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Thèse de doctorat de l'Université Paris-Saclay préparée à l'École Polytechnique

Ecole doctorale n° 578:

Sciences de l'Homme et de la Société

Spécialité de doctorat: Sciences Économiques

par

Robert Somogyi

Marchés caractérisés par des contraintes de capacité

Essays on capacity-constrained pricing

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Composition du Jury :

Eric Avenel	Université de Rennes 1	Président du Jury
Francis Bloch	Paris School of Economics	Directeur de thèse
Marc Möller	University of Bern	Rapporteur
Thibaud Vergé	Université Paris-Saclay	Examinateur
Xavier Wauthy	Université Saint-Louis - Bruxelles	Rapporteur

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Résumé

Cette thèse est composée d'une introduction et de trois chapitres.

L'introduction est composée d'une introduction générale suivie d'une revue de littérature sur les articles les plus pertinents concernant le sujet des trois chapitres de la thèse.

Le premier chapitre, "Bertrand-Edgeworth Competition with Substantial Product Differentiation", étudie le comportement d'un duopole lorsque les deux entreprises sont caractérisées par des contraintes de capacité et produisent un bien différencié à la Hotelling, dans la lignée de Boccard-Wauthy (2010). La littérature existante considère deux hypothèses extrêmes sur le degré de différentiation horizontale de produit. Avec différentiation horizontale forte, les entreprises ont un pouvoir de monopole local et se conduisent comme des entreprises monopolistiques. Avec différentiation faible, les entreprises sont en concurrence directe et le seul équilibre est en stratégies mixtes. Dans ce chapitre je limite l'analyse au cas d'un degré intermédiaire de différentiation horizontale de produits. Je démontre l'existence d'au moins un équilibre en stratégie pure pour tous les niveaux de capacités, complétant ainsi la littérature existante et montrant une continuité entre les situations extrêmes étudiées jusqu'à présent. Ensuite j'étends l'analyse au cas des entreprises asymétriques où l'une des entreprises est située à l'extrémité de l'intervalle [0,1] des consommateurs et l'autre est à l'extérieure de cette intervalle. Je montre que le résultat de l'existence d'équilibre en stratégie pure est robuste à ce type d'asymétrie si le degré de différentiation horizontale est suffisamment large. A des degrés de différentiation plus faible j'identifie des régions de paramètres où il n'existe pas d'équilibre en stratégie pure dans le modèle asymétrique.

Le deuxième chapitre, "Monopoly Pricing with Dual Capacity Constraints" analyse un monopole qui est contraint par deux types de contraintes de capacité : un sur les quantités

produites, l'autre sur le nombre des consommateurs servis. Ce modèle peut s'appliquer à plusieurs secteurs industriels : les restaurants (contraintes sur la capacité de la cuisine et le nombre des tables), les hôpitaux (contraintes sur la capacité des salles d'opération et le nombre des lits) et le transport par conteneurs (contraintes sur le volume et le poids des conteneurs). Je montre l'existence d'une région de paramètres « cœur » où les deux contraintes de capacité sont saturées, et que dans cette zone de paramètres, le prix n'est pas une fonction monotone du niveau des contraintes de capacité. A une capacité plus grande peut correspondre un prix plus élevé car la composition de la clientèle varie avec la capacité. Pour la même raison, le bien-être agrégé des consommateurs n'augmente pas nécessairement si une des contraintes de capacité est augmentée. Ce résultat contredit l'intuition obtenue en regardant des modèles de fixation de prix du monopole avec une seule contrainte de capacité. Je démontre ces résultats dans deux étapes. Premièrement, j'étudie un modèle simple où les dispositions à payer des consommateurs sont distribuées de façon uniforme qui engendre une demande linéaire et des profits quadratiques. Deuxièmement, je montre que les résultats obtenus dans le modèle simple sont généralisables au cas d'une distribution des dispositions à payer plus générale.

Le troisième chapitre, "Competition with Dual Capacity Constraints", étend le modèle de fixation de prix avec deux contraintes de capacité au cas du duopole symétrique. Je caractérise des conditions sous lesquelles un équilibre symétrique existe. Je démontre l'existence d'une région cœur où les contraintes de capacités sont saturées. Je montre qu'il existe des conditions sous lesquelles la non-monotonicité des prix et du bien-être des consommateurs observés dans le 2ème chapitre est également présente dans le cas du duopole. J'identifie six types d'équilibres en stratégie pure dans le duopole symétrique et j'analyse les propriétés de ces équilibres. Je démontre que certains équilibres donnent naissance à des prix de duopole égaux au prix de monopole. En outre, dans d'autres équilibres, les entreprises peuvent choisir des prix d'équilibre identiques sur les deux marchés malgré leur pouvoir de discrimination des prix. Toutefois, la complexité du modèle rend impossible une résolution analytique complète, et des résultats de statique comparée ne peuvent être obtenus que par simulations dans des exemples.

A la fin du troisième chapitre, à l'aide d'une simulation d'un exemple concret j'identifie les six régions de paramètres identifiées dans la partie théorique du chapitre ainsi que des régions où un tel équilibre en stratégie pure n'existe pas.

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Introduction

Economists have realized the importance of capacity constraints from very early on. At the end of the 19th century, Edgeworth (1897, translated in 1925) has noted that firms may not have the willingness or the ability to satisfy all the demand they face. This observation substantially changes how firms should optimally behave, in particular, the existence of capacity constraints offers a way to escape the Bertrand paradox. In honor of Franis Ysidro Edgeworth, the theory of capacity-constrained pricing is often referred to as Bertrand-Edgeworth competition.

The most well-known contribution to the theory of Bertrand-Edgeworth competition is the seminal article of Kreps and Scheinkman (1983, henceforth KS). They study a two-stage game where firms' simultaneous capacity-building if followed by their simultaneous pricing decision. Remarkably, under some conditions, the outcome of this game corresponds to the outcome of the standard Cournot duopoly. This finding provides a basis for justifying the use of the simple but more unrealistic Cournot model by viewing it as a shortcut for such a more realistic model.

This powerful result has generated a rich and exciting literature about Bertrand-Edgeworth oligopolies. Much of this literature has focused on relaxing the assumptions of the original model and exposed the fragility of the main result of KS, i.e., the outcome equivalence with the Cournot model.[1] However, this article remains a powerful benchmark for subsequent analysis. Moreover, Edgeworth's original insight remains true: the presence of capacity constraints tends to soften price competition.

Despite the central role of prices in economics in general, and in microeconomic theory in particular, in my view there are many remaining open questions. This is true for the theory of capacity-constrained pricing games as well. For example, as Wauthy (2014) points out:

> " The minimal core of strategic decisions a firm has to make is three-fold: What to produce? At which scale? At what price? A full-fledged theory of oligopolistic competition should be able to embrace these three dimensions jointly. [..] we do not have such a theory at our disposal. [..] it is urgent to devote more efforts to analyze in full depth the class of Bertrand-Edgeworth pricing games with product differentiation. "

[1] I discuss the subsequent literature in detail in the next section.

Chapter 1 of the present dissertation aims to make a small step in this direction. Specifically, it makes a contribution to the understanding of Bertrand-Edgeworth competition on markets characterized by a substantial level of product differentiation.

In addition, the literature so far has typically assumed one-dimensional capacity constraints. However, this is idealized because in real-world production processes firms typically face several capacity constraints (size of plants, inventories, workforce, etc.). Chapters 2 and 3 of this dissertation develop a theory of pricing in the presence of multiple capacity constraints. Chapter 2 investigates monopoly pricing under dual capacity constraints, whereas Chapter 3 examines a symmetric duopoly under dual capacity constraints.

Related literature

In order to position the contributions of this dissertation in the literature, the following section provides a short, non-exhaustive survey of the existing models of capacity-constrained pricing. Firstly, I present the most often used rationing rules as they play a central role in Bertrand-Edgeworth competition. Secondly, I review some of the seminal contributions to the theory of Bertrand-Edgeworth competition. Thirdly, I provide a separate overview of the literature of capacity-constrained pricing games with product differentiation. These results are of special significance as they are directly related to Chapter 1. Fourthly, I discuss some more recent research directions related to capacity constraints and demand uncertainty. Finally, as multiple capacity constraints have so far been relegated to the realms of operations research and revenue management, I review a few models from these fields that are relevant to Chapters 2 and 3. None of these sections have the intention of providing an exhaustive survey, the main purpose of the literature review being the positioning of the three chapters to the existing models.

Rationing rules

A Bertrand-Edgeworth oligopoly setting requires more specification than a Cournot or a Bertrand-type oligopoly. In particular, if the firm that charges the lowest price cannot serve all the demand at the price it named, a complete description of the model has to specify which buyers will it serve. This in turn characterizes the contingent demand that the more expensive firms face.

Most models in the literature use some kind of rationing rule to complete the characterization of the demand function. A rationing rule is a formula that specifies the residual demand of the more expensive firm in case the cheaper firm cannot supply the whole demand it faces. Thus the use of rationing rules can be viewed as a shortcut that circumvents a detailed analysis of consumer behavior.

The literature uses mainly two types of rationing rules (see e.g. Tirole (1988) or Vives (1999)): the efficient and the randomized rationing rules. Assume $p_1 < p_2$ and assume that Firm 1 is unable to satisfy the demand it faces given its preset capacity level, i.e. $D(p_1) > k_1$. In this case the efficient rationing rule describes the residual demand function of Firm 2 as

$$D_2(p_2) = \begin{cases} D(p_2) - k_1 & \text{if } D(p_2) - k_1 > 0, \\ 0 & \text{otherwise} \end{cases}$$

This means that the cheaper firm serves first the consumers with the highest willingness-to-pay for the good. It is called efficient because it is the rationing rule that maximizes consumer surplus. Indeed, it models a situation in which consumers can resell the product among themselves without any cost, i.e. they can engage in frictionless arbitrage.

The other broadly used rule is called randomized or proportional rationing rule. It assumes that the probability of being rationed (not being served by the cheaper firm) is uniform across consumers. Since this probability is given by $\frac{D(p_1)-k_1}{D(p_1)}$ the residual demand function of Firm 2 is

$$D_2(p_2) = \left(\frac{D(p_1) - k_1}{D(p_1)}\right) D(p_2).$$

Under this rationing rule consumers are overall worse-off than under the efficient one since some consumers with willingness-to-pay of less than p_2 can obtain the good at the lower price. On the other hand, Firm 2 always faces a higher demand under this rationing rule.

Furthermore, Tasnádi (1999) introduces a family of rules called combined rationing rules that embeds both the efficient and the randomized rationing rules as extreme cases. In the case of a combined rationing rule with parameter λ, the residual demand curve of the more expensive firm is given by

$$D_2(p_2) = \max\left\{ D(p_2) - \lambda k_1 - (1-\lambda)\frac{k_1}{D(p_1)} D(p_2) \quad , \quad 0 \right\}.$$

Clearly, $\lambda = 0$ corresponds to the randomized rationing rule, while $\lambda = 1$ coincides with the efficient rationing rule.

Seminal work on Bertrand-Edgeworth competition

In this subsection, I present some of the most influential articles in the literature on Bertrand-Edgeworth competition.

One of the earliest contributions to analyzing capacity-constrained pricing games is from **Levitan and Shubik (1972)**. They study price-setting in a symmetric duopoly where both firms are characterized by a capacity of the same size, $k_1 = k_2 = k$. Market demand is assumed to be linear, $q = a - p$ and rationing is efficient.

They first establish that pure-strategy equilibrium exists only for low and very high capacity levels. In particular, the equilibrium prices for low capacity levels ($k \leq a/3$) correspond to the optimal prices a monopoly of size k would choose, i.e., $p = a - 2k$. For very high levels of capacities, $k \geq a$, the Bertrand result of marginal cost pricing applies.

For $a/3 < k < a$, i.e., for intermediate capacity levels the equilibrium necessarily involves mixed strategies. One of the most important contributions of this article is the invention of the efficient rationing rule. Thanks to this novel assumption, the model becomes sufficiently simple to solve, which allows for the explicit characterization of the mixed-strategy equilibrium. Both firms will randomize on the interval

$$\left[\frac{1}{k}\left(\frac{a-k}{2}\right)^2; \frac{a-k}{2}\right]$$

according to the cumulative distribution function

$$F(p) = \frac{kp - \left(\frac{a-k}{2}\right)^2}{p(p-2k+a)}.$$

Next, I will describe in detail the setting of **Kreps and Scheinkman (1983)**, as it the most widely cited article in this strand of literature. It investigates a two-stage game where in the first stage two identical firms engage in simultaneous capacity decisions. In the second stage, firms play a simultaneous price-setting game with the restriction that they cannot sell more than the capacities they installed in the first stage.

In the first stage, building capacities is costly, the cost function is identical for both firms, twice continuously differentiable and convex. The capacity decisions are simultaneous. After paying for the cost of capacities, up to the level of their capacity they can produce at zero cost. Both firms can observe the capacity of the other firm before the pricing subgame. In the second stage firms name prices independently and simultaneously. The inverse demand $P(x) = D^{-1}(p)$ is assumed to be strictly positive on some bounded interval on which it is twice-continuously differentiable, strictly decreasing and concave. Similarly to LS, KS also uses the efficient rationing rule, i.e., consumers with the highest valuations are served first.

This means that if $p_1 < p_2$ i.e. Firm 1 charges a lower price, Firm 1 and Firm 2 will be able to sell

$$q_1 = \min\{D(p_1), k_1\} \quad \text{and} \quad q_2 = \min\{k_2, \max(D(p_2) - k_1)\}$$

respectively, where k_i denotes the capacity Firm i built in the first stage. Given these assumptions, the authors first show that the pricing subgame has an equilibrium in mixed strategies. More precisely both firms randomize on the coincident interval of $[\underline{p}, \overline{p}]$ according to strictly increasing and continuous cumulative distribution functions $\Psi_i(p)$:

$$\Psi_1(p) = \frac{(p - \underline{p}) \cdot k_2}{p \cdot (D(p) - k_1 - k_2)} \quad \text{and} \quad \Psi_2(p) = \frac{(p - \underline{p}) \cdot k_1}{p \cdot (D(p) - k_1 - k_2)}$$

It is easy to see that $k_1 > k_2$ implies $\Psi_1(p) < \Psi_2(p)$ i.e. the strategy of the firm that built a larger capacity in the first stage first-order stochastically dominates the strategy of the other firm. Moreover, the firm that fixed the larger capacity will put an atom at the top of the interval i.e. plays the uppermost price $\overline{p}$ with positive probability.

Finally, by tedious calculations the authors show that the capacity choice game and the whole game have a unique subgame-perfect Nash-equilibrium. Moreover, this equilibrium is in pure strategies: both firms choose the same capacity levels in the first stage then name the same price with probability one in the second stage. Furthermore, the (equal) quantities sold in this equilibrium and the equilibrium price exactly coincide with the outcome of a standard Cournot duopoly.

One of the main critiques of the KS result concerns the rationing rule. Namely, **Davidson and Deneckere (1986)** show that the only rationing rule that gives rise to the Cournot outcome of the two-stage Bertrand-Edgeworth game formulated by KS is the efficient one.

In the case of zero cost of capacities, the model is not robust even to the smallest changes in the rationing rule: If the residual demand is defined in any other way than the efficient rationing rule does, the outcome of the otherwise identical game will tend to be more competitive. The authors show this result by first defining a general class of rationing rules, characterized by the residual demand curve for Firm i, denoted $D(p_i|p_j)$ that must be twice differentiable except at the point $p_i = p_j$. The residual demand must satisfy the inequality

$$D(p_i) - k_j \leq D(p_i|p_j) \leq \min\{D(p_j) - k_j, D(p_i)\} \quad \text{for} \quad p_i > p_j$$

and the equality $D(p_i|p_j) = D(p_i)$ if $p_i \leq p_j$ and finally it is assumed to have a value of

$$D(p_i|p_j) = \min\{k_i, \max(D(p_i)/2, D(p_i) - k_j)\}$$

in the case of both firms charge the same price. The authors denote the residual demand of the efficient rationing rule $D_K(p_i|p_j)$. Next, they define that f is locally distinct to the right from g at the point y (where f, g are two continuously differentiable functions, except at a finite number of common discontinuity points) if $\lim_{x \downarrow y} f'(x) \neq \lim_{x \downarrow y} g'(x)$. The first main theorem of the paper states that if $D(p_i|p_j)$ is locally different to the right from $D_K(p_i|p_j)$ at $p_i = p_j = p^c$ when $K_1 = K_2 = K^c$ (where p^c and K^c denote the Cournot equilibrium price and quantity, respectively) then the Cournot outcome cannot emerge in the equilibrium of the two-stage game with residual demand curve $D(p_i|p_j)$.

In addition, if the costs of capacity building are relatively small and the rationing rule is sufficiently different from the efficient rule, the Cournot outcome is not an equilibrium either. Instead, there exist two asymmetric equilibria in which both firms choose higher capacities than the Cournot quantities.

The authors state that although it is naturally just an "accident" that the efficient rationing rule gives rise to the exact Cournot outcome, it could have been the case that rationing rules not very different from that one provide "sufficiently close" results. They find that this is not the case, *"when capacity is not overly expensive, equilibria may differ markedly from the Cournot outcome, both qualitatively and quantitatively. Plant sizes are much larger [...] Profits are substantially lower as well."* (p.411). To conclude, they find that the outcome of the game is generally more competitive than the Cournot-outcome.

Another important contribution to the literature of Bertrand-Edgeworth competition is made by **Deneckere and Kovenock (1996)**, as it highlights the lack of robustness of the KS result in another dimension. This article studies endogenous capacity choice followed by a pricing subgame, as in KS, however, it relaxes the symmetric cost assumption.

By studying unit cost asymmetry, the paper generalizes the model of KS to allow for differences in firms' production efficiency. Keeping the efficient rationing rule assumption and allowing for somewhat more general demand functions than KS, it shows that equilibrium price distributions of the two firms are not always connected and do not necessarily coincide. Choosing Cournot capacity levels do not necessary constitute a subgame-perfect equilibrium. Moreover, a pure-strategy equilibrium in the capacity setting stage may not exist for large cost asymmetries.

Bertrand-Edgeworth competition with horizontal product differentiation

Wauthy (2014) provides a recent survey of Bertrand-Edgeworth pricing games with product differentiation, hence this subsection is largely influenced by his paper. Although Hotelling (1929) has created a huge strand of theoretical literature, the vast majority of the ensuing models of horizontal product differentiation has maintained the assumption of constant marginal cost of production. In this subsection, I mention a few exceptions that jointly examine capacity constraints and horizontal product differentiation.

Canoy (1996) examines Bertrand-Edgeworth competition with a very particular form of product differentiation. Specifically, he assumes that there are an infinite number of infinitesimal consumers. A consumer's responsiveness to price differences is captured by parameter θ, i.e., it buys from firm 1 if and only if

$$p_2 - p_1 \geq \theta.$$

θ is uniformly distributed on the interval $[-\Delta\beta, \beta]$, where $\beta > 0$ is a measure of horizontal product differentiation, and $\Delta > 0$ measures the asymmetry between the two firms.

The first main result of the paper is the existence of a threshold level $\beta' > 0$ such that pure-strategy equilibrium fails to exist for lower levels of product differentiation, i.e., for any $\beta \leq \beta'$. Conversely, there is another threshold level $\beta'' \geq \beta'$ above which there is always an equilibrium in pure strategies. Therefore there exists an equilibrium in pure strategies if the products are sufficiently dissimilar. Finally, the paper shows that under log-concave demand the two threshold levels can coincide, moreover, if there exists a pure-strategy equilibrium then it is unique.

The first paper aiming to jointly examine Bertrand-Edgeworth competition with the most standard model of horizontal product differentiation, i.e., the Hotelling line, is **Wauthy(1996)**.

It considers firms being located symmetrically on the Hotelling segment. The main result is that the presence of capacity constraints may restore existence of pure strategy equilibrium for close locations where otherwise it does not exist (see d'Aspremont et al., 1979).

Boccard and Wauthy (2005) consider a game involving a capacity-constrained firm located at the opposite extremity of a Hotelling segment to an unconstrained firm. They assume efficient rationing as it has a natural interpretation in case of a Hotelling model. The main results of the paper is the complete characterization of the unconstrained firm's equilibrium payoff. Moreover, it provides a lower bound for the support of mixed strategies of both firms.

Boccard and Wauthy (2010) extend this work by allowing for both firms to be capacity constrained. First, in case of ample capacities, they show that the symmetric Hotelling equilibrium is the only pure-strategy equilibrium by using a fixed point argument to iteratively eliminate all the other, dominated strategies. Second, they show that the support of mixed-strategy equilibria of the pricing subgame is finite. The main proposition of the paper is that there exist only 3 types of equilibria in this model. The first type is the symmetric pure-strategy Hotelling equilibrium for high capacity levels. The second type is a mixed-strategy equilibria where one firm plays a strategy displaying $n + 1$ atoms, the other firm using $n \geq 1$ atoms. The third type is where both firms' strategy involve the same number of atoms.

Therefore, as common in the literature of Bertrand-Edgeworth competition, they find that there is no pure-strategy equilibrium for intermediate levels of capacity constraints. Although the paper does not provide an explicit characterization of the mixed-strategy equilibria, it demonstrates two features of it that are of particular interest to Chapter 1. Firstly, the support of the equilibrium distribution consists of a finite set of atoms, as opposed to KS where this support is an interval. Secondly, the iterative process used in the proof of the main proposition highlights the fact that the number of atoms in the support of the distribution must be decreasing in the level of product differentiation.

A common assumption these two models make is that consumers' valuation is large enough to ensure that the market be always covered. This corresponds to a relatively low level of product differentiation. Indeed, this is a very common assumption in the Hotelling literature.[2] However, as I show in Chapter 1, this assumption hides an interesting specification

[2]For an exception that discusses this issue in detail in the absence of capacity constraints, see Economides (1984).

of the model. By assuming a lower valuation for consumers in the setting of Boccard and Wauthy (2010), Chapter 1 restricts its attention to markets with substantial levels of product differentiation. Note that the level of product differentiation considered is lower than in the trivial case of local monopolies.

It shows that for a fixed value of transportation cost there is a threshold valuation of consumers under which an equilibrium in pure strategies exists for any capacity-pair. The existence result fills a gap in the literature in the following sense. Consider a pricing game between two firms of intermediate-sized capacities. The subgame of the KS model, that exhibits no product differentiation, has no pure-strategy equilibrium, and the firms use a continuum of prices in equilibrium. In the setting of Boccard and Wauthy (2010) with low but positive levels of product differentiation, still no pure strategy exists, however, firms mix over a finite number of atoms in equilibrium. Also, the number of atoms is decreasing in the level of product differentiation. Chapter 1 demonstrates that for substantial levels of product differentiation, the existence of pure-strategy equilibria is restored, which can be seen as the number of atoms being reduced to 1 for both firms. Finally, for very high levels of product differentiation, the trivial pure-strategy equilibrium consisting of local monopoly pricing prevails.

Capacity constraints and demand uncertainty

In this section, I present some more recent work related to capacity-constrained pricing under uncertain demand. A first strand of literature extends the original model of KS by introducing some form of demand uncertainty to the two-stage game. A second branch of literature deals with intertemporal pricing under capacity constraints, more specifically with pricing policies such as clearance sales and advance purchase discounts. Hence the main difference between the two groups of models is whether the pricing subgame following the capacity-setting stage is static or dynamic.

Static pricing under demand uncertainty .

Reynolds and Wilson (2000) is one of the first papers that study a Bertrand-Edgeworth model characterized by demand uncertainty. This article can be viewed as a straightforward generalization of Kreps and Scheinkman (1983) to include demand uncertainty as the authors consider the same two-stage model. Although the level of market demand is uncertain in the first, capacity-setting stage, the firms observe the actual realization of the random variable before making their pricing decisions. Formally, let a be the level of market demand, a realization of a random variable with a commonly known distribution over the support $[\underline{a}, \overline{a}]$.

Then the pricing subgame is characterized by the triple (a, k_1, k_2). Most of the assumptions the authors make coincide with the assumptions of Kreps and Scheinkman (1983) with the additional demand uncertainty, in particular, efficient rationing is used by the firms.

Their main finding is that despite the assumptions of symmetric firms, symmetric pure-strategy equilibria may not exist if the variability of the demand function is above a threshold level. They establish this result in two key steps. Theorem 1 of the paper states that given a strictly positive probability density function of demand with support $[\underline{a}, \overline{a}]$ (where $0 \leq \underline{a} < \overline{a} < \infty$)

1. If $\widehat{q}(\underline{a}) \geq \widetilde{q}_c$ then there is a symmetric equilibrium in capacity choices, with capacity equal to $\widetilde{q}_c$ for both firms. Capacity is fully utilized and prices are set equal to the market clearing level for every demand realization in this equilibrium.

2. If $\widehat{q}(\underline{a}) < \widetilde{q}_c$ then a symmetric equilibrium in pure strategies for capacity choices does not exist.

In this theorem $\widehat{q}(a)$ represents the unique equilibrium production of a Cournot duopoly with the level of demand a while $\widetilde{q}_c$ denotes the unique equilibrium production of a Cournot duopoly that faces the same demand uncertainty as the Bertrand-Edgeworth oligopoly, as described above.

By focusing on a simple two-point distribution of demand levels and linear demand functions, Theorem 2 establishes the relationship between the variance of demand uncertainty and equilibrium capacity choices. Formally, the theorem assumes that the level of demand equals $\underline{a}$ with probability $\theta \in (0, 1)$ and equals $\overline{a}$ with the complement probability and it also assumes inverse demand function $P(q, a) = a - q$ while $a \geq q$ and $P(q, a) = 0$ otherwise. Then

1. If $\underline{a} \geq E(a) - c$ where c is the marginal cost of production then there is a symmetric equilibrium in capacity choices, with capacity equal to $\widetilde{q}_c = (E(a) - c)/3$ for each firm. Capacity is fully utilized and prices are set equal to the market clearing level for every demand realization in this equilibrium.

2. If $\frac{\overline{a}}{2} \leq \underline{a} < E(a) - c) < \widetilde{q}_c$ then a symmetric equilibrium in pure strategies for capacity choices does not exist, although there are pure-strategy *asymmetric* equilibria in capacity investments.

The second case contradicts the results of Kreps and Scheinkman (1983) and is somewhat surprising given the symmetry of firms. The authors' interpretation of this result is that in asymmetric equilibria, when demand is low the small firm's expected marginal revenue is negative which implies that building additional capacities would create more loss for the small firm in the low demand state (given the capacity of the firm remains smaller than the other firm's capacity). It is exactly out of fear of the low demand periods that the small firm will refrain from expanding its capacity and the asymmetric equilibria are sustained.

Moreover, Theorem 2 implies that in periods of high demand the pricing subgame has a pure-strategy equilibrium and firms use all their capacities whereas in periods of low demand the equilibrium will be in mixed strategies and firms do not produce up to their full capacity levels. This provides an empirically verifiable prediction, namely that price volatility will be higher in periods of low demand. The authors use US manufacturing industry data to verify this result.

De Frutos and Fabra (2011) also investigate the interaction of demand uncertainty and capacity constraints in a similar game. The main difference with respect to Reynolds and Wilson (2000) is that they assume price-inelastic demand. Formally, there is a mass θ of infinitesimal and identical buyers, where θ is known to be distributed according to the cumulative distribution G.

Firstly, they solve the pricing subgame for every possible capacity-pair and demand. They show that similarly to LS, for very levels of capacity, $K < \theta$, where K is the total industry capacity, they act as monopolist's and charge the reservation price of consumers. For very high capacity levels (the smaller firm being able to serve the demand alone), the only equilibrium is marginal cost pricing. For intermediate capacity levels a pure-strategy equilibrium fails to exist and the mixed strategy equilibrium is unique.

Secondly, they investigate the capacity investment stage which precedes the pricing subgame. They show that a symmetric pure-strategy equilibrium does not exist under fairly general restrictions on the density function. Moreover, assuming that $\theta g(\theta)$ is non-decreasing allows the authors to apply the powerful theory of submodular games. It leads to the main proposition of the paper: the only pure-strategy equilibria of the capacity choice are asymmetric and they are outcome-equivalent.

Lepore (2012) also analyzes variants of the original Kreps and Scheinkman (1983) model. The main interest of the article is under which assumptions the equilibrium of the model coincides with the equilibrium of the Cournot game characterized by demand uncertainty. The author shows two necessary and sufficient conditions under which for a large variety of rationing rules those two equilibria coincide. The first condition requires that the variance in absolute market size be small relative to the cost of building capacity. The second is a situation in which demand uncertainty is such that the market demand is very high with high probability and with the remaining probability the market demand is extremely small.

Furthermore, he shows that under the efficient rationing rule the first condition is sufficient for the outcome of the game to coincide with the Cournot outcome under uncertainty. This result, contrary to most papers in this strand of literature, shows the robustness of Kreps and Scheinkman (1983).

Intertemporal pricing under demand uncertainty .

Nocke and Peitz (2007) examine monopolies' intertemporal pricing policy in a setting where the firms face demand uncertainty before making a capacity choice. The focus of the paper is on the clearance sales mechanism, in which the firm sells the product at the regular price in the first period, and if its capacity is not exhausted then it offers a marked-down price in the second period. Motivating examples include either seasonal products, as in the clearance sales in the apparel industry, or durable goods with a high storage costs, such as ski equipment.

Consumers are assumed to be forward-looking in the following sense: they anticipate that they get rationed in the sales period with a higher probability than in the first period. The main trade-off consumers face is thus the following: they can either buy the good in the first period for a higher price with probability 1, or wait for the lower price in the second period while risking to be rationed. Demand uncertainty is modeled by two demands states: with some probability there is a good state with more high-type consumers than in the bad state, occurring with the complement probability.

The paper compares the clearance sales mechanism with two benchmark mechanisms: the first is uniform pricing, the second is introductory offers policy. The former consists of setting the same price in both periods. The latter consists of the firm selling the good for a strictly lower price in the first period but restricting its available quantity. The first result of the paper is that uniform pricing is optimal in the absence of demand uncertainty.

The main result of the paper is that introductory offers are never optimal under demand uncertainty, therefore the optimal price path is necessarily non-increasing. Furthermore, it identifies necessary and sufficient conditions on consumers' valuation and on the mass of different types in different demand states for the clearance sales policy to be optimal.

Möller and Watanabe (2010) consider a similar intertemporal selling problem. The key differences compared to the previous paper are the presence of individual demand uncertainty and the lack of aggregate demand uncertainty. Consumers are assumed to be ignorant about their valuation of the product in the initial period. Moreover, they might face rationing if the aggregate demand exceeds capacity. The motivating examples are situations in which consumers must decide about the purchase of the good well in advance of its actual consumption, such as airplane tickets, sports events or theater tickets, where consumers become more knowledgeable with time about the product itself or their own utility for it. The main trade-off for the consumers is thus the following: buying early to avoid rationing or buying late to get informed about their valuation.

Consumers' individual demand uncertainty means that in the first period they do not know whether their valuation will take a high or low value. They all have unit demands and their mass is normalized to 1. A fraction of them are bad types who have a higher probability of having a low valuation than the good types. Bad types' valuation can be higher than good types' expected valuation.

The main objective of the article is the comparison of two commonly used pricing policies: clearance sales and advance purchase discounts. The former is defined by a strictly decreasing price path $p_1 > p_2$ whereas the latter is defined by strictly increasing prices $p_1 < p_2$. The monopoly also has the option to sell exclusively in the first or in the second period.

The first main result of the paper corresponds to the case of exogenously given and potentially binding capacity, random rationing rule, the monopoly being able to commit to any price schedule but being unable to set per period capacity limits, and consumers being unable to resell the product. Proposition 1 provides a complete partitioning of the parameter space, thus establishing the monopoly's optimal pricing schedule. Clearance sales can never be optimal if the fraction of good types is sufficiently large, otherwise any of the four options can be optimal for the firm depending on the exact parameter values.

The next results describe how the attractiveness of the different pricing schemes varies by changing the assumptions in the benchmark model one-by-one. When capacity is chosen endogenously, the monopoly will implement clearance sales when capacity costs are small but increasing rapidly, if they are high then it will implement an advance purchase discount, otherwise it will sell exclusively in the first or the second period. If the monopoly cannot commit to a second period price or cannot prohibit that the consumers engage in resale, the clearance sales mechanism becomes more attractive. However, when rationing is efficient instead of random, clearance sales can never be optimal.

Operations research models related to dual capacity constraints

Models where several capacity constraints co-exist have so far been relegated to the realms of operations research. However, most models in this field of research do not consider issues of price setting which is the main concern of this Ph.D. dissertation. One notable exception is Xiao and Yang (2010) which studies revenue management with dual capacity constraints that I describe in detail below. Therefore, I start this subsection by briefly mentioning a few of these applied models to highlight that the presence of multiple capacity constraints is an important issue in many real-world markets.

Patient admission planning for scheduled surgeries and patient mix optimization are both important problems hospitals have to face. The multidimensional nature of capacities in hospitals is crucial for such planning. Many recent papers in the operational research literature focus on solving a variety of problems that arise in a context where the treatment of different categories of patients require different levels of capacities.

Adan and Vissers (2002) take into account operating room time, intensive care unit beds, medium care unit beds and nurses' time to simulate the optimal schedule of a real-world hospital department. Testi et al. (2007) design a hierarchical scheduling of operating rooms based on constraints including regular operating room time, overtime, and surgical staff's time. Banditori and al. (2013) also consider multiple a capacity setting (wards, surgical staff, regular operating room time), in addition, they provide a comparison of the recent articles in this area.

The hospitality industry's capacity management literature has also recognized the importance of dual capacities. Kimes and Thompson (2004) optimize the table mix for restaurant revenue management taking into account not only the number and distribution of seats but also the size of the service areas. Bertsimas and Romy (2003) consider both sizes of parties to be seated and expected service duration to compare several optimization-based approaches to restaurant revenue management.

Xiao and Yang (2010) revisit traditional revenue management assuming dual capacity constraints. Thus they are interested in a dynamic pricing problem, a key difference compared to the present dissertation. Their motivating example is ocean container shipping, where container sizes are standardized (two varieties dominate the market) and maximal weights of all containers are given by on-road regulations. Hence the two constraints are of different dimensions: the first can be expressed in cubic meters, the second in kilograms.

Despite the additional complexity of a second capacity constraint, they are able to derive an analytical solution to the revenue maximization problem. The main finding of the paper is that under some mild conditions the presence of a second capacity constraint changes the optimal policy qualitatively. In particular, in one-dimensional models the optimal policy of which consumer groups to serve depends solely on the price and is independent of the consumer groups' demand . This does not hold for dual capacities where both prices and demand should be taken into account. Moreover, one-dimensional models typically exhibit a nested-fare structure, i.e., if a consumer group is served in a given period then all consumer groups with higher prices are also served. Remarkably, the optimal policy with dual capacity constraints does not exhibit this property. Finally, numerical simulations reveal that the optimal policy that takes into account both capacity constraints outperforms the most commonly used revenue management heuristics.

Chapter 1

Bertrand-Edgeworth Competition with Substantial Horizontal Product Differentiation

Abstract: Since Kreps and Scheinkman's seminal article (1983) a large number of papers have analyzed capacity constraints' potential to relax price competition. However, the majority of the ensuing literature has assumed that products are either perfect or very close substitutes. Therefore very little is known about the interaction between capacity constraints and local monopoly power. The aim of the present paper is to shed light on this question using a standard Hotelling setup. The high level of product differentiation results in a variety of equilibrium firm behavior and it generates at least one pure-strategy equilibrium for any capacity level.

