K^{\mathcal{C}}\left(\begin{array}{ll}\ulcorner & \boxed{\bullet}\end{array}\right)=d \mathbb{Z}$ using the Euklidean algorithm．Furthermore，note that $k(\mathcal{C}) \in$
 also $k(\mathcal{C})$ is a multiple of $d(\mathcal{C})$ ．

Definition 4.14. By $b_{s}$ we denote the partition in $P^{\circ \bullet}(0, s)$ consisting of a single block of length $s$ such that all points are white, hence $b_{2}=$ ○०, $b_{3}=$ ○о०, etc.

The next lemma is of quite technical nature, but it will be needed in this subsection as well as in the remainder of this chapter several times.

Lemma 4.15. Let $\mathcal{C} \subseteq N C^{\circ \bullet}$ be a category of noncrossing partitions.


 Then $\uparrow_{0}^{\otimes s} a^{\varepsilon_{1}} \uparrow_{\otimes s}^{\otimes-(s+c(q))} a^{\varepsilon_{2}} \in \mathcal{C}$, where $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ form the pair block $q$. Here, we use the notation $\uparrow^{\otimes s}=\uparrow^{\otimes-s}$ for $s<0$.
 Here $b_{-s}=\tilde{b}_{s}$ if $s<0$.
(c) Let $p \in \mathcal{C}$ be a partition such that two neighbouring points have the same color and belong to the same block. Then $\square \square \in \mathcal{C}$ or $\square \otimes \square \in \mathcal{C}$. In particular, if $\uparrow \otimes \uparrow \notin \mathcal{C}$ and $N D F^{\mathcal{C}}(\bullet \downarrow) \neq \emptyset$, then $\square_{\bullet} \bullet \bullet \mathcal{C}$ or $\square \otimes \square \in \mathcal{C}$. The same holds true, if $\uparrow \otimes \uparrow \notin \mathcal{C}$ and if $\mathcal{C}$ contains a partition $p_{1} \otimes p_{2}$ such that $c\left(p_{1}\right) \neq 0$.
 $N D F^{\mathcal{C}}(q)$ such that $c\left(p_{1}\right)=s$. Using the pair partitions and composing $p_{1}$ with the rotated
 points between the first and the last point of $p_{2}$ and we infer that $\uparrow_{0}^{\otimes s} a^{\varepsilon_{1}} \uparrow_{0}^{\otimes \alpha} a^{\varepsilon_{2}}$ is in $\mathcal{C}$ for some $\alpha \in \mathbb{Z}$. Here $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ form the pair block $q$. Now, disconnecting these two points using the partitions ${ }_{0}^{\circ}$ and $\stackrel{\bullet}{\bullet}$ again, we deduce $\uparrow_{0}^{\otimes s} \otimes \uparrow_{0}^{\otimes \alpha+c(q)} \in \mathcal{C}$. Using verticolor reflection and composition, this implies:

$$
\uparrow_{0}^{\otimes s} a^{\varepsilon_{1}} \uparrow_{0}^{\otimes \alpha} \uparrow_{0}^{\otimes-s} \uparrow_{0}^{\otimes-\alpha-c(q)} a^{\varepsilon_{2}} \in \mathcal{C}
$$

Using the pair partitions, we obtain $\uparrow^{\otimes s} a^{\varepsilon_{1}} \uparrow^{\otimes-s-c(q)} a^{\varepsilon_{2}} \in \mathcal{C}$.
(b) Let $p=p_{1} \otimes p_{2} \in \mathcal{C}$ be a partition in nest decomposed form such that $c\left(p_{1}\right)=s$. Assume $s>0$. Using the pair partitions and Lemma 4.2, we may connect all blocks in $p_{1}$ and we may erase all of its black points such that finally $b_{s} \otimes p_{2} \in \mathcal{C}$. By Lemma 4.1, $b_{s} \otimes \tilde{b}_{s} \in \mathcal{C}$; likewise if $s<0$.
(c) Let $p \in \mathcal{C}$ be a partition of the form $p=a a X$, where $a a$ are two points of the same color belonging to the same block. If this block is of size two, we have $\square_{\circ} \otimes \bullet \in \mathcal{C}$ by Lemma 4.1. Otherwise, we consider $p \otimes \tilde{p} \in \mathcal{C}$ and using the pair partition, we infer $\square_{\circ \bullet \bullet} \in \mathcal{C}$.

Now, if $\mathcal{C}$ contains a partition $p_{1} \otimes p_{2}$ such that $c\left(p_{1}\right)>0$, we use the pair partition to erase all black points in $p_{1}$. We obtain a partition $p_{1}^{\prime} \otimes p_{2} \in \mathcal{C}$ such that all points of $p_{1}^{\prime}$ are white. If $\uparrow \otimes \uparrow \notin \mathcal{C}$, all blocks in $p_{1}^{\prime}$ have size at least two. Since $p_{1}^{\prime}$ is noncrossing, we find two
neighbouring points belonging to the same block. By rotation, this yields a partition $p=a a X$ as above; likewise for $c\left(p_{1}\right)<0$.

Finally, if $N D F^{\mathcal{C}}(\mathfrak{\bullet}) \neq \emptyset$, then there is a $p=p_{1} \otimes p_{2} \in \mathcal{C}$ in nest decomposed form such that the first and the last point of $p_{2}$ are black. If $c\left(p_{1}\right)=0$, we erase $p_{1}$ using the pair partitions and we obtain $p_{2} \in \mathcal{C}$. By rotation, $p_{2}$ is of the form $a a X$. If $c\left(p_{1}\right) \neq 0$ we use the argument above.

In globally colorized categories, the local parameters $d(\mathcal{C})$ and $K^{\mathcal{C}}\left(\begin{array}{l}\text { ■ }\end{array}\right)$ are trivial. In par-
 colored categories.

Lemma 4.16. Let $\mathcal{C} \subseteq N C^{\bullet \bullet}$ be a globally colorized category of noncrossing partitions. The following holds true.
 $d=d(\mathcal{C})$.
(b) We have $d(\mathcal{C}) \in\{1,2\}$ with $d(\mathcal{C})=1$ if and only if $\hat{\text { ô }} \bullet \in \mathcal{C}$.

Proof. (a) Let $p=p_{1} \otimes p_{2} \in \mathcal{C}$ be a partition in nest decomposed form. Thus, $p_{2}=a^{\varepsilon_{1}} X a^{\varepsilon_{2}}$, where $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ belong to the same block, and $X$ is some subpartition (possibly connected to $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ ). Then, the following partition is in $\mathcal{C}$, by composition.

$$
p^{\prime}:=p_{1} \otimes a^{\varepsilon_{1}} \triangleright \circ \oslash X a^{\varepsilon_{2}}
$$

Using permutation of colors, we may change the colors of $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ arbitrarily. Moreover, $K^{\mathcal{C}}\left(\begin{array}{ll}\text { ᄃ } \\ 0 & \bullet\end{array}\right)=d \mathbb{Z}$ by Proposition 4.13 .
(b) By Example 4.12, we have $2 \in K^{\mathcal{C}}(\complement)$, which by (a) yields $2 \in K^{\mathcal{C}}\binom{\complement}{0}$. Thus, $d(\mathcal{C}) \in\{1,2\}$. If $\widehat{\hat{} \text {. }} \in \mathcal{C}$, then $d(\mathcal{C})=1$ by definition. Conversely, let $d(\mathcal{C})=1$. We thus find a partition $\hat{\jmath} \otimes p_{2}$ in $\mathcal{C}$. By Lemma 4.1, we deduce $\hat{\jmath} \otimes \hat{\imath} \in \mathcal{C}$, which by Lemma 4.15 implies给 $\in \mathcal{C}$.

We finally prove that our local parameters are given only by $d(\mathcal{C})$ and $K^{\mathcal{C}}\left(\complement_{\bullet}\right)$.
Proposition 4.17. We have $K^{\mathcal{C}}\binom{$ ■ }{ ■ }$=K^{\mathcal{C}}\left(\begin{array}{l}\text { ■ }\end{array}\right)=d \mathbb{Z}$ with $d=d(\mathcal{C})$ for all categories $\mathcal{C} \subseteq N C^{\circ \bullet}$. Furthermore, $K^{\mathcal{C}}\left(\begin{array}{ll}\ulcorner & 7 \\ 0 & 0\end{array}\right)=-K^{\mathcal{C}}\left(\begin{array}{ll}\text { ■ 〕 }\end{array}\right)$.

Proof. Let $\mathcal{C} \subseteq N C^{\circ \bullet}$ be a category of noncrossing partitions. If $p_{1} \otimes p_{2} \in \mathcal{C}$ is a partition in nest decomposed form such that the first and the last point of $p_{2}$ have the same color, then




Case 2. Let $\hat{\jmath} \otimes \hat{\bullet} \notin \mathcal{C}$ and let $N D F^{\mathcal{C}}(\boldsymbol{\complement}) \neq \emptyset$. By Lemma 4.15 we have $\Gamma_{0} \otimes \complement_{\bullet} \in \mathcal{C}$

 Lemma 4.15 this will finish the proof.

So，let $b_{s} \otimes \tilde{b}_{s} \in \mathcal{C}$ ．By rotation，we have $\left(\tilde{b}_{s}\right)^{*} \otimes \tilde{b}_{s} \in \mathcal{C}$ which is a partition in $P^{\circ \bullet}(s, s)$ ．Now，
 is black and $a^{-1}$ is white forming the pair block $\quad .0$ ．Hence $\left.s \in K^{\mathcal{C}}()_{0}\right)$ ．Similarly，we obtain $b_{s} \otimes a^{-1} \tilde{b}_{s} a \in \mathcal{C}$ and $s \in K^{\mathcal{C}}\left(\begin{array}{ll}\text { Г } \\ 0 & \bullet\end{array}\right)$ ．

Case 3．Let $\uparrow \otimes \uparrow \notin \mathcal{C}$ and let $N D F^{\mathcal{C}}\left(\begin{array}{l}\text { ■ })\end{array}\right)=\emptyset$ ．Then also $N D F^{\mathcal{C}}\left(\begin{array}{ll}〔 \\ 0 & \jmath\end{array}\right)=\emptyset$ ．Let $p_{1} \otimes p_{2} \in \mathcal{C}$ be a partition in nest decomposed form such that the first and the last point of $p_{2}$ have inverse colors．Assume $c\left(p_{1}\right)>0$ ．Using the pair partitions，we may erase all black points in $p_{1}$ and we obtain a partition $p_{1}^{\prime} \otimes p_{2}$ such that all points in $p_{1}^{\prime}$ are white．Since all blocks in $p_{1}^{\prime}$ have size at least two by Lemma 4．4．we obtain a partition in $N D F^{\mathcal{C}}\left(\begin{array}{lll}〔 \\ 0 & \jmath\end{array}\right)$ up to rotation which is a contradiction．Likewise we show that $c\left(p_{1}\right)$ cannot be strictly negative．We conclude that


## 4．2．5 Summary of the strategy for the classification

We now have all tools at hand for the classification of categories $\mathcal{C} \subseteq N C^{\circ}$ of noncrossing partitions．The general strategy is as follows．
－We study the cases $\mathcal{O}, \mathcal{H}, \mathcal{S}$ and $\mathcal{B}$（see Definition 4．5）step by step subdividing them again into the local and the global colorization（see Definition 4．6）respectively．
－In each of these cases we first determine all possible global and local parameters $k(\mathcal{C})$ ， $d(\mathcal{C})$ and $K^{\mathcal{C}}\left(\complement_{\bullet}\right)$ ．
－We then find characteristic sample partitions which somehow represent these param－ eters．
－Next，we isolate sets of partitions $M$ depending on the possible values of the parameters and we prove $M \subseteq\left\langle p_{1}, \ldots, p_{n}\right\rangle$ ，where $p_{1}, \ldots, p_{n}$ are the sample partitions．
－Finally，we prove that these are all possible categories in the considered case．To do so，if $\mathcal{C}$ has parameters $k, d$ and $K^{\mathcal{C}}(\smile)$ ），we have $\left\langle p_{1}, \ldots, p_{n}\right\rangle \subseteq \mathcal{C}$ ．On the other hand， $\mathcal{C} \subseteq M$ ，which proves $\mathcal{C}=\left\langle p_{1}, \ldots, p_{n}\right\rangle=M$ ．

## 4．3 Case $\mathcal{O}$

Let us first consider the case $\mathcal{O}$ ，i．e．the case of categories $\mathcal{C} \subseteq N C^{\bullet \bullet}$ of noncrossing partitions such that $\hat{\jmath} \otimes \hat{\bullet} \notin \mathcal{C}$ and $\because \notin \mathcal{C}$ ．By Lemma 4．4． $\mathcal{C}$ is a subset of the set $N C_{2}^{0 \bullet}$ of all noncrossing pair partitions．

## 4．3．1 Determining the parameters

Proposition 4．18．Let $\mathcal{C} \subseteq N C^{\circ \bullet}$ be a category of noncrossing partitions in case $\mathcal{O}$ ．
（a）If $\mathcal{C}$ is globally colorized，then $d(\mathcal{C})=2$ and $k(\mathcal{C}) \in 2 \mathbb{N}_{0}$ ．
（b）If $\mathcal{C}$ is locally colorized，then $d(\mathcal{C})=k(\mathcal{C})=0$ and $K^{\mathcal{C}}($ Ø 〕）$)=\emptyset$ ，i．e．：

Proof. (a) By Lemma 4.16, $d(\mathcal{C})=2$ and hence $k(\mathcal{C}) \in 2 \mathbb{N}_{0}$, using Proposition 4.13 .
(b) Let $p \in \mathcal{C}$ be a partition having no upper points. We prove that all blocks connect a white point with a black point from which the assertion easily follows. Assume that there is a block $V$ connecting two white points. We choose $V$ such that all blocks nested into it (if there are any) connect a white point with a black point. Using the pair partitions, we erase all those blocks and we end up with a partition in $\mathcal{C}$ such that two neighbouring points have the same color and belong to the same block. By Lemma 4.15(c), this is a contradiction.

### 4.3.2 Finding partitions realizing the parameters

Lemma 4.19. Let $\mathcal{C} \subseteq N C^{\circ \bullet}$ be a globally colorized category of noncrossing partitions in case $\mathcal{O}$ such that $k=k(\mathcal{C}) \neq 0$. Then ○○ $\sqcap^{\otimes \frac{k}{2}} \in \mathcal{C}$.

Proof. We find a partition $p \in \mathcal{C}$ having no upper points such that $c(p)=k$, i.e. there are $x+k$ white points and $x$ black points in $p$. Now, the partition $\square_{\bullet}^{\otimes \frac{k}{2}} \otimes p$ is in $\mathcal{C}$. By Lemma $x+k$ white points and $x$ black points in $p$. Now, the partition $\circ \bullet \otimes p$, we may permute the colors of this partition, and we infer that $\square^{\otimes \frac{k}{2}}$ 4.2. 4.2, we may permute the colors of this partition, and we infer that $\square_{0} \otimes p^{\prime}$ is in $\mathcal{C}$ for some partition $p^{\prime}$ with $c\left(p^{\prime}\right)=0$. Using the pair partitions $\circ \bullet$ and $\bullet \circ$ to erase $p^{\prime}$, we infer that $\square_{0}^{\otimes \frac{k}{2}} \in \mathcal{C}$.

### 4.3.3 Description of natural categories in case $\mathcal{O}$

The preceding subsection shows that there are natural categories $\left\langle\square_{0}^{\otimes \frac{k}{2}}, \square \otimes \bullet \bullet\right\rangle$ in case $\mathcal{O}$. Let us describe these categories and also the "empty category" $\langle\emptyset\rangle=\left\langle\square_{\bullet}, \square_{0}\right\rangle$. (Recall that the partitions $\square \bullet$ and $\bullet \circ$ are always contained in a category and that we omit to write them down explicitely as generators.)
Proposition 4.20. We have the following natural categories of partitions in case $\mathcal{O}$.
(a) The category $\mathcal{O}_{\mathrm{loc}}:=\langle\emptyset\rangle$ consists of all noncrossing pair partitions such that each block connects a white point with a black point, when the partition is rotated such that it has no upper points.
(b) Let $k \in 2 \mathbb{N}_{0}$. Then $\mathcal{O}_{\text {glob }}(k):=\left\langle\square^{\otimes \frac{k}{2}}, \square \otimes ■ \bullet\right.$ coincides with $\left\{p \in N C_{2}^{\circ} \bullet \mid c(p) \in k \mathbb{Z}\right\}$. Here, $\sqcap^{\otimes \frac{k}{2}}=\emptyset$ if $k=0$.

In particular, all these categories are pairwise different.
Proof. (a) We may construct all partitions from the assertion using $\circ \bullet, \square \circ$ and the category operations due to a simple inductive argument. Assume that $p$ has $m+1$ blocks. Since $p$ is noncrossing, it contains at least one block $q=\square \bullet$ or $q=\curvearrowleft \circ$. Removing $q$ yields a partition which is in $\langle\emptyset\rangle$ by induction hypothesis. Composing this partition with $r_{1} \otimes q \otimes r_{2}$ where $r_{i}$ are suitable tensor products of the identity partitions $\circ$ and - yields the partition $p$ which is hence in $\langle\emptyset\rangle$. Conversely, the set of all noncrossing pair partitions with the block rule of the assertion forms a category of partitions, hence containing $\langle\emptyset\rangle$.
(b) Let $p \in N C_{2}^{\circ \bullet}(0, l)$ be a partition with no upper points such that $c(p)=k m \geq 0$, for some $m \in \mathbb{N}_{0}$. Let $p^{\prime} \in N C_{2}^{\circ \bullet}(0, l)$ be the partition obtained from $p$ by replacing each unicolored block
 This implies that $p^{\prime} \otimes \Gamma_{\circ}^{\otimes \frac{k m}{2}}$ is in $\left\langle\Gamma_{\circ}^{\otimes \frac{k}{2}}\right.$, $\left.\circ \otimes \digamma_{\bullet}\right\rangle$, too, with $c\left(p^{\prime} \otimes \Gamma_{\circ}^{\otimes \frac{k m}{2}}\right)=c(p)$ by Lemma 4.9. Hence, permutation of colors yields that $p \otimes \bigcirc^{\otimes \frac{k m}{2}}$ is in $\left\langle\square^{\otimes \frac{k}{2}}, ~ \supset \otimes \otimes \bullet\right\rangle$. Using the pair partition, we infer $p \in\left\langle\Gamma_{0}^{\otimes \frac{k}{2}}, \Gamma_{\circ} \otimes \Gamma_{\bullet}\right\rangle$.

Conversely, the set $\left\{p \in N C_{2}^{\circ \bullet} \mid c(p) \in k \mathbb{Z}\right\}$ is a category of partitions due to Lemma 4.9 containing $\circ_{\circ}^{\otimes \frac{\kappa}{2}}$ and $\circ \otimes \odot \complement^{\circ}$.

### 4.3.4 Classification in the case $\mathcal{O}$

We are now ready to prove our first classification theorem.
Theorem 4.21. Let $\mathcal{C} \subseteq N C^{\bullet \bullet}$ be a category of noncrossing partitions in case $\mathcal{O}$. Then $\mathcal{C}$ coincides with one of the following categories.
(a) If $\mathcal{C}$ is globally colorized, then $\mathcal{C}=\mathcal{O}_{\text {glob }}(k)=\left\langle\square_{0}^{\otimes \frac{k}{2}}\right.$, $\circ \otimes$ ■๐ $\rangle$ for $k=k(\mathcal{C}) \in 2 \mathbb{N}_{0}$.
(b) If $\mathcal{C}$ is locally colorized, then $\mathcal{C}=\mathcal{O}_{\text {loc }}=\langle\emptyset\rangle$.

Proof. (a) By Propositions 4.10 and 4.20 , we have that $\mathcal{C}$ is contained in $\left\langle\square^{\otimes \frac{k}{2}}\right.$, $\left.\circ \otimes \square_{\bullet}\right\rangle$. Conversely, $\Gamma_{0} \otimes \square_{\bullet} \in \mathcal{C}$ by Definition 4.6 and $\square_{0}^{\otimes \frac{k}{2}} \in \mathcal{C}$ by Lemma 4.19.
(b) Let $\mathcal{C}$ be locally colorized and let $p \in \mathcal{C}$ be a partition with no upper points. Then, each block of $p$ connects a white point to a black point, see the proof of Proposition 4.18. By Proposition 4.20, $\mathcal{C}$ is contained in $\langle\emptyset\rangle$, hence they coincide.

Remark 4.22. For $k=2$, the category $\left\langle\square^{\otimes \frac{k}{2}}\right.$, $\left.\circ \otimes \square\right\rangle$ coincides with the non-colored category of partitions $\langle\Pi\rangle$ in the sense of Proposition 4.3.

### 4.4 Case $\mathcal{H}$

We now turn to the case $\mathcal{H}$, i.e. to categories $\mathcal{C} \subseteq N C^{\circ \bullet}$ such that $\hat{\jmath} \otimes \hat{\bullet} \notin \mathcal{C}$ but $\Gamma_{\bullet \bullet \bullet} \in \mathcal{C}$. By Lemma 4.4, no blocks of size one occur in any partition considered in this section. Recall, that due to Lemma 4.2 (d), we may connect neighbouring blocks of partitions in $\mathcal{C}$, if the blocks meet at two points with inverse colors.

### 4.4.1 Determining the parameters

Proposition 4.23. Let $\mathcal{C} \subseteq N C^{\bullet \bullet}$ be a category of noncrossing partitions in case $\mathcal{H}$.
(a) If $\mathcal{C}$ is globally colorized, then $d(\mathcal{C})=2$ and $k(\mathcal{C}) \in 2 \mathbb{N}_{0}$.
(b) If $\mathcal{C}$ is locally colorized, then
(i) either $K^{\mathcal{C}}($ © 〕) $=\emptyset$ and $k(\mathcal{C})=d(\mathcal{C})=0$,
(ii) or $k(\mathcal{C}), d(\mathcal{C}) \in \mathbb{N}_{0} \backslash\{1,2\}$ and $K^{\mathcal{C}}(\bullet \mathfrak{0})=d \mathbb{Z}$ for $d=d(\mathcal{C})$, i.e. in this case:

Moreover，we have $K^{\mathcal{C}}\left(\complement_{\bullet}\right) \neq \emptyset$ if and only if $\quad \circ \bullet \bullet \mathcal{C}$.
Proof．（a）This is analogue to Proposition 4．18．
（b）We first prove that $K^{\mathcal{C}}(\complement) \neq \emptyset$ holds if and only if $\Gamma_{0 \bullet \bullet} \in \mathcal{C}$ ．If $\sigma_{0 \bullet \bullet} \in \mathcal{C}$ ，then $\square \bullet \bullet \bullet \bullet \mathcal{C}$ from which we deduce $K^{\mathcal{C}}(\bullet \bullet) \neq \emptyset$ ．Conversely，if $K^{\mathcal{C}}(\complement \downarrow) \neq \emptyset$ ，then $N D F^{\mathcal{C}}(\bullet.) \neq \emptyset$ from which we deduce $\Gamma_{\bullet \bullet \bullet} \in \mathcal{C}$ by Lemma 4．4（c）．

Next，we prove that a locally colorized category can only be in the cases（i）or（ii）of the assertion．

Case 1：$K^{\mathcal{C}}\left(\right.$ 〔 〕）$=\emptyset$ ．Assume $d(\mathcal{C}) \neq 0$ ．Then there is a partition $p=p_{1} \otimes p_{2} \in \mathcal{C}$ such that $c\left(p_{1}\right) \neq 0$ ，which by Lemma 4.15 implies $\quad . \bullet \in \mathcal{C}$ ，a contradiction．Thus，we have $d(\mathcal{C})=0$ which implies $k(\mathcal{C})=0$ by Proposition 4．13．

Case 1：$K^{\mathcal{C}}(\smile) \neq \emptyset$ ．Assume $d(\mathcal{C})=2$ ．We find a partition $p=p_{1} \otimes p_{2} \in \mathcal{C}$ such that $c\left(p_{1}\right)=2$ ．Using the pair partition，we infer $\uparrow \otimes \uparrow \otimes p_{2} \in \mathcal{C}$ or $\Gamma_{\circ} \otimes p_{2} \in \mathcal{C}$ ．By Lemma 4．1．we have $\uparrow \otimes \uparrow \in \mathcal{C}$ or $\sqcap \otimes \sqcap \in \mathcal{C}$ ，both is a contradiction．Similarly，$d(\mathcal{C})=1$ implies $\uparrow \otimes p_{2} \in \mathcal{C}$ for some partition $p_{2}$ ，and the cases $k(\mathcal{C}) \in\{1,2\}$ can be excluded analoguously （with $p_{2}=\emptyset$ ）．

It remains to show that $K^{\mathcal{C}}\left(\complement_{\bullet}\right)=d \mathbb{Z}$ for $d=d(\mathcal{C})$ ．Since $K^{\mathcal{C}}\left(\complement_{\bullet}\right) \neq \emptyset$ ，we have $\Gamma_{0 \bullet \bullet} \in \mathcal{C}$ ． Thus，for $0 \neq s \in K^{\mathcal{C}}\binom{〔}{\bullet}$ ，we infer $b_{s} \otimes \tilde{b}_{s} \in \mathcal{C}$ by Lemma 4．15．This proves $s \in K^{\mathcal{C}}\binom{\bullet}{\bullet}$ ． Moreover，we have $0 \in K^{\mathcal{C}}\left(\boldsymbol{\bullet}^{\circ}\right)$ since a rotated version of $b_{s} \otimes \tilde{b}_{s}$ yields a partition $p_{2} \in \mathcal{C}$ such that the first and the last point are black and belong to the same block．Then $\quad \bullet \otimes p_{2}$ is in $\mathcal{C}$
 Lemma 4．15，we have $b_{s} \otimes \tilde{b}_{s} \in \mathcal{C}$ ．Like in the proof of Proposition 4．17，we obtain a partition $b_{s} \otimes a \tilde{b}_{s} a^{-1} \in \mathcal{C}$ ，where $a$ and $a^{-1}$ is a pair block $\square_{\bullet}$ ．This proves $s \in K^{\mathcal{C}}(\ulcorner\quad \downarrow)$ and hence $d \mathbb{Z}=K^{\mathcal{C}}($ ■ ！）.

## 4．4．2 Finding partitions realizing the parameters

Lemma 4．24．Let $\mathcal{C} \subseteq N C^{\bullet \bullet}$ be a category of noncrossing partitions in case $\mathcal{H}$ ．
（a）If $k=k(\mathcal{C}) \neq 0$ ，then $b_{k} \in \mathcal{C}$ ．
（b）If $d=d(\mathcal{C}) \neq 0$ ，then $b_{d} \otimes \tilde{b_{d}} \in \mathcal{C}$ ．
Proof．（a）By Proposition 4．23，we have $\quad 0 \bullet \bullet \mathcal{C}$ as soon as $k \neq 0$ ．Let $p \in \mathcal{C}$ be a partition with no upper points such that $c(p)=k$ ．Using the pair partition，we may assume that $p$ is a partition on $k$ white points．Using Lemma 4．2，we may connect all its blocks，ending up with $b_{k} \in \mathcal{C}$ ．
（b）Similar to（a），we deduce $b_{d} \otimes p_{2} \in \mathcal{C}$ from the existence of a partition $p_{1} \otimes p_{2} \in \mathcal{C}$ with $c\left(p_{1}\right)=d$ ．By Lemma 4．1，we have $b_{d} \otimes \tilde{b_{d}} \in \mathcal{C}$ ．

## 4．4．3 Description of natural categories

Motivated by Lemma 4.24 ，we want to describe the categories $\left\langle b_{k}, b_{d} \otimes \tilde{b_{d}}, \quad \square \bullet \bullet\right\rangle$ ．Note that for $k \geq 2$ or $d \geq 2$ ，we may always construct the partition $\Gamma \circ \bullet$ inside the category（Lemma 4．15）． Due to Proposition 4．23，this is a natural generator indeed，so we add it in the following lemma also for the cases $k=d=0$ and treat the case $\langle\downarrow \bullet\rangle\rangle$ separately．

Proposition 4.25. We have the following natural categories in case $\mathcal{H}$.
(a) The category $\mathcal{H}^{\prime}{ }_{\text {loc }}:=\left\langle\begin{array}{|}\square \\ \bullet \bullet \bullet\end{array}\right\rangle$ consists of all noncrossing partitions such that each block is of even length connecting white and black points in an alternating way, when the partition is rotated such that it has no upper points.
(b) Let $k, d \in \mathbb{N}_{0} \backslash\{1\}$ be such that $d$ is a divisor of $k$, if $k \neq 0$, and denote $b_{0}:=\emptyset$. Denote by $\mathcal{H}_{\mathrm{loc}}(k, d)$ the set of all partitions $p \in N C^{\circ \bullet}$ such that
(i) all blocks have length at least two,
(ii) $c(p) \in k \mathbb{Z}$,
(iii) if $p_{1} \otimes p_{2}$ is any rotated version of $p$ in nest decomposed form, then $c\left(p_{1}\right) \in d \mathbb{Z}$.

We have $\mathcal{H}_{\mathrm{loc}}(k, d) \subseteq\left\langle b_{k}, b_{d} \otimes \tilde{b_{d}}, \quad \sqcap \circ \bullet \bullet, \quad \square \bullet \bullet\right\rangle$
 Iteratively, we may construct all one block partitions of even length connecting white and black points in an alternating way, and using the category operations, we may construct all partitions of the assertion. Conversely, the set of all noncrossing partitions with the above block rule forms a category of partitions containing $\quad \square \bullet \bullet$.
(b) Denote the category $\left\langle b_{k}, b_{d} \otimes \tilde{b_{d}}\right.$, by induction on the number $m$ of blocks of $p$. Since $\mathcal{D}$ is closed under rotation, we may assume that $p$ has no upper points. For $m=1$, note that $\iota_{l} \otimes b_{k}^{\otimes t} \in \mathcal{D}$ by (a), where $\iota_{l} \in P^{0 \bullet}(0,2 l)$ consists of a single block on $2 l$ points with alternating colors, and $t \geq 0$. Using Lemma 4.2 (c), we infer that all one block partitions with $c(p) \in k \mathbb{Z}$ are in $\mathcal{D}$.

Let $m>1$. By rotation and since $p$ is noncrossing, $p$ is of the form $p=p_{1} \otimes p_{2}$ such that $p_{2}$ consists only of one block. Since we are in case $\mathcal{H}, p_{2}$ has length at least two and thus $p$ is in nest decomposed form. Thus $c\left(p_{1}\right) \in d \mathbb{Z}$ and hence also $c\left(p_{2}\right)=c(p)-c\left(p_{1}\right) \in d \mathbb{Z}$. Assume $c\left(p_{1}\right)=d s$ with some $s \geq 0$, by verticolor reflection. Let $p_{1}^{\prime}$ be the partition obtained from $p_{1} \otimes{\tilde{b_{d}}}^{\otimes s}$ by connecting all points of ${\tilde{b_{d}}}^{\otimes s}$ to the last point of $p_{1}$. Then $p_{1}^{\prime}$ is a partition with $m-1$ blocks and $c\left(p_{1}^{\prime}\right)=c\left(p_{1}\right)-d s=0$. Furthermore, any nest decomposed form $q_{1}^{\prime} \otimes q_{2}^{\prime}$ of $p_{1}^{\prime}$ yields a nest decomposed form $q_{1} \otimes q_{2}$ of $p$ such that $c\left(q_{1}^{\prime}\right) \in c\left(q_{1}\right)+d \mathbb{Z} \subseteq d \mathbb{Z}$ because $c\left(p_{2}\right) \in d \mathbb{Z}$ and $c\left({\tilde{b_{d}}}^{\otimes s}\right) \in d \mathbb{Z}$. Hence, $p_{1}^{\prime} \in \mathcal{H}_{\text {loc }}(k, d)$ and by induction hypothesis, $p_{1}^{\prime} \in \mathcal{D}$. Composing it with $r \otimes\left({\tilde{b_{d}}}^{*}\right)^{\otimes s} \otimes\left(\tilde{b_{d}}\right)^{\otimes s}$, where $r$ is a suitable tensor product of the identity partitions, yields $p_{1} \otimes \tilde{b}_{d}^{\otimes s} \in \mathcal{D}$. Similary $b_{d}^{\otimes s} \otimes p_{2} \in \mathcal{D}$, since the partition $p_{2}^{\prime}$ obtained from connecting all blocks of $b_{d}^{\otimes s} \otimes p_{2}$ is a one block partition with $c\left(p_{2}^{\prime}\right)=c(p)$. We conclude that $p_{1} \otimes{\tilde{b_{d}}}^{\otimes s} \otimes b_{d}^{\otimes s} \otimes p_{2} \in \mathcal{D}$ and using the pair partitions, we obtain $p=p_{1} \otimes p_{2} \in \mathcal{D}$.

### 4.4.4 Classification in the case $\mathcal{H}$

Theorem 4.26. Let $\mathcal{C} \subseteq N C^{\circ \bullet}$ be a category of noncrossing partitions in case $\mathcal{H}$. Then $\mathcal{C}$ coincides with one of the following categories.
(i) If $\mathcal{C}$ is globally colorized, then $\mathcal{C}=\mathcal{H}_{\mathrm{glob}}(k):=\left\langle b_{k}\right.$, П๑० , ○○ $\left.\otimes \bullet \bullet\right\rangle$ for $k=k(\mathcal{C}) \in 2 \mathbb{N}_{0}$.
(ii) If $\mathcal{C}$ is locally colorized, and $K^{\mathcal{C}}\left(\begin{array}{ll}\Gamma & 7 \\ 0 & 0\end{array}\right)=\emptyset$, then $\mathcal{C}=\mathcal{H}^{\prime}{ }_{\text {loc }}=\left\langle\begin{array}{ll}\square \bullet \bullet\end{array}\right\rangle$. Otherwise, we have $\mathcal{C}=\left\langle b_{k}, b_{d} \otimes \tilde{b}_{d},\left\lceil\square, ~\lceil\emptyset \bullet\rangle\right.\right.$ for $k=k(\mathcal{C}) \in \mathbb{N}_{0} \backslash\{1,2\}$ and $d=d(\mathcal{C}) \in \mathbb{N}_{0} \backslash\{1,2\}$. We use the notation $b_{0}=\emptyset$.
 $p \in \mathcal{C}$ ．Then，$c(p) \in k \mathbb{Z}$ by Proposition 4．10．Furthermore，if $p_{1} \otimes p_{2}$ is any rotated version of $p$ in nest decomposed form，then $c\left(p_{1}\right) \in 2 \mathbb{Z}$ by Lemma 4．16 and Proposition 4．23．By Proposition
 $\mathcal{C}=\left\langle b_{k}, \Gamma_{\bullet \bullet}, \Gamma_{0} \otimes \complement_{\bullet}\right\rangle$ ．
（ii）Let $\mathcal{C}$ be locally colorized and let $k:=k(\mathcal{C})$ and $d:=d(\mathcal{C})$ ．
Case 1．Let $K^{\mathcal{C}}(\bullet \mathfrak{\jmath})=\emptyset$ ．Since $\mathcal{C}$ is in case $\mathcal{H}$ ，we have $\langle\bullet \bullet\rangle \subseteq \mathcal{C}$ ．Conversely，let $p \in \mathcal{C}$ be a partition without upper points．Assume that there is a block of $p$ which does not connect white and black points in an alternating way．Using rotation，we can bring $p$ in nest decomposed form $p=p_{1} \otimes p_{2}$ such that the first and the last point of $p_{2}$ have the same color．
 white and black points in an alternating way．Furthermore，they are of even length．Otherwise， the first and the last point would have the same color，and again we would find an example of
 hence $\mathcal{C}=\langle\stackrel{\square}{\bullet \bullet 0}\rangle$ ．
 Lemma 4．24．Conversely，let $p \in \mathcal{C}$ ．By Lemma 4．10，we have $c(p) \in k \mathbb{Z}$ ．Furthermore，if $p^{\prime}=$ $p_{1} \otimes p_{2}$ is any rotated version of $p$ in nest decomposed form，we have $c\left(p_{1}\right) \in d \mathbb{Z}$ by Proposition 4．23．We thus have $p \in \mathcal{H}_{\mathrm{loc}}(k, d)$ ，which by Proposition 4.25 yields $p \in\left\langle b_{k}, b_{d} \otimes \tilde{b}_{d}, \quad\lceil\circ \bullet \bullet\right.$ ，厄๒८ $\rangle$ ． This shows $\left\langle b_{k}, b_{d} \otimes \tilde{b}_{d}\right.$ ，ர○•Ө，厄๑๐ $\rangle=\mathcal{C}$ ．

Corollary 4．27．We have $\mathcal{H}_{\mathrm{loc}}(k, d)=\left\langle b_{k}, b_{d} \otimes \tilde{b}_{d}\right.$ ，厄००७，ரぃ○ $\rangle$ in Proposition 4．25．In par－ ticular，all these categories are pairwise different．

Proof．In the above theorem，we showed $\left\langle b_{k}, b_{d} \otimes \tilde{b}_{d}, \quad \circ \bullet \bullet, \quad \bullet \bullet\right\rangle \subseteq \mathcal{C} \subseteq \mathcal{H}_{\text {loc }}(k, d)$ whenever $\mathcal{C}$ is a locally colorized category in case $\mathcal{H}$ with $k=k(\mathcal{C}), d=d(\mathcal{C})$ and $K^{\mathcal{C}}(\bullet . \emptyset) \neq \emptyset$ ．Together with $\mathcal{H}_{\text {loc }}(k, d) \subseteq\left\langle b_{k}, b_{d} \otimes \tilde{b}_{d},\lceil\circ \bullet \bullet, \quad \circ \bullet \bullet\rangle\right.$ of Proposition 4.25 ，we have equality here．Moreover， it can easily be seen that the sets $\mathcal{H}_{\text {loc }}(k, d)$ are distinct．

Remark 4．28．（a）One can show that the categories $\left\langle b_{k}, ~ \sqcap \bullet \bullet, ~ \sqcap \otimes \otimes\right\rangle$ are given by the set of all partitions $p \in N C^{\bullet \bullet}$ such that $c(p) \in k \mathbb{Z}$ and all blocks of $p$ have even length．
（b）The non－colored case $\langle\Pi \square\rangle$ is obtained from $\left\langle b_{k}, ~ \curvearrowleft \bullet \bullet, ~ \sqcap \otimes \bullet \bullet\right\rangle$ for $k=2$ in the sense of Proposition 4．3．

## 4．5 Case $\mathcal{S}$

We now consider the case $\mathcal{S}$ ，i．e．categories $\mathcal{C} \subseteq N C^{\circ \bullet}$ such that and $\uparrow \otimes \uparrow$ are in $\mathcal{C}$ ．

## 4．5．1 Determining the parameters

Proposition 4．29．Let $\mathcal{C} \subseteq N C^{\bullet \bullet}$ be a category of noncrossing partitions in case $\mathcal{S}$ ．
（a）We always have $\hat{\substack{\hat{\jmath}}} \in \mathcal{C}$ ．
（b）If $\mathcal{C}$ is globally colorized，then $d(\mathcal{C})=1$ and $k(\mathcal{C}) \in \mathbb{N}_{0}$ ．Moreover，$\hat{\circ} \hat{\circ} \in \mathcal{C}$ ．
（c）If $\mathcal{C}$ is locally colorized，then $k(\mathcal{C}), d(\mathcal{C}) \in \mathbb{N}_{0} \backslash\{1\}$ and $K^{\mathcal{C}}($ ■ $)=K^{\mathcal{C}}\binom{$ ■ }{ 〇 }$+1=d \mathbb{Z}+1$ for $d=d(\mathcal{C})$ ，i．e．：

Moreover，$\quad\lceil\bullet \notin \mathcal{C}$ ．
Proof．（a）By Lemma 4．2，we may disconnect the white points from $\quad$.
（b）By（a）and using color permutation，we have $\hat{0}$ ．$\in \mathcal{C}$ ，thus $d(\mathcal{C})=1$ ．
（c）If $k(\mathcal{C})=1$ ，then $\hat{\jmath} \in \mathcal{C}$ which allows us to erase arbitrary points of partitions in $\mathcal{C}$ ． Thus $\square \bullet \bullet \in \mathcal{C}$ implies $\square_{0} \in \mathcal{C}$ which is a contradiction to $\square_{0} \otimes \square_{\bullet} \notin \mathcal{C}$ ．If $d(\mathcal{C})=1$ ，then ţ̂．．$\in \mathcal{C}$ by Lemma 4.15 （a）．Using（a）and Lemma 4.2 （e），we infer $\hat{\jmath} \otimes \hat{\jmath} \otimes \Omega \in \mathcal{C}$ which implies $\Gamma_{\circ} \otimes \square \in \mathcal{C}$ by Lemma 4．1，a contradiction．

Now，let $s \in K^{\mathcal{C}}($ 〕 $), s>0$ ．We find a partition $p_{1} \otimes p_{2} \in \mathcal{C}$ in nest decomposed form such that the first and the last point of $p_{2}$ are black，and $c\left(p_{1}\right)=s$ ．We may assume that $p_{1}=\uparrow^{\otimes s}$ using Lemma 4.2 （b）and the pair partitions．Swaping one of the white singletons of $p_{1}$ with the first point of $p_{2}$（by（a）and Lemma 4．2（f））yields a partition $\uparrow^{\otimes s-1} \otimes p_{2}^{\prime} \in \mathcal{C}$ in nest decomposed form such that the first and the last point of $p_{2}^{\prime}$ have inverse colors．This shows $s-1 \in K^{\mathcal{C}}\binom{\complement}{\complement}$ ． We proceed similarly in the case $s<0$ and we deduce $K^{\mathcal{C}}\left(\begin{array}{l}\text { ■ }\end{array}\right) \subseteq K^{\mathcal{C}}\binom{$（ }{ ■ }$+1$ ．Note that $0 \notin K^{\mathcal{C}}\left(\right.$ 厄 〕）by Lemma 4.15 （c）．Conversely，if $s+1 \in K^{\mathcal{C}}\binom{$（ }{ し }$+1, s \geq 0$ ，we find a partition $\hat{\mathrm{O}}^{\otimes s} \otimes p_{2} \in \mathcal{C}$ in nest decomposed form such that the first point of $p_{2}$ is white and the last point



Finally，if $\Gamma_{\circ \bullet \bullet} \in \mathcal{C}$ ，then also $\Gamma_{\circ} \otimes \hat{\downarrow} \otimes \hat{\bullet} \in \mathcal{C}$ ，which implies $\Gamma_{\circ} \otimes \curvearrowleft_{\bullet} \in \mathcal{C}$ by Lemma 4.2 （b）．

## 4．5．2 Finding partitions realizing the parameters

Lemma 4．30．Let $\mathcal{C} \subseteq N C^{\bullet \bullet}$ be a category in case $\mathcal{S}$ ．
（a）If $k=k(\mathcal{C}) \neq 0$ ，then $\uparrow_{0}^{\otimes k} \in \mathcal{C}$ ．
（b）If $d=d(\mathcal{C}) \neq 0$ ，then ${ }^{p o d} \circ \int_{0}^{8 d} \cdot \in \mathcal{C}$ ．
Proof．（a）We find a partition $p \in \mathcal{C}$ such that $c(p)=k$ ．Using the pair partition we erase all black points and using $\uparrow \otimes \uparrow$ we know $\uparrow^{\otimes k} \in \mathcal{C}$ by Lemma 4.2 ．
（b）This follows from Lemma 4．15（a）．

## 4．5．3 Description of natural categories

Proposition 4．31．We have the following natural categories in case $\mathcal{S}$ ．Let $k, d \in \mathbb{N}_{0}$ such that $d$ is a divisor of $k$ ，if $k \neq 0$ ．Denote by $\mathcal{S}_{\text {loc }}(k, d)$ the set of all partitions $p \in N C^{\bullet \bullet}$ such that
（i）$c(p) \in k \mathbb{Z}$ ，
(ii) if $p_{1} \otimes p_{2}$ is any rotated version of $p$ in nest decomposed form such that the first and the last point of $p_{2}$
$\ldots$ have inverse colors, then $c\left(p_{1}\right) \in d \mathbb{Z}$,
$\ldots$ both are black, then $c\left(p_{1}\right) \in d \mathbb{Z}+1$,
$\ldots$ both are white, then $-c\left(p_{1}\right) \in d \mathbb{Z}+1$.

 induction on the number $m(p)$ of those blocks of $p$ which have length greater or equal two. By rotation, we may always assume that $p$ has no upper points.

Case 1. Let $m(p)=0$, i.e. $p$ consists only of singletons. Since $c(p) \in k \mathbb{Z}$, we have up to permutation of the colors (see Lemma 4.2(b)) $p=\hat{\jmath}^{\otimes k t} \otimes(\hat{\jmath} \otimes \uparrow)^{\otimes w}$ for some number $w$. Hence $p \in \mathcal{D}$.

Case 2. Let $m(p)=1$. Using rotation, $p$ is of the following form:

$$
p=a^{\varepsilon_{1}} X_{1} a^{\varepsilon_{2}} X_{2} \ldots a^{\varepsilon_{l}} X_{l}
$$

Here, the points $a^{\varepsilon_{i}}$ form a block of length $l \geq 2$, and the $X_{i}$ are some tensor products of the singletons $\hat{\delta}$ and $\hat{〔}$. If now all points $a^{\varepsilon_{i}}$ had alternating colors, we could first argue that the partition $a^{\varepsilon_{1}} \ldots a^{\varepsilon_{l}}$ is in $\langle\emptyset \bullet \bullet\rangle \subseteq \mathcal{D}$ and then insert the tensor products $X_{i}$ of singletons
 the case. We therefore construct a partition $p^{\prime}$ involving some "correction points". It will be of the form:

$$
p^{\prime}=A_{1}^{\prime} X_{1}^{\prime} A_{2}^{\prime} X_{2}^{\prime} \ldots A_{l}^{\prime} X_{l}^{\prime}
$$

The construction of $p^{\prime}$ is as follows. If $a^{\varepsilon_{i}}$ and $a^{\varepsilon_{i+1}}$ have different colors, then up to permutation of the colors, $X_{i}$ is of the form $X_{i}=\hat{\varsigma}^{\otimes d t_{i}} \otimes(\hat{\jmath} \otimes \hat{\bullet})^{\otimes w_{i}}$ for some $w_{i} \in \mathbb{N}_{0}, t_{i} \in \mathbb{Z}$, by condition (ii) of $\mathcal{S}_{\text {loc }}(k, d)$. We put $A_{i+1}^{\prime}:=a^{\varepsilon_{i+1}}$ and $X_{i}^{\prime}:=X_{i}$. On the other hand, if $a^{\varepsilon_{i}}$ and $a^{\varepsilon_{i+1}}$ both are black, then $X_{i}=\uparrow_{0}^{\otimes d t_{i}+1} \otimes(\hat{\jmath} \otimes \uparrow)^{\otimes w_{i}}$ up to permutation, and we put $A_{i+1}^{\prime}:=a^{-\varepsilon_{i+1}} a^{\varepsilon_{i+1}}$, $X_{i}^{\prime}:=X_{i} \otimes \uparrow, w_{i}^{\prime}:=w_{i}+1$. If $a^{\varepsilon_{i}}$ and $a^{\varepsilon_{i+1}}$ both are white, then $X_{i}^{\prime}:=X_{i} \otimes \uparrow$ instead. Finally, we put $A_{1}^{\prime}:=a^{\varepsilon_{1}}$ and $X_{l}^{\prime}:=X_{l}$ if $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{l}}$ have inverse colors and $A_{1}^{\prime}:=a^{-\varepsilon_{l}} a^{\varepsilon_{1}}$, $X_{l}^{\prime}:=X_{l} \otimes \widehat{\text { @ }} X_{l}^{\prime}:=X_{l} \otimes \uparrow$ otherwise.

Now, the partition $q_{1}:=A_{1}^{\prime} A_{2}^{\prime} \ldots A_{l}^{\prime}$ consists only of one block of even length with alternating colors, by construction. By Lemma 4.25 , it is contained in $\langle\hookleftarrow \bullet \bullet\rangle \subseteq \mathcal{D}$.

Let $q_{2}$ be the partition obtained from $q_{1}$ by inserting subpartitions $(\hat{0} \otimes \hat{\bullet}) \otimes w_{i}^{\prime}$ between $A_{i}^{\prime}$ and $A_{i+1}^{\prime}$, and $(\hat{0} \otimes \hat{\bullet}) \otimes w_{l}^{\prime}$ after $A_{l}^{\prime}$. Since $q_{2}$ can be obtained from $q_{1}$ using the category operations, we have $q_{2} \in \mathcal{D}$. Moreover, $c\left(p^{\prime}\right)=c(p)$ by construction and $c(p)=\sum_{i} d t_{i} \in k \mathbb{Z}$.
 the partitions $\uparrow^{\otimes d t_{i}}$ (or $\uparrow^{\otimes d t_{i}}$ resp.) at the right positions, and we infer $p^{\prime} \in \mathcal{D}$. Using the pair
partitions, we finally erase all extra points in $p^{\prime}$ together with the "correction singletons" and we deduce $p \in \mathcal{D}$.

Case 3. Let $m(p)>1$. By rotation, $p$ can be brought in nest decomposed form $p=p_{1} \otimes p_{2}$ such that $m\left(p_{2}\right)=1$. Such a decomposition exists since $p$ is noncrossing.

Case $3 a$. If the first and the last point of $p_{2}$ have inverse colors, we have $c\left(p_{1}\right) \in d \mathbb{Z}$ by condition (ii). We may assume $c\left(p_{1}\right) \geq 0$, i.e. $c\left(p_{1}\right)=d s$ for some $s \in \mathbb{N}_{0}$. Then, the partition $p_{1}^{\prime}:=p_{1} \otimes \uparrow^{\otimes d s}$ satisfies $c\left(p_{1}^{\prime}\right)=0$ and $m\left(p_{1}^{\prime}\right)=m\left(p_{1}\right)=m(p)-1$. As conditions (ii) and (iii) are fulfilled for $p_{1}^{\prime}$, we infer $p_{1}^{\prime} \in \mathcal{D}$ by induction hypothesis. By Case 2 , we also have $p_{2}^{\prime}:=\uparrow^{\otimes d s} \otimes p_{2} \in \mathcal{D}$, since $c\left(p_{2}^{\prime}\right)=c\left(p_{2}\right)+c\left(p_{1}\right)=c(p) \in k \mathbb{Z}$. Thus, we obtain $p_{1}^{\prime} \otimes p_{2}^{\prime} \in \mathcal{D}$ and hence $p=p_{1} \otimes p_{2} \in \mathcal{D}$ using the pair partitions.

Case 3b. On the other hand, if both the first and the last point of $p_{2}$ are black, we consider $p_{1}^{\prime}:=p_{1} \otimes \uparrow$. Furthermore, we consider the partition $p_{2}^{\prime}$ obtained from $p_{2}$ by changing the color of its first point from black to white and we insert a black singleton to the right of this first point. Then, $p_{1}^{\prime} \otimes p_{2}^{\prime}$ is a partition in nest decomposed form fulfilling the conditions (i), (ii) and (iii). By Case 3a, we infer $p_{1}^{\prime} \otimes p_{2}^{\prime} \in \mathcal{D}$. Since $\hat{\delta} \leqslant \bullet \in \mathcal{D}$, we may use Lemma 4.2 to infer that $p_{1} \otimes \hat{\bullet} \otimes \hat{\jmath} \otimes p_{2}$ is in $\mathcal{D}$, which yields $p \in \mathcal{D}$. We proceed in the same way, if the first and the last point of $p_{2}$ are white.

### 4.5.4 Classification in the case $\mathcal{S}$

Theorem 4.32. Let $\mathcal{C} \subseteq N C^{\circ}$ be a category of noncrossing partitions in case $\mathcal{S}$. Then $\mathcal{C}$ coincides with one of the following categories.
 $k(\mathcal{C}) \in \mathbb{N}_{0}$.
 and $d=d(\mathcal{C}) \in \mathbb{N}_{0} \backslash\{1\}$.

Proof. (i) By Lemma 4.30, we have $\uparrow^{\otimes k} \in \mathcal{C}$ for $k=k(\mathcal{C})$. Thus $\left\langle\uparrow^{\otimes k}, \Pi_{\bullet \circ}, \uparrow \otimes \uparrow\right.$, $\circ \otimes \otimes$
$\left.\square_{\bullet}\right\rangle \subseteq \mathcal{C}$. Conversely, let $p \in \mathcal{C}$. Then $c(p) \in k \mathbb{Z}$ by Proposition 4.10. Furthermore, $d=d(\mathcal{C})=1$


 for $k=k(\mathcal{C})$ and $d=d(\mathcal{C})$. Conversely, let $p \in \mathcal{C}$. Then $c(p) \in k \mathbb{Z}$ by Proposition 4.10. Let $p_{1} \otimes p_{2}$ be a rotated version of $p$ in nest decomposed form. If the first and the last point of $p_{2}$ have inverse colors, then $c\left(p_{1}\right) \in d \mathbb{Z}$ by Proposition 4.13. If the first and the last point of $p_{2}$ both are black, we have to prove $s-1 \in d \mathbb{Z}$ for $s:=c\left(p_{1}\right)$.

Assume that $s>0$. Using the pair partitions and Lemma 4.2, we may assume that $p_{1}$ is of the form $p_{1}=\uparrow^{\otimes s}$, hence $\uparrow_{0}^{\otimes s} \otimes p_{2} \in \mathcal{C}$. We have $s \neq 0$, since otherwise $p_{2} \in \mathcal{C}$ and rotation would yield a partition such that two neighbouring points have the same color and belong to the same block. By Lemma 4.15 (c) and Proposition 4.29 this would lead to a contradiction.

We thus have $s \geq 1$. Since $\uparrow \overparen{\imath} \bullet \in \mathcal{C}$ by Proposition 4.29, we may shift one of the white singletons to the right hand side of the first point of $p_{2}$, which inverts the colors of these two
points. We infer that the partition $\widehat{0}^{\otimes s-1} \otimes p_{2}^{\prime}$ is in $\mathcal{C}$, where $p_{2}^{\prime}$ is in nest decomposed form such that the first and the last point have inverse colors. By Proposition 4.13, we thus have $s-1 \in d \mathbb{Z}$.

As for $s<0$, we have $\uparrow^{\otimes-s} \otimes p_{2} \in \mathcal{C}$. By composition, we infer that also $\uparrow^{\otimes-s} \otimes \uparrow \otimes \uparrow \otimes p_{2} \in$ $\mathcal{C}$. Again, shifting the white singleton to the right hand side of the first point of $p_{2}$ yields $\uparrow^{\otimes-s+1} \otimes p_{2}^{\prime} \in \mathcal{C}$ where the first and the last point of $p_{2}^{\prime}$ have inverse colors belonging to the same block. Thus, $-s+1 \in d \mathbb{Z}$ and hence $s-1 \in d \mathbb{Z}$.

A similar proof shows that $s+1 \in d \mathbb{Z}$ if the first and the last point of $p_{2}$ are white. We


 particular, all these categories are pairwise different.
Remark 4.34. The non-colored case $\langle\uparrow \otimes \uparrow, \Pi \Pi\rangle$ is obtained from $\left\langle\uparrow_{0}^{\otimes k}, ~ 厄 \bullet \bullet, ~ \uparrow \otimes \uparrow, ~ \sqcap \otimes \otimes\right.$
-.) for $k=2$, whereas $\langle\uparrow, \Pi \Pi\rangle$ is the case $k=1$ (see Proposition 4.3).

### 4.6 Case $\mathcal{B}$

Finally, we turn to the case $\mathcal{B}$, i.e. to categories $\mathcal{C} \subseteq N C^{\bullet \bullet}$ such that $\bullet_{\bullet \bullet} \notin \mathcal{C}$ and $\uparrow \otimes \uparrow \in \mathcal{C}$, i.e. all blocks of partitions $p \in \mathcal{C}$ have length at most two (Lemma4.4). Like in the non-colored case, this is the most complicated situation, as we can already see when investigating which parameters can occur.

### 4.6.1 Determining the parameters

Proposition 4.35. Let $\mathcal{C} \subseteq N C^{\circ}$ be a category of noncrossing partitions in case $\mathcal{B}$.
(a) If $\mathcal{C}$ is globally colorized, then the cases $d(\mathcal{C})=1$ and $d(\mathcal{C})=2$ can occur.
(b) If $\mathcal{C}$ is locally colorized, then
(i) either $K^{\mathcal{C}}($ ■ $)=\emptyset$ and $k(\mathcal{C}), d(\mathcal{C}) \in \mathbb{N}_{0}$,
(ii) or $K^{\mathcal{C}}\left(\complement_{\bullet}\right)=d \mathbb{Z}+(r+1)$ for $r:=r(\mathcal{C}):=\min \left\{s \geq 1 \mid s \in K^{\mathcal{C}}(\right.$ ■ 〕) $\left.)\right\}-1$ and $k(\mathcal{C}) \in \mathbb{N}_{0} \backslash\{1\}, d=d(\mathcal{C}) \in \mathbb{N}_{0} \backslash\{1\}$. Furthermore, $r=0$ or $r=\frac{d}{2}$; and $r \neq 1$. Thus:

Proof. (a) This follows directly from Lemma 4.16 .
(b) Let $K^{\mathcal{C}}\left(\complement_{\bullet}\right) \neq \emptyset$. It is clear that $k(\mathcal{C}) \neq 1$, since $\uparrow \in \mathcal{C}$ (see Lemma 4.36) would



$$
Q_{0}^{Q_{s}} \cdot 0^{Q_{s-2}} \cdot \in \mathcal{C}
$$

Rotation yields $\left.-(s-2) \in K^{\mathcal{C}}(\complement\urcorner\right)$. We observe that $\left.0 \notin K^{\mathcal{C}}(\bullet\urcorner\right)$ and $\left.2 \notin K^{\mathcal{C}}(\bullet\urcorner\right)$, since otherwise $\wp \circ \otimes \square \in \mathcal{C}$, in contradiction to $\mathcal{C}$ being locally colorized. Thus $r \neq 1$. Moreover, we deduce that there is always a number $\left.s \in K^{\mathcal{C}}(\bullet\urcorner\right)$ such that $s \geq 1$, and hence the number $r$ is well-defined.

Like above, we have the following two partitions in $\mathcal{C}$, the latter one being obtained from the first one by verticolor reflection and rotation:


Forming the tensor product of these two partitions and composing it with a suitable tensor product of $\bullet \bigcirc$ and the identity partitions, we infer:


Thus, $2 r \in K^{\mathcal{C}}\left(\begin{array}{ll}\Gamma & \mathbf{~}\end{array}\right)=d \mathbb{Z}$. Let $r \neq 0$. Then $d \neq 0$. We now prove $2 r=d$. Put $r_{0}:=r+1$. Assume $2 r=d s$ for some $s \geq 2$. Then $2 d \leq d s=2 r$, hence $d<r_{0}$. Thus, $r^{\prime}:=r_{0}-d$ is a number $0<r^{\prime}<r_{0}$. Using the partition $0^{\otimes d} \sqrt{\otimes d} \cdot \in \mathcal{C}$ (which is in $\mathcal{C}$ by Lemma 4.15), we can shift exactly like in the proof of Lemma 4.2 (e) $d$ of the $r+1$ white singletons of $1^{\otimes r+1} \bullet 0^{\otimes r-1}$ from the outside of the pair to the inside. This yields a partition showing that $r^{\prime} \in K^{\mathcal{C}}\left(\complement_{\bullet} \downarrow_{\bullet}\right)$ in contradiction to the minimality of $r_{0}$. We conclude $2 r=d$ if $r \neq 0$.

It remains to show that $K^{\mathcal{C}}\binom{\complement}{\bullet}=d \mathbb{Z}+r_{0}$. Let $t \in \mathbb{Z}$. Since $d t \in d \mathbb{Z}=K^{\mathcal{C}}\left(\begin{array}{ll}\Gamma & \rceil \\ \circ & \bullet\end{array}\right)$ and $\uparrow^{\otimes r-1} \otimes \uparrow^{\otimes r-1} \in \mathcal{C}$, we have (by Lemma 4.15):

$$
p:=\emptyset_{0}^{\otimes d t} \bigcirc 0^{0^{\otimes r-1}} 0^{\otimes r-1} \emptyset^{\otimes d t} \cdot \in \mathcal{C}
$$

Furthermore, the following partition is in $\mathcal{C}$ since it is a rotated version of $0_{0}^{\otimes r+1} \cdot 0_{\bullet}^{\otimes r-1}$ :

$$
q:=0^{\otimes r+1} \oplus^{\circ} \stackrel{q}{\otimes r-1}
$$

Now, composing $p$ with $q$ we may shift $r-1$ white singletons from the inside of the pair of $p$ to the outside, by which their number increases to $r+1$ white singletons. Furthermore, the color of the first point of the pair changes from white to black. We conclude that $d t+(r+1)$ is in $K^{\mathcal{C}}\binom{\bullet}{\bullet}$, which proves $d \mathbb{Z}+r_{0} \subseteq K^{\mathcal{C}}\left(\begin{array}{l}\text { ■ })\end{array}\right.$.

Conversely, let $r_{0}>1$. Thus $r>0$ and hence $d=2 r>0$. Let $s \in K^{\mathcal{C}}\left(\begin{array}{l}\text { ■ }\end{array}\right)$ and assume
 shift clusters of $d$ singletons of the same color to arbitrary positions. We may thus assume $r_{0}<s<r_{0}+d$ which implies $0<s-r_{0}<d$. Like above, we have the following two partitions in $\mathcal{C}$ using verticolor reflection:


Composing the tensor product of them with $\bullet \cup$ yields $s-r_{0} \in K^{\mathcal{C}}\left(\begin{array}{ll}\lceil & \rceil\end{array}\right)=d \mathbb{Z}$ in contradication to $0<s-r_{0}<d$.

In the case $r_{0}=1$, we have $\uparrow \downarrow \uparrow \bullet \in \mathcal{C}$. If now $s \in K^{\mathcal{C}}\left(\begin{array}{ll}〔 & \imath\end{array}\right)$, then $s-1 \in K^{\mathcal{C}}\left(\begin{array}{ll}\Gamma & \rceil \\ \bullet & \bullet\end{array}\right)=d \mathbb{Z}$ again by applying $\downarrow$ ¢ $\uparrow$ to $1^{\otimes s} \bullet 1^{8 s-2}$.

Finally, if $d=1$, then $\uparrow \hat{\circ} \bullet \bullet \in \mathcal{C}$ and $r=0$ implies $\uparrow \widehat{\jmath} \bullet \in \mathcal{C}$. Hence $\square_{0} \otimes \bullet \in \mathcal{C}$, a contradiction.

### 4.6.2 Finding partitions realizing the parameters

Lemma 4.36. Let $\mathcal{C} \subseteq N C^{\circ}$ be a category in case $\mathcal{B}$.
(a) If $k=k(\mathcal{C}) \neq 0$, then $\uparrow_{\circ}^{\otimes k} \in \mathcal{C}$.
(b) If $d=d(\mathcal{C}) \neq 0$, then $\bigcirc \circ \bigcirc \bullet \bullet \in \mathcal{C}$.
(c) If $\left.K^{\mathcal{C}}(\bullet\urcorner\right) \neq \emptyset$ and $r(\mathcal{C})=0$, then $\uparrow \bullet \uparrow \bullet \in \mathcal{C}$; if $r=r(\mathcal{C}) \neq 0$, then $\stackrel{0}{0}_{\otimes r+1}^{\bullet^{\otimes r-1}} \bullet \in \mathcal{C}$.

Proof. (a) Using $\uparrow \otimes \uparrow \in \mathcal{C}$, we may disconnect any points from their blocks. Thus, we may assume that a partition $p \in \mathcal{C}$ with $c(p)=k$ is of the form $p=\uparrow^{\otimes k}$.
(b)\&(c) This is Lemma 4.15(a).

### 4.6.3 Description of natural categories

Proposition 4.37. We have the following natural categories in case $\mathcal{B}$.
(a) The category $\langle\uparrow \otimes \uparrow\rangle$ consists of all noncrossing partitions $p \in N C^{\circ \bullet}$ such that when $p$ is rotated to a partition having no upper points
(i) all blocks have size one or two,
(ii) the blocks of size two connect a black point and a white point,
(iii) the number of black singletons and the number of white singletons between two legs of every pair coincide, and on the global level, too.
(b) Let $k, d \in \mathbb{N}_{0}$ be such that $d$ is a divisor of $k$, if $k \neq 0$. Let $r \in\left\{0, \frac{d}{2}\right\} \backslash\{1\}$. Denote by $\mathcal{B}^{\prime}{ }_{\text {loc }}(k, d, r)$ the set of all noncrossing partitions $p \in N C^{\circ \bullet}$ such that
(i) all blocks have size one or two,
(ii) $c(p) \in k \mathbb{Z}$,
(iii) if $p_{1} \otimes p_{2}$ is any rotated version of $p$ in nest decomposed form such that the first and the last point of $p_{2}$
... have inverse colors, then $c\left(p_{1}\right) \in d \mathbb{Z}$,
$\ldots$ both are black, then $c\left(p_{1}\right) \in d \mathbb{Z}+r+1$,
$\ldots$ both are white, then $-c\left(p_{1}\right) \in d \mathbb{Z}+r+1$.

(c) Denote by $\mathcal{B}_{\text {loc }}(k, d)$ the set defined as $\mathcal{B}^{\prime}{ }_{\text {loc }}(k, d, r)$, but with the additional condition that all blocks of $p$ of size two are of the form $\ulcorner\bullet$ or $\bullet$ when being rotated to one line. We


Proof. (a) Denote the set of all partitions $p \in N C^{\circ \bullet}$ with (i), (ii) and (iii) by $\mathcal{E}$. It is easy to see that $\mathcal{E}$ is a category of partitions containing $\uparrow \otimes \uparrow$. So, we only need to prove $p \in\langle\uparrow \otimes \uparrow\rangle$ for all $p \in \mathcal{E}$. We do so by induction on the number $m$ of blocks of size two of $p$. If $m=0$, then $p$ consists of $l$ white singletons and $l$ black singletons, for some $l \in \mathbb{N}$. Hence, it is of the form $(\hat{\imath} \otimes \uparrow)^{\otimes l} \in\langle\hat{\varrho} \otimes \uparrow\rangle$ up to permutation of colors (see also Lemma 4.2). If $m=1$, the partition $p$ is of the form $p=X a Y a^{-1}$ up to rotation, where $X$ and $Y$ are tensor products of singletons and $a$ and $a^{-1}$ form a pair block $\square$ or $\quad$. . By assumption, the number of white singletons in $Y$ and the number of black singletons coincide, hence $Y \in\langle\uparrow \otimes \uparrow\rangle$ by case $m=0$. Likewise $X \in\langle\uparrow \otimes \uparrow\rangle$, by the assumption on the global color distribution of the singletons. We infer $p \in\langle\hat{0} \otimes \hat{\jmath}\rangle$.

If $m>1$, we can write $p=p_{1} \otimes p_{2}$ in nest decomposed form up to rotation, where $p_{2}$ consists of one pair block and some singletons. Then $p_{2} \in\langle\hat{0} \otimes \hat{\bullet}\rangle$ by case $m=1$ and $p_{1} \in\langle\hat{0} \otimes \hat{\jmath}\rangle$ by the induction hypothesis. Thus $p=p_{1} \otimes p_{2} \in\langle\hat{0} \otimes \hat{\bullet}\rangle$.
 We prove $p \in \mathcal{D}$ by induction on the number $m$ of blocks of $p$ of size two.

Case 1. Let $m=0$. Up to rotation and permutation of the colors, $p$ is of the form $p=$


Case 2. Let $m=1$. Up to rotation, $p$ is of the form $p=p_{1} \otimes a^{\varepsilon_{1}} p_{2}^{0} a^{\varepsilon_{2}}$ where $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ form a pair block, and $p_{1}$ and $p_{2}^{0}$ consist only of singletons respectively.

Case 2a. If the pair on $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ is of the form $\left\ulcorner\right.$. or $\bullet$, then $c\left(p_{1}\right) \in d \mathbb{Z}$. Consider $p_{1}^{\prime}:=p_{1} \otimes \widehat{\jmath}^{-c\left(p_{1}\right)}$. Then $p_{1}^{\prime} \in \mathcal{D}$ by Case 1 since $c\left(p_{1}^{\prime}\right)=0$. Furthermore, $p_{2}^{0^{\prime}}:=\widehat{\jmath}^{\otimes c\left(p_{1}\right)} \otimes p_{2}^{0}$ is in $\mathcal{D}$, again by Case 1 because $c\left(p_{2}^{0^{\prime}}\right)=c\left(p_{1}\right)+c\left(p_{2}^{0}\right)=c(p) \in k \mathbb{Z}$. Therefore, the partition $p_{1} \otimes \uparrow_{\circ}^{-c\left(p_{1}\right)} \otimes a^{\varepsilon_{1}} \uparrow^{\otimes c\left(p_{1}\right)} p_{2}^{0} a^{\varepsilon_{2}}$ is in $\mathcal{D}$. Since $c\left(p_{1}\right) \in d \mathbb{Z}$, we use the partition $\circ_{\circ}^{\otimes \alpha d} \circ \bullet \cdot \in \mathcal{D}$ to shift $c\left(p_{1}\right)$ singletons from inside the pair to the outside. Thus, $p_{1} \otimes \uparrow^{-c\left(p_{1}\right)} \otimes \uparrow^{\otimes c\left(p_{1}\right)} \otimes a^{\varepsilon_{1}} p_{2}^{0} a^{\varepsilon_{2}} \in$ $\mathcal{D}$ from which we infer $p \in \mathcal{D}$ using the pair partitions.

Case 2b. If the pair on $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ is of the form . . , then $c\left(p_{1}\right) \in d \mathbb{Z}+(r+1)$. Assume $c\left(p_{1}\right)=d s+r+1$ for some $s \geq 0$. Let $p_{1}^{\prime}$ be the partition obtained from $p_{1}$ by removing $c\left(p_{1}\right)$ white singletons. Then $p_{1}^{\prime} \in \mathcal{D}$ by Case 1 since $c\left(p_{1}^{\prime}\right)=0$. Furthermore, let $p_{2}^{\prime}$ be obtained from $p_{2}^{0} \otimes \uparrow^{\otimes d s}$ by adding $r-1$ white singletons. Then $p_{2}^{\prime} \in \mathcal{D}$ since $c\left(p_{2}^{\prime}\right)=c\left(p_{2}^{0}\right)+d s+(r-1)=$ $c(p) \in k \mathbb{Z}$. Finally, consider the partition $p_{1}^{\prime} \otimes \stackrel{o}{0}_{\otimes r+1}^{\bullet^{\otimes r-1}}$ • composed with $p_{2}^{\prime}$ in such a way that
 we can shift $d s$ white singletons from inside the pair to the outside. Up to permutation of colors of the singletons and using the pair partitions, this yields $p$ which is hence in $\mathcal{D}$. We proceed in a similar way for $s<0$ and likewise in the case that $a^{\varepsilon_{1}}$ and $a^{\varepsilon_{2}}$ form a pair $\Gamma_{0}$.

Case 3. Let $m>1$. Up to rotation, $p$ is in nest decomposed form $p=p_{1} \otimes p_{2}$ such that $p_{2}$
contains only one block of size two. Then, $p_{1}^{\prime}:=p_{1} \otimes \uparrow^{\otimes c\left(p_{2}\right)}$ is in $\mathcal{D}$ by induction hypothesis, since $c\left(p_{1}^{\prime}\right)=c(p)$. Likewise $p_{2}^{\prime}:=\uparrow_{0}^{\otimes-c\left(p_{2}\right)} \otimes p_{2}$ is in $\mathcal{D}$ by Case 2. Hence $p_{1} \otimes \uparrow^{\otimes c\left(p_{2}\right)} \otimes$ $\uparrow^{\otimes-c\left(p_{2}\right)} \otimes p_{2} \in \mathcal{D}$ from which we deduce $p \in \mathcal{D}$.

 form $\bigcirc \bullet$ or.$\bigcirc$.

### 4.6.4 Classification in the case $\mathcal{B}$

Theorem 4.38. Let $\mathcal{C} \subseteq N C^{\bullet \bullet}$ be a category of noncrossing partitions in case $\mathcal{B}$. Then $\mathcal{C}$ coincides with one of the following categories.
(a) If $\mathcal{C}$ is globally colorized and

$$
\begin{aligned}
& \ldots \text { if } d(\mathcal{C})=2 \text {, then } \mathcal{C}=\mathcal{B}_{\text {glob }}(k):=\left\langle\uparrow_{0}^{\otimes k}, \uparrow \otimes \uparrow,\lceil\otimes \oslash\rangle \text { for } k=k(\mathcal{C}) \in 2 \mathbb{N}_{0},\right.
\end{aligned}
$$

(b) If $\mathcal{C}$ is locally colorized and

 $\mathbb{N}_{0} \backslash\{1\}, d=d(\mathcal{C}) \in \mathbb{N}_{0} \backslash\{1\}$ and $r=r(\mathcal{C})=\frac{d}{2} \neq 1$ or $r(\mathcal{C})=0$.
Proof. (a) If $k(\mathcal{C}) \neq 0$, then $\uparrow^{\otimes k} \in \mathcal{C}$ by Lemma 4.36. Hence $\left\langle\uparrow_{0}^{\otimes k}, \uparrow \otimes \uparrow, \circ \otimes \otimes \bullet\right\rangle \subseteq \mathcal{C}$.
Case 1. Let $d(\mathcal{C})=2$. Let $p \in \mathcal{C}$ be a partition without upper points such that $c(p) \geq 0$. Then $c(p)=k s$ for some $s \in \mathbb{N}_{0}$ by Proposition 4.10. The number of points between two legs of a pair of $p$ is even, because $c\left(p_{1}\right) \in 2 \mathbb{Z}$ for all $p_{1} \otimes p_{2} \in N D F^{\mathcal{C}}\binom{\circ}{\circ}$ by Proposition 4.13. Consider $p^{\prime}:=p \otimes \widehat{\bullet}^{\otimes k s}$. Let $p^{\prime \prime}$ be the partition obtained from $p^{\prime}$ by replacing the colors of the points by the alternating color pattern white-black-white-black-etc. Then, all pair blocks are of the form $\square \bullet$ or $\square$, because there is an even number of points between two legs of a pair. Thus, $p^{\prime \prime} \in\langle\hat{0} \otimes \uparrow\rangle \subseteq\left\langle\hat{o}^{\otimes k}, \uparrow \otimes \uparrow, \square_{0} \otimes \complement\right\rangle$ by Proposition 4.37. Using permutation of colors, we infer $p^{\prime} \in \mathcal{C}$ since $c\left(p^{\prime}\right)=c\left(p^{\prime \prime}\right)=0$. This implies $p^{\prime} \otimes \uparrow_{0}^{\otimes k s} \in \mathcal{C}$ from which we deduce $p \in \mathcal{C}$ using the pair partitions.
 $\uparrow, ~ \sqcap \otimes \oslash \bullet) \subseteq \mathcal{C}$. Conversely, let $p \in \mathcal{C}$. Then $c(p) \in k \mathbb{Z}$ by Proposition 4.10 and thus
 ㄱ. $\otimes$.. since we may shift the singletons in $\hat{\jmath}$. arbitrarily, and hence it coincides with

 by Lemma 4.36 .
 Then $c(p) \in k \mathbb{Z}$ by Proposition 4.10 and $c\left(p_{1}\right) \in d \mathbb{Z}$ for all $p_{1} \otimes p_{2}$ in nest decomposed form by Proposition 4.13. Furthermore, all blocks of size two are of the form $\square \bullet$ or $\square \circ$ when being rotated to one line, since $K^{\mathcal{C}}\binom{\bullet}{\bullet}=K^{\mathcal{C}}\left(\begin{array}{ll}\ulcorner & \rceil\end{array}\right)=\emptyset$. Thus $p \in \mathcal{B}_{\mathrm{loc}}(k, d)$ which implies

 Proposition 4.35. Again, we use Proposition 4.37 to finish the proof.

 ent.
 Proposition 4.35.
(b) The non-colored case $\langle\uparrow \otimes \uparrow\rangle$ is obtained from $\left\langle\uparrow_{\circ}^{\otimes k}, \uparrow \otimes \uparrow\right.$, ■० $\left.\otimes \bullet\right\rangle$ for $k=2$, whereas
 $\uparrow$, ロ๐ $\otimes \curvearrowleft\rangle$ for the case $k=2$ (see Proposition 4.3).

### 4.7 Main result: Summary of the noncrossing case

We finally classified all categories $\mathcal{C} \subseteq N C^{\circ \bullet}$ of noncrossing (two-colored) partitions. This constitutes the main result of our chapter. Here is an overview on the results split into the globally colorized case and the locally colorized case. For the convenience of the reader we recall that the definition of a category of partitions may be found in Section 1.1.2, the cases $\mathcal{O}, \mathcal{H}, \mathcal{S}$ and $\mathcal{B}$ are defined in Definition 4.5, globally and locally colorization is given in Definition 4.6, the partition $b_{k}$ is defined in Definition 4.14 whereas the operation $p \mapsto \tilde{p}$ is the map giving the same partition with inversion of colors, as defined in Section 1.1.2, and the classification theorems are Theorems 4.21, 4.26, 4.32, and 4.38.

Theorem 4.41. Let $\mathcal{C} \subseteq N C^{\circ \bullet}$ be a globally colorized category of noncrossing partitions. Then it coincides with one of the following categories.

Case $\mathcal{O}: \mathcal{O}_{\text {glob }}(k)=\left\langle\square^{\otimes \frac{k}{2}}\right.$, П० $\otimes \bullet \bullet$ for $k \in 2 \mathbb{N}_{0}$
Case $\mathcal{H}: \mathcal{H}_{\text {glob }}(k)=\left\langle b_{k}, \quad \sqcap \bullet \bullet, ~ \sqcap \circ \otimes \curvearrowleft \bullet\right\rangle$ for $k \in 2 \mathbb{N}_{0}$



Theorem 4.42. Let $\mathcal{C} \subseteq N C^{\circ}$ be a locally colorized category of noncrossing partitions. Then it is of the following form:

Case $\mathcal{O}: \mathcal{O}_{\text {loc }}=\langle\emptyset\rangle$

Case $\mathcal{H}: \mathcal{H}^{\prime}{ }_{\text {loc }}=\left\langle\begin{array}{|}\square \bullet \circ\end{array}\right\rangle$

$$
\text { or } \mathcal{H}_{\mathrm{loc}}(k, d)=\left\langle b_{k}, b_{d} \otimes \tilde{b}_{d}, \quad \square_{0 \circ \bullet}, \quad\lceil\bullet \bullet\rangle \text { for } k, d \in \mathbb{N}_{0} \backslash\{1,2\}, d\right| k
$$



Case $\mathcal{B}: \mathcal{B}_{\mathrm{loc}}(k, d)=\left\langle\uparrow^{\otimes k}, \mathrm{o}^{\otimes d} \mid \varrho_{\bullet d}^{\otimes d}, \uparrow \otimes \uparrow\right\rangle$ for $k, d \in \mathbb{N}_{0}, d \mid k$
$d \mid k$

Here is a graphical overview of all categories of two-colored noncrossing partitions. The single framed categories are the locally colorized ones whose inclusions are indicated by single dashed lines (inclusions from top to bottom and from right to left, for fixed parameters $k$ and $d)$. Constraints for inclusions are marked in brackets. The double framed categories are the globally colorized ones with inclusion pattern according to the double dahed lines. The locally colorized categories are contained in the globally colorized ones according to the diagonal chain lines. In our graphic, we also included a cross marking the areas of the cases $\mathcal{B}, \mathcal{O}, \mathcal{S}$ and $\mathcal{H}$.


We also give the corresponding graphic in the orthogonal case ( $\square_{0} \in \mathcal{C}$ ), for comparison.


Remark 4.43. The constraints on the parameters $k$, $d$ and $r$ in the above theorems can be understood by the fact that we have the following equalities.

- $\mathcal{H}_{\text {glob }}(k)=\mathcal{H}_{\text {loc }}(k, 2)$ and $\mathcal{H}_{\text {glob }}(2 m+1)=\mathcal{S}_{\text {glob }}(2 m+1)$
- $\mathcal{S}_{\text {glob }}(k)=\mathcal{S}_{\text {loc }}(k, 1)=\mathcal{H}_{\text {loc }}(k, 1)$
- $\mathcal{B}_{\text {glob }}(k)=\mathcal{B}^{\prime}{ }_{\text {loc }}(k, 2 m, 1)$ and $\mathcal{B}_{\text {glob }}(2 m+1)=\mathcal{B}^{\prime}{ }_{\text {glob }}(2 m+1)$
- $\mathcal{B}^{\prime}{ }_{\text {glob }}(k)=\mathcal{B}^{\prime}{ }_{\text {loc }}(k, 1,0)=\mathcal{B}^{\prime}{ }_{\text {loc }}(k, 2 m+1,1)$


## $4.8 \quad C^{*}$-algebraic relations associated to partitions

We can associate $C^{*}$-algebras to categories of partitions by associating relations to partitions. This is the main step in the direction to defining unitary easy quantum groups.

Definition 4.44. Let $p \in P^{\bullet \bullet}(k, l)$ and let $\alpha=\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ and $\beta=\left(\beta_{1}, \ldots, \beta_{l}\right)$ be multi indices. We decorate the upper points of $p$ with $\alpha$ and the lower ones with $\beta$. If now for every block of $p$ all of the corresponding indices coincide, we put $\delta_{p}(\alpha, \beta):=1$; otherwise $\delta_{p}(\alpha, \beta):=0$.

Definition 4.45. Let $n \in \mathbb{N}$ and let $A$ be a $C^{*}$-algebra generated by $n^{2}$ elements $u_{i j}, 1 \leq i, j \leq n$. Let $p \in P^{\bullet \bullet}(k, l)$ be a partition and let $r=\left(r_{1}, \ldots, r_{k}\right) \in\{0, \bullet\}^{k}$ be its upper color pattern and $s=\left(s_{1}, \ldots, s_{l}\right) \in\{\circ, \bullet\}^{l}$ be its lower color pattern. We put $u_{i j}^{\circ}:=u_{i j}$ and $u_{i j}^{\bullet}:=u_{i j}^{*}$.

We say that the generators $u_{i j}$ fulfill the relations $R(p)$, if for all $\beta_{1}, \ldots, \beta_{l} \in\{1, \ldots, n\}$ and for all $i_{1}, \ldots, i_{k} \in\{1, \ldots, n\}$, we have:

$$
\sum_{\alpha_{1}, \ldots, \alpha_{k}=1}^{n} \delta_{p}(\alpha, \beta) u_{\alpha_{1} i_{1}}^{r_{1}} \ldots u_{\alpha_{k} i_{k}}^{r_{k}}=\sum_{\gamma_{1}, \ldots, \gamma_{l}=1}^{n} \delta_{p}(i, \gamma) u_{\beta_{1} \gamma_{1}}^{s_{1}} \ldots u_{\beta_{l} \gamma_{l}}^{s_{l}}
$$

The left-hand side of the equation is $\delta_{p}(0, \beta)$ if $k=0$ and analogous for the right-hand side.
Using this definition, we can give a list of relations associated to partitions that appeared throughout the classification of categories of noncrossing partitions. We denote by $u$ the matrix $u=\left(u_{i j}\right)_{1 \leq i, j \leq n}$, and $\bar{u}=\left(u_{i j}^{*}\right)$. Furthermore, $\operatorname{rot}_{t}(p) \in P^{\bullet \bullet}(t, k)$ denotes the partitions obtained from $p \in P^{\bullet \bullet}(0, k+t)$ by rotating the last $t$ points to the upper line. If we simply write $\operatorname{rot}(p)$, we do not specify which of the points are rotated. It is often more convenient to consider the relations of a partition in some rotated form rather than of the partition itself. By $\Gamma_{0}^{\text {nest }(k)}$ we denote the partition obtained from nesting the partition $\Gamma_{0} k$-times into itself, i.e.:


Note that $\square_{0}^{\otimes k} \in \mathcal{C}$ if and only if $\Gamma_{0}^{\text {nest }(k)} \in \mathcal{C}$, since $\square_{\bullet}^{\otimes k}$ and $\square_{\bullet}^{\text {nest }(k)}$ are in any category (then use Lemma 4.2(a)). The next relations can directly be derived from Definition 4.45
$R\left(\right.$ ऽ. $\left.^{\circ}\right): \sum_{k} u_{i k} u_{j k}^{*}=\delta_{i j}$, i.e. $u u^{*}=1$
$R(\curvearrowleft): \sum_{k} u_{i k}^{*} u_{j k}=\delta_{i j}$, i.e. $u^{*} u=1$
$R($ © $): \sum_{k} u_{k i} u_{k j}^{*}=\delta_{i j}$, i.e. $(\bar{u})^{*} \bar{u}=1$
$R\left(\bullet\right.$ ㅂ ) : $\sum_{k} u_{k i}^{*} u_{k j}=\delta_{i j}$, i.e. $\bar{u}(\bar{u})^{*}=1$
$R\left(\right.$ O. $\left._{0}^{0}\right)=R\left(\right.$ © $\left._{0}\right): u_{i j} u_{k l}=u_{k l} u_{i j}$


$R\left(\operatorname{rot}_{k}\left(\square_{0}^{\text {nest }(k)}\right)\right): u_{i_{1} j_{1}} \ldots u_{i_{k} j_{k}}=u_{i_{1} j_{1}}^{*} \ldots u_{i_{k} j_{k}}^{*}$
$R(\grave{\varrho} \otimes \emptyset)=R\left(\operatorname{rot}\left(\square_{\circ} \otimes \curvearrowleft\right)\right): u_{i j}^{*} u_{k l}=u_{i j} u_{k l}^{*}$
$R\left(\operatorname{rot}_{1}(\uparrow \otimes \uparrow)\right):\left(\sum_{k} u_{k j}\right)=\left(\sum_{l} u_{i l}\right)$
$R\left(\operatorname{rot}_{t}\left(\hat{0}^{\otimes s+t}\right)\right):\left(\sum_{k_{1}} u_{k_{1} j_{1}}\right) \ldots\left(\sum_{k_{s}} u_{k_{s} j_{s}}\right)=\left(\sum_{l_{1}} u_{i_{1} l_{1}}^{*}\right) \ldots\left(\sum_{l_{t}} u_{i_{t} l_{t}}^{*}\right)$
$R\left(\stackrel{\uparrow}{0}^{\otimes d}\right):\left(\sum_{k_{1}} u_{k_{1} j_{1}}\right) \ldots\left(\sum_{k_{d}} u_{k_{d} j_{d}}\right)=1$

$R\left(\right.$ •呙 ) : $u_{k i}^{*} u_{k j}=u_{i k}^{*} u_{j k}=0$ if $i \neq j$

$R\left(\operatorname{rot}_{d}\left(b_{d} \otimes \tilde{b}_{d}\right)\right): \sum_{k} \delta_{i_{1}=\ldots=i_{d}} u_{k j_{1}} \ldots u_{k j_{d}}=\sum_{l} \delta_{j_{1}=\ldots=j_{d}} u_{i_{1} l} \ldots u_{i_{d} l}$
$R\left(b_{k}\right): \sum_{l} u_{i_{1} l} \ldots u_{i_{k} l}=\delta_{i_{1}=\ldots=i_{k}}$
$R\left(\operatorname{rot}_{t}\left(b_{s+t}\right)\right): \delta_{i_{1}=\ldots=i_{s}} u_{i_{1} j_{1}} \ldots u_{i_{1} j_{s}}=\delta_{j_{1}=\ldots=j_{t}} u_{i_{1} j_{1}}^{*} \ldots u_{i_{t} j_{1}}^{*}$
$R\left(\operatorname{rot}_{d+1}\left(\delta^{\infty d} \circ \bullet^{\otimes^{d d}} \cdot\right)\right): u_{i j}\left(\sum_{k_{1}} u_{k_{1} j_{1}}\right) \ldots\left(\sum_{k_{d}} u_{k_{d} j_{d}}\right)=\left(\sum_{l_{1}} u_{i_{1} l_{1}}\right) \ldots\left(\sum_{l_{d}} u_{i_{d} l_{d}}\right) u_{i j}$
$\left.R\left(\operatorname{rot}_{r} \stackrel{8}{8 r+1}_{8}^{\bullet \bullet} \bullet^{8 r-1} \bullet\right)\right): u_{i j}\left(\sum_{k_{1}} u_{k_{1} j_{1}}\right) \ldots\left(\sum_{k_{r-1}} u_{k_{r-1} j_{r-1}}\right)=\left(\sum_{l_{1}} u_{i_{1} l_{1}}\right) \ldots\left(\sum_{l_{r+1}} u_{i_{r+1} l_{r+1}}\right) u_{i j}^{*}$

### 4.9 Free unitary easy group

### 4.9.1 Definition of $C_{n}^{+}$

In the sequel, the following quantum group will play an important role. It is a kind of a nonorthogonal version of $B_{n}^{+}$. It has been introduced in an unpublished paper of Banica, Curran and Speicher (see [83]).

Definition 4.46. Let $C_{n}^{+}$be the quantum group given by the universal $C^{*}$-algebra generated by $u_{i j}$ such that $u$ and $\bar{u}$ are unitaries and $\sum_{k} u_{i k}=\sum_{k} u_{k j}=1$ for all $i, j$.

Again, it can be read off directly from the relations in Section 4.8 that $C_{n}^{+}$is free easy with category $\mathcal{B}_{\text {loc }}(1,0)$.

### 4.9.2 Free and tensor complexifications with $\mathbb{Z}_{d}$

In [95], Wang proved the existence of a comultiplication on the free product as well as on the tensor product of the $C^{*}$-algebras associated to quantum groups. More precisely, let $G$ and $H$ be two compact (matrix) quantum groups with comultiplications $\Delta_{G}$ resp. $\Delta_{H}$. Let $C(G) \square C(H)$ either be the untail free product $C(G) * C(H)$ of the two $C^{*}$-algebras or the maximal tensor product $C(G) \otimes_{\max } C(H)$. Denote by $\iota_{C(G)}$ the embedding of $C(G)$ into $C(G) \square C(H)$ and likewise by $\iota_{C(G) \square C(G)}$ the embedding of $C(G) \otimes_{\text {min }} C(G)$ into $(C(G) \square C(H)) \otimes_{\text {min }}(C(G) \square C(H))$.

Proposition 4.47. Given two compact (matrix) quantum groups $G$ and $H$, there is always a comultiplication $\Delta$ on $C(G) \square C(H)$ for $\square \in\left\{*, \otimes_{\max }\right\}$ such that:

$$
\Delta \circ \iota_{C(G)}=\iota_{C(G) \square C(G)} \circ \Delta_{G} \quad \text { and } \quad \Delta \circ \iota_{C(H)}=\iota_{C(H) \square C(H)} \circ \Delta_{H}
$$

As a consequence, one can define the free product and the direct product of compact matrix quantum groups. The fundamental corepresentation is then given by the direct sum of these representations, thus by $\left(\begin{array}{ll}u & 0 \\ 0 & v\end{array}\right)$, where $u$ and $v$ are the matrices of generators for $G$ resp. $H$. We now define another kind of free resp. tensor product of two compact matrix quantum groups. Recall that for unital $C^{*}$-algebras $A$ and $B$, the maximal tensor product $A \otimes_{\max } B$ can be seen as the universal $C^{*}$-algebra generated by elements $a \in A$ (with the relations of $A$ ) and $b \in B$ (with the relations of $B$ ) such that all such $a$ and $b$ commute. We thus simply write $a b$ for elements $a \otimes b$.

Definition 4.48. Let $(G, u)$ and $(H, v)$ be two compact matrix quantum groups with $u$ of size $n$ and $v$ of size $m$.
(a) The glued free product $G \tilde{*} H$ of $G$ and $H$ is given by the $C^{*}$-subalgebra $C^{*}\left(u_{i j} v_{k l}, 1 \leq\right.$ $i, j \leq n, 1 \leq k, l \leq m) \subseteq C(G) * C(H)$.
(b) The glued direct product $G \tilde{\times} H$ of $G$ and $H$ is given by the $C^{*}$-subalgebra $C^{*}\left(u_{i j} v_{k l}, 1 \leq\right.$ $i, j \leq n, 1 \leq k, l \leq m) \subseteq C(G) \otimes_{\max } C(H)$.

As a simple consequence of Wang's result, the glued free product and the glued direct product are again compact matrix quantum groups.

Corollary 4.49. The $C^{*}$-subalgebra $C^{*}\left(u_{i j} v_{k l}, 1 \leq i, j \leq n, 1 \leq k, l \leq m\right)$ of $C(G) \square C(H)$, $\square \in\left\{*, \otimes_{\max }\right\}$ admits a comultiplication $\Delta\left(u_{i j} v_{k l}\right)=\Delta_{G}\left(u_{i j}\right) \Delta_{H}\left(v_{k l}\right)$.

Proof. Restriction of the comultiplication $\Delta$ of Proposition 4.47 yields the result.

To a discrete group $\Gamma$, we associate the universal $C^{*}$-algebra $C^{*}(\Gamma)$ generated by unitaries $u_{g}$, $g \in \Gamma$ with $u_{g} u_{h}=u_{g h}, u_{g}^{*}=u_{g^{-1}}$. It is well known that the comultiplication $\Delta\left(u_{g}\right)=u_{g} \otimes u_{g}$ turns it into a compact quantum group denoted by $\widehat{\Gamma}$.

Corollary 4.50. Let $\Gamma$ be a discrete group generated by a single element $g_{0}$, and denote by $z$ the generator $u_{g_{0}}$ of $C^{*}(\Gamma)$. Let $(G, u)$ be a compact matrix quantum group. Then $G \tilde{*} \widehat{\Gamma}$ and $G \tilde{\times} \widehat{\Gamma}$ are compact matrix quantum groups given by $C^{*}\left(u_{i j} z\right)$ in $C(G) * C^{*}(\Gamma)$ resp. in $C(G) \otimes_{\max } C^{*}(\Gamma)$ and $\Delta\left(u_{i j} z\right)=\sum_{k} u_{i k} z \otimes u_{k j} z$.

Proof. Since $\Gamma$ is generated by a single element, $\left(C^{*}(\Gamma), z\right)$ is a compact matrix quantum group of size 1. Using Corollary 4.49 we obtain the result.

As before, denote by $\mathbb{Z}_{d}$ the cyclic group $\mathbb{Z}_{d}:=\mathbb{Z} / d \mathbb{Z}$.
Definition 4.51. Let $G$ be a compact matrix quantum group.
(a) The quantum group $G \widehat{\overbrace{\mathbb{Z}}^{d}}$ is called the free $d$-complexification of $G$ and $G \widehat{\varkappa} \widehat{\mathbb{Z}}$ is called the free complexification.
(b) The quantum group $G \tilde{\times} \widehat{\mathbb{Z}_{d}}$ is called the tensor $d$-complexification of $G$ and $G \tilde{\times} \widehat{\mathbb{Z}}$ is called the tensor complexification.

The above definition is a generalization of Banica's free complexification [8].
[8]: $H_{N}^{\#}$, the free complexified of $H_{N}^{+}$, and $S_{N}^{\#}$, the free complexified of $S_{N}^{+}$. These two groups are free quantum groups corresponding respectively to the categories $\mathcal{H}^{\#}$ and $\mathcal{S}^{0,0}$.

Remark 4.52. The commutative complexification shouldn't be confused with the traditional complexification of Lie groups. In our case it is just the commutative counterpart of the free complexification introduced by Banica in[8]. This commutative complexification also appeared in the unpublished paper of Banica, Curran and Speicher [83]. Note that for a classical group G, the commutative d-complexification is just the product of $G$ with $\mathbb{Z}_{d}$.

The description of the free quantum groups is done in two steps. We first deal with the local parameters, and then with the global parameter.

### 4.9.3 Local colorization

In order to achieve the description of all free quantum groups, we have to describe the meaning of the local parameters $d, r$ for the cases $\mathcal{S}, \mathcal{B}$ and $\mathcal{H}$. In the case $\mathcal{H}$, this meaning has been already interpreted in [16]. The free complexification gives an interpretation in the cases $\mathcal{S}$, and $\mathcal{B}$ for $r=0$.

Proposition 4.53. The following correspondance holds between quantum groups and categories of partitions :

- $\mathcal{S}^{d, d}$ is the category of partitions corresponding to $S^{+,+d}$, the free $d$-complexification of $S^{+}$.
- $\mathcal{B}^{d, d}$ is the category of partitions corresponding to $C^{+,+d}$
- $\mathcal{B}^{d, d, 0}$ is the category of partitions corresponding to $B^{+,+d}$

Proof. For these three families, the proof follows exactly the same pattern. We sketch the proof and detail it for the case $\mathcal{S}$, the latter being the one needing more arguments.

1. Let $n \geq 1$. We indentify the category of partition $\mathcal{S}^{d, d}$ with its image as linear maps on tensor products of $\mathbb{C}^{n}$ through the construction of Section 1.1.3. Denote by $G_{n}$ the quantum group of dimension $n$ corresponding to the category of partition $\mathcal{S}^{d, d}, \mathcal{T}$ the Woronowicz tensor category associated to the quantum group $S_{n}^{+,+d}$. The goal is to prove that $G_{n}=S_{n}^{+,+d}$.
2. In order to prove that $G_{n}=S_{n}^{d,+d}$, it suffices to prove that on one hand $\mathcal{S}^{d, d} \subseteq \mathcal{T}$ (yielding $S_{n}^{+,+d} \subseteq G_{n}$ ), and that on the other hand there exist a $C^{*}-$ morphism sending the fundamental matrix of $S_{n}^{+,+d}$ on the one of $G_{n}$. The first step is just a verification that is left to the reader.
3. Write $G_{n}=\left(v_{i j}\right)_{1 \leq i, j \leq n}, S_{n}^{+,+d}=\left(u_{i j}\right)_{1 \leq i, j \leq n}$. We shall construct a surjective $C^{*}$-homorphism $\Phi: C\left(S_{n}^{+,+d}\right) \rightarrow C\left(G_{n}\right)$ sending $u_{i j} z$ on $v_{i j}$.
Recall that $C\left(S_{n}^{+,+d}\right)$ is the $C^{*}$-subalgebra of $C\left(S_{n}^{+}\right) * \mathbb{C}_{d}$ generated by $\left\{u_{i j} z\right\}_{1 \leq i, j \leq n}$. Since $\sum_{j} v_{i j}$ is independent of $i$ (thank to the partition $\uparrow \uparrow$ ), and $\left(\sum_{j} v_{i j}\right)^{d}=1$, there is a map from $C\left(\mathbb{Z}_{d}\right)$ to $C\left(G_{n}\right)$ sending $z$ to $\sum_{i} v_{i j}$. Set $\tilde{s}_{i j}=v_{i j}\left(\sum_{i} v_{i j}\right)^{-1}$. The partition ôt. implies that $\left(\sum_{i} v_{i j}\right)^{-1} v_{i j}=v_{i j}^{*}\left(\sum_{i} v_{i j}\right)$, and thus

$$
\begin{aligned}
\tilde{s}_{i j}^{*}=\left(\sum_{i} v_{i j}\right) v_{i j}^{*} & =\left(\sum_{i} v_{i j}\right) v_{i j}^{*}\left(\sum_{i} v_{i j}\right)\left(\sum_{i} v_{i j}\right)^{-1} \\
& =v_{i j}\left(\sum_{i} v_{i j}\right)^{-1}=\tilde{s}_{i j}
\end{aligned}
$$

Thus the matrix $\left(\tilde{s}_{i j}\right)$ is an orthogonal matrix fulfilling the relations

$$
\sum \tilde{s}_{i j}=1
$$

The expression of $\tilde{s}_{i j}$ together with the fact that if $k \neq l, v_{i k} v_{i l}^{*}=0$ (implied by the presence of the partition $\quad \bullet \bullet$ ) yields

$$
k \neq j \Rightarrow \tilde{s}_{i j} \tilde{s}_{i k}=0
$$

Due to these relations, there exists a $C^{*}$-homomorphism from $C\left(S_{n}^{+}\right)$to $C\left(G_{n}\right)$ sending $u_{i j}$ to $\tilde{s}_{i j}$. By the universality property, there exists a $C^{*}$-homorphism $\Phi$ from $C\left(S_{n}^{+}\right) * \mathbb{C} \mathbb{Z}_{d}$ to $C\left(G_{n}\right)$ sending $z$ to $\sum v_{i j}$ and $u_{i j}$ to $v_{i j}\left(\sum v_{i j}\right)^{-1}$. This homorphism sends thus $u_{i j} z$ to $v_{i j}$, which concludes the proof.

The proof for $B^{+,+d}$ is the same except that we don't need to prove the relation $k \neq j \Rightarrow$ $\tilde{s}_{i j} \tilde{s}_{i k}=0$. The one for $C^{+,+d}$ is the same as the latter except that we don't have to prove the self-adjointness of $\tilde{s}_{i j}$.

