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Yakine Bahri

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par

**Yakine Bahri**

**Stabilité des solitons et des multi-solitons pour l'équation  
de Landau-Lifshitz**

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**À la mémoire de :**

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**Bibliographie**

## Résumé

Dans cette thèse, nous étudions l'équation de Landau-Lifshitz avec une anisotropie planaire en dimension un. Cette équation décrit la dynamique de l'aimantation dans des matériaux ferromagnétiques. Elle admet des solutions particulières de type onde progressive appelées solitons.

D'abord, nous montrons la stabilité asymptotique des solitons de vitesse non nulle appelés solitons sombres dans l'espace d'énergie. Plus précisément, nous prouvons que toute solution correspondant à une donnée initiale proche du soliton de vitesse non nulle, converge faiblement dans l'espace d'énergie en temps long, vers un soliton de vitesse non nulle, sous les invariances géométriques de l'équation. Notre analyse repose sur les idées développées par Martel et Merle pour les équations de Korteweg-de Vries généralisées. Nous utilisons la transformée de Madelung pour étudier le problème dans le cadre hydrodynamique. Nous invoquons ensuite la stabilité orbitale des solitons et la continuité faible du flot afin de construire le profil limite. Nous établissons de plus une formule de monotonie pour le moment, ce qui nous permet d'avoir la localisation du profil limite. Sa régularité et sa décroissance exponentielle découlent d'un résultat de régularité pour les solutions localisées des équations de Schrödinger. Nous finissons la preuve par un théorème de type Liouville, qui nous indique que seuls les solitons vérifient ces propriétés dans leurs voisinages.

Nous nous intéressons également à la stabilité asymptotique d'une superposition de plusieurs solitons appelées multi-solitons. Les solitons de vitesse non nulle sont ordonnés selon leurs vitesses et sont initialement bien séparés. Nous démontrons la stabilité asymptotique autour et entre les solitons. Plus précisément, nous montrons que pour une donnée initiale proche de la somme de  $N$  solitons sombres, la solution correspondante converge faiblement vers un des solitons de la somme, quand elle est translatée au niveau du centre de ce soliton, et converge faiblement vers zéro quand elle est translatée entre les solitons.

## Abstract

In this thesis, we study the one-dimensional Landau-Lifshitz equation with an easy-plane anisotropy. This equation describes the dynamics of the magnetization in a ferromagnetic material. It owns travelling-wave solutions called solitons.

We begin by proving the asymptotic stability in the energy space of non-zero speed solitons. More precisely, we show that any solution corresponding to an initial datum close to a soliton with non-zero speed, is weakly convergent in the energy space as time goes to infinity, to a soliton with a possible different non-zero speed, up to the geometric invariances of the equation. Our analysis relies on the ideas developed by Martel and Merle for the generalized Korteweg-de Vries equations. We use the Madelung transform to study the problem in the hydrodynamical framework. In this framework, we rely on the orbital stability of the solitons and the weak continuity of the flow in order to construct a limit profile. We next derive a monotonicity formula for the momentum, which gives the localization of the limit profile. Its smoothness and exponential decay then follow from a smoothing result for the localized solutions of the Schrödinger equations. Finally, we prove a Liouville type theorem, which shows that only the solitons enjoy these properties in their neighbourhoods.

We also establish the asymptotic stability of multi-solitons. The solitons have non-zero speed, are ordered according to their speeds and have sufficiently separated initial positions. We provide the asymptotic stability around solitons and between solitons. More precisely, we show that for an initial datum close to a sum of  $N$  dark solitons, the corresponding solution converges weakly to one of the solitons in the sum, when it is translated to the centre of this soliton, and converges weakly to zero when it is translated between solitons.



# Chapitre 1

## Introduction.

## 1.1 Motivation physique

Dans un matériau ferromagnétique, les spins atomiques interagissent entre eux contrairement aux atomes d'un matériau paramagnétique qui n'interagissent qu'avec un champ magnétique extérieur. Plus précisément, chaque spin essaie d'aligner les autres spins dans sa direction propre. La mécanique quantique analyse ce phénomène comme un échange d'énergie entre les spins. L'interaction d'échange est décrite par le Hamiltonien d'échange de Heisenberg, qui est donné par

$$\mathcal{H} = -\frac{1}{2} \sum_{i \neq j} J_{i,j} S_i \cdot S_j.$$

Dans cette expression,  $S_i$  est l'opérateur de spin pour la  $i$ ème particule du réseau cristallin considéré et  $J_{i,j}$  désigne l'intégrale d'échange. Le cas  $i = j$  est exclu car les spins n'interagissent pas sur eux-mêmes. Dans le cas d'un matériau ferromagnétique, les nombres  $J_{i,j}$  sont strictement positifs.

La dynamique des spins en présence d'un champ magnétique est donnée par l'équation du mouvement

$$\hbar \partial_t S_i = -2\mu_0 S_i \times H_{\text{eff}}, \quad (1.1.1)$$

où  $\hbar$  est la constante de Planck et  $\mu_0$  est le magnéton de Bohr. Le champ magnétique effectif  $H_{\text{eff}}$  correspond à l'interaction des spins. Il s'écrit sous la forme

$$\mathcal{H} = \frac{2\mu_0}{\hbar} \sum_{i_0} S_{i_0} \cdot H_{\text{eff}}, \quad (1.1.2)$$

où

$$\frac{2\mu_0}{\hbar} H_{\text{eff}}^k = -\frac{1}{2} \sum_i \left( J_{i,i+i_0} S_{i+i_0}^k + J_{i,i-i_0} S_{i-i_0}^k \right) = \frac{\partial \mathcal{H}}{\partial S_{i_0}^k}, \quad k = 1, 2, 3, \quad (1.1.3)$$

et  $S_i := (S_i^1, S_i^2, S_i^3)$ . Ces équations conduisent à une description macroscopique du matériau ferromagnétique dans un régime de type ondes longues [16, 27, 33].

En 1935, Landau et Lifshitz [34] ont établi que le vecteur  $m$  de magnétisation, qui correspond à l'opérateur de spin  $S_i$  dans le cadre atomique, est solution de l'équation

$$\partial_t m = -\frac{2\mu_0}{\hbar} m \times H_{\text{eff}} - \gamma m \times (m \times H_{\text{eff}}), \quad m(x, t) \in \mathbb{S}^2, \quad (1.1.4)$$

où  $\gamma$  est la constante de relaxation qui caractérise l'amortissement du mouvement du vecteur  $m$ . Le champ magnétique effectif  $H_{\text{eff}}$  correspond à la dérivée variationnelle de l'énergie magnétique par rapport à  $m$

$$H_{\text{eff}} = -\frac{\delta E}{\delta m}.$$

L'équation de Landau-Lifshitz rend par exemple compte de la magnétisation d'une chaîne  $CsNiF_3$  [23, 35, 56] (Fluorure de Nickel(II)-Césium) en dimension un, d'une plaque ferromagnétique en dimension deux et des matériaux ferromagnétiques plus généraux en dimension trois.

Dans la suite, nous nous intéressons au cas non amorti, c'est-à-dire au cas où  $\gamma = 0$ , et nous nous restreignons à l'énergie

$$E(m) := \frac{1}{2} \int_{\mathbb{R}} \left[ |\nabla m(x, t)|^2 - \lambda m_3^2(x, t) \right] dx.$$

Le champ magnétique effectif associé est donné par l'expression

$$H_{\text{eff}} = \Delta m + \lambda m_3 e_3,$$

où  $e_3 = (0, 0, 1)$  et  $\lambda \in \mathbb{R}$ . Le paramètre  $\lambda$  rend compte de l'anisotropie du matériau. Dans le cas d'anisotropie planaire,  $\lambda$  est strictement négatif. Le cas  $\lambda > 0$  correspond à une anisotropie axiale. Dans le cas isotrope, i.e.  $\lambda = 0$ , le problème se réduit à l'équation des Schrödinger maps.

Dans la suite, nous nous plaçons dans le cas d'anisotropie planaire. Quitte à changer d'échelle, nous pouvons supposer que  $\lambda = -1$ . L'équation s'écrit alors de la façon suivante

$$\partial_t m + m \times (\Delta m - m_3 e_3) = 0, \quad (\text{LL})$$

pour une application  $m = (m_1, m_2, m_3) : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{S}^2$ . Cette équation est dispersive. La relation de dispersion de l'équation linearisée autour de la solution constante  $(1, 0, 0)$  est donnée par la formule

$$\omega(k)^2 = |k|^4 + |k|^2.$$

Les vitesses de groupe correspondantes sont égales à

$$\vartheta_g(k) := \nabla \omega(k) = \pm \frac{2|k|^2 + 1}{\sqrt{|k|^4 + |k|^2}} k.$$

La norme de la vitesse  $\vartheta_g$  est toujours supérieure à la vitesse du son  $c_s = 1$  [18].

Comme les équations de Schrödinger, l'équation de Landau-Lifshitz possède une forme hydrodynamique qui s'obtient par des changements de variables analogues à la transformation de Madelung [38].

La terminologie hydrodynamique vient du fait que l'équation ressemble alors à l'équation d'Euler pour un fluide irrotationnel. Sur le plan mathématique, ce cadre permet de s'abstenir des propriétés géométriques compliquées de l'équation et de se ramener à un système d'équations proches de l'équation de Korteweg-de Vries. Cette proximité permet d'espérer étendre les nombreux résultats de stabilité asymptotique pour cette dernière équation, qu'ont démontré Martel et Merle [41–45, 47, 51].

Soit  $m$  une solution de l'équation de Landau-Lifshitz d'énergie finie, c'est-à-dire qui appartient à l'espace

$$\mathcal{E}(\mathbb{R}) := \left\{ v : \mathbb{R} \rightarrow \mathbb{S}^2, \text{ t.q. } v' \in L^2(\mathbb{R}) \text{ et } v_3 \in L^2(\mathbb{R}) \right\}.$$

Puisque la fonction  $m_3$  appartient à  $H^1(\mathbb{R})$ , il résulte du théorème d'injection de Sobolev que l'application  $\check{m} := m_1 + im_2$  satisfait

$$|\check{m}(x)| = (1 - m_3^2(x))^{\frac{1}{2}} \rightarrow 1,$$

quand  $x \rightarrow \pm\infty$ . Dans les variables  $\check{m}$  et  $m_3$ , l'équation de Landau-Lifshitz s'écrit

$$\begin{cases} i\partial_t \check{m} - m_3 \partial_{xx} \check{m} + \check{m} \partial_{xx} m_3 - \check{m} m_3 = 0, \\ \partial_t m_3 + \partial_x \langle i\check{m}, \partial_x \check{m} \rangle_{\mathbb{C}} = 0. \end{cases}$$

Quand la fonction  $\check{m}$  ne s'annule pas, elle se relève sous la forme  $\check{m} = (1 - m_3^2)^{1/2} \exp i\varphi$ . Les variables hydrodynamiques  $v := m_3$  et  $w := \partial_x \varphi$  vérifient le système suivant

$$\begin{cases} \partial_t v = \partial_x \left( (v^2 - 1)w \right), \\ \partial_t w = \partial_x \left( \frac{\partial_{xx} v}{1 - v^2} + v \frac{(\partial_x v)^2}{(1 - v^2)^2} + v(w^2 - 1) \right). \end{cases} \quad (\text{HLL})$$

Dans ces nouvelles variables, l'énergie de Landau-Lifshitz est donnée par

$$E(\mathbf{v}) := \int_{\mathbb{R}} e(\mathbf{v}) := \frac{1}{2} \int_{\mathbb{R}} \left( \frac{(v')^2}{1 - v^2} + (1 - v^2)w^2 + v^2 \right), \quad (1.1.5)$$

où  $\mathbf{v} := (v, w)$  est le couple hydrodynamique associé à  $m$ . Le moment  $P$  est défini par

$$P(\mathbf{v}) := \int_{\mathbb{R}} vw. \quad (1.1.6)$$

Ces deux fonctionnelles sont conservées par le flot de l'équation. Elles jouent un rôle important dans l'analyse des solitons. Notons que l'espace d'énergie est donné dans le cadre hydrodynamique par l'ensemble

$$\mathcal{NV}(\mathbb{R}) := \left\{ \mathbf{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ t.q. } \max_{\mathbb{R}} |v| < 1 \right\},$$

et qu'il est possible de le munir de la structure métrique associée à la norme

$$\|\mathbf{v}\|_{H^1 \times L^2} := \left( \|v\|_{H^1}^2 + \|w\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

## 1.2 État de l'art mathématique

Nous commençons par rappeler les principaux résultats mathématiques quant à l'équation de Landau-Lifschitz, que ce soit dans les cadres anisotropes, ou dans le cadre isotrope. Nous présentons les résultats d'une part sur le problème de Cauchy, et d'autre part, sur les propriétés des solitons pour les cas d'anisotropie.

### 1.2.1 Au sujet du problème de Cauchy

#### Le cas isotrope : l'équation des Schrödinger maps

L'équation des Schrödinger maps est définie par

$$\begin{cases} \partial_t \phi = \phi \times \Delta \phi \text{ sur } \mathbb{R}^d \times \mathbb{R}, \\ \phi(0) = \phi_0, \end{cases} \quad (1.2.1)$$

où  $d \geq 1$  et  $\phi : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ . Cette équation généralise celle de Schrödinger dans un cadre géométrique. Différents travaux ont cherché à établir l'existence et l'unicité des solutions. Sulem, Sulem et Bardos [61] ont d'abord construit des solutions faibles.

**Théorème 1.2.1** ([61]). *Soit  $\phi_0 : \mathbb{R}^N \rightarrow \mathbb{S}^2$  avec  $\nabla \phi_0 \in L^2(\mathbb{R}^N)$ . Il existe une solution faible  $\phi$  de (1.2.1) pour la donnée initiale  $\phi^0$  telle que  $\phi \in L^\infty(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^N))$ .*

Dans le même papier, ils ont montré l'existence locale des solutions fortes dans des espaces de Sobolev très réguliers.

**Théorème 1.2.2.** *Soit  $k > 1 + \frac{N}{2}$  et  $\phi_0 : \mathbb{R}^N \rightarrow \mathbb{S}^2$  avec  $\nabla \phi_0 \in H^k(\mathbb{R}^N)$ . Il existe un nombre  $T > 0$  et une unique solution  $\phi$  de (1.2.1) pour la donnée initiale  $\phi^0$  tels que  $\nabla \phi \in L^\infty([0, T], H^k(\mathbb{R}^N))$ .*

Chang, Shatah et Uhlenbeck [10] ont montré l'existence globale et l'unicité en dimensions un et deux pour des solutions moins régulières, soit telles que la donnée initiale vérifie seulement  $\nabla \phi_0 \in H^1(\mathbb{R}^N)$ .

**Théorème 1.2.3** ([10]). *Soit  $N = 1, 2$  et  $\phi^0 \in H^2(\mathbb{R}^N)$ . Il existe une unique solution globale  $\phi \in C(\mathbb{R}, H^2(\mathbb{R}^N)) \cap L^4(\mathbb{R}, W^{2, \infty}(\mathbb{R}^N))$  de (1.2.1) pour la donnée initiale  $\phi^0$ .*

Bejenaru, Ionescu, Kenig et Tataru [3] ont étudié le problème de Cauchy en toute dimension dans le cas critique pour des données initiales petites. Plus précisément, ils introduisent, pour tout  $\sigma \geq 0$ , l'espace métrique

$$H_Q^\sigma = \{f : \mathbb{R}^d \rightarrow \mathbb{R}^3 : |f(x)| \equiv 1 \text{ p.p. et } f - Q \in H^\sigma\}, \quad (1.2.2)$$

où  $Q \in \mathbb{S}^2$ , qu'ils munissent de la distance induite  $d_Q^\sigma(f, g) := \|f - g\|_{H^\sigma}$ . Ils considèrent également les espaces métriques

$$H^\infty = \bigcap_{\sigma \in \mathbb{N}} H^\sigma \quad \text{et} \quad H_Q^\infty = \bigcap_{\sigma \in \mathbb{N}} H_Q^\sigma,$$

munis des distances induites canoniques.

Le théorème suivant résume leurs résultats d'existence et d'unicité des solutions régulières.

**Théorème 1.2.4** ([3]). *Soit  $d \geq 2$  et  $Q \in \mathbb{S}^2$ . Il existe un réel  $\varepsilon_0(d) > 0$  tel que pour toute fonction  $\phi_0 \in H_Q^\infty$  avec  $\|\phi_0 - Q\|_{\dot{H}^{d/2}} \leq \varepsilon_0(d)$ , l'équation (1.2.1) admet une unique solution  $\phi = S_Q(\phi_0) \in C(\mathbb{R}, H_Q^\infty)$  pour la donnée initiale  $\phi_0$ . De plus, cette solution satisfait*

$$\sup_{t \in \mathbb{R}} \|\phi(t) - Q\|_{\dot{H}^{d/2}} \leq C \|\phi_0 - Q\|_{\dot{H}^{d/2}}, \quad (1.2.3)$$

et pour tout  $T \geq 0$  et  $\sigma \in \mathbb{N}$ ,

$$\sup_{t \in [-T, T]} \|\phi(t)\|_{H_Q^\sigma} \leq C(\sigma, T, \|\phi_0 - Q\|_{\dot{H}^{d/2}}), \quad (1.2.4)$$

où  $\|\phi\|_{H_Q^\sigma} := d_Q^\sigma(\phi, Q)$ .

Ils ont aussi établi que le problème est bien posé dans l'ensemble  $\mathcal{B}_\varepsilon^\sigma$ , lequel est défini pour  $\sigma \geq d/2$  et  $\varepsilon$  suffisamment petit par

$$\mathcal{B}_\varepsilon^\sigma = \{\phi \in \dot{H}_Q^{d/2-1} \cap \dot{H}^\sigma : \|\phi - Q\|_{\dot{H}^{d/2}} \leq \varepsilon\}.$$

Cet espace est muni de la distance induite par l'espace  $\dot{H}_Q^{d/2-1} \cap \dot{H}^\sigma$ , où  $\dot{H}_Q^\sigma$  est défini par

$$\dot{H}_Q^\sigma = \{f : \mathbb{R}^d \rightarrow \mathbb{R}^3 : f - Q \in \dot{H}^\sigma, \quad \text{et} \quad |f| \equiv 1 \text{ p.p. dans } \mathbb{R}^d\}.$$

Le théorème suivant présente leur résultat de propagation de la régularité des solutions.

**Théorème 1.2.5** ([3]). *Soit  $d \geq 2$ ,  $Q \in \mathbb{S}^2$  et  $\sigma_1 \geq d/2$ . Il existe un réel  $\varepsilon_0(d, \sigma_1) > 0$  tel que pour tout  $\phi_0 \in H_Q^\infty$  vérifiant  $\|\phi_0 - Q\|_{\dot{H}^{d/2}} \leq \varepsilon_0(d, \sigma_1)$ , la solution globale  $\phi = S_Q(\phi_0) \in C(\mathbb{R}, H_Q^\infty)$  construite dans le Théorème 1.2.4 satisfait*

$$\sup_{t \in \mathbb{R}} \|\phi(t) - Q\|_{H^\sigma} \leq C_\sigma \|\phi_0 - Q\|_{H^\sigma}, \quad d/2 \leq \sigma \leq \sigma_1. \quad (1.2.5)$$

De plus, pour tout  $\sigma \in [d/2, \sigma_1]$  l'opérateur  $S_Q$  admet une extension continue

$$S_Q : \mathcal{B}_{\varepsilon_0(d, \sigma_1)}^\sigma \rightarrow C(\mathbb{R}, \dot{H}^\sigma \cap \dot{H}_Q^{d/2-1}).$$

Ces résultats reposent sur des transformations de jauges. La jauge calorique est introduite puis son flot thermique est analysé. Ensuite, en écrivant les premières dérivées de la solution  $\phi$  dans un nouveau repère contenant  $\phi$  et deux fonctions repérant l'espace tangent à la sphère, on obtient  $N$  équations de type NLS non homogène. Dans ce cadre, les estimées de Strichartz suffisent pour obtenir le résultat.

## Le cas anisotrope : l'équation de Landau-Lifshitz

De Laire et Gravejat [19] ont analysé le problème de Cauchy en dimension un dans le cadre hydrodynamique. Leur théorème d'existence et unicité dans ce cadre s'énonce de la façon suivante.

**Théorème 1.2.6** ([19]). Soit  $\mathbf{v}^0 = (v^0, w^0) \in \mathcal{NV}(\mathbb{R})$ .

(i) Il existe un nombre positif  $T_{\max}$  et une unique solution  $\mathbf{v} = (v, w) \in \mathcal{C}([0, T_{\max}), \mathcal{NV}(\mathbb{R}))$ , de (HLL) pour la donnée initiale  $\mathbf{v}^0$  tels qu'il existe des solutions régulières  $\mathbf{v}_n \in \mathcal{C}^\infty(\mathbb{R} \times [0, T])$  de (HLL) qui satisfont

$$\mathbf{v}_n \rightarrow \mathbf{v} \quad \text{dans } \mathcal{C}^0([0, T], \mathcal{NV}(\mathbb{R})), \quad (1.2.6)$$

quand  $n \rightarrow +\infty$ , pour tout  $T < T_{\max}$ .

(ii) Le temps maximal  $T_{\max}$  est caractérisé par la condition

$$\lim_{t \rightarrow T_{\max}} \max_{x \in \mathbb{R}} |v(x, t)| = 1, \quad \text{si } T_{\max} < +\infty.$$

(iii) L'énergie  $E$  et le moment  $P$  sont conservés par le flot.

(iv) Dans le cas où

$$\mathbf{v}_n^0 \rightarrow \mathbf{v}^0 \quad \text{dans } H^1(\mathbb{R}) \times L^2(\mathbb{R}), \quad (1.2.7)$$

quand  $n \rightarrow +\infty$ , le temps maximal d'existence  $T_n$  des solutions  $\mathbf{v}_n$  de (HLL) pour la donnée initiale  $\mathbf{v}_n^0$  satisfait

$$T_{\max} \leq \liminf_{n \rightarrow +\infty} T_n, \quad (1.2.8)$$

et

$$\mathbf{v}_n \rightarrow \mathbf{v} \quad \text{dans } \mathcal{C}^0([0, T], H^1(\mathbb{R}) \times L^2(\mathbb{R})), \quad (1.2.9)$$

quand  $n \rightarrow +\infty$ , pour tout  $0 \leq T < T_{\max}$ .

Dans les variables originelles, ils introduisent la distance suivante

$$d_{\mathcal{E}}(f, g) := |\check{f}(0) - \check{g}(0)| + \|f' - g'\|_{L^2(\mathbb{R})} + \|f_3 - g_3\|_{L^2(\mathbb{R})},$$

où  $f = (f_1, f_2, f_3)$  et  $\check{f} = f_1 + if_2$  (respectivement pour  $g$ ). Le résultat d'existence et d'unicité est alors donné par le théorème suivant.

**Théorème 1.2.7** ([19]). Soit  $m^0 \in \mathcal{E}(\mathbb{R})$  avec  $\max_{\mathbb{R}} |m_3^0| < 1$ . On note  $\mathbf{v}^0$  le couple hydrodynamique correspondant, et  $T_{\max}$  le temps d'existence maximal pour la solution

$\mathbf{v} \in \mathcal{C}^0([0, T_{\max}), \mathcal{NV}(\mathbb{R}))$  de (HLL) donnée par le Théorème 1.2.6 pour la condition initiale  $\mathbf{v}^0$ . Il existe une fonction  $m \in \mathcal{E}(\mathbb{R})$  satisfaisant les propriétés suivantes.

(i) Le couple hydrodynamique associé à  $m$  est bien défini sur  $\mathbb{R} \times [0, T_{\max})$  et il est égal à  $\mathbf{v}$ .

(ii) La fonction  $m$  est l'unique solution de (LL), de donnée initiale  $m^0$ , telle qu'il existe des solutions régulières  $m_n \in \mathcal{C}^\infty(\mathbb{R} \times [0, T])$  satisfaisant

$$m_n \rightarrow m, \quad \text{dans } \mathcal{C}^0([0, T], \mathcal{E}(\mathbb{R})),$$

quand  $n \rightarrow \infty$  et pour tout  $T \in (0, T_{\max})$ .

(iii) L'énergie  $E$  est conservée par le flot.

(iv) Si

$$m_n^0 \rightarrow m^0 \quad \text{dans } \mathcal{E}(\mathbb{R}), \quad (1.2.10)$$

alors la solution  $m_n$  de (LL) pour la donnée initiale  $m_n^0$  satisfait

$$m_n \rightarrow m \quad \text{dans } \mathcal{C}^0([0, T], \mathcal{E}(\mathbb{R})), \quad (1.2.11)$$

quand  $n \rightarrow +\infty$ , pour tout  $0 \leq T < T_{\max}$ .

Notons que l'étude du problème de Cauchy en dimension plus grande reste ouverte.

Cependant dans le cas d'anisotropie axiale, Gustafson et Shatah [28] ont montré que l'équation admet une unique solution globale pour des données initiales équivariantes dans  $H^2(\mathbb{R}^2)$  proches d'une onde solitaire dans  $H^1(\mathbb{R}^2)$ . Pour  $m \in \mathbb{Z} \setminus \{0\}$  et  $\omega > 0$ , une onde solitaire  $v_{m,\omega}$  est ici une fonction qui vérifie les propriétés suivantes :

- (i)  $u(x, t) = \mathcal{R}(\omega t)v_{m,\omega}(x)$  est une solution de (1.2.1), où  $\mathcal{R}(\nu)$  est la rotation d'angle  $\nu$  autour de l'axe  $\mathbb{R}e_3$ ,
- (ii)  $v_{m,\omega}$  est  $m$ -équivariante i.e.  $v_{m,\omega}(\mathcal{R}x) = \mathcal{R}^m v_{m,\omega}(x)$  pour tout  $x \in \mathbb{R}^2$  et toute rotation  $\mathcal{R}$  autour de l'axe  $\mathbb{R}e_3$ ,
- (iii)  $v_{m,\omega}(0) = -e_3$ ,
- (iv)  $(v_{m,\omega})_3$  est radiale et croissante,
- (v)  $(v_{m,\omega})_3$  converge exponentiellement vite vers  $e_3$  quand  $|x| \rightarrow \infty$ .

Gustafson et Shatah [28] ont montré l'existence de cette solution particulière. Leur résultat d'existence et unicité s'exprime alors sous la forme suivante.

**Théorème 1.2.8.** [28] *Si  $u_0 \in H^1(\mathbb{R}^2)$  est  $m$ -équivariante et proche d'une onde solitaire  $v_{m,\omega}$  dans  $H^1(\mathbb{R}^2)$  alors (1.2.1) admet une unique solution  $u \in \mathcal{C}([0, \infty), H^2(\mathbb{R}^2))$ .*

Ils ont aussi établi la stabilité orbitale des ondes solitaires en question.

**Théorème 1.2.9.** *Pour tout  $\epsilon > 0$ , il existe  $\delta > 0$  tel que, si  $u_0$  est  $m$ -équivariante et  $\|u_0 - v_{\omega,m}\|_{H^1} < \delta$ , alors*

$$\inf_{\mathcal{R} \in SO(2)} \|u(\cdot, t) - \mathcal{R}v_{\omega,m}\|_{H^1} < \epsilon.$$

## 1.2.2 Les solitons

### En dimension un

À l'échelle atomique, les seules solutions  $m$  de types ondes progressives de vitesse supersonique sont les ondes planaires appelées ondes de spins. Pour des vitesses non nulles et inférieures à la vitesse du son, une onde progressive  $m$  progresse sur la sphère dans un cône qui reste loin du pôle nord. Ce type d'onde s'appelle soliton sombre. Lorsque la vitesse est nulle, l'onde progressive, appelée soliton noir, tourne dans un plan vertical [33].

De manière plus précise, un soliton de vitesse  $c$  est une solution de (LL) de type onde progressive, soit de la forme

$$m(x, t) := u(x - ct).$$

Le profil  $u$  est solution de l'équation différentielle ordinaire

$$u'' + |u'|^2 u + u_3^2 u - u_3 e_3 + cu \times u' = 0. \quad (\text{TWE})$$

Les solutions de cette équation sont explicites. Si  $|c| < 1$ , il existe des solutions non constantes  $u_c$ , qui sont données par la formule suivante.

**Proposition 1.2.1** ([17]). *Soit  $c \in \mathbb{R}$  et  $u \in \mathcal{E}(\mathbb{R})$  une solution de (TWE).*

(i) *Si  $|c| \geq 1$ , alors  $u$  est constante.*

(ii) *Si  $|c| < 1$  et  $u$  est non constante, alors, aux invariances géométriques près,  $u$  est donnée par les formules*

$$[u_c]_1(x) = \frac{c}{\cosh\left((1-c^2)^{\frac{1}{2}}x\right)}, \quad [u_c]_2(x) = \tanh\left((1-c^2)^{\frac{1}{2}}x\right), \quad [u_c]_3(x) = \frac{(1-c^2)^{\frac{1}{2}}}{\cosh\left((1-c^2)^{\frac{1}{2}}x\right)}.$$

De plus,

$$E(u_c) = 2\sqrt{1-c^2}.$$

Notons que les invariances géométriques de (TWE) sont les translations, les rotations autour de l'axe  $x_3$  et la symétrie orthogonale par rapport au plan  $x_3 = 0$ . Dans le cas  $|c| \geq 1$ , les seules solutions d'énergie finie sont les vecteurs constants dans  $\mathbb{S}^1 \times \{0\}$ . Cette propriété a été démontré par de Laire [17]

Dans le cas où  $c \neq 0$  le soliton  $u_c$  ne s'annule donc pas. Il s'écrit sous forme hydrodynamique comme le couple  $Q_c := (v_c, w_c)$ , où les fonctions  $v_c$  et  $w_c$  sont égales à

$$v_c(x) = \frac{(1 - c^2)^{\frac{1}{2}}}{\cosh\left((1 - c^2)^{\frac{1}{2}}x\right)}, \quad \text{et} \quad w_c(x) = \frac{c v_c(x)}{1 - v_c(x)^2} = \frac{c(1 - c^2)^{\frac{1}{2}} \cosh\left((1 - c^2)^{\frac{1}{2}}x\right)}{\sinh\left((1 - c^2)^{\frac{1}{2}}x\right)^2 + c^2}. \quad (1.2.12)$$

Les sommes de solitons sont de la forme

$$S_{\mathbf{c}, \mathbf{a}, \mathbf{s}} := (V_{\mathbf{c}, \mathbf{a}, \mathbf{s}}, W_{\mathbf{c}, \mathbf{a}, \mathbf{s}}) := \sum_{j=1}^N Q_{c_j, a_j, s_j}, \quad (1.2.13)$$

où  $N \in \mathbb{N}^*$ ,  $\mathbf{c} = (c_1, \dots, c_N)$ , avec  $c_j \neq 0$ ,  $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$  et  $\mathbf{s} = (s_1, \dots, s_N) \in \{\pm 1\}^N$ . Dans les variables originelles, ceci peut être exprimé de la façon suivante

$$R_{\mathbf{c}, \mathbf{a}, \mathbf{s}} := \left( (1 - V_{\mathbf{c}, \mathbf{a}, \mathbf{s}}^2)^{\frac{1}{2}} \cos(\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}), (1 - V_{\mathbf{c}, \mathbf{a}, \mathbf{s}}^2)^{\frac{1}{2}} \sin(\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}), V_{\mathbf{c}, \mathbf{a}, \mathbf{s}} \right),$$

avec

$$\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}(x) := \int_0^x W_{\mathbf{c}, \mathbf{a}, \mathbf{s}}(y) dy,$$

pour tout  $x \in \mathbb{R}$ . De Laire et Gravejat [19] ont montré la stabilité orbitale des solitons, et des sommes bien préparées de solitons. Ils commencent par établir la stabilité dans le cadre hydrodynamique.

**Théorème 1.2.10** ([19]). *Soit  $\mathbf{s}^* \in \{\pm 1\}^N$  et  $\mathbf{c}^* = (c_1^*, \dots, c_N^*) \in (-1, 1)^N$ , avec  $c_j^* \neq 0$ , tel que*

$$c_1^* < \dots < c_N^*. \quad (1.2.14)$$

*Il existe des nombres positifs  $\alpha^*$ ,  $L^*$  et  $A^*$ , qui ne dépendent que de  $\mathbf{c}^*$ , tels que, si  $\mathbf{v}^0 \in \mathcal{NV}(\mathbb{R})$  satisfait la condition*

$$\alpha^0 := \left\| \mathbf{v}^0 - S_{\mathbf{c}^*, \mathbf{a}^0, \mathbf{s}^*} \right\|_{H^1 \times L^2} \leq \alpha^*, \quad (1.2.15)$$

*pour des points  $\mathbf{a}^0 = (a_1^0, \dots, a_N^0) \in \mathbb{R}^N$  tels que*

$$L^0 := \min \left\{ a_{j+1}^0 - a_j^0, 1 \leq j \leq N-1 \right\} \geq L^*,$$

*alors il existe une fonction  $\mathbf{a} = (a_1, \dots, a_N) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^N)$  telle que*

$$\sum_{j=1}^N \left| a_j'(t) - c_j^* \right| \leq A^* \left( \alpha^0 + \exp \left( - \frac{\nu_{\mathbf{c}^*} L^0}{65} \right) \right), \quad (1.2.16)$$

*et la solution globale  $\mathbf{v}$  de (HLL) dans  $\mathcal{NV}(\mathbb{R})$  pour la donnée initiale  $\mathbf{v}_0$  vérifie*

$$\left\| \mathbf{v}(\cdot, t) - S_{\mathbf{c}^*, \mathbf{a}(t), \mathbf{s}^*} \right\|_{H^1 \times L^2} \leq A^* \left( \alpha^0 + \exp \left( - \frac{\nu_{\mathbf{c}^*} L^0}{65} \right) \right), \quad (1.2.17)$$

*pour tout  $t \in \mathbb{R}_+$ .*

La stabilité orbitale d'un soliton dans le cadre hydrodynamique s'en déduit.



**Théorème 1.2.11** ([19]). Soit  $s^* \in \{\pm 1\}$  et  $c^* \in (-1, 1) \setminus \{0\}$ . Il existe des nombres positifs  $\alpha^*$  et  $A^*$ , qui ne dépendent que de  $c^*$  tels que, si  $\mathbf{v}^0 \in \mathcal{NV}(\mathbb{R})$  satisfait la condition

$$\alpha^0 := \left\| \mathbf{v}^0 - \mathbf{v}_{c^*, a^0, s^*} \right\|_{H^1 \times L^2} \leq \alpha^*, \quad (1.2.18)$$

pour un point  $a^0 \in \mathbb{R}$ , alors il existe une fonction  $a \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$  telle que

$$|a'(t) - c^*| \leq A^* \alpha^0, \quad (1.2.19)$$

et la solution globale  $\mathbf{v}$  de (HLL) dans  $\mathcal{NV}(\mathbb{R})$  pour la donnée initiale  $\mathbf{v}_0$  vérifie

$$\left\| \mathbf{v}(\cdot, t) - \mathbf{v}_{c^*, a(t), s^*} \right\|_{H^1 \times L^2} \leq A^* \alpha^0, \quad (1.2.20)$$

pour tout  $t \in \mathbb{R}_+$ .

La stabilité orbitale dans les variables de départ découle du cas hydrodynamique. Pour les multi-solitons, la stabilité orbitale est donnée par le théorème suivant.

**Théorème 1.2.12** ([19]). Soit  $\mathbf{s}^* \in \{\pm 1\}^N$  et  $\mathbf{c}^* = (c_1^*, \dots, c_N^*) \in (-1, 1)^N$ , avec  $c_j^* \neq 0$  et satisfaisant (1.2.14). Pour tout nombre positif  $\epsilon^*$ , il existe des nombres strictement positifs  $\rho^*$  et  $L^*$  tels que, si  $m^0 \in \mathcal{E}(\mathbb{R})$  satisfait la condition

$$d_{\mathcal{E}}(m^0, R_{\mathbf{c}^*, a^0, \mathbf{s}^*}) \leq \rho^*, \quad (1.2.21)$$

pour des points  $\mathbf{a}^0 = (a_1^0, \dots, a_N^0) \in \mathbb{R}^N$  tels que

$$\min \{a_{j+1}^0 - a_j^0, 1 \leq j \leq N-1\} \geq L^*,$$

alors il existe une fonction  $\mathbf{a} = (a_1, \dots, a_N) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^N)$  telle que

$$\sum_{j=1}^N |a'_j(t) - c_j^*| \leq \epsilon^*, \quad (1.2.22)$$

et la solution globale  $m$  de (LL) dans  $\mathcal{E}(\mathbb{R})$  pour la donnée initiale  $m^0$  vérifie

$$\begin{aligned} \sum_{j=1}^N \inf_{\theta_j \in \mathbb{R}} \left\{ \left| \check{m}(a_j(t), t) - \check{u}_{c^*, a_j(t), \theta_j, s_j^*}(a_j(t)) \right| + \left\| \partial_x m - u'_{c^*, a_j(t), \theta_j, s_j^*} \right\|_{L^2(I_j)} \right. \\ \left. + \left\| m_3 - [u_{c^*, a_j(t), \theta_j, s_j^*}]_3 \right\|_{L^2(I_j)} \right\} \leq \epsilon^*, \end{aligned} \quad (1.2.23)$$

pour tout  $t \in \mathbb{R}_+$ .

Ici, nous avons noté :

$$I_1 := \left( -\infty, \frac{a_1 + a_2}{2} \right], \quad I_j := \left[ \frac{a_{j-1} + a_j}{2}, \frac{a_j + a_{j+1}}{2} \right], \quad I_N := \left[ \frac{a_{N-1} + a_N}{2}, +\infty \right)$$

Pour un unique soliton, ce résultat s'écrit.

**Théorème 1.2.13** ([19]). Soit  $s^* \in \{\pm 1\}$  et  $c^* \in (-1, 1) \setminus \{0\}$ . Pour tout nombre positif  $\epsilon^*$ , il existe un nombre  $\rho^* > 0$  tel que, si  $m^0 \in \mathcal{E}(\mathbb{R})$  satisfait la condition

$$d_{\mathcal{E}}(m^0, u_{c^*, a^0, \theta^0, s^*}) \leq \rho^*, \quad (1.2.24)$$

pour des réels  $(\theta^0, a^0) \in \mathbb{R}^2$ , alors il existe une fonction  $a \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$  telle que,

$$|a'(t) - c^*| \leq \epsilon^*, \quad (1.2.25)$$

et la solution globale  $m$  de (LL) dans  $\mathcal{E}(\mathbb{R})$  pour la donnée initiale  $m^0$  vérifie

$$\inf_{\theta \in \mathbb{R}} \left\{ \left| \check{m}(a(t), t) - \check{u}_{c^*, a(t), \theta, s^*}(a(t)) \right| + \left\| \partial_x m - u'_{c^*, a(t), \theta, s^*} \right\|_{L^2} + \left\| m_3 - [u_{c^*, a(t), \theta, s^*}]_3 \right\|_{L^2} \right\} \leq \epsilon^*, \quad (1.2.26)$$

pour tout  $t \in \mathbb{R}_+$ .

## En dimension supérieure

Contrairement au cas unidimensionnel, les ondes progressives ne sont plus explicites. En dimension deux, Lin et Wei ont néanmoins montré l'existence d'ondes progressives non constantes de vitesses petites, et qui présentent de plus deux vortex de degré  $\pm 1$ .

**Théorème 1.2.14** ([36]). *Pour  $c$  suffisamment petit, il existe une solution  $u \in C^1(\mathbb{R}^2, \mathbb{S}^2)$  de (TWE) de vitesse  $c$ . De plus, la fonction  $u$  admet exactement deux vortex en  $(\pm a_c, 0) \in \mathbb{R}^2$  de degré  $\pm 1$ , où  $a_c \approx \frac{1}{2c}$  lorsque  $c \rightarrow 0$ .*

En dimension supérieure à deux, il n'existe aucune solution régulière statique de (TWE), i.e. de vitesse  $c = 0$ .

**Proposition 1.2.2** ([17]). *Soit  $N \geq 2$ . Supposons que  $u \in \mathcal{E}(\mathbb{R}^N)$  est une solution de (TWE) avec  $c = 0$ . On suppose de plus que  $u$  est uniformément continue si  $N \geq 3$ . Alors  $u$  est une solution constante.*

Plusieurs études qualitatives décrivent aussi les propriétés de ces ondes progressives. De Laire a montré qu'il n'existe pas d'ondes progressives d'énergie petite en dimensions 2, 3 et 4. Ce résultat est résumé par le théorème suivant.

**Théorème 1.2.15** ([17]). *Soit  $N \in \{2, 3, 4\}$ . Il existe une constante universelle  $\mu > 0$  telle que si  $u \in \mathcal{E}(\mathbb{R}^N)$  est une solution non triviale de (TWE) avec  $c \in [-1, 1] \setminus \{0\}$ , uniformément continue pour  $N \in \{3, 4\}$ , alors*

$$E(u) \geq \mu. \quad (1.2.27)$$

La stabilité des solitons en grande dimension est un problème ouvert. Les techniques utilisées en dimension un ne suffisent pas dans ce cas. L'analyse du problème de Cauchy est plus délicate, et la formulation hydrodynamique plus difficile à manipuler par manque de contrôle uniforme des solutions.

## 1.3 Stabilité asymptotique des solitons et multi-solitons de (LL) en dimension un

Dans cette section nous présentons les résultats prouvés pendant la thèse. Nous avons établi la stabilité asymptotique des solitons et des multi-solitons de (LL).

### 1.3.1 Cas d'un seul soliton

Le théorème de stabilité asymptotique est donné par

**Théorème 1.3.1** ([1]). *Soit  $\mathbf{c} \in (-1, 1) \setminus \{0\}$ . Il existe un nombre positif  $\delta_{\mathbf{c}}$ , qui ne dépend que de  $\mathbf{c}$ , satisfaisant les propriétés suivantes. Pour tout  $m^0 \in \mathcal{E}(\mathbb{R})$  vérifiant*

$$d_{\mathcal{E}}(m^0, u_{\mathbf{c}}) \leq \delta_{\mathbf{c}},$$

*il existe un nombre  $\mathbf{c}^* \in (-1, 1) \setminus \{0\}$ , et deux fonctions  $b \in C^1(\mathbb{R}, \mathbb{R})$  et  $\theta \in C^1(\mathbb{R}, \mathbb{R})$  tels que*

$$b'(t) \rightarrow \mathbf{c}^*, \quad \text{et} \quad \theta'(t) \rightarrow 0,$$

*quand  $t \rightarrow +\infty$ , et pour l'unique solution globale  $m \in C^0(\mathbb{R}, \mathcal{E}(\mathbb{R}))$ , l'application*

$$m_{\theta} := \left( \cos(\theta)m_1 - \sin(\theta)m_2, \sin(\theta)m_1 + \cos(\theta)m_2, m_3 \right),$$

satisfait les convergences

$$\partial_x m_{\theta(t)}(\cdot + b(t), t) \rightharpoonup \partial_x u_{\mathbf{c}^*} \quad \text{dans } L^2(\mathbb{R}), \quad m_{\theta(t)}(\cdot + b(t), t) \rightarrow u_{\mathbf{c}^*} \quad \text{dans } L_{\text{loc}}^\infty(\mathbb{R}),$$

et

$$m_3(\cdot + b(t), t) \rightharpoonup [u_{\mathbf{c}^*}]_3 \quad \text{dans } L^2(\mathbb{R}),$$

quand  $t \rightarrow +\infty$ .

Pour la preuve, on suit la stratégie de Martel et Merle pour l'équation de Korteweg-de Vries [45]. Elle consiste à construire un profil limite à l'aide de la stabilité orbitale. Ensuite, ils montrent que ce profil décroît exponentiellement pour tout temps en utilisant une formule de monotonie sur la masse. Ils concluent par un théorème de type Liouville indiquant que dans un voisinage du soliton, la seule solution qui est localisée exponentiellement est le soliton. Ils s'appuient alors sur une formule de monotonie pour l'énergie afin d'établir une convergence forte dans  $H^1(x \geq ct/10)$  où  $c$  est la vitesse du soliton. On ne peut pas avoir une convergence forte sur toute la ligne droite. En effet, si une solution converge fortement vers un soliton alors elle a la même masse et la même énergie que le soliton ce qui implique qu'elle est exactement un soliton.

Cette stratégie a été adaptée par Béthuel, Gravejat et Smets pour l'équation de Gross-Pitaevskii [6]. Ils se sont placés dans le cadre hydrodynamique pour pouvoir l'appliquer. Ils utilisent le moment qui est une quantité conservée. Cette quantité joue un rôle important dans la preuve de la décroissance exponentielle du profil limite. Par contre, ils ne montrent qu'une convergence faible. Il n'est pas clair qu'il y ait une monotonie pour l'énergie, car l'équation disperse dans deux sens contrairement au cas de Korteweg-de Vries où il y a un seul sens de dispersion. La situation est la même pour l'équation de Landau-Lifshitz. C'est pour cela que l'on n'établit que des limites faibles. Comprendre ce problème de dispersion reste toujours ouvert.

Le cadre hydrodynamique de Landau-Lifshitz ressemble à celui de Gross-Pitaevskii. Dans la preuve on suit les étapes de Béthuel, Gravejat et Smets dans [6]. Par contre, le cas de Landau-Lifshitz est plus difficile. En effet, en utilisant une transformation de type Hasimoto [29], nous relierions notre équation avec l'équation de Schrödinger non linéaire. Cependant, on perd de la régularité. On se place dans le cadre de l'équation NLS au niveau  $L^2$  et non pas au niveau  $H^1$  comme pour l'équation de Gross-Pitaevskii. Ceci entraîne des difficultés techniques importantes.

Le Théorème 1.3.1 est une conséquence de la stabilité asymptotique dans le cadre hydrodynamique, qui est donnée par le théorème suivant.

**Théorème 1.3.2** ([1]). *Soit  $\mathbf{c} \in (-1, 1) \setminus \{0\}$ . Il existe une constante positive  $\beta_{\mathbf{c}} \leq \alpha_{\mathbf{c}}$ , qui ne dépend que de  $\mathbf{c}$ , satisfaisant les propriétés suivantes. Pour tout  $(v_0, w_0) \in X(\mathbb{R})$  vérifiant*

$$\|(v_0, w_0) - Q_{\mathbf{c}, \mathbf{a}}\|_{X(\mathbb{R})} \leq \beta_{\mathbf{c}},$$

*pour un certain  $\mathbf{a} \in \mathbb{R}$ , il existe un nombre  $\mathbf{c}^* \in (-1, 1) \setminus \{0\}$  et une application  $b \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  tels que l'unique solution globale  $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  de (HLL) correspondant à la donnée initiale  $(v_0, w_0)$  satisfait*

$$(v(\cdot + b(t), t), w(\cdot + b(t), t)) \rightharpoonup Q_{\mathbf{c}^*} \quad \text{dans } X(\mathbb{R}), \quad (1.3.1)$$

et

$$b'(t) \rightarrow \mathbf{c}^*,$$

quand  $t \rightarrow +\infty$ .

Le cas du soliton de vitesse nulle a été exclu du résultat. En fait, la méthode utilisée ne permet pas d'analyser les solitons noirs. Plus précisément, ces derniers s'annulent donc n'admettent pas un

expression hydrodynamique. La stabilité orbitale des solitons noirs reste un problème ouvert pour l'équation de Landau-Lifshitz. Par contre, ce problème a été résolu par Gravejat et Smets dans [25] pour l'équation de Gross-Pitaevskii.

Le théorème de stabilité orbitale suivant qui a été prouvé par de Laire et Gravejat dans [19], nous assure que le soliton reste proche de l'orbite en tenant compte des invariances de l'équation.

**Théorème 1.3.3.** *Soit  $c \in (-1, 1) \setminus \{0\}$ . Il existe un nombre positif  $\alpha_c$ , qui ne dépend que de  $c$ , satisfaisant les propriétés suivantes. Pour tout  $(v_0, w_0) \in X(\mathbb{R}) := H^1(\mathbb{R}) \times L^2(\mathbb{R})$  vérifiant*

$$\alpha_0 := \|(v_0, w_0) - Q_{c,a}\|_{X(\mathbb{R})} \leq \alpha_c, \quad (1.3.2)$$

pour  $a \in \mathbb{R}$ , il existe une unique solution globale  $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  de (HLL) de donnée initiale  $(v_0, w_0)$ , et deux applications  $c \in \mathcal{C}^1(\mathbb{R}, (-1, 1) \setminus \{0\})$  et  $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  telles que la fonction  $\varepsilon$  définie par

$$\varepsilon(\cdot, t) := (v(\cdot + a(t), t), w(\cdot + a(t), t)) - Q_{c(t)}, \quad (1.3.3)$$

satisfait les conditions d'orthogonalité

$$\langle \varepsilon(\cdot, t), \partial_x Q_{c(t)} \rangle_{L^2(\mathbb{R})^2} = \langle \varepsilon(\cdot, t), \chi_{c(t)} \rangle_{L^2(\mathbb{R})^2} = 0, \quad (1.3.4)$$

pour tout  $t \in \mathbb{R}$ . De plus, il existe deux nombres positifs  $\sigma_c$  et  $A_c$ , dépendant continûment de  $c$ , tels que

$$\max_{x \in \mathbb{R}} v(x, t) \leq 1 - \sigma_c, \quad (1.3.5)$$

$$\|\varepsilon(\cdot, t)\|_{X(\mathbb{R})} + |c(t) - c| \leq A_c \alpha^0, \quad (1.3.6)$$

et

$$|c'(t)| + |a'(t) - c(t)| \leq A_c \|\varepsilon(\cdot, t)\|_{X(\mathbb{R})}, \quad (1.3.7)$$

pour tout  $t \in \mathbb{R}$ .