1.1 Introduction

The problem of capacity-constrained pricing decision in oligopolies has received considerable attention since Kreps and Scheinkman's seminal article (1983). Most of the work in the field of Bertrand-Edgeworth oligopolies focused on the case of homogeneous goods and the capacities' potential impact of relaxing price competition.[1] However, a large number of real-world industries characterized by capacity constraints offer differentiated products. Examples include the airline industry, where capacities clearly play a central role and different companies tend to include different services in the price of their ticket (checked-in luggage, seat reservation, in-flight meal etc.). In the telecommunication sector, mobile service operators are bound by the size of their 3G and 4G networks, and clearly offer differentiated products (monthly data cap, speed, network coverage etc.). In the hospitality industry, competing hotels tend to be differentiated (breakfast, reservation policy, amenities) and constrained by the number of available rooms.

Moreover, taking into account both horizontal product differentiation and the presence of capacity constraints might lead to novel and surprising theoretical results, as first demonstrated by Wauthy (1996). Despite the prevalence of such industries and the theoretical interest they present, the literature on Bertrand-Edgeworth oligopolies with product differentiation remains scarce. As Wauthy (2014) points out in a recent survey of this branch of literature:

> “ The minimal core of strategic decisions a firm has to make is three-fold: What to produce? At which scale? At what price? A full-fledged theory of oligopolistic competition should be able to embrace these three dimensions jointly. [..] we do not have such a theory at our disposal. [..] it is urgent to devote more efforts to analyze in full depth the class of Bertrand-Edgeworth pricing games with product differentiation. ”

This paper aims to make a step in this direction. Specifically, it analyzes Bertrand-Edgeworth competition on markets characterized by a substantial level of product differentiation. By restricting attention to relatively high levels of product differentiation in a standard Hotelling setup, it shows that there exists at least one pure-strategy equilibrium for any capacity-pair. This stands in contrast with most models of Bertrand-Edgeworth competition that typically find non-existence for intermediate capacity-levels. The main result of the paper is a complete characterization of the pure-strategy equilibria, which reveals a variety of equilibrium firm behavior in this setting. Note that an even higher level of product differentiation leads to a trivial pure-strategy equilibrium: non-interacting firms acting as local

[1] Recent examples include Acemoglu et al. (2009), de Frutos and Fabra (2011) and Lepore (2012).

monopolies.

Most closely related to this paper is Boccard and Wauthy (2011). They investigate the interaction between capacity constraints and Hotelling-type differentiation and find the absence of an equilibrium in pure strategies for intermediate capacity levels. Their main finding is that the support of equilibrium prices consists of a finite number of atoms, and the number of these atoms is decreasing in the level of product differentiation. An important assumption their paper makes is that consumers' valuation for the good is large compared to transportation costs, which results in the market always being covered in equilibrium. While this assumption prevails in the Hotelling literature[2], the present paper shows that it hides an interesting setting, namely the case of substantial product differentiation.

In earlier work, Benassy (1989) and Canoy (1996) also analyze Bertrand-Edgeworth models with horizontal product differentiation. The main difference with the present paper is that both of these papers use non-standard specifications of product differentiation. Specifically, Benassy (1989) captures product differentiation through demand elasticities in a model of monopolistic competition, whereas Canoy (1996) introduces asymmetries between the firms and allows consumers to buy several units of the good. A common finding of the papers is the existence of pure-strategy equilibrium for sufficiently high levels of product differentiation. The present paper reformulates this result in the more standard Hotelling framework. Furthermore, contrary to the papers above, the simplicity of the model allows for the complete characterization of pure-strategy equilibria for substantial levels of product differentiation.

The chapter is organized as follows. Section 1.2 describes the model, formulates the profit function and identifies the potential equilibrium strategies. Section 1.3 contains the main result of the paper, the complete characterization of the equilibria. Section 1.4 discusses the results in the light of the existing literature. Section 1.5 examines an asymmetric version of the baseline model. Section 1.6 concludes.

[2]For an exception that discusses this issue in detail, see Economides (1984). For more recent work making the same assumption implicitly or explicitly, see for example Gal-Or (1997), Lyon (1999), and Brekke et al. (2006) for models of the health care market, and Ishibashi and Kaneko (2008) for a model of a mixed duopoly.

1.2 The model

1.2.1 Setting

This paper analyzes a duopoly with firms denoted x and y that produce substitute products. They choose a price p_i ($i \in \{x, y\}$) for one unit of their product. Assume the firms are located on the two extreme points of a unit-length Hotelling-line (x at $\tau = 0$, y at $\tau = 1$) and transportation cost is linear. Moreover, consumers are uniformly distributed along the line but are otherwise identical. They all seek to buy one unit of the product which provides them a gross surplus v. The value of the outside option of not buying the product is normalized to 0. In addition, the firms face rigid capacity constraints k_x, k_y. For simplicity, assume that marginal costs of production are constant and normalized to zero. The size of the capacities as well as the value of the other parameters of the model are common knowledge. The firms' objective is to maximize their profit by choosing their price.

A consumer located at point τ purchasing from firm x has a net surplus of

$$v - p_x - t \cdot \tau$$

while purchasing from firm y provides her a net surplus of

$$v - p_y - t \cdot (1 - \tau)$$

where t is the per-unit transportation cost.

Assumption. Assume $v/t \leq 1.5$, i.e. the products of the firms are substantially different from one another. Furthermore, to get rid of some trivial cases I will assume $1 < v/t \leq 1.5$ and refer to it as *intermediate* level of product differentiation.

Boccard and Wauthy (2011) analyze a similar setting, the key difference being the level of product differentiation. They restrict their attention to situations in which products are relatively close substitutes, namely $v/t > 2$. Below I argue that this simplifying assumption has a surprisingly large impact on the nature of equilibria, hence extending the analysis to the case of intermediate capacity levels provides new insights into the mechanisms of capacity-constrained oligopolies.

Furthermore, it is easy to see that a very high level of product differentiation, i.e., $v/t < 1$, leads to the uninteresting case of firms behaving as local monopolists, never interacting.

1.2.2 The profit function

Assuming rational consumers the following two constraints are straightforward. The participation constraint (PC) ensures that a consumer located at point τ buys from firm x only if her net surplus derived from this purchase is non-negative:

$$v \geq p_x + t \cdot \tau \tag{PC}$$

The individual rationality constraint (IR) ensures that a consumer located at point τ buys from firm x only if this provides her a net surplus higher than buying from the competitor:

$$v - p_x - t \cdot \tau \geq v - p_y - t \cdot (1 - \tau) \tag{IR}$$

Let T_x be the marginal consumer who is indifferent whether to buy from firm x or not. In the absence of capacity constraints it is easy to see that T_x is the minimum of the solutions of the binding constraints (PC) and (IR).

Let $\overline{T}_x$ be the consumer for whom both of the above constraints are binding. Thus this consumer is indifferent among buying form x, buying from y and not buying at all. Formally,

$$v - p_x - t\overline{T}_x = v - p_y - t(1 - \overline{T}_x) = 0 \quad \Rightarrow \quad \overline{T}_x = \frac{p_y - v + t}{t}.$$

Thus $\overline{T}_x$ plays the role of partitioning the price space according to market coverage. The net surplus being decreasing in the distance from firm x implies that (PC) is binding for $T_x \leq \overline{T}_x$ and (IR) is binding if $T_x \geq \overline{T}_x$. Symmetric formulas apply to firm y. Therefore, in case capacities are abundant,

$$p_x = \begin{cases} v - t \cdot T_x & \text{if } T_x \leq \overline{T}_x, \\ p_y + t - 2 \cdot t \cdot T_x & \text{if } T_x \geq \overline{T}_x. \end{cases} \tag{1.1}$$

Naturally, the existence of capacity constraints means for firm x that it cannot serve more than k_x consumers. Assume that after each consumer chooses the firm to buy from (or to abstain from buying), firms have the possibility to select which consumers to serve and they serve those who are the closest to them. In our setting this corresponds to the assumption of efficient rationing rule, which is extensively used in the literature. Therefore the additional constraints caused by the fixed capacity levels can be written as:

$$T_x \leq k_x \quad \text{and} \quad 1 - T_y \leq k_y \tag{CC}$$

It is important to notice that in some cases, when firm y is capacity-constrained, firm x can extract a higher surplus from some consumers by knowing that they cannot purchase from the rival even if they wanted to since firm y does not serve them. Practically, this means that the participation constraint (PC) will always be binding on $\left[\overline{T}_x, 1 - k_y\right]$ whenever this interval is not empty, i.e. whenever the rival's capacity is sufficiently small: $k_y \leq 1 - \overline{T}_x$. Using this observation, one can reformulate (1.1) for any capacity level:

$$p_x = \begin{cases} v - t \cdot T_x & \text{if } T_x \leq \max\{\overline{T}_x, 1 - k_y\} \ , \\ p_y + t - 2 \cdot t \cdot T_x & \text{if } T_x > \max\{\overline{T}_x, 1 - k_y\} \end{cases} \tag{1.2}$$

Firm x's profit can be simply written as $\pi_x = p_x T_x$. Given the competitor's capacity and its price choice, determining the unit price p_x is equivalent to determining the marginal consumer T_x. The observation that prices and quantities can be used interchangeably will simplify the solution of the model.[3] Importantly, the firms decide about prices, however, the quantities those prices imply are more directly comparable with the size of capacities.

The profit can thus be rewritten as

$$\pi_x(T_x) = \begin{cases} \pi_x^{LM} = (v - t \cdot T_x) \cdot T_x & \text{if } T_x \leq \max\{\overline{T}_x, 1 - k_y\}, \\ \pi_x^{C} = (p_y + t - 2 \cdot t \cdot T_x) \cdot T_x & \text{if } T_x > \max\{\overline{T}_x, 1 - k_y\} \end{cases} \tag{1.3}$$

Note that this formula reveals another interpretation of $\overline{T}_x$: it is the point where the two quadratic curves that constitute the profit function cross (other than their crossing at 0).

The optimization problem of the firm consists of finding the value T_x which maximizes the above expression satisfying the capacity constraint (CC). The superscript LM stands for Local Monopoly because the firm extracts all the consumer surplus from the marginal consumer when (PC) binds. Similarly, the superscript C stands for Competition since the marginal consumer is indifferent between the offer of the two firms whenever (IR) binds.

[3] The technique of arguing in terms of quantities instead of prices is also used by Yin (2004).

1.2.3 Potential equilibrium strategies

Define $T_x^{LM} = \arg\max_{T_x} \pi_x^{LM}$ and $T_x^C = \arg\max_{T_x} \pi_x^C$, the values at which the two quadratic curves attain their maxima, hence they are local maxima of the profit function $\pi_x(T_x)$.

The relative order of the five variables

$$T_x^{LM}, \quad T_x^C, \quad \overline{T}_x, \quad 1 - k_y \quad \text{and} \quad k_x$$

is crucial in solving the maximization problem. The main difficulty in the solution of the firms' maximization program is twofold. On the one hand, the profit function is discontinuous whenever $k_y \leq 1 - \overline{T}_x$ and kinked otherwise. On the other hand, the values

$$\overline{T}_x = \frac{p_y - v + t}{t} \quad \text{and} \quad T_x^C = \frac{p_y + t}{4t}$$

depend on the choice of the other firm, p_y. The following lemma simplifies the solution considerably.

Lemma 1.1.
$T_x^{LM} \leq \overline{T}_x$ implies $T_x^C \leq \overline{T}_x$ and $T_x^C \geq \overline{T}_x$ implies $T_x^{LM} \geq T_x^C \geq \overline{T}_x$.

The proof of the lemma is relegated to the Appendix. The form of firm x's profit function hinges on the relative order of $\overline{T}_x$ and $1 - k_y$. Therefore in the following discussion Iwill separate two cases: In Case A the capacity of firm y is relatively large, $1 - k_y < \overline{T}_x$. In Case B $1 - k_y \geq \overline{T}_x$ which means that firm x may be able to take advantage of the fact that its adversary is relatively capacity-constrained.

Case A: $1 - k_y < \overline{T}_x$. When the capacity of firm y is relatively large, (1.1) shows the relation between the price p_x charged by firm x and its demand (captured by the marginal consumer T_x). Using Lemma 1.1 three different subcases can be identified depending on the parameter values of the model and the competitor's choice.

Lemma 1.2. *Assume $1 - k_y < \overline{T}_x$.*

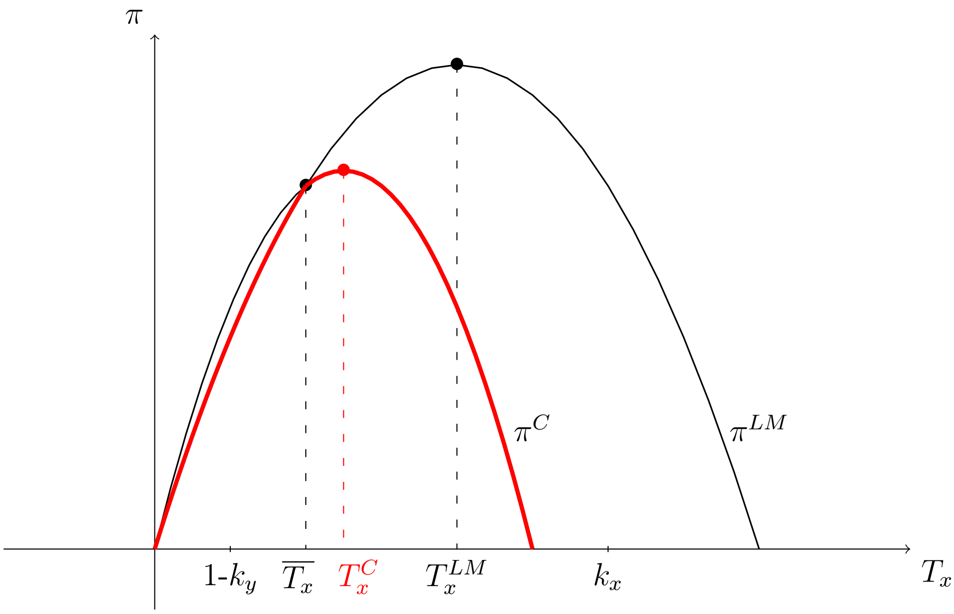
(A1) if $T_x^{LM} \leq \overline{T}_x$ then the optimal choice of firm x is $\min(T_x^{LM}, k_x)$,

(A2) if $T_x^C \geq \overline{T}_x$ then the optimal choice of firm x is $\min(T_x^C, k_x)$,

(A3) if $T_x^C \leq \overline{T}_x \leq T_x^{LM}$ then the optimal choice of firm x is $\min(\overline{T}_x, k_x)$.

Considering Lemma 1.1 it is easy to see that cases A1, A2 and A3 provide a complete partitioning of Case A. Hence for any parameter values in Case 1 and for every possible

Figure 1.1. Illustration of Case A2 ($T_x^C < k_x$)



behavior of the competitor, the lemma identifies the best response strategy of firm x. Symmetric formulas apply for firm y. The complete proof of this lemma is relegated to the Appendix.

However, for an intuition, first notice that the two branches of the profit function, π_x^{LM} and π_x^C are both quadratic functions of T_x that by definition cross each other at 0 and at $\overline{T}_x$. Then depending on the values t, v and T_y one of the three possibilities above will hold. As an illustration of Case A2 when $T_x^C < k_x$ see Figure 1.1. Using Lemma 1.1 the condition of the case $T_x^C \geq \overline{T}_x$ immediately implies $T_x^{LM} \geq \overline{T}_x$. We know that the profit function is composed of the function π_x^{LM} on the interval $[0, \overline{T}_x]$ then it switches to function π_x^C . The actual profit function is thus the thick (red) curve in the figure. Then using the figure it is straightforward to find the optimal choice of firm x. Since the two quadratic and concave functions cross each other before either of them reaches its maximum, the maximal profit will be attained on the second segment where $\pi_x = \pi_x^C$. By definition, $\arg\max_{T_x} \pi_x^C = T_x^C$ is the optimal choice, and the assumption $T_x^C < k_x$ makes this feasible.

Case B: $\overline{T}_x \leq 1 - k_y$. In Case B, the rival of firm x disposes of relatively low capacity. Therefore firm x might be inclined to take advantage of the fact that firm y is not capable of serving consumers located on the interval $[0, 1 - k_y]$. On this segment firm x does not have to care about its competitor's price and the individual rationality constraint (IR), it is only threatened by some consumers choosing the outside option of not buying the product (PC) and eventually by its own capacity constraint.

Lemma 1.3. *Assume $\overline{T}_x \leq 1 - k_y$. Then*

(B1) if $T_x^{LM} \leq \overline{T}_x$ then the optimal choice of firm x is $\min(T_x^{LM}, k_x)$,

(B2) if $\overline{T}_x \leq T_x^C \leq 1 - k_y$ then the optimal choice of firm x is $\min(1 - k_y, T_x^{LM}, k_x)$,

(B3) if $\overline{T}_x \leq 1 - k_y \leq T_x^C$ then the optimal choice of firm x is either $\min(1 - k_y, k_x)$ *or* $\min(T_x^C, k_x)$,

(B4) if $T_x^C \leq \overline{T}_x \leq 1 - k_y \leq T_x^{LM}$ then the optimal choice of firm x is $\min(1 - k_y, k_x)$.

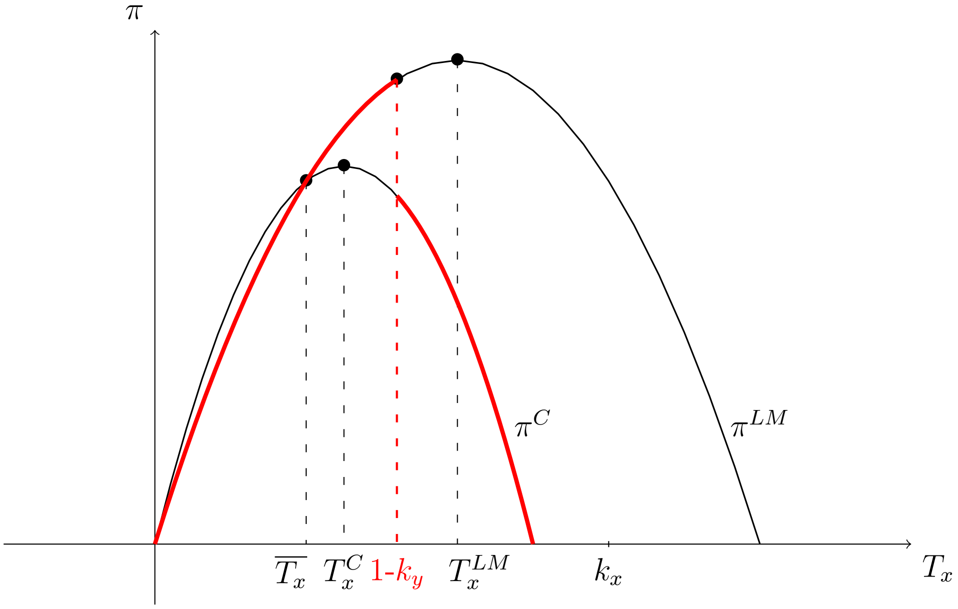
(B5) if $T_x^C \leq \overline{T}_x \leq T_x^{LM} \leq 1 - k_y$ then the optimal choice of firm x is $\min(T_x^{LM}, k_x)$.

Notice that case B1 corresponds exactly to case A1 of Lemma 1.2 and B5 also describes a very similar situation. However, the other cases are affected by the limited capacity of the rival firm. The case closest to case A2 pictured above is case B2. The only difference is in the size of the rival firm's capacity, here it is assumed to be much smaller. As an illustration of this situation, see Figure 1.2 (where k_x is assumed to be large in order to draw a clearer picture). As is clear from the figure and true in general, $\pi_x^{LM}(\tau) > \pi_x^C(\tau)$ whenever $\tau > \overline{T}_x$ i.e. to the right of the crossing point of the two curves. Hence the profit function is not only non-differentiable as in the above case, it is also discontinuous at $1 - k_y$. Therefore the assumption $T_x^C \leq 1 - k_y \leq T_x^{LM}$ immediately implies that $1 - k_y$ is the optimal choice of firm x, i.e. it produces up to the capacity of the other firm. The profit curve and the optimal solution are shown in thick (red) on Figure 1.2.

The most interesting case is arguably B3 where 3 different best replies may arise depending on the exact parameters of the model and the competitor's choice. This is also the most problematic case in Boccard and Wauthy (2011) in the sense that this discontinuity inhibits the possible existence of pure-strategy equilibrium. As Iwill show below, case B3 never arises in equilibrium when assuming intermediate levels of product differentiation.

The next section describes the numerous equilibria of the game using the conditional best replies of firms described above.

Figure 1.2. Illustration of Case B2 ($1 - k_y < T_x^{LM} < k_x$)



1.3 Equilibria

In this section I will determine which kinds of equilibria may arise in the intermediate product differentiation case as a function of firms' capacities and the other parameters of the model (v and t). The calculations will be based on the results of Lemmas 1.2 and 1.3 that describe the firms' conditional best responses.

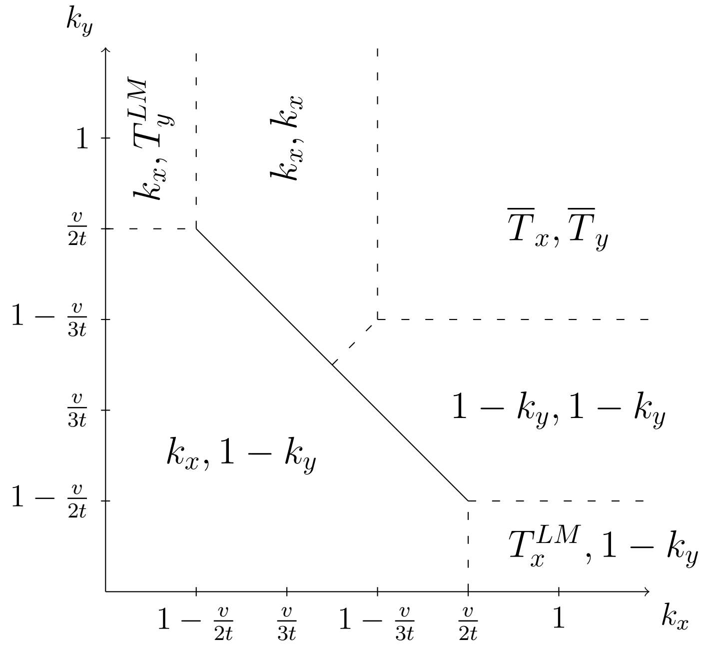
As is clear from those lemmas, there are 5 potential equilibrium strategies for firm x:

$$T_x^{LM}, \quad T_x^C, \quad \overline{T}_x, \quad 1 - k_y \quad \text{and} \quad k_x.$$

The exercise of finding all equilibria consists of comparing the conditions for potential equilibrium strategies (described in cases A1-A3 and B1-B5) of firm x to those of firm y one-by-one and determining whether the conditions are compatible. In case they are, one also has to formulate the conditions in terms of the parameters of the model. Since the cases described in the two lemmas are exhaustive, this method finds all the existing equilibria of the game. These case-by-case calculations are by nature tedious so they are relegated to the Appendix. The following proposition summarizes the main result of the paper.

Proposition 1.1. *For intermediate levels of product differentiation, i.e. for $1 < v/t \leq 1.5$ there exists at least one equilibrium in pure strategies for any capacity pair (k_x, k_y). The nature of the equilibria depends on the relative size of the capacity levels, and the relative value of consumers' willingness-to-pay v and their transportation cost t.*

Figure 1.3. Equilibria with substantial product differentiation ($1.2 < v/t \leq 1.5$)



Proposition 1.1 is in contrast to most of the existing results about Bertrand-Edgeworth oligopolies. The usual finding in the existing literature is that there is at least one region of capacity levels for which there does not exist a pure-strategy equilibrium. This clearly shows that the presence of substantial local monopoly power changes Bertrand-Edgeworth competition drastically. Even Boccard and Wauthy (2011) who investigate the case of slightly differentiated products face the problem of non-existence of pure-strategy equilibrium, indeed, their main contribution is a partial characterization of the mixed-strategy equilibrium.

By restricting attention to intermediate levels of product differentiation, one can provide a complete characterization of the equilibria of the model. Figure 1.3 illustrates the different types of equilibria that arise as a function of the parameters. For simplicity, the figure depicts the case of $1 < v/t \leq 1.2$. (The complement case of $1.2 < v/t \leq 1.5$ is qualitatively equivalent, the same type of equilibria arise, the only difference is in the ordering of the different values on the axes.)

The capacities of firm x and y are shown on the horizontal and the vertical axis, respectively. The values written in every parameter region show the equilibrium strategy of firm x and y, respectively. Note that symmetry of the figure is a direct consequence of firms being identical apart from their capacities.

Capacity-constrained equilibria The simplest case is the one where k_x and k_y are both very low ($k_x + k_y < 1$) which inhibits the interaction between the two firms. Consequently they maximize their profits independently by producing up to their capacity. Therefore $(k_x, 1 - k_y)$ is the unique equilibrium in this region. Assuming a similarly low capacity for firm y ($k_y < 1 - \frac{v}{2t}$) but a larger one for firm x ($k_x \geq \frac{v}{2t}$), one gets to the region where firm x cannot profitably increase its production and implements its unconstrained local monopoly profit $T_x^{LM} = \frac{v}{2t}$. Hence $(T_x^{LM}, 1 - k_y)$ is the unique equilibrium here.

Capacity-constrained secret handshake equilibria The most interesting region is arguably the one where the capacity of one firm is not very low but not very high either ($1 - \frac{v}{2t} < k_y < \min(1 - \frac{v}{3t}, \frac{v}{2t})$) and the industry capacity is sufficient to cover the market ($k_x + k_y \geq 1$). Firm y producing up to its capacity and firm x deciding to serve the remaining $1 - k_y$ consumers is a pure-strategy equilibrium of this region. Notice that the size of their capacity would allow firms to enter into direct competition, however, it would not be profitable for firm x. Instead it prefers to match the residual demand of the market. Essegaier et al. (2002) find similar equilibrium behavior in their model with heterogeneous demand and call it a "secret handshake" equilibrium. Notice that in the triangle-shaped region $k_x, k_y < \min(1 - \frac{v}{3t}, \frac{v}{2t})$ and $k_x + k_y \geq 1$ either firm producing up to its capacity with the other one engaging in the secret handshake constitutes an equilibrium. Thus in this region two pure-strategy equilibria co-exist.

Unconstrained secret handshake equilibria Lastly, when both capacities are large ($k_x, k_y > \min(1 - \frac{v}{3t}, \frac{v}{2t})$) there is a continuum of equilibria in pure strategies. As $\overline{T}_x$ depends on p_y and thus on T_y and vice versa, the location of the indifferent consumer ($\overline{T}_x = \overline{T}_y$) may take any values in between $1 - \frac{v}{2t}$ and $\min(1 - \frac{v}{3t}, \frac{v}{2t})$. Furthermore, these equilibria could also be described as a type of secret handshake since here $\overline{T}_x = \overline{T}_y$ holds so the market is exactly covered by the two firms. Note that the multiplicity of equilibria is a standard result for Hotelling models with substantial product differentiation without capacity constraints (Economides [1984]), so its presence is natural for the case of abundant capacities.

1.4 Discussion

To see how our results are related to the existing literature, it is worthwhile comparing the case of intermediate capacity levels with varying degrees of product differentiation:

(i) $v/t = \infty$: mixed-strategy equilibria with continuous support

(ii) $2 < v/t < \infty$: mixed-strategy equilibria with finite support

(iii) $1 < v/t \leq 1.5$: nontrivial pure-strategy equilibria

(iv) $v/t \leq 1$: trivial pure-strategy equilibrium

(i) is the case of homogeneous goods which is the seminal result of Kreps and Scheinkman (1983). (ii) is the main result of Boccard and Wauthy (2011). Furthermore, they prove that the number of atoms used in equilibrium is decreasing in v/t. I do not study the case of $1.5 < v/t \leq 2$, however, my conjecture is that the semi-mixed equilibrium in Boccard and Wauthy (2011, Lemma 4) will arise. This equilibrium consists of one firm mixing over two atoms while the other firm uses a pure strategy. (iii) is our main result. In light of the previous findings, one can view it as the number of atoms used in equilibrium being reduced to 1 for intermediate levels of product differentiation. Although (iv) is the trivial local monopoly case, it is worth mentioning here as it completes the picture of the nature of equilibria as a function of the degree of product differentiation.

1.5 An asymmetric model

In the next section, I will analyze a model where firms are asymmetric in the following sense: Firm y will be located at point $1 + a$, with $0 < a < 1$, while firm x remains at 0 and consumers are located uniformly on $[0, 1]$.[4] Thus firm y is disadvantaged: It is located on average a units farther from the consumers than its rival. This setup can also be thought of as a particular form of vertical product differentiation. Therefore this asymmetric model will serve as a robustness check for the baseline model of pure horizontal product differentiation.

The main difference with the baseline model is in the net surplus consumers derive from purchasing from firm y. For a consumer located at $0 \leq \tau \leq 1$, it is given by

$$v - p_y - (1 + a - \tau)t$$

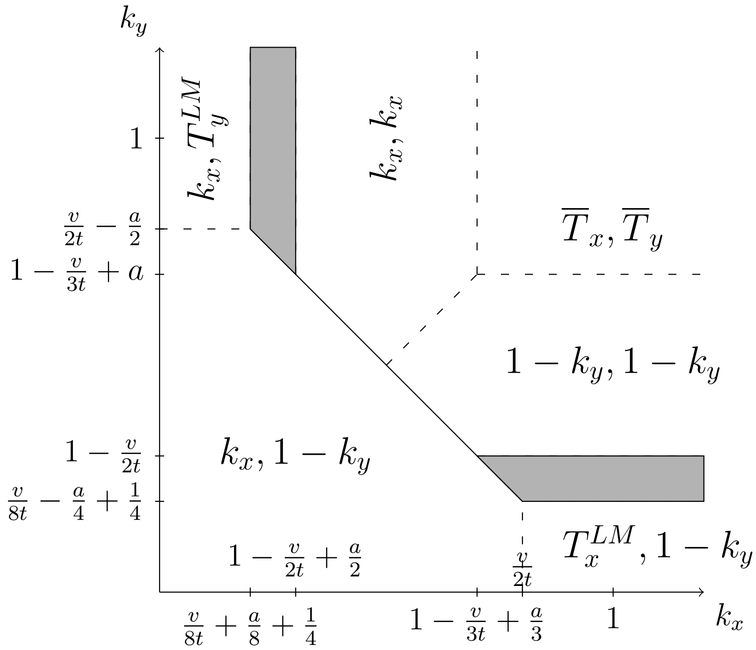
thus each consumer buying from y incurs an additional transportation cost of at compared to the baseline model. Consequently, both the participation constraint and the incentive compatibility constraint of consumers of y are changed:

$$v - p_y - (1 + a - \tau)t \geq 0 \qquad \text{(PC')}$$

and

[4] I would like to thank to Xavier Wauthy for the idea of this model variant.

Figure 1.4. Asymmetric equilibria ($1.2 + 0.6a < v/t \leq 1.5$)



$$v - p_x - t\tau \geq v - p_y - (1 + a - \tau)t \tag{IR'}$$

It is also important to note that the capacity constraints are unchanged. Despite being farther from the consumers, firm y still has the possibility of serving k_y consumers if it can attract them with a low price.

Naturally, the 5 potential equilibrium strategies of both firms are also affected by the asymmetry. Somewhat surprisingly, I can show that despite these changes, both Lemma 1.1 and Lemma 1.2 hold for $0 < a < 1$. The proofs are relegated to the Appendix. The next proposition summarizes the main result of the asymmetric model.

Proposition 1.2. *For $1 < v/t \leq 1.2 + 0.4a$ there exists at least one equilibrium in pure strategies for any capacity pair (k_x, k_y). For $1.2 + 0.4a < v/t \leq 1.5$ there is no pure strategy equilibrium for capacity levels satisfying $k_x + k_y > 1$ and $\frac{v}{8t} + \frac{1-a}{4} < k_y < 1 - \frac{v}{2t}$. Moreover, for $1.2 + 0.6a < v/t \leq 1.5$ there is no pure-strategy equilibrium for capacity-pairs satisfying $k_x + k_y > 1$ and $\frac{v}{8t} + \frac{a}{8} + \frac{1}{4} < k_x < 1 - \frac{v}{2t} + \frac{a}{2}$ either.*

Proposition 1.2 states that the main result of the baseline model holds in the asymmetric model as well for relatively high levels of product differentiation. However, for lower levels, it identifies two parameter regions without pure-strategy equilibrium. Hence the existence result of pure-strategy equilibria for any capacity level is only partially robust to the the introduction of vertical product differentiation.

Intuitively, if the level of product differentiation is high ($1 < v/t \leq 1.2 + 0.4a$), then the local monopoly power of firms is sufficiently strong to impede direct competition. Similarly to the baseline model, firms act as local monopolies if their capacity is small and engage in secret handshake equilibria for higher capacity levels.

However, for lower levels of product differentiation (but still assuming $v/t \leq 1.5$) the asymmetry in firms' location results in the lack of equilibria for intermediate capacity levels. As illustrated in Figure 1.4, the size of the two areas without pure-strategy equilibrium (shaded in the figure) depend on a in an intuitive way: the larger the asymmetry, the larger the regions without pure-strategy equilibria.

1.6 Conclusion

This paper analyzes a Bertrand-Edgeworth duopoly with exogenous capacity constraints and a non-negligible degree of product differentiation. The complete characterization of the model's equilibria was feasible and showed that there exists at least one pure-strategy equilibrium for any capacity level. This contrasts with the usual result of existing Bertrand-Edgeworth models that find nonexistence of such equilibria for some capacity levels. Thus the main finding of the paper illuminates the importance of local monopoly power in the price setting of capacity-constrained industries. Finally, by analyzing an asymmetric model, I show that the existence result is only partially robust to the introduction of vertical product differentiation.

1.7 Appendix of Chapter 1

Proof of Lemma 1.1 It is easy to see that

$$T_x^{LM} = \frac{v}{2t}, \quad \overline{T}_x = \frac{p_y - v + t}{t} \quad \text{and} \quad T_x^C = \frac{p_y + t}{4t}.$$

Then for any $t > 0$

$$T_x^{LM} \leq \overline{T}_x \iff \frac{v}{2t} \leq \frac{p_y - v + t}{t} \iff p_y \geq \frac{3}{2}v - t$$

and similarly

$$T_x^C \leq \overline{T}_x \iff \frac{p_y + t}{4t} \leq \frac{p_y - v + t}{t} \iff p_y \geq \frac{4}{3}v - t$$

also

$$T_x^{LM} \leq T_x^C \iff \frac{v}{2t} \leq \frac{p_y + t}{4t} \iff p_y \geq 2v - t$$

This proves the two parts of the lemma for any $v > 0$.