### 4.9.4 The family $\mathcal{B}^{+, d, d / 2-1}$

The free unitary group $U_{n}^{+}$has been first constructed in 95] as a free $0-$ complexification of $O_{n}^{+}$. This construction gives also an alternative description of the $C^{*}$-algebra underlying $C^{+}$. Let $O_{n}^{+}=\left(o_{i j}\right)_{1 \leq i, j \leq n}$ be the orthogonal group of dimension $n$. The following alternative description of $C^{+}$holds:

Proposition 4.54. Let $C_{n}^{+}=\left(u_{i j}\right)_{1 \leq i, j \leq n}$. Let $\mathbb{C} Z_{d}$ be the group algebra of $\mathbb{Z} / d \mathbb{Z}$ generated by the element $z$, with $d \in 2 \mathbb{N}$ non zero. Then we can write $u_{i j}=\tilde{o}_{i j} z^{d / 2}$ with:

- $\tilde{o}_{i j}$ is the image of $o_{i j}$ through the projection on $C\left(O_{n}^{+}\right) * \mathbb{C} Z_{d} /\left\langle\sum_{j} o_{i j}=z^{d / 2}\right\rangle$
- $C\left(C^{+}\right)$is defined as the $C^{*}$-subalgebra of $C\left(O^{+}\right) * \mathbb{C} Z_{d} /\left\langle\sum_{j} o_{i j}=z^{d / 2}\right\rangle$ generated by the elements $\tilde{o}_{i j} z^{d / 2}$
Proof. Note first that as a compact quantum group, $C_{n}^{+} \simeq U_{n-1}^{+} \oplus 1$ : indeed let $F \in \mathcal{U}_{n}$ sending $e_{n}$ to $\sum e_{i}$. Then $\left(v_{i j}\right)_{1 \leq i, j \leq n}=F^{*} U F$ is again a unitary quantum group. Moreover the condition $\sum u_{i j}=1$ translates into the condition $v_{i n}=v_{n j}=0$ for all $i, j<n$. Thus $C_{n}^{+}$ is a quantum subgroup of $F\left(U_{n-1}^{+} \oplus 1\right) F^{*}$. Since the intertwiners of $C_{n}^{+}$are also intertwiners of $F\left(U_{n-1}^{+} \oplus 1\right) F^{*}$, we deduce that $C_{n}^{+}=F\left(U_{n-1}^{+} \oplus 1\right) F^{*}$. Moreover the $C^{*}$-algebra defined as $C\left(O_{n}^{+}-* \mathbb{C} Z_{d} /\left\langle\sum_{j} o_{i j}=z^{-d / 2}\right\rangle\right.$, is isomorph to the $C^{*}-$ algebra of $O_{n-1}^{+} \oplus \varepsilon\left(\right.$ with $\left.\varepsilon=z^{d / 2}\right)$. Indeed the former is exatcly the $C^{*}$-algebra of the compact quantum group $B_{n}^{+\#}$ as described in 97, and $B_{n}^{+\#}$ has been shown by Raum in [70, Thm 4.1] to be isomorph to $O_{n-1}^{+} \oplus \varepsilon$. Since, from a result of Banica, $\left(o_{i j} \varepsilon\right)_{1 \leq i, j \leq n-1}$ is isomorph to $U_{n-1}^{+}, \tilde{O}_{n}^{+} \varepsilon \simeq C_{n}^{+}$.

Let us denote $\tilde{C}_{n}^{+, d}$ the quantum subgroup of $C^{+,+d}$ generated by the matrix $\left(\tilde{o}_{i j} z^{d / 2+1}\right)_{1 \leq i, j \leq n}$ (with the same notations we gave in the latter proposition).
Proposition 4.55. If $d$ is even, $\mathcal{B}^{d, d, d / 2}$ is the category of partition corresponding to $\tilde{C}_{n}^{+, d}$.
Proof. The pattern of the proof is the same as for the free complexification.
Let $n \geq 1$. We denote $\left.G_{n}=\left(v_{i j}\right)_{1 \leq i, j \leq n}\right)$ the quantum group of dimension $n$ corresponding to the category of partition $\mathcal{B}^{d, d, d / 2-1}, \mathcal{T}$ the Woronowicz tensor category associated to the quantum group $\tilde{C}_{n}^{+, d}$. Note first that $\mathcal{B}^{d, d, d / 2-1} \subseteq \mathcal{T}$, so that $\tilde{C}_{n}^{+, d} \subseteq G_{n}$.
Let us show that there exists a $C^{*}$-morphism from $\left.C\left(O_{n}^{+}\right) * \mathbb{C} \mathbb{Z}_{d}\right)$ to $C\left(G_{n}\right)$ sending $o_{i j} z^{d+1}$ to $v_{i j}$. Since $\left(\sum_{j} v_{i j}\right)^{d}=1$, there exists a $C^{*}$-morphism sending $z$ on $\sum_{j} v_{i j}$. Thank to the intertwiner $\uparrow \uparrow$, the sum $\sum_{j} v_{i j}$ is independent from $i$. Moreover the intertwiner associated with $\uparrow$ •

$$
v_{i j}\left(\sum_{j} v_{i j}\right)^{d / 2-1}=\left(\sum_{j} v_{i j}\right)^{d / 2+1} v_{i j}^{*}
$$

and thus since $\left(\sum_{j} v_{i j}\right)^{d / 2-1}=\left(\left(\sum_{j} v_{i j}\right)^{d / 2+1}\right)^{*}, v_{i j}\left(\sum_{j} v_{i j}\right)^{d / 2-1}$ is self-adjoint. Let $\tilde{o}_{i j}=$ $v_{i j}\left(\sum_{j} v_{i j}\right)^{d / 2-1}$. The matrix $\left(\tilde{o}_{i j}\right)_{1 \leq i, j \leq n}$ contains self-adjoint elements and

$$
\sum_{i} \tilde{o}_{i j} \tilde{o}_{i k}=\sum \tilde{o}_{j i} \tilde{o}_{k i}=\delta_{j k} \mathbf{1} .
$$

Thus there exist a $C^{*}$-morphism from $C\left(O_{n}^{+}\right)$to $C\left(G_{n}\right)$ sending $o_{i j}$ to $\tilde{o}_{i j}$. By universal property there exists a $C^{*}$-morphism $\Phi$ between $C\left(O_{n}^{+}\right) * \mathbb{C}_{d}$ extending the two latter morphisms. In particular we have $\Phi\left(o_{i j} z^{d / 2+1}\right)=v_{i j}$. By construction $\Phi\left(\sum o_{i j}-z^{d / 2}\right)=0$, and thus the morphism factorizes through $C\left(\tilde{C}_{n}^{+, d}\right)$.

### 4.9.5 Global colorization

It remains to interpret the global parameter $k$. For each words $r$ in $\circ, \bullet$, let $c(r)$ denote the quantity $\# \circ-\# \bullet$. The following result holds for every matrix compact quantum groups:

Proposition 4.56. Let $G=\left(u_{i j}\right)_{1 \leq i, j \leq n}$ be a compact matrix quantum group, $\mathcal{C}(G)=\left\{\operatorname{Mor}_{G}\left(r, r^{\prime}\right)\right\}_{r, r^{\prime}}$ the set of intertwiners associated to this matrix compact quantum group by the Tannaka-Krein duality. Then

$$
\mathcal{C}\left(G^{d}\right)=\left\{\operatorname{Mor}_{G}\left(r, r^{\prime}\right)\right\}_{c(r)-c\left(r^{\prime}\right)=0[d]}^{r, r^{\prime}}
$$

Proof. Let $\left(u_{i j}\right)_{1 \leq i, j \leq n}$ be the fundamental matrix of $G$, and $\left(v_{i j}\right)_{1 \leq i, j \leq n}$ be the one of $G^{d}$. Since $G$ is a quantum subgroup of $G^{d}$, for any word $r, r^{\prime}$ in $\circ, \bullet$,

$$
\operatorname{Mor}_{G^{d}}\left(r, r^{\prime}\right) \subseteq \operatorname{Mor}_{G}\left(r, r^{\prime}\right)
$$

By duality, it suffices to consider only the case $r^{\prime}=\emptyset$. Let $\left\{e_{i}\right\}$ be a basis of the fundamental representation $u$ of $G$. Let $X=\sum \lambda_{i} e_{i_{1}}^{r_{1}} \otimes \cdots \otimes e_{i_{t}}^{r_{t}}$ be a vector in $\operatorname{Mor}\left(u^{\otimes r}, \mathbf{1}\right)$. We note $\vec{i}$ for the tuple ( $i_{1}, \ldots, i_{r}$ ), and $u_{\overrightarrow{i j}}$ for the product $u_{i_{1} j_{1}}^{r_{1}} \ldots u_{i_{t} t}^{r_{t}}$. Then

$$
u_{G}^{\otimes r}(X)=\sum_{\vec{i}, \vec{j}} \lambda_{\vec{i}} e_{j_{1}}^{r_{1}} \otimes \cdots \otimes e_{j_{t}}^{r_{t}} \otimes u_{\vec{j} i}=X \otimes \mathbf{1}_{C(G)}
$$

And thus $\sum_{\vec{i}} \lambda_{\vec{i}} u_{\vec{j} i}=\lambda_{\vec{j}} \mathbf{1}_{C(G)}$. Applying this equality together with the expression $v_{i j}=u_{i j} z_{d}$ yields

$$
v^{r}(X)=\sum_{\vec{i}, \vec{j}} \lambda_{\vec{i}} e_{j_{1}}^{r_{1}} \otimes \cdots \otimes e_{j_{t}}^{r_{t}} \otimes u_{\overrightarrow{j i}} z^{c(r)}=X \otimes z^{c(r)}
$$

Finally $X$ is invariant under the action of $G^{d}$ if and only if $c(r)=0[d]$.

## Summary of the classification of free quantum groups

We can conclude by summarizing the previous description in the following theorems :
Theorem 4.57. The following correspondance holds between categories of partitions and unitary easy quantum groups :

1. the category $\mathcal{U}$ corresponds to the free unitary quantum group $U^{+}$.
2. the category $\mathcal{H}^{k, d}$ corresponds to the commutative $k$-complexification of $Z_{d} \imath S^{+}$.
3. the category $\mathcal{H}^{\#}$ corresponds to the free complexification of $H^{+}$.
4. the category $\mathcal{B}^{k, d}$ (resp $\mathcal{B}^{k, d, 0}$ ), corresponds to the commutative $k$-complexification of the free $d$-complexification of $C^{+}$(reps. $B^{+}$).
5. the category $\mathcal{B}^{k, d, d / 2}$ corresponds to the commutative $k$-complexification of $\tilde{C}^{+, d}$.
6. the category $\mathcal{S}^{k, d}$ corresponds to the commutative $k$-complexification of the free $d$-complexification of $S^{+}$.

Proof. The free quantum groups corresponding to $\mathcal{U}, \mathcal{O}, \mathcal{H}^{d, d}, \mathcal{H}^{\#}, \mathcal{B}^{d, d}, \mathcal{B}^{d, d, d / 2}, \mathcal{B}^{d, d, 0}$ and $\mathcal{S}^{d, d}$ have been already indetified in previous paragraph. Proposition 4.56 concludes the proof.

### 4.10 The group case and unitary easy groups

For general categories $\mathcal{C} \subseteq P^{0 \bullet}$ of two-colored partitions, there are two natural extreme cases. The first is the one of noncrossing partitions, completely classified in the preceding sections. The second is the one containing the crossing partitions 00,000 , which allow us to permute the points of a partition in an arbitrary way (without changing their colors). It is easy to see that one of these four partitions is in a category if and only if all are (by verticolor reflection and rotation).

Definition 4.58. A category of two colored partitions $\mathcal{C}$ is in the group case if one (and hence all) of the partitions ob, ©0, and 0 is in $\mathcal{C}$.

The name "group case" refers to the situation when a quantum group is associated to a category of partitions (see Section 3.2 ). If $\mathcal{C}$ is in the group case, the associated quantum group is in fact a group.

The classification of all categories in the group case follows directly from the classification of all categories of noncrossing partitions and the following lemma.

Lemma 4.59. Let $\mathcal{C}$ and $\mathcal{D}$ be categories of two-colored partitions.
(a) Then $\mathcal{C} \cap \mathcal{D}$ is again a category of partitions.
(b) Let $\mathcal{C}$ be in the group case and put $\mathcal{C}_{0}:=\mathcal{C} \cap N C^{\circ}$. Then $\mathcal{C}=\left\langle\mathcal{C}_{0}\right.$, oठ $\rangle$.

Proof. (a) This follows directly from the definition of a category.
(b) Let $p \in \mathcal{C}$. Using the four kinds of crossing partitions of Definition 4.58, we may permute the points of $p$ such that we obtain a noncrossing partition $p^{\prime}$. Since this can be done in $\mathcal{C}$, we have $p^{\prime} \in \mathcal{C}_{0} \subseteq\left\langle\mathcal{C}_{0}\right.$, $\left.\mathcal{O}_{0}\right\rangle$. Thus, we can also reconstruct $p$ in $\left\langle\mathcal{C}_{0}, \mathcal{O}_{0}\right\rangle$ doing all these operations backwards, so $\mathcal{C} \subseteq\left\langle\mathcal{C}_{0}\right.$, 〇o $\left.^{\circ}\right\rangle$. We deduce that equality holds.

For each category of partition $\mathcal{C}$, denote by $\mathcal{C}_{c}=\left\langle\mathcal{C}\right.$, O。 $\left._{\text {O }}\right\rangle$. Thus the latter Lemma says that any category of partition in the group case is of the form $\mathcal{C}_{c}$ for a category of non-crossing partition.

Theorem 4.60. The categories in the group case are the following.


- $\mathcal{O}_{\text {grp,loc }}:=\left\langle\begin{array}{l}0 \\ 0 \\ 0\end{array}\right\rangle$





- $\mathcal{B}_{\text {grp,loc }}(k)=\left\langle\uparrow_{0}^{\otimes k}, \uparrow \otimes \uparrow\right.$, ${ }_{0}$ oठ $\rangle$ for $k \in \mathbb{N}_{0}$

Proof. The crossing partition ${ }^{\circ}$ permutes the points of a partition. Applying this partitions on the generators of the categories of partitions in Section 4.7 yields that $\mathcal{H}_{c}^{\#}=\mathcal{H}_{c}^{0,0}$. Furthermore for the same reasons, $\mathcal{S}_{c}^{d, k}=\tilde{\mathcal{S}}_{c}^{k}, \mathcal{B}_{c}^{k, d}=\mathcal{B}_{c}^{k}$ and $\mathcal{B}_{c}^{k, d, 0}=\mathcal{B}_{c}^{k, d, d / 2}=\tilde{\mathcal{B}}_{c}^{k}$. Apart from these equalities, the commutative image of the categories of noncrossing partitions are different, yielding the result.

If $\mathcal{C}$ is a category containing the crossing partitions $0_{0}^{0}$ and $\underbrace{0}_{0}$, then the $C^{*}$-algebra associated to it is commutative (see the relations in Section 4.8). Hence, the associated quantum groups are in fact groups. They are listed in the next theorem.

Theorem 4.61. The groups corresponding to the catgories in the group case are the following :

- $\mathcal{U}_{c}$ corresponds to the unitary group $U_{n}$.
- $\mathcal{O}_{c}^{k}$ corresponds to the $k$-complexification of the orthogonal group $O_{n}$.
- $\mathcal{H}_{c}^{k, d}$ corresponds to the $k$-complexification of the wreath product $\mathcal{Z}_{d} \imath S_{n}$.
- $\tilde{\mathcal{S}}_{c}^{k}$ corresponds to the $k$-complexification of the permutation group $S_{n}$.
- $\tilde{\mathcal{B}}_{c}^{k}$ corresponds to the $k$-complexification of the group $\left(\left(\begin{array}{cc}O_{n-1} & 0 \\ 0 & 1\end{array}\right)\right)$.
- $\mathcal{B}_{c}^{k}$ corresponds to the $k$-complexification of the group $\left(\left(\begin{array}{cc}U_{n-1} & 0 \\ 0 & 1\end{array}\right)\right)$.

Remark 4.62. Note that $\left(\left(\begin{array}{cc}U_{n-1} & 0 \\ 0 & 1\end{array}\right)\right) \simeq U_{n-1}$ and $\left(\left(\begin{array}{cc}O_{n-1} & 0 \\ 0 & 1\end{array}\right)\right) \simeq O_{n-1}$. The category are still different because we singled out different fundamental representations of the same group.

## Chapter 5

## Stochastics on the free unitary easy groups

In this chapter we develop the Weingarten calculus for unitary easy quantum groups. This consists mainly in a rewriting of the classical method of Collins and Sniady (see Chapter 2 and [28, 29]) for a general unitary easy quantum groups. In the orthogonal case, this has already been done by Collins and Banica in [12] and by Banica, Curran and Speicher in [14]. As an application of this generalization, we recover the results of Diaconis and Shahshahani on the unitary group (see Chapter 2, Section 3 and [33]) and the second order freeness result of Mingo and Speicher (see Chapter 2, Section 5 and [63]), and extend both result to the free unitary group.

### 5.1 Weingarten calculus for easy quantum groups

In this section we introduce the Weingarten calculus for unitary easy quantum group, using the formalism developed by Banica and Speicher in [15]. Throughout this section $n$ is a fixed positive integer and $\left(C(G),\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)$ is a unitary easy quantum group, with associated category of partition $\mathcal{C}$. By Chapter 4, this implies that the vector space $\operatorname{Fix}_{G}(\varepsilon)$ of invariant vectors of $V^{\varepsilon}$ under the action of $G$ is given by $\left\langle T_{p}\right\rangle_{p \in \mathcal{C}(\varepsilon)}$, and $\left\{T_{p}\right\}_{p \in \mathcal{C}(\varepsilon)}$ is a basis of $\operatorname{Fix}_{G}(\varepsilon)$ if $n \geq|\varepsilon|$.
Therefore we can express integrals of polynomials in $u_{i j}, u_{i j}^{*}$ using the vectors $T_{p}^{\prime} s$, as in Chapter 2. We will start by giving a general formula to emphasize the geometric aspect of the Weingarten calculus, then we will specify this result to the unitary easy quantum groups. In the following expression we formally write $u_{i j}^{\circ}=u_{i j}$ and $u_{i j}^{\bullet}=u_{i j}^{*}$ :

Proposition 5.1. Let $\left(C(G),\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)$ be a compact matrix quantum group. Let $1 \leq r$ and let $\varepsilon$ be a word in $\circ, \bullet$ of length $r$. Suppose that $\left\{f_{i}\right\}_{1 \leq i \leq s}$ is a basis of $\operatorname{Fix}_{G}(\varepsilon)$. Then for each couple of sequences $1 \leq i_{1}, \ldots, i_{r} \leq n, 1 \leq j_{1}, \ldots, j_{r} \leq n$,

$$
\int_{G} u_{i_{1} j_{1}}^{\varepsilon_{1}} \cdots u_{i_{r} j_{r}}^{\varepsilon_{r}}=\left\langle p_{\mathrm{Fix}_{G}(\varepsilon)}\left(X_{j_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{j_{r}}^{\varepsilon_{r}}\right), X_{i_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{i_{r}}^{\varepsilon_{r}}\right\rangle,
$$

$p_{\text {Fix }_{G}(\varepsilon)}$ being the orthogonal projection on $\operatorname{Fix}_{G}(\varepsilon)$.
Proof. On one hand,

$$
\alpha_{\varepsilon}\left(X_{j_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{j_{r}}^{\varepsilon_{r}}\right)=\sum_{1 \leq i_{1}, \ldots, i_{r} \leq n} X_{i_{1}} \otimes \cdots \otimes X_{i_{r}} \otimes u_{i_{1} j_{1}}^{\varepsilon_{1}} \ldots u_{i_{r} j_{r}}^{\varepsilon_{r}},
$$

and thus

$$
\begin{equation*}
\left(I d \otimes \int_{G}\right) \alpha_{\varepsilon}\left(X_{j_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{j_{r}}^{\varepsilon_{r}}\right)=\sum\left(\int_{G} u_{i_{1} j_{1}}^{\varepsilon_{1}} \cdots u_{i_{r} j_{r}}^{\varepsilon_{r}}\right) X_{i_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{i_{r}}^{\varepsilon_{r}} . \tag{5.1.1}
\end{equation*}
$$

On the other hand, $V^{\varepsilon}$ decomposes into irreducible representations of $C(G)$ as $V^{\varepsilon}=\operatorname{Fix}_{G}(\varepsilon)+$ $\bigoplus_{\omega \text { irred, } \omega \neq 1} V_{\omega}^{\oplus n_{\omega}}$, and this decomposition is orthogonal. If $v \in \operatorname{Fix}_{G}(\varepsilon),\left(\operatorname{Id} \otimes \int_{G}\right) \alpha_{\varepsilon}(v)=$ $\left(I d \otimes \int_{G}\right)\left(v \otimes \mathbf{1}_{C(G)}\right)=v$. If $v \in V_{\omega}$ with $\omega \neq \mathbf{1}$, by Schur othogonality $\left(I d \otimes \int_{G}\right) \alpha_{\varepsilon}(v)=0$. Thus $\left(I d \otimes \int_{G}\right) \alpha_{\varepsilon}$ is an othogonal projection onto the vector space $\operatorname{Fix}_{G}(\varepsilon)$.
By (5.1.1), $\int_{G} u_{i_{1} j_{1}}^{\varepsilon_{1}} \ldots u_{i_{r} j_{r}}^{\varepsilon_{r}}$ is the coordinate $\left(i_{1}, \ldots, i_{r}\right)$ of the orthogonal projection of $X_{j_{1}}^{\varepsilon_{1}} \otimes$ $\cdots \otimes X_{j_{r}}^{\varepsilon_{r}}$ on $\operatorname{Fix}_{G}(\varepsilon)$.

In order to evaluate $\left\langle p_{\text {Fix }_{G}(\varepsilon)}\left(X_{j_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{j_{r}}^{\varepsilon_{r}}\right), X_{i_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{i_{r}}^{\varepsilon_{r}}\right\rangle$, a general framework is given by the Gram-Schmidt orthogonalization:

Proposition 5.2 (Gram-Schmidt orthogonalization). Let ( $V,\langle.,\rangle$.$) be a Hilbert space and W$ a finite dimensional vector subspace of $V$. Suppose that $\left(e_{i}\right)_{1 \leq i \leq s}$ is a basis of $W$, and let $x, y \in V$. The orthogonal projection $p_{W}$ on $W$ is given by the expression:

$$
\left\langle p_{W}(x), y\right\rangle=\sum_{i, j=1}^{s}\left\langle x, e_{i}\right\rangle\left\langle e_{j}, y\right\rangle K^{-1}(i, j),
$$

with $K(i, j)=\left\langle e_{i}, e_{j}\right\rangle$ the Gram-Schmidt matrix of the basis $\left(e_{i}\right)_{1 \leq i \leq s}$.
Proof. $K$ is the matrix of the scalar product $\langle.,$.$\rangle in the basis \left(e_{i}\right)_{1 \leq i \leq s}$. Let $\Lambda=\left(\lambda_{i}\right)_{1 \leq i \leq s}$ and $\Xi=\left(\xi_{i}\right)_{1 \leq i \leq s}$ be two vectors of $\mathbb{C}^{s}$ such that $x=h_{1}+\sum \lambda_{i} e_{i}$ and $y=h_{2}+\sum \mu_{i} e_{i}$ with $h_{1}, h_{2} \perp W$. Then

$$
\left\langle p_{W}(x), y\right\rangle=\left\langle p_{W}(x), p_{W}(y)\right\rangle=\Lambda^{t} K \Xi .
$$

On the other hand setting $\tilde{\Lambda}=\left(\left\langle e_{i}, x\right\rangle\right)_{1 \leq i \leq s}$ and $\tilde{\Xi}=\left(\left\langle e_{i}, y\right\rangle\right)_{1 \leq i \leq s}$ yields that $K \Lambda=\tilde{\Lambda}$ and $K \Xi=\tilde{\Xi}$. Therefore, since $K$ is invertible,

$$
\left\langle p_{W}(x), y\right\rangle=\tilde{\Lambda}^{t} K^{-1} \tilde{\Xi} .
$$

The combination of the two previous results gives a general Weingarten formula for compact matrix quantum groups :

Theorem 5.3 (Weingarten Formula). Let $\left(C(G),\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)$ be a compact matrix quantum group. Let $\varepsilon=\varepsilon_{1} \ldots \varepsilon_{r}$ be a word in $\circ, \bullet$, and $1 \leq i_{1}, \ldots, i_{r} \leq n, 1 \leq j_{1}, \ldots, j_{n} \leq n$ be two sequences of integers. Suppose that $\left\{f_{a}\right\}_{1 \leq a \leq s}$ is a basis of $\operatorname{Fix}_{G}(\varepsilon)$; then

$$
\int_{G} u_{i_{1} j_{1}}^{\varepsilon_{1}} \cdots u_{i_{r} j_{r}}^{\varepsilon_{r}}=\sum_{1 \leq a, b \leq s}\left\langle X_{i_{1}} \otimes \cdots \otimes X_{i_{r}}, f_{b}\right\rangle\left\langle f_{a}, X_{j_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{j_{r}}\right\rangle W g_{G}\left(f_{a}, f_{b}\right),
$$

where $W g_{G}=K_{G}^{-1}, K_{G}$ being the Gram-Schmidt matrix $K_{G}\left(f_{a}, f_{b}\right)=\left\langle f_{a}, f_{b}\right\rangle_{V^{\varepsilon}}$.
As it was already said in Chapter 2, this Theorem is really useful provided the set $\left(f_{a}\right)_{1 \leq a \leq s}$ has an explicit expression relatively to the basis $\left(X_{i_{1}}^{\varepsilon_{1}} \otimes \cdots \otimes X_{i_{r}}^{\varepsilon_{r}}\right)_{1 \leq i_{1}, \ldots, i_{r} \leq n}$ and the matrix $W g_{G}$ can be easily computed - or at least presents good approximations. The first condition is fulfilled in the case of a unitary easy quantum group, since an explicit basis is given by the vectors $T_{p}$ 's:

Corollary 5.4. Let $G_{n}$ be a unitary easy quantum group with associated category of partitions $\mathcal{C}$. With the notations above, if $n \geq|\varepsilon|$ (or $n \geq 4$ if $\mathcal{C}$ is non-crossing), then

$$
\int_{G_{n}} u_{i_{1} j_{1}}^{\varepsilon_{1}} \ldots u_{i_{r} j_{r}}^{\varepsilon_{r}}=\sum_{\substack{p, q \in \mathcal{C} \\ p \leq \operatorname{ker}(\vec{i}), q \leq \operatorname{ker}(\vec{j})}} W g_{G_{n}}(p, q),
$$

where $W g_{G_{n}}=K_{G_{n}}^{-1}(p, q)$ with $K_{G_{n}}(p, q)=n^{b(p \vee q)}$.
Proof. The condition $n \geq|\varepsilon|$ or $n \geq 4$ if $\mathcal{C}$ is non-crossing yields that the set $\left\{T_{p}\right\}_{p \in \mathcal{C}(\varepsilon)}$ is a basis of $\operatorname{Fix}_{G}(\varepsilon)$. Since $T_{p}=\sum_{p \leq \operatorname{ker}(\vec{i})} X_{i_{1}} \otimes \cdots \otimes X_{i_{r}}$, the scalar product of Theorem 5.3 is $\left\langle T_{p}, X_{i_{1}} \otimes \cdots \otimes X_{i_{r}}\right\rangle=\delta_{p \leq \operatorname{ker}(\vec{i})}$. The corollary is thus a direct application of this Theorem and the fact that $\left\langle T_{p}, T_{q}\right\rangle=n^{\bar{b}(p \vee q)}$.

The main problem remains the computation of the matrix $W g_{G_{n}}$. We have seen in Chapter 2 that in the case of $U_{n}$, the computation of $W g_{U_{n}}$ is already complicated. In the general case no explicit expression of this matrix has been found for a unitary easy quantum group; however a first order asymptotic of $W g_{G_{n}}$ allows to get several asymptotic probabilistic results as $n$ goes to $+\infty$.

Proposition 5.5 ([12]). As n goes to $+\infty$,

$$
W g_{G_{n}}(p, q)=(-1)^{\delta_{p \neq q}} n^{b(p \vee q)-b(p)-b(q)}(1+O(1 / n))
$$

Proof. For $n$ large enough, $\left\{T_{p}\right\}$ is a basis of $\operatorname{Fix}_{G_{n}}(\varepsilon)$. Let $K_{G_{n}}$ be the Gram-Schmidt matrix of the basis $\left\{T_{p}\right\}_{p \in \mathcal{C}}$. From the expression of $T_{p}, K_{G_{n}}(p, q)=n^{b(p \vee q)}$. Since $p \leq p \vee q, b(p \vee q) \leq b(p)$; for the same reasons, $b(p \vee q) \leq b(q)$ and thus $b(p \vee q) \leq \frac{b(p)+b(q)}{2}$. Moreover if $p \neq q$, the inequality becomes $b(p \vee q) \leq \frac{b(p)+b(q)}{2}-1 / 2$.
Let $\Delta$ be the diagonal matrix defined by $\Delta_{p q}=\delta_{p q} n^{b(p)}$. By the previous inequality

$$
\left(\Delta^{-1 / 2} K_{G_{n}} \Delta^{-1 / 2}\right)_{p q}=n^{b(p \vee q)-\frac{b(p)+b(q)}{2}}=I d+M
$$

where $M=O(1 / \sqrt{n})$. Thus $(I d+M)^{-1}=(I d-M)(1+O(\sqrt{n}))$ and $K_{G_{n}}^{-1}=\Delta^{-1 / 2}(I d-$ $M) \Delta^{-1 / 2}(1+O(\sqrt{n}))$. This yields

$$
\left(K_{G_{n}}^{-1}\right)_{p q}=(-1)^{\delta_{p \neq q}} n^{b(p \vee q)-b(p)-b(q)}(1+O(1 / \sqrt{n})) .
$$

### 5.2 Diaconis-Shahshahani results in the free case

The first application of the Weingarten calculus is a computation of the asymptotic law of the family $\left(\operatorname{Tr}\left(u^{k}\right)\right)_{k \geq 1}$, where $u$ is the fundamental matrix of a free unitary quantum group $G_{n}$ and $n$ goes to $+\infty$. For all easy unitary classical groups, this has already been done by Diaconis and Shahshahani in [33] (see Chapter 2, Section 3), or is a direct consequences of their results. In the free orthogonal case, this has been done in [14.
Recall that the list of free unitary easy groups is the following (we put the associated category of partitions in parenthesis) :

- $U_{n}^{+}(\mathcal{U})$,
- $O_{n}^{+} \tilde{\times} \mathbb{Z}_{k}\left(\mathcal{O}^{k}\right)$,
- $\left(H_{n} 2_{*} Z_{d}\right) \tilde{\times} \mathbb{Z}_{k}\left(\mathcal{H}^{d, k}\right)$,
- $H^{\#}\left(\mathcal{H}^{\#}\right)$,
- $\left(B_{n} \tilde{*} \mathbb{Z}_{d}\right) \tilde{\times} \mathbb{Z}_{k}\left(\mathcal{B}^{d, k}\right)$,
- $\left(C_{n} \tilde{{ }^{*}} \mathbb{Z}_{d}\right) \tilde{\times} \mathbb{Z}_{k}\left(\mathcal{B}^{d, k, 0}\right)$,
- $\tilde{C}_{n}^{d} \tilde{\times} \mathbb{Z}_{k}\left(\mathcal{B}^{d, k, d / 2)}\right)$,
- $\left(S_{n}^{+} \tilde{\mathbb{Z}_{d}}\right) \tilde{\times} \mathbb{Z}_{k}\left(\mathcal{S}^{d, k}\right)$

The reader should refer to Chapter 4 for a detailed description of these quantum groups. We will prove the following result:

Theorem 5.6. Let $\left(C\left(G_{n}\right),\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)$ be one of the quantum groups above, with category of partition $\mathcal{C}$. As $n$ goes to $+\infty$, the family $\left(\operatorname{Tr}\left(u^{k}\right)\right)_{k \geq 1}$ converges in moment with respect to $\int_{G_{n}}$ to a family of random variables $\left(u_{k}(\mathcal{C})\right)_{k \geq 1}$. The law of $\left(u_{k}(\mathcal{C})\right)_{k \geq 1}$ depends on $\mathcal{C}$ and is explicitly described in Section 2.2.

The proof of the Theorem is achieved in two parts. The first part is a proof of the convergence in law, which is a generalization of the proof of Theorem 2.5 in [14]. This proof gives also a combinatorial formula for the moments of the limit law. In the second part, this combinatorial formula is used to describe the law of the family $\left(u_{k}(\mathcal{C})\right)_{k \geq 1}$.
For $k>0, \operatorname{Tr}\left(u^{k}\right)^{*}=\operatorname{Tr}\left(u^{-k}\right)$, and thus the goal is therefore to prove the existence of the asymptotic moment

$$
m_{G, k_{1}, \ldots, k_{r}}:=\lim _{n \rightarrow+\infty} \int \operatorname{Tr}\left(u^{k_{1}}\right) \ldots \operatorname{Tr}\left(u^{k_{r}}\right)^{\varepsilon_{r}},
$$

with $k_{1}, \ldots, k_{r}$ non-zero integers. It suffices to only consider the cases without tensor complexification: indeed the Haar state on the tensor complexification $H \tilde{\times} \mathbb{Z}_{k}$ is the classical independence convolution of the Haar state on $(H, u)$ and the one on $\left(\mathbb{Z}_{k}, z\right)$. The law of $\operatorname{Tr}\left((u z)^{r}\right)$ can thus be deduced from the law of $\operatorname{Tr}\left(u^{r}\right)$ with the equality $\operatorname{Tr}\left((u z)^{r}\right) \simeq_{\text {law }} \operatorname{Tr}\left(u^{r}\right) \otimes z^{r}$.

## The asymptotic trace moment formula

We detailed here the generalization of a result of Banica, Curran and Speicher that relates $m_{G, \varepsilon, k_{1}, \ldots, k_{r}}$ with the cardinal of a set of partitions. The result was originally given in the framework of orthogonal easy group; since the statement and the proof for in unitary case are identical, so we present only this latter version.
Let $r \in \mathbb{Z}_{+}$and denote by $\mathbb{Z}_{*}^{r}$ the set of sequences $\vec{j}=\left(j_{1}, \ldots, j_{r}\right)$ of $r$ non-zero integers. To each sequence $\vec{j}$ of $\mathbb{Z}_{*}^{r}$ we associate the following objects:

- A vector $\vec{k}=\left(k_{1}, \ldots, k_{r}\right)$ of positive integers with $k_{s}=\left|j_{s}\right|$. This is the absolute part of $\vec{j}$. We set $k=\sum k_{i}$.
- A word $\varepsilon(\vec{j})$ in $\{0, \bullet\}$ of length $r$ by the condition that $\varepsilon_{s}(\vec{j})=\circ$ if and only if $j_{s}>0$.
- A word $w(\vec{j})$ in $\{\circ, \bullet\}$ of length $k$ such that $w_{s}(\vec{j})=\circ$ if and only if $\sum_{a=1}^{t-1} k_{a}+1 \leq s \leq$ $\sum_{a=1}^{t} k_{a}$ and $\varepsilon(t)=0$.
- A permutation $\gamma_{\vec{j}}$ described by the cycle decomposition

$$
\gamma_{\vec{j}}=\left(1, \ldots, k_{1}\right)^{\varepsilon_{1}(\vec{j})}\left(k_{1}+1, \ldots, k_{1}+k_{2}\right)^{\varepsilon_{2}(\vec{j})} \ldots\left(k-k_{r}+1, \ldots, k\right)^{\varepsilon_{j}(\vec{k})}
$$

with the convention that for a cycle $\tau, \tau^{\circ}=\tau$ and $\tau^{\bullet}=\tau^{-1}$.

- A two colored partition $p_{\vec{j}} \in P(0, w(\vec{j}))$ with $a \sim_{p_{\vec{j}}} b$ if and only if there exists $1 \leq t \leq r$ such that $\sum_{s=1}^{t-1} k_{s}+1 \leq a, b \leq \sum_{s=1}^{t} k_{s}$. From the definition of $w(\vec{j})$ and the one of $p_{\vec{j}}$, the partition $p_{\vec{j}}$ consists in $r$ interval blocks $\left\{B_{1}, \ldots, B_{r}\right\}$ such that $B_{i}$ has cardinal $k_{i}$ and all elements of $B_{i}$ have the color $\varepsilon_{i}(\vec{j})$.

The dependence on $\vec{j}$ of the latter object is omitted when the situation is clear.

Remark 5.7. Any $p \in P(0, w(\vec{j}))$ yields a partition $\tilde{p}$ in $P\left(0, \varepsilon_{\vec{j}}\right)$ by considering the set $\left\{B_{i}\right\}$ of blocks of $p_{\vec{j}}$ with the lexicographical order and the relation $B_{i} \sim_{\tilde{p}} B_{j}$ if $B_{i}$ and $B_{j}$ are in the same block of $p \vee p_{\vec{j}}$. Since $p_{\vec{j}}$ is non crossing, $p$ non-crossing implies that $p \vee p_{\vec{j}}$ is also non-crossing and therefore that $\tilde{p}$ is non-crossing.

For $p \in P^{\bullet \bullet}(k, 0)$ and $\sigma \in S_{k}, \sigma(p)$ denotes the unique partition such that $\sigma(i) \sim_{\sigma(p)} \sigma(j)$ if and only if $i \sim_{p} j$, the colors being also permuted by $\sigma$. For example $\gamma_{\vec{j}}\left(w_{\vec{j}}\right)=w_{\vec{j}}$. Moreover we take the convention that for $x \in C(G), x^{\circ}=x$ and $x^{\bullet}=x^{*}$.
Theorem 2.5 of [14] extends to the unitary case as follows:
Theorem 5.8. Let $G=\left(u_{i j}\right)_{1 \leq i, j \leq n}$ be an easy compact group with $\mathcal{C}$ its associated category of partitions, and $\vec{j} \in \mathbb{Z}_{*}^{r}$. With the same notations as before,

$$
\begin{equation*}
\int \operatorname{Tr}\left(u^{j_{1}}\right) \ldots \operatorname{Tr}\left(u^{j_{r}}\right)=\#\left\{p \in \mathcal{C}\left(w_{\vec{j}}\right) \mid p=\gamma_{\bar{j}}(p)\right\}+O(1 / \sqrt{n}) . \tag{5.2.1}
\end{equation*}
$$

Proof. The proof is a direct computation. Let $I=\int_{G} \operatorname{Tr}\left(u^{j_{1}}\right) \ldots \operatorname{Tr}\left(u^{j_{r}}\right)$. Then

$$
\begin{aligned}
I & =\sum_{1 \leq i_{1}, \ldots, i_{k} \leq n} \int_{G}\left(u_{i_{1} i_{2}} u_{i_{2} i_{3}} \ldots u_{i_{k_{1} i_{1}}}\right)^{\varepsilon_{1}} \ldots\left(u_{i_{k-k_{r}+1} i_{k-k_{r}+2}} \ldots u_{i_{k} i_{k-k_{r}+1}}\right)^{\varepsilon_{r}} \\
& =\sum_{1 \leq i_{1}, \ldots, i_{k} \leq n} \int_{G} u_{i_{1} i_{\gamma(1)}}^{\varepsilon_{1}} \ldots u_{i_{k_{1}} i_{\gamma\left(k_{1}\right)}^{\varepsilon_{1}}}^{\varepsilon_{i}} u_{i_{k_{1}+1} \varepsilon_{\gamma\left(k_{1}+1\right)}}^{\varepsilon_{2}} \ldots u_{i_{k} i_{\gamma(k)}}^{\varepsilon_{r}} .
\end{aligned}
$$

Applying the Weingarten formula to the latter expression yields

$$
\begin{aligned}
I & =\sum_{1 \leq i_{1}, \ldots, i_{k} \leq n}\left(\sum_{\substack{p, q \in \mathcal{C}\left(w_{\vec{J}}\right) \\
\operatorname{ker}\left(\left(i_{j}\right)_{1 \leq j \leq k} \leq p, \operatorname{ker}\left(i_{\gamma(j)}\right)_{1 \leq j \leq k}\right) \leq q}} W g_{G_{n}}(p, q)\right) \\
& =\sum_{1 \leq i_{1} \ldots, i_{k} \leq n} \sum_{\substack{p, q \in \mathcal{C}\left(w_{\vec{j}}\right) \\
\operatorname{ker}\left(\left(i_{j}\right)_{1 \leq j \leq s)} \leq p \vee \gamma(q)\right.}} W g_{G_{n}}(p, q) \\
& =\sum_{p, q \in \mathcal{C}\left(w_{\vec{j}}\right)} n^{b(p \vee(q))} W g_{G_{n}}(p, q),
\end{aligned}
$$

and thanks to the asymptotic formula of Proposition 5.5 we get

$$
I=\sum_{p, q \in \mathcal{C}\left(w_{\vec{j}}\right)} n^{b(p \vee \gamma(q))} n^{b(p \vee q)-b(p)-b(q)}(1+O(1 / \sqrt{n})) .
$$

The coefficient $n^{b(p \vee \gamma(q))+b(p \vee q)-b(p)-b(q)}$ doesn't vanish as $n$ goes to $+\infty$ if and only if $b(p \vee$ $\gamma(q))+b(p \vee q)-b(p)-b(q) \geq 0$. But $b(p \vee q) \leq \min (b(p), b(q))$ and since $b(q)=b(\gamma(q))$, $b(p \vee \gamma(q)) \leq \min (b(p), b(q))$. Therefore the positivity condition requires $b(p \vee q)=b(p), b(p)=$ $b(q)=b(\gamma(q))$, and $b(p \vee \gamma(q))=b(p)$. The first two equalities yield $p=q$ and then the last two equalities yield $p=\gamma(p)$; if the latter conditions are fulliflled then $n^{b(p \vee \gamma(q))+b(p \vee q)-b(p)-b(q)}=1$. Therefore,

$$
I=\#\{p \in \mathcal{C}(w), p=\gamma(p)\}+O(1 / \sqrt{n}) .
$$

Remark 5.9. Note that the latter Theorem actually yields that for any easy quantum groups, without any conditions on the crossings, the family $\left(\operatorname{Tr}(u), \operatorname{Tr}\left(u^{2}\right), \ldots\right)$ converges in law to $a$ random vector $\left(u_{1}, u_{2}, \ldots\right)$.

The remaining part of the method of [14] to describe the law of $\left(u_{1}, u_{2}, \ldots\right)$ doesn't apply here, since most of the categories of partitions considered in the present situation are not block stable (in particular Proposition 3.1 of [14] doesn't hold anymore). Therefore the goal is to better understand the set $\left\{p \in \mathcal{C}\left(w_{\vec{j}}\right), p=\gamma(p)\right\}$. Let us first simplify the condition $p=\gamma(p)$ :

Lemma 5.10. There is an equivalence between the condition $p=\gamma_{\vec{j}}(p)$ and $\gamma_{\vec{j}}(p) \leq p$.
Proof. Clearly $p=\gamma(p)$ yields $\gamma(p) \leq p$. Suppose that $\gamma(p) \leq p$. This means that if $\gamma(i) \sim_{\gamma(p)}$ $\gamma(j)$, then $\gamma(i) \sim_{p} \gamma(j)$. By definition $\gamma(i) \sim_{\gamma(p)} \gamma(j)$ if and only if $i \sim_{p} j$. Therefore if $\gamma(p) \leq p$, then $i \sim_{p} j$ yields $\gamma(i) \sim_{p} \gamma(j)$. Since the permutation group $S_{k}$ is finite, there exists $d$ such that $\gamma^{d}=\gamma^{-1}$. Thus iterating the latter implication yields also that if $i \sim_{p} j$, then $\gamma^{-1}(i) \sim_{p} \gamma^{-1}(j)$. Therefore if $i \sim_{p} j, i=\gamma\left(\gamma^{-1}(i)\right) \sim_{\gamma(p)} \gamma\left(\gamma^{-1}(j)\right)=j$. Thus $p \leq \gamma(p)$, and $p=\gamma(p)$.

In particular checking that $i \sim_{p} j$ implies $\gamma(i) \sim_{p} \gamma(j)$ is sufficient to know whether $p=\gamma(p)$. In the free case, we will see that the condition $p=\gamma(p)$ is very strong.
Let us first define a pairing between two blocks :
Definition 5.11. Let $p$ be a partition of $k$ and $B_{1}, B_{2}$ be two disjoint intervals of $\llbracket 1, k \rrbracket$ corresponding respectively to $\llbracket i_{1}, i_{1}+t \rrbracket$ and $\llbracket i_{2}-t, i_{2} \rrbracket$ with $i_{1}<i_{2}$. We say that $B_{1}$ and $B_{2}$ are block paired by $p$ if, for $0 \leq s \leq t$, the only element of $\llbracket 1, k \rrbracket$ linked to $i_{1}+s$ by $p$ is $i_{2}-s$.

The condition $i_{1}<i_{2}$ ensures that the pairing is non-crossing. For $1 \leq u \leq k$, denote by $B_{u}$ the blocks of $p_{\vec{j}}$ containing $u$. The rigidity of the condition $p=\gamma_{\vec{j}}(p)$ in the free case appears as follows:

Lemma 5.12. Let $\vec{j} \in \mathbb{Z}_{*}^{r}$ and let $p \in N C\left(w_{\vec{j}}\right)$. Then $\gamma_{\vec{j}}(p)=p$ if and only if for all $1 \leq a, b \leq r$ with $B_{a} \sim_{\tilde{p}} B_{b}$, one of the following conditions hold:

- $a=b$ and $B_{a}$ contains only singletons of $p$.
- $B_{a}$ and $B_{b}$ are contained in a same block of $p$.
- $B_{a}$ and $B_{b}$ are block paired. In this case $\left|B_{a}\right|=\left|B_{b}\right|$ and if $B_{a}$ has more than three points, $B_{a}$ and $B_{b}$ have opposite colors.

Proof. Suppose that $p=\gamma(p)$ and let $1 \leq a, b \leq r$ with $B_{a} \sim_{\tilde{p}} B_{b}$. Thus there exists $u \in B_{a}, v \in$ $B_{b}$ with $u \sim_{p} v$.

- Suppose that $a=b$ and $u$ is a singleton, $u=v$. The property $p=\gamma(p)$ yields that all elements of the orbit of $u$ under $\gamma$ are singletons. This orbit is precisely $B_{a}$, which proves the first point.
- Suppose that $a=b$ and $u \neq v$. We can assume without loss of generality that $B_{a}$ is colored $\circ$, with $u<v$. Then there exists $m \geq 0$ such that $v=\gamma^{m}(u)$. If $m= \pm 1\left[k_{a}\right]$, iterating $\gamma$ yields that for all $t \geq 1, \gamma^{t}(u) \sim_{p} \gamma^{t-1}(u) \cdots \sim_{p} u$ and thus $B_{a}$ is in a same block of $p$. Otherwise, $p=\gamma(p)$ implies $\gamma(u) \sim_{p} \gamma^{m+1}(u)$. The block $B_{a}$ is colored $\circ$ and $m \notin\{1,-1\}$, thus $u<\gamma(u)<v$, and $\gamma^{m+1}(u)$ is either greater than $v$ or lower than $u$. Since $p$ is non-crossing, this requires $u \sim_{p} \gamma(u)$ or $u \sim_{p} \gamma^{-1}(u)$. In both cases, iterating the equality $p=\gamma(p)$ yields that for all $t \geq 1, \gamma^{t}(u) \sim_{p} \gamma^{t-1}(u) \sim_{p} \cdots \sim_{p} \gamma(u) \sim_{p} u$. Therefore $B_{a}$ is contained in a block of $p$.
Suppose that $a \neq b$ and that $u, v$ are in a block of $p$ that contains a third element $t$, and let $1 \leq x \leq r$ be such that $B_{x}$ is the block of $p_{\vec{j}}$ containing $t$. After relabelling if necessary, we can suppose that $u<v<t$. If $b=x$, from the previous paragraph, all elements of $B_{b}$ are in a same block of $p$. The equality $p=\gamma(p)$ yields that any element of $B_{a}$ is linked by $p$ to an element of $B_{b}$, and thus $B_{a}$ and $B_{b}=B_{x}$ lie in a same block of $p$.
Since $\gamma(p)=p$, if $B_{a}$ is a singleton then every element in the orbit $B_{b}$ of $b$ is connected to $u$ by $p$, and the same holds for $B_{x}$. Therefore $B_{a}, B_{b}$ and $B_{x}$ are in the same block of $p$. Suppose that $b \neq x$ and $B_{a}$ is not a singleton. The latter conditions imply that $\gamma(u) \neq u$ and $B_{b} \neq B_{x}$. The condition $\gamma(p)=p$ yields that $\gamma(u)$ has to be connected to an element $v^{\prime}$ of $B_{b}$ and $t^{\prime}$ of $B_{x}$. Since any element of $B_{b}$ is lower than the elements of $B_{x}, v^{\prime}<t$, and $v<t^{\prime}$. Since $p$ is non-crossing, if $u<\gamma(u)$, the inequality $u<\gamma(u)<v<t^{\prime}$ yields that $\gamma(u) \sim_{p} u$. For the same reasons, if $\gamma(u)<u$, the inequality $\gamma(u)<u<v^{\prime}<t$ yields $\gamma(u) \sim_{p} u$. Thus iterating the equality $p=\gamma(p)$ yields that for all $m \geq 1, \gamma^{m}(u) \sim_{p} u$, and $B_{a}$ is contained in a block of $p$. Once again by the equality $\gamma(p)=p$, any element of $B_{b}$ or $B_{x}$ is linked to an element of $B_{a}$ through $p$ and $B_{a}, B_{b}, B_{x}$ are in a same block of $p$.
- Suppose that $B_{a} \neq B_{b}$ and that $u$ and $v$ are in a pair. We can assume that $u<v$. Since $\gamma(p)=p, \gamma^{m}(u)$ is paired with $\gamma^{m}(v)$. Thus any element of the orbit of $u$ is paired with an element of the orbit of $v$, and conversely. Therefore necessarily $\left|B_{a}\right|=\left|B_{b}\right|$ and there is a bijective map $\varphi: B_{a} \rightarrow B_{b}$ sending an element $x$ of $B_{a}$ to the unique element $y$ of $B_{b}$ such that $u \sim_{p} v$. Since $p$ is noncrossing, if $x_{1}<x_{2}$ in $B_{a}$ then $\varphi\left(x_{1}\right)>\varphi\left(x_{2}\right)$. Thus the map $\varphi$ is decreasing and this shows that there is a unique way to pair elements of $B_{a}$ with elements of $B_{b}$, which is the block pairing of $B_{a}$ with $B_{b}$. Suppose that $B_{a}$ has more than two elements, and let $x$ be the first element of $B_{a}, y$ the last one of $B_{b}$. By the previous reasoning, $x \sim_{p} y$ and $x+1 \sim_{p} y-1$. Thus if $x+1=\gamma(x)$ then $y-1=\gamma(y)$ and if $x+1=\gamma^{-1}(x)$ then $y-1=\gamma^{-1}(y)$. In any case if $B_{a}$ has more than three elements, the cycles of $\gamma$ have opposite direction on $B_{a}$ and $B_{b}$, and therefore $B_{a}$ and $B_{b}$ have opposite colors.

Conversely let $p$ be a partition which satisfies the conditions of the lemma. From Lemma 5.2 it suffices to prove that for all $1 \leq u \leq v \leq k, u \sim_{p} v$ implies $\gamma(u) \sim_{p} \gamma(v)$. From the conditions of the Lemma, either $B_{a}$ contains only singletons or $B_{a}$ and $B_{b}$ are in the same block of $p$ or $B_{a}$ and $B_{b}$ are block paired. If $B_{a}$ contains only singletons, $u=v$ and $\gamma(u) \sim_{p} \gamma(v)$. Since $B_{a}$ is the orbit of $u$ and $B_{b}$ the orbit of $v$, if $B_{a}$ and $B_{b}$ are in a same block of $p$ then $\gamma(u) \sim_{p} \gamma(v)$. Suppose that $B_{a}$ and $B_{b}$ are block paired and let $c$ be the cardinal of $B_{a}$ (which is also the
one of $B_{b}$ ). There exists $1 \leq i_{1} \leq i_{2} \leq k$ and $s \geq 0$ such that $u=i_{1}+s, v=i_{2}-s$ and for all $0 \leq x \leq c-1 i_{1}+x \sim_{p} i_{2}-x$. If $c=1, B_{a}$ and $B_{b}$ are singletons and therefore $\gamma(u)=u, \gamma(v)=v$ and $\gamma(u) \sim_{p} \gamma(v)$. If $c=2$, whatever the color of $B_{a}$ is, $\gamma(u)=i_{1}+(1-s)$ and $\gamma(v)=i_{2}-(1-s)$; thus once again $\gamma(u) \sim_{p} \gamma(v)$. If $c>3$, from the condition of the Lemma, $B_{a}$ and $B_{b}$ have opposite colors. We can assume without loss of generality that $B_{a}$ is colored $\circ$ and $B_{b}$ is colored $\bullet$. Thus $\gamma(u)=i_{1}+s+1$ and $\gamma(v)=i_{2}-s-1$ (resp. $\gamma(u)=i_{1}$ and $\gamma(v)=i_{2}$ ) if $s<c-1$ (resp. $s=c-1$ ). In any case $\gamma(u) \sim_{p} \gamma(v)$.

For $p$ a partition, let us denote by $\mathcal{B}_{p, 2}$ the set of blocks of $p$ of cardinal less than 2 , and for each $b \in \mathcal{B}_{p, 2}$, let $b_{1}$ and $b_{2}$ be respectively the first and second elements of $b$, with the convention that $b_{1}=b_{2}$ if $b$ is a singleton. For $\vec{j} \in \mathbb{Z}_{*}^{r}$, the set of non-crossing partitions of $r$ with point coloring $c(i)=j_{i}$ is denoted $N C\left(j_{1}, \ldots, j_{r}\right)$. Following the latter Lemma, we introduce the following set of colored partitions :

Definition 5.13. Let $r \geq 1$. A tracial partition is a partition in $N C\left(j_{1}, \ldots, j_{r}\right)$ with an additional block coloring $\star_{p}: \mathcal{B}_{p, 2}: \rightarrow\{1, *\}$ with the condition that for $b \in \mathcal{B}_{p, 2}, \star(b)=1$ if $\min \left(k_{b_{1}}, k_{b_{2}} \mid\right)=1$ or if $b$ is a pair with $\sup \left(k_{b_{1}}, k_{b_{2}}\right)>2$ and $j_{b_{1}}+j_{b_{2}} \neq 0$.

The set of tracial partitions in $N C\left(j_{1}, \ldots, j_{r}\right)$ is denoted $\mathcal{P}(\vec{j})$.
For convenience we extend the pair coloring $\star_{p}$ to a coloring of all the blocks by setting $\star_{p}(b)=1$ if $b$ is a block not belonging to $\mathcal{B}_{p, 2}$.
We define now a map $\Phi: \mathcal{P}_{\vec{j}} \rightarrow P\left(w_{\vec{j}}\right)$ as follows: let $p$ be a tracial partition in $\mathcal{P}(\vec{j})$ and let $1 \leq s \leq t \leq r$ with $s \sim_{p} t$. If $s$ is a singleton colored $*$, all elements of $B_{s}$ are singletons of $\Phi(p)$. If $s$ and $t$ are in a pair $b$ of $p$ colored $*$ then $B_{s}$ is block paired with $B_{t}$ in $\Phi(p)$. In all other cases, all elements of $B_{s}$ and $B_{t}$ are in a same block of $\Phi(p)$.

Proposition 5.14. The map $\Phi$ is injective and $\Phi\left(\mathcal{P}_{\vec{j}}\right)=\left\{p \in \operatorname{NC}\left(w_{\vec{j}}\right), \gamma_{\vec{j}}(p)=p\right\}$. In particular, if $G_{n}$ is an easy unitary free group with associated category of partitions $\mathcal{C}$, then for $n$ large enough

$$
m_{G}(\vec{j})=\# \Phi^{-1}\left(\mathcal{C}\left(w_{\vec{j}}\right)\right)
$$

Proof. Suppose that $p, q$ are two distinct tracial partitions of $\mathcal{P}_{\vec{j}}$. Then either $p, q$ have different block structures, or they have the same block structure and the block colorings differ. Suppose that they have a different block structure : by symmetry we can assume that there exists $1 \leq s_{1}<s_{2} \leq r$ with $s_{1}$ and $s_{2}$ in a same block $b$ of $p$, but $s_{1} \not \chi_{q} s_{2}$. In any case from the construction of $\Phi$, the first element of $B_{s_{1}}$ is linked to the last element of $B_{s_{2}}$ through $\Phi(p)$ but not through $q$. Therefore $\Phi(p) \neq \Phi(q)$.
Suppose that the block structure of $p$ and $q$ is the same but the block colorings differ. By symmetry we can assume that there exists a block $b=\left\{b_{1}, b_{2}\right\}$ such that $\star_{p}(b)=1$ and $\star_{q}(b)=*$. But from the definition of $\Phi$ this means that all elements of $B_{b_{1}}$ are in the same block of $\Phi(p)$, but they are each in distinct blocks of $\Phi(q)$. Therefore $\Phi(p) \neq \Phi(q)$ and $\Phi$ is injective.
By construction, $\Phi(p)$ verifies the conditions of Lemma 5.12 and thus $\gamma(\Phi(p))=p$.
Conversely, if $\gamma(q)=q, q$ verifies the conditions of Lemma 5.12. We construct $p \in \mathcal{P}_{\vec{j}}$ as follows: The block structure of $p$ is $\tilde{q} \in P\left(\varepsilon_{j}\right)$. Recall that the element $i$ of $\tilde{q}$ corresponds to the block $B_{i}$ of $p_{\vec{j}}$. Set $j_{i}=\left|B_{i}\right|$ if $\left(\varepsilon_{\tilde{j}}\right)_{i}=0$, and $j_{i}=-\left|B_{i}\right|$ if $\left(\varepsilon_{\vec{j}}\right)_{i}=\bullet$. Let $b=\left\{b_{1}<b_{2}\right\}$ be a pairing in $\tilde{q}$ with $\min \left(k_{b_{1}}, k_{b_{2}}\right)>1$. Since $\gamma(q)=q$, Lemma 5.12 yields that either $B_{b_{1}}$ and $B_{b_{2}}$ are in a same block of $q$, or $B_{b_{1}}$ and $B_{b_{2}}$ are block paired. In the first case, set $\star(b)=1$ and in the second case set $\star(b)=*$ (note that if $k_{b_{1}}>2$ then Lemma 5.12 enforces $j_{b_{1}}=-j_{b_{2}}$ ). Similarly if $i$ is a singleton of $\tilde{q}$, from Lemma 5.12 either $B_{i}$ contains only singletons, or it is a block of $q$ of cardinal greater than 2 . In the first case set $\star(\{i\})=*$ and the other case set $\star(\{i\})=1$. Set $\star(b)=1$ for all other singletons and pair. This gives a tracial partition $p \in \mathcal{P}(\vec{j})$ such that
$\Phi(p)=q$.
The last part of the proposition is straightforward.
The law of the family $\left(u_{k}(\mathcal{C})_{k \geq 1}\right.$ is therefore given by the cardinal of $\Phi^{-1}\left(\mathcal{C}\left(w_{\vec{j}}\right)\right)$ for various $\vec{j}$.

## Description of the law of $\left(u_{k}(\mathcal{C})\right)_{k \geq 1}$

As for the classification of categories in Chapter 4, we shall deal separately with the five cases $\mathcal{U}, \mathcal{O}, \mathcal{H}, \mathcal{B}$ and $\mathcal{S}$.

Partial partitions Let us first give a combinatorial tool:
Definition 5.15. A partial partition $(p, S)$ of $n$ is a set $S=\left\{1 \leq i_{1}<\cdots<i_{s} \leq n\right\}$ together with a partition $p$ of $S$ (identified with $\llbracket 1, s \rrbracket$ by the natural order on integers).

This is equivalent to the data of a set $\left\{B_{1}, \ldots, B_{r}\right\}$ of disjoint subsets of $\llbracket 1, n \rrbracket$, with $S=\bigcup B_{i}$. The natural order on $S$ yields a canonical bijection between the set of partial permutation $\{(p, S)\}_{p \in P(S)}$ and $P(s)$, and thus the order on $P(s)$ gives an order on the set of partial permutations $\{(p, S)\}_{p \in P(S)}$. $(p, S)$ is called a non-crossing partial partition if $p$ is a non-crossing partition, view as an element of $P(s)$; two partial partitions $(p, S)$ and $\left(p^{\prime}, S^{c}\right)$ yield a partition $(p, S) \vee\left(p^{\prime}, S^{c}\right)$ of $\llbracket 1, n \rrbracket$ simply by considering the reunion of the set of blocks of $p$ and the one of $p^{\prime}$. If $(p, S)$ is a non-crossing partial partition of $S$, the Kreweras complement $k r(p)$ of $p$ is the biggest partial partition $\left(p, S^{c}\right)$ such that $p \vee p^{\prime}$ is non-crossing.

Lemma 5.16. Let $(p, S)$ be a partial partition of $n$. Then $k r(p)$ is the partial partition with support $S^{c}$ defined by

$$
i \sim_{k r(p)} j \Leftrightarrow k \not \chi_{(p, S)} l, \text { for all } k \in \llbracket i, j \rrbracket \cap S, l \in S \backslash \llbracket i, j \rrbracket
$$

Proof. Since $(p, S) \vee\left(k r(p), S^{c}\right)$ is noncrossing the direct implication holds.
Suppose that for all $k \in \llbracket i, j \rrbracket \cap S, l \in S \backslash \llbracket i, j \rrbracket, k \not \chi_{(p, S)} l$; if $\pi$ is any noncrossing partition of $S^{c}$ such that $(p, S) \vee\left(\pi, S^{c}\right)$ is non crossing, then $\tilde{\pi}$ obtained from $\pi$ by the reunion of the block containing $i$ and the one containing $j$ is again noncrossing, and $(p, S) \vee\left(\pi, S^{c}\right)$ is again noncrossing. Thus by maximality of $k r(p), i \sim_{\left(k r(p), S^{c}\right)} j$.

We denote by $N C_{p}(n)$ (resp. $N C_{p}(\vec{j})$ ) the set of non-crossing partial partitions of $n$ (resp. non-crossing partition of $n$ with a coloring $\vec{j}$ on $\llbracket 1, n \rrbracket$ ). The set of non-crossing partial partitions with support $S$ is denoted $N C(S)$.

Case $\mathcal{U}$ There is a unique category of partition in the case $\mathcal{U}$, consisting only in pairings with endpoints of different colors.
Lemma 5.17. Let $\vec{j} \in \mathbb{Z}_{*}^{r}$. $\Phi^{-1}\left(\mathcal{U}\left(w_{\vec{j}}\right)\right)$ is isomorphic to the set of non-crossing pairings in $N C_{2}\left(j_{1}, \ldots, j_{r}\right)$ such that the endpoints of a pair have opposite colors.

Proof. Let $\vec{j} \in \mathbb{Z}_{*}^{r}$ and $p \in \mathcal{P}_{\vec{j}}$ such that $\Phi(p) \in \mathcal{U}\left(w_{\vec{j}}\right)$. If there exists a block $b$ of $p$ such that $\star(b)=1$, this means that all sets $B_{i}$ with $i \in b$ are in a same block of $\Phi(p)$. Since $\mathcal{U}$ consists only in pairings, $\left|\bigcup_{i} B_{i}\right|=\sum_{i \in b} k_{i}=2$. Thus either $b$ is a singleton $\{i\}$ with $k_{i}=2$, or $b$ is a pairing $\left\{i_{1}, i_{2}\right\}$ with $k_{i_{1}}=k_{i_{2}}=1$. The first case is not possible because both elements of $B_{i}$ have the same color, and the second case is possible only if $j_{i_{1}}=-j_{i_{2}}$.
If $\star(b)=*, b$ contains two elements $i_{1}$ and $i_{2}$ with $k_{i_{1}}=k_{i_{2}}$. But $B_{i_{1}}$ and $B_{i_{2}}$ are block paired
through $\Phi(p)$. Thus the condition $\Phi(p) \in \mathcal{U}\left(w_{\vec{j}}\right)$ yields that $j_{i_{1}}=-j_{i_{2}}$ (excluding only the case $j_{i_{1}}=j_{i_{2}}= \pm 2$ ).
Finally $p$ has only pairing $\left\{i_{1}, i_{2}\right\}$ between elements of opposite colors, which are colored $*$ if $k_{i_{1}} \geq 2$ and colored 1 if $k_{i_{1}}=1$.

Thus we can express the law of $\left(u_{1}(\mathcal{U}), u_{2}(\mathcal{U}), \ldots\right)$ as follows :
Proposition 5.18. $\left(u_{1}(\mathcal{U}), u_{2}(\mathcal{U}), \ldots\right)$ is a family of free independent variables, each of them being a circular variable with mean 0 and variance 1.

Proof. Let $\left(c_{1}, c_{2}, \ldots\right)$ be a family of free independent circular variables of variance 1 and mean 0 and set $c_{-i}=c_{i}^{*}$. The moment-cumulant formula shows that for $\vec{j} \in \mathbb{Z}_{*}^{r}, m\left(c_{j_{1}}, \ldots, c_{j_{r}}\right)$ is exactly the number of non-crossing pairings in $N C\left(j_{1}, \ldots, j_{r}\right)$ such that the endpoints of each pair have opposite colors (see also Chapter 1, Prop. 1.10).

Case $\mathcal{O}$ Since we only consider the categories without tensor complexification, there is only one category to study, namely the category $\mathcal{O}^{2}$ of all pairings. This case has already been done in [14]. We give the proof here for the sake completeness.

Lemma 5.19. Let $\vec{j} \in \mathbb{Z}_{*}^{r}$. $\Phi^{-1}\left(\mathcal{O}^{2}\left(w_{\vec{j}}\right)\right)$ is isomorphic to the set of non-crossing partitions of $N C\left(j_{1}, \ldots, j_{r}\right)$ consisting in pairs $b=\left\{i_{1}, i_{2}\right\}$ with either $j_{i_{1}}=-j_{i_{2}}$ or $j_{i_{1}}=j_{i_{2}} \in\{-2,-1,1,2\}$ and singletons $\{i\}$ with $j_{i}= \pm 2$.

Proof. Let $p \in \mathcal{P}_{\vec{j}}$ such that $\Phi(p) \in \mathcal{O}^{2}\left(w_{\vec{j}}\right)$, and let $b$ be a block in $p$. The same reasoning as in the $\mathcal{U}$-case shows that if $\star_{p}(b)=1,\left|\bigcup_{i \in b} B_{i}\right|=2$. Therefore in this case $b$ is a singleton $\{i\}$ with $k_{i}=2$ or a pair $\left\{i_{1}, i_{2}\right\}$ with $j_{i_{1}}, j_{i_{2}} \in\{-1,1\}$.
Any pair $\left\{i_{1}, i_{2}\right\}$ colored $*$ is allowed. For $k_{i_{1}}>2$, a block colored $*$ has endpoints of opposite colors, which concludes the proof.

Thus $\left(u_{1}, u_{2}, \ldots\right)$ is again a family of free (semi-)circular elements, the proof being the same as in the $\mathcal{U}$-case.

Proposition 5.20. $\left(u_{1}, u_{2}, \ldots\right)$ is a family of free variables such that $u_{k}$ is a circular variable with covariance 1 and mean 0 if $k \geq 3$, a semi-circular variable with variance 1 and mean 0 if $k=1$ and a semi-circular variable with variance 1 and mean 1 if $k=2$.

Case $\mathcal{H}$ Let $d \geq 2 . \mathcal{H}^{d, d}$ consists in non-crossing two-colored partitions such that each block has the same numbers of black and white points modulo $d . \mathcal{H}^{\#}$ is the category of non-crossing partitions whose blocks have an even number of elements with endpoints having alternating colors.

Lemma 5.21. Let $d \geq 3 . \Phi^{-1}\left(\mathcal{H}^{d, d}\right)$ is the set of non-crossing partitions of $\mathcal{P}_{\vec{j}}$ such that blocks $b=\left\{i_{1}, \ldots, i_{s}\right\}$ are either pair colored $\star$ with endpoints of opposite colors or block colored 1 with $\sum_{t=1}^{s} j_{i_{t}} \equiv 0[d]$.
If $d=2$, the conditions are the same except that any pair colored $*$ is allowed (which adds only the case where the two endpoints have the color $\pm 2$ ).
$\Phi^{-1}\left(\mathcal{H}^{\#}\right)$ is the set of partitions of $\mathcal{P}_{\vec{j}}$ such that blocks colored 1 have an even number of elements and have endpoints colored $\pm 1$ with alternating signs, and blocks colored $*$ are pairs with opposite colors.

Proof. $\mathcal{H}^{d, d}$ is the category of partitions whose blocks have the same number of white and black points modulo $d$. Let $p \in \mathcal{P}_{\vec{j}}$ such that $\Phi(p) \in \mathcal{H}^{d, d}\left(w_{\vec{j}}\right)$. If $b$ is a singleton of $p$ colored $*$, all the elements of $B_{b}$ are singletons in $\Phi(p)$, which is impossible since $d \geq 2$. If $b=\left\{b_{1}, b_{2}\right\}$ is a pair colored $*, B_{b_{1}}$ is block-paired to $B_{b_{2}}$ in $\Phi(p)$. If $j_{b_{1}}=-j_{b_{2}}$, the pairs between $B_{b_{1}}$ and $B_{b_{2}}$ have endpoints of different colors and thus verify the condition of $\mathcal{H}(d, d)$. If $j_{b_{1}}=j_{b_{2}}$, then $j_{b_{1}}=j_{b_{2}}= \pm 2$ and the pairs between $B_{b_{1}}$ and $B_{b_{2}}$ have endpoints with the same color; this is possible only if $d=2$.
If $b=\left\{i_{1}, \ldots, i_{s}\right\}$ is a block of $p$ colored 1 , then $B=B_{i_{1}} \cup B_{i_{2}} \cup \ldots B_{i_{s}}$ is a block of $\Phi(p)$. The difference between the number of white and black points in $B$ is exactly $\sum_{t=1}^{s} j_{i_{t}}$, thus the latter quantity has to be zero modulo $d$ in order to have $\Phi(p) \in \mathcal{H}^{d, d}$.
Let us consider the category $\mathcal{H}^{+}$and let $p \in \mathcal{P}_{\vec{j}}$ such that $\Phi(p) \in \mathcal{H}^{\#}$. A block $b=\left\{i_{1}, \ldots, i_{s}\right\}$ colored 1 with an endpoint $i_{t}$ such that $k_{i_{t}} \geq 2$ yields in $\Phi(p)$ a block with at least two consecutive endpoints of the same color; therefore, from the description of $\mathcal{H}^{\#}$, a block $b$ of $p$ colored 1 has only endpoints colored $\pm 1$, with two consecutive endpoints having opposite colors. Since any element of $b$ is colored $\pm 1, b$ yields a block with an even number of elements in $\Phi(p)$ if and only if $b$ has an even number of elements. The description of $\mathcal{H}^{\#}$ yields that pairs must also have endpoints with opposite colors. Thus the only blocks colored $*$ allowed are the pairs with endpoints of opposite color.

The description of $\left\{u_{1}(\mathcal{C}), u_{2}(\mathcal{C}) \ldots\right\}$ is the following:
Proposition 5.22. Let $d \geq 2 .\left(u_{i}\left(\mathcal{H}^{d, d}\right)\right)_{i \geq 1}$ has the same joint law as $\left(s z^{i} s+c_{i}\right)_{i \geq 1}$, where :

- $\left(c_{i}\right)_{i \geq 1}$ is a family of free random variables with $c_{1}=0, c_{2}$ a circular variable (resp. semicircular variable) of variance 1 and mean 0 for $d \geq 3$ (resp. $d=2$ ) and $c_{i}$ is a circular variable of variance 1 and mean 0 for $i \geq 3$.
- $s$ and $z$ are two free variables, also free from $\left(c_{i}\right)_{i \geq 1}, s$ being a semi-circular variable of variance 1 and mean 0 and $z$ a uniform variable on the complex d-roots of unity.
$\left(u_{i}\left(\mathcal{H}^{+}\right)\right)_{i \geq 1}$ is distributed as $u_{i}=c_{i}$ for $i \geq 2, u_{1}=p z$, with $c_{k}$ a family of free circular variables of variance 1 and mean $0, p$ a free poisson variable of mean 1 , free from $\left\{c_{k}\right\}_{k \geq 2}$, and $z$ a Bernoulli variable of mean 0 and variance 1 free from $p$ and $\left\{c_{k}\right\}_{k \geq 2}$.

Proof. Let $\left(v_{i}\right)_{i \geq 1}$ be a random vector with $v_{i}=s z^{i} s+c_{i}, s, z$ and $\left(c_{i}\right)_{i \geq 1}$ being as in the statement of the proposition. Let $\left(j_{1}, \ldots, j_{r}\right) \in \mathbb{Z}_{*}^{r}$ and set $Y_{i}=s z^{i} s$. The moment cumulant formula yields

$$
m_{\vec{v}}\left(j_{1}, \ldots, j_{r}\right)=\sum_{(p, S) \in N C_{p}(\vec{j})} k_{p}(\vec{c}) m_{k r(p)}(\vec{Y})
$$

From the law of $\left\{c_{i}\right\}_{i \geq 1}, k_{p}(\vec{c}) \in\{0,1\}$ and $k_{p}(\vec{c})$ is non-zero if and only if $(p, S)$ is a pair partition with each block $\{a, b\}$ satisfying $j_{a} \neq \pm 1$ and such that $j_{a}=-j_{b}$ or, if $d=2, j_{a}=j_{b}= \pm 2$. Let us denote by $\mathcal{P}(S)$ the set of such partial partitions.
For $i_{1}, \ldots, i_{t} \geq 1, m_{r}\left(s z^{i_{1}} s s z^{i_{2}} s \ldots s z^{i_{t}} s\right)=m\left(z^{i_{1}} s^{2} z^{i_{2}} s^{2} \ldots z^{i_{t}} s^{2}\right)$. Since $s^{2}$ is a free Poisson distribution of variance 1 , the moment-cumulant formula yields $m\left(z^{i_{1}} s^{2} z^{i_{2}} s^{2} \ldots z^{i_{t}} s^{2}\right)=\#\{p \in$ $N C\left(i_{1}, \ldots, i_{t}\right), p$ is $d$-balanced $\}$, where $p$ is said $d$-balanced if each block $\left\{x_{1}<\cdots<x_{q}\right\}$ satisfies the condition $\prod_{1 \leq \alpha \leq q} z^{i_{x_{\alpha}}}=1$. The latter condition is equivalent to $\sum_{\alpha=1}^{q} i^{x_{\alpha}}=0[d]$.

Thus

$$
\begin{aligned}
m_{\vec{v}}\left(j_{1}, \ldots, j_{r}\right) & =\sum_{(p, S) \in N C_{p}(\vec{j})} k_{p}(\vec{c}) \#\{\pi \leq k r(p), \pi \text { is } d \text {-balanced }\} \\
& =\#\left\{p \in \mathcal{P}(S), q \in N C\left(S^{c}\right) \mid p \vee q \in N C(\vec{j}), q d \text {-balanced }\right\}
\end{aligned}
$$

The latter set is exactly the set of non-crossing partitions of $N C\left(j_{1}, \ldots, j_{r}\right)$ with blocks either colored 1 (the ones coming from $\left(q, S^{c}\right)$ ) and being $d$-balanced, or colored $*$ (the ones coming from $(p, S)$ ) with the endpoints satisfying the conditions of Lemma 5.21. Therefore

$$
m_{\vec{v}}\left(j_{1}, \ldots, j_{r}\right)=m\left(u_{j_{1}}\left(\mathcal{H}^{d, d}\right), \ldots, u_{j_{r}}\left(\mathcal{H}^{d, d}\right)\right)
$$

The same proof yields the law of $\left\{u_{k}(\mathcal{H})^{\#}\right\}_{k \geq 1}$.

Case $\mathcal{B}$ In the case $\mathcal{B}$ there are three different families, depending on the value of the parameter $r \in\{* ; 0 ; d / 2\}$. If $p \in N C(\vec{j})$ and $b=\left\{i_{1}, i_{2}\right\}$ is a pair of $p$, denote by $c(b)$ the sum of colors between $i_{1}$ and $i_{2}$. Namely $c(b)=\sum_{s=i_{1}+1}^{i_{2}-1} j_{s}$.

Lemma 5.23. $\Phi^{-1}\left(\mathcal{B}^{d, r}\left(w_{\vec{j}}\right)\right)$ is non-empty only if $\sum j_{i}=0[d]$. If $\sum j_{i}=0[d], \Phi^{-1}\left(\mathcal{B}^{d, r}\left(w_{\vec{j}}\right)\right)$ is the set of partitions $p$ in $\mathcal{P}(\vec{j})$ such that:

1. $p$ contains only singletons $\{i\}$ (which are colored $*$ if $k_{i}>2$ ), pairs colored $*$ or pairs with endpoints colored $\pm 1$.
2. If $b$ is a pair of $p$ with endpoints of opposite colors, $c(b) \equiv 0[d]$.
3. Let $r \in\{0, d / 2\}$. If $b$ is a pair of $p$ with both endpoints colored either 1 or 2 (resp. -1 or -2 ), then $c(b) \equiv r-1[d]$ (resp. $c(b)=r+1[d]$ ). If $r \neq 1[d]$ (in particular $d \notin\{1,2\}$ ), $p$ has no singleton $\{i\}$ colored 1 with $k_{i}=2$. If $d \notin\{1,2\}, p$ has no pair $\left\{b_{1}, b_{2}\right\}$ with $j_{b_{1}}=j_{b_{2}}= \pm 2$.
4. if $r=*$, all pairs have endpoints of opposite color and $p$ has no singleton $\{i\}$ with $k_{i}=2$.

Recall that since $p \in \mathcal{P}(\vec{j})$, a pair $\left\{b_{1}, b_{2}\right\}$ colored $*$ has endpoints of the same color only if $k_{b_{1}}=2$.

Proof. From Chapter $4, \mathcal{B}^{d, r}\left(w_{\vec{j}}\right)$ is non-empty only if $\sum w_{\vec{j}}(s)=\sum j_{i}=0[d]$. If $\mathcal{B}^{d, r}\left(w_{\vec{j}}\right)$ is non-empty, it consists in partitions $q$ having the following properties:

1. $q$ contains only singletons and pairs.
2. For any pair $b$ of $q$ having endpoints of opposite colors, $c(b) \equiv 0[d]$.
3. If $r \in\{0, d / 2\}$, and $b$ is a pair having black endpoints (resp. white endpoints), $c(b) \equiv$ $r+1[d]($ resp. $c(b) \equiv r-1[d])$.
4. If $r=*$, any pair of $q$ has endpoints of opposite colors.

Suppose that $\sum j_{i}=0[d]$. We shall characterize the set of partitions $p$ of $\mathcal{P}_{\vec{j}}$ such that $\Phi(p)$ satisfies condition 1) -4 ).

1. From the definition of $\Phi$, a block $b$ of $p$ yields a block $B$ in $\Phi(p)$ having more than three elements if and only $b=\left\{i_{1}, \ldots, i_{s}\right\}$ is a block colored 1 such that $\left|\cup B_{i_{s}}\right|>2$. The only blocks colored 1 with $\left|\bigcup B_{i_{s}}\right| \leq 2$ are the singleton $\{i\}$ with $k_{i} \leq 2$ and the pairs $\left\{b_{1}, b_{2}\right\}$ with $k_{b_{1}}=k_{b_{2}}=1$. Thus condition 1) on $\Phi(p)$ is equivalent to the first condition of the Lemma on $p$. From now on by a pair $\left\{b_{1}, b_{2}\right\}$ of $p$, we mean a pair colored 1 if $k_{b_{1}}=1$ and colored $*$ otherwise: in any case this pair yields a block pairing in $\Phi(p)$.
2. An element of $p$ colored $j_{i}$ expands in $k_{i}$ singletons of color $\varepsilon_{i}$ in $\Phi(p)$. Moreover a pair $b=\left\{b_{1}, b_{2}\right\}$ of $p$ yields a block pairing in $\Phi(p)$ : if the endpoints of $b$ have opposite colors, each pair of the block pairing has also endpoints of opposite colors. Therefore the condition $c(\tilde{b}) \equiv 0[d]$ for all pairs $\tilde{b}$ of the block pairing between $B_{b_{1}}$ and $B_{b_{2}}$ is equivalent to the condition $c(b) \equiv 0[d]$.
3. Let $r \in\{0, d / 2\}$. The only blocks of $p$ yielding pairs with endpoints of the same colors in $\Phi(p)$ are the pairs $\left\{b_{1}, b_{2}\right\}$ with $j_{b_{1}}=j_{b_{2}} \in \llbracket-2,2 \rrbracket$ and the singletons $\{i\}$ colored 1 (which implies $k_{i}=2$ from the first point).
A pair $b=\left\{b_{1}, b_{2}\right\}$ with $j_{b_{1}}=j_{b_{2}}= \pm 2$ yields a block pairing in $\Phi(p)$ consisting in two pairs $\left\{a_{1}, a_{2}\right\}$ and $\left\{a_{1}+1, a_{2}-1\right\}$. Let us assume without loss of generality that $j_{b_{1}}=2$. Then $a_{1}, a_{1}+1, a_{2}-1, a_{2}$ are colored $\circ$. Since $\left\{a_{1}, a_{2}\right\}$ is a pair with endpoints colored $\circ$, the third condition of $\mathcal{B}^{d, r}$ yields that $\sum_{i=a_{1}+1}^{a_{2}-1} w(i)=r-1[d]$. But $\left\{a_{1}+1, a_{2}-1\right\}$ is also a pair with endpoints colored $\circ$. Therefore $\sum_{i=a_{1}+2}^{a_{2}-2} w(i)=r-1[d]$. This implies that $w_{a_{1}+1}+w_{a_{2}-1}=0[d]$. Since $a_{1}+1$ and $a_{2}-1$ are colored $\circ, w_{a_{1}+1}+w_{a_{2}-1}=2$. Thus $d$ has to be equal to 1 or 2 . Since $\sum_{i=a_{1}+2}^{a_{2}-2} w(i)=\sum_{i=b_{1}+1}^{b_{2-1}} j_{i}$, the condition $\sum_{i=a_{1}+2}^{a_{2}-2} w(i)=r[d]$ is equivalent to the condition $c(b)=r[d]$.
A singleton $\{i\}$ colored 1 with $k_{i}=2$ yields a pair $\tilde{b}=\left\{b_{1}, b_{1}+1\right\}$ with endpoints of the same color in $\Phi(p) . c(\tilde{b})=0$ means that $r+1=0[d]$ or $r-1=0[d]$. Since $r \in\{0, d / 2\}$, either $d=1$, or $r=1$ and $d=2$. In these two cases we have also $c(b)=0=r+1[d]$.
In any other cases, $b=\left\{b_{1}, b_{2}\right\}$ has endpoints of the same color if $c\left(b_{1}\right)=c\left(b_{2}\right)= \pm 1$. Let us assume that $c\left(b_{1}\right)=1$. Thus $b$ yields a pair $\tilde{b}=\left\{a_{1}, a_{2}\right\}$ in $\Phi(p)$ with endpoints colored $\circ$. Since $\sum_{a_{1}+1}^{a_{2}-1} w(i)=\sum_{i=b_{1}+1}^{i=b_{2}-1} j_{i}$, the condition $c(\tilde{b})=r-1[d]$ is equivalent to $c(b)=r-1[d]$.
4. A block $b$ of $p$ yields in $\Phi(p)$ pairs having endpoints of different colors if and only if $b$ is a pair having endpoints of opposite colors: thus if $r=*$, then $\Phi(p)$ satisfies the fourth property of $\mathcal{B}^{d, *}$ if and only if all pairs of $p$ have endpoints of opposite colors.

The latter lemma yields the law of $\left\{u_{i}\left(\mathcal{B}^{d, r}\right)\right\}$ :
Proposition 5.24. 1. Let $r=*$. The family $\left\{u_{i}\left(\mathcal{B}^{d, *}\right)\right\}_{i \geq 1}$ is distributed as $u_{i}=c_{i}+z^{i}$, with $\left\{c_{i}\right\}_{i \geq 1}$ a family of circular variables of mean 0 and variance 1 , and $z$ a variable free from $\left\{c_{i}\right\}_{i \geq 1}$ and distributed uniformly on the d-th roots of unity.
2. Let $r=0, d \geq 3$. The family $\left\{u_{i}\left(\mathcal{B}^{d, 0}\right)\right\}_{i \geq 1}$ is distributed as $u_{i}=c_{i}+z^{i}$ for $i \geq 2, u_{1}=c_{1} z$, where $\left\{c_{i}\right\}_{i \geq 2}$ is a family of circular variables of mean 0 and variance $1, z$ is a variable free from $\left\{c_{i}\right\}_{i \geq 2}$ and distributed uniformly on the $d$-th roots of unity, and $c_{1}$ is a free semi-circular variable of variance 1 and mean 0 . If $d=2$, the distribution is the same except that $u_{2}=c_{2} z+1$ with $c_{2}$ a semi-circular variable of variance 1 and mean 0 , free from the other variables.
3. Let $r=d / 2$. The family $\left\{u_{i}\left(\mathcal{B}^{d, d / 2}\right)\right\}_{i \geq 1}$ is distributed as $u_{i}=c_{i}+z^{i}$ for $i \geq 2$ and $u_{1}=c_{1} z^{d / 2+1}+z$, with $\left\{c_{i}\right\}_{i \geq 2}$ a family of circular variables of mean 0 and variance $1, z$ a variable free from $\left\{c_{i}\right\}_{i \geq 1}$ and distributed uniformly on the $d-$ th roots of unity, and $c_{1}$ a free semi-circular variable of mean 0 and variance 1 . If $d \in\{1,2\}$, the law is the same except that $u_{2}=c_{2}+1$, with $c_{2}$ is a semi-circular variable of mean 1 and variance 1 free from the other variables.

Proof. Let $r \in\{*, 0, d / 2\}$, and let $\left\{t_{k}\right\}_{k \geq 1}$ be a family of variables with the expected law. Set $t_{-k}=t_{k}^{*}, c_{-k}=c_{k}^{*}$. Thus we have to prove that the mixed moment $m\left(t_{j_{1}}, \ldots, t_{j_{s}}\right)$ is exactly the number of partitions of $\mathcal{P}_{\vec{j}}$ satisfying the conditions of Lemma 5.23 . On one hand, expanding the product in $t_{j_{1}} \ldots t_{j_{s}}$ yields:

$$
m\left(t_{j_{1}}, \ldots, t_{j_{s}}\right)=\sum_{S \subseteq \llbracket 1, s \rrbracket} m(\vec{\omega}),
$$

where $\omega_{i}=z^{j_{i}}$ if $i \notin S$ and $\omega_{i}=c_{j_{i}}$ if $i \in S$ (or $c_{j_{i}} z^{r+1}$ if $t_{j_{i}}=c_{j_{i}} z^{r+1}+z^{j_{i}}, z^{r-1} c_{j_{i}}$ if $\left.t_{j_{i}}=z^{r-1} c_{j_{i}}+z^{j_{i}}\right)$.
If $r=*$, we formally set $r+1=r-1=0$. Let us define a sequence $\tilde{\omega}$ of length $3 s$ and a coloring $\vec{h}$ of $\llbracket 1,3 s \rrbracket$ from $\omega$ as follows : for $1 \leq i \leq s$,

- $\tilde{\omega}(3 i-1)=c_{j_{i}}$ and $h(3 i-1)=0$ if $i \in S, \tilde{\omega}(3 i-1)=z^{j_{i}}$ and $h(3 i-1)=j_{i}$ else,
- $\tilde{\omega}(3 i-2)=z^{r-1}$ and $h(3 i-2)=r-1$ if $t_{i}=c_{-1}\left(\right.$ or $t_{i}=c_{-2}$ and $\left.d \in\{1,2\}\right), \tilde{\omega}(3 i-2)=1$ and $h(3 i-2)=0$ else,
- $\tilde{\omega}(3 i)=z^{r+1}$ and $h(3 i)=r+1$ if $t_{i}=c_{1}$ (or $t_{i}=c_{2}$ and $d \in\{1,2\}$ ), $\tilde{\omega}(3 i)=1$ and $h(3 i)=0$ else.

Let $\tilde{S}$ be the subset $\left\{3 i-1 \mid 1 \leq i \leq s, t_{i}=c_{j_{i}}\right\}$ of $\llbracket 1,3 s \rrbracket$. Since $\left\{c_{i}\right\}$ and $z$ are free, the moment-cumulant formula yields

$$
m(\vec{\omega})=\sum_{(\tilde{p}, \tilde{S})} k_{(\tilde{p}, \tilde{S})}(\tilde{\omega}) m_{k r(\tilde{p}, \tilde{S})}(\omega)
$$

where $\tilde{p}$ is a non-crossing partial partition whose support is $\tilde{S}$. Since $S \simeq \tilde{S}$, there is a bijection $p \mapsto \tilde{p}$ between partial partitions of $\llbracket 1, s \rrbracket$ with support $S$ and partial partitions of $\llbracket 1,3 s \rrbracket$ with support $\tilde{S}$, and thus

$$
m(\vec{\omega})=\sum_{p \in N C(S)} k_{(\tilde{p}, \tilde{S})}(\tilde{\omega}) m_{k r(\tilde{p}, \tilde{S})}(\omega)
$$

The elements outside $\tilde{S}$ consist only in powers of $z$, thus a block $B$ of $\operatorname{kr}(\tilde{p})$ yields a moment 1 if $\prod_{i \in B} \tilde{\omega}(i)=1$ (namely if $\sum_{i \in B} h(i)=0[d]$ ), and 0 else. Since $\left\{c_{i}\right\}$ is a collection of free semi-circular or circular variables, $k(\tilde{p}) \in\{0,1\}$ and $k(\tilde{p})$ is zero if $\tilde{p}$ has blocks with more than three elements or blocks containing different $c_{i}$. Thus

$$
m\left(t_{1}, \ldots, t_{s}\right)=\#\left\{(p, S) \mid k_{(\tilde{p}, \tilde{S})}(\tilde{\omega})=1, \forall B \in k r(\tilde{p}), \sum_{i \in B} h_{i}=0[d]\right\}
$$

The condition $\forall B \in k r(\tilde{p}), \sum_{i \in B} h(i)=0[d]$ is equivalent to the condition that $\sum h(i)=0[d]$ and that for any pair $\tilde{b}=\left\{\tilde{b}_{1}<\tilde{b}_{2}\right\}$ in $\tilde{p}, c(\tilde{b})=0[d]$. Moreover

- If $\tilde{b}_{1}$ and $\tilde{b}_{2}$ have opposite colors, then $h\left(\tilde{b}_{1}+1\right)=-h\left(\tilde{b}_{2}-1\right)^{-1}$. Thus $c(\tilde{b})=0[d]$ if and only if $\sum_{\tilde{b}_{1}+1<i<\tilde{b}_{2}-1, i \in \tilde{S}^{c}} h(i)=0[d]$.
- If $\tilde{b}_{1}$ and $\tilde{b}_{2}$ are colored white (resp. black), $h\left(\tilde{b}_{1}+1\right)=r+1$ and $h\left(\tilde{b}_{2}-1\right)=1$ (resp. $h\left(\tilde{b}_{1}+\right.$ $1)=1$ and $\left.h\left(\tilde{b}_{2}-1\right)=r-1\right)$. Thus $c(\tilde{b})=0[d]$ is equivalent to $\sum_{\tilde{b}_{1}+1<i<\tilde{b}_{2}-1, i \in \tilde{S}^{c}} h(i)=r-1$ if $\tilde{b}_{1}$ and $\tilde{b}_{2}$ are white and to $\sum_{\tilde{b}_{1}+1<i<\tilde{b}_{2}-1, i \in \tilde{S}^{c}} h(i)=r+1$ if $\tilde{b}_{1}$ and $\tilde{b}_{2}$ are black.

It remains to show that $\sum_{\tilde{b}_{1}+1<i<\tilde{b}_{2}-1, i \in \tilde{S}^{c}} h(i)=\sum_{b_{1}<i<b_{2}} j_{i}$, where $\left\{b_{1}, b_{2}\right\}$ is the pair of $p$ yielding $\tilde{b}$ in $\tilde{p}$. An element $i$ in $S^{c}$ colored $j_{i}$ yields in $(\tilde{p}, \tilde{S})$ three elements $3 i, 3 i+1,3 i+2$ colored respectively $0, j_{i}, 0$ : so the total contribution is $j_{i}$. An element $i$ is a singleton of $(p, S)$ only if $j_{i}= \pm 2$ and $r=1, d \in\{1,2\}$. It contributes in $\tilde{p}$ to three elements $3 i, 3 i+1,3 i+2$ colored 0 , whose sum is exactly $j_{i}$ modulo 1 or 2 .
A pair $\left\{b_{1}, b_{2}\right\}$ of $(p, S)$ with endpoints of opposite colors contributes to one pair, and four singletons, two of them colored 0 , and two having opposite colors, yielding a null contribution. A pair $\left\{b_{1}, b_{2}\right\}$ with endpoints of the same color only occurs if $j_{b_{1}} \in\{ \pm 1, \pm 2\}$ and $r \in\{0, d / 2\}$. If $j_{b_{1}}>0$, this pair contributes in $\tilde{p}$ to one pair and four singletons, two of them colored 0 and two of them colored $r+1$. Thus the total contribution is $2 r+2$; but $r \in\{0, d / 2\}$ and thus $2 r+2=2[d]$. If $j_{b_{1}}=<0$, the same reasoning yields a contribution equal to -2 modulo $d$. If $j_{b_{1}}= \pm 1$, this contribution is equal to $j_{b_{1}}+j_{b_{2}}$. If $j_{b_{1}}= \pm 2, d \in\{1,2\}$ and once again the contribution is equal to $j_{b_{1}}+j_{2}$.
To sum up, $(\tilde{p}, \tilde{S})$ verifies the conditions if and only if $\sum j_{i}=0$ and $(p, S)$ is such that $c(b)=0[d]$ if $b$ is a pair with endpoints of different colors and $c(b)=r-1[d]$ (resp. $c(b)=r+1[d])$ if $b$ is a pair with white (resp. black) endpoints. Filling $(p, S)$ with singletons on elements of $S^{c}$ gives exactly the partitions satisfying the conditions of Lemma 5.23 .

Case $\mathcal{S}$ The computation for the category $\mathcal{S}^{d}$ is a simpler version of the computation for the category $\mathcal{B}^{d, r}$. For $\left(c_{1}, \ldots, c_{r}\right)$ a sequence of integers (for example an element of $\mathbb{Z}_{*}^{r}$ or a word in $\circ$, $\bullet$ with the usual substitution $\circ \leftrightarrow 1, \bullet \leftrightarrow-1$ ), we set $c(s, t)=\sum_{s<x<t} c_{x}$.

Lemma 5.25. $\Phi^{-1}\left(\mathcal{S}^{d}\left(w_{\vec{j}}\right)\right)$ is the set of partitions $p$ in $\mathcal{P}(\vec{j})$ with:

1. If $d \geq 3$, blocks of $p$ colored 1 contain only elements colored $\pm 1$ and blocks colored $*$ have endpoints of opposite colors. If $d=2$ the same holds except that all pairs colored $*$ are allowed.
2. If $i_{1}, i_{2}$ are two consecutive elements of the same block and have positive (resp. negative) color, $j\left(i_{1}, i_{2}\right) \equiv-1[d]($ resp. +1$)$. If $i_{1} \sim_{p} i_{2}$ and $j_{i_{1}}=-j_{i_{2}}$, then $j\left(i_{1}, i_{2}\right) \equiv 0[d]$.

Proof. The category $\mathcal{S}^{d}\left(w_{\vec{j}}\right)$ is the set of non-crossing partitions $p$ having the following property: for $i_{1}$ and $i_{2}$ two consecutive elements of a block, $w_{\vec{j}}\left(i_{1}, i_{2}\right) \equiv 0[d]$ (resp. $w_{\vec{j}}\left(i_{1}, i_{2}\right) \equiv-1[d]$, resp. $w_{\vec{j}}\left(i_{1}, i_{2}\right) \equiv 1[d]$ ) if $\varepsilon\left(i_{1}\right)=-\varepsilon\left(i_{2}\right)$ (resp. $\left.\varepsilon\left(i_{1}\right)=\varepsilon\left(i_{2}\right)=1, \varepsilon\left(i_{1}\right)=\varepsilon\left(i_{2}\right)=-1\right)$. If $d=1$, this just means that any non-crossing partition is allowed, and thus $\Phi^{-1}\left(\mathcal{S}^{1}\left(w_{\vec{j}}\right)\right)=\Phi^{-1}(N C(\vec{j}))=\mathcal{P}(\vec{j})$. If $d \geq 2$, the same proof as the one of Lemma 5.23 yields the result.

The proof of the law of $\left(u_{i}\left(\mathcal{S}^{d}\right)\right)_{i \geq 1}$ is similar to the one of Proposition 5.24 , so we only state the result:

Proposition 5.26. Let $d \geq 3$. The family $\left(u_{i}\left(\mathcal{S}^{d}\right)\right)_{i \geq 1}$ is distributed as $u_{i}=c_{i}+z^{i}$ for $i \geq 2$, with $\left\{c_{i}\right\}_{i \geq 2}$ a family of free circular variables of variance 1 and mean 0 and $u_{1}=p z$ with $p$ a free Poisson distribution of mean 1 free from $\left\{c_{i}\right\}_{i \geq 2}$ and $z$ a uniform variable on the $d$-roots of unity, free from $p$ and $\left\{c_{i}\right\}_{i \geq 2}$.
If $d=2$, the law is the same as before, except that $u_{2}=c_{2} z+1$ with $c_{2}$ a semi-circular variable of variance 1 and mean 0 .
If $d=1$, for all $i \geq 1, u_{i}=p+c_{i}$, with $p$ a free poisson variable of mean $1,\left\{c_{i}\right\}_{i \geq 3}$ a family
of free circular variables withs mean 0 and variance $1, c_{2}$ a semi-circular variable with mean 1 and variance 1 and $c_{1}=0$.

### 5.3 Second-order freeness for the free unitary group

In [62], Cor. 15, the results of Diaconis and Shahshahani on the unitary group have been extended to the situation where traces of arbitrary cyclically reduced products of $u, \bar{u}, u^{t}, u^{*}$ were considered (see also [69]). We give here a similar result for the free unitary group. The method used in the present chapter differs from the one of 62] in order to take the non-commutativity of $U_{n}^{+}$into account. However both rely on the Weingarten formula, and the proof of the present section can be used to recover the results of 62]. In the case of $U_{n}^{+}$, the non-commutativity allows to consider the law of any reduced words in $u, \bar{u}, u^{t}, u^{*}$, instead of only considering cyclically reduced words. We present the result in the classical and free case.
To state the result and detail the proof, we shall use the notations introduced by Radulescu in [69]. Let $\mathbb{F}_{2}$ be the free group with two generators $a$ and $b$. Any element $f$ of $\mathbb{F}_{2}$ admits a unique representation as a reduced word $f_{1} f_{2} \ldots f_{r}$ in $a, b, a^{-1}$ and $b^{-1} . f$ is called cyclically reduced if $f_{r} \neq f_{1}^{-1}$.
For $n \geq 1$, we associate to each $f \in F_{2}$ the variable $X_{f}(n)=\operatorname{Tr}\left(u^{f_{1}} \ldots u^{f_{r}}\right)$, where $u$ is the fundamental matrix of $U_{n}^{+}$or $U_{n}$ and $u^{a}=u, u^{a^{-1}}=u^{*}, u^{b}=\bar{u}, u^{b^{-1}}=u^{t}$.
With these notations we have a main theorem describing the asymptotice laws of $\left\{X_{f}(n)\right\}_{f \in \mathbb{F}_{2}}$ for $U_{n}^{+}$and $U_{n}$. From now on, we always assume that elements of $\mathbb{F}_{2}$ are cyclically reduced in the classical case.

Theorem 5.27. In the free case (resp. classical case), when $n$ tends to infinity, the collection of variables $\left\{X_{f}(n)\right\}_{\tilde{X}}$ converges in law to a circular (resp. gaussian) system of free (resp. independent) variables $\left\{\tilde{X}_{f}\right\}$ whose covariance matrix can be explicitly described.

The description of the covariance matrix is given in Proposition 5.36. Throughout this section, $r \geq 1$ is fixed, and we use the convention $r+1:=1$.

### 5.3.1 Bidiagrams associated to $X_{f}(n)$

The proof of Theorem 5.27 relies on some properties of a combinatorial object called bidiagram. We present here this object and prove some useful combinatorial results.