Les conditions d'orthogonalité (1.3.4) ne sont pas les mêmes que celles dans [19]. On discutera de ce choix ultérieurement.

Par (1.3.6), on peut conclure que  $\varepsilon$  converge faiblement vers un certain  $\varepsilon_0^* \in X(\mathbb{R})$  et  $c(t)$  vers un nombre  $c_0^*$  pour une sous-suite de temps  $t_n \rightarrow \infty$ . Pour la donnée initiale  $(v_0^*, w_0^*) := \varepsilon_0^* + Q_{c_0^*}$ , on note  $(v^*, w^*)$  la solution globale correspondante de (HLL). Maintenant, on montre que  $(v_0^*, w_0^*) - Q_{c^*}$  satisfait (1.3.2) ce qui nous permet d'utiliser le théorème de stabilité orbitale pour établir que l'application

$$\varepsilon^*(\cdot, t) := (v^*(\cdot + a^*(t), t), w^*(\cdot + a^*(t), t)) - Q_{c^*(t)},$$

satisfait (1.3.4)–(1.3.7). Cela joue un rôle très important dans la preuve du théorème de type de Liouville et de la monotonie du moment. Ensuite, on montre la continuité faible du flot de (HLL) pour avoir la proposition suivante.

**Proposition 1.3.1.** *Fixons  $t \in \mathbb{R}$ . Alors,*

$$(v(\cdot + a(t_n), t_n + t), w(\cdot + a(t_n), t_n + t)) \rightharpoonup (v^*(\cdot, t), w^*(\cdot, t)) \quad \text{dans } X(\mathbb{R}), \quad (1.3.8)$$

et de plus

$$a(t_n + t) - a(t_n) \rightarrow a^*(t), \quad \text{et} \quad c(t_n + t) \rightarrow c^*(t), \quad (1.3.9)$$

quand  $n \rightarrow +\infty$ . En particulier, on a

$$\varepsilon(\cdot, t_n + t) \rightharpoonup \varepsilon^*(\cdot, t) \quad \text{dans } X(\mathbb{R}), \quad (1.3.10)$$

quand  $n \rightarrow +\infty$ .

La preuve de la continuité faible du flot hydrodynamique utilise la transformation de type Hasimoto introduite par de Laire et Gravejat dans [19], qui est donnée par

$$\Psi := \frac{1}{2} \left( \frac{\partial_x v}{(1-v^2)^{\frac{1}{2}}} + i(1-v^2)^{\frac{1}{2}} w \right) \exp i\theta, \quad (1.3.11)$$

où

$$\theta(x, t) := - \int_{-\infty}^x v(y, t) w(y, t) dy. \quad (1.3.12)$$

L'application  $\Psi$  est une solution de l'équation Schrödinger non-linéaire

$$i\partial_t \Psi + \partial_{xx} \Psi + 2|\Psi|^2 \Psi + \frac{1}{2} v^2 \Psi - \operatorname{Re} \left( \Psi (1 - 2F(v, \bar{\Psi})) \right) (1 - 2F(v, \Psi)) = 0, \quad (1.3.13)$$

avec

$$F(v, \Psi)(x, t) := \int_{-\infty}^x v(y, t) \Psi(y, t) dy, \quad (1.3.14)$$

tandis que la fonction  $v$  satisfait les deux équations

$$\begin{cases} \partial_t v = 2\partial_x \operatorname{Im} \left( \Psi (2F(v, \bar{\Psi}) - 1) \right), \\ \partial_x v = 2\operatorname{Re} \left( \Psi (1 - 2F(v, \bar{\Psi})) \right). \end{cases} \quad (1.3.15)$$

Le problème de Cauchy local pour (1.3.13)-(1.3.15) a été étudié par de Laire et Gravejat dans [19]. Ils ont montré la dépendance continue de la solution par rapport à la donnée initiale dans  $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ .

**Proposition 1.3.2** ([19]). *Soit  $(v^0, \Psi^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$  et  $(\tilde{v}^0, \tilde{\Psi}^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$  vérifiant*

$$\partial_x v^0 = 2\operatorname{Re} \left( \Psi^0 (1 - 2F(v^0, \bar{\Psi}^0)) \right), \quad \text{et} \quad \partial_x \tilde{v}^0 = 2\operatorname{Re} \left( \tilde{\Psi}^0 (1 - 2F(\tilde{v}^0, \bar{\tilde{\Psi}}^0)) \right).$$

*Soit  $(v, \Psi)$  et  $(\tilde{v}, \tilde{\Psi})$  deux solutions dans  $\mathcal{C}^0([0, T_*], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ , avec  $(\Psi, \tilde{\Psi}) \in L^4([0, T_*], L^\infty(\mathbb{R}))^2$ , de (1.3.13)-(1.3.15) de donnée initiale  $(v^0, \Psi^0)$ , respectivement  $(\tilde{v}^0, \tilde{\Psi}^0)$ , pour un temps positif  $T_*$ , il existe un nombre positif  $\tau$ , ne dépendant que de  $\|v^0\|_{L^2}$ ,  $\|\tilde{v}^0\|_{L^2}$ ,  $\|\Psi^0\|_{L^2}$  et  $\|\tilde{\Psi}^0\|_{L^2}$ , et une constante universelle  $A$  tels que l'on a*

$$\|v - \tilde{v}\|_{\mathcal{C}^0([0, T], L^2)} + \|\Psi - \tilde{\Psi}\|_{\mathcal{C}^0([0, T], L^2)} + \|\Psi - \tilde{\Psi}\|_{L^4([0, T], L^\infty)} \leq A \left( \|v^0 - \tilde{v}^0\|_{L^2} + \|\Psi^0 - \tilde{\Psi}^0\|_{L^2} \right), \quad (1.3.16)$$

*pour tout  $T \in [0, \min\{\tau, T_*\}]$ . De plus, il existe un nombre positif  $B$ , ne dépendant que de  $\|v^0\|_{L^2}$ ,  $\|\tilde{v}^0\|_{L^2}$ ,  $\|\Psi^0\|_{L^2}$  et  $\|\tilde{\Psi}^0\|_{L^2}$ , tel que*

$$\|\partial_x v - \partial_x \tilde{v}\|_{\mathcal{C}^0([0, T], L^2)} \leq B \left( \|v^0 - \tilde{v}^0\|_{L^2} + \|\Psi^0 - \tilde{\Psi}^0\|_{L^2} \right), \quad (1.3.17)$$

*pour tout  $T \in [0, \min\{\tau, T_*\}]$ .*

Cette proposition joue un rôle crucial dans la preuve de la continuité faible du flot hydrodynamique. En fait, elle nous permet d'affirmer que toute suite de solutions de (1.3.13)-(1.3.15) de donnée initiale faiblement convergente (donc bornée) est bornée. Ceci implique qu'elle converge faiblement dans  $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ . On montre après que cette limite faible est bien une solution de (1.3.13)-(1.3.15). Plus précisément, on utilise l'effet régularisant de Kato [37] pour voir que cette limite est une solution faible puis qu'elle est continue dans  $H^1(\mathbb{R}) \times L^2(\mathbb{R})$  par des estimations de Strichartz. Ceci finit la preuve de la Proposition 1.3.1.

L'étape suivante de la preuve du Théorème 1.3.2 est de montrer la décroissance exponentielle de la solution limite  $(v^*, w^*)$ . On commence par établir la formule de monotonie du moment. Soit  $R$  et  $t$  deux nombres réels, et notons

$$I_R(t) \equiv I_R^{(v,w)}(t) := \frac{1}{2} \int_{\mathbb{R}} [vw](x + a(t), t) \Phi(x - R) dx,$$

où  $\Phi$  est la fonction définie sur  $\mathbb{R}$  par

$$\Phi(x) := \frac{1}{2} \left( 1 + \operatorname{th}(\nu_{\mathfrak{c}} x) \right), \quad (1.3.18)$$

avec  $\nu_{\mathfrak{c}} := \sqrt{1 - \mathfrak{c}^2}/8$ . On a

**Proposition 1.3.3.** *Soit  $R \in \mathbb{R}$ ,  $t \in \mathbb{R}$ , et  $\sigma \in [-\sigma_{\mathfrak{c}}, \sigma_{\mathfrak{c}}]$ , avec  $\sigma_{\mathfrak{c}} := \sqrt{1 - \mathfrak{c}^2}/4$ . Pour tout  $(v, w) \in \mathcal{NV}(\mathbb{R})$  satisfaisant la conclusion du Théorème 1.3.3, il existe un nombre positif  $B_{\mathfrak{c}}$ , ne dépendant que de  $\mathfrak{c}$ , tel que*

$$\begin{aligned} \frac{d}{dt} [I_{R+\sigma t}(t)] &\geq \frac{1 - \mathfrak{c}^2}{8} \int_{\mathbb{R}} [(\partial_x v)^2 + v^2 + w^2](x + a(t), t) \Phi'(x - R - \sigma t) dx \\ &\quad - B_{\mathfrak{c}} e^{-2\nu_{\mathfrak{c}} |R + \sigma t|}. \end{aligned} \quad (1.3.19)$$

En particulier, on a

$$I_R(t_1) \geq I_R(t_0) - B_{\mathfrak{c}} e^{-2\nu_{\mathfrak{c}} |R|}, \quad (1.3.20)$$

quels que soient  $t_0 \leq t_1$ .

Ensuite, on montre que le moment du profil limite  $(v^*, w^*)$  est localisé. On note  $I_R^*(t) := I_R^{(v^*, w^*)}(t)$  pour tout  $R \in \mathbb{R}$  et  $t \in \mathbb{R}$ . On a

**Proposition 1.3.4** ([6]). *Soit  $\delta > 0$ . Il existe  $R_{\delta} > 0$ , ne dépendant que de  $\delta$ , tel que l'on a*

$$\begin{aligned} |I_R^*(t)| &\leq \delta, \quad \forall R \geq R_{\delta}, \\ |I_R^*(t) - P(v^*, w^*)| &\leq \delta, \quad \forall R \leq -R_{\delta}, \end{aligned}$$

pour tout  $t \in \mathbb{R}$ .

La preuve est la même que celle de la Proposition 3 dans [6]. Elle est basée sur un argument de contradiction. Plus précisément, on établit à l'aide de la formule de monotonie (1.3.19) et la Proposition 1.3.1 que si  $|I_R^*(t)| \geq \delta$  alors le moment  $I_R(t_n)$  tend vers  $+\infty$  quand  $n \rightarrow +\infty$ , ce qui n'est pas possible du fait que l'énergie de  $(v, w)$  est bornée.

Grâce aux Propositions 1.3.4 et 1.3.3, on obtient

**Proposition 1.3.5** ([6]). *Soit  $t \in \mathbb{R}$ . Il existe une constante positive  $\mathcal{A}_{\mathfrak{c}}$  telle que*

$$\int_t^{t+1} \int_{\mathbb{R}} [(\partial_x v^*)^2 + (v^*)^2 + (w^*)^2](x + a^*(s), s) e^{2\nu_{\mathfrak{c}} |x|} dx ds \leq \mathcal{A}_{\mathfrak{c}}.$$

Par suite, on établit que la dernière estimation reste vraie uniformément en temps et pour toutes les dérivées en espace de la solution limite  $(v^*, w^*)$ .

**Proposition 1.3.6.** *Le couple  $(v^*, w^*)$  est de classe  $\mathcal{C}^{\infty}$  et décroît exponentiellement en espace uniformément en temps. De plus, pour tout  $k \in \mathbb{N}$ , il existe une constante positive  $A_{k, \mathfrak{c}}$ , ne dépendant que de  $k$  et  $\mathfrak{c}$ , telle que*

$$\int_{\mathbb{R}} [(\partial_x^{k+1} v^*)^2 + (\partial_x^k v^*)^2 + (\partial_x^k w^*)^2](x + a^*(t), t) e^{\nu_{\mathfrak{c}} |x|} dx \leq A_{k, \mathfrak{c}}, \quad (1.3.21)$$

pour tout  $t \in \mathbb{R}$ .

Pour la preuve, on montre d'abord que la solution  $(v^*, \Psi^*)$  de (1.3.13)-(1.3.15) est de classe  $\mathcal{C}^\infty$  et décroît exponentiellement ainsi que toutes ses dérivées en utilisant une estimation de régularisation pour les solutions localisées de l'équation de Schrödinger linéaire.

**Proposition 1.3.7** ([6, 20]). *Soit  $\lambda \in \mathbb{R}$  et  $u \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}))$  une solution de l'équation de Schrödinger linéaire*

$$i\partial_t u + \partial_{xx} u = F, \quad (\text{LS})$$

avec  $F \in L^2(\mathbb{R}, L^2(\mathbb{R}))$ . Alors, il existe une constante positive  $K_\lambda$ , ne dépendant que de  $\lambda$ , telle que

$$\lambda^2 \int_{-T}^T \int_{\mathbb{R}} |\partial_x u(x, t)|^2 e^{\lambda x} dx dt \leq K_\lambda \int_{-T-1}^{T+1} \int_{\mathbb{R}} (|u(x, t)|^2 + |F(x, t)|^2) e^{\lambda x} dx dt, \quad (1.3.22)$$

pour tout  $T > 0$ .

La preuve consiste à montrer (1.3.22) pour des solutions régulières en dérivant deux fois par rapport à  $t$  la fonction  $t \mapsto \int_{\mathbb{R}} |u(t, x)|^2 \Phi(x) dx$ , où  $\Phi$  est à support compact. Puis, la conclusion se déduit d'un argument de densité.

Ensuite, on obtient (1.3.21) pour  $w^*$  en écrivant cette application en fonction de  $(\Psi^*, v^*)$ , la solution de (1.3.13)-(1.3.15) correspondant au profil limite  $(v^*, w^*)$  via la transformation (1.3.11).

La dernière étape est de montrer un théorème de classification de type Liouville. Il s'agit d'établir qu'au voisinage d'un soliton, toute solution de classe  $\mathcal{C}^\infty$  et qui décroît exponentiellement ainsi que toutes ses dérivées est un soliton pur. Ceci est donné par le théorème suivant.

**Théorème 1.3.4.** *Soit  $(v_0, w_0) \in X(\mathbb{R})$  vérifiant les hypothèses du Théorème 1.3.2. On note par  $(v, w)$  la solution globale de (HLL) correspondant à  $(v_0, w_0)$ . Si  $(v, w)$  vérifie (1.3.21), alors il existe  $c^* \in (-1, 1) \setminus \{0\}$  et  $x^* \in \mathbb{R}$  tels que*

$$(v, w)(t, x) = Q_{c^*}(x - x^* - c^*t) \quad \forall (x, t) \in \mathbb{R}^2.$$

La preuve repose sur les idées développées par Martel et Merle dans [45] (voir aussi [40]), qui ont été appliquées par Béthuel, Gravejat et Smets dans [6] pour l'équation de Gross-Pitaevskii.

On écrit l'équation vérifiée par  $\varepsilon^*$

$$\partial_t \varepsilon^* = J\mathcal{H}_{c^*(t)}(\varepsilon^*) + J\mathcal{R}_{c^*(t)}\varepsilon^* + (a^*(t) - c^*(t))(\partial_x Q_{c^*(t)} + \partial_x \varepsilon^*) - c^{*'}(t)\partial_c Q_{c^*(t)}, \quad (1.3.23)$$

où  $J$  est l'opérateur symplectique

$$J = -2S\partial_x := \begin{pmatrix} 0 & -2\partial_x \\ -2\partial_x & 0 \end{pmatrix}, \quad (1.3.24)$$

avec

$$S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

et le terme  $\mathcal{R}_{c^*(t)}\varepsilon^*$  est donné par

$$\mathcal{R}_{c^*(t)}\varepsilon^* := E'(Q_{c^*(t)} + \varepsilon^*) - E'(Q_{c^*(t)}) - E''(Q_{c^*(t)})(\varepsilon^*).$$

On introduit le couple

$$u^*(\cdot, t) := S\mathcal{H}_{c^*(t)}(\varepsilon^*(\cdot, t)). \quad (1.3.25)$$

Puisque  $S\mathcal{H}_{c^*(t)}(\partial_x Q_{c^*(t)}) = 0$ , on déduit par (1.3.23) que

$$\begin{aligned} \partial_t u^* &= S\mathcal{H}_{c^*(t)}(JSu^*) + S\mathcal{H}_{c^*(t)}(J\mathcal{R}_{c^*(t)}\varepsilon^*) - (c^*)'(t)S\mathcal{H}_{c^*(t)}(\partial_c Q_{c^*(t)}) \\ &\quad + (c^*)'(t)S\partial_c \mathcal{H}_{c^*(t)}(\varepsilon^*) + ((a^*)'(t) - c^*(t))S\mathcal{H}_{c^*(t)}(\partial_x \varepsilon^*). \end{aligned} \quad (1.3.26)$$

On a la formule de monotonie suivante.

**Proposition 1.3.8.** *Il existe deux nombres positifs  $A_*$  et  $R_*$ , ne dépendant que de  $\mathbf{c}$ , tels qu'on a<sup>1</sup>*

$$\frac{d}{dt} \left( \int_{\mathbb{R}} x u_1^*(x, t) u_2^*(x, t) dx \right) \geq \frac{1 - \mathbf{c}^2}{16} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 - A_* \|u^*(\cdot, t)\|_{X(B(0, R_*))}^2, \quad (1.3.27)$$

pour tout  $t \in \mathbb{R}$ .

Pour éliminer le terme négatif  $\|u^*(\cdot, t)\|_{X(B(0, R_*))}^2$  dans le terme droit de (1.3.27), on montre une deuxième formule de monotonie. L'idée consiste à voir que si on prend une matrice  $M$  carrée symétrique, bornée et de coefficients réguliers, alors on obtient

$$\frac{d}{dt} \langle M u^*, u^* \rangle_{L^2(\mathbb{R}^2)} = 2 \langle S M u^*, \mathcal{H}_{c^*}(-2\partial_x u^*) \rangle_{L^2(\mathbb{R}^2)} + \text{“ des termes super-quadratiques ”}. \quad (1.3.28)$$

Le choix de  $M$  vient du fait qu'on veut obtenir une forme quadratique positive pour le premier terme du côté droit dans (1.3.28) sous une hypothèse d'orthogonalité sur  $u^*$ . D'autre part, le noyau de l'opérateur  $\mathcal{H}_{c^*}(\partial_x \cdot)$  est engendré par  $Q_{c^*}$ . Si on choisit  $M$  de façon à ce qu'elle vérifie

$$M Q_{c^*} = S \partial_x Q_{c^*}, \quad (1.3.29)$$

le noyau de la forme quadratique  $\langle S M u^*, \mathcal{H}_{c^*}(-2\partial_x u^*) \rangle_{L^2(\mathbb{R}^2)}$  sera engendré par  $Q_{c^*}$ . Puisque  $Q_{c^*}$  ne s'annule pas, par le théorème de Sturm-Liouville, on conclut que l'opérateur associé n'admet pas de valeur propre négative. Par conséquent, la forme quadratique est définie positive sur  $\text{Vect}(Q_{c^*})^\perp$ . Ceci justifie le changement de variable  $u^*$ . Cet argument n'est pas en effet évident pour  $\varepsilon^*$  parce qu'on obtient une forme quadratique qui a pour noyau l'espace engendré par  $\partial_x Q_{c^*}$ . Cette fonction s'annule une fois donc l'opérateur admet une valeur propre négative mais ni celle-ci, ni la fonction propre associée, ne sont explicites. Par suite, on a besoin de faire face à deux conditions d'orthogonalités et un opérateur qui n'est pas facile à étudier.

Pour  $c \in (-1, 1) \setminus \{0\}$ , on note par  $M_c$  la matrice carrée qui vérifie (1.3.29) et qui est donnée par

$$M_c := \begin{pmatrix} -\frac{2cv_c \partial_x v_c}{(1-v_c)^2} & -\frac{\partial_x v_c}{v_c} \\ -\frac{\partial_x v_c}{v_c} & 0 \end{pmatrix}. \quad (1.3.30)$$

On a le lemme suivant.

**Lemme 1.3.1.** *Soit  $c \in (-1, 1) \setminus \{0\}$ ,  $u \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$  et*

$$G_c(u) := 2 \langle S M_c u, \mathcal{H}_c(-2\partial_x u) \rangle_{L^2(\mathbb{R})^2}.$$

Alors

$$G_c(u) = 2 \int_{\mathbb{R}} \mu_c \left( u_2 - \frac{cv_c^2}{\mu_c} u_1 - \frac{2cv_c \partial_x v_c}{\mu_c(1-v_c^2)} \partial_x u_1 \right)^2 + 3 \int_{\mathbb{R}} \frac{v_c^4}{\mu_c} \left( \partial_x u_1 - \frac{\partial_x v_c}{v_c} u_1 \right)^2, \quad (1.3.31)$$

où

$$\mu_c = 2(\partial_x v_c)^2 + v_c^2(1 - v_c^2) > 0. \quad (1.3.32)$$

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1. Dans (1.3.27), on utilise la notation

$$\|(f, g)\|_{X(\Omega)}^2 := \int_{\Omega} \left( (\partial_x f)^2 + f^2 + g^2 \right),$$

dans lequel  $\Omega$  est un sous-ensemble mesurable de  $\mathbb{R}$ .



La fonctionnelle  $G_c$  est une forme quadratique positive et son noyau est engendré par  $Q_c$ .

Par la deuxième condition d'orthogonalité dans (1.3.4) et le fait que  $\mathcal{H}_{c^*}(\chi_{c^*}) = -\tilde{\lambda}_{c^*}\chi_{c^*}$ , on a

$$0 = \langle \mathcal{H}_{c^*}(\chi_{c^*}), \varepsilon^* \rangle_{L^2(\mathbb{R})^2} = \langle \mathcal{H}_{c^*}(\varepsilon^*), \chi_{c^*} \rangle_{L^2(\mathbb{R})^2} = \langle u^*, S\chi_{c^*} \rangle_{L^2(\mathbb{R})^2}. \quad (1.3.33)$$

D'autre part, on sait que

$$\langle Q_{c^*}, S\chi_{c^*} \rangle = P'(Q_{c^*})(\chi_{c^*}) \neq 0, \quad (1.3.34)$$

de sorte que le couple  $u^*$  n'est pas proportionnel à  $Q_{c^*}$  sous la condition d'orthogonalité (1.3.33). On établit la propriété de coercivité de  $G_c$  sous cette condition d'orthogonalité.

**Proposition 1.3.9.** *Soit  $c \in (-1, 1) \setminus \{0\}$ . Il existe un nombre positif  $\Lambda_c$ , ne dépendant que de  $c$ , tel que*

$$G_c(u) \geq \Lambda_c \int_{\mathbb{R}} [(\partial_x u_1)^2 + (u_1)^2 + (u_2)^2](x) e^{-2|x|} dx, \quad (1.3.35)$$

pour tout  $u \in X(\mathbb{R})$  vérifiant

$$\langle u, S\chi_c \rangle_{L^2(\mathbb{R})^2} = 0. \quad (1.3.36)$$

La preuve est spectrale. Puisque les coefficients de  $G_c$  convergent vers 0 quand  $|x| \rightarrow +\infty$ , on effectue deux changements de variable afin d'obtenir une forme quadratique dont les coefficients ont des limites non nulles à l'infini. Son spectre essentiel se calcule par le critère de Weyl. Plus précisément, à la limite  $|x| \rightarrow +\infty$ , on obtient un opérateur à coefficients constants. Son spectre est égal au spectre essentiel de l'opérateur à coefficients variables. Nous obtenons 0 comme valeur propre simple et le spectre essentiel est inclus dans  $\mathbb{R}_+^*$ . Ceci nous permet de déduire que notre opérateur est strictement positif sous une condition d'orthogonalité. Ensuite, on revient à la formule initiale de  $G_c$  sous la condition d'orthogonalité (1.3.36). Cette condition d'orthogonalité est différente de celle choisie par Béthuel, Gravejat et Smets dans [6]. Dans la preuve on a besoin d'une condition d'orthogonalité avec  $v_c^{-1}\phi$  où  $\phi$  est une fonction telle que  $v_c^{-1}\phi \in L^2(\mathbb{R})$  pour donner un sens à la condition d'orthogonalité. On montre que la fonction propre associée à la valeur propre négative de  $\mathcal{H}_c$  vérifie bien cette propriété.

Revenant à (1.3.28), on a la proposition suivante.

**Proposition 1.3.10.** *Il existe un nombre positif  $B_*$ , ne dépendant que de  $c$ , tel que*

$$\begin{aligned} \frac{d}{dt} \left( \langle M_{c^*(t)} u^*(\cdot, t), u^*(\cdot, t) \rangle_{L^2(\mathbb{R})^2} \right) &\geq \frac{1}{B_*} \int_{\mathbb{R}} [(\partial_x u_1^*)^2 + (u_1^*)^2 + (u_2^*)^2](x, t) e^{-2|x|} dx \\ &\quad - B_* \left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2, \end{aligned} \quad (1.3.37)$$

pour tout  $t \in \mathbb{R}$ .

Pour la preuve, on écrit la forme explicite du terme droit de (1.3.37). Le premier terme s'exprime par la Proposition 1.3.9 puisque  $u^*$  vérifie (1.3.33), et on estime les autres termes de (1.3.37) par  $B_* \left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2$  en utilisant (1.3.6) et (1.3.7).

Maintenant, on note

$$N(t) := \frac{1}{2} \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} + A_* B_* e^{2R_*} M_{c^*(t)}.$$

Par les Propositions 1.3.8 and 1.3.10, il existe une constante positive  $\mathcal{A}_c$  telle qu'on a

$$\frac{d}{dt} \left( \langle N(t) u^*(\cdot, t), u^*(\cdot, t) \rangle_{L^2(\mathbb{R})^2} \right) \geq \mathcal{A}_c \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2, \quad (1.3.38)$$

pour tout  $t \in \mathbb{R}$ . Ainsi, en intégrant,

$$\int_{-\infty}^{+\infty} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 dt < +\infty, \quad (1.3.39)$$

car  $u^*$  est exponentiellement localisée par la Proposition 1.3.6. Il existe donc une suite de temps  $(t_k^*)_{k \in \mathbb{N}}$  telle que

$$\lim_{k \rightarrow +\infty} \|u^*(\cdot, t_k^*)\|_{X(\mathbb{R})}^2 = 0. \quad (1.3.40)$$

Par (1.3.4), (1.3.25) et la borne de l'inverse de l'opérateur  $\mathcal{H}_{c^*}$ , on a

$$\|\varepsilon^*(\cdot, t)\|_{X(\mathbb{R})} \leq A_{\mathbf{c}} \|u^*(\cdot, t)\|_{X(\mathbb{R})}. \quad (1.3.41)$$

Alors, on peut appliquer (1.3.40) et (1.3.41) pour obtenir

$$\lim_{k \rightarrow +\infty} \|\varepsilon^*(\cdot, t_k^*)\|_{X(\mathbb{R})}^2 = 0. \quad (1.3.42)$$

Par (1.3.42) et la stabilité orbitale dans le Théorème 1.3.3, on conclut que

$$\varepsilon_0^* \equiv 0,$$

ce qui termine la preuve du Théorème 1.3.4. À ce stade, nous avons montré le Théorème 1.3.2 pour une sous-suite de temps. Nous finissons la preuve en montrant que les convergences ne dépendent pas du choix de la suite de temps.

## 1.3.2 Cas des sommes de solitons

On se place dans une situation plus compliquée qu'un seul soliton. Le multi-soliton est une solution exacte de l'équation (LL) qui peut être vue comme une superposition non linéaire de plusieurs solitons découplés. Martel, Merle et Tsai [51] ont montré que si la donnée initiale est proche de la somme de  $N$  solitons alors la solution correspondante de (gKdV) converge fortement vers cette somme de soliton dans  $H^1(x \geq c_1 t/10)$  où  $c_1$  est la vitesse du premier soliton en utilisant deux formule de monotonie, une sur la masse et l'autre sur l'énergie. On ne peut pas établir une convergence forte sur toute la droite réelle car dans ce cas la solution est exactement un multi-soliton [39]. Dans notre cas, on ne peut pas aboutir à une convergence forte comme pour les équations (gKdV) par la méthode de Martel, Merle et Tsai [51] à cause de l'absence d'une formule de monotonie sur l'énergie. La formule de monotonie sur le moment ne permet que d'analyser ce qui se passe autour de chaque soliton. Pour cela, on montre pour (LL) que les solitons s'éloignent l'un de l'autre de plus en plus c'est-à-dire qu'on ne peut pas avoir une interaction entre eux. On établit ensuite la stabilité asymptotique autour de chaque soliton puis celle entre les solitons.

Soit  $N \in \mathbb{N}^*$ ,  $\mathbf{c} = (c_1, \dots, c_N)$ , avec  $c_j \neq 0$ ,  $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$  et  $\mathbf{s} = (s_1, \dots, s_N) \in \{\pm 1\}^N$ . On rappelle les notations pour les sommes de solitons dans le cadre hydrodynamique et dans le cadre de départ respectivement,

$$S_{\mathbf{c}, \mathbf{a}, \mathbf{s}} := (V_{\mathbf{c}, \mathbf{a}, \mathbf{s}}, W_{\mathbf{c}, \mathbf{a}, \mathbf{s}}) := \sum_{j=1}^N Q_{c_j, a_j, s_j},$$

et

$$R_{\mathbf{c}, \mathbf{a}, \mathbf{s}} := \left( (1 - V_{\mathbf{c}, \mathbf{a}, \mathbf{s}}^2)^{\frac{1}{2}} \cos(\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}), (1 - V_{\mathbf{c}, \mathbf{a}, \mathbf{s}}^2)^{\frac{1}{2}} \sin(\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}), V_{\mathbf{c}, \mathbf{a}, \mathbf{s}} \right),$$

avec

$$\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}(x) := \int_0^x W_{\mathbf{c}, \mathbf{a}, \mathbf{s}}(y) dy,$$

pour tout  $x \in \mathbb{R}$ .

Soit  $L > 0$ . On introduit l'ensemble des positions ordonnées et bien séparées

$$\text{Pos}(L) := \left\{ \mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N, \text{ t.q. } a_{j+1} > a_j + L \text{ pour } 1 \leq j \leq N-1 \right\},$$

et on pose

$$\mathcal{V}(\alpha, L) := \left\{ \mathbf{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ t.q. } \inf_{\mathbf{a} \in \text{Pos}(L)} \left\| \mathbf{v} - S_{\mathbf{c}^*, \mathbf{a}, \mathfrak{s}^*} \right\|_{H^1 \times L^2} < \alpha \right\},$$

pour  $\alpha > 0$ .

Le résultat de stabilité asymptotique est donné par le théorème suivant.

**Théorème 1.3.5.** [2] Soit  $\mathfrak{s} \in \{\pm 1\}^N$ ,  $\mathbf{c}^0 = (c_1^0, \dots, c_N^0) \in (-1, 1)^N$ , avec  $c_j^0 \neq 0$ , tel que

$$c_1^0 < \dots < c_N^0,$$

et  $\mathbf{a}^0 = (a_1^0, \dots, a_N^0) \in \mathbb{R}^N$ . Il existe un nombre positif  $\beta_{\mathbf{c}^0}$ , ne dépendant que de  $\mathbf{c}^0$ , et un nombre positif  $L^0$  tels que, si

$$d_{\mathcal{E}}(m^0, R_{\mathbf{c}^0, \mathbf{a}^0, \mathfrak{s}}) \leq \beta_{\mathbf{c}^0},$$

et

$$\mathbf{a}^0 \in \text{Pos}(L^0),$$

alors il existe  $\mathbf{c}^\infty := (c_1^\infty, \dots, c_N^\infty) \in (-1, 1)^N$ , avec  $c_j^\infty \neq 0$ , et  $2N$  fonctions  $a_j \in C^1(\mathbb{R}, \mathbb{R})$  et  $\theta_j \in C^1(\mathbb{R}, \mathbb{R})$ , telles que

$$a_j'(t) \rightarrow c_j^\infty, \quad \text{et} \quad \theta_j'(t) \rightarrow 0,$$

quand  $t \rightarrow +\infty$ , et pour lesquelles l'application

$$m_{\theta_j} := \left( \cos(\theta_j)m_1 - \sin(\theta_j)m_2, \sin(\theta_j)m_1 + \cos(\theta_j)m_2, m_3 \right),$$

correspondant à l'unique solution globale  $m \in C^0(\mathbb{R}, \mathcal{E}(\mathbb{R}))$  de donnée initiale  $m^0$ , satisfait les convergences

$$\begin{aligned} & \sum_{j=1}^N \left[ \partial_x m_{\theta_j(t)}(\cdot + a_j(t), t) - \partial_x u_{\tilde{c}_j} \right] \rightharpoonup 0 \quad \text{dans } L^2(\mathbb{R}), \\ & \sum_{j=1}^N \left[ m_{\theta_j(t)}(\cdot + a_j(t), t) - u_{\tilde{c}_j} \right] \rightarrow 0 \quad \text{dans } L_{\text{loc}}^\infty(\mathbb{R}), \\ \text{et} \quad & \sum_{j=1}^N \left[ m_3(\cdot + a_j(t), t) - [u_{\tilde{c}_j}]_3 \right] \rightharpoonup 0 \quad \text{dans } L^2(\mathbb{R}), \end{aligned} \tag{1.3.43}$$

quand  $t \rightarrow +\infty$ . De plus, pour toute fonction  $\mathbf{b} := (b_1, \dots, b_{N+1})$  avec  $b_j$  satisfaisant les conditions suivantes :

$$\left\{ \begin{array}{l} b_1(t) < a_j(t) \\ a_{j-1}(t) < b_j(t) < a_j(t) \quad \forall 2 \leq j \leq N \\ b_{N+1}(t) > a_N(t), \end{array} \right. \tag{1.3.44}$$

pour tout  $t \in \mathbb{R}_+$  et

$$\left\{ \begin{array}{l} \liminf_{t \rightarrow +\infty} \frac{b_j(t)}{t} > c_{j-1}^\infty, \\ \limsup_{t \rightarrow +\infty} \frac{b_j(t)}{t} < c_j^\infty, \end{array} \right. \tag{1.3.45}$$

avec

$$\begin{cases} c_0^\infty = -1, \\ c_{N+1}^\infty = 1, \end{cases}$$

nous avons les convergences :

$$\begin{aligned} \sum_{j=1}^N \partial_x m_{\theta_j(t)}(\cdot + b_j(t), t) &\rightharpoonup 0 \quad \text{dans } L^2(\mathbb{R}), \\ \sum_{j=1}^N \left[ m_{\theta_j(t)}(\cdot + b_j(t), t) - e_2 \right] &\rightarrow 0 \quad \text{dans } L_{\text{loc}}^\infty(\mathbb{R}), \\ \text{et } \sum_{j=1}^N m_3(\cdot + a_j(t), t) &\rightharpoonup 0 \quad \text{dans } L^2(\mathbb{R}), \end{aligned} \quad (1.3.46)$$

quand  $t \rightarrow +\infty$ , avec  $e_2 = (0, 1, 0)$ .

Comme chaque terme des sommes dans (1.3.43) et (1.3.46) converge faiblement vers zero, on procède de la même façon que pour un seul soliton. Le Théorème 1.3.5 est une conséquence directe du résultat suivant dans le cadre hydrodynamique.

**Théorème 1.3.6.** [2] Soit  $\mathbf{c}^0 = (c_1^0, \dots, c_N^0) \in (-1, 1)^N$ , avec  $c_j^0 \neq 0$  pour tout  $j = 1, \dots, N$ , tel qu'il existe  $L_0, \alpha_0 > 0$  satisfaisant les propriétés suivantes. Pour tout  $(v_0, w_0) \in X(\mathbb{R})$ , il existe  $L > L_0$  et  $\alpha < \alpha_0$  tels que si  $(v_0, w_0) \in \mathcal{V}(\alpha, L)$ , alors il existe  $\mathbf{a} := (a_1, \dots, a_N) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^N)$ ,  $\mathbf{c} := (c_1, \dots, c_N) \in \mathcal{C}^1(\mathbb{R}_+, (-1, 1) \setminus \{0\}^N)$  et des vitesses distinctes non nulles  $(c_1^{+\infty}, \dots, c_N^{+\infty})$  tels que l'unique solution globale  $(v, w) \in \mathcal{C}^0(\mathbb{R}_+, \mathcal{NV}(\mathbb{R}))$  de (HLL) de donnée initiale  $(v_0, w_0)$  satisfait, pour tout  $j \in \{1, \dots, N\}$ ,

$$\varepsilon(t, \cdot + a_j(t)) := (v, w)(t, x + a_j(t)) - \sum_{k=1}^N Q_{c_k(t)}(x + a_j(t) - a_k(t)) \rightharpoonup 0 \quad \text{dans } X(\mathbb{R}), \quad (1.3.47)$$

et

$$\varepsilon(t, \cdot + b_j(t)) := (v, w)(t, x + b_j(t)) - \sum_{k=1}^N Q_{c_k(t)}(x + b_j(t) - a_k(t)) \rightharpoonup 0 \quad \text{dans } X(\mathbb{R}), \quad (1.3.48)$$

pour tout  $\mathbf{b} := (b_1, \dots, b_{N+1}) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^{N+1})$  avec  $b_j$  satisfaisant (1.3.44) et

$$c_{j-1}^\infty < \lim_{t \rightarrow +\infty} b'_j(t) < c_j^\infty. \quad (1.3.49)$$

De plus, on a

$$c_j(t) \rightarrow c_j^{+\infty}, \quad a'_j(t) \rightarrow c_j^{+\infty}, \quad (1.3.50)$$

quand  $t \rightarrow +\infty$ .

**Remarque 1.3.1.** Dans la preuve de (1.3.48), on commence par montrer la convergence pour des fonctions  $b_j$  régulières et satisfaisant (1.3.44) et (1.3.49) puis, on en déduit le résultat pour des fonctions  $b_j$  quelconque vérifiant (1.3.45).

La preuve de (1.3.47) ressemble au cas d'un seul soliton et celle de (1.3.48) repose sur une stratégie similaire mais plus simple.

### Cas centré autour d'un soliton : Preuve de (1.3.47)

On remarque que tous les termes dans (1.3.47) convergent vers 0 pour  $k \neq j$ , puisque  $a_j(t) - a_k(t) \rightarrow \infty$  quand  $t \rightarrow \infty$  pour  $j \neq k$  et les solitons convergent vers zero à l'infini. Par suite, la preuve de (1.3.47) est la même que celle de (1.3.1). On commence par construire le profil limite. Puis on montre que cette solution translatée au centre du soliton  $j$  décroît exponentiellement pour tout temps, ainsi que toutes ses dérivées. Ceci nous permet de conclure en utilisant le théorème de Liouville, qui a été établi pour le cas d'un seul soliton, que ce profil limite est exactement un soliton.

### Cas entre les solitons : Preuve de (1.3.48)

Pour la preuve de (1.3.48), on construit le profil limite  $(v_j^*, w_j^*)$  et on montre que sa norme  $X(\mathbb{R})$  est petite et qu'il vérifie la proposition suivante.

**Proposition 1.3.11.** *Le couple  $(v_j^*, w_j^*)$  est infiniment dérivable et décroît exponentiellement sur  $\mathbb{R}$  pour tout temps. De plus, pour tout  $k \in \mathbb{N}$ , il existe une constante positive  $A_{k,c}$ , ne dépendant que de  $k$  et  $c$ , telle que*

$$\int_{\mathbb{R}} [(\partial_x^{k+1} v_j^*)^2 + (\partial_x^k v_j^*)^2 + (\partial_x^k w_j^*)^2](x + \tilde{b}_j(t), t) \exp\left(\frac{\nu_c}{16}|x|\right) dx \leq A_{k,c}, \quad (1.3.51)$$

pour tout  $t \in \mathbb{R}$ , où les fonctions  $\tilde{b}_j$  satisfont (1.3.44)–(1.3.49).

La preuve de cette proposition se déroule de la même façon que pour celle de la Proposition (1.3.6) en se fondant sur une formule de monotonie et la localisation du moment. La conclusion de (1.3.48) est donc obtenue par la proposition de classification suivante affirmant que toute solution petite et exponentiellement décroissante est en fait entièrement nulle.

**Proposition 1.3.12.** [2] *Il existe un nombre positif  $\alpha^*$  tel que, si  $(v, w)$  est une solution de (HLL) satisfaisant (1.3.51) et*

$$\|(v_0, w_0)\|_{X(\mathbb{R})} \leq \alpha^*,$$

alors,

$$(v, w)(t, x) = 0 \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}.$$

La preuve est plus simple que dans le cas d'un soliton. Elle consiste à établir une formule de monotonie en introduisant un opérateur dont il est aisé de voir qu'il est strictement positif. Ceci conduit à montrer que la norme de  $(v, w)$  dans  $X(\mathbb{R})$  est contrôlée par la dérivée en temps d'une fonction bornée. Nous concluons en intégrant en temps.

## 1.4 Conclusion

La stabilité des solitons n'est bien comprise qu'en dimension un. En dimension plus grande ce problème reste ouvert. En particulier, il faut d'abord résoudre le problème de Cauchy. Notons que Côte, Muñoz, Pilod et Simpson ont prouvé la stabilité asymptotique des solitons de l'équation de Zakharov-Kuznetsov en dimension grande [15].

Dans le cas de plusieurs solitons, nous n'avons traité que le cas des solitons découplés et de vitesses bien ordonnées. La collision n'est pas encore analysée. On ne sait pas ce qui se passe si deux solitons interagissent. Martel et Merle ont donné une description de la collision pour l'équation de Korteweg-de Vries [46, 48–50] que l'on peut espérer étendre ici.

# Chapitre 2

## Asymptotic stability in the energy space for dark solitons of the Landau-Lifshitz equation.

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### Abstract

We prove the asymptotic stability in the energy space of non-zero speed solitons for the one-dimensional Landau-Lifshitz equation with an easy-plane anisotropy

$$\partial_t m + m \times (\partial_{xx} m - m_3 e_3) = 0$$

for a map  $m = (m_1, m_2, m_3) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{S}^2$ , where  $e_3 = (0, 0, 1)$ . More precisely, we show that any solution corresponding to an initial datum close to a soliton with non-zero speed, is weakly convergent in the energy space as time goes to infinity, to a soliton with a possible different non-zero speed, up to the invariances of the equation. Our analysis relies on the ideas developed by Martel and Merle for the generalized Korteweg-de Vries equations. We use the Madelung transform to study the problem in the hydrodynamical framework. In this framework, we rely on the orbital stability of the solitons and the weak continuity of the flow in order to construct a limit profile. We next derive a monotonicity formula for the momentum, which gives the localization of the limit profile. Its smoothness and exponential decay then follow from a smoothing result for the localized solutions of the Schrödinger equations. Finally, we prove a Liouville type theorem, which shows that only the solitons enjoy these properties in their neighbourhoods.

## 2.1 Introduction

We consider the one-dimensional Landau-Lifshitz equation

$$\partial_t m + m \times (\partial_{xx} m + \lambda m_3 e_3) = 0, \quad (\text{LL})$$

for a map  $m = (m_1, m_2, m_3) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{S}^2$ , where  $e_3 = (0, 0, 1)$  and  $\lambda \in \mathbb{R}$ . This equation was introduced by Landau and Lifshitz in [34]. It describes the dynamics of magnetization in a one-dimensional ferromagnetic material, for example in CsNiF<sub>3</sub> or TMNC (see e.g. [30, 33] and the references therein). The parameter  $\lambda$  accounts for the anisotropy of the material. The choices  $\lambda > 0$  and  $\lambda < 0$  correspond respectively to an easy-axis and an easy-plane anisotropy. In the isotropic case  $\lambda = 0$ , the equation is exactly the one-dimensional Schrödinger map equation, which has been intensively studied (see e.g. [27, 31]). In this paper, we study the Landau-Lifshitz equation with an easy-plane anisotropy ( $\lambda < 0$ ). Performing, if necessary, a suitable scaling argument on the map  $m$ , we assume from now on that  $\lambda = -1$ . Our main goal is to prove the asymptotic stability for the solitons of this equation (see Theorem 2.1.1 below).

The Landau-Lifshitz equation is Hamiltonian. Its Hamiltonian, the so-called Landau-Lifshitz energy, is given by

$$E(m) := \frac{1}{2} \int_{\mathbb{R}} (|\partial_x m|^2 + m_3^2).$$

In the sequel, we restrict our attention to the Hamiltonian framework in which the solutions  $m$  to (LL) have finite Landau-Lifshitz energy, i.e. belong to the energy space

$$\mathcal{E}(\mathbb{R}) := \left\{ v : \mathbb{R} \rightarrow \mathbb{S}^2, \text{ s.t. } v' \in L^2(\mathbb{R}) \text{ and } v_3 \in L^2(\mathbb{R}) \right\}.$$

A soliton with speed  $c$  is a travelling-wave solution of (LL) having the form

$$m(x, t) := u(x - ct).$$

Its profile  $u$  is a solution to the ordinary differential equation

$$u'' + |u'|^2 u + u_3^2 u - u_3 e_3 + cu \times u' = 0. \quad (\text{TWE})$$

The solutions of this equation are explicit. When  $|c| \geq 1$ , the only solutions with finite Landau-Lifshitz energy are the constant vectors in  $\mathbb{S}^1 \times \{0\}$ . In contrast, when  $|c| < 1$ , there exist non-constant solutions  $u_c$  to (TWE), which are given by the formulae

$$[u_c]_1(x) = \frac{c}{\cosh\left((1-c^2)^{\frac{1}{2}}x\right)}, \quad [u_c]_2(x) = \tanh\left((1-c^2)^{\frac{1}{2}}x\right), \quad [u_c]_3(x) = \frac{(1-c^2)^{\frac{1}{2}}}{\cosh\left((1-c^2)^{\frac{1}{2}}x\right)},$$

up to the invariances of the problem, i.e. translations, rotations around the axis  $x_3$  and orthogonal symmetries with respect to the plane  $x_3 = 0$  (see [17] for more details).

Our goal is to study the asymptotic behaviour for solutions of (LL) which are initially close to a soliton in the energy space. We endow  $\mathcal{E}(\mathbb{R})$  with the metric structure corresponding to the distance introduced by de Laire and Gravejat in [19],

$$d_{\mathcal{E}}(f, g) := |\check{f}(0) - \check{g}(0)| + \|f' - g'\|_{L^2(\mathbb{R})} + \|f_3 - g_3\|_{L^2(\mathbb{R})},$$

where  $f = (f_1, f_2, f_3)$  and  $\check{f} = f_1 + if_2$  (respectively for  $g$ ). The Cauchy problem and the orbital stability of the travelling waves have been solved by de Laire and Gravejat in [19]. We are concerned the asymptotic stability of travelling waves. The following theorem is our main result.

**Théorème 2.1.1.** *Let  $\mathbf{c} \in (-1, 1) \setminus \{0\}$ . There exists a positive number  $\delta_{\mathbf{c}}$ , depending only on  $\mathbf{c}$ , such that, if*

$$d_{\mathcal{E}}(m^0, u_{\mathbf{c}}) \leq \delta_{\mathbf{c}},$$

*then there exist a number  $\mathbf{c}^* \in (-1, 1) \setminus \{0\}$ , and two functions  $b \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  and  $\theta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  such that*

$$b'(t) \rightarrow \mathbf{c}^*, \quad \text{and} \quad \theta'(t) \rightarrow 0,$$

*as  $t \rightarrow +\infty$ , and for which the map*

$$m_{\theta} := \left( \cos(\theta)m_1 - \sin(\theta)m_2, \sin(\theta)m_1 + \cos(\theta)m_2, m_3 \right),$$

*satisfies the convergences*

$$\partial_x m_{\theta(t)}(\cdot + b(t), t) \rightharpoonup \partial_x u_{\mathbf{c}^*} \quad \text{in } L^2(\mathbb{R}), \quad m_{\theta(t)}(\cdot + b(t), t) \rightarrow u_{\mathbf{c}^*} \quad \text{in } L_{\text{loc}}^{\infty}(\mathbb{R}),$$

*and*

$$m_3(\cdot + b(t), t) \rightharpoonup [u_{\mathbf{c}^*}]_3 \quad \text{in } L^2(\mathbb{R}),$$

*as  $t \rightarrow +\infty$ .*

**Remarques.** (i) *Note that the case  $\mathbf{c} = 0$ , that is black solitons, is excluded from the statement of Theorem 2.1.1. In this case, the map  $\check{u}_0$  vanishes and we cannot apply the Madelung transform and the subsequent arguments. Orbital and asymptotic stability remain open problems for this case. Note that, to our knowledge, there is currently no available proof of the local well-posedness of (LL) in the energy space, when  $u_0$  vanishes and so the hydrodynamical framework can no longer be used.* (ii) *Here, we state a weak convergence result and not a local strong convergence one, like the results given by Martel and Merle for the Korteweg-de Vries equation [45, 47]. In their situation, they can use two monotonicity formulae for the  $L^2$  norm and the energy. This heuristically originates in the property that dispersion has negative speed in the context of the Korteweg de Vries equation. In contrast, the possible group velocities for the dispersion of the Landau-Lifshitz equation are given by  $v_g(k) = \pm \frac{1+2k^2}{\sqrt{1+k^2}}$ , where  $k$  is the wave number. Dispersion has both negative and positive speeds. A monotonicity formula remains for the momentum due to the existence of a gap in the possible group velocities, which satisfy the condition  $|v_g(k)| \geq 1$ . However, there is no evidence that one can establish a monotonicity formula for the energy.*

Similar results were stated by Soffer and Weinstein in [58–60]. They provided the asymptotic stability of ground states for the nonlinear Schrödinger equation with a potential in a regime for which the nonlinear ground-state is a close continuation of the linear one. They rely on dispersive estimates for the linearized equation around the ground state in suitable weighted spaces, and they apply a fixed point argument. This strategy was successfully extended in particular by Buslaev, Perelman, C. Sulem and Cuccagna to the nonlinear Schrödinger equations without potential (see e.g. [7–9, 11]) and with a potential (see e.g. [22]). We refer to the detailed historical survey by Cuccagna [12] for more details. Later, Cuccagna proved in [13] stronger result for the ground state satisfying the sufficient conditions for orbital stability of M. Weinstein, for seemingly generic Non-Linear Schrödinger equation which has a smooth short range nonlinearity with the presence of a very short range and smooth linear potential. In addition, asymptotic stability in spaces of exponentially localized perturbations was studied by Pego and Weinstein in [54] (see also [53] for perturbations with algebraic decay).