Proof of Lemma 1.2

(A1) First assume $T_x^{LM} < k_x$. By Lemma 1.2 the condition $T_x^{LM} < \overline{T}_x$ implies $T_x^C < \overline{T}_x$. By definition T_x^{LM} is the profit maximizing quantity on the π_x^{LM} curve. Hence

$$\pi_x^{LM}(T_x^{LM}) \geq \pi_x^{LM}(\overline{T}_x) = \pi_x^C(\overline{T}_x) \geq \pi_x^C(\tau) \quad \text{for all } \tau > \overline{T}_x$$

where the last inequality holds because $T_x^C < \overline{T}_x$ means that π_x^C is decreasing on the interval in question.

k_x is clearly the optimal choice when $T_x^{LM} \geq k_x$ as π_x^{LM} is increasing up to T_x^{LM}.

(A2) is proved in the main text.

(A3) Assume $\overline{T}_x < k_x$. Firstly, $T_x^C \leq \overline{T}_x$ implies that

$$\pi_x^{LM}(\overline{T}_x) = \pi_x^C(\overline{T}_x) \geq \pi_x^C(\tau) \quad \text{for all } \tau > \overline{T}_x$$

Secondly, $\overline{T}_x \leq T_x^{LM}$ implies that

$$\pi_x^{LM}(\tau) \leq \pi_x^{LM}(\overline{T}_x) = \pi_x^C(\overline{T}_x) \quad \text{for all } \tau < \overline{T}_x$$

This means that the profit function is increasing up to $\overline{T}_x$ and then it is decreasing. Again, k_x is clearly the optimal choice when $\overline{T}_x \geq k_x$ as π_x^{LM} is increasing up to $\overline{T}_x$.

□

Proof of Lemma 1.3

(B1) The proof of case (B1) is identical to the proof of case (A1) above.

(B2) is proved in the main text.

(B3) $\overline{T}_x \leq 1 - k_y \leq T_x^C$ implies that firm x must compare $\pi_x^{LM}(1 - k_y)$ to $\pi_x^C(T_x^C)$ which are the two local maxima of the profit function, except if k_x is low, then the capacity might be the optimal choice.

(B4) Given the condition $\overline{T}_x < 1 - k_y$, the constraint (PC) binds on $[0, 1 - k_y]$. The profit function π_x^{LM} is increasing up to $1 - k_y$ since $T_x^{LM} > 1 - k_y$. Moreover, $\pi_x^{LM}(1 - k_y) > \pi_x^C(1 - k_y)$ and also π_x^C is decreasing above $1 - k_y$.

(B5) Given the condition $\overline{T}_x < 1 - k_y$, the constraint (PC) binds on $[0, 1 - k_y]$. The unconstrained optimum at $T_x^{LM}(< 1 - k_y)$ is feasible for x whenever its capacity is sufficiently large.

□

Proof of Proposition 1.1 The proof builds heavily on the results of Lemmas 1.2 and 1.3 that identify parameter regions in which one of the 5 potential equilibrium strategies dominate any other strategy for a given firm. In the following I check the conditions of the 15 possible combinations of the potentially dominating strategies of the two firms and determine whether they are compatible or not.

Firstly, notice that any case where $k_x + k_y \leq 1$ is trivial: the firms do not have sufficient capacity to cover the market, they can never enter into competition. Hence $\pi_i = \pi_i^{LM}$ and the only possible equilibrium is x playing $\min(T_x^{LM}, k_x)$ and similarly, y playing $\max(T_y^{LM}, 1 - k_y)$. In the following, I consider the case of $k_x + k_y > 1$.

Consider the 5 cases in which firm x plays T_x^{LM}:

T_y^{LM}: When firm y plays T_y^{LM} both firms serve $v/2t$ consumers and their price is equal to $p_x = p_y = v/2$. This may only happen if condition (A1) or (B1) or (B5) is satisfied for both firms. (A1) and (B1) imply $p_i > \frac{3}{2}v - t$ which in turn implies $v/t < 1$ which contradicts Assumption 1. The only remaining possibility is that (B5) holds for both firms. However, that cannot be, as it necessitates $k_x < T_y^{LM} < \overline{T}_y = \overline{T}_x < T_x^{LM} < 1 - k_y$, which in turn implies $k_x + k_y \leq 1$. Therefore this case will never arise in equilibrium if $k_x + k_y > 1$.

T_y^C: Firm x playing T_x^{LM} while firm y plays T_y^C can never happen since by definition this would entail (IR) binding for firm y and slack for firm x which is a contradiction.

$\overline{T}_y$: Firm y cannot play $\overline{T}_y$ for the same reason it cannot play T_y^C.

k_x: Firm y playing k_x is incompatible with x playing T_x^{LM}. To see this, notice that playing k_x can only be optimal for firm y if the condition of Case B is satisfied, namely $k_x < \overline{T}_y$, thus

$$\frac{v}{2t} < k_x < \overline{T}_y = \frac{v - p_x}{t} = \frac{v}{2t},$$

which is a contradiction. The last inequality is a result of $p_x = v/2$.

$1-k_y$: Next I show that firm y playing $1-k_y$ and firm x playing T_x^{LM} is an equilibrium if $k_x > v/2t$ and $k_y < 1-v/2t$. Notice that $p_x = v/2$ and $p_y = v - t\cdot k_y$. By replacing these values into the formulas, it is easy to see that

$$1 - T_y^C < 1 - \overline{T}_y < 1 - T_y^{LM}$$

which means by Lemma 1.3 that y should play $\max(\overline{T}_y, 1-k_y)$. Since $\overline{T}_y = v/2t < 1-k_y$ it is optimal for firm y to play $1-k_y$. Finally, notice that according to case (B5), $k_x > v/2t$ implies that playing T_x^{LM} is a best reply for firm x as well.

Now consider the 4 cases where firm x plays $1-k_y$. (The remaining fifth such case is symmetric to one case analyzed above.) This may only be optimal for the firm if one of the conditions (B2), (B3) or (B4) holds. Notice that it is common among these conditions that $\overline{T}_x \leq 1-k_y$, moreover, $1-k_y$ is only played when (PC) binds so $p_x = v - t\cdot(1-k_y)$.

$1-k_y$: If firm y plays $1-k_y$, $p_y = v - t\cdot k_y$ always holds. Conditions for (B2) imply $p_y < \frac{4}{3}v - t$ and $T_x^C < 1-k_y$ which imply $1 - v/3t < k_y < 1 - v/3t$ so (B2) is not compatible with k_y.

Conditions for (B3) require that $\pi_x^{LM}(1-k_y) > \pi_x^C(T_x^C)$ which is equivalent to

$$0 > \frac{(v + t(1-k_y))^2}{8t} - (v - (1-k_y)(1-k_y)) \iff 0 > [v - 3t(1-k_y)]^2$$

which is impossible, so (B3) is also incompatible with k_y.

Conditions for (B4) are in turn compatible with y playing $1-k_y$. The conditions for a $(1-k_y, 1-k_y)$-type equilibrium are the following:

$$1 - \frac{v}{2t} < k_y < \min(1 - \frac{v}{3t}, \frac{v}{2t}) \quad \text{and} \quad k_x + k_y > 1.$$

Firstly, it is optimal for firm x to play $1-k_y$ to firm y playing $1-k_y$ if and only if $1 - \frac{v}{2t} < k_y < 1 - \frac{v}{3t}$. Secondly, $1-k_y$ is a best reply for firm y to firm x playing $1-k_y$ if and only if $\frac{v}{3t} < k_y < \frac{v}{2t}$ or $\frac{v}{3t} \geq k_y$ which reduces to the additional constraint of $k_y < \frac{v}{2t}$.

$\overline{T}_y$: Notice that when firm y plays $\overline{T}_y$ and firm x plays $1-k_y$, $\overline{T}_y = 1-k_y$ so the cut-off value for firm y exactly coincides with it serving consumers up to capacity. This means that this case is identical to the one above.

T_y^C: Notice that T_y^C is only played by firm y if $T_y^C > \overline{T}_y$ which implies $p_x < \frac{4}{3}v - t$ which is equivalent to $k_y < v/3t$. However, $T_y^C < k_y$ which entails $k_y > v/3t$ is also necessary. This shows that T_y^C is incompatible with firm x playing $1 - k_y$.

k_x: Firm y playing k_x is incompatible with x playing $1 - k_y$. These quantities entail prices $p_x = v - t \cdot (1 - k_y)$ and $p_y = v - t(1 - k_x)$ which imply $\overline{T}_y = 1 - k_y$ and $\overline{T}_x = k_x$. However, conditions for x playing $1 - k_y$ (case B) require $k_x = \overline{T}_x \leq 1 - k_y$ which is ruled out by $k_x + k_y > 1$.

Now consider the 3 cases when firm x plays $\overline{T}_x$.

$\overline{T}_y$: There is an equilibrium where firm y plays $\overline{T}_y$ and firm x plays $\overline{T}_x$. The conditions of optimality translate to $p_x + p_y = 2v - t$ and also $\frac{4}{3}v - t < p_y < \frac{3}{2}v - t$. Furthermore, conditions concerning the capacities require $k_x, k_y \geq \min(1 - \frac{v}{3t}, \frac{v}{2t})$. Thus there is a continuum of of equilibria in this capacity range.

$1 - k_y$: Firm y playing $1 - k_y$ and firm x playing $\overline{T}_x$ is possible only if $1 - k_y = \overline{T}_y$ otherwise the (IR) constraint would bind for the one firm but not for the other. If this is true, the case is naturally identical to the case above.

T_y^C: Firm y playing T_y^C is impossible when firm plays $\overline{T}_x$ because then the constraint (IR) would be binding for firm x and slack for firm y which is a contradiction.

Now consider the 2 cases when firm x plays T_x^C.

T_y^C: Both firms playing the competitive strategy leads to $p_x = p_y = t$ and both firms serving exactly 1/2 of the market. However, this requires product differentiation to be low, $v/t > 1.5$ which case is not the object of the present paper.

$1 - k_y$: Firm y playing $1 - k_y$ and firm x playing T_y^C is possible only if $k_y = T_y^C$ otherwise the (IR) constraint would bind for the one firm but not for the other. If this is true, the case is naturally identical to the case above.

The remaining case is when both firms serve consumers up to their capacity. However, Lemma 1.3 ensures it can only be optimal for firms to do so if (PC) is binding for their marginal consumers. This is clearly impossible when $k_x + k_y > 1$.

$\square$

Proof of Proposition 1.2 Firstly, I show that Lemma 1.1 holds in the asymmetric model as well. The logic of the proof is similar to the original one. The new values of potential equilibrium strategies of firm x are the following:

$$T_x^{LM} = \frac{v}{2t}, \quad T_x^C = \frac{p_y + (1+a)t}{4t}, \quad \overline{T}_x = 1 + a - \frac{v - p_y}{t},$$

therefore

$$T_x^{LM} \leq \overline{T}_x \iff p_y \geq \frac{3}{2}v - (1+a)t$$

and

$$T_x^C \leq \overline{T}_x \iff p_y \geq \frac{4}{3}v - (1+a)t$$

and

$$T_x^{LM} \leq T_x^C \iff p_y \geq 2v - (1+a)t.$$

The above inequalities prove Lemma 1.1 holds for firm x. Similarly, the new potential equilibrium strategies of firm y are:

$$T_y^{LM} = 1 - \frac{v}{2t} + \frac{a}{2}, \quad T_y^C = 1 - \frac{p_x + t}{4t} + \frac{a}{4}, \quad \overline{T}_y = \frac{v - p_x}{t},$$

therefore

$$T_y^{LM} \geq \overline{T}_y \iff p_x \geq \frac{3}{2}v - t - \frac{at}{2}$$

and

$$T_y^C \geq \overline{T}_y \iff p_x \geq \frac{4}{3}v - t - \frac{at}{3}$$

and

$$T_y^{LM} \geq T_y^C \iff p_x \geq 2v - at.$$

It is easy to see that $a < v/t$ implies both

$$\frac{4}{3}v - t - \frac{at}{3} \leq \frac{3}{2}v - t - \frac{at}{2} \leq 2v - at.$$

Thus the assumptions of $a < 1$ and $1 \leq v/t$ together imply that Lemma 1.1 holds for firm y as well.

Secondly, Lemma 1.1 being satisfied in the asymmetric model directly imply that Lemmas 1.2 and 1.3 will also hold.

Thirdly, one must repeat the steps of the proof of Proposition 1.1 to find the pure-strategy equilibria of the asymmetric model. Below I will only show calculations for strategy-pairs forming an equilibrium or where the reasoning is different than the one in Proposition 1.1. For all other strategy-pairs the logic of the proof of Proposition 1.1 remains the same with obvious modifications. Let (T_x, T_y) denote a strategy-pair with x choosing T_x and y choosing T_y as its marginal consumer.

$(k_x, 1-k_y)$: Clearly, both firms serving up to capacity is still an equilibrium if $k_x + k_y \leq 1$, $k_x \leq T_x^{LM}$ and $k_y \leq 1 - T_y^{LM}$.

(T_x^{LM}, T_y^{LM}): Importantly, (T_x^{LM}, T_y^{LM}) can be an equilibrium of the asymmetric model even if

$$k_x + k_y > 1 \quad \text{and} \quad k_x \geq \frac{v}{2t} \quad \text{and} \quad k_y \geq 1 - \frac{v}{2t}$$

given that $v/t \leq 1 + a/2$. Indeed, if (A1) or (B1) holds for both firms, ensuring that the strategies are mutual best replies, then $p_x = v/2$ and $p_y = v/2 - a/2t$ imply $v/t \leq 1 + a/2$ and vica versa.

$(1-k_y, 1-k_y)$: Using the same arguments as in the proof of Proposition 1.1, one obtains that both firms choosing $1 - k_y$ is an equilibrium if and only if

$$1 - \frac{v}{2t} < k_y \leq \min(1 - \frac{v}{3t} + a,\ \frac{v}{2t} - \frac{a}{2}) \quad \text{and} \quad k_x + k_y > 1.$$

(k_x, k_x): Similarly to the equilibrium above, one can show that both firms choosing k_x as their marginal consumer is an equilibrium if and only if

$$1 - \frac{v}{2t} + \frac{a}{2} < k_x \leq \min(1 - \frac{v}{3t} + \frac{a}{3}, \frac{v}{2t}) \quad \text{and} \quad k_x + k_y > 1.$$

$(\overline{T}_x, \overline{T}_y)$: The conditions of optimality translate to $p_x + p_y = 2v - t - at$ and also $\frac{v}{2} < p_x < \frac{2}{3}v$. Consequently conditions concerning the capacities require

$$k_x, k_y \geq \min(1 - \frac{v}{3t}, \frac{v}{2t} - \frac{a}{2})$$

for this strategy-pair to constitute an equilibrium.

$(T_x^{LM}, 1-k_y)$: The capacity-pairs for which these strategies form an equilibrium are crucially different for $a > 0$ than for $a = 0$. Firstly, the condition for firm x playing T_x^{LM} being a best reply to firm y playing $1 - k_y$ is simply

$$k_y < 1 - \frac{v}{2t} \quad \text{and} \quad k_x \geq \frac{v}{2t}.$$

However, $1 - k_y$ being a best reply for firm y to T_x^{LM} depends on the degree of product differentiation and a. There are 3 cases:

(i) $v/t \leq 1 + 0.5a$: Then $1 - k_y$ is optimal to play if and only if $k_y \leq \frac{v}{2t} - \frac{a}{2}$. Thus $(T_x^{LM}, 1 - k_y)$ is an equilibrium if and only if

$$k_x \geq \frac{v}{2t} \quad \text{and} \quad k_y \leq \frac{v}{2t} - \frac{a}{2} < 1 - \frac{v}{2t},$$

where the last inequality stems from the assumption of case (i).

(ii) $1 + 0.5a < v/t \leq 1.2 + 0.4a$: Then $1 - k_y$ is optimal to play if and only if $k_y \leq 1 - \frac{v}{2t}$ which is exactly the condition for optimality for firm x, thus in this case $(T_x^{LM}, 1 - k_y)$ is an equilibrium if and only if

$$k_y < 1 - \frac{v}{2t} \quad \text{and} \quad k_x \geq \frac{v}{2t}.$$

(iii) $v/t > 1.2 + 0.4a$: Then $1 - k_y$ is optimal to play if and only if $k_y \leq \frac{v}{8t} + \frac{1-a}{4}$. Thus $(T_x^{LM}, 1 - k_y)$ is an equilibrium if and only if

$$k_x \geq \frac{v}{2t} \quad \text{and} \quad k_y \leq \frac{v}{8t} + \frac{1-a}{4} < 1 - \frac{v}{2t},$$

where the last inequality stems from the assumption of case (iii).

(k_x, T_y^{LM}): Similarly to the equilibrium above, three cases can be distinguished, and similar reasoning reveals that

(i) If $v/t \leq 1 + 0.5a$ then (k_x, T_y^{LM}) is an equilibrium if and only if

$$k_y \geq \frac{v}{2t} - \frac{a}{2} \quad \text{and} \quad k_x \leq \frac{v}{2t} \leq 1 - \frac{v}{2t} + \frac{a}{2}.$$

(ii) If $1 + 0.5a < v/t \leq 1.2 + 0.6a$ then (k_x, T_y^{LM}) is an equilibrium if and only if

$$k_y \geq \frac{v}{2t} - \frac{a}{2} \quad \text{and} \quad k_x \leq 1 - \frac{v}{2t} + \frac{a}{2}.$$

(iii) If $v/t > 1.2 + 0.6a$ then (k_x, T_y^{LM}) is an equilibrium if and only if

$$k_y \geq \frac{v}{2t} - \frac{a}{2} \quad \text{and} \quad k_x \leq \frac{v}{8t} + \frac{a}{8} + \frac{1}{4} < 1 - \frac{v}{2t} + \frac{a}{2}.$$

A comparison of the capacity thresholds delimiting the different kinds of equilibria reveals the first part of Proposition 1.2. Indeed, for $v/t \leq 1.2 + 0.4a$ the capacity-pairs for which there exist at least one equilibrium cover the whole positive quadrant. However, for $v/t > 1.2 + 0.4a$ there is no pure-strategy equilibrium for capacity-pairs satisfying

$$k_x + k_y > 1 \quad \text{and} \quad \frac{v}{8t} + \frac{1-a}{4} < k_y < 1 - \frac{v}{2t}.$$

Moreover, for $v/t > 1.2 + 0.6a$ there is no pure-strategy equilibrium for capacity-pairs satisfying

$$k_x + k_y > 1 \quad \text{and} \quad \frac{v}{8t} + \frac{a}{8} + \frac{1}{4} < k_x < 1 - \frac{v}{2t} + \frac{a}{2}$$

either.

□

Chapter 2

Monopoly Pricing with Dual Capacity Constraints

Abstract: This paper studies the price-setting behavior of a monopoly facing two capacity constraints: one on the number of consumers it can serve, the other on the total amount of products it can sell. Facing two consumer groups that differ in their demands and the distribution of their willingness-to-pay, the monopoly's optimal non-linear pricing strategy consists of offering one or two price-quantity bundles. The characterization of the firm's optimal pricing in the short run as a function of its two capacities reveals a rich structure that also gives rise to some surprising results. In particular, I show that prices are non-monotonic in capacity levels. Moreover, there always exists a range of parameters in which weakening one of the capacity constraints decreases consumer surplus. In the long run, when the firm can choose how much capacity to build, prices and consumer surplus are monotonic in capacity costs.

2.1 Introduction

The economics literature typically considers one-dimensional capacity constraints (Kreps and Scheinkman [1983]). This is idealized because in real-world production processes firms typically face several capacity constraints (size of plants, inventories, workforce, etc.). In general, due to the use of supply chains, firms are constrained on even more aspects of their production. The objective of this paper is to develop a theory of monopoly pricing in the presence of multiple capacity constraints.

Examples of industries characterized by dual capacity constraints range from hospitals, through restaurants, to the freight transport industry. Hospitals are constrained by the number of beds available in their intensive care unit on the one hand, and operating room time on the other hand. Each consumer (i.e. patient) needs one bed but consumers differ in their need of operating room time. Hospitals' patients have price-inelastic individual demands; even if a long surgery (e.g., a kidney transplant) is very cheap, someone in need of a short surgery (e.g., fixing of a broken arm) will never prefer having the longer one.

Restaurants constitute another prominent example for the co-existence of the two types of capacity constraints. Restaurants have to take into account in their pricing decisions both the number of tables they have at their disposal and the size of their kitchen, that limits the amount of food they can prepare. Patrons of restaurants arrive in groups of different sizes, hence they are typically heterogeneous in their consumption and also in their willingness-to-pay.

Dual capacities are key characteristics of the freight transport industry as well. Both in ocean container shipping and air cargo transport, an important concern of the transporting company is optimizing the mix of items according to both their size and their weight. As physical and regulatory limits are present in both dimensions, profit-maximizing firms cannot avoid taking into account both constraints.

Another class of examples includes markets where firms are constrained by some physical capacity constraint on the one hand and their workers' time on the other. For instance airplanes cannot fly at full capacity if the ratio of passengers to flight attendants exceeds a certain regulatory limit. Business class passengers tend to use up more of the flight attendants' time than economy class passengers and the willingness-to-pay of the two groups are obviously different.

Several questions arise if one wants to understand the consequences of the co-existence of the two types of capacity constraints. How much will the predictions of the model change compared to a model with only one capacity constraint? What are the optimal prices a firm must charge to different consumer groups? How will these optimal prices change as a function of the capacity levels? Under which conditions are both capacity constraints binding in optimum? How do aggregate consumer surplus, profit and total welfare vary with the capacity levels?

In order to answer these questions, I consider a price-setting monopolist facing two types of capacity constraints. The firm is unable to serve more than K consumers. In addition, it cannot sell more than a quantity Q of its products. Consumers differ in their price-inelastic individual demands and their willingness-to-pay (WTP). There are two consumer groups: high-types intend to buy a larger amount of the product than the low-types. The monopoly can observe the individual demand of each consumer but not their WTP. The WTP of high-types and low-types are distributed along two different intervals. Some high-types have a larger WTP than all the low-types while some low-types have a larger *per-unit* WTP than all the high-types. The monopoly is allowed to offer price-quantity bundles to discriminate between consumers.

The existence of a second type of capacity constraint fundamentally changes the monopoly's optimal behavior. The optimal pricing strategy in the short run (with exogenously given capacities) is the following. When K, the constraint on the mass of consumers, is very tight, the monopoly excludes low-types and serves only high-type consumers. Conversely, when the capacity constraint on total production, Q, is very small then only low-types are served. For larger levels of capacities, it is optimal for the monopoly to serve both types. When both constraints are very large, the firm chooses the unconstrained optimal prices. When K is of intermediate value while Q is very large, optimal prices are chosen in a way that K binds and Q does not. Conversely, if Q is of medium value and K is large, then Q is binding and K is slack. Importantly, there exist capacity levels for which the two prices are chosen so that both constraints bind.

Intuitively, when only very few consumers can be served, the monopoly will prefer serving those with the highest overall WTP so it excludes all the low-types. Conversely, when the total production is very limited, the firm is concerned by the per-unit surplus it can extract from the consumers, hence it excludes all the high-types. Even when capacities do not take extreme values and some consumers of both types are served, an increase in K *ceteris paribus* makes the low-types more attractive to the firm therefore it chooses its prices to attract more low-types.

In sharp contrast to the standard case with only one capacity constraint, prices are non-monotonic in the size of capacities. In particular, while prices are decreasing in both capacities in most parameter regions, for capacity levels when both constraints bind, the price charged for high-types increases in K while the price for low-types increases in Q.

The intuition for this result is the following. For very small values of K, the capacity on the mass of people served, the monopoly only serves high-types. As K increases low-types become relatively more valuable for the firm so after a threshold value it starts serving both consumer groups. Below the threshold value the firm decreases the price charged for the high-types so that their total demand equals K. Above the threshold, in order to make space for the low-types, the firm is interested in serving less high-types so it raises the price charged to them. A similar argument can be made to understand the price increase for the low-types as Q increases.

Furthermore, this price increase has a non-negligible effect on aggregate consumer surplus: I show that there always exists a range of parameters in which weakening one of the capacity constraints decreases aggregate consumer surplus. This phenomenon cannot occur in models that consider a single capacity constraint. The decrease of consumer surplus is a consequence of the monopoly adjusting its optimal mix of consumers in the following sense. When K is very small, only high-types get served. As it increases above a threshold value, the firm starts serving some low-type consumers as well. However, as the transition is smooth, close to the turning point the firm still serves many high-types and only a few low-types. The price increase suffered by the numerous high-types dominates the gain of the few low-types that start being served and the aggregate consumer surplus decreases.

I also investigate long-run behavior of the monopoly where in addition to prices, it can also choose endogenously how much capacity to build in each dimension. For any positive cost function of capacity building, the firm chooses capacity levels and prices so that both capacities bind. These are exactly the capacity levels for which the prices are increasing and aggregate consumer surplus is decreasing in the short run. I provide a complete characterization of optimal capacity levels for the case of linear cost functions. The outcome of the model gets close to the unconstrained optimum as both costs tend to zero. The optimal consumer mix depends on the relation of the capacity costs to the marginal benefit of serving an additional consumer of a given type. Therefore, depending on parameter values, both low-types and high-types can be excluded in optimum, or some consumers of both groups can be served. Both optimal prices and consumer surplus are monotonic in the capacity costs.

I also consider an extension which allows low-types to buy a large quantity of the product and freely dispose of the amount they do not consume. Such a model suits better some of the industries cited as examples, for instance the restaurants. I show that this opportunity does not alter any of the results obtained in the more restrictive basic model. Intuitively, consumers' possibility of free disposal may limit the monopoly's choice. In case it wants to sell the two quantities at different prices, it must charge a lower price for the smaller quantity, otherwise all consumers would buy the larger bundle for the lower price which is obviously unprofitable for the firm. However, the optimal prices of the basic model satisfy this condition, thus optimal firm behavior is not altered by free disposal.

Next, I investigate incentive compatibility for the high-type consumers. In this variant of the model, the high-types are allowed to buy several small bundles to satisfy their individual demand. A monopoly that wants to sell both types of bundles must choose lower per-unit prices for the large bundle than for the small one, otherwise it can sell exclusively small bundles. I show that the monopoly's optimal prices are not affected by high-types' ability to make repeat purchases.

Finally, I provide sufficient conditions on the distribution of consumers such that all the qualitative insights of the model with uniform distribution hold. In particular, I find that if both distributions satisfy the monotone hazard rate condition, then the region where both capacity constraints bind has a well-defined shape; moreover, prices are non-monotonic in capacity levels and I prove the existence of a region where consumer surplus is decreasing in a capacity level. Furthermore, I identify hazard rate dominance relations as sufficient conditions for the exclusion of a consumer group.

2.1.1 Related literature

The monopoly's problem of third-degree price discrimination with a single capacity constraint is a textbook exercise (see Besanko and Braeutigam, 2010, 507, Exercise 12.6). Therefore, the literature of capacity-constrained pricing has mainly focused on the case of competition. The seminal paper of Kreps and Scheinkman (1983) shows that under certain assumptions, the outcome of a two-stage game where firms first choose capacities then engage in price competition coincides with the Cournot outcome. Davidson and Deneckere (1986) were the first to point out that this result is not robust to the choice of rationing rule. In particular, they show that the results are more competitive than the Cournot outcome for virtually any rationing rule other than the efficient one.

Cripps and Ireland (1988) and Acemoglu et al. (2009) both consider capacity-constrained competition when firms face consumers with price-inelastic demands. They show the existence of pure-strategy subgame perfect equilibria for any capacity levels, which are supported by mixed strategy equilibria off-path, similarly to Kreps and Scheinkman (1983). Acemoglu et al. (2009) also analyze efficiency properties of the equilibria, i.e., they compare the total social welfare in different equilibria to the welfare maximizing first-best and they find that some equilibria can be arbitrarily inefficient.

Reynolds and Wilson (2000) introduce demand uncertainty into the two-stage game described by Kreps and Scheinkman (1983). They show that symmetric pure-strategy equilibria in the capacity choice game do not exist when the variability of demand is high, and they provide a set of assumptions that guarantee the existence of asymmetric pure-strategy equilibria. De Frutos and Fabra (2011) investigate the interaction of demand uncertainty and capacity constraints assuming price-inelastic demand. They identify submodularity of the demand distribution as a sufficient condition of the existence of pure-strategy equilibria and they show that these equilibria are asymmetric.

The welfare analysis of the model with general distribution function is related to another stream of recent literature which is investigating welfare effects of monopoly's third-degree price discrimination. Cowan (2007) derives two alternative sufficient conditions for the convexity of the slope of demand under which price discrimination enhances social welfare. Aguirre et al. (2010) analyze more general demand functions and identify sufficient conditions on the elasticities and demand curvatures of the two markets that make price discrimination reduce or increase welfare. Cowan (2012) shows how the the cost pass-through coefficient can determine the way discrimination effects aggregate consumer surplus. Finally, Bergemann et al. (2015) investigate welfare effects of additional information a monopoly can use to price discriminate and they show that any combination of consumer and producer surplus is achievable that satisfy some mild conditions.

Models where several capacity constraints co-exist have so far been relegated to the realms of operations research and revenue management. Patient admission planning for scheduled surgeries and patient mix optimization are both important problems hospitals have to face. The multidimensional nature of capacities in hospitals is crucial for such planning. Many recent papers in the operations research literature focus on solving a variety of problems that arise in a context where the treatment of different categories of patients require different levels of capacities. For example, Adan and Vissers (2002) take into account operating room time, intensive care unit beds, medium care unit beds and nurses' time to simulate the optimal schedule of a real-world hospital department. Banditori and al. (2013) also consider a multiple capacity setting for their simulation, in addition, they provide a comparison of the recent articles in this area (Banditori and al., 2013, Table 2).

Multidimensional capacities are crucial in freight transport, such as the air cargo industry or the container shipping industry. Since dynamic pricing is widespread in these industries, the literature models freight transport pricing by extending standard revenue management models to accommodate multiple capacities. Xiao and Yang (2010) consider revenue management with two capacity dimensions. Their focus is on ocean container shipping, where container sizes are standardized (two varieties dominate the market) and maximal weights of all containers are set by on-road regulations. They derive an analytical solution and show that under some conditions the optimal policy is qualitatively different when considering the second capacity constraint. Kasilingam (1997) describes how air cargo revenue management is different from air passenger revenue management, and one of the key differences he identifies is the multidimensional aspect of capacities: volume, weight and even cargo position may be constraining.

Finally, the hospitality industry's capacity management literature has also recognized the importance of dual capacities. Kimes and Thompson (2004) optimize the table mix for restaurant revenue management taking into account not only the number and distribution of seats but also the size of the service areas. Bertsimas and Romy (2003) consider both sizes of parties to be seated and expected service duration to compare several optimization-based approaches to restaurant revenue management.

The rest of the chapter is organized as follows. Section 2.2 discusses a simple benchmark, then outlines the main model. Section 2.3 describes the results of the main model, the monopoly's optimal behavior, and provides comparative statics for the capacity levels. Section 2.4 discusses the consequences of optimal pricing for consumer surplus. Section 2.5 generalizes the model by allowing capacity levels to be chosen endogenously. Section 2.6 investigates a variant of the baseline model with incentive-compatibility. Section 2.7 analyzes the model with general distribution of consumers and Section 2.8 concludes the chapter. All omitted proofs are relegated to the Appendix of Chapter 2.

2.2 The model

2.2.1 A simple benchmark

In this section I present a simple model of monopoly characterized by one capacity constraint facing only one consumer group.

Consider a price-setting monopolist that can produce for zero cost up to quantity Q of a good, then his costs become infinite, i.e., the monopoly is characterized by capacity constraint Q. All consumers have a unit demand. Consumers are heterogeneous in their willingness-to-pay (WTP), they are uniformly distributed on the interval $[0, 1]$. The monopoly can only observe the distribution but not the individual WTP values. The net consumer surplus of a buyer with $w \in [0, 1]$ is given by $w - p$ if he buys the product where p denotes its price. A consumer is willing to buy if and only if his net consumer surplus is positive.

In this setting, the mass of consumers willing to buy the product is given by $1 - p$: this is the fraction of consumers with a higher WTP than the price. Since each consumer has unit demand, the total demand for the product is also $1 - p$.

Hence the monopoly solves the following maximization problem:

$$\max_p \quad \pi = (1-p)p \qquad \text{s.t.} \qquad 1-p \leq Q$$

The profit-maximizing price is given by

$$p^* = \begin{cases} 1-Q & \text{if } Q < 1/2, \\ 1/2 & \text{otherwise.} \end{cases}$$

First notice that the capacity constraint is binding up to a certain level ($Q \leq 1/2$) then the monopoly can implement its unconstrained optimal strategy. The price is decreasing in the level of capacity, Q, as long as the capacity is small enough to bind, then the price becomes independent of it.

Given these prices, total demand is simply Q if $Q \leq 1/2$, for larger values of the capacity it is equal to $1/2$. Intuitively, the monopoly chooses prices such that the capacity binds when it is small, then it implements its unconditional optimum. Profit is $(1 - Q)Q$ if $Q \leq 1/2$; for larger capacities it is $1/4$. As one should expect, both total demand and the profit are increasing in capacity for binding levels of capacity.

Finally, consumer surplus in this setting is given by $(1 - p)^2/2$. Clearly, consumer surplus is always decreasing in the price (p never exceeds 1). Hence the fact that the optimal price is decreasing in Q means that consumer surplus is increasing in the level of capacity. This is not surprising: an increase in the level of capacity means more of the consumers served for a lower price.

The four main insights of this simple model are that an increase in the capacity level Q can decrease prices and can increase total demand, consumer surplus and profits. Out of these 4 insights only the one concerning profits will remain true for a model with dual capacity constraints presented in the next section.

2.2.2 The dual capacity model

Consider a market served by a monopoly consisting of two consumers groups. Each consumer is characterized by its individual demand and its total willingness-to-pay (WTP) for the product. Low-types want to consume a fix amount of $q_L > 0$ products while high-types want to consume a fix amount of $q_H > q_L$, i.e., individual demand is price-inelastic.

A consumer of type $i \in \{L, H\}$ with total WTP w has a net consumer surplus of $w - p_i$ if he buys a quantity q_i of the product for price p_i, and 0 otherwise. Total WTP of consumers of type i is uniformly distributed on the interval $[0, v_i]$. Consumers maximize their net surplus and they demand the good if and only if their net surplus is positive. Assume that the total mass of high-type and low-type consumers is αv_H and $(1-\alpha)v_L$, respectively, where $0 \leq \alpha \leq 1$ scales the relative weight of the two consumer groups.[1] The monopoly can observe α, the consumers' individual demand, and the distribution of their WTP but not their individual values.

Assumption 1. *Let the WTP of consumers satisfy the following conditions:*

$$0 < v_L < v_H \quad \text{and} \quad v_L/q_L > v_H/q_H$$

Assumption 1 guarantees that some high-type consumers' valuation always exceed all the low-types' valuation, whereas the per-unit-WTP of some low-type consumers is greater then the per-unit-WTP of all high-type consumers. This assumption restricts the analysis to the most interesting cases, because otherwise the monopoly would always prefer to serve consumers of one group first, irrespective of the size of capacity constraints. Also, this corresponds to decreasing marginal value of consumption in the present setting. Finally, notice that high-types are not necessarily more valuable for the firm, it simply refers to the high level of their individual demand.

[1]This normalization of the mass of consumers is made to simplify the exposition of results. In particular, it significantly shortens the formulas obtained for optimal prices and profits. The results of Section 2.7 prove that this simplification does not alter the qualitative properties of the model.