Bidiagrams Let $k \geq 1$. We define an involution $i \mapsto \bar{i}$ on $\{1, \ldots, 2 k\}$ by the formula $\bar{i}=2 k-i+1$.
We define $p_{k}$ as the partition of $P(k, k)$ defined by $p_{k}=\{\{i, \bar{i}\}, 1 \leq i \leq k\}$ and for $1 \leq i \leq k$, we denote by $S_{i}$ the subset $\{i, \bar{i}\}$.

Definition 5.28. Let $k \geq 1$. A cyclic partition $p$ of $P(k, k)$ is a two level pair partition such that $i$ is paired either with $i+1, i-1, \overline{i-1}$ or $\overline{i+1}$, and such that $p \vee_{P} p_{k}$ is the one block partition $\{\llbracket 1,2 k \rrbracket\}$.
A tracial diagram $\mathcal{D} \in P(k, k)$ is a tensor product of cyclic partitions $p_{1} \otimes \cdots \otimes p_{r}$ such that $p_{i} \in P\left(k_{i}, k_{i}\right)$ with $\sum k_{i}=k$.

We write $\mathcal{D}=p_{1} \otimes \cdots \otimes p_{r}$ to emphasize the unique decomposition into cyclic partitions. The conditions in the definition of a cyclic partition is equivalent to the fact that for any $1 \leq i \leq k$,
$p$ links exactly one element of $S_{i}$ with an element of $S_{i+1}$, where $i+1$ is understood modulo $k$. The same definition holds for colored cyclic partitions and colored tracial diagrams. A twocolored tracial diagram is represented in Figure 5.1.


Figure 5.1: Tracial diagram consisting in two cyclic partitions
We can now define a bidiagram :
Definition 5.29. Let $\varepsilon_{1}, \varepsilon_{2}$ be two words in $\circ$, . A bidiagram is the data of a tracial diagram $\mathcal{D} \in P\left(\varepsilon_{1}, \varepsilon_{2}\right)$ and two partitions $p, q$ such that $p \in P\left(\varepsilon_{1}\right)$ and $q \in P\left(\varepsilon_{2}\right)$.

A bidiagram is written $(p|\mathcal{D}| q)$. An example is shown in Figure 5.2,


Figure 5.2: Two-colored bidiagram
The natural bijection between $P\left(\varepsilon_{1}\right)$ and $P\left(\varepsilon_{1}, \emptyset\right)$, and between $P\left(\varepsilon_{2}\right)$ and $P\left(\varepsilon_{2}, \emptyset\right)$ identifies $(p, q)$ with an element of $P\left(\varepsilon_{1}, \varepsilon_{2}\right)$ through the composition $p R_{h}(q)$ (recall the definition of $R_{h}$ in Chapter 1). In particular the blocks of $b(p|\mathcal{D}| q)$ are defined as the blocks of the partition $\mathcal{D} \vee p R_{h}(q)$. We denote also by $b(p|\mathcal{D}| q)$ the number of blocks of $\mathcal{D} \vee p R_{h}(q)$.

Diagrams coming from $\left\{X_{f}(n)\right\}$ : A cyclic partition $\mathcal{D}_{f}$, independent of $n$, is associated to each variable $X_{f}(n)$ in the following way :

- Let $f \in \mathbb{F}_{2}$. We can write the reduced form of $f$ as

$$
f=x_{\eta(1)}^{\varepsilon(1)} x_{\eta(2)}^{\varepsilon(2)} \ldots x_{\eta(r)}^{\varepsilon(r)}=\prod_{\rightarrow} x_{\eta(i)}^{\varepsilon(i)}
$$

where $r$ is the length of the reduced word, $\prod_{\rightarrow}$ stands as an ordered product, and $x_{\eta_{i}}^{\varepsilon_{i}} \in$ $\left\{a, b, a^{-1}, b^{-1}\right\}$ according to the following rule :

| $\eta \backslash \varepsilon$ | 1 | -1 |
| :---: | :---: | :---: |
| 1 | $a$ | $b$ |
| -1 | $b^{-1}$ | $a^{-1}$ |

The fact that the above expression comes from a reduced word of $f$ implies that $\varepsilon_{i} \varepsilon_{i+1}+$ $\eta_{i} \eta_{i+1} \neq-2$ for $1 \leq i \leq r-1$. If $f$ is cyclically reduced, the same holds for $i=r$. The words $\eta(1) \ldots \eta(r)$ and $\varepsilon(1) \ldots \varepsilon(r)$ are denoted $\eta(f)$ and $\varepsilon(f)$.

- The partition $\mathcal{D}_{f}$ in $P(\varepsilon, \varepsilon)$ is constructed by pairing, for $1 \leq i \leq r$, exactly one element of $S_{i}$ with one element of $S_{i+1}$ with the rule:

| $\eta(i+1) \backslash \eta(i)$ | 1 | -1 |
| :---: | :---: | :---: |
| 1 | $(i, \bar{i}+1)$ | $(\bar{i}, i+1)$ |
| -1 | $(i, i+1)$ | $(\bar{i}, \bar{i}+1)$ |

For example the cyclic partition associated to $a b^{2} a^{-2} b$ is drawn in Figure 5.3:


Figure 5.3: The cyclic partition $\mathcal{D}_{a b^{2} a^{-2} b}$
The rules (5.3.1) and (5.3.2) yield that the partition $\mathcal{D}_{f}$ is indeed a cyclic partition. Thus if $f^{1}, \ldots, f^{s} \in \mathbb{F}_{2}, \mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{s}}$ is a tracial diagram. Actually one can prove that any tracial diagram such that the coloring is the same on each set $S_{i}$ can be written $\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{s}}$ for some $f^{1}, \ldots, f^{s} \in \mathbb{F}_{2}$.
We will need some basic facts on tracial diagrams. They are summed up in the following Lemma :

Lemma 5.30. Let $f=f_{1} \ldots f_{r} \in \mathbb{F}_{2}$ and let $D_{f}$ be the associated cyclic partition:

- For $1 \leq i \leq r-1$ two consecutive elements $i$ and $i+1$ (or $\bar{i}$ and $\overline{i+1}$ ) can be paired only if they have the same color. If $f$ is cyclically reduced, 1 is paired with $r$ (resp. $\overline{1}$ is paired with $\bar{r}$ ) only if both elements have the same color.
- There is the same number of pairs between elements of $\{1, \ldots, r\}$ and between elements of $\{\overline{1}, \ldots, \bar{r}\}$.

Proof. - Since $f$ is a reduced word, the rule (5.3.1) yields that for $1 \leq i \leq r-1, \eta(i) \eta(i+1)$ and $\varepsilon(i) \varepsilon(i+1)$ are not both equal to -1 . Therefore by (5.3.2), this means that two consecutive points of different colors cannot be paired. If $f$ is cyclically reduced, the same is truc for $i=r$ and $i+1=1$ ( or $\bar{r}$ and $\overline{1}$ ).

- By the rule (5.3.2), $\bar{i}$ is paired with $\overline{i+1}$ if and only if $\eta(i+1)=1$ and $\eta(i)=-1$, yielding $\eta(i+1)-\eta(i)=2 ; i$ is paired with $i+1$ if and only if $\eta(i+1)=-1$ and $\eta(i)=1$, yielding $\eta(i+1)-\eta(i)=-2$. In all other cases, $\eta(i)=\eta(i+1)$ and thus $\eta(i)-\eta(i+1)=0$. Since $\sum_{i=1}^{r} \eta(i+1)-\eta(i)=0$, the number of pairs inside $\{1, \ldots, r\}$ and inside $\{\overline{1}, \ldots, \bar{r}\}$ are the same.

A pair containing two points of the same row is called an horizontal strip. The other ones are called vertical strips.

Bidiagrams with pair partitions Since the intertwiners of $U_{n}^{+}$and $U_{n}$ are described by pair partitions, we will use bidiagrams $(p|\mathcal{D}| q)$ with $p, q$ being pairings; as in the previous section, the use of the Weingarten formula requires bounds on the number of blocks of $(p|\mathcal{D}| q)$. Recall that $\mathcal{U}$ (resp. $\mathcal{U}_{\text {class }}$ ) denotes the set of non-crossing pairings (resp. pairings) with pairs having endpoints of opposite colors.

Proposition 5.31. Let $f^{1}, \ldots, f^{s} \in \mathbb{F}_{2}$ and $p, q \in \mathcal{U}_{\text {class }}\left(\varepsilon\left(f^{1}\right) \ldots \varepsilon\left(f^{s}\right)\right)$. Let $\mathcal{D}=\mathcal{D}\left(f^{1}\right) \otimes \cdots \otimes$ $\mathcal{D}\left(f^{s}\right)$ We suppose that $p$ and $q$ are in $\mathcal{U}$ if at least one $f^{i}$ is not cyclically reduced. Then

$$
b(p|\mathcal{D}| q) \leq \frac{b(p)+b(q)}{2}
$$

In the case of non cyclically reduced terms, the statement needs a preliminary combinatorial result. From the definition of a bidiagram, a pair of $(p|\mathcal{D}| q)$ is a set $\left\{b_{1}, b_{2}\right\}$ in $\{1, \ldots, k, \overline{1}, \ldots, \bar{k}\}$ such that $b_{1} \sim_{p R_{h}(q)} b_{2}$ and $b_{1} \sim_{\mathcal{D}} b_{2}$. Since partitions $\mathcal{D}_{f^{i}}$ are disconnected from each other, $b_{1}$ and $b_{2}$ must be in a same block $\mathcal{D}_{f^{i}}$ to fulfill the condition $b_{1} \sim_{\mathcal{D}} b_{2}$. The notations in the following Lemma are the same as in 5.31 .

Lemma 5.32. Suppose that $p, q$ are in $\mathcal{U}\left(\varepsilon\left(f^{1}\right) \ldots \varepsilon\left(f^{s}\right)\right)$. For $1 \leq i \leq s$, there is at most one pair $t_{i}$ in $\mathcal{D}\left(f^{i}\right)$ which is also a block of $(p|\mathcal{D}| q)$. If this pair exists, there is a block of $(p|\mathcal{D}| q)$ containing at least 6 elements, among which at least 4 are in $\mathcal{D}\left(f^{i}\right)$.

Proof. Let $1 \leq i \leq s$ and suppose that such a $t_{i}$ exists in $D_{f^{i}}$, with endpoints $\left(a_{i}<b_{i}\right)$. By a rotation of all the partitions, we can suppose that $i=1$.
Since $t_{1}$ is a block of $(p|\mathcal{D}| q), t_{1}$ is also block of $p$ or $q$. By symmetry we can assume that $t_{1} \in p$ and $a_{1}, a_{2} \in\{1, \ldots, r\} . a_{1}$ and $b_{1}$ being in the same row, they are linked by a horizontal edge in $\mathcal{D}_{f^{1}}$. Since $\mathcal{U}=\left\langle\square \bullet, \square_{\circ}\right\rangle$, the two endpoints have opposite colors; therefore 5.30 yields that $a_{1}=1$ and $b_{1}=r$ in $\mathcal{D}\left(f^{1}\right)$. Thus, since $p$ is non crossing, any pair of $p$ having an endpoint in $\{2, r-1\}$ must have its other endpoint in $\{2, r-1\}$. This implies that $\mathcal{D}_{f^{1}}$ has as many points of both colors. Let us denote by $h_{u, w}$ (resp. $h_{u, b}, h_{d, w}, h_{d, b}$ ) the number of upper white (resp. upper black, lower white, lower black) points belonging to an horizontal strip of $\mathcal{D}_{f^{1}}$. Then we have :

- $\operatorname{By} 5.30, h_{u, b}+h_{u, w}=h_{d, w}+h_{d, b}$.
- By 5.30, since in the upper part all horizontal strips except $\left(a_{1}, b_{1}\right)$ link points of the same color, $h_{u, b}$ and $h_{u, w}$ are odd; in the lower part all horizontal strips link points of the same color, thus $h_{d, b}$ and $h_{d, w}$ are even.

Suppose that there is a pair $t^{\prime}=\{a, b\}$ in $p$ with $a$ in an horizontal strip $\left\{a, a^{\prime}\right\}$ and $b$ in a vertical strip $\left\{b, b^{\prime}\right\} ; a^{\prime}$ and $b^{\prime}$ cannot be in the same pair of $p$ or $q$, and thus $t^{\prime}$ is in a block of $(p|\mathcal{D}| q)$ with more than 6 elements among which at least 4 elements are in $\mathcal{D}\left(f^{1}\right)$.
Suppose that there is no pair $t \in p$ with one endpoint in an horizontal strip of $\mathcal{D}\left(f^{1}\right)$, and the other endpoint in a vertical strip. This means that we must have in the upper row as many horizontal strips with endpoints colored black and horizontal stips with endpoints colored white. Thus $h_{u, b}=h_{u, w}$. Since $h_{u, b}+h_{u, w}=h_{d, w}+h_{d, b}$, a parity argument yields that either $h_{d, b}>h_{u, b}=h_{u, w}>h_{d, w}$, either $h_{d, w}>h_{u, b}=h_{u, w}>h_{d, b}$. By a color symmetry we can assume that we are in the first case and $h_{d, b}+h_{u, b}>h_{u, w}+h_{d, w}$. Since there are as many points of both colors in $\mathcal{D}_{f^{1}}$, the number of white points linked by a vertical edge is strictly greater than the number of black points linked by a vertical edge. This implies that there must be a pair $t^{\prime} \in p$ whose endpoints are linked to vertical edges with white lower endpoints. Since these two lower endpoints have the same color, they cannot be in the same pair of $q$ and each of them are in
different pairs of $q$; therefore $t^{\prime}$ is in a block of $(p|\mathcal{D}| q)$ with more than 6 elements, among which at least 4 elements are in $\mathcal{D}\left(f^{1}\right)$.

Let us now prove Proposition 5.31.

Proof. Let $l_{i}$ be the length of the reduced word of $f^{i}$. If one of the pair partitions is crossing, from the hypothesis of the Proposition, all words are cyclically reduced. Therefore there is no horizontal strips between points of different colors. Since $p$ is in $\mathcal{U}$ or $\mathcal{U}_{\text {class }}$, each pair of $p$ has endpoints of opposite colors; thus endpoints of a pair of $p$ cannot be in the same pair of $\mathcal{D}$. In particular each block of $(p|\mathcal{D}| q)$ contains at least two pairs from $p$ or $q$, and $b(p|\mathcal{D}| q) \leq \frac{b(p)+b(q)}{2}$. From now on we assume that $p, q$ are non-crossing.
By the previous lemma there is at most one block of $\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)$ having two vertices in each $\mathcal{D}\left(f^{i}\right)$ : denote by $B_{i(1)}, \ldots, B_{i(k)}$ the blocks of $\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)$ having only two vertices (with $B_{i(j)}$ lying in $\left.\mathcal{D}_{f i(j)}\right)$. By the same lemma, one can associate to each block $B_{i(j)}$ a block $\tilde{B}_{i(j)}$ having at least 6 vertices and at least four vertices in $\mathcal{D}_{i(j)}$. Some blocks $\tilde{B}_{i(j)}$ may be the same: for $1 \leq s, t \leq k$, write $s \sim t$ if $\tilde{B}_{i(s)}=\tilde{B}_{i(t)}$. This gives a partition $A_{1} \amalg \cdots \amalg A_{l}$ of $\{1,2, \ldots, k\}$ and a bijective map $\varphi: \llbracket 1, l \rrbracket \rightarrow\left\{\tilde{B}_{i(j)}\right\}_{1 \leq i \leq k}$ sending $s$ to the block $\tilde{B}_{i(j)}$ such that $j \in A_{s}$. By Lemma 5.32, $\varphi(s)$ has at least $\min (6,4|A(s)|)$ elements. Moreover

$$
\begin{aligned}
& \sum_{j=1}^{k}\left|B_{i(j)}\right|+\sum_{j=1}^{l}|\varphi(j)|=2 k+\sum_{j, A_{j} \text { singleton }}|\varphi(j)|+\sum_{j,|A(j)| \geq 2}|\varphi(j)| \\
& \geq 2 k+6 \mid\{j, A(j) \text { singleton }\}\left|+\sum_{j,|A(j)| \geq 2} 4\right| A(j) \mid \\
& \geq \geq 2 k+\left(2 \mid\{j, A(j) \text { singleton }\}\left|+2 \sum_{j,|A(j)| \geq 2}\right| A(j) \mid\right) \\
&+4 \mid\{j, A(j) \text { singleton }\}\left|+2 \sum_{j,|A(j)| \geq 2}\right| A(j) \mid \\
& \geq 4 k+4 \mid\{j, A(j) \text { singleton }\}|+4|\{j, A(j) \geq 2\} \mid \\
& \geq 4(k+l) .
\end{aligned}
$$

On one hand, since $p$ and $q$ are pair partitions, $b(p)=b(q)=\frac{1}{2}\left(\sum_{i=1}^{r} l_{i}\right)$. On the other hand,

$$
\sum_{B \in\left\{\text { blocks of }\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f r}\right| q\right)\right\}}|B|=\sum_{i=1}^{r} 2 l_{i} .
$$

But from the previous computation,

$$
\begin{aligned}
\sum_{\substack{B \text { block of } \\
\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f r}\right| q\right)}}|B| & =\sum_{j=1}^{k}\left|B_{i(j)}\right|+\sum_{j=1}^{l}|\varphi(j)|+\sum_{\substack{B \text { other block of } \\
\left(p\left|\mathcal{D}_{f} \otimes \cdots \otimes \mathcal{D}_{f r}\right| q\right)}}|B| \\
& \geq 4(k+l)+4 \mid\left\{\text { other block of }\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)\right\} \mid \\
& \geq 4 b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right) & \leq \frac{1}{4} \sum_{B \in\left\{\text { blocks of }\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)\right\}}|B| \\
& \leq \frac{2 \sum l_{i}}{4} \leq \frac{b(p)+b(q)}{2} .
\end{aligned}
$$

### 5.3.2 Convergence and asymptotic law for $\left\{X_{f}(n)\right\}_{f \in \mathbb{F}_{2}}$

As in Section 5.2. the proof of Theorem 5.27 is done in two parts. The first part gives a combinatorial expression of the moments and the second part deduces the law of $\left\{X_{f}(n)\right\}_{f \in \mathbb{F}_{2}}$ from this expression.

Moments formula for $\left\{X_{f}\right\}$ Theorem 5.8 can be adapted in the present situation to get the following result:

Proposition 5.33. Let $G_{n}$ be either $U_{n}$ or $U_{n}^{+}$. Let $f^{1}, \ldots, f^{r}$ be elements of $\mathbb{F}_{2}$ such that $\sum l_{i} \leq n$. We assume that $f^{1}, \ldots, f^{r}$ are cyclically reduced if $G=U_{n}$. Then

$$
\begin{equation*}
\int_{G_{n}} X_{f^{1}}(n) \ldots X_{f^{r}}(n)=\operatorname{Card}\left\{p \in \mathcal{C} \mid b\left(p\left|D_{f^{1}} \otimes \cdots \otimes D_{f^{r}}\right| p\right)=b(p)\right\} \tag{5.3.3}
\end{equation*}
$$

with $\mathcal{C}=\mathcal{U}$ if $G_{n}=U_{n}^{+}$and $\mathcal{C}=\mathcal{U}_{\text {class }}$ if $G_{n}=U_{n}$.
Proof. The proof of the proposition follows the proof of Theorem 5.8, with the help of the previous combinatorial results.
Let $1 \leq m \leq r$. The definition of $X_{f^{m}}(n)$ yields:

$$
X_{f^{m}}(n)=\sum_{j=1}^{n}\left(\underset{1 \leq t \leq l_{m}}{\overrightarrow{\prod_{n}}} u^{x_{\eta(t)}^{\varepsilon(t)}}\right)_{j j}=\sum_{j_{1}, \ldots, j_{i}=0}^{n} \prod_{1 \leq t \leq l_{m}}^{\overrightarrow{ }}\left(u^{x_{\eta(i)}^{\varepsilon(i)}}\right)_{j_{t} j_{t+1}},
$$

with $j_{l_{m+1}}=j_{1}$, and $\eta, \varepsilon$ the words associated to $f^{m}$ in Section 5.3.1. From the definition of $u^{x}$ for $x \in\left\{a, b, a^{-1}, b^{-1}\right\}$, the values of $u_{i j}^{x_{\eta}^{\varepsilon(t)}}$ depend on $\{\varepsilon(i), \eta(i)\}$ following the present rule:

| $\eta \backslash \varepsilon$ | 1 | -1 |
| :---: | :---: | :---: |
| 1 | $\left(u_{i j}\right)$ | $\left(\overline{u_{i j}}\right)$ |
| -1 | $\left(u_{j i}\right)$ | $\left(\overline{u_{j i}}\right)$ |

If we write $u_{i j}^{-1}=\overline{u_{i j}}$ and $u_{(i j)^{-1}}=u_{j i}$, the expression of $X_{f^{m}}(n)$ becomes :

$$
X_{f^{m}}(n)=\sum_{j_{1}, \ldots, j_{l_{m}}=0}^{n} \prod_{1 \leq t \leq l_{m}}^{\overrightarrow{ }} u_{\left(j_{i} j_{i+1}\right)^{\eta(i)}}^{\varepsilon(i)} \quad\left(\text { with } j_{l_{m}+1}=j_{1}\right) .
$$

This expression can be translated in terms of diagrams. Indeed, the right term is exactly :

$$
X_{f^{m}}(n)=\sum_{\vec{i}, \vec{j} \in[1, n]^{l_{m}, \operatorname{ker}(\vec{i}, \vec{j}) \geq \mathcal{D}_{f}}} u_{i_{11} j_{1}}^{\varepsilon(1)} \ldots u_{i_{l_{m}} j_{l_{m}}}^{\varepsilon\left(l_{m}\right)} .
$$

A product of different $X_{f^{m}}$ is just a concatenation of these expressions. With $L=\sum_{t=1}^{r} l_{t}$ and $\varepsilon$ the concatenation of the words $\varepsilon_{f m}$ for $1 \leq m \leq r$, this yields:

$$
X_{f^{1}}(n) \ldots X_{f^{r}}(n)=\sum_{\substack{\vec{i}, \vec{j} \in[1, n]^{L} \\ \operatorname{ker}(\vec{i}, \vec{j}) \geq \mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}}} u_{i_{1} j_{1}}^{\varepsilon(1)} \ldots u_{i_{L} j_{L}}^{\varepsilon(L)}
$$

And this sum can be integrated with the Weingarten formula to get for $n \geq L$ :

$$
\begin{aligned}
\int X_{f^{1}}(n) \ldots X_{f^{r}}(n)= & \sum_{\substack{\vec{i} \vec{j} \in[1, n]^{L} \\
\operatorname{ker}(\vec{i}, \vec{j}) \geq \mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}}} u_{i_{1} j_{1}}^{\varepsilon(1)} \ldots u_{i_{L} j_{L}}^{\varepsilon(L)} \\
= & \sum_{\substack{\vec{i}, \vec{j} \in[1, n]^{L}}} \sum_{\substack{p \in \mathcal{C}(\varepsilon), p \leq \operatorname{ker}(\vec{i}) \\
\operatorname{ker}(\vec{i}, \vec{j}) \geq \mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}}} W \sum_{G_{G_{n}}(p, q)} \sum_{p, q \in \mathcal{C}(\varepsilon), q \leq \operatorname{ker}(\vec{j})} \sum_{\substack{\vec{i}, \vec{j} \in[1, n]^{L}}}^{\operatorname{ker}(\vec{i}, \vec{j}) \geq\left(p\left|\mathcal{D}_{f} 1 \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)} \\
& =\sum_{p, q \in \mathcal{C}(\varepsilon)} n^{b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)} W_{G_{n}}(p, q)
\end{aligned}
$$

Proposition 5.5 gives an asymptotic formula for $W g_{G_{n}}$ :

$$
W g_{G_{n}}(p, q)=(-1)^{\delta_{p \neq q}} n^{b(p \vee q)-b(p)-b(q)}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)
$$

Applying this to the previous computation yields:

$$
\begin{equation*}
\int X_{f^{1}}(n) \ldots X_{f^{r}}(n)=\sum_{p, q \in \mathcal{C}(\varepsilon)}(-1)^{\delta_{p, q}} n^{b(p|\mathcal{D}| q)+b(p \vee q)-b(p)-b(q)}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right) \tag{5.3.4}
\end{equation*}
$$

Since $\mathcal{C} \in\left\{\mathcal{U}, \mathcal{U}_{\text {class }}\right\}$, Proposition 5.31 yields the following inequalities:

$$
\left\{\begin{array}{c}
b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right) \leq \frac{b(p)+b(q)}{2} \leq \max (b(p), b(q)) \\
b(p \vee q) \leq \min (b(p), b(q))
\end{array}\right.
$$

In the large $n$ limit, the non-vanishing terms in 5.3.4) are the ones such that $b\left(p \mid \mathcal{D}_{f^{1}} \otimes \cdots \otimes\right.$ $\left.\mathcal{D}_{f^{r}} \mid q\right)+b(p \vee q)-b(p)-b(q) \geq 0$. By the previous inequalities, these terms must verify:

$$
\left\{\begin{array}{c}
b(p)=b(q) \\
b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)=b(p) \\
b(p \vee q)=b(q)
\end{array}\right.
$$

This implies that $p=q$ and $b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| p\right)=b(p)$. In this case $b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| q\right)+$ $b(p \vee q)-b(p)-b(q)=0$, which yields the final expression:

$$
\lim _{n \rightarrow \infty} \int X_{f^{1}}(n) \ldots X_{f^{r}}(n)=\#\left\{p \in \mathcal{C}(\varepsilon), b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| p\right)=b(p)\right\}
$$

Law of the $\left\{X_{f}\right\}$ It remains to describe the law of $\left\{X_{f}\right\}_{f \in \mathbb{F}_{2}}$. Note first that expanding $f$ in the reduced word $f=\prod_{\rightarrow} x_{\eta(i)}^{\varepsilon(i)}$ yields

$$
\begin{aligned}
& \overline{X_{f}(n)}=\overline{\sum_{j_{1}, \ldots, j_{r}=0}^{n} \prod_{\rightarrow} u_{\left(j_{i} j_{i+1}\right)^{\eta(i)}}^{\varepsilon(i)}} \quad\left(\text { with } j_{r+1}=j_{1}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{j_{1}, \ldots, j_{r}=0}^{n} \prod_{\rightarrow} u_{\left(j_{i} j_{i-1}\right)}^{\tilde{\varepsilon}(i)}
\end{aligned}
$$

with $\tilde{\eta}(i)=-\eta(r+1-i), \tilde{\varepsilon}(i)=-\tilde{\varepsilon}(r+1-i)$. Therefore for $f \in \mathbb{F}_{2}$,

$$
\overline{X_{f}(n)}=X_{f^{-1}}(n)
$$

Following a method already used in [14] we first compute the cumulants of the family $\left\{X_{f}\right\}_{f \in \mathbb{F}_{2}}$. Let us associate to each sequence of words $\left(f^{1}, \ldots, f^{r}\right)$, with $f^{i}$ having length $l_{i}$, the partition $p_{\vec{f}}$ of $P(\varepsilon)$, whose blocks are the sets $B_{a}=\left\{\sum_{i=0}^{a-1} l_{i}+1, \sum_{i=0}^{a-1} l_{i}+l_{a}\right\}$ for $1 \leq a \leq r$.

Lemma 5.34. Let $\mathcal{C} \in\left\{\mathcal{U}, \mathcal{U}_{\text {class }}\right\}$. Let $r \geq 1, f^{1}, \ldots, f^{r} \in \mathbb{F}_{2}$ such that $f^{i}$ is cyclically reduced if $\mathcal{C}=\mathcal{U}_{\text {class }}$. If $\mathcal{C}=\mathcal{U}\left(\right.$ resp $\left.. \mathcal{C}=\mathcal{U}_{\text {class }}\right)$ the free cumulant (resp classical cumulant) of $\left(f^{1}, \ldots, f^{r}\right)$ is

$$
c_{r}\left(f^{1}, \ldots, f^{r}\right)=\#\left\{p \in \mathcal{C}(\varepsilon), p \vee p_{\vec{f}}=\mathbf{1}, b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f^{r}}\right| p\right)=b(p)\right\}
$$

Proof. The proof is essentially an adaptation of the proof of [14] in the context of bidiagrams. We write the proof in the free case, since the proof in the classical one is exactly the same. For each tuple $\vec{f}=\left(f^{1}, \ldots, f^{k}\right)$, let $\int\left(f^{1}, \ldots, f^{k}\right)=\int X_{f^{1}} \ldots X_{f^{k}}$, and for $\sigma$ a partition of $\{1, \ldots, r\}$, let

$$
\int_{\sigma}\left(f^{1}, \ldots, f^{k}\right)=\prod_{\left\{i_{1}<\cdots<i_{s}\right\} \in \sigma} \int\left(f^{i_{1}}, \ldots, f^{i_{s}}\right)
$$

For $\sigma$ partition of $\{1, \ldots, r\}$, let $\sigma^{p_{\vec{f}}}$ be the the partition of $P(\varepsilon)$ obtained by linking the $r$ blocks of $p_{\vec{f}}$ according to $\sigma$. Since $\mathcal{U}$ is block stable, Proposition 5.33 yields

$$
\int_{\sigma}\left(f^{1}, \ldots, f^{r}\right)=\#\left\{p \leq \sigma^{p_{\vec{f}}}, b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f^{r}}\right| p\right)=b(p)\right\}
$$

Therefore

$$
\begin{aligned}
c_{r}\left(f^{1}, \ldots, f^{r}\right) & =\sum_{\sigma} \mu\left(\sigma, \mathbf{1}_{r}\right) \int_{\sigma}\left(f^{1}, \ldots, f^{r}\right) \\
& =\sum_{\sigma} \mu\left(\sigma, \mathbf{1}_{r}\right) \#\left\{p \leq \sigma^{p_{\vec{f}}}, b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f^{r}}\right| p\right)=b(p)\right\} \\
& =\sum_{p_{\vec{f}} \leq \tau} \mu\left(\tau, \mathbf{1}_{L}\right) \#\left\{p \leq \tau, b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f^{r}}\right| p\right)=b(p)\right\}
\end{aligned}
$$

where we use on the third equality the invariance property of the Moebius function $\mu$. Writing $\#\left\{p \leq \tau, b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f^{r}}\right| p\right)=b(p)\right\}$ as $\sum_{\substack{p \leq \tau \\ b\left(p \mid \mathcal{D}_{f 1} \ldots \overline{\mathcal{D}} \\ f r r \\ \hline \\ \hline\right.}} 1$ and interverting the sums yields

$$
\begin{aligned}
c_{r}\left(f^{1}, \ldots, f^{r}\right) & =\sum_{b\left(p\left|\mathcal{D}_{f 1} \ldots \mathcal{D}_{f r}\right| p\right)=b(p)} \sum_{\tau \geq p \vee p_{\vec{f}}} \mu\left(\tau, \mathbf{1}_{L}\right) \\
& =\sum_{b\left(p\left|\mathcal{D}_{f} 1 \ldots \mathcal{D}_{f} r\right| p\right)=b(p)} \delta_{p \vee p_{\vec{f}}, \mathbf{1}_{\varepsilon}} \\
& =\#\left\{p \vee p_{\vec{f}}=\mathbf{1}_{\varepsilon}, b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f r}\right| p\right)=b(p)\right\},
\end{aligned}
$$

where the second equality uses again the properties of the Moebius function.
The last step is the computation of $\left|\left\{p \vee p_{\vec{f}}=\mathbf{1}, b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f^{r}}\right| p\right)=b(p)\right\}\right|$ for arbitrary $f^{1}, \ldots, f^{r}$. From now on, $c_{r}\left(f^{1}, \ldots, f^{r}\right)$ denotes the classical or the free cumulant, depending on the situation. $f^{1}, \ldots, f^{r}$ are assumed cyclically reduced if we are in the classical case. As in Section 2, the condition $b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| p\right)=b(p)$ yields restrictive properties on $p$. For $t \in \mathbb{Z} / l_{a} \mathbb{Z}$, let us denote by $x_{t}^{a}$ the element $\sum_{i=0}^{a-1} l_{i}+t$ of $B_{a}$. We have the following result:

Lemma 5.35. Let $r \geq 2$. Suppose that $p$ is a pairing such that $b\left(p\left|\mathcal{D}_{f^{1}} \otimes \cdots \otimes \mathcal{D}_{f^{r}}\right| p\right)=b(p)$ and $p \vee p_{\vec{f}}=\mathbf{1}_{\varepsilon}$. Let $x_{t}^{a} \in B_{a}$ and $x_{t^{\prime}}^{b} \in B_{b}$ such that $a \neq b$ and $x_{t}^{a} \sim_{p} x_{t^{\prime}}^{b}$. Then $l_{a}=l_{b}$, and there exists $\tau \in\{1,-1\}$ such that for all $1 \leq h \leq l_{a}, x_{t+h}^{a} \sim x_{t^{\prime}+\tau h}^{b}$. In particular such $p$ exists only if $r=2$ and $c_{r}\left(f_{1}, \ldots, f_{r}\right)=0$ if $r \geq 3$.

To simplify the proof, we use the same notation to denote elements of the partitions in $P(\varepsilon)$ and elements of the upper row of partitions in $P(\varepsilon, \varepsilon)$.
Proof. Let $r \geq 2, f^{1}, \ldots, f^{r} \in \mathbb{F}_{2}$. Let $p \in \mathcal{C}$ be such that $b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f^{r}}\right| p\right)=b(p)$ and $p \vee p_{\vec{f}}=\mathbf{1}_{\varepsilon}$. In the classical case, the proof of Proposition 5.31 yields that $(p|\mathcal{D}| p)$ has no block with only two elements. In the free case, this could happen only if these two elements are extremal in a particular block $B$ of $p_{\vec{f}}$; in particular since $p$ is non-crossing, any element of $B$ is also paired to an element of $B$ through $p$. Since $r \geq 2$, this contradicts the fact $p \vee p_{\vec{f}}=\mathbf{1}$.
Therefore any block of $(p|\mathcal{D}| p)$ has at least 4 elements. Since $b\left(p\left|\mathcal{D}_{f^{1}} \ldots \mathcal{D}_{f^{r}}\right| p\right)=b(p)=\frac{L}{2}$, each block of $(p|\mathcal{D}| p)$ has exactly 4 elements. Let $t \in \mathbb{Z} / l_{a} \mathbb{Z}$ and $t^{\prime} \in \mathbb{Z} / l_{b} \mathbb{Z}$ such that $a \neq b$ and $x_{t}^{a} \sim_{p} x_{t^{\prime}}^{b}$. Thus $x_{t}^{a} \sim_{p R_{h}(p)} x_{t^{\prime}}^{b}$ and $x_{t}^{a}$ and $x_{t^{\prime}}^{b}$ are in a same block $B$ of $(p|\mathcal{D}| p)$. By $\mathcal{D}$, $x_{t}^{a}$ is linked to exactly one element $u$ among $\left\{x_{t-1}^{a}, x_{t+1}^{a}, \overline{x_{t-1}^{a}}, \overline{x_{t-1}^{a}}\right\}$. The same is true for $x_{t^{\prime}}^{b}$ and one element $v$ among $\left\{x_{t^{\prime}-1}^{b}, x_{t^{\prime}+1}^{b}, \overline{x_{t^{\prime}-1}^{b}}, \overline{x_{t^{\prime}-1}^{b}}\right\}$. Thus $B=\left\{u, v, x_{t}^{a}, x_{t^{\prime}}^{b}\right\}$ forms a block of $(p|\mathcal{D}| p)$ and $u$ is linked to an element of $B$ through the pairing $p R_{h}(p)$. Since we already have $x_{t}^{a} \sim_{p} x_{t^{\prime}}^{b}$, the only possibility is $u \sim_{p R_{h}(p)} v$. Doing the same for $\overline{x_{t}^{a}}$ and $\overline{x_{t^{\prime}}^{b}}$ yields that either $x_{t+1}^{a} \sim_{p} x_{t^{\prime}+1}^{b}$ or $x_{t+1}^{a} \sim_{p} x_{t-1}^{a}$.
Let us assume without loss of generality that $x_{t+1}^{a} \sim_{p} x_{t+1}^{b}$. Doing the same reasoning with $\left\{x_{t+1}^{a}, x_{t+1}^{b}\right\}$ yields that either $x_{t+2}^{a} \sim_{p} x_{t^{\prime}+2}^{b}$ or $x_{t+2}^{a} \sim_{p} x_{t^{\prime}}^{b}$. But the latter is impossible (except if $l_{a}=2$ ) since $x_{t^{\prime}}^{b}$ is already linked to $x_{t}^{a}$ through $p$. Thus $x_{t+2}^{a} \sim_{p} x_{t^{\prime}+2}^{b}$. By recursion, for all $1 \leq h \leq l_{a}, x_{t+h}^{a} \sim_{p} x_{t^{\prime}+h}^{b}$ and $l_{a} \leq l_{b}$; by symmetry $l_{a}=l_{b}$.
Therefore any element of $B_{a}$ is paired to an element of $B_{b}$ through $p$, and $B_{a} \cup B_{b}$ is a block of $p \vee p_{\vec{l}}$. In particular if $r>2$, this contradicts the assumption $p \vee p_{\vec{l}}=\mathbf{1}_{\varepsilon}$.

If $r \leq 2$, the situation is more evolved. First notice that $X_{a b}=X_{b^{-1} a^{-1}}, X_{b a}=X_{a^{-1} b^{-1}}$, $X_{a b^{-1}}=X_{b^{-1} a}, X_{a^{-1} b}=X_{b a^{-1}}, X_{a}=X_{b^{-1}}$ and $X_{a^{-1}}=X_{b}$; moreover $X_{f}=X_{f^{\prime}}$ with $f \neq f^{\prime}$ only in the previous cases. Let us write $f \sim f^{\prime}$ if $X_{f}=X_{f^{\prime}}$.

Proposition 5.36. If $\mathcal{C}=\mathcal{U}$ (resp. $\left.\mathcal{C}=\mathcal{U}_{\text {class }}\right),\left\{X_{f}\right\}_{f \in \mathbb{F}^{2}}$ is a family of free (resp. independent) (semi-) circular (resp. gaussian) variables. Let $f_{1}, f_{2} \in \mathbb{F}_{2}$. In the free case, $c_{2}\left(f_{1}, f_{2}\right)=0$ unless $f_{2} \sim f_{1}^{-1}$, and in this case $k\left(f_{1}, f_{1}^{-1}\right)=1 . k(f)=0$ unless $f \sim a b$ or $f \sim a^{-1} b^{-1}$, and in the latter case $k(f)=1$.
In the classical case,

$$
c_{2}\left(f_{1}, f_{2}\right)=\sum_{1 \leq k \leq r, \tau \in\{1,-1\}} \delta_{c^{k, \tau}\left(f_{1}\right) f_{2}=1}
$$

where $c^{k, \tau}$ acts on a reduced word $x_{1} \ldots x_{t} \in \mathbb{F}_{2}$ as

$$
c\left(x_{1} \ldots x_{t}\right)=x_{k+\tau} x_{k+2 \tau} \ldots x_{k+t \tau}
$$

$k_{f}=0$ unless $f \sim a b$ or $f \sim a^{-1} b^{-1}$, where $k(f)=1$.
Proof. The first part of the proposition is a straightforward application of Lemma 5.35.
Let $f_{1}, f_{2} \in \mathbb{F}_{2}$ with $f_{1}=x_{\eta_{1}}^{\varepsilon_{1}} \ldots x_{\eta_{r}}^{\varepsilon_{r}}$ and $f_{2}=x_{\eta_{1}^{\prime}}^{\varepsilon_{1}^{\prime}} \ldots x_{\eta_{r^{\prime}}}^{\varepsilon_{r^{\prime}}^{\prime}}$. Let $p$ be such that $b\left(p \mid \mathcal{D}\left(f_{1}\right) \otimes\right.$ $\left.\mathcal{D}\left(f_{2}\right) \mid p\right)=b(p)$ and $p_{\vec{f}} \vee p=\mathbf{1}_{\varepsilon \varepsilon^{\prime}}$.
The condition $p_{\vec{f}} \vee p=\mathbf{1}_{\varepsilon \varepsilon^{\prime}}$ means that there exist $a \in B_{f_{1}}, b \in B_{f_{2}}$ with $a \sim_{p} b$. By the previous Lemma, this implies that $r=r^{\prime}$, and there exists a bijection $\varphi:\{1, \ldots, r\} \rightarrow\{1, \ldots r\}$ such that $i \sim_{p} r+\varphi(i)$. By the same Lemma, there exist $r_{0} \in \llbracket 0, r-1 \rrbracket$ and $\tau \in\{-1,1\}$ such that $\varphi(x)=\tau x+r_{0}$ for all $1 \leq x \leq r$. Since $p$ is in $\mathcal{U}$ or $\mathcal{U}_{\text {class }}, i$ and $r+\varphi(i)$ have opposite colors and thus $\varepsilon^{\prime}(\varphi(i))=-\varepsilon(i)$. Having each block of $\left(p\left|\mathcal{D}_{f_{1}} \otimes \mathcal{D}_{f_{2}}\right| p\right)$ with exactly 4 elements requires that $\eta^{\prime}(\varphi(i))=-\eta(i)$ (except if $r=2$, in which case the condition is $\eta^{\prime}(1) \eta^{\prime}(2)=\eta(1) \eta(2)$ ).

We must now distinguish the classical and the free case.

- In the free case there is only one way to achieve a non-crossing pairing in this way, namely $\varphi_{0}(x)=r+1-x$. This proves that $c_{2}\left(f_{1}, f_{2}\right) \leq 1$. From the first part of the proof this partition occurs if and only if $\eta^{\prime}(r+1-i)=-\eta(i)$ and $\varepsilon^{\prime}(r+1-i)=-\varepsilon^{\prime}(i)$, which is equivalent to $f_{1} \sim f_{2}^{-1}$.
- In the classical case, any map $\varphi$ could arise. It remains to find which $\varphi$ yields a pairing with the condition that each block of $\left(p\left|\mathcal{D}_{f_{1}} \otimes \mathcal{D}_{f_{2}}\right| p\right)$ has 4 elements. Suppose that $\varphi$ is given by $r_{0} \in \llbracket 0, r-1 \rrbracket$ and $\tau \in\{-1,1\}$. By the first part of the proof the condition is satisfied if and only if $c^{r-r_{0}+1,-\tau}\left(f_{1}\right)=f_{2}^{-1}$. Summing on all $r_{0}, \tau$ yields the result.

The same proof yields that $c_{1}(f)$ is non-zero only if the length of $f$ is at most 2 and $f \sim f^{-1}$.
Theorem 5.27 is the combination of Proposition 5.33 and Proposition 5.36 ,

## Part III

## Free wreath product

## Chapter 6

## Free wreath product with the free symmetric group

## Introduction

In this chapter, we will consider the case of the free wreath product quantum groups defined by Bichon in [21. This product was introduced as the most natural way to build the quantum automorphism group of the $n$-times disjoint union of a finite connected graph. The free wreath product $i_{*}$ associates to a compact quantum group $\mathbb{G}$ and a compact subgroup $\mathbb{F}$ of $S_{N}^{+}$a new compact quantum group $\mathbb{G} \imath_{*} \mathbb{F}$. It is constructed as an analogue of the wreath products of classical groups. An example of this construction was studied by Banica and Vergnioux in [16], and then by Banica, Belinschi, Capitaine and Collins in [9]: they focused on the free wreath product of the dual of the cyclic group $\mathbb{Z} / s \mathbb{Z}$ with $S_{N}^{+}$. Banica and Vergnioux obtained the fusion rules and Banica, Belinschi, Capitaine and Collins obtained interesting probability results involving free compound Poisson variables.

François Lemeux generalized these results in [55] to the case of a free wreath product between the dual $\hat{\Gamma}$ of a discrete groupe $\Gamma$ and $S_{N}^{+}$. Once again he was able to find the fusion rules of the quantum group as well as some operator algebraic properties by using certain results of Brannan on $S_{N}^{+}$(see [25]). We investigate here the general problem of the free wreath product of any compact quantum group of Kac type $\mathbb{G}$ with $S_{N}^{+}$. In particular we construct the intertwiner spaces of $\mathbb{G} 2_{*} S_{N}^{+}$from the knowledge of the intertwiner spaces of $\mathbb{G}$, (see Theorem 6.16). We give also an expression of the Haar state of $\mathbb{G} \imath_{*} S_{N}^{+}$from the Haar state of $S_{N}^{+}$. This yields the equality in law

$$
\chi_{u l_{* v}} \sim \chi_{u} \boxtimes \chi_{v},
$$

with $u$ being a representation of $\mathbb{G}, v$ the fundamental representation of $S_{N}^{+}$(with $\chi$ denoting in each case the associated character) and $\boxtimes$ is the free multiplicative convolution of two noncommutative variables. This is a positive answer to a special case of a conjecture raised by Banica and Bichon in [10] (See Subsection 6.4.1).
Using the description of the intertwiner spaces, François Lemeux and Jonas Wahl also obtained in [94, [56] further interesting results on the von Neumann algebra and $C^{*}$-algebras associated to $\mathbb{G} \imath_{*} S_{N}^{+}$. The chapter is organised as follows : the first section is dedicated to some preliminaries and notations. The second section gives some classical results and proofs that provide some insight into the final general description of the intertwiner spaces for the free wreath products. The description of the intertwiner spaces for the free wreath products $\mathbb{G} \imath_{*} S_{N}^{+}$is the main result of the third section. In the fourth section, we give the probabilistic applications that one can deduce from the latter description.

### 6.1 Preliminaries

In this section, we recall a few facts and results about compact quantum groups and about free wreath products by the quantum permutation groups $S_{N}^{+}$, and we set the notations.
A compact quantum group (see [100]), or Woronowicz- $C^{*}$-algebra, is a pair $\mathbb{G}=(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital separable $C^{*}$-algebra and $\Delta: C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes_{\min } C(\mathbb{G})$ is a unital $*$ homomorphism (i.e. it satisfies the coassociativity relation $(\mathrm{id} \otimes \Delta) \circ \Delta=(\Delta \otimes \mathrm{id}) \circ \Delta$ ), and such that the cancellation property holds (i.e $\operatorname{span}\{\Delta(a)(b \otimes 1): a, b \in C(\mathbb{G})\}$ and $\operatorname{span}\{\Delta(a)(1 \otimes$ $b): a, b \in C(\mathbb{G})\}$ are norm dense in $C(\mathbb{G}) \otimes C(\mathbb{G}))$. These assumptions allow to prove the existence and uniqueness of a Haar state $h: C(\mathbb{G}) \rightarrow \mathbb{C}$ satisfying the bi-invariance relations $(h \otimes \mathrm{id}) \circ \Delta(\cdot)=h(\cdot) 1=(\mathrm{id} \otimes h) \circ \Delta(\cdot)$. In this chapter we will deal with compact quantum groups of Kac type, which means that their Haar state $h$ is a trace. Let $\lambda_{h}: C(\mathbb{G}) \rightarrow \mathcal{B}\left(L^{2}(\mathbb{G}, h)\right)$ be the GNS representation associated to the Haar state $h$ of $\mathbb{G}=(C(\mathbb{G}), \Delta)$ (also called the left regular representation). The reduced $C^{*}$-algebra associated to $\mathbb{G}$ is then defined by $C_{r}(\mathbb{G})=$ $\lambda_{h}(C(\mathbb{G})) \simeq C(\mathbb{G}) / \operatorname{Ker}\left(\lambda_{h}\right)$ and the von Neumann algebra of $\mathbb{G}$ by $L^{\infty}(\mathbb{G})=C_{r}(\mathbb{G})^{\prime \prime}$. One can prove that $C_{r}(\mathbb{G})$ is again a Woronowicz- $C^{*}$-algebra whose Haar state extends to $L^{\infty}(\mathbb{G})$. We will denote simply by $\Delta$ and $h$ the coproduct and Haar state on $C_{r}(\mathbb{G})$.

An $N$-dimensional (unitary) correpresentation $u=\left(u_{i j}\right)_{i j}$ of $\mathbb{G}$ is a (unitary) matrix $u \in$ $M_{N}(C(\mathbb{G})) \simeq C(\mathbb{G}) \otimes \mathcal{B}\left(\mathbb{C}^{N}\right)$ such that for all $i, j \in\{1, \ldots, N\}$, one has

$$
\Delta\left(u_{i j}\right)=\sum_{k=1}^{N} u_{i k} \otimes u_{k j} .
$$

The matrix $\bar{u}=\left(u_{i j}^{*}\right)$ is called the conjugate of $u \in M_{N}(C(\mathbb{G}))$ and in general it is not necessarily unitary (even if $u$ is). However all the compact quantum groups we will deal with are of Kac type and in this case the conjugate of a unitary correpresentation is also unitary (see 65]).

An intertwiner between two correpresentations

$$
u \in M_{N_{u}}(C(\mathbb{G})) \text { and } v \in M_{N_{v}}(C(\mathbb{G}))
$$

is a matrix $T \in M_{N_{u}, N_{v}}(\mathbb{C})$ such that $v(1 \otimes T)=(1 \otimes T) u$. We say that $u$ is equivalent to $v$, and we note $u \sim v$, if there exists an invertible intertwiner between $u$ and $v$. We denote by $\operatorname{Hom}_{\mathbb{G}}(u, v)$ the space of intertwiners between $u$ and $v$. A correpresentation $u$ is said to be irreducible if $\operatorname{Hom}_{\mathbb{G}}(u, u)=\mathbb{C}$ id. We denote by $\operatorname{Irr}(\mathbb{G})$ the set of equivalence classes of irreducible correpresentations of $\mathbb{G}$.

As a Woronowicz-C $C^{*}$-algebra, $C(\mathbb{G})$ contains a dense $*$-subalgebra denoted by $\operatorname{Pol}(\mathbb{G})$ and linearly generated by the coefficients of the irreducible correpresentations of $\mathbb{G}$ (see 65 for details on the subject). The coefficients of a $\mathbb{G}$-representation $r$ acting on a Hilbert space $H_{r}$ are given by $(\mathrm{id} \otimes \varphi)(r)$ for some $\varphi \in \mathcal{B}\left(H_{r}\right)^{*}$. This algebra has a Hopf- $*$-algebra structure and in particular there is a $*$-antiautomorphism $\kappa: \operatorname{Pol}(\mathbb{G}) \rightarrow \operatorname{Pol}(\mathbb{G})$ which acts on the coefficients of an irreducible correpresentation $r=\left(r_{i j}\right)$ as $\kappa\left(r_{i j}\right)=r_{j i}^{*}$. This algebra is also dense in $L^{2}(\mathbb{G}, h)$. Since $h$ is faithful on the $*$-algebra $\operatorname{Pol}(\mathbb{G})$, one can identify $\operatorname{Pol}(\mathbb{G})$ with its image in the GNSrepresentation $\lambda_{h}(C(\mathbb{G}))$. We will denote by $\chi_{r}$ the character of the irreducible correpresentation $r \in \operatorname{Irr}(\mathbb{G})$, that is $\chi_{r}=(\mathrm{id} \otimes \operatorname{Tr})(r)$.

A fundamental and basic family of examples of compact quantum groups is recalled in the following definition:

Definition 6.1. ([96]) Let $N \geq 2 . S_{N}^{+}$is the compact quantum group $\left(C\left(S_{N}^{+}\right), \Delta\right)$, where $C\left(S_{N}^{+}\right)$is the universal $C^{*}$-algebra generated by $N^{2}$ elements $u_{i j}$ such that the matrix $u=\left(u_{i j}\right)$ is unitary and $u_{i j}=u_{i j}^{*}=u_{i j}^{2}, \forall i, j$ (i.e. $u$ is a magic unitary) and such that the coproduct $\Delta$ is
given by the usual relations making $u$ a finite dimensional correpresentation of $C\left(S_{N}^{+}\right)$, that is $\Delta\left(u_{i j}\right)=\sum_{k=1}^{N} u_{i k} \otimes u_{k j}, \forall i, j \leq N$.

In the cases $N=2,3$, one obtains the usual algebras $C\left(\mathbb{Z}_{2}\right), C\left(S_{3}\right)$. If $N \geq 4$, one can find an infinite dimensional quotient of $C\left(S_{N}^{+}\right)$so that $C\left(S_{N}^{+}\right)$is not isomorphic to $C\left(S_{N}\right)$, see e.g. [96, 7].

In [95], Wang defined the free product $\mathbb{G}=\mathbb{G}_{1} * \mathbb{G}_{2}$ of compact quantum groups, showed that $\mathbb{G}$ is still a compact quantum group and gave a description of the irreducible correpresentations of $\mathbb{G}$ as alternating tensor products of nontrivial irreducible correpresentations of $\mathbb{G}_{1}$ and $\mathbb{G}_{2}$. For a correpresentation $v$ of $\mathbb{G}$, denote by $\bar{v}$ the contragredient correpresentation.

Theorem 6.2. (95]) Let $\mathbb{G}_{1}$ and $\mathbb{G}_{2}$ be compact quantum groups. Then the set $\operatorname{Irr}(\mathbb{G})$ of irreducible correpresentations of the free product of quantum groups $\mathbb{G}=\mathbb{G}_{1} * \mathbb{G}_{2}$ can be identified with the set of alternating words in $\operatorname{Irr}\left(\mathbb{G}_{1}\right) * \operatorname{Irr}\left(\mathbb{G}_{2}\right)$ and the fusion rules can be recursively described as follows:

- If the words $x, y \in \operatorname{Irr}(\mathbb{G})$ end and start in $\operatorname{Irr}\left(\mathbb{G}_{i}\right)$ and $\operatorname{Irr}\left(\mathbb{G}_{j}\right)$ respectively with $j \neq i$ then $x \otimes y$ is an irreducible correpresentation of $\mathbb{G}$ corresponding to the concatenation $x y \in \operatorname{Irr}(\mathbb{G})$.
- If $x=v z$ and $y=z^{\prime} w$ with $z, z^{\prime} \in \operatorname{Irr}\left(\mathbb{G}_{i}\right)$, then there is a recurrence formula

$$
x \otimes y=\bigoplus_{1 \neq t \subseteq z \otimes z^{\prime}} v t w \oplus \delta_{\bar{z}, z^{\prime}}(v \otimes w)
$$

where the sum runs over all non-trivial irreducible correpresentations $t \in \operatorname{Irr}\left(\mathbb{G}_{i}\right)$ contained in $z \otimes z^{\prime}$, with multiplicity.

In this chapter, we are interested in the free wreath product of quantum groups:
Definition 6.3. ([21, Definition 2.2]) Let $A$ be a Woronowicz-C ${ }^{*}$-algebra, $N \geq 2$ and $\nu_{i}: A \rightarrow$ $A^{* N}$ be the canonical inclusion of the $i$-th copy of $A$ in the free product $A^{* N}, i=1, \ldots, N$.

The free wreath product of $A$ by $C\left(S_{N}^{+}\right)$is the quotient of the $C^{*}$-algebra $A^{* N} * C\left(S_{N}^{+}\right)$by the two-sided ideal generated by the elements

$$
\nu_{k}(a) u_{k i}-u_{k i} \nu_{k}(a), \quad 1 \leq i, k \leq N, \quad a \in A
$$

It is denoted by $A *_{w} C\left(S_{N}^{+}\right)$.
In the next result, we use the Sweedler notation $\Delta_{A}(a)=\sum a_{(1)} \otimes a_{(2)} \in A \otimes A$.
Theorem 6.4. ([21, Theorem 2.3]) Let $A$ be a Woronowicz-C*-algebra, then the free wreath product $A *_{w} C\left(S_{N}^{+}\right)$admits a Woronowicz- $C^{*}$-algebra structure: if $a \in A$, then

$$
\begin{gathered}
\Delta\left(u_{i j}\right)=\sum_{k=1}^{N} u_{i k} \otimes u_{k j}, \forall i, j \in\{1, \ldots, N\}, \\
\Delta\left(\nu_{i}(a)\right)=\sum_{k=1}^{N} \nu_{i}\left(a_{(1)}\right) u_{i k} \otimes \nu_{k}\left(a_{(2)}\right), \\
\varepsilon\left(u_{i j}\right)=\delta_{i j}, \varepsilon\left(\nu_{i}(a)\right)=\varepsilon_{A}(a), S\left(u_{i j}\right)=u_{j i}, S\left(\nu_{i}(a)\right)=\sum_{k=1}^{N} \nu_{k}\left(S_{A}(a)\right) u_{k i}, \\
u_{i j}^{*}=u_{i j}, \nu_{i}(a)^{*}=\nu_{i}\left(a^{*}\right) .
\end{gathered}
$$

Moreover, if $\mathbb{G}$ is a full compact quantum group, then $\mathbb{G} \imath_{*} S_{N}^{+}=\left(A *_{w} C\left(S_{N}^{+}\right), \Delta\right)$ is also a full compact quantum group.

Remark 6.5. The homomorphisms $\nu_{i}: A \rightarrow A^{* N} \subseteq A *_{w} C\left(S_{N}^{+}\right)$are injective and we have $\nu_{i}=\pi \circ \bar{\nu}_{i}$, where

$$
\bar{\nu}_{i}=q \circ \nu_{i}: A \rightarrow A *_{w} C\left(S_{N}^{+}\right)
$$

$q: A^{* N} * C\left(S_{N}^{+}\right) \rightarrow A *_{w} C\left(S_{N}^{+}\right)$is the quotient map and $\pi: A *_{w} C\left(S_{N}^{+}\right)=i d * \varepsilon$. Hence the morphisms $\bar{\nu}_{i}: A \rightarrow A *_{w} C\left(S_{N}^{+}\right)$are injective.

Recall that the case of the dual of a discrete group $\mathbb{G}=\widehat{\Gamma}$ is investigated in [55]. In particular, a description of the irreducible representations is given and several operator algebraic properties are obtained from this description.

### 6.2 Classical wreath products by permutation groups.

In this section we provide a probabilistic formula for the moments of the character coming from certain wreath products of classical groups. This is in particular a hint for the formula in the free case. Recall that we denote by $\mathcal{P}(k)$ the set of all partitions of the set $\{1, \ldots, k\}$.

Let $G$ be a classical group, $n \geq 1$. Then $S_{n}$ acts on $G^{n}$ by the automorphisms

$$
\begin{equation*}
s: \sigma \in S_{n} \mapsto s(\sigma) \cdot\left(g_{1}, \ldots, g_{n}\right)=\left(g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}\right) \tag{*}
\end{equation*}
$$

Definition 6.6. The wreath product between $G$ and $S_{n}$, denoted by $G \backslash S_{n}$, is the semi-direct product of $G^{n}$ and $S_{n}$, where $S_{n}$ acts on $G^{n}$ by $(*)$. In other words,

$$
G \imath S_{n}=\left\{\left(\left(g_{1}, \ldots, g_{n}\right), \sigma\right), g_{i} \in G, \sigma \in S_{n}\right\}
$$

with the product

$$
\left(\left(g_{1}, \ldots, g_{n}\right), \sigma\right) \cdot\left(\left(g_{1}^{\prime}, \ldots, g_{n}^{\prime}\right), \mu\right)=\left(\left(g_{1} g_{\sigma^{-1}(1)}^{\prime}, \ldots, g_{n} g_{\sigma^{-1}(n)}^{\prime}\right), \sigma \mu\right)
$$

If $G$ is a compact group, $G \backslash S_{n}$ is compact as well and thus there exists a Haar measure on $G \imath S_{n}$. It is easy to see that $\left.G\right\} S_{n}$ is isomorphic to $G \times \cdots \times G \times S_{n}$ as a measure space and that the Haar measure on $G \imath S_{n}$ is given by $d \lambda_{G \imath S_{n}}=\bigotimes_{i} d g_{i} \otimes d \sigma$, where $d_{g}$ denotes the Haar measure on $G$ and $d \sigma$ the normalized counting measure on $S_{n}$.
If $\alpha: G \rightarrow U(V)$ is a unitary representation of $G$, then $G \imath S_{n}$ acts on $V^{\otimes n}$ via

$$
\alpha^{n}\left(\left(g_{1}, \ldots, g_{n}\right), \sigma\right)\left(v_{1} \otimes \cdots \otimes v_{n}\right):=\alpha\left(g_{1}\right)\left(v_{\sigma^{-1}(1)}\right) \otimes \cdots \otimes \alpha\left(g_{n}\right)\left(v_{\sigma^{-1}(n)}\right)
$$

We will use the following notation in the sequel:
Notation 6.7. Let $\beta: G \rightarrow U(H)$ be a unitary representation of a compact group $G$; then:

- $\chi_{\beta}$ denotes the character of $\beta$,
- $F_{\beta}$ is the exponential generating serie of the moments of $\chi_{\beta}$ with respect to the Haar measure

The purpose of this section is to describe the distribution of $\chi_{\alpha^{n}}$ under $d \lambda_{G l S_{n}}$, when $\alpha$ is a represention of $G$. We will assume that $\alpha(G) \subseteq G L_{p}(\mathbb{R})$ for some $p \geq 1$. In particular $\chi_{\alpha^{n}}$ is real. The computations are similar in the complex setting; we just have to deal separately with the real and imaginary part of $\chi_{\alpha}$.

Notation 6.8. For each partition $\nu \in \mathcal{P}(k)$ with blocks $B_{1}, \ldots B_{r}$ and sequence of numbers $\left(c_{1}, \ldots, c_{n}, \ldots\right)$ of length greater than $k$ we write

$$
c_{\nu}=c_{\left|B_{1}\right|} c_{\left|B_{2}\right|} \ldots c_{\left|B_{r}\right|},
$$

with $\left|B_{i}\right|$ being the cardinal of the block $B_{i}$.
Proposition 6.9. The exponential serie of the moments of $\chi_{\alpha^{n}}$ is given by

$$
F_{\alpha^{n}}(x)=\sum m_{\alpha^{n}}(k) \frac{x^{k}}{k!}
$$

with

$$
m_{\alpha^{n}}(k)=\sum_{\nu \in \mathcal{P}(k), l(\nu) \leq n} m_{\alpha}(\nu)
$$

where $l(\pi)$ is the length of a partition $\pi$ (namely the number of blocks of $\pi$ ).
Proof. Let $t=\left(\left(g_{1}, \ldots, g_{n}\right), \sigma\right) \in G \backslash S_{n}$, we have for $x>0$ small enough. Writing the action of $t$ through $\alpha^{n}$ in block matrices yields the following result

$$
\begin{aligned}
F_{\alpha^{n}}(x) & =\mathbb{E}_{G 2 S_{n}}\left(\exp \left(x \operatorname{Tr}\left(\alpha^{n}(t)\right)\right)\right)=\int_{G l S_{n}} \exp \left(x \sum_{\substack{1 \leq i \leq n \\
\sigma(i)=i}} \operatorname{Tr}\left(\alpha\left(g_{i}\right)\right)\right) \prod d g_{i} d \sigma \\
& =\int_{S_{n}} \prod_{i \text { fixed point of } \sigma}\left(\int_{G_{i}} \exp \left(x \times \operatorname{Tr}\left(\alpha\left(g_{i}\right)\right)\right) d g_{i}\right) d \sigma \\
& =\int_{S_{n}} \prod_{i \text { fixed point of } \sigma} F_{\alpha}(x) d \sigma \\
& =\int_{S_{n}} F_{\alpha}(x)^{\# \text { fixed points of } \sigma} d \sigma \\
& =\int_{S_{n}} \exp \left(\log \left(F_{\alpha}(x)\right) \# \text { fixed points of } \sigma\right) d \sigma
\end{aligned}
$$

Considering $\log \left(F_{\alpha}(x)\right)$ as fixed in the last integral yields the equality

$$
\begin{equation*}
F_{\alpha^{n}}(x)=F_{S_{n}}\left(\log \left(F_{\alpha}(x)\right)\right), \tag{6.2.1}
\end{equation*}
$$

where $F_{S_{n}}$ denotes the exponential generating serie of the moments of the fundamental representation $S_{n} \hookrightarrow M_{n}(\mathbb{C})$. Now, we can exploit the general facts that

$$
\begin{equation*}
F_{\beta}(x)=\sum m_{\beta}(k) \frac{x^{k}}{k!} \tag{6.2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\log F_{\beta}(x)=\sum c_{\beta}(k) \frac{x^{k}}{k!} \tag{6.2.3}
\end{equation*}
$$

where $\left(m_{\beta}(k)\right)_{k>1}$ are the moments of the law of $\chi_{\beta}$ and $\left(c_{\beta}(k)\right)_{k>1}$ are the classical cumulants of this law. The latter is the only sequence of real numbers satisfying

$$
\begin{equation*}
m_{\beta}(k)=\sum_{\pi \in \mathcal{P}(k)} c_{\beta}(\pi) \tag{6.2.4}
\end{equation*}
$$

for all $k \geq 1$. From the left-hand side of (6.2.1) and 6.2.2 we get

$$
F_{\alpha^{n}}(x)=\sum_{k} m_{\alpha^{n}}(k) \frac{x^{k}}{k!},
$$

and from the right-hand of (6.2.1) with (6.2.3) we compute

$$
\begin{aligned}
F_{\alpha^{n}}(x) & =\sum_{r} m_{S_{n}}(r) \frac{\left(\sum c_{\alpha}(u) \frac{x^{u}}{u!}\right)^{r}}{r!} \\
& =\sum_{k} \frac{x^{k}}{k!} \sum_{r} \frac{m_{S_{n}}(r)}{r!} \sum_{\substack{u_{1} \times 1+\cdots+u_{k} \times k=k \\
\sum u_{i}=r}} k!\frac{r!}{\prod u_{i}!}\left(\frac{c_{\alpha}(1)}{1!}\right)^{u_{1}} \ldots\left(\frac{c_{\alpha}(k)}{k!}\right)^{u_{k}}
\end{aligned}
$$

The last equality above being due to the multinomial expansion. Hence, after identifying coefficients we obtain:

$$
\begin{equation*}
m_{\alpha^{n}}(k)=\sum_{r} m_{S_{n}}(r) \sum_{\substack{u_{1} \times 1+\ldots+u_{k} \times k=k \\ \sum u_{i}=r}} k!\frac{1}{u_{1}!\ldots u_{r}!}\left(\frac{c_{\alpha}(1)}{1!}\right)^{u_{1}} \ldots\left(\frac{c_{\alpha}(k)}{k!}\right)^{u_{k}} \tag{6.2.5}
\end{equation*}
$$

We say that a partition $p \in \mathcal{P}(k)$ is of type $\left(1^{u_{1}}, \ldots, k^{u_{r}}\right)$, if it is a partition having $u_{1}$ blocks of cardinal $1, u_{2}$ of cardinal 2 and so on. The number of partitions of $\{1, \ldots, k\}$ of type $\left(1^{u_{1}}, \ldots, k^{u_{r}}\right)$ is exactly

$$
\frac{k!}{u_{1}!\ldots u_{k}!} \frac{1}{1!{ }^{u_{1}} \ldots k!!_{k}}
$$

(see e.g. page 22 in [60]). Thus summing over every types of partitions in (6.2.5) yields:

$$
\begin{equation*}
m_{\alpha^{n}}(k)=\sum_{r} m_{S_{n}}(r) \sum_{\pi \in \mathcal{P}(k), l(\pi)=r} c_{\alpha}(\pi) \tag{6.2.6}
\end{equation*}
$$

Using the fact that (see [15], 72])

$$
\begin{equation*}
m_{S_{n}}(r)=\#\{\text { partitions of }\{1, \ldots, r\} \text { having at most } n \text { blocks }\} \tag{6.2.7}
\end{equation*}
$$

we can transform (6.2.6) into

$$
\begin{aligned}
m_{\alpha^{n}}(k) & =\sum_{r} \sum_{\substack{\nu \leq \mathbf{1}_{r}}} \sum_{\substack{\pi \leq \mathbf{1}_{k} \\
l(\nu) \leq n \\
l(\pi)=r}} c_{\alpha}(\pi)=\sum_{r} \sum_{\substack{\pi \leq \mathbf{1}_{k}}} \sum_{\substack{\pi \leq \nu \leq 1_{k} \\
l(\pi)=r \\
l(\nu) \leq n}} c_{\alpha}(\pi) \\
& =\sum_{\substack{\pi \leq 1_{k}}} \sum_{\substack{\pi \leq \leq \leq \mathbf{1}_{k} \\
l(\nu) \leq n}} c_{\alpha}(\pi)=\sum_{\substack{\nu \leq \mathbf{1}_{k} k \\
l(\nu) \leq n}} \sum_{\pi \leq \nu} c_{\alpha}(\pi) \\
& =\sum_{\substack{\nu \leq \leq_{k} \\
l(\nu) \leq n}} m_{\alpha}(\nu) .
\end{aligned}
$$

We can deduce from Proposition 6.9 the aymptotic law of $\chi_{\alpha^{n}}$ when $n$ goes to infinity :

Corollary 6.10. The following convergence in moments holds:

$$
\chi_{\alpha^{n}} \underset{n \rightarrow \infty}{\longrightarrow} \mathcal{P}\left(\chi_{\alpha}\right),
$$

where $\mathcal{P}\left(\chi_{\alpha}\right)$ is the compound Poisson law with respect to the parameter 1 and the law $\alpha$.
Proof. We have

$$
\begin{aligned}
m_{\alpha^{n}}(k) & =\sum_{\nu \leq \mathbf{1}_{k}, l(\nu) \leq n} m_{\alpha}(\nu) \\
& \underset{n \rightarrow \infty}{\longrightarrow} \sum_{\nu \leq \mathbf{1}_{k}} m_{\alpha}(\nu)=m_{\mathcal{P}\left(\chi_{\alpha}\right)}
\end{aligned}
$$

Remark 6.11. In the next section we will describe the intertwiner spaces for a free wreath product $\mathbb{G} 2_{*} S_{N}^{+}$. The result and proofs can be easily adapted to get the same result in the classical case; one only needs to use all partitions instead of non-crossing ones.

### 6.3 Intertwiner spaces in $\mathbb{G} \imath_{*} S_{N}^{+}$

Let $\mathbb{G}=(C(\mathbb{G}), v)$ be a compact matrix quantum group of Kac type, generated by a unitary $v$ acting on $H$. In this section, the $C^{*}$-algebras associated with compact quantum groups are considered in their maximal versions. A generating magic unitary $u$ of the free quantum permutation group $S_{N}^{+}$acting on $\mathbb{C}^{N}$ is a matrix $\left(u_{i j}\right)_{1 \leq i, j \leq N}$ of orthogonal projections of $C\left(S_{N}^{+}\right)$, such that the algebra generated by $\left\{u_{i j}\right\}$ is dense in $\bar{C}\left(S_{N}^{+}\right)$and such that

$$
\sum_{i} u_{i j}=\sum_{j} u_{i j}=1, u_{i j} u_{i k}=\delta_{j k} u_{i j}, u_{i j} u_{l j}=\delta_{i l} u_{i j}
$$

We recall that the correpresentation

$$
\omega:=\left(\omega_{i j k l}\right)_{1 \leq i, j \leq N}^{1 \leq k, l \leq d_{G}}=\left(u_{i j} v_{k l}^{(i)}\right)_{i, j, k, l}
$$

acting on $W:=\mathbb{C}^{N} \otimes H$, is the generating matrix of the free wreath product quantum groups $\mathbb{G} z_{*} S_{N}^{+}$, see [21].

Let $\operatorname{Rep}(\mathbb{G})$ be the set of equivalence classes of unitary finite dimensional (not necessarily irreducible) correpresentations of $\mathbb{G}$ and we denote by $H^{\alpha}=\left\langle Y_{1}^{\alpha}, \ldots, Y_{d_{\alpha}}^{\alpha}\right\rangle$ the representation space of $\alpha$, for $\alpha \in \operatorname{Rep}(\mathbb{G})$. We have a natural family of $\mathbb{G} z_{*} S_{N}^{+}$-representations (see the proof of Theorem 2.3 in [21]) given by

$$
\begin{equation*}
\left\{r(\alpha):=\left(u_{i j} \alpha_{k l}^{(i)}\right): \alpha \in I\right\} \tag{6.3.1}
\end{equation*}
$$

Notice that $r(\alpha)$ acts on the vector space $\mathbb{C}^{N} \otimes H^{\alpha}$. These correpresentations will be called basic correpresentations for $\mathbb{G} 2_{*} S_{N}^{+}$.

Let $\left\{T_{p}\right\}_{p \in N C(k, l)}$ be the basis of $\operatorname{Hom}\left(u^{\otimes k}, u^{\otimes l}\right)$ for $k, l \in \mathbb{N}$. For each partition $p=$ $\left\{B_{1}, \ldots, B_{r}\right\}$, the blocks of $p$ are ordered by the lexicographical order.

We want to describe the intertwiner spaces between tensor products of basic correpresentations of $\mathbb{G} z_{*} S_{N}^{+}$. These spaces will be described by linear maps associated with certain noncrossing partitions and with $\mathbb{G}$-morphisms. Indeed, let $[\alpha]:=\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ and $[\beta]:=\left(\beta_{1}, \ldots, \beta_{l}\right)$
be tuples of $\mathbb{G}$-representations such that the points of $p$ are decorated by these correpresentations. This means that in each block $B_{i}$, certain correpresentations $\alpha_{1}^{i}, \ldots, \beta_{1}^{i}, \ldots$, are attached to the upper and lower points respectively. We make the convention that if $k=0$, then the trivial correpresentation decorates the upper part of $p \in N C(0, l)$ and an similar convention if $l=0$. The non-crossing partitions describing intertwiners in $\mathbb{G} \imath_{*} S_{N}^{+}$will also be such that their blocks are decorated by $\mathbb{G}$-morphisms. To be more precise, let us introduced some notation.
Notation 6.12. Let $p \in N C(k, l)$; its blocks are denoted by $B_{i}$ for $1 \leq i \leq r$. We will simplify the notation $B_{i}$ into $B$ when the context is clear. We denote by:

- $B=U_{B} \cup L_{B}$ the upper and lower parts of each block $B$.
- $H^{U_{B}}=\bigotimes_{i \in U_{B}} H^{\alpha_{i}}$ the tensor product of spaces $H^{\alpha_{i}}$, and similarly we write $H^{L_{B}}=$ $\otimes_{j \in L_{B}} H^{\beta_{j}}$.
- $\alpha\left(U_{B}\right)=\bigotimes_{i \in U_{B}} \alpha_{i}$ the tensor product of correpresentations $\alpha_{i}$ and similarly we write $\beta\left(L_{B}\right)=\bigotimes_{j \in L_{B}} \beta_{j}$.
Furthermore, we assume that "attached" to each block $B$ there is a $\mathbb{G}$-morphism

$$
\begin{equation*}
S_{B}=\alpha\left(U_{B}\right) \rightarrow \beta\left(L_{B}\right) \in \mathcal{B}\left(H^{U_{B}}, H^{L_{B}}\right) \tag{6.3.2}
\end{equation*}
$$

and we put

$$
\begin{equation*}
S=\bigotimes_{B} S_{B}: \bigotimes_{B} \alpha\left(U_{B}\right) \rightarrow \bigotimes_{B} \beta\left(L_{B}\right) \tag{6.3.3}
\end{equation*}
$$

with the order on the blocks we gave above. We say that the blocks of $p$ are decorated by $[S]=\left(S_{1}, \ldots, S_{r}\right)$ where $r$ is the number of blocks in $p$.

Definition 6.13. We say that the partition $p$ decorated by representations $[\alpha],[\beta]$ is admissible if $\forall B \in p, \operatorname{Hom}_{\mathbb{G}}\left(\alpha\left(U_{B}\right) ; \alpha\left(L_{B}\right)\right) \neq 0$.

Therefore, we can consider

$$
T_{p} \otimes S \in \mathcal{B}\left(\left(\mathbb{C}^{N}\right)^{\otimes k} \otimes \bigotimes_{B} H^{U_{B}} ;\left(\mathbb{C}^{N}\right)^{\otimes l} \otimes \bigotimes_{B}^{r} H^{L_{B}}\right)
$$

Remark 6.14. Notice that if the $\mathbb{G}$-morphisms in $\mathcal{B}\left(H^{U_{B}}, H^{L_{B}}\right)$ run over a basis of intertwiners $\alpha\left(U_{B}\right) \rightarrow \beta\left(L_{B}\right)$ then the family $\left(T_{p} \otimes S\right)_{p, S}$ is free.

We shall twist this linear map to obtain a morphism

$$
\widetilde{T_{p} \otimes S} \in \operatorname{Hom}_{\mathbb{G}_{2} S_{N}^{S}}\left(r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right), r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right) .
$$

Notation 6.15. Let $p \in N C(k, l)$ be decorated by $\mathbb{G}$-representations $[\alpha],[\beta]$ and morphisms $[S]$ as in the above notation. One can consider a unitary $t_{p}^{U}$ acting on vectors $x_{i} \in \mathbb{C}^{N}, y_{i} \in H^{\alpha_{i}}$, $i=1, \ldots, k$

$$
\begin{aligned}
t_{p}^{U}:\left(\mathbb{C}^{N} \otimes H^{\alpha_{1}}\right) \otimes \cdots \otimes\left(\mathbb{C}^{N} \otimes H^{\alpha_{k}}\right) & \rightarrow\left(\mathbb{C}^{N}\right)^{\otimes k} \otimes \bigotimes_{B} H^{U_{B}}, \\
\bigotimes_{i=1}^{k}\left(x_{i} \otimes y_{i}\right) & \mapsto \bigotimes_{i=1}^{k} x_{i} \otimes \bigotimes_{B} \bigotimes_{i^{\prime} \in U_{B}} y_{i^{\prime}}
\end{aligned}
$$

and a unitary $t_{p}^{L}$ acting on vectors $x_{j} \in \mathbb{C}^{N}, y_{j} \in H^{\beta_{j}}, j=1, \ldots, l$

$$
\begin{aligned}
t_{p}^{L}:\left(\mathbb{C}^{N} \otimes H^{\beta_{1}}\right) \otimes \cdots \otimes\left(\mathbb{C}^{N} \otimes H^{\beta_{l}}\right) & \rightarrow\left(\mathbb{C}^{N}\right)^{\otimes l} \otimes \bigotimes_{B} H^{L_{B}}, \\
\bigotimes_{j=1}^{l}\left(x_{j} \otimes y_{j}\right) & \mapsto \bigotimes_{j=1}^{l} x_{j} \otimes \bigotimes_{B} \bigotimes_{j^{\prime} \in L_{B}} y_{j^{\prime}} .
\end{aligned}
$$

We set

$$
\begin{aligned}
U^{p, S} & :=\left(t_{L}^{p}\right)^{*} \circ\left(T_{p} \otimes S\right) \circ t_{U}^{p} \\
& \in \mathcal{B}\left(\left(\mathbb{C}^{N} \otimes H^{\alpha_{1}}\right) \otimes \cdots \otimes\left(\mathbb{C}^{N} \otimes H^{\alpha_{k}}\right),\left(\mathbb{C}^{N} \otimes H^{\beta_{1}}\right) \otimes \cdots \otimes\left(\mathbb{C}^{N} \otimes H^{\beta_{l}}\right)\right)
\end{aligned}
$$

We can now prove the following result:
Theorem 6.16. Let $\mathbb{G}=(C(\mathbb{G}), \Delta)$ be a compact quantum group of Kac type. Let $\alpha_{1}, \ldots, \alpha_{k}$ and $\beta_{1}, \ldots, \beta_{l}$ be finite dimensional correpresentations in $\operatorname{Rep}(\mathbb{G})$. We set $[\alpha]=\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ and $[\beta]=\left(\beta_{1}, \ldots, \beta_{l}\right)$. Then

$$
\begin{align*}
\operatorname{Hom}_{\mathfrak{G} \ell * S_{N}^{+}}\left(r\left(\alpha_{1}\right) \otimes \ldots \otimes\right. & \left.r\left(\alpha_{k}\right) ; r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right)  \tag{6.3.4}\\
& =\operatorname{span}\left\{U^{p, S}: p \in N C_{\mathbb{G}}([\alpha],[\beta]), S \text { as below }\right\} \tag{6.3.5}
\end{align*}
$$

where $U^{p, S}=\left(t_{L}^{p}\right)^{*} \circ\left(T_{p} \otimes S\right) \circ t_{U}^{p}$ with

- the isomorphisms $t_{U}^{p}, t_{L}^{p}$ defined in Notation 6.15,
- $N C_{\mathbb{G}}([\alpha],[\beta])$ consists of non-crossing partitions in $N C(k, l)$ decorated with correpresentations $[\alpha],[\beta]$ on the upper and lower points respectively,
- $S=\bigotimes_{B} S_{B}: \otimes_{B} \alpha\left(U_{B}\right) \rightarrow \bigotimes_{B} \beta\left(L_{B}\right)$ as in (6.3.3), where the $\mathbb{G}$-morphisms in $\mathcal{B}\left(U_{B}, L_{B}\right)$ which decorate the blocks $B \in p$ run over intertwiners in $\operatorname{Hom}_{\mathbb{G}}\left(\alpha\left(U_{B}\right), \beta\left(L_{B}\right)\right)$.

Proof. We first prove that

$$
U^{p, S} \in \operatorname{Hom}_{\mathbb{G} \psi S_{N}^{+}}\left(r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right) ; r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right),
$$

which is the inclusion of the right hand space (6.3.5) in the left hand space (6.3.4).
The Frobenius reciprocity for $C^{*}$-tensor categories with conjugates provide the following isomorphisms:

$$
\begin{aligned}
\operatorname{Hom}_{G_{\imath} S_{N}^{+}}\left(r\left(\alpha_{1}\right)\right. & \left.\otimes \cdots \otimes r\left(\alpha_{k}\right) ; r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right) \\
& \simeq \operatorname{Hom}_{G_{l} * S_{N}^{+}}\left(1 ; \overline{r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right)} \otimes r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right) \\
& \simeq \operatorname{Hom}_{G_{2} * S_{N}^{+}}\left(1 ; r\left(\bar{\alpha}_{k}\right) \otimes \cdots \otimes r\left(\bar{\alpha}_{1}\right) \otimes r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right),
\end{aligned}
$$

$$
\operatorname{Hom}_{\mathbb{G}}\left(\alpha_{1} \otimes \cdots \otimes \alpha_{k} ; \beta_{1} \otimes \cdots \otimes \beta_{l}\right) \simeq \operatorname{Hom}_{\mathbb{G}}\left(1 ; \bar{\alpha}_{k} \otimes \cdots \otimes \bar{\alpha}_{1} \otimes \beta_{1} \otimes \cdots \otimes \beta_{l}\right)
$$

Hence, it is enough to prove that

$$
\begin{equation*}
t_{L}^{p}\left(T_{p} \otimes \xi\right) \in \operatorname{Hom}_{G \in \nleftarrow S_{N}^{+}}\left(1 ; r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right)\right) \tag{6.3.6}
\end{equation*}
$$

for all $k \in \mathbb{N}, p \in N C(k)$ and all fixed vectors

$$
\xi=\bigotimes_{B} \xi\left(L_{B}\right): \mathbb{C} \rightarrow \bigotimes_{B} H^{L_{B}} .
$$

Moreover it is enough to prove 6.3.6) for the one block partition $1_{k}$ since one can recover any $p \in N C$ by tensor products and compositions of partitions $1_{k}, k \geq 1$ and id.

We now fix $\left(e_{i}\right)_{i=1}^{N}$ a basis of $\mathbb{C}^{N}$ and $\left(Y_{j}^{\alpha}\right)_{j=1}^{d_{\alpha}}$ a basis of $H^{\alpha}$, for any $\alpha \in \operatorname{Rep}(\mathbb{G})$. Let us prove taht $t_{L}^{1_{k}}\left(T_{p} \otimes \xi\right) \in \operatorname{Hom}_{\mathscr{G} l_{*} S_{N}^{+}}\left(1 ; r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right)\right)$ for any

$$
\xi=\sum_{[j]} \lambda_{[j]}^{k} Y_{j_{1}}^{\alpha_{1}} \otimes \cdots \otimes Y_{j_{k}}^{\alpha_{k}} \in \operatorname{Hom}_{\mathbb{G}}\left(1 ; \alpha_{1} \otimes \cdots \otimes \alpha_{k}\right)
$$

Setting $T_{\xi}^{1_{k}}:=t_{L}^{1_{k}}\left(T_{p} \otimes \xi\right)$ yields

$$
T_{\xi}^{1_{k}} \equiv \sum_{i,[j]} \lambda_{[j]}^{k}\left(e_{i} \otimes Y_{j_{1}}^{\alpha_{1}}\right) \otimes \cdots \otimes\left(e_{i} \otimes Y_{j_{k}}^{\alpha_{k}}\right)
$$

so that

$$
\begin{aligned}
r_{\alpha_{1}} \otimes \cdots \otimes r_{\alpha_{k}}\left(T_{\xi}^{11_{k}} \otimes 1\right)=\sum_{i,[j]} \lambda_{[j]}^{k} \sum_{[r],[s]}\left(e_{s_{1}} \otimes\right. & \left.Y_{r_{1}}^{\alpha_{1}}\right) \otimes \cdots \otimes\left(e_{s_{k}} \otimes Y_{r_{k}}^{\alpha_{k}}\right) \\
& \otimes\left(u_{s_{1} i}\left(\alpha_{1}\right)_{r_{1} j_{1}}^{\left(s_{1}\right)} \ldots u_{s_{k} i}\left(\alpha_{k}\right)_{r_{k} j_{k}}^{\left(s_{k}\right)}\right)
\end{aligned}
$$

But the magic unitary $u$ satisfies for all $s, t, u_{s i} u_{t i}=\delta_{s t} u_{s i}, \sum_{i} u_{s i}=1$ and then combining this with the commuting relations in the free wreath product $C(\mathbb{G}) *_{w} C\left(S_{N}^{+}\right)$, we get

$$
\begin{align*}
r_{\alpha_{1}} \otimes \cdots \otimes r_{\alpha_{k}}\left(T_{\xi}^{1 k_{k}} \otimes 1\right)= & \sum_{[j]} \lambda_{[j]}^{k} \sum_{[r], s_{1}}\left(e_{s_{1}} \otimes Y_{r_{1}}^{\alpha_{1}}\right) \otimes \cdots \otimes\left(e_{s_{1}} \otimes Y_{r_{k}}^{\alpha_{k}}\right)  \tag{6.3.7}\\
& \otimes\left(\left(\alpha_{1}\right)_{r_{1} j_{1}}^{\left(s_{1}\right)} \cdots\left(\alpha_{k}\right)_{r_{k} j_{k}}^{\left(s_{1}\right)} 1\right) \\
= & \sum_{s_{1}} \sum_{[j]} \lambda_{[j]}^{k} \sum_{[r]}^{[r]}\left(e_{s_{1}} \otimes Y_{r_{1}}^{\alpha_{1}}\right) \otimes \cdots \otimes\left(e_{s_{1}} \otimes Y_{r_{k}}^{\alpha_{k}}\right)  \tag{6.3.8}\\
& \otimes\left(\left(\alpha_{1}\right)_{r_{1} j_{1}}^{\left(s_{1}\right)} \ldots\left(\alpha_{k}\right)_{r_{k} j_{k}}^{\left(s_{1}\right)}\right) . \tag{6.3.9}
\end{align*}
$$

Now, since

$$
\begin{equation*}
\xi=\sum_{[j]} \lambda_{[j]}^{k} Y_{j_{1}}^{\alpha_{1}} \otimes \cdots \otimes Y_{j_{k}}^{\alpha_{k}} \in \operatorname{Hom}_{\mathbb{G}}\left(1 ; \alpha_{1} \otimes \cdots \otimes \alpha_{k}\right) \tag{6.3.11}
\end{equation*}
$$

applying $\left(t_{L}^{1_{k}}\right)^{-1}$ in 6.3.7), using 6.3.11 and then applying $t_{L}^{1_{k}}$ yield

$$
\begin{aligned}
r_{\alpha_{1}} \otimes \cdots \otimes r_{\alpha_{k}}\left(T_{\xi_{1_{k}}}^{1_{k}} \otimes 1\right) & =\sum_{[r], s_{1}} \lambda_{[r]}^{k}\left(e_{s_{1}} \otimes Y_{r_{1}}^{\alpha_{1}}\right) \otimes \cdots \otimes\left(e_{s_{1}} \otimes Y_{r_{k}}^{\alpha_{k}}\right) \otimes 1 \\
& =T_{\xi}^{1_{k}} \otimes 1
\end{aligned}
$$

This proves that $T_{\xi}^{1_{k}} \in \operatorname{Hom}_{\mathbb{G} \downarrow * S_{N}^{+}}\left(1 ; r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right)\right)$.

A straightforward computation shows that the collection of $\mathbb{G} 2_{*} S_{N}^{+}$-intertwiners spaces $\operatorname{span}\left\{U^{p, S}: p, S\right.$ as in 6.3 .5$\}$ is stable by composition, tensor product and duality. Therefore this collection of vector spaces defines a rigid monoidal $C^{*}$-tensor category $\mathcal{T}$, with objects indexed by families $[\alpha]$ of $\mathbb{G}$-representations.
If one applies Woronowicz's Tannaka-Krein duality to this category $\mathcal{T}$, we get a compact matrix quantum group $(\mathbb{H}, \Omega)$ generated by a unitary $\Omega$ corresponding to $r(v) \in \mathcal{B}\left(\mathbb{C}^{N} \otimes H\right) \otimes C(\mathbb{H})$ and a family of correpresentations $\left(R_{\alpha_{i}}\right)_{i \in I}$ such that

$$
\begin{aligned}
\operatorname{Hom}_{\mathbb{H}}\left(R_{\alpha_{1}} \otimes \ldots\right. & \left.\otimes R_{\alpha_{k}} ; R_{\beta_{1}} \otimes \cdots \otimes R_{\beta_{l}}\right) \\
& \left.=\operatorname{span}\left\{U^{p, S}: p, S \text { as in } 6.3 .5\right)\right\}
\end{aligned}
$$

with $p \in N C(k, l), S: \bigotimes_{B} \alpha\left(U_{B}\right) \rightarrow \bigotimes_{B} \beta\left(L_{B}\right),[\alpha]=\left(\alpha_{1}, \ldots, \alpha_{k}\right),[\beta]=\left(\beta_{1}, \ldots, \beta_{l}\right)$.
We proved above that

$$
U^{p, S} \in \operatorname{Hom}_{\mathbb{G} l_{*} S_{N}^{+}}\left(r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right) ; r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right)
$$

In particular, there is by universality a (surjective) morphism

$$
\pi_{1}: C(\mathbb{H}) \rightarrow C\left(\mathbb{G}{l_{*}} S_{N}^{+}\right), \quad \Omega_{i j k l} \mapsto \omega_{i j k l}
$$

To prove the theorem we shall construct a surjective morphism $\pi_{2}: C\left(\mathbb{G} z_{*} S_{N}^{+}\right) \rightarrow C(\mathbb{H})$ such that

$$
\pi_{1} \circ \pi_{2}=\mathrm{id}=\pi_{2} \circ \pi_{1}
$$

We define the following elements in $C(\mathbb{H})$

$$
\begin{equation*}
V_{k l}^{(i)}:=\sum_{j} \Omega_{i j k l} \quad \text { and } \quad U_{i j k}=\sum_{l} \Omega_{i j k l} \Omega_{i j k l}^{*} \tag{6.3.12}
\end{equation*}
$$

We shall prove that the generating relations in $C\left(\mathbb{G} 2_{*} S_{N}^{+}\right)$are also satisfied by the elements $V_{k l}^{(i)}$ and $U_{i j k}$ in $C(\mathbb{H})$.

Since the generating matrix $v$ of $\mathbb{G}$ is unitary, we get that $\xi=\sum_{b} Y_{b} \otimes \bar{Y}_{b}$ is a fixed vector of $v \otimes \bar{v}$ and thus $\xi \otimes \xi \in \operatorname{Hom}(1 ; v \otimes \bar{v} \otimes v \otimes \bar{v}) \simeq \operatorname{Hom}(v ; v \otimes \bar{v} \otimes v)$. Via this isomorphism, we identify $\xi \otimes \xi$ with $Y \mapsto \sum_{c} Y_{c} \otimes \overline{Y_{c}} \otimes Y$.

We then have an intertwiner $T:=t_{L}^{p}\left(T_{p} \otimes \xi \otimes \xi\right) \in \operatorname{Hom}(\Omega ; \Omega \otimes \bar{\Omega} \otimes \Omega) \subseteq \mathcal{T}$ with

$$
p=\left\{\begin{array}{l} 
\\
|\quad|
\end{array}\right\}
$$

i.e. with Notation 6.12 and making plain the $\mathbb{G}$-morphisms on the block $p$,

$$
P=\left\{\begin{array}{c}
\prod_{v}^{v} \prod_{v}^{v} \|_{v}
\end{array}\right\}
$$

corresponding to the linear map

$$
T\left(Y \otimes e_{a}\right)=\sum_{c}\left(e_{a} \otimes Y_{c}\right) \otimes \overline{\left(e_{a} \otimes Y_{c}\right)} \otimes\left(e_{a} \otimes Y\right)
$$

We obtain for all $a=1, \ldots, N$ and $b=1, \ldots, d_{\mathbb{G}}$ :

$$
\begin{aligned}
& \sum_{[i],[k], c}\left(e_{i_{1}} \otimes Y_{k_{1}}\right) \otimes\left(e_{i_{2}} \otimes Y_{k_{2}}\right) \otimes\left(e_{i_{3}} \otimes Y_{k_{3}}\right) \otimes \Omega_{i_{1} a k_{1} c} \Omega_{i_{2} a k_{2} c}^{*} \Omega_{i_{3} a k_{3} b} \\
& =\sum_{i, k, r}\left(e_{i} \otimes Y_{r}\right) \otimes\left(e_{i} \otimes Y_{r}\right) \otimes\left(e_{i} \otimes Y_{k}\right) \otimes \Omega_{i a k b}
\end{aligned}
$$

so that for all $[i] \in\{1, \ldots, N\}^{3},[k] \in\left\{1, \ldots, d_{\mathbb{G}}\right\}^{2}, a \in\{1, \ldots, N\}$ and $b \in\left\{1, \ldots, d_{\mathbb{G}}\right\}$ :

$$
\begin{equation*}
\left(\sum_{c} \Omega_{i_{1} a k_{1} c} \Omega_{i_{2} a k_{2} c}^{*}\right) \Omega_{i_{3} a k_{3} b}=\delta_{i_{1}, i_{2}, i_{3}} \delta_{k_{1}, k_{2}} \Omega_{i_{3} a k_{3} b} . \tag{6.3.13}
\end{equation*}
$$

and taking adjoints:

$$
\begin{equation*}
\Omega_{i_{3} a k_{3} b}^{*}\left(\sum_{c} \Omega_{i_{2} a k_{2} c} \Omega_{i_{1} a k_{1} c}^{*}\right)=\delta_{i_{1}, i_{2}, i_{3}} \delta_{k_{1}, k_{2}} \Omega_{i_{3} a k_{3} b}^{*} . \tag{6.3.14}
\end{equation*}
$$

Considering now

$$
P^{\prime}=\left\{\begin{array}{l}
\prod_{v}^{v} \\
\prod_{v} \prod_{\bar{v}}
\end{array}\right\}
$$

we can get the same way, for all $[i] \in\{1, \ldots, N\}^{3},[k] \in\left\{1, \ldots, d_{\mathbb{G}}\right\}^{2}, a \in\{1, \ldots, N\}$ and $b \in\left\{1, \ldots, d_{\mathbb{G}}\right\}$, using $t_{L}^{p}\left(T_{p^{\prime}} \otimes \xi \otimes \xi\right) \in \operatorname{Hom}(\Omega ; \Omega \otimes \Omega \otimes \bar{\Omega}) \subseteq \mathcal{T}$,

$$
\begin{equation*}
\Omega_{i_{3} a k_{3} b}\left(\sum_{c} \Omega_{i_{1} a k_{1} c} \Omega_{i_{2} a k_{2} c}^{*}\right)=\delta_{i_{1}, i_{2}, i_{3}} \delta_{k_{1}, k_{2}} \Omega_{i_{3} a k_{3} b}, \tag{6.3.15}
\end{equation*}
$$

and taking adjoints:

$$
\begin{equation*}
\left(\sum_{c} \Omega_{i_{2} a k_{2} c} \Omega_{i_{1} a k_{1} c}^{*}\right) \Omega_{i_{3} a k_{3} b}^{*}=\delta_{i_{1}, i_{2}, i_{3}} \delta_{k_{1}, k_{2}} \Omega_{i_{3} a k_{3} b}^{*} . \tag{6.3.16}
\end{equation*}
$$

We shall obtain from (6.3.13), (6.3.14), (6.3.15), 6.3.16) all the necessary relations in $C(\mathbb{H})$ to reconstruct the free wreath product $\mathbb{G} \imath_{*} S_{N}^{+}$.

From these relations, we see in particular that the elements $U_{i j k}=\sum_{c} \Omega_{i j k c} \Omega_{i j k c}^{*}$ do not depend on $k$ since

$$
\begin{aligned}
U_{i j k} U_{i j k^{\prime}} & =\sum_{c, d} \Omega_{i j k c} \Omega_{i j k c}^{*} \Omega_{i j k^{\prime} d} \Omega_{i j k^{\prime} d}^{*} \\
& =\sum_{d}\left(\sum_{c} \Omega_{i j k c} \Omega_{i j k c}^{*} \Omega_{i j k^{\prime} d}\right) \Omega_{i j k^{\prime} d}^{*} \\
& =\sum_{d} \Omega_{i j k^{\prime} d} \Omega_{i j k^{\prime} d}^{*}=U_{i j k^{\prime}}
\end{aligned}
$$

and similarly $U_{i j k} U_{i j k^{\prime}}=U_{i j k}$, using (6.3.15). We then obtain $U_{i j k}=U_{i j k^{\prime}}$. Let us simply write $U_{i j}:=U_{i j k}$. Notice that the case $k=k^{\prime}$ above shows that $U_{i j}$ is an orthogonal projection (the
relation $U_{i j}^{*}=U_{i j}$ is clear). In fact, the matrix $\left(U_{i j}\right)$ is a magic unitary, since it is a unitary whose entries are orthogonal projections.

We now prove that for all $i=1, \ldots, N$ and all $\varepsilon_{j}, \varepsilon_{k}^{\prime} \in\{1, *\}$,

$$
\operatorname{Hom}_{\mathbb{G}}\left(v^{\varepsilon_{1}} \otimes \cdots \otimes v^{\varepsilon_{k}} ; v^{\varepsilon_{1}^{\prime}} \otimes \cdots \otimes v^{\varepsilon_{l}^{\prime}}\right) \subseteq \operatorname{Hom}_{\mathbb{H}_{i}}\left(V^{(i) \varepsilon_{1}} \otimes \cdots \otimes V^{(i) \varepsilon_{k}} ; V^{(i) \varepsilon_{1}^{\prime}} \otimes \cdots \otimes V^{(i) \varepsilon_{l}^{\prime}}\right)
$$

where $\mathbb{H}_{i}$ is the compact matrix quantum groups whose underlying Woronowicz- $C^{*}$-algebra is generated by the coefficients of $V^{(i)}$. By Frobenius reciprocity, it is enough to prove that any fixed vector in $\mathbb{G}$ is fixed in $\mathbb{H}_{i}$.