Our strategy for establishing the asymptotic stability result in Theorem 2.1.1 is reminiscent from ideas developed by Martel and Merle for the Korteweg-de Vries equation [40, 45, 47], and successfully adapted by Béthuel, Gravejat and Smets in [5] for the Gross-Pitaevskii equation.



The main steps of the proof are similar to the ones for the Gross-Pitaevskii equation in [6]. Indeed, the solitons of the Landau-Lifshitz equation share many properties with the solitons of the Gross-Pitaevskii equation. In fact, the stereographic variable  $\psi$  defined by

$$\psi = \frac{u_1 + iu_2}{1 + u_3},$$

verifies the following equation

$$\partial_{xx}\psi + \frac{1 - |\psi|^2}{1 + |\psi|^2}\psi - ic\partial_x\psi = \frac{2\bar{\psi}}{1 + |\psi|^2}(\partial_x\psi)^2,$$

which can be seen as a perturbation of the equation for the travelling waves of the Gross-Pitaevskii equation, namely

$$\partial_{xx}\Psi + (1 - |\Psi|^2)\Psi - ic\partial_x\Psi = 0.$$

However, the analysis of the Landau-Lifshitz equation is much more difficult. Indeed, we rely on a Hasimoto like transform in order to relate the Landau-Lifshitz equation with a nonlinear Schrödinger equation. During so, we lose some regularity. We have to deal with a nonlinear equation at the  $L^2$ -level and not at the  $H^1$ -level as in the case of the Gross-Pitaevskii equation. This leads to important technical difficulties.

Coming back to the proof of Theorem 2.1.1, we first translate the problem into the hydrodynamical formulation. Then, we prove the asymptotic stability in that framework. In fact, we begin by refining the orbital stability. Next, we construct a limit profile, which is smooth and localized. For the proof of the exponential decay of the limit profile, we cannot rely on the Sobolev embedding  $H^1$  into  $L^\infty$  as it was done in [6]. We use instead the results of Kenig, Ponce and Vega in [32], and the Gagliardo-Nirenberg inequality (see the proof of Proposition 2.2.7 for more details). We also have to deal with the weak continuity of the flow in order to construct the limit profile. For the Gross Pitaevskii equation, this property relies on the uniqueness in a weaker space (see [6]). There is no similar result at the  $L^2$ -level. Instead, we use the Kato smoothing effect. The asymptotic stability in the hydrodynamical variables then follows from a Liouville type theorem. It shows that the only smooth and localized solutions in the neighbourhood of the solitons are the solitons. Finally, we deduce the asymptotic stability in the original setting from the result in the hydrodynamical framework.

In Section 2 below, we explain the main tools and different steps for the proof. First, we introduce the hydrodynamical framework. Then, we state the orbital stability of the solitons under a new orthogonality condition. Next, we sketch the proof of the asymptotic stability for the hydrodynamical system and we state the main propositions. We finally complete the proof of Theorem 2.1.1.

In Section 3 to 5, we give the proofs of the results stated in Section 2. In Section 3, We deal with the orbital stability in the hydrodynamical framework. In Section 4, we prove the localization and the smoothness of the limit profile. In the last section, we prove a Liouville type theorem. In a separate appendix, we show some facts used in the proofs, in particular, the weak continuity of the (HLL) flow.

## 2.2 Main steps for the proof of Theorem 2.1.1

### 2.2.1 The hydrodynamical framework

We introduce the map  $\check{m} := m_1 + im_2$ . Since  $m_3$  belongs to  $H^1(\mathbb{R})$ , it follows from the Sobolev embedding theorem that

$$|\check{m}(x)| = (1 - m_3^2(x))^{\frac{1}{2}} \rightarrow 1,$$

as  $x \rightarrow \pm\infty$ . As a consequence, the Landau-Lifshitz equation shares many properties with the Gross-Pitaevskii equation (see e.g. [4]). One of these properties is the existence of an hydrodynamical framework for the Landau-Lifshitz equation. In terms of the maps  $\check{m}$  and  $m_3$ , this equation may be written as

$$\begin{cases} i\partial_t\check{m} - m_3\partial_{xx}\check{m} + \check{m}\partial_{xx}m_3 - \check{m}m_3 = 0, \\ \partial_t m_3 + \partial_x \langle i\check{m}, \partial_x \check{m} \rangle_{\mathbb{C}} = 0. \end{cases}$$

When the map  $\check{m}$  does not vanish, one can write it as  $\check{m} = (1 - m_3^2)^{1/2} \exp i\varphi$ . The hydrodynamical variables  $v := m_3$  and  $w := \partial_x \varphi$  verify the following system

$$\begin{cases} \partial_t v = \partial_x((v^2 - 1)w), \\ \partial_t w = \partial_x \left( \frac{\partial_{xx} v}{1 - v^2} + v \frac{(\partial_x v)^2}{(1 - v^2)^2} + v(w^2 - 1) \right). \end{cases} \quad (\text{HLL})$$

This system is similar to the hydrodynamical Gross-Pitaevskii equation (see e.g. [6]).<sup>1</sup> We first study the asymptotic stability in the hydrodynamical framework.

In this framework, the Landau-Lifshitz energy is expressed as

$$E(\mathbf{v}) := \int_{\mathbb{R}} e(\mathbf{v}) := \frac{1}{2} \int_{\mathbb{R}} \left( \frac{(v')^2}{1 - v^2} + (1 - v^2)w^2 + v^2 \right), \quad (2.2.1)$$

where  $\mathbf{v} := (v, w)$  denotes the hydrodynamical pair. The momentum  $P$ , defined by

$$P(\mathbf{v}) := \int_{\mathbb{R}} vw, \quad (2.2.2)$$

is also conserved by the Landau-Lifshitz flow. The momentum  $P$  and the Landau-Lifshitz energy  $E$  play an important role in the study of the asymptotic stability of the solitons. When  $c \neq 0$ , the function  $\check{u}_c$  does not vanish. The hydrodynamical pair  $Q_c := (v_c, w_c)$  is given by

$$v_c(x) = \frac{(1 - c^2)^{\frac{1}{2}}}{\cosh\left((1 - c^2)^{\frac{1}{2}}x\right)}, \quad \text{and} \quad w_c(x) = \frac{c v_c(x)}{1 - v_c(x)^2} = \frac{c(1 - c^2)^{\frac{1}{2}} \cosh\left((1 - c^2)^{\frac{1}{2}}x\right)}{\sinh\left((1 - c^2)^{\frac{1}{2}}x\right)^2 + c^2}. \quad (2.2.3)$$

The only invariances of (HLL) are translations and the opposite map  $(v, w) \mapsto (-v, -w)$ . We restrict our attention to the translation invariances. All the analysis developed below applies when the opposite map is also taken into account. For  $a \in \mathbb{R}$ , we denote

$$Q_{c,a}(x) := Q_c(x - a) := (v_c(x - a), w_c(x - a)),$$

a non-constant soliton with speed  $c$ . We also set

$$\mathcal{NV}(\mathbb{R}) := \left\{ \mathbf{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \max_{\mathbb{R}} |v| < 1 \right\}.$$

This non-vanishing space is endowed in the sequel with the metric structure provided by the norm

$$\|\mathbf{v}\|_{H^1 \times L^2} := \left( \|v\|_{H^1}^2 + \|w\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

---

1. The hydrodynamical terminology originates in the fact that the hydrodynamical Gross-Pitaevskii equation is similar to the Euler equation for an irrotational fluid (see e.g. [5]).

## 2.2.2 Orbital stability

A perturbation of a soliton is provided by another soliton with a slightly different speed. This property follows from the existence of a continuum of solitons with different speeds. A solution corresponding to such a perturbation at initial time diverges from the soliton due to the different speeds of propagation, so that the standard notion of stability does not apply to solitons. The notion of orbital stability is tailored to deal with such situations. The orbital stability theorem below shows that a perturbation of a soliton at initial time remains a perturbation of the soliton, up to translations, for all time.

The following theorem is a variant of the result by de Laire and Gravejat [19] concerning sums of solitons. It is useful for the proof of the asymptotic stability.

**Théorème 2.2.1.** *Let  $c \in (-1, 1) \setminus \{0\}$ . There exists a positive number  $\alpha_c$ , depending only on  $c$ , with the following properties. Given any  $(v_0, w_0) \in X(\mathbb{R}) := H^1(\mathbb{R}) \times L^2(\mathbb{R})$  such that*

$$\alpha^0 := \left\| (v_0, w_0) - Q_{c,a} \right\|_{X(\mathbb{R})} \leq \alpha_c, \quad (2.2.4)$$

for some  $a \in \mathbb{R}$ , there exist a unique global solution  $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  to (HLL) with initial datum  $(v_0, w_0)$ , and two maps  $c \in \mathcal{C}^1(\mathbb{R}, (-1, 1) \setminus \{0\})$  and  $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  such that the function  $\varepsilon$  defined by

$$\varepsilon(\cdot, t) := \left( v(\cdot + a(t), t), w(\cdot + a(t), t) \right) - Q_{c(t)}, \quad (2.2.5)$$

satisfies the orthogonality conditions

$$\langle \varepsilon(\cdot, t), \partial_x Q_{c(t)} \rangle_{L^2(\mathbb{R})^2} = \langle \varepsilon(\cdot, t), \chi_{c(t)} \rangle_{L^2(\mathbb{R})^2} = 0, \quad (2.2.6)$$

for any  $t \in \mathbb{R}$ . Moreover, there exist two positive numbers  $\sigma_c$  and  $A_c$ , depending only and continuously on  $c$ , such that

$$\max_{x \in \mathbb{R}} v(x, t) \leq 1 - \sigma_c, \quad (2.2.7)$$

$$\left\| \varepsilon(\cdot, t) \right\|_{X(\mathbb{R})} + |c(t) - c| \leq A_c \alpha^0, \quad (2.2.8)$$

and

$$|c'(t)| + |a'(t) - c(t)| \leq A_c \left\| \varepsilon(\cdot, t) \right\|_{X(\mathbb{R})}, \quad (2.2.9)$$

for any  $t \in \mathbb{R}$ .

**Remarque 2.2.1.** *In this statement, the function  $\chi_c$  is a normalized eigenfunction associated to the unique negative eigenvalue of the linear operator*

$$\mathcal{H}_c := E''(Q_c) + cP''(Q_c).$$

The operator  $\mathcal{H}_c$  is self-adjoint on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ , with domain  $\text{Dom}(\mathcal{H}_c) := H^2(\mathbb{R}) \times L^2(\mathbb{R})$  (see (2.6.42) for its explicit formula). It has a unique negative simple eigenvalue  $-\tilde{\lambda}_c$ , and its kernel is given by

$$\text{Ker}(\mathcal{H}_c) = \text{Span}(\partial_x Q_c). \quad (2.2.10)$$

Our statement of orbital stability relies on a different decomposition from that proposed by Grillakis, Shatah and Strauss in [26]. This modification is related to the proof of asymptotic stability. A key ingredient in the proof is the coercivity of the quadratic form  $G_c$ , which is defined in (2.2.46), under a suitable orthogonality condition. In case we use the orthogonality conditions in [26], the corresponding orthogonality condition for  $G_c$  is provided by the function  $v_c^{-1} S \partial_c Q_c$  (see (2.2.40) for the definition of  $S$ ), which does not belong to  $L^2(\mathbb{R})$ . In order to by-pass this difficulty,

we use the second orthogonality condition in (2.2.6) for which the corresponding orthogonality condition for  $G_c$  is given by the function  $v_c^{-1}S\chi_c$ , which does belong to  $L^2(\mathbb{R})$  (see the appendix for more details). This alternative decomposition is inspired from the one used by Martel and Merle in [45].

Concerning the proof of Theorem 2.2.1, we first establish an orbital stability theorem with the classical decomposition of Grillakis, Shatah and Strauss [26]. This appears as a particular case of the orbital stability theorem in [19] for sum of solitons. We next show that, if we have orbital stability for some decomposition and orthogonality conditions, then we also have it for different decomposition and orthogonality conditions (see Section 2 for the detailed proof of Theorem 2.2.1).

### 2.2.3 Asymptotic stability for the hydrodynamical variables

The following theorem shows the asymptotic stability result in the hydrodynamical framework.

**Théorème 2.2.2.** *Let  $\mathbf{c} \in (-1, 1) \setminus \{0\}$ . There exists a positive constant  $\beta_{\mathbf{c}} \leq \alpha_{\mathbf{c}}$ , depending only on  $\mathbf{c}$ , with the following properties. Given any  $(v_0, w_0) \in X(\mathbb{R})$  such that*

$$\|(v_0, w_0) - Q_{\mathbf{c}, \mathbf{a}}\|_{X(\mathbb{R})} \leq \beta_{\mathbf{c}},$$

for some  $\mathbf{a} \in \mathbb{R}$ , there exist a number  $\mathbf{c}^* \in (-1, 1) \setminus \{0\}$  and a map  $b \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  such that the unique global solution  $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  to (HLL) with initial datum  $(v_0, w_0)$  satisfies

$$(v(\cdot + b(t), t), w(\cdot + b(t), t)) \rightarrow Q_{\mathbf{c}^*} \quad \text{in } X(\mathbb{R}), \quad (2.2.11)$$

and

$$b'(t) \rightarrow \mathbf{c}^*,$$

as  $t \rightarrow +\infty$ .

Theorem 2.2.2 establishes a convergence to some orbit of the soliton. This result is stronger than the one given by Theorem 2.2.1 which only shows that the solution stays close to that orbit.

In the next subsections, we explain the main ideas of the proof, which follows the strategy developed by Martel and Merle for the Korteweg-de Vries equation [45, 47].

#### Construction of a limit profile

Let  $\mathbf{c} \in (-1, 1) \setminus \{0\}$ , and  $(v_0, w_0) \in X(\mathbb{R})$  be any pair satisfying the assumptions of Theorem 2.2.2. Since  $\beta_{\mathbf{c}} \leq \alpha_{\mathbf{c}}$  in the assumptions of Theorem 2.2.2, we deduce from Theorem 2.2.1 that the unique solution  $(v, w)$  to (HLL) with initial datum  $(v_0, w_0)$  is global.

We take an arbitrary sequence of times  $(t_n)_{n \in \mathbb{N}}$  tending to  $+\infty$ . In view of (2.2.8) and (2.2.9), we may assume, up to a subsequence, that there exist a limit perturbation  $\varepsilon_0^* \in X(\mathbb{R})$  and a limit speed  $c_0^* \in [-1, 1]$  such that

$$\varepsilon(\cdot, t_n) = (v(\cdot + a(t_n), t_n), w(\cdot + a(t_n), t_n)) - Q_{c(t_n)} \rightarrow \varepsilon_0^* \quad \text{in } X(\mathbb{R}), \quad (2.2.12)$$

and

$$c(t_n) \rightarrow c_0^*, \quad (2.2.13)$$

as  $n \rightarrow +\infty$ . Our main goal is to show that

$$\varepsilon_0^* \equiv 0,$$

(see Corollary 2.2.2). For that, we establish smoothness and rigidity properties for the solution of (HLL) with the initial datum  $Q_{c_0^*} + \varepsilon_0^*$ .

First, we impose the constant  $\beta_c$  to be sufficiently small so that, when the number  $\alpha^0$  which appears in Theorem 2.2.1 satisfies  $\alpha^0 \leq \beta_c$ , then we infer from (2.2.8) and (2.2.9) that

$$\min \{c(t)^2, a'(t)^2\} \geq \frac{\mathfrak{c}^2}{2}, \quad \max \{c(t)^2, a'(t)^2\} \leq \frac{1 + \mathfrak{c}^2}{2}, \quad (2.2.14)$$

and

$$\|v_c(\cdot) - v(\cdot + a(t), t)\|_{L^\infty(\mathbb{R})} \leq \min \left\{ \frac{\mathfrak{c}^2}{4}, \frac{1 - \mathfrak{c}^2}{16} \right\}, \quad (2.2.15)$$

for any  $t \in \mathbb{R}$ . This yields, in particular, that  $c_0^* \in (-1, 1) \setminus \{0\}$ , and then, that  $Q_{c_0^*}$  is well-defined and different from the black soliton.

By (2.2.8), we also have

$$|c_0^* - \mathfrak{c}| \leq A_c \beta_c, \quad (2.2.16)$$

and, applying again (2.2.8), as well as (2.2.12), and the weak lower semi-continuity of the norm, we also know that the function

$$(v_0^*, w_0^*) := Q_{c_0^*} + \varepsilon_0^*,$$

satisfies

$$\|(v_0^*, w_0^*) - Q_c\|_{X(\mathbb{R})} \leq A_c \beta_c + \|Q_c - Q_{c_0^*}\|_{X(\mathbb{R})}. \quad (2.2.17)$$

We next impose a supplementary smallness assumption on  $\beta_c$  so that

$$\|(v_0^*, w_0^*) - Q_c\|_{X(\mathbb{R})} \leq \alpha_c. \quad (2.2.18)$$

By Theorem 2.2.1, there exists a unique global solution  $(v^*, w^*) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  to (HLL) with initial datum  $(v_0^*, w_0^*)$ , and two maps  $c^* \in \mathcal{C}^1(\mathbb{R}, (-1, 1) \setminus \{0\})$  and  $a^* \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  such that the function  $\varepsilon^*$  defined by

$$\varepsilon^*(\cdot, t) := (v^*(\cdot + a^*(t), t), w^*(\cdot + a^*(t), t)) - Q_{c^*(t)}, \quad (2.2.19)$$

satisfies the orthogonality conditions

$$\langle \varepsilon^*(\cdot, t), \partial_x Q_{c^*(t)} \rangle_{L^2(\mathbb{R})^2} = \langle \varepsilon^*(\cdot, t), \chi_{c^*(t)} \rangle_{L^2(\mathbb{R})^2} = 0, \quad (2.2.20)$$

as well as the estimates

$$\|\varepsilon^*(\cdot, t)\|_{X(\mathbb{R})} + |c^*(t) - \mathfrak{c}| + |a^{*'}(t) - c^*(t)| \leq A_c \|(v_0^*, w_0^*) - Q_c\|_{X(\mathbb{R})}, \quad (2.2.21)$$

for any  $t \in \mathbb{R}$ .

We may take  $\beta_c$  small enough such that, combining (2.2.16) with (2.2.17) and (2.2.21), we obtain

$$\min \{c^*(t)^2, (a^*)'(t)^2\} \geq \frac{\mathfrak{c}^2}{2}, \quad \max \{c^*(t)^2, (a^*)'(t)^2\} \leq \frac{1 + \mathfrak{c}^2}{2}, \quad (2.2.22)$$

and

$$\|v_c(\cdot) - v^*(\cdot + a^*(t), t)\|_{L^\infty(\mathbb{R})} \leq \min \left\{ \frac{\mathfrak{c}^2}{4}, \frac{1 - \mathfrak{c}^2}{16} \right\}, \quad (2.2.23)$$

for any  $t \in \mathbb{R}$ .

Finally, we use the weak continuity of the flow map for the Landau-Lifshitz equation. The proof relies on Proposition 2.6.1 and follows the lines the proof of Proposition 1 in [6].

**Proposition 2.2.1.** *Let  $t \in \mathbb{R}$  be fixed. Then,*

$$\left(v(\cdot + a(t_n), t_n + t), w(\cdot + a(t_n), t_n + t)\right) \rightharpoonup \left(v^*(\cdot, t), w^*(\cdot, t)\right) \quad \text{in } X(\mathbb{R}), \quad (2.2.24)$$

while

$$a(t_n + t) - a(t_n) \rightarrow a^*(t), \quad \text{and} \quad c(t_n + t) \rightarrow c^*(t), \quad (2.2.25)$$

as  $n \rightarrow +\infty$ . In particular, we have

$$\varepsilon(\cdot, t_n + t) \rightharpoonup \varepsilon^*(\cdot, t) \quad \text{in } X(\mathbb{R}), \quad (2.2.26)$$

as  $n \rightarrow +\infty$ .

### Localization and smoothness of the limit profile

Our proof of the localization of the limit profile is based on a monotonicity formula.

Consider a pair  $(v, w)$  which satisfies the conclusions of Theorem 2.2.1 and suppose that (2.2.14) and (2.2.15) are true. Let  $R$  and  $t$  be two real numbers, and set

$$I_R(t) \equiv I_R^{(v,w)}(t) := \frac{1}{2} \int_{\mathbb{R}} [vw](x + a(t), t) \Phi(x - R) dx,$$

where  $\Phi$  is the function defined on  $\mathbb{R}$  by

$$\Phi(x) := \frac{1}{2} \left(1 + \text{th}(\nu_{\mathfrak{c}} x)\right), \quad (2.2.27)$$

with  $\nu_{\mathfrak{c}} := \sqrt{1 - \mathfrak{c}^2}/8$ . We have

**Proposition 2.2.2.** *Let  $R \in \mathbb{R}$  and  $\sigma \in [-\sigma_{\mathfrak{c}}, \sigma_{\mathfrak{c}}]$ , with  $\sigma_{\mathfrak{c}} := \sqrt{1 - \mathfrak{c}^2}/4$ . Under the above assumptions, there exists a positive number  $B_{\mathfrak{c}}$ , depending only on  $\mathfrak{c}$ , such that*

$$\begin{aligned} \frac{d}{dt} [I_{R+\sigma t}(t)] &\geq \frac{1 - \mathfrak{c}^2}{8} \int_{\mathbb{R}} [(\partial_x v)^2 + v^2 + w^2](x + a(t), t) \Phi'(x - R - \sigma t) dx \\ &\quad - B_{\mathfrak{c}} e^{-2\nu_{\mathfrak{c}} |R + \sigma t|}, \end{aligned} \quad (2.2.28)$$

for any  $t \in \mathbb{R}$ . In particular, we have

$$I_R(t_1) \geq I_R(t_0) - B_{\mathfrak{c}} e^{-2\nu_{\mathfrak{c}} |R|}, \quad (2.2.29)$$

for any real numbers  $t_0 \leq t_1$ .

For the limit profile  $(v^*, w^*)$ , we set  $I_R^*(t) := I_R^{(v^*, w^*)}(t)$  for any  $R \in \mathbb{R}$  and any  $t \in \mathbb{R}$ . We claim

**Proposition 2.2.3** ([6]). *Given any positive number  $\delta$ , there exists a positive number  $R_{\delta}$ , depending only on  $\delta$ , such that we have*

$$\begin{aligned} |I_R^*(t)| &\leq \delta, \quad \forall R \geq R_{\delta}, \\ |I_R^*(t) - P(v^*, w^*)| &\leq \delta, \quad \forall R \leq -R_{\delta}, \end{aligned}$$

for any  $t \in \mathbb{R}$ .

The proof of Proposition 2.2.3 is the same as the one of Proposition 3 in [6].

From Propositions 2.2.2 and 2.2.3, we derive as in [6] that

**Proposition 2.2.4** ([6]). *There exists a positive constant  $\mathcal{A}_\epsilon$  such that*

$$\int_t^{t+1} \int_{\mathbb{R}} [(\partial_x v^*)^2 + (v^*)^2 + (w^*)^2](x + a^*(s), s) e^{2\nu_\epsilon |x|} dx ds \leq \mathcal{A}_\epsilon,$$

for any  $t \in \mathbb{R}$ .

We next consider the following map which was introduced by de Laire and Gravejat in [19],

$$\Psi := \frac{1}{2} \left( \frac{\partial_x v}{(1-v^2)^{\frac{1}{2}}} + i(1-v^2)^{\frac{1}{2}} w \right) \exp i\theta, \quad (2.2.30)$$

where

$$\theta(x, t) := - \int_{-\infty}^x v(y, t) w(y, t) dy. \quad (2.2.31)$$

The map  $\Psi$  solves the nonlinear Schrödinger equation

$$i\partial_t \Psi + \partial_{xx} \Psi + 2|\Psi|^2 \Psi + \frac{1}{2} v^2 \Psi - \operatorname{Re} \left( \Psi (1 - 2F(v, \bar{\Psi})) \right) (1 - 2F(v, \Psi)) = 0, \quad (2.2.32)$$

with

$$F(v, \Psi)(x, t) := \int_{-\infty}^x v(y, t) \Psi(y, t) dy, \quad (2.2.33)$$

while the function  $v$  satisfies the two equations

$$\begin{cases} \partial_t v = 2\partial_x \operatorname{Im} \left( \Psi (2F(v, \bar{\Psi}) - 1) \right), \\ \partial_x v = 2\operatorname{Re} \left( \Psi (1 - 2F(v, \bar{\Psi})) \right). \end{cases} \quad (2.2.34)$$

The local Cauchy problem for (2.2.32)-(2.2.34) was analyzed by de Laire and Gravejat in [19]. We recall the following proposition which shows the continuous dependence with respect to the initial datum of the solutions to the system of equations (2.2.32)-(2.2.34) (see [19] for the proof).

**Proposition 2.2.5** ([19]). *Let  $(v^0, \Psi^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$  and  $(\tilde{v}^0, \tilde{\Psi}^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$  be such that*

$$\partial_x v^0 = 2\operatorname{Re} \left( \Psi^0 (1 - 2F(v^0, \bar{\Psi}^0)) \right), \quad \text{and} \quad \partial_x \tilde{v}^0 = 2\operatorname{Re} \left( \tilde{\Psi}^0 (1 - 2F(\tilde{v}^0, \bar{\tilde{\Psi}}^0)) \right).$$

*Given two solutions  $(v, \Psi)$  and  $(\tilde{v}, \tilde{\Psi})$  in  $\mathcal{C}^0([0, T_*], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ , with  $(\Psi, \tilde{\Psi}) \in L^4([0, T_*], L^\infty(\mathbb{R}))^2$ , to (2.2.32)-(2.2.34) with initial datum  $(v^0, \Psi^0)$ , resp.  $(\tilde{v}^0, \tilde{\Psi}^0)$ , for some positive time  $T_*$ , there exist a positive number  $\tau$ , depending only on  $\|v^0\|_{L^2}$ ,  $\|\tilde{v}^0\|_{L^2}$ ,  $\|\Psi^0\|_{L^2}$  and  $\|\tilde{\Psi}^0\|_{L^2}$ , and a universal constant  $A$  such that we have*

$$\begin{aligned} \|v - \tilde{v}\|_{\mathcal{C}^0([0, T], L^2)} + \|\Psi - \tilde{\Psi}\|_{\mathcal{C}^0([0, T], L^2)} + \|\Psi - \tilde{\Psi}\|_{L^4([0, T], L^\infty)} \\ \leq A \left( \|v^0 - \tilde{v}^0\|_{L^2} + \|\Psi^0 - \tilde{\Psi}^0\|_{L^2} \right), \end{aligned} \quad (2.2.35)$$

for any  $T \in [0, \min\{\tau, T_*\}]$ . In addition, there exists a positive number  $B$ , depending only on  $\|v^0\|_{L^2}$ ,  $\|\tilde{v}^0\|_{L^2}$ ,  $\|\Psi^0\|_{L^2}$  and  $\|\tilde{\Psi}^0\|_{L^2}$ , such that

$$\|\partial_x v - \partial_x \tilde{v}\|_{\mathcal{C}^0([0, T], L^2)} \leq B \left( \|v^0 - \tilde{v}^0\|_{L^2} + \|\Psi^0 - \tilde{\Psi}^0\|_{L^2} \right), \quad (2.2.36)$$

for any  $T \in [0, \min\{\tau, T_*\}]$ .

This proposition plays an important role in the proof of not only the smoothing of the limit profile, but also the weak continuity of the hydrodynamical Landau-Lifshitz flow.

In order to prove the smoothness of the limit profile, we rely on the following smoothing type estimate for localized solutions of the linear Schrödinger equation (see [6, 20] for the proof of Proposition 2.2.6).

**Proposition 2.2.6** ([6, 20]). *Let  $\lambda \in \mathbb{R}$  and consider a solution  $u \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}))$  to the linear Schrödinger equation*

$$i\partial_t u + \partial_{xx} u = F, \quad (\text{LS})$$

with  $F \in L^2(\mathbb{R}, L^2(\mathbb{R}))$ . Then, there exists a positive constant  $K_\lambda$ , depending only on  $\lambda$ , such that

$$\lambda^2 \int_{-T}^T \int_{\mathbb{R}} |\partial_x u(x, t)|^2 e^{\lambda x} dx dt \leq K_\lambda \int_{-T-1}^{T+1} \int_{\mathbb{R}} \left( |u(x, t)|^2 + |F(x, t)|^2 \right) e^{\lambda x} dx dt, \quad (2.2.37)$$

for any positive number  $T$ .

We apply Proposition 2.2.6 to  $\Psi^*$  as well as all its derivatives, where  $\Psi^*$  is the solution to (2.2.32) associated to the solution  $(v^*, w^*)$  of (HLL), and then we express the result in terms of  $(v^*, w^*)$  to obtain

**Proposition 2.2.7.** *The pair  $(v^*, w^*)$  is indefinitely smooth and exponentially decaying on  $\mathbb{R} \times \mathbb{R}$ . Moreover, given any  $k \in \mathbb{N}$ , there exists a positive constant  $A_{k, \mathfrak{c}}$ , depending only on  $k$  and  $\mathfrak{c}$ , such that*

$$\int_{\mathbb{R}} \left[ (\partial_x^{k+1} v^*)^2 + (\partial_x^k v^*)^2 + (\partial_x^k w^*)^2 \right] (x + a^*(t), t) e^{\nu_{\mathfrak{c}} |x|} dx \leq A_{k, \mathfrak{c}}, \quad (2.2.38)$$

for any  $t \in \mathbb{R}$ .

## The Liouville type theorem

We next establish a Liouville type theorem, which guarantees that the limit profile constructed above is exactly a soliton. In particular, we will show that  $\varepsilon_0^* \equiv 0$ .

The pair  $\varepsilon^*$  satisfies the equation

$$\partial_t \varepsilon^* = J\mathcal{H}_{c^*(t)}(\varepsilon^*) + J\mathcal{R}_{c^*(t)}\varepsilon^* + \left( a^{*\prime}(t) - c^*(t) \right) \left( \partial_x Q_{c^*(t)} + \partial_x \varepsilon^* \right) - c^{*\prime}(t) \partial_c Q_{c^*(t)}, \quad (2.2.39)$$

where  $J$  is the symplectic operator

$$J = -2S\partial_x := \begin{pmatrix} 0 & -2\partial_x \\ -2\partial_x & 0 \end{pmatrix}, \quad (2.2.40)$$

and the remainder term  $\mathcal{R}_{c^*(t)}\varepsilon^*$  is given by

$$\mathcal{R}_{c^*(t)}\varepsilon^* := E'(Q_{c^*(t)} + \varepsilon^*) - E'(Q_{c^*(t)}) - E''(Q_{c^*(t)})(\varepsilon^*).$$

We rely on the strategy developed by Martel and Merle in [45] (see also [40]), and then applied by Béthuel, Gravejat and Smets in [6] to the Gross-Pitaevskii equation. We define the pair

$$u^*(\cdot, t) := S\mathcal{H}_{c^*(t)}(\varepsilon^*(\cdot, t)). \quad (2.2.41)$$

Since  $S\mathcal{H}_{c^*(t)}(\partial_x Q_{c^*(t)}) = 0$ , we deduce from (2.2.39) that

$$\begin{aligned} \partial_t u^* &= S\mathcal{H}_{c^*(t)}(JSu^*) + S\mathcal{H}_{c^*(t)}(J\mathcal{R}_{c^*(t)}\varepsilon^*) - (c^*)'(t)S\mathcal{H}_{c^*(t)}(\partial_c Q_{c^*(t)}) \\ &\quad + (c^*)'(t)S\partial_c \mathcal{H}_{c^*(t)}(\varepsilon^*) + \left( (a^*)'(t) - c^*(t) \right) S\mathcal{H}_{c^*(t)}(\partial_x \varepsilon^*). \end{aligned} \quad (2.2.42)$$

Decreasing further the value of  $\beta_{\mathfrak{c}}$  if necessary, we have



**Proposition 2.2.8.** *There exist two positive numbers  $A_*$  and  $R_*$ , depending only on  $\mathbf{c}$ , such that we have<sup>2</sup>*

$$\frac{d}{dt} \left( \int_{\mathbb{R}} x u_1^*(x, t) u_2^*(x, t) dx \right) \geq \frac{1 - \mathbf{c}^2}{16} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 - A_* \|u^*(\cdot, t)\|_{X(B(0, R_*))}^2, \quad (2.2.43)$$

for any  $t \in \mathbb{R}$ .

We give a second monotonicity type formula to dispose of the non-positive local term  $\|u^*(\cdot, t)\|_{X(B(0, R_*))}^2$  in the right-hand side of (2.2.43). If  $M$  is a smooth, bounded, two-by-two symmetric matrix-valued function, then

$$\frac{d}{dt} \langle M u^*, u^* \rangle_{L^2(\mathbb{R})^2} = 2 \langle S M u^*, \mathcal{H}_{c^*}(-2\partial_x u^*) \rangle_{L^2(\mathbb{R})^2} + \text{“super-quadratic terms”}, \quad (2.2.44)$$

where  $S$  is the matrix

$$S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For  $c \in (-1, 1) \setminus \{0\}$ , let  $M_c$  be given by

$$M_c := \begin{pmatrix} -\frac{2cv_c \partial_x v_c}{(1-v_c)^2} & -\frac{\partial_x v_c}{v_c} \\ -\frac{\partial_x v_c}{v_c} & 0 \end{pmatrix}. \quad (2.2.45)$$

We have the following lemma.

**Lemma 2.2.1.** *Let  $c \in (-1, 1) \setminus \{0\}$  and  $u \in X^3(\mathbb{R})$ . Then,*

$$\begin{aligned} G_c(u) &:= 2 \langle S M_c u, \mathcal{H}_c(-2\partial_x u) \rangle_{L^2(\mathbb{R})^2} \\ &= 2 \int_{\mathbb{R}} \mu_c \left( u_2 - \frac{cv_c^2}{\mu_c} u_1 - \frac{2cv_c \partial_x v_c}{\mu_c(1-v_c^2)} \partial_x u_1 \right)^2 + 3 \int_{\mathbb{R}} \frac{v_c^4}{\mu_c} \left( \partial_x u_1 - \frac{\partial_x v_c}{v_c} u_1 \right)^2, \end{aligned} \quad (2.2.46)$$

where

$$\mu_c = 2(\partial_x v_c)^2 + v_c^2(1 - v_c^2) > 0. \quad (2.2.47)$$

The functional  $G_c$  is a non-negative quadratic form, and

$$\text{Ker}(G_c) = \text{Span}(Q_c). \quad (2.2.48)$$

We have indeed chosen the matrix  $M_c$  such that  $M_c Q_c = \partial_x Q_c$  to obtain (2.2.48). Since  $Q_c$  does not vanish, we deduce from standard Sturm-Liouville theory, that  $G_c$  is non-negative, which is confirmed by the computation in Lemma 2.2.1.

By the second orthogonality condition in (2.2.20) and the fact that  $\mathcal{H}_{c^*}(\chi_{c^*}) = -\tilde{\lambda}_{c^*} \chi_{c^*}$ , we have

$$0 = \langle \mathcal{H}_{c^*}(\chi_{c^*}), \varepsilon^* \rangle_{L^2(\mathbb{R})^2} = \langle \mathcal{H}_{c^*}(\varepsilon^*), \chi_{c^*} \rangle_{L^2(\mathbb{R})^2} = \langle u^*, S \chi_{c^*} \rangle_{L^2(\mathbb{R})^2}. \quad (2.2.49)$$

On the other hand, we know that

$$\langle Q_{c^*}, S \chi_{c^*} \rangle = P'(Q_{c^*})(\chi_{c^*}) \neq 0, \quad (2.2.50)$$

so that the pair  $u^*$  is not proportional to  $Q_{c^*}$  under the orthogonality condition in (2.2.49). We claim the following coercivity property of  $G_c$  under this orthogonality condition.

---

2. In (2.2.43), we use the notation

$$\|(f, g)\|_{X(\Omega)}^2 := \int_{\Omega} \left( (\partial_x f)^2 + f^2 + g^2 \right),$$

in which  $\Omega$  denotes a measurable subset of  $\mathbb{R}$ .

**Proposition 2.2.9.** *Let  $c \in (-1, 1) \setminus \{0\}$ . There exists a positive number  $\Lambda_c$ , depending only and continuously on  $c$ , such that*

$$G_c(u) \geq \Lambda_c \int_{\mathbb{R}} [(\partial_x u_1)^2 + (u_1)^2 + (u_2)^2](x) e^{-2|x|} dx, \quad (2.2.51)$$

for any pair  $u \in X(\mathbb{R})$  verifying

$$\langle u, S\chi_c \rangle_{L^2(\mathbb{R})^2} = 0. \quad (2.2.52)$$

Coming back to (2.2.44), we can prove

**Proposition 2.2.10.** *There exists a positive number  $B_*$ , depending only on  $\mathbf{c}$ , such that*

$$\begin{aligned} \frac{d}{dt} \left( \langle M_{c^*(t)} u^*(\cdot, t), u^*(\cdot, t) \rangle_{L^2(\mathbb{R})^2} \right) &\geq \frac{1}{B_*} \int_{\mathbb{R}} [(\partial_x u_1^*)^2 + (u_1^*)^2 + (u_2^*)^2](x, t) e^{-2|x|} dx \\ &\quad - B_* \left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2, \end{aligned} \quad (2.2.53)$$

for any  $t \in \mathbb{R}$ .

Using Propositions 2.2.8 and 2.2.10, we claim

**Corollaire 2.2.1.** *Set*

$$N(t) := \frac{1}{2} \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} + A_* B_* e^{2R_*} M_{c^*(t)}.$$

There exists a positive constant  $\mathcal{A}_c$  such that we have

$$\frac{d}{dt} \left( \langle N(t) u^*(\cdot, t), u^*(\cdot, t) \rangle_{L^2(\mathbb{R})^2} \right) \geq \mathcal{A}_c \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2, \quad (2.2.54)$$

for any  $t \in \mathbb{R}$ . Since

$$\int_{-\infty}^{+\infty} \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2 dt < +\infty, \quad (2.2.55)$$

there exists a sequence  $(t_k^*)_{k \in \mathbb{N}}$  such that

$$\lim_{k \rightarrow +\infty} \left\| u^*(\cdot, t_k^*) \right\|_{X(\mathbb{R})}^2 = 0. \quad (2.2.56)$$

In view of (2.2.20), (2.2.41) and the bound for  $\mathcal{H}_{c^*}$  in (2.6.43), we have

$$\left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})} \leq A_c \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}, \quad (2.2.57)$$

Hence, we can apply (2.2.56) and (2.2.57) in order to obtain

$$\lim_{k \rightarrow +\infty} \left\| \varepsilon^*(\cdot, t_k^*) \right\|_{X(\mathbb{R})}^2 = 0. \quad (2.2.58)$$

By (2.2.58) and the orbital stability in Theorem 2.2.1, this yields

**Corollaire 2.2.2.** *We have*

$$\varepsilon_0^* \equiv 0.$$

At this stage we obtain (2.2.11) for some subsequence. We should extend this result for any sequence. The proof is exactly the same as the one done by Béthuel, Gravejat and Smets in [6] (see Subsection 1.3.4 in [6] for the details).

## 2.2.4 Proof of Theorem 2.1.1

We choose a positive number  $\delta_{\mathbf{c}}$  such that  $\|(v_0, w_0) - Q_{\mathbf{c}}\|_{X(\mathbb{R})} \leq \beta_{\mathbf{c}}$ , whenever  $d_{\mathcal{E}}(m^0, u_{\mathbf{c}}) \leq \delta_{\mathbf{c}}$ . We next apply Theorem 2.2.2 to the solution  $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  to (HLL) corresponding to the solution  $m$  to (LL). This yields the existence of a speed  $\mathbf{c}^*$  and a position function  $b$  such that the convergences in Theorem 2.2.2 hold. In particular, since the weak convergence for  $m_3$  is satisfied by Theorem 2.2.2, it is sufficient to show the existence of a phase function  $\theta$  such that  $\exp(i\theta(t))\partial_x \check{m}(\cdot + b(t), t)$  is weakly convergent to  $\partial_x \check{u}_{\mathbf{c}^*}$  in  $L^2(\mathbb{R})$  as  $t \rightarrow \infty$ . The locally uniform convergence of  $\exp(i\theta(t))\check{m}(\cdot + b(t), t)$  towards  $\check{u}_{\mathbf{c}^*}$  then follows from the Sobolev embedding theorem. We begin by constructing this phase function.

We fix a non-zero function  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, [0, 1])$  such that  $\chi$  is even. Using the explicit formula of  $\check{u}_{\mathbf{c}^*}$ , we have

$$\int_{\mathbb{R}} \check{u}_{\mathbf{c}^*}(x)\chi(x) dx = 2\mathbf{c}^* \int_{\mathbb{R}} \frac{\chi(x)}{\cosh(\sqrt{1 - (\mathbf{c}^*)^2}x)} dx \neq 0. \quad (2.2.59)$$

Decreasing the value of  $\beta_{\mathbf{c}}$  if needed, we deduce from the orbital stability in [19] that

$$\left| \int_{\mathbb{R}} \check{m}(x + b(t), t)\chi(x) dx \right| \geq |\mathbf{c}^*| \int_{\mathbb{R}} \frac{\chi(x)}{\cosh(\sqrt{1 - (\mathbf{c}^*)^2}x)} dx \neq 0, \quad (2.2.60)$$

for any  $t \in \mathbb{R}$ .

Let  $\Upsilon : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the  $\mathcal{C}^1$  function defined by

$$\Upsilon(t, \theta) := \operatorname{Im} \left( e^{-i\theta} \int_{\mathbb{R}} \check{m}(x + b(t), t)\chi(x) dx \right).$$

From (2.2.60) we can find a number  $\theta_0$  such that  $\Upsilon(0, \theta_0) = 0$  and  $\partial_{\theta}\Upsilon(0, \theta_0) > 0$ . Then, using the implicit function theorem, there exists a  $\mathcal{C}^1$  function  $\theta : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\Upsilon(t, \theta(t)) = 0$ . In addition, using (2.2.60) another time, we can fix the choice of  $\theta$  so that there exists a positive constant  $A_{\mathbf{c}^*}$  such that

$$\partial_{\theta}\Upsilon(t, \theta(t)) = \operatorname{Re} \left( e^{-i\theta(t)} \int_{\mathbb{R}} \check{m}(x + b(t), t)\chi(x) dx \right) \geq A_{\mathbf{c}^*} > 0. \quad (2.2.61)$$

This implies, differentiating the identity  $\Upsilon(t, \theta(t)) = 0$  with respect to  $t$ , that

$$|\theta'(t)| = \left| \frac{\partial_t \Upsilon(t, \theta(t))}{\partial_{\theta} \Upsilon(t, \theta(t))} \right| \leq \frac{1}{A_{\mathbf{c}^*}} |\partial_t \Upsilon(t, \theta(t))|, \quad (2.2.62)$$

for all  $t \in \mathbb{R}$ . Now, we differentiate the function  $\Upsilon$  with respect to  $t$ , and we use the equation of  $\check{m}$  to obtain

$$\begin{aligned} \partial_t \Upsilon(t, \theta(t)) &= \operatorname{Im} \left( e^{-i\theta} \int_{\mathbb{R}} \chi(x) \left( \partial_x \check{m}(x + b(t), t)b'(t) - im_3(x + b(t), t)\partial_{xx} \check{m}(x + b(t), t) \right. \right. \\ &\quad \left. \left. + i\check{m}(x + b(t), t)\partial_{xx} m_3(x + b(t), t) - im_3(x + b(t), t)\check{m}(x + b(t), t) \right) dx \right). \end{aligned} \quad (2.2.63)$$

Since  $b \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{R})$ , and since both  $\partial_x \check{m}$  and  $\partial_t \check{m}$  belong to  $\mathcal{C}_b^0(\mathbb{R}, H^{-1}(\mathbb{R}))$ , it follows that the derivative  $\theta'$  is bounded on  $\mathbb{R}$ .

We denote by  $\varphi$  the phase function defined by

$$\varphi(x + b(t), t) := \varphi(b(t), t) + \int_0^x w(y + b(t), t) dy,$$

with  $\varphi(b(t), t) \in [0, 2\pi]$ , which is associated to the function  $\check{m}(x + b(t), t)$  for any  $(x, t) \in \mathbb{R}^2$  in the way that

$$\check{m}(x + b(t), t) = \left(1 - m_3^2(x + b(t), t)\right)^{\frac{1}{2}} \exp\left(i\varphi(x + b(t), t)\right).$$

It is sufficient to prove that

$$\exp\left(i\left(\varphi(b(t), t) - \theta(t)\right)\right) \longrightarrow 1, \quad (2.2.64)$$

as  $t \rightarrow \infty$  to obtain

$$\exp\left(i\left(\varphi(\cdot + b(t), t) - \theta(t)\right)\right) \longrightarrow \exp\left(i\varphi_{c^*}(\cdot)\right) := \exp\left(i\int_0^\cdot w_{c^*}(y) dy\right) \quad \text{in } L_{loc}^\infty(\mathbb{R}),$$

as  $t \rightarrow \infty$ . This implies, using Theorem 2.2.2 once again, and the Sobolev embedding theorem, that

$$\begin{aligned} e^{-i\theta(t)}\partial_x\check{m}(\cdot + b(t), t) &\rightarrow \partial_x\check{u}_{c^*} \quad \text{in } L^2(\mathbb{R}), \\ e^{-i\theta(t)}\check{m}(\cdot + b(t), t) &\rightarrow \check{u}_{c^*} \quad \text{in } L_{loc}^\infty(\mathbb{R}), \end{aligned} \quad (2.2.65)$$

as  $t \rightarrow \infty$ . Now, let us prove (2.2.64). We have

$$\begin{aligned} &e^{-i\theta(t)} \int_{\mathbb{R}} \check{m}(x + b(t), t)\chi(x) dx \\ &= \exp\left(i[\varphi(b(t), t) - \theta(t)]\right) \int_{\mathbb{R}} \left(1 - m_3^2(x + b(t), t)\right)^{\frac{1}{2}} \exp\left(i\int_0^x w(y + b(t), t) dy\right)\chi(x) dx. \end{aligned}$$

We use the fact that  $\Upsilon(t, \theta(t)) = 0$  to obtain

$$\begin{aligned} &\cos\left(\varphi(b(t), t) - \theta(t)\right) \operatorname{Im}\left(\int_{\mathbb{R}} \left(1 - m_3^2(x + b(t), t)\right)^{\frac{1}{2}} \exp\left(i\int_0^x w(y + b(t), t) dy\right)\chi(x) dx\right) \\ &+ \sin\left(\varphi(b(t), t) - \theta(t)\right) \operatorname{Re}\left(\int_{\mathbb{R}} \left(1 - m_3^2(x + b(t), t)\right)^{\frac{1}{2}} \exp\left(i\int_0^x w(y + b(t), t) dy\right)\chi(x) dx\right) \\ &= 0. \end{aligned}$$

On the other hand, by (2.2.61), we have

$$\begin{aligned} &\cos\left(\varphi(b(t), t) - \theta(t)\right) \operatorname{Re}\left(\int_{\mathbb{R}} \left(1 - m_3^2(x + b(t), t)\right)^{\frac{1}{2}} \exp\left(i\int_0^x w(y + b(t), t) dy\right)\chi(x) dx\right) \\ &- \sin\left(\varphi(b(t), t) - \theta(t)\right) \operatorname{Im}\left(\int_{\mathbb{R}} \left(1 - m_3^2(x + b(t), t)\right)^{\frac{1}{2}} \exp\left(i\int_0^x w(y + b(t), t) dy\right)\chi(x) dx\right) \\ &> 0. \end{aligned}$$

We derive from Theorem 2.2.2 and (2.2.59) that

$$\operatorname{Im}\left(\int_{\mathbb{R}} \left(1 - m_3^2(x + b(t), t)\right)^{\frac{1}{2}} \exp\left(i\int_0^x w(y + b(t), t) dy\right)\chi(x) dx\right) \rightarrow \operatorname{Im}\left(\int_{\mathbb{R}} \check{u}_{c^*}(x)\chi(x) dx\right) = 0,$$

and

$$\operatorname{Re}\left(\int_{\mathbb{R}} \left(1 - m_3^2(x + b(t), t)\right)^{\frac{1}{2}} \exp\left(i\int_0^x w(y + b(t), t) dy\right)\chi(x) dx\right) \rightarrow \operatorname{Re}\left(\int_{\mathbb{R}} \check{u}_{c^*}(x)\chi(x) dx\right) > 0.$$

This is enough to derive (2.2.64).