I analyze a monopoly facing two types of capacity constraints:

- K denotes the maximal mass of consumers the firm can serve
- Q denotes the maximal total production of the firm

Both constraints are exogenously given. For simplicity, production is costless up to capacity then it becomes impossible.

The monopoly has an optimal pricing structure that consist of offering at most 2 price-quantity bundles. Given the price-inelastic individual demands, no consumer would buy any bundle that offers them a quantity different from their desired demand. Moreover, if the firm were to offer several bundles with the same quantity for a different price, consumers would only buy the cheapest one. Let p_H and p_L denote the price of the bundle with high and low quantities, respectively.

2.3 Results

2.3.1 Optimal monopoly pricing

In this section I describe and solve the monopoly's profit maximization problem.

For any price $p_H \in [0, v_H]$, the high-type consumers willing to buy are the ones who have a higher WTP than p_H. They represent a fraction $\frac{v_H - p_H}{v_H}$ of the high-types, which means that the total mass of high-type consumers who demand the good is given by

$$\alpha(v_H - p_H).$$

Similarly, for any price $p_L \in [0, v_L]$, the total mass of low-types willing to buy is

$$(1 - \alpha)(v_L - p_L).$$

Hence the total demand the monopoly faces at such prices is given by

$$\alpha(v_H - p_H)q_H + (1 - \alpha)(v_L - p_L)q_L$$

The monopoly also has the option of not serving one of the consumer groups. Hence, it must choose between serving both consumer groups, excluding low-type consumers, or excluding high-type consumers. Notice that in some cases the latter possibility can be profitable since some low-type consumers have a higher per-unit WTP than all the high-types. I will break down the general optimization problem into 3 separate maximization problems and compare the locally optimal profits to find the monopoly's profit-maximizing strategy.

The maximization program of the firm if it decides to exclude low-types writes as[2]

$$\text{(P-EL)} \qquad \max_{p_H} \quad \pi = \alpha(v_H - p_H)p_H \qquad \text{s.t.}$$

$$\alpha(v_H - p_H) \leq K \tag{1a}$$

$$\alpha(v_H - p_H)q_H \leq Q \tag{1b}$$

$$p_H \geq 0 \tag{1c}$$

The maximization program if the monopoly serves only low-types is given by

$$\text{(P-EH)} \qquad \max_{p_L} \quad \pi = (1-\alpha)(v_L - p_L)p_L \qquad \text{s.t.}$$

$$(1-\alpha)(v_L - p_L) \leq K \tag{2a}$$

$$(1-\alpha)(v_L - p_L)q_L \leq Q \tag{2b}$$

$$p_L \geq 0 \tag{2c}$$

The maximization program of the firm when serving some consumers of both groups writes as

$$\text{(P-LH)} \qquad \max_{p_L, p_H} \quad \pi = \alpha(v_H - p_H)p_H + (1-\alpha)(v_L - p_L)p_L \qquad \text{s.t.}$$

$$\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) \leq K \tag{3a}$$

$$\alpha(v_H - p_H)q_H + (1-\alpha)(v_L - p_L)q_L \leq Q \tag{3b}$$

$$p_L \geq 0\ , \quad p_H \geq 0 \tag{3c}$$

$$p_L < v_L\ , \quad p_H < v_H \tag{3d}$$

In each case, the firm maximizes the product of the mass of consumers buying and the price. Constraints 1a, 2a and 3a constitute the upper bound on the maximal mass of people

[2]Throughout the paper, EL stands for "excluding low-types" and EH stands for "excluding high-types".

that the monopoly can serve. 1b, 2b and 3b are the capacity constraints on total production: The mass of people buying times their individual demand cannot exceed Q. 1c, 2c and 3c are non-negativity constraints for the prices. Finally, 3d ensures that some consumers are both served in the last case.

2.3.1.1 Excluding one group of consumers

In problem (P-EL) when the monopoly excludes low types, first notice that the 2 capacity constraints 1a and 1b can be rewritten as

$$\alpha(v_H - p_H) \leq \min(K, Q/q_H)$$

Obviously, the unconstrained maximum is attained at $p_H = v_H/2$. Thus, the profit equals $\alpha v_H^2/4$ if $\min(K, Q/q_H) > \alpha v_H/2$. Otherwise the firm sells up to the tighter capacity constraint for a price $p_H = v_H - \min(K, Q/q_H)/\alpha$, consequently its profit equals to

$$v_H \min(K, Q/q_H) - \frac{(\min(K, Q/q_H))^2}{\alpha}$$

The non-negativity constraint is trivially satisfied.

The solution of the second maximization problem, (P-EH), when the firm excludes high-types, is analogous. Hence the optimal price is given by

$$p_L = \begin{cases} v_L - \frac{\min(K,Q/q_L)}{1-\alpha} & \text{if } \min(K, Q/q_L) \leq (1-\alpha)v_L/2, \\ v_L/2 & \text{otherwise.} \end{cases}$$

The optimal profit is

$$\pi = \begin{cases} v_L \min(K, Q/q_L) - \frac{(\min(K,Q/q_L))^2}{1-\alpha} & \text{if } \min(K, Q/q_L) \leq (1-\alpha)v_L/2, \\ (1-\alpha)v_L^2/4 & \text{otherwise.} \end{cases}$$

2.3.1.2 Serving both consumer groups

Solving the remaining maximization problem, (P-LH) is more complex and requires writing the Karush-Kuhn-Tucker conditions. I omit the constraints 3c and 3d on prices and show ex-post that the solution of the relaxed problem satisfies them. Let λ_1 denote the multiplier of constraint 3a, and let λ_2 denote the multiplier of 3b. Therefore, λ_1 and λ_2 can be interpreted as the shadow prices of the customer constraint and the production constraint, respectively.

The objective function is

$$
\begin{aligned}
L(p_L, p_H, \lambda_1, \lambda_2) = \alpha(v_H - p_H)p_H + (1-\alpha)(v_L - p_L)p_L - \\
\lambda_1 \left[\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) - K\right] - \\
\lambda_2 \left[\alpha(v_H - p_H)q_H + (1-\alpha)(v_L - p_L)q_L - Q\right]
\end{aligned}
$$

The first order conditions can be written as

$$
\begin{aligned}
2p_H = v_H + \lambda_1 + \lambda_2 q_H \quad \text{and} \\
2p_L = v_L + \lambda_1 + \lambda_2 q_L.
\end{aligned}
$$

There are 4 cases depending on the sign of the multipliers, i.e., depending on which of the two constraints is binding.

The following notation simplifies the exposition of results: let E(x) denote the weighted average of any 2 variables x_L and x_H, i.e., $E(x) = \alpha x_H + (1-\alpha)x_L$.

Case 1 When the capacity constraint on the mass of people served is binding while the other constraint is slack ($\lambda_1 > 0$ and $\lambda_2 = 0$), the optimal pricing strategy of the monopoly is given by

$$
p_L = v_L + \alpha\frac{v_H - v_L}{2} - K \quad \text{and} \quad p_H = v_H - (1-\alpha)\frac{v_H - v_L}{2} - K
$$

and its optimal profit is

$$
\pi_K \equiv \alpha(1-\alpha)\left(\frac{v_H - v_L}{2}\right)^2 + KE(v) - K^2
$$

Optimality conditions of this local optimum require that K, the capacity constraint on the mass of people served, be of an intermediate size with respect to the other parameters of the model:

$$
\frac{\alpha}{2}(v_H - v_L) \le K \le \min\left(\frac{E(v)}{2},\, g(Q)\right)
$$

where

$$g(Q) = \frac{1}{E(q)}\left(Q - \alpha(1-\alpha)(q_H - q_L)\frac{v_H - v_L}{2}\right).$$

The non-negativity constraints are satisfied at this solution.

Case 2 When the capacity constraint on the total production is binding while the other constraint is slack ($\lambda_1 = 0$ and $\lambda_2 > 0$), the optimal pricing strategy of the monopoly is given by

$$p_L = \frac{q_L}{E(q^2)}\left(\frac{\alpha}{2}(v_L\frac{q_H}{q_L} + v_H)q_H + (1-\alpha)v_L q_L - Q\right)$$

and

$$p_H = \frac{q_H}{E(q^2)}\left(\frac{\alpha}{2}(v_L\frac{q_H}{q_L} + v_H)q_H + (1-\alpha)v_L q_L - Q\right) - \frac{1}{2}(v_L\frac{q_H}{q_L} - v_H)$$

Its optimal profit is

$$\pi_Q \equiv \frac{\alpha}{4}\left(v_H^2 - \left(v_L\frac{q_H}{q_L}\right)^2\right) + \\ + \frac{1}{E(q^2)}\left(\frac{\alpha}{2}(v_L\frac{q_H}{q_L} + v_H)q_H + (1-\alpha)v_L q_L - Q\right)\left(Q + \frac{\alpha}{2}(v_L\frac{q_H}{q_L} - v_H)q_H\right)$$

Optimality conditions of this local optimum require that Q, the capacity constraint on the total production, be of an intermediate size with respect to the other parameters of the model:

$$\frac{1-\alpha}{2}q_L\left(v_L - v_H\frac{q_L}{q_H}\right) \leq Q \leq \frac{E(vq)}{2} \quad \text{and} \quad K \geq f(Q)$$

where

$$f(Q) = \frac{E(q)}{E(q^2)}Q + \frac{1}{2E(q^2)}\alpha(1-\alpha)(v_L q_H - v_H q_L)(q_H - q_L)$$

Case 3 When both capacity constraints are binding ($\lambda_1 > 0$ and $\lambda_2 > 0$), the optimal pricing strategy of the monopoly is given by

$$p_L = v_L - \frac{Kq_H - Q}{(1-\alpha)(q_H - q_L)}$$

and

$$p_H = v_H - \frac{Q - Kq_L}{\alpha(q_H - q_L)}$$

Its optimal profit is

$$\pi_{KQ} \equiv \frac{1}{q_H - q_L}\left((v_H - v_L)Q + (v_L q_H - v_H q_L)K - \frac{Q^2 + E(q^2)K^2 - 2KQE(q)}{\alpha(1-\alpha)(q_H - q_L)}\right)$$

Optimality conditions of this local optimum require that

$$\max\left(Q/q_H,\, g(Q)\right) \leq K \leq \min\left(Q/q_L,\, f(Q)\right)$$

Case 4 Clearly, the global unconstrained optimum ($\lambda_1 = 0$ and $\lambda_2 = 0$) is attainable whenever $K \geq E(v)/2$ and $Q \geq E(vq)/2$. It consists of the firm choosing prices $v_L/2$ and $v_H/2$, and its value is

$$\pi^U = \alpha\frac{v_H^2}{4} + (1-\alpha)\frac{v_L^2}{4}$$

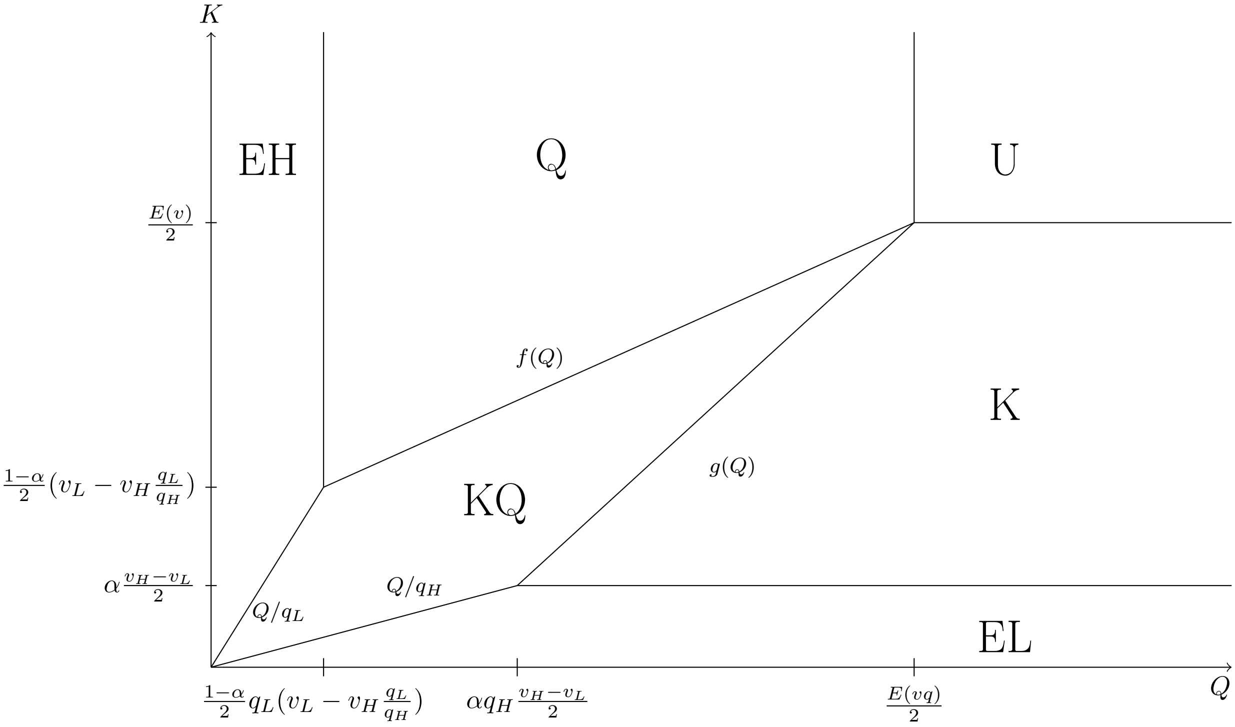
I characterize the monopoly's optimal pricing strategy by comparing the local maxima obtained above for all possible range of parameters. Figure 2.1 depicts the firm's optimal division of the K-Q space.[3] The monopoly's optimal behavior is different in each of these regions. The firm chooses optimal prices in such a way that only capacity K binds in region K (Case 1), only capacity Q binds in region Q (Case 2), both K and Q bind in region KQ (Case 3). None of the constraints bind in region U which corresponds to the unconstrained optimum (Case 4). Only capacity K binds and the monopoly excludes low-types in region EL. Conversely, only capacity Q binds and the monopoly excludes high-types in region EH. Notice that each of the 4 lines bordering the core region KQ have a different slope.

Proposition 2.1 provides a complete characterization of the firm's optimal choice for any combination of its capacity levels.

Proposition 2.1. *The optimal prices of the monopoly are*

[3]Figure 2.1 depicts the case of $(1-\alpha)q_L\left(v_L - v_H\frac{q_L}{q_H}\right) < \alpha q_H(v_H - v_L)$. When this ordering is reversed the figure changes accordingly. However, the coordinates of all lines and critical points remain the same.

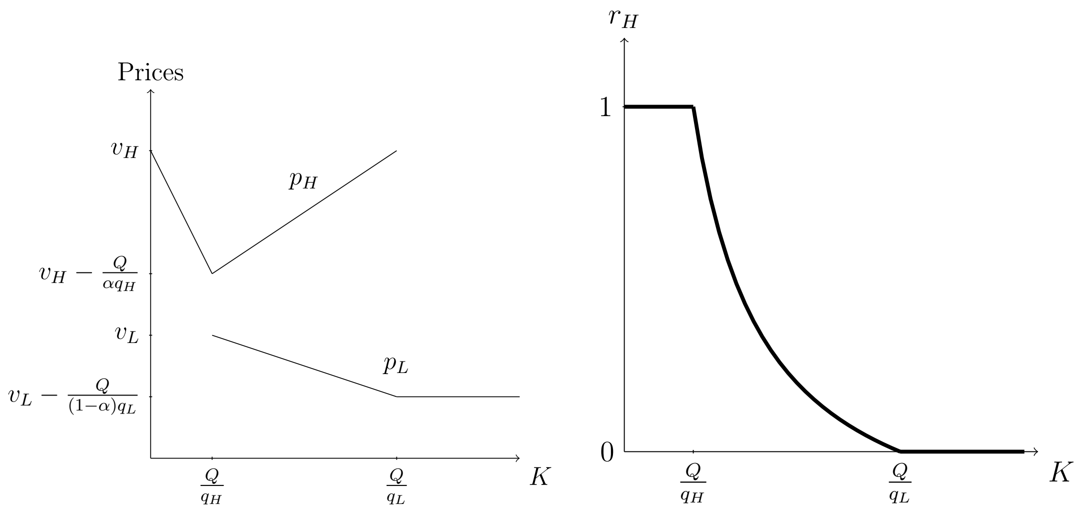
Figure 2.1. Optimal division of the parameter space



- $p_L = v_L + \alpha \frac{v_H - v_L}{2} - K$ *and* $p_H = v_H - (1-\alpha)\frac{v_H - v_L}{2} - K$ *in region K where K binds, Q is slack, some consumers of both groups are served;*
- $p_L = \frac{q_L}{E(q^2)}\left(\frac{\alpha}{2}(v_L \frac{q_H}{q_L} + v_H) q_H + (1-\alpha) v_L q_L - Q\right)$ *and* $p_H = p_L \frac{q_H}{q_L}$ *in region Q where Q binds, K is slack, some consumers of both groups are served;*
- $p_L = v_L - \frac{K q_H - Q}{(1-\alpha)(q_H - q_L)}$ *and* $p_H = v_H - \frac{Q - K q_L}{\alpha(q_H - q_L)}$ *in region KQ where both K and Q bind, some consumers of both groups are served;*
- $p_H = v_H - \frac{K}{\alpha}$ *in region EL where K binds, Q is slack, low-types are excluded;*
- $p_L = v_L - \frac{Q}{q_L(1-\alpha)}$ *in region EH where Q binds, K is slack, high-types are excluded;*
- $p_L = v_L/2$ *and* $p_H = v_H/2$ *in region U where both K and Q are slack, some consumers of both groups are served.*

Assumption 1 guarantees the existence of all 6 regions enumerated in Proposition 2.1. The main intuition driving the results is the two consumer groups' varying relative attractiveness for the firm. For any given Q, an increase in K makes the constraint on total production, Q, tighter. This in turn makes low-types, that consume less of the tighter capacity, relatively more valuable for the firm, so in optimum the firm adjusts its prices to attract more of them (except for extreme values of K).

Figure 2.2. Small Q



(a) Prices (b) Share of high-types

2.3.2 Comparative statics

In this section I investigate the comparative statics properties of the monopoly's optimal behavior. The variables of interest are optimal prices charged for the two consumer groups and the share of high-types among all consumers served. I present all the results as a function of the constraint on the mass of people served, K, however, analogous statements can be made for the other constraint, Q, as the model is almost symmetric in K and Q.

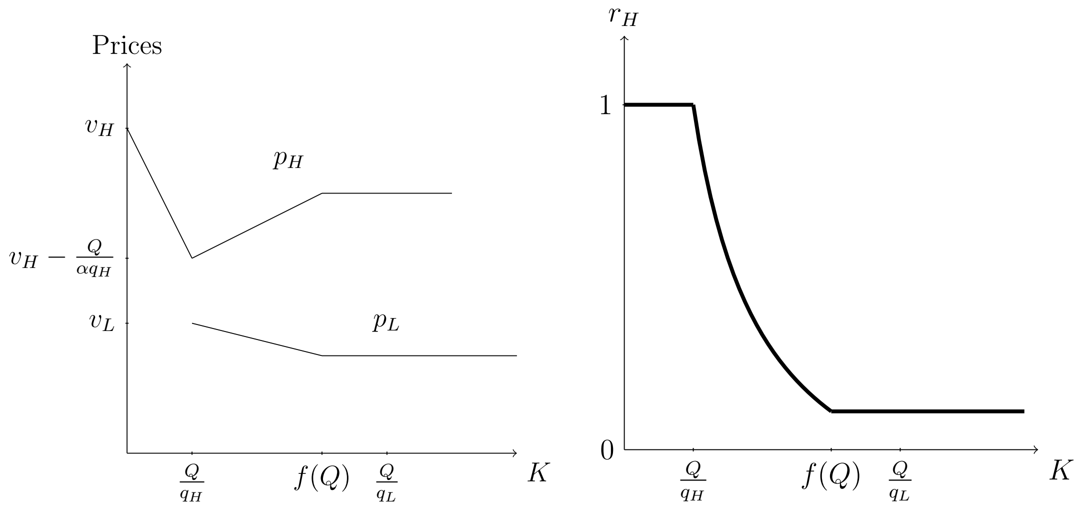
For a detailed analysis of the firm's pricing behavior, it will be useful to divide the Q-K plane into 4 regions by 3 vertical lines going through the 3 values that appear on the Q axis. I call the 4 resulting cases "small Q", "medium Q", "large Q", and "very large Q".

Small Q : The first region of interest is delimited by

$$0 < Q \leq \min\left(\frac{1-\alpha}{2} q_L \left(v_L - v_H \frac{q_L}{q_H}\right), \quad \alpha q_H \frac{v_H - v_L}{2}\right)$$

For very low levels of K the firm excludes low-types and sells up to K exclusively to high-type consumers, Q is slack here. As K grows larger, the monopoly starts serving some low-types as well in such a way that both capacity constraints bind. This means that as K

Figure 2.3. Medium Q, Case A



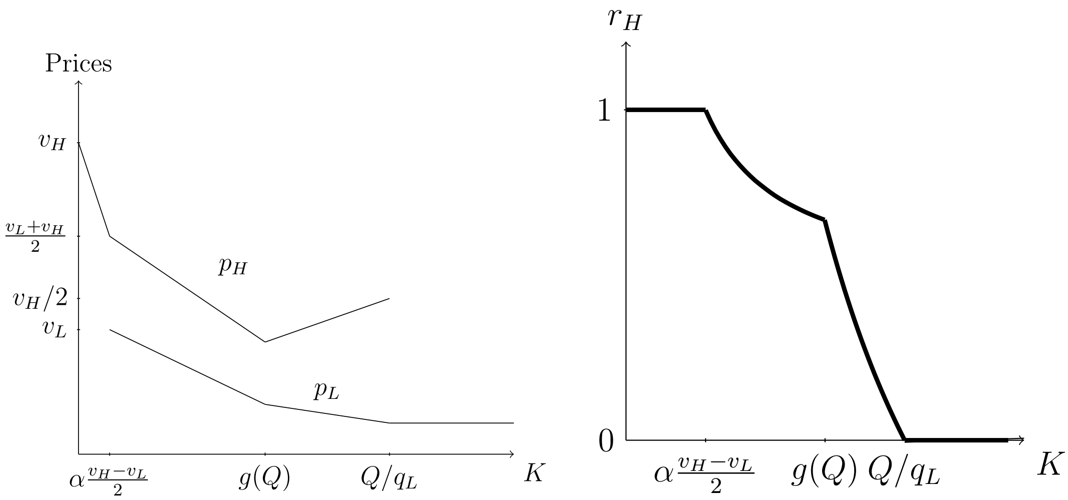
(a) Prices (b) Share of high-types

grows the firm serves more and more low-types and less and less high-types. Finally, as K becomes even larger high-types get excluded and the firm only caters to low-types.

The intuition is the following. As K grows, the shadow price of K (λ_1) decreases while the shadow price of Q (λ_2) increases. The marginal cost of serving an additional high-type consumer in terms of shadow prices, $\lambda_1 + \lambda_2 q_H$ is thus increased relative to the marginal cost of serving a low-type, $\lambda_1 + \lambda_2 q_L$. This means that low-type consumers who use up less of the production constraint become more and more valuable for the firm. This is indeed reflected in the share of high-type consumers served decreasing from 1 all the way to 0 as depicted in the right panel of Figure 2.2.

It is interesting to note that while p_L is decreasing in K, p_H first decreases then increases(see left panel of Figure 2.2). The decrease occurs in region EL where only high-types are served: the monopoly decreases its prices so that the total demand of high-types equals the capacity level. The increase is again a consequence of low-types becoming more valuable for the firm: In order to accommodate more low-types, the firm must serve less high-types so it raises its price for them.

Figure 2.4. Medium Q, Case B



(a) Prices

(b) Share of high-types

Medium Q : The monopoly's pricing strategy changes somewhat when Q is of intermediate size, i.e.,

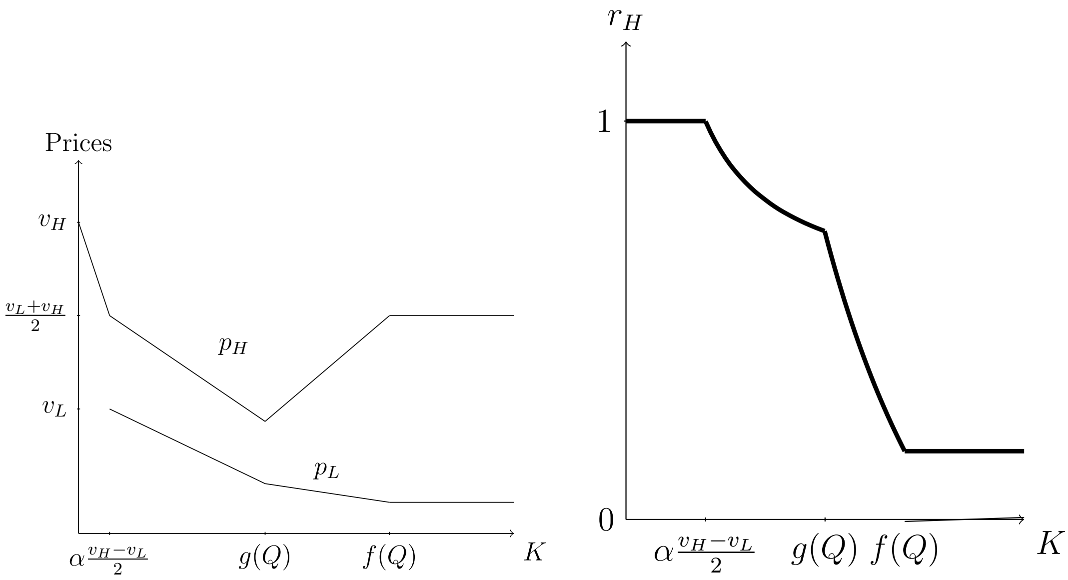
$$\min\left(\frac{1-\alpha}{2}q_L\left(v_L - v_H\frac{q_L}{q_H}\right), \alpha q_H\frac{v_H - v_L}{2}\right) < Q \leq \max\left(\frac{1-\alpha}{2}q_L\left(v_L - v_H\frac{q_L}{q_H}\right), \alpha q_H\frac{v_H - v_L}{2}\right)$$

Case A: $(1-\alpha)q_L\left(v_L - v_H\frac{q_L}{q_H}\right) < \alpha q_H(v_H - v_L)$:

Case A represents the parameter setting depicted on Figure 2.1. As before, for very low levels of K the firm excludes low-types. For larger levels of K the firm starts serving low-types and serves less and less high-types. However, when K hits the threshold value of $f(Q)$, the prices become independent of the size of K. For these capacity levels the firm serves some consumers of both groups in such a way that Q binds and K is slack.

Intuitively, as K grows serving low-types becomes relatively more profitable so there is a region where the mass of high-types served decreases. The profit is always the sum of the profit the firm makes on serving low-types and high-types. It is easy to see that both of those partial profit curves are concave. The profit the monopoly squeezes out of high-types is decreasing at an increasing rate while the profit made on low-types is increasing at a decreasing rate. This means that there is a point $(f(Q))$ where the marginal revenue of high-types equals the marginal loss on low-types. At this point the monopoly prefers to switch to the Q regime where the share of high-types does not decrease more in K. As $f(K) < Q/q_L$, the share of high-types

Figure 2.5. Large Q



(a) Prices (b) Share of high-types

does not decrease all the way to 0 (see Figure 2.3).

Case B: $\alpha q_H(v_H - v_L) \leq (1-\alpha)q_L\left(v_L - v_H\frac{q_L}{q_H}\right)$:

As before, for very low levels of K the firm excludes low-types. However, as K grows the capacity levels enter the K region which means that K still remains binding while both types start getting served and both prices keep decreasing in K. Next, K reaches the KQ region where the monopoly starts raising p_H and both constraints bind. The increase in K continues until none of the high-types are willing to buy, only low-types are served and Q binds.

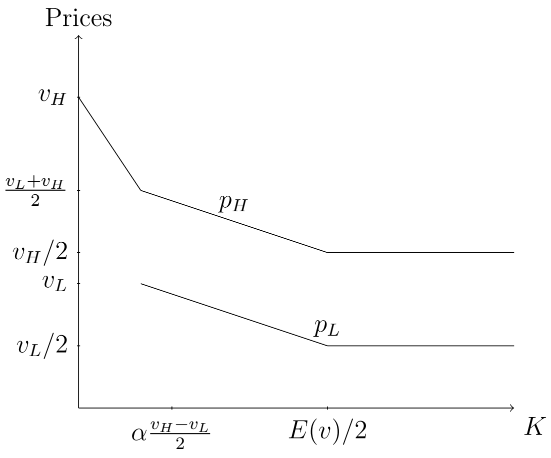
The mass of high-types firstly increases with K, then the mass still increases but their share starts decreasing hyperbolically. When both constraints bind even the absolute mass of high-types starts decreasing so their share drops even further, according to a steeper hyperbola until it reaches zero, see Figure 2.4.

Large Q : The next region of interest is

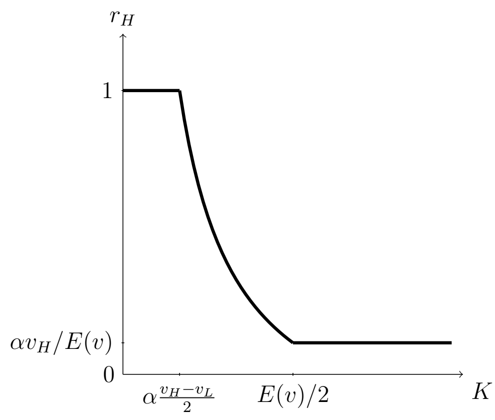
$$\max\left(\frac{1-\alpha}{2}q_L\left(v_L - v_H\frac{q_L}{q_H}\right), \quad \alpha q_H\frac{v_H - v_L}{2}\right) < Q \leq \frac{E(v)}{2}$$

The evolution of optimal prices and consumer shares is very similar to the one for medium

Figure 2.6. Very large Q



(a) Prices



(b) Share of high-types

Q, case B, as one can see from Figure 2.5. Indeed, as K grows the optimal prices are chosen from regions EL, K, KQ, respectively. However, as Q is larger here, the monopoly does not exclude high-types even for very large values of K, the last region the prices are chosen from is Q. Accordingly, the share of high-types varies as described in Case B up to $K = f(Q)$ where it becomes independent of K and levels off at a strictly positive value.

Very large Q : When $Q > E(vq)/2$, the capacity constraint on the amount of production will never be binding.[4]

For very low levels of K the firm excludes low-types and serves only high-type consumers. As K grows larger the monopoly starts to serve some low-types as well, although the constraint on the mass of consumers still binds. Finally, as K becomes very large neither of the constraints will bind and the monopoly can achieve the unconstrained optimum.

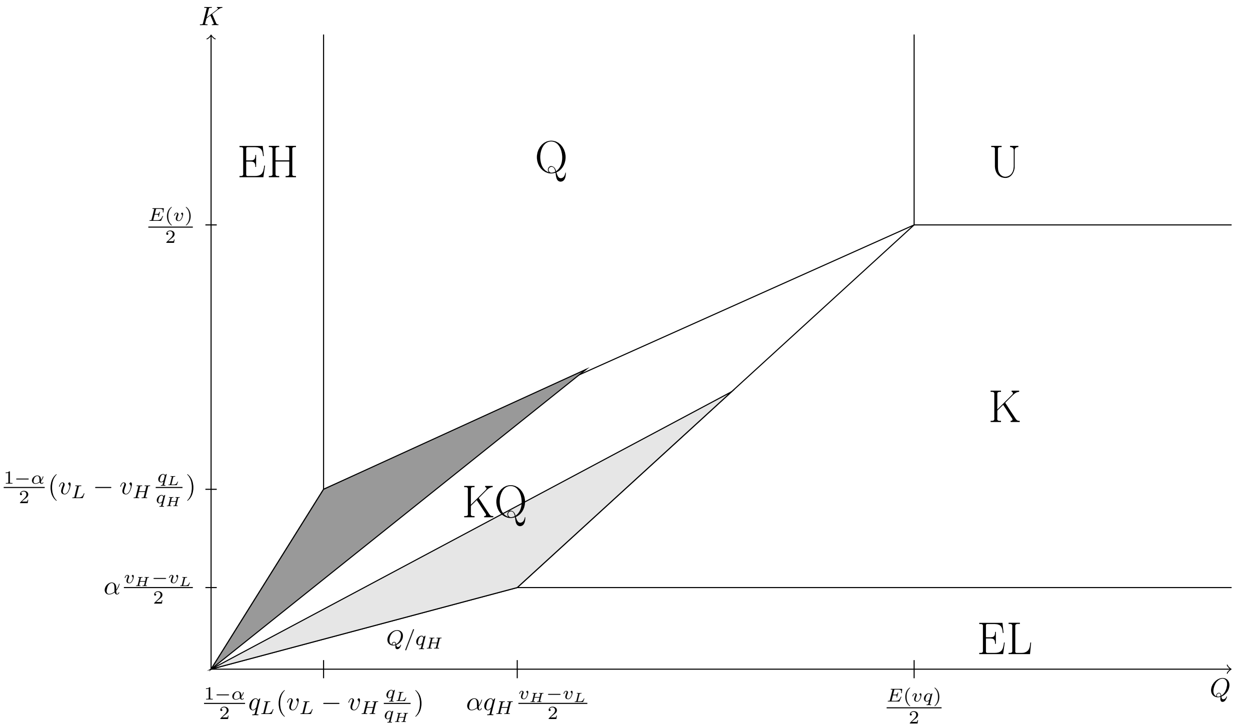
Clearly, both prices p_H and p_L are weakly decreasing in K in this region (Figure 2.6a). The share of high-types served decreases from 1 to $\alpha v_H/E(v)$ which corresponds exactly to the proportion of high-types in the whole market (Figure 2.6b).

[4] One can see the case of very large Q as an alternative benchmark. As Q never binds, this region describes the monopoly's optimal strategy when there is only capacity constraint and two consumer groups.

2.4 Welfare

In this section, I investigate welfare properties of the monopoly's optimal pricing behavior in the presence of dual capacity constraints. I first analyze aggregate consumer surplus as a function of the capacity constraints given that the monopoly chooses its profit-maximizing prices described in Proposition 2.1. Next, I calculate total welfare as the sum of consumer surplus and the monopoly's profit, although most of the interest comes form the study of consumer surplus.

Figure 2.7. Non-monotonicity of consumer surplus



When both consumer groups are served, the general form of total consumer surplus writes as

$$CS = \frac{\alpha}{2}(v_H - p_H)^2 + \frac{1-\alpha}{2}(v_L - p_L)^2.$$

In the EL region, where the monopoly only caters to high-types, consumer surplus equals $\frac{\alpha}{2}(v_H - p_H)^2$. In the EH region, where the firm serves exclusively the low-types, consumer surplus equals $\frac{1-\alpha}{2}(v_L - p_L)^2$.

Thus consumer surplus is weakly decreasing in both prices p_L and p_H. As shown in the previous section, with the exception of the KQ region where both constraints bind, prices are decreasing in the size of capacities. Hence consumer surplus is increasing in capacity levels for any capacity-pairs outside of the KQ region.

However, the problem is more complicated in the KQ region where both capacities bind. In this region p_H is increasing while p_L is decreasing in K. The following proposition sheds light on the effect of this trade-off.