If $\xi_{k}=\sum_{[j]} \lambda_{[j]} Y_{j_{1}} \otimes \cdots \otimes Y_{j_{k}} \in \operatorname{Hom}\left(1 ; v^{\varepsilon_{1}} \otimes \cdots \otimes v^{\varepsilon_{k}}\right)$, we have:

$$
\sum_{[r][j]} \lambda_{[j]} Y_{r_{1}} \otimes \cdots \otimes Y_{r_{k}} \otimes v_{r_{1} j_{1}}^{\varepsilon_{1}} \ldots v_{r_{k} j_{k}}^{\varepsilon_{k}}=\sum_{[r]} \lambda_{[r]} Y_{r_{1}} \otimes \cdots \otimes Y_{r_{k}} \otimes 1
$$

i.e. $\forall[r] \in\left\{1, \ldots, d_{\mathbb{G}}\right\}^{k}$, we have the following relations in $C(\mathbb{G})$ :

$$
\begin{equation*}
\sum_{[j]} \lambda_{[j]} v_{r_{1} j_{1}}^{\varepsilon_{1}} \ldots v_{r_{k} j_{k}}^{\varepsilon_{k}}=\lambda_{[r]} . \tag{6.3.17}
\end{equation*}
$$

Now, we use the morphism $\left(t_{L}^{p}\right)^{*} \circ\left(T_{p} \otimes \xi_{k}\right) \in \mathcal{T}$, with $p=1_{k} \in N C(k)$ i.e.

$$
\begin{aligned}
\left(t_{L}^{p}\right)^{*} \circ\left(\xi_{k} \otimes T_{p}\right) & =\sum_{i[j]} \lambda_{[j]}\left(e_{i} \otimes Y_{j_{1}}\right) \otimes \cdots \otimes\left(e_{i} \otimes Y_{j_{k}}\right) \\
& \in \operatorname{Hom}\left(1 ; \Omega^{\varepsilon_{1}} \otimes \cdots \otimes \Omega^{\varepsilon_{k}}\right) \subseteq \mathcal{T}
\end{aligned}
$$

We get

$$
\begin{gather*}
\sum_{[r][t]}\left(e_{r_{1}} \otimes Y_{t_{1}}\right) \otimes \cdots \otimes\left(e_{r_{k}} \otimes Y_{t_{k}}\right) \otimes \sum_{i[j]} \lambda_{[j]} \Omega_{r_{1} i t_{1} j_{1}}^{\varepsilon_{1}} \ldots \Omega_{r_{k} i t_{k} j_{k}}^{\varepsilon_{k}}  \tag{6.3.18}\\
=\sum_{r[t]} \lambda_{[t]}\left(e_{r} \otimes Y_{t_{1}}\right) \otimes \cdots \otimes\left(e_{r} \otimes Y_{t_{k}}\right) \otimes 1 . \tag{6.3.19}
\end{gather*}
$$

Notice that the relations (6.3.13, 6.3.14), 6.3.15, 6.3.16 yield for $\varepsilon=1, *$ and all $i, j, k, l$ :

$$
\begin{equation*}
U_{i j} \Omega_{i j k l}^{\varepsilon}=\Omega_{i j k l}^{\varepsilon}=\Omega_{i j k l}^{\varepsilon} U_{i j} \tag{6.3.20}
\end{equation*}
$$

Then using these commuting relations and the fact that $\left(U_{i j}\right)$ is a magic unitary, we get from
(6.3.18):

$$
\begin{aligned}
\sum_{[r][t]}\left(e_{r_{1}} \otimes Y_{t_{1}}\right) & \otimes \cdots \otimes\left(e_{r_{k}} \otimes Y_{t_{k}}\right) \otimes \sum_{i[j]} \lambda_{[j]} \Omega_{r_{1} i t_{1} j_{1}}^{\varepsilon_{1}} \ldots \Omega_{r_{k} i t_{k} j_{k}}^{\varepsilon_{k}} \\
& =\sum_{[r][t]}\left(e_{r_{1}} \otimes Y_{t_{1}}\right) \otimes \cdots \otimes\left(e_{r_{k}} \otimes Y_{t_{k}}\right) \otimes \sum_{i[j]} \lambda_{[j]}\left(\Omega_{r_{1} i t_{1} j_{1}}^{\varepsilon_{1}} U_{r_{1} i}\right) \ldots\left(\Omega_{r_{k} i t_{k} j_{k}}^{\varepsilon_{k}} U_{r_{k} i}\right) \\
& =\sum_{r_{1}[t]}\left(e_{r_{1}} \otimes Y_{t_{1}}\right) \otimes \cdots \otimes\left(e_{r_{1}} \otimes Y_{t_{k}}\right) \otimes \sum_{i[j]} \lambda_{[j]}\left(\Omega_{r_{1} t_{1} j_{1}}^{\varepsilon_{1}} \ldots \Omega_{r_{1} i t_{k} j_{k}}^{\varepsilon_{k}}\right)\left(U_{r_{1} i} \ldots U_{r_{1} i}\right) \\
& =\sum_{r_{1}[t]}\left(e_{r_{1}} \otimes Y_{t_{1}}\right) \otimes \cdots \otimes\left(e_{r_{1}} \otimes Y_{t_{k}}\right) \otimes \sum_{[i][j]} \lambda_{[j]}\left(\Omega_{r_{1} i t_{1} j_{1}}^{\varepsilon_{1}} \ldots \Omega_{r_{1} i t_{k} j_{k}}^{\varepsilon_{k}}\right)\left(U_{r_{1} i_{1}} \ldots U_{r_{1} i_{k}}\right) \\
& =\sum_{r_{1}[t]}\left(e_{r_{1}} \otimes Y_{t_{1}}\right) \otimes \cdots \otimes\left(e_{r_{1}} \otimes Y_{t_{k}}\right) \otimes \sum_{[i][j]} \lambda_{[j]}\left(\Omega_{r_{1} i_{1} t_{1} j_{1}}^{\varepsilon_{1}} U_{r_{1} i_{1}}\right) \ldots\left(\Omega_{r_{1} i_{k} t_{k} j_{k}}^{\varepsilon_{k}} U_{r_{1} i_{k}}\right) \\
& =\sum_{r_{1}[t]}\left(e_{r_{1}} \otimes Y_{t_{1}}\right) \otimes \cdots \otimes\left(e_{r_{1}} \otimes Y_{t_{k}}\right) \otimes \sum_{[j]} \lambda_{[j]} V_{t_{1} j_{1}}^{\left(r_{1}\right) \varepsilon_{1}} \ldots V_{t_{k} j_{k}}^{\left(r_{1}\right) \varepsilon_{k}}
\end{aligned}
$$

Hence with 6.3.19, we obtain $\forall[t] \in\left\{1, \ldots, d_{\mathbb{G}}\right\}^{k}$

$$
\sum_{[j]} \lambda_{[j]} V_{t_{1} j_{1}}^{\left(r_{1}\right) \varepsilon_{1}} \ldots V_{t_{k} j_{k}}^{\left(r_{1}\right) \varepsilon_{k}}=\lambda_{[t]}
$$

so that $\xi_{k}=\sum_{[j]} \lambda_{[j]} Y_{j_{1}} \otimes \cdots \otimes Y_{j_{k}} \in \operatorname{Hom}_{\mathbb{H}_{r}}\left(1 ; V^{(r) \varepsilon_{1}} \otimes \cdots \otimes V^{(r) \varepsilon_{k}}\right)$ for all $r=1, \ldots, N$.
Then, we obtain that $\operatorname{Rep}(\mathbb{G}) \subseteq \operatorname{Rep}\left(\mathbb{H}_{i}\right) \subseteq \operatorname{Rep}(\mathbb{H})$ as full sub-categories. Woronowicz's Tannaka-Krein duality theorem then implies that for all $i=1, \ldots, N$ there exists a morphism

$$
\pi_{i}: C(\mathbb{G}) \rightarrow C\left(\mathbb{H}_{i}\right) \subseteq C(\mathbb{H})
$$

sending $v$ to $V^{(i)}$.
Now, we prove that $V_{k l}^{(i)} U_{i j}=\Omega_{i j k l}=U_{i j} V_{k l}^{(i)}$. This follows from 6.3.20):

$$
V_{k l}^{(i)} U_{i j}=\sum_{J} \Omega_{i J k l} U_{i j}=\Omega_{i j k l} U_{i j}=\Omega_{i j k l}
$$

and similarly

$$
U_{i j} V_{k l}^{(i)}=\Omega_{i j k l}
$$

It follows from what we have proved above that there exist morphisms

- $\pi_{i}: C(\mathbb{G}) \rightarrow C\left(\mathbb{H}_{i}\right)$ such that $\pi_{i}\left(v_{k l}^{(i)}\right)=V_{k l}^{(i)}$, for all $i=1, \ldots, N$,
- $\pi_{N+1}: C\left(S_{N}^{+}\right) \rightarrow C(\mathbb{H})$ such that $\pi_{N+1}\left(u_{i j}\right)=U_{i j}$.

Thanks to the commuting relations we obtained above, these morphisms induce a morphism $\pi_{2}: C\left(\mathbb{G} l_{*} S_{N}^{+}\right) \rightarrow C(\mathbb{H})$, such that $\pi_{2}\left(v_{k l}^{(i)} u_{i j}\right)=V_{k l}^{(i)} U_{i j}$. By construction, we then get $\pi_{1} \circ \pi_{2}=\mathrm{id}=\pi_{2} \circ \pi_{1}$ and the proof is complete.

Remark 6.17. In the case where $\mathbb{G}$ is the dual of a discrete (classical) group $\mathbb{G}=\widehat{\Gamma}$, we recover the results of [16] and [55]. Indeed, in this case, the irreducible correpresentations of $\mathbb{G}=\left(C^{*}(\Gamma), \Delta\right)$ are the one-dimensional group like correpresentations $\Delta(g)=g \otimes g, g \in \Gamma$, the
trivial one is the neutral element $e$ and the tensor product of two irreducible correpresentations is their product in $\Gamma$. Any morphism

$$
S_{[g],[h]}: \mathbb{C} \simeq \mathbb{C}^{\otimes k} \rightarrow \mathbb{C}^{\otimes l} \simeq \mathbb{C}, g_{1} \ldots g_{k} \rightarrow h_{1} \ldots h_{l}
$$

is determined by the image of $1 \in \mathbb{C}$ and the tensor products $S_{[g],[h]} \otimes T_{p}$ are scalar multiplication of the linear maps $T_{p}$. The space

$$
\operatorname{Hom}_{\widehat{\Gamma} l_{*} S_{N}^{+}}\left(r\left(g_{1}\right) \otimes \cdots \otimes r\left(g_{k}\right) ; r\left(h_{1}\right) \otimes \cdots \otimes r\left(h_{l}\right)\right)
$$

is generated by the maps $T_{p}$ where $p \in N C(k, l)$ is an admissible diagram in $N C_{\widehat{\Gamma}}$ as in Definition 6.13. In this setting, $p$ is admissible if $p \in N C(k, l)$ has the additional rules that if one decorates the points of $p$ by the elements $g_{i}, h_{j}$ then in each block, the product on top is equal to the product on bottom in $\Gamma$.

In the sequel, we denote by $1_{\mathbb{G}}$ the trivial $\mathbb{G}$-representation and simply by 1 the one of $\mathbb{G} \imath_{*} S_{N}^{+}$.
Corollary 6.18. Let $N \geq 4$, then:

1. For all $\alpha_{1}, \ldots, \alpha_{k}, \beta_{1}, \ldots, \beta_{l} \in \operatorname{Rep}(\mathbb{G})$, we have

$$
\begin{aligned}
\operatorname{dim} \operatorname{Hom}_{\mathbb{G} \imath_{*} S_{N}^{+}} & \left(r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right) ; r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right) \\
& =\sum_{p \in N C_{\mathbb{G}}([\alpha],[\beta])} \prod_{B \in p} \operatorname{dim} \operatorname{Hom}_{\mathbb{G}}\left(\alpha\left(U_{B}\right), \beta\left(L_{B}\right)\right) .
\end{aligned}
$$

2. If $\alpha \in \operatorname{Irr}(\mathbb{G})$ is non-equivalent to $1_{\mathbb{G}}$ then $r(\alpha)$ is an irreducible $\mathbb{G} 2_{*} S_{N}^{+}$-representation.
3. $r\left(1_{\mathbb{G}}\right)=\left(u_{i j}\right)=1 \oplus \omega\left(1_{\mathbb{G}}\right)$ for some $\omega\left(1_{\mathbb{G}}\right) \in \operatorname{Irr}\left(\mathbb{G} \imath_{*} S_{N}^{+}\right)$.
4. Denoting $\omega(\alpha):=r(\alpha) \ominus \delta_{\alpha, 1_{\mathbb{G}}} 1$ then $(\omega(\alpha))_{\alpha \in \operatorname{Irr}(\mathbb{G})}$ is a family of pairwise non-equivalent $\mathbb{G} \nu_{*} S_{N}^{+}$-irreducible correpresentations.

Proof. We use Theorem 6.16 and the independence of the linear maps

$$
T_{p} \in \mathcal{B}\left(\left(\mathbb{C}^{N}\right)^{\otimes k},\left(\mathbb{C}^{N}\right)^{\otimes l}\right), p \in N C(k, l)
$$

for all $N \geq 4$. The first assertion follows from this linear independence of the maps $T_{p}$. Indeed, we have

$$
\begin{aligned}
\operatorname{Hom}_{\mathbb{G} \tau_{*} S_{N}^{+}} & \left(r\left(\alpha_{1}\right) \otimes \cdots \otimes r\left(\alpha_{k}\right) ; r\left(\beta_{1}\right) \otimes \cdots \otimes r\left(\beta_{l}\right)\right) \\
& =\bigoplus_{p \in N C_{\mathbb{G}}([\alpha],[\beta])} \operatorname{span}\left\{U^{p, S}: \forall B, S_{B} \in \operatorname{Hom}_{\mathbb{G}}\left(\alpha\left(U_{B}\right), \beta\left(L_{B}\right)\right)\right\}
\end{aligned}
$$

and the first assertion follows by computing the dimension of the spaces on each side.
Now we prove simultaneously the last three relations. For $\alpha, \beta \in \operatorname{Irr}(\mathbb{G})$, the intertwiner space

$$
\operatorname{Hom}_{\mathbb{G} \imath_{*} S_{N}^{+}}(r(\alpha), r(\beta))
$$

is encoded by the following candidate diagrams:

$$
p_{1}=\left\{\begin{array}{c}
\alpha \\
\mid \\
\beta
\end{array}\right\} \text { and } p_{2}=\left\{\begin{array}{c}
\alpha \\
1 \\
1 \\
\beta
\end{array}\right\} .
$$

Since $\alpha$ and $\beta$ are irreducible, we see that $p_{1}$ is an admissible diagram if and only if $\alpha \simeq \beta$ and $p_{2}$ is admissible if and only if $\alpha \simeq \beta \simeq 1_{\mathbb{G}}$.

Therefore, if $\alpha$ is not equivalent to $\beta$ :

$$
\operatorname{dim} \operatorname{Hom}_{\mathbb{G} \imath * S_{N}^{+}}(r(\alpha), r(\beta))=0 .
$$

If $\alpha \simeq \beta$ are not the trivial correpresentation $1_{\mathbb{G}}$ then the only intertwiner $r(\alpha) \rightarrow r(\beta)$ arises from $p_{1}$ :

$$
\operatorname{dim} \operatorname{Hom}_{\mathbb{G}\left\langle * S_{N}^{+}\right.}(r(\alpha), r(\alpha))=1
$$

If $\alpha \simeq \beta \simeq 1_{\mathbb{G}}$, then the diagram $p_{2}$ also gives rise to an intertwiner $U_{\left(1_{\mathbb{G}}\right),\left(1_{\mathbb{G}}\right)}^{p_{2}, S}$ with $S: 1_{\mathbb{G}} \rightarrow$ $1_{\mathbb{G}}$ the trivial inclusion. The independence of $T_{\{\mid\}}=\mathrm{id}_{\mathbb{C}^{N}}$ and $T_{\{i\}}$ allows to conclude

$$
\operatorname{dim} \operatorname{Hom}_{\mathbb{G} \psi * S_{N}^{+}}\left(r\left(1_{\mathbb{G}}\right), r\left(1_{\mathbb{G}}\right)\right)=2
$$

### 6.4 Probabilistic aspects of the free wreath product

We provide here some probabilistic consequences of the description of the intertwiner spaces of $\mathbb{G})_{*} S_{N}^{+}$. In this section we are mainly interested in the non-commutative probability space arising from the Haar state on $C\left(\mathbb{G} \iota_{*} S_{N}^{+}\right)$and the behavior of the coefficients of a correpresentation as random variables in this setting. Since most of the results involve the law of free compound Poisson laws, we shall recall its definition. We refer to [66] for an introductory course on noncommutative variables.

### 6.4.1 Laws of characters

Notation 6.19. In the sequel $\varepsilon=\varepsilon(1) \ldots \varepsilon(r)$ denotes a word in $\{1, *\}$ and $N C(\varepsilon)$ is the set of noncrossing partitions with each endpoint $i$ colored with $\varepsilon(i)$. For $p \in N C(\varepsilon)$ and $B$ a block of $p, \varepsilon(B)$ denotes the subword of $\varepsilon$ coming from the points in the block $B$ (with the same order as in $p$ ).

Let $(A, \varphi)$ be a noncommutative probability space, $X$ an element of $A$ with $*$-distribution $\mu_{X}$ depicted by all of its moments

$$
m_{X}(\varepsilon)=\varphi\left(X^{\varepsilon(1)} \ldots X^{\varepsilon(r)}\right)
$$

Similarly as in (6.2.4), the free cumulants of $X,\left\{k_{X}(\varepsilon)\right\}_{\varepsilon}$ is the unique collection of complex numbers such that the following moment-cumulant formula holds for all $\varepsilon$ :

$$
m_{X}(\varepsilon)=\sum_{p \in N C(\varepsilon)} \prod_{B} k_{X}(\varepsilon(B)) .
$$

The existence and uniqueness of such a collection is easily proved by induction on the length of $\varepsilon$ [66].

Definition 6.20. The free compound Poisson distribution $\mathcal{P}_{\lambda}\left(\mu_{X}\right)$ with law $\mu_{X}$ and parameter $\lambda>0$ is the $\star$-distribution defined by its free cumulants

$$
\begin{equation*}
k_{\mathcal{P}_{\lambda}\left(\mu_{X}\right)}(\varepsilon)=\lambda m_{X}(\varepsilon) \tag{6.4.1}
\end{equation*}
$$

In particular, if $Y$ is a random variable following a free compound Poisson distribution with law $\mu_{X}$ and parameter 1, then we have the following moment formula :

$$
m_{Y}(\varepsilon)=\sum_{p \in N C(\varepsilon)} \prod_{B} m_{X}(\varepsilon(B)) .
$$

We refer to [66] for the proof that there exists actually a propability space and a random variable on it with such a distribution.
The first result is a direct application of the Corollary 6.18. We refer to Definition 6.3.1 for the definition of the correpresentation $r(\alpha)$.
Proposition 6.21. Let $\mathbb{G}$ be a compact quantum group of Kac type, $\alpha \in \operatorname{Rep}(\mathbb{G}), n \geq 4$. Then the law of the character $\chi(r(\alpha))$, with respect to the Haar state $h$, is a free compound Poisson with law $\chi(\alpha)$ and parameter 1 .
Proof. Let $\varepsilon$ be a word in $\{1, \star\}$. Then the law of a free compound Poisson with law $\chi(\alpha)$ and parameter $1, \mathcal{P}(\chi(\alpha))$ is described by its free cumulants, with the formula 6.4.1):

$$
k_{\mathcal{P}(\chi(\alpha))}(\varepsilon(1) \ldots \varepsilon(r))=m_{\chi(\alpha)}(\varepsilon(1) \ldots \varepsilon(r) .
$$

With the moment-cumulant formula, this is equivalent to the following expression for the moments of $\mathcal{P}(\chi(\alpha))$ :

$$
m_{\mathcal{P}(\chi(\alpha))}=\sum_{p \in N C_{\varepsilon}} \prod_{B} m_{\chi(\alpha)}(\varepsilon(B)) .
$$

By the Corollary 6.18 we have

$$
\begin{aligned}
h\left(\chi\left(r(\alpha)^{\varepsilon(1)} \cdots \chi_{n}(r(\alpha))^{\varepsilon(r)}\right)\right. & =\operatorname{dim}_{\operatorname{Hom}_{\mathbb{G} \imath_{*} S_{N}^{+}}\left(1 ; r(\alpha)^{\varepsilon(1)} \otimes \cdots \otimes r(\alpha)^{\varepsilon(r)}\right)} \\
& =\sum_{p \in N C_{\varepsilon}} \prod_{B} \operatorname{dim} \operatorname{Hom}_{\mathbb{G}}\left(1, \alpha\left(L_{B}\right)\right) \\
& =\sum_{p \in N C_{\varepsilon}} \prod_{B} m_{\chi(\alpha)}(\varepsilon(B)) .
\end{aligned}
$$

The second equality is given by Corollay 6.18, and the third one by the definition of $\alpha\left(L_{B}\right)$ and the tensor product structure.

A consequence of this result is a partial answer to the free product conjecture given by Banica and Bichon (see [10]) : for each compact matrix quantum group ( $A, v$ ) we denote by $\mu(A, v)$ the law of the character of the fundamental representation with respect to the Haar measure. A quantum permutation group is a quantum subgroup of $S_{N}^{+}$for some $N \geq 0$, in the following sense : it is a compact matrix quantum group $(A, v)$ such that there exists a surjective $C^{*}$-morphism $\Phi: C\left(S_{N}^{+}\right) \rightarrow A$ sending the elements $u_{i j}$ of $C\left(S_{N}^{+}\right)$to $v_{i j}$ (see 82 for a survey on the subject).
Corollary 6.22. Let $(A, v)$ be a quantum permutation group, and $S_{N}^{+}=\left(C\left(S_{N}^{+}, u\right), n \geq 4\right.$. Then

$$
\mu\left(A \imath_{*} B, w\right)=\mu(A, v) \boxtimes \mu\left(C\left(S_{N}^{+}, u\right) .\right.
$$

Proof. It is a direct consequence of the last proposition and the fact that in the law of a free compound poisson with law $\mu$ is the same as the free multiplicative convolution of $\mu$ with the free Poisson distribution.

The conjecture asserts that this formula still holds when replacing $S_{N}^{+}$with certain quantum subgroups of $S_{N}^{+}$. See [10] for more details.

### 6.4.2 Weingarten calculus

Let us construct a Weingarten calculus for a free wreath product. Weingarten calculus has been mainly developped in the framework of compact quantum groups and permutation quantum groups by Banica and Collins (see [12,,[13]). This tool has mainly two advantages : on one hand it allows us sometimes to get some interesting formulae for the Haar state on the matrix entries of a correpresentation, and on the other hand it yields some asymptotic results on the joint law of a finite set of elements when the dimension of the quantum group goes to infinity.

Let us first sum up the pattern of this method coming from [12]: let $\mathbb{G}=\left(A,\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)$ be a matrix compact quantum group acting on $V^{\otimes k}=\left\langle X_{i}\right\rangle_{1 \leq i \leq n}^{\otimes k}$ with the correpresentation $\alpha_{k}$, and $h$ the associated Haar measure. We will assume that $\mathbb{G}$ is orthogonal to simplify the notations, although it could be easily generalized to the general Kac type case : that means that the elements $u_{i j}$ are all self-adjoint in $A$ (see [99]). By the property of the Haar state,

$$
(I d \otimes h) \circ \alpha_{k}\left(X_{i_{1}} \otimes \cdots \otimes X_{i_{k}}\right)=P\left(X_{i_{1}} \otimes \cdots \otimes X_{i_{k}}\right)
$$

with $P$ the orthogonal projection of $V^{\otimes k}$ on the invariant subspace of $\alpha_{k}$. On the other hand,

$$
(I d \otimes h) \circ \alpha_{k}\left(X_{i_{1}} \otimes \cdots \otimes X_{i_{k}}\right)=\sum h\left(u_{j_{1} i_{1}} \ldots u_{j_{k} i_{k}}\right)\left(X_{j_{1}} \otimes \cdots \otimes X_{j_{k}}\right)
$$

We get thus the following expression for the Haar state on $u_{j_{1} i_{1}} \ldots u_{j_{k} i_{k}}$ :

$$
h\left(u_{j_{1} i_{1}} \ldots u_{j_{k} i_{k}}\right)=\left\langle P\left(X_{i_{1}} \otimes \cdots \otimes X_{i_{k}}\right), X_{j_{1}} \otimes \cdots \otimes X_{j_{k}}\right\rangle
$$

The right-hand side may be hard to compute. Hopefully the Gram-Schmidt orthogonalisation process yields a nicer expression if we already know a basis of the invariant subspace $S_{k}$ of $\alpha_{k}$. Let $\left\{S_{k}(i)\right\}$ be a basis of this subspace, $G_{k}$ being the Gram-Schmidt matrix of $\left\{S_{k}(i)\right\}$ defined by $G_{k}(i, j)=\left\langle S_{k}(i), S_{k}(j)\right\rangle$ and $W_{k}=G_{k}^{-1}$. A standard computation yields:

$$
h\left(u_{j_{1} i_{1}} \ldots u_{j_{k} i_{k}}\right)=\sum_{i, j}\left\langle X_{i_{1}} \otimes \cdots \otimes X_{i_{k}}, S_{k}(i)\right\rangle W_{k}(i, j)\left\langle S_{k}(j), X_{j_{1}} \otimes \cdots \otimes X_{j_{k}}\right\rangle
$$

Of course the matrix $W_{k}(i, j)$ is hard to compute.
Let us see nonetheless what it gives in the case of a free wreath product $\left(\mathbb{G} \imath_{*} S_{N}^{+},\left(w_{i j, k l}\right)\right)$, with $\mathbb{G}$ an orthogonal matrix quantum group. A basis of $S_{k}$ is given by the vectors $U^{p, S}, p \in N C(k)$, as defined in 6.3.5). The first task is to compute the matrix $W_{k}(i, j)$. Consider the following map

$$
\begin{gathered}
t_{k}:\left(\mathbb{C}^{N} \otimes V\right) \otimes \cdots \otimes\left(\mathbb{C}^{N} \otimes V\right) \rightarrow\left(\mathbb{C}^{N}\right)^{\otimes k} \otimes V \otimes \cdots \otimes V \\
\bigotimes_{i=1}^{k}\left(x_{i} \otimes y_{i}\right) \mapsto \bigotimes_{i=1}^{k} x_{i} \otimes \bigotimes_{i=1}^{k} y_{i} .
\end{gathered}
$$

$t_{k}$ is unitary and and by definition of $U^{p, S}$,

$$
t_{k}\left(U^{p, S}\right)=T_{p} \otimes S
$$

Recall that $S$ depends implicitly on $p$ through the definition 6.3.5: the latter is an invariant vector of the $k$-tensor product representation of $\mathbb{G}$ having the block structure of $p$. Nevertheless $S$ is independent of $N$ and in particular we have the expression

$$
\begin{aligned}
\left\langle U^{p, S}, U^{q, S^{\prime}}\right\rangle & =\left\langle t_{k}\left(U^{p, S}\right), t_{k}\left(U^{q, S^{\prime}}\right)\right\rangle \\
& =\left\langle T_{p}, T_{q}\right\rangle\left\langle S, S^{\prime}\right\rangle=N^{b(p \vee q)}\left\langle S, S^{\prime}\right\rangle .
\end{aligned}
$$

Remark 6.23. Easy quantum groups form a particular family of compact quantum groups whose associated intertwiners spaces can be combinatorially described. Namely if $\mathbb{G}$ is an easy quantum group, the invariant subspace of the $k$-tensor-product representation is spanned by the vectors $T_{p}$, as defined in Section 1.1.3, with $p$ belonging to a subcategory of $\mathcal{P}(k)$. See [15], [71] for more informations on the subject, and [50], [40] and [25] for some applications. In this case, the scalar product matrix has a simpler form. Indeed if $\mathbb{G}$ is an easy quantum group of dimension $s$ and with category of partition $\mathcal{C}$, then a direct computation yields for $\alpha \leq p, \beta \leq q$ two partitions in $\mathcal{C}$ :

$$
\left\langle U^{p, \alpha}, U^{q, \beta}\right\rangle=N^{b(p \vee q)} s^{b(\alpha \vee \beta)} .
$$

The Weingarten formula has also a more combinatorial form since we can write:

$$
h\left(w_{i_{1} j_{1}, k_{1} l_{1}} \ldots w_{i_{r} j_{r}, k_{r} l_{r}}\right)=\sum_{\substack{\alpha \leq \operatorname{ker}(\vec{i}), \beta \leq \operatorname{ker}(\vec{j}) \\ \alpha \leq p \leq \operatorname{ker}(\vec{k}), \beta \leq q \leq \operatorname{ker}(\vec{l})}} G_{k}^{-1}((p, \alpha),(q, \beta)),
$$

where $\operatorname{ker}(\vec{i})$ is the partition whose blocks are the set of indices on which $i$ has the same value.
The scalar product matrix $G_{k}=\left(\left\langle U^{p, S}, U^{q, S^{\prime}}\right\rangle\right)_{(p, S),\left(q, S^{\prime}\right)}$ is a block matrix, the blocks $G_{k}^{p q}$ being indexed by $p, q \in N C(k)$. Note that as in [12], one can factorize this matrix as follows:

$$
G_{k}=\Delta_{n k}^{1 / 2} \tilde{G} \Delta_{n k}^{1 / 2}
$$

where $\Delta_{n k}$ is the diagonal matrix with diagonal coefficients

$$
\Delta_{n k}((p, S),(p, S))=N^{b(p)}
$$

and

$$
\tilde{G}_{k}\left((p, S),\left(q, S^{\prime}\right)=N^{b(p \vee q)-\frac{b(p)+b(q)}{2}}\left\langle S, S^{\prime}\right\rangle .\right.
$$

Asymptotically with $n$ going to infitiny, $\tilde{G}_{k}=D_{k}\left(1+o\left(\frac{1}{\sqrt{n}}\right)\right), D_{k}$ being the block diagonal matrix

$$
D_{k}\left((p, S),\left(q, S^{\prime}\right)\right)=\delta_{p, q}\left\langle S, S^{\prime}\right\rangle
$$

Finally we can remark that restricted on the subspace $V_{p_{0}}=V e c t\left(\left(U_{p_{0}, S}\right)_{S}\right)$, the matrix $\left(\left\langle S, S^{\prime}\right\rangle\right)_{S, S^{\prime}}$ is the tensor product of the Gram-Schmidt matrices of $\mathbb{G} G_{\mathbb{G},\left|B_{i}\right|}$, for each block $\left|B_{i}\right|$ of $p_{0}$. If we put all these considerations together, we get that

$$
W_{n}\left((p, S),\left(q, S^{\prime}\right)\right)=\delta_{p, q} N^{-b(p)}\left(\bigotimes_{B \in p} W_{\mathbb{G}}^{-1}\right)\left(S, S^{\prime}\right)\left(1+o\left(\frac{1}{\sqrt{n}}\right)\right)
$$

This formula allows to generalize the results in [12] to the free wreath product case. Define the following partial trace:

Definition 6.24. Let $0 \leq s \leq n$ the partial trace of order $s$ of the matrix $w=\left(w_{i j, k l}\right)_{1 \leq i, j \leq r, 1 \leq k, l \leq n}$ is

$$
\chi^{w}(s)=\sum_{i=1}^{r} \sum_{k=1}^{s} w_{i i, k k}
$$

Let $t \in(0,1]$ and let $\mathbb{G}$ be a matrix compact quantum group of Kac type and dimension $r$. Denote by $\chi_{\mathbb{G}}$ the law of the character of its fundamental representation. Let ( $\mathbb{G} \imath_{*}$ $\left.S_{n}^{+},\left(w_{i j, k l}\right)_{1 \leq i, j \leq r, 1 \leq k, l \leq n}\right)$ be the matrix quantum group $\mathbb{G} \imath_{*} S_{n}^{+}$with its fundamental representation $w$.

Theorem 6.25. With respect to the Haar measure, if $s \sim t n$ for $n$ going to infinity,

$$
\chi_{w}(s) \rightarrow \mathcal{P}_{t}\left(\chi_{\mathbb{G}}\right),
$$

where $\mathcal{P}_{t}\left(\chi_{\mathbb{G}}\right)$ is the free compound Poisson with parameter $t$ and original law $\chi_{\mathbb{G}}$.
Proof. A similar computation as in [12, Theorem 5.1 gives

$$
\int\left(\chi^{w}(s)\right)^{k}=\operatorname{Tr}\left(G_{k, n}^{-1} G_{k, s}\right)
$$

and with the asymptotic form of $G_{k, n}$ this gives us:

$$
G_{k, n}^{-1} G_{k, s}=\Delta_{n k}^{-1 / 2} D_{k}^{-1} \Delta_{n k}^{-1 / 2}\left(I d+o\left(\frac{1}{\sqrt{n}}\right)\right) \Delta_{s k}^{1 / 2} D_{k} \Delta_{s k}^{1 / 2}\left(I d+o\left(\frac{1}{\sqrt{n}}\right)\right) .
$$

Since $D_{k}$ is block diagonal and $\Delta_{n k}, \Delta_{s k}$ are diagonal, and equal to the identity on each block, these three matrices commute, and

$$
\operatorname{Tr}\left(G_{k, n}^{-1} G_{k, s}\right)=\operatorname{Tr}\left(\Delta_{s / n, k}\left(I d+o\left(\frac{1}{\sqrt{n}}\right)\right)\right) \rightarrow \operatorname{Tr}\left(\Delta_{t, k}\right) .
$$

Since

$$
\operatorname{Tr}\left(\Delta_{t, k}\right)=\sum_{p \in N C(k)} t^{b(p)} \operatorname{dim} V_{p}=\sum_{p \in N C(k)} t^{b(p)} \prod_{B \in p} m_{|B|}\left(\chi_{\mathbb{G}}\right),
$$

$\operatorname{Tr}\left(\Delta_{t, k}\right)$ is exactly the $k$-th moment of the law $\mathcal{P}_{t}\left(\chi_{\mathbb{G}}\right)$.
Remark 6.26. All the results of this section can be transposed to the classical case. One just has to substitute classical compound Poisson laws for free compound Poisson laws, and use crossing partitions instead of non-crossing ones.

### 6.4.3 Non-commutative symmetric functions as a probability space

We expose in this subsection a relation between the ring of non-commutative symmetric functions (as defined in Definition 3.38 ) and the representation theory of the free wreath product $\mathbb{U} \imath_{*} S_{n}^{+}$ for $n \geq 4$. This free wreath product is also the easy quantum group $H_{n}^{+, \infty}$ of Chapter 4 and 5 . This quantum group has been deeply studied by Banica and Verginoux in [16]. In particular, they found the fusion rules of the irreducible representations:
Theorem 6.27 ([16]). Let $n \geq 4$. The irreducible representations of $\left(C\left(H_{n}^{+, \infty}\right),\left(u_{i j}\right)_{1 \leq i, j \leq n}\right)$ are indexed by finite sequences of integers, with the fusion rules given by the recursive formula

$$
\begin{gathered}
\left(j_{1}, \ldots, j_{r}\right) \otimes\left(i_{1}, \ldots, i_{r^{\prime}}\right)=\left(j_{1}, \ldots, j_{r}, i_{1}, \ldots, i_{r^{\prime}}\right)+\left(j_{1}, \ldots, j_{r-1}, j_{r}+i_{1}, i_{2}, \ldots, i_{r^{\prime}}\right) \\
+\delta_{j_{r}+i_{1}=0}\left(j_{1}, \ldots, j_{r-1}\right) \otimes\left(i_{2}, \ldots, i_{r^{\prime}}\right),
\end{gathered}
$$

such that ( 0 ) is the trivial representation and for $k$ non zero, $(k)$ is the irreducible representation given by the matrix $\left(u_{i j}^{k}\right)_{1 \leq i j \leq n}$.

This yields the following embedding of NSym in $\mathrm{Cl}\left(H_{n}^{+, \infty}\right)$ :
Proposition 6.28. The map $\Phi: \mathbf{N S y m} \longrightarrow C l\left(H_{n}^{+, \infty}\right)$ defined by $\Phi\left(S_{k}\right)=\sum_{1 \leq i \leq k} u_{i i}^{k}$ is an embedding of NSym in $\mathrm{Cl}\left(\mathrm{H}_{n}^{+, \infty}\right)$ such that for any composition I of $n$,

$$
\Phi\left(R_{I}\right)=\chi_{I},
$$

where $\chi_{I}$ is the character of the irreducible representation of $H_{n}^{+, \infty}$ indexed by I in Theorem 6.27 .

This result can be seen as a non-commutative version of Theorem 2.23. The ring of noncommutative symmetric functions has already been related with the representations of a specialization at $q=0$ of the quantum linear group and the Hecke algebra [52]. Proposition 6.28 gives a semisimple version of this result.

Proof. Since NSym is the free ring generated by the variables $S_{k}, k \geq 1$, there is a morphism of algebra $\Phi:$ NSym $\longrightarrow C l\left(H_{n}^{+, \infty}\right)$ sending $S_{i}$ to $\chi_{(k)}=\sum_{1 \leq i \leq k} u_{i i}^{k}$. We prove by recursion of the length $l$ of a composition $I$ that $\Phi\left(R_{I}\right)=\chi_{(I)}$. If $l=1$, this is true by the definition of $I$ and the fact that $R_{(k)}=S_{k}$ for $k \geq 1$. Suppose the result true for $l-1$ and let $I=\left(i_{1}, \ldots, i_{l}\right)$ be a composition of length $l$. On one hand, the product formula on the ribbon Schur basis yields

$$
R_{\left(i_{1}, \ldots, i_{l-1}\right)} R_{\left(i_{l}\right)}=R_{\left(i_{1}, \ldots, i_{l}\right)}+R_{\left(i_{1}, \ldots, i_{l-2}, i_{l-1}+i_{l}\right)} .
$$

Thus applying the map $\Phi$ on both sides and using the induction hypothesis yields

$$
\chi_{\left(i_{1}, \ldots, i_{l-1}\right)} \chi_{\left(i_{l}\right)}=\Phi\left(R_{\left(i_{1}, \ldots, i_{l}\right)}\right)+\chi_{\left(i_{1}, \ldots, i_{l-2}, i_{l-1}+i_{l}\right)} .
$$

Since each $i_{j}$ is positive, the fusion rules of Theorem 6.27 give on the left hand side of the latter equation

$$
\chi_{\left(i_{1}, \ldots, i_{l-1}\right)} \chi_{\left(i_{l}\right)}=\chi_{\left(i_{1}, \ldots, i_{l-1}+i_{l}\right)}+\chi_{\left(i_{1}, \ldots, i_{l}\right)} .
$$

Therefore, $\Phi\left(R_{\left(i_{1}, \ldots, i_{l}\right)}\right)=\chi_{\left(i_{1}, \ldots, i_{l}\right)}$. Since the set $\left\{\chi_{(I)}\right\}_{I \text { compositions }}$ is a set of characters of distinct irreducible representations (and thus linearly independent), $\Phi$ is injective.

In particular, the Haar state on $\mathrm{Cl}\left(H_{n}^{+, \infty}\right)$ yields a scalar product on NSym by the formula

$$
\langle F, G\rangle=h_{H_{n}^{+, \infty}}\left(\Phi(F) \Phi(G)^{*}\right),
$$

and the basis of ribbon Schur functions is an orthonormal basis with respect to this scalar product.
From now on, NSym is identified with its image in $\mathrm{Cl}\left(H_{n}^{+, \infty}\right)$. By this identification, each variable $S_{i}$ is a random variable with respect to the Haar measure on $C\left(H_{n}^{+, \infty}\right)$. The law of the random vector $\left(S_{i}\right)_{i \geq 1}$ can be computed thanks to the description of the intertwiners of $\mathbb{U} \imath_{*} S_{n}^{+}$ in Section 6.3 (we could also directly use a result of [16], which already gives the description of the intertwiners of $\hat{\mathbb{Z}} 2_{*} S_{n}^{+}$).
Proposition 6.29. The family $\left\{S_{i}\right\}_{i \geq 1}$ is distributed as $\left(s z^{i} s\right)_{i \geq 1}$, where $s$ is a semi-circular element and $z$ is a uniform variable on the unit circle free from $s$.
Proof. Let $i_{1}, \ldots, i_{r}$ be integers distinct from zero, and write $S_{-i}=S_{i}^{*}$ for $i$ positive. Then by the Tannaka-Krein duality,

$$
h\left(S_{i_{1}} \ldots S_{i_{r}}\right)=\operatorname{dim} \operatorname{Mor}_{H_{n}^{+}, \infty}\left(\mathbf{1}, \alpha\left(i_{1}\right) \otimes \cdots \otimes \alpha\left(i_{r}\right)\right),
$$

where $\alpha(k)$ is the irreducible representation $\left(u_{i j}^{k}\right)_{1 \leq i, j \leq n}$.
By Theorem 6.16 and the fact that the tensor product of the representations $z^{j_{1}}, \ldots, z^{j_{s}}$ of $\mathbb{U}$ is trivial if and only if $\sum_{1 \leq m \leq s} j_{m}=0$,

$$
h\left(S_{i_{1}} \ldots S_{i_{r}}\right)=\#\left\{\pi \in N C\left(i_{1}, \ldots, i_{r}\right) \mid \forall B \in \pi, B \text { is balanced }\right\},
$$

where as in Chapter $5, N C\left(i_{1}, \ldots, i_{r}\right)$ denotes the set of non-crossing partitions with the element $1 \leq m \leq r$ colored with $i_{m}$, and where a block $\left\{j_{1}, \ldots, j_{s}\right\}$ of a partition in $N C\left(i_{1}, \ldots, i_{r}\right)$ is balanced if and only if $\sum_{1 \leq m \leq s} j_{i_{m}}=0$. In the proof of Proposition 5.22, it has been proven that

$$
\#\left\{\pi \in N C\left(i_{1}, \ldots, i_{r}\right) \mid \forall B \in \pi, B \text { is balanced }\right\}=m\left(s z^{i_{1}} s, s z^{i_{2}} s, \ldots, s z^{i_{r}} s\right),
$$

with $s$ a semi-circular variable and $z$ a uniform variable on the unit circle free from $s$.

## Chapter 7

## Planar algebra of a free wreath product

This chapter is devoted to the general description of the intertwiner spaces for the free wreath product of two free permutation groups. This description leads to the proof of the following result, which was conjectured by Banica and Bichon in [10]:

Theorem 7.1 ([10], Conj 3.1). Let $F$ and $G$ be two non-commutative permutation groups such that $\operatorname{dim}_{F} \operatorname{Mor}(0,1)=\operatorname{dim}_{G} \operatorname{Mor}(0,1)=1$. Then

$$
\mu\left(F 乙_{*} G\right)=\mu(F) \boxtimes \mu(G),
$$

where $\mu(\mathbb{G})$ denotes the law (with respect to the Haar measure) of the character of the fundamental representation of a matrix compact quantum group $\mathbb{G}$.

The main ingredient in the proof of Theorem 7.1 is the notion of free product of planar algebras, which has been introduced by Bisch and Jones in [23]. Section 1 is an introduction to the concept of planar algebra; several basic combinatorial results are given on planar tangles, the main objects in the construction of planar algebras. Section 2 gives an isomorphism between the intertwiner spaces of a free wreath product and the free product of certain planar algberas. Section 3 is devoted to the combinatorial proof of a dimension formula which has been found by Bisch and Jones in an unpublished paper [23]: this result yields the proof of Theorem 7.1 thanks to the isomorphism given in Section 2.

### 7.1 Planar algebras

As already said in Chapter 3, a planar algebra is a collection $\left(V_{k}\right)_{k \geq 1}$ of vector spaces with an action of planar tangles on thess vector spaces.

### 7.1.1 Definition of planar tangles

By a diffeomorphism of $\mathbb{R}^{2}$ we mean an orientation preserving diffeormophism defined on a domain $D$ of $\mathbb{R}^{2}$. In this subsection we define the notion of irreducible planar tangle and we associate a non-crossing partition to each planar tangle.

Disk and intervals By a disk we mean an open subset of $\mathbb{R}^{2}$ whose boundary is a smooth Jordan curve.
Let $\mathbb{U}$ be the unit circle on $\mathbb{R}^{2}$. For $\omega \in \mathbb{U}$, we denote by $[1, \omega[$ (resp. $[1, \omega])$ the subset of $\mathbb{U}$
whose argument is less (resp. less or equal) to the one of $\omega$ (or to $2 \pi$ if $\omega=1$ ). If $\omega^{\prime} \in[1, \omega[$, $\left[\omega^{\prime}, \omega\right]$ (resp. $] \omega^{\prime}, \omega\left[\right.$ ) denotes the set $[1, \omega] \backslash\left[1, \omega^{\prime}\left[\right.\right.$ (resp. $\left[1, \omega\left[\backslash\left[1, \omega^{\prime}\right]\right)\right.$.
Let $D$ be a disk and let $\partial D$ denote its boundary. For each finite subset $S \subseteq \partial D$ of cardinal $k$ with a distinguished element $i_{*} \in S$, there is a canonical bijection from $S$ to $\llbracket 1, k \rrbracket$ obtained by numbering the elements of $S$ counterclockwise, starting at $i_{*}$. The element of $S$ which is numbered $i$ is denoted by $i_{D}$.
Since $\partial D$ is a Jordan curve, there is an orientation preserving diffeomorphism $\varphi_{D}$ mapping $\partial D$ to the unit circle $\mathbb{U}$. For each couple $(i, j)$ of elements of $S$ with $i<j$, the set $\varphi_{D}^{-1}(] \varphi_{D}(i), \varphi_{D}(j)[)$ is denoted by $(i, j)$ and called the interval component of $\partial D$ between $i$ and $j$; the set $\varphi_{D}^{-1}(\mathbb{U} \backslash$ $\left.\left[\varphi_{D}(i), \varphi_{D}(j)\right]\right)$ is denoted by $(j, i)$ and called the interval component of $\partial D$ between $j$ and $i$.
When there is no possible confusion, $\llbracket 1, k \rrbracket$ is always identified with $\mathbb{Z} /(k \mathbb{Z})$ and thus $k+1=1$. When the bounds of an interval component is not specified, this interval is always assume to be of type $(i, i+1)$ for some $i \in \llbracket 1, k \rrbracket$.
By a curve $\gamma$, we mean either a Jordan curve or a injective smooth map $\gamma:[0,1] \rightarrow \mathbb{R}^{2}$. If $\gamma$ is a Jordan curve, $\gamma$ is called a closed curve.

Planar tangle A planar tangle is a particular collection of subsets of $\mathbb{R}^{2}$ which has been introduced by Jones in [46]. We can see it as a generalization of the two-level noncrossing partitions of Section 1.1 .2 to a kind of multilevel noncrossing partition.

Definition 7.2. A planar tangle $P$ of degree $k \geq 0$ consists of the following objects:

- $A$ disk $D_{0}$ of $\mathbb{R}^{2}$, called the outer disk.
- Some disjoint disks $D_{1}, \ldots, D_{n}$ in the interior of $D_{0}$ which are called the inner disks.
- For each $0 \leq i \leq n$, a finite subset $S_{i} \in \partial D_{i}$ of cardinal $2 k_{i}$ (such that $k_{0}=k$ ) with a particular element $i_{*} \in S_{i}$. The elements of $S_{i}$ are called the distinguished points of $D_{i}$ and numbered counterclockwise starting from $i_{*}$. $k_{i}$ is called the degree of the inner disk $D_{i}$.
- A finite set of disjoint smooth curves $\left\{\gamma_{j}\right\}_{1 \leq j \leq r}$ such that each $\dot{\gamma}_{j}$ lies in the interior of $D_{0} \backslash \bigcup_{i \geq 1} D_{i}$ and such that $\bigcup_{1 \leq j \leq r} \partial \gamma_{j}=\bigcup_{0 \leq i \leq n} S_{i}$; it is also required that each curve meets a disk boundary orthogonally, and that its endpoints have opposite (resp. same) parity if they both belong to inner disks or both belong to the outer disk (resp. one belongs to an inner disk and the other one to the outer disk).
- A region of $P$ is a connected component of $D_{0} \backslash\left(\bigcup_{i \geq 1} D_{i} \cup\left(\bigcup \gamma_{j}\right)\right)$. Give a chessboard shading on the regions of $P$ in such a way that the interval components of type $(2 i+1,2 i)$ are boundaries of shaded regions.

The skeleton of $P$, denoted by $\Gamma P$, is the set $\left(\bigcup \partial D_{i}\right) \cup\left(\bigcup \gamma_{j}\right)$.
In the sequel, $D_{0}$ denotes always the outer disk of a planar tangle. If the degree of $P$ is 0 , then $P$ is of degree $0_{+}$(resp. $0_{-}$) if the boundary of the outer disk is the boundary of an unshaded region (resp. shaded). An example of planar tangle with its associated shading is given in Figure 7.1.

An isotopy of $P$ is the image of $P$ (i.e the family of images of all the given sets in the definition, such that a region and its image have same shading) by a diffeomorphism, and two planar tangles $P$ and $P^{\prime}$ are said equivalent if there exists an isotopy $\varphi$ such that $\varphi(P)=P^{\prime}$. For any disk $D$ of $\mathbb{R}^{2}$ with a set $S$ of $2 k$ distinguished points on $\partial D$ (numbered counterclockwise from 1


Figure 7.1: : Planar tangle of degree 4 with 4 inner disks.
to $2 k)$, there is an isotopy $\varphi_{D, S}$ of $P$ such that $\varphi_{D, S}\left(\partial D_{0}\right)=\partial D$ and $\varphi_{D, S}\left(i_{D_{0}}\right)=i_{D}$ for each distinguished point of $D$.
A connected planar tangle is a planar tangle whose regions are simply connected; this implies that for any inner disk $D$ and any element $x \in \partial D$, there is a path from $x$ to the boundary of the outer disk which is contained in $\left(\bigcup \partial D_{i}\right) \cup\left(\bigcup \gamma_{j}\right)$. An irreducible planar tangle is a connected planar tangle such that each curve has an endpoint being a distinguished point of $D_{0}$ and the other one being on an inner disk. An example of connected planar tangle and of an irreducible planar tangle is given in 7.2 .


Figure 7.2: : A connected and an irreducible planar tangle.

Composition of planar tangles Let $P$ and $P^{\prime}$ be two planar tangles of respective degree $k$ and $k^{\prime}$, and let $D$ be an inner disk of $P$. We assume that the degree of $D$ is also $k^{\prime}$. Let $S$ be the set of distinguished points of $D$ and let $\varphi_{D, S}$ be an isotopy of $P^{\prime}$ to $\partial D$ respecting the distinguished points. For each distinguished point $i$ of $D$, the union of the curve of $P$ and the one of $\varphi_{D, S}\left(P^{\prime}\right)$ ending at $i$ yields a new smooth curve. Thus, the union of $P$ and $\varphi_{D, S}\left(P^{\prime}\right)$ with the disk $D$ removed is a new planar tangle which is denoted by $P \circ_{D} P^{\prime}$ and called the composition of $P$ and $P^{\prime}$ with respect to $D$. An example of composition the planar tangle of Figure 7.1 with the second planar tangle of Figure 7.2 is given in Figure 7.3 (only an equivalent planar tangle is displayed in orther to get a clear picture).


Figure 7.3: : Composition of two planar tangles.

Note that if $D_{1}$ and $D_{2}$ are two distinct inner disks of $P$ and $P_{1}$ and $P_{2}$ are two planar tangles having respectively the same degree as $D_{1}$ and $D_{2}$, then $\left(P \circ_{D_{1}} P_{1}\right) \circ_{D_{2}} P_{2}=\left(P \circ_{D_{2}} P_{2}\right) \circ_{D_{1}} P_{1}$. If $P_{1}, \ldots, P_{s}$ are distinct planar tangles and $D_{i_{1}}, \ldots, D_{i_{n}}$ are distinct inner disks of $P$ such that $\operatorname{deg} P_{j}=\operatorname{deg} D_{i_{j}}$ for all $1 \leq i \leq s$, we denote by $P \circ_{\left(D_{i_{1}}, \ldots, D_{i_{s}}\right)}\left(P_{1}, \ldots, P_{s}\right)$ the planar tangle obtained by iterating the composition with respect to the different inner disks.

### 7.1.2 Non-crossing partition and irreducible planar tangles

Let $P$ be a planar tangle of degree $k . \Gamma P \backslash \partial D_{0}$ is the union of a finite number of connected components $C_{1}, \ldots, C_{r}$. For $1 \leq i \leq 2 k$, the distinguished point $i_{D_{0}}$ of the outer boundary belongs to the closure of a unique connected component $C_{f(i)}$. We define an equivalence relation $\sim_{P}$ on $\llbracket 1,2 k \rrbracket$ by saying that $i \sim_{P} j$ if and only if $f_{i}=f_{j}$. We denote by $\pi_{P}$ the partition associated to $\sim_{P}$ through the correspondence between partitions and equivalence relations which has been established in Section 1.1.1.

Lemma 7.3. $\pi_{P}$ is a non-crossing partition of $2 k$ with even blocks.
Recall that a block is called even if its cardinal is even.
Proof. Suppose that $1 \leq i<j<k<l \leq 2 k$ with $i \sim_{P} k$ and $j \sim_{P} l$. Thus there exist a path $\gamma_{1}$ in $\Gamma P \backslash D_{0}$ between $i$ and $k$ and a path $\gamma_{2}$ in $\Gamma P \backslash D_{0}$ between $j$ and $l$. Since $j \in(i, k)$ and $l \in(k, i), \gamma_{1}$ and $\gamma_{2}$ intersect. Therefore, the four points are in the same connected component of $\Gamma P \backslash D_{0}$ and $i \sim_{P} j \sim_{P} k \sim_{P} l . \pi_{P}$ is thus non-crossing.
Since each inner disk has an even number of distinguished points and each curve connects two distinguished points, a counting argument yields the parity of the size of the blocks.

Reciprocally, a noncrossing partition $\pi$ of $2 k$ with even blocks yields an irreducible planar tangle $P_{\pi}$ such that $\pi_{P_{\pi}}=\pi$. The construction is done recursively on the number of blocks as follows:

1. If $\pi$ is the one block partition, $P_{\pi}$ is the planar tangle with one outer disk $D_{0}$ of degree $2 k$, one inner disk $D_{1}$ of degree $2 k$, and a curve between the point $i$ of $D_{0}$ and the point $i$ of $D_{1}$.
2. Suppose that $P_{\pi}$ is constructed for all partitions having less than $r$ blocks. Let $B=\llbracket i_{1}, i_{2} \rrbracket$ be an interval block of $\pi$, and let $\pi^{\prime}$ be the non-crossing partition obtained by removing this block (and relabelling increasingly the integers). $\pi^{\prime}$ has also even blocks.
Let $P_{\pi, B}$ be the planar tangle consisting of an outer disk $D_{0}$ of degre $2 k$ and two inner disks $D_{1}$ and $D_{2}$ of respective degree $i_{2}-i_{1}+1$ and $2 k-\left(i_{2}-i_{1}+1\right)$, and curves connecting

- $i_{D_{0}}$ to $i_{D_{2}}$ for $i<i_{1}$.
- $i_{D_{0}}$ to $\left(i-i_{1}+1\right)_{D_{1}}$ for $i_{1} \leq i \leq i_{2}$ if $i_{1}$ is odd, and $i_{D_{0}}$ to $\left(i-i_{1}\right)_{D_{1}}$ for $i_{1} \leq i \leq i_{2}$ if $i_{1}$ is even.
- $i_{D_{0}}$ to $\left(i-\left(i_{2}-i_{1}+1\right)\right)$ for $i>i_{2}$.

Set $P_{\pi}=P_{\pi, B}{ }^{\circ} D_{2} P_{\pi^{\prime}}$. Note that the resulting planar tangle doesn't depend on the choice of the interval block $B$.

By construction, $P_{\pi}$ is irreducible and $\pi_{P_{\pi}}=\pi$. The inner disk of $P_{\pi}$ corresponding to the block $B$ of $\pi$ is denoted by $D_{B}$.
These particular irreducible planar tangles yield a decomposition of connected planar tangles. Let $P$ be a connected planar tangle. Let $B_{1}, \ldots, B_{r}$ be the blocks of $\pi_{P}$ ordered lexicographically and let $C_{1}, \ldots, C_{r}$ be the corresponding connected components of $P$. For $1 \leq i \leq r, P_{i}$ is defined as the planar tangle $P \backslash\left(\bigcup_{j \neq i} C_{j}\right)$, where the distinguished points of the outer boundary of $P$ which are in $B_{i}$ have been counterclockwise relabelled in such a way that the first odd point is labelled 1. The planar tangle $P_{1}$ of the connected planar tangle of Figure 7.2 is depicted in Figure 7.4.


Figure 7.4: : First connected component of the first planar tangle of Figure 7.2.

Proposition 7.4. Let $P$ be a connected planar tangle, and set $\pi=\pi_{P}$. Then

$$
P=P_{\pi} \circ_{D_{B_{1}}, \ldots, D_{B_{r}}}\left(P_{1}, \ldots, P_{r}\right) .
$$

Proof. It is possible to draw $r$ disjoint Jordan curves $\left\{\gamma_{i}\right\}_{1 \leq i \leq r}$ such that $\gamma_{i}$ intersects $P\left|B_{i}\right|$ times, once at each curve of $C_{i}$ connected to a distinguished point of the outer boundary (or two times at a curve joining two distinguished points of the outer boundary). The intersection points are labelled counterclockwise around $\gamma_{i}$, in such a way that the intersection point with the curve coming from the first odd point of $B_{i}$ is labelled 1 .
Let $\Gamma_{i}$ be the closed region delimited by $\gamma_{i}$ and set $\tilde{P}_{i}=\left(P \cap \Gamma_{i}\right) \cup \gamma_{i}$, with the labelling of the distinguished points of $\gamma_{i}$ given above. Then $\tilde{P}_{i}$ is a planar tangle which is an isotopy of $P_{i}$. Figure 7.5 shows a possible choice of Jordan curves for the connected planar tangle of Figure 7.2 ,

Let $\tilde{P}$ be the planar tangle whose inner disks are $\left\{\Gamma_{i}\right\}_{1 \leq i \leq r}$, with the distinguished points being the ones of $\gamma_{i}$, and whose skeleton is $\Gamma P \backslash\left(\bigcup \stackrel{\circ}{\Gamma}_{i}\right)$. Then $\tilde{P}$ is equivalent to $P_{\pi}$ and

$$
\tilde{P} \circ_{\Gamma_{1}, \ldots, \Gamma_{r}}\left(\tilde{P}_{1}, \ldots, \tilde{P}_{r}\right)=P
$$



Figure 7.5: : Jordan curves surrounding the connected components of a planar tangle.

Shaded regions and Kreweras complement We refer to Section 5.2 for the notations on the Kreweras complement related to partial set partitions. If $S$ is a set of cardinal $k, f: \llbracket 1, k \rrbracket \longrightarrow$ is a bijective function and $\pi \in P(k)$, then $f(\pi)$ is the partition of $S$ defined by $f(i) \sim_{f(\pi)} f(j)$ if and only if $i \sim_{\pi} j$. Let $k \geq 1$, and for $i \geq 1$, set $\delta(i)=1$ if $i$ is odd and 0 else. A partial partition $(\tilde{\pi}, S)$ of $4 k$ is associated to each $\pi \in N C(2 k)$ as follows:

- $S=\{2 i-\delta(i)\}_{1 \leq i \leq 2 k}$.
- $\tilde{\pi}=f(\pi)$ where $f: \llbracket 1,2 k \rrbracket \longrightarrow \llbracket 1,4 k \rrbracket$ given by $f(i)=2 i-\delta(i)$.
$\underset{\sim}{S}$ is the set $\{1,4,5,8, \ldots, 4 k-3,4 k\}$. Let $\tilde{f}$ be the map from $\llbracket 1,2 k \rrbracket$ to $\llbracket 1,4 k \rrbracket \backslash S$ defined by $\tilde{f}(i)=2 i-(1-\delta(i))$.

Definition 7.5. Let $\pi \in N C(2 k)$. The nested Kreweras complement of $\pi$, denoted by $k r^{\prime}(\pi)$, is the partition of $2 k$ such that $\tilde{f}\left(k r^{\prime}(\pi)\right)=k r(\tilde{\pi}, S)$.

The nested Kreweras complement of the partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ is the partition $\{\{1,2\},\{3,4\},\{5,6\}\}$, as shown in Figure 7.6 .


Figure 7.6: The partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ and its nested Kreweras complement.
Contratry to the usual Kreweras complement, the nested Kreweras complement is not bijective. Let $\pi_{0}\left(\operatorname{resp} \pi_{1}\right)$ be the partition of $2 k$ with block $(2 i, 2 i+1)$ (resp. $(2 i+1,2 i+2)$ ).

Lemma 7.6. $k r^{\prime}(\pi)=k r^{\prime}\left(\pi \vee \pi_{0}\right)$
Proof. Since $\pi \leq\left(\pi \vee \pi_{0}\right), k r^{\prime}\left(\pi \vee \pi_{0}\right) \leq k r^{\prime}(\pi)$.
Suppose that $i \sim_{k r^{\prime}(\pi)} j$. Then $2 i-(1-\delta(i)) \sim_{k r\left(\tilde{\pi}, S^{c}\right)} 2 j-(1-\delta(j))$. By Lemma 5.16, this implies that for all $k \in \llbracket 2 i-(1-\delta(i)), 2 j-(1-\delta(j)) \rrbracket \cap S, l \in S \backslash \llbracket 2 i-(1-\delta(i)), 2 j-(1-\delta(j)) \rrbracket$, $k \not \chi_{\pi} l$. If $k \in \llbracket 2 i-(1-\delta(i)), 2 j-(1-\delta(j)) \rrbracket \cap S$ and $l \notin \llbracket 2 i-(1-\delta(i)), 2 j-(1-\delta(j)) \rrbracket$,
then $l \neq k+1$ and thus $k \not \chi_{\pi_{0}} l$; therefore, for all $k \in \llbracket 2 i-(1-\delta(i)), 2 j-(1-\delta(j)) \rrbracket \cap S$, $l \in S \backslash \llbracket 2 i-(1-\delta(i)), 2 j-(1-\delta(j)) \rrbracket, k \not \chi_{\pi \vee \pi_{0}} l$. By Lemma 5.16, this implies that $i \sim_{k r^{\prime}\left(\pi \vee \pi_{0}\right)} j$ : $k r^{\prime}(\pi) \leq k r^{\prime}\left(\pi \vee \pi_{0}\right)$.

The nested Kreweras complement is involved in the description of planar tangles in the following way:

Proposition 7.7. Let $P$ be a planar tangle of degree $k$. $i$ and $j$ are in the same block of $k r^{\prime}\left(\pi_{P}\right)$ if and only if $i_{D_{0}}$ and $j_{D_{0}}$ are boundary points of the same shaded region.

Proof. Let $P$ be a planar tangle. Relabel $i$ with $2-\delta(i)$ : with this relabelling each interval of type $(4 i+1,4 i+4)$ is the boundary interval of a shaded region, and $\pi(P)$ becomes a partial non-crossing partition of $4 k$ with support $S$. Add two points $4 i+2$ and $4 i+3$ in $(4 i+1,4 i+4)$ in such a way that $4 i+2 \in(4 i+1,4 i+3)$. Let $\sim$ be the relation on $S^{c}$ defined by $i \sim j$ if and only if $i$ and $j$ are boundary points of a same shaded region. Let $\left(\pi^{\prime}, S^{c}\right)$ be the partial partition associated to $\sim: \pi^{\prime}$ is non-crossing since two regions that intersect are the same.
Let $\pi=\left(\left(\pi_{P}, S\right) \vee\left(\pi^{\prime}, S^{c}\right)\right.$. Let $1 \leq i<j<r<s \leq 4 k$ with $i \sim_{\pi} r$ and $j \sim_{\pi} s$. Since $\pi_{P}$ is non-crossing, if $i, j, r, s$ are all in $S$ then $i \sim_{\pi} j \sim_{\pi} r \sim_{\pi} s$. Assume from now on that they are not all in $S$, and suppose without loss of generality that $i \in S^{c} . i \sim_{\pi} r$, thus $r$ is also in $S^{c}$ and $i$ and $r$ are boundary points of a same shaded region $\sigma$. Since $j \in(i, r)$ and $s \in(r, i)$, any path on $\Gamma P$ between $j$ and $s$ would cut $\sigma$ in two distinct regions: thus if $j, s \in S$, then $j \not \chi_{\pi} s$. Therefore, the hypothesis $j \sim_{\pi} s$ yields that $j, s \in S^{c} . \pi^{\prime}$ being non-crossing, $i, j, r, s$ are in the same block of $\pi$. Finally, $\pi_{P} \vee \pi^{\prime}$ is non-crossing and thus $\pi^{\prime} \leq k r^{\prime}\left(\pi_{P}\right)$.
Let $\pi_{2}$ be a partial partition with support $S^{c}$, such that $\pi_{P} \vee \pi_{2}$ is non-crossing. Suppose that $i \sim_{\pi_{2}} j$, with $i, j \in S^{c}$. Let $\sigma_{i}$ (resp. $\sigma_{j}$ ) be the shaded region having $i$ (resp. $j$ ) as boundary point. $\pi_{P} \vee \pi_{2}$ is non-crossing, thus for all $r, s \in S$ such that $i \leq r \leq j$ and $j \leq s \leq i, r \not \chi_{\pi_{P}} s$. Thus, there is no path in $\Gamma P$ between $(i, j)$ and $(j, i)$, and $\sigma_{i}=\sigma_{j}$ : this yields $i \sim_{\pi^{\prime}} j$. Therefore, $\left(\pi_{2}, S^{c}\right) \leq\left(\pi^{\prime}, S^{c}\right)$ and $\left(\pi^{\prime}, S^{c}\right)=k r\left(\pi_{P}, S\right)$.
Let $1 \leq i, j \leq 2 k$. By the two previous paragraphs, $i$ and $j$ are in the same block of $k r^{\prime}\left(\pi_{P}\right)$ and only if $2 i-(1-\delta(i))$ and $2 j-(1-\delta(j))$ are boundary points of a same shaded region. Since the points $2 i-(1-\delta(i))$ and $2 i-\delta(i)$ both belong to the interval $(2(i+\delta(i))-3,2(i+\delta(i)))$ (which is part of the boundary of a shaded region), $2 i-(1-\delta(i))$ and $2 j-(1-\delta(j))$ are boundary points of a same shaded region if and only if $2 i-\delta(i)$ and $2 j-\delta(j)$ are boundary points of the same shaded region. Since we had relabelled $i$ by $2-\delta(i)$, this yields the result.

### 7.1.3 Planar algebra

Let us recall the definition of a planar algebra as given in Section 3.3.2. Planar algebras have been introduced by Jones in 46] in order to study the structure of subfactors.

Definition 7.8. A planar algebra $\mathcal{P}$ is a collection of finite dimensional vector spaces $\left\{\mathcal{P}_{n}\right\}_{n \in \mathbb{N}^{*} \cup\{-,+\}}$ with $\operatorname{dim} \mathcal{P}_{+}=\operatorname{dim} \mathcal{P}_{-}=1$, such that each planar tangle $P$ of order $k$ with $n$ inner disks of respective order $k_{1}, \ldots, k_{n}$ yields a multilinear map:

$$
T_{P}: \bigotimes_{1 \leq i \leq n} \mathcal{P}_{k_{i}} \longrightarrow \mathcal{P}_{k}
$$

which is compatible with the composition of planar tangles and invariant under isotopy. Namely, if $Q$ is another tangle of order $k_{i_{0}}$ for some $1 \leq i_{0} \leq n$ and $\varphi$ is a diffeomorphism, then

$$
T_{P} \circ\left(\bigotimes_{i \neq i_{0}} I d_{\mathcal{P}_{k_{i}}} \otimes T_{Q}\right)=T_{P \circ D_{i_{0}}} Q \text { and } T_{\varphi(P)}=T_{P},
$$

where $P \circ_{i_{0}} Q$ is the composition of tangles defined in Section 7.1.1 and $\varphi(P)$ is the isotopy of $P$ with respect to $\varphi$.

In particular the compatibility with the composition of tangles yields that if $P$ contains a close curve $\gamma$ which delimits a simply connected region $\sigma, T_{P}=\delta_{\sigma} T_{\tilde{P}}$, where $\tilde{P}$ is the planar tangle $P$ with the curve $\gamma$ removed, and $\delta_{\sigma}$ is a scalar that depends only on the shading of $\sigma$. In the sequel $\delta_{1}\left(\right.$ resp. $\left.\delta_{2}\right)$ will denote the value of $\delta_{\sigma}$ for a shaded (resp. unshaded) region.

Several operations are defined by the action of particular planar tangles : Each $\mathcal{P}_{n}$ is an algebra with multiplication given by the planar tangle of Figure 7.7.


Figure 7.7: : Multiplication tangle of degree $\mathcal{P}_{6}$.
The is an inclusion of algebras $\mathcal{P}_{n} \subseteq \mathcal{P}_{n+1}$ with the planar tangle of Figure 7.8.


Figure 7.8: : Inclusion tangle from $\mathcal{P}_{4}$ to $\mathcal{P}_{6}$.
There exist two linear functionals $\operatorname{Tr}_{R}$ and $\operatorname{Tr}_{L}$ respectively defined by the first and second planar tangles of Figure 7.9 .


Figure 7.9: : Right trace tangle and Left trace tangle on $\mathcal{P}_{4}$.
Note that $\operatorname{Tr}_{L}\left(I d_{\mathcal{P}_{1}}\right)=\delta_{2}$ and $\operatorname{Tr}_{R}\left(I d_{\mathcal{P}_{1}}\right)=\delta_{1}$.
A planar subalgebra of $\mathcal{P}$ is a collection of vector subspaces $W_{n} \subseteq \mathcal{P}_{n}$ which is stable under the action of the planar tangles. If $\mathcal{P}$ is a planar algebra, there is a minimal planar subalgebra
of $\mathcal{P}$ denoted by $T L(\mathcal{P})$ and given by the image of all planar tangles without inner disk. A morphism $\Phi$ between two planar algebras $\mathcal{P}:\left\{\mathcal{P}_{n}\right\}$ and $\mathcal{Q}=\left\{\mathcal{Q}_{n}\right\}$ is a collection of linear maps $\Phi_{n}: \mathcal{P}_{n} \longrightarrow \mathcal{Q}_{n}$ that commute with the action of the planar tangles. Namely $T_{P} \circ$ $\left(\otimes_{D_{i} \text { interior disk of } P} \Phi_{i}\right)=\Phi_{k} \circ T_{P}$ for any planar tangle of order $k$.

Subfactor planar algebra Two planar tangles $P$ and $Q$ of degree $0_{+}$or $0_{-}$are related by a spherical symmetry if $P$ can be obtained from $Q$ by the composition of a Moebius transformation and a diffeomorphism. An example of two planar tangles related by a spherical symmetry is shown in Figure 7.10 .


Figure 7.10: : Two planar tangles related by a spherical symmetry.

A planar algebra is called spherical if the action of a planar tangle $P$ of degree $0_{+}$or $0_{-}$is invariant under spherical symmetries. In such planar algebra, the linear functionals $\operatorname{Tr}_{L}$ and $\operatorname{Tr}_{R}$ are equal and simply denoted by $\operatorname{Tr}$; in particular $\delta_{1}=\delta_{2}$ and $\operatorname{Tr}$ is a trace on each algebra $\mathcal{P}_{n}$.

The conjugate $P^{*}$ of a planar tangle $P$ is defined as the image of $P$ by any axial symmetry $s$, with the rule that the first distinguished point of a disk $s(D)$ is the last distinguished point of the inital disk $D$. The conjugate of the planar tangle of Figure 7.1 is drawn in Figure 7.11 .


Figure 7.11: : Axial symmetry of the planar tangle of Figure 7.1

Suppose that there exists an involutive antilinear map $*$ on each vector space $\mathcal{P}_{n}$. The planar algebra is called a $*$ planar algebra if $T_{P^{*}}\left(v_{1}^{*} \otimes \cdots \otimes v_{n}^{*}\right)=\left(T_{P}\left(v_{1}, \ldots, v_{n}\right)\right)^{*}$ for any $v_{1}, \ldots, v_{n}$
with $v_{i} \in \mathcal{P}_{k_{i}}$.

Definition 7.9. A planar algebra $\mathcal{P}$ is called a subfactor planar algebra if $\mathcal{P}$ is a spherical *-planar algebra such that the bilinear product $\langle.,$.$\rangle defined on each \mathcal{P}_{n}$ by the formula $\langle x, y\rangle=$ $\operatorname{Tr}\left(y^{*} x\right)$ is an hermitian form.

Subfactor planar algebras have a very rich structure: a complete review of their properties can be found in [46]. The main result that will be needed in the present chapter is the following:

Theorem 7.10. [Jones, [46]] Let $\mathcal{P}$ be a subfactor planar algebra. There exists a bipartite graph $G_{\mathcal{P}}$ with root vertex $*$ such that:

$$
\operatorname{dim} \mathcal{P}_{k}=\#\left\{\text { walk of length } 2 k \text { on } G_{\mathcal{P}} \text { starting and ending at } *\right\} .
$$

### 7.2 Intertwiner spaces of a free wreath product

A non-commutative permutation group $f$ is called irreducible if $\operatorname{dim}\left(\operatorname{Mor}_{F}(0,1)\right)=1$. This section is devoted to the description of the intertwiner spaces of the free wreath product of two irreducible non-commutative permutations groups $F$ and $G$;

### 7.2.1 Intertwiner spaces of non-commutative permutation groups

Spin planar algebra As planar tangles are generalization of noncrossing partitions to higher dimensions, the spin planar algebra is a way to generalize the maps $T_{P}$ 's (as defined in Section 1.1.2) in order to build multilinear maps. This forms a planar algebra which has been introduced by Jones in [46]. Some cares are needed in order to define properly a planar algebra that possesses all the properties of a subfactor planar algebra.
Let $V$ be a $d$-dimensional Hilbert space with a distinguished orthonormal basis $\left(e_{i}\right)_{1 \leq i \leq d}$. We denote by $V_{n}$ the vector space $V^{\otimes n}$ and we set $V_{+}=\mathbb{C}, V_{-}=V$. For each $n \geq 1$, a basis of $V^{\otimes n}$ is given by $\left\{e_{i_{1}} \otimes \cdots \otimes e_{i_{n}}\right\}_{1 \leq i_{1}, \ldots, i_{n} \leq d}$. The action of planar tangle will be described on these bases. Namely let $P$ be a planar tangle of degree $k$ with $r$ inner disks $D_{i}, 1 \leq i \leq r$ of respective degree $k_{i}$. $P$ defines a map $T_{P}$ from $\otimes V_{k_{i}}$ to $V_{k}$ whose expression on the respective bases is the following:

1. For each $1 \leq i \leq r$, let $e_{\vec{j}^{i}}=e_{j_{1}^{i}} \otimes \cdots \otimes e_{j_{k_{i}}^{i}}$ be a basis element of $V_{k_{i}}$.
2. Each inner disk $D_{i}$ has $k_{i}$ boundary components which are also the boundary of a shaded region (each interval $(2 l-1,2 l)$ for $\left.1 \leq l \leq k_{i}\right)$ : label the interval component between the points $2 l-1$ and $2 l$ of $D_{i}$ with the value $j_{l}^{i}$.
3. A function $f:\{$ shaded regions of $P\} \rightarrow \llbracket 1 ; d \rrbracket$ is called compatible if the label of the boundary of an inner disk which is also the boundary of a shaded region $\sigma$ is equal to $f(\sigma)$.
4. A compatible function $f$ yields a labelling of the outer boundary by setting $j_{l}=f(\sigma)$ if the interval component $(2 l-1,2 l)$ of $D_{0}$ is the boundary of the shaded region $\sigma$. The resulting vector $e_{j_{1}} \otimes \cdots \otimes e_{j_{l}}$ is denoted by $e_{f}$.
5. Set

$$
T_{P}\left(e_{\vec{j}_{1}} \otimes \cdots \otimes e_{\vec{j}_{r}}\right)=\sum_{f \text { compatible }} e_{f}
$$

Pictorially, this means that we impose the indices of the tensor products to be the same on the boundaries of a shaded regions. An example of such condition is given in Figure 7.12.


Figure 7.12: : Spin action a planar tangle.
The planar tangle $P$ of Figure 7.12 yields the map $T_{P}: V^{\otimes 2} \otimes V \otimes V^{\otimes 2} \otimes V^{\otimes 2} \otimes V \otimes V^{\otimes 3} \longrightarrow V^{\otimes 5}$ defined by

$$
\begin{aligned}
& T_{P}\left(\left(e_{j_{2}^{1}} \otimes e_{j_{2}^{1}}\right) \otimes e_{j_{1}^{2}} \otimes\left(e_{j_{1}^{3}} \otimes e_{j_{2}^{3}}\right) \otimes\left(e_{j_{1}^{4}} \otimes e_{j_{2}^{4}}\right) \otimes e_{j_{1}^{5}} \otimes\left(e_{j_{1}^{6}} \otimes e_{j_{2}^{6}} \otimes e_{j_{3}^{6}}\right)\right) \\
& =\delta_{j_{1}^{1} j_{1}^{4} j_{3}^{6}} \delta_{j_{1}^{2} j_{1}^{3}} \delta_{j_{1}^{5} j_{1}^{6}} e_{j_{2}^{1}} \otimes e_{j_{2}^{3}} \otimes e_{j_{2}^{4}} \otimes e_{j_{1}^{5}} \otimes e_{j_{2}^{2}} .
\end{aligned}
$$

This action of planar tangles is clearly invariant under isotopy and is compatible with the composition of tangles. The spin planar algebra is not spherical since $\delta_{1}=d$ and $\delta_{2}=1$. However it is possible to multiply each $T_{P}$ by a scalar $\mu(P)$ in such a way that the resulting action still yields a planar algebra structure with $\delta_{1}=\delta_{2}=\sqrt{d}$.

Definition 7.11 ([46). The collection of vector spaces given by $V_{n}=V^{\otimes n}, V_{-}=V, V_{+}=\mathbb{C}$, with the action of a planar tangle $P$ given by $\mu(P) T_{P}$ as defined above, is called the spin planar algebra and denoted by $\mathcal{P}(V)$.

Note that for $d>1$, the spin planar algebra is not a planar algebra since $\operatorname{dim} V_{-}=d>1$. However the following result of [46] holds:

Lemma 7.12 (Jones, [46]). A planar algebra contained in the spin planar algebra is a spherical planar algebra.

A planar algebra contained in spin planar algebra is called a spin planar subalgebra.
In particular the spin planar subalgebra $T L(V) \subseteq \mathcal{P}(V)$ which is given by the image of all planar tangles without inner disk is a spherical planar algebra. Constructing $T_{P}$ for each of these planar tangles gives that

$$
T L(V)_{n}=\operatorname{span}\left(T_{p}, p \in N C(n)\right) .
$$

This planar algebra of noncrossing partitions has been introduced and studied by Sunder and Kodiyalam in [49]. It also yields the first connection between spin planar algebras and intertwiner spaces of non-commutative permutation groups, since by a result of [5]

$$
T L(V)=\left\{\operatorname{Mor}_{S_{n}^{+}}(0, n)\right\}_{n \geq 1} .
$$

Let $n \geq 1$. As explained in Definition 3.30, a non-commutative permutation group of size $n$ is a quantum subgroup of the free symmetric group $S_{n}^{+}$. Therefore by the Tannaka-Krein duality of 3.15

$$
\begin{equation*}
\text { For any } k, k^{\prime} \geq 1, \operatorname{Mor}_{S_{n}^{+}}\left(k, k^{\prime}\right) \subseteq \operatorname{Mor}_{F}\left(k, k^{\prime}\right) \tag{7.2.1}
\end{equation*}
$$

We have seen in last section that the intertwiner spaces of $S_{n}^{+}$form a particular planar subalgebra of the spin planar algebra. The relation (7.2.1) extends to the following result due to Banica in [6]:

Proposition 7.13 ([6],Sec.5). Let $F$ be a quantum permutation group of size n. If $F$ is irreducible, then $\left\{\operatorname{Mor}_{F}(k, 0)\right\}_{k \geq 1}$ is a sub-planar algebra of the spin planar algebra $\mathcal{P}(V), V$ being the vector space of the fundamental representation of $F$.
Moreover any sub-planar algebra of $\mathcal{P}(V)$ is of the form $\left\{\operatorname{Mor}_{F}(k, 0)\right\}_{k \geq 1}$ for an irreducible quantum permutation group $F$ of size $n$, and the correspondence is bijective.

We denote by $\mathcal{P}(F)$ the planar algebra associated to the quantum permutation $F$ by the previous Proposition. Recall that given a compact quantum subgroup $G$ of $O_{n}^{+}$, the knowledge of $\left\{\operatorname{Mor}_{G}(k, 0)\right\}_{k \geq 1}$ is enough to get all the intertwiner spaces $\operatorname{Mor}_{G}(k, l)$ for $1 \leq k, l$, because of the isomorphism $\operatorname{Mor}_{G}(k, l) \simeq \operatorname{Mor}_{G}(k+l, 0)$. Therefore, $\mathcal{P}(F)$ describes all the representations of $F$.

### 7.2.2 Case of a free wreath product

If $G$ and $F$ are two irreducible quantum permutation groups of respective size $d$ and $n$, then the free wreath product $G \imath_{*} F$ is again a quantum permutation group by the inclusion $G \imath_{*} F \subseteq S_{d n}^{+}$. The problem is to relate the intertwiner spaces of $G \tau_{*} F$ with the ones of $G$ and $F$. As it will be proven in this subsection, $G \imath_{*} F$ is again irreducible; therefore by Proposition 7.13, the problem is equivalent to relating $\mathcal{P}\left(G \imath_{*} F\right)$ to $\mathcal{P}(G)$ and $\mathcal{P}(F)$. This relation involves the free product of planar algebras, a construction which has been done by Bisch and Jones in [23].

## Tensor and free products of planar algebras

Definition 7.14. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. The tensor product planar algebra $\mathcal{P} \otimes \mathcal{Q}$ is the collection of vector spaces $(\mathcal{P} \otimes \mathcal{Q})_{i}=\mathcal{P}_{i} \otimes \mathcal{Q}_{i}$, with the action of any planar tangle given by the tensor product of the action on each component: Namely for a planar tangle $P$,

$$
T_{P}\left(\bigotimes_{D_{i}}\left(p_{i} \otimes q_{i}\right)\right)=T_{P}\left(\bigotimes_{D_{i}} p_{i}\right) \otimes T_{P}\left(\bigotimes_{D_{i}} q_{i}\right),
$$

where $p_{i} \in \mathcal{P}_{k_{i}}$ and $q_{i} \in \mathcal{Q}_{k_{i}}$.
If $\mathcal{P}(V)$ and $\mathcal{P}(W)$ are two spin planar algebras respectively associated to the vector spaces $V$ and $W$, then $\mathcal{P}(V) \otimes \mathcal{P}(W)$ is isomorphic to the spin planar algebra $\mathcal{P}(V \otimes W)$, with the isomorphism $V^{\otimes k} \otimes W^{\otimes k} \simeq(V \otimes W)^{\otimes k}$. The free product of two planar algebras $\mathcal{P}$ and $\mathcal{Q}$ is a subplanar algebra of $\mathcal{P} \otimes \mathcal{Q}$ defined by the image of certains planar tangles.
A pair $(P, Q)$ of planar tangles of degree $k$ is called free if there exists a planar tangle $R$ of degree $2 k$ and two isotopies $\varphi_{1}$ and $\varphi_{2}$, respectively of $P$ and $Q$, such that:

- $\Gamma R=\Gamma \varphi_{1}(P) \cup \Gamma \varphi_{2}(Q)$, and the set of distinguished points of $R$ is the image through $\varphi_{1}$ and $\varphi_{2}$ of the set of distinguished points of $P$ and $Q$.
- $\varphi_{1}\left(P \backslash D_{0}(P)\right) \cap \varphi_{2}\left(Q \backslash D_{0}(Q)\right)=\emptyset$. This means that a connected component of $R$ is the image of a connected component of either $P$ or $Q$.
- The distinguished point numbered $i$ of $\partial D_{0}(P)$ is sent by $\varphi_{1}$ to the distinguished point numbered $2 i-\delta(i)$ of $\partial D_{0}(R)$.
- The distinguished point numbered $i$ of $\partial D_{0}(Q)$ is sent by $\varphi_{2}$ to the distinguished point numbered $2 i-(1-\delta(i))$.
- A distinduished point of an inner disk coming from $P$ is labelled as in $P$; a distinguished point $i$ of an inner disk coming from $Q$ is labelled $i-1$.

The last condition ensure that curves of $R$ have endpoints with correct parities. If $P$ and $Q$ are connected planar tangles and $R$ exists, then $R$ is unique up to orientation preserving diffeomorphism: this planar tangle is called the free composition of $P$ and $Q$ and denoted by $P * Q$. An example of a free pair of planar tangles, with the resulting free composition, is drawn in Figure 7.13


Figure 7.13: : Free composition of two planar tangles.

Definition 7.15. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. The free product planar algebra $\mathcal{P} * \mathcal{Q}$ is the collection of vector subspaces $(\mathcal{P} * \mathcal{Q})_{k}$ of $(\mathcal{P} \otimes \mathcal{Q})_{k}$ spanned by the image of the maps $T_{P} \otimes T_{Q}$ for all free pairs of planar tangles of degree $k$.
$\left.\{\mathcal{P} * \mathcal{Q})_{k}\right\}_{k \geq 1}$ is stable under the action of planar tangles:
Lemma 7.16. $\mathcal{P} * \mathcal{Q}$ is a planar sub-algebra of $\mathcal{P} \otimes \mathcal{Q}$.
Proof. It suffices to check the stability on the generating sets of the vector spaces $(\mathcal{P} * \mathcal{Q})_{k}$ given in Definition 7.15. Let $P$ be a planar tangle, and for each inner disk $D_{i}$ of $P$, let $v_{i}$ be an element of $(\mathcal{P} * \mathcal{Q})_{k_{i}}$ of the form $T_{P_{i}} \otimes T_{Q_{i}}\left(\left(\otimes_{D_{j}\left(P_{i}\right)} v_{j}^{i}\right) \otimes\left(\otimes_{D_{j}\left(Q_{i}\right)} w_{j}^{i}\right)\right.$ (with ( $\left.P_{i}, Q_{i}\right)$ free pair). By composition of actions of planar tangles,

$$
\begin{aligned}
T_{P}\left(\bigotimes_{D_{i}(P)} v_{i}\right)= & T_{P_{\left(D_{1}, \ldots D_{n}\right)}}\left(P_{1}, \ldots, P_{n}\right)\left(\bigotimes_{D_{i}(P)} \bigotimes_{D_{j}\left(P_{i}\right)} v_{j}^{i}\right) \\
& \otimes T_{P_{\left(D_{1}, \ldots D_{n}\right)}}\left(Q_{1}, \ldots, Q_{n}\right)\left(\bigotimes_{D_{i}(P)} \bigotimes_{D_{j}\left(Q_{i}\right)} w_{j}^{i}\right) .
\end{aligned}
$$

Thus, it is enough to prove that $S=P \circ_{\left(D_{1}, \ldots D_{n}\right)}\left(P_{1}, \ldots, P_{n}\right)$ and $T=P \circ_{\left(D_{1}, \ldots D_{n}\right)}\left(Q_{1}, \ldots, Q_{n}\right)$ form a free pair. Let $\tilde{P}$ be the planar tangle of order $2 k$ obtained from $P$ by doubling all the curves of $P$ and all the distinguished points (in such a way that the tangle still remains planar).

By this construction, a curve joining the point $j$ of $D_{i}$ to the point $j^{\prime}$ of $D_{i^{\prime}}$ in $P$ yields in $\tilde{P}$ two curves: one joining the point $2 j-1$ of $D_{i}$ to the point $2 j^{\prime}-1$ of $D_{i^{\prime}}$ and the other one joining the point $2 j$ of $D_{i}$ to the point $2 j^{\prime}$ of $D_{i^{\prime}}$. Since $P$ is a planar tangle, the conditions on the parities of $j$ and $j^{\prime}$ yield that in $\tilde{P}$, the curves join points labelled 0 or 1 modulo $4($ resp. 2 or $3 \bmod 4$ ) to points labelled 0 or 1 modulo 4 (resp. 2 or $3 \bmod 4$ ). Reciprocally, by removing from $\tilde{P}$ all the distinguished points labelled 0 and 1 modulo 4 and the curves joining them, we recover the planar tangle $P$. The same holds for the distinguished points labelled 2 and 3 . Therefore, if we compose the tangle $P_{i} * Q_{i}$ inside each disk $D_{i}$, the resulting tangle is exactly $S * T$ (up to a relabelling). Thus $\tilde{P}{ }_{\circ_{1}, \ldots, D_{n}}\left(P_{1} * Q_{1}, \ldots, P_{n} * Q_{n}\right)=S * T$ and $\mathcal{P} * \mathcal{Q}$ is stable by the action of planar tangles.

There is also a combinatorial way to characterize free pairs of planar tangles.
Lemma 7.17. If $(P, Q)$ is a free pair, then $\pi(P * Q)=\left(\pi_{P}, S\right) \vee\left(\pi_{Q}, S^{c}\right)$.
In particular $(P, Q)$ is a free pair if and only if $\left(P_{\pi_{P}}, P_{\pi_{Q}}\right)$ is a free pair.
Proof. The first statement of the lemma is a direct consequence of the definition of $S$ and the fact that $i \sim_{P} j$ (resp. $i \sim_{Q} j$ if and only if $2 i-\delta(i) \sim_{P * Q} 2 j-\delta(j)$ (resp. $2 i-(1-\delta(i)) \sim_{P * Q}$ $2 j-(1-\delta(j))$.
Thus, if $(P, Q)$ is free pair, then $P_{\pi(P * Q)}$ is exactly the free composition of $P_{\pi_{P}}$ with $P_{\pi_{Q}}$ and $\left(P_{\pi_{P}}, P_{\pi_{Q}}\right)$ is also a free pair.
Suppose that $\left(P_{\pi_{P}}, P_{\pi_{Q}}\right)$ is a free pair. There exist $P_{1}, \ldots, P_{r}, Q_{1}, \ldots, Q_{r^{\prime}}$ such that $P=$
 $\left(P_{1}, \ldots, P_{r}, Q_{1}, \ldots, Q_{r^{\prime}}\right)$ gives the free composition of $P$ and $Q$.

Proposition 7.18. Let $P$ and $Q$ be two connected planar tangles. Then $(P, Q)$ is a free pair if and only if $\pi_{Q} \leq k r^{\prime}\left(\pi_{P}\right)$. In particular if $P, P^{\prime}$ and $Q, Q^{\prime}$ satisfy $\pi_{P}=\pi_{P^{\prime}}$ and $\pi_{Q}=\pi_{Q^{\prime}}$, then $(P, Q)$ is a free pair if and only if $\left(P^{\prime}, Q^{\prime}\right)$ is a free pair.
Proof. If $(P, Q)$ is a free pair, then by the previous Lemma $\left(\pi_{P}, S\right) \vee\left(\pi_{Q}, S^{c}\right)$ is non-crossing. Therefore $\pi_{Q} \leq k r^{\prime}\left(\pi_{P}\right)$.
If $\pi_{Q} \leq k r^{\prime}\left(\pi_{P}\right), \pi_{0}=\left(\pi_{P}, S\right) \vee\left(\pi_{Q}, S^{c}\right)$ is noncrossing with even blocks. Therefore, $P_{\pi_{0}}$ is a well-defined planar tangle. Let $\left\{C_{i}\right\}$ be the connected components of $\pi_{0}$ coming from blocks of $\left(\pi_{P}, S\right)$ and $\left\{D_{i}\right\}$ the ones coming from blocks of $\left(\pi_{Q}, S^{c}\right)$. Then $P_{\pi_{0}} \backslash \bigcup C_{i}$ is an irreducible planar tangle and $\pi\left(P_{\pi_{0}} \backslash \bigcup C_{i}\right)=\pi_{Q}$. Thus $P_{\pi_{0}} \backslash \bigcup C_{i}=P_{\pi_{Q}}$ up to a relabelling of the distinguished points, and similarly $P_{\pi_{0}} \backslash \bigcup D_{i}=P_{\pi_{P}}$ up to a relabelling. Therefore, $P_{\pi_{0}}$ is, up to a relabelling, the free composition of $P_{\pi_{P}}$ with $P_{\pi_{Q}}$. Thus, $\left(P_{\pi_{P}}, P_{\pi_{Q}}\right)$ is a free pair and by the previous lemma, $(P, Q)$ is also a free pair.

## Reduced free pair

Definition 7.19. A free pair $(P, Q)$ of planar tangles is called reduced if $P$ and $Q$ are irreducible, and any region of $P * Q$ is bounded by at most one connected component from $P$ and one from $Q$.

By a connected component from $P$ (resp. $Q$ ) in $P * Q$, we mean the image by $\varphi_{1}$ (resp. $\varphi_{2}$ ) of a connected component of $\Gamma P \backslash \partial D_{0}(P)$ (resp. $\Gamma Q \backslash \partial D_{0}(Q)$ ), where $\varphi_{1}$ and $\varphi_{2}$ are the maps involved in the construction of $P * Q$. An example of reduced free pair is given in Figure 7.14.

In the sequel, a distinguished point is called an outer boundary point of a region $\sigma$ if it is both a point of the boundary of $\sigma$ and a distinguished point of the outer disk.


Figure 7.14: : Example of reduced free pair.
Lemma 7.20. A free pair $(P, Q)$ of degree $k$ is reduced if and only if $P$ and $Q$ are irreducible and

- For $0 \leq i \leq k-1,\{(4 i+2),(4 i+3)\}$ is the set of outer boundary points of a region of $P * Q$. The same holds true for $\{4 i,(4 i+1)\}$.
- The set of outer boundary points of any other region has the form $\{4 i-1,4 i, 4 j+1,4 j+2\}$ for $1 \leq i, j \leq k$.

Proof. Suppose that $(P, Q)$ is reduced. Let $0 \leq i \leq k-1.4 i+2$ and $4 i+3$ are two consecutive points of the outer boundary, thus they are boundary points of a same unshaded region of $P * Q$. They are both coming from $Q$ and the region having $(4 i+2,4 i+3)$ as boundary component has at most one boundary component coming from $Q$, thus $(4 i+2)$ and $(4 i+3)$ are in the same connected component. Since $Q$ is irreducible, they are both connected to the same inner disk: therefore, $\{(4 i+2),(4 i+3)\}$ is the set of outer boundary points of a region of $P * Q$. By symmetry between $P$ and $Q$, the same holds true for $4 i$ and $(4 i+1)$.
Let $\sigma$ be a region which has no outer boundary interval of the form $(4 i, 4 i+1)$ or $(4 i+2,4 i+3)$. Since $P * Q$ is irreducible, $\sigma$ has an outer boundary interval, which is by hypothesis of the form ( $4 i-1,4 i$ ) or ( $4 i+1,4 i+2$ ) (in particular $\sigma$ is shaded). Let us assume without loss of generality that this boundary interval is of the form $(4 i-1,4 i)$, and order counterclockwise the distinguished points on the boundary of $\sigma$. Since $4 i-1$ and $4 i$ are not coming from the same tangle, they are connected to different inner disks. Moreover, the next outer boundary point of $\sigma$ after $4 i$ is in the same connected component as $4 i$, and thus it is of the form $4 j+1$ or $4 j$. But consecutive boundary points of a connected component have to be of opposite parity (for example as a consequence to the fact that $\pi_{P}$ has even blocks); thus the next outer boundary point of sigma is of the form $4 j+1$ for some $0 \leq j \leq k$. Since $\sigma$ is not bounded by other connected component coming from $P$, the only outer boundary points of $\sigma$ coming from $P$ are $4 i$ and $4 j+1$. Since $\sigma$ is shaded and $(4 j+1)$ is a boundary point of $\sigma,(4 j+1,4 j+2)$ is a boundary interval of $\sigma$ and thus $4 j+2$ is also a boundary point of $\sigma$. Let $x$ be the next outer boundary point of $\sigma$ after $4 j+2$; by the same arguments, $x=4 i^{\prime}-1$ for some $1 \leq i^{\prime} \leq k$. This implies also that $4 i^{\prime}$ is an outer boundary point of $\sigma$. Since $4 i^{\prime}$ comes from $P, 4 i^{\prime}=4 i$ and $x=4 i-1$. The set of outer boundary points of $\sigma$ is thus exactly $\{4 i-1,4 i, 4 j+1,4 j+2\}$.
Reciprocally, suppose that $P * Q$ satisfies the two conditions of the lemma. By the first condition, any unshaded region is bounded by only one connected component. If $\sigma$ is unshaded, then $\{4 i-1,4 i, 4 j+1,4 j+2\}$ is the set of outer boundary points of $\sigma$ : thus, $\sigma$ has exactly one connected component coming from each tangle and $(P * Q)$ is reduced.

Recall that $\pi_{0}$ (resp. $\pi_{1}$ ) is the pair partition of $2 k$ with blocks $\{(2 i, 2 i+1)\}$ (resp. $\{2 i+$ $1,2 I)$ ). The latter Lemma can be rephrased as follows:

Lemma 7.21. A free pair $(P, Q)$ is reduced if and only if $P$ and $Q$ are irreducible and $\pi_{P} \geq \pi_{0}$, $\pi_{Q}=k r^{\prime}\left(\pi_{P}\right)$.

Proof. Let $P, Q$ be two irreducible tangles. It suffices to show that the two conditions of Lemma 7.20 is equivalent to the two conditions $\pi_{P} \geq \pi_{0}$ and $\pi_{Q}=k r^{\prime}\left(\pi_{P}\right)$. The first condition is equivalent $\pi_{P} \geq \pi_{0}, \pi_{Q} \geq \pi_{1}$. The second condition of Lemma 7.20 is equivalent to the condition : if $4 i \sim_{P * Q} 4 j+1$ and for all $4 i<k<4 j+1,4 j+1<l<4 i, k \not \chi_{P * Q} l$, then $4 j+2 \sim_{P * Q} 4 i-1$. By Lemma 5.16 this is equivalent to $\left(\pi_{Q}, S^{c}\right)=k r\left(\pi_{P}, S\right)$ (which implies also that $\pi_{Q} \geq \pi_{1}$ ).

Despite this rigid structure, only considering reduced free pairs is nonetheless enough to describe free product of planar algebras.

Proposition 7.22. The free planar algebra $\mathcal{P} \otimes \mathcal{Q}$ is spanned by the images of $T_{P} \otimes T_{Q}$ for all reduced free pairs $(P, Q)$.

Proof. It suffices to prove that the image of $T_{P} \otimes T_{Q}$ with $(P, Q)$ a free pair is contained in the image of $\left(P^{\prime}, Q^{\prime}\right)$ with $\left(P^{\prime}, Q^{\prime}\right)$ a reduced free pair.
The image of $T_{P} \otimes T_{Q}$ is contained in the image of $T_{P_{\pi_{P}}} \otimes T_{P_{\pi_{Q}}}$ by Proposition 7.4 and by Lemma $7.17\left(P_{\pi_{P}}, P_{\pi_{Q}}\right)$ is again a free pair. We can thus assume that $P=P_{\pi}$ and $Q=P_{\pi^{\prime}}$, with the condition $\pi^{\prime} \leq k r^{\prime}(\pi)$.
Suppose that $\mu \leq \nu$ are two noncrossing partitions of $k$. Let $B_{1}, \ldots, B_{r}$ be the blocks of $\nu$ in the lexicographical order. Since $\mu \leq \nu, \mu=\bigvee\left(\mu_{\mid B_{i}}, B_{i}\right)$. Therefore,

$$
P_{\mu}=P_{\nu} \circ{ }_{D_{1}, \ldots, D_{r}}\left(P_{\mu_{\mid B_{1}}}, \ldots, P_{\mu_{\mid B_{r}}}\right)
$$

and the image of $T_{P_{\mu}}$ is contained in the one of $T_{P_{\nu}}$.
Since $k r^{\prime}(\pi)=k r^{\prime}\left(\pi_{0} \vee \pi\right), \pi^{\prime} \leq k r^{\prime}\left(\pi_{0} \vee \pi\right) . \pi \leq \pi \vee \pi_{0}$ and $\pi^{\prime} \leq k r^{\prime}\left(\pi \vee \pi_{0}\right)$, thus the image of $T_{P_{\pi}}$ is included in the image of $T_{P_{\pi \vee \pi_{0}}}$ and the image of $T_{P_{\pi^{\prime}}}$ is included in the image of $T_{P_{k r^{\prime}\left(\pi \vee \pi_{0}\right)}}$. By Lemma 7.21. $\left(P_{\pi \vee \pi_{0}}, P_{k r^{\prime}\left(\pi_{0} \vee \pi\right)}\right)$ is reduced.

Generating subset of a free product of planar algebras Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. We denote by $S_{\mathcal{P}}(k)$ (resp. $U_{\mathcal{P}}(k)$ ) the image in $\mathcal{P}_{k}$ of $S_{k}$ (resp. $U_{k}$ ), the tangle without inner disk where $2 i-1$ is linked to $2 i$ (resp. $2 i$ is linked to $2 i+1$ ) for all $1 \leq i \leq k$. The picture of both tangles is drawn in Figure 7.15 for $k=4$.


Figure 7.15: : Tangles $S_{4}$ and $U_{4}$.

Proposition 7.23. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. Then $\mathcal{P} * \mathcal{Q}$ is the subplanar algebra of $\mathcal{P} \otimes \mathcal{Q}$ generated by $\left\{U_{\mathcal{P}}(k) \otimes \mathcal{Q}_{k}\right\}_{k \geq 1}$ and $\left\{\mathcal{P}_{k} \otimes S_{\mathcal{Q}}(k)\right\}_{k \geq 1}$.