Finally, we claim that  $\theta'(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Indeed, we can introduce (2.2.65) into (2.2.63), and we then obtain, using the equation satisfied by  $\check{u}_{c^*}$ , that

$$\partial_t\Upsilon(t, \theta(t)) \longrightarrow 0,$$

as  $t \rightarrow \infty$ . By (2.2.62), this yields  $\theta'(t) \rightarrow 0$  as  $t \rightarrow \infty$ , which finishes the proof of Theorem 2.1.1.  $\square$

## 2.3 Proof of the orbital stability

First, we recall the orbital stability theorem, which was established in [19] (see Corollary 2, Propositions 2 and 4 in [19]).

**Théorème 2.3.1.** *Let  $c \in (-1, 1) \setminus \{0\}$  and  $(v_0, w_0) \in X(\mathbb{R})$  satisfying (2.2.4). There exist a unique global solution  $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  to (HLL) with initial datum  $(v_0, w_0)$ , and two maps  $c_1 \in \mathcal{C}^1(\mathbb{R}, (-1, 1) \setminus \{0\})$  and  $a_1 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  such that the function  $\varepsilon_1$ , defined by (2.2.5), satisfies the orthogonality conditions*

$$\langle \varepsilon_1(\cdot, t), \partial_x Q_{c_1(t)} \rangle_{L^2(\mathbb{R})^2} = P'(Q_{c_1(t)})(\varepsilon_1(\cdot, t)) = 0, \quad (2.3.1)$$

for any  $t \in \mathbb{R}$ . Moreover,  $\varepsilon_1(\cdot, t)$ ,  $c_1(t)$  and  $a_1(t)$  satisfy (2.2.7), (2.2.8) and (2.2.9) for any  $t \in \mathbb{R}$ .

With Theorem 2.3.1 at hand, we can provide the proof of Theorem 2.2.1.

*Proof.* We consider the following map

$$\Xi((v, w), \sigma, \mathbf{b}) := \left( \langle \partial_x Q_{\sigma, \mathbf{b}}, \varepsilon \rangle_{L^2 \times L^2}, \langle \chi_{\sigma, \mathbf{b}}, \varepsilon \rangle_{L^2 \times L^2} \right),$$

where we have set  $\varepsilon = (v, w) - Q_{\sigma, \mathbf{b}}$ , and  $\chi_{\sigma, \mathbf{b}} = \chi_\sigma(\cdot - \mathbf{b})$  (we recall that  $\chi_\sigma$  is the eigenfunction associated to the unique negative eigenvalue  $-\tilde{\lambda}_\sigma$  of the operator  $\mathcal{H}_\sigma$ ). The map  $\Xi$  is well-defined for, and depends smoothly on,  $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ ,  $\sigma \in (-1, 1) \setminus \{0\}$ , and  $\mathbf{b} \in \mathbb{R}$ .

We fix  $t \in \mathbb{R}$ . In order to simplify the notation, we substitute  $(c_1(t), a_1(t))$  by  $(c_1, a_1)$ . We check that

$$\Xi(Q_{c_1, a_1}, c_1, a_1) = 0,$$

and we compute

$$\begin{cases} \partial_\sigma \Xi_1(Q_{c_1, a_1}, c_1, a_1) = 0, \\ \partial_\sigma \Xi_2(Q_{c_1, a_1}, c_1, a_1) = -\langle \chi_{c_1, a_1}, \partial_\sigma Q_{c_1, a_1} \rangle_{L^2 \times L^2}. \end{cases}$$

Let  $c \in (-1, 1) \setminus \{0\}$  and suppose by contradiction that

$$\langle \chi_c, \partial_c Q_c \rangle_{L^2 \times L^2} = 0.$$

Using the fact that  $\mathcal{H}_c(\partial_c Q_c) = P'(Q_c)$ , it comes

$$0 = \langle \chi_c, \partial_c Q_c \rangle_{L^2 \times L^2} = -\frac{1}{\tilde{\lambda}_c} \langle \chi_c, \mathcal{H}_c(\partial_c Q_c) \rangle_{L^2 \times L^2} = -\frac{1}{\tilde{\lambda}_c} \langle \chi_c, P'(Q_c) \rangle_{L^2 \times L^2}.$$

Since  $\mathcal{H}_c$  is self-adjoint, we also have

$$\langle \chi_c, \partial_x Q_c \rangle_{L^2 \times L^2} = 0.$$

By Proposition 1 in [19], we infer that

$$0 > -\tilde{\lambda}_c \|\chi_c\|_{L^2 \times L^2}^2 = \langle \chi_c, \mathcal{H}_c(\chi_c) \rangle_{L^2 \times L^2} \geq \Lambda_c \|\chi_c\|_{L^2 \times L^2}^2 > 0,$$

which provides the contradiction and shows that

$$\langle \chi_c, \partial_c Q_c \rangle_{L^2 \times L^2} \neq 0, \quad (2.3.2)$$

for all  $c \in (-1, 1) \setminus \{0\}$ . In addition, we have

$$\begin{cases} \partial_b \Xi_1(Q_{c_1, a_1}, c_1, a_1) = \left\| \partial_x Q_{c_1} \right\|_{L^2}^2 = 2(1 - c_1^2)^{\frac{1}{2}} > 0, \\ \partial_b \Xi_2(Q_{c_1, a_1}, c_1, a_1) = 0. \end{cases}$$

Therefore, the matrix

$$d_{\sigma, \mathbf{b}} \Xi(Q_{c_1, a_1}, c_1, a_1) = \begin{pmatrix} 0 & \langle \chi_{c_1, a_1}, \partial_\sigma Q_{c_1, a_1} \rangle_{L^2 \times L^2} \\ 2(1 - c_1^2)^{\frac{1}{2}} & 0 \end{pmatrix}$$

is an isomorphism from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .

Then, we can apply the version of the implicit function theorem in [5] in order to find a neighbourhood  $\mathcal{V}$  of  $Q_{c_1, a_1}$ , a neighbourhood  $\mathcal{U}$  of  $(c_1, a_1)$ , and a map  $\gamma_{c_1, a_1} : \mathcal{U} \rightarrow \mathcal{V}$  such that

$$\Xi((v, w), \sigma, \mathbf{b}) = 0 \Leftrightarrow (c(v, w), a(v, w)) := (\sigma, \mathbf{b}) = \gamma_{c, a}(v, w) \quad \forall (v, w) \in \mathcal{V}, \quad \forall (\sigma, \mathbf{b}) \in \mathcal{U}.$$

In addition, there exists a positive constant  $\Lambda$ , depending only on  $c_1$  such that

$$\|\varepsilon(t)\|_X + |c(t) - c_1(t)| + |a(t) - a_1(t)| \leq \Lambda \|\varepsilon_1(t)\|_X \leq \Lambda_{c_1} A_c \alpha_0, \quad (2.3.3)$$

where  $c(t) := c(v(t), w(t))$ ,  $a(t) := a(v(t), w(t))$  and  $\varepsilon(t) := (v(t), w(t)) - Q_{c(t), a(t)}$ , for any fixed  $t \in \mathbb{R}$ . Using the fact that  $(v(t), w(t))$  stays into a neighbourhood of  $Q_{c_1(t), a_1(t)}$  for all  $t \in \mathbb{R}$  by Theorem 2.3.1, and also the fact that  $c_1$  satisfies (2.2.8), we are led to the following lemma.

**Lemma 2.3.1.** *Under the assumptions of Theorem 2.3.1, there exists a unique pair of functions  $(a, c) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^2)$  such that*

$$\varepsilon(t) := (v(t), w(t)) - Q_{c(t), a(t)},$$

*verifies the two following orthogonality conditions*

$$\langle \varepsilon(t), \partial_x Q_{c(t), a(t)} \rangle_{L^2 \times L^2} = \langle \chi_{c(t), a(t)}, \varepsilon(t) \rangle_{L^2 \times L^2} = 0. \quad (2.3.4)$$

Moreover, we have (2.2.8).

This completes the proof of orbital stability. Now, let us prove the continuous differentiability of the functions  $a$  and  $c$ , as well as the inequality

$$|c'(t)| + |a'(t) - c(t)| \leq A_c \|\varepsilon(\cdot, t)\|_{X(\mathbb{R})}, \quad (2.3.5)$$

for all  $t \in \mathbb{R}$ . The  $\mathcal{C}^1$  nature of  $a$  and  $c$  can be derived from a standard density argument as in [19]. Concerning (2.3.5), we can write the equation verified by  $\varepsilon$ , namely

$$\partial_t \varepsilon_v = \left( (a'(t) - c(t)) \partial_x v_{c, a} - c'(t) \partial_c v_{c, a} \right) + \partial_x \left( ((v_{c, a} + \varepsilon_v)^2 - 1)(v_{c, a} + \varepsilon_w) - (v_{c, a}^2 - 1)w_{c, a} \right), \quad (2.3.6)$$

and

$$\begin{aligned} \partial_t \varepsilon_w &= (a'(t) - c(t)) \partial_x w_{c, a} - c'(t) \partial_c w_{c, a} \\ &+ \partial_x \left( \frac{\partial_{xx} v_{c, a} + \partial_{xx} \varepsilon_v}{1 - (v_{c, a} + \varepsilon_v)^2} + (v_{c, a} + \varepsilon_v) \frac{(\partial_x v_{c, a} + \partial_x \varepsilon_v)^2}{(1 - (v_{c, a} + \varepsilon_v)^2)^2} \right. \\ &\quad \left. - \frac{\partial_{xx} v_{c, a}}{1 - v_{c, a}^2} - v_{c, a} \frac{(\partial_x v_{c, a})^2}{(1 - v_{c, a}^2)^2} \right) \\ &+ \partial_x \left( (v_{c, a} + \varepsilon_v) ((w_{c, a} + \varepsilon_w)^2 - 1) - v_{c, a} (w_{c, a}^2 - 1) \right). \end{aligned} \quad (2.3.7)$$

We differentiate with respect to time the orthogonality conditions in (2.2.6) and we invoke equations (2.3.6) and (2.3.7) to write the identity

$$M \begin{pmatrix} c' \\ a' - c \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix}. \quad (2.3.8)$$

Here,  $M$  refers to the matrix of size 2 given by

$$\begin{aligned} M_{1,1} &= \langle \partial_c Q_c, \chi_c \rangle_{L^2 \times L^2} + \langle \partial_c \chi_{c,a}, \varepsilon \rangle_{L^2 \times L^2}, \\ M_{1,2} &= \langle \chi_c, \partial_x Q_c \rangle_{L^2 \times L^2} - \langle \partial_x \chi_{c,a}, \varepsilon \rangle_{L^2 \times L^2}, \\ M_{2,1} &= -\langle \partial_x Q_c, \partial_c Q_c \rangle_{L^2 \times L^2} + \langle \partial_c \partial_x Q_{c,a}, \varepsilon \rangle_{L^2 \times L^2}, \\ M_{2,2} &= \left\| \partial_x Q_c \right\|_{L^2 \times L^2}^2 - \langle \partial_{xx} Q_{c,a}, \varepsilon \rangle_{L^2 \times L^2}. \end{aligned}$$

The vectors  $Y$  and  $Z$  are defined by

$$\begin{aligned} Y &= \left\langle \partial_x w_{c,a}, \left( (v_{c,a} + \varepsilon_v)^2 - 1 \right) (w_{c,a} + \varepsilon_w) - (v_{c,a}^2 - 1) w_{c,a} \right\rangle_{L^2} \\ &\quad + \left\langle \partial_x v_{c,a}, \left( (w_{c,a} + \varepsilon_w)^2 - 1 \right) (v_{c,a} + \varepsilon_v) - (w_{c,a}^2 - 1) v_{c,a} \right\rangle_{L^2} \\ &\quad - \left\langle \partial_{xx} v_{c,a}, \frac{\partial_{xx} v_{c,a} + \partial_{xx} \varepsilon_v}{1 - (v_{c,a} + \varepsilon_v)^2} - \frac{\partial_{xx} v_{c,a}}{1 - v_{c,a}^2} \right\rangle_{L^2} + c \langle \partial_x \chi_{c,a}, \varepsilon \rangle_{L^2 \times L^2}, \end{aligned}$$

and

$$\begin{aligned} Z &= \left\langle \partial_{xx} v_{c,a}, \left( (v_{c,a} + \varepsilon_v)^2 - 1 \right) (w_{c,a} + \varepsilon_w) - (v_{c,a}^2 - 1) w_{c,a} \right\rangle_{L^2} \\ &\quad + \left\langle \partial_{xx} w_{c,a}, \left( (w_{c,a} + \varepsilon_w)^2 - 1 \right) (v_{c,a} + \varepsilon_v) - (w_{c,a}^2 - 1) v_{c,a} \right\rangle_{L^2} \\ &\quad - \left\langle \partial_{xxx} w_{c,a}, \frac{\partial_{xx} v_{c,a} + \partial_{xx} \varepsilon_v}{1 - (v_{c,a} + \varepsilon_v)^2} - \frac{\partial_{xx} v_{c,a}}{1 - v_{c,a}^2} \right\rangle_{L^2} + c \langle \partial_{xx} Q_{c,a}, \varepsilon \rangle_{L^2 \times L^2}. \end{aligned}$$

We next decompose the matrix  $M$  as  $M = D + H$ , where  $D$  is the diagonal matrix of size 2 with diagonal coefficients

$$D_{1,1} = \langle \partial_c Q_c, \chi_c \rangle_{L^2 \times L^2} \neq 0,$$

by (2.3.2), and

$$D_{2,2} = \left\| \partial_x Q_{c(t)} \right\|_{L^2}^2 = 2(1 - c(t)^2)^{\frac{1}{2}},$$

so that  $D$  is invertible. Concerning the matrix  $H$ , we check that

$$\langle P'(Q_c), \partial_x Q_c \rangle_{L^2 \times L^2} = \langle \partial_x Q_c, \partial_c Q_c \rangle_{L^2 \times L^2} = 0.$$

Then,

$$H = \begin{pmatrix} \langle \partial_c \chi_{c,a}, \varepsilon \rangle_{L^2 \times L^2} & -\langle \partial_x \chi_{c,a}, \varepsilon \rangle_{L^2 \times L^2} \\ \langle \partial_c \partial_x Q_{c,a}, \varepsilon \rangle_{L^2 \times L^2} & -\langle \partial_{xx} Q_{c,a}, \varepsilon \rangle_{L^2 \times L^2} \end{pmatrix}.$$

It follows from the exponential decay of  $Q_{c,a}$  and its derivatives that

$$|H| \leq A_c \|\varepsilon\|_{L^2 \times L^2}.$$

We can make a further choice of the positive number  $\alpha_c$ , such that the operator norm of the matrix  $D^{-1}H$  is less than 1/2. In this case, the matrix  $M$  is invertible and the operator norm of its inverse is uniformly bounded with respect to  $t$ . Coming back to (2.3.8), we are led to the estimate

$$|c'(t)| + |a'(t) - c(t)| \leq A_c \left( |Y(t)| + |Z(t)| \right). \quad (2.3.9)$$

It remains to estimate the quantities  $Y$  and  $Z$ . We write

$$\begin{aligned} & \left| \left\langle \partial_x w_{c,a}, \left( (v_{c,a} + \varepsilon_v)^2 - 1 \right) (w_{c,a} + \varepsilon_w) - (v_{c,a}^2 - 1) w_{c,a} \right\rangle_{L^2} \right| \\ &= \left| \left\langle \partial_x w_{c,a}, (\varepsilon_v^2 + 2v_{c,a}\varepsilon_v) w_{c,a} + \varepsilon_w \left( (\varepsilon_v + v_{c,a})^2 - 1 \right) \right\rangle_{L^2} \right| \leq A_c \|\varepsilon\|_{L^2 \times L^2}. \end{aligned}$$

Arguing in the same way for the other terms in  $Y$  and  $Z$ , we obtain

$$|Y| + |Z| = \mathcal{O}\left(\|\varepsilon\|_{L^2 \times L^2}\right),$$

which is enough to deduce (2.3.5) from (2.3.9).

To achieve the proof, we show (2.2.7). Using the Sobolev embedding theorem of  $H^1(\mathbb{R})$  into  $\mathcal{C}^0(\mathbb{R})$ , we can write

$$\max_{x \in \mathbb{R}} v(x, t) \leq \|v_{c(t)}\|_{L^\infty(\mathbb{R})} + \|v(\cdot, t) - v_{c(t), a(t)}\|_{L^\infty(\mathbb{R})} \leq \|v_{c(t)}\|_{L^\infty(\mathbb{R})} + \|\varepsilon(t)\|_{X(\mathbb{R})}.$$

By (2.2.3),  $\|v_c\|_{L^\infty(\mathbb{R})} < 1$ , so that by (2.2.8) there exists a small positive number  $\gamma_c$  such that  $\|v_{c(t)}\|_{L^\infty(\mathbb{R})} \leq 1 - \gamma_c$ . We obtain

$$\max_{x \in \mathbb{R}} v(x, t) \leq 1 - \gamma_c + \|\varepsilon(t)\|_{X(\mathbb{R})} \leq 1 - \gamma_c + \alpha_c.$$

For  $\alpha_c$  small enough, estimate (2.2.7) follows, with  $\sigma_c := -\alpha_c + \gamma_c$ .  $\square$

## 2.4 Proofs of localization and smoothness of the limit profile

### 2.4.1 Proof of Proposition 2.2.2

The proof relies on the conservation law for the density of momentum  $vw$ . Let  $R$  and  $t$  be two real numbers, and recall that

$$I_R(t) \equiv I_R^{(v,w)}(t) := \frac{1}{2} \int_{\mathbb{R}} [vw](x + a(t), t) \Phi(x - R) dx,$$

where  $\Phi$  is the function defined on  $\mathbb{R}$  by

$$\Phi(x) := \frac{1}{2} \left( 1 + \text{th}(\nu_c x) \right),$$

with  $\nu_c := \sqrt{1 - c^2}/8$ . First, we deduce from the conservation law for  $vw$  (see Lemma 3.1 in [19] for more details) the identity

$$\begin{aligned} \frac{d}{dt} [I_{R+\sigma t}(t)] &= -(a'(t) + \sigma) \int_{\mathbb{R}} [vw](x + a(t), t) \Phi'(x - R - \sigma t) dx \\ &+ \int_{\mathbb{R}} \left[ v^2 + w^2 - 3v^2 w^2 + \frac{3 - v^2}{(1 - v^2)^2} (\partial_x v)^2 \right] (x + a(t), t) \Phi'(x - R - \sigma t) dx \\ &+ \int_{\mathbb{R}} [\ln(1 - v^2)] (x + a(t), t) \Phi'''(x - R - \sigma t) dx. \end{aligned} \quad (2.4.1)$$

Our goal is to provide a lower bound for the integrand in the right-hand side of (2.4.1).



Notice that the function  $\Phi$  satisfies the inequality

$$|\Phi'''| \leq 4\nu_c^2 \Phi'. \quad (2.4.2)$$

In view of the bound (2.2.14) on  $a'(t)$  and the definition of  $\sigma_c$ , we obtain that

$$|a'(t) + \sigma|^2 \leq \frac{9 + 7\mathfrak{c}^2}{8}. \quad (2.4.3)$$

Hence, we deduce

$$\begin{aligned} \frac{d}{dt} [I_{R+\sigma t}(t)] &\geq \int_{\mathbb{R}} \left[ 4\nu_c^2 \ln(1 - v^2) + v^2 + w^2 - 3v^2 w^2 \right. \\ &\quad \left. + (\partial_x v)^2 - \sqrt{\frac{9 + 7\mathfrak{c}^2}{8}} |vw| \right] (x + a(t), t) \Phi'(x - R - \sigma t) dx := J_1 + J_2. \end{aligned} \quad (2.4.4)$$

At this step, we decompose the real line into two domains,  $[-R_0, R_0]$  and its complement, where  $R_0$  is to be defined below and we denote  $J_1$  and  $J_2$  the value of the integral in the right-hand side of (2.4.4) on each region. On  $\mathbb{R} \setminus [-R_0, R_0]$ , we bound the integrand pointwise from below by a positive quadratic form in  $(v, w)$ . Exponentially small error terms arise from integration on  $[-R_0, R_0]$ .

For  $|x| \geq R_0$ , using Theorem 2.2.1, the Sobolev embedding theorem, and choosing  $\alpha_0$  small enough and  $R_0$  large enough, we obtain

$$|v(x + a(t), t)| \leq |\varepsilon_v(x, t)| + |v_{c(t)}(x)| \leq A_c(\alpha_0 + \exp(-\sqrt{1 - \mathfrak{c}^2} R_0)) \leq \frac{1}{12}, \quad (2.4.5)$$

for any  $t \in \mathbb{R}$ . Using the fact that  $\ln(1 - s) \geq -2s$  for all  $s \in [0, \frac{1}{2}]$  and introducing (2.4.5) in (2.4.4), we obtain

$$J_1 \geq \frac{1 - \mathfrak{c}^2}{8} \int_{|x| \geq R_0} \left[ v^2 + w^2 + (\partial_x v)^2 \right] (x + a(t), t) \Phi'(x - R - \sigma t) dx. \quad (2.4.6)$$

We next consider the case  $x \in [-R_0, R_0]$ . In that region, we have

$$|x - R - \sigma t| \geq -R_0 + |R + \sigma t|.$$

Hence,

$$\Phi'(x - R - \sigma t) \leq 2\nu_c e^{2\nu_c R_0} e^{-2\nu_c |R + \sigma t|}. \quad (2.4.7)$$

Since the function  $|\ln|$  is decreasing on  $(0, 1]$ , in view of (2.2.7) and (2.4.4),

$$|J_2| \leq A_c \int_{|x| \leq R_0} \left[ v^2 + w^2 + (\partial_x v)^2 \right] (x + a(t), t) \Phi'(x - R - \sigma t) dx.$$

Then, by (2.4.7) and the control on the norm of  $(v, w)$  in  $X(\mathbb{R})$  provided by the conservation of the energy, we obtain

$$|J_2| \leq B_c e^{-2\nu_c |R + \sigma t|}.$$

This finishes the proof of (2.2.28). It remains to prove (2.2.29). For that, we distinguish two cases. If  $R \geq 0$ , we integrate (2.2.28) from  $t = t_0$  to  $t = (t_0 + t_1)/2$ , choosing  $\sigma = \sigma_c$  and  $R = R - \sigma_c t_0$ , and then from  $t = (t_0 + t_1)/2$  to  $t = t_1$  choosing  $\sigma = -\sigma_c$  and  $R = R + \sigma_c t_1$ . If  $R \leq 0$ , we use the same arguments for the reverse choices  $\sigma = -\sigma_c$  and  $\sigma = \sigma_c$ . This implies (2.2.29), and finishes the proof of Proposition 2.2.2.  $\square$

## 2.4.2 Proof of Proposition 2.2.7

Let  $\Psi^*$  and  $v^*$  be the solutions of (2.2.32)-(2.2.34) expressed in terms of the hydrodynamical variables  $(v^*, w^*)$  as in (2.2.30). We split the proof into five steps.

**Step 1.** *There exists a positive number  $A_{\mathbf{c}}$ , depending only on  $\mathbf{c}$ , such that*

$$\int_t^{t+1} \int_{\mathbb{R}} \left| \partial_x \Psi^*(x + a^*(t), s) \right|^2 e^{\nu_{\mathbf{c}}|x|} dx ds \leq A_{\mathbf{c}}, \quad (2.4.8)$$

for any  $t \in \mathbb{R}$ .

By (2.2.23) and (2.2.30),

$$|\Psi^*| \leq A_{\mathbf{c}} (|\partial_x v^*| + |w^*|). \quad (2.4.9)$$

In view of Proposition 2.2.4 and the fact that  $|a^*(t) - a^*(s)|$  is uniformly bounded for  $s \in [t-1, t+2]$  by (2.2.22), this yields

$$\int_{t-1}^{t+2} \int_{\mathbb{R}} \left| \Psi^*(x + a^*(t), s) \right|^2 e^{2\nu_{\mathbf{c}}|x|} dx ds \leq A_{\mathbf{c}}. \quad (2.4.10)$$

We denote

$$F^* := -\frac{1}{2}(v^*)^2 \Psi^* + \operatorname{Re} \left( \Psi^* (1 - 2F(v^*, \bar{\Psi}^*)) \right) (1 - 2F(v^*, \Psi^*)).$$

We recall that  $\|v^*\|_{L^\infty(\mathbb{R} \times \mathbb{R})} < 1 - \sigma_{\mathbf{c}}$  by (2.2.23). Using the Cauchy-Schwarz inequality, the Sobolev embedding theorem and the control of the norm in  $X(\mathbb{R})$  provided by the conservation of energy, we have  $F(v^*, \Psi^*) \in L^\infty(\mathbb{R} \times \mathbb{R})$ . Hence,

$$|F^*| \leq A_{\mathbf{c}} |\Psi^*|, \quad (2.4.11)$$

where  $A_{\mathbf{c}}$  is a positive number depending only on  $\mathbf{c}$ . Then, by (2.4.10),

$$\int_{t-1}^{t+2} \int_{\mathbb{R}} \left| F^*(x + a^*(t), s) \right|^2 e^{2\nu_{\mathbf{c}}|x|} dx ds \leq A_{\mathbf{c}}, \quad (2.4.12)$$

for any  $t \in \mathbb{R}$ . Next, by Proposition 2.2.5, we have

$$\left\| \Psi^* \right\|_{L^4([t-1, t+2], L^\infty)} \leq A_{\mathbf{c}}. \quad (2.4.13)$$

Indeed, we fix  $t \in \mathbb{R}$  and we denote  $(\Psi_1^0, v_1^0) := (\Psi^*(\cdot + a^*(t-1), t-1), v^*(\cdot + a^*(t-1), t-1))$  and  $(\Psi_1(s), v_1(s)) := (\Psi^*(\cdot + a^*(t-1), t-1+s), v^*(\cdot + a^*(t-1), t-1+s))$  the corresponding solution to (2.2.32)-(2.2.34). Denote also  $(\Psi_2^0, v_2^0) := (\Psi_{c^*(t-1)}, v_{c^*(t-1)})$  and  $(\Psi_2(s), v_2(s)) := (\Psi_{c^*(t-1)}(x - c^*(t-1)s), v_{c^*(t-1)}(x - c^*(t-1)s))$  the corresponding solution to (2.2.32)-(2.2.34), where  $\Psi_{c^*(t)}$  is the solution to (2.2.32) associated to the soliton  $Q_{c^*(t)}$ . We have, by (2.2.35),

$$\left\| \Psi_1(s) - \Psi_2(s) \right\|_{L^4([0, \tau_{\mathbf{c}}], L^\infty)} \leq A \left( \left\| v_1^0 - v_2^0 \right\|_{L^2} + \left\| \Psi_1^0 - \Psi_2^0 \right\|_{L^2} \right).$$

Using (2.2.21), we obtain

$$\left\| \Psi_1(s) - \Psi_2(s) \right\|_{L^4([0, \tau_{\mathbf{c}}], L^\infty)} \leq A_{\mathbf{c}},$$

where  $\tau_{\mathbf{c}} = \tau_{\mathbf{c}}(\|v_1^0\|_{L^2}, \|v_2^0\|_{L^2}, \|\Psi_1^0\|_{L^2}, \|\Psi_2^0\|_{L^2})$  depend only on  $\mathbf{c}$ . Since  $[0, 3] \subseteq \bigcup_{0 \leq k \leq 3/\tau_{\mathbf{c}}} [k\tau_{\mathbf{c}}, (k+1)\tau_{\mathbf{c}}]$ ,

we can infer (2.4.13) inductively .

In addition, by (2.4.9), we have

$$\left\| \Psi^*(\cdot + a^*(t), \cdot) \right\|_{L^\infty([t-1, t+2], L^2)} \leq A_{\mathbf{c}}. \quad (2.4.14)$$

Hence, applying the Cauchy-Schwarz inequality to the integral with respect to the time variable, (2.4.10), (2.4.13) and (2.4.14),

$$\begin{aligned}
& \int_{t-1}^{t+2} \int_{\mathbb{R}} \left| \Psi^*(x + a^*(t), s) \right|^4 e^{\nu_c |x|} dx ds \\
& \leq \int_{t-1}^{t+2} \int_{\mathbb{R}} \left| \Psi^*(x + a^*(t), s) \right|^2 e^{\nu_c |x|} dx \|\Psi^*(s)\|_{L^\infty(\mathbb{R})}^2 ds \\
& \leq \left\| \Psi^*(\cdot + a^*(t), \cdot) e^{\frac{\nu_c}{2} |\cdot|} \right\|_{L^4([t-1, t+2], L^2(\mathbb{R}))}^2 \left\| \Psi^*(\cdot + a^*(t), \cdot) \right\|_{L^4([t-1, t+2], L^\infty(\mathbb{R}))}^2 \\
& \leq \left\| \Psi^*(\cdot + a^*(t), \cdot) e^{\nu_c |\cdot|} \right\|_{L^2([t-1, t+2], L^2(\mathbb{R}))} \left\| \Psi^*(\cdot + a^*(t), \cdot) \right\|_{L^\infty([t-1, t+2], L^2(\mathbb{R}))} \\
& \quad \left\| \Psi^*(\cdot + a^*(t), \cdot) \right\|_{L^4([t-1, t+2], L^\infty(\mathbb{R}))}^2 \\
& \leq A_c.
\end{aligned} \tag{2.4.15}$$

In order to use Proposition 2.2.6 on  $\Psi^*$ , it is sufficient to verify

$$\sup_{s \in [t-1, t+2]} \int_{\mathbb{R}} \left| \Psi^*(x + a^*(t), s) \right|^2 e^{2\nu_c |x|} dx ds \leq A_c. \tag{2.4.16}$$

Indeed, using (2.4.16) and (2.4.13), we can write

$$\begin{aligned}
& \int_{t-1}^{t+2} \int_{\mathbb{R}} \left| \Psi^*(x + a^*(t), s) \right|^6 e^{2\nu_c |x|} dx ds \\
& \leq \left\| \Psi^*(\cdot + a^*(t), \cdot) e^{\nu_c |\cdot|} \right\|_{L^\infty([t-1, t+2], L^2(\mathbb{R}))}^2 \left\| \Psi^*(\cdot + a^*(t), \cdot) \right\|_{L^4([t-1, t+2], L^\infty(\mathbb{R}))}^4 \\
& \leq A_c,
\end{aligned} \tag{2.4.17}$$

which proves that  $\Psi^*$  satisfies the assumptions of Proposition 2.2.6. Then, we apply Proposition 2.2.6 with  $u := \Psi^*(\cdot + a^*(t), \cdot + (t + 1/2))$ ,  $T := 1/2$ ,  $F := |u|^2 u + F^*(\cdot, t + 1/2)$  and successively  $\lambda := \pm \nu_c$  and we use (2.4.10) and (2.4.12) to obtain (2.4.8).

Now let us prove (2.4.16). First, we recall the next lemma stated by Kenig, Ponce and Vega [32].

**Lemma 2.4.1.** *Let  $a \in [-2, -1]$  and  $b \in [2, 3]$ . Assume that  $u \in C^0([a, b] : L^2(\mathbb{R}))$  is a solution of the inhomogeneous Schrödinger equation*

$$i\partial_t u + \partial_{xx} u = H, \tag{2.4.18}$$

with  $H \in L^1([a, b] : L^2(e^{\beta x} dx))$ , for some  $\beta \in \mathbb{R}$ , and

$$u_a \equiv u(\cdot, a), \quad u_b \equiv u(\cdot, b) \in L^2(e^{\beta x} dx). \tag{2.4.19}$$

There exist a positive number  $K$  such that

$$\begin{aligned}
& \sup_{a \leq t \leq b} \|u(\cdot, t)\|_{L^2(e^{\beta x} dx)} \\
& \leq K(\|u_a\|_{L^2(e^{\beta x} dx)} + \|u_b\|_{L^2(e^{\beta x} dx)} + \|H\|_{L^1([a, b], L^2(e^{\beta x} dx))}).
\end{aligned} \tag{2.4.20}$$

In order to apply the lemma, we need to verify the existence of numbers  $a$  and  $b$  such that (2.4.19) holds for  $u := \Psi^*(\cdot + a^*(t), \cdot + t)$  and such that  $H := |u|^2 u + F^*(\cdot, \cdot + t) \in L^1([a, b], L^2(e^{\beta x} dx))$ , for  $\beta = \pm \nu_c$  respectively and any  $t \in \mathbb{R}$ . Our first claim is a consequence of (2.4.10) and the Markov inequality. Indeed, there exist  $s_0 \in [-2, -1]$  and  $s_1 \in [2, 3]$  such that

$$\int_{\mathbb{R}} \left| \Psi^*(x + a^*(t), s_j + t) \right|^2 e^{2\nu_c |x|} dx \leq A_c \quad \text{for } j = 0, 1.$$

For the second claim, by (2.4.12) and the Cauchy-Schwarz estimate, it is sufficient to show that  $|u|^2u \in L^1([-2, 3], L^2(e^{\nu_\epsilon|x|}dx))$ . To prove this we use the Cauchy-Schwarz inequality for the time variable, (2.4.10) and (2.4.13),

$$\begin{aligned} & \int_{-2}^3 \left( \int_{\mathbb{R}} |\Psi^*(x + a^*(t), s + t)|^6 e^{2\nu_\epsilon|x|} dx \right)^{\frac{1}{2}} ds \\ & \leq \left\| \Psi^*(\cdot + a^*(t), \cdot + t) e^{\nu_\epsilon|\cdot|} \right\|_{L^2([-2,3], L^2)} \left\| \Psi^*(\cdot + a^*(t), \cdot + t) \right\|_{L^4([-2,3], L^\infty)}^2 \\ & \leq A_\epsilon. \end{aligned}$$

Now, we are allowed to apply Lemma 2.4.1 with  $a = s_0$  and  $b = s_1$  to deduce (2.4.16). This finishes the proof of this first step.

In the next step, we prove that (2.4.8) remains true for all the derivatives of  $\Psi^*$  and  $v^*$ .

**Step 2.** Let  $k \geq 1$ . There exists a positive number  $A_{k,\epsilon}$ , depending only on  $k$  and  $\epsilon$ , such that

$$\int_t^{t+1} \int_{\mathbb{R}} \left| \partial_x^k \Psi^*(x + a^*(t), s) \right|^2 e^{\nu_\epsilon|x|} dx ds \leq A_{k,\epsilon}, \quad (2.4.21)$$

and

$$\int_t^{t+1} \int_{\mathbb{R}} \left| \partial_x^k v^*(x + a^*(t), s) \right|^2 e^{\nu_\epsilon|x|} dx \leq A_{k,\epsilon}, \quad (2.4.22)$$

for any  $t \in \mathbb{R}$ .

The proof of Step 2 is by induction on  $k \geq 1$ . We are going to differentiate (2.2.32)  $k$  times with respect to the space variable and write the resulting equation as

$$i\partial_t \left( \partial_x^k \Psi^* \right) + \partial_{xx} \left( \partial_x^k \Psi^* \right) = R_k(v^*, \Psi^*). \quad (2.4.23)$$

where  $R_k(v^*, \Psi^*) = \partial_x^k (|\Psi^*|^2 \Psi^*) + \partial_x^k F^*$ . We are going to prove by induction that (2.4.21), (2.4.22) and

$$\int_t^{t+1} \int_{\mathbb{R}} \left| R_k(v^*, \Psi^*)(x + a^*(t), s) \right|^2 e^{\nu_\epsilon|x|} dx ds \leq A_{k,\epsilon}, \quad (2.4.24)$$

hold simultaneously for any  $t \in \mathbb{R}$ . Notice that (2.4.21) implies that  $\partial_x^k \Psi^* \in L_{\text{loc}}^2(\mathbb{R}, L^2(\mathbb{R}))$ , while (2.4.24) implies that  $R_k(v^*, \Psi^*) \in L_{\text{loc}}^2(\mathbb{R}, L^2(\mathbb{R}))$ . Therefore, if (2.4.21), (2.4.22) and (2.4.24) are established for some  $k \geq 1$ , then applying Proposition 2.2.6 to  $\partial_x^k \Psi^*$  can be justified by a standard approximation procedure.

For  $k = 1$ , (2.4.21) is exactly (2.4.8). (2.4.22) holds from Proposition 2.2.4 and the fact that  $|a^*(t) - a^*(s)|$  is uniformly bounded for  $s \in [t - 1, t + 2]$ . Next, we write

$$\begin{aligned} R_1(v^*, \Psi^*) &= -v^* \partial_x v^* \Psi^* - \frac{1}{2} (v^*)^2 \partial_x \Psi^* + \text{Re} \left( \partial_x \Psi^* \left( 1 - 2F(v^*, \bar{\Psi}^*) \right) \right) \left( 1 - 2F(v^*, \Psi^*) \right) \\ &\quad - 2v^* |\Psi^*|^2 \left( 1 - 2F(v^*, \Psi^*) \right) - 2v^* \Psi^* \text{Re} \left( \Psi^* \left( 1 - 2F(v^*, \bar{\Psi}^*) \right) - 2\partial_x (\Psi^* |\Psi^*|^2) \right). \end{aligned}$$

We will show that

$$\Psi^* \in L^\infty([t - 1, t + 2], L^\infty(\mathbb{R})), \quad (2.4.25)$$

in order to control the derivative of the cubic non-linearity by  $|\partial_x \Psi^*|$  and then we will use the fact that  $F(v^*, \Psi^*) \in L^\infty(\mathbb{R} \times \mathbb{R})$ ,  $\|v^*\|_{L^\infty(\mathbb{R} \times \mathbb{R})} < 1$  and the second equation in (2.2.34) to get

$$R_1(v^*, \Psi^*) \leq K \left( |\partial_x \Psi^*| + |\partial_x v^*| |\Psi^*| + |\Psi^*|^2 \right). \quad (2.4.26)$$

Let us prove (2.4.25). We define the function  $H$  on  $\mathbb{R}$  by

$$H(s) := \frac{1}{2} \int_{\mathbb{R}} (|\partial_x \Psi^*(x, s)|^2 - |\Psi^*(x, s)|^4) dx.$$

We differentiate it with respect to  $s$ , integrate by part and use (2.2.32) to obtain

$$\begin{aligned} H'(s) &= -\operatorname{Re} \left( \int_{\mathbb{R}} \partial_s \Psi^*(x, s) \left[ \overline{\partial_{xx} \Psi^* + 2\Psi^* |\Psi^*|^2} \right] (x, s) dx \right) \\ &= \operatorname{Re} \left( \int_{\mathbb{R}} \partial_s \Psi^*(x, s) F^*(x, s) dx \right) \\ &\leq \|\partial_s \Psi^*(s)\|_{H^{-1}(\mathbb{R})} \|F^*(s)\|_{H^1(\mathbb{R})}. \end{aligned} \quad (2.4.27)$$

We have

$$|\partial_x F^*| \leq K \left( |\partial_x \Psi^*| + |\partial_x v^*| |\Psi^*| + |\Psi^*|^2 \right),$$

using the fact that  $F(v^*, \Psi^*) \in L^\infty(\mathbb{R} \times \mathbb{R})$ ,  $\|v^*\|_{L^\infty(\mathbb{R} \times \mathbb{R})} < 1$  and the second equation in (2.2.34).

Hence, by (2.4.8), (2.4.10), (2.4.15), and the fact that  $|\partial_x v^*| \leq |\Psi^*|$  on  $\mathbb{R} \times \mathbb{R}$ , we obtain

$$\|\partial_x F^*\|_{L^2([t-1, t+2], L^2(\mathbb{R}))} \leq A_c. \quad (2.4.28)$$

On the other hand, we infer from (2.2.32), (2.4.8), (2.4.12) and the fact that  $\Psi^* \in L^4([t-1, t+2], L^\infty(\mathbb{R})) \cap L^8([t-1, t+2], L^4(\mathbb{R}))$ , that

$$\|\partial_s \Psi^*\|_{L^2([t-1, t+2], H^{-1}(\mathbb{R}))} \leq A_c. \quad (2.4.29)$$

Next, we integrate (2.4.27) between  $t-1$  and  $t+2$  and we apply the Cauchy-Schwarz inequality to obtain  $H \in W^{1,1}([t-1, t+2])$  for all  $t \in \mathbb{R}$  using (2.4.28) and (2.4.29). Notice that all these computations can be justified by a standard approximation procedure. This yields, by the Sobolev embedding theorem, that  $H \in L^\infty([t-1, t+2])$ . We conclude that the derivative  $\partial_x \Psi^* \in L^\infty([t-1, t+2], L^2(\mathbb{R}))$ . Indeed, we can use the Gagliardo-Nirenberg inequality and the fact that  $\Psi^*$  is uniformly bounded in  $L^2(\mathbb{R})$  by a positive number to write

$$\begin{aligned} H(s) &\geq \frac{1}{2} \int_{\mathbb{R}} |\partial_x \Psi^*(x, s)|^2 dx - A \|\Psi^*(s)\|_{L^2(\mathbb{R})}^3 \|\partial_x \Psi^*(\cdot)\|_{L^2(\mathbb{R})} \\ &\geq \frac{1}{2} \int_{\mathbb{R}} |\partial_x \Psi^*(x, s)|^2 dx - AK^3 \|\partial_x \Psi^*(\cdot)\|_{L^2(\mathbb{R})}. \end{aligned}$$

The function  $x \mapsto \frac{1}{2}x^2 - AM^3x$  diverges to  $+\infty$  when  $x$  goes to  $+\infty$ . Since  $H$  is bounded, we infer that  $\|\partial_x \Psi^*(\cdot)\|_{L^2(\mathbb{R})}$  is uniformly bounded on  $[t-1, t+2]$  for all  $t \in \mathbb{R}$ . This finishes the proof of (2.4.25) by the Sobolev embedding theorem. Then, by (2.4.26), (2.4.24) for  $k=1$  is a consequence of (2.4.8), (2.4.15) and the fact that  $|\partial_x v^*| \leq |\Psi^*|$  on  $\mathbb{R} \times \mathbb{R}$ .

Assume now that (2.4.21), (2.4.22) and (2.4.24) are satisfied for any integer  $1 \leq k \leq k_0$  and any  $t \in \mathbb{R}$ . Let us prove these three estimates for  $k = k_0 + 1$ . We apply Proposition 2.2.6 with  $u := \partial_x^{k_0} \Psi^*(\cdot + a^*(t), \cdot + (t+1/2))$ ,  $T := 1/2$  and successively  $\lambda := \pm \nu_c$ . In view of (2.4.21), (2.4.23), (2.4.24), and the fact that  $|a^*(t) - a^*(s)|$  is uniformly bounded for  $s \in [t-1, t+2]$ , this yields

$$\int_t^{t+1} \int_{\mathbb{R}} |\partial_x^{k_0+1} \Psi^*(x + a^*(t), s)|^2 e^{\nu_c |x|} dx ds \leq A_c, \quad (2.4.30)$$

so that (2.4.21) is satisfied for  $k = k_0 + 1$ .

Let  $k \in \{1, \dots, k_0\}$ . We use the induction hypothesis and (2.4.30) to infer that

$$\partial_x^{k-1} \Psi^* \in L^2([t, t+1], H^2(\mathbb{R})).$$

Also, we have

$$\partial_x^{k-1}\Psi^* \in H^1([t, t+1], L^2(\mathbb{R}))$$

using (2.4.23) and (2.4.24). This yields, by interpolation,

$$\partial_x^{k-1}\Psi^* \in H^{\frac{2}{3}}([t, t+1], H^{\frac{2}{3}}(\mathbb{R})).$$

Hence, using the Sobolev embedding theorem, we obtain

$$\partial_x^{k-1}\Psi^* \in L^\infty([t, t+1], L^\infty(\mathbb{R})) \quad \text{for all } t \in \mathbb{R}. \quad (2.4.31)$$

On the other hand, since  $|\partial_x v^*| \leq |\Psi^*|$ , we have, by (2.4.25),  $\partial_x v^* \in L^\infty([t, t+1], L^\infty(\mathbb{R}))$ . For  $k \in \{2, \dots, k_0\}$ , we differentiate the second equation in (2.2.34)  $k$  times and we use (2.4.31) to obtain

$$|\partial_x^k v^*| \leq K \left( \sum_{j=1}^{k-1} |\partial_x^j \Psi^*| + \sum_{j=0}^{k-2} |\partial_x^j v^*| \right), \quad (2.4.32)$$

where  $K$  is a positive constant. We infer from (2.4.31) by induction that

$$\partial_x^k v^* \in L^\infty([t, t+1], L^\infty(\mathbb{R})) \quad \text{for all } t \in \mathbb{R}, \quad (2.4.33)$$

for all  $k \in \{2, \dots, k_0\}$ . Then, we just compute explicitly  $R_{k_0+1}(v^*, \Psi^*)$  and we use (2.4.31) and (2.4.33) to obtain

$$|R_{k_0+1}(v^*, \Psi^*)| \leq A_{k_0+1, c, K} \left( \sum_{j=0}^{k_0+1} |\partial_x^j \Psi^*| + \sum_{j=1}^{k_0} |\partial_x^j v^*| \right).$$

Hence, by (2.4.21) for all  $k \leq k_0$ , (2.4.22) and (2.4.30), we obtain (2.4.24) for  $k = k_0 + 1$ . Finally, we introduce (2.4.21) for all  $k \leq k_0 + 1$  and (2.4.22) for all  $k \leq k_0$  into (2.4.32) to deduce (2.4.22) for  $k = k_0 + 1$ . This finishes the proof of this step.

In order to finish the proof of Proposition 2.2.7, we now turn these  $L^2_{\text{loc}}$  in time estimates into  $L^\infty$  in time estimates, and then into uniform estimates.

**Step 3.** Let  $k \geq 0$ . There exists a positive number  $A_{k, c}$ , depending only on  $k$  and  $c$ , such that

$$\int_{\mathbb{R}} \left| \partial_x^k \Psi^*(x + a^*(t), t) \right|^2 e^{\nu_c |x|} dx \leq A_{k, c}, \quad (2.4.34)$$

for any  $t \in \mathbb{R}$ . In particular, we have

$$\left\| \partial_x^k \Psi^*(\cdot + a^*(t), t) e^{\frac{\nu_c}{2} |\cdot|} \right\|_{L^\infty(\mathbb{R})} \leq A_{k, c}, \quad (2.4.35)$$

for any  $t \in \mathbb{R}$ , and a further positive constant  $A_{k, c}$ , depending only on  $k$  and  $c$ .

Here, we use the Sobolev embedding theorem in time and (2.4.23) for the proof. By the Sobolev embedding theorem, we have

$$\begin{aligned} \left\| \partial_x^k \Psi^*(\cdot + a^*(t), t) e^{\frac{\nu_c}{2} |\cdot|} \right\|_{L^2(\mathbb{R})}^2 &\leq K \left( \left\| \partial_s \left( \partial_x^k \Psi^*(\cdot + a^*(t), s) e^{\frac{\nu_c}{2} |\cdot|} \right) \right\|_{L^2([t-1, t+1], L^2(\mathbb{R}))}^2 \right. \\ &\quad \left. + \left\| \partial_x^k \Psi^*(\cdot + a^*(t), s) e^{\frac{\nu_c}{2} |\cdot|} \right\|_{L^2([t-1, t+1], L^2(\mathbb{R}))}^2 \right), \end{aligned}$$

while, by (2.4.23),

$$\begin{aligned} \left\| \partial_s \left( \partial_x^k \Psi^*(\cdot + a^*(t), s) e^{\frac{\nu_c}{2} |\cdot|} \right) \right\|_{L^2([t-1, t+1], L^2(\mathbb{R}))}^2 &\leq 2 \left( \left\| \partial_x^{k+2} \Psi^*(\cdot + a^*(t), s) e^{\frac{\nu_c}{2} |\cdot|} \right\|_{L^2([t-1, t+1], L^2(\mathbb{R}))}^2 \right. \\ &\quad \left. + \left\| R_k(\Psi^*)(\cdot + a^*(t), s) e^{\frac{\nu_c}{2} |\cdot|} \right\|_{L^2([t-1, t+1], L^2(\mathbb{R}))}^2 \right), \end{aligned}$$

so that we finally deduce (2.4.34) from (2.4.21) and (2.4.24). Estimate (2.4.35) follows from applying the Sobolev embedding theorem to (2.4.34).