Proposition 2.2. *Aggregate consumer surplus is non-monotonic in the size of the capacity constraints in the parameter region where both capacities are binding. In particular, there always exists a region of capacity pairs inside KQ where consumer surplus is decreasing in K and increasing in Q. Moreover, there exists a second, disjoint region inside KQ where consumer surplus is decreasing in Q and increasing in K.*

The proof of Proposition 2.2 shows that consumer surplus is decreasing in K in the KQ region iff $K \leq Q\frac{E(q)}{E(q^2)}$ (light grey area in Figure 2.7). The intuition for this result is the clearest when Q is relatively low and one can consider the transition between the regions EL and KQ. When K is increasing from a value lower than Q/q_H the monopoly first excludes all low-types and serves only high-types. As K reaches Q/q_H it starts to be profitable to serve some low-type consumers as well such that both constraints bind. To accommodate low-types, the firm starts serving less high-types thus it increases p_H while it lowers p_L.

Consumer surplus is affected by 3 factors. Firstly, as K is binding in both regions, the total mass of consumers served goes up which ceteris paribus increases consumer surplus. Secondly, the decrease in p_L has the same effect also, however, the increase of p_H goes in the opposite direction. To see that this last effect dominates the first two, one should consider the mass of consumers affected. Indeed, when K is relatively small, the firm serves a lot more high-types than low-types, so the loss suffered by the high-types dominates the gain of the few low-types.

Similarly, consumer surplus is decreasing in Q in the KQ region iff $K \geq Q/E(q)$ (dark grey area in Figure 2.7) and the arguments are analogous to the ones described above. The next proposition provides comparative statics results for total welfare, which in this context simply equals to the sum of the monopoly's profit and consumer surplus, i.e.,

$$TW = CS + \pi = \frac{\alpha}{2}(v_H^2 - p_H^2) + \frac{1-\alpha}{2}(v_L^2 - p_L^2)$$

Proposition 2.3. *Total welfare, i.e., the sum of consumer surplus and the monopoly's profit, is increasing in both capacity levels for every parameter value.*

I show in the Appendix that in both areas where consumer surplus is decreasing, profit increases faster than consumer surplus decreases. This property is obviously true for other capacity pairs where both consumer surplus and profit are increasing in capacities.

Finally, total welfare under the monopolistic allocation can be compared with the welfare maximizing allocation for every capacity-pair. Given the formula for total welfare above, finding the welfare maximizing allocations boils down to solving the following optimization problem:

$$\text{(P-TW)} \qquad \begin{aligned} \min_{p_L,p_H} \quad \alpha p_H^2 + (1-\alpha)p_L^2 \qquad \text{s.t.} \\ \alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) \leq K \\ \alpha(v_H - p_H)q_H + (1-\alpha)(v_L - p_L)q_L \leq Q \\ 0 \leq p_L \leq v_L \quad , \quad 0 \leq p_H \leq v_H \end{aligned}$$

Clearly, when capacities are abundant, the solution of the above problem is $p_L = p_H = 0$, i.e., marginal cost pricing. The monopoly's optimal prices being positive create a classic dead-weight loss. As I show in the Appendix, this observation holds for most capacity-pairs. Moreover, in the K region the monopoly serves more low-types and fewer high-types than socially optimal, and the converse is true in the Q region. In particular, in parts of the K and Q regions the welfare maximizing allocation would exclude one group of consumers, unlike the monopoly that serves some consumers of both types. Remarkably, the monopoly's optimal allocation throughout the KQ region coincides with the welfare maximizing allocation.

2.5 Endogenous capacity choice

In this section I investigate the monopoly's optimal choice of capacity levels on the long run. Although in the short run it is reasonable to assume that the monopoly chooses its prices facing fixed capacity levels, in the long run firms can extend or shrink both of their capacities.

Let $c_K(K)$ denote the cost of building capacity K and let $c_Q(Q)$ denote the cost of building Q. I assume that the costs are separable. In the hospital example, although there is a fixed cost of constructing the hospital building, the additional costs of adding beds and equipping the operating rooms are separable. Assume that these costs are strictly positive whenever the capacity levels are strictly positive. The monopoly maximizes its profit which is now a function of its two capacities as well as its prices. The optimal choice of prices for any capacity-pair is the one described in Proposition 2.1. The following Lemma holds under these very general conditions.

Lemma 2.1. *If capacity choice is endogenous and the cost of building capacities is strictly positive then the monopoly chooses its prices and capacity levels in such a way that both constraints bind, i.e., the optimal capacities are always chosen from the KQ region.*

An alternative interpretation of Lemma 2.1 is that the monopoly never chooses capacities in such a way that only one of the capacities be binding. Intuitively, in a world of deterministic demand it is never profitable for a monopoly to build unused capacity. Notice that choosing capacities from the KQ region does not necessarily mean that the monopoly serves both types of consumers. The firm may choose its capacities at the limit of the region in a way to exclude all consumers of one or the other type.

In order to obtain closed-form results and hence a clear intuition, I will focus on the case of linear costs in the remainder of the section, i.e., let

$$c_K(K) = cK \quad \text{and} \quad c_Q(Q) = dQ.$$

The following proposition provides a complete characterization of the monopoly's optimal capacity choice given the linear cost functions.

Proposition 2.4. *If cost functions are linear, i.e.,* $c_K(K) = cK$ *and* $c_Q(Q) = dQ$ *then the optimal capacity levels are given by*

1. $K = \frac{1}{2}(E(v) - c - dE(q))$ *and* $Q = \frac{1}{2}(E(vq) - cE(q) - dE(q^2))$
   if $v_H > c + dq_H$ *and* $v_L > c + dq_L$
2. $K = \frac{\alpha}{2}(v_H - c - dq_H)$ *and* $Q = \frac{\alpha}{2}q_H(v_H - c - dq_H)$
   if $v_H > c + dq_H$ *and* $v_L \leq c + dq_L$
3. $K = \frac{1-\alpha}{2}(v_L - c - dq_L)$ *and* $Q = \frac{1-\alpha}{2}q_L(v_L - c - dq_L)$
   if $v_H \leq c + dq_H$ *and* $v_L > c + dq_L$
4. $K = Q = 0$ *if* $v_H \leq c + dq_H$ *and* $v_L \leq c + dq_L$

The resulting optimal prices are $p_i = (v_i + c + dq_i)/2, \quad i \in \{L, H\}$ *whenever i-types are not excluded.*

Notice that $c + dq_H$ corresponds to the marginal cost of building capacity to serve an additional high-type, while $c + dq_L$ is the cost of serving an additional low-type consumer. Some high-type consumers are served if and only if the marginal cost of capacity necessary to serve a high-type is lower than the WTP of the most valuable high-type consumer, i.e., $v_H > c + dq_H$. An analogous result holds for low-types. Hence the four cases of Proposition 2.4: depending on the relative sizes of the two marginal costs with respect to the WTP of the most valuable consumers, the firm either serves both types or excludes one or both groups of consumers.

The first case in Proposition 2.4 corresponds to the situation when some consumers of both types are served, i.e., capacities are chosen from the inside of the KQ region.

The second case describes a situation where the monopoly prefers excluding low-type consumers. This arises when the capacity constraint on production, Q is relatively cheap to build, which makes K relatively stricter which in turn increases the attractiveness of high-types. Indeed, the two conditions imply $d < (v_H - v_L)/(q_H - q_L)$. The optimal capacities satisfy $K = Q/q_H$ which means that the monopoly chooses its capacities from exactly one side of the KQ quadrilateral. Notice also that the resulting price for high-types, $p_H = (v_H + c + dq_H)/2$, corresponds exactly to the optimal price of a monopoly facing only high-type consumers and whose production costs are equal to $c + dq_H$.

The third case is analogous to the second one: the firm prefers excluding high-types and chooses its capacity from the $K = Q/q_L$ side of the KQ quadrilateral. The fourth case completes the discussion: if both capacities are very expensive to build with respect to the WTP of all consumers then the monopoly prefers to exit the market.

As one should expect, capacity levels are decreasing in costs, moreover, transitions between the different cases are smooth in the sense that K and Q are continuous in both c and d. In the limit as both costs go to 0, the monopoly builds enough capacities to achieve its unconstrained optimum, i.e., $K \to E(v)/2$ and $Q \to E(vq)/2$.

Both prices p_L and p_H are an increasing function of both cost parameters c and d. Therefore, the consumer surplus is always decreasing in both c and d when capacities are chosen endogenously. Hence, in the long run both prices and consumer surplus are monotonic in capacity costs.

2.6 Incentive compatibility

In this section I relax the assumption that the monopoly is able to observe the quantity demanded by each consumer. Given the self-selection of consumers, the firm faces new incentive compatibility constraints. I investigate two different extensions of the baseline model. In the first scenario low-types are allowed to buy the large bundle and freely dispose of the unused part. In the second scenario, I check the robustness of the baseline model by assuming that q_H is a multiple of q_L, and that high-types are allowed to buy several small bundles .

2.6.1 Incentive compatibility for low-types

In this section I consider an extension of the model where low-types are allowed to buy quantity q_H for p_H and throw away the quantity $q_H - q_L$ they do not consume. One can think of this scenario as the case with free disposal. This assumption is realistic if the monopoly cannot tell consumers apart according to their individual demand.

How does consumers' possibility of free disposal alter the monopoly's incentives? Firstly notice that free disposal only alters consumers' (low-types') incentives in case $p_H < p_L \leq v_L$, i.e., whenever the larger quantity is cheaper. Problem (P-EH) where only low-types are served will thus remain unaffected by free disposal.

The maximization problem of serving both consumer groups becomes more complicated in the presence of free disposal. The monopoly must decide whether it chooses its prices in a way that makes the consumers buy both the quantity intended for them, or alternatively, it may choose a price such that all consumers buy the larger quantity, q_H. In the former case, the maximization problem is very similar to (P-LH) described previously, with one additional constraint:

$$
\text{(P-ICL)} \qquad \max_{p_L, p_H} \quad \pi = \alpha(v_H - p_H)p_H + (1-\alpha)(v_L - p_L)p_L \qquad \text{s.t.}
$$

$$
\begin{aligned}
\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) &\leq K \\
\alpha(v_H - p_H)q_H + (1-\alpha)(v_L - p_L)q_L &\leq Q \\
0 \leq p_L < v_L \quad , \quad p_L \leq p_H &< v_H
\end{aligned}
$$

The monopoly must choose a lower price for the smaller quantity if it wants the two consumer groups separated, hence the new, stricter lower bound on the high-price: $p_L \leq p_H$. However, we know from the previous section that the solution of (P-LH) for any parameter region satisfies this stricter condition. This means that the solution of (P-ICL) will exactly coincide with the solution of (P-LH) described by Proposition 2.1.

In addition, the monopoly might now choose a price $p_H < p_L \leq v_L$ that makes all consumers buy q_H for p_H. In this case the mass of buyers is

$$\alpha(v_H - p_H) + (1 - \alpha)(v_L - p_H) = E(v) - p_H$$

and total demand is equal to $(E(v) - p_H)q_H$. Hence the maximization problem writes as

$$\text{(P-ICL2)} \qquad \max_{p_H} \quad \pi = (E(v) - p_H)p_H \qquad \text{s.t.}$$
$$E(v) - p_H \leq \min(K, Q/q_H)$$
$$p_H < v_L$$

Notice that the two constraints imply that the following two inequalities must be satisfied for (P-ICL2) to have a feasible solution:

$$K > \alpha(v_H - v_L) \quad \text{and} \quad Q > \alpha q_H(v_H - v_L)$$

Intuitively, the capacity constraints must be relatively large if the monopoly is able to serve some consumers of both types for a relatively low price.

Finally, if the monopoly wants to exclude low-types from buying its products, it must satisfy the stricter condition of $v_L < p_H$:

$$\text{(P-ICLEL)} \qquad \max_{p_H} \quad \pi = \alpha(v_H - p_H)p_H \qquad \text{s.t.}$$
$$\alpha(v_H - p_H) \leq \min(K, Q/q_H)$$
$$v_L < p_H$$

However, the optimal solution of (P-ICLEL) is obviously dominated by the solution of (P-EL) as it is a constrained version of it.

Proposition 2.5. *The monopoly's optimal pricing strategy is not affected by the possibility of free disposal.*

Proposition 2.5 states that the model of dual capacities is robust to the introduction of the free disposal assumption. The new strategies of the monopoly consist of pooling both types at p_H and selling them the large quantity q_H. The proof in the Appendix shows that this pooling is always dominated by some strategy that was already available in the baseline model.

2.6.2 Incentive compatibility for high-types

In this section I consider a model in which the monopoly cannot observe consumers' individual demand, moreover, high-types are allowed to satisfy their individual demand q_H by buying multiple small bundles. Clearly, in this context high-types must make a choice between buying one bundle of q_H or $\left\lceil \frac{q_H}{q_L} \right\rceil$ bundles of q_L, where $\lceil x \rceil$ denotes the smallest integer larger or equal to x. In the latter case, I assume free disposal, i.e. the consumer can dispose freely of the unused amount $\left\lceil \frac{q_H}{q_L} \right\rceil q_L - q_H$.

I proceed in two steps. Firstly, I assume the special case where individual demand of high-types is a multiple of the individual demand of low-types, i.e., $\left\lceil \frac{q_H}{q_L} \right\rceil = \frac{q_H}{q_L} = k$ with k being an integer. I show that the possibility of repeat purchase does not alter the monopoly's optimal pricing behavior. Secondly, I show that this first result implies that the monopoly's behavior will be unchanged in the more general case of free disposal as well. This second step is rather intuitive: if it is not in the consumers' interest to buy several small bundles when they can do it without waste, then even under free disposal they will not be tempted to do it.

The monopoly must decide whether to choose prices such that high-types prefer buying the large bundle (q_H) or to choose them in a way that everyone buys small bundles (q_L). Given that the monopoly is unable to tell apart consumers, it must choose a low enough price for q_H if it wants high-types to buy it instead of several small bundles. In particular, the unit price of the large bundle cannot exceed the unit price of the small bundle:

$$p_H \leq kp_L \iff p_H/q_H \leq p_L/q_L.$$

Notice that the optimal solutions of (P-LH), as described in Proposition 2.1 satisfy this condition for any capacity-pair. Hence, similarly to the case discussed in the previous section, the additional incentive compatibility constraint does not alter the firm's optimal pricing behavior if it wants the two consumer groups to buy different bundles.

The other option of the firm is to choose a relatively low price for the small quantity, resulting in all consumers buying that variety. The total price a high-type buyer must pay to satisfy its individual demand q_H is kp_L. The maximization problem when serving some consumers of both types then writes as

$$\text{(P-ICH)} \qquad \max_{p_L} \quad \pi = \alpha(v_H - q_H \frac{p_L}{q_L})q_H \frac{p_L}{q_L} + (1-\alpha)(v_L - p_L)p_L \qquad \text{s.t.}$$

$$\alpha(v_H - q_H \frac{p_L}{q_L}) + (1-\alpha)(v_L - p_L) \leq K$$

$$\alpha(v_H - q_H \frac{p_L}{q_L})q_H + (1-\alpha)(v_L - p_L)q_L \leq Q$$

$$0 \leq kp_L < v_H$$

The last inequality ensures a low enough price so that some high-types consume the product. Notice that this maximization problem coincides with (P-LH) with the additional constraint of $p_H = kp_L = q_H \frac{p_L}{q_L}$. As shown above, the monopoly's optimal choice with perfect divisibility coincides with the solution of (P-LH). (P-ICH) being a more restricted problem, it can never be more profitable for the firm to make all consumers buy the small bundle than to separate the two consumer groups.

Furthermore, the monopoly can choose to exclude one group of consumers. The firm faces exactly the same problem as (P-EL) if it serves only q_H bundles to high-types. Serving only q_L bundles does not necessarily exclude the high-types, if the monopoly wants to serve only low-types, it must choose a relatively high unit price: $\frac{v_H}{q_H} < \frac{p_L}{q_L} < \frac{v_L}{q_L}$. However, the solution of this sub-problem either coincides with the solution of (P-EH), or it is dominated by serving both consumer groups.

These considerations lead to the conclusion that the firm's optimal behavior is unaffected by the possibility of repeat purchases if $q_H = kq_L$. The remaining step consists in showing that the result of this special case implies the same result in the more general specification when high-types must buy $\left\lceil \frac{q_H}{q_L} \right\rceil$ small bundles. To see this, it is sufficient to note that in each optimization problem the constraints in the general case are stricter then in the integer case. Therefore the option of repeat purchase is even less attractive for high-types, hence the following proposition:

Proposition 2.6. *The monopoly's optimal pricing strategy is not affected by the possibility of repeat purchase despite the monopoly's inability to observe consumers' individual demand.*

Proposition 2.6 concludes the analysis of incentive compatibility. Jointly with Proposition 2.5, it suggests that the results of the model are robust to the relaxation of the observable individual demand assumption.

2.7 General distribution of consumers

In the baseline model presented in Section 2.2, consumers' WTP is distributed uniformly within each group, which implies linear total demand that in turn leads to the clear-cut results presented in Proposition 2.1. In this section I generalize the baseline model by assuming a general distribution function of consumers' WTP. I identify sufficient conditions on the distribution functions of the two consumer groups that guarantee the existence of the regions discovered in the baseline model, illustrated by Figure 2.1. Moreover, I show that under these conditions the main results carry over to the general distribution case.

Let the WTP of consumers of type $i \in \{L, H\}$ be distributed according to the twice continuously differentiable cumulative distribution function F_i with support $[0,\ \theta_i]$. Let f_i denote the density function i.e., the first derivative of F_i. For any $p \geq 0$ let $D_i(p) = 1 - F_i(p)$ be the total demand function that measures the proportion of i-type consumers willing to buy at price p. Let α and $1 - \alpha$ denote the total mass of high-types and low-types, respectively. Individual demand of consumers is the same as in the baseline model: $q_H > q_L$. As before, the monopoly is constrained by K, the total mass of consumers it can serve on the one hand, and by Q, its maximal production on the other hand. The maximization problem of the monopoly writes as

$$\text{(P-GEN)} \qquad \max_{p_L, p_H} \quad \pi = \alpha p_H D_H(p_H) + (1-\alpha) p_L D_L(p_L) \qquad \text{s.t.}$$

$$\alpha D_H(p_H) + (1-\alpha) D_L(p_L) \leq K \qquad (\lambda_1)$$

$$\alpha q_H D_H(p_H) + (1-\alpha) q_L D_L(p_L) \leq Q \qquad (\lambda_2)$$

$$p_L \geq 0\ , \quad p_H \geq 0$$

where λ_1 is the multiplier of the constraint on the mass of consumers served and λ_2 is the multiplier of the constraint on total production. The only restrictions on the total demand functions so far are that they be decreasing, $D_i(0) = 1$ and $D_i(\theta_i) = 0$. Notice that $D_i(p_i) = 0$ for any $p_i \geq \theta_i$, so choosing a large enough p_i excludes consumer group i. Assume that the profits derived from serving the two groups, i.e., $p_i D_i(p_i)$, is concave. The *monotone hazard rate condition*, which in this context translates to the function

$$\phi_i(p) \equiv \frac{f_i(p)}{1-F_i(p)} = -\frac{D_i'(p)}{D_i(p)}$$

being non-decreasing, is a sufficient condition for the concavity of $p_i D_i(p_i)$. The Karush-Kuhn-Tucker conditions of the maximization problem write as

$$(p_L - \lambda_1 - \lambda_2 q_L)D_L'(p_L) + D_L(p_L) = 0 \quad \text{and} \quad (p_H - \lambda_1 - \lambda_2 q_H)D_H'(p_H) + D_H(p_H) = 0.$$

These conditions are trivially satisfied for a consumer group which is excluded from the market. Otherwise, when prices are chosen such that some consumers of both types are served (i.e., $p_L < \theta_L$ and $p_H < \theta_H$) the first order conditions can be rewritten using the cumulative distribution functions:

$$\lambda_1 + \lambda_2 q_L = p_L - \frac{1-F_L(p_L)}{f_L(p_L)} \quad \text{and} \quad \lambda_1 + \lambda_2 q_H = p_H - \frac{1-F_H(p_H)}{f_H(p_H)}.$$

Notice that the term $p - \frac{1-F_i(p)}{f_i(p)}$ which appears in both equations is the *virtual valuation function* that is widely used in the mechanism design literature.[5]

The first order conditions imply that in optimum the virtual valuation of the two consumer groups must coincide whenever the capacity on the mass of people served (K) binds and the other constraint is slack ($\lambda_1 > 0,\ \lambda_2 = 0$). Conversely, if Q binds and K is slack ($\lambda_1 = 0,\ \lambda_2 > 0$) then the *per-unit* virtual valuation of the two groups must be equal. The monopoly can charge its unconstrained optimal prices (where both virtual valuations equal zero) if none of the constraints bind. Replacing the optimal prices into the two capacity constraints, one gets a unique threshold level for both capacity levels, $\overline{K}$ and $\overline{Q}$ that delimit region U'.

Next, I identify sufficient conditions for the existence of a core region KQ' where both constraints bind. Both constraints binding immediately imply that the optimal prices must satisfy

$$D_L(p_L) = \frac{Kq_H - Q}{(1-\alpha)(q_H - q_L)} \quad \text{and} \quad D_H(p_H) = \frac{Q - Kq_L}{\alpha(q_H - q_L)}.$$

This means that 2 of the 4 curves delimiting the core region KQ' remain the same as in the

[5] The connection between auction theory and the monopoly's problem of third-degree price discrimination was first revealed by Bulow and Roberts (1989).

baseline model: in order to serve both consumer groups, $K \geq Q/q_H$ and $K \leq Q/q_L$ must be satisfied. The two other frontiers of the KQ' region are given by capacity-pairs within the region that also satisfy the equations $\lambda_1 = 0$ and $\lambda_2 = 0$. Let $K = g(Q)$ denote the curve above which capacity K is slack ($\lambda_1 = 0$), and let $K = h(Q)$ be the curve that specifies whether Q is binding or slack ($\lambda_2 = 0$). Finding an explicit formula for these curves would necessitate knowing the value of $D_i'(p_i)$ for the optimal prices. Although total demand levels at the optimal prices are simple, there is no direct formula to express the derivatives of the demands at the optimal prices. However, using the implicit function theorem, one can prove the following Lemma.

Lemma 2.2.

(i) Assume that both distribution functions satisfy Myerson's regularity condition, i.e., the virtual valuation functions $p - \frac{1-F_i(p)}{f_i(p)}$ *,* $i \in \{L, H\}$ *are strictly increasing in* p*. Then* $g'(Q) > 0$ *and* $h'(Q) > 0$*.*

(ii) $g(Q) < h(Q)$ *for any* $0 \leq Q < \overline{Q}$ *and* $g(\overline{Q}) = h(\overline{Q}) = \overline{K}$*.*

(iii) $\exists\, \widetilde{Q} > 0$ *such that* $g(Q) \leq Q/q_H$ *for* $\forall\, 0 \leq Q < \widetilde{Q}$*.*

(iv) $\exists\, \widetilde{K} > 0$ *such that* $h(Q) \geq Q/q_L$ *for* $\forall\, 0 \leq h(Q) < \widetilde{K}$*.*

(i) identifies regularity of the cumulative distribution functions as sufficient conditions for curves g and h to be increasing. Notice that the monotone hazard rate condition implies Myerson's regularity condition. (ii) states that g is always below h when at least one of the capacities are binding, and they cross exactly at point $(\overline{Q}, \overline{K})$. (iii) and (iv) state that $g(Q) < Q/q_H$ for small values of Q and conversely, $h(Q) > Q/q_L$ for small values of K. The combination of these results imply the existence of a KQ' region delimited by two increasing curves in addition to the two straight lines.

Next, I show that the existence of region EL' where the monopoly excludes low-types is guaranteed if the cumulative distribution of high-types *hazard rate dominates* the cumulative distribution of low-types, i.e.,

$$\frac{f_H(p)}{1 - F_H(p)} < \frac{f_L(p)}{1 - F_L(p)} \quad \text{for all } p.$$

Notice that the hazard rate dominance relation rewrites in terms of demand as $\varepsilon_H(p) < \varepsilon_L(p)$ for all p, where $\varepsilon_i(p)$ is the elasticity of demand:

$$\varepsilon_i(p) = -\frac{pD_i'(p)}{D_i(p)}.$$

Therefore, hazard rate dominance is equivalent to the low-types having a larger demand elasticity than high-types at each price. As hazard rate dominance implies first order stochastic dominance, it also implies $\theta_L < \theta_H$. The proof consists of showing that firstly, prices are decreasing and total demands are increasing in K in the K' region, and secondly, there is a positive threshold level of K below which optimal prices induce zero demand for the low-types and strictly positive demand for the high-types. Intuitively, the curve separating regions K and EL is a horizontal line because in both regions only constraint K binds and the prices are profits are independent of Q.

Conversely, I show that a sufficient condition for the existence of an EH' region where the monopoly serves exclusively the low-type consumers is

$$q_L \frac{f_L(p)}{1-F_L(p)} < q_H \frac{f_H(p)}{1-F_H(p)} \quad \Longleftrightarrow \quad q_L \varepsilon_L(p) < q_H \varepsilon_H(p) \quad \text{for all } p.$$

The sufficient condition requires that for every price, the demand elasticity weighted by the individual demand be higher for the high-types than for the low-types. This ensures the existence of a threshold level of Q at which the optimal prices in the Q' region induce a zero demand for the high-types while demand of low-types remain positive. The curve separating regions Q' and EH' is a vertical line because prices and profits in both regions only depend on Q and are independent of K.

Consumer surplus in the context of general distributions can be written as

$$CS = \alpha \int_{p_H}^{\theta_H} (w - p_H) f_H(w) dw + (1-\alpha) \int_{p_L}^{\theta_L} (w - p_L) f_L(w) dw.$$

Obviously, the consumer surplus is a decreasing function of both prices. In most regions both prices are decreasing in both capacity levels, thus aggregate consumer surplus is increases in both K and Q. However, in the KQ' region p_H is increasing in K and decreasing in Q and conversely, p_L is increasing in Q and decreasing in K. Therefore, consumer surplus of the high-types is decreasing while consumer surplus of low-types is increasing in K. Proposition 2.7 states that there always exist regions of capacity-pairs inside of KQ' where the first effect is dominant.

Proposition 2.7. *Assume that both cumulative distribution functions satisfy the monotone hazard rate condition. Then there exist two disjoint regions inside of KQ' such that the consumer surplus decreases in K in one of the regions, and it decreases in Q in the other region. In particular,*

$\frac{\partial CS}{\partial Q} < 0 \Leftrightarrow \frac{\varepsilon_L(p_L)}{\varepsilon_H(p_H)} < \frac{p_L}{p_H}$ *and* $\frac{\partial CS}{\partial K} < 0 \Leftrightarrow \frac{\varepsilon_L(p_L)}{\varepsilon_H(p_H)} > \frac{p_L/q_L}{p_H/q_H}$.

The monotone hazard rate condition implies both the concavity of profits and increasing virtual valuation functions. In the Appendix, I show that

$$\frac{\partial CS}{\partial Q} < 0 \Leftrightarrow \frac{D_L(p_L)}{D_L'(p_L)} < \frac{D_H(p_H)}{D_H'(p_H)} \quad \text{and} \quad \frac{\partial CS}{\partial K} < 0 \Leftrightarrow q_L \frac{D_L'(p_L)}{D_L(p_L)} < q_H \frac{D_H'(p_H)}{D_H(p_H)}.$$

It follows from Lemma 2.2 that part of the Q/q_L and Q/q_H lines border the KQ' region. From the demand functions, it is obvious that the first condition is satisfied for $K = Q/q_L$ and the second for $K = Q/q_H$. Existence of the two regions can be proved by continuity arguments.

The formula in Proposition 2.7 corresponds to the previous formulas, rewritten in terms of demand elasticities. It reveals that consumer surplus is decreasing in Q whenever *at the optimal prices* the demand elasticity to price ratio is larger for high-types than for low-types. Furthermore, consumer surplus is decreasing in K when *at the optimal prices* the demand elasticity to *unit* price ratio is higher for the low-types than for the high-types. This formulation also shows clearly that there is no capacity-pair that satisfies both conditions, i.e., there is no situation in which an increase in both capacities decreases aggregate consumer surplus.

2.7.1 A simple model of take-out restaurants

In this section, I present an alternative model of a firm constrained by dual capacities with general distribution function.[6] It has the merit of providing some clear intuition by being simpler than the most general model and still exhibiting its main features.

Assume a restaurant providing only falafel balls being constrained both by the number of seats it has at its disposal and the number of falafel balls it can produce. There are two types of consumers: the first type buys the meal as take-out, the other type prefers to eat it in the restaurant. For simplicity, assume everyone wants to eat the same number of falafel balls, and without loss of generality this amount can be normalized to 1. The other assumptions of the model are identical to the main model presented above.

In this set-up high-types are the ones who prefer eating in the restaurant, thus occupying a seat, as opposed to low-types, whose consumption of seats is 0. This is the first difference

[6] I would like to thank to Marc Möller for the idea of this model variant.

compared to the main model: consumers are heterogeneous in their consumption of the constraint on the mass of people served, and homogeneous in their consumption of the other constraint. In the main model the opposite is true. A second important difference is that low-types consume 0 chairs, something which is not allowed in the main model where $q_L > 0$ is extensively used.

This firm's optimization problem is the following:

$$\text{(P-OUT)} \qquad \max_{p_L, p_H} \quad \pi = \alpha p_H D_H(p_H) + (1-\alpha) p_L D_L(p_L) \qquad \text{s.t.}$$
$$\alpha D_H(p_H) \leq K$$
$$\alpha D_H(p_H) + (1-\alpha) D_L(p_L) \leq Q$$
$$p_L \geq 0 \ , \quad p_H \geq 0$$

The first equation is the new constraint on the mass of people served, with K being the number of seats in the restaurant. Notice that only high-types contribute to it. The second constraint is on the total number of falafel balls, Q that the restaurant can produce. In order to avoid the trivial case where the first constraint never binds, assume $Q > K$.

In the Appendix I show that despite its differences with the main model, the main results of the baseline model are present in the take-out model. In particular, there exists a core region of capacity levels in which optimal prices are chosen in such a way that some consumers of both types are served and both capacities are exactly exhausted. In the core region, optimal prices imply that total demands are given by

$$D_H(p_H) = \frac{K}{\alpha} \quad \text{and} \quad D_L(p_L) = \frac{Q-K}{1-\alpha}.$$

Thus in the core region the monopoly fills its seats with high-type consumers, then sells the remaining falafel balls as take-out to low-types. Clearly, the mass of high-type consumers that the monopoly prefers to attract depends solely on the number of available seats in an intuitive way: the more seats it has, the more high-types it wants to attract. This gives rise to the standard comparative statics result: the larger the capacity K the lower the optimal prices are for high-types.

Therefore the non-monotonicity property can only arise for low-types in this setting. As the restaurant's optimal strategy is to serve the *remaining* falafel balls as take-out after serving the seated costumers, the mass of low-types the monopoly wants to serve decreases in the mass of its high-type consumers. Since the mass of high-types served is exactly equal to the number of seats, the more seats the restaurant has, the fewer falafel balls it has left, so it can rise the price of take-out and still sell the remaining falafel balls. Thus in the core region p_L is an increasing function of K, the first key result of the main model.

The second key result of the main model, i.e. the observation that increasing one capacity may harm consumers on the aggregate is also present in the take-out model. To see this, first consider the case of a pure take-out restaurant that does not have any seats, i.e. $K = 0$. Naturally, in this case the restaurant can only serve low-types and no high-type consumer. Assume the restaurant is expanding and adding ε seats without increasing the size of its kitchen. According to the logic described above,[7] it will then serve a mass ε of high-types, which will in turn decrease the mass of low-types served and increase the price they pay. Therefore, a large mass of low-types will be worse-off while only a small mass of high-types will gain thanks to this capacity expansion. Overall, aggregate consumer surplus will decrease.

2.8 Conclusion

Several capacity constraints co-exist in various real-world industries. The present paper provides a formal economic analysis of the effects of dual capacity constraints on optimal firm behavior. It reveals a rich structure of optimal monopoly pricing which in the short run is qualitatively different from the predictions of models of firms bound by a single capacity. In particular, prices charged for some consumers increase for some capacity pairs as one of the capacities is enlarged. Moreover, aggregate consumer surplus is decreased by an increase of one capacity level for some capacity pairs. These results are robust to observability of individual demand and also for a fairly general class of distribution of consumers. In the long run, when capacity building is endogenous, prices and consumer surplus are monotonic in capacity costs.

Future research can extend the results in several important aspects. One could verify the model's robustness by approximating the capacity constraints with convex and continuous cost functions. Moreover, the model could accommodate more than two consumer groups. Finally, by extending the model to the case of a duopoly, it would become directly comparable with the various models of Bertrand-Edgeworth competition with one capacity constraint.

[7] In the Appendix I show that the a segment of the K axis constitutes one of the borders of the core region, thus for $K = 0$ and for very small K the logic can indeed be used.

2.9 Appendix of Chapter 2

Proof of Proposition 2.1

Consider the following maximization problem that encompasses (P-LH), (P-EL) and (P-EH):

$$
\text{(P-GEN)} \qquad \max_{p_L, p_H} \quad \pi = \alpha(v_H - p_H)p_H + (1-\alpha)(v_L - p_L)p_L \qquad \text{s.t.}
$$

$$
\begin{aligned}
\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) &\leq K && (\lambda_1)\\
\alpha(v_H - p_H)q_H + (1-\alpha)(v_L - p_L)q_L &\leq Q && (\lambda_2)\\
p_L &\leq v_L && (\lambda_3)\\
p_H &\leq v_H && (\lambda_4)\\
p_L \geq 0\,, \quad p_H &\geq 0
\end{aligned}
$$

The non-negativity constraints are omitted and verified ex-post. Multiplying the third and fourth constraint by $1-\alpha$ and α, respectively, the objective function for deriving the Karush-Kuhn-Tucker conditions writes as

$$
\begin{aligned}
&L(p_L, p_H, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \alpha(v_H - p_H)p_H + (1-\alpha)(v_L - p_L)p_L - \\
&\lambda_1\left[\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) - K\right] - \lambda_2\left[\alpha(v_H - p_H)q_H + (1-\alpha)(v_L - p_L)q_L - Q\right] - \\
&\lambda_3(1-\alpha)(v_L - p_L) - \lambda_4\alpha(v_H - p_H)
\end{aligned}
$$

Hence the two first order conditions are

$$
2p_L = v_L + \lambda_1 + \lambda_3 + \lambda_2 q_L \quad \text{and} \quad 2p_H = v_H + \lambda_1 + \lambda_4 + \lambda_2 q_H .
$$

As $\lambda_3 > 0$ implies $p_L = v_L$, this case corresponds to excluding the low-types. Indeed, (P-GEN) is reduced to (P-EL) whose solution is described in the main text. Similarly, $\lambda_4 > 0$ implies $p_H = v_H$ i.e., the exclusion of high-types which corresponds to the maximization problem (P-EH), also solved in the main text. Obviously, $\lambda_3 > 0$ and $\lambda_4 > 0$ leads to zero profit hence it is never optimal. Finally, $\lambda_3 = \lambda_4 = 0$ corresponds to the case of serving both consumer groups, described in (P-LH). In the following, I prove the formulas for optimal prices and the borders of the optimal regions described in Case 1 - Case 3. Case 4 is described in the main body of the paper.