Proof. $\left(U_{k}, I d_{k}\right)$ and $\left(I d_{k}, S_{k}\right)$ are two free pairs of planar tangles, thus for all $k \geq 1, U_{\mathcal{P}}(k) \otimes \mathcal{Q}_{k}$ and $\mathcal{P}_{k} \otimes S_{\mathcal{Q}}(k)$ are subspaces of $\mathcal{P} * \mathcal{Q}$. In particular the subplanar algebra they generate is also a subplanar algebra of $\mathcal{P} * \mathcal{Q}$.
Reciprocally let $(P, Q)$ be a reduced free pair of degree $k$. Let us show that there exists a planar tangle $R$ of degree $k$ with $r$ inner disks $D_{i}$ of respective degree $k_{i}$, such that $P=$ $R_{D_{1}, \ldots, D_{i}}\left(X_{1}, \ldots, X_{r}\right)$ and $Q=R_{D_{1}, \ldots, D_{i}}\left(\tilde{X}_{1}, \ldots, X_{r}\right)$, where for each $1 \leq i \leq r,\left(X_{i}, \tilde{X}_{i}\right)$ is either $\left(U_{k_{i}}, I d_{k_{i}}\right)$ or $\left(I d_{k_{i}}, S_{k_{i}}\right)$.
Since $(P, Q)$ is a free pair, the free composition $P * Q$ exists; since this pair is reduced, each inner disk of $P * Q$ is only connected to the outer boundary, and, by Lemma 7.20, both elements of $\{4 i, 4 i+1\}$ (resp. $\{4 i-2,4 i-1\}$ ) are connected to a same inner disk. Color an inner disk $D_{i}$ of $P * Q$ with 1 if it comes from $P$ and with 2 if it comes from $Q$. We denote by $\gamma_{i}$ the curve arriving on the distinguished point $i$ of the outer boundary and by $\bar{i}$ the distinguished point of an inner disk which is connected to $i$.
We operate the following operation on $P * Q$ : for each interval ( $4 i-1,4 i$ ), let $\sigma$ be the region for which $(4 i-1,4 i)$ is a boundary interval; add a curve $\tilde{\gamma}$ in this region connecting $\overline{4 i-1}$ to $\overline{4 i}$ and erase $\gamma_{4 i-1}, \gamma_{4 i}$ and the distinguished points $4 i-1$ and $4 i$.
The degrees of the inner disks don't change and this yields a planar tangle $R$ with $2 k$ boundary points and $r$ inner disks (where $r$ is the sum of the number of inner disks in $P$ and in $Q$ ). In the resulting planar tangle $R$, an odd point $i$ of the outer boundary is still connected to the point $\bar{i}$ on a disk colored 1 , and an even point $i$ of the outer boundary is still connected to the point $\bar{i}$ on a disk colored 2. The construction of the tangle $R$ is shown in Figure 7.16.


Figure 7.16: : Construction of the planar tangle $R$ for the reduced free pair of Figure 7.14 .
Set $X_{i}=U_{k_{i}}, \tilde{X}_{i}=I d_{k_{i}}$ if $D_{i}$ is colored 2, and $X_{i}=I d_{k_{i}}, \tilde{X}_{i}=S_{k_{i}}$ if $D_{i}$ is colored 1. Consider $R_{1}=R_{D_{1}, \ldots, D_{r}}\left(X_{1}, \ldots, X_{r}\right)$. Each disk of $R$ colored 2 is replaced by a planar tangle without inner disk, and thus disappears in $R_{1}$. A disk of $R$ colored 1 is composed with the identity, and thus remains the same in $R_{1}$. An odd point $4 i+1$ is already connected to $\overline{4 i+1}$. An even point $4 i+2$ is connected to an odd point $\overline{4 i+2}$ of a disk $D$ colored 2 . Therefore since $D$ is composed with $U_{k_{i}}, \overline{4 i+2}$ is connected in $R_{1}$ by a curve to the following point of $D$ in the counterclockwise order: since $4 i+2$ and $4 i+3$ are in the same connected component, the following point is exactly $\overline{4 i+3}$. By the modification we made on $P * Q, \overline{4 i+3}$ is connected by a curve to the point $\overline{4 i+4}$. Therefore in $R_{1}, 4 i+2$ is connected to $\overline{4 i}$. Thus relabelling the distinguished point $4 i+2$ by $4 i+4$ yields exactly the image of $P$ in $P * Q$. This reconstruction of $P$ is shown in Figure 7.17 .

Likewise $R_{2}=R_{D_{1}, \ldots, D_{r}}\left(\tilde{X}_{1}, \ldots, \tilde{X}_{r}\right)$ is equal to the image of $Q$ in $P * Q$.


Figure 7.17: : Reconstruction of $P$ from $R$.

Remark 7.24. In [54], the free product of two planar algebras $\mathcal{P} * \mathcal{Q}$ is directly defined as the subplanar algebra of $\mathcal{P} \otimes \mathcal{Q}$ generated by $\left\{U_{\mathcal{P}}(k) \otimes \mathcal{Q}_{k}\right\}_{k \geq 1}$ and $\left\{\mathcal{P}_{k} \otimes S_{\mathcal{Q}}(k)\right\}_{k \geq 1}$.

## Free product of spin planar algebras

Let us now consider the free product of two spin planar algebras $\mathcal{P}(V)$ and $\mathcal{P}(W)$; note that $\mathcal{P}(V) * \mathcal{P}(W) \subseteq \mathcal{P}(V) \otimes \mathcal{P}(W) \simeq \mathcal{P}(V \otimes W) . \quad \Phi=\left\{t_{k}\right\}$ denotes the isomorphism of planar algebras $\mathcal{P}(V) \otimes \mathcal{P}(W) \longrightarrow \mathcal{P}(V \otimes W)$ given by

$$
\left.t_{k}\left(\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right) \otimes\left(f_{j_{1}} \otimes \cdots \otimes f_{j_{k}}\right)\right)=\left(e_{i_{1}} \otimes f_{j_{1}}\right) \otimes \cdots \otimes\left(e_{i_{k}} \otimes f_{j_{k}}\right)\right)
$$

where $\left(e_{i}\right)_{1 \leq i \leq n}$ is an orthonormal basis of $V,\left(f_{j}\right)_{1 \leq j \leq m}$ is an orthonormal basis of $W$ and $1 \leq i_{1}, \ldots, i_{k} \leq n, 1 \leq j_{1}, \ldots, j_{k} \leq m$.
We will give a more precise description of a free product of spin planar algebras.
Let $S_{0}$ be the set $\{2 i+1\}_{0 \leq i \leq k-1}$ and let $f: \llbracket 1, k \rrbracket \longrightarrow S_{0}$ be the function $f(i)=2 i+1$. Let $\tilde{f}: \llbracket 1, k \rrbracket \longrightarrow S_{0}^{c}$ be the function $\tilde{f}(i)=2 i$. For $\pi$ a noncrossing partition of $k$, the Kreweras complement $\operatorname{Kr}(\pi)$ of $\pi$ (see [66]) is the non-crossing partition of $N C(k)$ such that $\left(\tilde{f}(\operatorname{Kr}(\pi)), S_{0}^{c}\right)=$ $\operatorname{kr}\left(f(\pi), S_{0}\right)$. The Kreweras complement of the non-crossing partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ is the partition $\{\{1,2\},\{3\},\{4,6\},\{5\}\}$, as shown in Figure 7.18 .


Figure 7.18: The partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ and its Kreweras complement.
Let $V$ be an $n$-dimensional vector space with basis $\left(e_{i}\right)_{1 \leq i \leq n}$.
Definition 7.25. Let $\pi$ be a non-crossing partition of $k$. For each block $B$ of $\pi$, let $\lambda^{B} \in V^{\otimes|B|}$. The composition of $\pi$ with $\left\{\lambda^{B}\right\}_{B \in \pi}$ is the vector $T\left(\pi,\left\{\lambda^{B}\right\}_{B \in \pi}\right)$ of $V^{\otimes k}$ defined by

$$
T\left(\pi,\left\{\lambda^{B}\right\}_{B \in \pi}\right)=\sum_{1 \leq i(1) \cdots, i(k) \leq n}\left(\prod_{\substack{B \in \pi \\ B=\left\{r_{1}<\cdots<r_{|B|}\right\}}} \lambda_{i\left(r_{1}\right), \ldots, i\left(r_{|B|}\right)}^{B}\right) e_{i(1)} \otimes \cdots \otimes e_{i(k)}
$$

The dual composition of $\pi$ with $\left\{\lambda^{B}\right\}_{B \in \pi}$ is the vector $\tilde{T}\left(\pi,\left\{\lambda^{B}\right\}_{B \in \pi}\right)$ of $V^{\otimes k}$ defined by

$$
\tilde{T}\left(\pi,\left\{\lambda^{B}\right\}_{B \in \pi}\right)=\sum_{\substack{1 \leq i(1), \ldots, i(k) \leq n \\ \operatorname{ker}(i) \leq K r^{\prime}(\pi)}}\left(\prod_{\substack{B \in \pi \\ B=\left\{r_{1}<\cdots<r_{|B|}\right\}}} \lambda_{i\left(r_{1}\right), \ldots, i\left(r_{|B|}\right.}^{B}\right) e_{i(1)} \otimes \cdots \otimes e_{i(k)},
$$

where $\operatorname{ker}(\vec{i})$ is as defined in 1.1.3.

Example 7.26. Let $\pi=\{\{1,4,7\},\{2,3\},\{5,6\}\}$. Thus $\operatorname{Kr}(\pi)=\{\{1,3\},\{2\},\{4,6\},\{5\},\{7\}\}$. In this case,

$$
T\left(\pi,\left\{\lambda^{1,4,7}, \lambda^{2,3}, \lambda^{5,6}\right\}\right)=\sum \lambda_{i_{1}, i_{2}, i_{3}}^{1,4,7} \lambda_{j_{1}, j_{2}}^{2,3} \lambda_{k_{1}, k_{2}}^{5,6} e_{i_{1}} \otimes e_{j_{1}} \otimes e_{j_{2}} \otimes e_{i_{2}} \otimes e_{k_{1}} \otimes e_{k_{2}} \otimes e_{i_{3}}
$$

and

$$
\tilde{T}\left(\pi,\left\{\lambda^{1,4,7}, \lambda^{2,3}, \lambda^{5,6}\right\}\right)=\sum \lambda_{i_{1}, i_{3}, i_{5}}^{1,4,7} \lambda_{i_{2}, i_{1}}^{2,3} \lambda_{i_{4}, i_{3}}^{5,6} e_{i_{1}} \otimes e_{i_{2}} \otimes e_{i_{1}} \otimes e_{i_{3}} \otimes e_{i_{4}} \otimes e_{i_{3}} \otimes e_{i_{5}}
$$

We denote by $T_{\pi}$ the linear map from $\bigotimes_{B \in \pi} V^{\otimes|B|}$ to $V^{\otimes k}$ defined by $T_{\pi}\left(\bigotimes_{B \in \pi} \lambda_{B}\right)=$ $T\left(\pi,\left\{\lambda_{B}\right\}_{B \in \pi}\right)$ and by $\tilde{T}_{\pi}$ the linear map from $\bigotimes_{B \in \pi} V^{\otimes|B|}$ to $V^{\otimes k}$ defined by $\tilde{T}_{\pi}\left(\bigotimes_{B \in \pi} \lambda_{B}\right)=$ $\tilde{T}\left(\pi,\left\{\lambda_{B}\right\}_{B \in \pi}\right)$. Compositions and dual compositions appear in the action of planar tangles on spin planar algebras. A partition $\pi$ of $2 k$ such that $\pi \geq \pi_{1}$ yields a partition of $k$ denoted by $\pi / 2$ and defined by $i \sim_{\pi / 2} j$ if and only of $2 i \sim_{\pi} 2 j$. A partition $\pi$ of $2 k$ such that $\pi \geq \pi_{0}$ yields a partition of $k$ also denoted by $\pi / 2$ and defined by $i \sim_{\pi / 2} j$ if and only of $2 i-1 \sim_{\pi} 2 j-1$. If $\pi \geq \pi_{1}$ (resp. $\pi \geq \pi_{0}$ ) and $B$ is a block of $\pi / 2, \tilde{B}$ denotes the block of $\pi$ containing $2 i-1$ and $2 i$ (resp. $2 i-2$ and $2 i-1$ ) for each $i \in B$.

Lemma 7.27. If $\pi \geq \pi_{0}, \operatorname{Kr}(\pi / 2)=k r^{\prime}(\pi) / 2$.
Proof. Suppose that $i \sim_{K r(\pi / 2)} j$. For all $i<k \leq j, j<l \leq i, k \not \chi_{\pi / 2} l$, thus for all $i<k \leq$ $j, j<l \leq i, 2 k-1 \not \chi_{\pi} 2 l-1$. Since $\pi \geq \pi_{0}$, for all $2 i-1<k \leq 2 j-1,2 j-1<l \leq 2 i-1$, $k \not \chi_{\pi} l$.
Thus for all $4(i-1)+1<k \leq 4(j-1)+1,4(j-1)+1<l \leq 4(i-1)$ with $k, l \in S$, $k \not \chi_{(f(\pi), S)} l$, where $f$ is the map $f(i)=2 i-\delta(i)$ defined in Section 7.1.2 this implies that $4(i-1)+2 \sim_{k r(f(\pi), S)} 4(j-1)+2$ and thus $2(i-1)+1 \sim_{k r^{\prime}(\pi)} 2(j-1)+1$. Since $k r^{\prime}(\pi) \geq \pi_{1}$, $2 i \sim_{k r^{\prime}(\pi)} 2 j$ and $i \sim_{k r^{\prime}(\pi) / 2} j$. Thus $\operatorname{Kr}(\pi / 2) \leq k r^{\prime}(\pi) / 2$. The same proof yields the converse inequality.

Proposition 7.28. Let $\pi$ be a partition of $2 k$ such that $\pi \geq \pi_{0}$. The action of the irreducible planar tangle $P_{\pi}$ on $\mathcal{P}(V)$ is $\tilde{T}_{\pi / 2}$, and the action of $P_{k r^{\prime}(\pi)}$ is $T_{K r(\pi / 2)}$.

Proof. By Proposition 7.7, $(2 i-1,2 i)$ and $(2 j-1,2 j)$ are outer boundary components of the same shaded region of $P_{\pi}$ if and only if $2 i \sim_{k r^{\prime}(\pi)} 2 j$. By Lemma 7.27 , this is equivalent to $i \sim_{K r(\pi / 2)} j$. For each block $B$ of $P_{\pi}$, let $D_{B}$ denote the inner disk of $P_{\pi}$ corresponding to $B$.
Let us compute $T_{P_{\pi}}$. By the construction of the planar tangle $P_{\pi}$ in Section 7.1.2, if $2 i-1$ is the $(2 s-1)$-th or $2 s$-th element of a block $B$ of $\pi$ then $2 i-1$ is linked by a curve to the point $2 s-1$ of $D_{B}$. If $i$ is the $s$-th element in the block $B$ of $\pi / 2$, then $2 i-1$ is the $2 s$-th element of the corresponding block $\tilde{B}$ in $\pi$ if $1 \notin B$ and the $(2 s-1)$-element of $\tilde{B}$ if $1 \in B$. In any case, $(2 i-1,2 i)$ is a boundary interval of the same shaded region as the interval $(2 s-1,2 s)$ of $D_{\tilde{B}}$. Let $f$ be a function from $\llbracket 1, k \rrbracket$ to $\llbracket 1, n \rrbracket$. For each block $B=\left\{i_{1}^{B}, \ldots, i_{|B|}^{B}\right\}$ of $\pi / 2$, let $e_{B}=$ $e_{f\left(i_{1}^{B}\right)} \otimes \cdots \otimes e_{f\left(i_{|B|}^{B}\right)}$ be the corresponding element of $V^{\otimes|B|}$. Then by the expression of $T_{P_{\pi}}$ in the spin planar algebra and the above remarks,

$$
T_{P_{\pi}}\left(\bigotimes_{B \in \pi / 2} e_{B}\right)=\delta_{\operatorname{ker}(f(1), \ldots, f(k)) \leq K r(\pi / 2)} e_{f(1)} \otimes \cdots \otimes e_{f(k)}
$$

This is exactly the expression of $\tilde{T}_{\pi / 2}\left(\bigotimes_{B \in \pi / 2} e_{B}\right)$.
Let us compute $T_{P_{k r^{\prime}(\pi)}}$. Let $B$ be a block of $k r^{\prime}(\pi)$. It corresponds to an inner disk $D_{B}$ of $P_{k r^{\prime}(\pi)}$. Label $i$ the interval $(2 i-1,2 i)$ of the boundary of $D_{B}$, and do the same for the intervals of $D_{0}$.
$k r^{\prime}(\pi) \geq \pi_{1}$, thus for all $1 \leq i \leq k$, the outer distinguished points $2 i-1$ and $2 i$ are in the same connected component of $P_{k r^{\prime}(\pi)}$. Since $P_{k r^{\prime}(\pi)}$ is also irreducible, each shaded region $\sigma$ has exactly one boundary component which is an outer interval $i_{\sigma}$, and one boundary component which is an interval $j_{\sigma}$ of an inner disk $D_{B(\sigma)}$. Therefore, there are as many shaded region as outer intervals of type $(2 i-1,2 i)$, and we can identify $i_{\sigma}$ with $i$.
Let $f$ be a function from $\llbracket 1, k \rrbracket$ to $\llbracket 1, n \rrbracket$. For each block $B=\left\{i_{1}^{B}, \ldots, i_{|B|}^{B}\right\}$ of $k r^{\prime}(\pi) / 2$, let $e_{B}=e_{f\left(i_{1}^{B}\right)} \otimes \cdots \otimes e_{f\left(i_{|B|}^{B}\right)}$ be the corresponding element of $V^{\otimes|B|}$. Then by the spin action of $T_{P_{k r^{\prime}(\pi)}}$,

$$
T_{P_{k r^{\prime}(\pi)}}\left(\bigotimes_{B \in k r^{\prime}(\pi)} e_{B}\right)=e_{f_{1}} \otimes \cdots \otimes e_{f_{k}}
$$

and $T_{P_{k r^{\prime}(\pi)}}=T_{k r^{\prime}(\pi) / 2}$. Since $k r^{\prime}(\pi) / 2=\operatorname{Kr}(\pi / 2)$, this gives the second result of the Proposition.

This yields a simpler description of a free product of spin planar subalgebras:
Corollary 7.29. Let $\mathcal{P}, \mathcal{Q}$ be two spin planar subalgebras. Then $(\mathcal{P} * \mathcal{Q})_{k}$ is spanned by the union of $\tilde{T}_{\pi}\left(\otimes_{B \in \pi} \mathcal{P}_{|B|}\right) \otimes T_{K r(\pi)}\left(\otimes_{B \in K r(\pi)} \mathcal{Q}_{|B|}\right)$ for all $\pi \in N C(k)$.
Proof. By Proposition 7.22 (and its proof), $(\mathcal{P} * \mathcal{Q})_{k}$ is spanned by the images of $\left(T_{P_{\pi}}, T_{P_{k r^{\prime}(\pi)}}\right)$ for all $\pi \in N C(2 k)$ with $\pi \geq \pi_{0}$. Proposition 7.28 yields the result.

In particular, for $\mathcal{P}=T L(V),(\mathcal{P} * \mathcal{Q})_{k}$ is spanned by $\left\{T_{p} \otimes T_{p}\left(\mathcal{Q}_{k}\right)\right\}_{p \in N C(k)}$, where $T_{p}$ is the vector defined from $p$ in Chapter 1. Therefore if $\mathcal{Q}=\mathcal{P}(G)$ for an irreducible noncommutative permutation group $G$, then by Theorem 6.16 $\mathcal{P}\left(G 2_{*} S_{n}^{+}\right) \simeq T L(V) * \mathcal{P}(G)$, the isomorphism being given by the collection of maps $\Phi=\left\{t_{k}\right\}_{k \geq 1}$ of Section 7.2.2.

## Intertwiner spaces of a free wreath product

The following Theorem is the main result of the section: this was originally conjectured by Banica and Bichon in [10].

Theorem 7.30. Let $F$ and $G$ be two irreducible free permutation groups. Then

$$
\mathcal{P}\left(G \imath_{*} F\right) \simeq \mathcal{P}(F) * \mathcal{P}(G)
$$

with the isomorphism $\Phi$ given in Section 7.2.2.
Proof. The proof follows the same pattern as the proof of Theorem6.16. Suppose that $F$ acts on $V$ and $G$ on $W$ : let $\left(u_{i j}\right)_{1 \leq i \leq n}$ be the fundamental representation of $F$ and $\left(v_{k l}\right)_{1 \leq k, l \leq m}$ the one of $G$. Recall that the free wreath product $G 2_{*} F$ is defined by the fundamental representation $\left(u_{i j} v_{k l}^{i}\right)_{\substack{1 \leq i, j \leq n \\ 1 \leq k, l \leq m}}$ with the commutation relations $u_{i j} v_{k l}^{i}=v_{k l}^{i} u_{i j}$ (see 3.3.1).
Step 1: $\overline{\mathcal{P}}(\bar{F}) * \mathcal{P}(G)$ is isomorphic through $\Phi$ to a spin planar subalgebra of $\mathcal{P}(V \otimes W)$, thus by Theorem 7.13 there exists an irreducible permutation group $H$ acting on $V \otimes W$, such that $(\mathcal{P}(F) * \mathcal{P}(G))_{k} \simeq \operatorname{Mor}_{H}(0, k)$ for all $k \geq 1$, each isomorphism map being given by $t_{k}$.
Step 2: Let us prove that $\mathcal{P}(F) * \mathcal{P}(G) \subseteq \mathcal{P}\left(G 2_{*} F\right)$. Since both are planar algebras, it is enough to prove that a set of generating elements of $\mathcal{P}(F) * \mathcal{P}(G)$ is in $\mathcal{P}\left(G z_{*} F\right)$. By Proposition 7.23, a generating set is given by elements of two kinds:

- $\left(T_{\mathbf{1}_{k}} \otimes w_{k}\right)$ with $w_{k} \in \operatorname{Mor}_{G}(0, k)$. It has been proven in Theorem 6.16 that $t_{k}\left(T_{\mathbf{1}_{k}} \otimes w_{k}\right) \in$ Mor $_{G \imath_{*} S_{n}^{+}}(0, k)$, where $t_{k}$ is the map defined in Section 7.2 .2 . Since $F \subseteq S_{n}^{+}, G \imath_{*} F \subseteq G \imath_{*} S_{n}^{+}$ and thus $t_{k}\left(T_{\mathbf{1}_{k}} \otimes w_{k}\right) \in \operatorname{Mor}_{G \imath_{*} F}(0, k)$.
- $t_{k}\left(w_{k} \otimes T_{\mathbf{0}_{k}}\right)$ with $w_{k} \in \operatorname{Mor}_{F}(0, k)$. Let us expand $w_{k}$ in the basis of $V^{\otimes k}$ as $\sum_{1 \leq i_{1}, \ldots, i_{k} \leq n} w_{\vec{i}} e_{i_{1}} \otimes$ $\cdots \otimes e_{i_{k}}$. Then

$$
t_{k}\left(w_{k} \otimes T_{\mathbf{0}_{k}}\right)=\sum_{\substack{1 \leq i_{1}, \ldots, i_{k} \leq n \\ 1 \leq r_{1}, \ldots, r_{k} \leq m}} w_{\vec{i}}\left(e_{i_{1}} \otimes f_{r_{1}}\right) \otimes \cdots \otimes\left(e_{i_{k}} \otimes f_{r_{k}}\right)
$$

Since $w_{k} \in \operatorname{Mor}_{F}(0, k), \sum w_{\vec{i}} u_{j_{1} i_{1}} \ldots u_{j_{k} i_{k}}=w_{\vec{j}}$. Applying the action of $G{z_{*}} F$ yields (we use the abbreviated notations $e_{\vec{i}}=e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}$ )

$$
\begin{aligned}
\alpha_{G l_{*} F}\left(t_{k}\left(w_{k} \otimes T_{\mathbf{0}_{k}}\right)\right) & =\sum_{\substack{\vec{i}, \vec{j} \\
\vec{r}, \vec{s}}} w_{\vec{i}} t_{k}\left(e_{\vec{j}} \otimes f_{\vec{s}}\right) \otimes v_{s_{1} r_{1}}^{j_{1}} u_{j_{1} i_{1}} \ldots v_{s_{k} r_{k}}^{j_{k}} u_{j_{k} i_{k}} \\
& =\sum_{\vec{i}, \vec{j}} w_{\vec{i}} t_{k}\left(e_{\vec{j}} \otimes f_{\vec{s}}\right) \otimes\left(\sum_{r_{1}} v_{s_{1} r_{1}}^{j_{1}}\right) u_{j_{1} i_{1}} \ldots\left(\sum_{r_{k}} v_{s_{k} r_{k}}^{j_{k}}\right) u_{j_{k} i_{k}} \\
& =\sum_{\substack{\vec{i}, \vec{j}}} w_{\vec{i}} t_{k}\left(e_{\vec{j}} \otimes f_{\vec{s}}\right) \otimes u_{j_{1} i_{1}} \ldots u_{j_{k} i_{k}} \\
& =\sum_{\vec{j}, \vec{s}} w_{\vec{j}} t_{k}\left(e_{\vec{j}} \otimes f_{\vec{s}}\right) \otimes \mathbf{1}=t_{k}\left(w_{k} \otimes T_{\mathbf{0}_{k}}\right) \otimes \mathbf{1}
\end{aligned}
$$

where the third equality is due to the fact that $G$ is a free permutation group (yielding the equality $\sum_{r} v_{s r}^{j}=1$ for all $1 \leq j \leq n, 1 \leq s \leq m$ ).

Therefore $\mathcal{P}(F) * \mathcal{P}(G) \subseteq \mathcal{P}\left(G 2_{*} F\right)$ and by the Tannaka-Krein duality, $G \imath_{*} F$ is a quantum subgroup of $H$.
Step 3: Let $\left(w_{i j k l}\right)_{\substack{1 \leq i, j \leq n \\ 1 \leq k, \leq m}}$ be the fundamental representation of $H$. Let us show that $H \subseteq G \imath_{*} F$ by showing that all the relations satisfied by the fundamental representation of $G 2_{*} F$ are also satisfied by the fundamental representation of $H$. Since $\mathcal{P}\left(S_{n}^{+}\right) * \mathcal{G} \subseteq \mathcal{P}(F) * \mathcal{P}(G)$ and from the end of Section $\left.7.2 .2 \mathcal{P}\left(S_{n}^{+}\right) * \mathcal{G}\right)=\mathcal{P}\left(G \imath_{*} S_{n}^{+}\right), H \subseteq G \imath_{*} S_{n}^{+}$.
Therefore, all the relations satisfied by the fundamental representation of $G 2_{*} S_{n}^{+}$are also satisfied by the fundamental representation of $H$. In particular from the proof of Theorem 6.16. $w_{i j k l}=\tilde{u}_{i j} \tilde{v}_{k l}^{j}$ for $\tilde{u}_{i j}=\sum_{l} w_{i j k l}$ and $\tilde{v}_{k l}^{i}=\sum_{j} w_{i j k l}$. Moreover from the same proof, $\tilde{u}_{i j} \tilde{v}_{k l}^{j}=\tilde{v}_{k l}^{j} \tilde{u}_{i j}$ and there exist a $C^{*}-$ morphism $\pi_{n+1}: C\left(S_{n}^{+}\right) \rightarrow C(H)$ defined by $\pi_{n+1}\left(s_{i j}\right)=\tilde{u_{i j}}$ and $n C^{*}$-morphisms $\pi_{i}: C(G) \rightarrow C(H)$ defined by $\pi_{i}\left(v_{k l}\right)=\tilde{v}_{k l}^{i}$.
Moreover,

$$
\begin{aligned}
\Delta \tilde{u}_{i j}=\Delta\left(\sum_{l} w_{i j k l}\right) & =\sum_{l, r, s} w_{i r k s} \otimes w_{r j s l} \\
& =\sum_{r, s} \tilde{u}_{i r} \tilde{v}_{k s}^{j} \otimes \tilde{u}_{r j}\left(\sum_{l} \tilde{v}_{s l}^{j}\right) \\
& =\sum_{r} \tilde{u}_{i r} \sum_{s} \tilde{v}_{k s}^{j} \otimes \tilde{u}_{r j}=\sum_{r} \tilde{u}_{i r} \otimes \tilde{u}_{r j}
\end{aligned}
$$

where the second and third equalities are due to the fact that $\sum_{l} v_{s l}^{j}=\sum_{s} v_{s l}^{j}=\mathbf{1}$. Thus $\left(\tilde{u}_{i j}\right)_{1 \leq i, j \leq n}$ is a representation of the compact quantum group $H$ : let $\tilde{F}$ the $C^{*}$-algebra generated by the representation $\left(\tilde{u}_{i j}\right)_{1 \leq i, j \leq n}$. Let us prove that there is a surjective morphism
from $C(F)$ to $C(\tilde{F})$ sending $u_{i j}$ to $\tilde{u}_{i j}$. It suffices to show that $\operatorname{Mor}_{F}(0, k) \subseteq \operatorname{Mor}_{\tilde{F}}(0, k)$. Let $v_{k}=\sum_{\vec{i}} v_{\vec{i}} e_{\vec{i}}$ be in $\operatorname{Mor}_{F}(0, k)$. Then $t_{k}\left(v_{k} \otimes \mathbf{0}_{k}\right) \in \operatorname{Mor}_{H}(0, k)$. On one hand the action of $H$ gives:

$$
\begin{aligned}
& \sum_{\vec{i}, \vec{j}, \vec{r}, \vec{s}} v_{\vec{i}} t_{k}\left(e_{\vec{j}} \otimes f_{\vec{s}}\right) \otimes \tilde{u}_{j_{1} i_{1}} \tilde{v}_{s_{1} r_{1}}^{j_{1}} \ldots \tilde{u}_{j_{k} i_{k}} \tilde{v}_{s_{k} r_{k}}^{s_{k}} \\
& \quad=\sum_{\vec{i}_{\vec{j}, \vec{s}, \vec{i}}} v_{\vec{i}} t_{k}\left(e_{\vec{j}} \otimes f_{\vec{s}}\right) \otimes \tilde{u}_{j_{1} i_{1}}\left(\sum_{r_{1}} \tilde{v}_{s_{1} r_{1}}^{j_{1}}\right) \ldots \tilde{u}_{j_{k} i_{k}}\left(\sum_{r_{k}} \tilde{v}_{s_{k} r_{k}}^{s_{k}}\right) \\
& \quad=\sum_{\vec{i}, \vec{j}, \vec{s}} v_{\vec{i}} t_{k}\left(e_{\vec{j}} \otimes f_{\vec{s}}\right) \otimes \tilde{u}_{j_{1} i_{1}} \ldots \tilde{u}_{j_{k} i_{k}} .
\end{aligned}
$$

Since $t_{k}\left(v_{k} \otimes \mathbf{0}_{k}\right) \in \operatorname{Mor}_{H}(0, k)$, this implies the equality:

$$
\sum_{\vec{i}, \vec{s}} v_{\vec{i}} t_{k}\left(e_{\vec{i}} \otimes f_{\vec{s}}\right) \otimes \mathbf{1}=\sum_{\vec{i}, \vec{j}, \vec{s}} v_{\vec{i}} t_{k}\left(e_{\vec{j}} \otimes f_{\vec{s}}\right) \otimes \tilde{u}_{j_{1} i_{1}} \ldots \tilde{u}_{j_{k} i_{k}},
$$

which yields the equality $\sum_{\vec{i}} v_{\vec{i}} \tilde{u}_{j_{1} i_{1}} \ldots \tilde{u}_{j_{k} i_{k}}=v_{\vec{j}} \mathbf{1}$. In particular $v_{k} \in \operatorname{Mor}_{\tilde{F}}(0, k)$.
Thus, there is a $*$-morphism $\Phi_{0}: C_{0}(F) \longrightarrow C(H)$ sending $u_{i j}$ to $\tilde{u}_{i j}$ and $n *$-morphisms $\Phi_{i}: C_{0}(G) \longrightarrow C(H)$ sending $v_{k l}$ to $\tilde{v}_{k l}^{i}$. Since for all $1 \leq i \leq n, \tilde{u}_{i j} \tilde{v}_{k l}^{i}=\tilde{v}_{k l}^{i} \tilde{u}_{i j}$, there exists a *-morphism

$$
\Phi: C_{0}(G)^{* n} * C_{0}(F) /\left\langle u_{i j} v_{k l}^{i}=v_{k l}^{i} u_{i j}\right\rangle \longrightarrow C(H)
$$

sending $u_{i j} v_{k l}^{i}$ to $\tilde{u}_{i j} \tilde{v}_{k l}^{i}$. Since $C_{0}(G)^{* n} * C_{0}(F) /\left\langle u_{i j} v_{k l}^{i}=v_{k l}^{i} u_{i j}\right\rangle=C_{0}\left(G z_{*} F\right)$ and $\Phi$ sends the fundamental representation of $G \imath_{*} F$ to the one of $H$, this yields that $H \subseteq G \imath_{*} F$.
Step 4: Since $G \imath_{*} F \subseteq H$ from Step 2 and $H \subseteq G \imath_{*} F$ from Step $3, H=G \imath_{*} F$ and by the Tannaka-Krein duality ,

$$
\mathcal{P}\left(G \imath_{*} F\right)=\mathcal{P}(H)=\Phi(\mathcal{P}(F) * \mathcal{P}(G)) .
$$

Theorem 7.30 with Corollary 7.29 yields a description of the intertwiner spaces of a free wreath product $G z_{*} F$. Namely,

$$
\operatorname{Mor}_{G l_{*} F}(0, k)=\left\langlet _ { k } \left(\tilde { T } _ { \pi } \left(\bigotimes_{B \in \pi} \operatorname{Mor}_{F}(0,|B|) \otimes T_{K r(\pi)}\left(\bigotimes_{B \in K r(\pi)} \operatorname{Mor}_{G}(0,|B|)\right\rangle\right.\right.\right.
$$

The main problem to deduce the law of $\chi_{G \imath * F}$ is that it is difficult to extract a basis from these sets. In the next section, we will explain a result of Bisch and Jones that gives the dimension of each vector space $(\mathcal{P} * \mathcal{Q})_{k}$ thanks to the ones of $\mathcal{P}$ and $\mathcal{Q}$.

### 7.3 Free product formula of Bisch and Jones

Each planar algebra $\mathcal{P}$ yields a probability measure $\mu(\mathcal{P})$ by saying that the $k$-th moment $\mu(\mathcal{P})_{k}$ of $\mu(\mathcal{P})$ is $\operatorname{dim}\left(\mathcal{P}_{k}\right)$. It will be later clear that this sequence of moments actually defines a measure. In this section we review the following result of Bisch and Jones ([23]), which yields the proof of Theorem 7.1:
Theorem 7.31 (Bisch and Jones, [23]). Let $\mathcal{P}, \mathcal{Q}$ be two planar algebras with $\mu(\mathcal{P})=\mu(\mathcal{Q})=1$. Then

$$
\mu(\mathcal{P} * \mathcal{Q})=\mu(\mathcal{P}) \boxtimes \mu(\mathcal{Q}) .
$$

Since the proof the this result still doesn't exist in the literature, we give a combinatorial proof of Theorem 7.31 which is based on the computation of the principal graph of a free product by Landau (see [54]).

## Principal graph of a free product

One of the major consequence of the axioms of a subfactor planar algebra is the Theorem 7.10 of Section 7.1.3

Theorem (Jones, [46]). Let $\mathcal{P}$ be a subfactor planar algebra. There exists a bipartite graph $G_{\mathcal{P}}$ with root vertex $*$ such that:

$$
\operatorname{dim} \mathcal{P}_{k}=\#\left\{\text { walk of length } 2 k \text { on } G_{\mathcal{P}} \text { starting and ending at } *\right\} .
$$

The principal graph of a free product $\mathcal{P} * \mathcal{Q}$ can be obtained from the principal graph of $\mathcal{P}$ and the one of $\mathcal{Q}$. This has been done by Landau in [54. We will only give the result when $\operatorname{dim} \mathcal{P}_{1}=\operatorname{dim} \mathcal{Q}_{1}=1$, since this is the only interesting case in this chapter.
If $\mathcal{P}$ is a planar algebra such that $\operatorname{dim} \mathcal{P}_{1}$, the root vertex $*$ of $G_{\mathcal{P}}$ is linked to only one other vertex $\tilde{*}$ by a unique edge $e_{*}$ called the root edge. Let $G_{\mathcal{P}}$ and $G_{\mathcal{Q}}$ be respectively the principal graph of $P$ and $Q$. Let $\tilde{G}_{\mathcal{P}}$ (resp. $\tilde{G}_{\mathcal{Q}}$ ) be the graph $G_{\mathcal{P}}$ with the root vertex and the root edge removed, and with the new root being $\tilde{*}$. We build recursively a graph $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and a height function on edges of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ as follows:

1. Let $G_{1}$ be a copy of $G_{\mathcal{P}}$. The root of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ is the root $*$ of $G_{1}$. For all edges $e$ of $G_{1}$, set $h(e)=0$.
2. On each vertex at odd distance to $*$, add a copy of $\tilde{G}_{\mathcal{Q}}$ (that means that we identify the root of $\tilde{G}_{\mathcal{Q}}$ with the given vertex). For each edge $e$ newly added, set $h(e)=1$.
3. On each vertex added during the previous step which is at even distance to $*$, add a copy of $\tilde{G}_{\mathcal{P}}$. For each edge $e$ belonging to a new copy $G$ of $\tilde{G}_{\mathcal{P}}$, set $h(e)=h\left(e^{\prime}\right)+1$, where $e^{\prime}$ is any edge not belonging to $G$ and having the root of $G$ as endpoint.
4. On each vertex added during the previous step which is at odistance to $*$, add a copy of $\tilde{G}_{\mathcal{Q}}$. For each edge $e$ belonging to a new copy $G$ of $\tilde{G}_{\mathcal{Q}}$, set $h(e)=h\left(e^{\prime}\right)+1$, where $e^{\prime}$ is any edge not belonging to $G$ and having the root of $G$ as endpoint.
5. Return to step 3.

By a copy $G$ of a graph $X$ in $G_{\mathcal{P}} \star G_{\mathcal{Q}}$, we mean the set of edges and vertices of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ which were added by pasting one graph of type $X$ at one particular vertex: $X$ is called the initial graph of $G$. The height of a copy of graph is the height of any edge belonging to this copy of graph.
Note that each copy of graph is connected to exactly one copy of graph of lower height. Let $\mathcal{G}$ denotes the graph whose set of vertices is the set of copies of graph $\{G\}$, and such that there is an edge between $G$ and $G^{\prime}$ if and only both share a vertex. By the previous remark $\mathcal{G}$ is tree; let the only copy of $G_{\mathcal{P}}$ be the root of this tree. For $G$ a vertex of $\mathcal{G}$, the subtree $T_{G}$ of $\mathcal{G}$ rooted at $G$ is the subtree of $\mathcal{G}$ consisting of vertices whose unique path to the root goes through $G$.
There exists a map $\varphi$ from the set of edges of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ to the set of vertices of $\mathcal{G}$ such that $\varphi(e)$ is the copy of graph $G$ such that $e \in G$.
Landau proved in [54] the following result:
Proposition 7.32 (Landau, [54]). $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ is the principal graph of $\mathcal{P} * \mathcal{Q}$.

We will describe combinatorially the set of loops $\gamma$ of length $2 k$ on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ which start at the root. Such a loop is called a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$.
Since $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ is bipartite, the set of rooted loops of length $2 k$ is exactly the set of words $\gamma=\gamma_{1} \ldots \gamma_{2 k}$ in the edges of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ of length $2 k$, such that $\gamma_{1}$ and $\gamma_{2 k}$ has the root vertex as endpoint, and such that for all $1 \leq i \leq 2 k-1, \gamma_{i}$ and $\gamma_{i+1}$ share a common vertex; in particular $\left|h\left(\gamma_{i+1}\right)-h\left(\gamma_{i}\right)\right| \leq 1$. Moreover each walk $\gamma$ yields a lazy walk $\varphi(\gamma)$ of length $2 k-1$ on $\mathcal{G}$ defined by $\varphi(\gamma)(i)=\varphi\left(\gamma_{i}\right)$ (a lazy walk is a walk that can be stationary).

Lemma 7.33. Let $\gamma$ be a rooted loop of length $2 k$ on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and let $1 \leq i \leq j \leq 2 k$. If $h\left(\gamma_{i}\right)=h\left(\gamma_{j}\right)$ and for all $i \leq k \leq j, h\left(\gamma_{k}\right) \geq h\left(\gamma_{i}\right)$, then $i$ and $j$ are in the same copy of graph.

Proof. Let $G$ be the copy of graph containing $\gamma_{i}$. Since for all $i \leq k \leq j, h\left(\gamma_{k}\right) \geq h\left(\gamma_{i}\right)$, the lazy walk $\varphi(\gamma)$ restricted to $\llbracket i, j \rrbracket$ is a lazy walk on $T_{G}$. Since the only vertex of height $h\left(\gamma_{i}\right)$ of this subtree is $G$ itself and $h\left(\gamma_{j}\right)=h\left(\gamma_{i}\right), \gamma_{j}$ belongs to $G$.

Each rooted loop $\gamma$ defines a symmetric relation $\sim_{\gamma}$ on $2 k$ as follows: for $i \leq j, i \sim_{\gamma} j$ if and only if $\gamma_{i}$ and $\gamma_{j}$ belong to the same copy of graph, and for all $i \leq k \leq j, h\left(\gamma_{k}\right) \geq h\left(\gamma_{j}\right)$.

Lemma 7.34. $\sim_{\gamma}$ is an equivalence relation and the associated partition $\pi_{\gamma}$ is a non-crossing partition of $2 k$.

Proof. The reflexivity is assumed by definition and $i \sim_{\gamma} i$ for all $1 \leq i \leq 2 k$. Let $1 \leq i, j, k \leq 2 k$ be such that $i \sim_{\gamma} j$ and $j \sim_{\gamma} k . \gamma_{i}$ and $\gamma_{j}$ are in the same copy of graph and $\gamma_{j}$ and $\gamma_{k}$ are in the same copy of graph, thus $\gamma_{i}$ and $\gamma_{k}$ are in the same copy of graph and $h\left(\gamma_{i}\right)=h\left(\gamma_{j}\right)=h\left(\gamma_{k}\right)$. $h\left(\gamma_{s}\right) \geq h\left(\gamma_{j}\right)$ for $s \in \llbracket i, j \rrbracket$ and $h\left(\gamma_{s}\right) \geq h\left(\gamma_{j}\right)$ for $s \in \llbracket j, k \rrbracket$, thus $h\left(\gamma_{s}\right) \geq h\left(\gamma_{i}\right)$ for $s \in \llbracket i, k \rrbracket$. Therefore, the relation $\sim_{\gamma}$ is transitive and thus an equivalence relation. Let $\pi_{\gamma}$ denote the associated partition of $2 k$.
Let $1 \leq i<j<k<l \leq 2 k$ be such that $i \sim_{\gamma} k$ and $j \sim_{\gamma} l$. Since $i \sim_{\gamma} k$ and $i \leq j \leq k$, $h\left(\gamma_{j}\right) \geq h\left(\gamma_{k}\right)$. Since $j \sim_{\gamma} l$ and $j \leq k \leq l, h\left(\gamma_{k}\right) \geq h\left(\gamma_{l}\right)$. Therefore $h\left(\gamma_{k}\right)=h\left(\gamma_{j}\right)$ and for all $j \leq r \leq k, h\left(\gamma_{r}\right) \geq h\left(\gamma_{k}\right)$. By Lemma 7.33, $j$ and $k$ are in the same copy of graph. $\pi_{\gamma}$ is thus non-crossing.

We define the value $h(B)$ of a block of $\pi_{\gamma}$ as the height of the corresponding copy of graph. Since two neighbouring blocks correspond to adjacent copies of graph, they must have consecutive value (and thus opposite parity).
Let $\pi$ be a non-crossing partition. An interval of a block $B$ of $\pi$ is a maximal interval subset $\llbracket i, j \rrbracket \subseteq B$. In particular $i-1 \notin B$ and $j+1 \notin B$. The set of intervals of a block $B$ is ordered by the lexicographical order; if $B$ has a unique interval, this interval is called a block interval. Otherwise, the first interval is called the initial interval of $B$ and the last interval is called the final interval of $B$. The other ones are called intermediate intervals.

Lemma 7.35. Let $\gamma$ be a rooted loop of length $2 k$ and let $B=\left\{i_{1}<\cdots<i_{r}\right\}$ be a block of $\pi_{\gamma}$. Then $i_{1}=1$ if and only if $i_{r}=2 k$, and if $i_{1} \neq 1$ then $h\left(i_{1}-1\right)=h\left(i_{2}+1\right)=h\left(i_{1}\right)-1$ and $i_{1}-1 \sim_{\gamma} i_{2}+1$.

Proof. $h\left(\gamma_{1}\right)=h\left(\gamma_{2 k}\right)=0$ and for all $1 \leq i \leq 2 k, h\left(\gamma_{i}\right) \geq 0$, thus 1 and $2 k$ are in the same block.
Let $i_{0}=\sup \left\{i<i_{1}, h(i)=h\left(i_{1}\right)\right\} . i_{0} \neq i_{1}-1$ because $i_{0} \not \chi_{\gamma} i_{1}$. On $\llbracket i_{0}+1, i_{1}-1 \rrbracket$, either $h>h\left(i_{1}\right)$ or $h<h\left(i_{1}\right)$. Since $i_{0} \not \chi_{\gamma} i_{1}, h<h\left(i_{1}\right)$ on $\llbracket i_{0}+1, i_{1}-1 \rrbracket$ and thus $h\left(i_{1}-1\right)=h\left(i_{1}\right)-1$. Likewise, $h\left(i_{2}+1\right)=h\left(i_{2}\right)-1=h\left(i_{1}\right)-1$.
Moreover on $\llbracket i_{1}-1, i_{2}+1 \rrbracket, h \geq h\left(i_{1}-1\right)$, thus by Lemma $7.33 i_{1}-1$ and $i_{2}+1$ are in the same copy of graph and $i_{1}-1 \sim_{\gamma} i_{2}+1$.

A partition $\pi$ is called irreducible if $1 \sim_{\pi} 2 k$ (see [2]). A subpartition of $\pi$ is a subset $\mathcal{A} \subseteq \pi$ of blocks of $\pi$ such that $\bigcup_{B \in \mathcal{A}} B$ is an interval.

Lemma 7.36. Let $\gamma$ be a rooted loop of length $2 k$. Then $\pi_{\gamma}$ and every subpartitions of $\pi_{\gamma}$ are irreducible. The first and last elements of a block having even value are respectively odd and even. The first and last elements of a block having odd value are respectively even and odd.

Proof. By Lemma 7.35, 1 and $2 k$ are in the same block, thus $\pi$ is irreducible.
Let $\tilde{\pi}=\left\{B_{i}\right\}_{i \in I_{0}}$ be a strict subpartition of $\pi$ and set $\llbracket i_{1}, i_{2} \rrbracket=\bigcup_{i \in I_{0}} B_{i}$. Since $\tilde{\pi} \neq \pi, i_{1} \neq 1$ and $i_{2} \neq 2 k$. Let $x=\inf \left\{i_{1}-1<i, h(i)=h\left(i_{1}-1\right)\right\}$. On $\llbracket i_{1}-1, x \rrbracket$, either $h>h\left(i_{1}-1\right)$ or $h<h\left(i_{1}-1\right)$. By Lemma 7.35, $h\left(i_{1}\right)>h\left(i_{1}-1\right)$, thus $h>h\left(i_{1}-1\right)$ on $\llbracket i_{1}-1, x \rrbracket$ and $i_{1}-1 \sim x$. Since $x \notin \llbracket i_{1}, i_{2} \rrbracket, h\left(i_{2}\right)>h\left(i_{1}-1\right)$ and thus $h\left(i_{2}\right) \geq h\left(i_{1}\right)$. By symmetry, $h\left(i_{1}\right) \geq h\left(i_{2}\right)$ and finally $h\left(i_{1}\right)=h\left(i_{2}\right)$. Since $h>h\left(i_{1}\right)-1$ on $\llbracket i_{1}, i_{2} \rrbracket, h \geq h\left(i_{1}\right)$ on $\llbracket i_{1}, i_{2} \rrbracket$ : Lemma 7.33 yields that $i_{1} \sim i_{2}$.
1 is the first element of a block with even value and is odd. Let $i>1$ be the first element of a block $B$ with even value. This means that $h\left(\gamma_{i}\right)$ is odd, and thus $\gamma_{i}$ belong to a copy $G$ of $\tilde{G}_{\mathcal{P}}$. Since $i$ is the first element of $B, h(i-1)=h(i)-1$, and thus $\gamma_{i-1}$ belong to the only copy of $\tilde{G}_{\mathcal{Q}}$ with lower height than $G$ : thus the vertex between $\gamma_{i-1}$ and $\gamma_{i}$ is the root vertex $v$ of $G$. Since $G$ is a copy of $\tilde{G}_{\mathcal{P}}, v$ is at even distance to the root. $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ being bipartite, $i-1$ is also even and thus $i$ is odd. The same proof holds for the three other cases.

Definition 7.37. A non-crossing partition $\pi$ is of type $\star$ if $\pi$ is irreducible, all its subpartitions are irreducible, all its blocks have an even number of elements, and the first elements of two neighbouring blocks have opposite parities.
The value $h(B)$ of a block $B$ in $\pi$ is 0 if $B$ contains 1,1 if the first element of $B$ is even and 2 if the first element of $B$ is odd and distinct from 1.

Let $P_{\star}(k)$ the set of partitions of type $\star$.
Lemma 7.38. $\left|P_{\star}(2 k)\right|=N C(k)$.
Proof. Let $P_{k}=\left|P_{\star}(2 k)\right|$. It suffices to show that $P_{k}=\sum_{i=1}^{k} P_{i-1} P_{k-i}$ with $P_{0}=P_{1}=1$. We set a partition of $P_{\star}=\amalg P_{x}$ depending on the position of the second element $x$ of the block containing 1. Since the blocks are even, $x \in\{2 i \mid 1 \leq i \leq k\}$. Let $x_{0}=2 i_{0}$ and $\pi \in P_{x_{0}}$. If $x_{0} \geq 4$, $\pi_{\llbracket 2, x_{0}-1 \rrbracket}$ is again completely irreducible with blocks of even numbers, and the first elements of two neighbouring blocks of $\pi_{\llbracket 2, x_{0}-1 \rrbracket}$ have opposite parities: thus $\pi_{\llbracket 2, x_{0}-1 \rrbracket} \in P_{\star}\left(i_{0}-1\right)$. On the other a hand since the first point of the first block after $x_{0}$ is even, $x_{0}+1$ belongs again to the block containing 1 (and thus $2 k$ ): thus likewise if $x_{0} \leq 2 k-2, \pi_{\llbracket x_{0}+1,2 k \rrbracket} \in P_{\star}\left(k-i_{0}\right)$. Therefore, there exists a map $\varphi: P_{x} \longrightarrow P_{\star}(k-x / 2) \times P_{\star}(x / 2-1)$; this map is clearly bijective and thus $\left|P_{x}\right|=\left|P_{\star}(k-x / 2)\right| \times\left|P_{\star}(x / 2-1)\right|$. Summing on $2 \leq 2 x \leq 2 k$ yields the result.

Let $\gamma$ be a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$. For $B=\left\{i_{1}<\cdots<i_{r}\right\}$ a block of $\pi_{\gamma}$, we denote $\gamma_{B}$ the word $\gamma_{i_{1}} \gamma_{i_{2}} \ldots \gamma_{i_{r}}$.

Lemma 7.39. For $\gamma$ a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and $B$ a block of $\pi_{\gamma}, \gamma_{B}$ is a rooted loop on $G_{B}$, where $G_{B}$ is the copy of graph containing $\left\{\gamma_{i}\right\}_{i \in B}$.

Note that $G_{B}$ is well-defined, since all edges of $B$ are in the same copy of graph.
Proof. We have to show that $\gamma_{B}$ defines indeed a walk, and that the first and last vertices of $\gamma_{B}$ are the root vertex of $G_{B}$. Let $B=\llbracket r_{1}, s_{1} \rrbracket \cup \llbracket r_{2}, s_{2} \rrbracket \cup \cdots \cup \llbracket r_{t}, s_{t} \rrbracket$ with $r_{i} \leq s_{i}, s_{i}<r_{i+1}-1$. For $r_{i} \leq j<s_{i}, \gamma_{j}$ and $\gamma_{j+1}$ are consecutive edges of $\gamma_{B}$ that are also consecutive edges of $\gamma$ : therefore, they share a vertex. Let $j=s_{i}$ with $i<t$. Since any subpartition of $\pi_{\gamma}$ is
also irreducible, $x=s_{i}+1$ and $y=r_{i+1}-1$ are in a same block $B^{\prime}$ of $\pi$. By Lemma 7.35, $h\left(B^{\prime}\right)=h(B)+1$, and thus the vertex between $\gamma_{s_{i}}$ and $\gamma_{x}$ is the root vertex between $G_{B^{\prime}}$ and $G_{B}$. Likewise, the vertex between $\gamma_{y}$ and $\gamma_{r_{i+1}}$ is the root vertex between $G_{B^{\prime}}$ and $G_{B}$. Therefore, the final vertex of $\gamma_{s_{i}}$ and the first vertex of $\gamma_{r_{i+1}}$ is the same. $\gamma_{B}$ is thus a walk on $G_{B}$.
Since $h\left(r_{1}-1\right)=h\left(r_{1}\right)-1$, the root vertex between $\gamma_{r_{1}-1}$ and $\gamma_{r_{1}}$ is the root vertex of $G_{B}$. For the same reason, the root vertex between $\gamma_{s_{t}}$ and $\gamma_{s_{t}+1}$ is the root vertex of $G_{B}$. Thus, $\gamma_{B}$ is a rooted loop on $G_{B}$.

This implies in particular that the blocks of $\pi(\gamma)$ are even (this could have been proven directly from Lemma 7.36). The latter result yields also a combinatorial description of the rooted loops on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$.
Let $\psi_{G}$ be the canonical bijection sending an edge of the copy of graph $G$ to the same edge in the initial graph (which is either $G_{\mathcal{P}}, \tilde{G}_{\mathcal{P}}$ or $\tilde{G}_{\mathcal{Q}}$, depending on the value of $B$ ). $\psi_{G}^{-1}$ maps an edge of the initial graph of $G$ to the same edge in $G$. For each rooted loop $\gamma$ and block $B$ of $\pi_{\gamma}$, let $\tilde{\gamma}_{B}$ denote the image of $\gamma_{B}$ by $\psi_{G_{B}}$.

Proposition 7.40. There is a bijection between rooted loops of length $2 k$ on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and pairs $\left(\pi,\left\{\zeta_{B}\right\}_{B \in \pi}\right)$, where

- $\pi \in P_{\star}(2 k)$.
- $\zeta_{B}$ is a reduced loop of length $|B|$ on $\tilde{G}_{\mathcal{P}}\left(\right.$ resp. $\tilde{G}_{\mathcal{Q}}$, resp. $\left.G_{\mathcal{P}}\right)$ if $h(B)=2($ resp. $h(B)=1$, resp. $h(B)=0$ ).

Proof. Let $A$ be the set of rooted loops of length $2 k$ on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and let $B$ be the set of pairs $\left(\pi,\left\{\zeta_{B}\right\}_{B \in \pi}\right)$ as in the statement of the proposition.
Let $\Phi: A \longrightarrow B$ be the map defined by $\Phi(\gamma)=\left(\pi_{\gamma},\left\{\tilde{\gamma}_{B}\right\}_{B \in \pi}\right)$. By Lemmas 7.36 and 7.39 , this map is well defined.
Let $\gamma \neq \gamma^{\prime}$. If $\pi_{\gamma} \neq \pi_{\gamma^{\prime}}$ then $\Phi(\gamma) \neq \Phi\left(\gamma^{\prime}\right)$. Suppose that $\pi_{\gamma}=\pi_{\gamma^{\prime}}$ and let $1 \leq i \leq 2 k$ be the first element such that $\gamma_{i} \neq \gamma_{i}^{\prime}$. Let $B$ be the block containing $i$. For all $t<i, \gamma_{t}=\gamma_{t}^{\prime}$, thus $G_{B}(\gamma)=G_{B}\left(\gamma^{\prime}\right)$. Thus $\gamma_{B}$ and $\gamma_{B}^{\prime}$ are both walks on $G_{B}$. Since $\gamma_{i} \neq \gamma_{i}^{\prime}$, these walks are distinct and thus $\tilde{\gamma}_{B} \neq \tilde{\gamma}_{B}^{\prime}$ : therefore, $\Phi(\gamma) \neq \Phi\left(\gamma^{\prime}\right)$.
Let ( $\pi,\left\{\zeta_{B}\right\}_{B \in \pi}$ ) be a pair as in the statement of the proposition. We define recursively on each block a copy of graph $G_{B}$ and a rooted loop $\gamma$ as follows:

1. Let $B_{0}=\left\{i_{1}=1<\cdots<i_{r}=2 k\right\}$ be the block containing 1. We set $G_{B_{0}}=G_{\mathcal{P}}$. For $i_{s} \in B_{0}$, we set $\gamma_{i_{s}}=\psi_{G_{B_{0}}}\left(\zeta_{B_{0}}(s)\right)$. Since $\zeta_{B_{0}}$ is a reduced loop on $G_{\mathcal{P}}, \gamma_{B_{0}}$ is a reduced loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ (which is contained in the only copy of $G_{\mathcal{P}}$ ). Set $t=0$.
2. Let $B=\left\{i_{1}<\cdots<i_{r}\right\}$ be the first block neighbouring $B_{t}$ : note that $\llbracket 1, i_{1}-1 \rrbracket \subseteq B_{t}$. Suppose that $h(B)=1$. Then $i_{1}$ is even and thus the length of $\gamma_{B_{t} \cap \llbracket 1, i_{1}-1 \rrbracket}$ is odd. Therefore, the walk $\gamma_{B_{t} \cap \llbracket 1, i_{1}-1 \rrbracket}$ ends at a vertex $v$ which is at odd distance from the root, and thus there exists a copy $G$ of $\tilde{G}_{\mathcal{Q}}$ whose root is exactly $v$. This implies that the height of $h(G)$ is $h\left(i_{1}-1\right)+1$. Let $G_{B}=G$. For $i_{s} \in B$, set $\gamma_{i_{s}}=\psi_{G_{B}}\left(\zeta_{B}(s)\right), h(i)=h(G)$. Since by irreducibility $i_{1}-1$ and $i_{r}+1$ belongs to a same block of $\pi$ (which is thus included in $B_{t}$ ) and since $\gamma_{B_{t}}$ is a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$, the final vertex of $\gamma_{i_{1}-1}$ and the first vertex of $i_{r}+1$ is the same vertex $v$ which is the root of $G_{B} . \psi_{G_{B}}\left(\zeta_{B}\right)$ is a rooted loop on $G_{B}$, thus the final vertex of $\gamma_{i_{r}}$ coincides also with the root $v$. Therefore $\gamma_{B_{t} \cup B}$ is again a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$. Let $B_{t+1}:=B_{t} \cup B$.
Do the same construction (with a copy of $\tilde{G}_{\mathcal{P}}$ ) if $h(B)=2$.
3. Increase $t$ by one and return to Step 2 , until $t$ is the number of blocks of $\pi$.

By construction $\gamma$ is a rooted loop on $G_{\mathcal{P} \star} G_{\mathcal{Q}}$. Let $B=\left\{i_{1}<\cdots<i_{r}\right\}$ and $B^{\prime}=\left\{j_{1}<\cdots<j_{r^{\prime}}\right\}$ be two blocks of $\pi$ such that $i_{1}<j_{1}<j_{r^{\prime}}<i_{r}$ (written $B^{\prime} \leq B$ ): in this case the construction yields that $h\left(G_{B^{\prime}}\right)>h\left(G_{B}\right)$.
If $i \sim_{\pi} j$, then $i$ and $j$ are in the same copy of graph $G_{B}$ and for all $i \leq k \leq j h\left(\gamma_{k}\right) \geq h\left(\gamma_{i}\right)$ : this implies $i \sim_{\gamma} j$ and $\pi \leq \pi_{\gamma}$. Suppose that $1 \leq i<j \leq 2 k$ are such that $i \not \chi_{\pi} j$, and let $B_{i}=\left\{i_{1}<\cdots<i_{r}\right\}$ (resp. $B_{j}=\left\{j_{1}<\cdots<j_{r}\right\}$ ) be the block of $i$ (resp. $j$ ) in $\pi$. If $\gamma_{i}$ is in a different copy of graph than $\gamma_{j}, i \not \chi_{\gamma} j$. Let us assume that they are in the same copy of graph: since $h\left(G_{B_{i}}\right)=h\left(G_{B_{j}}\right), i_{r}<j_{1}$. Since $\pi$ is completely irreducible and $i_{1} \not \chi_{\pi} j_{r^{\prime}}, \pi_{\llbracket i_{1}, j_{r^{\prime}} \rrbracket}$ is not a subpartition of $\pi$ and thus there exist $i_{r}<k<j_{1}$ and $l \in \llbracket 1,2 k \rrbracket \backslash \llbracket i_{1}, j_{r^{\prime}} \rrbracket$ such that $k \sim_{\pi} l$. Assume without loss of generality that $l \leq i_{1}$ and let $B$ be the block of $l$ and $k$. Since $B_{i} \leq B$, $h\left(G_{B}\right)>h\left(G_{B_{i}}\right)$. Thus $h\left(\gamma_{l}\right)<h\left(\gamma_{i}\right)$ and $i \not \chi_{\pi_{\gamma}} j$. Therefore, $\pi=\pi_{\gamma}$. Since $\pi_{\gamma}=\pi$ and by construction $\gamma_{B}=\zeta_{B}$ for each block $B$ of $\pi, \Phi(\gamma)=\left(\pi,\left\{\zeta_{B}\right\}_{B \in \pi}\right)$. Therefore, $\Phi$ is surjective and bijective.

## Free product formula

The proof of Theorem 7.31 relies on the combinatorics of the free multiplicative convolution. We will first construct a bijection between $P_{\star}$ and $N C(k)$ that respect the cardinal of the blocks.
Let $p \in P_{\star}(2 k)$ and let $l(p)$ denote the number of blocks of $p$. Two blocks $B$ and $B^{\prime}$ are said neighbors if there exists $i \in B$ such that $i+1$ or $i-1$ belongs to $B^{\prime}$.

Lemma 7.41. Let $p \in P_{\star}(2 k)$. If $B$ and $B^{\prime}$ are two neighbouring blocks, then either $B \leq B^{\prime}$ or $B^{\prime} \leq B$.

Proof. Suppose that $B \not \leq B^{\prime}$ and $B^{\prime} \not \leq B$. Let $i_{B}$ and $i_{B^{\prime}}$ (resp. $f_{B}$ and $f_{B^{\prime}}$ ) be the first (resp. last) elements of $B$ and $B^{\prime}$. Since $B \not \leq B^{\prime}, B^{\prime} \not \leq B$ and $p$ is non crossing, either $f_{B}<i_{B^{\prime}}$ or $f_{B^{\prime}}<i_{B}$. Suppose without loss of generality that $f_{B}<i_{B^{\prime}}$. Then $f_{B} \neq i_{B^{\prime}}-1$, since otherwise $p_{\llbracket i_{B}, f_{B^{\prime}} \rrbracket}$ would be a subpartition of $p$ with $i_{B} \nsim f_{B^{\prime}}$. Therefore $B$ and $B^{\prime}$ are not neighbors.

Let $p \in P_{\star}$. For $B$ a block of $p$ and $i \in B$, we set $h(i)=0$ if $h(B) \equiv 0[2]$ and $h(i)=1$ if $h(B) \equiv 1[2]$. Let $f_{p}: \llbracket 1,2 k \rrbracket \longrightarrow \llbracket 1,4 k \rrbracket$ be the function defined by $f_{p}(2 i)=4 i-h(i)$ and $f_{p}(2 i-1)=4 i-3+h(i)$. Since $f_{p}(2 i) \in\{4 i-1,4 i\}$ and $f_{p}(2 i-1) \in\{4 i-3,4 i-2\}, f_{p}$ is strictly increasing and thus $(f(p), f(\llbracket 1,2 k \rrbracket))$ defines a partial partiton of $4 k$. Complete this partition by saying that the elements of $\llbracket 1,4 k \rrbracket \backslash f(\llbracket 1,2 k \rrbracket)$ are singletons: this yields a non-crossing partition of $4 k$ denoted by $\tilde{p}$.
We define also an involutive map . on $\llbracket 1,4 k \rrbracket$ by saying that $\overline{4 i+1}=4 i, \overline{4 i+2}=4 i+3, \overline{4 i+3}=$ $4 i+2$ and $\overline{4 i}=4 i+1$ for $0 \leq i \leq k-1$.

Lemma 7.42. Let $1 \leq i \leq k$ and suppose that $4 i-2 \not \chi_{\tilde{p}} 4 i-1$. Then at least one of them is a singleton. The same holds for $4 i$ and $4 i+1$.

Proof. Let $1 \leq i \leq k$. Suppose that $4 i-2$ and $4 i-1$ are not singletons. Thus $4 i-2$ and $4 i-1$ are in $f_{p}(\llbracket 1,2 k \rrbracket)$. This means that $2 i-1$ and $2 i$ are both in a block of $p$ of odd value. Since two neighbouring blocks have values of opposite parities, $2 i-1$ and $2 i$ are in the same block, and $4 i-2 \sim_{\tilde{p}} 4 i-1$. The same holds for $4 i$ and $4 i+1$ and $1 \leq i \leq k-1$. Since $1 \sim_{\tilde{p}} 4 k$, the implication holds also for $i=k$.

Lemma 7.43. $4 i-2$ (resp. $4 i+1$ ) is a singleton of $\tilde{p}$ and $4 i-1$ (resp. $4 i$ ) is not a singleton of $\tilde{p}$ if and only if $2 i$ (resp. $2 i+1$ ) is the first element of a block of $p$ with value 1 (resp. 2).
$4 i-1$ (resp. $4 i$ ) is a singleton of $\tilde{p}$ and $4 i-2$ (resp. $4 i+1$ ) is not a singleton of $\tilde{p}$ if and only if $2 i-1$ (resp. 2i) is the last element of a block of $p$ with value 1 (resp. 2).

Proof. $p$ has no singleton, thus $f(p)$ has no singleton. Therefore, $4 i-2$ is a singleton of $\tilde{p}$ and $4 i-1$ is not if and only if $f(2 i)=4 i-1$ and $f(2 i-1)=4 i-3$. By construction of $f$, the latter means that $h(2 i) \equiv 1, h(2 i-1) \equiv 0$, and the elements $2 i-1$ and $2 i$ belong to distinct neighbouring blocks $B_{2 i-1}$ and $B_{2 i}$ of $p$. The two blocks are neighbors and $p$ is non-crossing, thus $B_{2 i-1}$ is distinct from $B_{2 i}$ if and only if either $2 i-1$ is the last element of $B_{2 i-1}$ or $2 i$ is the first of $B_{2 i}$. Since $p \in P_{\star}(2 k)$ the last element of a block of even value is even. Therefore, $2 i-1$ is never the last element of $B_{2 i-1}$, and thus $B_{2 i}$ and $B_{2 i-1}$ are distinct blocks if and only if $2 i$ is the first element of $B_{2 i}$.
The proof is the same in the three other cases (note that $4 k$ and 1 are never singleton).
By the definition of $f$, elements of a block of $p$ of even value are mapped to $S$ and elements of a block of $p$ of odd value are mapped to $S^{c}$. Thus $\tilde{p}$ has the form $\left((\pi(p), S) \vee\left(\pi^{\prime}(p), S^{c}\right)\right)$ for some partitions $\pi(p), \pi^{\prime}(p)$ of $2 k$. We define $\Theta(p)$ as the partition $\left(\left(\pi(p) \vee \pi_{0}, S\right) \vee\left(\pi^{\prime}(p) \vee \pi_{1}, S^{c}\right)\right)$. By construction $\Theta(p)$ is a non-crossing partition of $4 k$ of the form $\left((\pi, S) \vee\left(\pi^{\prime}, S^{c}\right)\right)$ for some partition $\pi \geq \pi_{0}, \pi^{\prime} \geq \pi_{1}$. Let $B$ be a block of $p$. There exists a unique block $\vartheta_{p}(B)$ of $\Theta(p)$ such that $f(B) \subseteq \vartheta_{p}(B)$.

Lemma 7.44. $\vartheta_{p}$ is an injective map. Moreover $\left|\vartheta_{p}(B)\right|=|B|+2$ if $1 \neq B$, and $\left|\vartheta_{p}(B)\right|=|B|$ if $1 \in B$. For all block $B$ not being in the image of $\vartheta_{p},|B|=2$.

Proof. Let $\tilde{B}$ be a block of $\tilde{p}$ which is not a singleton. And let us assume that $\tilde{B}=f(B)$ for a block $B$ of $p$. By Lemma 7.42, if $x \in \tilde{B}$ then either $\bar{x}$ is a singleton or $\bar{x} \in \tilde{B}$. Therefore, each block of $\Theta(p)$ contains at most one block of $\tilde{p}$ which is not a singleton, and $\vartheta(p)$ is injective.
By Lemma 7.43, $x \in \tilde{B}$ and $\bar{x}$ is a singleton if and only if $x$ is the first or last point of $\tilde{B}$ and $\tilde{B}$ doesn't contain 1. Therefore $\vartheta_{p}(B)=\tilde{B}$ if $1 \in B$ and $\vartheta_{p}(B)$ is the union of $\tilde{B}$ and two singletons if $1 \notin B$ : this yields $\left|\vartheta_{p}(B)\right|=|B|+2$ if $1 \notin B$, and $\left|\vartheta_{p}(B)\right|=|B|$ if $1 \in B$.
Let $B$ be a block not being in the image of $\vartheta_{p}$. Then $B$ is a union of singletons of $\tilde{p}$. Since the blocks of $\left(\pi_{0}, S\right)$ and $\left(\pi_{1}, S^{c}\right)$ have size $2,|B|=2$.

Lemma 7.45. Let $p \in P_{\star}$. Then $\Theta(p)$ is of the form $\left((\pi, S) \vee\left(k r(\pi), S^{c}\right)\right)$ for some partition $\pi \geq \pi_{0}$ in $N C(2 k)$.

Proof. By construction, $\Theta(p)$ is of the form $\left((\pi, S) \vee\left(\pi^{\prime}, S^{c}\right)\right)$ for some partitions $\pi \geq \pi_{0}, \pi^{\prime} \geq \pi_{1}$ in $N C(2 k)$. Thus $\pi^{\prime} \leq k r^{\prime}(\pi)$. Since $\pi \geq \pi_{0}$, Lemma 7.27 yields $k r^{\prime}(\pi) / 2=K r(\pi / 2)$. The Kreweras complement $\operatorname{Kr}(\pi)$ of a partition $\pi$ of $k$ satisfies the relation $l(K r(\pi))+l(\pi)=k+1$ (see [66], Ch. 9). Therefore, $l\left(k r^{\prime}(\pi) / 2\right)+l(\pi / 2)=k+1$. Since $l\left(k r^{\prime}(\pi / 2)\right)=l\left(k r^{\prime}(\pi)\right)$ and $l(\pi / 2)=l(\pi), l(\pi)+l\left(k r^{\prime}(\pi)\right)=k+1$.
On the other hand, $\left|B_{1}\right|+\sum_{\substack{B \in p \\ 1 \notin B}}|B|=2 k$, thus by Lemma $7.44 \sum_{B \in p}\left|\vartheta_{p}(B)\right|=2 k+2(l(p)-1)$. Since all other blocks of $\Theta(p)$ have cardinal 2,

$$
l(\Theta(p))=l(p)+\frac{4 k-(2 k+2(l(p)-1))}{2}=k+1
$$

Therefore, $l(\pi)+l\left(\pi^{\prime}\right)=k+1=l(\pi)+l\left(k r^{\prime}(\pi)\right)$ and $l\left(\pi^{\prime}\right)=l\left(k r^{\prime}(\pi)\right)$. Thus, $\pi^{\prime}=k r^{\prime}(\pi)$.

Recall that a partition is formally defined as a set of subsets of $\{1, \ldots, k\}$. The former results give the desired bijection:

Proposition 7.46. There exists a bijection $\Lambda: P_{\star} \rightarrow N C(k)$, and for each $p \in P_{\star}$ an injective map $\lambda_{p}: \longrightarrow \Lambda(p) \cup K r(\Lambda(p))$ such that $\left|\lambda_{p}(B)\right|=|B| / 2+1$ if $1 \notin B$ and $\left|\lambda_{p}(B)\right|=|B| / 2$ if $1 \in B$; moreover if $B$ is not in the image of $\lambda_{p}$, then $B$ is a singleton.

Proof. Let $p$ and $p^{\prime}$ be two distinct partitions of $P_{\star}(2 k)$. Let $1 \leq i, j \leq 2 k$ such that $i$ and $j$ are in the same block $B$ of $p$ but in distinct blocks $B_{i}$ and $B_{j}$ of $p^{\prime}$. Therefore, $f_{p}(i)$ and $f_{p}(j)$ are in the same block of $\Theta(p)$. But $f_{p^{\prime}}(i)$ and $f_{p^{\prime}}(j)$ are not in the same block of $\Theta\left(p^{\prime}\right)$ : this would contradict the injectivity of $\vartheta_{p^{\prime}}$. Thus $\Theta(p) \neq \Theta\left(p^{\prime}\right)$. $\Theta$ is injective and by Lemma 7.38, $\left|P_{\star}(2 k)\right|=|N C(k)|=\left|\left\{\pi \in N C(2 k), \pi \geq \pi_{0}\right\}\right|$, thus $\Theta$ is bijective.
If $\Theta(p)=\left((\pi, S),\left(k r^{\prime}(\pi), S^{c}\right)\right)$, set $\Lambda(p)=\pi / 2$. For each $B \in p$, set $\lambda_{p}(B)=\vartheta_{p}(B) / 2$, where $B / 2$ is the image of $B \in \pi$ through the map $\pi \longrightarrow \pi / 2 . \Lambda$ and $\lambda_{p}$ have all the desired properties.

Denote by $M_{k}(G)$ the number of rooted loops of length $2 k$ on a rooted graph $G$. If $p$ is a partition, $B_{1}(p)$ denotes the block of $p$ containing 1. Proposition 7.46 yields the following formula for the dimension of $\mathcal{P} * \mathcal{Q}$.

Lemma 7.47. For all $k \geq 1$.

$$
\begin{equation*}
\mathcal{P} * \mathcal{Q}_{k}=\sum_{p \in N C(k)} M_{\left|B_{1}\right|}\left(G_{\mathcal{P}}\right) \prod_{B \in p, 1 \notin B} M_{|B|-1}\left(\tilde{G}_{\mathcal{P}}\right) \prod_{B \in K r(p)} M_{|B|-1}\left(\tilde{G}_{\mathcal{Q}}\right) . \tag{7.3.1}
\end{equation*}
$$

Proof. By Proposition 7.40 ,

$$
\mathcal{P} * \mathcal{Q}_{k}=\sum_{p \in P_{\star}(k)} M_{\left|B_{1}\right| / 2}\left(G_{\mathcal{P}}\right) \prod_{B \in p, h(B)=2} M_{|B| / 2}\left(\tilde{G}_{\mathcal{P}}\right) \prod_{B \in p h(B)=1} M_{|B| / 2}\left(\tilde{G}_{\mathcal{Q}}\right) .
$$

Applying the map $\lambda_{p}$ on each block of $p$ and using the results of Proposition 7.46 yield

$$
\mathcal{P} * \mathcal{Q}_{k}=\sum_{p \in P_{*}(k)} M_{\left|\lambda_{p}\left(B_{1}\right)\right|}\left(G_{\mathcal{P}}\right) \prod_{B \in p, h(B)=2} M_{\left|\lambda_{p}(B)\right|-1}\left(\tilde{G}_{\mathcal{P}}\right) \prod_{B \in p, h(p)=1} M_{\left|\lambda_{p}(B)\right|-1}\left(\tilde{G}_{\mathcal{Q}}\right) .
$$

By Proposition 7.46, any other block $B$ of $\Lambda(p)$ is a singleton, therefore for such a $B, M_{|B|-1}(G)=$ $M_{0}(G)=1$. Thus, the above formula can be completed as

$$
\mathcal{P} * \mathcal{Q}_{k}=\sum_{p \in P_{\star}(k)} M_{\left|\lambda_{p}\left(B_{1}\right)\right|}\left(G_{\mathcal{P}}\right) \prod_{B \in \Lambda(p), 1 \notin p} M_{|B|-1}\left(\tilde{G}_{\mathcal{P}}\right) \prod_{B \in \operatorname{Kr}(p)} M_{|B|-1}\left(\tilde{G}_{\mathcal{Q}}\right) .
$$

Summing on $\Lambda(p)$ instead of $p$ yields the result.
Since the graph $\tilde{G}(\mathcal{P})$ is the graph $G_{\mathcal{P}}$ without the root vertex, the number of rooted loops of length $2 k$ on $\tilde{G}(\mathcal{P})$ is equal to the number of rooted loops of length $2 k+2$ on $G_{\mathcal{P}}$ that never pass through the vertex root (except at the first and last vertex).
If $\gamma$ is a rooted loop of length $2 k$ on a rooted bipartite graph $G$, we define an interval partition $I_{\gamma}$ of $2 k$ by saying that $i \sim_{I_{\gamma}} j$ if and only if the walk between $i$ and $j$ ( $i$ and $j$ excluded) doesn't pass through the root: note that the blocks of $I$ are necessarily even. For $I$ an interval partition of $2 k$ with even blocks, the number of rooted loops $\gamma$ with $I_{\gamma}=I$ is exactly $\prod_{B \in I} \tilde{M}_{|I| / 2}(G)$, where $\tilde{M}_{k}(G)$ is the number of rooted loops of length $2 k$ that only pass through the root vertex at the first and last point. Thus,

$$
M_{k}(G)=\sum_{I \in \mathcal{I}(k)} \prod_{B \in I} \tilde{M}_{|B|}(G),
$$

where $\mathcal{I}(k)$ denotes the set of interval partitions of $k$. Therefore, $\left(\tilde{M}_{k}(G)\right)_{k \geq 1}$ is the sequence of Boolean cumulants associated to $\left(M_{k}(G)\right) \geq k \geq 1$.

Denote by $b_{k}(\mu)$ the $k$-th boolean cumulant of a distribution $\mu$. The formula 7.3.1) is thus

$$
\begin{equation*}
m_{k}(\mu(\mathcal{P} * \mathcal{Q}))=\sum_{p \in N C(k)} m_{\left|B_{1}\right|}\left(\mu_{\mathcal{P}}\right) \prod_{B \in p, 1 \notin B} b_{|B|}\left(\mu_{\mathcal{P}}\right) \prod_{B \in K r(p)} b_{|B|}\left(\mu_{\mathcal{Q}}\right) \tag{7.3.2}
\end{equation*}
$$

Let $I=\left(I_{1}, \ldots, I_{r}\right)$ be an interval partition with $I_{1}=\llbracket 1, i_{1} \rrbracket, \ldots, I_{r}=\llbracket i_{r-1}+1, k \rrbracket$. The descent set $D(I)$ of $I$ is the set $\left\{i_{1}+1, \ldots, i_{r-1}+1\right\}$.

Lemma 7.48. Let $I=\left(I_{1}, \ldots, I_{r}\right)$ be an interval partition of $k$. There is a bijection $\varphi$ from the set $\left\{p \in N C(k), D(I) \subseteq B_{1}(p)\right\}$ to $\prod_{j} N C\left(\left|I_{j}\right|\right)$. such that $\varphi$ preserves the size of the block not containing 1 and the Kreweras complement in the following sense: if $\varphi(p)=\left(\pi_{1}, \ldots, \pi_{r}\right)$, any block of $\pi_{j}$ not containing $i_{j-1}+1$ is a block of $\pi$, and

$$
k r(p)=\left(k r\left(\pi_{1}\right), I_{1}\right) \vee\left(k r\left(\pi_{2}\right), I_{2}\right) \cdots \vee\left(k r\left(\pi_{r}\right), I_{r}\right) .
$$

Proof. Let $\varphi(p)=\left(p_{\mid I_{1}}, \ldots, p_{\mid I_{r}}\right) . \varphi$ preserves the size of the blocks not containing 1 in the sense of the statement of the lemma. Let $\pi_{I}$ be the partition $\left(\mathbf{1}_{|D(I)|+1}, D(I) \cup\{1\}\right) \vee\left(\mathbf{0}_{k-|D(I)|-1}, \llbracket 1, k \rrbracket \backslash\right.$ $(D(I) \cup\{1\}))$ : namely, $\pi_{I}$ is the partition with only singletons, except one block $D(I) \cup\{1\}$. We define $\varphi^{-1}$ by the formula

$$
\varphi^{-1}\left(\pi_{1}, \ldots, \pi_{r}\right)=\left(\left(\pi_{1}, I_{1}\right) \vee\left(\pi_{2}, I_{2}\right) \cdots \vee\left(\pi_{r}, I_{r}\right)\right) \vee \pi_{I}
$$

Note that $\varphi^{-1}\left(\left(\pi_{1}, \ldots, p i_{r}\right)\right)$ is a non-crossing partition such that $D(I) \subseteq B_{1}$. Since each element of $D(I) \cup\{1\}$ is in a different interval of $I, \varphi \circ \varphi^{-1}=I d$.
Let $i$ and $j$ be in the same block $I_{s}$ of $I$. Then $i \sim_{\varphi^{-1} \odot \varphi(p)} j$ if and only if $i \sim_{p_{I_{s}}} j$ (which is equivalent to $i \sim_{p} j$ ). If $i$ and $j$ are in different blocks, then $i \sim_{\varphi^{-1} \circ \varphi(p)} j$ if and only if $i, j \in D(I) \cup\{1\} \subseteq B_{1}$ (which is again equivalent to $i \sim_{p} j$ ).
Since $D(I) \subseteq B_{1}(p), k r(\pi) \leq I$, which gives the result.
The proof of Theorem 7.31 is based on the rewriting of Equation (7.3.2) with Lemma 7.48;
Proof of Theorem 7.31. Applying the Boolean moment-cumulant formula to $m_{\left|B_{1}\right|}\left(\mu_{\mathcal{P}}\right)$ in (7.3.2) yields

$$
m_{k}(\mu(\mathcal{P} * \mathcal{Q}))=\sum_{p \in N C(k)} \sum_{I \in \mathcal{I}\left(\left|B_{1}\right|\right)} b_{I}\left(\mu_{\mathcal{P}}\right) \prod_{B \in p, 1 \notin B} b_{|B|}\left(\mu_{\mathcal{P}}\right) \prod_{B \in K r(p)} b_{|B|}\left(\mu_{\mathcal{Q}}\right)
$$

Let $p$ be a partition and $B_{1}$ the block containing 1. Restricting an interval partition to $B_{1}$ yields a bijection $\psi$ between interval partitions of $k$ such that $D(I) \subseteq B_{1}$ and interval partitions of $B_{1}$. Moreover if $I=\left(I_{1}, \ldots, I_{r}\right)$ is an interval partition such that $D(I) \subseteq B_{1}$, the blocks of $\psi(I)$ are exactly the set $B_{1} \cap I_{j}$ for $1 \leq j \leq r$. Thus

$$
m_{k}(\mu(\mathcal{P} * \mathcal{Q}))=\sum_{p \in N C(k)} \sum_{I \in \mathcal{I}(k), D(I) \subseteq B_{1}}\left(\prod_{1 \leq j \leq r} b_{\left|B_{1} \cap I_{j}\right|}\left(\mu_{\mathcal{P}}\right) \prod_{B \in p, B \neq B_{1}} b_{|B|}\left(\mu_{\mathcal{P}}\right)\right) b_{K r(p)}\left(\mu_{\mathcal{Q}}\right),
$$

where we use the notation $b_{\pi}=\prod_{B \in \pi} b_{|B|}$ for $\pi=K r(p)$. Inverting the sums and using Lemma
7.48 yields

$$
\begin{aligned}
m_{k}(\mu(\mathcal{P} * \mathcal{Q}) & =\sum_{\substack{I \in \mathcal{I}(k) \\
I=\left(I_{1}, \ldots, I_{r}\right)}} \sum_{\substack{p \in N C(k) \\
(I) \subseteq B_{1}(p)}}\left(\prod_{1 \leq j \leq r} b_{\left|B_{1} \cap I_{j}\right|}\left(\mu_{\mathcal{P}}\right) \prod_{B \in p, B \neq B_{1}} b_{\left|B \cap I_{j}\right|}\left(\mu_{\mathcal{P}}\right)\right) b_{k r(p)}\left(\mu_{\mathcal{Q}}\right) \\
& =\sum_{\substack{I \in \mathcal{I}(k) \\
I=\left(I_{1}, \ldots, I_{r}\right)}} \sum_{\substack{\left(p_{1}, \ldots, p_{r}\right) \in \prod N C\left(I_{j} \mid\right)}} \prod_{1 \leq j \leq r} b_{p_{j}}\left(\mu_{\mathcal{P}}\right) b_{k r\left(p_{j}\right)}\left(\mu_{\mathcal{Q}}\right) . \\
& =\sum_{\substack{I \in \mathcal{I}(k) \\
I=\left(I_{1}, \ldots, I_{r}\right)}} \prod_{1 \leq j \leq r}\left(\sum _ { p \in N C ( | I _ { j } | ) } \left(b_{p}\left(\mu_{\mathcal{P}}\right) b_{k r(p)}\left(\mu_{\mathcal{Q}}\right)\right.\right.
\end{aligned}
$$

Therefore, the Boolean cumulants of $\mu(\mathcal{P} * \mathcal{Q})$ are:

$$
\begin{equation*}
b_{k}=\sum_{p \in N C(k)}\left(b_{p}\left(\mu_{\mathcal{P}}\right) b_{k r(p)}\left(\mu_{\mathcal{Q}}\right)\right), \tag{7.3.3}
\end{equation*}
$$

for $k \geq 1$.
Let $\mathbb{B}$ be the Boolean Bercovici-Pata bijection (see [19) from Boolean infinite divisible distributions to free infinite divisible distributions. $\mathbb{B}$ maps in particular a law $\mu$ having Boolean cumulants $\left(c_{k}\right)_{k \geq 1}$ to a law $\nu$ having free cumulants $\left(c_{k}\right)_{k \geq 1}$. Equation (7.3.3) together with the cumulant formula of the free multiplicative convolution (see [66]) yields

$$
\mathbb{B}(\mu(\mathcal{P} * \mathcal{Q}))=\mathbb{B}\left(\mu_{\mathcal{P}}\right) \boxtimes \mathbb{B}\left(\mu_{\mathcal{Q}}\right) .
$$

But from [17], $\mathbb{B}$ is a semi-group homomorphism with respect to the free multiplicative convolution. Thus, $\mathbb{B}\left(\mu_{\mathcal{P}}\right) \boxtimes \mathbb{B}\left(\mu_{\mathcal{Q}}\right)=\mathbb{B}\left(\mu_{\mathcal{P}} \boxtimes \mu_{\mathcal{Q}}\right)$, and applying $\mathbb{B}^{-1}$ yields

$$
\mu(\mathcal{P} * \mathcal{Q})=\mu_{\mathcal{P}} \boxtimes \mu_{\mathcal{Q}} .
$$

## Part IV

## The multiplicative graph $\mathcal{Z}$ of quasisymmetric functions

## Chapter 8

## Combinatorics of large compositions

### 8.1 Introduction

A descent of a permutation $\sigma$ of $n \in \mathbb{N}^{*}$ is an integer $i$ such that $\sigma(i)>\sigma(i+1)$. For each permutation $\sigma$, the corresponding descent set $D(\sigma)$ is the set of all the descents of $\sigma$. Since descents can be located everywhere except on $n$, a descent set is just a subset of $\{1, \ldots, n-1\}$. Let us call a composition of $n$ the data of $n$ and a subset of $\{1, \ldots, n-1\}$. A composition $D$ is represented by a ribbon Young diagram $\lambda_{D}$ of $n$ cells labelled 1 to $n$ by the following rule : cells $i$ and $i+1$ are neighbors and the cell $i+1$ is right to $i$ if $i \notin D$, below $i$ otherwise. Therefore, the descent set of a permutation $\sigma$ is $D$ if and only if inserting $\sigma(i)$ in each cell $i$ of $\lambda_{D}$ yields a standard ribbon Young tableau. For example, the composition $D=\{10,(3,5,9)\}$ gives the following ribbon Young diagram:


Figure 8.1: Ribbon Young diagram $\lambda_{D}$ of to the composition $D=\{10,(3,5,9)\}$
The permutation $\sigma=(3,5,8,4,7,1,6,9,10,2)$ has the descent set $D$ since the associated filling of $\lambda_{D}$ yields a ribbon Young tableau, as shown in figure 8.2.


Figure 8.2: Standard filling of the composition (3, 2, 4, 1)

The descent statistic of a composition $D$ is the number of standard fillings of the associated ribbon Young tableau $\lambda_{D}$ (or, equivalently, the number of permutations having $D$ as descent set). This latter number, denoted by $\beta(D)$, has been intensively studied in the last decades (see Viennot [87] and [88], Niven [67], de Bruijn [31] , ...). Two main questions arose in this study: the first one is to find the compositions of $n$ having a maximum descent statistic, and the second one is to find exact or asymptotic formulae for the descent statistic of large compositions having a given shape. For example, Niven and de Bruijn proved in 67] and 31 that
the two compositions of $n$ maximizing the descent statistic are $D_{1}(n)=\{1,3,5, \ldots\} \cap[1, n]$ and $D_{2}(n)=\{2,4,6, \ldots,\} \cap[1, n]$ : permutations having such descent sets are called alternating permutations. Désiré André already gave in [1] an asymptotic formula for the number of alternating permutations by showing that $\beta\left(D_{1}\right)(n) \sim 2(2 / \pi)^{n} n$ ! as $n$ goes to infinity.
In order to evaluate the descent statistic of a broad class of compositions, Ehrenborg, Levin and Readdy formalized in [37] a probabilistic approach to the counting problem, by relating each permutation of $n$ to a particular simplex of $[0,1]^{n}$. Since the Lebesgue measure yields a probability measure on $[0,1]^{n}$, it is possible to use probabilistic tools to get interesting results on descent statistics. Ehrenborg obtained in [36] an asymptotic formula for the descent statistics of the so-called nearly periodic permutations: the latter consist in permutations having the same descent pattern repeated several times, with some local perturbations. As for alternating permutations, the asympotic formula has the shape $K \lambda^{n} n$ !, with $K$ and $\lambda$ being some constants depending on the situation. Using the approach of [37] with functional analysis tools, Bender, Helton and Richmond extended in [18] the previous results to a broader class of descent sets, and they found asymptotic formulae of the same shape as before.
The factorial term of the asymptotic formula is easy to understand, since it comes from the cardinality of the set of permutations of $n$ elements. However, the term $\lambda^{n}$ seems more mysterious. In [18], the authors identified in their examples the phenomenon that makes the term $\lambda^{n}$ appear: namely, if we consider a large uniform random permutation with a fixed descent set, then the value of $\sigma(1)$ and $\sigma(n)$ are nearly independent, which causes a factorization in the asymptotic counting. Thus, the natural question is to know which compositions induce this phenomenon; it has been conjectured in [18] that every composition have this property as they become large. In the present chapter we construct a family of probabilistic models, called sawtooth models, which extend the probabilistic approach of Ehrenborg, Readdy and Levin. These models are more general than the ones used in [18], but the combinatorial properties of the large descent sets appear more clearly in this broader case; thus, we first study these models in their full generality, before deducing some specific results on descent sets. A main consequence of the latter work is an affirmative answer to Conjecture 1 on asymptotic independence from Bender, Helton and Richmond ([18]). We are also able to give by the following intuitive result on compositions: In the random filling of a composition, the contents of two distant cells are almost independent. In the next chapter, we will use the results of this chapter to study an analog of the Young lattice that was introduced by Gnedin and Olshanski in [42].

### 8.2 Preliminaries and results

### 8.2.1 Compositions

This paragraph gives definitions and notations concerning compositions.
Definition 8.1. Let $n \in \mathbb{N}$. A composition $\lambda$ of $n$ is a sequence of positive integers $\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ such that $\sum \lambda_{j}=n$.

A unique ribbon Young diagram with $n$ cells is associated to each composition: each row $j$ has $\lambda_{j}$ cells, and the first cell of the row $j+1$ is just below the last cell of the row $j$. For example the composition of $10,(3,2,4,1)$ is represented as in figure 8.1. This picture shows directly the link between Definition 8.1 and the definition we stated in the introduction : a composition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ of $n$ yields a subset $D_{\lambda}$ of $\{1, \ldots, n-1\}$, namely the subset $\left\{\lambda_{1}, \lambda_{1}+\lambda_{2}, \ldots, \lambda_{1}+\cdots+\lambda_{r-1}\right\}$. The latter correspondence is clearly bijective.
The size $|\lambda|$ of a composition is the sum of the $\lambda_{j}$. When nothing is specified, $\lambda$ will always be assumed to have the size $n$, and $n$ will always denote the size of the composition $\lambda$.

A standard filling of a composition $\lambda$ of size $n$ is a standard filling of the associated ribbon Young diagram: this is an assignement of a number between 1 and $n$ for each cell of the composition, such that every cells have different entries, and the entries are increasing to the right along the rows and decreasing to the bottom along the columns. An example for the composition of figure 8.1 is shown in figure 8.2 .

In particular, reading the tableau from left to right and from top to bottom associates a permutation $\sigma$ to each standard filling; moreover, the descent set of such a permutation $\sigma$, namely the set of indices $i$ such that $\sigma(i+1)<\sigma(i)$, is exactly the set

$$
D_{\lambda}=\left\{\lambda_{1}, \lambda_{1}+\lambda_{2}, \ldots, \sum_{1}^{r-1} \lambda_{i}\right\} .
$$

There is a bijection between the standard fillings of $\lambda$ and the permutations of $|\lambda|$ with descent set $D_{\lambda}$. For example the filling in figure 8.2 yields the permutation (3, 5, 8, 4, 7, 1, 6, 9, 10, 2).

### 8.2.2 Result on asymptotic independence

We present here the main results that are proven in the present chapter.
Notation 8.2. Let $\lambda$ be a composition. Let $\Sigma_{\lambda}$ denote the set of all permutations with descent set $D_{\lambda}$. With the uniform counting measure $\mathbb{P}_{\lambda}$, it becomes a probability space, and $\sigma_{\lambda}$ denotes the random permutation coming from this probability space. As usual $\left|\Sigma_{\lambda}\right|$ is the cardinality of the set $\Sigma_{\lambda}$.
$\left|\Sigma_{\lambda}\right|$ is thus the descent statistic associated to the composition $\lambda$.
Denote for each random variable $X$ by $\mu(X)$ its law and by $d_{X}$ its density, and write $\mu \otimes \nu$ for the independent product of two laws. The goal of the chapter is to prove that distant cells in a composition have independent entries, namely:

Theorem 8.3. Let $\varepsilon, r \in \mathbb{N}$. Then there exists $k \geq 0$ such that if $\lambda$ is a composition of $n$ and $0<i_{1}<\cdots<i_{r} \leq n$ are indices with $i_{j+1}-i_{j} \geq k$,

$$
\pi\left(\mu\left(\frac{\sigma_{\lambda}\left(i_{1}\right)}{n}, \ldots, \frac{\sigma_{\lambda}\left(i_{r}\right)}{n}\right), \mu\left(\frac{\sigma\left(i_{1}\right)}{n}\right) \otimes \cdots \otimes \mu\left(\frac{\sigma\left(i_{r}\right)}{n}\right)\right) \leq \varepsilon,
$$

with $\pi$ denoting the Levy-Prokhorov metric on the set of measures of $[0,1]^{r}$.
If the variance of $\frac{\sigma_{\lambda}\left(i_{1}\right)}{n}$ and $\frac{\sigma_{\lambda}\left(i_{n}\right)}{n}$ remain bounded from below by a positive constant, then the approximate independence of $\frac{\sigma_{\lambda}\left(i_{1}\right)}{n}$ and $\frac{\sigma_{\lambda}\left(i_{n}\right)}{n}$ can be given with a stronger metric than the Levy-Prokhorov metric. This is the content of Conjecture 1 of [18], which is proven in this chapter and formulated in Theorem 8.36.

### 8.2.3 Runs of a composition

Let $\lambda$ be a composition. We number the cells as we read them, from left to right and from top to bottom. The cells are identified with integers from 1 to $n$ through this numbering. For example in the standard filling of figure 8.2 , the number 7 is in the cell 5 .
We call run any set consisting in all the cells of a given column or row. The set of runs is ordered with the lexicographical order. In the same example as before the runs are

$$
s_{1}=(1,2,3), s_{2}=(3,4), s_{3}=(4,5), s_{4}=(5,6), s_{5}=(6,7,8,9), s_{6}=(9,10),
$$

where we put in the parenthesis the cells of each run.
Note that inside each run the cells are ordered by the natural order on integers. We call extreme cell a cell that is an extremum in a run with respect to this order, and denote by $\mathcal{E}_{\lambda}$ the set of extreme cells of $\lambda$. Apart from the first and last cells of the composition, each extreme cell belongs to two consecutive runs. Let $P_{\lambda}$ be the set of extreme cells followed by a column, or preceeded by a row and $V_{\lambda}$ the set of extreme cells followed by a row or preceeded by a column. The elements of $P_{\lambda}$ are called peaks and the ones of $V_{\lambda}$ valleys. The sets $V_{\lambda}$ and $P_{\lambda}$ are also ordered with the natural order:

$$
P_{\lambda}=\left\{p_{1}<\cdots<p_{k}\right\}, V_{\lambda}=\left\{q_{1}<\cdots<q_{k^{\prime}}\right\},
$$

with $k-1 \leq k^{\prime} \leq k+1$.
The first and last cells are always extreme points. A composition is said being of type ++ (resp. ,,+--+-- ) if the first cell is a peak and the last cell is a peak (resp peak-valley, valley-peak, valley-valley).
Finally, let $l(s)$, the length of a run $s$, be the cardinality of $s$, and $L(\lambda)$, the amplitude of $\lambda$, be the supremum of all lengths $l(s)$.

### 8.2.4 The coupling method

In this paragraph we introduce a probabilistic tool called the coupling method, and set the relative notations for the sequel. We refer to [58] for a review on the subject. We will present the notions in the framework of random variables but we could have done the same with probability laws as well.

Definition 8.4. Let $(E, \mathcal{E})$ be a probability space and $X, Y$ two random variables on $E$. $A$ coupling of $(X, Y)$ is a random variable $\left(Z_{1}, Z_{2}\right)$ on $(E \times E, \mathcal{E} \otimes, \mathcal{E})$ such that

$$
Z_{1} \sim_{\text {law }} X, Z_{2} \sim_{\text {law }} Y .
$$

Such a coupling always exists: it suffices to consider two independent random variables $Z_{1}$ and $Z_{2}$ with respective law $\mu_{X}$ and $\mu_{Y}$. However, a coupling is often useful precisely when the resulting random variables $Z_{1}$ and $Z_{2}$ are far from being independent. In particular, in this chapter we are mainly interested in the case where $Z_{1}$ and $Z_{2}$ respect a certain order on the set $E$. From now on $E$ is a Polish space considered with its Borel $\sigma$-algebra $\mathcal{E}$, and $\triangleleft$ is a partial order on $E$ such that the graph $\mathcal{G}=\{(x, y), x \triangleleft y\}$ is $\mathcal{E}$-measurable.
Definition 8.5. Let $X, Y$ be two random variables on $E$. $Y$ stochastically dominates $X$ (denoted $Y \succeq X$ ) if and only if

$$
\mathbb{P}(X \in A) \leq \mathbb{P}(Y \in A)
$$

for any Borel set A such that

$$
x \in A \Rightarrow\{y \in E, x \triangleleft y\} \subseteq A .
$$

For example if $E=\mathbb{R}$ with the canonical order $\leq$ and $\sigma$-algebra $\mathcal{B}(\mathbb{R})$, then $Y$ stochastically dominates $X$ if and only if for all $x \in \mathbb{R}$,

$$
\mathbb{P}(X \in[x,+\infty[) \leq \mathbb{P}(Y \in[x,+\infty[)
$$

or equivalently, if we denote their respective cumulative distribution function by $F_{X}(t)$ and $F_{Y}(t)$ :

$$
F_{Y}(t) \leq F_{X}(t) \quad \text { for all } t \in \mathbb{R}
$$

There are several ways to characterize the stochastic dominance:

Proposition 8.6. The three following statements are equivalent :

- $Y$ stochastically dominates $X$
- there exists a coupling $\left(Z_{1}, Z_{2}\right)$ of $X, Y$ such that $Z_{1} \triangleleft Z_{2}$ almost surely.
- for any positive measurable bounded function $f$ that is non-decreasing with respect to $\triangleleft$,

$$
\mathbb{E}(f(X)) \leq \mathbb{E}(f(Y))
$$

The proof is straightforward and can be found in [58. This yields the following intuitive Lemma :

Lemma 8.7. Let $\left(X_{1}, X_{2}, Y_{1}, Y_{2}\right)$ be a random variable on $E^{4}$ such that :

- $X_{1} \preceq Y_{1}$ and $Y_{2} \preceq X_{2}$,
- $\left(X_{1}, Y_{1}\right)$ is independent from $\left(X_{2}, Y_{2}\right)$.