Similarly, the function  $v^*$  satisfies

**Step 4.** Let  $k \in \mathbb{N}$ . There exists a positive number  $A_{k,c}$ , depending only on  $k$  and  $\mathbf{c}$ , such that

$$\int_{\mathbb{R}} \left( \partial_x^k v^*(x + a^*(t), t) \right)^2 e^{\nu_c |x|} dx \leq A_{k,c}, \quad (2.4.36)$$

and

$$\left\| \partial_x^k v^*(\cdot + a^*(t), t) e^{\frac{\nu_c}{2} |\cdot|} \right\|_{L^\infty(\mathbb{R})} \leq A_{k,c}, \quad (2.4.37)$$

for any  $t \in \mathbb{R}$ .

The proof is similar to the proof of Step 3 using the first equation in (2.2.34) instead of (2.2.32). We use the Sobolev embedding theorem to write

$$\begin{aligned} \left\| \partial_x^k v^*(\cdot + a^*(t), t) e^{\nu_c |\cdot|} \right\|_{L^2(\mathbb{R})}^2 &\leq K \left( \left\| \partial_s \left( \partial_x^k v^*(\cdot + a^*(t), s) e^{\nu_c |\cdot|} \right) \right\|_{L^2([t-1, t+1], L^2(\mathbb{R}))}^2 \right. \\ &\quad \left. + \left\| \partial_x^k v^*(\cdot + a^*(t), s) e^{\nu_c |\cdot|} \right\|_{L^2([t-1, t+1], L^2(\mathbb{R}))}^2 \right). \end{aligned}$$

By the first equation in (2.2.34), (2.4.21), (2.4.23) and (2.4.33), we have

$$\left\| \partial_s \left( \partial_x^k v^*(\cdot + a^*(t), s) e^{\nu_c |\cdot|} \right) \right\|_{L^2([t-1, t+1], L^2(\mathbb{R}))}^2 \leq A_{\mathbf{c}}.$$

This leads to (2.4.36). The uniform bound in (2.4.37) is then a consequence of the Sobolev embedding theorem.

Finally, we provide the estimates for the function  $w^*$ .

**Step 5.** Let  $k \in \mathbb{N}$ . There exists a positive number  $A_{k,c}$ , depending only on  $k$  and  $\mathbf{c}$ , such that

$$\int_{\mathbb{R}} \left| \partial_x^k w^*(x + a^*(t), t) \right|^2 e^{\nu_c |x|} dx \leq A_{k,c}, \quad (2.4.38)$$

and

$$\left\| \partial_x^k w^*(\cdot + a^*(t), t) e^{\frac{\nu_c}{2} |\cdot|} \right\|_{L^\infty(\mathbb{R})} \leq A_{k,c}, \quad (2.4.39)$$

for any  $t \in \mathbb{R}$ .

The proof relies on the last two steps. First, we write

$$v^* \Psi^* = -\frac{1}{2} \partial_x \left( (1 - (v^*)^2)^{\frac{1}{2}} \exp i\theta^* \right).$$

Since  $(1 - v^*(x, t)^2)^{1/2} \exp i\theta^*(x, t) \rightarrow 1$  as  $x \rightarrow -\infty$  for any  $t \in \mathbb{R}$ , we obtain the formula

$$2F(v^*, \Psi^*) = 1 - (1 - (v^*)^2)^{\frac{1}{2}} \exp i\theta^*. \quad (2.4.40)$$

Hence, using (2.2.30), we have

$$w^* = 2 \operatorname{Im} \left( \frac{\Psi^* (1 - 2F(v^*, \Psi^*))}{1 - (v^*)^2} \right). \quad (2.4.41)$$

Combining (2.2.7) and (2.4.40), we recall that

$$\frac{|1 - 2F(v^*, \Psi^*)|}{1 - (v^*)^2} \leq A_{\mathbf{c}}. \quad (2.4.42)$$

Hence, we obtain

$$|w^*| \leq A_{\mathbf{c}} |\Psi^*|.$$

Then, (2.4.38) and (2.4.39) follow from (2.4.34) and (2.4.35) for  $k = 0$ . For  $k \geq 1$ , we differentiate (2.4.41)  $k$  times with respect to the space variable, and using (2.4.35), (2.4.37) and (2.4.42), we are led to

$$|\partial_x^k w^*| \leq A_{k,c} \left( \sum_{j=0}^k |\partial_x^j \Psi^*| + \sum_{j=1}^{k-1} |\partial_x^j v^*| \right).$$

We finish the proof of this step using Steps 3 and 4. This achieves the proof of Proposition 2.2.7.  $\square$

## 2.5 Proof of the Liouville theorem

### 2.5.1 Proof of Proposition 2.2.8

First, by (2.2.38) and the explicit formula for  $v_c$  and  $w_c$  in (2.2.3), there exists a positive number  $A_{k,c}$  such that

$$\int_{\mathbb{R}} \left( \left( \partial_x^k \varepsilon_v^*(x, t) \right)^2 + \left( \partial_x^k \varepsilon_w^*(x, t) \right)^2 \right) e^{\nu_c |x|} dx \leq A_{k,c}, \quad (2.5.1)$$

for any  $k \in \mathbb{N}$  and any  $t \in \mathbb{R}$ . In view of the formulae of  $\mathcal{H}_c$  in (2.6.42) and for  $u^*$  in (2.2.41), a similar estimate holds for  $u^*$ , for a further choice of the constant  $A_{k,c}$ . As a consequence, we are allowed to differentiate with respect to the time variable the quantity

$$\mathcal{I}^*(t) := \int_{\mathbb{R}} x u_1^*(x, t) u_2^*(x, t) dx,$$

in the left-hand side of (2.2.43). Moreover, we can compute

$$\begin{aligned} \frac{d}{dt} (\mathcal{I}^*) &= -2 \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(\partial_x u^*), u^* \rangle_{\mathbb{R}^2} + \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(J\mathcal{R}_{c^*} \varepsilon^*), u^* \rangle_{\mathbb{R}^2} \\ &\quad - (c^*)' \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(\partial_c Q_{c^*}), u^* \rangle_{\mathbb{R}^2} + (c^*)' \int_{\mathbb{R}} \mu \langle \partial_c \mathcal{H}_{c^*}(\varepsilon^*), u^* \rangle_{\mathbb{R}^2} \\ &\quad + ((a^*)' - c^*) \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(\partial_x \varepsilon^*), u^* \rangle_{\mathbb{R}^2}, \end{aligned} \quad (2.5.2)$$

where we have set  $\mu(x) = x$  for any  $x \in \mathbb{R}$ .

At this stage, we split the proof into five steps. The proof of these steps is similar to the proof of Proposition 7 in [6]. We first show

**Step 1.** *There exist two positive numbers  $A_1$  and  $R_1$ , depending only on  $\mathbf{c}$ , such that*

$$\mathcal{I}_1^*(t) := -2 \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(\partial_x u^*), u^* \rangle_{\mathbb{R}^2} \geq \frac{1 - \mathbf{c}^2}{8} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 - A_1 \|u^*(\cdot, t)\|_{X(B(0, R_1))}^2, \quad (2.5.3)$$

for any  $t \in \mathbb{R}$ .

We introduce the explicit formulae of the operator  $\mathcal{H}_{c^*}$  in the definition of  $\mathcal{I}_1^*(t)$  to obtain

$$\begin{aligned} \mathcal{I}_1^*(t) &= 2 \int_{\mathbb{R}} \mu \partial_x \left( \frac{\partial_{xx} u_1^*}{1 - v_{c^*}^2} \right) u_1^* - 2 \int_{\mathbb{R}} \mu \left( 1 - (c^*)^2 - (5 + (c^*)^2) v_{c^*}^2 + 2v_{c^*}^4 \right) \frac{\partial_x u_1^*}{(1 - v_{c^*}^2)^2} u_1^* \\ &\quad + 2 \int_{\mathbb{R}} \mu c^* \frac{1 + v_{c^*}^2}{1 - v_{c^*}^2} (\partial_x u_2^*) u_1^* - 2 \int_{\mathbb{R}} \mu (c^*)^2 \frac{(1 + v_{c^*}^2)^2}{(1 - v_{c^*}^2)^3} (\partial_x u_1^*) u_1^* \\ &\quad + 2 \int_{\mathbb{R}} \mu c^* \frac{1 + v_{c^*}^2}{1 - v_{c^*}^2} (\partial_x u_1^*) u_2^* - 2 \int_{\mathbb{R}} \mu (1 - v_{c^*}^2) (\partial_x u_2^*) u_2^*. \end{aligned}$$

Integrating by parts each term, we obtain

$$\mathcal{I}_1^*(t) = \int_{\mathbb{R}} \iota_1^*(x, t) dx,$$

with

$$\begin{aligned} \iota_1^* &= \left( \frac{2}{1 - v_{c^*}^2} + 2x \frac{\partial_x v_{c^*} v_{c^*}}{1 - v_{c^*}^2} \right) (\partial_x u_1^*)^2 - 2c^* \left( \frac{1 + v_{c^*}^2}{1 - v_{c^*}^2} + \frac{4x \partial_x v_{c^*} v_{c^*}}{(1 - v_{c^*}^2)^2} \right) u_2^* u_1^* \\ &\quad + \left( 1 - v_{c^*}^2 - 2x \partial_x v_{c^*} v_{c^*} \right) (u_2^*)^2 + \frac{1 + 2((c^*)^2 - 3)v_{c^*}^2 + (2(c^*)^2 - 3)v_{c^*}^4 - 2v_{c^*}^6}{(1 - v_{c^*}^2)^3} (u_1^*)^2 \\ &\quad + 4x \partial_x v_{c^*} v_{c^*} \frac{((c^*)^2 - 3) + (2(c^*)^2 - 3)v_{c^*}^2 - 3v_{c^*}^4}{(1 - v_{c^*}^2)^4} (u_1^*)^2. \end{aligned}$$



Let  $\delta$  be a small positive number. We next use the exponential decay of the function  $v_c$  and its derivatives to guarantee the existence of a radius  $R$ , depending only on  $\mathbf{c}$  and  $\delta$  (in view of the bound on  $c^* - \mathbf{c}$  in (2.2.21)), such that

$$\iota_1^*(x, t) \geq (2 - \delta) (\partial_x u_1^*)^2(x, t) + \left( \frac{1 - \mathbf{c}^2}{4} - \delta \right) \left( (u_1^*)^2(x, t) + (u_2^*)^2(x, t) \right),$$

when  $|x| \geq R$ .

Then, we choose  $\delta$  small enough and fix the number  $R_1$  according to the value of the corresponding  $R$ , to obtain

$$\int_{|x| \geq R_1} \iota_1^*(x, t) dx \geq \frac{1 - \mathbf{c}^2}{8} \int_{|x| \geq R_1} \left( (\partial_x u_1^*(x, t))^2 + u_1^*(x, t)^2 + u_2^*(x, t)^2 \right) dx. \quad (2.5.4)$$

On the other hand, it follows from (2.2.3), and again (2.2.8), that

$$\int_{|x| \leq R_1} \iota_1^*(x, t) dx \geq \left( \frac{1 - \mathbf{c}^2}{8} - A_1 \right) \int_{|x| \leq R_1} \left( (\partial_x u_1^*(x, t))^2 + u_1^*(x, t)^2 + u_2^*(x, t)^2 \right) dx,$$

for a positive number  $A_1$  depending only on  $\mathbf{c}$ . Combining with (2.5.4), we obtain (2.5.3).

**Step 2.** *There exist two positive numbers  $A_2$  and  $R_2$ , depending only on  $\mathbf{c}$ , such that*

$$|\mathcal{I}_2^*(t)| := \left| \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(\mathcal{J}\mathcal{R}_{c^*}\varepsilon^*), u^* \rangle_{\mathbb{R}^2} \right| \leq \frac{1 - \mathbf{c}^2}{64} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 + A_2 \|u^*(\cdot, t)\|_{X(B(0, R_2))}^2, \quad (2.5.5)$$

for any  $t \in \mathbb{R}$ .

We refer to the proof of Step 2 in the proof of Proposition 7 in [6] for more details.

Next, we infer from (2.2.9), (2.2.57), the explicit formula of  $\mathcal{H}_{c^*}$  in (2.6.42) and the exponential decay of the function  $\partial_c Q_{c^*}$  and its derivatives, that

**Step 3.** *There exist two positive numbers  $A_3$  and  $R_3$ , depending only on  $\mathbf{c}$ , such that*

$$|\mathcal{I}_4^*(t)| := \left| (c^*)' \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(\partial_c Q_{c^*}), u^* \rangle_{\mathbb{R}^2} \right| \leq \frac{1 - \mathbf{c}^2}{64} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 + A_3 \|u^*(\cdot, t)\|_{X(B(0, R_3))}^2, \quad (2.5.6)$$

for any  $t \in \mathbb{R}$ .

We decompose the real line into two regions  $[-R, R]$  and its complement for any  $R > 0$ . We use the fact that  $|x| \leq e^{\frac{\nu_c |x|}{4}}$  for all  $|x| \geq R$ , to write

$$\begin{aligned} |\mathcal{I}_4^*(t)| &\leq R |(c^*)'(t)| \int_{|x| \leq R} \left| \mathcal{H}_{c^*(t)}(\partial_c Q_{c^*(t)})(x) \right| |u^*(x, t)| dx \\ &\quad + \delta |(c^*)'(t)| \int_{|x| \geq R} \left| \mathcal{H}_{c^*(t)}(\partial_c Q_{c^*(t)})(x) \right| |u^*(x, t)| e^{\frac{\nu_c |x|}{4}} dx, \end{aligned}$$

for any  $t \in \mathbb{R}$ . We deduce from (2.2.9), the explicit formula of  $\mathcal{H}_{c^*}$  in (2.6.42) and the exponential decay of the function  $\partial_c Q_{c^*}$  and its derivatives that

$$|\mathcal{I}_4^*(t)| \leq A_c \left( R \|u^*(\cdot, t)\|_{X(B(0, R))} + \delta \|u^*(\cdot, t)\|_{X(\mathbb{R})} \right) \|\varepsilon^*(\cdot, t)\|_{L^2(\mathbb{R})^2},$$

for any  $t \in \mathbb{R}$ . Hence, by (2.2.57),

$$|\mathcal{I}_4^*(t)| \leq A_c \left( \frac{R^2}{\delta} \|u^*(\cdot, t)\|_{X(B(0, R))}^2 + 2\delta \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 \right).$$

We choose  $\delta$  so that  $2A_c \delta \leq (1 - \mathbf{c}^2)/64$ , and we denote by  $R_4$  the corresponding number  $R$ , we obtain (2.5.6), with  $A_4 = A_c R_4^2 / \delta$ .

Similarly, we use (2.2.9), (2.2.21) and (2.2.57) to obtain

**Step 4.** *There exists two positive numbers  $A_4$  and  $R_4$ , depending only on  $\mathbf{c}$ , such that*

$$|\mathcal{I}_3^*(t)| := \left| (c^*)' \int_{\mathbb{R}} \mu \langle \partial_c \mathcal{H}_{c^*}(\varepsilon^*), u^* \rangle_{\mathbb{R}^2} \right| \leq \frac{1 - \mathbf{c}^2}{64} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 + A_4 \|u^*(\cdot, t)\|_{X(B(0, R_4))}^2, \quad (2.5.7)$$

for any  $t \in \mathbb{R}$ .

We argue as in Steps 3 to show

**Step 5.** *There exist two positive numbers  $A_5$  and  $R_5$ , depending only on  $\mathbf{c}$ , such that*

$$\begin{aligned} |\mathcal{I}_5^*(t)| &:= \left| ((a^*)' - c^*) \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(\partial_x \varepsilon^*), u^* \rangle_{\mathbb{R}^2} \right| \\ &\leq \frac{1 - \mathbf{c}^2}{64} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 + A_5 \|u^*(\cdot, t)\|_{X(B(0, R_5))}^2, \end{aligned} \quad (2.5.8)$$

for any  $t \in \mathbb{R}$ .

Finally, combining the estimates in Steps 1 to 5 with the identity (2.5.2), we obtain

$$\frac{d}{dt} \left( \mathcal{I}^*(t) \right) \geq \frac{1 - \mathbf{c}^2}{16} \|u^*(\cdot, t)\|_{X(\mathbb{R})}^2 - (A_1 + A_2 + A_3 + A_4 + A_5) \|u^*(\cdot, t)\|_{X(B(0, R_*))}^2,$$

this allow us to conclude the proof of (2.2.43) with  $R_* = \max\{R_1, R_2, R_3, R_4, R_5\}$  and  $A_* = A_1 + A_2 + A_3 + A_4 + A_5$ .

## 2.5.2 Proof of Lemma 2.2.1

When  $u \in H^3(\mathbb{R}) \times H^1(\mathbb{R})$ , the function  $\partial_x u$  is in the space  $H^2(\mathbb{R}) \times L^2(\mathbb{R})$  which is the domain of  $\mathcal{H}_c$ . The scalar product in the right-hand side of (2.2.46) is well-defined in view of (2.2.45). Next, we use the formula for  $\mathcal{H}_c$  in (2.6.42) to express  $G_c(u)$  as

$$\begin{aligned} &\left\langle SM_c u, \mathcal{H}_c(-2\partial_x u) \right\rangle_{L^2(\mathbb{R})^2} \\ &= 2 \int_{\mathbb{R}} \frac{\partial_x v_c}{v_c} \left( \frac{1 - c^2 - (5 + c^2)v_c^2 + 2v_c^4}{(1 - v_c^2)^2} + c^2 \frac{(1 + v_c^2)^2}{(1 - v_c^2)^3} - 2c^2 \frac{v_c^2(1 + v_c^2)}{(1 - v_c^2)^3} \right) u_1 \partial_x u_1 \\ &\quad - 2 \int_{\mathbb{R}} \frac{\partial_x v_c}{v_c} \partial_x \left( \frac{\partial_{xx} u_1}{1 - v_c^2} \right) + 2 \int_{\mathbb{R}} \frac{\partial_x v_c (1 - v_c^2)}{v_c} u_2 \partial_x u_2 \\ &\quad + 2c \int_{\mathbb{R}} \left( 2 \frac{v_c \partial_x v_c}{1 - v_c^2} u_1 \partial_x u_2 - \frac{\partial_x v_c (1 + v_c^2)}{v_c (1 - v_c^2)} \partial_x (u_1 u_2) \right). \end{aligned} \quad (2.5.9)$$

We recall that  $v_c$  solves the equation

$$\partial_{xx} v_c = (1 - c^2 - 2v_c^2)v_c, \quad (2.5.10)$$

which leads to

$$(\partial_x v_c)^2 = (1 - c^2 - v_c^2)v_c^2, \quad \text{and} \quad \partial_x \left( \frac{\partial_x v_c}{v_c} \right) = -v_c^2. \quad (2.5.11)$$

Then, the third integral in the right-hand side of (2.5.9) can be written as

$$2 \int_{\mathbb{R}} \frac{\partial_x v_c (1 - v_c^2)}{v_c} u_2 \partial_x u_2 = \int_{\mathbb{R}} \mu_c u_2^2, \quad (2.5.12)$$

with  $\mu_c := 2(\partial_x v_c)^2 + (1 - v_c^2)v_c^2$ . Similarly, the last integral is given by

$$\int_{\mathbb{R}} \left( 2 \frac{v_c \partial_x v_c}{1 - v_c^2} u_1 \partial_x u_2 - \frac{\partial_x v_c (1 + v_c^2)}{v_c (1 - v_c^2)} \partial_x (u_1 u_2) \right) = - \int_{\mathbb{R}} \left( v_c^2 u_1 u_2 + 2 \frac{v_c \partial_x v_c}{1 - v_c^2} u_2 \partial_x u_1 \right). \quad (2.5.13)$$

Combining (2.5.12) and (2.5.13) with (2.5.9), we obtain the identity

$$\left\langle SM_c u, \mathcal{H}_c(-2\partial_x u) \right\rangle_{L^2(\mathbb{R})^2} = I + \int_{\mathbb{R}} \mu_c \left( u_2 - \frac{cv_c^2}{\mu_c} u_1 - \frac{2cv_c \partial_x v_c}{\mu_c (1 - v_c^2)} \partial_x u_1 \right)^2,$$

where

$$\begin{aligned} I = & \int_{\mathbb{R}} 2 \left( \frac{\partial_x v_c}{v_c} \left( \frac{1 - c^2 - (5 + c^2)v_c^2 + 2v_c^4}{(1 - v_c^2)^2} + c^2 \frac{1 + v_c^2}{(1 - v_c^2)^2} \right) - 2c^2 \frac{v_c^3 \partial_x v_c}{\mu_c (1 - v_c^2)} \right) u_1 \partial_x u_1 \\ & - \int_{\mathbb{R}} \frac{\partial_x v_c}{v_c} u_1 \partial_x \left( \frac{\partial_{xx} u_1}{1 - v_c^2} \right) - c^2 \int_{\mathbb{R}} \frac{v_c^4}{\mu_c} u_1^2 - 4c^2 \int_{\mathbb{R}} \frac{(\partial_x v_c)^2 v_c^2}{\mu_c (1 - v_c^2)^2} (\partial_x u_1)^2. \end{aligned}$$

Using (2.5.10) and (2.5.11), we finally deduce that

$$I = \frac{3}{2} \int_{\mathbb{R}} \frac{v_c^4}{\mu_c} \left( \partial_x u_1 - \frac{\partial_x v_c}{v_c} u_1 \right)^2,$$

which finishes the proof of (2.2.46).

### 2.5.3 Proof of Proposition 2.2.9

We first rely on (2.2.3) and (2.2.46) to check that the quadratic form  $G_c$  is well-defined and continuous on  $X(\mathbb{R})$ . Next, setting

$$v = (v_c u_1, v_c u_2), \quad (2.5.14)$$

and using (2.5.10), we can express it as

$$G_c(u) = K_c(v) := \int_{\mathbb{R}} \frac{v_c^2}{\mu_c} \left( \partial_x v_1 - \frac{2\partial_x v_c}{v_c} v_1 \right)^2 + \int_{\mathbb{R}} \frac{\mu_c}{v_c^2} \left( v_2 + \frac{c\lambda_c}{\mu_c (1 - v_c^2)} v_1 - 2 \frac{cv_c \partial_x v_c}{\mu_c (1 - v_c^2)} \partial_x v_1 \right)^2, \quad (2.5.15)$$

where we have set  $\lambda_c := -\mu_c + 4(\partial_x v_c)^2$ . From (2.2.48) and (2.5.14) we deduce that

$$\text{Ker}(K_c) = \text{Span}(v_c Q_c). \quad (2.5.16)$$

Let  $w$  be the pair defined in the following way

$$w = \left( v_1, v_2 - 2 \frac{cv_c \partial_x v_c}{\mu_c (1 - v_c^2)} \partial_x v_1 \right).$$

We compute

$$K_c(v) = \left\langle \mathcal{T}_c(w), w \right\rangle_{L^2(\mathbb{R})^2}, \quad (2.5.17)$$

with

$$\begin{aligned} \mathcal{T}_c(w) = & \left( -3\partial_x \left( \frac{v_c^2}{\mu_c} \partial_x w_1 \right) + \left( \frac{8v_c^4 (\partial_x v_c)^2 - 2v_c^6 (1 - v_c^2)}{\mu_c^2} + \frac{4(\partial_x v_c)^2}{\mu_c} + \frac{c^2 (2c^2 - 1 + v_c^2)^2 v_c^2}{\mu_c (1 - v_c^2)^2} \right) w_1 - \frac{c(2c^2 - 1 + v_c^2)}{(1 - v_c^2)} w_2 \right. \\ & \left. - \frac{c(2c^2 - 1 + v_c^2)}{(1 - v_c^2)} w_1 + \frac{\mu_c}{v_c^2} w_2 \right). \end{aligned} \quad (2.5.18)$$

The operator  $\mathcal{T}_c$  in (2.5.18) is self-adjoint on  $L^2(\mathbb{R})^2$ , with domain  $\text{Dom}(\mathcal{T}_c) = H^2(\mathbb{R}) \times L^2(\mathbb{R})$ . In addition, combining (2.5.15) with (2.5.17) we deduce that  $\mathcal{T}_c$  is non-negative, with a kernel equal to

$$\text{Ker}(\mathcal{T}_c) = \text{Span} \left\{ \left( v_c^2, \frac{2cv_c^2 (\partial_x v_c)^2}{\mu_c (1 - v_c^2)} \right) \right\}.$$

At this stage, we divide the proof into three steps.

**Step 1.** Let  $c \in (-1, 1) \setminus \{0\}$ . There exists a positive number  $\Lambda_1$ , depending continuously on  $c$ , such that

$$\langle \mathcal{T}_c(w), w \rangle_{L^2(\mathbb{R})^2} \geq \Lambda_1 \int_{\mathbb{R}} (w_1^2 + w_2^2), \quad (2.5.19)$$

for any pair  $w \in X^1(\mathbb{R})$  such that

$$\left\langle w, \left( v_c^2, \frac{2cv_c^2(\partial_x v_c)^2}{\mu_c(1-v_c^2)} \right) \right\rangle_{L^2(\mathbb{R})^2} = 0. \quad (2.5.20)$$

We claim that the essential spectrum of  $\mathcal{T}_c$  is given by

$$\sigma_{\text{ess}}(\mathcal{T}_c) = [\tau_c, +\infty), \quad (2.5.21)$$

with

$$\tau_c = \tau_{1,c} - \frac{1}{2}\tau_{2,c}^{\frac{1}{2}} > 0. \quad (2.5.22)$$

Here, we have set

$$\tau_{1,c} = \frac{4(1-c^2) + c^2(2c^2-1)^2}{2(3-2c^2)} + \frac{3-2c^2}{2}$$

and

$$\tau_{2,c} = \left( \frac{4(1-c^2) + c^2(2c^2-1)^2}{3-2c^2} - (3-2c^2) \right)^2 + 4c^2(2c^2-1)^2.$$

In particular, 0 is an isolated eigenvalue in the spectrum of  $\mathcal{T}_c$ . Inequality (2.5.19) follows with  $\Lambda_1$  either equal to  $\tau_c$ , or to the smallest positive eigenvalue of  $\mathcal{T}_c$ . In view of the analytic dependence on  $c$  of the operator  $\mathcal{T}_c$ ,  $\Lambda_1$  depends continuously on  $c$ .

Now, let us prove (2.5.21). We rely on the Weyl criterion. It follows from (2.2.47) and (2.5.10) that

$$\frac{\mu_c(x)}{v_c^2(x)} \rightarrow 3 - 2c^2, \quad \text{and} \quad \frac{(\partial_x v_c)^2(x)}{\mu_c(x)} \rightarrow \frac{1 - c^2}{3 - 2c^2},$$

as  $x \rightarrow \pm\infty$ . Coming back to (2.5.18), we introduce the operator  $\mathcal{T}_\infty$  given by

$$\mathcal{T}_\infty(w) = \begin{pmatrix} -\frac{3}{3-2c^2}\partial_{xx}w_1 + \frac{4(1-c^2)+c^2(2c^2-1)^2}{3-2c^2}w_1 - c(2c^2-1)w_2 \\ -c(2c^2-1)w_1 + (3-2c^2)w_2 \end{pmatrix}.$$

By the Weyl criterion, the essential spectrum of  $\mathcal{T}_c$  is equal to the spectrum of  $\mathcal{T}_\infty$ .

We next apply again the Weyl criterion to establish that a real number  $\lambda$  belongs to the spectrum of  $\mathcal{T}_\infty$  if and only if there exists a complex number  $\xi$  such that

$$\lambda^2 - \left( \frac{3}{3-2c^2}|\xi|^2 + \frac{4(1-c^2) + c^2(2c^2-1)^2}{3-2c^2} + 3 - 2c^2 \right) \lambda + 3|\xi|^2 + 4(1-c^2) = 0.$$

This is the case if and only if

$$\begin{aligned} \lambda &= \frac{4(1-c^2) + c^2(2c^2-1)^2 + 3|\xi|^2}{2(3-2c^2)} + \frac{3-2c^2}{2} \\ &\pm \frac{1}{2} \left( \left( \frac{4(1-c^2) + c^2(2c^2-1)^2 + 3|\xi|^2}{3-2c^2} - (3-2c^2) \right)^2 + 4c^2(2c^2-1)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

This leads to

$$\sigma_{\text{ess}}(\mathcal{T}_c) = \sigma(\mathcal{T}_\infty) = [\tau_c, +\infty),$$

with  $\tau_c$  as in (2.5.22). This completes the proof of Step 1.

**Step 2.** *There exists a positive number  $\Lambda_2$ , depending continuously on  $c$ , such that*

$$K_c(v) \geq \Lambda_2 \int_{\mathbb{R}} \left( (\partial_x v_1)^2 + v_1^2 + v_2^2 \right), \quad (2.5.23)$$

for any pair  $v \in X^1(\mathbb{R})$  such that

$$\langle v, v_c^{-1} S \chi_c \rangle_{L^2(\mathbb{R})^2} = 0. \quad (2.5.24)$$

We start by improving the estimate in (2.5.19). Given a pair  $w \in X^1(\mathbb{R})$ , we observe that

$$\left| \langle \mathcal{T}_c(w), w \rangle_{L^2(\mathbb{R})^2} - 3 \int_{\mathbb{R}} \frac{v_c^2}{\mu_c} (\partial_x w_1)^2 \right| \leq A_c \int_{\mathbb{R}} (w_1^2 + w_2^2).$$

Here and in the sequel,  $A_c$  refers to a positive number, depending continuously on  $c$ . For  $0 < \tau < 1$ , we have

$$\langle \mathcal{T}_c(w), w \rangle_{L^2(\mathbb{R})^2} \geq (1 - \tau) \langle \mathcal{T}_c(w), w \rangle_{L^2(\mathbb{R})^2} + 3\tau \int_{\mathbb{R}} \frac{v_c^2}{\mu_c} (\partial_x w_1)^2 - A_c \tau \int_{\mathbb{R}} (w_1^2 + w_2^2).$$

Since  $v_c^2/\mu_c \geq 1/(3 - 2c^2)$ , this yields

$$\langle \mathcal{T}_c(w), w \rangle_{L^2(\mathbb{R})^2} \geq \left( (1 - \tau) \Lambda_1 - A_c \tau \right) \int_{\mathbb{R}} (w_1^2 + w_2^2) + \frac{3\tau}{3 - 2c^2} \int_{\mathbb{R}} (\partial_x w_1)^2,$$

under condition (2.5.20). For  $\tau$  small enough, this leads to

$$\langle \mathcal{T}_c(w), w \rangle_{L^2(\mathbb{R})^2} \geq A_c \int_{\mathbb{R}} \left( (\partial_x w_1)^2 + w_1^2 + w_2^2 \right), \quad (2.5.25)$$

when  $w$  satisfies condition (2.5.20).

Since the pair  $w$  depends on the pair  $v$ , we can write (2.5.25) in terms of  $v$ . By (2.5.17),  $K_c(v)$  is equal to the left-hand side of (2.5.25). We deduce that (2.5.25) may be expressed as

$$K_c(v) \geq A_c \int_{\mathbb{R}} \left( (\partial_x v_1)^2 + v_1^2 \right) + A_c \int_{\mathbb{R}} \left( v_2 - \frac{2cv_c(\partial_x v_c)}{\mu_c(1 - v_c^2)} \partial_x v_1 \right)^2.$$

We recall that, given two vectors  $a$  and  $b$  in a Hilbert space  $H$ , we have

$$\|a - b\|_H^2 \geq \tau \|a\|_H^2 - \frac{\tau}{1 - \tau} \|b\|_H^2,$$

for any  $0 < \tau < 1$ . Then, we deduce that

$$K_c(v) \geq A_c \int_{\mathbb{R}} \left( (\partial_x v_1)^2 + v_1^2 + \tau v_2^2 \right) - \frac{\tau A_c}{1 - \tau} \int_{\mathbb{R}} \left( \frac{v_c(\partial_x v_c)}{\mu_c(1 - v_c^2)} \partial_x v_1 \right)^2.$$

We choose  $\tau$  small enough so that we can infer from (2.2.3) that

$$K_c(v) \geq A_c \int_{\mathbb{R}} \left( (\partial_x v_1)^2 + v_1^2 + v_2^2 \right), \quad (2.5.26)$$

when  $w$  satisfies condition (2.5.20), i.e. when  $v$  is orthogonal to the pair

$$\mathbf{v}_c = \left( v_c^2 - \partial_x \left( \frac{2cv_c^2(\partial_x v_c)^2}{\mu_c(1 - v_c^2)} \right), \frac{2cv_c^2(\partial_x v_c)^2}{\mu_c(1 - v_c^2)} \right). \quad (2.5.27)$$

Next, we verify that (2.5.26) remains true, decreasing possibly the value of  $A_c$ , when we replace this orthogonality condition by

$$\langle v, v_c Q_c \rangle_{L^2(\mathbb{R})^2} = 0. \quad (2.5.28)$$

We remark that

$$\langle \mathbf{v}_c, v_c Q_c \rangle_{L^2(\mathbb{R})^2} \neq 0.$$

Indeed, we would deduce from (2.5.26) that

$$0 = K_c(v_c Q_c) \geq A_c \int_{\mathbb{R}} \left( (\partial_x v_c^2)^2 + v_c^4 + (v_c w_c)^2 \right) > 0,$$

which is impossible. In addition, the number  $\langle \mathbf{v}_c, v_c Q_c \rangle_{L^2(\mathbb{R})^2}$  depends continuously on  $c$  in view of (2.5.27). Given a pair  $\tilde{v}$  satisfying (2.5.28), we denote by  $\lambda$  the real number such that  $\mathbf{v} = \lambda v_c Q_c + \tilde{v}$  is orthogonal to  $\mathbf{v}_c$ . Since  $v_c Q_c$  belongs to the kernel of  $K_c$ , we obtain using (2.5.26),

$$K_c(\tilde{v}) = K_c(\mathbf{v}) \geq A_c \int_{\mathbb{R}} \left( (\partial_x \mathbf{v}_1)^2 + \mathbf{v}_1^2 + \mathbf{v}_2^2 \right). \quad (2.5.29)$$

On the other hand, since  $\tilde{v}$  satisfies (2.5.28), we have

$$\lambda = \frac{\langle \mathbf{v}, v_c Q_c \rangle_{L^2(\mathbb{R})^2}}{\|v_c Q_c\|_{L^2(\mathbb{R})^2}^2}.$$

Using the Cauchy-Schwarz inequality, this yields

$$\lambda^2 \leq A_c \left( \int_{\mathbb{R}} (v_c^4 + (v_c w_c)^2) \right) \left( \int_{\mathbb{R}} (\mathbf{v}_1^2 + \mathbf{v}_2^2) \right),$$

hence, by (2.2.3) and (2.5.29),

$$\lambda^2 \leq A_c K_c(\mathbf{v}) = A_c K_c(\tilde{v}).$$

Using (2.5.29), this leads to

$$\begin{aligned} \int_{\mathbb{R}} \left( (\partial_x \tilde{v}_1)^2 + \tilde{v}_1^2 + \tilde{v}_2^2 \right) &\leq 2 \left( \lambda^2 \int_{\mathbb{R}} v_c^2 \left( (\partial_x v_c)^2 + v_c^2 + w_c^2 \right) + \int_{\mathbb{R}} \left( (\partial_x \mathbf{v}_1)^2 + \mathbf{v}_1^2 + \mathbf{v}_2^2 \right) \right) \\ &\leq A_c K_c(\tilde{v}), \end{aligned}$$

We finish the proof of this step applying again the same argument. We write  $v = \lambda v_c S Q_c + \tilde{v}$ , with  $\langle \tilde{v}, v_c Q_c \rangle_{L^2(\mathbb{R})^2} = 0$ . Since  $v_c Q_c$  belongs to the kernel of  $K_c$ , we infer from the same argument that

$$K_c(v) = K_c(\tilde{v}) \geq \Lambda_2 \int_{\mathbb{R}} (\partial_x \tilde{v}_1)^2 + \tilde{v}_1^2 + \tilde{v}_2^2. \quad (2.5.30)$$

Using the orthogonality condition in (2.5.24), we obtain

$$\lambda = - \frac{\langle \tilde{v}, v_c^{-1} S \chi_c \rangle_{L^2(\mathbb{R})^2}}{\langle Q_c, S \chi_c \rangle_{L^2(\mathbb{R})^2}}.$$

By the Cauchy-Schwarz inequality, we are led to

$$\lambda^2 \leq A_c \|v_c^{-1} S \chi_c\|_{L^2 \times L^2}^2 \int_{\mathbb{R}} (\tilde{v}_1^2 + \tilde{v}_2^2).$$

Invoking the exponential decay of  $\chi_c$  in (2.6.46), we deduce

$$\|v_c^{-1} S \chi_c\|_{L^2 \times L^2}^2 \leq A_c.$$

As a consequence, we can derive from (2.5.30) that

$$\lambda^2 \leq A_c K_c(\tilde{v}) = A_c K_c(v).$$

Combining again with (2.5.30), we are led to

$$\begin{aligned} \int_{\mathbb{R}} \left( (\partial_x v_1)^2 + v_1^2 + v_2^2 \right) &\leq 2 \left( \lambda^2 \int_{\mathbb{R}} v_c^2 \left( (\partial_x v_c)^2 + v_c^2 + w^2 \right) + \int_{\mathbb{R}} \left( (\partial_x \tilde{v}_1)^2 + \tilde{v}_1^2 + \tilde{v}_2^2 \right) \right) \\ &\leq A_c K_c(v). \end{aligned}$$

which completes the proof of Step 2.

**Step 3.** *End of the proof.*

Since the pair  $v$  depends on the pair  $u$  as in (2.5.14), we can write (2.5.23) in terms of  $u$ . The left hand side of (2.5.23) is equal to  $G_c(u)$  by (2.5.15). Moreover, for the right-hand side, we have

$$\int_{\mathbb{R}} \left( (\partial_x v_1)^2 + v_1^2 + v_2^2 \right) = \int_{\mathbb{R}} v_c^2 \left( (\partial_x u_1)^2 + (2v_c^2 + c^2)u_1^2 + u_2^2 \right).$$

We deduce that (2.5.23) may be written as

$$G_c(u) \geq A_c \int_{\mathbb{R}} v_c^2 \left( (\partial_x u_1)^2 + u_1^2 + u_2^2 \right), \quad (2.5.31)$$

when  $v_c u$  verifies the orthogonality condition (2.5.24), which means that  $u$  verifies the orthogonality condition (2.2.52). We recall that

$$v_c(x) \geq A_c e^{-|x|},$$

by (2.2.3), which is sufficient to obtain (2.2.51). This completes the proof of Proposition 2.2.9.

## 2.5.4 Proof of Proposition 2.2.10

First we check that we are allowed to differentiate the quantity

$$\mathcal{J}^*(t) := \left\langle M_{c^*(t)} u^*(\cdot, t), u^*(\cdot, t) \right\rangle_{L^2(\mathbb{R})^2}.$$

Indeed, by (2.2.41), (2.5.1), and (2.6.42), there exists a positive number  $A_{k,c}$  such that

$$\int_{\mathbb{R}} \left( \left( \partial_x^k u_1^*(x, t) \right)^2 + \left( \partial_x^k u_2^*(x, t) \right)^2 \right) e^{\nu_c |x|} dx \leq A_{k,c}. \quad (2.5.32)$$

Next, using (2.2.42) and (2.2.45), we obtain

$$\begin{aligned} \frac{d}{dt}(\mathcal{J}^*) &= 2 \left\langle S M_{c^*} u^*, \mathcal{H}_{c^*}(J S u^*) \right\rangle_{L^2(\mathbb{R})^2} + 2 \left\langle S M_{c^*} u^*, \mathcal{H}_{c^*}(J \mathcal{R}_{c^*} \varepsilon^*) \right\rangle_{L^2(\mathbb{R})^2} \\ &\quad + 2 \left( (a^*)' - c^* \right) \left\langle S M_{c^*} u^*, \mathcal{H}_{c^*}(\partial_x \varepsilon^*) \right\rangle_{L^2(\mathbb{R})^2} \\ &\quad - 2 (c^*)' \left\langle S M_{c^*} u^*, \mathcal{H}_{c^*}(\partial_c Q_{c^*}) \right\rangle_{L^2(\mathbb{R})^2} \\ &\quad + (c^*)' \left\langle \partial_c M_{c^*} u^*, u^* \right\rangle_{L^2(\mathbb{R})^2} + 2 (c^*)' \left\langle M_{c^*} u^*, S \partial_c \mathcal{H}_{c^*}(\varepsilon^*) \right\rangle_{L^2(\mathbb{R})^2}. \end{aligned} \quad (2.5.33)$$

The proof of (2.2.53) is the same as in [6]. We will give only the main ideas of the proof. We will estimate all the terms in the right-hand side of (2.5.33) except the fourth term which vanishes.

For the first one, we infer from Proposition 2.2.9 the following estimate.

**Step 1.** *There exists a positive number  $B_1$ , depending only on  $\mathbf{c}$ , such that*

$$\mathcal{J}_1^*(t) := 2 \left\langle S M_{c^*} u^*, \mathcal{H}_{c^*}(J S u^*) \right\rangle_{L^2(\mathbb{R})^2} \geq B_1 \int_{\mathbb{R}} \left[ (\partial_x u_1^*)^2 + (u_1^*)^2 + (u_2^*)^2 \right] (x, t) e^{-2|x|} dx,$$

for any  $t \in \mathbb{R}$ .

For the second term, by (2.2.21), (2.2.57) and (2.5.1), we have

**Step 2.** *There exists a positive number  $B_2$ , depending only on  $\mathfrak{c}$ , such that*

$$|\mathcal{J}_2^*(t)| := 2 \left| \left\langle SM_{c^*} u^*, \mathcal{H}_{c^*}(\mathcal{J}\mathcal{R}_{c^*} \varepsilon^*) \right\rangle_{L^2(\mathbb{R})^2} \right| \leq B_2 \left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2,$$

for any  $t \in \mathbb{R}$ .

For the third one, we use (2.2.21) to obtain

**Step 3.** *There exists a positive number  $B_3$ , depending only on  $\mathfrak{c}$ , such that*

$$|\mathcal{J}_3^*(t)| := 2 \left| (a^*)' - c^* \right| \left| \left\langle SM_{c^*} u^*, \mathcal{H}_{c^*}(\partial_x \varepsilon^*) \right\rangle_{L^2(\mathbb{R})^2} \right| \leq B_3 \left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2,$$

for any  $t \in \mathbb{R}$ .

We now prove the following statement for the fourth term.

**Step 4.** *We have*

$$\mathcal{J}_4^*(t) := 2(c^*)' \left\langle SM_{c^*} u^*, \mathcal{H}_{c^*}(\partial_c Q_{c^*}) \right\rangle_{L^2(\mathbb{R})^2} = 0,$$

for any  $t \in \mathbb{R}$ .

Since  $\mathcal{H}_{c^*}(\partial_c Q_{c^*}) = P'(Q_{c^*}) = SQ_{c^*}$  and  $M_{c^*} Q_{c^*} = S\partial_x Q_{c^*}$ , we have

$$\begin{aligned} \left\langle SM_{c^*} u^*, \mathcal{H}_{c^*}(\partial_c Q_{c^*}) \right\rangle_{L^2(\mathbb{R})^2} &= \left\langle M_{c^*} u^*, Q_{c^*} \right\rangle_{L^2(\mathbb{R})^2} = \left\langle u^*, S\partial_x Q_{c^*} \right\rangle_{L^2(\mathbb{R})^2} \\ &= \left\langle \varepsilon^*, \mathcal{H}_{c^*}(\partial_x Q_{c^*}) \right\rangle_{L^2(\mathbb{R})^2} = 0. \end{aligned}$$

This is the reason why we do not need to establish a quadratic dependence of  $(c^*)'(t)$  on  $\varepsilon^*$ .

Next, we use (2.2.3), (2.2.9), (2.2.21) and (2.2.45) to bound the fifth term in the following way.

**Step 5.** *There exists a positive number  $B_5$ , depending only on  $\mathfrak{c}$ , such that*

$$|\mathcal{J}_5^*(t)| := |(c^*)'| \left| \left\langle \partial_c M_{c^*} u^*, u^* \right\rangle_{L^2(\mathbb{R})^2} \right| \leq B_5 \left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2,$$

for any  $t \in \mathbb{R}$ .

Finally, we have in the same way.

**Step 6.** *There exists a positive number  $B_6$ , depending only on  $\mathfrak{c}$ , such that*

$$|\mathcal{J}_6^*(t)| := |(c^*)'| \left| \left\langle M_{c^*} u^*, S\partial_c \mathcal{H}_{c^*}(\varepsilon^*) \right\rangle_{L^2(\mathbb{R})^2} \right| \leq B_6 \left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2,$$

for any  $t \in \mathbb{R}$ .

We conclude the proof of Proposition 2.2.10 by combining the six previous steps to obtain (2.2.53), with  $B_* := \max \{1/B_1, B_2 + B_3 + B_5 + B_6\}$ .  $\square$



## 2.5.5 Proof of Corollary 2.2.1

Corollary 2.2.1 is a consequence of Propositions 2.2.8 and 2.2.10. We combine the two estimates (2.2.43) and (2.2.53) with the definition of  $N(t)$  to obtain

$$\frac{d}{dt} \left( \langle N(t)u^*(\cdot, t), u^*(\cdot, t) \rangle_{L^2(\mathbb{R}^2)} \right) \geq \left( \frac{1 - \mathfrak{c}^2}{16} - A_* B_*^2 e^{2R_*} \left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \right) \left\| u^*(\cdot, t) \right\|_{X(\mathbb{R})}^2,$$

for any  $t \in \mathbb{R}$ . In view of (2.2.21), we fix the parameter  $\beta_{\mathfrak{c}}$  such that

$$\left\| \varepsilon^*(\cdot, t) \right\|_{X(\mathbb{R})}^{\frac{1}{2}} \leq \frac{1 - \mathfrak{c}^2}{32A_* B_*^2 e^{2R_*}},$$

for any  $t \in \mathbb{R}$ , to obtain (2.2.54). In view of (2.2.3), (2.2.21) and (2.2.45), we notice that there exists a positive number  $A_{\mathfrak{c}}$ , depending only on  $\mathfrak{c}$ , such that

$$\left\| M_{c^*(t)} \right\|_{L^\infty(\mathbb{R})} \leq A_{\mathfrak{c}}, \quad (2.5.34)$$

for any  $t \in \mathbb{R}$ . Moreover, since the map  $t \mapsto \langle N(t)u^*(\cdot, t), u^*(\cdot, t) \rangle_{L^2(\mathbb{R}^2)}$  is uniformly bounded by (2.5.32) and (2.5.34), estimate (2.2.55) follows by integrating (2.2.54) from  $t = -\infty$  to  $t = +\infty$ . Finally, statement (2.2.56) is a direct consequence of (2.2.55).

## 2.6 Appendix

### 2.6.1 Weak continuity of the hydrodynamical flow

In this section, we prove the weak continuity of the hydrodynamical flow which is stated in the following proposition.

**Proposition 2.6.1.** *We consider a sequence  $(v_{n,0}, w_{n,0})_{n \in \mathbb{N}} \in \mathcal{NV}(\mathbb{R})^{\mathbb{N}}$ , and a pair  $(v_0, w_0) \in \mathcal{NV}(\mathbb{R})$  such that*

$$v_{n,0} \rightharpoonup v_0 \quad \text{in } H^1(\mathbb{R}), \quad \text{and} \quad w_{n,0} \rightharpoonup w_0 \quad \text{in } L^2(\mathbb{R}), \quad (2.6.1)$$

as  $n \rightarrow +\infty$ . We denote by  $(v_n, w_n)$  the unique solution to (HLL) with initial datum  $(v_{n,0}, w_{n,0})$  and we assume that there exists a positive number  $T_n$  such that the solutions  $(v_n, w_n)$  are defined on  $(-T_n, T_n)$ , and satisfy the condition

$$\sup_{n \in \mathbb{N}} \sup_{t \in (-T_n, T_n)} \max_{x \in \mathbb{R}} v_n(x, t) \leq 1 - \sigma, \quad (2.6.2)$$

for a given positive number  $\sigma$ . Then, the unique solution  $(v, w)$  to (HLL) with initial datum  $(v_0, w_0)$  is defined on  $(-T_{\max}, T_{\max})$ , with<sup>3</sup>

$$T_{\max} = \liminf_{n \rightarrow +\infty} T_n,$$

and for any  $t \in (-T_{\max}, T_{\max})$ , we have

$$v_n(t) \rightharpoonup v(t) \quad \text{in } H^1(\mathbb{R}), \quad \text{and} \quad w_n(t) \rightharpoonup w(t) \quad \text{in } L^2(\mathbb{R}), \quad (2.6.3)$$

as  $n \rightarrow +\infty$ .

---

3. See Theorem 1 in [19] for more details.

First we prove a weak continuity property of the flow of equations (2.2.32)–(2.2.34). Next, we deduce the weak convergence of  $w_n$  from (2.4.41).