Case 1 : $\lambda_1 > 0$ and $\lambda_2 = 0$

From the FOCs: $\lambda_1 = 2p_L - v_L = 2p_H - v_H$ and $\lambda_1 > 0$ implies that K binds: $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) = K$. The optimal prices can be calculated from these 2 equations:

$$p_L = v_L + \alpha\frac{v_H - v_L}{2} - K \quad \text{and} \quad p_H = v_H - (1-\alpha)\frac{v_H - v_L}{2} - K$$

The capacity constraint on production rewrites as

$$\alpha(K + (1-\alpha)\frac{v_H - v_L}{2})q_H + (1-\alpha)(K - \alpha\frac{v_H - v_L}{2})q_L \leq Q,$$

which is equivalent to $K \leq g(Q)$ by definition. $\lambda_1 > 0$ implies $K < E(v)/2$ and finally, $p_L \leq v_L$ implies $\frac{\alpha}{2}(v_H - v_L) \leq K$. The non-negativity constraints and $p_H \leq v_H$ are always satisfied in this optimum.

Case 2 : $\lambda_1 = 0$ and $\lambda_2 > 0$

From the FOCs: $\lambda_2 = \frac{2p_L - v_L}{q_L} = \frac{2p_H - v_H}{q_H}$ and $\lambda_2 > 0$ implies that Q binds: $\alpha(v_H - p_H)q_H + (1-\alpha)(v_L - p_L)q_L = Q$. The optimal prices can be calculated from these 2 equations. The capacity constraint K must be slack, replacing the optimal prices into the constraint leads to $f(Q) \leq K$. Moreover, $p_H \leq v_H$ implies $\frac{1-\alpha}{2}q_L\left(v_L - v_H\frac{q_L}{q_H}\right) \leq Q$ and finally, $\lambda_2 > 0$ implies $Q < E(vq)/2$.

Case 3 : $\lambda_1 > 0$ and $\lambda_2 > 0$

Both K and Q bind, hence the optimal values of the prices follow directly from the two equations. The four borders of the KQ regions can be calculated as follows. The values of the two multipliers can be calculated from the FOCs by replacing the optimal prices:

$$\lambda_2 = \frac{v_H - v_L}{q_H - q_L} + 2\frac{Kq_H - Q}{(1-\alpha)(q_H - q_L)^2} - 2\frac{Q - Kq_L}{\alpha(q_H - q_L)^2} \quad \text{and}$$

$$\lambda_1 = v_H - 2\frac{Q - Kq_L}{\alpha(q_H - q_L)} - q_H\left(\frac{v_H - v_L}{q_H - q_L} + 2\frac{Kq_H - Q}{(1-\alpha)(q_H - q_L)^2} - 2\frac{Q - Kq_L}{\alpha(q_H - q_L)^2}\right).$$

It follows that $\lambda_1 > 0$ is equivalent to $K < f(Q)$ and $\lambda_2 > 0$ is equivalent to $K > g(Q)$. Furthermore, $p_L \leq v_L$ implies $K \leq Q/q_L$ and $p_H \leq v_H$ implies $K \geq Q/q_H$.

Excluding one consumer group

In the parameter regions delimited by one of the Cases 1 - 4, the optimal prices are such that some consumers of both groups are served. There are two remaining regions where one group of consumers will be excluded. If $K \leq \min\left(Q/q_H, \frac{\alpha}{2}(v_H - v_L)\right)$ then K must bind, so one must compare the profit of $v_H K - \frac{K^2}{\alpha}$ when excluding low-types with the profit of $v_L K - \frac{K^2}{1-\alpha}$ obtained by excluding the high-types. $K \leq \frac{\alpha}{2}(v_H - v_L)$ implies that the first profit, i.e., excluding the low-types is always more profitable for small values of K. An analogous argument shows why excluding high-types is more profitable than excluding low-types when $Q \leq \min\left(Kq_L, \frac{1-\alpha}{2} q_L \left(v_L - v_H \frac{q_L}{q_H}\right)\right)$. ■

Proof of Proposition 2.2

Replacing the optimal prices p_L and p_H of region KQ into the general formula, the consumer surplus equals

$$
\begin{aligned}
CS =& \frac{\alpha}{2}\left(\frac{Q - Kq_L}{\alpha(q_H - q_L)}\right)^2 + \frac{1-\alpha}{2}\left(\frac{Kq_H - Q}{(1-\alpha)(q_H - q_L)}\right)^2 = \\
=& \frac{1}{2\alpha(1-\alpha)(q_H - q_L)^2}\left(Q^2 - 2E(q)KQ + E(q^2)K^2\right).
\end{aligned}
$$

Therefore the first derivative of the consumer surplus with respect to K and Q are

$$
\begin{aligned}
\frac{\partial CS}{\partial K} &= \frac{1}{\alpha(1-\alpha)(q_H - q_L)^2}\left(-E(q)Q + E(q^2)K\right) \quad \text{and} \\
\frac{\partial CS}{\partial Q} &= \frac{1}{\alpha(1-\alpha)(q_H - q_L)^2}\left(-E(q)K + Q\right).
\end{aligned}
$$

Thus consumer surplus is decreasing in K whenever $K \leq E(q)/E(q^2)Q$ and it is decreasing in Q if $K \geq Q/E(q)$. The two regions delimited by these lines are disjoint since they both cross the origin and the slope of $K = E(q)/E(q^2)Q$ is smaller than the slope of $K = Q/E(q)$ as $(E(q))^2 < E(q^2)$. ■

Proof of Proposition 2.3

Total welfare is given by the sum of consumer surplus and profit:

$$TW = CS + \pi_{KQ} = \frac{1}{(q_H - q_L)}\left((v_H - v_L)Q + (v_L q_H - v_H q_L)K\right) - \\ - \frac{1}{2\alpha(1-\alpha)(q_H - q_L)^2}\left(Q^2 - 2E(q)KQ + E(q^2)K^2\right).$$

Notice that wherever consumer surplus is decreasing in either K or Q, the third term of the above equation is increasing, so do the first two terms which means that total welfare is always an increasing function of both capacity levels. ■

Welfare maximizing allocation

Deriving the welfare maximizing allocation for every capacity-pair consists of solving (P-TW). Let λ_1 be the shadow price of the costumer constraint K and λ_2 the shadow price of the production constraint Q. Then the Karush-Kuhn-Tucker conditions imply

$$2p_L = \lambda_1 + \lambda_2 q_L \quad \text{and} \quad 2p_H = \lambda_1 + \lambda_2 q_H.$$

When $\lambda_1 > 0$ and $\lambda_2 = 0$, i.e., K binds and Q is slack, the welfare maximizing prices are equal on the two markets and equal $E(v) - K$. Straightforward calculations show that this common price is higher than the monopoly's corresponding $p_L^K = v_L + \alpha\frac{v_H - v_L}{2} - K$ and lower than $p_H^K = v_H - (1-\alpha)\frac{v_H - v_L}{2} - K$. This proves that the monopoly serves more low-types and less high-types than in the social optimum. Moreover,

$$E(v) - K < v_L \quad \Leftrightarrow \quad K > \alpha(v_H - v_L)$$

delimits the region where K binding is socially optimal. For lower levels of K, excluding low-types is the social optimum. Hence for $\alpha(v_H - v_L)/2 < K < \alpha(v_H - v_L)$ the monopoly serves some low-types while excluding them would be socially optimal. By symmetry, the converse holds for $\lambda_1 = 0$ and $\lambda_2 > 0$, the per-unit prices are equal in the social optimum. Finally, in case $\lambda_1 > 0$ and $\lambda_2 > 0$, both constraints bind in the social optimum, therefore the prices coincide with the monopoly's optimum. Thus

$$\lambda_2 = 2\frac{v_H - v_L}{q_H - q_L} + 2\frac{KE(q) - Q}{\alpha(1-\alpha)(q_H - q_L)^2} \quad \text{and}$$

$$\lambda_1 = 2v_H - 2\frac{Q - Kq_L}{\alpha(q_H - q_L)} - q_H\lambda_2.$$

In the parameter region where both constraints binding constitutes the welfare maximum, $\lambda_1 > 0$ and $\lambda_2 > 0$ must hold, which are equivalent to

$$K < f(Q) + \frac{1}{2E(q^2)}\alpha(1-\alpha)(v_L q_H - v_H q_L)(q_H - q_L) \quad \text{and} \quad K > g(Q) - \alpha(1-\alpha)(q_H - q_L)\frac{v_H - v_L}{2E(q)},$$

respectively. Similarly to the monopoly case, the positivity of prices is ensured by $Q/q_H < K < Q/q_L$. The second terms on the right-hand side of the above inequalities are positive which ensures that this region encompasses the KQ region of the monopoly, proving the last statement of Section 2.4.

Proof of Proposition 2.4

Given Lemma 2.1, optimal capacities are chosen from the KQ region which the following maximization problem:

$$\max_{K,Q} \quad \pi = \frac{(v_H - v_L)Q}{q_H - q_L} + \frac{(v_L q_H - v_H q_L)K}{q_H - q_L} - \frac{Q^2 + E(q^2)K^2 - 2KQE(q)}{\alpha(1-\alpha)(q_H - q_L)^2} - cK - dQ \qquad \text{s.t.}$$

$$\begin{aligned} Q/q_H &\le K && (\lambda_1) \\ K &\le Q/q_L && (\lambda_2) \\ g(Q) \le &K \le h(Q) \\ K \ge 0\,, &\quad Q \ge 0 \end{aligned}$$

Firstly, consider the interior solution where $\lambda_1 = \lambda_2 = 0$.From $\frac{\partial \pi_{KQ}}{\partial K} = c$ and $\frac{\partial \pi_{KQ}}{\partial Q} = d$ one immediately gets the optimal capacity levels described in part 1 of Proposition 2.4. Replacing these optimal values into conditions $Q/q_H \le K$ and $K \le Q/q_L$ imply $v_H > c + dq_H$ and $v_L > c + dq_L$, respectively. The remaining four feasibility conditions are satisfied at this solution.

Secondly, consider $\lambda_1 > 0$ and $\lambda_2 = 0$ that corresponds to the exclusion of low-types. $K = Q/q_H$ and the two first order conditions provide the optimal capacity levels described in part 2 of Proposition 2.4. Positivity of λ_1 requires $v_L \le c + dq_L$, and the positivity of Q implies $v_H > c + dq_H$.The remaining four primal feasibility conditions are satisfied.

Thirdly, consider $\lambda_2 > 0$ and $\lambda_1 = 0$ that corresponds to the exclusion of high-types. $K = Q/q_L$ and the two first order conditions provide the optimal capacity levels described in part 3 of Proposition 2.4. Positivity of λ_2 requires $v_H \leq c + dq_H$, in addition, the positivity of K implies $v_L > c + dq_L$. The remaining four primal feasibility conditions are satisfied.

Both capacity levels are zero if the marginal cost of serving each consumer group is prohibitively high, which corresponds to part 4 of Proposition 2.4.

Finally, replacing the optimal capacity levels into the optimal prices in region KQ, one immediately gets that $p_i = (v_i + c + dq_i)/2, \quad i \in \{L, H\}$. ■

Proof of Proposition 2.5

Showing that the outcome of (P-ICL) coincides with the outcome of (P-LH) consists of showing that in each local optimum, $p_L \leq p_H$ is satisfied. Optimal prices in regions K, KQ and U trivially satisfy this condition. In region Q, it is equivalent to

$$v_L - \frac{Kq_H - Q}{(1-\alpha)(q_H - q_L)} \leq v_H - \frac{Q - Kq_L}{\alpha(q_H - q_L)} \Leftrightarrow K \geq \frac{1}{E(q)}\left(Q - \alpha(1-\alpha)(q_H - q_L)(v_H - v_L)\right),$$

which by the definition of $g(Q)$ is equivalent to

$$K \geq g(Q) - \frac{1}{E(q)}\alpha(1-\alpha)(q_H - q_L)\frac{v_H - v_L}{2}.$$

This condition is always satisfied in region Q, since in this region the stronger condition of $K \geq g(Q)$ is also satisfied.

Next, I show that the solutions of (P-ICL2) are always dominated by some solution of (P-ICL). Notice that $p_H = E(v)/2$ is only attainable if $K > E(v)/2$ and $Q > E(v)q_H/2$ meaning that the capacities are from the U region, where the solution of (P-ICL2) is clearly dominated by π^U. The two conditions can be rewritten as

$$E(v) - \min(K, Q/q_H) \leq p_H < v_L,$$

which immediately implies that a necessary condition for existence of a solution is

$\min(K, Q/q_H) > \alpha(v_h - v_L)$. This in turn implies that the solution cannot be in the EL region. By concavity of the objective function, whenever solution exists, it is given by $p_H = E(v) - \min(K, Q/q_H)$ thus the profit equals $E(v)K - K^2$. For $K < Q/q_H$ the necessary condition implies that capacity levels fall in the K region, where π_K is attainable in (P-ICL). By definition,

$$\pi_K = \alpha(1-\alpha)\left(\frac{v_H - v_L}{2}\right)^2 + KE(v) - K^2 > KE(v) - K^2,$$

so if $K < Q/q_H$, the optimal solution is dominated. Next, I show that the same is true for $K \geq Q/q_H$. In this case, for all K we have

$$KE(v) - K^2 > E(v)Q/q_H - (Q/q_H)^2,$$

where the right-hand side is the optimal profit of (P-ICL2). The left-hand side is smaller than π_K, as shown above, therefore it is also dominated by the optimal profits attainable in regions KQ and Q in the (P-ICL) problem.■

Proof of Lemma 2.2

(i) Myerson's regularity condition, i.e., $p - \frac{1-F_i(p)}{f_i(p)}$, $i \in \{L, H\}$ being strictly increasing in p can be rewritten in terms of the demand function as

$$D_i''(p) < \frac{2(D_i'(p))^2}{D_i(p)}.$$

Firstly, consider the slope of curve $h(Q)$, defined as the part of the KQ' region where $\lambda_2 = 0$. Substituting $\lambda_2 = 0$ in the first order conditions, one gets

$$p_H + \frac{D_H(p_H)}{D_H'(p_H)} = p_L + \frac{D_L(p_L)}{D_L'(p_L)} \quad \Longleftrightarrow \quad \xi(K, Q) \equiv D_L D_H' - D_H D_L' + (p_L - p_H) D_L' D_H' = 0$$

The implicit function theorem ensures that $h'(Q) = \frac{\partial K}{\partial Q} = -\frac{\partial \xi/\partial Q}{\partial \xi/\partial K}$. In the following I show that this derivative is positive if the regularity condition is satisfied. Let $\overline{x} = \frac{\partial x}{\partial Q}$, i.e., for any function x, $\overline{x}$ denotes its partial derivative with respect to Q. We know that $D_L(p_L) = \frac{Kq_H - Q}{(1-\alpha)(q_H - q_L)}$ and $D_H(p_H) = \frac{Q - Kq_L}{\alpha(q_H - q_L)}$, which implies

$$\overline{D_L} = \frac{-1}{(1-\alpha)(q_H - q_L)} \quad \text{and} \quad \overline{D_H} = \frac{1}{\alpha(q_H - q_L)} \quad \text{and} \quad \overline{p_i} = \frac{\overline{D_i}}{D_i'} \quad \text{and} \quad \overline{D_i'} = D_i'' \frac{\overline{D_i}}{D_i'}$$

for $i \in \{L, H\}$. We have

$$\overline{\xi} = \frac{\partial \xi}{\partial Q} = \frac{\partial D_L}{\partial Q} \left(2D_H' + ((p_L - p_H)D_H' - D_H) \frac{D_L''}{D_L'} \right) - \frac{\partial D_H}{\partial Q} \left(2D_L' - ((p_L - p_H)D_L' + D_L) \frac{D_H''}{D_H'} \right)$$

Similarly,

$$\frac{\partial \xi}{\partial K} = \frac{\partial D_L}{\partial K} \left(2D_H' + ((p_L - p_H)D_H' - D_H) \frac{D_L''}{D_L'} \right) - \frac{\partial D_H}{\partial K} \left(2D_L' - ((p_L - p_H)D_L' + D_L) \frac{D_H''}{D_H'} \right).$$

From the implicit function theorem:

$$\frac{\partial K}{\partial Q} = -\frac{\frac{-A}{(1-\alpha)(q_H - q_L)} - \frac{B}{\alpha(q_H - q_L)}}{\frac{A q_H}{(1-\alpha)(q_H - q_L)} + \frac{B q_L}{\alpha(q_H - q_L)}} = \frac{\alpha A + (1-\alpha)B}{\alpha A q_H + (1-\alpha) B q_L}$$

where

$$A = 2D_H' + ((p_L - p_H)D_H' - D_H) \frac{D_L''}{D_L'} \quad \text{and} \quad B = 2D_L' - ((p_L - p_H)D_L' + D_L) \frac{D_H''}{D_H'}$$

A sufficient condition for $\frac{\partial K}{\partial Q} > 0$ is that both A and B be negative. Next, I show that the regularity condition for F_L and F_H are equivalent to $A < 0$ and $B < 0$, respectively.

$$A < 0 \quad \Longleftrightarrow \quad D_L''(D_H - (p_L - p_H)D_H') < 2D_L'D_H' \quad \Longleftrightarrow \quad D_L'' < \frac{2D_L'D_H'}{D_H + p_H D_H' - p_L D_H'}$$

The first inequality comes from $D_L' < 0$. The second inequality's direction follows from the positive slope of the profit curve that guarantees $D_H + p_H D_H' > 0$ which in turn implies that $D_H + p_H D_H' - p_L D_H' > 0$. Simplifying the ratio by D_H' leads to

$$D_L'' < \frac{2D_L'}{D_H/D_H' + p_H - p_L} \iff D_L'' < \frac{2D_L'}{D_L/D_L' + p_L - p_L} \iff D_L'' < \frac{2(D_L')^2}{D_L}$$

where the first inequality comes from the first order condition of $p_H + \frac{D_H}{D_H'} = p_L + \frac{D_L}{D_L'}$. The last inequality corresponds exactly to the regularity of F_L. Similar arguments prove that $B < 0$ is equivalent to the regularity of F_H, which concludes the proof of $h'(Q) > 0$.

Analogous arguments can be made with obvious modifications that show that g is also increasing under the regularity assumption. ■

(ii) By definition K is slack for $K > h(Q)$ and Q is slack for $K < g(Q)$. By contradiction, assume that $h < g$ at some point $\widehat{Q} < \overline{Q}$. Then by continuity there exists a neighborhood of $\widehat{Q}$ where $h < g$. For any Q in this interval and for any $K < h(Q)$ we have none of the constraints binding. However, this is impossible, as the unconstrained optimum can only be achieved for $(Q, K) \geq (\overline{Q}, \overline{K})$. Moreover, $g(\overline{Q}) = h(\overline{Q}) = \overline{K}$ follows directly from the definition of $\overline{Q}$ and $\overline{K}$. ■

(iii) By contradiction, assume that $\forall\ Q > 0:\ g(Q) > Q/q_H$. This means that $\forall\ Q > 0:$ $\exists\ \epsilon(Q) > 0$ such that $g(Q) > Q/q_H + \epsilon(Q)$. By definition of g, the constraint on Q is slack for all capacity-pairs $(Q, Q/q_H + \epsilon(Q)$. The constraint on Q being slack translates to

$$\alpha q_H D_H(p_H) + (1-\alpha) q_L D_L(p_L) < Q$$

and the constraint on K at point $(Q, Q/q_H + \epsilon(Q))$ writes as

$$\alpha D_H(p_H) + (1-\alpha) D_L(p_L) \leq K = Q/q_H + \epsilon(Q)$$

Taking the limit of $Q \to 0$, the first inequality implies D_L and D_H must also tend to 0. Thus the second constraint must also be slack as the left hand side tends to zero while the right hand side equals $\epsilon(Q)$ which is always strictly positive. However, this is a contradiction as both constraints cannot be slack unless they are larger then $(\overline{Q}, \overline{K})$. (iv) can be proven with arguments analogous to (iii). ■

Proof of Proposition 2.6

It is sufficient to show that the optimal solutions of (P-LH) satisfy the additional constraint of $p_H/q_H \leq p_L/q_L$. In the K region, this condition writes as

$$\frac{1}{q_H}\left(v_H-(1-\alpha)\frac{v_H-v_L}{2}-K\right)\leq\frac{1}{q_L}\left(v_L+\alpha\frac{v_H-v_L}{2}-K\right),$$

which can be rewritten as

$$K<\frac{v_Lq_H-v_Hq_L}{q_H-q_L}+\frac{E(q)(v_H-v_L)}{2(q_H-q_L)}.$$

Elementary algebra shows that the expression on the right-hand side is greater than $E(v)/2$, so the condition is always satisfied in the K region. In the KQ region, we have

$$p_H/q_H\leq p_L/q_L \;\Leftrightarrow\; K<f(Q)+\frac{1}{2E(q^2)}\alpha(1-\alpha)(v_Lq_H-v_Hq_L)(q_H-q_L),$$

which is always satisfied as the KQ region is delimited by $K\leq f(Q)$ and the right-hand side is greater than $f(Q)$. It is straightforward to see that the constraint is also satisfied in regions Q and U. ■

Proof of Proposition 2.7

As the consumer surplus is additive in the consumer surplus of the consumer groups, we have

$$\frac{\partial CS}{\partial K}=\frac{\partial p_H}{\partial K}\frac{\partial}{\partial p_H}\int_{p_H}^{\theta_H}\alpha(w-p_H)f_H(w)dw+\frac{\partial p_L}{\partial K}\frac{\partial}{\partial p_L}\int_{p_L}^{\theta_L}(1-\alpha)(w-p_L)f_L(w)dw.$$

Using the Leibniz-rule it follows that

$$\frac{\partial CS}{\partial K}=\frac{q_H}{(1-\alpha)(q_H-q_L)D_L'}(-(1-\alpha)D_L)+\frac{-q_L}{\alpha(q_H-q_L)D_H'}(-\alpha D_H)$$

which implies that

$$\frac{\partial CS}{\partial K}<0 \quad\Longleftrightarrow\quad q_L\frac{D_L'(p_L)}{D_L(p_L)}<q_H\frac{D_H'(p_H)}{D_H(p_H)}.$$

Analogous steps prove the second statement. ■

The take-out restaurant model

I show that the proof of Lemma 2.2 can be modified to fit the take-out model in order to show existence of a core region. The core region in the take-out model is delimited by the lines $K = 0$ and $K = Q$, ensuring that some consumers of both types are served, moreover, by curves $K = g(Q)$ and $K = h(Q)$ ensuring that the 2 constraints are binding. To derive properties of g and h, the steps of Lemma 2.2 can be reproduced with the necessary modifications.

To prove part (i) of the lemma, notice the main difference with respect to the main model is that $D_H(p_H) = K/\alpha$ is independent of Q. Thus the second term in the formula of $\frac{\partial \xi}{\partial Q}$ is zero. Everything else in the proof of part (i) holds in the take-out model as well, thus both the nominator and the denominator of the formula for $h'(Q)$ are still negative, proving part (i) of the lemma.

Part (ii) applies without modifications. Part (iii) of Lemma 2.2 rewrites in the take-out model as

$$\exists\, \widetilde{Q} > 0 \text{ such that } g(Q) \leq 0 \text{ for } \forall\, 0 \leq Q < \widetilde{Q}.$$

To see that this statement also holds, replace the term Q/q_H with 0 in the original proof. Then the constraint on Q being slack translates to

$$\alpha D_H(p_H) + (1-\alpha) D_L(p_L) < Q$$

and the constraint on K at point $(Q, \epsilon(Q))$ writes as

$$\alpha D_H(p_H) \leq K = \epsilon(Q)$$

Taking the limit of $Q \to 0$, the first inequality implies that D_H must also tend to 0. Thus the second constraint must also be slack as the left hand side tends to zero while the right hand side equals $\epsilon(Q)$ which is always strictly positive. However, this is a contradiction as both constraints cannot be slack unless they are larger then $(\overline{Q}, \overline{K})$. Part (iv) can be proven analogously. Similarly to the main model, Lemma 2.2 directly implies the existence of a core region in the take-out model. ■

Chapter 3

Competition with Dual Capacity Constraints

Abstract: This paper studies duopoly pricing under dual capacity constraints, limiting both the total quantity and the number of consumers served. By isolating parameter regions where a symmetric pure-strategy equilibrium exists, I find that several types of equilibria are possible, depending on the model's specifications. For some of them, duopoly prices are identical to monopoly prices. Equilibrium prices are non-monotonic in capacity levels if consumers' valuations are sufficiently heterogeneous. Moreover, I show that despite their ability to price discriminate, competition may lead firms to charge identical prices across markets.

3.1 Introduction

The objective of this paper is to extend the theory of monopoly pricing in the presence of dual capacity constraints developed in Chapter 2 to a competitive setting. Numerous real-world markets are characterized by multiple capacity constraints, especially in the short run; many firms are limited both by the number of consumers they can serve and the number of products they can supply. The literature on capacity-constrained pricing has focused so far on oligopolistic competition as the case of a monopoly constrained by a single capacity is a textbook exercise.[1] The study of a monopoly bound by dual capacities reveals some qualitatively novel, surprising results; the present paper aims to extend that analysis for the case of a duopoly, which seems even more relevant in many of the real-world examples.

Firms characterized by dual capacity constraints that may also be competitive abound in the hospitality industry and the health care industry. Restaurants cannot serve more consumers than the number of seats they have, and they are also bound by the size of their kitchen. Consumers are heterogeneous in the ratio of capacity they use: they all need one seat, whereas large eaters use up more kitchen time than small eaters. Hospitals are bound by the number of surgeons on the one hand and the number of beds in the intensive care unit on the other hand. Clearly, the ratio of time patients spend in surgery over the time of recovery is heterogeneous. More generally, most firms bound by a physical capacity constraint are also bound by the availability of their workforce, which can be thought of as a second capacity constraint.

Many questions arise in a competitive setting that the analysis of the monopoly case cannot answer. Firstly, are the surprising comparative statics results of the monopoly case also present in the duopoly setting? In particular, are prices and consumer surplus monotonic in the level of capacities? Secondly, how does dividing a monopoly into two identical and competing firms affect prices and welfare? Thirdly, by investigating a duopoly where both firms are bound by two capacities of the same size, this paper generalizes the model of Levitan and Shubik (1972, henceforth LS). Given this connection, a natural question is how the presence of a second capacity constraint changes the results of LS.

To answer these questions, I model a symmetric, price-setting duopoly where both firms face two exogenous capacities of the same size: one on the mass of consumers they can serve, the other on the amount of goods they can produce. Consumers are heterogeneous in the number of units they want to buy: high-type consumers demand k units, whereas low-type consumers

[1] See Besanko and Braeutigam, 2010, p. 507, Exercise 12.6

demand 1 unit of the good. The two consumer groups also differ in their total willingness-to-pay (WTP) for the bundle they demand: their distributions have different supports. Specifically, I assume that the *total WTP* of the most valuable high-types always exceeds the total WTP of all low-type consumers, whereas the *per-unit WTP* of some low-types exceeds the per-unit WTP of all high-types. Finally, the model must specify which consumers get served in case of rationing. For this purpose I define the high-type preferring rationing rule, a generalization of the standard efficient rationing rule to allow for 2 consumer groups and 2 capacity constraints.

Given that this model compounds the complexities of Bertrand-Edgeworth oligopoly games with the co-existence of two capacity constraints, I restrict the analysis to isolating symmetric pure-strategy equilibria. In particular, I identify and study six plausible equilibria that are qualitatively different from one another and never co-exist. I find two exclusionary equilibria consisting of firms serving exclusively consumers of one group if one of the capacity levels is very low and the other capacity level is relatively high. I find and characterize prices and profits in two equilibria where exactly one constraint binds. I show that prices on both markets are equal in the equilibrium where the capacity on the number of people served binds. Analogously, *unit prices* charged for the two consumer groups are equal in the equilibrium where the capacity on the number of products binds.

I restrict the analysis of the equilibrium where both constraints bind to the study of numerical examples due to the complexities of the model. The examples show existence of such an equilibrium for certain parameter values. I analytically show that similarly to the monopoly case, prices are increasing in one capacity level while decreasing in the other. Moreover, the numerical examples suggest the existence of a region of capacity-pairs where increasing the capacity on the number of products decreases aggregate consumer surplus. Therefore, whenever such an equilibrium exists, both prices are non-monotonic in the level of one of the capacities. Furthermore, aggregate consumer surplus is also non-monotonic for certain parameter values. Finally, the last type of equilibrium I identify arises for very large capacities. I show that the Bertrand-outcome of marginal cost pricing prevails, and it is the unique pure-strategy equilibrium for very large capacities.

Moreover, I compare the results of the model to two monopoly benchmarks. The first benchmark is a small monopoly that is identical to the firms in the duopoly. I show that in all five non-trivial duopoly equilibria I identify, prices are lower, aggregate consumer surplus is higher and industry profit is lower than in the corresponding small monopoly. The second benchmark is a larger monopoly whose capacities are equal to the sum of the capacities of the two firms in the duopoly. In most cases, prices are lower, aggregate consumer surplus is higher and industry profit is lower in the duopoly than their monopoly counterparts. However, I also show that there exist duopoly equilibria where for certain capacity-pairs the equilibrium prices are equal to the corresponding large monopoly's optimal prices. Thus dividing monopolies into two, identical and competing firms may have no effect on prices, consumer surplus and industry profit.

Finally, I also compare the results to the symmetric single capacity model of Levitan and Shubik (1972) and its extension for two consumer groups. The comparison reveals that the non-monotonicity results are a direct consequence of the co-existence of the two capacities and not merely an artifact of assuming two consumer groups.

3.1.1 Related literature

This paper builds on the model of a monopoly constrained by dual capacities, as analyzed in Chapter 2, extending it to allow competition. The main findings of the monopoly case are the non-monotonicity of optimal prices and aggregate consumer surplus in capacity levels. In the present paper I show that these findings can be generalized to a symmetric duopoly under some conditions. Section 5.1 provides a detailed comparison of the results of the duopoly to two monopoly benchmarks.

Most closely related to this paper in the literature of Bertrand-Edgeworth competition is the classic article of Levitan and Shubik (1972). They study a symmetric duopoly where both firms are constrained by the same capacity constraint. Moreover, they introduce the efficient rationing rule, which later becomes widely used in the literature (e.g. Kreps and Sheinkman [1983], Reynolds and Wilson [2000], Boccard and Wauthy [2010], Lepore [2012] etc.). This paper generalizes the efficient rationing rule for the case of 2 consumer groups and 2 capacity constraints. Section 5.4 provides a detailed comparison of my results to Levitan and Shubik (1972).

Finally, it is worth noting that applied operations research models of the health care industry routinely take into account multiple capacity constraints. Indeed, multiple capacities seems to play a central role in patient admission planning and patient mix optimization. Banditori et al. (2013) provide a recent survey of this literature. In the revenue management literature, Xiao and Yang (2010) study a model with two capacity constraints. The main difference from the present model is that they investigate a dynamic pricing problem. Their main finding is that under some conditions the existence of a second capacity qualitatively changes the optimal pricing policy of a firm.

The rest of the chapter is organized as follows. Section 3.2 outlines the model. Section 3.3 describes the five equilibria where at most one capacity constraint binds. Section 3.4 discusses the equilibrium where both capacity constraints bind. Section 3.5 provides a general discussion of the results, first by studying comparative statics results then by comparing the model's results to 3 different benchmarks. Section 3.6 concludes. All omitted proofs are relegated to the Appendix of Chapter 3.

3.2 The model

Consider a market consisting of two consumer groups served by two price-setting firms. Each consumer is characterized by its demand and its total willingness-to-pay (WTP) for the product. Low-types want to consume one unit of the product while high-types want to consume a fix amount of k products, i.e., individual demand is price-inelastic.

A consumer of type $i \in \{L, H\}$ with total WTP w has a net consumer surplus of $w - p_i$ if he buys its desired quantity of the product for price p_i, and 0 otherwise. Total WTP of consumers of type i is uniformly distributed on the interval $[0, v_i]$. Consumers maximize their net surplus and they demand the good if and only if their net surplus is positive. Assume that the total mass of high-type and low-type consumers is αv_H and $(1-\alpha)v_L$, respectively.[2] For simplicity, I assume $\alpha \geq 0.5$ throughout the paper. The firms can observe the consumers' individual demand, and the distribution of their WTP but not their individual values.

Furthermore, I assume that the WTP of consumers satisfy the following conditions:

$$0 < v_L < v_H \quad \text{and} \quad v_L > v_H/k$$

These assumptions guarantee that some high-type consumers' valuation always exceeds all the low-types' valuation, whereas the per-unit-WTP of some low-type consumers is greater then the per-unit-WTP of all high-type consumers. These conditions restrict the analysis to the most interesting case, firms would otherwise find consumers of one group strictly more valuable, irrespective of their capacity constraints. In addition, the second condition corresponds to decreasing marginal value of consumption in the present setting. Finally, notice that high-types are not necessarily more valuable for the firms, the terminology simply refers to the high level of their demand.

[2]This normalization of the mass of consumers is made to simplify the exposition of results. In particular, it shortens significantly the formulas obtained for equilibrium prices and profits without altering the qualitative properties of the model.

I analyze a symmetric duopoly situation where both firms face the same capacity constraints. They are constrained by the number of people they can serve, $K_1 = K_2 = K$ and they are also bound by the maximal amount of goods they can serve: $Q_1 = Q_2 = Q$.

Both capacity constraints are exogenously given. For simplicity, marginal cost of production is constant and normalized to 0 up to capacity then production becomes impossible.

None of the firms finds it optimal to use a pricing structure consisting of more than 2 price-quantity bundles. Given the price-inelastic individual demands, no consumer would buy any bundle that offers them a quantity different from their desired demand. Moreover, if the same firm were to offer several bundles with the same quantity for a different price, consumers would only buy the cheapest one. Let p_j^H and p_j^L denote Firm j's price of the bundle with k and 1 units, respectively.

3.2.1 Rationing rule

The model must also specify how residual demand is calculated when demand is rationed. Assume that for some exogenous reason (e.g., a social norm) high-type consumers are always served first. I define the *high-type preferring efficient rationing rule* as follows:

- After observing the prices, each high-type consumer goes first to the firm that charges a lower price p_H.
- The cheaper firm serves the resulting demand from high-types if its capacities allow it to do so. In this case the more expensive firm cannot serve any high-type consumers. If the resulting total demand from high-types exceeds any of its capacities, the cheaper firm serves exclusively high-types. The high-type consumers served are the ones with the largest WTP for the product (*efficient rationing among high-types*). The more expensive firm faces the resulting residual demand, the unserved high-types willing to buy at the higher price, if any.
- In case the firms charge the same price p_H, they share the demand at that price equally.
- After high-type consumers are served, all low-type consumers go to the firm that charges a lower price p_L. It serves all the demand from low-types if it has enough remaining capacity to do so, otherwise it fills one of its capacities. In this case, the low-type consumers served are the ones with the largest WTP for the product (*efficient rationing among low-types*). The more expensive firm serves the residual demand, if positive, and if its capacities

allow it to do so. In case both firms charge the same price for low-types, their demand is shared equally between the two firms, if one is capacity-constrained, the other also gets the residual demand.

This rationing rule generalizes the efficient rationing rule, standard in the literature of Bertrand-Edgeworth competition for the presence of 2 consumer groups. In particular, rationing is efficient *within* consumer groups. The sharing rule specifying an equal split of demand in case of identical prices is also standard.