Then

$$
\mathbb{P}\left(X_{1} \triangleleft X_{2}\right) \geq \mathbb{P}\left(Y_{1} \triangleleft Y_{2}\right) .
$$

Proof. Let $\ll$ be the partial order on $E \times E$ defined by

$$
(x, y) \ll\left(x^{\prime}, y^{\prime}\right) \leftrightarrow x \triangleleft x^{\prime} \text { and } y^{\prime} \triangleleft y .
$$

Since $Y_{1} \succeq X_{1}$ and $X_{2} \succeq Y_{2}$, there exists a coupling $\left(\hat{X}_{1}, \hat{Y}_{1}\right)$ (resp. $\left.\left(\hat{X}_{2}, \hat{Y}_{2}\right)\right)$ of $X_{1}, Y_{1}$ (resp. $\left.X_{2}, Y_{2}\right)$ such that almost surely $\hat{X}_{1} \triangleleft \hat{Y}_{1}\left(\operatorname{resp} \hat{X}_{2} \triangleright \hat{Y}_{2}\right)$. The random variables ( $\hat{X}_{1}, \hat{Y}_{1}$ ) and $\left(\hat{X}_{2}, \hat{Y}_{2}\right)$ can be chosen independent one from each other. Since $\left(X_{1}, Y_{1}\right)$ and $\left(X_{2}, Y_{2}\right)$ are also independent, this implies that $\left(\left(\hat{X}_{1}, \hat{X}_{2}\right),\left(\hat{Y}_{1}, \hat{Y}_{2}\right)\right)$ is a coupling of $\left(\left(X_{1}, X_{2}\right),\left(Y_{1}, Y_{2}\right)\right)$ with almost surely

$$
\left(\hat{X}_{1}, \hat{X}_{2}\right) \ll\left(\hat{Y}_{1}, \hat{Y}_{2}\right) .
$$

But if $\hat{Y}_{1} \triangleleft \hat{Y}_{2}$, then $\hat{X}_{1} \triangleleft \hat{Y}_{1} \triangleleft \hat{Y}_{2} \triangleleft \hat{X}_{1}$ and thus

$$
\mathbb{P}\left(Y_{1} \triangleleft Y_{2}\right)=\mathbb{P}\left(\hat{Y}_{1} \triangleleft \hat{Y}_{2}\right) \leq \mathbb{P}\left(\hat{X}_{1} \triangleleft \hat{X}_{2}\right)=\mathbb{P}\left(X_{1} \triangleleft X_{2}\right)
$$

These results will be concretely applied on $\mathbb{R}^{p}, p \geq 1$, and thus we need to define a family of partial orders on those sets.

Definition 8.8. Let $p \geq 1$. The partial order $\leq$ on $\mathbb{R}^{p}$ is the natural order on $\mathbb{R}$ for $p=1$, and for $p \geq 2$ if $\left(x_{i}\right)_{1 \leq i \leq p},\left(y_{i}\right)_{1 \leq i \leq p} \in \mathbb{R}^{p}$,

$$
\left(x_{i}\right)_{1 \leq i \leq p} \leq\left(y_{i}\right)_{1 \leq i \leq p} \Leftrightarrow \forall i \in[1 ; p], x_{i} \leq y_{i} .
$$

For any word $\varepsilon$ of length $p$ in $\{+1,-1\}$ (or simply in $\{+,-\}$ ), the modified partial order $\leq_{\varepsilon}$ is defined as

$$
\left(x_{i}\right)_{1 \leq i \leq p} \leq_{\varepsilon}\left(y_{i}\right)_{1 \leq i \leq p} \Leftrightarrow \forall i \in[1 ; p], \varepsilon_{i} x_{i} \leq \varepsilon_{i} y_{i} .
$$

The easiest way to check the stochastic dominance is to look at the cumulative distribution function. The proof of the following Lemma is a direct application of Proposition 8.6.

Lemma 8.9. Let $\left(X_{i}\right)_{1 \leq i \leq p}$ and $\left(Y_{i}\right)_{1 \leq i \leq p}$ be two random variables on $\left(\mathbb{R}^{p}, \leq_{\varepsilon}\right)$. Then $\left(Y_{i}\right)_{1 \leq i \leq p}$ stochastically dominates $\left(X_{i}\right)_{1 \leq i \leq p}$ if and only if for all $\left(t_{i}\right)_{1 \leq i \leq p} \in \mathbb{R}^{p}$,

$$
F_{\left(X_{i}\right)}^{\varepsilon}\left(t_{1}, \ldots, t_{p}\right) \geq F_{\left(Y_{i}\right)}^{\varepsilon}\left(t_{1}, \ldots, t_{p}\right),
$$

with $F_{\left(X_{i}\right)}^{\varepsilon}$ being the modified cumulative distribution function defined by

$$
F_{\left(X_{i}\right)}^{\varepsilon}\left(t_{1}, \ldots, t_{p}\right)=\mathbb{P}\left(\left(X_{i}\right) \leq_{\varepsilon}\left(t_{i}\right)\right) .
$$

The stochastic dominance in the case $\left(\mathbb{R}^{p}, \leq_{\varepsilon}\right)$ is denoted as $\left(X_{1}, \ldots, X_{p}\right) \preceq_{\varepsilon}\left(Y_{1}, \ldots, Y_{p}\right)$. A consequence of the previous result is that if $\left(Y_{1}, \ldots, Y_{p}\right)$ stochastically dominates ( $X_{1}, \ldots, X_{p}$ ), then for all subsets $I=\left(i_{1}, \ldots, i_{r}\right)$ of $\{1, \ldots, p\},\left(Y_{i_{1}}, \ldots, Y_{i_{r}}\right)$ also stochastically dominates $\left(X_{i_{1}}, \ldots, X_{i_{r}}\right)$.
Applying Lemma 8.9 to the case $p=2$ yields the following Lemma:
Lemma 8.10. Let $\left(U_{1}, V_{1}\right),\left(U_{2}, V_{2}\right)$ be two random variables on $[0,1]$ such that $U_{2}$ and $V_{2}$ are independent. Suppose that for all $0 \leq t \leq 1$,

$$
F_{V_{1}}(t) \leq F_{V_{2}}(t)
$$

and for all $v \in[0,1]$,

$$
F_{U_{1} \mid V_{1}=v}(t) \leq F_{U_{2}}(t) .
$$

There exists a coupling $\left(\left(Z_{1}, \tilde{Z}_{1}\right),\left(Z_{2}, \tilde{Z}_{2}\right)\right)$ of $\left(U_{1}, V_{1}\right)$ and $\left(U_{2}, V_{2}\right)$ such that almost surely

$$
\left(Z_{1}, \tilde{Z}_{1}\right) \geq\left(Z_{2}, \tilde{Z}_{2}\right)
$$

### 8.3 Sawtooth model

### 8.3.1 Definition of the model

In this section we introduce a statistical model of particles in a tube, which is a generalization of the probabilistic approach of Ehrenborg, Levin and Readdy in [37]. The model consists in a sequence of particles, each of them moving vertically in an horizontal two-dimensional tube. Each particle has a repulsive action on the two neighbouring particles, and moreover, the set of particles splits into two groups: the upper particles and the lower particles. The upper particles are always above the lower ones. The model is depicted in Figure 8.3 .


Figure 8.3: Upper particles $\left\{p_{1}, p_{2}, p_{3}\right\}$ and lower particles $\left\{q_{1}, q_{2}, q_{3}\right\}$ in a tube.
Such a system is called a Sawtooth model in the sequel.

Remark 8.11. If there are $k$ upper-particles, there must be $k^{\prime}$ lower particles with $k^{\prime} \in\{k-$ $1, k, k+1\}$, depending on the type of the first and the last particles. We define therefore the type $\varepsilon(\mathcal{S})$ of the model $\mathcal{S}$ as the word $\varepsilon_{I} \varepsilon_{F}$, with $\varepsilon_{I}=+$ (resp. $\varepsilon_{F}=+$ ) if the first (resp. last) particle is an upper one, and $\varepsilon_{I}=-$ (resp. $\varepsilon_{F}=-$ ) otherwise.

Unless specified otherwise, the first particle is a lower particle (as in the picture). The particles are ordered from the left, and following this order the upper particles are written $\left\{p_{1}<p_{2}<\cdots<p_{k}\right\}$ and the lower particles $\left\{q_{1}<\cdots<q_{k^{\prime}}\right\}$. Since the nature of our results won't depend on the type of the model, we will also assume that there are $k+1$ lower particles, yielding that the last particle is a lower one too.
Denote by $x_{i}$ the position of $q_{i}$ and by $y_{i}$ the position of $p_{i}$ : by a rescaling, we can assume that $x_{i}, y_{i} \in[0,1]$. These positions are considered as random, and each configuration of positions is weighted according to repulsive interactions between neighbouring particles. This yields the following definition:

Definition 8.12. A Sawtooth model $\mathcal{S}$ is the union of two families of random variables $\left\{X_{i}\right\}_{1 \leq i \leq k+1}$ and $\left\{Y_{j}\right\}_{1 \leq j \leq k}$ on $[0,1]$ with the multivariate density

$$
\begin{equation*}
\mathbb{P}\left(\left\{X_{i}=x_{i}, Y_{j}=y_{j}\right\}\right)=\frac{1}{\mathcal{V}} \prod \mathbf{1}_{x_{i} \leq y_{i} \geq x_{i+1}} f_{i}\left(y_{i}-x_{i}\right) g_{i}\left(y_{i}-x_{i+1}\right) \prod d x_{i} \prod d y_{j} \tag{8.3.1}
\end{equation*}
$$

where $\left\{f_{i}, g_{i}\right\}_{1 \leq i \leq k}$ is a family of increasing positive $C^{1}$ functions on $[0,1]$.
The quantity $\mathcal{V}$ is called the volume of $\mathcal{S}$ and is sometimes denoted by $\mathcal{V}(\mathcal{S})$ to avoid confusion. $\mathcal{S}$ is said normalized if $\int f_{i}=\int g_{i}=1$ for $1 \leq i \leq k$.

The volume has the following expression:

$$
\begin{equation*}
\mathcal{V}(\mathcal{S})=\int_{[0,1]^{2 k+1}} \prod \mathbf{1}_{x_{i} \leq y_{i} \geq x_{i+1}} f_{i}\left(y_{i}-x_{i}\right) g_{i}\left(y_{i}-x_{i+1}\right) \prod d x_{i} d y_{i} \tag{8.3.2}
\end{equation*}
$$

In particular, an appropriate rescaling of the functions $f_{i}, g_{i}$ can transform any Sawtooth model into a normalized one, without changing the probability space. Thus, from now on and unless stated otherwise, the model is assumed normalized. In case we are considering non-normalized models, we will use the notation $f_{i}, g_{i}$, etc. for the normalized quantities, and $\tilde{f}_{i}, \tilde{g}_{i}$, etc. for the non-normalized ones.
Aiming the results we stated on compositions, we should answer these questions :

1. As the number of particles goes to infinity, is there some independence between $X_{1}$ and $X_{k+1}$ ?
2. It is possible to estimate the behavior of a particle $X_{r}$ by only considering its neighbouring particles?

For each subset of particles $\Omega=\left(q_{i_{1}}, \ldots, q_{i_{r}}, p_{j_{1}}, \ldots, p_{j_{r^{\prime}}}\right)$ and measurable event $\mathcal{X}$, denote by

$$
d_{\Omega \mid \mathcal{X}}\left(x_{i_{1}}, \ldots, x_{i_{r}}, y_{j_{1}}, \ldots, y_{j_{r^{\prime}}}\right)
$$

the marginal density of $\Omega$ conditioned on $\mathcal{X}$. The subscripts will be dropped when there is no confusion, and we denote by $X_{I}$ the first variable $X_{1}$ and $X_{F}$ the last variable $X_{k+1}$. Finally, since the system is fully described by the functions $\left\{f_{i}, g_{j}\right\}$, we will refer sometimes to a particular system just by mentioning this set of functions.
The definition of a Sawtooth model yields directly two first results which are given in Lemma 8.13 and Lemma 8.14. The first one stresses the Markovian aspect of a Sawtooth model :

Lemma 8.13. Let $\mathcal{S}$ be a Sawtooth model of size $k$, and $1 \leq i \leq k$. Let $Z$ be the position of $a$ particle right to $X_{i}$ (namely $Z=X_{j}$ for $j>i$ or $Z=Y_{j}$ for $j \geq i$ ) and $\mathcal{X}$ be an event depending on the positions of particles right to $Z$. Then for $0 \leq z \leq 1$,

$$
d_{X_{i} \mid Z=z, \mathcal{X}}=d_{X_{i} \mid Z=z}
$$

Proof. It suffices to prove that the particles left to $Z$ are independent of the particles right to $Z$ conditionally on the value of $Z$. This is implied by the form of the density of the model, since the latter splits between the density of the particles left to $Z$ and the ones right to $Z$.

The second one is a generalization of Lemma 3-(a) in [18]. :
Lemma 8.14. Let $1 \leq r \leq k+1$, and let $\mathcal{X}$ be an event depending on the position of all particles except $X_{r}$. Then $d_{X_{r} \mid \mathcal{X}}\left(x_{r}\right)$ is decreasing in $x_{r}$.
Proof. Let $a$ be in $[0,1]$. By Lemma 8.13 ,

$$
\begin{aligned}
d_{X_{r} \mid \mathcal{X}}(a) & =\int_{[0,1]^{2}} d_{\left(X_{r} \mid \mathcal{X}\right) \mid Y_{r-1}=z, Y_{r+1}=z^{\prime}}(a) d_{Y_{r-1}, Y_{r+1} \mid \mathcal{X}}\left(z, z^{\prime}\right) d z d z^{\prime} \\
& =\int_{[0,1]^{2}} d_{X_{r} \mid Y_{r-1}=z, Y_{r+1}=z^{\prime}}(a) d_{Y_{r-1}, Y_{r+1} \mid \mathcal{X}}\left(z, z^{\prime}\right) d z d z^{\prime} .
\end{aligned}
$$

Thus, it is enough to prove the monotonicity in the case of a conditioning on $Y_{r-1}=z, Y_{r+1}=z^{\prime}$. In this case

$$
d_{X_{r} \mid Y_{r-1}=z, Y_{r+1}=z^{\prime}}(a)=\mathbf{1}_{z \geq a, z^{\prime} \geq a} \frac{1}{R}\left(g_{r-1}(z-a) f_{r}\left(z^{\prime}-a\right)\right),
$$

with $R$ a normalizing constant. Since $g_{r-1}$ and $f_{r}$ are increasing, this concludes the proof.
The same result holds for upper particles, but in this case the density is increasing.

### 8.3.2 The processes $\mathcal{S}_{\lambda}$ and $\Sigma_{\lambda}$

Let us see how these definitions fit into the framework of compositions. The main idea from [37] is to consider the set of all permutations with a given descent set $D_{\lambda}$ as a probability space.
$\left|\Sigma_{\lambda}\right|$ can indeed be related to the volume of a polytope in $[0,1]^{n}$ (see for example the survey of Stanley on alternating permutations, [78]) . For each sequence of distinct elements $\vec{z}=$ $\left(z_{1}, \ldots, z_{n}\right)$ in $[0,1]$, the ranking permutation of $\vec{z}$ is the permutation $\sigma(\vec{z})$ that assigns to each $j$ the position of $z_{j}$ in the ordered sequence $\left(z_{i_{1}}<\cdots<z_{i_{n}}\right)$ : namely, $\sigma(\vec{z})(j)=k$ if and only if $\#\left\{1 \leq i \leq n \mid z_{i} \leq z_{j}\right\}=k$.
Proposition 8.15 ( 37$]$ ). The law of $\sigma_{\lambda}$ is the law of the ranking permutation for a sequence of independent uniform variables $Z_{1}, \ldots, Z_{n}$ in $[0,1]$ conditioned on the event

$$
\left\{Z_{i}>Z_{i+1} \text { if and only if } i \in D_{\lambda}\right\} .
$$

In particular, the following expression of the number of permutations with descent set $D_{\lambda}$ holds

$$
\left|\Sigma_{\lambda}\right|=n!\int_{[0,1]^{n}} \prod_{i \in D_{\lambda}} \mathbf{1}_{z_{i} \geq z_{i+1}} \prod_{i \notin D_{\lambda}} \mathbf{1}_{z_{i} \leq z_{i+1}} \prod d z_{i},
$$

with $z_{n+1}=1$.
The proof of the latter proposition is straightforward as soon as we remark that the volume of the polytope $\left\{0 \leq z_{1}, \ldots, z_{n} \leq 1\right\}$ is exactly $\frac{1}{n!}$. The processus $\left\{Z_{i}\right\}_{1 \leq i \leq n}$ in the previous proposition is denoted by $\tilde{\mathcal{S}}_{\lambda}$. Since the indicator function in the integrand depends on conditions between neighbouring points, this result can be rephrased in terms of Sawtooth model.
Regrouping the inequalities between elements of the same run of $\lambda$ yields:

$$
\begin{equation*}
\left|\Sigma_{\lambda}\right|=n!\int_{[0,1]^{n}} \mathbf{1}_{z_{1} \leq z_{2} \leq \cdots \leq z_{i_{1}}} \mathbf{1}_{z_{i_{1}} \geq z_{i_{1}+1} \geq \cdots \geq z_{i_{1}+i_{2}}} \ldots \mathbf{1}_{z_{n-i_{2 r}} \leq \cdots \leq z_{n}} \prod d z_{i}, \tag{8.3.3}
\end{equation*}
$$

and by integrating over all the coordinates that do not correspond to extreme cells, we get

$$
\begin{aligned}
\left|\Sigma_{\lambda}\right|= & n!\int_{[0,1]^{n}} \mathbf{1}_{x_{1}^{-} \leq x_{1}^{+} \geq x_{2}^{-} \leq \ldots\left(\frac{1}{\left(l\left(s_{1}\right)-2\right)!}\left|x_{1}^{+}-x_{1}^{-}\right|^{l\left(s_{1}\right)-2}\right.} \\
& \frac{1}{\left(l\left(s_{2}\right)-2\right)!}\left|x_{1}^{+}-x_{2}^{-}\right|^{l\left(s_{2}\right)-2} \cdots \frac{1}{\left(l\left(s_{2 r}\right)-2\right)!}\left|x_{k}^{+}-x_{k+1}^{-}\right|^{l\left(s_{k}\right)-2} \prod_{i=1}^{k} d x_{i}^{+} \prod_{i=1}^{k+1} d x_{i}^{-} .
\end{aligned}
$$

Let $\mathcal{S}_{\lambda}$ be the non-normalized Sawtooth model with the non-normalized density functions $\left\{\tilde{f}_{j}, \tilde{g}_{j}\right\}_{1 \leq i \leq r}$ such that

$$
\tilde{f}_{j}(t)=\frac{1}{\left(l\left(s_{2 j-1}\right)-2\right)!} t^{l\left(s_{2 j-1}\right)-2}, \tilde{g}_{j}(t)=\frac{1}{\left(l\left(s_{2 j}\right)-2\right)!} t^{l\left(s_{2 j}\right)-2} .
$$

A comparison between the latter expression of $\left|\Sigma_{\lambda}\right|$ and the expression 8.3.2) of the volume of a Sawtooth model gives

$$
\left|\Sigma_{\lambda}\right|=|\lambda|!\mathcal{V}\left(\mathcal{S}_{\lambda}\right)
$$

To sum up, three processes are constructed from $\lambda$. The first one, $\sigma_{\lambda}$ comes from the uniform random standard filling of the ribbon Young tableau $\lambda$, the second one, $\tilde{\mathcal{S}}_{\lambda}$, comes from the probabilistic approach of [37], and the third one, $\mathcal{S}_{\lambda}$, is obtained from $\tilde{\mathcal{S}}_{\lambda}$ by considering only the extreme particles. They are of course intimately related, even if the first one is discrete and the second and third ones are continuous. $\sigma_{\lambda}$ can be recovered from $\tilde{\mathcal{S}}_{\lambda}$ by the associated ranking permutation, and when $|\lambda|$ goes to infinity $\frac{\sigma_{\lambda}(i)}{n}$ and $Z_{i}$ are approximately the same :

Lemma 8.16. The following inequality always holds for $0<A, n \in \mathbb{N}$ :

$$
\mathbb{P}\left(\max \left(\left|\frac{\sigma_{\lambda}(i)}{n+1}-Z_{i}\right|>\frac{A}{\sqrt{n+2}}\right) \leq \frac{1}{A^{2}}\right.
$$

In particular, if the densities of $Z_{i}$ remains bounded by a constant $B$,

$$
\left\|F_{Z_{i}}-F_{\frac{\sigma(i)}{n}}\right\|_{\infty} \rightarrow_{|\lambda| \rightarrow+\infty} 0 .
$$

Proof. Let us evaluate $\mathbb{P}\left(\left|\left|\frac{\sigma_{\lambda}(i)}{n+1}-Z_{i}\right|>\frac{A}{n+2}\right)\right.$. Condition the event $\left\{\left|\frac{\sigma_{\lambda}(i)}{n+1}-Z_{i}\right|>\frac{A}{n+2}\right\}$ on a particular realization $\sigma$ of $\sigma_{\lambda}$, and suppose that $\sigma(i)=k$. In this case, the conditional density of $Z_{i}$ is :

$$
\begin{aligned}
d_{Z_{i} \mid \sigma_{\lambda}=\sigma}\left(z_{i}\right)= & n!\left(\int_{0 \leq z_{\sigma-1}(1)} \leq \cdots \leq z_{\sigma-1}(k-1) \leq z_{i}\right. \\
& \left(\prod_{1 \leq \sigma(j) \leq k-1} d z_{j}\right) \\
= & \frac{n!}{(k-1)!(n-k)!} z_{i}^{k-1}\left(1-z_{i}\right)^{n-k} .
\end{aligned}
$$

Computing the conditional expectation yields $\mathbb{E}\left(Z_{i} \mid \sigma_{\lambda}=\sigma\right)=\frac{k}{n+1}$ and

$$
\operatorname{Var}\left(Z_{i} \mid \sigma_{\lambda}=\sigma\right)=\left(\frac{k}{n+1} \frac{n+1-k}{n+1}\right) \frac{1}{n+2} \leq \frac{1}{n+2} .
$$

Thus, by the Chebyshev's inequality,

$$
\mathbb{P}_{Z_{i} \mid \sigma_{\lambda}=\sigma}\left(\left|Z_{i}-\frac{\sigma(1)}{n+1}\right|>\frac{A}{\sqrt{n+2}}\right) \leq \frac{1}{A^{2}}
$$

Integrating this inequality on all the disjoint events $\sigma$ on which $Z_{i}$ can be conditioned yields the first part of the Lemma.

From now on, let $\tilde{\gamma}_{r}$ denote for $r \geq 2$ the function $\tilde{\gamma}_{r}(t)=\frac{1}{(r-2)!} t^{r-2}$, and $\gamma_{r}(t)=(r-1) t^{r-2}$ its normalized density function.

### 8.4 Convex Sawtooth Model

In this section, we study the behavior of the extreme particles for a Sawtooth model respecting a particular convexity property. The results of this section are much easier to get in the particular case of the Sawtooth models $\mathcal{S}_{\lambda}$ of the last section, since the density functions $\left\{f_{i}, g_{i}\right\}$ are explicitly given. We will use this particular Sawtooth models as examples for our more general computations.

### 8.4.1 Log-concave densities

To be able to get some results on the behavior of the particles, it is necessary to impose some conditions on the density functions $\left\{f_{i}, g_{i}\right\}$. Actually the condition we need is quite natural from a physical point of view, since we will require that the repulsive forces in the definition of the Sawtooth model come from a convex potential : the consequence is that the density functions should be log-concave. This motivates the following definition :

Definition 8.17. A Sawtooth model is called convex if all the functions $\left(f_{i}, g_{i}\right)_{1 \leq i \leq k}$ are $\log$ concave. This means that for all $1 \leq i \leq k, \frac{f_{i}^{\prime}(t)}{f_{i}(t)}$ and $\frac{g_{i}^{\prime}(t)}{g_{i}(t)}$ are decreasing.

The main advantage of the log-concavity is that the behavior of the particles becomes monotone in a certain sense.
For $1 \leq s \leq k+1$, let $\mathcal{S}_{\rightarrow X_{s}}$ (resp. $\mathcal{S}_{X_{s} \leftarrow}$ ) denote the Sawtooth model obtained by keeping only the particles between $X_{I}$ and $X_{s}$ (resp. between $X_{s}$ and $X_{F}$ ) and the functions $\left\{f_{i}, g_{i}\right\}_{i \leq s}$ (resp. $\left\{f_{i}, g_{i}\right\}_{i \geq s+1}$ ). Likewise, let $\mathcal{S}_{\rightarrow Y_{s}}$ (resp. $\mathcal{S}_{Y_{s} \leftarrow}$ ) denote the Sawtooth model obtained by keeping only the particles between $X_{I}$ and $Y_{s}$ (resp. between $Y_{s}$ and $X_{F}$ ) and the functions $\left\{f_{i}, g_{j}\right\}_{\substack{i \leq s \\ j \leq s-1}}$ (resp. $\left.\left\{f_{i}, g_{j}\right\}_{\substack{\geq s+1 \\ j \geq s}}\right)$.
In order to emphasize a specific Sawtooth model $\mathcal{S}$, we write $X_{i}^{\mathcal{S}}$ to denote the particle $X_{i}$ in $\mathcal{S}$, and $F_{X_{i}, \mathcal{S}}$ to denote the cumulative distribution function of $X_{i}$ in $\mathcal{S}$ (and the same for $Y_{i}$ ).

Proposition 8.18. Let $\left\{f_{i}, g_{i}\right\}$ be a convex Sawtooth model. Then for $1 \leq s \leq k,\left(X_{s} \mid Y_{s}=y\right)$ is increasing with $y$ (in terms of stochastic dominance) and $\left(Y_{s} \mid X_{s+1}=x\right)$ is increasing with $x$. Moreover,

$$
X_{s}^{\mathcal{S}_{\rightarrow X}} \succeq\left(X_{s} \mid Y_{s}=y\right), Y_{s}^{\mathcal{S}_{\rightarrow Y_{s}}} \succeq\left(Y_{s} \mid X_{s+1}=x\right) .
$$

Proof. Let $1 \leq s \leq k$. To prove the first part of the proposition, it is enough to show that for $0 \leq t \leq 1, F_{X_{s} \mid Y_{s}=y}(t)$ is decreasing in $y$ and $F_{Y_{s} \mid X_{s+1}=x}(t)$ is decreasing in $x$.
Let $d(x)$ be the density of $X_{s}$ in $\mathcal{S}_{\rightarrow X_{s}}$. Then by the definition of the probability density of $\mathcal{S}$,
the density of $X_{s}$ in $\mathcal{S}$ conditioned on the value of $Y_{s}$ is $\mathbf{1}_{x \leq y} \frac{d(x) f_{s}(y-x)}{A}$, with $A$ a normalizing constant. Thus, the cumulative distribution function $F_{y}($.$) of X_{s}$ conditioned on $Y_{s}=y$ is

$$
F_{y}(t)=\frac{\int_{0}^{t \wedge y} d(x) f_{s}(y-x) d x}{\int_{0}^{y} d(x) f_{s}(y-x) d x}
$$

For $t>y$ it is clear that $\frac{\partial}{\partial y} F_{y}(t)=0$, and from now on we only consider $t \leq y$. Since the logarithm function is increasing, it is enough to show that $\frac{\partial}{\partial y} \log \left(F_{y}(t)\right) \leq 0$. This derivative is equal to

$$
\frac{\partial}{\partial y} \log \left(F_{y}(t)\right)=\frac{\int_{0}^{t} d(x) f_{s}^{\prime}(y-x) d x}{\int_{0}^{t} d(x) f_{s}(y-x) d x}-\frac{\int_{0}^{y} d(x) f_{s}^{\prime}(y-x) d x}{\int_{0}^{y} d(x) f_{s}(y-x) d x}-\frac{d(y) f_{s}(0)}{\int_{0}^{y} d(x) f_{s}(y-x) d x}
$$

Since $\left(-\frac{d(y) f_{s}(0)}{\int_{0}^{y} d(x) f_{s}(y-x) d x}\right) \leq 0$, the non-positivity of the remaining part of the sum suffices. Denote

$$
\Delta=\int_{0}^{t} d(x) f_{s}^{\prime}(y-x) d x \int_{0}^{y} d(x) f_{s}(y-x) d x-\int_{0}^{y} d(x) f_{s}^{\prime}(y-x) d x \int_{0}^{t} d(x) f_{s}(y-x) d x
$$

Thus, we have to show that $\Delta \leq 0$. For $t \leq y$,

$$
\begin{aligned}
\Delta= & \int_{0}^{t} d(x) f_{s}^{\prime}(y-x) d x\left(\int_{0}^{t} d(x) f_{s}(y-x) d x+\int_{t}^{y} d(x) f_{s}(y-x) d x\right) \\
& \quad-\left(\int_{0}^{t} d(x) f_{s}^{\prime}(y-x) d x+\int_{t}^{y} d(x) f_{s}^{\prime}(y-x) d x\right) \int_{0}^{t} d(x) f_{s}(y-x) d x \\
= & \int_{0}^{t} d(x) f_{s}^{\prime}(y-x) d x \int_{t}^{y} d(x) f_{s}(y-x) d x \\
& \quad-\int_{t}^{y} d(x) f_{s}^{\prime}(y-x) d x \int_{0}^{t} d(x) f_{s}(y-x) d x
\end{aligned}
$$

Expressing products of integrals as double integrals yields

$$
\begin{aligned}
\Delta & =\int_{\substack{0 \leq z_{1} \leq t \\
t \leq z_{2} \leq y}} d\left(z_{1}\right) d\left(z_{2}\right) f_{s}^{\prime}\left(y-z_{1}\right) f_{s}\left(y-z_{2}\right) d z_{1} d z_{2} \\
& -\int_{\substack{0 \leq z_{1} \leq t \\
t \leq z_{2} \leq y}} d\left(z_{1}\right) d\left(z_{2}\right) f_{s}\left(y-z_{1}\right) f_{s}^{\prime}\left(y-z_{2}\right) d z_{1} d z_{2} \\
& =\int_{\substack{0 \leq z_{1} \leq t \\
t \leq z_{2} \leq y}} d\left(z_{1}\right) d\left(z_{2}\right)\left(f_{s}^{\prime}\left(y-z_{1}\right) f_{s}\left(y-z_{2}\right)-f_{s}\left(y-z_{1}\right) f_{s}^{\prime}\left(y-z_{2}\right)\right) d z_{1} d z_{2}
\end{aligned}
$$

Since $d\left(z_{1}\right) d\left(z_{2}\right)$ is positive and $\frac{f_{s}^{\prime}(t)}{f_{s}(t)}$ is decreasing, $\Delta \leq 0$ and the first part of the proposition is proven.
The second part of the proposition is equivalent to the inequalities

$$
F_{X_{s} \mid Y_{s}=y}(t) \geq F_{X_{s}, \mathcal{S}_{\rightarrow X_{s}}}(t)
$$

and

$$
F_{Y_{s} \mid X_{s+1}=x}(t) \leq F_{Y_{s}, \mathcal{S}_{\rightarrow Y_{s}}}(t)
$$

for all $0 \leq t \leq 1$.
From the first part of the Proposition, it suffices to prove the first inequality only for $y=1$. Since $f_{s}$ is increasing, there exists a measure $\mu$ on $[0,1]$ such that $f_{s}(x)=\int_{0}^{x} d \mu(u)$. Thus,

$$
F_{1}(t)=\frac{\int_{0}^{t} d(x)\left(\int_{0}^{1-x} d \mu(u)\right) d x}{\int_{0}^{1} d(x)\left(\int_{0}^{1-x} d \mu(u)\right) d x}=\frac{\int_{[0,1]^{2}} \mathbf{1}_{x \leq t, u \leq 1-x} d(x) d \mu(u) d x}{\int_{[0,1]^{2}} \mathbf{1}_{u \leq 1-x} d(x) d \mu(u) d x} .
$$

The main point is to express the latter quantity as the expectation of a random variable almost surely greater than $\int_{0}^{t} d(x) d x$. Changing the order of the integrals yields

$$
F_{1}(t)=\frac{\int_{0}^{1}\left(\int_{0}^{t \wedge(1-u)} d(x) d x\right) d \mu(u)}{\int_{0}^{1}\left(\int_{0}^{1-u} d(x) d x\right) d \mu(u)}=\frac{\int_{0}^{1}\left(\int_{0}^{t \wedge(1-u)} \frac{d(x)}{\left.\int_{0}^{1-u} d x\right) d x} d x\right)\left(\int_{0}^{1-u} d(x) d x\right) d \mu(u)}{\int_{0}^{1}\left(\int_{0}^{1-u} d(x) d x\right) d \mu(u)}
$$

Let $\tilde{U}$ be a random variable absolutely continuous with respect to $\mu$ and having the density

$$
d_{\tilde{U}}(u)=\frac{\left(\int_{0}^{1-u} d(x) d x\right) d \mu(u)}{\int_{0}^{1}\left(\int_{0}^{1-u} d(x) d x\right) d \mu(u)}
$$

Then

$$
F_{1}(t)=\mathbb{E}_{\tilde{U}}\left(\frac{\int_{0}^{t \wedge(1-\tilde{U})} d(x) d x}{\int_{0}^{1-\tilde{U}} d(x) d x}\right)
$$

Since for each $u \geq 0$

$$
\frac{\int_{0}^{t \wedge 1-u} d(x) d x}{\int_{0}^{1-u} d(x) d x} \geq \int_{0}^{t} d(x) d x
$$

this concludes the proof.
It is exactly the same for $F_{Y_{s} \mid X_{s+1}=x}(t)$.
Remark 8.19. In the case of a Sawtooth model $\mathcal{S}_{\lambda}$, a simpler proof of the monotonicity result of Proposition 8.18 can be done by induction on the length of the run of $\lambda$ between $x_{s}^{-}$and $x_{s}^{+}$. Namely, if the run has length 2,

$$
F_{X_{s} \mid Y_{s}=y}(t)=\frac{\int_{0}^{t \wedge y} d_{X_{s}, \mathcal{S}_{\lambda} \rightarrow X_{s}}(x) d x}{\int_{0}^{y} d_{X_{s}, \mathcal{S}_{\lambda} \rightarrow X_{s}}(x) d x},
$$

which is decreasing in $y$. If the run has length $r>2$, the expression of the density in the integral of (8.3.3) yields

$$
F_{X_{s} \mid Y_{s}=y}(t)=\frac{\int_{0}^{y} F_{\tilde{X}_{s}} \tilde{Y}_{s}=y^{\prime}}{}(t) d_{\tilde{Y}_{s}, \mathcal{S}_{\tilde{\lambda}} \rightarrow \tilde{Y}_{s}}\left(y^{\prime}\right) d y^{\prime},
$$

where $\tilde{\lambda}$ is the composition $\lambda$ with the run between $x_{s}^{-}$and $x_{s}^{+}$reduced to $r-1$, and $\tilde{X}_{s}$ and $\tilde{Y}_{s}$ correspond to the variables $x_{s}^{-}$and $x_{s}^{+}$in $\mathcal{S}_{\tilde{\lambda}}$. By recurrence hypothesis, $F_{\tilde{X}_{s} \mid \tilde{Y}_{s}=y^{\prime}}(t)$ is decreasing in $y^{\prime}$, and thus, $\frac{\int_{0}^{y} F_{\tilde{X}_{s}} \tilde{Y}_{s}=y^{\prime}(t) d_{\tilde{Y}_{s}}, s_{\overparen{\wedge}}+\tilde{Y}_{s}\left(y^{\prime}\right) d y^{\prime}}{\int_{0}^{y} d_{\tilde{Y}_{s}, S_{\tilde{\lambda}} \rightarrow \bar{Y}_{s}}\left(y^{\prime}\right) d y^{\prime}}$ is decreasing in $y$.

### 8.4.2 Alternating pattern of a convex sawtooth model

Proposition 8.18 yields two main features for the model. The first one is an extension of the previous result.

Proposition 8.20. Let $1 \leq s, 0 \leq t \leq 1$. Then for $r \geq s, F_{X_{s} \mid X_{r}=x}(t)$ is decreasing in $x$ and $F_{X_{s} \mid Y_{r}=y}(t)$ is decreasing in $y$. Likewise, $F_{X_{s} \mid X_{r}=0}(t)$ is decreasing in $r$ and $F_{X_{s} \mid Y_{r}=1}(t)$ is increasing in $r$.
Moreover,

$$
F_{X_{s}, \mathcal{S}_{\rightarrow X_{r}}}(t) \leq F_{X_{s} \mid Y_{r}=y}(t)
$$

and

$$
F_{X_{s}, \mathcal{S}_{\rightarrow Y_{r}}}(t) \geq F_{X_{s} \mid X_{r+1}=x}(t) .
$$

Proof. Let $s \geq 1$ and let us prove the monotonicty on $x$ and $y$ by recurrence on $r$, starting at $s=r . F_{X_{s} \mid X_{s}=x}(t)$ is clearly decreasing in $x$ and from Proposition 8.18, $F_{X_{s} \mid Y_{s}=y}(t)$ is decreasing in $y$. Thus, the initialization is done.
Suppose the result proved until $X_{r}$. Then

$$
F_{X_{s} \mid X_{r+1}=x}(t)=\int_{0}^{1} F_{X_{s} \mid Y_{r}=y, X_{r+1}=x}(t) d_{Y_{r} \mid X_{r+1}=x}(y) d y
$$

and by an integration by part, since from Lemma $8.13 F_{X_{s} \mid Y_{r}=y, X_{r+1}=x}(t)=F_{X_{s} \mid Y_{r}=y}(t)$,

$$
F_{X_{s} \mid X_{r+1}=x}(t)=F_{X_{s} \mid Y_{r}=1}(t)-\int_{0}^{1} \frac{\partial}{\partial y} F_{X_{s} \mid Y_{r}=y}(t) F_{Y_{r} \mid X_{r+1}=x}(y) d y .
$$

Thus,

$$
\frac{\partial}{\partial x} F_{X_{s} \mid X_{r+1}=x}(t)=-\int_{0}^{1} \frac{\partial}{\partial y} F_{X_{s} \mid Y_{r}=y}(t) \frac{\partial}{\partial x} F_{Y_{r} \mid X_{r+1}=x}(y) d y
$$

By recurrence $\frac{\partial}{\partial y} F_{X_{s} \mid Y_{r}=y}(t)$ is negative and by Proposition $8.18 \frac{\partial}{\partial x} F_{Y_{r} \mid X_{r+1}=x}(y)$ is negative, thus $\frac{\partial}{\partial x} F_{X_{s} \mid X_{r+1}=x}(t)$ is also negative. It is exactly the same for $F_{X_{s} \mid Y_{r+1}=y}(t)$.
Let $r \geq s . F_{X_{s} \mid X_{r+1}=0}(t)=\int_{0}^{1} F_{X_{s} \mid X_{r}=x}(t) d_{X_{r} \mid X_{r+1}=0}(x) d x$, thus by Proposition 8.18

$$
F_{X_{s} \mid X_{r+1}=0}(t) \leq \int_{0}^{1} F_{X_{s} \mid X_{r}=0}(t) d_{X_{r} \mid X_{r+1}=0}(x) d x \leq F_{X_{s} \mid X_{r}=0}(t) .
$$

The same proof holds to show that $F_{X_{s} \mid Y_{r}=1}(t)$ is increasing in $r$.
Let us prove the second part of the proposition and let $y \in[0,1]$. Conditioning $X_{s}$ on $X_{r}$ in $\mathcal{S}_{\rightarrow X_{r}}$ yields

$$
F_{X_{s}, \mathcal{S}_{\rightarrow X_{r}}}(t)=\mathbb{E}\left(F_{X_{s} \mid X_{r}=\tilde{X}_{r}}(t)\right),
$$

with $\tilde{X}_{r}$ following the law of $q_{r}$ in $\mathcal{S}_{\rightarrow X_{r}}$.
On one hand from the first part of the proposition, $F_{X_{s} \mid X_{r}=x}(t)$ is decreasing in $x$. On the other hand from Proposition 8.18, $\tilde{X}_{r}$ stochastically dominates $\left(X_{r} \mid Y_{r}=y\right)$. Thus, from Proposition 8.6

$$
F_{X_{s}, \mathcal{S}_{\rightarrow X_{r}}}(t)=\mathbb{E}\left(F_{X_{s} \mid X_{r}=\tilde{X}_{r}}(t)\right) \leq F_{X_{s} \mid Y_{r}=y}(t) .
$$

The same pattern proves the second inequality.
There is an immediate consequence of this Proposition on the behavior of $F_{X_{s}, \mathcal{S}_{\rightarrow X_{u}}}(t)$ with $u \geq s$.

Corollary 8.21. The following inequalities hold for $k \geq s$ :

$$
F_{X_{s}, \mathcal{S}_{\rightarrow X_{s}}}(t) \leq \cdots \leq F_{X_{s}, \mathcal{S}_{\rightarrow X_{u}}}(t) \leq \cdots \leq F_{X_{s}, \mathcal{S}_{\rightarrow Y_{u}}}(t) \cdots \leq F_{X_{s}, \mathcal{S}_{\rightarrow Y_{s}}}(t)
$$

Proof. The previous Proposition yields directly the following inequalities :

$$
F_{X_{s}, \mathcal{S}_{\rightarrow Y_{r}}}(t) \geq F_{X_{s} \mid Y_{r}=1}(t) \geq F_{X_{s}, \mathcal{S}_{\rightarrow X_{r}}}(t)
$$

Moreover,

$$
\begin{aligned}
F_{X_{s}, \mathcal{S}_{\rightarrow X_{u+1}}}(t) & =\int_{[0,1]} F_{X_{s} \mid Y_{u}=y}(t) d_{Y_{n}, \mathcal{S}_{\rightarrow X_{u+1}}}(y) d y \\
& \geq \int_{[0,1]} F_{X_{s}, \mathcal{S}_{\rightarrow X_{u}}}(t) d_{Y_{u}, \mathcal{S}_{\rightarrow X_{u+1}}}(y) d y \\
& \geq F_{X_{s}, \mathcal{S}_{\rightarrow X_{u}}}(t)
\end{aligned}
$$

the first inequality being due to Proposition 8.18. By symmetry between $X_{u}$ and $Y_{u}$ the general result holds.

### 8.4.3 Estimates on the behavior of extreme particles

As a second consequence of Proposition 8.18 we can get a more accurate estimate on the behavior of the first and last particles of $\mathcal{S}$. In particular, we can achieve a coupling of $\left(X_{I}, X_{F}\right)$ with two couples of random variables, which only depend on $f_{1}$ and $g_{n}$ and give some bounds on $\left(X_{I}, X_{F}\right)$ in the sense of the stochastic domination.
In this paragraph we will not assume that the first and last particles are lower ones, and deal with model of any type (refer to Remark 8.11 for the definition of the type of a model). Moreover, to describe the bounding random variables, we introduce two particular transforms $\Gamma^{+}$and $\Gamma^{-}$:

Definition 8.22. Let $f$ be a positive function on $[0,1]$. Then $\Gamma^{+}(f)$ and $\Gamma^{-}(f)$ are the functions defined on $[0,1]$ as :

$$
\Gamma^{-}(f)(t)=\frac{\int_{1-t}^{1} f(u) d u}{\int_{0}^{1} f(u) d u},
$$

and

$$
\Gamma^{+}(f)(t)=\frac{\int_{0}^{t} f(u) d u}{\int_{0}^{1} f(u) d u}
$$

Remark that $\Gamma^{-}(f)(t)$ (resp. $\Gamma^{+}(f)(t)$ ) is the cumulative distribution function of the random variable $1-Z$ (resp. $Z), Z$ being the random variable with density $\frac{f(x)}{\int_{0}^{1} f(x) d x}$.
Proposition 8.23. Let $\mathcal{S}$ be a convex Sawtooth model of type $\varepsilon$ with density functions $\left\{f_{i}, g_{i}\right\}_{1 \leq i \leq k}$ and at least four particles. There exists a probability space and two couples of random variables $\left(X_{+}, Y_{+}\right),\left(X_{-}, Y_{-}\right)$on it, such that :

- $\left(X_{-}, Y_{-}\right) \preceq_{\varepsilon}\left(X_{I}, X_{F}\right) \preceq_{\varepsilon}\left(X_{+}, Y_{+}\right)$.
- $X_{+}$and $Y_{+}$are independent with distribution function

$$
F_{X_{+}, Y_{+}}(s, t)=\Gamma^{\varepsilon_{1}}\left(f_{1}\right)(s) \Gamma^{\varepsilon_{2}}\left(g_{n}\right)(t)
$$

- $X_{-}$and $Y_{-}$are independent with distribution function

$$
F_{X_{-}, Y_{-}}(s, t)=\left(\Gamma^{\varepsilon_{1}} \circ \Gamma^{\varepsilon_{1}^{*}}\left(f_{1}\right)\right)(s)\left(\Gamma^{\varepsilon_{2}} \circ \Gamma^{\varepsilon_{2}^{*}}\left(g_{n}\right)\right)(t)
$$

with $-*=+$ and $+^{*}=-$.
Proof. We assume without loss of generality that each $f_{i}, g_{i}$ is normalized and, since the type of the Sawtooth model doesn't change the pattern of the proof, we assume that $\mathcal{S}$ is of type -- . On one hand the conditional law of $\left(X_{I}, X_{F}\right)$ given the value of $Y_{1}=y_{1}, Y_{k}=y_{k}$ has for cumulative distribution function :

$$
\begin{aligned}
F_{X_{I}, X_{F} \mid Y_{1}=y_{1}, Y_{k}=y_{k}}\left(t_{1}, t_{2}\right) & =\frac{\left(\int_{0}^{t_{1} \wedge y_{1}} f_{1}\left(y_{1}-x\right) d x\right)\left(\int_{0}^{t_{2} \wedge y_{k}} g_{k}\left(y_{k}-y\right) d y\right)}{\left(\int_{0}^{y_{1}} f_{1}(x) d x\right)\left(\int_{0}^{y_{k}} g_{k}(x) d x\right)} \\
& =F_{X_{I} \mid Y_{1}=y_{1}}\left(t_{1}\right) F_{X_{F} \mid Y_{k}=y_{k}}\left(t_{2}\right)
\end{aligned}
$$

This together with Proposition 8.18 gives the bound

$$
\begin{aligned}
F_{X_{I}, X_{F} \mid Y_{1}=y_{1}, Y_{k}=y_{k}}\left(t_{1}, t_{2}\right) & =F_{X_{I} \mid Y_{1}=y_{1}}\left(t_{1}\right) F_{X_{F} \mid Y_{k}=y_{k}}\left(t_{2}\right) \\
& \geq F_{X_{I} \mid Y_{1}=1}\left(t_{1}\right) F_{X_{F} \mid Y_{k}=1}\left(t_{2}\right) .
\end{aligned}
$$

Since

$$
F_{X_{I} \mid Y_{1}=1}\left(t_{1}\right) F_{X_{F} \mid Y_{k}=1}\left(t_{2}\right)=\left(1-F_{f_{1}}\left(1-t_{1}\right)\right)\left(1-F_{g_{k}}\left(1-t_{2}\right)\right)=\Gamma^{-}\left(f_{1}\right)\left(t_{1}\right) \Gamma^{-}\left(g_{k}\right)\left(t_{2}\right)
$$

this gives the upper part of the stochastic bound.
On the other hand, the density of $\left(Y_{1}, Y_{k}\right)$ conditioned on the value of $\left(X_{2}, X_{k}\right)$ is

$$
\begin{aligned}
& d_{Y_{1}, Y_{k} \mid X_{2}=x_{2}, X_{k}=x_{k}}\left(y_{1}, y_{k}\right) \\
= & \mathbf{1}_{y_{1} \geq x_{2}, y_{k} \geq x_{k}} \frac{\left(\int_{0}^{y_{1}} f_{1}\left(y_{1}-x\right) d x\right) g_{1}\left(y_{1}-x_{2}\right)}{\int_{x_{2}}^{1}\left(\int_{0}^{z} f_{1}(z-x) d x\right) g_{1}\left(z-x_{2}\right) d z} \frac{\left(\int_{0}^{y_{k}} g_{k}\left(y_{k}-x\right) d x\right) f_{k}\left(y_{k}-x_{k}\right)}{\int_{x_{k}}^{1}\left(\int_{0}^{z} g_{k}(z-x) d x\right) f_{k}\left(z-x_{k}\right) d z} \\
= & \mathbf{1}_{y_{1} \geq x_{2}, y_{k} \geq x_{k}} \frac{F_{f_{1}}\left(y_{1}\right) g_{1}\left(y_{1}-x_{2}\right)}{\int_{x_{2}}^{1} F_{f_{1}}(z) g_{1}\left(z-x_{2}\right) d z} \frac{F_{g_{k}}\left(y_{k}\right) f_{k}\left(y_{k}-x_{k}\right)}{\int_{x_{k}}^{1} F_{g_{k}}(z) f_{k}\left(z-x_{k}\right) d z} .
\end{aligned}
$$

Factorizing the latter density yields

$$
d_{Y_{1}, Y_{k} \mid X_{2}=x_{2}, X_{k}=x_{k}}\left(y_{1}, y_{k}\right)=d_{Y_{1} \mid X_{2}=x_{2}}\left(y_{1}\right) d_{Y_{k} \mid X_{k}=x_{k}}\left(y_{k}\right)
$$

Let us first consider $Y_{1}$. Recall that $g_{1}$ is an increasing $\mathcal{C}^{1}$ function. This means in particular that

$$
g_{1}(x)=\frac{1}{K} \int_{0}^{x} d \lambda(u)
$$

with $\lambda$ a probability measure on $[0,1]$ having eventually a dirac mass at 0 and then a continuous density function on $] 0,1]$. Thus, the density of $Y_{1}$ conditioned on the value of $X_{2}$ is

$$
d_{Y_{1} \mid X_{2}=x_{2}}\left(y_{1}\right)=\frac{1}{A} \mathbf{1}_{y_{1} \geq x_{2}} F_{f_{1}}\left(y_{1}\right) \int_{x_{2}}^{y_{1}} d \lambda\left(u-x_{2}\right),
$$

with $A$ a normalizing constant. Let $d_{u}$ be the density function defined for $0 \leq u \leq 1$ by

$$
d_{u}(y)=\frac{1}{A_{u}} \mathbf{1}_{y \geq u} F_{f_{1}}(y),
$$

with $A_{u}$ a normalizing constant depending on $u$ and let $F_{u}(t)$ be the associated cumulative distribution function. On one hand

$$
\begin{aligned}
F_{Y_{1} \mid X_{2}=x_{2}}(t) & =\frac{\int_{0}^{t} \mathbf{1}_{y_{1} \geq x_{2}} F_{f_{1}}\left(y_{1}\right) \int_{x_{2}}^{y_{1}} d \lambda\left(u-x_{2}\right) d y_{1}}{\int_{0}^{1} \mathbf{1}_{y_{1} \geq x_{2}} F_{f_{1}}\left(y_{1}\right) \int_{x_{2}}^{y_{1}} d \lambda\left(u-x_{2}\right) d y_{1}} \\
& =\frac{\int_{0}^{t} \int_{x_{2}}^{1} \mathbf{1}_{y_{1} \geq u} F_{f_{1}}\left(y_{1}\right) d \lambda\left(u-x_{2}\right) d y_{1}}{\int_{0}^{1} \int_{x_{2}}^{1} \mathbf{1}_{y_{1} \geq u} F_{f_{1}}\left(y_{1}\right) d \lambda\left(u-x_{2}\right) d y_{1}},
\end{aligned}
$$

and after changing the order of the integrals, since $F_{u}(1)=1$,

$$
\begin{aligned}
F_{Y_{1} \mid X_{2}=x_{2}}(t) & =\frac{\int_{x_{2}}^{1}\left(\int_{0}^{t} \mathbf{1}_{y_{1} \geq u} F_{f_{1}}\left(y_{1}\right) d y_{1}\right) d \lambda\left(u-x_{2}\right)}{\int_{x_{2}}^{1}\left(\int_{0}^{1} \mathbf{1}_{y_{1} \geq u} F_{f_{1}}\left(y_{1}\right) d y_{1}\right) d \lambda\left(u-x_{2}\right)} \\
& =\frac{\int_{x_{2}}^{1} A_{u} F_{u}(t) d \lambda\left(u-x_{2}\right)}{\int_{x_{2}}^{1} A_{u} d \lambda\left(u-x_{2}\right)} \\
& =\mathbb{E}_{\tilde{U}}\left(F_{\tilde{U}}(t)\right)
\end{aligned}
$$

with $\tilde{U}$ a random variable with law $d \tilde{U}(u)=\mathbf{1}_{u \geq x_{2}} \frac{A_{u} d \lambda\left(u-x_{2}\right)}{\int_{x_{2}}^{1} A_{u} d \lambda\left(u-x_{2}\right)}$.
On the other hand

$$
F_{u}(t)=\mathbf{1}_{t \geq u} \frac{\int_{u}^{t} F_{f_{1}}(u) d u}{\int_{u}^{1} F_{f_{1}}(u) d u}=\mathbf{1}_{t \geq u} \frac{\mathcal{F}_{f_{1}}(t)-\mathcal{F}_{f_{1}}(u)}{\mathcal{F}_{f_{1}}(1)-\mathcal{F}_{f_{1}}(u)}
$$

with $\mathcal{F}_{f_{1}}$ being the primitive of $F_{f_{1}}$ taking the value 0 at 0 . This yields

$$
\begin{aligned}
\frac{\partial}{\partial u} F_{u}(t) & =\frac{\partial}{\partial u}\left(\mathbf{1}_{u \leq t} \frac{\mathcal{F}_{f_{1}}(t)-\mathcal{F}_{f_{1}}(u)}{\mathcal{F}_{f_{1}}(1)-\mathcal{F}_{f_{1}}(u)}\right) \\
& =\mathbf{1}_{u \leq t} \frac{\partial}{\partial u}\left(\left(\mathcal{F}_{f_{1}}(t)-\mathcal{F}_{f_{1}}(1)\right) \frac{1}{\mathcal{F}_{f_{1}}(1)-\mathcal{F}_{f_{1}}(u)}+1\right) \\
& =\mathbf{1}_{u \leq t}\left(\mathcal{F}_{f_{1}}(t)-\mathcal{F}_{f_{1}}(1)\right) \frac{\partial}{\partial u}\left(\frac{1}{\mathcal{F}_{f_{1}}(1)-\mathcal{F}_{f_{1}}(u)}\right) \\
& =\mathbf{1}_{u \leq t}\left(\mathcal{F}_{f_{1}}(t)-\mathcal{F}_{f_{1}}(1)\right) \frac{F_{f_{1}}(u)}{\left(\mathcal{F}_{f_{1}}(1)-\mathcal{F}_{f_{1}}(u)\right)^{2}} \leq 0,
\end{aligned}
$$

and thus

$$
F_{u}(t) \leq F_{0}(t)=\frac{\mathcal{F}_{f_{1}}(t)}{\mathcal{F}_{f_{1}}(1)}
$$

Integrating with respect to $\tilde{U}$ yields

$$
F_{Y_{1} \mid X_{2}=x_{2}}(t)=\mathbb{E}_{\tilde{U}}\left(F_{\tilde{U}}(t)\right) \leq \mathbb{E}_{\tilde{U}}\left(F_{0}(t)\right)
$$

and finally, $F_{Y_{1} \mid X_{2}=x_{2}}(t) \leq \frac{\mathcal{F}_{f_{1}}(t)}{\mathcal{F}_{f_{1}}(1)}$. We can now integrate this inequality to get a bound on the
cumulative distribution function of $X_{I}$ conditioned on $X_{2}$ :

$$
\begin{aligned}
F_{X_{I} \mid X_{2}=x_{2}}(t) & =\int_{0}^{1} F_{X_{I} \mid Y_{1}=y}(t) d_{Y_{1} \mid X_{2}=x_{2}}(y) d y \\
& =F_{X_{I} \mid Y_{1}=1}(t)-\int_{0}^{1} \frac{\partial}{\partial y} F_{X_{I} \mid Y_{1}=y}(t) F_{Y_{1} \mid X_{2}=x_{2}}(y) d y \\
& \leq F_{X_{I} \mid Y_{1}=1}(t)-\int_{0}^{1} \frac{\partial}{\partial y} F_{X_{I} \mid Y_{1}=y}(t) \frac{\mathcal{F}_{f_{1}}(y)}{\mathcal{F}_{f_{1}(1)}} d y \\
& \leq \int_{0}^{1} F_{X_{I} \mid Y_{1}=y}(t) \frac{F_{f_{1}}(y)}{\mathcal{F}_{f_{1}(1)}} d y
\end{aligned}
$$

Note that the direction of the inequality on the third line is due to the negative sign of $\frac{\partial}{\partial y} F_{X_{I} \mid Y_{1}=y}(t)$. Since

$$
\begin{aligned}
\int_{0}^{1} F_{X_{I} \mid Y_{1}=y}(t) \frac{F_{f_{1}}(y)}{\mathcal{F}_{f_{1}(1)}} d y & =\int_{0}^{1} \frac{\int_{0}^{t \wedge y} f_{1}(y-u) d u}{F_{f_{1}}(y)} \frac{F_{f_{1}}(y)}{\mathcal{F}_{f_{1}}(1)} d y \\
& =\int_{0}^{t} \int_{u}^{1} \frac{f_{1}(y-u)}{\mathcal{F}_{f_{1}}(1)} d y d u \\
& =\frac{\int_{0}^{t} F_{f_{1}}(1-u) d u}{\mathcal{F}_{f_{1}}(1)}=\Gamma^{-}\left(F_{f_{1}}\right)(t)
\end{aligned}
$$

this yields the inequality

$$
F_{X_{I} \mid X_{2}=x_{2}}(t) \leq \Gamma^{-} \circ \Gamma^{+}\left(f_{1}\right)(t)
$$

Note that the latter inequality is valid even if the model has only three particles (see the next Corollary). Finally, since in our case there are at least four particles, $X_{F} \neq X_{2}$, and thus $F_{X_{I} \mid X_{2}=x_{2}, X_{F}=y}(t)=F_{X_{1} \mid X_{2}=x_{2}}(t)$. Therefore

$$
F_{X_{I} \mid X_{F}=y}(t) \leq \Gamma^{-} \circ \Gamma^{+}\left(f_{1}\right)(t)
$$

and by averaging on $y$,

$$
F_{X_{I}}(t) \leq \Gamma^{-} \circ \Gamma^{+}\left(f_{1}\right)(t)
$$

Doing the same with $X_{F}$ gives the bound :

$$
F_{X_{F}}(t) \leq \Gamma^{-} \circ \Gamma^{+}\left(g_{k}\right)(t)
$$

The result follows from Lemma 8.10.
Remark 8.24. The case of a Sawtooth model $\mathcal{S}_{\lambda}$ illustrates the pattern of the proof in the general case. Namely, suppose that $\lambda$ has a first run of length $r$ which is increasing. Then, conditioning the law of $x_{1}$ on the value of the first particle after the first peak (which is $x_{r+1}$ in this case) yields the formula:

$$
F_{x_{1} \mid x_{r+1}=z}(t)=\frac{\int_{0}^{t}\left(\int_{x \wedge z}^{1}(y-x)^{r} d y\right) d x}{\int_{0}^{1}\left(\int_{x \wedge z}^{1}(y-x)^{r} d y\right) d x}
$$

Computing the integral in the numerator and in the denominator yields

$$
\begin{equation*}
F_{x_{1} \mid x_{r+1}=z}(t)=\frac{\left[1-(1-t)^{r}\right]-\left[z^{r}-((t \vee z)-t)^{r}\right]}{1-z^{r}} \tag{8.4.1}
\end{equation*}
$$

By Proposition 8.18, $F_{x_{1} \mid x_{r+1}=1}(t) \leq F_{x_{1} \mid x_{r+1}=z}(t) \leq F_{x_{1} \mid x_{r+1}=0}(t)$ : therefore, the bounds are given by the cases $z=1$ and $z=0$. By Equation 8.4.1), $F_{x_{1} \mid x_{r+1}=0}(t)=1-(1-t)^{r}=$ $\Gamma^{-} \Gamma^{+}\left(\tilde{\gamma}_{r}\right)(t)$. Suppose that $z \geq t$ : rewriting the right hand side of 8.4.1 as $\frac{h(1)-h\left(z^{r}\right)}{1-z^{r}}$ with $h(x)=x-\left(x^{1 / r}-t\right)$ yields

$$
F_{x_{1} \mid x_{r+1}=1}(t)=h^{\prime}(1)=1-(1-t)^{r-1}=\Gamma^{-}\left(\gamma_{r}\right)(t)
$$

The proof of Proposition 8.23 is actually a generalization of the proof in the case $\mathcal{S}_{\lambda}$.
In particular, as a corollary of Proposition 8.23 (and as a corollary of the proof in the case $k=2$ ), the following result holds :

Corollary 8.25. Let $\mathcal{S}$ be a convex Sawtooth model of type $\varepsilon$ with density functions $\left\{f_{i}, g_{i}\right\}_{1 \leq i \leq k}$ There exists a couple of random variables $\left(Z^{(1)}, Z^{(2)}\right)$ such that for $y \in[0,1]$,

- $Z^{(1)} \preceq_{\varepsilon(1)}\left(X_{I} \mid X_{F}=y\right) \preceq_{\varepsilon(1)} Z^{(2)}$,
- The cumulative distribution function of $Z^{(2)}$ is :

$$
F_{Z^{(2)}}(t)=\Gamma^{\varepsilon(1)}\left(f_{1}\right)(t)
$$

- The cumulative distribution function of $Z^{(1)}$ is

$$
F_{Z^{(1)}}(t)=\Gamma^{\varepsilon(1)} \circ \Gamma^{\varepsilon(1)^{*}}\left(f_{1}\right)(t)
$$

Proof. For $k \geq 3$, the result is deduced from the latter Proposition. In the case $k=2$, the proof is exactly the same as in the latter Proposition, except that we only deal with the left case, and thus we don't need anymore the fact that $X_{2} \neq X_{F}$.

In the case of a composition $\lambda$ with first run of length $R+1$, the latter corollary yields that for $\mathcal{S}_{\lambda}$ :

$$
1-(1-t)^{R} \leq F_{X_{I}}(t) \leq 1-(1-t)^{R+1}
$$

if the first run is increasing, and

$$
t^{R+1} \leq F_{X_{I}}(t) \leq t^{R}
$$

if the first run is decreasing.

### 8.5 The independence theorem in a bounded Sawtooth Model

This section is devoted to the proof of the approximate independence of $X_{I}$ and $X_{F}$ when the number of particles grows whereas the repulsion forces remain bounded. In this section the Sawtooth model is assumed normalized.

### 8.5.1 Decorrelation principle and bounding Lemmas

Definition 8.26. Let $A>0$. A Sawtooth model $\mathcal{S}$ with density functions $\left\{f_{i}, g_{i}\right\}$ is bounded by $A$ if

$$
\max \left(\left\|f_{i}\right\|_{[0,1]},\left\|g_{i}\right\|_{[0,1]}\right) \leq A
$$

The purpose is to prove the following Theorem :

Theorem 8.27. Let $A>0$. For all $\varepsilon>0$ there exists $N_{A} \geq 0$ such that for any Sawtooth model $\mathcal{S}$ bounded by $A$ and with $2 k \geq N_{A}$ particles we have :

$$
\left\|d_{X_{I}, X_{F}}(x, y)-d_{X_{I}}(x) d_{X_{F}}(y)\right\|_{\infty} \leq \varepsilon
$$

The pattern of the proof is the following : conditioned on the fact that a particle $P$ - from now on called a splitting particle - is close to the boundary of the domain, the left part $\mathcal{S}_{\rightarrow P}$ and the right part $\mathcal{S}_{\leftarrow P}$ of the system are almost not correlated anymore (see Figure 8.4).


Figure 8.4: Decorrelation of the process
However, we may still not have independence if the law of $X_{I}$ and $X_{F}$ depends on which particle splits the system. Thus, we have to find a set of particles that is large enough, so that with probability close to one an element of this set is close to the boundary, and such that nonetheless conditioning on having any particle from this set close to the boundary yields the same law on $\left(X_{I}, X_{F}\right)$.
Let us first begin by bounding the density of $\left(X_{I}, X_{F}\right)$.
Lemma 8.28. Suppose that $\left\|f_{1}\right\|_{\infty} \leq A$ and let $\mathcal{S}$ be a Sawtooth model larger than 2 . Then there exists $K_{A}$ only depending on $A$ such that for any event $\mathcal{X}$ depending on $\left\{X_{i}, Y_{i}\right\}_{i \geq 2}$,

$$
\left\|d_{X_{I} \mid \mathcal{X}}\right\|_{\infty} \leq K_{A}
$$

More precisely $K_{A}=4 A^{2}$ fits.
This Lemma was already mentioned in the specific context of compositions in [18]. We provide here a different proof.

Proof. By Lemma 8.13, it suffices to prove it for a conditioning on $\left\{X_{2}=x_{2}\right\}$. From Lemma $8.14 d_{X_{I} \mid X_{2}=x_{2}}(x)$ is decreasing in $x$ and thus it is enough to bound $d_{X_{I} \mid X_{2}=x_{2}}(0)$. We have

$$
d_{X_{I} \mid X_{2}=x_{2}}(0)=\frac{\int_{x_{2}}^{1} f_{1}(z) g_{1}\left(z-x_{2}\right) d z}{\int_{x_{2}}^{1} F_{f_{1}}(z) g_{1}\left(z-x_{2}\right) d z} \leq A \frac{\int_{x_{2}}^{1} g_{1}\left(z-x_{2}\right) d z}{\int_{x_{2}}^{1} F_{f_{1}}(z) g_{1}\left(z-x_{2}\right) d z}
$$

Remark that

$$
\frac{\int_{x_{2}}^{1} g_{1}\left(z-x_{2}\right) d z}{\int_{x_{2}}^{1} F_{f_{1}}(z) g_{1}\left(z-x_{2}\right) d z}=\frac{1}{\mathbb{E}_{\tilde{Z}}\left(F_{f_{1}}(\tilde{Z})\right)}
$$

with $\tilde{Z}$ being a random variable with density $\mathbf{1}_{z \geq x_{2}} g_{1}\left(z-x_{2}\right)$. Since $\left\|F_{f_{1}}^{\prime}\right\| \leq A$ and $F_{f_{1}}(1)=1$, $F_{f_{1}}(t) \geq 1 / 2$ on $[1-1 /(2 A)]$; moreover, $z \mapsto g_{1}\left(z-x_{2}\right)$ is increasing, thus $\mathbb{P}(\tilde{Z} \in[1-1 /(2 A), 1]) \geq$ $\frac{1}{2 A}$ and by Markov's inequality $\mathbb{E}_{\tilde{Z}}\left(F_{f_{1}}(\tilde{Z})\right) \geq 1 / 4 A$. Finally,

$$
d_{X_{I} \mid X_{2}=x_{2}}(0) \leq 4 A^{2} .
$$

The next step is to get a bound on the first derivative of $d_{X_{I}}$. This is possible only if $g_{1}$ is also bounded by $A$ and the model is large enough.

Lemma 8.29. Suppose that $\max \left(\left\|f_{1}\right\|_{\infty},\left\|g_{1}\right\|_{\infty}\right) \leq A$ and that $\mathcal{S}$ is a Sawtooth model with at least four particles. Then there exists a constant $R_{A}$ only depending on $A$ such that for any event $\mathcal{X}$ depending on $\left\{X_{i+1}, Y_{i}\right\}_{i \geq 2}$,

$$
\left\|\left(d_{X_{I} \mid \mathcal{X}}\right)^{\prime}\right\|_{\infty} \leq R_{A} .
$$

Proof. For exactly the same reasons as in the previous proof, it suffices to bound the derivative of the density conditioned on $\mathcal{X}=\left\{Y_{2}=y_{2}\right\}$. The expression of the density probability yields

$$
d_{X_{I} \mid Y_{2}=y_{2}}(x)=\frac{\int_{x}^{1} f_{1}\left(y_{1}-x\right) d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1}}{\int_{0}^{1}\left(\int_{x}^{1} f_{1}\left(y_{1}-x\right) d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1}\right) d x} .
$$

Let $\Delta=\int_{0}^{1}\left(\int_{x}^{1} f_{1}\left(y_{1}-x\right) d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1}\right) d x$, which is independent of $x$. Then

$$
\begin{aligned}
\left|\frac{\partial}{\partial x} d_{X_{I} \mid Y_{2}=y_{2}}(x)\right| & =\frac{1}{\Delta}\left|\frac{\partial}{\partial x} \int_{x}^{1} f_{1}\left(y_{1}-x\right) d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1}\right| \\
& =\frac{1}{\Delta} \left\lvert\, \int_{x}^{1}\left(\frac{\partial}{\partial x} f_{1}\left(y_{1}-x\right)\right) d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1}-f_{1}(0) d_{Y_{1}, S_{Y_{2} \leftarrow \mid Y_{2}=y_{2}}(x) \mid}\right. \\
& \leq \frac{1}{\Delta}\left(\left|\int_{x}^{1}-\left(\frac{\partial}{\partial x} f_{1}\right)\left(y_{1}-x\right) d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1}\right|+\left|f_{1}(0) d_{Y_{1} \mid Y_{2}=y_{2}}(x)\right|\right),
\end{aligned}
$$

Let us first bound the numerator. By the expression of the density of $Y_{1}$ conditioned on $Y_{2}=y_{2}$,

$$
d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right)=\frac{F_{f_{1}}\left(y_{1}\right) d_{Y_{1}, \mathcal{S}_{Y_{1} \leftarrow \mid} \mid Y_{2}=y_{2}}\left(y_{1}\right)}{\mathbb{E}_{\tilde{Y}_{1}}\left(F_{f_{1}}\left(\tilde{Y}_{1}\right)\right)},
$$

with $\tilde{Y}_{1}$ having the density $d_{Y_{1}, \mathcal{S}_{Y_{1} \leftarrow \mid Y_{2}=y_{2}}}$. Since $g_{1}$ is bounded by $A$, from Lemma 8.28 . $\left|d_{Y_{1}, \mathcal{S}_{Y_{1} \leftarrow} \leftarrow \mid Y_{2}=y_{2}}\right| \leq K_{A}$. From Lemma 8.14 . $d_{Y_{1}, \mathcal{S}_{Y_{1} \leftarrow} \mid Y_{2}=y_{2}}(y)$ is increasing in $y$, and $\left|F_{f_{1} \mid}^{\prime}\right| \leq A$, thus $\mathbb{E}_{\tilde{Y}_{1}}\left(F_{f_{1}}\left(\tilde{Y}_{1}\right)\right) \geq \frac{1}{4 A^{2}}$ and

$$
\left|f_{1}(0) d_{Y_{1}, \mathcal{S}_{Y_{2} \leftarrow \mid} \mid Y_{2}=y_{2}}(x)\right| \leq 4 A^{2} K_{A}^{2} .
$$

Let us bound also the first term of the sum: $f_{1}$ being increasing, $\frac{\partial}{\partial x} f_{1}\left(y_{1}-x\right) \leq 0$ and we can thus remove the absolute value in this first term. An other application of Lemma 8.28 yields:

$$
\begin{aligned}
\int_{x}^{1}-\left(\frac{\partial}{\partial x} f_{1}\right)\left(y_{2}-x\right) d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1} & \leq K_{A}\left(\int_{x}^{1}\left(\frac{\partial}{\partial x} f_{1}\right)\left(y_{2}-x\right) d y_{2}\right) \\
& \leq K_{A}\left(\left(f_{1}(1-x)-f_{1}(0)\right) \leq A \times K_{A}\right.
\end{aligned}
$$

The numerator is thus bounded by $A K_{A}+4 A^{2} K_{A}^{2}$.
Changing the order of the integrals in $\Delta$ yields :

$$
\Delta=\int_{0}^{1} F_{f_{1}}\left(y_{1}\right) d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1}
$$

Since $F_{f_{1}}^{\prime}$ is bounded by $A$ and $F_{f_{1}}(1)=1$, we can conclude as in the previous proof that $F_{f_{1}}(t) \geq \frac{1}{2 A}$ on $[1-1 /(2 A), 1]$. Moreover, $Y_{1}$ is an upper particle, and thus by Lemma 8.14. $d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right)$ is increasing in $y_{2}$. Since $\int_{[0,1]} d_{Y_{1} \mid Y_{2}=y_{2}}=1$, this implies that

$$
\int_{1-1 /(2 A)}^{1} d_{Y_{1} \mid Y_{2}=y_{2}}\left(y_{1}\right) d y_{1} \geq \frac{1}{2 A},
$$

and yields $\Delta \geq \frac{1}{4 A^{2}}$. The bounds on the numerator and on $\Delta$ yield :

$$
\left|\frac{\partial}{\partial x} d_{X_{I} \mid Y_{2}=y_{2}}(x)\right| \leq 4 A^{3}\left(K_{A}+4 A K_{A}^{2}\right)
$$

As an application of Lemma 8.29, we can also prove that $y \mapsto F_{X_{I} \mid X_{F}=y}(t)$ is Lipchitz :
Proposition 8.30. Let $\mathcal{S}$ be a Sawtooth model with $k \geq 3$ lower particles. Suppose that $\left\{f_{1}, g_{1}, f_{k}, g_{k}\right\}$ are bounded by $A>0$. Let $R_{A}$ be the constant of Lemma 8.29 (with $R_{A} \geq 1$ ). Then on a neighbourhood $\left[0,1 / R_{A}\right]$ of 0 ,

$$
\mathcal{F}:\left\{\begin{array}{ccc}
{\left[0,1 / R_{A}\right]} & \rightarrow & (\mathcal{C}([0,1], \mathbb{R}),\|\cdot\|) \\
y & \mapsto & F_{X_{I} \mid X_{F}=y}
\end{array}\right.
$$

is Lipschitz with a Lipschitz constant $B_{A}$ only depending on $A$.
Proof. It suffices to prove that for $x \in[0,1], y \mapsto d_{X_{I} \mid X_{F}=y}(x)$ is Lipschitz on $\left[0,1 / R_{A}\right]$ with a Lipschitz constant independent of $x$.
From Lemma 8.14, $d_{X_{F}}$ is decreasing and thus on $\left[1 / R_{A}, 1\right], d_{X_{F}} \leq d_{X_{F}}\left(1 / R_{A}\right)$. From Lemma 8.29. $\left|\frac{\partial}{\partial y} d_{X_{F}}(y)\right| \leq R_{A}$ and thus on $\left[0,1 / R_{A}\right], d_{X_{F}}(y) \leq d_{X_{F}}\left(1 / R_{A}\right)+R_{A}\left(1 / R_{A}-y\right)$. This implies that

$$
\begin{aligned}
\int_{[0,1]} d_{X_{F}}(y) d y & \leq \int_{0}^{1 / R_{A}} d_{X_{F}}\left(1 / R_{A}\right)+R_{A}\left(1 / R_{A}-y\right) d y+\int_{1 / R_{A}}^{1} d_{X_{F}}\left(1 / R_{A}\right) \\
& \leq d_{X_{F}}\left(1 / R_{A}\right)+\frac{1}{2 R_{A}}
\end{aligned}
$$

Since $\int_{[0,1]} d_{X_{F}}=1$, this implies that $d_{X_{F}}\left(1 / R_{A}\right) \geq 1-\frac{1}{2 R_{A}}$, and thus that $d_{X_{F}} \geq 1-\frac{1}{2 R_{A}}$ on $\left[0,1 / R_{A}\right]$.
From Lemma $8.29,\left\|\frac{\partial}{\partial y} d_{X_{F} \mid X_{I}=x}\right\| \leq R_{A}$. Thus, since $\left\|f_{1}\right\| \leq A$, this yields by applying Lemma 8.28 on $d_{X_{I}, X_{F}}(x, y)=d_{X_{F} \mid X_{I}=x}(y) d_{X_{I}}(x)$ :

$$
\left|\frac{\partial}{\partial y} d_{X_{I}, X_{F}}(x, y)\right| \leq K_{A} R_{A} .
$$

Thus, on $\left[0,1 / R_{A}\right]$,

$$
\begin{aligned}
\left|\frac{\partial}{\partial y} d_{X_{I} \mid X_{F}=y}(x)\right| & =\frac{1}{d_{X_{F}}(y)}\left|\frac{\partial}{\partial y} d_{X_{I}, X_{F}}(x, y)-\frac{d_{X_{I}, X_{F}}(x, y) \frac{\partial}{\partial y} d_{X_{F}}(y)}{d_{X_{F}}(y)}\right| \\
& \leq \frac{1}{1-1 /\left(2 R_{A}\right)}\left(K_{A} R_{A}+\frac{R_{A} K_{A}^{2}}{1-1 /\left(2 R_{A}\right)}\right) .
\end{aligned}
$$

Set $B_{A}=\frac{1}{1-1 /\left(2 R_{A}\right)}\left(K_{A} R_{A}+\frac{R_{A} K_{A}^{2}}{1-1 /\left(2 R_{A}\right)}\right)$. Then $\mathcal{F}$ is $B_{A}-$ Lipschitz on $\left[0,1 / R_{A}\right]$.

### 8.5.2 Behavior of $\left\{X_{i}\right\}$ for large models

The purpose of this subsection is to find for a model $\mathcal{S}$ a large set of intermediate particles $\left\{X_{r}\right\}$ for which one of these particles is close to 0 with high probability and such that $F_{X_{I} \mid X_{r}=0}$ is essentially the same for all particles of this set.
The first part is a essentially probability computation :
Proposition 8.31. Let $\eta>0, \varepsilon>0$. There exists $N_{0}$ such that for any model $\mathcal{S}$ of size $N$ larger than $N_{0}+4$ and for any $2 \leq r \leq N-N_{0}, y_{r+N_{0}} \in[0,1]$,

$$
\mathbb{P}\left(\bigcup_{r \leq i \leq r+N_{0}}\left\{X_{i}<\eta\right\} \mid Y_{r+N_{0}}=y_{r+N_{0}}\right) \geq 1-\varepsilon .
$$

Proof. Let $N_{0}$ be an integer to specify later and let $\mathcal{S}, r$ be as in the statement of the Proposition. Let $\tilde{P}=\mathbb{P}\left(\bigcap_{r \leq i \leq r+N_{0}}\left\{X_{i} \geq \eta\right\} \mid Y_{r+N_{0}}=y_{r+N_{0}}\right)$.
Let $0 \leq y_{r-1}, \ldots, y_{r+N_{0}} \leq 1$ and condition $\left(\bigcap_{r \leq i \leq r+N_{0}}\left\{X_{i} \geq \eta\right\} \mid Y_{r+N_{0}}=y_{r+N_{0}}\right)$ on the event $\bigcap_{r-1 \leq i<r+N_{0}-1}\left\{Y_{i}=y_{i}\right\}$. We denote by $P_{\vec{y}}$ the probability of this conditioned event. By Lemma 8.13, the random variables $\left\{X_{i}\right\}_{r \leq i \leq r+N_{0}}$ are conditionally independent given the value of $\left\{Y_{i}\right\}_{r-1 \leq i \leq N_{0}}$; therefore,

$$
P_{\vec{y}}=\prod_{i=r}^{r+N_{0}} \mathbb{P}\left(X_{i} \geq \eta \mid Y_{i-1}=y_{i-1}, Y_{i}=y_{i}\right) .
$$

Moreover, Lemma 8.14 yields that $d_{X_{i} \mid Y_{i-1}=y_{i-1}, Y_{i}=y_{i}}$ is decreasing: thus, $\mathbb{P}\left(X_{i} \geq \eta \mid Y_{i-1}=\right.$ $\left.y_{i-1}, Y_{i}=y_{i}\right) \leq(1-\eta)$. This yields

$$
P_{\vec{y}} \leq(1-\eta)^{N_{0}+1} .
$$

Integrating $P_{\vec{y}}$ with respect to $y_{r-1}, \ldots, y_{N_{0}-1}$ gives $\tilde{P} \leq(1-\eta)^{N_{0}+1}$. Let $N_{0}$ be such that $(1-\eta)^{N_{0}+1} \leq \varepsilon$. For $N \geq N_{0}$,

$$
\mathbb{P}\left(\bigcup_{r \leq i \leq r+N_{0}}\left\{X_{i}<\eta\right\} \mid Y_{r+N_{0}}=y_{r+N_{0}}\right) \geq 1-\varepsilon .
$$

As said before, it is also necessary that $F_{X_{I} \mid X_{r}=0}$ remains almost constant among this subset of particles. This is possible for large Sawtooth models, thank to the monotonicity results of Proposition 8.20:

Proposition 8.32. Let $A, \varepsilon>0, M \in \mathbb{N}^{*}$. There exists $N_{\varepsilon, A, M}$ such that for any Sawtooth model bounded by $A$ and of size $N \geq N_{\varepsilon, A, M}$, there exists $1 \leq r \leq N-M$ such that for $r \leq i, j \leq r+M$,

$$
\left\|F_{X_{I} \mid X_{i}=0}-F_{X_{I} \mid X_{j}=0}\right\|_{\infty} \leq \varepsilon .
$$

Proof. Let $\mathcal{S}$ be a Sawtooth model bounded by $A$ and of size $N$.
Denote by $F_{i}$ the function $t \mapsto F_{X_{I} \mid X_{i}=0}(t)$ for $2 \leq i \leq N$. By Lemma 8.28, all the $F_{i}$ are $K_{A}$-Lipschitz. Let $K=\left\lfloor\frac{2 K_{A}}{\varepsilon}\right\rfloor$. It suffices to find $r \geq 2$ such that for all $r \leq i, j \leq r+M$, and all $0 \leq k \leq K$,

$$
\left|F_{i}\left(\frac{k}{K}\right)-F_{j}\left(\frac{k}{K}\right)\right| \leq \frac{\varepsilon}{3} .
$$

Denote by $v_{i} \in[0,1]^{K+1}$ the vector $\left(F_{i}\left(\frac{k}{K}\right)\right)_{0 \leq k \leq K}$ and let $N_{\varepsilon, A, M}=(M+1)\left(\left\lfloor\frac{3}{\varepsilon}\right\rfloor+1\right)^{K+1}$. Suppose that $N \geq N_{\varepsilon, A, M}$. For $\vec{m} \in \llbracket 0,\left\lfloor\frac{3}{\varepsilon}\right\rfloor \rrbracket^{K+1}$, denote by $C_{\vec{m}}$ the hypercube $\left\{\vec{x} \in[0,1]^{K+1} \mid \forall 1 \leq\right.$
$\left.i \leq K+1, m_{i} \frac{\varepsilon}{3} \leq x_{i}<\left(m_{i}+1\right) \frac{\varepsilon}{3}\right\} .\left\{C_{\vec{m}}\right\}_{\vec{m} \in \llbracket 0,\left\lfloor\frac{3}{\varepsilon}\right\rfloor \rrbracket^{K+1}}$ is a partition of $[0,1]^{K+1}$ in $\left(\left\lfloor\frac{3}{\varepsilon}\right\rfloor+1\right)^{K+1}$ subsets. If $v_{i}$ and $v_{j}$ are both in a same $C_{\vec{m}}$, then for all $0 \leq k \leq K,\left|v_{i}(k)-v_{j}(k)\right| \leq \frac{\varepsilon}{3}$.
Since $N \geq(M+1)\left(\left\lfloor\frac{3}{\varepsilon}\right\rfloor+1\right)^{K+1}$, Dirichlet's principle yields the existence of $\vec{m}_{0} \in \llbracket 0,\left\lfloor\frac{3}{\varepsilon}\right\rfloor \rrbracket^{K+1}$ such that $\#\left(\left\{v_{i}\right\}_{1 \leq i \leq N} \cap C_{\vec{m}_{0}}\right) \geq M+1$. Let $i_{0}<\cdots<i_{M}$ be such that for all $0 \leq j \leq M$, $v_{i_{j}} \in C_{\vec{m}_{0}}$; in particular, $i_{M} \geq i_{0}+M$. From the previous paragraph, for all $0 \leq k \leq K$, $\left|v_{i_{M}}(k)-v_{i_{0}}(k)\right| \leq \frac{\varepsilon}{3}$. By Proposition $8.20, F_{i}\left(\frac{k}{K}\right)$ is decreasing in $i$; thus, since $v_{i}(k)=F_{i}\left(\frac{k}{K}\right)$, for all $i_{0} \leq j \leq i_{M}$ and all $0 \leq k \leq K$

$$
v_{i_{0}}(k) \geq v_{i}(k) \geq v_{i_{M}}(k)
$$

Since $i_{0}+M \leq i_{M}$, this yields $\left\|v_{i}-v_{j}\right\|_{\infty} \leq \frac{\varepsilon}{3}$ for $i_{0} \leq i, j \leq i_{0}+M$.

### 8.5.3 Proof of Theorem 8.27

Theorem 8.27 is a consequence of the following proposition :
Proposition 8.33. Let $A>0$. For all $\varepsilon>0$, there exists a number $N_{A, \varepsilon} \geq 0$ such that for any Sawtooth model $\mathcal{S}$ bounded by $A$ and with $2 k \geq N_{A, \varepsilon}$ particles, the following inequality holds:

$$
\left|F_{X_{I} \mid X_{F}=y}(t)-F_{X_{I}}(t)\right| \leq \varepsilon
$$

for all $t, y \in[0,1]$.
Proof. Set $\eta=\inf \left(\frac{1}{R_{A}}, \frac{\varepsilon}{B_{A}}\right)$ with $R_{A}, B_{A}$ the constants given respectively by Lemma 8.29 and Proposition 8.30. Let $N_{0}$ be the constant given for $\eta$ and $\varepsilon$ by Proposition 8.31. Finally, set $N_{A, \varepsilon}=N_{\varepsilon / 4, A, N_{0}}+4$ given by Proposition 8.32 .
Let $\mathcal{S}$ be a Sawtooth model bounded by $A$ of size larger than $N_{A, \varepsilon}$. Then by Proposition 8.32, there exists $2 \leq r \leq N_{A, \varepsilon}-2-N_{0}$ such that for all $r \leq i, j \leq r+N_{0}$,

$$
\left\|F_{X_{I} \mid X_{i}=0}-F_{X_{I} \mid X_{j}=0}\right\|_{\infty} \leq \varepsilon .
$$

Denote $t=r+N_{0}$ and let $y_{t} \in[0,1]$. For $r \leq i \leq r+N_{0}$, set $L_{i}=\left\{X_{i} \leq \eta \cap\left\{\forall s>i, X_{s}>\eta\right\}\right\}$. Note that $L_{i} \cap L_{j}=\emptyset$ for all $i \neq j$ and $\bigcup L_{i}=L$ with $L=\bigcup_{r \leq i \leq r+N_{0}}\left\{X_{i} \leq \eta\right\}$. Moreover, since $L_{i}$ is $\left(X_{s}, Y_{s}\right)_{s \geq i}-$ measurable, by Lemma 8.13, conditioning $\bar{X}_{I}$ on $\left\{X_{i}=u, Y_{t}=y_{t}\right\} \cap L_{i}$ is the same as conditioning $X_{I}$ on $\left\{X_{i}=u\right\}$. Thus,

$$
\begin{aligned}
\left\|F_{X_{I} \mid L_{i}, Y_{t}=y_{t}}-F_{X_{I} \mid X_{r}=0}\right\|_{\infty} & =\left\|\int_{0}^{\eta}\left(F_{X_{I} \mid X_{i}=u}-F_{X_{I} \mid X_{r}=0}\right) d_{X_{i} \mid L_{i}, Y_{t}=y_{t}}(u) d u\right\|_{\infty} \\
& \leq \int_{0}^{\eta}\left\|F_{X_{I} \mid X_{i}=u}-F_{X_{I} \mid X_{r}=0}\right\|_{\infty} d_{X_{i} \mid L_{i}, Y_{t}=y_{t}}(u) d u \\
& \leq 2 \varepsilon
\end{aligned}
$$

by the choice of $\eta$. Recall that if $A=\bigcup A_{i}$, with $A_{i}$ disjoint events, then for any event $C$,

$$
\mathbb{P}(C \mid A)=\sum \mathbb{P}\left(C \mid A_{i}\right) \mathbb{P}\left(A_{i} \mid A\right)
$$

In particular, for $L=\bigcup_{i} L_{i}$ this yields

$$
\begin{aligned}
\left\|F_{X_{I} \mid L, Y_{t}=y_{t}}-F_{X_{I} \mid X_{r}=0}\right\| & =\left\|\sum_{i}\left(F_{X_{I} \mid L_{i}, Y_{t}=y_{t}}-F_{X_{I} \mid X_{r}=0}\right) \mathbb{P}\left(L_{i} \mid L, Y_{t}=y_{t}\right)\right\|_{\infty} \\
& \leq \sum_{i} \|\left(F_{X_{I} \mid L_{i}, Y_{t}=y_{t}}-F_{X_{I} \mid X_{r}=0} \|_{\infty} \mathbb{P}\left(L_{i} \mid L, Y_{t}=y_{t}\right)\right. \\
& \leq 2 \varepsilon .
\end{aligned}
$$

By Proposition 8.31 and the choice of $N_{0}, \mathbb{P}\left(L \mid Y_{t}=y_{t}\right) \geq 1-\varepsilon$, and thus

$$
\left\|F_{X_{I} \mid Y_{t}=y_{t}}-F_{X_{I} \mid X_{r}=0}\right\|_{\infty} \leq 3 \varepsilon
$$

By averaging on $y_{t}$ with the density $d_{Y_{t} \mid X_{F}=y}$ we get

$$
\left\|F_{X_{I} \mid X_{F}=y}-F_{X_{I}}\right\|_{\infty} \leq 4 \varepsilon
$$

Let us end the proof of the Theorem 8.27, which consists essentially in a rewriting in terms of densities of the latter Proposition.

Proof. Let $A>0, \varepsilon>0$. Set $\varepsilon_{1}=\frac{\left(\varepsilon / K_{A)}^{2}\right.}{4 R_{A}}$ and let $\mathcal{S}$ be a Sawtooth model bounded by $A$ of size larger than $N_{A, \varepsilon_{1}}\left(N_{A, \varepsilon_{1}}\right.$ being given by Proposition 8.33). Then from Proposition 8.33, for $y \in[0,1]$,

$$
\begin{equation*}
\left\|F_{X_{I} \mid X_{F}=y}-F_{X_{I}}\right\|_{\infty} \leq \frac{\left(\varepsilon / K_{A}\right)^{2}}{4 R_{A}} \tag{8.5.1}
\end{equation*}
$$

Moreover, the following result holds for $C^{1}$-functions on $[0,1]$ :
Lemma 8.34. Let $f, g:[0,1] \rightarrow[0,1]$ be two $C^{1}-$ functions, such that $\left\|f^{\prime}\right\|_{\infty},\left\|g^{\prime}\right\|_{\infty} \leq M$. Then for $\varepsilon>0$ small enough, if $F, G$ are two primitives of $f, g$ and

$$
\|F-G\|_{\infty} \leq \frac{\varepsilon^{2}}{4 M}
$$

then $\|f-g\|_{\infty} \leq \varepsilon$.

Proof. This is implied by proving that if $f:[0,1] \longrightarrow \mathbb{R}$ verifies $\|f\|_{\infty} \leq \frac{\varepsilon^{2}}{4 M}$ and $\left\|f^{\prime \prime}\right\|_{\infty} \leq M$, then $\left\|f^{\prime}\right\|_{\infty} \leq \varepsilon$. But the majoration on $f^{\prime \prime}$ yields that if $\left|f^{\prime}(x)\right| \geq \varepsilon$,

$$
\max \left(\left|\int_{x}^{x+\varepsilon / M} f^{\prime}(x) d x\right|,\left|\int_{x-\varepsilon / M}^{x} f^{\prime}(x) d x\right|\right) \geq \frac{\varepsilon^{2}}{2 M}
$$

Thus,

$$
\max (|f(x+\varepsilon / M)|,|f(x)|,|f(x-\varepsilon / m)|) \geq \frac{\varepsilon^{2}}{4 M}
$$

Applying this Lemma to 8.5.1 yields for $y \in[0,1]$,

$$
\left\|d_{X_{I} \mid X_{F}=y}-d_{X_{i}}\right\|_{\infty} \leq \varepsilon / K_{A} .
$$

Finally,

$$
\left.\left|d_{X_{I}, X_{F}}(x, y)-d_{X_{I}}(x) d_{X_{F}}(y)\right|=\left|d_{X_{F}}(y) \|\right| d_{X_{I} \mid X_{F}=y}(x)-d_{X_{I}}(x)\right) \left\lvert\, \leq K_{A} \frac{\varepsilon}{K_{A}} \leq \varepsilon\right.
$$

### 8.6 Application to compositions

Theorem 8.27 can be applied to the framework of compositions :
Corollary 8.35. Let $A \geq 0, \varepsilon>0$. There exists $n \geq 0$ such that for any composition $\lambda$ of size larger than $n$ with every runs bounded by $A$,

$$
\left\|d_{\mathcal{S}_{\lambda}}(x, y)-d_{\mathcal{S}_{\lambda}}(x) d_{\mathcal{S}_{\lambda}}(y)\right\|<\varepsilon
$$

Proof. Each run of $\lambda$ of length $l$ yields a density function $\gamma_{l}$ in $\mathcal{S}_{\lambda}$, and $\left\|\gamma_{l}\right\|_{\infty}=l-1$. Thus, if any run of $\lambda$ is bounded by $A$, then all the density functions $\left\{f_{i}, g_{i}\right\}$ in $\mathcal{S}_{\lambda}$ are bounded by $A-1$. It suffices then to apply Theorem 8.27.

The purpose of this section is to strengthen Corollary 8.35 and to prove the following Theorem :

Theorem 8.36. Let $\varepsilon>0, A \geq 0$. There exists $n \geq 0$ such that for any composition $\lambda$ of size larger than $n$ with first and last run bounded by $A$,

$$
\begin{equation*}
\left\|d_{\mathcal{S}_{\lambda}}(x, y)-d_{\mathcal{S}_{\lambda}}(x) d_{\mathcal{S}_{\lambda}}(y)\right\|<\varepsilon . \tag{8.6.1}
\end{equation*}
$$

This Theorem was Conjecture 1 in [18]. The proof of Theorem 8.36 is followed by some applications.

### 8.6.1 Effect of a large run on the law of $\left(X_{I}, X_{F}\right)$

From Corollary 8.35, it is enough to prove that the presence of a large run inside the composition disconnects the behaviors of $X_{I}$ and $X_{F}$. The main reason for this is the Lemma below: for each composition $\lambda$, denote by $\lambda^{+}$the composition $\lambda$ with a cell added on the last run, and by $\lambda^{-}$the composition $\lambda$ with a cell removed on the last run.

Lemma 8.37. Let $A>0$ and let $\lambda$ be a composition with more than three runs and with the first run smaller than $A$. If the last run of $\lambda$ is of size $R$,

$$
\left\|d_{X_{I}, \mathcal{S}_{\lambda}}-d_{X_{I}, \mathcal{S}_{\lambda}}\right\|_{\infty} \leq \frac{K_{A}}{R-1}
$$

where $K_{A}$ is the bound on the density of $X_{I}$ as defined in Lemma 8.28.
Proof. Let us prove it in the case where the first run of $\lambda$ is increasing and the last run decreasing, the other cases having the same proof. The expression (8.3.3) yields

$$
d_{\left(X_{I}, X_{F}\right), \mathcal{S}_{\lambda}+}(x, y)=\frac{\int_{y}^{1} d_{\left(X_{I}, X_{F}\right), \mathcal{S}_{\lambda}}(x, z) d z}{\int_{[0,1]^{2}}\left(\int_{y}^{1} d_{\left(X_{I}, X_{F}\right), \mathcal{S}_{\lambda}}(x, z)\right) d x d y} .
$$

Thus, by integrating with respect to $y$ and then changing the order of the integrals, this yields

$$
\begin{aligned}
d_{X_{I}, \mathcal{S}_{\lambda}+}(x) & =\frac{\int_{0}^{1}\left(\int_{0}^{1} d_{\left(X_{I}, X_{F}\right), \mathcal{S}_{\lambda}}(x, z) \mathbf{1}_{y \leq z} d y\right) d z}{\int_{[0,1]^{2}}\left(\int_{0}^{1} d_{\left(X_{I}, X_{F}\right), \mathcal{S}_{\lambda}}(x, z) \mathbf{1}_{y \leq z} d y\right) d x d z} \\
& =\frac{\int_{0}^{1} d_{\left(X_{I}, X_{F}\right), \mathcal{S}_{\lambda}}(x, z) z d z}{\int_{[0,1]^{2}} d_{\left(X_{I}, X_{F}\right), \mathcal{S}_{\lambda}}(x, z) z d z d x} .
\end{aligned}
$$

Factorizing by $d_{X_{I}, \mathcal{S}_{\lambda}}(x)$ makes a conditional expectation appear and thus

$$
d_{X_{I}, \mathcal{S}_{\lambda^{+}}}(x)=d_{X_{I}, \mathcal{S}_{\lambda}}(x) \frac{\mathbb{E}_{\mathcal{S}_{\lambda}}\left(X_{F} \mid X_{I}=x\right)}{\mathbb{E}_{\mathcal{S}_{\lambda}}\left(X_{F}\right)}
$$

Moreover, Proposition 8.23 yields

$$
F_{Z_{1}} \leq F_{X_{F} \mid X_{I}=x} \leq F_{Z_{2}},
$$

with $F_{Z_{1}}=\Gamma^{-}\left(F_{\gamma_{R}}\right)$ and $F_{Z_{2}}=\Gamma^{-}\left(\gamma_{R}\right)$. Since $\Gamma^{-}\left(F_{\gamma_{R}}\right)(t)=1-(1-t)^{R}$ and $\Gamma^{-}\left(\gamma_{R}\right)(t)=$ $1-(1-t)^{R-1}$, by stochastic dominance, applying Proposition 8.6 gives

$$
\frac{1}{R} \leq \mathbb{E}_{\mathcal{S}_{\lambda}}\left(X_{F} \mid X_{I}=x\right) \leq \frac{1}{R-1}
$$

Integrating the latter result on $x$ yields $\frac{1}{R} \leq \mathbb{E}_{\mathcal{S}_{\lambda}}\left(X_{F}\right) \leq \frac{1}{R-1}$, and thus

$$
\frac{R-1}{R} \leq \frac{\mathbb{E}_{\mathcal{S}_{\lambda}}\left(X_{F} \mid X_{I}=x\right)}{\mathbb{E}_{\mathcal{S}_{\lambda}}\left(X_{F}\right)} \leq \frac{R}{R-1} .
$$

This yields

$$
\left|d_{X_{I}, \mathcal{S}_{\lambda+}}(x)-d_{X_{I}, \mathcal{S}_{\lambda}}(x)\right| \leq\left|d_{\mathcal{S}_{\lambda}}(x)\right| \frac{1}{R-1} \leq \frac{K_{A}}{R-1} .
$$

In particular, the previous Lemma can be used to bound the conditional law of the first particle with respect to the last one. For each composition $\lambda$, and any cells $i, j \in \lambda$, denote by $\lambda_{\rightarrow i}\left(\right.$ resp $\lambda_{i \rightarrow}$, resp $\left.\lambda_{i \rightarrow j}\right)$ the composition consisting of the cells of $\lambda$ from 1 to $i$ (resp. from $i$ to $n$, resp. from $i$ to $j$ ). Moreover, denote by $R_{\text {int }}(\lambda)$ the set of all runs of $\lambda$ except the first and last ones.