More precisely, we consider now a sequence of initial conditions  $(\Psi_{n,0}, v_{n,0}) \in L^2(\mathbb{R}) \times H^1(\mathbb{R})$ , such that the norms  $\|\Psi_{n,0}\|_{L^2}$  and  $\|v_{n,0}\|_{L^2}$  are uniformly bounded with respect to  $n$  and we assume that

$$\sup_{n \in \mathbb{N}} \|v_{n,0}\|_{L^\infty(\mathbb{R})} < 1. \quad (2.6.4)$$

Then, there exist two functions  $\Psi_0 \in L^2(\mathbb{R})$  and  $v_0 \in H^1(\mathbb{R})$  such that, going possibly to a subsequence,

$$\Psi_{n,0} \rightharpoonup \Psi_0 \quad \text{in } L^2(\mathbb{R}), \quad (2.6.5)$$

$$v_{n,0} \rightharpoonup v_0 \quad \text{in } H^1(\mathbb{R}), \quad (2.6.6)$$

and, for any compact subset  $K$  of  $\mathbb{R}$ ,

$$v_{n,0} \rightarrow v_0 \quad \text{in } L^\infty(K), \quad (2.6.7)$$

as  $n \rightarrow +\infty$ . We claim that this convergence is conserved along the flow corresponding to equations (2.2.32)–(2.2.34)<sup>4</sup>.

**Proposition 2.6.2.** *We consider two sequences  $(\Psi_{n,0})_{n \in \mathbb{N}} \in L^2(\mathbb{R})^{\mathbb{N}}$  and  $(v_{n,0})_{n \in \mathbb{N}} \in H^1(\mathbb{R})^{\mathbb{N}}$ , and two functions  $\Psi_0 \in L^2(\mathbb{R})$  and  $v_0 \in H^1(\mathbb{R})$ , such that assumptions (2.6.4)–(2.6.7) are satisfied, and we denote by  $(\Psi_n, v_n)$ , respectively  $(\Psi, v)$ , the unique global solutions to (2.2.32)–(2.2.34) with initial datum  $(\Psi_{n,0}, v_{n,0})$ , respectively  $(\Psi_0, v_0)$ , which we assume to be defined on  $[0, T]$  for a positive number  $T$ . For any fixed  $t \in [0, T]$ , we have*

$$\Psi_n(\cdot, t) \rightharpoonup \Psi(\cdot, t) \quad \text{in } L^2(\mathbb{R}), \quad (2.6.8)$$

and

$$v_n(\cdot, t) \rightharpoonup v(\cdot, t) \quad \text{in } H^1(\mathbb{R}), \quad (2.6.9)$$

when  $n \rightarrow +\infty$ .

*Proof.* We split the proof into four steps.

**Step 1.** *There exist three functions  $\Phi \in L^2([0, T], L^2(\mathbb{R}))$  and  $\mathbf{v} \in L^2([0, T], H^1(\mathbb{R}))$  such that, up to a further subsequence,*

$$\Psi_n(t) \rightharpoonup \Phi(t) \quad \text{in } L^2(\mathbb{R}), \quad (2.6.10)$$

$$v_n(\cdot, t) \rightharpoonup \mathbf{v}(\cdot, t) \quad \text{in } H^1(\mathbb{R}), \quad (2.6.11)$$

$$v_n(\cdot, t) \rightarrow \mathbf{v}(\cdot, t) \quad \text{in } L_{loc}^\infty(\mathbb{R}), \quad (2.6.12)$$

for all  $t \in [0, T]$ , and

$$|\Psi_n|^2 \Psi_n \rightharpoonup |\Phi|^2 \Phi \quad \text{in } L^2([0, T], L^2(\mathbb{R})), \quad (2.6.13)$$

when  $n \rightarrow +\infty$ .

---

4. We only consider here positive time but the proof remains available for negative time.

*Proof.* We recall that there exists a constant  $M$  such that

$$\|\Psi_{n,0}\|_{L^2} \leq M \quad \text{and} \quad \|v_{n,0}\|_{H^1} \leq M,$$

uniformly on  $n$ . Applying Proposition 2.2.5 to the pairs  $(\Psi_n, v_n)$  and  $(0, 0)$ , we obtain

$$\|\Psi_n\|_{\mathcal{C}_T^0 L_x^2} + \|v_n\|_{\mathcal{C}_T^0 H_x^1} + \|\Psi_n\|_{L_T^4 L_x^\infty} \leq A \left( \|\Psi_{n,0}\|_{L^2} + \|v_{n,0}\|_{H^1} \right).$$

This leads to

$$\|\Psi_n\|_{L_T^4 L_x^\infty} \leq 2AM, \quad \|\Psi_n\|_{L_T^\infty L_x^2} \leq 2AM, \quad \text{and} \quad \|v_n\|_{L_T^\infty H_x^1} \leq 2AM. \quad (2.6.14)$$

Hence, there exist two functions  $\Phi \in L^\infty([0, T], L^2(\mathbb{R})) \cap L^4([0, T], L^\infty(\mathbb{R}))$  and  $\mathbf{v} \in L^\infty([0, T], H^1(\mathbb{R}))$  such that

$$\Psi_n \xrightarrow{*} \Phi \quad \text{in } L^\infty([0, T], L^2(\mathbb{R}))$$

and

$$v_n \xrightarrow{*} \mathbf{v} \quad \text{in } L^\infty([0, T], H^1(\mathbb{R}))$$

Let us prove (2.6.10) and (2.6.11). We argue as in [6] and we introduce a cut-off function  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  in the way that  $\chi \equiv 1$  on  $[-1, 1]$  and  $\chi \equiv 0$  on  $(-\infty, 2] \cup [2, +\infty)$ . Denote  $\chi_p(\cdot) := \chi(\cdot/p)$  for any integer  $p \in \mathbb{N}^*$ . By (2.6.14), the sequences  $(\chi_p \Psi_n)_{n \in \mathbb{N}}$  and  $(\chi_p v_n)_{n \in \mathbb{N}}$  are bounded in  $\mathcal{C}^0([0, T], L^2(\mathbb{R}))$  and  $\mathcal{C}^0([0, T], H^1(\mathbb{R}))$  respectively. In view of the Rellich-Kondrachov theorem, the sets  $\{\chi_p \Psi_n(\cdot, t), n \in \mathbb{N}\}$  and  $\{\chi_p v_n(\cdot, t), n \in \mathbb{N}\}$  are relatively compact in  $H^{-2}(\mathbb{R})$  and  $H^{-1}(\mathbb{R})$  respectively, for any fixed  $t \in [0, T]$ . In addition, since the couple  $(\Psi_n, v_n)$  is solution to (2.2.32)-(2.2.34), we have  $(\partial_t \Psi_n, \partial_t v_n)$  belongs to  $\mathcal{C}^0([0, T], H^{-2}(\mathbb{R}) \times H^{-1}(\mathbb{R}))$  and satisfies

$$\|\partial_t \Psi_n(\cdot, t)\|_{H^{-2}(\mathbb{R})} \leq K_M \quad \text{and} \quad \|\partial_t v_n(\cdot, t)\|_{H^{-1}(\mathbb{R})} \leq K_M.$$

This leads to the fact that the couple  $(\chi_p \Psi_n, \chi_p v_n)$  is equicontinuous in  $\mathcal{C}^0([0, T], H^{-2}(\mathbb{R}) \times H^{-1}(\mathbb{R}))$ . Then, we apply the Arzela-Ascoli theorem and the Cantor diagonal argument, to find a further subsequence (independent of  $p$ ), such that, for each  $p \in \mathbb{N}^*$ ,

$$\chi_p \Psi_n \rightarrow \chi_p \Phi \quad \text{in } \mathcal{C}^0([0, T], H^{-2}(\mathbb{R})), \quad (2.6.15)$$

and

$$\chi_p v_n \rightarrow \chi_p \mathbf{v} \quad \text{in } \mathcal{C}^0([0, T], H^{-1}(\mathbb{R})), \quad (2.6.16)$$

as  $n \rightarrow +\infty$ . Combining this with (2.6.14) we infer that (2.6.10) and (2.6.11) hold. By the Sobolev embedding theorem, (2.6.12) is a consequence of (2.6.11).

Now, let us prove (2.6.13). Using the Hölder inequality, we infer that

$$\int_0^T \int_{\mathbb{R}} |\Psi_n(x, t)|^6 dx dt \leq \|\Psi_n\|_{L^\infty L_x^2}^2 \|\Psi_n\|_{L_T^4 L_x^\infty}^4.$$

By (2.6.14), we conclude that

$$\| |\Psi_n|^2 \Psi_n \|_{L_T^2 L_x^2} \leq M. \quad (2.6.17)$$

Then, there exists a function  $\Phi_1 \in L^2(\mathbb{R} \times [0, T])$  such that up to a further subsequence,

$$|\Psi_n|^2 \Psi_n \rightharpoonup \Phi_1 \quad \text{in } L^2(\mathbb{R} \times [0, T]).$$

Let us prove that  $\Phi_1 \equiv |\Phi|^2 \Phi$ . To obtain this it is sufficient to prove that, up to a subsequence,

$$\Psi_n \longrightarrow \Phi \quad \text{in } L^2([0, T], L^2([-R, R])), \quad (2.6.18)$$

for any  $R > 0$ , i.e the sequence  $(\Psi_n)$  is relatively compact in  $L^2([-R, R] \times [0, T])$ . Indeed, using the Holder inequality, we obtain

$$\begin{aligned} \|\Psi_n^2\Psi_n - |\Phi|^2\Phi\|_{L_{T,R}^{\frac{6}{5}}} &= \|(\Psi_n - \Phi)(|\Psi_n|^2 + |\Phi|^2) + \Psi_n\Phi(\overline{\Psi_n} - \overline{\Phi})\|_{L_{T,R}^{\frac{6}{5}}} \\ &\leq 2\|\Psi_n - \Phi\|_{L_{T,R}^{\frac{6}{5}}}(\|\Psi_n\|_{L_{T,R}^6}^2 + \|\Phi\|_{L_{T,R}^6}^2) \\ &\leq 2\|\Psi_n - \Phi\|_{L_{T,R}^2}(\|\Psi_n\|_{L_{T,R}^6}^2 + \|\Phi\|_{L_{T,R}^6}^2), \end{aligned} \quad (2.6.19)$$

for any  $R > 0$ . By (2.6.17),  $(\Psi_n)$  is uniformly bounded in  $L^6(\mathbb{R} \times [0, T])$  and  $\Phi \in L^6(\mathbb{R} \times [0, T])$ . Then

$$|\Psi_n|^2\Psi_n \longrightarrow |\Phi|^2\Phi \quad \text{in } L^{\frac{6}{5}}([-R, R] \times [0, T]).$$

So that  $\Phi_1 \equiv |\Phi|^2\Phi$ . Now, let us prove that the sequence  $(\Psi_n)$  is relatively compact in  $L^2([-R, R] \times [0, T])$ . The main point of the proof is the following claim.

**Fait 1.** *Let  $\Psi$  be a solution of (2.2.32) in  $\mathcal{C}^0([0, T], L^2(\mathbb{R})) \cap L^4([0, T], L^\infty(\mathbb{R}))$ . Then,  $\Psi \in L^2([0, T], H_{loc}^{\frac{1}{2}}(\mathbb{R}))$ .*

*Proof.* The proof relies on the Kato smoothing effect for the linear Schrödinger group (see [37]). Denote  $S(t) = e^{it\partial_{xx}}$ , and

$$\mathcal{F}(\Psi, v) := \frac{1}{2}v^2\Psi - \text{Re}\left(\Psi(1 - 2F(v, \overline{\Psi}))\right)(1 - 2F(v, \Psi)). \quad (2.6.20)$$

We recall that there exists a positive constant  $M$  such that

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{+\infty} |D_x^{\frac{1}{2}}S(t)f(x)|^2 dt \leq M\|f\|_{L^2}^2, \quad (2.6.21)$$

and

$$\left\| \int_{\mathbb{R}} S(-t')D_x^{\frac{1}{2}}h(\cdot, t')dt' \right\|_{L^2} \leq M\|h\|_{L_x^1L_t^2}, \quad (2.6.22)$$

when  $f \in L^2(\mathbb{R})$  and  $h \in L^1(\mathbb{R}, L^2(\mathbb{R}))$  (see [37] for more details). We prove that there exists a positive constant  $M$  such that

$$\|D_x^{\frac{1}{2}}\Psi\|_{L_x^\infty L_T^2} \leq M\|\Psi_0\|_{L^2} + M\|\Psi\|_{L_{T,x}^2} \left( \|\Psi\|_{L_{T,x}^6}^2 + T^{\frac{1}{2}}(\|v\|_{L_{T,x}^\infty}^2 + \|1 - 2F(v, \Psi)\|_{L_{T,x}^\infty}^2) \right). \quad (2.6.23)$$

The claim is a consequence of this estimate, so that it is sufficient to prove (2.6.23).

We write

$$\Psi(x, t) = S(t)\Psi_0(x) + i \int_0^t S(t-t') \left( 2(|\Psi|^2\Psi)(x, t') + \mathcal{F}(\Psi, v)(x, t') \right) dt',$$

for all  $(x, t) \in \mathbb{R}$ . First, using (2.6.21), we obtain

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{+\infty} |D_x^{\frac{1}{2}}S(t)\Psi_0(x)|^2 dt \leq M\|\Psi_0\|_{L^2}^2.$$

For the nonlinear term, we can argue as in [24] to prove that

$$\left\| \int_0^t S(t-t')D_x^{\frac{1}{2}}g(\cdot, t')dt' \right\|_{L_x^\infty L_T^2} \leq M\|g\|_{L_T^1 L_x^2}. \quad (2.6.24)$$

Using a duality argument, it is equivalent to prove that for any smooth function  $h$  that satisfies  $\|h\|_{L_x^1 L_t^2} \leq 1$ , we have

$$\left| \int_{\mathbb{R} \times [0, T]^2} S(t-t') D_x^{\frac{1}{2}} g(x, t') \bar{h}(x, t) dt' dx dt \right| \leq M \|g\|_{L_T^1 L_x^2}. \quad (2.6.25)$$

The left-hand side can be written, using the Cauchy-Schwarz and Strichartz estimates, and (2.6.22), as

$$\begin{aligned} & \left| \int_{\mathbb{R}} \left( \int_0^T S(-t') D_x^{\frac{1}{2}} g(x, t') dt' \right) \left( \int_0^T \overline{S(-t) h(x, t)} dt \right) dx \right| \\ &= \left| \int_{\mathbb{R}} \left( \int_0^T S(-t') g(x, t') dt' \right) \left( \int_0^T \overline{S(-t) D_x^{\frac{1}{2}} h(x, t)} dt \right) dx \right| \\ &\leq M \left\| \int_0^T S(-t') g(x, t') dt' \right\|_{L^2} \leq M \|g\|_{L_T^1 L_x^2}. \end{aligned}$$

This achieves the proof of (2.6.24). Similarly, we have

$$\left\| \int_0^t S(t-t') D_x^{\frac{1}{2}} g(\cdot, t') dt' \right\|_{L_x^\infty L_T^2} \leq M \|g\|_{L_{T,x}^{\frac{5}{6}}}. \quad (2.6.26)$$

We next apply (2.6.24) and (2.6.26) on the nonlinear terms to obtain, using the Cauchy-Schwarz and Hölder estimates,

$$\left\| \int_0^t D_x^{\frac{1}{2}} S(t-t') (|\Psi|^2 \Psi)(\cdot, t') dt' \right\|_{L_x^\infty L_T^2} \leq M \|\Psi^3\|_{L_{T,x}^{\frac{6}{5}}} \leq M \|\Psi\|_{L_{T,x}^2} \|\Psi\|_{L_{T,x}^6}^2,$$

and

$$\begin{aligned} & \left\| \int_0^t D_x^{\frac{1}{2}} S(t-t') \mathcal{F}(\Psi, v)(\cdot, t') dt' \right\|_{L_x^\infty L_T^2} \leq M \|\mathcal{F}(\Psi, v)\|_{L_T^1 L_x^2} \\ &\leq M \|\Psi\|_{L_T^1 L_x^2} \left( \|v\|_{L_{T,x}^\infty}^2 + \|1 - 2F(v, \Psi)\|_{L_{T,x}^\infty}^2 \right) \\ &\leq MT^{\frac{1}{2}} \|\Psi\|_{L_{T,x}^2} \left( \|v\|_{L_{T,x}^\infty}^2 + \|1 - 2F(v, \Psi)\|_{L_{T,x}^\infty}^2 \right). \end{aligned}$$

Since  $v \in L^\infty([0, T], H^1(\mathbb{R}))$  and  $\Psi \in L^\infty([0, T], L^2(\mathbb{R}))$ , we know that  $\Psi \in L^\infty([0, T], L^2(\mathbb{R}))$  and  $F(\Psi, v) \in L^\infty(\mathbb{R} \times [0, T])$ . Using the fact that  $\Psi \in L^6(\mathbb{R} \times [0, T])$ , we finish the proof of this claim.  $\square$

Applying this claim to the sequence  $(\Psi_n)$  yields that  $(\Psi_n)$  is uniformly bounded in the space  $L^2([0, T], H_{loc}^{\frac{1}{2}}(\mathbb{R}))$ . On the other hand, we have  $\mathcal{F}(\Psi_n, v_n) \in L^\infty([0, T], L^2(\mathbb{R}))$ , since  $v_n \in L^\infty([0, T], H^1(\mathbb{R}))$ ,  $\Psi_n \in L^\infty([0, T], L^2(\mathbb{R}))$  and  $F(\Psi_n, v_n) \in L^\infty(\mathbb{R} \times [0, T])$ . Then, using (2.2.32) and (2.6.17), we obtain that  $(\Psi_n)$  is uniformly bounded in  $H^1([0, T], H^{-2}(\mathbb{R}))$ . Hence, by interpolation  $(\Psi_n) \in H^{\frac{1}{10}}([0, T], H_{loc}^{\frac{1}{4}}(\mathbb{R}))$  so that it converges in  $L^2([-R, R] \times [0, T])$  for any  $R > 0$ . This finishes the proofs of (2.6.18) and of Step 1.  $\square$

**Step 2.** *We have*

$$\mathcal{F}(\Psi_n, v_n) \rightharpoonup \mathcal{F}(\Phi, \mathbf{v}) \quad \text{in } L^2(\mathbb{R}), \quad (2.6.27)$$

for any  $t \in [0, T]$ , and

$$\mathcal{F}(\Psi_n, v_n) \longrightarrow \mathcal{F}(\Phi, \mathbf{v}) \quad \text{in } L^1([0, T], L_{loc}^2(\mathbb{R})). \quad (2.6.28)$$

*Proof.* Let  $\phi \in L^2(\mathbb{R})$ . We compute

$$\begin{aligned} & \int_{\mathbb{R}} \left( v_n^2(x, t) \Psi_n(x, t) - \mathbf{v}^2(x, t) \Phi(x, t) \right) \phi(x) dx \\ &= \int_{\mathbb{R}} \left( v_n^2(x, t) - \mathbf{v}^2(x, t) \right) \Psi_n(x, t) \phi(x) dx + \int_{\mathbb{R}} \left( \Psi_n(x, t) - \Phi(x, t) \right) \mathbf{v}^2(x, t) \phi(x) dx. \end{aligned} \quad (2.6.29)$$

The second term in the right-hand side goes to 0 when  $n$  goes to  $+\infty$ , since  $\mathbf{v}^2(t)\phi \in L^2(\mathbb{R})$  for all  $t$  on one hand and using (2.6.10) on the other hand. For the first term in the right-hand side, we consider a cut-off function  $\chi$  with support into  $[-1, 1]$  and denote  $\chi_R(x) = \chi(\frac{x}{R})$  for all  $(x, R) \in \mathbb{R} \times (0, +\infty)$ . We set

$$I_n(t) := \int_{\mathbb{R}} \left( v_n^2(x, t) - \mathbf{v}^2(x, t) \right) \Psi_n(x, t) \phi(x) dx,$$

$$I_n^{(1)}(t) := \int_{\mathbb{R}} \left( v_n^2(x, t) - \mathbf{v}^2(x, t) \right) \Psi_n(x, t) \chi_R(x) \phi(x) dx,$$

and

$$I_n^{(2)}(t) := \int_{\mathbb{R}} \left( v_n^2(x, t) - \mathbf{v}^2(x, t) \right) \Psi_n(x, t) \left( 1 - \chi_R(x) \right) \phi(x) dx,$$

so that  $I_n(t) = I_n^{(1)}(t) + I_n^{(2)}(t)$ . By the Cauchy-Schwarz inequality, we have

$$|I_n^{(1)}(t)| \leq \|\Psi_n(t)\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|v_n^2(t) - \mathbf{v}^2(t)\|_{L^\infty([-R, R])}. \quad (2.6.30)$$

Using (2.6.12) and (2.6.14), we infer that

$$I_n^{(1)}(t) \rightarrow 0 \quad \text{for any } t \in [0, T], \quad (2.6.31)$$

as  $n$  goes to  $+\infty$ . Next, we write

$$|I_n^{(2)}(t)| \leq \left( \|v_n(t)\|_{L^\infty(\mathbb{R})}^2 + \|\mathbf{v}(t)\|_{L^\infty(\mathbb{R})}^2 \right) \|\Psi_n(t)\|_{L^2(\mathbb{R})} \|(1 - \chi_R)\phi\|_{L^2(\mathbb{R})}.$$

Since  $\phi \in L^2(\mathbb{R})$ , we have

$$\lim_{R \rightarrow \infty} \|(1 - \chi_R)\phi\|_{L^2(\mathbb{R})} = 0.$$

In view of (2.6.14), this is sufficient to prove that

$$I_n(t) \rightarrow 0 \quad (2.6.32)$$

as  $n$  goes to  $+\infty$ , for all  $t \in [0, T]$ . This yields

$$(v_n^2 \Psi_n)(t) \rightharpoonup (\mathbf{v}^2 \Phi)(t) \quad \text{in } L^2(\mathbb{R}), \quad (2.6.33)$$

for any  $t \in [0, T]$ . Now, we prove

$$v_n^2 \Psi_n \longrightarrow \mathbf{v}^2 \Phi \quad \text{in } L^1([0, T], L_{loc}^2(\mathbb{R})). \quad (2.6.34)$$

We write as in (2.6.29),

$$\|v_n^2 \Psi_n - \mathbf{v}^2 \Phi\|_{L_T^1 L_R^2} \leq \|(v_n^2 - \mathbf{v}^2) \Psi_n\|_{L_T^1 L_R^2} + \|(\Psi_n - \Phi) \mathbf{v}^2\|_{L_T^1 L_R^2}.$$

For the first term in the right-hand side, we infer from the Cauchy-Schwarz inequality, that

$$\begin{aligned} \|(v_n^2 - \mathbf{v}^2) \Psi_n\|_{L_T^1 L_R^2} &\leq \|v_n^2 - \mathbf{v}^2\|_{L_T^2 L_R^2} \|\Psi_n\|_{L_T^2 L_R^\infty} \\ &\leq \|v_n - \mathbf{v}\|_{L_T^4 L_R^4} \left( \|v_n\|_{L_T^4 L_R^4} + \|\mathbf{v}\|_{L_T^4 L_R^4} \right) T^{\frac{1}{2}} \|\Psi_n\|_{L_T^4 L_R^\infty}. \end{aligned}$$

On the other hand, by (2.6.14),  $v_n$  is uniformly bounded on  $L^2([0, T], H^1(\mathbb{R}))$ . By the first equation of (2.2.34) and (2.6.14),  $v_n$  is uniformly bounded in  $H^1([0, T], H^{-1}(\mathbb{R}))$ . We deduce that  $v_n$  is uniformly bounded in  $H^{\frac{1}{3}}([0, T], H^{\frac{1}{3}}(\mathbb{R}))$  and so that  $v_n$  converges to  $\mathbf{v}$  in  $L^4([0, T], L^4([-R, R]))$  when  $n$  goes to  $+\infty$ . Hence, using (2.6.14) once again, we obtain

$$\|(v_n^2 - \mathbf{v}^2)\Psi_n\|_{L_T^1 L_R^2} \rightarrow 0,$$

as  $n$  goes to  $+\infty$ . For the second term we have by the Cauchy-Schwarz inequality and the Sobolev embedding theorem,

$$\|(\Psi_n - \Phi)\mathbf{v}^2\|_{L_T^1 L_R^2} \leq \|\Psi_n - \Phi\|_{L_T^2 L_R^2} \|\mathbf{v}^2\|_{L_T^2 L_R^\infty} \leq M^2 T^{1/2} \|\Psi_n - \Phi\|_{L_T^2 L_R^2}.$$

This yields using (2.6.18),

$$\|(\Psi_n - \Phi)\mathbf{v}^2\|_{L_T^1 L_R^2} \rightarrow 0,$$

as  $n$  goes to  $+\infty$ , which proves (2.6.34). Next, we set

$$\mathcal{G}(v_n, \Psi_n) = \Psi_n \left(1 - F(v_n, \overline{\Psi}_n)\right) \left(1 - F(v_n, \Psi_n)\right).$$

We have by (2.2.33),

$$\partial_x F(v_n, \Psi_n) = v_n \Psi_n \quad \text{and} \quad \partial_x F(\mathbf{v}, \Phi) = \mathbf{v} \Phi.$$

Using the same arguments as in the proof of (2.6.32), we obtain

$$\partial_x F(v_n, \Psi_n) \rightharpoonup \partial_x F(\mathbf{v}, \Phi) \quad \text{in} \quad L^2(\mathbb{R}),$$

for any  $t \in [0, T]$ . Hence,

$$F(v_n, \Psi_n) \longrightarrow F(\mathbf{v}, \Phi) \quad \text{in} \quad L_{loc}^\infty(\mathbb{R}), \tag{2.6.35}$$

for any  $t \in [0, T]$ . Using (2.6.10), (2.6.35) and the same arguments as in the proof of (2.6.33), we conclude that

$$\mathcal{G}(v_n, \Psi_n) \rightharpoonup \mathcal{G}(\mathbf{v}, \Phi) \quad \text{in} \quad L^2(\mathbb{R}), \tag{2.6.36}$$

for any  $t \in [0, T]$ . Next, we use (2.6.18) and (2.6.35) to prove that

$$\mathcal{G}(v_n, \Psi_n) \longrightarrow \mathcal{G}(\mathbf{v}, \Phi) \quad \text{in} \quad L^1([0, T], L_{loc}^2(\mathbb{R})). \tag{2.6.37}$$

This finishes the proof of this step.  $\square$

**Step 3.**  $(\Phi, \mathbf{v})$  is a weak solution of (2.2.32)–(2.2.34).

*Proof.* By (2.6.18), we have

$$i\partial_t \Psi_n \rightarrow i\partial_t \Phi \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times [0, T]), \quad \text{and} \quad \partial_{xx}^2 \Psi_n \rightarrow \partial_{xx}^2 \Phi \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times [0, T]),$$

as  $n \rightarrow +\infty$ . It remains to invoke (2.6.13) and (2.6.35) and to take the limit  $n \rightarrow +\infty$  in the expression

$$\int_0^T \int_{\mathbb{R}} \left( i\partial_t \Psi_n + \partial_{xx}^2 \Psi_n + 2|\Psi_n|^2 \Psi_n + \frac{1}{2} v_n^2 \Psi_n - \text{Re} \left( \Psi \left(1 - 2F(v_n, \overline{\Psi}_n)\right) \right) \left(1 - 2F(v_n, \Psi_n)\right) \right) \bar{h} = 0,$$

where  $h \in \mathcal{C}_c^\infty(\mathbb{R} \times [0, T])$ , in order to establish that  $(\Phi, \mathbf{v})$  is solution to (2.2.32) in the sense of distributions. In addition, using the same arguments as above and (2.6.35) we prove that  $(\Phi, \mathbf{v})$  is solution to (2.2.34) in the sense of distributions. Moreover, we infer from (2.6.5) that  $\Phi(\cdot, 0) = \Psi_0$  and from (2.6.6) that  $\mathbf{v}(\cdot, 0) = v_0$ .  $\square$

In order to prove that the function  $(\Phi, \mathbf{v})$  coincides with the solution  $(\Psi, v)$  in Proposition 2.6.2, it is sufficient, in view of the uniqueness result given by Proposition 2.2.5, to establish that

**Step 4.**  $\Phi \in \mathcal{C}([0, T], L^2(\mathbb{R}))$  and  $\mathbf{v} \in \mathcal{C}([0, T], H^1(\mathbb{R}))$ .

*Proof.* First, we prove that  $\Phi \in \mathcal{C}([0, T], L^2(\mathbb{R}))$ . This is a direct consequence of the identity

$$\Phi(x, t) = S(t)\Phi_0 + \int_0^t S(t-t') \left( 2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathbf{v})(\cdot, t') \right) dt'. \quad (2.6.38)$$

Indeed, let us denote

$$G(\Phi, \mathbf{v})(t) = \int_0^t S(t-t') \left( 2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathbf{v})(\cdot, t') \right) dt'.$$

Since  $S(t)\Phi_0 \in \mathcal{C}([0, T], L^2(\mathbb{R}))$ , it is enough to show that  $G(\Phi, \mathbf{v}) \in \mathcal{C}([0, T], L^2(\mathbb{R}))$ . We take  $(t_1, t_2) \in [0, T]^2$  and we write

$$\begin{aligned} G(\Phi, \mathbf{v})(t_1) - G(\Phi, \mathbf{v})(t_2) &= \int_0^{t_1} \left( S(t_1-t') - S(t_2-t') \right) \left( 2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathbf{v})(\cdot, t') \right) dt' \\ &\quad - \int_{t_1}^{t_2} S(t-t') \left( 2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathbf{v})(\cdot, t') \right) dt'. \end{aligned}$$

For the second term in the right-hand side, we use the Strichartz and Cauchy-Schwarz inequalities to obtain

$$\begin{aligned} &\left\| \int_{t_1}^{t_2} S(t-t') \left( 2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathbf{v})(\cdot, t') \right) dt' \right\|_{L^2} \\ &\leq M \| 2|\Phi|^2\Phi + \mathcal{F}(\Phi, \mathbf{v}) \|_{L^1([t_1, t_2], L^2(\mathbb{R}))} \\ &\leq M |t_1 - t_2|^{\frac{1}{2}} \| |\Phi|^2\Phi \|_{L^2_{T,x}} + M |t_1 - t_2| \| \mathcal{F}(\Phi, \mathbf{v}) \|_{L^\infty_T L^2_x}. \end{aligned} \quad (2.6.39)$$

For the first term, we write

$$S(t_1-t') - S(t_2-t') = S(t_1-t') \left( 1 - S(t_2-t_1) \right).$$

Hence,

$$\begin{aligned} &\left\| \int_0^{t_1} \left( S(t_1-t') - S(t_2-t') \right) \left( 2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathbf{v})(\cdot, t') \right) dt' \right\|_{L^2} \\ &= \left\| \left( 1 - S(t_2-t_1) \right) G(\Phi, \mathbf{v})(t_1) \right\|_{L^2}. \end{aligned} \quad (2.6.40)$$

Taking the limit  $t_2 \rightarrow t_1$  in (2.6.39) and (2.6.40), we obtain that  $\Phi \in \mathcal{C}([0, T], L^2(\mathbb{R}))$ .

Now, let us prove (2.6.38). Denote  $\tilde{\Phi}$  the function given by the right-hand side of (2.6.38). We will prove that

$$\Psi_n(t) \rightharpoonup \tilde{\Phi}(t) \quad \text{in } L^2(\mathbb{R}), \quad (2.6.41)$$

for all  $t \in \mathbb{R}$ . This yields  $\Phi \equiv \tilde{\Phi}$  by uniqueness of the weak limit. Let  $R > 0$  and denote by  $\chi_R$  the function defined in Step 2. Set

$$\begin{aligned} G_n^{(1)}(\cdot, t) &= \int_0^t S(t-t') \chi_R \left( 2(|\Psi_n|^2\Psi_n)(\cdot, t') + \mathcal{F}(\Psi_n, v_n)(\cdot, t') \right) dt', \\ G_n^{(2)}(\cdot, t) &= \int_0^t S(t-t') (1 - \chi_R) \left( 2(|\Psi_n|^2\Psi_n)(\cdot, t') + \mathcal{F}(\Psi_n, v_n)(\cdot, t') \right) dt', \\ G^{(1)}(\cdot, t) &= \int_0^t S(t-t') \chi_R \left( 2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathbf{v})(\cdot, t') \right) dt', \end{aligned}$$



and

$$G^{(2)}(\cdot, t) = \int_0^t S(t-t')(1 - \chi_R) \left( 2(|\Phi|^2 \Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathbf{v})(\cdot, t') \right) dt',$$

for all  $t \in \mathbb{R}$ , so that  $G(\Phi, \mathbf{v}) = G^{(1)} + G^{(2)}$  and  $G(\Psi_n, v_n) = G_n^{(1)} + G_n^{(2)}$ . Since  $S(t)\Psi_{n,0} \rightharpoonup S(t)\Phi_0$  in  $L^2(\mathbb{R})$  as  $n \rightarrow +\infty$  for all  $t \in \mathbb{R}$ , it is sufficient to show that  $G(\Psi_n, v_n)(t) \rightarrow G(\Phi, \mathbf{v})(t)$  in  $L^2(\mathbb{R})$  as  $n \rightarrow +\infty$  for all  $t \in \mathbb{R}$ . We write

$$\begin{aligned} \left( G(\Psi_n, v_n)(t) - G(\Phi, \mathbf{v})(t), \varphi \right)_{L^2} &= \int_{-\infty}^{+\infty} \left[ G_n^{(1)}(x, t) - G^{(1)}(x, t) \right] \overline{\varphi(x)} dx \\ &\quad + \int_{-\infty}^{+\infty} \left[ G_n^{(2)}(x, t) - G^{(2)}(x, t) \right] \overline{\varphi(x)} dx \\ &= I_n^R(t) + J_n^R(t). \end{aligned}$$

For the first integral, using the Cauchy-Schwartz inequality, the Strichartz estimates for the admissible pairs (6, 6) and  $(\infty, 2)$ , the Hölder inequality as well as (2.6.19), there exists a positive constant  $M$  such that for all  $t \in [0, T]$  we have

$$\begin{aligned} |I_n^R(t)| &\leq \|G_n^{(1)}(t) - G^{(1)}(t)\|_{L^2} \|\varphi\|_{L^2} \\ &\leq M \|\varphi\|_{L^2} \left( \|\Psi_n|^2 \Psi_n - |\Phi|^2 \Phi\|_{L_{T,R}^{\frac{6}{5}}} + \|\mathcal{F}(\Psi_n, v_n) - \mathcal{F}(\Phi, \mathbf{v})\|_{L_T^1 L_R^2} \right) \\ &\leq M \|\varphi\|_{L^2} \left( \|\mathcal{F}(\Psi_n, v_n) - \mathcal{F}(\Phi, \mathbf{v})\|_{L_T^1 L_R^2} + \|\Psi_n - \Phi\|_{L_{T,R}^2} \left( \|\Psi_n\|_{L_{T,R}^6}^2 + \|\Phi\|_{L_{T,R}^6}^2 \right) \right). \end{aligned}$$

Then, using (2.6.18) and (2.6.28), we obtain for all  $t \in \mathbb{R}$

$$|I_n^R(t)| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Next, using the Hölder inequality we have

$$\begin{aligned} &|J_n^R(t)| \\ &\leq 2 \left( \int_0^T \int_{-\infty}^{\infty} \left| |\Psi_n|^2 \Psi_n(x, t') - |\Phi|^2 \Phi(x, t') \right|^{\frac{6}{5}} dx dt' \right)^{\frac{5}{6}} \left( \int_0^T \int_{|x| \geq R} |S(t-t') \varphi|^6 dx dt' \right)^{\frac{1}{6}} \\ &\quad + \int_0^T \left( \int_{-\infty}^{\infty} \left| \mathcal{F}(\Psi_n, v_n)(x, t') - \mathcal{F}(\Phi, \mathbf{v})(x, t') \right|^2 dx \right)^{\frac{1}{2}} dt' \sup_{t' \in [0, T]} \left( \int_{|x| \geq R} |S(t-t') \varphi(x)|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

The terms in the right-hand side are bounded by a constant independent of  $n$ . Besides, since (6, 6) and  $(\infty, 2)$  are admissible pairs, we have  $\|S(t)\varphi\|_{L_{T,x}^6} \leq M \|\varphi\|_{L^2(\mathbb{R})}$  and  $\|S(t)\varphi\|_{L_T^\infty L^2(\mathbb{R})} \leq M \|\varphi\|_{L^2(\mathbb{R})}$ , so that, by the dominated convergence theorem and the fact that  $t \mapsto S(t)$  is uniformly continuous from  $[0, T]$  to  $L^2(\mathbb{R})$ , we obtain

$$\lim_{R \rightarrow \infty} \int_0^T \int_{|x| \geq R} |S(t)\varphi|^6 dx dt = \lim_{R \rightarrow \infty} \sup_{t \in [0, T]} \left( \int_{|x| \geq R} |S(t)\varphi(x)|^2 dx \right)^{\frac{1}{2}} = 0.$$

Hence,

$$\lim_{R \rightarrow \infty} |J_n^R(t)| = 0 \quad \text{uniformly with respect to } n \in \mathbb{N},$$

for any  $t \in [0, T]$ . This completes the proof of (2.6.41) and then of (2.6.38). This leads to the fact that  $\Phi \in \mathcal{C}^0([0, T], L^2(\mathbb{R}))$ .

Now, let us prove that  $\mathbf{v} \in \mathcal{C}^0([0, T], H^1(\mathbb{R}))$ . Since  $(\Phi, \mathbf{v})$  verifies the first equation in (2.2.34),  $\Phi \in L^\infty([0, T], L^2(\mathbb{R}))$  and  $F(\Psi, \mathbf{v}) \in L^\infty([0, T], L^\infty(\mathbb{R}))$ , we have  $\mathbf{v} \in H^1([0, T], H^{-1}(\mathbb{R}))$ . This

yields, using the Sobolev embedding theorem,  $\mathbf{v} \in \mathcal{C}^0([0, T], H^{-1}(\mathbb{R}))$ . Let  $(t_1, t_2) \in [0, T]^2$ . We can write that

$$\begin{aligned} \int_{\mathbb{R}} |\mathbf{v}(t_1, x) - \mathbf{v}(t_2, x)|^2 dx &= \left\langle \mathbf{v}(t_1, x) - \mathbf{v}(t_2, x), \mathbf{v}(t_1, x) - \mathbf{v}(t_2, x) \right\rangle_{H^{-1}, H^1} \\ &\leq \|\mathbf{v}(t_1, x) - \mathbf{v}(t_2, x)\|_{H^{-1}} \|\mathbf{v}(t_1, x) - \mathbf{v}(t_2, x)\|_{H^1}. \end{aligned}$$

Since  $\mathbf{v} \in \mathcal{C}^0([0, T], H^{-1}(\mathbb{R})) \cap L^\infty([0, T], H^1(\mathbb{R}))$ , we obtain  $\mathbf{v} \in \mathcal{C}^0([0, T], L^2(\mathbb{R}))$ . Next, we write

$$\begin{aligned} &\|F(\mathbf{v}, \Phi)(t_1) - F(\mathbf{v}, \Phi)(t_2)\|_{L^\infty(\mathbb{R})} \\ &\leq \|\mathbf{v}(t_1) - \mathbf{v}(t_2)\|_{L^2} \|\Phi(t_1)\|_{L^2} + \|\Phi(t_2) - \Phi(t_1)\|_{L^2} \|\mathbf{v}(t_2)\|_{L^2}. \end{aligned}$$

Using the fact that  $\Phi, \mathbf{v} \in \mathcal{C}^0([0, T], L^2(\mathbb{R}))$ , we infer that  $F(\mathbf{v}, \Phi) \in \mathcal{C}^0([0, T], L^\infty(\mathbb{R}))$ . Then, by the second equation in (2.2.34),  $\mathbf{v} \in \mathcal{C}^0([0, T], H^1(\mathbb{R}))$ . This finishes the proof of this step.  $\square$

This achieves the proof of Proposition 2.6.2.  $\square$

Finally, we give the proof of Proposition 2.6.1.

*Proof.* In view of Proposition 2.6.2, it is sufficient to prove the convergence of  $w_n$ . The proof follows the arguments in the proof of (2.6.27). Let  $\phi \in L^2(\mathbb{R})$ . We rely on (2.4.41) to write

$$\begin{aligned} &\int_{\mathbb{R}} [w^*(t, x) - w_n(t, x)] \phi(x) dx \\ &= 2 \int_{\mathbb{R}} \operatorname{Im} \left( \frac{\Psi^*(t, x) (1 - 2F(v^*, \Psi^*)(t, x))}{1 - (v^*)^2(t, x)} - \frac{\Psi_n(t, x) (1 - 2F(v_n, \Psi_n)(t, x))}{1 - (v_n)^2(t, x)} \right) \phi(x) dx \\ &= 2 \int_{\mathbb{R}} \operatorname{Im} \left( \frac{\Psi^*(t, x)}{1 - (v^*)^2(t, x)} - \frac{\Psi_n(t, x)}{1 - (v_n)^2(t, x)} \right) \phi(x) dx \\ &\quad - 4 \int_{\mathbb{R}} \operatorname{Im} \left( \frac{\Psi^*(t, x) F(v^*, \Psi^*)(t, x)}{1 - (v^*)^2(t, x)} - \frac{\Psi_n(t, x) F(v_n, \Psi_n)(t, x)}{1 - (v_n)^2(t, x)} \right) \phi(x) dx, \end{aligned}$$

for all  $t \in [0, T]$ . Then, we use the same arguments as in the proof of (2.6.27) to show that the two last terms in the right-hand side go to 0 when  $n$  goes to  $+\infty$ . This finishes the proof of the proposition.  $\square$

## 2.6.2 Exponential decay of $\chi_c$

In this subsection, we recall the explicit formula and some useful properties of the operator  $\mathcal{H}_c$ , and then study its negative eigenfunction  $\chi_c$ . For  $c \in (-1, 1) \setminus \{0\}$ , the operator  $\mathcal{H}_c$  is given in explicit terms by

$$\mathcal{H}_c(\varepsilon) = \begin{pmatrix} \mathcal{L}_c(\varepsilon_v) + c^2 \frac{(1+v_c^2)^2}{(1-v_c^2)^3} \varepsilon_v - c \frac{1+v_c^2}{1-v_c^2} \varepsilon_w \\ -c \frac{1+v_c^2}{1-v_c^2} \varepsilon_v + (1-v_c^2) \varepsilon_w \end{pmatrix}, \quad (2.6.42)$$

where  $\varepsilon = (\varepsilon_v, \varepsilon_w)$ , and

$$\mathcal{L}_c(\varepsilon_v) = -\partial_x \left( \frac{\partial_x \varepsilon_v}{1-v_c^2} \right) + \left( 1 - c^2 - (5+c^2)v_c^2 + 2v_c^4 \right) \frac{\varepsilon_v}{(1-v_c^2)^2}.$$

In view of (2.6.42), the operator  $\mathcal{H}_c$  is an isomorphism from  $H^2(\mathbb{R}) \times L^2(\mathbb{R}) \cap \operatorname{Span}(\partial_x Q_c)^\perp$  onto  $\operatorname{Span}(\partial_x Q_c)^\perp$ . In addition, there exists a positive number  $A_c$ , depending continuously on  $c$ , such that

$$\|\mathcal{H}_c^{-1}(f, g)\|_{H^{k+2}(\mathbb{R}) \times H^k(\mathbb{R})} \leq A_c \|(f, g)\|_{H^k(\mathbb{R})^2}, \quad (2.6.43)$$

for any  $(f, g) \in H^k(\mathbb{R})^2 \cap \text{Span}(\partial_x Q_c)^\perp$  and any  $k \in \mathbb{N}$ .

The following proposition establishes the coercivity of the quadratic form  $H_c$  under suitable orthogonality conditions.

**Proposition 2.6.3.** *Let  $c \in (-1, 1) \setminus \{0\}$ . There exists a positive number  $\Lambda_c$ , depending only on  $c$ , such that*

$$H_c(\varepsilon) \geq \Lambda_c \|\varepsilon\|_{H^1 \times L^2}^2, \quad (2.6.44)$$

for any pair  $\varepsilon \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$  satisfying the two orthogonality conditions

$$\langle \partial_x Q_c, \varepsilon \rangle_{L^2 \times L^2} = \langle \chi_c, \varepsilon \rangle_{L^2 \times L^2} = 0. \quad (2.6.45)$$

Moreover, the map  $c \mapsto \Lambda_c$  is uniformly bounded from below on any compact subset of  $(-1, 1) \setminus \{0\}$ .

The proof relies on standard Sturm-Liouville theory (see e.g. the proof of Proposition 1 in [19] for more details).

Now, we turn to the analysis of the pair  $\chi_c$ .

**Lemme 2.6.1.** *The pair  $\chi_c$  belongs to  $\mathcal{C}^\infty(\mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R})$ . In addition, there exist two positive numbers  $A_c$  and  $a_c$ , depending continuously on  $c$ , such that  $a_c > \sqrt{1 - c^2}$  and*

$$|\partial_x^k \chi_c| \leq A_c e^{-a_c |x|} \quad \text{on } \mathbb{R} \quad \text{for } k \in \{0, 1, 2\}. \quad (2.6.46)$$

*Proof.* We denote  $\chi_c := (\zeta_c, \xi_c)$ . Since  $\mathcal{H}_c(\chi_c) = -\tilde{\lambda}_c \chi_c$ , we have the following system

$$\begin{aligned} -\partial_x \left( \frac{\partial_x \zeta_c}{1 - v_c^2} \right) + \left( 1 - c^2 - (5 + c^2)v_c^2 + 2v_c^4 \right) \frac{\zeta_c}{(1 - v_c^2)^2} + c^2 \frac{(1 + v_c^2)^2}{(1 - v_c^2)^3} \zeta_c \\ - c \frac{1 + v_c^2}{1 - v_c^2} \xi_c = -\tilde{\lambda}_c \zeta_c, \end{aligned} \quad (2.6.47)$$

$$c \frac{1 + v_c^2}{1 - v_c^2} \zeta_c = (1 - v_c^2 + \tilde{\lambda}_c) \xi_c. \quad (2.6.48)$$

It follows from standard elliptic theory that  $\chi_c \in H^2(\mathbb{R}) \times L^2(\mathbb{R})$ . Since the coefficients in (2.6.48) are smooth, bounded from below and above, we infer from a standard bootstrap argument that  $\chi_c \in \mathcal{C}^\infty(\mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R})$ . Notice in particular that, by the Sobolev embedding theorem,  $\chi_c$  and  $\partial_x \chi_c$  are bounded on  $\mathbb{R}$ . Then, we deduce from the first statement in (2.5.11) that <sup>5</sup>

$$-\partial_{xx} \zeta_c + (1 + \tilde{\lambda}_c) \zeta_c - c \xi_c = \mathcal{O}(v_c^2), \quad (2.6.49)$$

$$\zeta_c = \frac{1 + \tilde{\lambda}_c}{c} \xi_c + \mathcal{O}(v_c^2). \quad (2.6.50)$$

Note that, we have

$$B_c \exp(-\sqrt{1 - c^2}|x|) \leq v_c(x) \leq A_c \exp(-\sqrt{1 - c^2}|x|) \quad \text{for all } x \in \mathbb{R}, \quad (2.6.51)$$

where  $B_c$  and  $A_c$  are two positive numbers.

In order to prove (2.6.46), we now introduce (2.6.50) into (2.6.49) to obtain,

$$-\partial_{xx} \zeta_c + b_c^2 \zeta_c = \mathcal{O}(\exp(-2\sqrt{1 - c^2}|x|)), \quad (2.6.52)$$

$$\zeta_c = \frac{c}{1 + \tilde{\lambda}_c} \xi_c + \mathcal{O}(\exp(-2\sqrt{1 - c^2}|x|)), \quad (2.6.53)$$

---

5. The notation  $\mathcal{O}(v_c^2)$  refers to a quantity, which is bounded by  $A_c v_c^2$  (pointwisely), where the positive number  $A_c$  depends only on  $c$ .

with  $b_c^2 = \frac{1-c^2+2\tilde{\lambda}_c+(\tilde{\lambda}_c)^2}{1+\tilde{\lambda}_c} > 1 - c^2$ . Next, we set

$$g_c := -\partial_{xx}\zeta_c + b_c^2\zeta_c, \quad (2.6.54)$$

so that  $g_c(x) = \mathcal{O}(\exp(-2\sqrt{1-c^2}|x|))$  for all  $x \in \mathbb{R}$ . Using the variation of constant method, we obtain, for all  $x \in \mathbb{R}$ ,

$$\zeta_c(x) = A(x)e^{b_c x} + A_c e^{b_c x} + B(x)e^{-b_c x} + B_c e^{-b_c x},$$

with

$$A(x) = \frac{-1}{2b_c} \int_0^x e^{-b_c t} g_c(t) dt,$$

and

$$B(x) = \frac{-1}{2b_c} \int_0^x e^{b_c t} g_c(t) dt.$$

Since  $\zeta_c \in L^2(\mathbb{R})$ , this leads to

$$\zeta_c(x) = \mathcal{O}\left(\exp\left(-2\sqrt{1-c^2}|x|\right) + \exp(-b_c|x|)\right).$$

Hence, we can take  $a_c = \min\{2\sqrt{1-c^2}, b_c\}$  and invoke (2.6.50) to obtain (2.6.46) for  $k = 0$ . Using (2.5.10), (2.5.11), (2.6.47), (2.6.48) and (2.6.51), we extend (2.6.46) to  $k \in 1, 2$ . □

## Chapitre 3

# On the asymptotic stability in the energy space for multi-solitons of the Landau-Lifshitz equation.

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### Abstract

We establish the asymptotic stability of multi-solitons for the one-dimensional Landau-Lifshitz equation with an easy-plane anisotropy. The solitons have non-zero speed, are ordered according to their speeds and have sufficiently separated initial positions. We provide the asymptotic stability around solitons and between solitons. More precisely, we show that for an initial datum close to a sum of  $N$  dark solitons, the corresponding solution converges weakly to one of the solitons in the sum, when it is translated to the centre of this soliton, and converges weakly to zero when it is translated between solitons.