Throughout the paper, I look for capacity-pairs for which symmetric pure-strategy Nash equilibria exist, i.e., there exists a price-pair (p_L, p_H) which is the best response for the other firm charging the same price-pair.

3.3 Symmetric pure-strategy equilibria with at most one binding capacity

In this section, I study 5 types of equilibria in which at most one of the two capacity constraints binds. I start the analysis by establishing that marginal cost pricing is the unique pure-strategy equilibrium if both capacities are very large. Next, I isolate the 2 cases where the firms serve exclusively low-type or high-type consumers and completely exclude the consumers of the other group. Finally, I identify 2 equilibria where some consumers of both groups are served, exactly one capacity binds while the other is slack.

Lemma 3.1 establishes that for very large capacity constraints, the standard undercutting argument makes the Bertrand-outcome of 0 prices the unique pure-strategy equilibrium.

Lemma 3.1.

For $K \geq \alpha v_H + (1-\alpha)v_L$ and $Q \geq \alpha v_H k + (1-\alpha)v_L$, the unique pure-strategy equilibrium is $p_L = p_H = 0$.

Indeed, if firms are not capacity constrained, for any positive price, the rival can profitably deviate by slightly undercutting that price, which results in a discontinuous increase in its sales.

3.3.1 Excluding low-types

In this subsection I look for a symmetric pure-strategy equilibrium where only high-types are served. The firms can exclude low-types by charging any price $p_L \geq v_L$. The next Lemma identifies the parameter region where such an equilibrium can arise.

Lemma 3.2. *A symmetric pure-strategy equilibrium where only high-types are served exists if and only if $K \leq \min\left(\frac{Q}{k},\ \frac{\alpha}{3}(v_H - v_L)\right)$. In this equilibrium the prices are $p_H^{EL} = v_H - \frac{2K}{\alpha}$ and $p_L \geq v_L$. Both firms' demand from high-types equals their capacity K and their profit is $Kv_H - \frac{2K^2}{\alpha}$.*

Lemma 3.2 shows that in the equilibrium with low-types excluded the total demand from high-types must equal the total industry capacity, $2K$. Intuitively, lower prices would create a total demand larger than the industry capacity which would in turn create an incentive to slightly increase prices as the deviating firm could still sell up to capacity. Conversely, if prices were higher than p_H^{EL} then both firms would sell below capacity. This would create an incentive for firms to slightly undercut their rival, thus increasing their sales discontinuously.

Intuitively, the existence of such an exclusionary equilibrium necessitates a low level of K, otherwise the firms would find it profitable to allocate some of the capacity for serving low-types consumers as well. The relatively high level of Q ensures that the capacity on the number of products sold never binds.

3.3.2 Excluding high-types

In this subsection I look for a symmetric equilibrium where only low-types are served. The firms can exclude low-types by charging any price $p_H \geq v_H$.

Lemma 3.3. *A symmetric pure-strategy equilibrium where only low-types are served exists if and only if $Q \leq \min\left(K,\ \frac{1-a}{3}(v_L k - v_H)\right)$. Then prices are $p_L^{EH} = v_L - \frac{2Q}{(1-\alpha)}$ and $p_H \geq v_H$, both firms' demand from low-types equals their capacity Q and their profit equals $v_L Q - \frac{2Q^2}{(1-\alpha)}$.*

Lemma 3.3 shows that in the equilibrium with high-types excluded, analogously to the case of excluding low-types, total demand from low-types must equal the total industry capacity, $2Q$. Lower prices would create a total demand larger than the industry capacity which would create an incentive to increase prices. Conversely, if prices were higher than p_L^{EH} then both firms would sell below capacity which would create an incentive for firms to slightly undercut their rival, increasing their sales discontinuously.

Intuitively, the existence of such an exclusionary equilibrium necessitates a low level of Q otherwise the firms would find it profitable to allocate some of the capacity for serving high-types consumers. The relatively high level of K guarantees that the capacity on the number of people served never binds.

3.3.3 Serving some consumers of both types

Lemma 3.4 establishes connections between the equilibrium prices charged on the two markets depending on which constraints bind. Firms get an equal share of consumers in any symmetric pure-strategy equilibrium as the two firms charge the same prices on each market. Thus, by symmetry the same constraint must bind for both firms.

Lemma 3.4.

Any symmetric pure-strategy equilibrium with strictly positive prices $(p_L;\ p_H)$ where some consumers of both types are served satisfy at least one of the following condition-pairs:

1. $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) = 2K$ *and* $\alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L) \leq 2Q$

2. $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) \leq 2K$ *and* $\alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L) = 2Q$.

Lemma 3.4 states that there cannot exist an equilibrium in which total market demand exceeds any of the two capacities, thus rationing never occurs in equilibrium. Furthermore, both capacity constraints cannot be slack at the same time. Therefore there are at most 3 types of equilibria in this duopoly characterized by both consumer groups at least partially served at positive prices:

1. Both firms' capacity constraint on the number of people served (K) is exactly covered while the constraint on production (Q) is slack;

2. Both firms' capacity constraint on production (Q) is exactly covered while the constraint on the number of people served (K) is slack;

3. Both capacity constraints of both firms are exactly covered.

The following two subsections deal with the first and second case, respectively, while the third case is investigated in Section 3.4.

3.3.3.1 Symmetric pure-strategy equilibrium with K binding, Q slack

In this subsection I establish a series of lemmas that investigate the equilibrium prices in case capacity K binds and capacity Q is slack.

The following Lemma shows that the price charged for high-types must exceed the price charged for low-types in any equilibrium where the capacity constraint on the number of people served binds.

Lemma 3.5. *$p_L \leq p_H < v_H$ holds in any symmetric pure-strategy equilibrium where some consumers of both types are served and (i) $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) = 2K$ and (ii) $(v_H - p_H)k + (1-\alpha)(v_L - p_L) \leq 2Q$.*

The intuition for this Lemma is the following. Firms sell up to capacity K, some of this capacity is allocated for serving low-types, the remaining for serving high-types. In case of $p_L > p_H$, serving low-types would be strictly more profitable for the firms than selling to high-types. In this case they would benefit from serving more low-types and less high-types, and they have the means to do so. Indeed, slightly overpricing their rival on the high-types market while undercutting their rival on the low-types market is always profitable. Notice that such a deviation cannot result in a binding Q, as it attracts more low-types who use less of the Q capacity.

Stronger results can be obtained about the equilibrium price-pair by restricting one's attention to the parameter region $K < Q/k$. The next Lemma shows that in this region equilibrium prices must be equal.

Lemma 3.6. *$p_L = p_H = \alpha v_H + (1-\alpha)v_L - 2K$ holds in any symmetric pure-strategy equilibrium where some consumers of both types are served and $K < Q/k$.*

Given the condition of the Lemma, $K < Q/k$, capacity Q can never bind, thus according to Lemma 3.4, K must be exactly covered in equilibrium. Hence, Lemma 3.5 applies so $p_L \leq p_H$ must hold. Due to the stronger condition of Q never binding, one can show that p_H cannot be strictly larger than p_L. In that case serving high-types would be strictly more profitable for the firms. Slightly undercutting their rival on the high-market would be a profitable deviation as it could discontinuously increase the mass of high-types served. Notice that Q never binding is necessary for this argument to hold: otherwise the deviating firm might hit its capacity constraint on production by serving more high-types.

Thus Lemma 3.6 shows uniqueness of symmetric pure-strategy equilibrium for large Q, however, one also need to tackle the question of the existence of equilibria. The next Proposition provides necessary and sufficient conditions for the existence of an equilibrium in which both firms charge the same prices $p_L = p_H = \alpha v_H + (1-\alpha)v_L - 2K$.

Proposition 3.1.

A symmetric pure-strategy equilibrium where some consumers of both types are served and firms charge the same prices

$$p_L = p_H = \alpha v_H + (1-\alpha)v_L - 2K$$

on both markets exists if and only if

$$\begin{aligned}\alpha(v_H - v_L) \leq K \leq \min\Bigg(&\max\left(\frac{\alpha(2\alpha-1)v_H + 2\alpha(1-\alpha)v_L}{4\alpha-1}; \frac{\alpha v_H + (1-\alpha)v_L}{3}\right); \\ &\frac{1}{4}(\alpha v_H + (1-\alpha)v_L) + \frac{1}{4}\sqrt{(\alpha v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2}; \quad f(Q); \\ &\frac{\alpha(v_H - v_L)}{2} + \frac{v_L}{3}\Bigg) \end{aligned} \tag{3.1}$$

$$\textit{where} \quad f(Q) = \frac{2Q - \alpha(1-\alpha)(k-1)(v_H - v_L)}{2(\alpha k + 1 - \alpha)}.$$

The profit of each firm equals

$$\big(\alpha v_H + (1-\alpha)v_L\big)K - 2K^2.$$

This equilibrium is unique if $K \leq \frac{Q}{k}$ also holds.

The most remarkable finding in Proposition 3.1 is that equilibrium prices across markets are identical, despite the firms' ability to price discriminate. The intuition is the following. The capacity on the number of consumers served, K, becomes the only scarce resource of the firms for a large enough Q. Therefore, the marginal gain of serving each consumer must be equal, otherwise there is a profitable deviation of serving more valuable consumers. The marginal gain of serving a consumer is exactly the price charged for them, which explains why equilibrium prices must be equal.

Proposition 3.1 establishes that the capacity-pairs leading to the equalization of prices across markets are characterized by a large Q and a K of intermediate size. Intuitively, if K is very low then the firms find it profitable to deviate to only serve high-types. If K is very high then there is a profitable deviation consisting of either only serving low-types or to increase prices charged for low-types while keeping the prices constant for high-types. The capacity constraint on production, Q, must be large otherwise firms would deviate to choosing prices such that it binds.

3.3.3.2 Symmetric pure-strategy equilibrium with Q binding, K slack

In this section I analyze the case of capacity Q binding and capacity K being slack.

Lemma 3.7 is analogous to Lemma 3.5; it shows that whenever Q binds in an equilibrium, the unit price p_L charged for low-types must be larger than p_H/k, the unit price charged for high-types.

Lemma 3.7. *$p_H/k \leq p_L < v_L$ in any symmetric pure-strategy equilibrium where some consumers of both types are served and (i) $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) \leq 2K$ and (ii) $(v_H - p_H)k + (1-\alpha)(v_L - p_L) = 2Q$.*

The intuition for this Lemma is the following. Firms sell up to capacity Q, some of this capacity is allocated for serving low-types, the remaining for serving high-types. In case of $p_H/k > p_L$, serving high-types would be strictly more profitable for the firms than selling to low-types. In this case they would benefit from serving more high-types and less low-types. This could be achieved by a firm slightly undercutting their rival on the high-types market, which would thus always be profitable. Notice that such a deviation cannot result in a binding K, as it attracts more high-types who use relatively less of the K capacity.

Stronger results can be obtained about the equilibrium price-pair if one restricts their attention to the parameter region of $K > Q$, i.e., when the capacity on the number of consumers is very large. The next Lemma shows that in this region, in equilibrium the unit prices must be equal. In this sense, Lemma 3.8 is analogous to Lemma 3.6.

Lemma 3.8. *$p_L = p_H/k = \frac{\alpha k v_H + (1-\alpha)v_L - 2Q}{\alpha k^2 + 1 - \alpha}$ holds in any symmetric pure-strategy equilibrium where some consumers of both types are served and $Q < K$.*

The condition of very large K in Lemma 3.8, $Q < K$, ensures that capacity K can never bind. According to Lemma 3.4, Q must be exactly covered in equilibrium. Hence, Lemma 3.7 applies so $p_H/k \leq p_L$ must hold. Due to the stronger condition of K never binding, one can also show that p_H/k cannot be strictly smaller than p_L. In that case serving low-types would be strictly more profitable for the firms. Slightly undercutting their rival on the low-market while deviating upward on the high-market would be a profitable deviation as it could discontinuously increase the mass of low-types served. Notice that K never binding is necessary for this argument to hold: otherwise the deviating firm might hit its capacity K by serving more low-types.

Proposition 3.2.

Assume $v_L \leq \frac{2\alpha k v_H}{\alpha k^2+1-\alpha}$. Then a symmetric pure-strategy equilibrium where some consumers of both types are served and firms charge the same unit prices

$$p_L = p_H/k = \frac{\alpha k v_H + (1-\alpha)v_L - 2Q}{\alpha k^2 + 1 - \alpha}$$

on both markets exists if and only if $\quad K > g(Q)$ *and*

$$\frac{1-\alpha}{2}(v_L - v_H/k) \leq Q \leq \min\Bigg(\frac{\alpha k v_H(\alpha k^2+1-\alpha) + 2\alpha k^2(1-\alpha)v_L}{3\alpha k^2+1-\alpha};\ \frac{1}{4}\Big(\alpha k v_H + (1-\alpha)v_L + \sqrt{(\alpha k v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2(\alpha k^2+1-\alpha)}\Big);\ \frac{\alpha k v_H}{2} + \frac{2(1-\alpha)-\alpha k^2}{6}v_L\Bigg)$$

$$\text{where} \quad g(Q) = \frac{2(\alpha k + 1 - \alpha)Q + \alpha(1-\alpha)(k-1)(kv_L - v_H)}{2(\alpha k^2 + 1 - \alpha)}.$$

The profit of each firm equals

$$\frac{\big(\alpha k v_H + (1-\alpha)v_L\big)Q - 2Q^2}{\alpha k^2 + 1 - \alpha}.$$

This equilibrium is unique if $Q \leq K$ also holds.

Proposition 3.2 identifies capacity-pairs for which an equilibrium with identical *unit prices* exists. The intuition behind the equalization of unit prices for a binding Q is analogous to the intuition for the equalization of prices for a binding K. Specifically, if K is large enough then the total amount of products becomes the scarce resource. The marginal gain of serving a consumer thus depends on how many units they want to buy. There is a profitable deviation consisting of serving more consumers of one group whenever *unit prices* are different.

The assumption on v_L is purely technical: it simplifies the exposition of the set of capacity-pairs where the equilibrium in question exists. Proposition 3.2 establishes that Q must be of intermediate size while K must be relatively larger for this kind of equilibrium to exist.

Intuitively, a lower Q would create incentives for firms to deviate by only serving low-types. A higher Q could result in firms deviating by serving only high-types or by substantially increasing one of the prices. Furthermore, a low K would make firms choose prices in such a way that both constraints bind.

The next section investigates the parameter region in which both constraints binding can be a symmetric equilibrium.

3.4 Symmetric pure-strategy equilibrium with both capacity constraints binding

This section analyzes the capacity-pairs where a symmetric equilibrium with both K and Q binding exists. The next Proposition shows that both capacities binding pins down the equilibrium prices and profit.

Proposition 3.3.

In any symmetric pure-strategy equilibrium where some consumers of both types are served and K and Q both bind, the prices are

$$p_L^{KQ} = v_L - \frac{2kK - 2Q}{(1-\alpha)(k-1)}$$

and

$$p_H^{KQ} = v_H - \frac{2Q - 2K}{\alpha(k-1)}.$$

and the profit of each firm equals

$$\frac{(kK-Q)v_L + (Q-K)v_H}{(k-1)} - \frac{2}{\alpha(1-\alpha)(k-1)^2}(Q^2 + (\alpha k^2 + 1 - \alpha)K^2 - 2KQ(\alpha k + 1 - \alpha)).$$

Lemma 3.4 excludes the possibility of rationing in equilibrium and shows that if both constraints bind then both constraints must exactly be covered. Hence the number of people served in such an equilibrium must equal $2K$, while their total demand must be $2Q$. The only price-pair satisfying both of these conditions is the one derived in Proposition 3.3. These equilibrium prices correspond exactly to the optimal prices that a monopoly with capacities $2K$ and $2Q$ chooses for corresponding capacity levels. Intuitively, it is in the monopoly's interest to choose prices so that demands exactly cover the capacities, and competition makes this same strategy an equilibrium in a duopoly.

Furthermore, as I am looking for an equilibrium where some consumers of both groups are served, $p_L < v_L$ and $p_H < v_H$ are necessary. These conditions rewrite as

$$Q/k < K < Q. \tag{3.2}$$

Moreover, equilibrium prices must satisfy $p_L \leq p_H \leq kp_L$ as both Lemma 3.5 and Lemma 3.7 apply. These two conditions rewrite as

$$g(Q) \leq K \leq f(Q). \tag{3.3}$$

Thus any equilibrium where both constraints bind must be in the quadrilateral area in the K-Q capacity space delimited by the 4 inequalities in (3.2) and (3.3). To ensure the existence of such an equilibrium, one must verify that no profitable deviations exist. All such calculations are relegated to the Appendix.

Given the number and complexity of equilibrium conditions, I restrict the analysis of this type of equilibria to 4 examples. I give numerical values to all parameters: $\alpha = 0.5$, $k = 5$, $v_L = 1$, then examine 4 different scenarios depending on whether v_H, the WTP of the most valuable high-type is of very low, low, intermediate or high level. Notice that according to the assumption of the model,

$$v_L = 1 < v_H < 5 = kv_L$$

must be satisfied, which provides limits for the choice of v_H.

3.4.1 Low levels of v_H

I start the analysis with the case that provides the richest set of equilibria described by Proposition 3.3. It is the case of a low v_H, specifically, I first examine $v_H = 1.2$.

Figure 3.1 depicts the K-Q capacity space with $v_H = 1.2$, the equilibria where both constraints bind and the different kinds of potential profitable deviations.

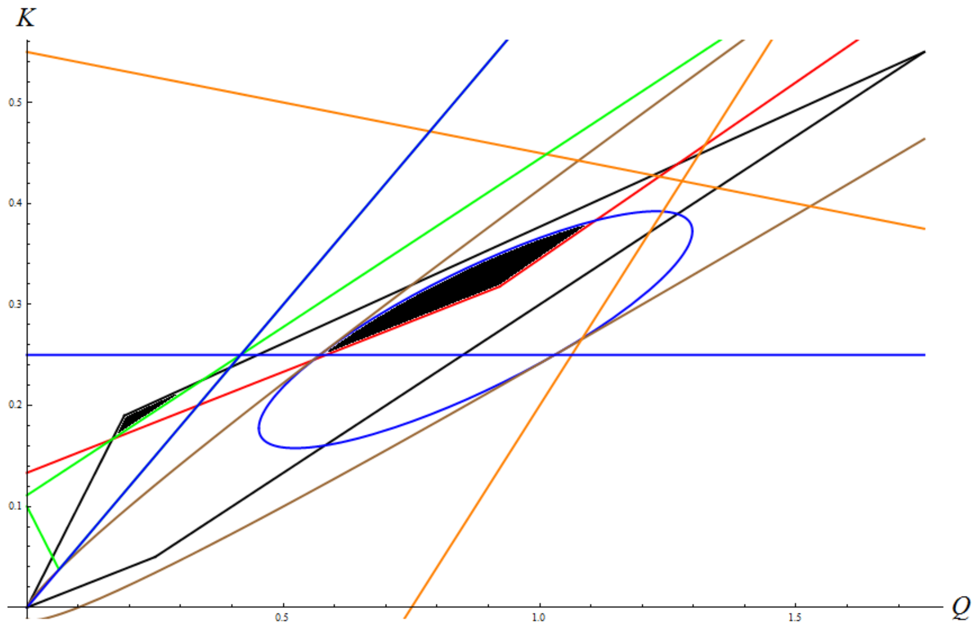


Figure 3.1. Equilibria with low levels of v_H

All equilibria where both constraints bind must be *inside the black quadrilateral area* in Figure 3.1 as it corresponds to the four constraints in (3.2) and (3.3).

There are various kinds of potential deviations from the equilibrium prices described in Proposition 3.3. One must examine each type of deviation and exclude capacity-pairs where there is at least one profitable deviation from the equilibrium. The detailed analysis is relegated to the Appendix, here I describe the different deviations resulting in profitable deviations and the corresponding curves in the figures.

1. There is a profitable deviation *below the red lines.* These lines represent two types of deviations. One consists of keeping the price charged for high-types unchanged while choosing a higher price than in equilibrium for low-types. The other red line corresponds to increasing the prices charged for high-types and choosing a price for low-types that exactly exhausts Q.

2. There is a profitable deviation *between the two orange lines after their crossing* consisting of choosing a higher price for high-types and serving the resulting positive residual demand, while simultaneously choosing a relatively high price for low-types that leaves both capacities slack.

3. There is a profitable deviation *inside the area delimited by the two green lines and the diagonal blue line* consisting of an upward deviation on both markets, that results in the deviating firm serving 0 high-types and the residual demand from low-types.

4. There is a profitable deviation in the area *outside the blue ellipse, above the blue horizontal line and to the right of the blue diagonal line* consisting of serving only low-type consumers in such a way that K is slack.

5. There is a profitable deviation in the area *outside the brown ellipse, below the blue horizontal line and to the right of the blue diagonal line* consisting of serving only low-type consumers in such a way that K binds.

Figure 3.1 shows that for low levels of v_H there are 2 disjoint regions in the K-Q space where the equilibrium described in Proposition 3.3 exists. These are the regions filled with black. All the other capacity-pairs inside the black quadrilateral, i.e., the set of potential equilibrium capacity-pairs, are excluded by one of the profitable deviations enumerated above.

3.4.2 Very low levels of v_H

In this subsection, I investigate the case of a very low WTP of high-types, specifically, I analyze the case of $v_H = 1.05$.

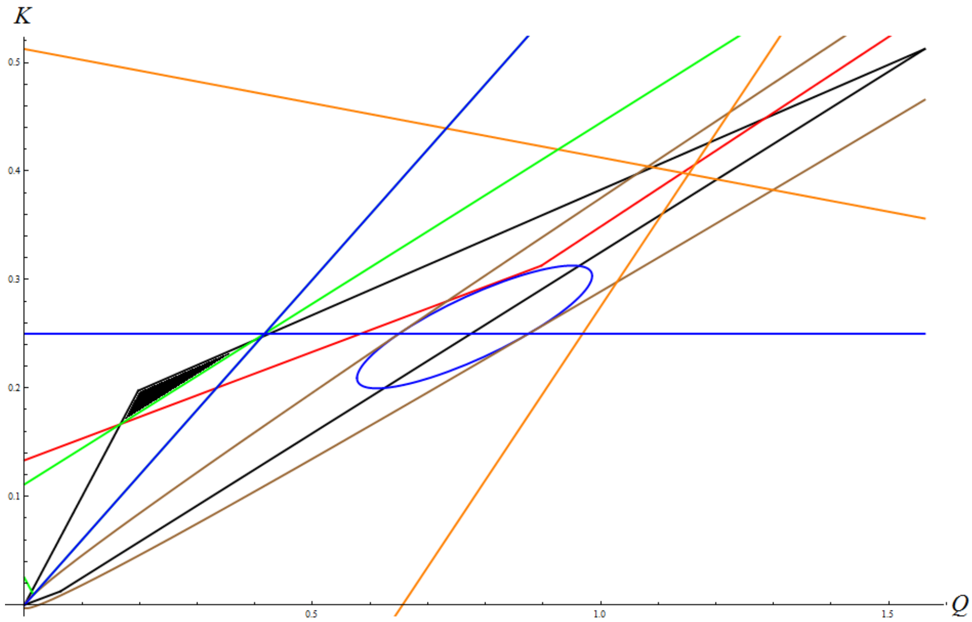


Figure 3.2. Equilibria with very low levels of v_H

Figure 3.2 shows that the only region of capacity-pairs where the equilibrium described in Proposition 3.3 exists is inside the shaded triangle. Compared to the case of low v_H, reducing v_H even further makes one of the equilibrium regions disappear. Figure 3.2 shows that the blue ellipse shrinks as v_H gets smaller, hence the area outside the ellipse containing a profitable deviation grows. Intuitively, as v_H gets very close to v_L, low-types become relatively more profitable, which in turn makes a deviation consisting of serving only low-types very attractive. Therefore one of the equilibrium areas present for higher values of v_H disappears.

3.4.3 Medium levels of v_H

In this subsection, I investigate the case of a medium v_H, specifically, I analyze the case of $v_H = 2$.

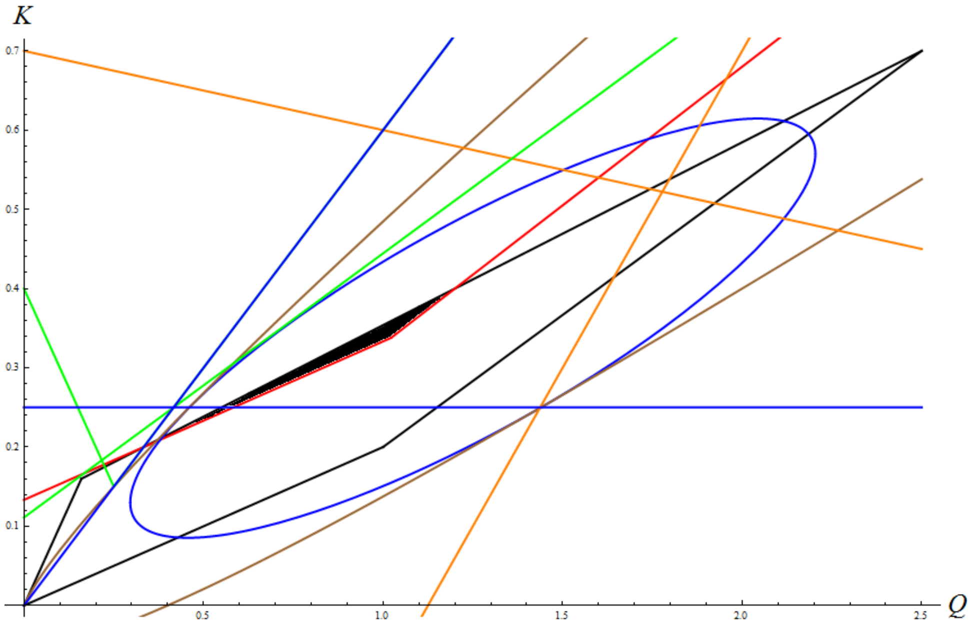


Figure 3.3. Equilibria with intermediate levels of v_H

Figure 3.3 shows the disappearance of the triangular equilibrium region present for lower levels of v_H. Thus, similarly to the case of very low v_H, there is exactly one region where the equilibrium exists, however, it is the other area present for low levels of v_H.

Increasing v_H shrinks the set of potential equilibria, i.e., the black quadrilateral region. The reason is that an increased v_H leads to an increased equilibrium p_H, thus the set of capacity-pairs satisfying the $p_H \leq kp_L$ equilibrium condition gets smaller. As the set of potential equilibria gets smaller, the deviation consisting of increasing the price for low-types and keeping the price for high-types unchanged (red line) gets more effective, making the triangular equilibrium region disappear.

3.4.4 High levels of v_H

In this subsection, I investigate the case of a high v_H, specifically, I analyze the case of $v_H = 4.8$.

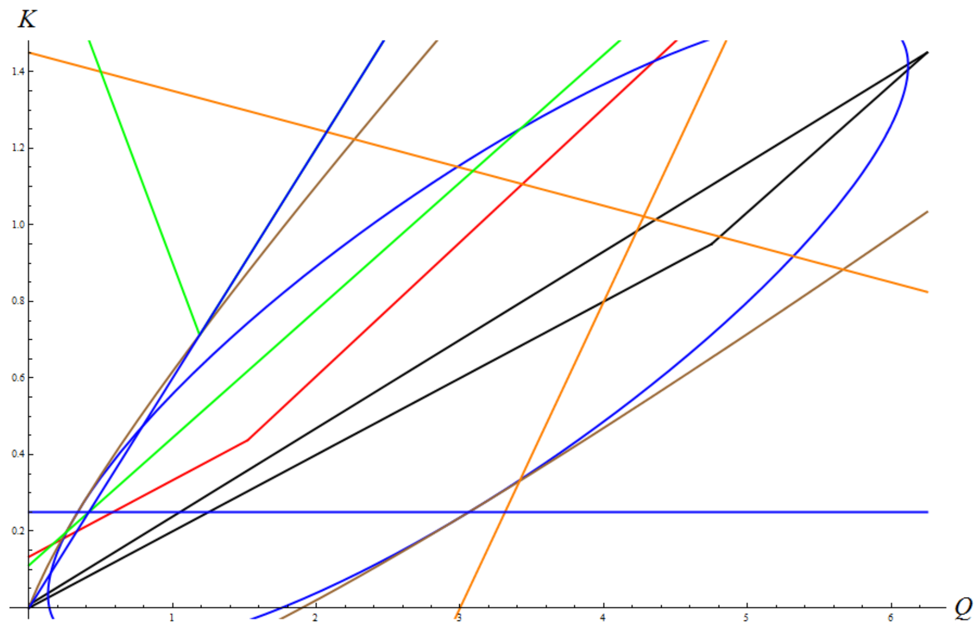


Figure 3.4. Lack of equilibria for high levels of v_H

Figure 3.4 shows the lack of equilibria for a large v_H. The intuition is similar to the disappearance of the first equilibrium region for medium v_H. As v_H becomes even larger, the equilibrium condition $p_H \leq kp_L$ becomes even more binding, further reducing the set of potential equilibria. Therefore the deviation represented by the red lines becomes even more effective, to the point that no equilibrium persists.

This concludes the analysis of equilibria where both capacity constraints bind and some consumers of both types are served.

3.5 Discussion

I start this section by presenting a numerical example with all pure-strategy equilibrium regions I can identify. Next, I examine comparative statics results on equilibrium prices, profits, consumer surplus and total surplus derived from Lemmas 3.1 - 3.2 and Propositions 3.1 - 3.3. Moreover, I compare the results of the symmetric dual capacity duopoly model to three benchmark cases. The first benchmark is the symmetric duopoly constrained by a single capacity analyzed in Levitan and Shubik (1972). The two other benchmarks come from the dual-constrained monopoly analyzed in Chapter 2. Firstly, one can compare the duopoly model to either a monopoly whose capacities equal the total industry capacities ($2K$ and $2Q$). Such a comparison sheds light to how prices, consumer surplus and profits change if a large monopoly is divided into 2 symmetric and competing firms. Secondly, the duopoly results can be compared to a monopoly whose capacities equal the capacity of one firm in the duopoly (K and Q). This analysis reveals the impact of the entry of a second, identical firm in a previously monopolistic market.

3.5.1 A numerical example

In this subsection, I present a numerical example to illustrate the symmetric pure-strategy equilibria identified above. The choice of parameters is identical to the one in Section 3.4.1: $\alpha = 0.5$, $k = 5$, $v_L = 1$, and $v_H = 1.2$. This configuration has the merit of resulting in a rather rich set of equilibria.

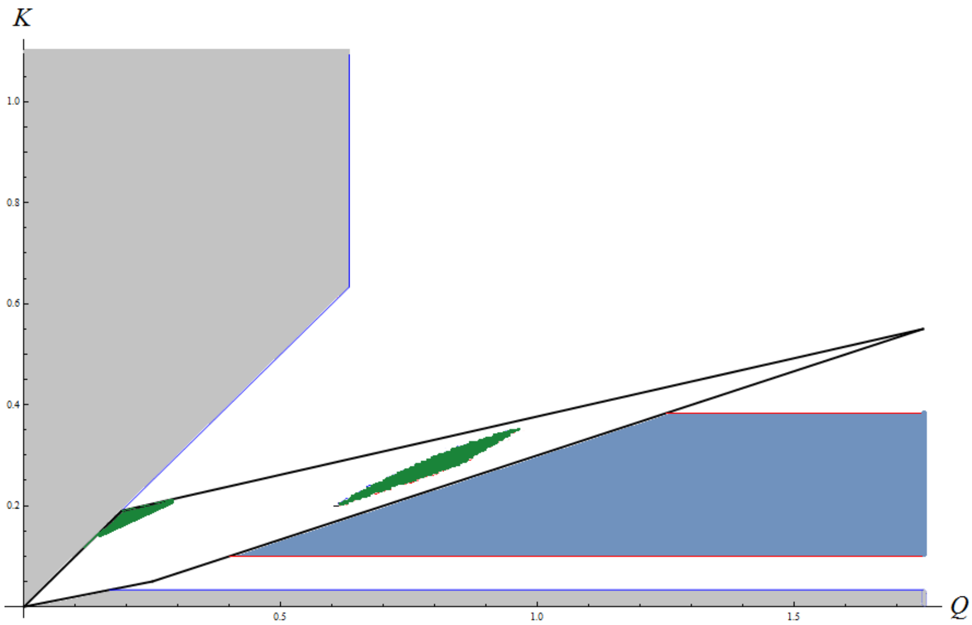


Figure 3.5. Symmetric pure-strategy equilibrium regions

Figure 3.5 shows the different equilibria of the model. The black quadrilateral in the middle is the core region, delimited by lines Q, $Q/5$, $(Q-0.1)/3$ and $(3Q+1.9)/13$. According the Lemma 3.2, there is a symmetric pure-strategy equilibrium excluding low-type consumers if $K \leq \min(Q/5;\ 1/30)$. This is the area is shaded by gray at the bottom of the figure. According the Lemma 3.3, there is a symmetric pure-strategy equilibrium excluding high-type consumers if $Q \leq \min(K;\ 19/30)$. This is the area is shaded by gray at the left of the figure.

The green patches inside the core region correspond to the two equilibrium regions where both constraints bind, identified in Section 3.4.1. The triangular area on the left is given by

$$\frac{Q}{3} + \frac{1}{3} \leq K \leq \min\left(Q;\ \frac{3Q+1.9}{13}\right).$$

The other region is composed by capacity-pairs satisfying

$$K \geq \max\left(\frac{1}{4};\ \frac{3Q+2}{15};\ \frac{7Q-0.1}{20}\right) \quad \text{and}$$

$$0 > K^2 - K\left(\frac{6}{13}Q + \frac{19}{130}\right) + \frac{Q^2}{13} - \frac{Q}{130} + \frac{1}{52}.$$

Given that the assumption at the beginning of Proposition 3.2 does not hold, I cannot identify symmetric pure-strategy equilibria where capacity Q binds. Finally, according to Proposition 3.1, there is a symmetric pure-strategy equilibria where capacity K binds if

$$0.1 \leq K \leq \min\left(\frac{Q}{3} - \frac{1}{30};\ \frac{23}{60}\right).$$

This corresponds to the area shaded in blue in Figure 3.5.

3.5.2 Comparative statics

In this subsection I investigate how a change in the level of capacities affects the equilibria derived above. I have obtained results on 5 types of non-trivial equilibria: two involving the exclusion of one group of consumers, two where one capacity constraint is exactly covered while the other is slack, and one where both constraints are exactly exhausted. The sixth type of equilibrium is the marginal cost pricing equilibrium, less interesting in terms of comparative statics.

3.5.2.1 Prices

It is easy to see that prices are a weakly decreasing function of both capacities in all of these equilibria with the exception of the one where both constraints bind. This is the standard result found in all Bertrand-Edgeworth oligopolies with a single capacity. However, in the equilibria where both constraints bind, p_H^{KQ} is increasing in K and similarly, p_L^{KQ} is increasing in Q. Hence, whenever an equilibrium with both capacities binding exists, *prices are non-monotonic in capacity levels.*

If such an equilibrium does not exist, the four types of equilibrium prices identified above are decreasing in capacity levels. A similar non-monotonicity result also appears in the monopoly case described in Chapter 2. However, in a dual-constrained monopoly there always exists a region where the non-monotonicity arises.