Proposition 8.38. Let $A \geq 0$ and $\lambda$ a composition with first run bounded by $A$. Then

$$
\left\|F_{X_{I} \mid X_{F}=x}-F_{X_{I}}\right\|_{\infty} \leq \frac{K_{A}}{\max _{s \in R_{\text {int }}(\lambda)} l(s)-2} .
$$

Proof. Let $t \in[0,1]$. Let $s_{0}$ be the run with maximal length $R$ in $R_{\text {int }}$ and let $i_{0}$ be the rightest cell of this run. This cell corresponds to a particle $X_{i}$ or $Y_{i}$ in $\mathcal{S}_{\lambda}$. Let us assume without loss of generality that this particle is a lower one. From Proposition 8.18, $F_{X_{1} \mid X_{r}=x}(t)$ is decreasing in $x$ and thus

$$
\begin{aligned}
\left|F_{X_{I} \mid X_{F}=x}(t)-F_{X_{I}}(t)\right| & =\left|F_{X_{I} \mid X_{F}=x}(t)-\int_{X_{F}} F_{X_{I} \mid X_{F}=x}(t) d_{X_{F}}(x) d x\right| \\
& \leq\left|F_{X_{I} \mid X_{F}=0}(t)-F_{X_{I} \mid X_{F}=1}(t)\right| \\
& \leq F_{X_{I} \mid X_{F}=0}(t)-F_{X_{I} \mid Y_{k}=1}(t) .
\end{aligned}
$$

Moreover, from Proposition 8.18 and Proposition 8.20 ,

$$
F_{X_{I} \mid X_{F}=0}(t) \leq F_{X_{I}, \mathcal{S}_{\lambda} \rightarrow Y_{k}}(t) \leq F_{X_{I}, \mathcal{S}_{\lambda} \rightarrow Y_{i}}(t) \leq F_{X_{I} \mid X_{i}=0},
$$

and

$$
F_{X_{I} \mid Y_{k}=1}(t) \geq F_{X_{I}, \mathcal{S}_{\lambda} \rightarrow X_{k}}(t) \geq F_{X_{I}, \mathcal{S}_{\lambda} \rightarrow X_{i}}(t) .
$$

These inequalities imply

$$
\left|F_{X_{I} \mid X_{F}=x}(t)-F_{X_{I}}(t)\right| \leq F_{X_{I} \mid X_{i}=0}(t)-F_{X_{I}, \mathcal{S}_{\lambda} \rightarrow X_{i}}(t) .
$$

From the expression 8.3.3), $F_{X_{I}, \mathcal{S}_{\lambda} \rightarrow X_{i}}(t)=F_{X_{I}, \mathcal{S}_{\lambda_{\rightarrow i}}}(t)$ and $F_{X_{I} \mid X_{i}=0}(t)=F_{X_{1}, \mathcal{S}_{\lambda_{\rightarrow i_{0}}}}(t)$. Thus, with the previous Lemma, since the last run of $\lambda_{\rightarrow i_{0}}^{-}$is of size $R-1$,

$$
\left|F_{X_{I} \mid X_{F}=x}(t)-F_{X_{I}}(t)\right| \leq\left|F_{X_{I}, \mathcal{S}_{\lambda \rightarrow i_{0}}}(t)-F_{X_{I}, \mathcal{S}_{\lambda_{\rightarrow i_{0}}}}(t)\right| \leq \frac{K_{A}}{R-2} .
$$

### 8.6.2 Proof of Theorem 8.36

The latter Proposition together with Lemma 8.34 yields Theorem 8.36 in case $d_{X_{I}}^{\prime}$ remains bounded. However, the bound of the derivative in Lemma 8.29 requires also a bound on the second run, and the latter is not assumed in our case. We should thus deal with this case before getting the general proof. Let us first consider a particular case.

Lemma 8.39. Let $\lambda_{b}$ be the composition with three runs of respective length $2, b$ and 2 , and $d_{b}(x, y)=d_{X_{I}, \mid Y_{2}=y}(x)$. Then the following convergence holds:

$$
\lim _{b \rightarrow \infty} \sup _{[0,1]^{2}}\left(d_{b}(x, y)-\left(1-x^{b}\right)\right)=0 .
$$

In particular, the asymptotic independence :

$$
\begin{equation*}
\lim _{b \rightarrow \infty} \sup _{x, y, y^{\prime}}\left(d_{b}(x, y)-d_{b}\left(x, y^{\prime}\right)\right)=0 . \tag{8.6.2}
\end{equation*}
$$

is valid.
Proof. After integrating in 8.3.3) the coordinates of the particles inside the composition :

$$
\begin{equation*}
d_{b}(x, y)=\frac{1-x^{b}-(1-y)^{b}+((x-y) \wedge 0)^{b}}{(1-1 /(b+1))\left(1-(1-y)^{b}\right)+y /(b+1)(1-y)^{b}} . \tag{8.6.3}
\end{equation*}
$$

Let us show that $\lim _{b \rightarrow \infty} d_{b}(x, y)-\left(1-x^{b+1}\right)=0$ uniformly in $x$ and $y$. In the denominator of (8.6.3), letting $b$ go to $+\infty$ yields

$$
\left(1-\frac{1}{b+1}\right)\left(1-(1-y)^{b}\right)+y /(b+1)(1-y)^{b} \sim_{b \rightarrow \infty} 1-(1-y)^{b},
$$

with the equivalent being uniform in $x$ and $y$. Indeed

$$
\frac{y /(b+1)(1-y)^{b}}{1-(1-y)^{b}}=\frac{1}{b+1} \frac{(1-y)^{b}}{\sum_{k=0}^{b-1}(1-y)^{k}} \leq \frac{1}{b+1} .
$$

Since for $x \in[0,1 / 2], y \in[1 / 2,1], d_{b}(x, y)$ converges uniformly to 1 , it suffices to consider in the sequel that $x \in[1 / 2,1]$ and $y \in[0,1 / 2]$. Let $\Delta$ be defined as

$$
\begin{aligned}
\Delta(x, y) & =\frac{1-x^{b}-(1-y)^{b}+(x-y)^{b}}{1-(1-y)^{b}}-\left(1-x^{b}\right) \\
& =\left(1-\frac{x^{b}-(x-y)^{b}}{1-(1-y)^{b}}\right)-\left(1-x^{b}\right)=\frac{(x-y)^{b}-(1-y)^{b} x^{b}}{1-(1-y)^{b}} .
\end{aligned}
$$

A derivative computation shows that $\Delta(x, y) \leq \frac{1}{b}$, which proves the uniform convergence. Since $\lim _{b \rightarrow \infty}\left\|d_{b}(x, y)-\left(1-x^{b+1}\right)\right\|_{\infty,[0,1]^{2}}=0$,

$$
\lim _{b \rightarrow \infty} \sup _{y, y^{\prime}, x}\left(d_{b}(x, y)-d_{b}\left(x, y^{\prime}\right)\right)=0
$$

From the latter result can be deduced the asymptotic independence with a large second run :

Lemma 8.40. Let $A, \varepsilon>0$. There exist $B_{A} \in \mathbb{N}$ such that if $\lambda$ is a composition with at least three runs, the extreme runs bounded by $A$ and the second run larger than $B_{A}$, then

$$
\left\|d_{X_{I}, X_{F}}-d_{X_{I}} d_{X_{F}}\right\|_{\infty} \leq \varepsilon
$$

Proof. Let $\lambda$ be a composition with first run of length $a$ and second run of length $b$. From the definition of the density $d_{X_{I}, X_{F}}$ in 8.3.3), conditioning the law of $X_{I}$ on the position $x_{P}$ of the particle $P=a+b$ yields

$$
d_{X_{I} \mid x_{p}=y}(x)=\frac{\int_{x}^{1}\left(\int_{0}^{z_{1} \wedge y}\left(z_{1}-x\right)^{a-2}\left(z_{1}-z_{2}\right)^{b-2} d z_{2}\right) d z_{1}}{\mathcal{Z}}
$$

Let $2 \leq a \leq A$. Then

$$
d_{X_{I} \mid x_{p}=y}(x)=\frac{\int_{x}^{1}(u-x)^{a-3} d_{b}(u, y) d u}{\frac{1}{a-2} \int_{0}^{1} u^{a-2} d_{b}(u, y) d u}
$$

From the first part of Lemma 8.39, $\left|d_{b}(u, y)-\left(1-u^{b}\right)\right| \rightarrow_{b \rightarrow \infty} 0$ uniformly in $u$ and $y$, and thus

$$
\frac{1}{a-2} \int_{0}^{1} u^{a-2} d_{b}(u, y) d u \rightarrow_{b \rightarrow \infty} \frac{1}{(a-2)(a-1)}
$$

uniformly in $y$. Since $a$ is bounded by $A$, and from the second part of Lemma 8.39,

$$
\left\|d_{X_{I} \mid x_{p}=y}-d_{X_{I} \mid x_{p}=y^{\prime}}\right\|_{\infty} \leq A^{2} \sup _{y, y^{\prime}, x}\left(d_{b}(x, y)-d_{b}\left(x, y^{\prime}\right)\right) \rightarrow 0
$$

uniformly in $y$. Thus, for $b$ large enough, $\left\|d_{X_{I} \mid x_{p}=y}-d_{X_{I} \mid x_{p}=y^{\prime}}\right\|<\varepsilon / A$ for all $y$, $y^{\prime}$; then averaging on the law of $x_{p}$ conditioned on $X_{F}=y$ yields $\left|d_{X_{I} \mid X_{F}=y}-d_{X_{I} \mid X_{F}=y^{\prime}}\right|<\varepsilon / A$ for all $y, y^{\prime}$. Finally, this implies that

$$
\left\|d_{X_{I}, X_{F}}-d_{X_{I}} d_{X_{F}}\right\|_{\infty} \leq \varepsilon
$$

The proof of Theorem 8.36 is just a gathering of all the previous results :
Proof. Let $A, \varepsilon>0$. Since the first and last runs are bounded by $A$, any composition large enough has at least three runs. Let $B_{A}$ be given by Lemma 8.40, $R$ be the associate constante given by Lemma 8.29 for $B_{A}$, and set $C=\frac{4 K_{A} R}{(\varepsilon / A)^{2}}$. Finally, let $n$ be the integer given by Corollary 8.35 for compositions of runs bounded by $C$. Suppose that $\lambda$ is a composition larger than $n$. By Lemma 8.40, if the second run is larger than $B_{A}, 8.6 .1$ is verified. Thus, we can suppose that
the second run is bounded by $B_{A}$.
If $\lambda$ has a run larger than $C$, then from Proposition 8.38 ,

$$
\left\|F_{X_{I} \mid X_{F}=x}-F_{X_{I}}\right\|_{\infty} \leq \frac{K_{A}}{C-1} \leq \frac{(\varepsilon / A)^{2}}{4 R}
$$

But from Lemma 8.29, $d_{X_{I}}^{\prime}$ is bounded by $R$, thus the latter inequality yields with Lemma 8.34

$$
\left\|d_{X_{I} \mid X_{F}=y}-d_{X_{I}}\right\| \leq \varepsilon / A
$$

And $d_{X_{I}}$ being bounded by $A$, this yields 8.6.1).
Thus, we can assume that all the runs of $\lambda$ are bounded by $C$. Once again by the choice of $n$ and Corollary 8.35, 8.6.1) is verified.

Note that we actually proved something stronger than Theorem 8.36, namely :
Corollary 8.41. Let $A, \varepsilon>0$. There exists $n_{0}$ such that for every composition $\lambda$ of size larger than $n_{0}$ and first run bounded by $A$, and for all $y, y^{\prime} \in[0,1]$,

$$
\left\|d_{X_{I} \mid X_{F}=y}-d_{X_{I} \mid X_{F}=y^{\prime}}\right\| \leq \varepsilon
$$

### 8.6.3 Consequences and proof of Theorem 8.3

Here are some interesting consequences of Theorem 8.36. Let us first remove the constraints on the extreme runs.

Lemma 8.42. Let $\varepsilon>0$. There exists $n \geq 0$ such that for all compositions larger than $n$ with at least two runs,

$$
\sup _{\left(y, y^{\prime}\right) \in[0,1]^{2}}\left(\left\|F_{X_{I} \mid X_{F}=y}-F_{X_{I} \mid X_{F}=y^{\prime}}\right\|_{\infty}\right) \leq \varepsilon
$$

Proof. Let $R$ be the length of the first run of a composition $\lambda$. From Proposition 8.23 applied to $\mathcal{S}_{\lambda}$,

$$
1-(1-t)^{R} \leq F_{X_{I} \mid X_{F}=y}(t) \leq 1-(1-t)^{R-1}
$$

Since $\sup _{[0,1]}\left(u^{R-1}-u^{R}\right) \rightarrow_{R \rightarrow \infty} 0$, there exists $A$ such that for any composition with first run larger than $A$,

$$
\sup _{[0,1]^{2}}\left\|F_{X_{I} \mid X_{F}=y}-F_{X_{I} \mid X_{F}=y^{\prime}}\right\|_{\infty} \leq \varepsilon
$$

Applying Corollary 8.41 to $A, \varepsilon$ yields that there exists $n$ such that for any composition larger than $n$,

$$
\sup _{[0,1]^{2}}\left\|F_{X_{I} \mid X_{F}=y}-F_{X_{I} \mid X_{F}=y^{\prime}}\right\|_{\infty} \leq \varepsilon
$$

This result can be adapted to show that the law of the first particle depends only on the neighbouring particles : for any composition $\lambda$ of size $N$, and $n \leq N$, denote by $\lambda(n)$ the composition $\lambda$ containing only the $n$ first cells.

Proposition 8.43. Let $\varepsilon>0$. There exists $n_{0} \geq 1$ such that for any $n \geq n_{0}$ and any composition $\lambda$ of size larger than $n$ with first run smaller than $n$,

$$
\left\|F_{X_{I}}^{\mathcal{S}_{\lambda}}-F_{X_{I}}^{\mathcal{S}_{\lambda(n)}}\right\|_{\infty} \leq \varepsilon
$$

The proof consists only in an averaging of the inequality of the previous Lemma.
We will close this chapter by proving Theorem 8.3.
Let $\lambda$ be a composition and let $s=\llbracket i_{1}, i_{2} \rrbracket$ be a run of $\lambda$. For a cell $i$ in $s$, the position of $i$ in $s$, denoted by $a_{i}$, is the ratio $a_{i}=\frac{i-i_{1}}{i_{2}-i_{1}}$ (resp. $\left.\frac{i_{2}-i}{i_{2}-i_{1}}\right)$ if the run is increasing (resp. decreasing). When a run is large, the behavior of a cell in this run is approximately frozen:

Lemma 8.44. Let $\varepsilon>0$. There exists $R_{\varepsilon}>0$ such that for any composition $\lambda$ of $n$ and $1 \leq i \leq n$ such that $i$ is in a run $s$ of size larger than $R_{\varepsilon}$,

$$
\mathbb{P}\left(\left|\frac{\sigma_{\lambda}(i)}{n}-a_{i}\right| \geq \varepsilon\right) \leq \varepsilon
$$

where $a_{i}$ is the position of $i$ in $s$ as previously defined.
Proof. Let $\lambda$ be a composition of $n$, and let $1 \leq i \leq n$ be a cell of $\lambda$ in a run $s$ of length $R$. Let $i_{1} \leq i_{2}$ be the extreme cells of the run $s$ and suppose without loss of generality that $s$ is increasing. We use the probabilistic model $\tilde{\mathcal{S}}_{\lambda}$ of Section 8.3.2. By Lemma 8.16, it suffices to prove that for $R$ large enough,

$$
\mathbb{P}\left(\left|Z_{i}-a_{i}\right| \geq \varepsilon\right) \leq \varepsilon .
$$

Conditioning $Z_{i_{1}}$ on the value of $Z_{i_{1}-1}$ and $Z_{i_{2}}$ gives the conditional expectation:

$$
\mathbb{E}\left(Z_{i_{1}} \mid Z_{i_{1}-1}=z, Z_{i_{2}}=z^{\prime}\right)=\frac{\int_{0}^{z \wedge z^{\prime}} x\left(z^{\prime}-x\right)^{R-2} d x}{\int_{0}^{z \wedge z^{\prime}}\left(z^{\prime}-x\right)^{R-2} d x} \leq \frac{1}{R},
$$

where the last bound is given by a computation of the integral. Since the bound is independent of $z$ and $z^{\prime}$, for $R$ large enough $\mathbb{P}\left(Z_{i_{1}} \geq \varepsilon\right) \leq \varepsilon$. Likewise, for $R$ large enough, $\mathbb{P}\left(Z_{i_{2}} \leq 1-\varepsilon\right) \leq \varepsilon$. This gives the result if $i=i_{1}$ or $i=i_{2}$. Suppose that $i \neq i_{1}$ and $i \neq i_{2}$.
Conditioned on the value of $Z_{i_{1}}$ and $Z_{i_{2}}$, the law of $Z_{i}$ is

$$
d_{Z_{i} \mid Z_{i_{1}}=z, Z_{i_{2}}=z^{\prime}}(x)=\frac{\mathbf{1}_{z \leq x \leq z^{\prime}}\left(z^{\prime}-x\right)^{i_{2}-i-1}(x-z)^{i-i_{1}-1}}{\int_{z}^{z^{\prime}}\left(z^{\prime}-x\right)^{i_{2}-i-1}(x-z)^{i-i_{1}-1} d x} .
$$

Thus, by a computation, the conditional expectation of $Z_{i}-z$ is

$$
\mathbb{E}\left(Z_{i}-z \mid Z_{i_{1}}=z, Z_{i_{2}}=z^{\prime}\right)=\left(z^{\prime}-z\right) \frac{i-i_{1}}{i_{2}-i_{1}},
$$

and the conditional variance of $Z_{i}-z$ is

$$
\operatorname{Var}\left(Z_{i}-z \mid Z_{i_{1}}=z, Z_{i_{2}}=z^{\prime}\right)=\left(z^{\prime}-z\right)^{2} \frac{i-i_{1}}{i_{2}-i_{1}}\left(\frac{i-i_{1}+1}{i_{2}-i_{1}+1}-\frac{i-i_{1}}{i_{2}-i_{1}}\right) \leq\left(z^{\prime}-z\right)^{2} \frac{1}{R} .
$$

Thus, for $R$ large enough, $\mathbb{P}\left(\left|Z_{i}-\left(Z_{i_{1}}+a_{i}\left(Z_{i_{2}}-Z_{i_{1}}\right)\right)\right| \geq \varepsilon\right) \leq \varepsilon$.
By the first part of the proof, for $R$ large enough $\mathbb{P}\left(Z_{i_{1}} \geq \varepsilon\right) \leq \varepsilon$ and $\mathbb{P}\left(Z_{i_{2}} \leq 1-\varepsilon\right) \leq \varepsilon$; thus, for $R$ large enough,

$$
\mathbb{P}\left(\left|Z_{i}-a_{i}\right| \geq \varepsilon\right) \leq \varepsilon .
$$

We can improve the result of Corollary 8.42 by considering the case of a cell in the middle of a composition.

Lemma 8.45. Let $\varepsilon>0, R>0$. There exists $k_{R} \geq 1$ such that for any composition $\lambda$ and $1 \leq j_{1}<i<j_{2} \leq n$ such that $i$ is in a run bounded by $R$ and $\left|i-j_{1}\right|,\left|j_{2}-i\right| \geq k_{R}$, then

$$
\left\|d_{Z_{i} \mid Z_{j_{1}}=z_{1}, Z_{j_{2}}=z_{2}}-d_{Z_{i} \mid Z_{j_{1}}=z_{1}^{\prime}, Z_{j_{2}}=z_{2}^{\prime}}\right\|_{\infty} \leq \varepsilon
$$

for all $0 \leq z_{1}, z_{2}, z_{1}^{\prime}, z_{2}^{\prime} \leq 1$, where $Z_{i}$ is the random variable corresponding to the particle $i$ in $\tilde{\mathcal{S}}_{\lambda}$. Likewise,

$$
\left\|d_{Z_{i} \mid Z_{j_{1}}=z_{1}}-d_{Z_{i} \mid Z_{j_{1}}=z_{1}^{\prime}}\right\|_{\infty} \leq \varepsilon
$$

and

$$
\left\|d_{Z_{i} \mid Z_{j_{2}}=z_{2}}-d_{Z_{i} \mid Z_{j_{2}}=z_{2}^{\prime}}\right\|_{\infty} \leq \varepsilon
$$

for all $0 \leq z_{1}, z_{2}, z_{1}^{\prime}, z_{2}^{\prime} \leq 1$.
Proof. We will only prove the first part of the Lemma, since the proof of the second part is a simpler version of the one of the first part.
Let $\lambda$ be a composition and let $1 \leq j_{1}<i<j_{2} \leq n$ be three cells of $\lambda$. By the expression of the density in 8.3.3,

$$
d_{Z_{i} \mid Z_{j_{1}}=z_{1}, Z_{j_{2}}=z_{2}}(x)=\frac{d_{X_{F} \mid X_{I}=z_{1}, \mathcal{S}_{\nu_{1}}}(x) d_{X_{I} \mid X_{F}=z_{2}, \mathcal{S} \nu_{2}}(x)}{\int_{0}^{1} d_{X_{F} \mid X_{I}=z_{1}, \mathcal{S}_{\nu_{1}}}(x) d_{X_{I} \mid X_{F}=z_{2}, \mathcal{S} \nu_{2}}(x) d x}
$$

where $\nu_{1}=\lambda_{j_{1} \rightarrow i}$ and $\nu_{2}=\lambda_{i \rightarrow j_{2}}$. Since $i$ is in a run bounded by $R$ in $\lambda, i$ is in a run bounded by $R$ in $\nu_{1}$ and in $\nu_{2}$. Therefore by Corollary 8.41, there exists $n_{\varepsilon}$ such that if $\left|\nu_{1}\right| \geq n_{\varepsilon}$ and $\left|\nu_{2}\right| \geq n_{\varepsilon}$, then

$$
\left\|d_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}-d_{X_{F} \mid X_{I}=z_{1}^{\prime}, \nu_{1}}\right\|_{\infty} \leq \varepsilon
$$

and

$$
\left\|d_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}-d_{X_{I} \mid X_{F}=z_{2}^{\prime}, \nu_{2}}\right\|_{\infty} \leq \varepsilon
$$

for all $0 \leq z_{1}, z_{2}, z_{1}^{\prime}, z_{2}^{\prime} \leq 1$. Moreover, by Lemma $8.28, d_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}$ is bounded by some constant $K$ only depending on $R$, and the same holds for $d_{X_{I} \mid X_{F}}=z_{2}, \nu_{2}$. Therefore

$$
\left\|d_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}(x) d_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}(x)-d_{X_{F} \mid X_{I}=z_{1}^{\prime}, \nu_{1}}(x) d_{X_{I} \mid X_{F}=z_{2}^{\prime}, \nu_{2}}(x)\right\|_{\infty} \leq 2 A \varepsilon
$$

for $0 \leq z_{1}, z_{1}^{\prime}, z_{2}, z_{2}^{\prime} \leq 1$. In particular,

$$
\left|\int_{0}^{1} d_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}(x) d_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}(x)-d_{X_{F} \mid X_{I}=z_{1}^{\prime}, \nu_{1}}(x) d_{X_{I} \mid X_{F}=z_{2}^{\prime}, \nu_{2}}(x) d x\right| \leq 2 A \varepsilon
$$

Set

$$
A_{z_{1}, z_{2}}=\int_{0}^{1} d_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}(x) d_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}(x) d x, B_{z_{1}, z_{2}}=d_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}(x) d_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}(x)
$$

By the above computations,

$$
\begin{aligned}
\left|\frac{B_{z_{1}, z_{2}}}{A_{z_{1}, z_{2}}}-\frac{B_{z_{1}^{\prime}, z_{2}^{\prime}}}{A_{z_{1}^{\prime}, z_{2}^{\prime}}}\right| & \leq\left|\frac{B_{z_{1}, z_{2}}}{A_{z_{1}, z_{2}}}-\frac{B_{z_{1}^{\prime}, z_{2}^{\prime}}}{A_{z_{1}, z_{2}}}\right|+\left|\frac{B_{z_{1}^{\prime}, z_{2}^{\prime}}}{A_{z_{1}, z_{2}}}-\frac{B_{z_{1}^{\prime}, z_{2}^{\prime}}}{A_{z_{1}^{\prime}, z_{2}^{\prime}}}\right| \\
& \leq \frac{1}{A_{z_{1}, z_{2}}}(2 R \varepsilon)+\frac{B_{z_{1}^{\prime}, z_{2}^{\prime}}}{A_{z_{1}, z_{2}} A_{z_{1}^{\prime}, z_{2}^{\prime}}}(2 R \varepsilon)
\end{aligned}
$$

It remains to show that $\frac{1}{A_{z_{1}, z_{2}}}$ and $\frac{B_{z_{1}^{\prime}, z_{2}^{\prime}}}{A_{z_{1}, z_{2}} A_{z_{1}^{\prime}, z_{2}^{\prime}}}$ are bounded by a constant only depending on $R$. Since $i$ is in a run bounded by $R$ in $\nu_{1}$ and $\nu_{2},\left|B_{z_{1}, z_{2}}\right|$ is bounded by $K^{2}$, where $K$ is the
constant given Lemma 8.28 for a run of size $R$.
Let us show that $A_{z_{1}, z_{2}}$ admits a lower bound only depending on $R$; suppose without loss of generality that the run of $\lambda$ containing $i$ is increasing and that $i$ is not an extreme cell. Let $R_{1}$ be the length of the run containing $i$ in $\nu_{1}$ and let $R_{2}$ be the length of the run containing $i$ in $\nu_{2}$; since these both runs are part of the run of $i$ in $\lambda$, they are both increasing and $R_{1}+R_{2}=R+1$. By Corollary $8.25, t^{R_{1}} \leq F_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}(t) \leq t^{R_{1}-1}$ and $1-(1-t)^{R_{2}-1} \leq F_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}(t) \leq 1-$ $(1-t)^{R_{2}}$ for $0 \leq t \leq 1$. By Lemma 8.14. $d_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}$ is increasing and $d_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}$ is decreasing, thus $F_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}$ is convex and $F_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}$ is concave. The convexity of $F_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}$ yields that

$$
F_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}^{\prime}(t) \geq \frac{F_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}(t)-F_{X_{F} \mid X_{I}=z_{1}, \nu_{1}}(0)}{t-0} \geq t^{R_{1}-1}
$$

Likewise, the concavity of $F_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}$ yields that

$$
F_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}^{\prime}(t) \geq \frac{F_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}(1)-F_{X_{I} \mid X_{F}=z_{2}, \nu_{2}}(t)}{1-t} \geq(1-t)^{R_{2}-1}
$$

Therefore,

$$
A_{z_{1}, z_{2}} \geq \int_{0}^{1} x^{R_{1}-1}(1-x)^{R_{2}-1} d x=\frac{\left(R_{1}-1\right)!\left(R_{2}-1\right)!}{\left(R_{1}+R_{2}-1\right)!} \geq \frac{1}{\left(R_{1}+R_{2}-1\right)!}
$$

Since $R_{1}+R_{2}-1=R, A_{z_{1}, z_{2}} \geq \frac{1}{R!}$. This yields

$$
\left|\frac{B_{z_{1}, z_{2}}}{A_{z_{1}, z_{2}}}-\frac{B_{z_{1}^{\prime}, z_{2}^{\prime}}}{A_{z_{1}^{\prime}, z_{2}^{\prime}}}\right| \leq(2 R \varepsilon)\left(R!+K^{2}(R!)^{2}\right)
$$

Thus, if $\min \left(\left|\nu_{1}\right|,\left|\nu_{2}\right|\right) \geq n_{\varepsilon}$, then

$$
\left\|d_{Z_{i} \mid Z_{j_{1}}=z_{1}, Z_{j_{2}}=z_{2}}-d_{Z_{i} \mid Z_{j_{1}}=z_{1}^{\prime}, Z_{j_{2}}=z_{2}^{\prime}}\right\|_{\infty} \leq(2 R \varepsilon)\left(R!+K^{2}(R!)^{2}\right)
$$

for all $0 \leq z_{1}, z_{2}, z_{1}^{\prime}, z_{2}^{\prime} \leq 1$. Setting $k_{R}=n_{\varepsilon /\left(2 R\left(R!+K^{2}(R!)^{2}\right)\right.}$ gives the appropriate constant for the statement of the Lemma.

We can now prove Theorem 8.3 .
Proof of Theorem 8.3. The proof is done by recurrence on $r$.
Let $r=2$. Let $\varepsilon>0$ and $R_{\varepsilon}$ be the constant from Lemma 8.44. Let $\lambda$ be a composition of $n$ and let $1<i<j \leq n$ be two cells of $\lambda$. If $i$ and $j$ are both in runs larger than $R_{\varepsilon}$, then by Lemma 8.44. $\mathbb{P}\left(\left|\frac{\sigma_{\lambda}(\bar{i})}{n}-a_{i}\right| \geq \varepsilon\right) \leq \varepsilon$ and $\mathbb{P}\left(\left|\frac{\sigma_{\lambda}(j)}{n}-a_{j}\right| \geq \varepsilon\right) \leq \varepsilon$. Therefore,

$$
\begin{aligned}
\pi\left(\mu\left(\frac{\sigma_{\lambda}(i)}{n}, \frac{\sigma_{\lambda}(j)}{n}\right), \mu\left(\frac{\sigma_{\lambda}(i)}{n}\right) \otimes \mu\left(\frac{\sigma_{\lambda}(j)}{n}\right)\right) \leq \pi\left(\mu\left(\frac{\sigma_{\lambda}(i)}{n}, \frac{\sigma_{\lambda}(j)}{n}\right), \delta_{a_{i}} \otimes \delta_{a_{j}}\right) \\
+\pi\left(\delta_{a_{i}} \otimes \delta_{a_{j}}, \mu\left(\frac{\sigma_{\lambda}(i)}{n}\right) \otimes \mu\left(\frac{\sigma_{\lambda}(j)}{n}\right)\right) \leq 2 \varepsilon
\end{aligned}
$$

Suppose without loss of generality that $i$ is in a run smaller than $R_{\varepsilon}$. On the one hand, for $0 \leq t_{1}, t_{2} \leq 1$,

$$
F_{Z_{i}, Z_{j}}\left(t_{1}, t_{2}\right)-F_{Z_{i}}\left(t_{1}\right) F_{Z_{j}}\left(t_{2}\right)=\int_{0}^{t_{2}}\left(\int_{0}^{t_{1}} d_{Z_{i} \mid Z_{j}=y}(x)-d_{Z_{i}}(x) d x\right) d_{Z_{j}}(y) d y
$$

On the other end, by Lemma 8.45, there exists $k$ such that if $|j-i| \geq k$,

$$
\left\|d_{Z_{i} \mid Z_{j}=z}-d_{Z_{i} \mid Z_{j}=z^{\prime}}\right\|_{\infty} \leq \varepsilon
$$

for any $0 \leq z, z^{\prime} \leq 1$. Therefore, for $|j-i| \geq k,\left\|d_{Z_{i} \mid Z_{j}=y}-d_{Z_{i}}\right\|_{\infty} \leq \varepsilon$ for $0 \leq y \leq 1$. This yields

$$
\left|F_{Z_{i}, Z_{j}}\left(t_{1}, t_{2}\right)-F_{Z_{i}}\left(t_{1}\right) F_{Z_{j}}\left(t_{2}\right)\right| \leq \int_{0}^{t_{2}} t_{1} \varepsilon d_{Z_{j}}(y) d y \leq \varepsilon .
$$

In particular,

$$
\pi\left(\mu\left(Z_{i}, Z_{j}\right), \mu\left(Z_{i}\right) \otimes \mu\left(Z_{j}\right)\right) \leq \varepsilon
$$

Lemma 8.16 concludes the case $r=2$.
Suppose that $r>2$. Let $\lambda$ be a composition and let $1 \leq i_{1}, \ldots, i_{r} \leq n$ be distinct cells of $\lambda$. If $i_{1}, \ldots, i_{r}$ are all in runs larger than $R_{\varepsilon}$, by the same reason as before,

$$
\pi\left(\mu\left(\frac{\sigma_{\lambda}\left(i_{1}\right)}{n}, \ldots, \frac{\sigma_{\lambda}\left(i_{r}\right)}{n}\right), \mu\left(\frac{\sigma_{\lambda}\left(i_{1}\right)}{n}\right) \otimes \cdots \otimes \mu\left(\frac{\sigma_{\lambda}\left(i_{r}\right)}{n}\right)\right) \leq 2 \varepsilon .
$$

Suppose without loss of generality that $i_{r}$ is in a run bounded by $R_{\varepsilon}$, and let $k$ be the constant associated to $R_{\varepsilon}$ in Lemma 8.45. By recurrence hypothesis, there exists $k_{1}$ such that if $i_{j}-i_{j-1} \geq$ $k_{1}$ for $2 \leq j \leq r-1$, then

$$
\pi\left(\mu\left(Z_{i_{1}}, \ldots, Z_{i_{r-1}}\right), \mu\left(Z_{i_{1}} \otimes \cdots \otimes \mu\left(Z_{i_{r-1}}\right)\right) \leq \varepsilon .\right.
$$

On the one hand for $\vec{t} \in[0,1]^{r}$,

$$
\begin{aligned}
F_{\left(Z_{i}\right)_{1 \leq i \leq r}}(\vec{t}) & -F_{Z_{i_{r}}}\left(t_{r}\right) F_{\left(Z_{i_{s}}\right)_{s<r}}\left(\left(t_{s}\right)_{s<r}\right) \\
& =\int_{x_{s} \in\left[0, t_{s}\right]}\left(d_{Z_{i_{r}} \mid Z_{i_{s}}=x_{s}, s<r}\left(x_{r}\right)-d_{Z_{i_{r}}}\left(x_{r}\right)\right) d_{\left(Z_{i_{s}}\right)_{s<r}}\left(\left(x_{s}\right)_{s<r}\right) \prod_{s=1}^{r} d x_{s}
\end{aligned}
$$

By Formula (8.3.3), $d_{Z_{i_{r}} \mid Z_{i_{1}}=x_{1}, . . . Z_{i_{r}-1}=x_{r-1}}\left(x_{r}\right)=d_{Z_{i_{r}} \mid Z_{i_{a}}=x_{a}, Z_{i_{b}}=x_{b}}\left(x_{r}\right)$, where $a$ and $b$ are such that $i_{a}$ is the cell of $\left\{i_{1}, \ldots, i_{r-1}\right\}$ directly before $i_{r}$ and $i_{b}$ is the cell of $\left\{i_{1}, \ldots, i_{r-1}\right\}$ directly after $i_{r}$. By Lemma 8.45, if $i_{r}-i_{a} \geq k$ and $i_{b}-i_{r} \geq k$, then

$$
\left\|d_{Z_{i_{r}} \mid Z_{i_{a}}=x_{a}, Z_{i_{b}}=x_{b}}-d_{Z_{i_{r}}}\right\|_{\infty} \leq \varepsilon .
$$

Thus,

$$
\left|F_{\left(Z_{i_{s}}\right)_{1 \leq s \leq r}}(\vec{t})-F_{Z_{i_{r}}}\left(t_{r}\right) F_{\left(Z_{i_{s}}\right)_{s<r}}\left(\left(t_{i}\right)_{i<r}\right)\right| \leq \int_{x_{s} \in\left[0, t_{s}\right], s<r} \varepsilon d_{\left(Z_{i_{s}}\right)_{s<r}}\left(\left(x_{s}\right)_{s<r} \prod_{s<r} d x_{s} \leq \varepsilon,\right.
$$

which yields

$$
\pi\left(\mu\left(\left(Z_{i_{1}}, \ldots, Z_{i_{r}}\right), \mu\left(Z_{i_{r}}\right) \otimes \mu\left(\left(Z_{i_{s}}\right)_{s<r}\right)\right) \leq \varepsilon\right.
$$

Finally,

$$
\begin{array}{r}
\pi\left(\mu\left(Z_{i_{1}}, \ldots, Z_{i_{r}}\right), \mu\left(Z_{i_{1}}\right) \otimes \cdots \otimes \mu\left(Z_{i_{r}}\right)\right) \leq \pi\left(\mu\left(Z_{i_{1}}, \ldots, Z_{i_{r}}\right), \mu\left(Z_{i_{r}}\right) \otimes \mu\left(\left(Z_{i_{s}}\right)_{s<r}\right)\right. \\
+\pi\left(\mu\left(Z_{i_{r}}\right) \otimes \mu\left(\left(Z_{i_{s}}\right)_{s<r}\right), \mu\left(Z_{i_{1}}\right) \otimes \cdots \otimes \mu\left(Z_{i_{r}}\right)\right) \leq \varepsilon+\varepsilon \leq 2 \varepsilon .
\end{array}
$$

## Chapter 9

## Martin bounday of $\mathcal{Z}$

### 9.1 Introduction

The lattice $\mathcal{Z}$ of zigzag diagrams is a graded graph whose vertices of degree $n$ are labelled by compositions of $n$ (which can be seen as ribbon Young diagrams). The study of this kind of lattices drew increasing interests these last decades, due to their interactions with representations of semi-simple algebras and with discrete random walks. In particular an other example of graded graph, the Young lattice $\mathcal{Y}$, has been deeply studied by Vershik, Kerov and other authors (see [47] for a review on the subject), yielding major breakthroughs on the representation theory of $S_{\infty}$ and on the asymptotic study of certain particle systems. As explained in [42], the lattices $\mathcal{Z}$ and $\mathcal{Y}$ are somehow related, since the latter can be seen as a projection of the former.
The connection between the lattice structure and its probabilistic applications is made through the study of harmonic functions on the associated graph. One of the first tasks is therefore the characterization of harmonic functions on the lattice ; it is then possible to single out particular harmonic functions and study the random variables they generate. A general framework for the representation of harmonic functions on a graph $E$ has been initiated by Martin in 61, with the concept of Martin boundary $\partial_{M} E$ and minimal boundary $\partial E_{\min }$ : the Martin boundary is a topological space coming from the graph and allowing to establish a bijection between positive harmonic functions and measures whose support is included in a particular subset of $\partial_{M} E$. The latter subset is precisely the minimal boundary $\partial E_{\min }$. It is therefore important to know both $\partial_{M} E$ and $\partial E_{\min }$ to provide a topological and measure theoretic approach to the kernel representation of harmonic functions (see 34 for an exhaustive review on the subject).
In general $\partial E_{\text {min }}$ is strictly included in $\partial_{M} E$. However in many cases the two coincide, as it happens for example for the lattice $\mathcal{Y}$. In this chapter we prove that the two boundaries also coincide for the lattice $\mathcal{Z}$. The minimal boundary of $\mathcal{Z}$ has already been described by Gnedin and Olshanski in 42, through the so-called oriented paintbox construction, and thus it remains to prove that any element of the Martin boundary fits in this construction. As an application we provide a precise link between harmonic measures on $\mathcal{Y}$ and harmonic measures on $\mathcal{Z}$ : this link was already exposed in [42], and in the present chapter we explain this relation by mapping directly paths on $\mathcal{Z}$ to paths on $\mathcal{Y}$. Finally we study the behavior of a random path with respect to the Plancherel measure by providing a Central Limit Theorem.
Section 2 and 3 are devoted to preliminaries : the first gives necessary backgounds on Martin boundary, and the second describes the graph $\mathcal{Z}$ together with its link with compositions. The results of Gnedin and Olshanski on this graph are given in Section 4. In this section we provide also the pattern of the proof for the identification of the Martin boundary.
The proof heavily relies on combinatorics of compositions. In particular the Martin kernel of $\mathcal{Z}$, a two parameters function that characterizes the Martin boundary, is related to standard
fillings of compositions. Two constructions are needed in order to identify the Martin boundary: Section 5 deals with the first one, which is the construction of a sequence of random variables that relates the Martin kernel to the oriented Paintbox construction of Gnedin and Olshanski. The second one has been done in Chapter 8 and is a general framework that gives combinatorial estimates on compositions. Some results of Chapter 8 are recalled in Section 6. Section 7 and 8 show the identification of the Martin boundary. Finally Section 9 gives the map between paths on $\mathcal{Z}$ and paths on $\mathcal{Y}$ and exposes probabilistic results associated to a particular point of the Martin boundary, called the Plancherel measure (due to its relations with the Plancherel measure on the graph $\mathcal{Y}$ ).
We should stress that, as it has been explained to us by Jean-Yves Thibon, the map between the paths on the two graphs appears clearly by using the algebra FQSym of Free Quasi-Symmetric functions; although this algebra won't be described in this chapter, the interested reader should refer to the Chapter 3 of [35] for an introduction to FQSym and an explanation from a Hopf algebraic point of view to the construction we are doing in Section 9.2 of the present chapter.

### 9.2 Graded graphs and Martin boundary

This section is a discussion that introduces the concept of Martin boundary and motivates its role in the framework of graded graphs. All these results and proofs can be found in [34.

### 9.2.1 Graded graphs and random walks

The notations used here are from [79. A rooted graded graph $\mathcal{G}$ is the data of a triple ( $V, \rho, E$ ) where:

- $V$ is a denumerable set of vertices with a distinguished element $\mu_{0}$.
- $\rho: V \rightarrow \mathbb{N}$ is a rank function with $\rho^{-1}(\{0\})=\left\{\mu_{0}\right\}$.
- The adjacency matrix E is a $V \times V$-matrix with entries in $\{0,1\}$, such that $E(\mu, \nu)$ is zero if $\rho(\nu) \neq \rho(\mu)+1$.

We write $\mu \nearrow \nu$ if $E(\mu, \nu)=1$. A path on $\mathcal{G}$ is sequence of vertices $\left(\mu_{1}, \ldots, \mu_{n}, \ldots\right)$ of increasing degree such that for all $i \geq 1, \mu_{i} \nearrow \mu_{i+1}$. For a given graded graph the paths counting function $d: \mathcal{V} \rightarrow \mathbb{N}^{*}$ is the function that gives for each vertex $\mu \in \mathcal{G}$ the number of paths between $\mu_{0}$ and $\mu$. When the endpoints of a path are not specified, the path is considered as an infinite path starting at the root.
There is a natural way of constructing random walks that respect the structure of the graph $\mathcal{G}$ : such a random walk starts at $\mu_{0}$, and at each step the successor is chosen according to a transition matrix $P$, with the condition that $P(\mu, \nu)=0$ if $E(\mu, \nu)=0$ and $\sum_{\nu} P(\mu, \nu)=1$. Thus each transition matrix $P$ associates to any path $\lambda=\left(\mu_{0} \nearrow \mu_{1} \cdots \nearrow \mu_{n}\right)$ a weight $p_{\lambda}$ which is the probability of the realization of $\lambda$, namely

$$
p_{\lambda}=\mathbb{P}\left(X_{0}=\mu_{0}, X_{1}=\mu_{1}, \ldots, X_{n}=\mu_{n}\right)=\frac{1}{Z} P\left(\mu_{0}, \mu_{1}\right) \ldots P\left(\mu_{n-1}, \mu_{n}\right) .
$$

For some transition matrices $P$ on $\mathcal{G}$, the weight $p(\lambda)$ only depends on the final vertex of the path (in this case we write $p(\lambda)=p(\mu)$ for any path $\lambda$ between $\mu_{0}$ and $\mu$ ); such a transition matrix is called a harmonic matrix. In this case, a staightforward computation shows that $p$, the associated weight function, must verify

$$
\begin{equation*}
p(\mu)=\sum_{\mu \nearrow \nu} p(\nu), \tag{9.2.1}
\end{equation*}
$$

and conversely, any positive solution $p$ of (9.2.1) such that $p\left(\mu_{0}\right)=1$ yields a harmonic matrix. This can be interpreted in terms of potential theory.
Let $X$ be a denumerable states space with transition matrix $P$. Let $H(X, P)^{+}($resp. $M(X, P))$ denote the set of positive harmonic functions (resp. positive harmonic measures), which is the set of functions $f: X \rightarrow \mathbb{R}^{+}$satisfying $\sum_{y} P(x, y) f(y)=f(x)$ (resp. $\sum_{x} f(x) P(x, y)=f(y)$ ). For each $\alpha \in M(X, P)$ let the dual transition matrix $P_{\alpha}^{t}$ be defined by the expression

$$
P_{\alpha}^{t}(x, y)=\mathbf{1}_{\alpha(x) \neq 0} \frac{\alpha(y)}{\alpha(x)} P(y, x),
$$

if $x \neq y$, and $P^{t}(x, x)=1-\sum_{x \neq y} P^{t}(x, y)$. Then $P_{\alpha}^{t}$ is indeed a transition matrix on $X$ and the following maps are well-defined:

$$
H_{\alpha}:\left\{\begin{array}{ccc}
H(X, P)^{+} & \rightarrow & M\left(X, P_{\alpha}^{t}\right) \\
h & \mapsto & \left(x \mapsto \mathbf{1}_{\alpha(x)>0} \frac{1}{\alpha(x)} h(x)\right),
\end{array}\right.
$$

and

$$
M_{\alpha}:\left\{\begin{array}{ccc}
M(X, P) & \rightarrow & H\left(X, P_{\alpha}^{t}\right)^{+} \\
m & \mapsto & (x \mapsto \alpha(x) m(x)) .
\end{array}\right.
$$

The two maps are bijective if $\alpha>0$ on $X$.
Let $P$ be a transition matrix on a graded graph $\mathcal{G}$; by a recursive computation there exists a unique invariant measure $\alpha_{P}$ with respect to $P$ such that $\alpha_{P}\left(\mu_{0}\right)=1$. If $P_{p}$ is a harmonic matrix, with $p$ the associated weight function, then $P(\mu, \nu)=\mathbf{1}_{\mu \nearrow \nu} \frac{p(\nu)}{p(\mu)}$ and $\alpha_{p}=d(\mu) p(\mu)$. Thus the dual transition matrix is

$$
P_{\alpha_{p}}^{t}(\nu, \mu)=\mathbf{1}_{\mu \nearrow \nu} \frac{d(\mu) p(\mu)}{d(\nu) p(\nu)} \frac{p(\nu)}{p(\mu)}=\mathbf{1}_{\mu \nearrow \nu} \frac{d(\mu)}{d(\nu)} .
$$

In particular $P_{\alpha_{p}}^{t}$ is independent of $p$ and, by $H_{\alpha_{p}}$, any harmonic function of $P$ comes from an invariant measure of $P^{t}$. Conversely let $\alpha$ be an invariant measure of $P^{t}$. Then the dual matrix $\left(P^{t}\right)_{\alpha}^{t}$ is exactly $P_{\alpha / d}$, the harmonic matrix associated to the weight function $p=\alpha / d$. We can check that the duality yields indeed a bijection between harmonic matrices of $\mathcal{G}$ and elements of $M\left(\mathcal{G}, P^{t}\right)$ taking the value 1 on $\mu_{0}$.
Thus the problem of finding the harmonic matrices on $\mathcal{G}$ is equivalent to the dual problem of finding harmonic measures with respect to $P^{t}$. Moreover an answer to the latter problem gives also by duality all the harmonic functions with respect to a harmonic matrix.
Fortunately a general framework, the Martin entrance boundary, describes exactly the harmonic measures associated to a transition matrix.

### 9.2.2 Martin entrance boundary

Let us take a closer look at the Markov chain $\left(\mathcal{G}, P^{t}\right)$. Let $n_{0} \geq 1$ and $\nu$ a vertex of degree $n_{0}$. The random walk $X=\left(X_{n}\right)_{n \geq 0}$ with transition matrix $P^{t}$ and initial distribution $\delta_{\nu}$ goes backward from $\nu$ to $\mu_{0}$ and stops at $\mu_{0}$ at the times $n_{0}$. Let $\lambda$ be a path between $\mu_{0}$ and $\nu$; from the definition of the kernel $P^{t}$, the probability that $X$ follows the path $\lambda$ is independent of $\lambda$ and is therefore $\frac{1}{d(\nu)}$.
For $\mu$ of degree $m \leq n_{0}$, denote by $d(\mu, \nu)$ the number of paths between $\mu$ and $\nu$ (and by extension $d(\mu, \nu)=0$ if the degree of $\mu$ is larger than the one of $\nu$ ). By counting the paths going from $\mu_{0}$ to $\nu$ and passing through $\mu$, the probability that $X_{n_{0}-m}=\mu$ is thus

$$
\mathbb{P}\left(X_{n_{0}-m}=\mu\right)=\frac{d(\mu) d(\mu, \nu)}{d(\nu)} .
$$

In particular setting $\alpha_{\nu}(\mu)=\frac{d(\mu) d(\mu, \nu)}{d(\nu)}$ yields a measure $\alpha_{\nu}$ that is harmonic with respect to $P^{t}$, except on the vertex $\nu$. To construct actual harmonic measures, it seems thus natural to look at the behavior of $\alpha_{\nu}$ when $\nu \rightarrow \infty$. Making the latter rigorous requires to specify a convergence mode for sequences of vertices of increasing degree. Let $K_{\mu}(\nu)=\frac{d(\mu, \nu)}{d(\nu)}$ be the Martin kernel of $\mathcal{G}$, and define on $\mathcal{G}$ the metric :

$$
D\left(\nu_{1}, \nu_{2}\right)=\sum_{\mu} \frac{1}{2^{\Gamma(\mu)}}\left|K_{\mu}\left(\nu_{1}\right)-K_{\mu}\left(\nu_{2}\right)\right|,
$$

$\Gamma$ being any injective function $V \rightarrow \mathbb{N}$. Identifying $\nu \in V$ with $K .(\nu), V$ is seen through this metric as a subset of the space of functions from $V$ to $[0,1]$ with the pointwise convergence topology. Thus by Tychonoff's Theorem the completion $\tilde{V}$ of $V$ with respect to $D$ is a compact space, and by construction $K_{\mu}$ extends continuously on this completion. Actually the completion is exactly the set of sequences $\left(\nu_{n}\right)_{n \geq 1}$ such that for each $\mu, K_{\mu}\left(\nu_{n}\right)$ converges, with two sequences $\left(\nu_{n}^{1}\right)_{n \geq 1},\left(\nu_{n}^{2}\right)_{n \geq 1}$ being identified whenever for each $\mu, K_{\mu}\left(\nu_{n}^{1}\right)$ and $K_{\mu}\left(\nu_{n}^{2}\right)$ have the same limit. Denote by $\partial_{M} \mathcal{G}$ the set $\tilde{V} \backslash V$. The latter is called the Martin entrance boundary of the graded graph $\mathcal{G}$ and is a compact subset of $\tilde{V}$. Each element $\omega=\lim _{n \rightarrow \infty} \nu_{n}$ in $\partial_{M} \mathcal{G}$ defines a function on $V$ by the formula

$$
\omega(\mu)=\lim K_{\mu}\left(\nu_{n}\right) .
$$

The following Theorem is a special case of a Theorem from Doob ([34).
Theorem 9.1. With the notations above, the two following results hold:

- There exists a Borel subset $\partial_{\min } \mathcal{G} \subseteq \partial_{M} \mathcal{G}$ (called minimal boundary) such that for any measure $\alpha$ harmonic with respect to $P^{t}$, there exists a unique measure $\lambda_{\alpha}$ on $\partial_{\min } \mathcal{G}$ giving the kernel representation

$$
\alpha(\mu)=\int_{\partial_{\min } \mathcal{G}} K_{\mu}(x) d \lambda_{\alpha}(x) .
$$

- For any reverse random walk $\left(X_{n}\right)_{n \leq 0}$ that respects $P^{t}$, the path $\left(X_{0}, X_{-1}, \ldots\right)$ converges almost surely to a $\partial_{\min } \mathcal{G}$-valued random variable $X_{-\infty}$. Moreover the probability that $\left(X_{n}\right)_{n \leq 0}$ goes through $\mu$ is exactly $d(\mu) \mathbb{E}\left(K_{\mu}\left(X_{-\infty}\right)\right)$.

There exists a more general construction of the Martin boundary from Kunita and Watanabe in [53], which encompasses the case of discrete random walks as well as more general Markov processes (including the Brownian motion on a domain). However our situation is much simpler and the previous Theorem is enough.
To summarize, the Martin entrance boundary gives a topological framework to represent harmonic measures, whereas the minimal entrance boundary gives the corresponding measure theoretic framework. The situation is simpler when the two boundaries coincide. In the case of the graph $\mathcal{Z}$ that we are studying, the minimal entrance boundary was already described by Gnedin and Olshanski in [42]. The purpose of the present chapter is to extend this desciption to the Martin entrance boundary by proving that the two boundary coincide.

### 9.3 The graph $\mathcal{Z}$

This section is devoted to an introduction to the graph $\mathcal{Z}$ and its relation with sequences of permutations. All the results from this section can de found in 42.

### 9.3.1 Compositions

Let us first recall the definition of a composition:
Definition 9.2. Let $n \in \mathbb{N}$. A composition $\lambda$ of $n$, written $\lambda \vdash n$, is a sequence of positive integers $\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ such that $\sum \lambda_{j}=n$.

Let $D_{\lambda}$ be the subset of $[1 ; n]$ defined by $D_{\lambda}=\left\{\lambda_{1}, \lambda_{1}+\lambda_{2}, \ldots, \sum_{1}^{r-1} \lambda_{i}\right\}$. Since there is a bijection between subsets of $[1 ; n-1]$ and compositions of $n, D_{\lambda}$ is often simply denoted by $\lambda$. To a composition is also associated a unique ribbon Young diagram with $n$ cells: each row $j$ has $\lambda_{j}$ cells, and the first cell of the row $j+1$ is just below the last cell of the row $j$. For example the composition $(3,2,4,1)$ of 10 is represented in Figure 9.1.


Figure 9.1: Skew Young tableau associated to the composition $\lambda=(3,2,4,1)$.
The size $n$ is included in the definition of composition itself, since $n$ is equal to the sum of all $\lambda_{j}$. If we want to emphasize the size of a compostion $\lambda$, we denote it as $|\lambda|$. When nothing is specified, $\lambda$ is always assumed to have the size $n$, and $n$ always denotes the size of the composition $\lambda$.
A standard filling of a composition $\lambda$ of size $n$ is a standard filling of the associated ribbon Young diagram: it is the assignement of an integer from 1 to $n$ to each cell of the composition, such that every cells have different entries, and the entries are increasing to the right along the rows and decreasing to the bottom along the columns. An example for the composition of Figure 9.1 is shown in Figure 9.2.


Figure 9.2: Standard filling of the composition $(3,2,4,1)$.
In particular, reading the tableau from left to right and from top to bottom gives for each standard filling a permutation $\sigma$; moreover the descent set $\operatorname{des}(\sigma)$ of $\sigma$, namely the set of indices $i$ such that $\sigma(i+1)<\sigma(i)$, is exactly the set $D_{\lambda}$. There is a bijection between the standard fillings of $\lambda$ and the permutations of $|\lambda|$ with descent set $D_{\lambda}$. For example the filling in Figure 9.2 yields the permutation ( $3,5,8,4,7,1,6,9,10,2$ ).

### 9.3.2 The graph $\mathcal{Z}$

The graded graph $\mathcal{Z}$, which was introduced by Viennot in [89], is defined as follows:

1. The set $\mathcal{Z}_{n}$ of vertices of degree $n$ of $\mathcal{Z}$ is the set of compositions of $n$. The vertex of degree 0 is denoted $\emptyset$.
2. Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ and $\mu=\left(\mu_{1}, \ldots, \mu_{s}\right)$ be two compositions. There is an edge between $\mu$ and $\lambda$ if and only if $|\lambda|=|\mu|+1$ and

- either $r=s$ and for each $i$ except one $\mu_{i}=\lambda_{i}$ (thus exactly one $\mu_{i_{0}}$ is increased by one)
- either $r=s+1$, and there exists $j$ such that: for $k<j, \lambda_{k}=\mu_{k}, \lambda_{j}+\lambda_{j+1}-1=\mu_{j}$, and for $k>j, \lambda_{k+1}=\mu_{k}$ (namely one $\mu_{i}$ is split, and one cell is added at the end of the first piece).

The first four levels of $\mathcal{Z}$ are displayed in Figure 9.3 .


Figure 9.3: Vertices of $\mathcal{Z}$ of degree 0 to 3 .
For a composition $\lambda$, let $\Omega_{\lambda}$ be the set of paths between $\emptyset$ and $\lambda$. It has been shown in [89] that $\Omega_{\lambda} \simeq\left\{\sigma \in S_{|\lambda|}, \operatorname{des}(\sigma)=D_{\lambda}\right\}$. One way to see this is to remark that $\Omega_{\lambda}$ is the set of all standard fillings of the ribbon diagram associated to $\lambda$. Thus these sets have same cardinality and

$$
d(\lambda)=\left|\Omega_{\lambda}\right|=\#\left\{\sigma \in S_{|\lambda|}, \operatorname{des}(\sigma)=D_{\lambda}\right\} .
$$

Let $\mathbb{P}_{\lambda}$ denote the uniform distribution on $\Omega_{\lambda}$; from Section 2 , this is equivalent to considering the random walk starting at $\lambda$ with transition matrix $P^{t}$. This random walk gives $n$ random variables $\sigma_{k}^{\lambda}, 1 \leq k \leq n$, each of them being the random path restricted to the vertices of degree smaller than $k$.
Since there is a bijection between paths on $\mathcal{Z}$ from $\emptyset$ to $\mu$ and permutations of $|\mu|$ with descent set $D_{\mu}$, each $\sigma_{k}^{\lambda}$ is a random permutation in $S_{k}$, and the law of $\sigma_{\lambda}=\sigma_{n}^{\lambda}$ is the uniform distribution on the set of permutations with descent set $D_{\lambda}$. Moreover a counting argument yields that for $\sigma \in \mathfrak{S}_{k}$ with $\operatorname{des}(\sigma)=D_{\mu}$, under the probability $\mathbb{P}_{\lambda}$,

$$
\begin{equation*}
\mathbb{P}_{\lambda}\left(\sigma_{\lambda}^{k}=\sigma\right)=\frac{d(\mu, \lambda)}{d(\lambda)} \tag{9.3.1}
\end{equation*}
$$

By abuse of notation a finite path starting at $\emptyset$ on $\mathcal{Z}$ and the corresponding permutation are both usually denoted by $\sigma$. In particular if $\sigma \in \Omega_{\lambda}, \sigma_{k}$ denotes the path after $k$ steps, whereas $\sigma(i)$ will denote the image of $i$ by the permutation associated to $\sigma$ (the same for $\sigma(A)$ with $A$ a subset of $\{1, \ldots, n\}$ ).

### 9.3.3 Arrangement on $\mathbb{N}$

In this paragraph a permutation $\sigma \in \mathfrak{S}_{k}$ is written as a word in the alphabet $\{1, \ldots, k\}$, where $i_{j}=\sigma(j)$. For $k \geq 2$ and $\sigma=\left(i_{1} \ldots i_{k}\right), \sigma_{\downarrow} \in \mathfrak{S}_{k-1}$ is defined as the permutation $\left(i_{1} \ldots k \ldots i_{k}\right)$. If $\sigma \in \mathfrak{S}_{n}, \sigma_{\downarrow k}$ denotes the $(n-k)$-iteration of the $\downarrow$-operation : namely all the indices between $k+1$ and $n$ have been erased.
An arrangement of $\mathbb{N}$ is a sequence $\left(\sigma_{1}, \ldots, \sigma_{k}, \ldots\right)$ such that for all $k \geq 1, \sigma_{k} \in \mathfrak{S}_{k}$, and such that the following compatibility condition holds :

$$
\left(\sigma_{k}\right)_{\downarrow}=\sigma_{k-1} .
$$

For example the following sequence is the first part of an arrangement :

$$
((1),(21),(231),(2341),(52341), \ldots) .
$$

The set of all arrangements is denoted $\mathcal{A}$. For $k \geq 1$, let $\pi_{k}: \mathcal{A} \rightarrow \mathfrak{S}_{k}$ be the map which consists in the projection of the sequence ( $\sigma_{1}, \sigma_{2}, \ldots$ ) on the $k$-th element $\sigma_{k}$. $\mathcal{A}$ is considered with the initial topology with respect to the maps $\pi_{k}$, and with the corresponding borelian $\sigma$-algebra. Thus by the Kolmogorov's extension Theorem, any random variable $\Pi$ on $\mathcal{A}$ is uniquely determined by the law of its finite-dimensional projections $\left(\pi_{1}(\Pi), \ldots, \pi_{k}(\Pi)\right.$ ).
The result of the previous subsection yields that there is a bijection between infinite paths on $\mathcal{Z}$ and arrangements of $\mathbb{N}$, and from Section 2 this bijection extends to a bijection between harmonic measures $\alpha$ with respect to $P^{t}$ and random arrangements $\Pi$ such that

$$
\mathbb{P}\left(\pi_{1}(\Pi)=\sigma_{1}, \ldots, \pi_{k}(\Pi)=\sigma_{k}\right)=p\left(\operatorname{des}\left(\sigma_{k}\right)\right),
$$

with $p$ a positive function on $\mathcal{Z}$ given by $p=\frac{\alpha}{d}$. This correspondance is convenient since it allows to describe the solutions of the problem (9.2.1) in terms of random arrangements.

### 9.4 Paintbox construction and Minimal boundary

Thanks to the latter correspondance, Gnedin and Olshanski described the minimal entrance boundary of $\mathcal{Z}$ in terms of random arrangements. This description is the purpose of the following paragraph.

### 9.4.1 Paintbox construction

The description is based on a topological space consisting in pairs of disjoint open sets of $[0,1]$ :
Definition 9.3. The topological space $\mathcal{U}^{(2)}$ is the space

$$
\left(\left\{\left(U_{\uparrow}, U_{\downarrow}\right) \mid U_{\uparrow} \text { and } U_{\downarrow} \text { disjoint open sets of }\right] 0,1[ \}, d\right) \text {, }
$$

with the distance $d$ between $\left(U_{\uparrow}, U_{\downarrow}\right)$ and $\left(V_{\uparrow}, V_{\downarrow}\right)$ given by

$$
d\left(\left(U_{\uparrow}, U_{\downarrow}\right),\left(V_{\uparrow}, V_{\downarrow}\right)\right)=\sup \left(d_{\text {Haus }}\left(U_{\uparrow}^{c}, V_{\uparrow}^{c}\right), d_{\text {Haus }}\left(U_{\downarrow}^{c}, V_{\downarrow}^{c}\right)\right) .
$$

Let $M_{1}\left(\mathcal{U}^{(2)}\right)$ denote the set of probability measures with respect to the $\sigma$-algebra coming from the above topology.
From the definition of the metric, $\left.\left(U_{\uparrow}(j), U_{\downarrow}(j)\right)\right)_{j \geq 1}$ converges to $\left.\left(V_{\uparrow}, V_{\downarrow}\right)\right)$ if and only if, for each $\varepsilon>0$, all of the following phenomena occur:

- for $j$ large enough, the number of connected components of size larger than $\varepsilon$ in $U_{\uparrow}$ and $V_{\uparrow}$ are the same,
- the boundaries of the connected components of size larger than $\varepsilon$ in $U_{\uparrow}$ converge to the ones of $V_{\uparrow}$,
- the same holds by switching $\uparrow$ and $\downarrow$.

In particular $\left(U_{\uparrow}(j), U_{\downarrow}(j)\right)$ converges to $(\emptyset, \emptyset)$ if and only if the size of the largest components in $U_{\uparrow}(j)$ and $U_{\downarrow}(j)$ tends to 0 . The following important result holds for $\mathcal{U}^{(2)}$ :
Proposition 9.4 ([42]). $\mathcal{U}^{(2)}$ is compact space.
The minimal entrance boundary of $\mathcal{Z}$ is described by random arrangements constructed from elements of $\mathcal{U}^{(2)}$.

Definition 9.5. Let $U=\left(U_{\uparrow}, U_{\downarrow}\right)$ be fixed, $\left(X_{1}, \ldots, X_{k}, \ldots\right)$ a sequence of $[0,1]$. For each $k \geq 1$, $\sigma_{U}\left(X_{1}, \ldots, X_{k}\right) \in \mathfrak{S}_{k}$ is defined by the following rule:
$\left(\sigma_{U}\left(X_{1}, \ldots, X_{k}\right)\right)^{-1}(i)$ is less than $\left(\sigma_{U}\left(X_{1}, \ldots, X_{k}\right)\right)^{-1}(j)$ if and only if one of the three following situations arises :

- $X_{i}$ and $X_{j}$ are not in the same connected component of $U_{\uparrow}$ or $U_{\downarrow}$ and $X_{i}<X_{j}$
- $X_{i}$ and $X_{j}$ are in the same connected component of $U_{\uparrow}$ and $i<j$
- $X_{i}$ and $X_{j}$ are in the same connected component of $U_{\downarrow}$ and $j<i$.

The random variable $\sigma_{U}\left(X_{1}, \ldots, X_{k}\right)$ defined for an infinite family $\left(X_{1}, \ldots, X_{k}, \ldots\right)$ of independent uniform variables on $[0,1]$ is denoted $\sigma_{U}(k)$. The sequence $\left(\sigma_{U}(1), \sigma_{U}(2), \ldots\right)$ is denoted $\sigma_{U}$.

The construction of $\sigma_{U}\left(X_{1}, \ldots, X_{k}\right)$ from $\left(X_{1}, \ldots, X_{k}\right)$ and $U \in \mathcal{U}^{(2)}$ is well-defined and unique. If $U=(\emptyset, \emptyset), \sigma_{(\emptyset, \emptyset)}\left(X_{1}, \ldots, X_{k}\right)$ is just the permutation associated to the reordering $\left(X_{i_{1}}<X_{i_{2}}<\ldots X_{i_{k}}\right)$. This permutation is denoted by $\operatorname{Std}^{-1}\left(X_{1}, \ldots, X_{k}\right)$. For each $k$, the random variable $\sigma_{(\emptyset, \emptyset)}(k)$ has a uniform distribution on $\mathfrak{S}_{k}$.
The next Theorem is due to Gnedin and Olshanski in [42] (based on an important work of Jacka and Warren in [45]) and identifies $\mathcal{U}^{(2)}$ with the minimal entrance boundary of the graded graph $\mathcal{Z}:$

Theorem 9.6. Each random variable $\sigma_{U}$ defines a random arrangement $\mathcal{A}$ that comes from a harmonic probability measure on $\left(\mathcal{Z}, P^{t}\right)$, and there is an isomorphism:

$$
\Phi: M_{1}\left(\mathcal{U}^{(2)}\right) \longrightarrow M_{1}\left(\partial_{\min } \mathcal{Z}\right)
$$

which restricts to a bijective map $p: \mathcal{U}^{(2)} \longrightarrow \partial_{\min } \mathcal{Z}$ mapping $\delta_{\left(U_{\uparrow}, U_{\downarrow}\right)}$ to $\sigma_{\left(U_{\uparrow}, U_{\downarrow}\right)}$.
In particular for each $k \geq 1$ and $\sigma \in \mathfrak{S}_{k}, \mathbb{P}\left(\sigma_{U}(k)=\sigma\right)$ only depends on the descent set $\mu$ of $\sigma$ and is thus denoted by $p_{U}(\mu)$.

### 9.4.2 Martin entrance boundary of $\mathcal{Z}$

The question is to know if $\partial_{\min } \mathcal{Z}=\partial_{M} \mathcal{Z}$. The problem is summed up in Conjecture 45 of [42]. To each composition $\lambda$ of $n$ is associated an element $U_{\lambda}=\left(U_{\uparrow}(\lambda), U_{\downarrow}(\lambda)\right)$ of $\mathcal{U}^{(2)}$ as follows : for each $s \leq n-1$ set $I_{s}=\left[\frac{s-1}{n-1}, \frac{s}{n-1}\right]$, and define

$$
U_{\uparrow}(\lambda)=\operatorname{int}\left(\bigcup_{i \notin \operatorname{des}(\lambda)} I_{s}\right), U_{\downarrow}(\lambda)=\operatorname{int}\left(\bigcup_{i \in \operatorname{des}(\lambda)} I_{s}\right),
$$

with int denoting the interior of a set. Then the conjecture states the following :

Conjecture 9.7. a) $A$ sequence $\left(\lambda_{n}\right)_{n \geq 1}$ is in $\partial_{M} \mathcal{Z}$ if and only if $U_{\lambda_{n}}$ converges in $\mathcal{U}^{(2)}$.
b) $U_{\lambda_{n}} \rightarrow_{\mathcal{U}^{(2)}}\left(U_{\uparrow}, U_{\downarrow}\right)$ is equivalent to $K_{\mu}\left(\lambda_{n}\right) \rightarrow p_{\left(U_{\uparrow}, U_{\downarrow}\right)}(\mu)$ for all $\mu \in \mathcal{Z}$.
c) The Martin boundary of the graph $\mathcal{Z}$ actually coincides with its minimal boundary : $\partial_{M} \mathcal{Z}=$ $\mathcal{U}^{(2)}$

Actually, the only difficult part is to prove the first implication of $b$ ):

$$
\begin{equation*}
\left(U_{\lambda_{n}} \rightarrow_{\mathcal{U}^{(2)}}\left(U_{\uparrow}, U_{\downarrow}\right)\right) \Longrightarrow\left(\forall \mu \in \mathcal{Z}, K_{\mu}\left(\lambda_{n}\right) \rightarrow p_{\left(U_{\uparrow}, U_{\downarrow}\right)}(\mu)\right) . \tag{9.4.1}
\end{equation*}
$$

Indeed suppose that the latter is true :
Proof. a) Let $\omega=\left(\lambda_{n}\right)_{n \geq 1}$ be in $\partial_{M} \mathcal{Z}$. Since $\mathcal{U}^{(2)}$ is compact, proving the convergence of $U_{\lambda_{n}}$ in $\mathcal{U}^{(2)}$ is the same as proving that every convergent subsequences of $U_{\lambda_{n}}$ have the same limit. Let $\left(\lambda_{\varphi(n)}\right)_{n \geq 1}$ and $\left(\lambda_{\varphi^{\prime}(n)}\right)_{n \geq 1}$ be such that

$$
U_{\lambda_{\varphi(n)}} \rightarrow\left(U_{\uparrow}^{1}, U_{\downarrow}^{2}\right), U_{\lambda_{\varphi^{\prime}(n)}} \rightarrow\left(U_{\uparrow}^{2}, U_{\downarrow}^{2}\right)
$$

Then by 9.4.1, for all $\mu \in \mathcal{Z}, \omega(\mu)=p_{U_{\uparrow}^{1}, U_{\downarrow}^{1}}(\mu)$ and $\omega(\mu)=p_{U_{\uparrow}^{2}, U_{\downarrow}^{2}}(\mu)$. Since $p: \mathcal{U}^{(2)} \rightarrow$ $\partial_{\min } \mathcal{Z}$ is injective, necessarily $\left(U_{\uparrow}^{1}, U_{\downarrow}^{1}\right)=\left(U_{\uparrow}^{2}, U_{\downarrow}^{2}\right)$. This shows that $U_{\lambda_{n}}$ converges. Conversely if $U_{\lambda_{n}}$ converges in $\mathcal{U}^{(2)}$, the assumption (9.4.1) implies directly that $\left(\lambda_{n}\right) \in$ $\partial_{M} \mathcal{Z}$.
b) The direct implication is exactly (9.4.1); for the converse implication, the convergence of $K_{\mu}\left(\lambda_{n}\right)$ for all $\mu \in \mathcal{Z}$ implies that $\left(\lambda_{n}\right)_{n \geq 1} \in \partial_{M} \mathcal{Z}$. Thus from $\left.a\right), U_{\lambda_{n}}$ converges in $\mathcal{U}^{(2)}$. By injectivity of $p, U_{\lambda_{n}}$ converges to $\left(U_{\uparrow}, U_{\downarrow}\right)$.
c) This is the summary of 1) and 2).

The following sections are devoted to the proof of the implication (9.4.1), which implies Conjecture 9.7:

Theorem 9.8. Let $\lambda_{n}$ be a sequence of compositions such that $\lambda_{n} \vdash n$. If $U_{\lambda_{n}}$ converges to $\left(U_{\uparrow}, U_{\downarrow}\right)$, then for all $\mu \in \mathcal{Z}$,

$$
K_{\mu}\left(\lambda_{n}\right) \rightarrow p_{\left(U_{\uparrow}, U_{\downarrow}\right)}(\mu)
$$

The result of Theorem 9.8 roughly means that, for $k$ fixed and $\lambda$ a large composition such that $U_{\lambda}$ is close to $\left(U_{\uparrow}, U_{\downarrow}\right)$, the restriction to $\{1, \ldots, k\}$ of the uniform random filling $\lambda$ yields a random variable on $\mathfrak{S}_{k}$ close to the Paintbox construction $\sigma_{\left(U_{\uparrow}, U_{\downarrow}\right)}(k)$. Since the Paintbox construction involves for each integer $1 \leq i \leq k$ an independent uniform random variable on $[0,1]$, we will also create in Section 5 a random variable $\xi_{i}^{\lambda}$ which mimicks the position of $i$ in the uniform random filling of $\lambda$. The proof of Theorem 9.8 consists then essentially in proving that the family $\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k}$ becomes a family of independent uniform random variables on $[0,1]$ when $\lambda$ becomes large. The latter convergence implies that the permutation $\sigma_{\lambda}$ is close to $\sigma_{U_{\lambda}}$. The fact that $U_{\lambda}$ is approximately $\left(U_{\uparrow}, U_{\downarrow}\right)$ will conclude the proof.
Note that there are two kinds of limit involved in the proof: the limit of the law of $\left(\xi_{i}^{\lambda_{n}}\right)_{1 \leq i \leq k}$ and the topological limit of $U_{\lambda_{n}}$. In order to finalize the proof, we need a final result showing that the order of the limits does not matter. Since the proof of the latter fact is straightforward but lengthy, it is postponed to the Appendix.
The convergence to the family of independent uniform random variables is not clear and explained in Sections 7 and 8. The proof uses the results of Chapter 8, which deal with combinatorics of large compositions and which are summarized in Section 6.

### 9.5 The familiy $\left(\xi_{i}^{\lambda}\right)_{i \geq 1}$

Some definitions on compositions are needed before defining the family $\left(\xi_{i}^{\lambda}\right)_{i \geq 1}$.

### 9.5.1 Combinatorics of compositions

Let $\lambda$ be a composition of $n$ with its associated descent set $D_{\lambda}=\left(\lambda_{1}, \lambda_{1}+\lambda_{2}, \ldots, \lambda_{1}+\cdots+\lambda_{r-1}\right)$. An integer $i \in[1 ; n]$ is a peak of $\lambda$ if $i \in D_{\lambda} \cup\{n\}$ and $i-1 \notin D_{\lambda}$, and $i \in[1 ; n]$ is a valley if $i \notin D_{\lambda}$ and $i-1 \in D_{\lambda} \cup\{0\}$. This definition makes sense if we consider any standard filling $\sigma$ of $\lambda$ : $\sigma(i)$ is a local maximum (resp. minimum) of $\sigma=\sigma(1) \ldots \sigma(n)$ if and only if $i$ is a peak (resp. valley) of $\lambda$. Let $V$ denote the set of valleys, $P$ the set of peaks, and $\mathcal{E}=V \cup P$ the set of extreme cells.
A run $s$ of $\lambda$ is an interval $[a ; b]$ of $[1 ; n]$ such that $a, b$ are consecutive integers of $\mathcal{E}$. A run $[a ; b]$ is called descending if $a \in P$ and ascending if $a \in V$. The runs are ordered by the lower endpoint of the corresponding interval, and this yields a total ordered set $S=\left\{s_{i}\right\}_{1 \leq i \leq t}$. Each element $s_{i}$ of $S$ corresponds to an interval $\left[a_{i} ; a_{i+1}\right]$, with $a_{1}=1$ and $a_{t+1}=n$. In particular two consecutive runs $s_{i}$ and $s_{i+1}$ overlap on $a_{i+1}$. The length of a run $s_{i}$ is defined as the value $l_{i}=a_{i+1}-a_{i}$. For example if $\lambda=(3,2,4,1), V=\{1,4,6,10\}, P=\{3,5,9\}$ and $S=\{[1 ; 3],[3 ; 4],[4 ; 5],[5,6],[6 ; 9],[9,10]\}$.
For any cell $i$ of $\lambda$, the slope of $i, \mathfrak{s}(i)=[x(i) ; y(i)]$, is defined as the maximum subinterval of $[1 ; n]$ that contains $i$ and no other peak or valley. In the previous example, $\mathfrak{s}(7)=[7 ; 8]$ and $\mathfrak{s}(6)=[6 ; 8]$.

### 9.5.2 Definition of $\left(\xi_{i}^{\lambda}\right)_{i \geq 1}$

Let $\left(X^{i}(p, q)\right) \quad i \geq 1$ be a family of independent variables such that $X^{i}(p, q)$ follows the uniform law on $[p, q]$.

Definition 9.9. Let $\lambda$ be a composition of $n, \sigma \in \Omega_{\lambda}$. The averaged coordinate of $i$ with respect to $\sigma$ is the random variable defined by $\xi_{i}(\sigma)=0$ if $i>n$, and

$$
\xi_{i}(\sigma)=X^{i}\left(\frac{x\left(\sigma^{-1}(i)\right)-1}{n}, \frac{y\left(\sigma^{-1}(i)\right)}{n}\right),
$$

for $1 \leq i \leq n$.
For $\sigma_{\lambda}$ chosen uniformly among $\Omega_{\lambda}, \xi_{i}\left(\sigma_{\lambda}\right)$ is denoted $\xi_{i}^{\lambda}$. $\xi^{\lambda}(k)$ denotes the vector $\left(\xi_{1}^{\lambda}, \ldots, \xi_{k}^{\lambda}\right)$ and $\xi^{\lambda}(n)$ is simply written $\xi^{\lambda}$.

Basically constructing $\xi_{i}^{\lambda}$ means that we sample a uniformly random standard filling $\sigma_{\lambda}$ of $\lambda$, we look at the cell containing $i$ with respect to this filling, and then sample a random variable uniformly distributed on the rescaled slope of this cell. The advantage is that the knowledge of $\xi^{\lambda}(k)$ is enough to reconstruct $\sigma_{k}^{\lambda}$. This reconstruction needs a slightly modified version of $U_{\lambda}$ :
Definition 9.10. The run paintbox $\tilde{U}_{\lambda}$ associated to $\lambda$ is an element of $\mathcal{U}^{(2)}$ consisting in the following open subsets:

- $\left.\tilde{U}_{\uparrow}(\lambda)=\bigcup_{a_{i} \in V}\right] \frac{a_{i}-1}{n}, \frac{a_{i+1}-1}{n}[$
- $\left.\tilde{U}_{\downarrow}(\lambda)=\bigcup_{a_{i} \in P}\right]^{\frac{a_{i}-1}{n}}, \frac{a_{i+1}-1}{n}[$
with $a_{i+1}=n+1$ if $a_{i}=n$.
The run paintbox $\tilde{U}_{\lambda}$ becomes close to $U_{\lambda}$ when $n$ goes to infinity:

Lemma 9.11. Let $\lambda$ be a composition of $n$. With respect to the distance on $\mathcal{U}^{(2)}$,

$$
d\left(U_{\lambda}, \tilde{U}_{\lambda}\right) \leq \frac{1}{n}
$$

Proof. The definition of $U_{\lambda}$ yields the following open sets:

$$
\left.U_{\uparrow}(\lambda)=\bigcup_{a_{i} \in V, a_{i} \neq n}\right] \frac{a_{i}-1}{n-1}, \frac{a_{i+1}-1}{n-1}[,
$$

and

$$
\left.U_{\downarrow}(\lambda)=\bigcup_{a_{i} \in P, a_{i} \neq n}\right] \frac{a_{i}-1}{n-1}, \frac{a_{i+1}-1}{n-1}[.
$$

Let us show that $U_{\uparrow}(\lambda)^{c}$ is included in the $\frac{1}{n}$-inflation of $\tilde{U}_{\uparrow}(\lambda)^{c}$ and conversely (the proof for $U_{\downarrow}(\lambda)^{c}$ and $\tilde{U}_{\downarrow}(\lambda)^{c}$ is the same). The $\frac{1}{n}$-inflation of $U_{\uparrow}(\lambda)^{c}$ is

$$
U_{\uparrow}(\lambda)^{c, 1 / n}=\left(\bigcup_{a_{i} \in P, a_{i} \neq n}\left[\left(\frac{a_{i}-1}{n-1}-\frac{1}{n}\right) \vee 0,\left(\frac{a_{i+1}-1}{n-1}+\frac{1}{n}\right) \wedge 1\right]\right) \cup\left[0, \frac{1}{n}\right] \cup\left[1-\frac{1}{n}, 1\right] .
$$

On the other hand

$$
\tilde{U}_{\uparrow}(\lambda)^{c}=\left(\bigcup_{a_{i} \in P}\left[\frac{a_{i}-1}{n}, \frac{a_{i+1}-1}{n}\right]\right) \cup\{0\} \cup\{1\} .
$$

Suppose that $a_{i} \neq n$. Then for all $1 \leq k \leq n-1, \frac{k}{n-1}-\frac{1}{n} \leq \frac{k}{n} \leq \frac{k}{n-1}+\frac{1}{n}$, thus

$$
\left[\frac{a_{i}-1}{n}, \frac{a_{i+1}-1}{n}\right] \subseteq\left[\left(\frac{a_{i}-1}{n-1}-\frac{1}{n}\right) \vee 0,\left(\frac{a_{i+1}-1}{n-1}+\frac{1}{n}\right) \wedge 1\right] \subseteq U_{\uparrow}(\lambda)^{c, 1 / n} .
$$

If $a_{i}=n,\left[\frac{a_{i}-1}{n}, \frac{a_{i+1}-1}{n}\right]=[1-1 / n, 1] \subseteq U_{\uparrow}(\lambda)^{c, 1 / n}$. Finally $\tilde{U}_{\uparrow}(\lambda)^{c} \subseteq U_{\uparrow}(\lambda)^{c, 1 / n}$.
For the converse inclusion the $\frac{1}{n}$-inflation of $\tilde{U}_{\uparrow}(\lambda)^{c}$ is

$$
\tilde{U}_{\uparrow}(\lambda)^{c, 1 / n}=\left(\bigcup_{a_{i} \in P}\left[\left(\frac{a_{i}-1}{n}-\frac{1}{n}\right) \vee 0,\left(\frac{a_{i+1}-1}{n}+\frac{1}{n}\right) \wedge 1\right]\right) \cup\left[0, \frac{1}{n}\right] \cup\left[1-\frac{1}{n}, 1\right],
$$

and

$$
U_{\uparrow}(\lambda)^{c}=\left(\bigcup_{a_{i} \in P, a_{i} \neq n}\left[\frac{a_{i}-1}{n-1}, \frac{a_{i+1}-1}{n-1}\right]\right) \cup\{0\} \cup\{1\} .
$$

Since for $1 \leq k \leq n-1, \frac{k}{n}-\frac{1}{n} \leq \frac{k}{n-1} \leq \frac{k}{n}+\frac{1}{n}$, for each $a_{i} \neq n$,

$$
\left[\frac{a_{i}-1}{n-1}, \frac{a_{i+1}-1}{n-1}\right] \subseteq\left[\left(\frac{a_{i}-1}{n}-\frac{1}{n}\right) \vee 0,\left(\frac{a_{i+1}-1}{n}+\frac{1}{n}\right) \wedge 1\right]
$$

and therefore $U_{\uparrow}(\lambda)^{c} \subseteq \tilde{U}_{\uparrow}(\lambda)^{c, 1 / n}$.
Doing the same for $U_{\downarrow}(\lambda)$ and $\tilde{U}_{\downarrow}(\lambda)$ concludes the proof.
The previous Lemma implies that for any sequence $\left(\lambda_{n}\right)_{n \geq 1}$ with $\left|\lambda_{n}\right| \rightarrow \infty$, the convergence of $U_{\lambda_{n}}$ is equivalent to the convergence of $\tilde{U}_{\lambda_{n}}$, and both have the same limit. The advantage is that the knowledge of $\xi_{k}^{\lambda}$ and $\tilde{U}_{\lambda_{n}}$ is enough to recover $\sigma_{k}^{\lambda}$. Recall that $\sigma_{\tilde{U}_{\lambda}}\left(\left(\xi_{i}(\sigma)\right)_{1 \leq i \leq k}\right)$ is the paintbox construction relative to the tuple $\left(\xi_{i}(\sigma)\right)_{1 \leq i \leq k}$ and the paintbox $\tilde{U}_{\lambda}$ as defined in subsection 9.4.1.

Proposition 9.12. For each $1 \leq k \leq n, \sigma \in \Omega_{\lambda}$,

$$
\sigma_{\tilde{U}_{\lambda}}\left(\left(\xi_{i}(\sigma)\right)_{1 \leq i \leq k}\right)=\sigma_{\downarrow k}
$$

In particular the random variables $\sigma_{k}^{\lambda}$ and $\sigma_{\tilde{U}_{\lambda}}\left(\xi^{\lambda}(k)\right)$ have the same law.
Proof. It is enough to prove it for $k=n$. Denote $\xi_{i}(\sigma)=\xi_{i}$ and $\xi=\left(\xi_{i}(\sigma)\right)_{1 \leq i \leq n}$. It is equivalent to prove that for $1 \leq i, j \leq n$,

$$
\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(i)<\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(j) \Leftrightarrow \sigma_{\lambda}^{-1}(i)<\sigma_{\lambda}^{-1}(j)
$$

Let $1 \leq i<j \leq n$. Then $\sigma_{\lambda}^{-1}(i)<\sigma_{\lambda}^{-1}(j)$ implies that $i$ is left to $j$ in the associated filling of $\lambda$. This is possible in one of the two following situations :

1. $\mathfrak{s}(i)$ and $\mathfrak{s}(j)$ are disjoint and $\mathfrak{s}(i)$ is left to $\mathfrak{s}(j)$. In this case $\xi_{i}$ and $\xi_{j}$ are not in the same interval component of $\tilde{U}_{\lambda}$ and $\xi_{i}$ is in an interval component left to the one of $\xi_{j}$. By the run Paintbox construction,

$$
\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(i)<\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(j)
$$

2. $\mathfrak{s}(i)$ and $\mathfrak{s}(j)$ overlap. This implies that $i$ and $j$ are in a same run $s=\left[a_{i} ; a_{i+1}\right]$ of $\lambda$. Let $\left.I_{s}=\right] \frac{a_{i}-1}{n}, \frac{a_{i+1}-1}{n}\left[\right.$. Since $i<j$ and $\sigma_{\lambda}^{-1}(i)<\sigma_{\lambda}^{-1}(j)$, the run $s$ has to be an ascending one and thus $a_{i} \in V$ and $a_{i+1} \in P$. In particular $\sigma_{\lambda}^{-1}(i)$ cannot be a peak, and $\sigma_{\lambda}^{-1}(j)$ cannot be a valley. Thus $\xi_{i}$ is either in an interval component left to $I_{s}$, either in $I_{s}$. For similar reasons, $\xi_{j}$ is either in an interval component right to $I_{s}$, either in ${ }_{s}$. This implies that if $\xi_{i}$ or $\xi_{j}$ is not in $I_{s},\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(i)<\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(j)$. But if $\xi_{i}$ and $\xi_{j}$ are both in $I_{s}$, since the latter is in $\tilde{U}_{\uparrow}(\lambda)$, the same inequality holds.

Finally, in any case,

$$
\sigma_{\lambda}^{-1}(i)<\sigma_{\lambda}^{-1}(j) \Longrightarrow\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(i)<\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(j)
$$

The pattern is exactly the same to prove that

$$
\sigma_{\lambda}^{-1}(i)>\sigma_{\lambda}^{-1}(j) \Longrightarrow\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(i)>\left(\sigma_{\tilde{U}_{\lambda}}(\xi)\right)^{-1}(j)
$$

yielding the first part of the Proposition. This first part implies clearly the second one.
It is also possible to recover exactly the position of $\{1, \ldots, k\}$ in the filling $\sigma$ of $\lambda$ from $\left(\xi_{i}(\sigma)\right)_{1 \leq i \leq k}$ :
Lemma 9.13. Let $\sigma, \sigma^{\prime}$ be two permutations of $\Omega_{\lambda}$. If $\left(\sigma^{-1}(1), \ldots, \sigma^{-1}(k)\right)$ is not equal to $\left(\sigma^{\prime-1}(1), \ldots, \sigma^{\prime-1}(k)\right)$, then $\left(\xi_{1}(\sigma), \ldots, \xi_{k}(\sigma)\right)$ and $\left(\xi_{1}\left(\sigma^{\prime}\right), \ldots, \xi_{k}\left(\sigma^{\prime}\right)\right)$ have disjoint supports.

Proof. The proof is done by recurrence on $k \geq 1$. Let $k=1$. 1 has to be located in a valley of $\lambda$. If $\sigma^{-1}(1) \neq \sigma^{\prime-1}(1), 1$ is located in a different valley in $\sigma$ and $\sigma^{\prime}$. Thus the slopes of $\sigma^{-1}(1)$ and $\sigma^{\prime-1}(1)$ are disjoint, and $\xi_{1}(\sigma)$ and $\xi_{1}(\sigma)$ have disjoint supports.
Let $k>1$. Suppose that $\left(\sigma^{-1}(1), \ldots, \sigma^{-1}(k)\right) \neq\left(\sigma^{\prime-1}(1), \ldots, \sigma^{\prime-1}(k)\right)$. By recurrence hypothesis, if $\left(\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1)\right)$ is not equal to $\left(\sigma^{\prime-1}(1), \ldots, \sigma^{\prime-1}(k-1)\right),\left(\xi_{1}(\sigma), \ldots, \xi_{k-1}(\sigma)\right)$ and $\left(\xi_{1}\left(\sigma^{\prime}\right), \ldots, \xi_{k-1}\left(\sigma^{\prime}\right)\right)$ have disjoint supports. This yields also that $\left(\xi_{1}(\sigma), \ldots, \xi_{k}(\sigma)\right)$ and $\left(\xi_{1}\left(\sigma^{\prime}\right), \ldots, \xi_{k}\left(\sigma^{\prime}\right)\right)$ have disjoint supports.
Thus let us assume that $\left(\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1)\right)=\left(\sigma^{\prime-1}(1), \ldots, \sigma^{\prime-1}(k-1)\right)$. This implies that $\sigma^{-1}(k) \neq \sigma^{\prime-1}(k)$; since the position of $\{1, \ldots, k-1\}$ is the same in the fillings $\sigma$ and $\sigma^{\prime}$ of $\lambda$, the cell containing $k$ in $\sigma$ is in a different run than the cell containing $k$ in $\sigma^{\prime}$. Therefore their slopes are disjoint, and $\left(\xi_{1}(\sigma), \ldots, \xi_{k}(\sigma)\right)$ and $\left(\xi_{1}\left(\sigma^{\prime}\right), \ldots, \xi_{k}\left(\sigma^{\prime}\right)\right)$ have disjoint supports.

This section ends by a convergence result. Although this result is crucial for the proof of Theorem 9.4.1, its proof is rather technical and has been postponed to the Appendix.
Proposition 9.14. Let $U_{n}$ be a sequence of $\mathcal{U}^{(2)}$ and $\left(\left(X^{n}(i)\right)_{i \geq 1}\right)_{n \geq 1}$ a sequence of random infinite vectors on $[0,1]$. Let $\left(X^{0}(1), \ldots, X^{0}(n), \ldots\right)$ be a random infinite vector on $[0,1]$. Suppose that each finite dimensional marginal law of any of these random vectors admits a density with respect to the Lebesgue measure. If $U_{n} \rightarrow U \in \mathcal{U}^{(2)}$ and for each $k \geq 1, X_{k}^{n}=\left(X^{n}(1), \ldots, X^{n}(k)\right)$ converges in law to $X_{k}^{0}=\left(X^{0}(1), \ldots, X^{0}(k)\right)$, then for each $k \geq 1$,

$$
\sigma_{U_{n}}\left(X_{k}^{n}\right) \longrightarrow_{\text {law }} \sigma_{U}\left(X_{k}^{0}\right)
$$

### 9.6 Combinatorics of large compositions

The purpose of this section is to introduce the background material to prove that the family $\left(\xi_{u}^{\lambda}\right)_{1 \leq u \leq k}$ converges in law to a family of independent uniform random variables on $[0,1]$. Since $\xi_{u}^{\lambda}$ depends uniquely on the runs in which $u$ is located in a random filling $\sigma_{\lambda}$ of $\lambda$, it is necessary to evaluate the probability for $u$ to be located in a particular run $s$ of $\lambda$. For a composition $\lambda$ and $i \in \lambda$ a fixed cell, denote by $\lambda_{\leq i}$ (resp. $\lambda_{\geq i}, \lambda_{<i}, \lambda_{>i}$ ) the composition $\lambda$ restricted to cells left (resp. right, res. strictly left, resp. strictly right) to $i$. Recall that $d(\lambda)$ denotes the number of standard fillings of the ribbon Young diagram associated to $\lambda$.
Let us focus here on the location of 1 in $\sigma_{\lambda}$. Since 1 is necessary a local minimum in any filling of $\lambda$, it has to be located in a valley $v \in V$. For a fixed valley $v$ of $\lambda$, the cardinal of standard fillings of $\lambda$ such that 1 is located in $v$ is exactly the number of possibilities to fill in the part of $\lambda$ left of $v$, with any subset $S$ of cardinal $\left|\lambda_{<v}\right|$ of $[2, n]$, and to fill in independently the part of $\lambda$ right to $v$ with the complementary subset of $S$ in $[2, n]$. Thus

$$
\begin{equation*}
\mathbb{P}_{\sigma_{\lambda}}(1 \in v)=\frac{(|\lambda|-1)!}{\left|\lambda_{<v}\right|!\left|\lambda_{>v}\right|!} \frac{d\left(\lambda_{>v}\right) d\left(\lambda_{<v}\right)}{d(\lambda)} . \tag{9.6.1}
\end{equation*}
$$

The problem is therefore essentially to relate $d\left(\lambda_{>v}\right) d\left(\lambda_{<v}\right)$ to $d(\lambda)$.

### 9.6.1 Probabilistic approach to descent combinatorics

Ehrenborg, Levin and Readdy (see [37]) formalized in the context of descent sets an old relation between permutations of $n$ and polytopes in $[0,1]^{n}$. Namely since the volume of the set $R_{\sigma}=\left\{x_{\sigma(1)}<\cdots<x_{\sigma(n)}\right\}$ is exactly $\frac{1}{n!}$, it is possible to determine probabilistic quantities on $\mathfrak{S}_{n}$ by integrating certain functions that are constant on each region $R_{\sigma}$. The appropriate functions for descent sets were found in [37], yielding some new estimates as in 36] and [18]. The model of Ehrenborg, Levin and Readdy is exposed in this paragraph, but in a modified way to focus only on the set of extreme cells $\mathcal{E}$ (as defined in the paragraph 5.1). This yields the following framework : let $\lambda$ be a composition of $n \geq 2$ with set of extreme cells $\mathcal{E}=\left\{a_{1}=\right.$ $\left.1, a_{2}, \ldots, a_{r}=n\right\}$. Suppose for example that the first cell is a valley (namely $a_{1} \in V$ ) and denote by $s_{j}$ the run between $a_{j}$ and $a_{j+1}$, with $l_{j}$ its length. To each $\lambda$ is associated the couple of random variable $\left(X_{\lambda}, Y_{\lambda}\right)$ on $[0,1]^{2}$ with density

$$
\begin{equation*}
d_{X_{\lambda}, Y_{\lambda}}\left(x_{1}, x_{r}\right)=\frac{1}{\mathcal{V}_{\lambda}} \int_{[0,1]^{r-2}} \prod \mathbf{1}_{x_{2 i-1}<x_{2 i}>x_{2 i+1}} \prod_{1 \leq i \leq r-1} \frac{\left|x_{i}-x_{i+1}\right|^{l_{i}-1}}{\left(l_{i}-1\right)!} \prod_{2 \leq i \leq r-1} d x_{i} . \tag{9.6.2}
\end{equation*}
$$

If the first cell is a peak (i.e $a_{1} \in P$ ), the inequalities in the expression of the density are reversed. If $\lambda=\square$, the expression for the distribution of ( $X_{\square}, Y_{\square}$ ) (in the distributional sense) is simply:

$$
d_{X_{\square}, Y_{\square}}(u, v)=\delta_{u=v} .
$$

The latter probabilistic model is related to the combinatorics of descent sets through the equality

$$
\begin{equation*}
d(\lambda)=|\lambda|!\mathcal{V}_{\lambda}, \tag{9.6.3}
\end{equation*}
$$

whose proof can be found in [37.
The first advantage of this model is that it behaves simply with respect to concatenation of compositions.
Definition 9.15. Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ and $\mu=\left(\mu_{1}, \ldots, \mu_{s}\right)$ be two compositions of $m$ and $n$. The concatenated composision $\lambda+\mu$ is the composition of $n+m$

$$
\lambda+\mu=\left(\lambda_{1}, \ldots, \lambda_{r}+\mu_{1}, \mu_{2}, \ldots, \mu_{s}\right),
$$

and the concatenated composition $\lambda-\mu$ is the composition of $n+m$

$$
\lambda-\mu=\left(\lambda_{1}, \ldots, \lambda_{r}, \mu_{1}, \mu_{2}, \ldots, \mu_{s}\right) .
$$

This definition has a simple meaning in terms of associated ribbon Young diagrams: namely the diagram of $\lambda+\mu$ (resp. $\lambda-\mu$ ) is the juxtaposition of the one of $\lambda$ and the one of $\mu$ such that the last cell of $\lambda$ is left to (resp. above) the first cell of $\mu$. An application of the section 2 of [37] (see also Lemma 2 in [18]) implies the following expression of the concatenation in the probabilistic framework :
Proposition 9.16. Let $\lambda, \mu$ be two compositions, $\varepsilon \in\{-,+\}$. Then

$$
\mathcal{V}_{\lambda \varepsilon \mu}=\mathcal{V}_{\lambda} \mathcal{V}_{\mu} \mathbb{E}\left(Y_{\lambda} \leq_{\varepsilon} X_{\mu}\right)
$$

and

$$
d_{X_{\lambda \varepsilon \mu}, Y_{\lambda \varepsilon \mu}}(x, y)=\frac{1}{\mathbb{E}\left(Y_{\lambda} \leq_{\varepsilon} X_{\mu}\right)} \int_{[0,1]^{2}} d_{X_{\lambda}, Y_{\lambda}}(x, u) \mathbf{1}_{u \leq_{\varepsilon} v} d_{X_{\mu}, Y_{\mu}}(v, y) d u d v,
$$

where $\leq_{-}=\geq$and $\leq_{+}=\leq$, and the couples $\left(X_{\lambda}, Y_{\lambda}\right)$ and $\left(X_{\mu}, Y_{\mu}\right)$ are considered as independent.
The previous Proposition yields a particular case that helps to compute the law of $\xi_{1}^{\lambda}$. Denote by $F_{X}$ the distributive cumulative function of a random variable $X$.
Corollary 9.17. Let $\lambda$ be a composition of $n$ and $v$ a valley of $\lambda$. Then

$$
\mathbb{P}_{\lambda}(1 \in v)=\frac{1}{n} \frac{1}{\int_{0}^{1}\left(1-F_{Y_{\lambda<v}}(t)\right)\left(1-F_{X_{\lambda>v}}(t)\right) d t},
$$

with the convention $X_{\lambda>n}=\delta_{1}$ and $Y_{\lambda_{<1}}=\delta_{1}$.
Proof. Since $v$ is a valley, $\lambda$ can be written $\lambda_{<v}-\square+\lambda_{>v}$. Thus the previous Proposition yields

$$
\mathcal{V}_{\lambda_{<v}-\square+\lambda>v}=\mathcal{V}_{\lambda_{<v}} \mathcal{V}_{\lambda_{\geq v}} \mathbb{E}\left(Y_{\lambda_{<v}} \geq X_{\lambda_{\geq v}}\right) .
$$

Conditioning the expectation on the value of $X_{\lambda \geq v}$ gives by independence,

$$
\mathbb{E}\left(Y_{\lambda_{<v}} \geq X_{\lambda_{\geq v}}\right)=\int_{0}^{1}\left(1-F_{Y_{\lambda_{<v}}}(t)\right) d_{X_{\lambda_{\geq v}}}(t) d t .
$$

On the other hand from the previous Proposition, since $X_{\lambda_{\geq v}}=X_{\square+\lambda>v}$,

$$
\begin{aligned}
d_{X_{\lambda \geq v}}(t) & =\frac{1}{\mathbb{E}\left(X_{\lambda>v} \geq Y_{\square}\right)} \int_{[0,1]^{3}} \delta(t, u) \mathbf{1}_{u \leq v} d_{X_{\lambda>v}, Y_{\lambda>v}}(v, y) d u d v d y \\
& =\frac{\mathcal{V}_{\lambda>v} \mathcal{V}_{\square}}{\mathcal{V}_{\lambda_{\geq v}}}\left(1-F_{X_{\lambda>v}}(t)\right),
\end{aligned}
$$

and finally

$$
\mathcal{V}_{\lambda_{<v}-\square+\lambda>v}=\mathcal{V}_{\lambda_{<v}} \mathcal{V}_{\lambda>v} \int_{0}^{1}\left(1-F_{Y_{\lambda_{<v}}}(t)\right)\left(1-F_{X_{\lambda>v}}(t)\right) d t
$$

Using the latter result in the equalities (9.6.1) and 9.6.3 yields

$$
\begin{aligned}
\mathbb{P}_{\sigma_{\lambda}}(1 \in v) & =\frac{(|\lambda|-1)!}{\left|\lambda_{<v}\right|!\left|\lambda_{>v}\right|!} \frac{d\left(\lambda_{>v}\right) d\left(\lambda_{<v}\right)}{d(\lambda)} \\
& =\frac{(|\lambda|-1)!}{\left|\lambda_{<v}\right|!\left|\lambda_{>v}\right|!} \frac{\left|\lambda_{<v}\right|!\left|\lambda_{>v}\right|!}{|\lambda|!} \frac{\mathcal{V}_{\lambda_{>v}} \mathcal{V}_{\lambda_{<v}}}{\mathcal{V}_{\lambda}} \\
& =\frac{1}{|\lambda|} \frac{\mathcal{V}_{\lambda_{>v}} \mathcal{V}_{\lambda_{<v}}}{\mathcal{V}_{\lambda_{<v}} \mathcal{V}_{\lambda_{>v}} \int_{0}^{1}\left(1-F_{Y_{\lambda_{<v}}}(t)\right)\left(1-F_{X_{\lambda>v}}(t)\right) d t} \\
& =\frac{1}{|\lambda|} \frac{1}{\int_{0}^{1}\left(1-F_{Y_{\lambda<v}}(t)\right)\left(1-F_{X_{\lambda>v}}(t)\right) d t}
\end{aligned}
$$

### 9.6.2 Estimates on $\left(X_{\lambda}, Y_{\lambda}\right)$

The latter corollary shows that the knowledge of $F_{X_{\mu}}$ and $F_{Y_{\mu}}$ for any subcomposition $\mu$ of $\lambda$ yields estimates on the location of 1 in $\sigma_{\lambda}$. The results on the behavior of $F_{X_{\mu}}, F_{Y_{\mu}}$ obtained in Chapter 8 are summarized in this paragraph, and the reader should refer to this chapter for the corresponding proofs. The first result is a bound of $F_{X_{\lambda}}$ depending on the length of the first run of $\lambda$ and corresponds to Corollary 2 in Chapter 8 (and the following paragraph):

Proposition 9.18 (Cor. 2 Chapter 8). Let $\lambda$ be a composition with at least two runs, and with first run of length $R$. If the first run is increasing, the following inequality holds :

$$
1-(1-t)^{R} \leq F_{X_{\lambda}}(t) \leq 1-(1-t)^{R+1}
$$

If the first run is decreasing, the inequality is

$$
t^{R+1} \leq F_{X_{\lambda}}(t) \leq t^{R}
$$

The same result holds for $Y_{\lambda}$ after exchanging increasing run and decreasing run. The latter inequalities are very accurate when $R$ is large, but when the runs remain bounded, the result is not so useful. It is still possible to show that the distribution of $X_{\lambda}$ only depends on the first cells of the composition. This corresponds to Proposition 11 in Chapter 8.