### 3.1 Introduction

We consider the one-dimensional Landau-Lifshitz equation

$$\partial_t m + m \times (\partial_{xx} m + \lambda m_3 e_3) = 0, \tag{LL}$$

for a map  $m = (m_1, m_2, m_3) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{S}^2$ , where  $e_3 = (0, 0, 1)$  and  $\lambda \in \mathbb{R}$ . This equation, which was introduced by Landau and Lifshitz in [34], describes the dynamics of magnetization in a one-dimensional ferromagnetic material, for example in CsNiF<sub>3</sub> or TMNC (see e.g. [30, 33] and the references therein).  $\lambda$  is the anisotropy parameter of the material. The case  $\lambda > 0$  gives account for an easy-axis anisotropy and the case  $\lambda < 0$  of an easy-plane anisotropy. The equation reduces to the one-dimensional Schrödinger map equation in the isotropic case  $\lambda = 0$ . This equation has been intensively studied (see e.g. [3, 27, 31]). In this paper, we are interested in the easy-plane anisotropy case ( $\lambda < 0$ ). Scaling the map  $m$ , if necessary, we can assume from now on  $\lambda = -1$ .

The Hamiltonian for the Landau-Lifshitz equation, the so-called Landau-Lifshitz energy, is given by

$$E(m) := \frac{1}{2} \int_{\mathbb{R}} (|\partial_x m|^2 + m_3^2).$$

In this paper, we study the solutions  $m$  to (LL) with finite Landau-Lifshitz energy, i.e. which belong to the energy space

$$\mathcal{E}(\mathbb{R}) := \left\{ v : \mathbb{R} \rightarrow \mathbb{S}^2, \text{ s.t. } v' \in L^2(\mathbb{R}) \text{ and } v_3 \in L^2(\mathbb{R}) \right\}.$$

A soliton with speed  $c$  is a travelling-wave solution of (LL) which has the form

$$m(x, t) := u(x - ct).$$

Its profile  $u$  is solution to the ordinary differential equation

$$u'' + |u'|^2 u + u_3^2 u - u_3 e_3 + cu \times u' = 0. \tag{TWE}$$

The solutions of this equation are explicit. If  $|c| < 1$ , there exist non-constant solutions  $u_c$  to (TWE), which are given by the formulae

$$[u_c]_1(x) = \frac{c}{\cosh\left(\frac{1-c^2}{2}x\right)}, \quad [u_c]_2(x) = \tanh\left(\frac{1-c^2}{2}x\right), \quad [u_c]_3(x) = \frac{(1-c^2)^{\frac{1}{2}}}{\cosh\left(\frac{1-c^2}{2}x\right)},$$

up to the invariances of the problem, i.e. translations, rotations around the axis  $x_3$  and the orthogonal symmetry with respect to the plane  $x_3 = 0$  (see [17] for more details). Else, when  $|c| \geq 1$ , the only solutions with finite Landau-Lifshitz energy are the constant vectors in  $\mathbb{S}^1 \times \{0\}$ .

In dimension one the equation is completely integrable using the inverse scattering method (see e.g. [21]). This method allows to justify the existence of multi-solitons for (LL) and to compute their expression (see [52, 57]). Multi-solitons, which can be considered as a nonlinear superposition of single solitons, are exact solutions to (LL). Our main goal is to prove the asymptotic stability of multi-solitons (see Theorem 3.1.1 below).

Martel, Merle and Tsai proved the asymptotic stability for multi-solitons of the subcritical gKdV equations in [51]. Martel and Merle stated this result for one soliton of the generalized KdV equation in [42] and then they refined the results for multi-solitons in [47]. This method was successfully adapted by Bethuel, Gravejat and Smets to prove the asymptotic stability for a dark soliton of the Gross-Pitaevskii equation in [6] and then in [1] to show the same result for the Landau-Lifshitz equation. Cuccagna and Jenkins proved similar results for the Gross-Pitaevskii equation in [14] using the inverse scattering method. Perelman established the asymptotic stability of multi-solitons for the nonlinear Schrödinger equation in [55].

In the next subsections, we first introduce the hydrodynamical framework in which we provide all the analysis and we provide our main result.

### 3.1.1 The hydrodynamical framework

We denote by  $\check{m}$  the map defined by  $\check{m} := m_1 + im_2$ . We have

$$|\check{m}(x)| = (1 - m_3^2(x))^{\frac{1}{2}} \rightarrow 1,$$

as  $x \rightarrow \pm\infty$ , using the fact that  $m_3$  belongs to  $H^1(\mathbb{R})$ , and the Sobolev embedding theorem. This allows us, as in the case of the Gross-Pitaevskii equation (see e.g. [4]), to consider the hydrodynamical framework for the Landau-Lifshitz equation. In terms of the maps  $\check{m}$  and  $m_3$ , this equation may be written as

$$\begin{cases} i\partial_t\check{m} - m_3\partial_{xx}\check{m} + \check{m}\partial_{xx}m_3 - \check{m}m_3 = 0, \\ \partial_t m_3 + \partial_x \langle i\check{m}, \partial_x \check{m} \rangle_{\mathbb{C}} = 0. \end{cases}$$

When the map  $\check{m}$  does not vanish, one can write it as  $\check{m} = (1 - m_3^2)^{1/2} \exp i\varphi$ . The hydrodynamical variables  $v := m_3$  and  $w := \partial_x\varphi$  verify the following system

$$\begin{cases} \partial_t v = \partial_x((v^2 - 1)w), \\ \partial_t w = \partial_x\left(\frac{\partial_{xx}v}{1 - v^2} + v\frac{(\partial_x v)^2}{(1 - v^2)^2} + v(w^2 - 1)\right). \end{cases} \quad (\text{HLL})$$

This system is similar to the hydrodynamical Gross-Pitaevskii equation (see e.g. [6]).<sup>1</sup> The Cauchy problem in the space  $X(\mathbb{R}) := H^1(\mathbb{R}) \times L^2(\mathbb{R})$  for this system was solved by de Laire and Gravejat in [19], where local well-posedness is established.

In this framework, the Landau-Lifshitz energy is expressed as

$$E(\mathbf{v}) := \int_{\mathbb{R}} e(\mathbf{v}) := \frac{1}{2} \int_{\mathbb{R}} \left( \frac{(v')^2}{1 - v^2} + (1 - v^2)w^2 + v^2 \right), \quad (3.1.1)$$

where  $\mathbf{v} := (v, w)$  denotes the hydrodynamical pair. The momentum  $P$ , defined by

$$P(\mathbf{v}) := \int_{\mathbb{R}} vw, \quad (3.1.2)$$

is also conserved by the Landau-Lifshitz flow. When  $c \neq 0$ , the function  $\check{u}_c$  does not vanish. The hydrodynamical pair  $Q_c := (v_c, w_c)$  is given by

$$v_c(x) = \frac{(1 - c^2)^{\frac{1}{2}}}{\cosh\left((1 - c^2)^{\frac{1}{2}}x\right)}, \quad \text{and} \quad w_c(x) = \frac{c v_c(x)}{1 - v_c(x)^2} = \frac{c(1 - c^2)^{\frac{1}{2}} \cosh\left((1 - c^2)^{\frac{1}{2}}x\right)}{\sinh\left((1 - c^2)^{\frac{1}{2}}x\right)^2 + c^2}. \quad (3.1.3)$$

The flow of (HLL) is invariant by translations and the opposite map  $(v, w) \mapsto (-v, -w)$ . These geometric transformations play an important role in the stability statement. We will show that the stability depends on these invariances.

We denote

$$Q_{c,a,s}(x) := sQ_c(x - a) := (sv_c(x - a), sw_c(x - a)),$$

for  $a \in \mathbb{R}$  and  $s \in \{\pm 1\}$ . We also define

$$S_{c,a,s} := (V_{c,a,s}, W_{c,a,s}) := \sum_{j=1}^N Q_{c_j, a_j, s_j}, \quad (3.1.4)$$

---

1. The hydrodynamical terminology originates in the fact that the hydrodynamical Gross-Pitaevskii equation is similar to the Euler equation for an irrotational fluid (see e.g. [5]).

with  $N \in \mathbb{N}^*$ ,  $\mathbf{c} = (c_1, \dots, c_N)$ , with  $c_j \neq 0$ ,  $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$  and  $\mathbf{s} = (s_1, \dots, s_N) \in \{\pm 1\}^N$ . In the original framework, this can be translated in the following way

$$R_{\mathbf{c}, \mathbf{a}, \mathbf{s}} := \left( (1 - V_{\mathbf{c}, \mathbf{a}, \mathbf{s}}^2)^{\frac{1}{2}} \cos(\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}), (1 - V_{\mathbf{c}, \mathbf{a}, \mathbf{s}}^2)^{\frac{1}{2}} \sin(\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}), V_{\mathbf{c}, \mathbf{a}, \mathbf{s}} \right),$$

where we have denoted

$$\Theta_{\mathbf{c}, \mathbf{a}, \mathbf{s}}(x) := \int_0^x W_{\mathbf{c}, \mathbf{a}, \mathbf{s}}(y) dy,$$

for any  $x \in \mathbb{R}$ . In this paper, we provide the proof of the asymptotic stability around any soliton and between any two solitons of a sum of well-separated solitons with ordered speed, i.e.

$$a_j - a_{j-1} \geq L, \quad \text{for any } j \in \{2, \dots, N\}, \quad \text{where } L > 0, \quad \text{and } c_1 < \dots < c_N.$$

Multi-solitons are orbitally stable under these invariance parameters (see [19] for more details). We recall this result in the next section (see Theorem 3.2.1 below).

### 3.1.2 Asymptotic stability in the original framework

In this subsection, we provide our main result. First, we introduce a metric structure on the energy space  $\mathcal{E}(\mathbb{R})$  in order to establish them. As it was done by de Laire and Gravejat in [19], we define the following distance

$$d_{\mathcal{E}}(f, g) := |\check{f}(0) - \check{g}(0)| + \|f' - g'\|_{L^2(\mathbb{R})} + \|f_3 - g_3\|_{L^2(\mathbb{R})},$$

where  $f = (f_1, f_2, f_3)$  and  $\check{f} = f_1 + if_2$  (respectively for  $g$ ). With this choice,  $(\mathcal{E}(\mathbb{R}), d_{\mathcal{E}})$  is a metric space. The following theorem shows the asymptotic stability around each soliton and between the solitons.

**Théorème 3.1.1.** *Let  $\mathbf{s} \in \{\pm 1\}^N$ ,  $\mathbf{c}^0 = (c_1^0, \dots, c_N^0) \in (-1, 1)^N$ , with  $c_j^0 \neq 0$ , such that*

$$c_1^0 < \dots < c_N^0,$$

*and  $\mathbf{a}^0 = (a_1^0, \dots, a_N^0) \in \mathbb{R}^N$ . There exist a positive number  $\beta_{\mathbf{c}^0}$ , depending only on  $\mathbf{c}^0$ , and a positive number  $L^0$  such that, if*

$$d_{\mathcal{E}}(m^0, R_{\mathbf{c}^0, \mathbf{a}^0, \mathbf{s}}) \leq \beta_{\mathbf{c}^0},$$

*and*

$$\mathbf{a}^0 \in \text{Pos}(L^0),$$

*then there exist  $N$  numbers  $\tilde{\mathbf{c}} := (\tilde{c}_1, \dots, \tilde{c}_N) \in (-1, 1)^N$ , with  $\tilde{c}_j \neq 0$ , and  $2N$  functions  $a_j \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$  and  $\theta_j \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$ , such that*

$$a'_j(t) \rightarrow \tilde{c}_j, \quad \text{and} \quad \theta'_j(t) \rightarrow 0,$$

*as  $t \rightarrow +\infty$ , and for which the map*

$$m_{\theta_j} := \left( \cos(\theta_j)m_1 - \sin(\theta_j)m_2, \sin(\theta_j)m_1 + \cos(\theta_j)m_2, m_3 \right),$$

*corresponding to the unique global solution  $m \in \mathcal{C}^0(\mathbb{R}, \mathcal{E}(\mathbb{R}))$  with initial datum  $m^0$ , satisfies the convergences*

$$\begin{aligned} & \sum_{j=1}^N \left[ \partial_x m_{\theta_j(t)}(\cdot + a_j(t), t) - \partial_x u_{\tilde{c}_j} \right] \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}), \\ & \sum_{j=1}^N \left[ m_{\theta_j(t)}(\cdot + a_j(t), t) - u_{\tilde{c}_j} \right] \rightarrow 0 \quad \text{in } L_{\text{loc}}^\infty(\mathbb{R}), \\ & \text{and} \quad \sum_{j=1}^N \left[ m_3(\cdot + a_j(t), t) - [u_{\tilde{c}_j}]_3 \right] \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}), \end{aligned} \tag{3.1.5}$$



as  $t \rightarrow +\infty$ . In addition, for any map  $b_j$  satisfying the following conditions :

$$\left\{ \begin{array}{l} b_1(t) < a_1(t), \\ a_{j-1}(t) < b_j(t) < a_j(t) \quad \forall 2 \leq j \leq N, \\ b_{N+1}(t) > a_N(t), \end{array} \right. \quad (3.1.6)$$

for all  $t \in \mathbb{R}_+$  and

$$\left\{ \begin{array}{l} \liminf_{t \rightarrow +\infty} \frac{b_j(t)}{t} > c_{j-1}^\infty, \\ \limsup_{t \rightarrow +\infty} \frac{b_j(t)}{t} < c_j^\infty, \end{array} \right. \quad (3.1.7)$$

with

$$\left\{ \begin{array}{l} c_0^\infty = -1, \\ c_{N+1}^\infty = 1, \end{array} \right.$$

we have

$$\begin{aligned} \sum_{j=1}^N \partial_x m_{\theta_j(t)}(\cdot + b_j(t), t) &\rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}), \\ \sum_{j=1}^N [m_{\theta_j(t)}(\cdot + b_j(t), t) - e_2] &\rightarrow 0 \quad \text{in } L_{\text{loc}}^\infty(\mathbb{R}), \\ \text{and } \sum_{j=1}^N m_3(\cdot + a_j(t), t) &\rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}), \end{aligned} \quad (3.1.8)$$

as  $t \rightarrow +\infty$ , with  $e_2 = (0, 1, 0)$ .

The proof of this theorem is similar to the one of Theorem 1.1 in [1]. It relies on a modulation argument and Theorem 3.1.2. The proof still applies for our case of  $N$  solitons since each term of the sums in (3.1.5) and (3.1.8) converges to zero. It remains to deal with each term separately and apply the arguments used for the case of one soliton  $N$  times. In particular, (3.1.5) and (3.1.8) are direct consequences of (3.1.9) and (3.1.10) respectively (see Subsection 2.4 in [1] for more details).

**Remarque 3.1.1.** *The locally strong asymptotic stability result for multi-solitons, as stated by Martel, Merle and Tsai in [51] for the KdV equation, is stronger than the two weak asymptotic stability results stated in this paper. It is still an open problem for this equation. As a matter of fact, the method used by Martel, Merle and Tsai is based on a monotonicity argument for the localized energy. This argument is not obvious in our case, since dispersion has both positive and negative speeds in contrast with the KdV case in which dispersion has only negative speeds.*

### 3.1.3 Asymptotic stability in the hydrodynamical framework

The following theorem shows the asymptotic stability of multi-solitons in the hydrodynamical framework. We show the asymptotic stability around and between solitons.

**Théorème 3.1.2.** *Let  $\mathbf{c}^0 = (c_1^0, \dots, c_N^0) \in (-1, 1)^N$ , with  $c_j^0 \neq 0$  for all  $j = 1, \dots, N$ , such that there exist  $L_0, \alpha_0 > 0$  with the following properties. Given any  $(v_0, w_0) \in X(\mathbb{R})$ , there exist  $L > L_0$  and  $\alpha < \alpha_0$  such that if  $(v_0, w_0) \in \mathcal{V}(\alpha, L)$ , then there exist  $\mathbf{a} := (a_1, \dots, a_N) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^N)$ ,  $\mathbf{c} := (c_1, \dots, c_N) \in \mathcal{C}^1(\mathbb{R}_+, (-1, 1) \setminus \{0\}^N)$  and non zero different speeds  $\mathbf{c}^{+\infty} = (c_1^{+\infty}, \dots, c_N^{+\infty})$  such that the unique global solution  $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  to (HLL) with initial datum  $(v_0, w_0)$  satisfies, for all  $j \in \{1, \dots, N\}$ ,*

$$\varepsilon(t, \cdot + a_j(t)) := (v, w)(t, x + a_j(t)) - \sum_{k=1}^N Q_{c_k(t)}(x + a_j(t) - a_k(t)) \rightharpoonup 0 \quad \text{in } X(\mathbb{R}), \quad (3.1.9)$$

as well as

$$\varepsilon(t, \cdot + b_j(t)) := (v, w)(t, x + b_j(t)) - \sum_{k=1}^N Q_{c_k(t)}(x + b_j(t) - a_k(t)) \rightharpoonup 0 \quad \text{in } X(\mathbb{R}), \quad (3.1.10)$$

for any  $\mathbf{b} := (b_1, \dots, b_{N+1}) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^{N+1})$  with  $b_j$  satisfying (3.1.6) and

$$c_{j-1}^\infty < \lim_{t \rightarrow +\infty} b_j'(t) < c_j^\infty. \quad (3.1.11)$$

Moreover, we have

$$c_j(t) \rightarrow c_j^{+\infty}, \quad a_j'(t) \rightarrow c_j^{+\infty}, \quad (3.1.12)$$

as  $t \rightarrow +\infty$ .

In fact, all the solitons in (3.1.9) with speed  $c_k$  for  $k \neq j$  are weakly convergent to 0 in  $X(\mathbb{R})$  as  $t \rightarrow +\infty$ , due to (3.1.12), so that (3.1.9) truly provides the asymptotic stability of the soliton with speed  $c_j$ . For (3.1.10), all the solitons are weakly convergent to 0 in  $X(\mathbb{R})$  as  $t \rightarrow +\infty$ , so that (3.1.10) provides the asymptotic stability of the zero solution between the solitons.

**Remarque 3.1.2.** (i) For (3.1.10), we begin by proving the convergence for  $\mathbf{b} := (b_1, \dots, b_{N+1}) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^{N+1})$  with  $b_j$  satisfying (3.1.6) and (3.1.11). Then, we show that it remains also true for any  $b_j$  verifying (3.1.7) in order to deduce (3.1.8) (see the end of Subsection 4.1 for the proof).

(ii) The case when  $c_j^0 \neq 0$  is excluded from the statement. In fact, we cannot use the hydrodynamical formulation in that case because the solitons can vanish. In addition, the Liouville type theorem cannot be applied as well as the orbital stability theorem. To our knowledge, this is still an open problem.

The proof relies on the strategy developed by Martel, Merle and Tsai in [51].

### 3.1.4 Plan of the paper

In the second section, we recall the orbital stability result for the multi-solitons, stated by de Laire and Gravejat in [19], which is an important tool to prove our results.

In the third section, we prove the asymptotic stability around solitons. More precisely, we show that any solution close to the sum of  $N$  solitons is weakly convergent to a soliton in the translating neighbourhood of each soliton. We state that all other solitons stay far in the way that in this region the problem reduces to the asymptotic stability for a single soliton. This is the reason why we can use the Liouville type theorem proved in [1].

In the last section, we change the translation parameter to show that any solution, corresponding to an initial datum close to the sum of  $N$  solitons, converges weakly to zero when it is moving in the core of the region separating two solitons. For this, we establish a Liouville type theorem, which affirms that small solutions which are smooth and exponentially localized are zero solutions. As a consequence, (3.1.10) claims that there is no interaction between well separated solitons with ordered speed.

## 3.2 Orbital stability in the hydrodynamical framework

In this section, we first recall the orbital stability result proved by de Laire and Gravejat in [19]. In order to quantify it precisely, we set

$$\mathcal{NV}(\mathbb{R}) := \left\{ \mathbf{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \max_{\mathbb{R}} |v| < 1 \right\}.$$

In the sequel we consider this space as a metric space equipped with the metric structure provided by the norm

$$\|\mathbf{v}\|_{H^1 \times L^2} := \left( \|v\|_{H^1}^2 + \|w\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

**Théorème 3.2.1.** [19] *Let  $\mathfrak{s}^* \in \{\pm 1\}^N$  and  $\mathfrak{c}^* = (c_1^*, \dots, c_N^*) \in (-1, 1)^N$ , with  $c_j^* \neq 0$ , such that*

$$c_1^* < \dots < c_N^*. \quad (3.2.1)$$

*There exist positive numbers  $\alpha^*$ ,  $L^*$  and  $A^*$ , depending only on  $\mathfrak{c}^*$  such that, if  $\mathbf{v}^0 \in \mathcal{NV}(\mathbb{R})$  satisfies the condition*

$$\alpha^0 := \left\| \mathbf{v}^0 - S_{\mathfrak{c}^*, \mathbf{a}^0, \mathfrak{s}^*} \right\|_{H^1 \times L^2} \leq \alpha^*, \quad (3.2.2)$$

*for points  $\mathbf{a}^0 = (a_1^0, \dots, a_N^0) \in \mathbb{R}^N$  such that*

$$L^0 := \min \left\{ a_{j+1}^0 - a_j^0, 1 \leq j \leq N-1 \right\} \geq L^*,$$

*then the solution  $\mathbf{v}$  to (HLL) with initial datum  $\mathbf{v}^0$  is globally well-defined on  $\mathbb{R}_+$ , and there exists a function  $\mathbf{a} = (a_1, \dots, a_N) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}^N)$  such that*

$$\sum_{j=1}^N \left| a_j'(t) - c_j^* \right| \leq A^* \left( \alpha^0 + \exp \left( - \frac{\nu_{\mathfrak{c}^*} L^0}{65} \right) \right), \quad (3.2.3)$$

*and*

$$\left\| \mathbf{v}(\cdot, t) - S_{\mathfrak{c}^*, \mathbf{a}(t), \mathfrak{s}^*} \right\|_{H^1 \times L^2} \leq A^* \left( \alpha^0 + \exp \left( - \frac{\nu_{\mathfrak{c}^*} L^0}{65} \right) \right), \quad (3.2.4)$$

*for any  $t \in \mathbb{R}_+$ .*

Given a positive number  $L > 0$ , we introduce the set of well-separated and ordered positions

$$\text{Pos}(L) := \left\{ \mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N, \text{ s.t. } a_{j+1} > a_j + L \text{ for } 1 \leq j \leq N-1 \right\},$$

and we set

$$\mathcal{V}(\alpha, L) := \left\{ \mathbf{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \text{ s.t. } \inf_{\mathbf{a} \in \text{Pos}(L)} \left\| \mathbf{v} - S_{\mathfrak{c}^*, \mathbf{a}, \mathfrak{s}^*} \right\|_{H^1 \times L^2} < \alpha \right\},$$

for  $\alpha > 0$ . We also define

$$\mu_{\mathfrak{c}} := \min_{1 \leq j \leq N} |c_j|, \quad \text{and} \quad \nu_{\mathfrak{c}} := \min_{1 \leq j \leq N} \left( 1 - c_j^2 \right)^{\frac{1}{2}},$$

for any  $\mathfrak{c} \in (-1, 1)^N$ . The following proposition provides some details contained in the proof of Theorem 3.2.1. In particular, it shows the existence of the speed and the translation parameters for each soliton (see [19] for the proof). It is an important tool for the proof of the asymptotic stability result.

**Proposition 3.2.1.** [19] *There exist positive numbers  $\alpha_1^*$  and  $L_1^*$ , depending only on  $\mathfrak{c}^*$  and  $\mathfrak{s}^*$ , such that we have the following properties.*

(i) *Any pair  $\mathbf{v} = (v, w) \in \mathcal{V}(\alpha_1^*, L_1^*)$  belongs to  $\mathcal{NV}(\mathbb{R})$ , with*

$$1 - v^2 \geq \frac{1}{8} \mu_{\mathfrak{c}^*}^2. \quad (3.2.5)$$

(ii) There exist two maps  $\mathbf{c} \in \mathcal{C}^1(\mathcal{V}(\alpha_1^*, L_1^*), (-1, 1)^N)$  and  $\mathbf{a} \in \mathcal{C}^1(\mathcal{V}(\alpha_1^*, L_1^*), \mathbb{R}^N)$ , and a positive number  $A^*$ , depending only on  $\mathbf{c}^*$  and  $\mathbf{s}^*$ , such that, if

$$\left\| \mathbf{v} - S_{\mathbf{c}^*, \mathbf{a}^*, \mathbf{s}^*} \right\|_{H^1 \times L^2} < \alpha,$$

for  $\mathbf{a}^* \in \text{Pos}(L)$ , with  $L > L_1^*$  and  $\alpha < \alpha_1^*$ , then we have

$$\|\varepsilon\|_{H^1 \times L^2} + \sum_{j=1}^N |c_j(\mathbf{v}) - c_j^*| + \sum_{j=1}^N |a_j(\mathbf{v}) - a_j^*| \leq A^* \left( \alpha + \exp \left( -\frac{\nu_{\mathbf{c}^*} L}{32} \right) \right), \quad (3.2.6)$$

as well as

$$\mathbf{a}(\mathbf{v}) \in \text{Pos}(L-1), \quad \mu_{\mathbf{c}(\mathbf{v})} \geq \frac{1}{2} \mu_{\mathbf{c}^*} \quad \text{and} \quad \nu_{\mathbf{c}(\mathbf{v})} \geq \frac{1}{2} \nu_{\mathbf{c}^*}, \quad (3.2.7)$$

where

$$\varepsilon = \mathbf{v} - S_{\mathbf{c}(\mathbf{v}), \mathbf{a}(\mathbf{v}), \mathbf{s}^*},$$

satisfies the orthogonality conditions

$$\langle \varepsilon, \partial_x Q_{c_k(\mathbf{v})} \rangle_{L^2(\mathbb{R})^2} = \langle \varepsilon, \chi_{c_k(\mathbf{v})} \rangle_{L^2(\mathbb{R})^2} = 0, \quad (3.2.8)$$

for any  $k \in \{1, \dots, N\}$ . The function  $\chi_{c_k(\mathbf{v})}$  stands here for an eigenvector of the quadratic form  $\mathcal{H}_{c_k(\mathbf{v})} := E''(Q_{c_k(\mathbf{v})}) - c_k(\mathbf{v})P''(Q_{c_k(\mathbf{v})})$  associated to its unique negative eigenvalue.

**Remarque 3.2.1.** The second orthogonality condition in (3.2.8) is not the same as the one used by de Laire and Gravejat in [19]. However, the result remains true by the same argument used in [1] (see Section 3 in [1] for more details). Moreover, we need this orthogonality condition in order to apply the Liouville type theorem (Theorem 3.3.1 below) (see Subsection 2.3.3 in [1] for more details).

Next, we recall the result for only one soliton which is a direct consequence of Theorem 3.2.1. It is an important tool for the proof of (3.1.5) since we analyse the soliton around each soliton.

**Théorème 3.2.2.** [19] Let  $c \in (-1, 1) \setminus \{0\}$ . There exists a positive number  $\alpha_c$ , depending only on  $c$ , with the following properties. Given any  $(v_0, w_0) \in \mathcal{NV}(\mathbb{R})$  such that

$$\alpha_0 := \left\| (v_0, w_0) - Q_{c, a} \right\|_{X(\mathbb{R})} \leq \alpha_c, \quad (3.2.9)$$

for some  $a \in \mathbb{R}$ , there exist a unique global solution  $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$  to (HLL) with initial datum  $(v_0, w_0)$ , two maps  $c \in \mathcal{C}^1(\mathbb{R}, (-1, 1) \setminus \{0\})$  and  $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ , and two positive numbers  $\sigma_c$  and  $A_c$ , depending only and continuously on  $c$ , such that

$$\max_{x \in \mathbb{R}} v(x, t) \leq 1 - \sigma_c, \quad (3.2.10)$$

$$\left\| \varepsilon(\cdot, t) \right\|_{X(\mathbb{R})} + |c(t) - c| \leq A_c \alpha^0, \quad (3.2.11)$$

and

$$\left| c'(t) \right| + \left| a'(t) - c(t) \right| \leq A_c \left\| \varepsilon(\cdot, t) \right\|_{X(\mathbb{R})}, \quad (3.2.12)$$

for any  $t \in \mathbb{R}$ , where the function  $\varepsilon$  is defined by

$$\varepsilon(\cdot, t) := \left( v(\cdot + a(t), t), w(\cdot + a(t), t) \right) - Q_{c(t)}, \quad (3.2.13)$$

and satisfies the orthogonality conditions

$$\langle \varepsilon(\cdot, t), \partial_x Q_{c(t)} \rangle_{L^2(\mathbb{R})^2} = \langle \varepsilon(\cdot, t), \chi_{c(t)} \rangle_{L^2(\mathbb{R})^2} = 0, \quad (3.2.14)$$

for any  $t \in \mathbb{R}$ .

Set

$$\mathbf{c}(t) := \mathbf{c}(\mathbf{v}(\cdot, t)) := (c_1(t), \dots, c_N(t)) \quad \text{and} \quad \mathbf{a}(t) := \mathbf{a}(\mathbf{v}(\cdot, t)) := (a_1(t), \dots, a_N(t)),$$

as well as

$$\varepsilon(\cdot, t) := (\varepsilon_1(\cdot, t), \varepsilon_2(\cdot, t)) = \mathbf{v}(\cdot, t) - S_{\mathbf{c}(t), \mathbf{a}(t), \mathbf{s}^*}. \quad (3.2.15)$$

The pair  $\varepsilon$  is well defined and satisfies the orthogonality conditions

$$\langle \varepsilon(\cdot, t), \partial_x Q_{c_k(t)} \rangle_{L^2(\mathbb{R})^2} = \langle \varepsilon(\cdot, t), \chi_{c_k(t)} \rangle_{L^2(\mathbb{R})^2} = 0, \quad (3.2.16)$$

for any  $t \in \mathbb{R}_+$  and for any  $k \in \{1, \dots, N\}$  (see [19] for more details). For  $\alpha$  and  $L$  given by Proposition 3.2.1, we also infer from the results in [19] that

$$\|\varepsilon(\cdot, t)\|_{H^1 \times L^2} + \sum_{j=1}^N |c_j(t) - c_j^*| \leq A^* \left( \alpha + \exp\left(-\frac{\nu_{\mathbf{c}^*} L}{65}\right) \right), \quad (3.2.17)$$

and

$$\mathbf{a}(t) \in \text{Pos}(L-1), \quad \mu_{\mathbf{c}(t)} \geq \frac{1}{2}\mu_{\mathbf{c}^*} \quad \text{and} \quad \nu_{\mathbf{c}(t)} \geq \frac{1}{2}\nu_{\mathbf{c}^*}. \quad (3.2.18)$$

### 3.3 Asymptotic stability around the solitons in the hydrodynamical variables

#### 3.3.1 Proofs of (3.1.9) and (3.1.12)

Let  $\mathbf{c}^0$  be as in Theorem 3.1.2 and  $\mathbf{v}_0$  be any pair which belongs to the set  $\mathcal{V}(\alpha, L)$  with  $\alpha$  and  $L$  as in the hypothesis of Theorem 3.1.2.

Let  $j \in \{1, \dots, N\}$ . By (3.2.17), the functions  $\varepsilon$  and  $c_j$  are uniformly bounded in  $X(\mathbb{R})$ , respectively in  $\mathbb{R}$ . Then, there exist  $\tilde{\varepsilon}_{j,0} \in X(\mathbb{R})^2$  and  $\tilde{c}_{j,0} \in (-1, 1) \setminus \{0\}$  such that, up to a subsequence,

$$\varepsilon(t_n, \cdot + a_j(t_n)) \rightharpoonup \tilde{\varepsilon}_{j,0} \quad \text{in} \quad X(\mathbb{R}) \quad \text{and} \quad c_j(t_n) \rightarrow \tilde{c}_{j,0} \quad \text{as} \quad n \rightarrow +\infty. \quad (3.3.1)$$

Indeed, the bounds in (3.2.17) and the possibility to choose  $\alpha$  small enough guarantee that  $\tilde{c}_{j,0}$  stays always close to  $c_j^0$  which prevents  $\tilde{c}_{j,0}$  to be in  $\{-1, 0, 1\}$  for any  $j \in \{1, \dots, N\}$ .

We set  $\tilde{\mathbf{v}}_{j,0} = (\tilde{v}_{j,0}, \tilde{w}_{j,0}) := Q_{\tilde{c}_{j,0}} + \tilde{\varepsilon}_{j,0}$  and denote by  $\tilde{\mathbf{v}}_j = (\tilde{v}_j, \tilde{w}_j)$  the unique global solution to (HLL) corresponding to this initial datum  $\tilde{\mathbf{v}}_{j,0}$ . We claim that this solution exponentially decays with respect to the space variable for any time, as well as all its space derivatives. More precisely, we have

**Proposition 3.3.1.** *The pair  $(\tilde{v}_j, \tilde{w}_j)$  is indefinitely smooth and exponentially decaying on  $\mathbb{R} \times \mathbb{R}$ . Moreover, given any  $k \in \mathbb{N}$ , there exist a positive constant  $A_{k,\mathbf{c}}$ , depending only on  $k$  and  $\mathbf{c}$ , and a function  $\tilde{a}_j \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  such that*

$$\int_{\mathbb{R}} \left[ (\partial_x^{k+1} \tilde{v}_j)^2 + (\partial_x^k \tilde{v}_j)^2 + (\partial_x^k \tilde{w}_j)^2 \right] (x + \tilde{a}_j(t), t) \exp\left(\frac{\nu_{\mathbf{c}}}{16}|x|\right) dx \leq A_{k,\mathbf{c}}, \quad (3.3.2)$$

for any  $t \in \mathbb{R}$ .

With this proposition at hand, we can finish the proof of (3.1.9). We recall the Liouville type theorem stated in [1].

---

2. In view of (3.2.17), the norm of  $\tilde{\varepsilon}_{j,0}$  in  $X(\mathbb{R})$  is small.

**Théorème 3.3.1.** [1] Let  $j \in \{1, \dots, N\}$ ,  $c_j \in (-1, 1) \setminus \{0\}$  and  $(\tilde{v}_j, \tilde{w}_j)$  a solution of (HLL) satisfying (3.3.2) and

$$\|(\tilde{v}_{j,0}, \tilde{w}_{j,0}) - Q_{c_j}\|_{X(\mathbb{R})} \leq \alpha. \quad (3.3.3)$$

Then, there exist two numbers  $x^* \in \mathbb{R}$  and  $c^* \in (-1, 1) \setminus \{0\}$  such that

$$(\tilde{v}_j, \tilde{w}_j)(t, x) = Q_{c^*}(x - x^* - c^*t) \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}.$$

Due to the orbital stability of  $Q_{\tilde{c}_{j,0}}$ , condition (3.3.3) is satisfied when  $\alpha_0$  is small enough. Applying Theorem 3.3.1, we get  $x^* \in \mathbb{R}$  and  $c^* \in (-1, 1) \setminus \{0\}$  such that we have

$$\tilde{\mathbf{v}}_j(t, x) = Q_{c^*}(x - x^* - c^*t), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}.$$

In particular, we have  $Q_{\tilde{c}_{j,0}}(x) + \tilde{\varepsilon}_{j,0}(x) = Q_{c^*}(x - x^*)$ . We claim that  $x^* = 0$ . Indeed, we use the fact that  $\|\tilde{\varepsilon}_{j,0}\|_{X(\mathbb{R})} \leq \alpha$  and a modulation argument to obtain  $|c^* - \tilde{c}_{j,0}| \leq A_c \alpha$  and  $|x^*| \leq A_c \alpha$ . We define

$$h(c^*, x^*) = \int_{\mathbb{R}} \langle Q_{c^*}(x - x^*), Q'_{\tilde{c}_{j,0}} \rangle.$$

We have

$$\partial_{x^*} h(\tilde{c}_{j,0}, 0) = - \int_{\mathbb{R}} |Q'_{\tilde{c}_{j,0}}|^2 \neq 0.$$

From the implicit function theorem, there exist a neighbourhood  $V$  of  $(\tilde{c}_{j,0}, 0)$  and a function  $\phi$  such that  $(c^*, x^*) \in V$  and  $h(c^*, x^*) = 0$  if and only if  $x^* = \phi(c^*)$ . Since, by parity,  $h(c^*, 0) = 0$ , we infer that  $x^* = 0$ .

Next, we set  $g(c^*) = \int_{\mathbb{R}} \langle Q_{c^*} - Q_{\tilde{c}_{j,0}}, Q_{\tilde{c}_{j,0}} \rangle$ . Since  $g'(\tilde{c}_{j,0}) \neq 0$ , we can prove that  $c^* = \tilde{c}_{j,0}$ , which leads to the fact that  $\tilde{\varepsilon}_0 \equiv 0$ . This allows us to deduce the convergence (3.1.9) for a subsequence of  $(t_n)_{n \in \mathbb{N}}$ .

Finally, we prove (3.1.9) and (3.1.12) for  $t \rightarrow +\infty$ . Since  $a_l(t_{n_k}) - a_j(t_{n_k}) \rightarrow \infty$  for all  $l \neq j$ , the solution converges to only one soliton because the other solitons converges to zero. This means that we have

$$\left( v(\cdot + a_j(t_{n_k}), t_{n_k}), w(\cdot + a_j(t_{n_k}), t_{n_k}) \right) - Q_{c_j(t_{n_k})} \rightharpoonup 0 \quad \text{in } X(\mathbb{R}),$$

as  $k \rightarrow +\infty$ . This restricts the problem to the case of only one soliton. The proof is then similar to the one stated by Béthuel, Gravejat and Smets in [6]. It relies on the monotonicity formula for the quantities  $\mathcal{I}_{j,y_0}$  in Proposition 3.3.3.

The main idea is to show that  $\tilde{c}_{j,0}$  is independent of the sequence  $(t_n)_{n \in \mathbb{N}}$ . Assume by contradiction that for two different sequences  $(t_n)_{n \in \mathbb{N}}$  and  $(s_n)_{n \in \mathbb{N}}$ , both tending to  $+\infty$ , we have

$$c_j(t_n) \rightarrow c_{j,1} \quad \text{and} \quad c_j(s_n) \rightarrow c_{j,2},$$

as  $n \rightarrow +\infty$ , with  $c_{j,1} \neq c_{j,2}$  satisfying (3.2.17). In addition, we suppose that we have

$$\left( v(\cdot + a(t_n), t_n), w(\cdot + a(t_n), t_n) \right) - Q_{c_j(t_n)} \rightharpoonup 0 \quad \text{in } X(\mathbb{R}), \quad (3.3.4)$$

and

$$\left( v(\cdot + a(s_n), s_n), w(\cdot + a(s_n), s_n) \right) - Q_{c_j(s_n)} \rightharpoonup 0 \quad \text{in } X(\mathbb{R}). \quad (3.3.5)$$

Note that these two convergences are different since  $Q_{c_j(t_n)} \rightarrow Q_{c_{j,1}}$  and  $Q_{c_j(s_n)} \rightarrow Q_{c_{j,2}}$  as  $n \rightarrow \infty$ . We may assume, without loss of generality, that  $c_{j,1} < c_{j,2}$  and that the sequences  $(t_n)_{n \in \mathbb{N}}$  and  $(s_n)_{n \in \mathbb{N}}$  are strictly increasing and are taken such that

$$t_n + 1 \leq s_n \leq t_{n+1} - 1, \quad (3.3.6)$$

for any  $n \in \mathbb{N}$ . Let  $\delta > 0$ . For  $y_0$  sufficiently large, we can define the quantities  $\mathcal{I}_{j,y_0}$  as in (3.3.28), and deduce from (3.3.6) and (3.3.30) that

$$\mathcal{I}_{j,\pm y_0}(s_n) \geq \mathcal{I}_{j,\pm y_0}(t_n) - \frac{\delta}{10} \quad \text{and} \quad \mathcal{I}_{j,\pm y_0}(t_{n+1}) \geq \mathcal{I}_{j,\pm y_0}(s_n) - \frac{\delta}{10}, \quad (3.3.7)$$

for any  $n \in \mathbb{N}$ . On the other hand, by (3.3.4) and (3.3.5), there exists an integer  $n_0$  such that

$$\left| \mathcal{I}_{j,-y_0}(t_n) - \mathcal{I}_{j,y_0}(t_n) - P(Q_{c_j(t_n)}) \right| \leq \frac{\delta}{5}, \quad (3.3.8)$$

and

$$\left| \mathcal{I}_{j,-y_0}(s_n) - \mathcal{I}_{j,y_0}(s_n) - P(Q_{c_j(s_n)}) \right| \leq \frac{\delta}{5}, \quad (3.3.9)$$

for any  $n \geq n_0$  and for  $y_0$  large enough. From (3.3.7), (3.3.8) and (3.3.9), we have

$$\mathcal{I}_{j,y_0}(s_n) \geq \mathcal{I}_{j,y_0}(t_n) + \frac{\delta}{2},$$

for any  $n \geq n_0$ , this yields, using (3.3.7) again, that

$$\mathcal{I}_{j,y_0}(t_{n+1}) \geq \mathcal{I}_{j,y_0}(t_n) + \frac{2\delta}{5},$$

for any  $n \geq n_0$ . Therefore, the sequence  $(\mathcal{I}_{j,y_0}(t_n))_{n \in \mathbb{N}}$  is unbounded, which leads to a contradiction with the fact that the pair  $(v, w)$  has a bounded energy.

The second convergence in (3.1.12) follows from the fact that

$$a_j(t_n + t) - a_j(t_n) \rightarrow c_j^{+\infty} t,$$

for any fixed  $t \in \mathbb{R}$  and any sequence  $(t_n)_{n \in \mathbb{N}}$  tending to  $+\infty$  (due to (3.3.21)), and Lemma 2 in [6] (see [6] for more details).  $\square$

### 3.3.2 Localization and smoothness of the limit profile

In this section, we prove Proposition 3.3.1. First, we use (3.2.3) and (3.2.17) to claim that

$$\min_{j=1,\dots,N} \{c_j(t)^2, a'_j(t)^2\} \geq \frac{\mu_c^2}{2}, \quad \max_{j=1,\dots,N} \{c_j(t)^2, a'_j(t)^2\} \leq 1 + \frac{\mu_c^2}{2}, \quad (3.3.10)$$

and

$$\left\| V_{c,a(t),s} - v(t) \right\|_{L^\infty(\mathbb{R})} \leq \min \left\{ \frac{\mu_c^2}{4}, \frac{\nu_c^2}{16} \right\}, \quad (3.3.11)$$

for any  $t \in \mathbb{R}$ . In particular, we conclude that  $\tilde{c}_{j,0} \in (-1, 1) \setminus \{0\}$ , so that  $Q_{\tilde{c}_{j,0}}$  is a dark soliton.

In addition, for  $j \in \{1, \dots, N\}$ , we have

$$\left| \tilde{c}_{j,0} - c_j \right| \leq A_{\mu_c} \alpha. \quad (3.3.12)$$

On the other hand, by the weak lower semi-continuity of the norm, (3.2.17) and (3.3.1), we infer that

$$\left\| (\tilde{v}_{j,0}, \tilde{w}_{j,0}) - Q_{c_j} \right\|_{X(\mathbb{R})} \leq A_{\mu_c} \alpha + \left\| Q_{c_j} - Q_{\tilde{c}_{j,0}} \right\|_{X(\mathbb{R})} \leq A_{\mu_c} \alpha. \quad (3.3.13)$$

Now, we suppose that  $\alpha$  is sufficiently small so that, by (3.3.13),

$$\left\| (\tilde{v}_{j,0}, \tilde{w}_{j,0}) - Q_{c_j} \right\|_{X(\mathbb{R})} \leq \alpha_c. \quad (3.3.14)$$

By Theorem 3.2.2, there exist two maps  $\tilde{c}_j \in \mathcal{C}^1(\mathbb{R}, (-1, 1) \setminus \{0\})$  and  $\tilde{a}_j \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  such that the function  $\tilde{\varepsilon}_j$  defined by

$$\tilde{\varepsilon}_j(\cdot, t) := \left( \tilde{v}_j(\cdot + \tilde{a}_j(t), t), \tilde{w}_j(\cdot + \tilde{a}_j(t), t) \right) - Q_{\tilde{c}_j(t)}, \quad (3.3.15)$$

satisfies the estimates

$$\left\| \tilde{\varepsilon}_j(\cdot, t) \right\|_{X(\mathbb{R})} + \left| \tilde{c}_j(t) - c_j \right| + \left| \tilde{a}'_j(t) - \tilde{c}_j(t) \right| \leq A_\varepsilon \left\| (\tilde{v}_{j,0}, \tilde{w}_{j,0}) - Q_{c_j} \right\|_{X(\mathbb{R})}, \quad (3.3.16)$$

and the orthogonality conditions

$$\langle \tilde{\varepsilon}_j(\cdot, t), \partial_x Q_{\tilde{c}_j(t)} \rangle_{L^2(\mathbb{R})^2} = \langle \tilde{\varepsilon}_j(\cdot, t), \chi_{\tilde{c}_j(t)} \rangle_{L^2(\mathbb{R})^2} = 0, \quad (3.3.17)$$

for any  $t \in \mathbb{R}$ .

Using (3.3.13) and (3.3.16), and choosing  $\alpha$  small enough we claim that

$$\min \left\{ \tilde{c}_j(t)^2, \tilde{a}'_j(t)^2 \right\} \geq \frac{\mu_c^2}{4}, \quad \max \left\{ \tilde{c}_j(t)^2, \tilde{a}'_j(t)^2 \right\} \leq \frac{1 + \mu_c^2}{2}, \quad (3.3.18)$$

and

$$\left\| v_{c_j}(\cdot) - \tilde{v}_j(\cdot + \tilde{a}_j(t), t) \right\|_{L^\infty(\mathbb{R})} \leq \min \left\{ \frac{\mu_c^2}{4}, \frac{1 - \mu_c^2}{16} \right\}, \quad (3.3.19)$$

for any  $t \in \mathbb{R}$ . We then prove the following weak continuity property in the hydrodynamical framework.

**Proposition 3.3.2.** *Let  $j \in \{1, \dots, N\}$  and  $t \in \mathbb{R}$  be fixed. Then,*

$$(v, w)(\cdot + a_j(t_n), t_n + t) \rightharpoonup (\tilde{v}_j, \tilde{w}_j)(\cdot, t) \quad \text{in } X(\mathbb{R}), \quad (3.3.20)$$

while

$$a_j(t_n + t) - a_j(t_n) \rightarrow \tilde{a}_j(t), \quad \text{and} \quad c_j(t_n + t) \rightarrow \tilde{c}_j(t), \quad (3.3.21)$$

as  $n \rightarrow +\infty$ . In particular, we have

$$(v, w)(\cdot + a_j(t_n + t), t_n + t) \rightharpoonup (\tilde{v}_j, \tilde{w}_j)(\cdot + \tilde{a}_j(t), t) \quad \text{in } X(\mathbb{R}), \quad (3.3.22)$$

as  $n \rightarrow +\infty$ .

The weak continuity of the flow and of the modulation parameters were proved in [1] in the case of a simple soliton. The proof of Proposition 3.3.2 is similar.

*Proof.* Let  $j \in \{1, \dots, N\}$  be a fixed integer. First, we prove (3.3.20). By the second convergence in (3.3.1) and the explicit formula of  $Q_{c_j(t_n)}$  in (3.1.3), we can infer that

$$Q_{c_j(t_n)} \rightarrow Q_{\tilde{c}_j,0} \quad \text{in } X(\mathbb{R}),$$

as  $n \rightarrow +\infty$ . This leads, using the first convergence in (3.3.1), to

$$\left( v(\cdot + a_j(t_n), t_n), w(\cdot + a_j(t_n), t_n) \right) \rightharpoonup \tilde{\varepsilon}_{j,0} + Q_{\tilde{c}_j,0} \quad \text{in } X(\mathbb{R}),$$

as  $n \rightarrow +\infty$ . In view of the fact that  $t \mapsto (v(\cdot + a_j(t_n), t_n + t), w(\cdot + a_j(t_n), t_n + t))$  and  $(\tilde{v}_j, \tilde{w}_j)$  are the solutions to (HLL) with initial data  $(v(\cdot + a_j(t_n), t_n), w(\cdot + a_j(t_n), t_n))$ , respectively  $\varepsilon_0^* + Q_{c_0^*}$ , we deduce (3.3.20) from the weak continuity of the flow (see Proposition A.1 in [1] for more details.)



Next, let us prove (3.3.21). By (3.2.11) and (3.2.12) the maps  $a'_j$  and  $c_j$  are bounded on  $\mathbb{R}$ , so that the sequences  $(a_j(t_n + t) - a_j(t_n))_{n \in \mathbb{N}}$  and  $(c_j(t_n + t))_{n \in \mathbb{N}}$  are bounded. Hence it is sufficient to prove that the unique possible accumulation points for these sequences are  $\tilde{a}_j(t)$ , respectively  $\tilde{c}_j(t)$ .

We suppose now that, up to a possible subsequence, we have

$$a_j(t_n + t) - a_j(t_n) \rightarrow \alpha_j, \quad \text{and} \quad c_j(t_n + t) \rightarrow \sigma_j, \quad (3.3.23)$$

as  $n \rightarrow +\infty$ . Given a function  $\phi \in H^1(\mathbb{R})$ , we write

$$\begin{aligned} & \left\langle v(\cdot + a_j(t_n + t), t_n + t), \phi \right\rangle_{H^1(\mathbb{R})} \\ &= \left\langle v(\cdot + a_j(t_n), t_n + t), \phi(\cdot - a_j(t_n + t) + a_j(t_n)) - \phi(\cdot - \alpha_j) \right\rangle_{H^1(\mathbb{R})} \\ &+ \left\langle v(\cdot + a_j(t_n), t_n + t), \phi(\cdot - \alpha_j) \right\rangle_{H^1(\mathbb{R})}. \end{aligned}$$

Since we know that

$$\phi(\cdot + h) \rightarrow \phi \quad \text{in } H^1(\mathbb{R}),$$

when  $h \rightarrow 0$ , we can use (3.3.20) and (3.3.23) to infer that

$$v(\cdot + a_j(t_n + t), t_n + t) \rightharpoonup \tilde{v}_j(\cdot + \alpha_j, t) \quad \text{in } H^1(\mathbb{R}),$$

as  $n \rightarrow +\infty$ . Similarly, we obtain

$$w(\cdot + a_j(t_n + t), t_n + t) \rightharpoonup \tilde{w}_j(\cdot + \alpha_j, t) \quad \text{in } L^2(\mathbb{R}).$$

By (3.3.23) we also have

$$Q_{c_j(t_n+t)} \rightarrow Q_{\sigma_j} \quad \text{in } X(\mathbb{R}),$$

as  $n \rightarrow +\infty$ . This leads to

$$\varepsilon(\cdot, t_n + t) \rightharpoonup \left( \tilde{v}_j(\cdot + \alpha_j, t), \tilde{w}_j(\cdot + \alpha_j, t) \right) - Q_{\sigma_j} \quad \text{in } X(\mathbb{R}), \quad (3.3.24)$$

as  $n \rightarrow +\infty$ .

Now, we use the fact that the function  $\chi_c$  is continuous with respect to the parameter  $c$ , (3.1.3) and the second convergence in (3.3.23) to prove that

$$\partial_x Q_{c_j(t_n+t)} \rightarrow \partial_x Q_{\sigma_j} \quad \text{and} \quad \chi_{c_j(t_n+t)} \rightarrow \chi_{\sigma_j} \quad \text{in } L^2(\mathbb{R})^2,$$

as  $n \rightarrow +\infty$ . Combining this with (3.3.24), we can take the limit  $n \rightarrow +\infty$  in the two orthogonality conditions in (3.3.17) to obtain

$$\left\langle (\tilde{v}_j(\cdot + \alpha_j, t), \tilde{w}_j(\cdot + \alpha_j, t)) - Q_{\sigma_j}, \partial_x Q_{\sigma_j} \right\rangle_{L^2(\mathbb{R})^2} = \left\langle (\tilde{v}_j(\cdot + \alpha_j, t), \tilde{w}_j(\cdot + \alpha_j, t)) - Q_{\sigma_j}, \chi_{\sigma_j} \right\rangle_{L^2(\mathbb{R})^2} = 0.$$

Since the parameters  $\tilde{a}_j(t)$  and  $\tilde{c}_j(t)$  are uniquely defined in (3.3.15), we infer that

$$\alpha_j = \tilde{a}_j(t), \quad \text{and} \quad \sigma_j = \tilde{c}_j(t), \quad (3.3.25)$$

which is enough to complete the proof of (3.3.21). Convergence (3.3.22) follows combining (3.3.15) with (3.3.24) and (3.3.25).  $\square$

Now, we consider the function  $\Phi$ , which is defined on  $\mathbb{R}$  by

$$\Phi(x) := \frac{1}{2} \left( 1 + \tanh \left( \frac{\nu_{\mathbf{c}}}{16} x \right) \right). \quad (3.3.26)$$

Recall that  $\Phi'$  verifies the following property

$$\left| \Phi'''(x) \right| \leq \frac{\nu_{\mathbf{c}}^2}{64} \Phi'(x) \leq \frac{\nu_{\mathbf{c}}^3}{512} \exp \left( - \frac{\nu_{\mathbf{c}}}{16} |x| \right). \quad (3.3.27)$$

We set

$$\delta_{\mathbf{c}} := \frac{1}{2} \min \{ 1 + c_1, c_2 - c_1, c_3 - c_2, \dots, c_N - c_{N-1}, 1 - c_N \}$$

for any  $\mathbf{c} \in (-1, 1)^N$ .

Let  $(v, w)$  be a pair given by Theorem 3.2.1,  $j \in \{1, \dots, N\}$  and  $y_0 \in \mathbb{R}$ . Denote

$$\mathcal{I}_{j, y_0}(t) := \int_{\mathbb{R}} \Phi(x - (a_j(t) + y_0)) [vw](x, t) dx. \quad (3.3.28)$$

We prove a monotonicity formula for these localized versions of the momentum following the ideas used by Martel, Merle and Tsai in the proof of Lemma 3 in [51].

**Proposition 3.3.3.** *Let  $y_0 \in \mathbb{R}$ ,  $t \in \mathbb{R}_+$  and  $\sigma \in [-\delta_{\mathbf{c}}, \delta_{\mathbf{c}}]$ . There exist positive numbers  $\alpha_1 \leq \alpha$ ,  $L_1 \geq L^*$  and  $A_1, A_1^* > 0$ , depending only on  $\mathbf{c}$  and  $\mathfrak{s}$ , such that, if  $\alpha_0 \leq \alpha_1$  and  $L \geq L_1$ , then the map  $\mathcal{I}_j$  is of class  $\mathcal{C}^1$  on  $\mathbb{R}$ , and it satisfies*

$$\begin{aligned} \frac{d}{dt} [\mathcal{I}_{j, y_0 + \sigma t}(t)] &\geq \frac{\nu_{\mathbf{c}}^2}{32} \int_{\mathbb{R}} [(\partial_x v)^2 + v^2 + w^2](x, t) \Phi'(x - (a_j(t) + y_0 + \sigma t)) dx \\ &\quad - A_1 \exp \left( - \frac{\nu_{\mathbf{c}}}{16} |y_0 + \sigma t| \right), \end{aligned} \quad (3.3.29)$$

for any  $1 \leq j \leq N$  and any  $t \in \mathbb{R}_+$ . In particular, we have

$$\mathcal{I}_{j, y_0}(t_1) \geq \mathcal{I}_{j, y_0}(t_0) - A_1^* \exp \left( - \frac{\nu_{\mathbf{c}}}{16} |y_0| \right), \quad (3.3.30)$$

for any real numbers  $t_1 \geq t_0 \geq 0$ .

**Remarque 3.3.1.** *In view of the proof below, Proposition 3.3.3 holds for any time  $t \in \mathbb{R}$ , when there is only one soliton in the sum. In particular, this further property is true for the limit solution  $(\tilde{v}_j, \tilde{w}_j)$ .*

*Proof.* We differentiate the quantities  $\mathcal{I}_{j, y_0 + \sigma t}$  with respect to  $t$  in order to obtain

$$\begin{aligned} \frac{d}{dt} [\mathcal{I}_{j, y_0 + \sigma t}(t)] &= \frac{1}{2} \int_{\mathbb{R}} \Phi'(\cdot - (a_j(t) + y_0 + \sigma t)) \times \\ &\quad \times \left( v^2 + w^2 - (a_j'(t) + \sigma)vw - 3v^2w^2 + \frac{3 - v^2}{(1 - v^2)^2} (\partial_x v)^2 \right) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \Phi'''(\cdot - (a_j(t) + y_0 + \sigma t)) \ln(1 - v^2), \end{aligned} \quad (3.3.31)$$

for any  $t \in \mathbb{R}_+$ . We decompose the real line into two regions,

$$R_j(t) = \left[ a_j(t) - \frac{L-1}{4}, a_j(t) + \frac{L-1}{4} \right],$$

and its complementary set. We set

$$\frac{d}{dt} [\mathcal{I}_{j,y_0+\sigma t}(t)] = \mathcal{I}_j^1(t) + \mathcal{I}_j^2(t),$$

where

$$\begin{aligned} \mathcal{I}_j^2(t) &= \frac{1}{2} \int_{R_j(t)} \Phi'(\cdot - (a_j(t) + y_0 + \sigma t)) \times \\ &\quad \times \left( v^2 + w^2 - (a'_j(t) + \sigma)vw - 3v^2w^2 + \frac{3-v^2}{(1-v^2)^2} (\partial_x v)^2 \right) \\ &\quad + \frac{1}{2} \int_{R_j(t)} \Phi'''(\cdot - (a_j(t) + y_0 + \sigma t)) \ln(1-v^2). \end{aligned}$$

When  $x \in R_j(t)$ , we have

$$|x - a_j(t) - y_0 - \sigma t| \geq -\frac{L}{4} + |y_0 + \sigma t|.$$

Hence, using (3.2.5), (3.3.19), and (3.3.27), we obtain

$$|\mathcal{I}_j^2(t)| \leq A_{\mathfrak{c}} \exp\left(-\frac{\nu_{\mathfrak{c}}}{16}|y_0 + \sigma t|\right), \quad (3.3.32)$$

where  $A_{\mathfrak{c}}$  denotes, here as in the sequel, a positive number depending only on  $\mathfrak{c}$  and  $\mathfrak{s}$ .

Next, we use (3.2.5) and (3.3.27) to bound  $\mathcal{I}_j^1(t)$  from below by

$$\begin{aligned} \mathcal{I}_j^1(t) &\geq \frac{1}{2} \int_{\mathbb{R} \setminus R_j(t)} \Phi'(\cdot - (a_j(t) + y_0 + \sigma t)) \times \\ &\quad \times \left( (\partial_x v)^2 + v^2 + w^2 - 2\left(1 - \frac{\nu_{\mathfrak{c}}^2}{4}\right)^{\frac{1}{2}} |v||w| - 3v^2w^2 + \frac{\nu_{\mathfrak{c}}^2}{64} \ln(1-v^2) \right). \end{aligned} \quad (3.3.33)$$

For any  $x \in \mathbb{R} \setminus R_j(t)$ , we have

$$|x - a_k(t)| \geq \frac{L}{4},$$

for any  $1 \leq k \leq N$ . This yields, by (3.2.15), (3.2.17), the Sobolev embedding theorem, the exponential decay of the solitons and (3.2.18), that

$$|v(x, t)| \leq |\varepsilon_v(x, t)| + \sum_{k=1}^N |v_{c_k(t)}(x - a_k(t))| \leq A_{\mathfrak{c}} \left( \alpha + \exp\left(-\frac{\nu_{\mathfrak{c}}}{16}L\right) \right),$$

for any  $x \in \mathbb{R} \setminus R_j(t)$ . For  $\alpha$  small enough and  $L$  big enough, we have

$$v^2 \leq \min\left\{\frac{1}{2}, \frac{\nu_{\mathfrak{c}}^2}{96}\right\}, \quad (3.3.34)$$

on  $\mathbb{R} \setminus R_j(t)$ . We conclude from (3.3.33), (3.3.34) and the fact that  $\ln(1-s) \geq -2s$  for  $0 \leq s \leq 1/2$ , that

$$\mathcal{I}_j^1(t) \geq \frac{1}{2} \left( 1 - \left( 1 - \frac{\nu_{\mathfrak{c}}^2}{4} \right)^{\frac{1}{2}} - \frac{\nu_{\mathfrak{c}}^2}{32} \right) \int_{\mathbb{R} \setminus R_j(t)} \Phi'(\cdot - (a_j(t) + y_0 + \sigma t)) \left( (\partial_x v)^2 + v^2 + w^2 \right).$$

Then, using the fact that  $1 - (1-s)^{1/2} \geq s/2$  for  $0 \leq s \leq 1$ , we obtain

$$\begin{aligned} \mathcal{I}_j^1(t) &\geq \frac{1}{2} \left( 1 - \left( 1 - \frac{\nu_{\mathfrak{c}}^2}{4} \right)^{\frac{1}{2}} - \frac{\nu_{\mathfrak{c}}^2}{32} \right) \int_{\mathbb{R} \setminus R_j(t)} \Phi'(\cdot - (a_j(t) + y_0 + \sigma t)) \left( (\partial_x v)^2 + v^2 + w^2 \right) \\ &\geq \frac{3\nu_{\mathfrak{c}}^2}{64} \int_{\mathbb{R} \setminus R_j(t)} \Phi'(\cdot - (a_j(t) + y_0 + \sigma t)) \left( (\partial_x v)^2 + v^2 + w^2 \right). \end{aligned}$$

This concludes the proof of (3.3.29). Now let us prove (3.3.30). When  $y_0 \geq 0$ , we integrate (3.3.29) from  $t_0$  to  $\frac{t_1+t_0}{2}$  taking  $\sigma = \frac{\delta_\epsilon}{2}$  and  $y_0 = y_0 - \frac{\delta_\epsilon}{2}t_0$  and from  $\frac{t_1+t_0}{2}$  to  $t_1$  taking  $\sigma = -\frac{\delta_\epsilon}{2}$  and  $y_0 = y_0 + \frac{\delta_\epsilon}{2}t_1$ , to obtain (3.3.30). The proof is similar when  $y_0 < 0$ . This finishes the proof of this proposition.  $\square$

Using Propositions 3.3.2 and 3.3.3 and Remark 3.3.1, we claim as in [1] that

**Proposition 3.3.4** ([1]). *Let  $t \in \mathbb{R}$ . There exists a positive constant  $\mathcal{A}_{c^0}$  such that*

$$\int_t^{t+1} \int_{\mathbb{R}} [(\partial_x \tilde{v}_j)^2 + \tilde{v}_j^2 + \tilde{w}_j^2](x + \tilde{a}_j(s), s) e^{\frac{\nu_c}{16}|x|} dx ds \leq \mathcal{A}_{c^0}.$$

The two lemmas below are the main ingredients for the proof of this proposition. For the limit profile  $(\tilde{v}_j, \tilde{w}_j)$ , we set  $\tilde{\mathcal{I}}_{j, \pm y_0}(t) := \mathcal{I}_{j, \pm y_0}^{(\tilde{v}_j, \tilde{w}_j)}(t)$  for any  $t \in \mathbb{R}$  and any  $y_0 > 0$ .

**Lemma 3.3.1** ([6]). *For any positive number  $\delta$ , there exists a positive number  $y_\delta$ , depending only on  $\delta$ , such that for any  $t \in \mathbb{R}$  we have*

$$\left| \tilde{\mathcal{I}}_{j, y_0}(t) \right| \leq \delta \quad \text{and} \quad |P(\tilde{v}_j, \tilde{w}_j) - \tilde{\mathcal{I}}_{j, -y_0}(t)| \leq \delta, \quad (3.3.35)$$

for any  $y_0 \geq y_\delta$ .

This lemma shows that the momentum of the limit profile is localized in a compact region of the real line. This is a key point to claim that this momentum is exponentially decaying with respect to  $y_0$ .

*Proof.* The proof of this lemma is by contradiction. We assume that there exists a positive number  $\delta_0$  such that, for any positive number  $y_0$ , there exists a number  $t_0 \in \mathbb{R}$  such that either  $|\tilde{\mathcal{I}}_{j, y_0}(t_0)| \geq \delta_0$  or  $|\tilde{\mathcal{I}}_{j, -y_0}(t_0) - P(\tilde{v}_j, \tilde{w}_j)| \geq \delta_0$ .

At initial time  $t = 0$ , we have  $\lim_{y_0 \rightarrow +\infty} \tilde{\mathcal{I}}_{j, y_0}(0) = \lim_{y_0 \rightarrow +\infty} \tilde{\mathcal{I}}_{j, -y_0}(0) - P(\tilde{v}_j, \tilde{w}_j) = 0$ . Hence, there exists  $y_0 > 0$  such that

$$|\tilde{\mathcal{I}}_{j, y_0}(0)| + |\tilde{\mathcal{I}}_{j, -y_0}(0) - P(\tilde{v}_j, \tilde{w}_j)| \leq \frac{\delta_0}{4} \quad \text{and} \quad A_c \exp\left(-\frac{\nu_c}{16}y_0\right) \leq \frac{\delta_0}{32}. \quad (3.3.36)$$

Now, we prove that the case  $\tilde{\mathcal{I}}_{j, y_0}(t_0) \geq \delta_0$  cannot hold for this choice of  $y_0$ . The proof of the other cases can be written in a very similar manner.

First, we deduce from (3.3.36) that

$$\tilde{\mathcal{I}}_{j, y_0}(t_0) \geq \delta_0 \geq \frac{\delta_0}{4} + \frac{\delta_0}{16} \geq \tilde{\mathcal{I}}_{j, y_0}(0) + A_c \exp\left(-\frac{\nu_c}{16}y_0\right),$$

Using (3.3.30), we conclude that  $t_0 > 0$ . Next, from the fact that  $\lim_{y_0 \rightarrow +\infty} \tilde{\mathcal{I}}_{j, -y_0}(t_0) - P(\tilde{v}_j, \tilde{w}_j) = 0$  we can choose  $y'_0 \geq y_0$  such that

$$\left| \tilde{\mathcal{I}}_{j, -y'_0}(t_0) - P(\tilde{v}_j, \tilde{w}_j) \right| \leq \frac{\delta_0}{4}. \quad (3.3.37)$$

The choice of  $y'_0$  can be done to conserve (3.3.36) and to obtain

$$\left| \tilde{\mathcal{I}}_{j, -y'_0}(t_0) - \tilde{\mathcal{I}}_{j, y_0}(t_0) - P(\tilde{v}_j, \tilde{w}_j) \right| \geq \frac{3\delta_0}{4} \quad \text{and} \quad \left| \tilde{\mathcal{I}}_{j, -y'_0}(0) - \tilde{\mathcal{I}}_{j, y_0}(0) - P(\tilde{v}_j, \tilde{w}_j) \right| \leq \frac{\delta_0}{2},$$

and therefore

$$\left| \left( \tilde{\mathcal{I}}_{j, -y'_0}(0) - \tilde{\mathcal{I}}_{j, y_0}(0) \right) - \left( \tilde{\mathcal{I}}_{j, -y'_0}(t_0) - \tilde{\mathcal{I}}_{j, y_0}(t_0) \right) \right| \geq \frac{\delta_0}{4}.$$

Using the fact that the integrands of the expressions between parenthesis are compactly supported in the space, we infer from Proposition 3.3.2 that there exists an integer  $n_0$  such that

$$\left| \left( \mathcal{I}_{j,-y'_0}(t_n) - \mathcal{I}_{j,y_0}(t_n) \right) - \left( \mathcal{I}_{j,-y'_0}(t_n + t_0) - \mathcal{I}_{j,y_0}(t_n + t_0) \right) \right| \geq \frac{\delta_0}{8},$$

for any  $n \geq n_0$ . Ordering well the terms in the previous inequality, we obtain

$$\max \left\{ \left| \mathcal{I}_{j,-y'_0}(t_n) - \mathcal{I}_{j,-y'_0}(t_n + t_0) \right|, \left| \mathcal{I}_{j,y_0}(t_n) - \mathcal{I}_{j,y_0}(t_n + t_0) \right| \right\} \geq \frac{\delta_0}{16}. \quad (3.3.38)$$

Since  $t_0 \geq 0$ , by (3.3.30), and (3.3.36), we deduce

$$\mathcal{I}_{j,-y'_0}(t_n) - \mathcal{I}_{j,-y'_0}(t_n + t_0) \leq \frac{\delta_0}{32} \quad \text{and} \quad \mathcal{I}_{j,y_0}(t_n) - \mathcal{I}_{j,y_0}(t_n + t_0) \leq \frac{\delta_0}{32},$$

and then we infer from (3.3.38) that, for any  $n \geq n_0$ ,

$$\text{either } \mathcal{I}_{j,-y'_0}(t_n + t_0) - \mathcal{I}_{j,-y'_0}(t_n) \geq \frac{\delta_0}{16}, \quad \text{or } \mathcal{I}_{j,y_0}(t_n + t_0) - \mathcal{I}_{j,y_0}(t_n) \geq \frac{\delta_0}{16}.$$

This leads us to the possibility of choosing an increasing sequence  $(n_k)_{k \in \mathbb{N}}$  such that  $t_{n_{k+1}} \geq t_{n_k} + t_0$  for any  $k \in \mathbb{N}$ , and either

$$\mathcal{I}_{j,y_0}(t_{n_k} + t_0) - \mathcal{I}_{j,y_0}(t_{n_k}) \geq \frac{\delta_0}{16}, \quad (3.3.39)$$

for any  $k \in \mathbb{N}$ , or

$$\mathcal{I}_{j,-y'_0}(t_{n_k} + t_0) - \mathcal{I}_{j,-y'_0}(t_{n_k}) \geq \frac{\delta_0}{16},$$

for any  $k \in \mathbb{N}$ . Next, we suppose that (3.3.39) holds, the proof of the other case being exactly the same. From the fact that  $t_{n_{k+1}} \geq t_{n_k} + t_0$ , we conclude using (3.3.30), (3.3.36) and (3.3.39), that

$$\mathcal{I}_{j,y_0}(t_{n_{k+1}}) \geq \mathcal{I}_{j,y_0}(t_{n_k} + t_0) - \frac{\delta_0}{32} \geq \mathcal{I}_{j,y_0}(t_{n_k}) + \frac{\delta_0}{32}, \quad (3.3.40)$$

for any  $k \in \mathbb{N}$ . Now, we recall that  $\mathcal{I}_{j,y_0}(t_{n_k})$  is bounded by the energy of the initial datum. This yields a contradiction with (3.3.40) and finishes the proof.  $\square$

At this stage, the problem reduces to the case of one soliton. The proof of the next statement is exactly the same as the one given by the author in [1] for that case (see also [6] for more details).

**Lemme 3.3.2** ([1]). *Let  $y_0 > 0$ . For any  $t \in \mathbb{R}$  we have*

$$\tilde{\mathcal{I}}_{j,y_0}(t) \leq A_\epsilon \exp\left(-\frac{\nu_\epsilon}{16} y_0\right) \quad \text{and} \quad |P(\tilde{v}_j, \tilde{w}_j) - \tilde{\mathcal{I}}_{j,-y_0}(t)| \leq A_\epsilon \exp\left(-\frac{\nu_\epsilon}{16} y_0\right). \quad (3.3.41)$$

The proof of Proposition 3.3.1 is then exactly the same as the one of Proposition 2.7 in [1].

## 3.4 Asymptotic stability between the solitons in the hydrodynamical framework

### 3.4.1 Proof of (3.1.10)

Let  $\mathbf{c}^0$  be as in Theorem 3.1.2 and  $\mathbf{v}_0$  be any pair which belongs to the set  $\mathcal{V}(\alpha, L)$  with  $\alpha$  and  $L$  as in the hypothesis of Theorem 3.1.2.

Let  $j \in \{1, \dots, N\}$  and  $b_j$  satisfying (3.1.6)–(3.1.11). By (3.2.17),  $\varepsilon$  is uniformly bounded in  $X(\mathbb{R})$ . Then, there exists  $\varepsilon_{j,0}^* \in X(\mathbb{R})$  such that, up to a subsequence,

$$\varepsilon(t_n, \cdot + b_j(t_n)) \rightharpoonup \varepsilon_{j,0}^* \quad \text{in } X(\mathbb{R}) \quad \text{as } n \rightarrow +\infty. \quad (3.4.1)$$

We set  $\mathbf{v}_{j,0}^* = (v_{j,0}^*, w_{j,0}^*) := \varepsilon_{j,0}^*$  and denote by  $\mathbf{v}_j^* = (v_j^*, w_j^*)$  the unique global solution to (HLL) corresponding to this initial datum  $\mathbf{v}_{j,0}^*$ . We claim that this solution exponentially decays with respect to the space variable for any time, as well as all its space derivatives. More precisely, we have

**Proposition 3.4.1.** *The pair  $(v_j^*, w_j^*)$  is indefinitely smooth and exponentially decaying on  $\mathbb{R} \times \mathbb{R}$ . Moreover, given any  $k \in \mathbb{N}$ , there exists a positive constant  $A_{k,c}$ , depending only on  $k$  and  $c$ , such that*

$$\int_{\mathbb{R}} \left[ (\partial_x^{k+1} v_j^*)^2 + (\partial_x^k v_j^*)^2 + (\partial_x^k w_j^*)^2 \right] (x + \tilde{b}_j(t), t) \exp\left(\frac{\nu_c}{16}|x|\right) dx \leq A_{k,c}, \quad (3.4.2)$$

for any  $t \in \mathbb{R}$ , where  $\tilde{b}_j$  satisfies (3.1.6)–(3.1.11).

In view of this proposition, we can establish a Liouville type theorem in order to finish the proof of Theorem 3.1.2.

**Proposition 3.4.2.** *There exists a positive number  $\alpha^*$  such that, if  $(v, w)$  is a solution of (HLL) satisfying (3.4.2) and*

$$\|(v_0, w_0)\|_{X(\mathbb{R})} \leq \alpha^*,$$

then,

$$(v, w)(t, x) = 0 \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}.$$

This result concludes the proof of Theorem 3.1.2 since  $\varepsilon_{j,0}^* \equiv 0$  for any sequence  $(t_n)_{n \in \mathbb{N}}$ . Indeed, if we suppose that there exists a sequence of time  $(s_n)$  such that  $\varepsilon_{j,0} \neq 0$ , then, in view of the previous analysis, we get a contradiction from Proposition 3.4.2.

Now, we will show that (3.1.8) holds also when  $b_j$  is an arbitrary map satisfying (3.1.6) and (3.1.7) instead of (3.1.11).

*Proof.* Let  $(t_n)$  be a sequence of time such that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ . It follows from (3.1.7), up to a subsequence,  $\frac{b_j(t_n)}{t_n}$  has a limit  $l_j$  as  $n \rightarrow \infty$  and  $c_{j-1}^\infty < l_j < c_j^\infty$ . Next, we take  $\tilde{b}_j$  a smooth extension of  $b_j$  such that  $\tilde{b}_j(t_n) = b_j(t_n)$  for all  $n \in \mathbb{N}$ . More precisely,  $\tilde{b}_j \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$  verifies (3.1.6), and, from (3.1.7), we have

$$\lim_{t \rightarrow \infty} \tilde{b}_j'(t) = \lim_{n \rightarrow \infty} \frac{\tilde{b}_j(t_n)}{t_n} = l_j.$$

Hence,  $\tilde{b}_j$  satisfies (3.1.11). Then, by (3.1.10), we obtain

$$(v, w)(t_n, \cdot + \tilde{b}_j(t_n)) \rightharpoonup 0 \quad \text{in } X(\mathbb{R}),$$

as  $n \rightarrow \infty$ . This leads to

$$(v, w)(t_n, \cdot + b_j(t_n)) \rightharpoonup 0 \quad \text{in } X(\mathbb{R}),$$

as  $n \rightarrow \infty$ . This finishes the proof since this convergence holds for any sequence  $(t_n)$  such that  $t_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ .  $\square$

In the next two subsections we begin by proving Proposition 3.4.2 and then we give the proof of Proposition 3.4.1.

### 3.4.2 Proof of the Liouville type theorem

First, we verify that our limit solution has a small norm. This is a direct consequence of the conservation of the energy, (3.4.1), Theorem 3.2.1 and equivalence between the energy and the norm of  $X(\mathbb{R})$ . More precisely, we have

$$\|(v_{j,0}^*, w_{j,0}^*)\|_{X(\mathbb{R})} \leq \liminf_{n \rightarrow \infty} \|\varepsilon(t_n)\|_{X(\mathbb{R})} \leq A_c \alpha,$$

and then,

$$\|(v_j^*, w_j^*)(t)\|_{X(\mathbb{R})} \leq A_c \mathcal{E}(v_j^*, w_j^*)(t) = A_c \mathcal{E}(v_{j,0}^*, w_{j,0}^*) \leq A_c \|(v_{j,0}^*, w_{j,0}^*)\|_{X(\mathbb{R})} \leq A_c \alpha,$$

for all  $t \in ]T_-, T_+[$ , where  $]T_-, T_+[$  denotes the maximal interval of existence for the solution  $(v_j^*, w_j^*)$ . We derive from this inequality the existence of a number  $0 < \delta < 1$  such that

$$\|v_j^*(t)\|_{L^\infty} \leq \delta < 1,$$

for all  $t \in ]T_-, T_+[$ . It then follows from the result in [19] that the solution  $(v_j^*, w_j^*)$  is actually global, and that it satisfies

$$\|(v_j^*, w_j^*)(t)\|_{X(\mathbb{R})} \leq A_c \mathcal{E}(v_j^*, w_j^*)(t) \leq A_c \alpha. \quad (3.4.3)$$

for all  $t \in \mathbb{R}$ .

Next, we linearise (HLL) around zero. Let  $\mathbf{v} := (v, w)$  be a solution of (HLL) verifying (3.4.3). We obtain

$$\partial_t \mathbf{v} = J L \mathbf{v} + J B \mathbf{v}, \quad (3.4.4)$$

where we have denoted

$$J = S \partial_x := \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad (3.4.5)$$

$$L \mathbf{v} := \begin{pmatrix} -v + \partial_{xx} v \\ -w \end{pmatrix},$$

and

$$B \mathbf{v} := \begin{pmatrix} \frac{(\partial_{xx} v) v^2}{1-v^2} + \frac{(\partial_x v)^2 v}{(1-v^2)^2} + v w^2 \\ v^2 w \end{pmatrix}.$$

Now, we consider the following quantity

$$U(t) := \int_{\mathbb{R}} x [v_j^* w_j^*](t, x) dx,$$

for any  $t \in \mathbb{R}$ . Since  $(v_j^*, w_j^*)$  is a smooth solution of (HLL) which satisfies (3.4.2), the map  $U$  is of class  $\mathcal{C}^1$  and it is possible to differentiate the integrand with respect to the time variable. Hence, we deduce from (3.4.4) and an integration by parts that

$$U'(t) = -\langle L \mathbf{v}_j^*(t), \mathbf{v}_j^*(t) \rangle_{L^2(\mathbb{R})} - \langle L \mathbf{v}_j^*(t), \mu \partial_x \mathbf{v}_j^*(t) \rangle_{L^2(\mathbb{R})} + \langle \mu \partial_x B \mathbf{v}_j^*, \mathbf{v}_j^* \rangle_{L^2(\mathbb{R})}, \quad (3.4.6)$$

where  $\mu(x) = x$  for all  $x \in \mathbb{R}$ . For the linear terms, we integrate by parts to write

$$-\langle L \mathbf{v}_j^*(t), \mathbf{v}_j^*(t) \rangle_{L^2(\mathbb{R})} - \langle L \mathbf{v}_j^*(t), \mu \partial_x \mathbf{v}_j^*(t) \rangle = \int_{\mathbb{R}} \left[ \frac{3}{2} (\partial_x v_j^*(t))^2 + \frac{1}{2} (v_j^*(t))^2 + \frac{1}{2} (w_j^*(t))^2 \right]. \quad (3.4.7)$$

For the other term, we use the Cauchy-Schwarz inequality, the Sobolev embedding theorem, (3.4.2) and (3.4.3) to infer that

$$\left| \langle \mu \partial_x B \mathbf{v}_j^*, \mathbf{v}_j^* \rangle_{L^2(\mathbb{R})} \right| \leq A_c \alpha \|\mathbf{v}_j^*\|_{X(\mathbb{R})}^2. \quad (3.4.8)$$

Indeed, let us estimate two terms of the right hand side. The other ones can be estimated in a very similar way. Performing integrations by parts, and using the Cauchy-Schwarz inequality and (3.2.10), we can write

$$\left| \int_{\mathbb{R}} x \partial_x \left( (v_j^*(t, x))^2 w_j^*(t, x) \right) w_j^*(t, x) dx \right| \leq \|\mu \partial_x v_j^*(t)\|_{L^\infty} \|v_j^*(t)\|_{L^\infty} \|v_j^*(t)\|_{L^2} \|w_j^*(t)\|_{L^2} + \|v_j^*(t)\|_{L^\infty}^2 \|w_j^*(t)\|_{L^2}^2,$$

and

$$\left| \int_{\mathbb{R}} x \partial_x \left( \frac{(\partial_{xx} v_j^*(t, x))(v_j^*(t, x))^2}{1 - (v_j^*(t, x))^2} \right) v_j^*(t, x) dx \right| \leq A_c \|\mu \partial_{xx} v_j^*(t)\|_{L^\infty} \|\partial_x v_j^*(t)\|_{L^2} \|v_j^*(t)\|_{L^2} \|v_j^*(t)\|_{L^\infty} + A_c \|\partial_{xx} v_j^*(t)\|_{L^\infty} \|v_j^*(t)\|_{L^\infty} \|v_j^*(t)\|_{L^2}^2.$$

Then, by the Sobolev embedding theorem, 3.4.2 and (3.4.3), we obtain

$$\left| \int_{\mathbb{R}} x \partial_x \left( (v_j^*(t, x))^2 w_j^*(t, x) \right) w_j^*(t, x) dx \right| \leq A_c \alpha \|\mathbf{v}_j^*(t)\|_{X(\mathbb{R})}^2,$$

and

$$\left| \int_{\mathbb{R}} x \partial_x \left( \frac{(\partial_{xx} v_j^*(t, x))(v_j^*(t, x))^2}{1 - (v_j^*(t, x))^2} \right) v_j^*(t, x) dx \right| \leq A_c \alpha \|\mathbf{v}_j^*(t)\|_{X(\mathbb{R})}^2.$$

Now, we introduce (3.4.7) and (3.4.8) into (3.4.6) and we choose  $\alpha$  small enough to claim that

$$U'(t) \geq \frac{1}{4} \|\mathbf{v}_j^*(t)\|_{X(\mathbb{R})}^2. \quad (3.4.9)$$

Since  $U$  is uniformly bounded on  $\mathbb{R}$ , we infer that the map  $t \mapsto \|\mathbf{v}_j^*(t)\|_{X(\mathbb{R})}$  belongs to  $L^2(\mathbb{R})$ . This yields the existence of a sequence of positive times  $(s_n)_{n \in \mathbb{N}}$ , which goes to  $+\infty$  as  $n \rightarrow +\infty$ , such that we have

$$\lim_{n \rightarrow \infty} \|\mathbf{v}_j^*(\pm s_n)\|_{X(\mathbb{R})} = 0. \quad (3.4.10)$$

In view of (3.4.2), this gives

$$\lim_{n \rightarrow \infty} U(\pm s_n) = 0.$$

Integrating (3.4.9) from  $-s_n$  to  $s_n$  and taking the limit  $n \rightarrow +\infty$ , we deduce that

$$\int_{\mathbb{R}} \|\mathbf{v}_j^*(t)\|_{X(\mathbb{R})}^2 dt = 0.$$

Hence,

$$\mathbf{v}_j^* \equiv 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}.$$

This finishes the proof of Theorem 3.4.2.  $\square$

### 3.4.3 Proof of Proposition 3.4.1

In this section, we prove the exponential decay of the limit solution  $\mathbf{v}_j^*$ . First, we state the monotonicity of the momentum. Let  $(v, w)$  be a pair given by Theorem 3.2.1,  $j \in \{1, \dots, N\}$  and  $y_0 \in \mathbb{R}$ . Denote

$$\mathcal{I}_{j, y_0}(t) := \int_{\mathbb{R}} \Phi(x - (b_j(t) + y_0)) [vw](x, t) dx,$$

for  $b_j$  satisfying (3.1.6) and (3.1.11) and set

$$\lambda_{\mathbf{c}, \gamma} := \frac{1}{2} \min \left\{ 1 + \gamma_1, \gamma_2 - c_1, c_2 - \gamma_2, \dots, \gamma_{N+1} - c_N, 1 - \gamma_{N+1} \right\}$$

for any  $\mathbf{c} \in (-1, 1)^N$ , where  $\gamma := (\gamma_1, \dots, \gamma_{N+1}) := \lim_{t \rightarrow \infty} (b'_1(t), \dots, b'_{N+1}(t))$ . We claim the following monotonicity formula for this localized version of the momentum.



**Proposition 3.4.3.** *There exist positive numbers  $\alpha_2 \leq \alpha$ ,  $L_2 \geq L^*$ ,  $T > 0$  and  $A_2, A_2^* > 0$ , depending only on  $\mathbf{c}$  and  $\mathfrak{s}$ , such that, if  $\alpha_0 \leq \alpha_2$  and  $L \geq L_2$ , then the map  $\mathcal{I}_{j,y_0}$  is of class  $\mathcal{C}^1$  on  $\mathbb{R}$ , and it satisfies*

$$\begin{aligned} \frac{d}{dt} [\mathcal{I}_{j,y_0}(t)] &\geq \frac{\nu_{\mathbf{c}}^2}{32} \int_{\mathbb{R}} [(\partial_x v)^2 + v^2 + w^2](x, t) \Phi'(x - (b_j(t) + y_0)) dx \\ &\quad - A_2 \exp\left(-\frac{\nu_{\mathbf{c}}}{16}(|y_0 + \lambda_{\mathbf{c},\gamma} t|)\right), \end{aligned} \quad (3.4.11)$$

for any  $1 \leq j \leq N$  and any  $t \geq T$ . In particular, we have

$$\mathcal{I}_{j,y_0}(t_1) \geq \mathcal{I}_{j,y_0}(t_0) - A_2^* \exp\left(-\frac{\nu_{\mathbf{c}}}{16}|y_0|\right), \quad (3.4.12)$$

for any real numbers  $t_1 \geq t_0 \geq T$ .

The proof is very similar to the one of Proposition 5 in [19]. We will only sketch it.

*Proof.* As in the proof of Proposition 3.3.3, we write

$$\mathcal{I}'_{j,y_0}(t) = \mathcal{I}_1(t) + \mathcal{I}_2(t),$$

decomposing the real line into the region  $I_j(t)$  and its complementary set, where  $I_j(t)$  is the interval defined by

$$I_j(t) = \left[ b_j(t) - \frac{1}{4}(L + \lambda_{\mathbf{c},\gamma} t), b_j(t) + \frac{1}{4}(L + \lambda_{\mathbf{c},\gamma} t) \right].$$

For  $\mathcal{I}_2$ , we have (see the proof of Proposition 3.3.3 for more details)

$$|\mathcal{I}_2(t)| \leq A^* \exp\left(-\frac{1}{32}(L + \lambda_{\mathbf{c},\gamma} t)\right).$$

For  $\mathcal{I}_1(t)$ , we first infer from (3.1.6) that there exists  $T > 0$  sufficiently large such that for all  $t \geq T$ ,

$$c_{j-1}^\infty < b'_j(t) < c_j^\infty,$$

and then

$$b'_j(t)^2 \leq 1 - \frac{\nu_{\mathbf{c}}^2}{4}.$$

This leads, using (3.2.5) and (3.3.27), to

$$\mathcal{I}_1(t) \geq \frac{1}{2} \int_{I_j(t)} \Phi'(\cdot - (b_j(t) + y_0)) \left( (\partial_x v)^2 + v^2 + w^2 - 2 \left(1 - \frac{\nu_{\mathbf{c}}^2}{4}\right)^{\frac{1}{2}} |v||w| - 3v^2 w^2 + \frac{\nu_{\mathbf{c}}^*}{64} \ln(1 - v^2) \right) dx.$$

Now, increasing the value of  $T > 0$  if necessary, we infer from (3.1.11) that

$$|a_k(t) - b_j(t)| \geq \frac{1}{2}(L + \lambda_{\mathbf{c},\gamma} t),$$

for any  $t \geq T$  and  $1 \leq k \leq N$ . When  $x \in I_j(t)$ , we have

$$|x - a_k(t)| \geq |a_k(t) - b_j(t)| - \frac{1}{4}(L + \lambda_{\mathbf{c},\gamma} t) \geq \frac{1}{4}(L + \lambda_{\mathbf{c},\gamma} t),$$

for any  $1 \leq k \leq N$ . This yields, using (3.2.15), (3.2.17) (and the Sobolev embedding theorem), (3.2.18) and the exponential decay of the solitons,

$$|v(x, t)| \leq |\varepsilon_1(x, t)| + \sum_{k=1}^N |v_{c_k(t)}(x - a_k(t))| \leq A^* \left( \alpha + \exp\left(-\frac{\nu_{\mathbf{c}}^*}{16}(L + \lambda_{\mathbf{c},\gamma} t)\right) \right),$$

for any  $x \in I_j(t)$ . We now decrease  $\alpha$  and increase  $L$ , if necessary, to guarantee that  $|v|$  is sufficiently small on the interval  $I_j(t)$ . Then we can finish the proof as the one of Proposition 3.4.3.  $\square$

**Remarque 3.4.1.** *In view of the proof below, the limit solution  $(v_j^*, w_j^*)$  satisfies the conclusions of Proposition 3.4.3 for any time  $t \in \mathbb{R}$ , .*

The following claim contains the weak continuity of the flow and the convergence of the parameter  $b_j$ .

**Fait 2.** *Let  $j \in \{1, \dots, N\}$  and  $t \in \mathbb{R}$  be fixed. Then, there exist a map  $b_j^* \in C^1(\mathbb{R}, \mathbb{R})$  verifying (3.1.6)-(3.1.11) such that*

$$\left(v(\cdot + b_j(t_n), t_n + t), w(\cdot + b_j(t_n), t_n + t)\right) \rightharpoonup \left(v_j^*(\cdot, t), w_j^*(\cdot, t)\right) \quad (3.4.13)$$

and

$$\left(v(\cdot + b_j(t_n + t), t_n + t), w(\cdot + b_j(t_n + t), t_n + t)\right) \rightharpoonup \left(v_j^*(\cdot + b_j^*(t), t), w_j^*(\cdot + b_j^*(t), t)\right) \quad (3.4.14)$$

in  $X(\mathbb{R})$ , while

$$b_j(t_n + t) - b_j(t_n) \rightarrow b_j^*(t), \quad (3.4.15)$$

as  $n \rightarrow +\infty$ .

*Proof.* We take  $b_j^*(t) := \gamma_j t$ , for all  $t \in \mathbb{R}$ , where  $\gamma_j := \lim_{t \rightarrow +\infty} b_j'(t)$ . Clearly,  $b_j^*$  satisfies (3.1.6)-(3.1.11). Then, the proof remains exactly the same as the one of Proposition 3.3.2.  $\square$

As in the previous section, we claim the following lemma which shows the localisation of the momentum for the limit solution. For the limit profile  $(v_j^*, w_j^*)$ , we set

$$\mathcal{I}_{j, \pm y_0}^*(t) := \mathcal{I}_{j, \pm y_0}^{(v_j^*, w_j^*)}(t) = \int_{\mathbb{R}} [v_j^* w_j^*](t) \Phi(\cdot - (\pm y_0 + b_j^*(t))),$$

for any  $t \in \mathbb{R}$  and  $y_0 > 0$ .

**Lemme 3.4.1** ([6]). *For any positive number  $\delta$ , there exists a positive number  $y_\delta$ , depending only on  $\delta$ , such that for any  $t \in \mathbb{R}$  we have*

$$\left| \mathcal{I}_{j, y_0}^*(t) \right| \leq \delta \quad \text{and} \quad |P(v_j^*, w_j^*) - \mathcal{I}_{j, -y_0}^*(t)| \leq \delta, \quad (3.4.16)$$

for any  $y_0 \geq y_\delta$ .

In view of Remark 3.4.1, the proof is similar to the one of Lemma 3.3.1.

We also have

**Lemme 3.4.2** ([6]). *Let  $y_0 > 0$ . For any  $t \in \mathbb{R}$ , we have*

$$\mathcal{I}_{j, y_0}^*(t) \leq A_c \exp\left(-\frac{\nu_c}{16} y_0\right) \quad \text{and} \quad |P(v_j^*, w_j^*) - \mathcal{I}_{j, -y_0}^*(t)| \leq A_c \exp\left(-\frac{\nu_c}{16} y_0\right). \quad (3.4.17)$$

Using Proposition 3.4.3, we claim as in [6] that

**Proposition 3.4.4** ([1]). *Let  $t \in \mathbb{R}$ . There exists a positive constant  $\mathcal{A}_{c^0}$  such that*

$$\int_t^{t+1} \int_{\mathbb{R}} \left[ (\partial_x v_j^*)^2 + (v_j^*)^2 + (w_j^*)^2 \right] (x + b_j^*(s), s) e^{\frac{\nu_c}{16} |x|} dx ds \leq \mathcal{A}_{c^0}.$$

At this stage, the proof of Proposition 3.4.1 remains exactly the same as in [1] (see Section 4.2 for more details).

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**Titre** : Stabilité des solitons et des multi-solitons pour l'équation de Landau-Lifshitz.

**Mot clés** : Stabilité, solitons, Landeau-Lifschitz.

**Résumé** : Dans cette thèse, nous étudions l'équation de Landau-Lifshitz avec une anisotropie planaire en dimension un. Cette équation décrit la dynamique de l'aimantation dans des matériaux ferromagnétiques. Elle admet des solutions particulières de type onde progressive appelées solitons. D'abord, nous montrons la stabilité asymptotique des solitons de vitesse non nulle appelés solitons sombres dans l'espace d'énergie. Plus précisément, nous prouvons que toute solution correspondant à une donnée initiale proche du soliton de vitesse non nulle, converge faiblement dans l'espace d'énergie en temps long, vers un soliton de vitesse non nulle, sous les invariances géométriques de l'équation. Notre analyse repose sur les idées développées par Martel et Merle pour les équations de Korteweg-de Vries généralisées. Nous utilisons la transformée de Madelung pour étudier le problème dans le cadre hydrodynamique. Nous invoquons ensuite la stabilité orbitale des solitons et la continuité faible du flot afin de construire le profil limite. Nous établissons de plus une formule

de monotonie pour le moment, ce qui nous permet d'avoir la localisation du profil limite. Sa régularité et sa décroissance exponentielle découlent d'un résultat de régularité pour les solutions localisées des équations de Schrödinger. Nous finissons la preuve par un théorème de type Liouville, qui nous indique que seuls les solitons vérifient ces propriétés dans leurs voisinages. Nous nous intéressons également à la stabilité asymptotique d'une superposition de plusieurs solitons appelées multi-solitons. Les solitons de vitesse non nulle sont ordonnés selon leurs vitesses et sont initialement bien séparés. Nous démontrons la stabilité asymptotique autour et entre les solitons. Plus précisément, nous montrons que pour une donnée initiale proche de la somme de  $N$  solitons sombres, la solution correspondante converge faiblement vers un des solitons de la somme, quand elle est translatée au niveau du centre de ce soliton, et converge faiblement vers zéro quand elle est translatée entre les solitons.

**Title** : Stability of solitons and multi-solitons for Landau-Lifshitz equation.

**Key words** : Stability, solitons, Landeau-Lifschitz.

**Abstract** : In this thesis, we study the one-dimensional Landau-Lifshitz equation with an easy-plane anisotropy. This equation describes the dynamics of the magnetization in a ferromagnetic material. It owns travelling-wave solutions called solitons. We begin by proving the asymptotic stability in the energy space of non-zero speed solitons. More precisely, we show that any solution corresponding to an initial datum close to a soliton with non-zero speed, is weakly convergent in the energy space as time goes to infinity, to a soliton with a possible different non-zero speed, up to the geometric invariances of the equation. Our analysis relies on the ideas developed by Martel and Merle for the generalized Korteweg-de Vries equations. We use the Madelung transform to study the problem in the hydrodynamical framework. In this framework, we rely on the orbital stability of the solitons and the weak continuity of the flow in order to construct

a limit profile. We next derive a monotonicity formula for the momentum, which gives the localization of the limit profile. Its smoothness and exponential decay then follow from a smoothing result for the localized solutions of the Schrödinger equations. Finally, we prove a Liouville type theorem, which shows that only the solitons enjoy these properties in their neighbourhoods. We also establish the asymptotic stability of multi-solitons. The solitons have non-zero speed, are ordered according to their speeds and have sufficiently separated initial positions. We provide the asymptotic stability around solitons and between solitons. More precisely, we show that for an initial datum close to a sum of  $N$  dark solitons, the corresponding solution converges weakly to one of the solitons in the sum, when it is translated to the centre of this soliton, and converges weakly to zero when it is translated between solitons.