Proposition 9.19 (Prop. 11,Ch. 8). Let $\varepsilon>0$. There exists $n_{0} \geq 1$ such that for any $n \geq n_{0}$ and any composition $\lambda$ of size larger than $n$ with first run smaller than $n$,

$$
\left\|F_{X_{\lambda_{\leq n}}}-F_{X_{\lambda}}\right\|_{\infty} \leq \varepsilon
$$

In the latter result, $n_{0}$ depends only on $\varepsilon$, and not on the shape of $\lambda$.

### 9.7 Asymptotic law of $\xi_{1}^{\lambda}$

This section is devoted to the asymptotic law of $\xi_{1}^{\lambda}$.

### 9.7.1 Preliminary results

Propositions 9.18 and 9.19 imply that $\mathbb{P}_{\lambda}(1 \in v)$ only depends on the shape of $\lambda$ around $v$.
Lemma 9.20. Let $\varepsilon>0$. There exists $n_{\varepsilon} \in \mathbb{N}$ such that for $n_{0} \geq n_{\varepsilon}$ and two compositions $\lambda \vdash n$ and $\mu \vdash m$ with the first run of $\mu$ smaller than $n_{0}$,

$$
1-\varepsilon \leq \frac{\int_{0}^{1}\left(1-F_{Y_{\lambda}}(t)\right)\left(1-F_{X_{\mu}}(t)\right) d t}{\int_{0}^{1}\left(1-F_{Y_{\lambda}}(t)\right)\left(1-F_{X_{\mu_{\leq n}}}(t)\right) d t} \leq 1+\varepsilon .
$$

Proof. Let $\lambda, \mu$ be two compositions, with $L$ the size of the last run of $\lambda$ and $R$ the size of the first run of $\mu$. Set $\varepsilon_{1}=+$ if the last run of $\lambda$ is increasing, $\varepsilon_{1}=-$ else, and the same with $\varepsilon_{2}$ and the first run of $\mu$. Let $\Delta=\int_{0}^{1}\left(1-F_{Y_{\lambda}}(t)\right)\left(1-F_{X_{\mu}}(t)\right) d t$. From Proposition 9.18, integrating the inequalities yields the following bounds on $\Delta$ :

- If $\varepsilon_{1}=+, \varepsilon_{2}=+$,

$$
\frac{1}{R+2}\left(1-\frac{1}{(R+3) \ldots(R+L+2)}\right) \leq \Delta \leq \frac{1}{R+1}\left(1-\frac{1}{(R+2) \ldots(R+L+2)}\right),
$$

- if $\varepsilon_{1}=-, \varepsilon_{2}=+$,

$$
\frac{1}{L+R+3} \leq \Delta \leq \frac{1}{L+R+1},
$$

- if $\varepsilon_{1}=+, \varepsilon_{2}=-$,

$$
1-\frac{1}{R+1}-\frac{1}{L+1}+\frac{1}{R+L+1} \leq \Delta \leq 1-\frac{1}{R+2}-\frac{1}{L+2}+\frac{1}{L+R+3},
$$

- if $\varepsilon_{1}=-, \varepsilon_{2}=-$,

$$
\frac{1}{L+2}\left(1-\frac{1}{(L+3) \ldots(L+R+2)}\right) \leq \Delta \leq \frac{1}{L+1}\left(1-\frac{1}{(L+2) \ldots(L+R+2)}\right) .
$$

The latter bounds are independent of the shape of $\lambda, \mu$ apart from the lengths of the last run of $\lambda$ and the first run of $\mu$. Denote by $A_{L, R}^{\varepsilon_{1}, \varepsilon_{2}}$ each upper bound in the previous list, and $B_{L, R}^{\varepsilon_{1}, \varepsilon_{2}}$ each lower bound. Then as $\min (L, R) \rightarrow \infty$,

$$
\frac{B_{L, R}^{\varepsilon_{1}, \varepsilon_{2}}}{A_{L, R}^{\varepsilon_{1,2}, \varepsilon_{2}}} \rightarrow 1 .
$$

Thus there exists $K$ such that if $\min (L, R) \geq K$, whatever is the shape of $\lambda, \mu$ outside these runs and $n_{0} \geq R$,

$$
1-\varepsilon \leq \frac{\int_{0}^{1}\left(1-F_{Y_{\lambda}}(t)\right)\left(1-F_{X_{\mu}}(t)\right) d t}{\int_{0}^{1}\left(1-F_{Y_{\lambda}}(t)\right)\left(1-F_{X_{\mu^{\prime}}}(t)\right) d t} \leq 1+\varepsilon .
$$

From now on, let us assume that the last run of $\lambda$ and the first run of $\mu$ are bounded by $K$. Set
 being finite, this yields $\eta>0$.
By Proposition 9.19, there exists $n_{\varepsilon} \geq 1$ such that for $n_{0} \geq n_{\varepsilon}$ and any composition $\nu$ of size $n \geq n_{0}$ and first run smaller than $n_{0}, F_{X_{\nu \leq n_{0}}}=F_{X_{\nu}}+g$ with $\|g\|_{\infty} \leq \varepsilon \eta$. Let $n_{0} \geq n_{\varepsilon}$ and
suppose that $\lambda \vdash n, \mu \vdash m$ with the first run of $\mu$ smaller than $n_{0}$. Then there exists $g$ such that $\|g\|_{\infty} \leq \varepsilon \eta$, and $F_{X_{\mu_{\leq n}}}=F_{X_{\mu}}+g$. This implies

$$
\begin{array}{r}
\int_{0}^{1}\left(1-F_{Y_{\lambda}}(t)\right)\left(1-F_{X_{\mu_{\leq n_{0}}}}(t)\right) d t=\int_{0}^{1}\left(1-F_{Y_{\lambda}}(t)\right)\left(1-F_{X_{\mu}}(t)-g(t)\right) d t \\
=\Delta-\int_{0}^{1} g(t)\left(1-F_{Y_{\lambda}}(t)\right) d t .
\end{array}
$$

Since $\left|\int_{0}^{1} g(t)\left(1-F_{Y_{\lambda}}(t)\right) d t\right| \leq \varepsilon \eta$ and $\Delta \geq \eta$,

$$
1-\varepsilon \leq \frac{\int_{0}^{1}\left(1-F_{Y_{\lambda}}(t)\right)\left(1-F_{X_{\mu_{\leq n_{0}}}}(t)\right) d t}{\Delta} \leq 1+\varepsilon .
$$

A corollary of the previous lemma yields a precise estimate of the probability that 1 is located in a particular valley $v$ when the length of the slope of $v$ goes to $+\infty$.

Corollary 9.21. Let $\varepsilon>0$. There exists $n_{0} \geq 1$ such that if $\lambda$ is a composition, and $v \in \lambda$ is a valley with slope $\mathfrak{s}(v)=[a ; b]$ of size $b-a \geq n_{0}$, then

$$
(1-\varepsilon) \frac{b-a}{n} \leq \mathbb{P}_{\lambda}(1 \in v) \leq(1+\varepsilon) \frac{b-a}{n} .
$$

Proof. Since $v$ is a valley, $v$ belongs to two runs $s_{i}, s_{i+1}$. If $b-a \geq 1$, then at least one run containing $v$ is of size larger than 2. Assume without loss of generality that $l\left(s_{i+1}\right) \geq 2$. This means that the first run of $\lambda_{>v}$ is increasing. Let $\Delta=\int_{0}^{1}\left(1-F_{Y_{\lambda<v}}(t)\right)\left(1-F_{X_{\lambda>v}}(t)\right) d t$. Let $L$ denote the length of the last run of $\lambda_{<v}$, and $R$ the length of the first run of $\lambda_{>v}$.
If $l\left(s_{i}\right)=1$, the last run of $\lambda_{<v}$ is increasing. Moreover in this case $b-a=R$. Thus the bounds on $\Delta$ from the proof of the last Lemma yield

$$
\frac{1}{b-a+2}\left(1-\frac{1}{\prod_{i=3}^{L+2}(b-a+i)}\right) \leq \Delta \leq \frac{1}{b-a+1}\left(1-\frac{1}{\prod_{i=2}^{L+2}(b-a+i)}\right) .
$$

Thus independently from $L$, there exists $n_{1}$ such that if $l\left(s_{i}\right)=1$ and $b-a \geq n_{1}$, then ( $1-$ $\varepsilon)(b-a) \leq \Delta^{-1} \leq(1+\varepsilon)(b-a)$.
If $l\left(s_{i}\right)>1$, the last run of $\lambda_{<v}$ is decreasing. Then $b-a=L+R-1$, and the bounds on $\Delta$ from the proof of the previous Lemma yield

$$
\frac{1}{b-a+2} \leq \Delta \leq \frac{1}{b-a} .
$$

There exists $n_{2}$ such that if $l\left(s_{i}\right)>1$ and $b-a \geq n_{2}$, then $(1-\varepsilon)(b-a) \leq \Delta^{-1} \leq(1+\varepsilon)(b-a)$. Set $n_{0}=\max \left(n_{1}, n_{2}\right)$, and let $n \geq n_{0}$. From Corollary $9.17 \mathbb{P}_{\lambda}(1 \in v)=\frac{1}{n \Delta}$, and thus

$$
(1-\varepsilon) \frac{b-a}{n} \leq \mathbb{P}_{\lambda}(1 \in v) \leq(1+\varepsilon) \frac{b-a}{n} .
$$

From the bounds $\left\{A_{R, L}^{\varepsilon_{1}, \varepsilon_{2}}, B_{R, L}^{\varepsilon_{1}, \varepsilon_{2}}\right\}$ on $\Delta$ that were found in the proof of Lemma 9.20 , it is also possible to deduce a bound on the location probability of 1 in $\sigma_{\lambda}$ :

Lemma 9.22. Let $\lambda$ be a composition of $n$, and $a<b$ be two peaks of $\lambda$. Then

$$
\mathbb{P}_{\lambda}\left(1 \in \lambda_{>a,<b}\right) \leq 3 \frac{b-a}{n}
$$

Proof. Since 1 has to be located in a valley of $\lambda$,

$$
\mathbb{P}\left(1 \in \lambda_{>a,<b}\right)=\sum_{\substack{v \in V \\ a<v<b}} \mathbb{P}(1 \in v)
$$

From Corollary 9.17, for each $v \in V$,

$$
\mathbb{P}(1 \in v)=\frac{1}{|\lambda|} \frac{1}{\int_{0}^{1}\left(1-F_{Y_{\lambda_{<v}}}(t)\right)\left(1-F_{X_{\lambda>v}}(t)\right) d t}
$$

Suppose that $v \in s_{i} \cap s_{i+1}$. By the bounds on $\Delta$ from the proof of Lemma 9.20 ,

$$
\frac{1}{\int_{0}^{1}\left(1-F_{Y_{\lambda_{<v}}}(t)\right)\left(1-F_{X_{\lambda>v}}(t)\right) d t} \leq l\left(s_{i}\right)+l\left(s_{i+1}\right)+3 \leq 3\left(l\left(s_{i}\right)+l\left(s_{i+1}\right)\right)
$$

The latter inequality yields

$$
\mathbb{P}\left(1 \in \lambda_{>a,<b}\right) \leq \frac{3}{|\lambda|} \sum_{\substack{v \in V \\ a<v<b}} l\left(s_{i}\right)+l\left(s_{i+1}\right) \leq \frac{3(b-a)}{n}
$$

### 9.7.2 Convergence to a uniform distribution

Let us show the convergence in law of $\xi_{1}^{\lambda}$. Let $\pi$ denote the Levy-Prokhorov metric on the set $\mathcal{M}_{1}[0,1]$ of probability measures on $[0,1]$.

Proposition 9.23. Let $\varepsilon>0$. There exists $n_{0}$ such that for $n \geq n_{0}, \lambda \vdash n$,

$$
\pi\left(\xi_{1}^{\lambda}, \mathcal{U}([0,1])\right) \leq \varepsilon
$$

Proof. Let $\varepsilon>0$. Since $F_{\mathcal{U}([0,1])}=I d_{[0,1]}$ is continuous, it is enough to prove that for $s \in[0,1]$ and for $\lambda$ large enough,

$$
\left|\mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right)-s\right| \leq \varepsilon
$$

Let $0<s<1$ and $n_{\varepsilon}$ be the constant given by Lemma 9.20 for $\varepsilon$. Let $\lambda \vdash n$ and let $v_{n}$ denote the last valley such that the associated slope intersects $[0, n s]$, namely $\mathfrak{s}\left(v^{n}\right) \cap[0, n s] \neq \emptyset$ : since $0<s$, such $v_{n}$ always exists for $n$ large enough. Let $\left[a_{n} ; b_{n}\right]$ denote the slope of $v_{n}$. Thus $a_{n} \leq n s<b_{n}+1$.
If $1 \in \lambda_{<a_{n}}, \xi_{1}^{\lambda} \in\left[0, \frac{a_{n}}{n}[\subseteq[0, s]\right.$. Moreover

$$
\begin{aligned}
\mathbb{P}_{\lambda}\left(1 \in \lambda_{<a_{n}}\right) & =\sum_{\substack{v \in V \\
v<a_{n}}} \mathbb{P}_{\lambda}(1 \in v) \\
& =\sum_{\substack{v \in V \\
v<a_{n}}} \frac{1}{|\lambda|} \frac{1}{\int_{0}^{1}\left(1-F_{Y_{\lambda_{<v}}}(t)\right)\left(1-F_{X_{\lambda_{>v}}}(t)\right) d t}
\end{aligned}
$$

If $v<a_{n}$ is a valley, there is necessarily a peak between $v$ and $v_{n}$, and thus the first run of $\lambda_{>v}$ is of size smaller than $v_{n}-v$. Therefore from Lemma 9.20 ,

$$
\begin{aligned}
\mathbb{P}_{\lambda}\left(1 \in \lambda_{<a_{n}}\right) & \leq \sum_{\substack{v \in V \\
v<a_{n}}} \frac{1}{n} \frac{1+\varepsilon}{\int_{0}^{1}\left(1-F_{Y_{\lambda_{<v}}}(t)\right)\left(1-F_{X_{\lambda_{>v,<v_{n}+n_{\varepsilon}}}}(t)\right) d t} \\
& \leq(1+\varepsilon) \frac{v_{n}+n_{\varepsilon}}{n} \mathbb{P}_{\lambda_{<v_{n}+n_{\varepsilon}}}\left(1 \in \lambda_{<a_{n}}\right),
\end{aligned}
$$

and for the same reasons,

$$
\mathbb{P}_{\lambda}\left(1 \in \lambda_{<a_{n}}\right) \geq(1-\varepsilon) \frac{v_{n}+n_{\varepsilon}}{n} \mathbb{P}_{\lambda_{<v_{n}+n_{\varepsilon}}}\left(1 \in \lambda_{<a_{n}}\right)
$$

The proof is now divided into two complementary cases :

- $\lambda$ is such that $\frac{a_{n}}{v_{n}+n_{\varepsilon}}>1-\varepsilon$ : in this case, $\frac{v_{n}+n_{\varepsilon}-a_{n}}{v_{n}+n_{\varepsilon}}<\varepsilon$. Thus from Lemma 9.22 , $\mathbb{P}_{\lambda_{<v_{n}+n_{\varepsilon}}}\left(1 \in \lambda \geq a_{n}\right) \leq 3 \varepsilon$. This yields

$$
1-3 \varepsilon \leq \mathbb{P}_{\lambda<v_{n}+n_{\varepsilon}}\left(1 \in \lambda_{<a_{n}}\right) \leq 1
$$

and thus

$$
(1-3 \varepsilon)^{2} \frac{v_{n}+n_{\varepsilon}}{n} \leq \mathbb{P}_{\lambda}\left(1 \in \lambda_{<a_{n}}\right) \leq(1+\varepsilon) \frac{v_{n}+n_{\varepsilon}}{n}
$$

The hypothesis $\frac{a_{n}}{v_{n}+n_{\varepsilon}}>1-\varepsilon$ yields also

$$
(1-3 \varepsilon)^{2} \frac{a_{n}}{n} \leq \mathbb{P}_{\lambda}\left(1 \in \lambda_{<a_{n}}\right) \leq \frac{1+\varepsilon}{1-\varepsilon} \frac{a_{n}}{n}
$$

On the other hand by independence between $\sigma_{\lambda}$ and the family $\left(X_{(p, q)}^{1}\right)_{p, q \in \mathbb{Q}}$,

$$
\begin{aligned}
\mathbb{P}\left(1 \in v_{n} \cap \xi_{1}^{\lambda} \in[0, s]\right) & =\mathbb{P}_{\lambda}\left(1 \in v_{n}\right) \frac{\operatorname{Leb}\left([0, s] \cap\left[\left(a_{n}-1\right) / n, b_{n} / n\right]\right)}{\left(b_{n}-a_{n}+1\right) / n} \\
& =\mathbb{P}_{\lambda}\left(1 \in v_{n}\right)\left(\frac{s-a_{n}+1}{b_{n}-a_{n}+1} \wedge 1\right)
\end{aligned}
$$

From the latter computation and from the bounding Lemma 9.22, $\mathbb{P}\left(1 \in v \cap \xi_{1}^{\lambda} \in[0, s]\right) \leq$ $\frac{3\left(b_{n}-a_{n}\right)}{n}$. Thus the latter quantity doesn't become negligible for $n$ large only if at least one of the two runs $s_{i}$ or $s_{i+1}$ surrounding $v_{n}$ tends to infinity when $n$ grows. But in this case from Corollary 9.21 ,

$$
\mathbb{P}\left(1 \in v_{n}\right) \sim_{b_{n}-a_{n} \rightarrow \infty}\left(b_{n}-a_{n}\right) / n .
$$

Therefore in any case:

$$
\begin{aligned}
\mathbb{P}\left(1 \in v_{n} \cap \xi_{1}^{\lambda} \in[0, s]\right) & ={ }_{n \rightarrow \infty}\left(b_{n}-a_{n}\right) / n \frac{s-a_{n} / n}{b_{n} / n-a_{n} / n}+o(1) \\
& ={ }_{n \rightarrow+\infty} \operatorname{Leb}\left(\left[a_{n} / n, s\right]\right)+o(1)
\end{aligned}
$$

with $o(1)$ being a quantity converging to zero with $n$, independently of the shape of $\lambda$. Summing the probabilities yields for $n$ large enough

$$
\begin{aligned}
\mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right) & \leq \frac{1+\varepsilon}{1-\varepsilon} \operatorname{Leb}\left(\left[0, \frac{a_{n}}{n}\right]\right)+\operatorname{Leb}\left(\left[\frac{a_{n}}{n}, s\right]\right)+o(1) \\
& \leq \frac{1+\varepsilon}{1-\varepsilon} \operatorname{Leb}([0, s])+o(1),
\end{aligned}
$$

and for the same reasons

$$
(1-3 \varepsilon)^{2} \operatorname{Leb}([0, s])+o(1) \leq \mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right) .
$$

There exists thus $n_{1}$ such that for $n \geq n_{1}$, if $\frac{a_{n}}{v_{n}+n_{\varepsilon}}>1-\varepsilon$,

$$
(1-3 \varepsilon)^{2} \operatorname{Leb}([0, s])-\varepsilon \leq \mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right) \leq \frac{1+\varepsilon}{1-\varepsilon} \operatorname{Leb}([0, s])+\varepsilon .
$$

- $\lambda$ is such that $\frac{a_{n}}{v_{n}+n_{\varepsilon}} \leq 1-\varepsilon$ : this implies that either $v_{n}$ remains bounded or $v_{n}-a_{n}$ goes to $+\infty$ as $n$ grows.
Suppose that $v_{n}$ remains bounded by $K$ as $n$ grows. In this case by Lemma $9.22, \mathbb{P}(1 \in$ $\left.\lambda_{<v_{n}}\right) \rightarrow 0$. Thus

$$
\mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right)=\mathbb{P}\left(1 \in v \cap \xi_{1}^{\lambda} \in[0, s]\right)+o(1) .
$$

Since $v_{n}$ remains bounded by $K$ and $b_{n}+1>n s$, the slope of $v_{n}$ tends to $+\infty$, and therefore from Corollary 9.21,

$$
\mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right)={ }_{n \rightarrow+\infty}\left(\frac{b_{n}-a_{n}}{n}+o(1)\right) \frac{n s-a_{n}}{b_{n}-a_{n}}=s+o(1) .
$$

Suppose that $v_{n}-a_{n}$ goes to $+\infty$. Since $n_{\varepsilon}$ is a fixed integer and $v_{n}-a_{n}$ goes to $+\infty$, the size of the slope of $v_{n}$ in $\lambda_{<v_{n}+\varepsilon}$ is equivalent to $v_{n}+n_{\varepsilon}-a_{n}$ as $n$ goes to $+\infty$. Thus from Corollary 9.21 ,

$$
\mathbb{P}_{\lambda_{<v_{n}+n_{\varepsilon}}}\left(1 \in v_{n}\right)={ }_{n \rightarrow \infty} \frac{v_{n}+n_{\varepsilon}-a_{n}}{v_{n}+n_{\varepsilon}}+o(1) .
$$

The same Corollary yields moreover

$$
\mathbb{P}_{\lambda}\left(1 \in v_{n}\right)=_{n \rightarrow \infty} \frac{b_{n}-a_{n}}{n}+o(1) .
$$

From Lemma 9.22. $\mathbb{P}_{\lambda_{<v_{n}+n_{\varepsilon}}}\left(1 \in \lambda_{>v_{n},<v_{n}+n_{\varepsilon}}\right) \leq \frac{3 n_{\varepsilon}}{v_{n}+n_{\varepsilon}}=o(1)$ and thus

$$
\begin{aligned}
\mathbb{P}_{\lambda_{<v_{n}+n_{\varepsilon}}}\left(1 \in \lambda_{<a_{n}}\right) & =1-\mathbb{P}_{\lambda_{<v_{n}+n_{\varepsilon}}}\left(1 \in v_{n}\right)-\mathbb{P}_{\lambda_{<v_{n}+n_{\varepsilon}}}\left(1 \in \lambda_{>v_{n},<v_{n}+n_{\varepsilon}}\right) \\
& =\frac{a_{n}}{v_{n}+n_{\varepsilon}}+o(1) .
\end{aligned}
$$

Thus

$$
\mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right) \leq(1+\varepsilon) \frac{a_{n}}{n}+\frac{b_{n}-a_{n}}{n} \frac{n s-a_{n}}{b_{n}-a_{n}}+o(1) \leq(1+\varepsilon) s+o(1),
$$

and for the same reasons $(1-\varepsilon) s+o(1) \leq \mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right)$. This yields the existence of $n_{2}$ such that if $n \geq n_{2}$ and $\frac{a_{n}}{v_{n}+n_{\varepsilon}} \leq 1-\varepsilon$,

$$
(1-\varepsilon) \operatorname{Leb}([0, s])-\varepsilon \leq \mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right) \leq(1+\varepsilon) \operatorname{Leb}([0, s])+\varepsilon .
$$

By the results from both cases, there exists $n_{0}$ such that for $n \geq n_{0}, \lambda \vdash n$,

$$
\left|\mathbb{P}\left(\xi_{1}^{\lambda} \in[0, s]\right)-s\right| \leq \varepsilon
$$

### 9.8 Martin boundary of $\mathcal{Z}$

This section is devoted to the proof of Conjecture 9.7, yielding the identification of the Martin boundary of $\mathcal{Z}$ with its minimal boundary.

### 9.8.1 Generalization of Proposition 9.23

The result of the previous section can be generalized for $k \geq 2$ :
Proposition 9.24. Let $\lambda_{n}$ be a sequence of compositions of size tending to infinity. Then for $k \geq 1$,

$$
\left(\xi_{i}^{\lambda_{n}}\right)_{1 \leq i \leq k} \rightarrow_{\text {law }}\left(X_{1}, \ldots, X_{k}\right)
$$

with $\left(X_{1}, \ldots, X_{k}\right)$ a vector of $k$ independent uniform random variables on $[0,1]$.
Proof. Let us prove by recurrence on $k \geq 1$ that for $\varepsilon>0$, there exists $n_{k} \in \mathbb{N}$ such that for $n \geq n_{k}, \lambda \vdash n$,

$$
\pi\left(\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k},\left(X_{1}, \ldots, X_{k}\right)\right) \leq \varepsilon
$$

$\pi$ denoting the Levy-Prokhorov metric on $[0,1]^{k}$.
The initialization of the recurrence is done by Proposition 9.23 . Let $k \geq 2$. It suffices to show that the law $\xi_{k}^{\lambda}$ conditioned on $\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k-1}$ is close to the uniform law on $[0,1]$ when $n$ becomes large.
Let $s \in[0,1] \backslash \mathbb{Q}, \varepsilon>0$. Let

$$
\Upsilon_{\eta}=\cap_{1 \leq i \leq k-1}\left\{\left(x_{1}, \ldots, x_{k-1}\right) \in[0,1]^{k-1} \mid x_{i} \notin[s-\eta, s+\eta]\right\}
$$

For all $\eta, \operatorname{Leb}\left(\partial \Upsilon_{\eta}\right)=0$ and $\operatorname{Leb}\left(\lim _{\eta \rightarrow 0} \Upsilon_{\eta}\right)=1$, thus by the recurrence hypothesis and the portemanteau theorem, there exists $\eta>0$ such that for $\lambda$ large enough,

$$
\begin{equation*}
\mathbb{P}\left(\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k-1} \in \Upsilon_{\eta}\right) \geq 1-\varepsilon \tag{9.8.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\pi\left(\left(\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k-1} \mid A_{\eta}\right),\left(\left(X_{i}\right)_{1 \leq i \leq k-1} \mid B_{\eta}\right)\right) \leq \varepsilon \tag{9.8.2}
\end{equation*}
$$

with $A_{\eta}=\left\{\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k-1} \in \Upsilon_{\eta}\right\}$ and $B_{\eta}=\left\{\left(X_{i}\right)_{1 \leq i \leq k-1} \in \Upsilon_{\eta}\right\}$.
Let $\lambda \vdash n$ and $\vec{i}=\left(i_{1}, \ldots, i_{k-1}\right)$ such that $\mathbb{P}\left(\sigma_{\lambda}^{-1}(1)=i_{1}, \ldots, \sigma_{\lambda}^{-1}(k-1)=i_{k-1}\right) \neq 0$. Let us further assume that $\vec{i}$ satisfies the following condition :

$$
\begin{equation*}
\forall 1 \leq j \leq k-1, \mathfrak{s}\left(i_{j}\right) \nsubseteq[n(s-\eta), n(s+\eta)] \tag{*}
\end{equation*}
$$

where $\mathfrak{s}\left(i_{j}\right)$ denotes the slope of $i_{j}$ as defined in Section 9.5.1. Then $\lambda$ can be decomposed as

$$
\lambda=\lambda_{1}-\mu_{1}+\lambda_{2}-\cdots-\mu_{r}+\lambda_{r+1}
$$

with $\mu_{i}$ consisting only in cells included in $\vec{i}$. From the latter construction, each run of $\lambda$ intersects at most one $\lambda_{i}$.
Conditioned on $\mathcal{X}_{\vec{i}}=\left\{\sigma_{\lambda}^{-1}(1)=i_{1}, \ldots, \sigma_{\lambda}^{-1}(k-1)=i_{k-1}\right\}$, the random filling of $\lambda$ consists in sampling a uniformly random multiset $\vec{R}=\left(R_{1}, \ldots, R_{r+1}\right)$ of cardinal $\left(\left|\lambda_{1}\right|, \ldots,\left|\lambda_{r+1}\right|\right)$ among $[k ; n]$, and then independently filling each subcomposition $\lambda_{1}, \ldots, \lambda_{r+1}$ respectively with $R_{1}, \ldots, R_{r+1}$. Since $k$ is the lowest element of $[k ; n]$, for $v \in \lambda_{i}, \mathbb{P}\left(k \in v \mid \mathcal{X}_{\vec{i}}\right) \neq 0$ if and only if $v$ is a valley of $\lambda_{i}$, and if this is the case,

$$
\mathbb{P}\left(k \in v \mid \mathcal{X}_{\vec{i}}\right)=\mathbb{P}_{\left(R_{1}, \ldots, R_{r+1}\right)}\left(k \in R_{i}\right) \mathbb{P}_{\lambda_{i}}(1 \in v)
$$

Let $s_{p}=\left[a_{p} ; a_{p+1}\right]$ be the run of $\lambda$ such that $s \in\left[\frac{a_{p}-1}{n} ; \frac{a_{p+1}}{n}\right]$. If $i_{j}$ is a peak, necessarily the two runs overlapping on $i_{j}$ contain only elements lower than $j$, and thus the slope of $i_{j}$ is smaller than $j$. Thus since for all $1 \leq j \leq k-1, \mathfrak{s}\left(i_{j}\right) \nsubseteq[n(s-\eta), n(s+\eta)]$, for $n$ large enough, the peak of $s_{p}$ cannot be in any $\mu_{i}$ for $1 \leq i \leq r$. This yields the existence of a unique $i_{0}$ such that $s_{p} \cap \lambda_{i_{0}} \neq \emptyset$.
Set $\lambda_{i_{0}}=\lambda_{\geq a,<b}$. Since the cells $a-1$ and $b$ (if they exist) are in compositions of type $\mu$, their content is smaller than $k-1$ and therefore smaller than the content of the cells of $\lambda_{i_{0}}$. Thus a valley $v$ of $\lambda_{i_{0}}$ such that the slope $\mathfrak{s}_{\lambda}(v)$ of $v$ in $\lambda$ is not included in $\lambda_{i_{0}}$ has to be $a$ or $b-1$. In particular if $v$ is a valley of $\lambda_{i_{0}}$ different from $a$ and $b-1$, a rescaling of $\xi_{1}^{\lambda_{i_{0}}}$ yields that

$$
\left(\xi_{k}^{\lambda} \mid k \in v\right)={ }_{\text {law }}\left(\left.\frac{a}{n}+\frac{b-a}{n} \xi_{1}^{\lambda_{i 0}} \right\rvert\, 1 \in v\right) .
$$

Thus

$$
\mathbb{P}_{\lambda}\left(\{k \in v\} \cap\left\{\xi_{k}^{\lambda} \leq s\right\} \mid \mathcal{X}_{\vec{i}}\right)=\mathbb{P}_{\vec{R}}\left(k \in R_{i_{0}}\right) \mathbb{P}_{\lambda_{i_{0}}}\left(\{1 \in v\} \cap\left\{\xi_{1}^{\lambda_{i_{0}}} \in\left[0, \frac{n s-a}{b-a}\right]\right\}\right) .
$$

Suppose that $a$ is a valley of $\lambda_{i_{0}}$. This implies that in $\lambda, a$ is neither a peak nor a valley. Therefore the slope of $a$ in $\lambda_{i_{0}}$ is $[1, x]$ for some integer $x$, and the slope of $a$ in $\lambda$ is $[a-r, a+x]$ for the same $x$ and some integer $1 \leq r \leq k-1$. Thus as $n$ goes to infinity, this implies that

$$
\pi\left(\left(\xi_{k}^{\lambda} \mid k \in a\right),\left(\left.\frac{a}{n}+\frac{b-a}{n} \xi_{1}^{\lambda_{i}} \right\rvert\, 1 \in a\right)\right) \rightarrow_{n \rightarrow+\infty} 0
$$

and the rate of convergence only depends on $n$ and $k$ (and is therefore independent of the shape of $\lambda$ and the choice of $\vec{i}$ ).
The same holds for $b-1$. Thus if $v=a$ or $b$, since $\left|\lambda_{i_{0}}\right|=b-a$,

$$
\mathbb{P}_{\lambda}\left(\{k \in v\} \cap\left\{\xi_{k}^{\lambda} \leq s\right\} \mid \mathcal{X}_{\vec{i}}\right)=\mathbb{P}_{\vec{R}}\left(k \in R_{i_{0}}\right)\left(\mathbb{P}_{\lambda_{i_{0}}}\left(\{1 \in v\} \cap\left\{\xi_{1}^{\lambda_{i_{0}}} \in\left[0, \frac{n s-a}{b-a}\right]\right\}\right)+o(1)\right) .
$$

Summing the probabilities yields

$$
\mathbb{P}\left(\left\{\xi_{k}^{\lambda} \in[0, s]\right\} \cap\left\{k \in \lambda_{i_{0}}\right\} \mid \mathcal{X}_{\vec{i}}\right)=\mathbb{P}_{\vec{R}}\left(k \in R_{i_{0}}\right) \mathbb{P}_{\lambda_{i_{0}}}\left(\xi_{1}^{\lambda_{i_{0}}} \leq \frac{n s-a}{b-a}\right)+o(1) .
$$

If $i<i_{0}, k \in \lambda_{i}$ implies that $\xi_{k}^{\lambda} \in[0, s]$. A standard counting argument shows that

$$
\mathbb{P}_{\vec{R}}\left(k \in R_{i}\right)=\frac{R_{i}}{\sum_{j} R_{j}},
$$

and thus,

$$
\begin{aligned}
\mathbb{P}\left(\xi_{k}^{\lambda} \in[0, s] \mid \mathcal{X}_{\vec{i}}\right) & =\left(\sum_{i<i_{0}} \mathbb{P}_{\vec{R}}\left(k \in R_{i}\right)+\mathbb{P}_{\vec{R}}\left(k \in R_{i_{0}}\right) \mathbb{P}_{\lambda_{i_{0}}}\left(\xi_{1}^{\lambda_{i_{0}}} \in\left[0, \frac{n s-a}{R_{i_{0}}}\right]\right)\right)+o(1) \\
& =\left(\sum_{i<i_{0}} \frac{R_{i}}{n}+\frac{R_{i_{0}}}{n} \mathbb{P}_{\lambda_{i_{0}}}\left(\xi_{1}^{\lambda_{i_{0}}} \in\left[0, \frac{n s-a}{R_{i_{0}}}\right]\right)\right)+o(1) \\
& =\frac{a}{n}+\frac{R_{i_{0}}}{n} \mathbb{P}_{\lambda_{i_{0}}}\left(\xi_{1}^{\lambda_{i_{0}}} \in\left[0, \frac{n s-a}{R_{i_{0}}}\right]\right)+o(1) .
\end{aligned}
$$

Either $R_{i_{0}}$ remains bounded as $n \rightarrow \infty$ and $\frac{R_{i_{0}}}{n} \mathbb{P}_{\lambda_{i_{0}}}\left(\xi_{1}^{\lambda_{i_{0}}} \in\left[0, \frac{n s-a}{b-a}\right]\right) \rightarrow 0$, either $R_{i_{0}}$ goes to infinity, and by Proposition 9.23 ,

$$
\mathbb{P}_{\lambda_{i_{0}}}\left(\xi_{1}^{\lambda_{i_{0}}} \in\left[0, \frac{n s-a}{b-a}\right]\right)=\frac{n s-a}{R_{i_{0}}}+o(1) .
$$

Thus in any case,

$$
\mathbb{P}\left(\xi_{k}^{\lambda} \in[0, s] \mid \mathcal{X}_{\vec{i}}\right)=\frac{a}{n}+\frac{R_{i_{0}}}{n} \frac{n s-a}{R_{i_{0}}}+o(1) \rightarrow s,
$$

and the convergence is uniform in $\lambda, \vec{i}$.
Let $\left(x_{i}\right)_{1 \leq i \leq k-1} \in \Upsilon_{\eta}$. If $\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k-1}=\left(x_{i}\right)_{1 \leq i \leq k-1}$, then $\left(\sigma_{\lambda}^{-1}(1), \ldots, \sigma_{\lambda}^{-1}(k-1)\right)$ verifies the condition *. Moreover from Lemma 9.13. $\left(\xi_{1}^{\lambda}, \ldots, \xi_{k-1}^{\lambda}\right) \mapsto\left(\sigma_{\lambda}^{-1}(1), \ldots, \sigma_{\lambda}^{-1}(k-1)\right)$ is well-defined and

$$
\left(\xi_{k}^{\lambda} \mid\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k-1}\right)=\left(\xi_{k}^{\lambda} \mid \sigma_{\lambda}^{-1}(\{1, \ldots, k-1\}) .\right.
$$

Thus for $n$ going to $+\infty$,

$$
\mathbb{P}\left(\xi_{k}^{\lambda} \in[0, s] \mid\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k-1}=\left(x_{i}\right)_{1 \leq i \leq k-1}\right) \rightarrow s,
$$

and the convergence is uniform in $\left(x_{i}\right)_{1 \leq i \leq k-1} \in \Upsilon_{\eta}$.
From the latter convergence and from 9.8.2), for $n$ large enough,

$$
\left.\pi\left(\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k} \mid A_{\eta}\right),\left(\left(X_{i}\right)_{1 \leq i \leq k} \mid B_{\eta}\right)\right) \leq \varepsilon .
$$

If $\varepsilon$ is small enough, then $\mathbb{P}\left(A_{\eta}\right) \geq 1-\varepsilon$ and $\mathbb{P}\left(B_{\eta}\right) \geq 1-\varepsilon$ imply that

$$
\pi\left(\left(\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k} \mid A_{\eta}\right),\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k}\right) \leq 2 \varepsilon,
$$

and

$$
\pi\left(\left(\left(X_{i}\right)_{1 \leq i \leq k} \mid B_{\eta}\right),\left(X_{i}\right)_{1 \leq i \leq k}\right) \leq 2 \varepsilon .
$$

Thus for $n$ large enough,

$$
\pi\left(\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k},\left(X_{i}\right)_{1 \leq i \leq k}\right) \leq 5 \varepsilon
$$

This concludes the proof of the proposition.

### 9.8.2 Proof of Theorem 9.8

Proof. Let $\left(\lambda_{n}\right)_{n \geq 1}$ be a sequence of compositions and $U=\left(U_{\uparrow}, U_{\downarrow}\right) \in \mathcal{U}^{(2)}$ such that $\lambda_{n} \vdash n$ and $U_{\lambda_{n}} \rightarrow U$ in $\mathcal{U}^{(2)}$. By Lemma 9.11, $\tilde{U}_{\lambda_{n}} \rightarrow U$, with $\tilde{U}_{\lambda}$ the run paintbox defined for $\lambda$ in Section 5.
Let $\mu \in \mathcal{Z}, \mu \vdash k$. Since $K_{\mu}\left(\lambda_{n}\right)=\frac{d\left(\mu, \lambda_{n}\right)}{d\left(\lambda_{n}\right)}$, by equality 9.3.1,

$$
K_{\mu}\left(\lambda_{n}\right)=\mathbb{P}\left(\sigma_{k}^{\lambda_{n}}=\sigma\right),
$$

$\sigma$ being any permutation such that $\operatorname{des}(\sigma)=\mu$. By Proposition 9.12.

$$
\left.\mathbb{P}\left(\sigma_{k}^{\lambda_{n}}=\sigma\right)=\mathbb{P}\left(\sigma_{\tilde{U}_{\lambda_{n}}}\left(\xi_{i}^{\lambda_{n}}\right)_{1 \leq i \leq k}\right)=\sigma\right) .
$$

By Proposition 9.24, as $n$ goes to $+\infty,\left(\xi_{i}^{\lambda_{n}}\right)_{1 \leq i \leq k}$ converges in law to a sequence $\left(X_{1}, \ldots, X_{k}\right)$ of uniform independent random variables on $[0,1]$.
Thus since $\tilde{U}_{\lambda_{n}} \rightarrow U$, by Proposition 9.14

$$
\sigma_{\tilde{U}_{\lambda_{n}}}\left(\left(\xi_{i}^{\lambda}\right)_{1 \leq i \leq k}\right) \rightarrow_{\text {law }} \sigma_{U}\left(\left(X_{1}, \ldots, X_{k}\right)\right)=\sigma_{U} .
$$

Therefore

$$
K_{\mu}\left(\lambda_{n}\right)=\mathbb{P}\left(\sigma_{\lambda_{n}}=\sigma\right) \rightarrow p_{\left(U_{\uparrow}, U_{\downarrow}\right)}(\mu) .
$$

As explained in Section 4, the latter Theorem implies Conjecture 9.7 .
Corollary 9.25. Conjecture 9.7 is true and for the graded graph $\mathcal{Z}$,

$$
\partial_{\min } \mathcal{Z}=\partial_{M} \mathcal{Z}
$$

We end this section by showing that the topology on $\hat{\mathcal{Z}}=\mathcal{Z} \cup \partial_{M} \mathcal{Z}$, abstractly constructed in Section 2.1, can be concretely described in terms of oriented Paintbox construction. From the work of Gnedin and Olshanski in [42] (Proposition 36), $\partial_{\min } \mathcal{Z}$ with the induced topology of Section 2.1 is homeomorphic to $\mathcal{U}^{(2)}$. Since from the latter Corollary, $\partial_{\min } \mathcal{Z}=\partial_{M} \mathcal{Z}$, as topological spaces

$$
\partial_{M} \mathcal{Z}=\mathcal{U}^{(2)}
$$

It remains to describe the topology of $\hat{\mathcal{Z}}=\mathcal{Z} \cup \partial_{M} \mathcal{Z}$. Let $\mathcal{U}_{n} \subseteq \mathcal{U}^{(2)}$ be the set of $\left(U_{\uparrow}, U_{\downarrow}\right)$ such that $[0,1] \backslash U_{\uparrow} \cup U_{\downarrow} \subseteq\left\{\frac{k}{n-1}, 0 \leq k \leq n-1\right\}$. Then $\hat{\mathcal{Z}}$ is characterized as follows:

Corollary 9.26. Let $\mathcal{T}=[0,1] \times \mathcal{U}^{(2)}$ with the product topology. As topological spaces,

$$
\hat{\mathcal{Z}} \simeq\left(\{0\} \times \mathcal{U}^{(2)}\right) \cup \bigcup_{n \geq 1}\left(\left\{\frac{1}{n}\right\} \times \mathcal{U}_{n}\right) \subseteq \mathcal{T}
$$

the space on the right being considered with the induced topology from $\mathcal{T}$.
Proof. The bijection $\Phi$ is achieved by sending $\lambda \vdash n$ to $\frac{1}{n} \times U_{\lambda}$ and $\omega=\lim \lambda_{n} \in \partial_{M} \mathcal{Z}$ to $0 \times \lim U_{\lambda_{n}}$. Since $\hat{\mathcal{Z}}$ is compact, the only thing to prove is the continuity of the map.
Let $x_{n} \rightarrow \omega \in \hat{\mathcal{Z}}$. If $\omega \in \mathcal{Z}$, the sequence is stationary and the convergence is straightforward. Suppose that $\omega \in \partial_{M} \mathcal{Z}$, and divide $x_{n}$ into two complementary subsequences $\left(x_{\varphi(n)}\right)$ and $\left(x_{\varphi^{c}(n)}\right)$ such that $x_{\varphi(n)} \in \mathcal{Z}$ and $x_{\varphi^{c}(n)} \in \partial_{M} \mathcal{Z}$.
By Proposition 36 of 42],

$$
\Phi\left(x_{\varphi^{c}(n)}\right) \rightarrow \Phi(\omega) .
$$

By Corollary 9.25 , since $x_{\varphi(n)} \rightarrow_{\hat{\mathcal{Z}}} w$,

$$
\Phi\left(x_{\varphi(n)}\right)=U_{x_{\varphi(n)}} \rightarrow \Phi(\omega),
$$

which concludes the proof.

### 9.9 The Plancherel measure

The purpose of this section is to investigate the Plancherel measure on the graph $\mathcal{Z}$, which is the point $(\emptyset, \emptyset)$ of $\partial_{M} \mathcal{Z}$. We first recall the link between $\mathcal{Z}$ and the Young graph $\mathcal{Y}$ to justify the name of Plancherel measure. This link was already explained in 42 in terms of associated algebras of functions, but it seems to us that no direct link on the level of paths was clearly defined. It is the purpose of the second paragraph to clearly establish this link on the level of paths.

### 9.9.1 The graph $\mathcal{Y}$

A partition $\rho$ of $n$ is the data of a decreasing sequence of positive integers ( $\rho_{1} \geq \cdots \geq \rho_{r}$ ) such that $\sum \rho_{i}=n$. $n$ is called the degree of $\rho$ and is denoted $\rho \vdash n$. Let us denote by $l(\rho)$ the length of the sequence. The set of partitions of $n$ is denoted $\mathcal{Y}_{n}$, and the set of all partitions $\mathcal{Y}$. $\mathcal{Y}$ is ordered by saying that $\rho \preceq \tau$ if and only if $l(\rho) \leq l(\tau)$ and for all $1 \leq i \leq l(\rho), \rho_{i} \leq \tau_{i}$.

As for compositions, a Young diagram is associated to each partition by drawing $\rho_{1}$ cells on the first row, $\rho_{2}$ cells on the second row and so on, such that the first cell of the row $i+1$ is just below the first cell of the row $i$. A standard filling of $\rho$ is a filling of $\rho$ with elements of $\{1, \ldots, n\}$, such that the filling is increasing to the right and to the bottom. We denote by $T_{\rho}$ the set of standard fillings of $\rho$ (also called stantard tableau of shape $\rho$ ). Here is an example of a partition $\rho=(7,4,2,1)$ and a standard filling of the associated diagram.


Figure 9.4: Young diagram of $(7,4,2,1)$ and an example of standard filling
We say that $\rho \nearrow \tau$ if and only if $\operatorname{deg} \tau=\operatorname{deg} \rho+1$ and $\rho \preceq \tau$. When $T \in T_{\tau}$ is a standard tableau of shape $\tau \vdash n, T_{\downarrow}$ is defined as the standard tableau obtained by deleting the cell containing $n$. In particular $T_{\downarrow}$ has a shape $\rho$ such that $\rho \nearrow \tau$. Adding an edge from $\rho$ to $\tau$ if and only if $\rho \nearrow \tau$ transforms $\mathcal{Y}$ into a graded graph. The latter graph is a major construction for the representation theory of the symmetric groups $\left(\mathfrak{S}_{n}\right)_{n \geq 1}$, since the irreducible representations $V_{\tau}$ of $\mathfrak{S}_{n}$ are indexed by elements $\tau$ of $\mathcal{Y}_{n}$, and there is a decomposition

$$
\operatorname{Res}\left(V_{\tau}\right)_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_{n}}=\sum_{\rho \nearrow \tau} V_{\rho} .
$$

As for the graph $\mathcal{Z}$, the set of paths on $\mathcal{Y}$ between the root $\emptyset$ and a partition $\rho$ is in bijection with the set of standard tableaux of shape $\rho$, and each element of $\partial_{\min } \mathcal{Y}$ yields a random path on the graph $\mathcal{Y}$ (namely an infinite standard tableau). The minimal and Martin boundaries of $\mathcal{Y}$ have been intensively studied (see [76, ,48, [85]) and fully described. In particuler the equality $\partial_{\min } \mathcal{Y}=\partial_{M} \mathcal{Y}$ holds also in this setting, and

$$
\partial_{M} \mathcal{Y}=\left\{\left(a_{1} \geq a_{2} \geq \cdots \geq 0\right),\left(b_{1} \geq b_{2} \geq \cdots \geq 0\right), \sum a_{j}+b_{j} \leq 1\right\} .
$$

For each $\omega \in \partial_{M} \mathcal{Y}, \rho_{\omega}$ denotes the random path on $\mathcal{Y}$ according to the harmonic measure $\omega$. The next paragraph establishes a link between $\mathcal{Y}$ and $\mathcal{Z}$ based on the algorithm RSK of Robinson, Schensted and Knuth. The relation between both graphs has been already established through the ring of symmetric functions and the one of quasisymmetric functions. The reader should refer to [42] for a complete review of the subject.

### 9.9.2 RSK algorithm and the projection $\mathcal{Z} \rightarrow \mathcal{Y}$

Let us first recall the RSK algorithm in the special case of permutations. This algorithm, initiated by Robinson in [30] and created by Schensted in [75, establishes a bijection between $\mathfrak{S}_{n}$ and pairs of standard tableaux of $n$ of the same shape. Let $\sigma=(\sigma(1), \ldots, \sigma(n)) \in \mathfrak{S}_{n}$. The algorithm constructs a pair of standard tableaux from $\sigma$ as follows :

1. Start with an infinite array $A^{0}=\left(a_{k, l}^{0}\right)_{k, l \geq 1}$ such that each cell is filled with the entry $n+1$ (namely $a_{k, l}^{0}=n+1$ ), and an infinite array $B=\left(b_{k, l}\right)_{k, l \geq 1}$ such that each cell is empty ( $B$ is called the recording tableau).
2. At each step $i, 1 \leq i \leq n$, the following insertion is done on the array $A^{i-1}$ :

- Let $\left(1, l_{1}\right)$ be the first cell (starting from the left) on the first row of $A^{i-1}$ such that $\sigma(i) \leq a_{1, l_{1}}^{i-1}$. Set $a_{1, l_{1}}^{i}=\sigma(i)$.
- Let $\left(2, l_{2}\right)$ be the first cell on the second row of $A^{i-1}$ such that $a_{1, l_{1}}^{i-1} \leq a_{2, l_{2}}^{i-1}$. Set $a_{2, l_{2}}^{i}=a_{1, l_{1}}^{i-1}$.
- Continue the process until the step $k_{0}$ where $a_{k_{0}, l_{k_{0}}}^{i-1}>n$. For $k>k_{0}$ or $k \leq k_{0}, l \neq$ $l(k)$, define $a_{k, l}^{i}=a_{k, l}^{i-1}$. Return $A^{i}=\left(A_{k, l}^{i}\right)_{k, l \geq 1}$. Set $b_{k_{0}, l_{k_{0}}}=i$.

3. Let $P(\sigma)$ be the part of the array $A^{n}$ containing entries lower or equal to $n$, and $Q(\sigma)$ the part of the array $B$ consisting in non empty cells.

Then the following Theorem holds ([75], [30]):
Theorem 9.27. The map $S: \sigma \mapsto(P(\sigma), Q(\sigma))$ is a bijection between $\mathfrak{S}_{n}$ and pairs of standard tableaux of $n$ of the same shape. Moreover

$$
\left(P\left(\sigma^{-1}\right), Q\left(\sigma^{-1}\right)\right)=(Q(\sigma), P(\sigma))
$$

From now on $\rho(\sigma)$ denotes the shape of $P(\sigma)$ (or $Q(\sigma)$ ).
The link between $\mathcal{Z}$ and $\mathcal{Y}$ resides in the following proposition, mapping paths on $\mathcal{Z}$ to paths on $\mathcal{Y}$.

Proposition 9.28. Let $\left(\sigma_{k}\right)_{k \geq 1}$ be a path on $\mathcal{Z}$. Then $\left(\rho\left(\sigma_{k}\right)\right)_{k \geq 1}$ is a path on $\mathcal{Y}$. Moreover if $\sigma=\left(\sigma_{k}\right)_{k \geq 1}$ is a random path on $\mathcal{Z}$, then $\rho(\sigma)=\left(\rho\left(\sigma_{k}\right)\right)_{k \geq 1}$ is a random path on $\mathcal{Y}$ and for $P \in \mathcal{T}_{\tau}$ a path on $\mathcal{Y}$ between $\emptyset$ and $\tau \vdash k_{0}$,

$$
\mathbb{P}\left(\left(\rho\left(\sigma_{1}\right), \ldots, \rho\left(\sigma_{k_{0}}\right)\right)=P\right)=\sum_{\substack{\sigma \in \mathfrak{S}_{k_{0}} \\ P(\sigma)=P}} \mathbb{P}\left(\sigma_{k_{0}}=\sigma\right)
$$

Proof. Let $\sigma=\left(i_{1}, \ldots, i_{k-1}, n, i_{k+1}, \ldots\right) \in \mathfrak{S}_{n}$. If suffices to prove that

$$
P\left(\sigma_{\downarrow}\right)=P(\sigma)_{\downarrow}
$$

Although the latter equality appears clearly in the algorithm, the proof is easier to write by using $\sigma^{-1}$ : indeed write $\sigma^{-1}=\left(j_{1}, \ldots, j_{n-1}, k\right)$. Since $\sigma_{\downarrow}=\left(i_{1}, \ldots, i_{k-1}, i_{k+1}, \ldots\right),\left(\sigma_{\downarrow}\right)^{-1}=$ $\left(j_{1}^{*}, \ldots, j_{n-1}^{*}\right)$, with $j_{l}^{*}=j_{l}$ if $j_{l}<k$ and $j_{l}^{*}=j_{l}-1$ if $j_{l}>k$. All the $j_{l}^{*}\left(\right.$ resp $\left.j_{l}\right)$ are distinct and thus

$$
\operatorname{std}\left(\left(j_{1}^{*}, \ldots, j_{n-1}^{*}\right)\right)=\operatorname{std}\left(\left(j_{1}, \ldots, j_{n-1}\right)\right)
$$

Since the Schensted algorithm only depends on the relative values of the entries, the recording tableaux $B$ of the algorithm for $\left(\sigma_{\downarrow}\right)^{-1}$ and $\sigma^{-1}$ after $n-1$ steps are the same. Therefore

$$
Q\left(\left(\sigma_{\downarrow}\right)^{-1}\right)=Q\left(\sigma^{-1}\right)_{\downarrow}
$$

Thus from Theorem 9.27 ,

$$
P\left(\sigma_{\downarrow}\right)=Q\left(\sigma_{\downarrow}^{-1}\right)=Q\left(\sigma^{-1}\right)_{\downarrow}=P(\sigma)_{\downarrow}
$$

This yields that $\rho\left(\sigma_{\downarrow}\right) \nearrow \rho(\sigma)$ and for any arrangement $\left(\sigma_{k}\right)_{k \geq 1}$, the sequence $\left(\rho\left(\sigma_{k}\right)\right)_{k \geq 1}$ is a well-defined path on $\mathcal{Y}$.
In particular if $\left(\sigma_{k}\right)_{k \geq 1}$ is a random path on $\mathcal{Z}$ and $P \in T_{\tau}, \tau \vdash k_{0}$, summing the probabilities of each path yields

$$
\mathbb{P}\left(\left(\rho\left(\sigma_{1}\right), \ldots, \rho\left(\sigma_{k_{0}}\right)\right)=P\right)=\sum_{\substack{\alpha \in \mathfrak{S}_{k_{0}} \\ P(\alpha)=P}} \mathbb{P}_{\sigma_{k}}\left(\sigma_{k_{0}}=\alpha\right)
$$

The important fact is that harmonic measures on $\mathcal{Z}$ yield harmonic measures on the graph $\mathcal{Y}$.

Corollary 9.29. Let $\sigma=\left(\sigma_{k}\right)_{k \geq 1}$ be a random arrangement such that $\mathbb{P}\left(\sigma_{k}=\alpha\right)$ depends only on $Q(\alpha)$. Then $\rho(\sigma)$ yields a harmonic measure on $\mathcal{Y}$.
In particular harmonic measures on $\mathcal{Z}$ yield harmonic measures on the graph $\mathcal{Y}$.
Proof. From Section 2, a random path $\rho=\left(\rho_{k}\right)_{k \geq 1}$ on $\mathcal{Y}$ comes from a harmonic measure if and only if for any partition $\tau \vdash n$, and $P_{1}, P_{2} \in T_{\tau}$,

$$
\mathbb{P}\left(\left(\rho_{1}, \ldots, \rho_{n}\right)=P_{1}\right)=\mathbb{P}\left(\left(\rho_{1}, \ldots, \rho_{n}\right)=P_{2}\right) .
$$

Let $\sigma=\left(\sigma_{k}\right)_{k \geq 1}$ be a random arrangement such that $\mathbb{P}\left(\sigma_{k}=\alpha\right)=p(Q(\alpha))$, with $p$ a positive function on standard Young tableaux. From Proposition 9.28, for $k_{0} \geq 1, \tau \vdash k_{0}$ and $P \in T_{\tau}$,

$$
\begin{aligned}
\mathbb{P}\left(\left(\rho_{1}, \ldots, \rho_{k_{0}}\right)=P\right) & =\sum_{\substack{\alpha \in \mathfrak{S}_{k_{0}} \\
P(\alpha)=P}} \mathbb{P}\left(\sigma_{k_{0}}=\alpha\right) \\
& =\sum_{\substack{\alpha \in \mathfrak{S}_{k_{0}} \\
P(\alpha)=P}} p(Q(\alpha))=\sum_{Q \in T_{\tau}} p(Q),
\end{aligned}
$$

the last equality being due to Theorem 9.27. Thus $\mathbb{P}\left(\left(\rho_{1}, \ldots, \rho_{k_{0}}\right)=P\right)$ is independent of $P \in T_{\tau}$.
Let $\varphi$ be a harmonic measure on $\mathcal{Z}$. From Section $3, \varphi$ yields a random arrangement $\sigma=\left(\sigma_{k}\right)_{k \geq 1}$ such that $\mathbb{P}\left(\sigma_{k}=\alpha\right)=p(\operatorname{des}(\alpha))$, for a particular function $p: \mathcal{Z} \rightarrow \mathbb{R}^{+}$. By a standard combinatorial result (see [79]), $i$ is a descent of $\alpha$ if and only if $i+1$ is in a strictly lower row than $i$ in $Q(\alpha)$. Thus if $Q(\alpha)=Q\left(\alpha^{\prime}\right)$, then $\operatorname{des}(\alpha)=\operatorname{des}\left(\alpha^{\prime}\right)$ and $\mathbb{P}\left(\sigma_{k}=\alpha\right)=\mathbb{P}\left(\sigma_{k}=\alpha^{\prime}\right)$. From the first part of the Corollary, $\rho(\sigma)$ yields a harmonic measure on $\mathcal{Y}$.

In general, for $Q$ a standard tableau, $\operatorname{des}(Q)$ denotes the set of indices $i$ such that $i+1$ is in a strictly lower row than $i$. This yields in particular the following equality for $\lambda \in \mathcal{Z}$ :

$$
\begin{equation*}
d_{\mathcal{Z}}(\emptyset, \lambda)=\sum_{\tau \in \mathcal{Y}} d_{\mathcal{Y}}(\emptyset, \tau) \#\left\{Q \in T_{\tau}, \operatorname{des}(Q)=D_{\lambda}\right) . \tag{9.9.1}
\end{equation*}
$$

The latter equation yields the law of $\rho\left(\sigma_{\lambda}\right)$, when $\sigma_{\lambda}$ is chosen uniformly on the set of paths on $\mathcal{Z}$ between $\emptyset$ and $\lambda$. Let $K_{\tau}^{\mathcal{Y}}(\rho)=\frac{d \mathcal{y}(\tau, \rho)}{d \mathcal{y}((,))}$ denote the Martin kernel on $\mathcal{Y}$.

Lemma 9.30. Let $\lambda \vdash n$ be a composition and $\sigma_{\lambda}$ be a uniform random path between $\emptyset$ and $\lambda$. Then $\rho\left(\sigma_{\lambda}\right)$ is a random path on $\mathcal{Y}$ with law

$$
\mathbb{P}\left(\rho\left(\sigma_{\lambda}\right)(k)=\tau\right)=d_{\mathcal{Y}}(\emptyset, \tau) K_{\tau}^{\mathcal{Y}}\left(\rho_{\lambda}\right),
$$

for $\tau \in \mathcal{Y}_{k}$ and $\rho_{\lambda}$ a random element of $\mathcal{Y}_{n}$ with law

$$
\mathbb{P}\left(\rho_{\lambda}=\rho\right)=\mathbb{P}\left(Q\left(\sigma_{\lambda}\right) \in T_{\rho}\right) .
$$

Proof. Let us apply the Schensted algorithm to $\sigma_{\lambda}$. If $P \in T_{\tau}$, with $\tau$ a Young diagram of $k$ cells $(k \leq n)$,

$$
\mathbb{P}\left(\rho\left(\sigma_{\lambda}\right)(k)=\tau\right)=\frac{1}{d_{\mathcal{Z}}(\lambda)} \#\left\{\sigma \in \mathfrak{S}_{n}, \operatorname{des}(\sigma)=\lambda, P\left(\sigma_{\downarrow k}\right)=\tau\right\} .
$$

Equation (9.9.1) transforms the latter expression into

$$
\begin{aligned}
\mathbb{P}\left(\rho\left(\sigma_{\lambda}\right)(k)=\tau\right) & =\frac{\left.\left.\sum_{\rho \in \mathcal{Y}} \#\left\{Q \in T_{\rho}, \operatorname{des}(Q)=\lambda\right)\right\} \#\left\{\sigma, Q(\sigma)=Q, P\left(\sigma_{\downarrow k}\right)=\tau\right)\right\}}{\sum_{\rho \in \mathcal{Y}} d \mathcal{Y}(\emptyset, \rho) \#\left\{Q \in T_{\rho}, \operatorname{des}(Q)=\lambda\right\}} \\
& =\sum_{\rho \in \mathcal{Y}} \frac{d_{\mathcal{Y}}(\emptyset, \rho) \#\left\{Q \in T_{\rho}, \operatorname{des}(Q)=\lambda\right\}}{\sum_{\rho \in \mathcal{Y}} d_{\mathcal{Y}}(\emptyset, \rho) \#\left\{Q \in T_{\rho}, \operatorname{des}(Q)=\lambda\right\}} \frac{d_{\mathcal{Y}}(\emptyset, \tau) d_{\mathcal{Y}}(\tau, \rho)}{d \mathcal{Y}(\emptyset, \rho)} \\
& =d_{\mathcal{Y}}(\emptyset, \tau) K_{\tau}\left(\rho_{\lambda}\right),
\end{aligned}
$$

with $\rho_{\lambda}=\rho\left(\sigma_{\lambda}\right)$ a random variable on $\mathcal{Y}_{n}$ with law

$$
\mathbb{P}\left(\rho_{\lambda}=\rho\right)=\frac{d_{\mathcal{Y}}(\emptyset, \rho) \#\left\{Q \in T_{\rho}, \operatorname{des}(Q)=\lambda\right\}}{\sum_{\rho \in \mathcal{Y}} d \mathcal{y}(\emptyset, \rho) \#\left\{Q \in T_{\rho}, \operatorname{des}(Q)=\lambda\right\}}=\mathbb{P}\left(Q\left(\sigma_{\lambda}\right) \in T_{\rho}\right) .
$$

We finally prove that on $\partial_{M} \mathcal{Z}, \rho$ restricts to a surjective map $p: \partial_{M} \mathcal{Z} \rightarrow \partial_{M} \mathcal{Y}$.
Proposition 9.31. Let $U=\left(U_{\uparrow}, U_{\downarrow}\right) \in \partial_{M} \mathcal{Z}$. Let $\left(a_{1} \geq a_{2} \geq \cdots \geq 0\right)$ (resp. $\left(b_{1} \geq b_{2} \geq\right.$ $\cdots \geq 0)$ ) be the lengths of the interval components of $U_{\uparrow}$ (resp. $U_{\downarrow}$ ) in decreasing order. Then $\omega(U)=\left(\left(a_{1} \geq a_{2} \geq \cdots \geq 0\right),\left(\left(b_{1} \geq b_{2} \geq \cdots \geq 0\right)\right) \in \partial_{M} \mathcal{Y}\right.$ and $\rho\left(\sigma_{U}\right)=\rho_{\omega(U)}$. Moreover the induced map

$$
p: \partial_{M} \mathcal{Z} \rightarrow \partial_{M} \mathcal{Y}
$$

is surjective.
Proof. Since the Schensted algorithm relates a finite number of permutations to each Young diagram, the map $\rho$ sending random paths of $\mathcal{Z}$ to random paths of $\mathcal{Y}$ is clearly continuous with respect to the topology of convergence in law. Thus it is enough to prove the result on a dense subset of $\partial_{M} \mathcal{Z}$. Let $U=\left(U_{\uparrow}, U_{\downarrow}\right) \in \partial_{M} \mathcal{Z}$ be such that $\bar{U}=[0,1]$ and $U$ has a finite number of interval components. Denote by ( $a_{1} \geq a_{2} \geq \cdots \geq a_{n}$ ) (resp. $\left(b_{1} \geq b_{2} \geq \cdots \geq b_{m}\right)$ ) the lengths of the interval components of $U_{\uparrow}$ (resp. $\left.U_{\downarrow}\right)$ in decreasing order, and $\omega_{U}=\left(\left(a_{1} \geq\right.\right.$ $\left.\left.a_{2} \geq \cdots \geq a_{n}\right),\left(b_{1} \geq b_{2} \geq \cdots \geq b_{m}\right)\right)$. Then $\sigma_{U}$ can be approximated by a sequence $\sigma_{\lambda_{n}}$ with $\lambda_{n} \vdash n, U\left(\lambda_{n}\right) \rightarrow U$. By Greene's Theorem (see [43]) and Lemma 2 of the paper [86] of Kerov and Vershik, almost surely $\rho\left(\sigma_{\lambda_{n}}\right)$ converges in $\mathcal{Y} \cup \partial_{M} \mathcal{Y}$ to $\omega_{U}$. Thus by identification of the Martin boundary on $\mathcal{Y}$ and Lemma 9.30 ,

$$
\mathbb{P}\left(\rho\left(\sigma_{\lambda_{n}}\right)(k)=\tau\right) \rightarrow d(\emptyset, \tau) K_{\omega_{U}}(\tau),
$$

for $\tau \vdash k$.
In particular $\rho\left(\sigma_{U}\right)=\rho_{\omega(U)}$. Since the subset

$$
\left\{U \in \mathcal{U}^{(2)}, \bar{U}=[0,1], U \text { has a finite number of components }\right\}
$$

is dense in $\mathcal{U}^{(2)}$, the latter equality holds on $\mathcal{U}^{(2)}$.
For any element $\omega=\left(\left(a_{1} \geq a_{2} \geq \cdots \geq 0\right),\left(b_{1} \geq b_{2} \geq \cdots \geq 0\right)\right) \in \partial_{M} \mathcal{Y}$, it is possible to find $U \in \mathcal{U}^{(2)}$ such that $\omega_{U}=\omega$, thus the map

$$
p:\left\{\begin{array}{clc}
\partial_{M} \mathcal{Z} & \longrightarrow & \partial_{M} \mathcal{Y} \\
U & \mapsto & \omega_{U}
\end{array}\right.
$$

is surjective.

### 9.9.3 Asymptotic of $\lambda_{n}$ under the Plancherel measure

The purpose of this subsection is to explore further the behavior of $\sigma_{(\emptyset, \emptyset)} \cdot \sigma_{(\emptyset, \emptyset)}$ is called the Plancherel measure on $\mathcal{Z}$ since it is the only element of $\partial_{M} \mathcal{Z}$ that yields the Plancherel measure on $\mathcal{Y}$ through the map $p$ of the last paragraph.
In order to describe the descent set of a permutation $\sigma \in \mathfrak{S}_{n+1}$ we introduce the following notations. Let $f_{\sigma}$ be the piecewise linear function on $[0, n-1]$ such that $f_{\sigma}(0)=0$, and

$$
f_{\sigma}(i)-f_{\sigma}(i-1)=\left\{\begin{array}{lc}
-1 & \text { if } i \text { is a descent of } \sigma \\
+1 & \text { othewise }
\end{array} .\right.
$$

To describe the asymptotic value of $f_{\sigma}$ for $\sigma$ following the probability measure $p_{\left(U_{\uparrow}, U_{\downarrow}\right)}\left(\left(U_{\uparrow}, U_{\downarrow}\right) \in\right.$ $\mathcal{U}^{(2)}$, we define also the following function $f_{\left(U_{\uparrow}, U_{\downarrow}\right)}$ : it is the unique a.e differentiable function on $[0,1]$ such that

$$
\left\{\begin{array}{cc}
f_{\left(U_{\uparrow}, U_{\downarrow}\right)}(0)=0 & \\
f_{\left(U_{\uparrow}, U_{\downarrow}\right)}^{\prime}(t)=1 & \text { if } t \in U_{\uparrow} \\
f_{\left(U_{\uparrow}, U_{\downarrow}\right)}^{\prime}(t)=-1 & \text { if } t \in U_{\downarrow} \\
f_{\left(U_{\uparrow}, U_{\downarrow}\right)}^{\prime}(t)=0 & \text { if } t \in[0,1] \backslash U
\end{array} .\right.
$$

The map $\left(U_{\uparrow}, U_{\downarrow}\right) \mapsto f_{\left(U_{\uparrow}, U_{\downarrow}\right)}$ is continuous from $\mathcal{U}^{(2)}$ to $\mathcal{C}([0,1], \mathbb{R})$, and the following result holds :

Proposition 9.32. Let $U \in \mathcal{U}^{(2)}$. Then

$$
\left(t \mapsto \frac{1}{n} f_{\sigma_{U}(n)}(n t)\right) \rightarrow_{p . s,\| \| \|_{\infty}} f_{\left(U_{\uparrow}, U_{\downarrow}\right)} .
$$

The proof is a deduction from Theorem 9.1, since $U\left(\sigma_{U}(n)\right) \rightarrow_{\mathcal{U}^{(2)}} U$.
The next step is to get the fluctuations of $f_{\sigma_{U}}$. Only the case $U=(\emptyset, \emptyset)$ is done here. The result consists mainly in a mathematical formalization of the results obtained by Oshanin and Voituriez from a physical point of view in [68]. The reader should refer to the latter paper for interesting additional informations on the process $f_{\sigma \emptyset, \theta}$.

Theorem 9.33. For $\sigma_{n}$ being uniformly sampled among $\mathfrak{S}_{n}$,

$$
\left(t \mapsto \frac{1}{\sqrt{n}} f_{\sigma_{n}}(n t)\right) \rightarrow \frac{1}{\sqrt{3}} \mathcal{B},
$$

$\mathcal{B}$ denoting the Brownian motion on $[0,1]$.
Proof. Recall from Section 4 that $\sigma_{n}$ can be sampled from a family of independent uniform random variables $\left(x_{i}\right)_{i \geq 1}$ on $[0,1]$ by applying the map $s t d^{-1}$ on the sequence $\left(x_{i}\right)_{1 \leq i \leq n}$. Since $\sigma \mapsto \sigma^{-1}$ is a measure preserving map (uniquely for the uniform measure), $f_{\sigma_{n}} \sim f_{\sigma_{n}^{-1}}$. The property noticed by Oshanin and Voituriez is that $\left(\left(f_{\sigma_{n}^{-1}}, x_{n}\right)\right)_{n \geq 1}$ is a Markov chain : indeed $i$ is a descent of $\sigma_{n}^{-1}$ if and only if $x_{i}>x_{i+1}$. Therefore, $\operatorname{Des}\left(\sigma_{n+1}^{-1}\right) \cap\{1, \ldots, n-1\}=\operatorname{Des}\left(\sigma_{n}^{-1}\right)$, and $n \in \operatorname{Des}\left(\sigma_{n+1}^{-1}\right)$ if and only if $x_{n}>x_{n+1}$. In the sequel $f_{\sigma_{n}^{-1}}(i)$ is denoted by $Y_{i}$ (the subscript $n$ is dropped, since this depends only on $\left.\left(\sigma_{n}\right)_{\downarrow i}\right)$.
This yields that for $R=\left[r_{1} ; r_{1}+r_{2}\right]$ and $S=\left[s_{1} ; s_{1}+s_{2}\right]$, with $s_{1} \geq r_{1}+r_{2}+2, n \geq s_{1}+s_{2}$, we have

$$
\left.\left(\# \operatorname{Des}\left(\sigma_{n}\right) \cap R, \# \operatorname{Des}\left(\sigma_{n}\right) \cap S\right)\right) \sim X_{1} \otimes X_{2},
$$

with $X_{1} \sim \# \operatorname{Des}\left(\sigma_{r_{2}+1}\right)$ and $X_{2} \sim \# \operatorname{Des}\left(\sigma_{s_{2}+1}\right)$. Moreover the number of permutations of $n$ with $k$ descents is given by the Eulerian number $A_{k}^{n}$, and its asymptotic value (see [79]) gives:

Lemma 9.34. For $n$ going to infinity, and $\sigma_{n}$ uniform on $\mathfrak{S}_{n}$,

$$
\frac{1}{\sqrt{n}}\left(\# \text { Des }-\frac{n}{2}\right) \rightarrow_{\text {law }} \mathcal{N}(0,1 / 12) .
$$

The latter Lemma together with the strong Markov property shows that if we write $\tilde{f}_{\sigma_{n}}(t)=$ $\frac{1}{\sqrt{n}} f_{\sigma_{n}}(n t)$, the marginal distributions of $\tilde{f}_{\sigma_{n}}$ converge towards the ones of $\frac{1}{\sqrt{3}} \mathcal{B}$. An adequate bound for $\left\|f_{\sigma_{n}}\right\|$ is needed to be able to conclude by stantard tightness arguments. We follow Theorem 8.4 of the book [22] of Billingsley :
Theorem 9.35. Let $\left(Y_{i}\right)_{i \geq 0}$ be a real random process. Let $f_{n}:[0,1] \rightarrow \mathbb{R}$ define the linear interpolation between the points

$$
f_{n}\left(\frac{i}{n}\right)=\frac{1}{\sqrt{n}} Y_{i} .
$$

Suppose that for all $\varepsilon>0$, there exists $\lambda>0, n_{0} \geq 0$ such that for all $k \in \mathbb{N}, n \geq n_{0}$,

$$
\mathbb{P}\left(\max _{i \leq n}\left|Y_{k+i}-Y_{k}\right| \geq \lambda \sigma \sqrt{n}\right) \leq \varepsilon / \lambda^{2} .
$$

Then the sequence $f_{n}$ is tight.
The hypothesis of the Theorem is verified through the following Lemma, that mimicks the situation coming from a usual random walk.
Lemma 9.36. Let $S_{n}=\sup _{[0, n]} Y_{n}, a>0$ and $b \leq a-2$. Then

$$
\mathbb{P}\left(S_{n} \geq a, Y_{n} \leq b\right) \leq \mathbb{P}\left(Y_{n} \geq 2 a-b-2\right),
$$

and $F_{S_{n}}(t) \geq F_{\left|Y_{n}\right|}(t)$ for all $t \in \mathbb{R}$.
Proof. In the Markov chain $\left(Y_{n}, x_{n}\right), T=\inf \left(u \in \mathbb{N}, Y_{u}=a\right)$ is a stopping time. Since $\left\{S_{n} \geq\right.$ $a\}=\{T \leq n\},\left\{S_{n} \geq a\right\} \in \mathcal{F}_{T}$ and by the strong Markov property,

$$
\begin{aligned}
& \mathbb{P}\left(S_{n} \geq a, Y_{n} \leq b\right)=\mathbb{P}\left((T \leq n) \cap\left(Y_{n}-Y_{T} \leq b-a\right)\right) \\
&\left.=\mathbb{E}\left(\mathbf{1}_{T \leq n} \mathbb{P}_{\left(Y_{T}, x_{T}\right)}\right)\left(\tilde{Y}_{n-T}-\tilde{Y}_{0} \leq b-a\right)\right) \\
& \leq \mathbb{E}\left(\mathbf{1}_{T \leq n} \mathbb{P}\left(Y_{T}, x_{T}\right)\right. \\
&\left.\left.\tilde{Y}_{n-T}-\tilde{Y}_{1} \leq b-a+1\right)\right),
\end{aligned}
$$

with $\left(\tilde{Y}, \tilde{x}_{i}\right)$ being an independent random walk starting at $\left(Y_{T}, x_{T}\right)$. Since $\tilde{Y}_{n-T}-\tilde{Y}_{1}$ is independent of the value $\tilde{Y}_{0}=Y_{T}$,

$$
\mathbb{E}\left(\mathbf{1}_{T \leq n} \mathbb{P}_{\left(Y_{T}, x_{T}\right)}\left(\tilde{Y}_{n-T}-\tilde{Y}_{1} \leq b-a+1\right)\right)=\mathbb{E}\left(\mathbf{1}_{T \leq n} \mathbb{P}_{\left(0, x_{T}\right)}\left(\tilde{Y}_{n-T}-\tilde{Y}_{1} \leq b-a+1\right)\right)
$$

Moreover $\tilde{Y}_{n-T}-\tilde{Y}_{1} \sim-\left(\tilde{Y}_{n-T}-\tilde{Y}_{1}\right)$, thus

$$
\begin{aligned}
\mathbb{P}\left(S_{n} \geq a, Y_{n} \leq b\right) & \leq \mathbb{E}\left(\mathbf{1}_{T \leq n} \mathbb{P}_{\left(0, x_{T}\right)}\left(-\left(\tilde{Y}_{n-T}-\tilde{Y}_{1}\right) \leq b-a+1\right)\right) \\
& =\mathbb{E}\left(\mathbf{1}_{T \leq n} \mathbb{P}_{\left(0, x_{T}\right)}\left(\tilde{Y}_{n-T} \geq a-(b+1)+\tilde{Y}_{1}\right)\right) \\
& \leq \mathbb{E}\left(\mathbf{1}_{T \leq n} \mathbb{P}_{\left(0, x_{T}\right)}\left(\tilde{Y}_{n-T} \geq a-(b+1)-1\right)\right) \\
& \leq \mathbb{P}\left(\left((T \leq n) \cap\left(Y_{n} \geq 2 a-b-2\right)\right) \leq \mathbb{P}\left(Y_{n} \geq 2 a-b-2\right),\right.
\end{aligned}
$$

the last equality being due to the fact that $\left(Y_{n} \geq 2 a-b-2\right) \subseteq(T \leq n)$. This yields

$$
\begin{aligned}
\mathbb{P}\left(S_{n} \geq a\right) & \leq \mathbb{P}\left(\left(S_{n} \geq a\right) \cap\left(Y_{n} \leq a-2\right)\right)+\mathbb{P}\left(\left(S_{n} \geq a\right) \cap\left(Y_{n} \geq a\right)\right) \\
& \leq \mathbb{P}\left(Y_{n} \geq a\right)+\mathbb{P}\left(Y_{n} \geq a\right) \leq \mathbb{P}\left(\left|Y_{n}\right| \geq a\right),
\end{aligned}
$$

the last equality being due to the fact that the law of $Y_{n}$ is symmetric. This yields

$$
F_{S_{n}}(u) \geq F_{\mid Y_{n \mid}}(u)
$$

In particular from the latter Lemma, for $\varepsilon>0$ and $\lambda$ such that $\mathbb{P}(\mathcal{N}(0,1 / 3) \geq \lambda) \leq \frac{\varepsilon}{4 \lambda^{2}}$, and for $n$ large enough, $k \geq 0$,

$$
\begin{aligned}
\mathbb{P}\left(\max _{i \leq n}\left|Y_{k+i}-Y_{k}\right| \geq \lambda \sqrt{n}\right) & \leq 2 \mathbb{P}\left(\left|Y_{k+n}-Y_{k}\right| \geq \lambda \sqrt{n}\right) \\
& \leq 2 \mathbb{P}(\mathcal{N}(0,1 / 3) \geq \lambda)+\frac{\varepsilon}{2 \lambda^{2}} \\
& \leq \frac{\varepsilon}{\lambda^{2}}
\end{aligned}
$$

And this concludes the proof of the Proposition.

## Appendix: convergence result for the Paintbox construction

This appendix is dedicated to the proof of Proposition 9.14. Some notations and two preliminary results are first given.

### 9.9.4 Cluster sets

Let $k \geq 1$ and $A$ a given set. We define an $A$-cluster of $k$ as a map $f: A \rightarrow \mathcal{P}([1 ; k])$ such that $f\left(a_{1}\right) \cap f\left(a_{2}\right)=\emptyset$ for $a_{1} \neq a_{2}$. The residue of $f$ is the set $R_{f}=[1 ; k] \backslash \bigcup f(a)$ and the support of $f$ is the set $S_{f}$ of $a \in A$ such that $f(a) \neq \emptyset$. The degree of $f$ is the minimum of the cardinals of non-empty sets $f(a)$. The set of $A$-clusters (resp. $A$-clusters of degree larger than $s)$ is denoted $C^{k}(A)\left(\operatorname{resp} C_{s}^{k}(A)\right)$. For $1 \leq s \leq k$, the $s$-level of the $A$-cluster $f$, denoted $f^{s}$, is the $A$-cluster of $C_{s}^{k}(A)$ defined by :

$$
f^{s}(a)=\left\{\begin{array}{cc}
f(a) & \text { if }|f(a)| \geq s \\
\emptyset & \text { else }
\end{array}\right.
$$

Let $\vec{x}=\left(x_{1}, \ldots, x_{k}\right)$ be a sequence on a space $\left(\Omega^{k}, \mathcal{A}^{\otimes k}\right)$. Then any set $J$ and any collection of disjoint subsets $\mathcal{A}=\left(\mathcal{A}_{j}\right)_{j \in J}$ of $\Omega$ yields a $J$-cluster map of $k$

$$
f_{\vec{x}, \mathcal{A}}(j)=\left\{i \mid x_{i} \in \mathcal{A}_{j}\right\}
$$

For $U \in \mathcal{U}^{(2)}$, denote by $\mathcal{U}=\left\{U_{\alpha}\right\}_{\alpha \in A_{U}}$ the collection of interval components of $U_{\uparrow} \cup U_{\downarrow}$. We say that $\alpha \in A_{U}^{+}$(resp. $A_{U}^{-}$) if $U_{\alpha} \subseteq U_{\uparrow}$ (resp. $U_{\alpha} \subseteq U_{\downarrow}$ ).

Lemma 9.37. Let $U=\left(U_{\uparrow}, U_{\downarrow}\right) \in \mathcal{U}^{(2)}$. For $\sigma \in \mathfrak{S}_{k}$ and $f \in C_{2}^{k}\left(A_{U}\right)$ define the sets

$$
X_{\sigma, f}(U)=\left\{\vec{x} \in[0,1]^{k} \mid s t d^{-1}(\vec{x})=\sigma\right\} \cap\left\{\left(f_{\vec{x}, \mathcal{U}}\right)^{2}=f\right\}
$$

Then the sets $X_{\sigma, f}(U)$ are disjoint open sets and $\sigma_{U}$ is constant on each of these sets. In particular if $X_{k}=(X(i))_{1 \leq i \leq k}$ is a random variable with density on $[0,1]^{k}, \sigma_{U}\left(X_{1}, \ldots, X_{k}\right)$ is $\left(\left(X_{\sigma, f}(U)\right)_{f, \sigma}\right.$ measurable.

Proof. Let us write simply $X_{\sigma, f}$ instead of $X_{\sigma, f}(U)$. Suppose that $X_{k} \in X_{\sigma, f} \cap X_{\sigma^{\prime}, f^{\prime}}$. Then $s^{\prime 2} d^{-1}\left(X_{k}\right)=\sigma=\sigma^{\prime}$. Moreover $f=\left(f_{X_{k}}, \mathcal{U}\right)^{2}=f^{\prime}$ and $X_{\sigma, f}=X_{\sigma^{\prime}, f^{\prime}}$. The events $X_{\sigma, f}$ are thus disjoint for distinct pairs $(\sigma, f)$. They are open from their definition and the fact that $U_{\uparrow}, U_{\downarrow}$ are open sets.
Each $\tau \in \mathfrak{S}_{k}$ is entirely defined by the set $S_{\tau}=\left\{(i, j) \mid i<j, \tau^{-1}(i) \leq \tau^{-1}(j)\right\}$. Let $\sigma \in \mathfrak{S}_{k}$. Then from the Paintbox construction, $\sigma_{U}^{-1}(\{\tau\}) \cap\left\{X_{k} \mid s t d^{-1}\left(X_{k}\right)=\sigma\right\}$ is precisely the set of $X_{k}$ such that by writing $f=f_{X_{k}}, \mathcal{u}$ :

- $(i, j) \in S_{\sigma} \cap S_{\tau} \Rightarrow \forall a \in A_{U}^{-},\{i, j\} \nsubseteq f(a)$,
- $(i, j) \in S_{\sigma} \backslash S_{\tau} \Rightarrow \exists a \in A_{U}^{-},\{i, j\} \subseteq f(a)$,
- $(i, j) \in S_{\tau} \backslash S_{\sigma} \Rightarrow \exists a \in A_{U}^{+},\{i, j\} \subseteq f(a)$,
- $(i, j) \notin S_{\sigma} \cup S_{\tau} \Rightarrow \forall a \in A_{U}^{+},\{i, j\} \nsubseteq f(a)$.

Define by $D(\sigma, \tau)$ the set of $A(U)$-clusters that respect the above four conditions. Thus

$$
\sigma_{U}^{-1}(\{\tau\}) \cap\left\{X_{k} \mid s t d^{-1}\left(X_{k}\right)=\sigma\right\}=\bigcup_{f \in D(\sigma, \tau)} X_{\sigma, f} .
$$

Since $X_{k}$ admits a density function, $[0,1]^{k} \backslash \bigcup\left\{X_{k} \mid s t d^{-1}\left(X_{k}\right)=\sigma\right\}$ is a null-set and thus :

$$
\sigma_{U}^{-1}(\{\tau\})=\bigcup_{\sigma} \bigcup_{f \in D(\sigma, \tau)} X_{\sigma, f},
$$

which proves the Lemma.

### 9.9.5 Convergence in law with conditioning

The pattern of the proof of Proposition 9.14 implies the following question: suppose that $A, B$ are metric spaces, considered as measure spaces with a given measure on each associated borelian $\sigma$-algebra. Let $f_{n}: A \rightarrow B$ be a sequence of measurable functions that converges pointwise almost surely to a continuous function $f: A \rightarrow B$. Let $\left(X_{n}\right)_{n \geq 1}$ be a sequence of random variables on $A$ that converges in law to a random variable $X$. Do we have the convergence in law $f_{n}\left(X_{n}\right) \rightarrow f(X)$ ? The answer is negative in general, but in a very particular case the result holds.

Lemma 9.38. Let $\left(\mathcal{X}_{m}\right)_{m \geq 1}$ be a family of measurable spaces of $A$ with the following conditions :

- $\lim _{m \rightarrow \infty} \mathbb{P}\left(X \in \mathcal{X}_{m}\right)=1$.
- $\forall m \geq 1, \mathbb{P}\left(X \in \partial \mathcal{X}_{m}\right)=0$.
- For all $m \geq 1, f_{n \mid \mathcal{X}_{m}} \rightarrow f_{\mid \mathcal{X}_{m}}$ uniformly.

Then $f_{n}\left(X_{n}\right)$ converges in law to $f(X)$.
Proof. Let $g: B \rightarrow \mathbb{R}$ be a 1 -Lipschitz function bounded by 1 . It suffices to show that $\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right)\right)-\mathbb{E}(g \circ f(X)) \longrightarrow 0$. For each $m \geq 1$, the difference can be bounded by

$$
\begin{aligned}
& \left|\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right)\right)-\mathbb{E}(g \circ f(X))\right| \leq \\
& \quad\left|\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right)\right)-\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right) \mid \mathcal{X}_{m}\right)\right|+\left|\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right) \mid \mathcal{X}_{m}\right)-\mathbb{E}\left(g \circ f\left(X_{n}\right) \mid \mathcal{X}_{m}\right)\right| \\
& +\left|\mathbb{E}\left(g \circ f\left(X_{n}\right) \mid \mathcal{X}_{m}\right)-\mathbb{E}\left(g \circ f(X) \mid \mathcal{X}_{m}\right)\right|+\left|\mathbb{E}\left(g \circ f(X) \mid \mathcal{X}_{m}\right)-\mathbb{E}(g \circ f(X))\right| .
\end{aligned}
$$

Let $m$ be such that $\mathbb{P}\left(X \in \mathcal{X}_{m}\right) \geq 1-\varepsilon$. Since $\mathbb{P}\left(X \in \partial \mathcal{X}_{m}\right)=0$, by the convergence in law there exists $n_{0}$ such that for $n \geq n_{0}, \mathbb{P}\left(X_{n} \in \mathcal{X}_{m}\right) \geq 1-2 \varepsilon$. For $n \geq n_{0}$,

$$
\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right)\right)=\mathbb{P}\left(X_{n} \in \mathcal{X}_{m}\right) \mathbb{E}\left(g \circ f_{n}\left(X_{n}\right) \mid \mathcal{X}_{m}\right)+\mathbb{P}\left(X_{n} \notin \mathcal{X}_{m}\right) \mathbb{E}\left(g \circ f_{n}\left(X_{n}\right) \mid \mathcal{X}_{m}^{c}\right) .
$$

Since $g$ is bounded by 1 and $\mathbb{P}\left(X_{n} \notin \mathcal{X}_{m}\right) \leq 2 \varepsilon$,

$$
\begin{aligned}
\left|\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right)\right)-\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right) \mid \mathcal{X}_{m}\right)\right| & \leq 2 \varepsilon+\left|1-\mathbb{P}\left(X_{n} \in \mathcal{X}_{m}\right)\right| \\
& \leq 4 \varepsilon .
\end{aligned}
$$

For the same reasons,

$$
\left|\mathbb{E}\left(g \circ f(X) \mid \mathcal{X}_{m}\right)-\mathbb{E}(g \circ f(X))\right| \leq 2 \varepsilon .
$$

Let $n_{1} \geq n_{0}$ such that for $n \geq n_{1},\left\|f_{n \mid \mathcal{X}_{m}}-f_{\mid \mathcal{X}_{m}}\right\| \leq \varepsilon$. Since $g$ is 1 -Lipschitz, for $n \geq n_{1}$,

$$
\left|\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right) \mid \mathcal{X}_{m}\right)-\mathbb{E}\left(g \circ f\left(X_{n}\right) \mid \mathcal{X}_{m}\right)\right| \leq \varepsilon .
$$

Since $\mathbb{P}\left(X \in \partial \mathcal{X}_{m}\right)=0,\left(X_{n} \mid \mathcal{X}_{m}\right)$ converges in law to $\left(X \mid \mathcal{X}_{m}\right)$ and thus there exists $n_{2} \geq n_{1}$ such that for $n \geq n_{2}$,

$$
\left|\mathbb{E}\left(g \circ f\left(X_{n}\right) \mid \mathcal{X}_{m}\right)-\mathbb{E}\left(g \circ f(X) \mid \mathcal{X}_{m}\right)\right| \leq \varepsilon .
$$

Therefore for $n \geq n_{2}$,

$$
\left|\mathbb{E}\left(g \circ f_{n}\left(X_{n}\right)\right)-\mathbb{E}(g \circ f(X))\right| \leq 5 \varepsilon,
$$

which implies the Lemma.

### 9.9.6 Proof of Proposisition 9.14

Let us recall here the statement of Proposition 9.14.
Proposition. Let $U_{n}$ be a sequence of $\mathcal{U}^{(2)}$ and $\left(\left(X^{n}(i)\right)_{i \geq 1}\right)_{n \geq 1}$ a sequence of random infinite vectors on $[0,1]$. Let $\left(X^{0}(1), \ldots, X^{0}(n), \ldots\right)$ be a random infinite vector on $[0,1]$. Suppose that each finite dimensional marginal law of any of these random vectors admits a density with respect to the Lebesgue measure. If $U_{n} \rightarrow U \in \mathcal{U}^{(2)}$ and for each $k \geq 1, X_{k}^{n}=\left(X^{n}(1), \ldots, X^{n}(k)\right)$ converges in law to $X_{k}^{0}=\left(X^{0}(1), \ldots, X^{0}(k)\right)$, then for each $k \geq 1$,

$$
\sigma_{U_{n}}\left(X_{k}^{n}\right) \longrightarrow_{\text {law }} \sigma_{U}\left(X_{k}^{0}\right) .
$$

Proof. Let $k \geq 1$ and set $X=X_{k}^{0}, X_{n}=X_{k}^{n}$.
Let $A=\bigcup \underset{\substack{\sigma \in \mathcal{C}_{2}\left(A_{U}\right)}}{ } X_{f, \sigma}$ (refer to Lemma 9.37 for the definition of $X_{f, \sigma}$ ) with the induced topology from $[0,1]^{k}$, and $B=\mathfrak{S}_{k}$ with the discrete topology. Then from Lemma $9.37, \sigma_{U}$ : $A \rightarrow B$ is constant on each connected component $X_{f, \sigma}$ of $A$, thus $\sigma_{U}$ is continuous.
By the definition of the convergence on $\mathcal{U}^{(2)}$, for $\vec{X}=\left(X_{1}, \ldots, X_{k}\right) \in A, \sigma_{U_{n}}(\vec{X})$ converges to $\sigma_{U}(\vec{X})$.
Since $[0,1]^{k} \backslash A$ is of Lebesgue measure 0 , we can suppose that $X_{n}, X$ are random variables on
$A$. It remains to build a sequence of measurable sets $\mathcal{X}_{m}$ that respects the hypothesis of Lemma 9.38

Let $m \geq 1$. For $\eta>0$, define $\Delta_{\eta}=\bigcup_{1 \leq i, j \leq k}\left\{\left(x_{1}, \ldots, x_{k}\right) \in[0,1]^{k},\left|x_{i}-x_{j}\right| \leq \eta\right\}$. Then $\partial_{[0,1]^{k}} \Delta_{\eta} \subseteq \bigcup_{1 \leq i, j \leq k}\left\{\left(x_{1}, \ldots, x_{k}\right) \in[0,1]^{k},\left|x_{i}-x_{j}\right|=\eta\right\}$. Since the latter is of Lebesgue measure $0, \mathbb{P}\left(X \in \partial_{[0,1]^{k}} \Delta_{\eta}\right)=0$. Since $\Delta_{\eta}$ is decreasing in $\eta$ and $\operatorname{Leb}\left(\bigcap \Delta_{\eta}\right)=0$, there exists $\eta_{1}^{m}>0$ such that $\mathbb{P}\left(X \in \Delta_{\eta_{1}^{m}}\right) \leq \frac{1}{m}$.

Denote by $\mathcal{U}=\left\{U_{\alpha}=\right] r_{\alpha}, s_{\alpha}[ \}_{\alpha \in \mathcal{A}}$ the finite ordered collection of interval components of $U_{\downarrow} \cup U_{\uparrow}$ of size larger than $\eta_{1}^{m}$. $\mathcal{A}$ is a subset of $A(U)$. For $\eta>0$, let

$$
B_{\eta}=\bigcup_{i, \alpha}\left\{\left(x_{1}, \ldots, x_{k}\right) \in[0,1]^{k}, x_{i} \in\right] r_{\alpha}-\eta, r_{\alpha}+\eta[\cup] s_{\alpha}-\eta, s_{\alpha}+\eta[ \} .
$$

Once again $\left.\operatorname{Leb}\left(\partial_{[0,1]^{k}} B_{\eta}\right)=0\right)$, and since $\operatorname{Leb}\left(\bigcap_{\eta} B_{\eta}\right)=0$, there exists $\eta_{2}^{m}$ such that $\mathbb{P}(X \in$ $\left.B_{\eta_{2}^{m}}\right) \leq \frac{1}{m}$. Let $K_{m}=B_{\eta_{2}^{m}} \cup \Delta_{\eta_{1}^{m}}, \mathcal{X}_{m}$ be the set $\{\vec{x} \notin K\}$. Then $\mathbb{P}\left(X \in \partial \mathcal{X}_{m}\right)=0$ and $\lim _{m \rightarrow+\infty} \mathbb{P}\left(X \in \mathcal{X}_{m}\right)=1$.
Let $\mathcal{X}_{m}$ be fixed, with associated complementary set $K_{m}=B_{\eta_{2}^{m}} \cup \Delta_{\eta_{1}^{m}}$. Set $\eta_{m}=\inf \left(\eta_{1}^{m}, \eta_{2}^{m}\right)$, and let $n_{m}$ be such that for $n \geq n_{0}, d_{\mathcal{U}^{(2)}}\left(U_{n}, U\right) \leq \eta_{m}$. Suppose from now on that $n \geq n_{m}$. Since $d_{\mathcal{U}^{(2)}}\left(U_{n}, U\right) \leq \eta_{m} \leq \eta_{1}^{m}$ the interval components of $U_{\downarrow}^{n}$ (resp. $U_{\uparrow}^{n}$ ) of size larger than $\eta_{1}^{m}$ are in order respecting bijection with those of $U_{\downarrow}$ (resp. $U_{\uparrow}$ ). Denote these interval components of $U^{n}$ by $\mathcal{U}_{n}=\left\{U_{\alpha}^{n}=\right] r_{\alpha}^{n}, s_{\alpha}^{n}[ \}_{\alpha \in \mathcal{A}}$, with $\mathcal{A} \subseteq A\left(U_{n}\right)$. Moreover since $d_{\mathcal{U}^{(2)}}\left(U_{n}, U\right) \leq \eta_{m} \leq \eta_{2}^{m}$, $\left|r_{\alpha}^{n}-r_{\alpha}\right|<\eta_{2}^{m}$ and $\left|s_{\alpha}^{n}-s_{\alpha}\right|<\eta_{2}^{m}$.
Since on $\mathcal{X}_{m},\left|x_{i}-x_{j}\right| \geq \eta_{1}$, if $f \in C_{2}^{k}(A(U))$ and $S(f) \nsubseteq \mathcal{A}$, then $X_{\sigma, f}(U) \cap \mathcal{X}_{m}=\emptyset$. Thus we can consider that $f \in C_{2}^{k}(\mathcal{A})$. The same is true for $f \in C_{2}^{k}\left(A\left(U_{n}\right)\right), S(f) \nsubseteq \mathcal{A}$ with $\left(X_{\sigma, f}\left(U_{n}\right) \cap\right.$ A) $\cap \mathcal{X}_{m}$.

Let $f \in C_{2}^{k}(\mathcal{A})$ and suppose that $\vec{x} \in X_{\sigma, f}(U) \cap \mathcal{X}_{m}$. Let $\alpha \in S_{f}$ and suppose that $x_{i} \in U_{\alpha}=$ $] r_{\alpha}, s_{\alpha}\left[\right.$; since $\left.\vec{x} \in \mathcal{X}_{m}, x_{i} \in\right] r_{\alpha}+\eta_{2}^{m}, s_{\alpha}-\eta_{2}^{m}\left[\right.$. But $\left|r_{\alpha}^{n}-r_{\alpha}\right|<\eta_{2}^{m}$ and $\left|s_{\alpha}^{n}-s_{\alpha}\right|<\eta_{2}^{m}$, thus $x_{i} \in U_{\alpha}^{n}$. Conversely is $\alpha \in S_{f}$ and $x_{i} \in U_{\alpha}^{n}$, for the same reasons $x_{i} \in U_{\alpha}$. This shows that $X_{\sigma, f}(U) \cap \mathcal{X}_{m}=X_{\sigma, f}\left(U_{n}\right) \cap \mathcal{X}_{m}$. This yields that $\sigma_{U \mid X_{\sigma, f}(U)}=\sigma_{U_{n} \mid X_{\sigma, f}\left(U_{n}\right)}$. Finally we have proven that for $n \geq n_{m}, \sigma_{U_{n} \mid \mathcal{X}_{m}}=\sigma_{U \mid \mathcal{X}_{m}}$, which implies obviously the uniform convergence $\sigma_{U_{n} \mid \mathcal{X}_{m}} \rightarrow_{n \rightarrow \infty} \sigma_{U \mid \mathcal{X}_{m}}$.
The application of Lemma 9.38 concludes the proposition.

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