3.5.2.2 Consumer surplus, profits and total welfare

Next, I analyze aggregate consumer surplus. As Lemma 3.4 guarantees that there is no rationing in equilibrium, the aggregate consumer surplus is simply given by

$$CS = \frac{\alpha}{2}(v_H - p_H)^2 + \frac{1-\alpha}{2}(v_L - p_L)^2$$

if some consumers of both groups are served. If the firms only serve high-types, consumer surplus equals $\frac{\alpha}{2}(v_H - p_H)^2$. If the firms serve exclusively the low-types, consumer surplus equals $\frac{1-\alpha}{2}(v_L - p_L)^2$.

Naturally, consumer surplus is weakly decreasing in both prices. Therefore, given that prices are decreasing in capacities, *aggregate consumer surplus is increasing in both capacity levels in all four types of equilibria where at most one constraint binds.* This is the standard result obtained in oligopolies characterized by a single capacity for each firm.

The results are more complex for equilibria where both constraints bind. As the equilibrium prices p_L^{KQ} and p_H^{KQ} equal their monopoly counterparts, and similarly to the monopoly case there is no rationing in this equilibrium, the consumer surplus is equal to the consumer surplus of a monopoly of size $(2K, 2Q)$. Hence the results obtained in the monopoly case directly apply for this equilibrium. I showed in Chapter 2 that aggregate consumer surplus is decreasing in Q for capacity-pairs satisfying $Q \leq K \leq Q/(\alpha k + 1 - \alpha)$. Graphically this corresponds to the area close to the $K = Q$ line, one of the borders of the region where both constraints bind in equilibrium. The four examples examined in Section 3.4 suggest that the equilibrium exists for relatively low levels of v_H in this region. Specifically, the triangular-shaped region in Figures 3.1 and 3.2 must have an intersection with the region where *consumer surplus is decreasing in Q.*

The intuition for the decreasing consumer surplus is the following. Whenever an equilibrium exists in the triangular-shaped area discussed in Section 3.4, the price charged for low-type consumers increases as Q grows. The $K = Q$ line is the turning point between the equilibrium where high-types are excluded and the equilibrium where some high-types are served. However, as the prices and demands are continuous, the mass of low-types served compared to the mass of high-types served remains very large close to the turning point. Therefore there are many low-type consumers that lose because their price increases and only a few high-types that gain by being served at a lower price, which explains the decrease in aggregate consumer surplus.

It is straightforward to check that *profits in all the 5 equilibria are weakly increasing in both capacity levels.* Therefore total welfare, being defined as the sum of aggregate consumer surplus and the profits, is also increasing in capacities for the 4 equilibria where exactly one constraint binds. In the equilibrium where both constraints bind, given the equal prices, it is easy to see that the aggregate profit of the two firms equals the profit of a monopoly of size $(2K, 2Q)$. Thus the result obtained in the monopoly case (Chapter 2) directly applies and establishes that *total welfare is increasing* even for parameter values where consumer surplus is decreasing.

3.5.3 Comparison to monopoly benchmarks

In this subsection I compare the results of the duopoly model to the 2 different monopolies as benchmarks. Firstly I compare it to a monopoly characterized by capacities of the size of the industry capacity of the duopoly, i.e., $2K$ and $2Q$. For simplicity, I will refer to this monopoly as the *large monopoly.* Secondly, I also compare the duopoly results to a monopoly with a capacity which is the size of the capacity of a single firm in the duopoly, i.e., K and Q, and refer to it as the *small monopoly.*

The following remark will be helpful for a comparison of the duopoly results to the results obtained in the two monopoly cases.

Remark. For each of the five types of equilibria identified above, if a capacity constraint binds then the same capacity constraint also binds under the small monopoly's optimal pricing structure. Similarly, if a capacity constraint is slack in a duopoly equilibrium, then the same constraint must be slack in the corresponding optimal region of the small monopoly.

Indeed, the borders of the quadrilateral region defined by (3.2) and(3.3) (depicted with black lines in the figures of Section 3.4) correspond exactly to the "KQ region's" borders in the small monopoly. A corollary of this observation is that all five equilibrium regions are contained in the corresponding optimal region of the small monopoly. This makes the comparison of prices relatively easy for the small monopoly and somewhat more complicated for the large monopoly.

As the results of the comparison to the large monopoly facilitate some of the comparisons to the small monopoly, I start the discussion with the former.

3.5.3.1 Comparison to the large monopoly

The comparison of the duopoly to the large monopoly can shed light to the welfare effects of dividing a large monopoly into two identical, competing firms.

Equilibria excluding one group of consumers

If capacities are such that an equilibrium with excluding low-types exists, then the corresponding large monopoly's optimum involves either excluding low-types ("EL region") or serving some consumers of both types with K binding ("K region"). This is easy to see by observing the monopoly's optimal partitioning of the capacity-space (Chapter 2, p. 27, Figure 1.3).

For small capacity levels, in particular, for any $K \leq \frac{\alpha}{4}(v_H - v_L)$ the corresponding large monopoly also excludes low-types, and the equilibrium prices are exactly equal to the large monopoly's optimal price.

For larger capacities, the corresponding large monopoly serves some consumers of both-types, thus one has to compare the equilibrium price of $p_H^{EL} = v_H - 2K/\alpha$ to the monopoly price of $p_H^{mon} = v_H - \frac{1-\alpha}{2}(v_H - v_L) - K$. However, direct calculations show that

$$p_H^{EL} < p_H^{mon} \Leftrightarrow K > \frac{\alpha}{4}(v_H - v_L),$$

which always holds otherwise we are in the case of small capacity levels. Therefore, in this case the duopoly's equilibrium price is strictly smaller than the large monopoly's price.

Analogous results obtain for the equilibrium excluding high-types. To summarize, exclusionary equilibrium prices are either exactly equal to or strictly smaller than the corresponding large monopoly's optimal price. In the second case, aggregate consumer surplus in the duopoly is higher, and industry profit is lower than in the large monopoly.

Equilibria where K binds and Q is slack

For capacity-pairs giving rise to an equilibrium where K binds and Q is slack, the corresponding large monopoly's optimal prices either entail a binding K and a Q that is slack ("K region"), or they entail none of the constraints binding ("U region"). I restrict the analysis to the former case, due to the computational complexities of the latter.

For capacity-pairs satisfying the conditions in Proposition 3.1 and either $K \leq \frac{1}{4}(\alpha v_H + (1-\alpha)v_L)$ or $Q \leq \frac{1}{4}(\alpha k v_H + (1-\alpha)v_L)$ or both conditions, the corresponding large monopoly's prices entail K binding and Q slack. The optimal prices of a large monopoly are thus given by

$$p_L^{mon} = v_L + \frac{\alpha}{2}(v_H - v_L) - 2K \quad \text{and} \quad p_H^{mon} = v_H - v_L + \frac{1-\alpha}{2}(v_H - v_L) - 2K,$$

whereas the equilibrium prices are equal for the 2 consumer groups and given by

$$p_L^K = p_H^K = \alpha v_H + (1-\alpha)v_L - 2K.$$

In the Appendix I show that $p_L^{mon} < p_L^K = p_H^K < p_H^{mon}$, i.e., dividing a large monopoly into two identical parts has an ambiguous effect on prices; it increases the price the low-types must pay while decreases the price for high-type consumers. One must explicitly compute the consumer surpluses and profits in the duopoly and monopoly case to obtain a comparison. The large monopoly's profit is always greater than the combined profit of the two firms in the duopoly:

$$\pi^{mon} - 2\pi^{duop} =$$
$$\left(\alpha(1-\alpha)\left(\frac{v_H - v_L}{2}\right)^2 + 2K(\alpha v_H + (1-\alpha)v_L) - 4K^2\right) - 2\left(K(\alpha v_H + (1-\alpha)v_L) - 2K^2\right) > 0$$

In the Appendix, I also show that aggregate consumer surplus is larger in the duopoly. Therefore, despite the ambiguous price effect of dividing a large monopoly into two equal parts, I find the usual comparative statics results. Namely, in the duopoly aggregate consumer surplus is always higher and industry profit is always lower than in the large monopoly benchmark.

Equilibria where Q binds and K is slack

For capacity-pairs giving rise to an equilibrium where Q binds and K is slack, the corresponding large monopoly's optimal prices either entail a binding Q and K slack ("Q region"), or they entail none of the constraints binding ("U region"). I restrict the analysis to the former case, due to the computational complexities of the latter.

For capacity-pairs satisfying the conditions in Proposition 3.2 and either $K \leq \frac{1}{4}(\alpha v_H + (1-\alpha)v_L)$ or $Q \leq \frac{1}{4}(\alpha k v_H + (1-\alpha)v_L)$ or both, the corresponding large monopoly's prices entail Q binding and K slack. The optimal prices of a large monopoly are thus given by

$$p_L^{mon} = \frac{1}{\alpha k^2 + 1 - \alpha}\left(\frac{\alpha k}{2}(kv_L + v_H) + (1-\alpha)v_L - 2Q\right) \quad \text{and} \quad p_H^{mon} = kp_L^{mon},$$

whereas the equilibrium prices are

$$p_L^Q = \frac{1}{\alpha k^2 + 1 - \alpha}\left(\alpha k v_H + (1-\alpha)v_L - 2Q\right) \quad \text{and} \quad p_H^Q = kp_L^Q.$$

Clearly, the $kv_L > v_H$ assumption of the model guarantees $p_L^{mon} > p_L^Q$, i.e. both duopoly prices are lower than the prices of the large monopoly.

Equilibria where both K and Q bind

For these kind of equilibria, I also restrict the analysis to the case where the same constraints bind in the large monopoly as in the duopoly. In this case, as discussed before, equilibrium prices when both constraints bind are equal to the optimal prices that the large monopoly selects. Therefore, in this capacity region, aggregate consumer surplus and industry profit remain unaffected by cutting the large monopoly into two, competing firms.

3.5.3.2 Comparison to the small monopoly

The comparison of duopoly equilibria to the small monopoly's prices can reveal the effect of the exit of one of the firms from the duopoly, or alternatively, the entry of an identical firm to an incumbent, dual-constrained monopoly. Using the Remark and the results obtained above about the large monopoly, the comparison is straightforward.

Equilibria where at most one capacity binds

The Remark ensures that the equilibrium prices are directly comparable with the small monopoly's prices as the some constraints will bind in the two setups. Moreover, one result of the above comparison with the large monopoly is that the equilibrium prices are either equal to either strictly lower than the large monopoly's prices in the same region if at most one capacity binds. In addition, in these 4 regions prices are a decreasing function of both capacity levels. Hence the small monopoly's prices are always strictly above the large monopoly's prices.

Therefore, if at most one capacity binds then the duopoly's equilibrium prices are always strictly lower than the small monopoly's optimal prices. This entails that aggregate consumer surplus is higher and industry profit is lower in the duopoly than in the small monopoly.

Equilibria where both constraints bind

The previous logic cannot be applied to the equilibria where both constraints bind as prices are not decreasing in both capacity levels. However, the comparison is straightforward:

$$p_L^{KQ} = v_L - 2\frac{kK - 2Q}{(1-\alpha)(k-1)} < v_L - \frac{kK - 2Q}{(1-\alpha)(k-1)} = p_L^{mon}$$

and

$$p_H^{KQ} = v_H - 2\frac{Q - K}{\alpha(k-1)} < v_H - \frac{Q - K}{\alpha(k-1)} = p_H^{mon}.$$

Clearly, the duopoly prices are strictly lower than their counterparts in the small monopoly in the equilibria with both constraints binding. This translates into a higher aggregate consumer surplus and lower industry profit of the duopoly.

3.5.4 Comparison to the single capacity benchmark

This paper investigates the case of symmetric firms, both constrained by the same capacity levels. Thus, the natural benchmark to compare my results with is the Levitan and Shubik (1972) model as it investigates a symmetric duopoly where both firms are constrained by the same, singular capacity. Moreover, it analyzes efficient rationing of consumers. In order to study the effects of the co-existence of two capacity constraints, I introduced two consumer groups as opposed to the one, homogenous group of consumers in the LS model. Therefore I also had to generalize the efficient rationing rule to the case of multiple consumer groups and multiple capacity constraints.

Several comparisons can be of interest. Firstly, I verify that the predictions of LS coincide with the predictions of my model with only one consumer group and one capacity constraint. For this purpose, I choose $\alpha = 1$ and $Q = \infty$. The first condition ensures that only high-type consumers are present on the market, the second one ensures that the monopoly is bound solely by K. It is straightforward to check that all equilibria disappear except for the one described in Lemma 3.2. It predicts that a pure-strategy equilibrium exist if $K \leq v_H/3$ and the prices are given by $v_H - 2K$. This is exactly what the LS model predicts.

Secondly, I study the LS model with 2 consumer groups, which corresponds to my model with one single capacity ($Q = \infty$) and $\alpha < 1$. Naturally, the introduction of a second group of consumers changes some results. The region where only high-types are served ($K \leq \alpha(v_H - v_L)/3$) shrinks as there is more room for deviation: firms also have the possibility to serve some low-type consumers. Moreover, for some parameter values there is a new region of pure-strategy equilibria where some consumers of both types are served and K binds. Similarly to the original LS model, for very large K the only pure-strategy equilibrium consists of choosing zero prices on both markets.

Arguably the most interesting comparison is the one between the LS model with 2 consumer groups and 1 capacity constraint, and my model where the two capacities co-exist ($Q < \infty$). As demonstrated in Section 3.4, the co-existence of the 2 capacities may give rise to qualitatively novel equilibria, such as the one characterized by non-monotonicity of prices. Moreover, aggregate consumer surplus may also be a non-monotonic function of one of the capacities. Prices and aggregate consumer surplus, on the contrary, are always monotonic in the capacity level in both the original LS model and its extension to 2 consumer groups. This shows that the novel comparative statics results are a consequence of the dual capacity constraints and not merely an artifact of assuming 2 consumer groups.

3.6 Conclusion

This paper studies duopoly pricing under symmetric and dual capacity constraints that limit both the total quantity and the number of consumers the firms can serve. Beside the region of very large capacity-pairs where marginal cost pricing prevails, I identify 5 types of non-trivial, symmetric pure-strategy equilibria that are qualitatively different from one another. Rationing never occurs in equilibrium. There exist two exclusionary equilibria where solely one group of consumer is served by the firms. I also find two equilibria where one capacity constraint is exactly exhausted while the other is slack. Finally, I study the equilibrium where both capacities are covered.

I show that despite their ability to price discriminate, competition might force firms to charge identical prices across markets. In other equilibria, unit prices are equal across markets. Equilibrium prices and aggregate consumer surplus are non-monotonic in capacity levels in some of the equilibria where both capacities bind. Moreover, in some of the equilibria, duopoly prices are identical to the corresponding monopoly prices. Finally, by comparing my model to an extended version of Levitan and Shubik (1972), I show that the non-monotonicity results are a direct result of the co-existence of the two capacities and not merely an artifact of the two consumer groups assumption.

Directions of further research include finding more pure-strategy equilibria in the model. For instance, my results suggest the possible existence of an equilibrium where $p_H > p_L$, K is binding and some consumers of both types are served. In addition, a complete characterization of the set of equilibria would also require the analysis of capacity-pairs where only mixed-strategy equilibria can exist. However, given the complexities of the model, these tasks may be intractable.

Assuming other rationing rules constitutes another avenue for further research. It seems necessary to assume some sort of efficient rationing for the sake of tractability. Instead of the high-preferring efficient rationing rule, one may define an efficient rationing rule that does not prioritize high-type consumers.

Finally, an obvious next step would be the analysis of firms with asymmetric capacity constraints. This would also allow for endogenizing the choice of capacities, thus potentially providing very interesting insights. Arguably, a first step in the study of asymmetric dual capacity-constrained duopolies should be the analysis of the extreme case of one unconstrained firm competing with a firm constrained in both dimensions.

3.7 Appendix of Chapter 3

Proof of Lemma 3.2

I prove this Lemma in 2 steps. First, I show that $p_H = v_H - \frac{2K}{\alpha}$ must hold in any symmetric pure-strategy equilibrium where low-types are excluded, then I show that $K \leq \frac{\alpha}{3}(v_H - v_L)$ is a necessary and sufficient condition for the existence of such an equilibrium.

Firstly, assume to get a contradiction that charging $p_H < v_H - \frac{2K}{\alpha}$ is a symmetric equilibrium. This means that the industry demand at that price is strictly larger than $2K$: it equals $\alpha(v_H - p_H) > 2K$. Therefore firms can serve up to capacity and their profit equals Kp_H. Then there always exists a $p_H^d > p_H$ such that $\alpha(v_H - p_H^d) > 2K$, which means that the residual demand at price p_H^d is still larger than the binding capacity K. Hence the deviating profit is $Kp_H^d > Kp_H$ which proves that there always exists a profitable deviation if $p_H < v_H - \frac{2K}{\alpha}$.

Secondly, assume to get a contradiction that charging $p_H > v_H - \frac{2K}{\alpha}$ is a symmetric equilibrium. At p_H firms cannot serve up to capacity, their demand is $\frac{1}{2}\alpha(v_H - p_H) < K$. By undercutting its rival, a firm can increase its sales discontinuously: it can sell to $\min(K, \alpha(v_H - p_H))$ consumers as $Q/k \geq K$ by assumption. By a continuity argument, there exists a price p_H^d close enough to p_H such that the discontinuous increase in sales makes the deviation profitable. This proves $p_H^{EL} = v_H - \frac{2K}{\alpha}$.

Next, I show that $K \leq \frac{\alpha}{3}(v_H - v_L)$ is necessary and sufficient for the existence of this equilibrium by considering all possible profitable deviations from p_H^{EL}. No deviations with $p_H^D < p_H^{EL}$ can be profitable as the firm would then still be bound by the capacity constraint and it would sell all its capacity at a lower price. Similarly, no deviation consisting of keeping the same price for high-types, $p_H^D = p_H^{EL}$, while also serving some low-types ($p_L < v_L$) can be profitable. Indeed, given the high-preferring rationing rule, such a deviation would not change the consumers that are served and hence the profit would remain unchanged.

The only kind of potentially profitable deviation is thus increasing the price for high-types $p_H^D > p_H^{EL}$, while also serving some low-types: $p_L < v_L$. Firstly, I consider deviating prices for high-types that are low enough to attract some high-types:

Case 1: $p_H^D < v_H - K/\alpha$

In this case the deviating firm can serve $\alpha(v_H - p_H^D) - K > 0$ high-types. By choosing $p_L < v_L$, it faces a demand of $(1-\alpha)(v_L - p_L)$ since the non-deviating firm does not serve any low-types. Given the rationing rule, it ends up serving $\min((1-\alpha)(v_L - p_L);\ 2K - \alpha(v_H - p_H^D))$ low-types, hence its deviating profit is given by

$$\pi^D(p_L, p_H^D) = (\alpha(v_H - p_H^D) - K)p_H^D + \min((1-\alpha)(v_L - p_L);\ 2K - \alpha(v_H - p_H^D))p_L$$

This profit is linear in p_L up to $\overline{p_L} = v_L - \frac{2K}{1-\alpha} + \frac{\alpha}{1-\alpha}(v_H - p_H^D)$ then it is quadratic in p_L. The optimal value of p_L depends on the relative order of $\overline{p_L}$ and $v_L/2$.

Sub-case 1a: $v_L/2 < \overline{p_L}$

In this sub-case the optimal choice of p_L as a function of p_H^D is $\overline{p_L}$. Replacing this expression to the profit function and maximizing in p_H^D gives the optimal value of $p_H^* = (1+\alpha)v_H/2 + (1-\alpha)v_L/2 - \frac{1+3\alpha}{2\alpha}K$ for the high price. Notice that π^D is concave and quadratic in p_H^D, moreover, $\pi^D(p_H^{EL})$ equals the equilibrium profit and $\alpha \geq 0.5$ ensures that the condition of Case 1 is satisfied for p_H^*. This implies that this is a profitable deviation if and only if $p_H^* > p_H^{EL}$ and the condition of sub-case 1a are satisfied. Hence there is a profitable deviation whenever

$$\frac{\alpha(v_H - v_L)}{3} < K < \frac{\alpha v_H + (1-\alpha)v_L}{3}.$$

Sub-case 1b: $v_L/2 \geq \overline{p_L}$

In this sub-case the optimal deviating prices are $p_L = v_L/2$ and $p_H^* = \frac{v_H}{2} - \frac{K}{2\alpha}$. Notice that the condition of sub-case 1b always implies an upward deviation in the high price and rewrites as $K \geq \frac{\alpha v_H + (1-\alpha)v_L}{3}$. Furthermore, the condition of Case 1 rewrites as $K < \alpha v_H$. Finally, the deviating profit is again quadratic and concave in p_H^D and equals the equilibrium profit in p_H^{EL}, therefore the deviation is profitable if and only if all the conditions are satisfied, i.e.,

$$\frac{\alpha v_H + (1-\alpha)v_L}{3} \leq K < \alpha v_H.$$

To conclude the analysis of Case 1, notice that in any equilibrium $K < \alpha v_H/2$ must hold, a larger capacity would lead to negative equilibrium prices. Thus there is a profitable deviation in Case 1 if and only if $\frac{\alpha(v_H - v_L)}{3} < K$.

Case 2: $p_H^D \geq v_H - K/\alpha$

In Case 2 the deviating firm chooses a prohibitively high price for high-types and only serves low-types. I will show that this choice is always dominated by the equilibrium. The deviating firm's demand from low-types is $(1-\alpha)(v_L - p_L)$, its optimal choice of p_L depends on the size of K.

Case 2a: $K \geq (1-\alpha)v_L/2$

For large K, the optimal choice of the deviating firm is $p_L^* = v_L/2$ and its profit is $(1-\alpha)v_L^2/4$. However, this is always dominated by the equilibrium profit as

$$K(v_H - \frac{2K}{\alpha}) > \frac{(1-\alpha)v_L}{2}(v_H - \frac{2K}{\alpha}) > \frac{(1-\alpha)v_L}{2}\frac{v_L}{2}$$

since the last inequality rewrites as $K < \frac{\alpha v_H}{2} - \frac{\alpha v_L}{4}$ which is satisfied as in any equilibrium $K \leq \frac{\alpha(v_H - v_L)}{3}$ holds (from Case 1).

Case 2b: $K < (1-\alpha)v_L/2$

In this sub-case, the deviating firm fills its capacity K with low-types, its optimal price is $v_L - K/(1-\alpha)$. Such a deviation is profitable if this deviating price is higher than p_H^{EL}, however, straightforward computations show that this never happens. This concludes the analysis of Case 2 and shows that the only potential profitable deviations are the kind investigated in Case 1.

Therefore, there is a profitable deviation from p_H^{EL} if and only if $K > \frac{\alpha(v_H - v_L)}{3}$. $\square$

Proof of Lemma 3.3

I prove this Lemma in 2 steps, first by showing that $p_L = v_L - \frac{2Q}{1-\alpha}$ must hold in any symmetric pure-strategy equilibrium where high-types are excluded, then by showing that $Q \leq \frac{1-\alpha}{3}(v_L - v_H/k)$ is a necessary and sufficient condition for the existence of such an equilibrium.

Much of this proof in analogous to the proof of Lemma 3.2. The first claim, i.e., $p_L^{EH} = v_L - \frac{2Q}{1-\alpha}$, can be shown by using the same continuity arguments as before.

To see that $Q \leq \frac{1-\alpha}{3}(v_L - v_H/k)$ is necessary and sufficient for p_L^{EH} to constitute a symmetric equilibrium, one must consider each possible deviation from p_L^{EH}. First, consider the deviation of serving some high-type consumers. Being the only firm serving high-types, the deviating firm faces a demand of $\alpha(v_H - p_H^D)$ high-types for any $p_H^D < v_H$. Given the rationing rule, it serves these consumers first, then fills its remaining capacity with low-types. The deviating profit is thus given by

$$\alpha(v_H - p_H^D)p_H^D + (Q - \alpha(v_H - p_H^D)k)p_L^D$$

for any choice of p_H^D such that $\alpha(v_H - p_H^D)k \leq Q$ and $\alpha(v_H - p_H^D)k + (1-\alpha)(v_L - p_L^D) \geq Q$. Similarly to the analogous part of the proof of Lemma 3.2, this deviation is profitable if and only if $Q > \frac{1-\alpha}{3}(v_L - v_H/k)$. Furthermore, calculations analogous to the proof of Lemma 3.2 show that this deviation dominates any other potential deviations. □

Proof of Lemma 3.4

First notice that $p_L < v_L$ and $p_H < v_H$ are necessary conditions for some consumers of both types to be served. I will prove the Lemma by showing that there exists a profitable deviation for any $(p_L;\ p_H)$ that do not satisfy one of the condition-pairs.

Case 1: $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) > 2K$ and $\alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L) \leq 2Q$

In such a candidate equilibrium, given the high-preferring rationing rule, firms serve $\frac{1}{2}\alpha(v_H - p_H)$ high-types each. This quantity is strictly smaller than K otherwise firms would only serve high-types which violates the assumptions of Lemma (3.4). They fill their remaining capacity with $K - \frac{1}{2}\alpha(v_H - p_H)$ low-types, hence the profit in the candidate equilibrium is

$$\frac{1}{2}\alpha(v_H - p_H)p_H + \left(K - \frac{1}{2}\alpha(v_H - p_H)\right)p_L.$$

For such $(p_L;\ p_H)$ I show that slightly increasing the price charged for low-types is a profitable deviation. Consider charging $p_L^D = p_L + \delta$ where $\delta > 0$. Notice that this upward deviation cannot increase the demand the deviating firm faces, so the capacity constraint Q remains slack. Given the rationing rule, such a deviation does not change the number of high-types served by the firms; they each continue serving $\frac{1}{2}\alpha(v_H - p_H)$ high-types. Although the number of low-types willing to buy from the non-deviating firm increases in case of a deviation, as it is capacity constrained it will continue serving $K - \frac{1}{2}\alpha(v_H - p_H)$ low-types. Hence the residual demand of low-types for the deviating firm is $(1-\alpha)(v_L - p_L^D) - K + \frac{1}{2}\alpha(v_H - p_H)$. For a small enough deviation, the deviating firm remains capacity constrained as its total demand writes as

$$\frac{1}{2}\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L^D) - \left(K - \frac{1}{2}\alpha(v_H - p_H)\right) = \alpha(v_H - p_H) + (1-\alpha)(v_L - p_L^D) - K,$$

and this demand is greater than K for $\delta < \frac{\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) - 2K}{1-\alpha}$. Thus the deviating firm will serve the same mass of low-type consumers for a higher price, the profit writes as

$$\frac{1}{2}\alpha(v_H - p_H)p_H + \left(K - \frac{1}{2}\alpha(v_H - p_H)\right)p_L < \frac{1}{2}\alpha(v_H - p_H)p_H + \left(K - \frac{1}{2}\alpha(v_H - p_H)\right)p_L^D,$$

so the deviation is clearly profitable.

Case 2: $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) \leq 2K$ and $\alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L) > 2Q$

I show that there is a profitable upward deviation in the price charged on the low-types market. The argument is analogous to the one used in the proof of Case 1. The binding constraint is now Q, the upper bound on production. A price increase cannot increase the mass of people willing to buy so K remains slack for the deviating firm. The price increase for low-types keeps the mass of high-types served by each firm unchanged. There exists a small enough price increase so that the sum of residual demand from low-types plus the demand from high-types exceeds Q. This means that the deviating firm makes the same profit on high-types, and it sells to the same number of low-types for an increased price. Thus the deviation is profitable.

Case 3: $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) > 2K$ and $\alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L) > 2Q$

According to the rationing rule, K is effective, firms serve $\frac{1}{2}\alpha(v_H - p_H)$ high-types and $K - \frac{1}{2}\alpha(v_H - p_H)$ low-types whenever $K + \frac{1}{2}(k-1)\alpha(v_H - p_H) < Q$. Then one can use the arguments of Case 1 to show that a small price increase for low-types is a profitable deviation. Similarly, Q is effective if $K + \frac{1}{2}(k-1)\alpha(v_H - p_H) \geq Q$ and the arguments of Case 2 apply in this case to prove existence of a profitable deviation.

Case 4: $\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) < 2K$ and $\alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L) < 2Q$

In this candidate equilibrium none of the constraints bind. The firms can thus serve all the demand they face, hence their profit equals

$$\frac{1}{2}\alpha(v_H - p_H)p_H + \frac{1}{2}(1-\alpha)(v_L - p_L)p_L.$$

I will show that a small decrease in the price for low-types is a profitable deviation. Consider the deviating firm charging $p_L^D = p_L - \delta$ where $\delta > 0$. The demand from high-types is unchanged; it is $\frac{1}{2}\alpha(v_H - p_H)$. Then depending on the size of the capacity constraints, the deviating firm will be able to serve $m \equiv \min\left((1-\alpha)(v_L - p_L^D);\ K;\ Q\right)$ low-types. The assumptions of Case 4 guarantee that $m > 0$ for $\delta < v_L - p_L$. The deviation is profitable if

$$\frac{1}{2}\alpha(v_H - p_H)p_H + mp_L > \frac{1}{2}\alpha(v_H - p_H)p_H + \frac{1}{2}(1-\alpha)(v_L - p_L)p_L$$

which simplifies to

$$m(p_L - \delta) > \frac{1}{2}(1-\alpha)(v_L - p_L)p_L \quad \Leftrightarrow \quad \delta < (m - \frac{1}{2}(1-\alpha)(v_L - p_L))\frac{p_L}{m}.$$

Clearly, such a $\delta > 0$ exists if $m > \frac{1}{2}(1-\alpha)(v_L - p_L)$. If $m \in \{K, Q\}$ then this inequality is satisfied due to the assumptions of Case 4. It is also satisfied for $m = (1-\alpha)(v_L - p_L^D)$ since $(1-\alpha)(v_L - p_L^D) > (1-\alpha)(v_L - p_L) > \frac{1}{2}(1-\alpha)(v_L - p_L)$.

Together, Cases 1 - 4 exclude all the possible scenarios except for the 2 cases enumerated in Lemma (3.4), which concludes the proof. □

Proof of Lemma 3.5

To get a contradiction, assume $p_H = p_L + \varepsilon$ where $\varepsilon > 0$. I will show that for any $\varepsilon > 0$ there always exists $\delta_L > 0$ and $\delta_H > 0$ such that a deviation of $p_L^D = p_L - \delta_L$ and $p_H^D = p_H + \delta_H$ is profitable. Given the rationing rule, $\alpha(v_H - p_H)$ high-types aim to buy from the firm that does not deviate as it now offers a lower price on the market of high-types. There are 3 cases depending on the relative sizes of the capacity constraints: this firm will serve $\min\left(\alpha(v_H - p_H);\ K;\ Q/k\right)$ high-type consumers.

Case 1: $\alpha(v_H - p_H) \leq \min(K;\ Q/k)$

This is the case when the capacities are large enough so that the non-deviating firm can serve all the high-types at price p_H. According to the rationing rule, this means zero residual demand from the high-types for the deviating firm that is more expensive on this market. Hence it can only serve low-types. Its demand is $(1-\alpha)(v_L - p_L^D)$ since it is cheaper on the market for low-types. In addition, (ii) implies that Q remains slack for the deviating price-pair because the deviating firm serves more low-types and less high-types than it does in the candidate equilibrium. As a result, the deviating firm can serve $\min\big(K;\ (1-\alpha)(v_L - p_L^D)\big)$ low-types. However,

$$(1-\alpha)(v_L - p_L^D) > (1-\alpha)(v_L - p_L) = 2K - \alpha(v_H - p_H) \geq 2K - K = K.$$

The first inequality is a result of $\delta_L > 0$, the equality comes from (i) and the second inequality is given by the condition of Case 1. Hence the deviating profit is simply Kp_L^D. To conclude the proof of Case 1, I show that there exists a deviation δ_L which is small enough so that the deviating profit be larger than the profit in the candidate equilibrium:

$$Kp_L^D > \frac{1}{2}\left((1-\alpha)(v_L - p_L)p_L + \alpha(v_H - p_H)p_H\right).$$

Using the definition of p_L^D, (i) and the definition of ε this inequality rewrites as:

$$2K(p_L - \delta_L) > (2K - \alpha(v_H - p_H))p_L + \alpha(v_H - p_H)p_H = 2Kp_L - \varepsilon\alpha(v_H - p_H),$$

which in turn rewrites as

$$\delta_L < \varepsilon\frac{\alpha(v_H - p_H)}{2K}.$$

Such a δ_L always exists as $\frac{\alpha(v_H - p_H)}{2K} > 0$, which concludes the proof of Case 1.

Case 2: $K < \min(\alpha(v_H - p_H);\ Q/k)$

Now the deviating firm serves $\min(K,;\ \max(0, \alpha(v_H - p_H^D) - K))$ high-types. To see that the minimum is never K consider the following series of inequalities:

$$\alpha(v_H - p_H^D) - K < \alpha(v_H - p_H) - K < 2K - K = K.$$

The first inequality is a direct result of the definition of p_H^D whereas the second one stems from (i). Given that in this case $K < \alpha(v_H - p_H)$, the deviating firm serves $\alpha(v_H - p_H^D) - K > 0$ high-types if $\delta_H < v_H - p_H - K/\alpha$. Thus the deviating firm can serve $\min\left(2K - \alpha(v_H - p_H^D);\ (1-\alpha)(v_L - p_L^D)\right)$ low-types. Next I find conditions on δ_L and δ_H such that $2K - \alpha(v_H - p_H^D) < (1-\alpha)(v_L - p_L^D)$. This inequality rewrites as

$$\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) = 2K < \alpha(v_H - p_H - \delta_H) + (1-\alpha)(v_L - p_L + \delta_L),$$

so the required condition is $\delta_H < \frac{1-\alpha}{\alpha}\delta_L$. If this condition holds, the profit of the deviating firm is given by $(\alpha(v_H - p_H^D) - K)p_H^D + (2K - \alpha(v_H - p_H^D))p_L^D$. This profit is larger than the one obtained in the candidate equilibrium if

$$(\alpha(v_H - p_H^D) - K)p_H^D + (2K - \alpha(v_H - p_H^D))p_L^D > \frac{1}{2}\left((1-\alpha)(v_L - p_L)p_L + \alpha(v_H - p_H)p_H\right).$$

Using the definitions of p_L^D, p_H^D and ε; and (i) one can rewrite this inequality as

$$(\alpha(v_H - p_H) - \alpha\delta_H - K)(p_H + \delta_H) + (2K - \alpha(v_H - p_H) + \alpha\delta_H)(p_L - \delta_L) > Kp_H + \frac{\varepsilon}{2}(1-\alpha)(v_L - p_L).$$

By using (i) repeatedly, this translates to

$$(K - (1-\alpha)(v_L - p_L) - \alpha\delta_H)(p_H + \delta_H) + ((1-\alpha)(v_L - p_L) + \alpha\delta_H)(p_L - \delta_L) > Kp_H + \frac{\varepsilon}{2}(1-\alpha)(v_L - p_L),$$

which can be simplified to

$$(1-\alpha)(v_L - p_L)(p_L - p_H - \frac{\varepsilon}{2} - \delta_L - \delta_H) + K\delta_H + \alpha\delta_H(p_L - p_H - \delta_L - \delta_H) > 0$$

which finally rewrites as

$$(1-\alpha)(v_L - p_L)(\frac{\varepsilon}{2} - \delta_L - \delta_H) + K\delta_H + \alpha\delta_H(\varepsilon - \delta_L - \delta_H) > 0.$$

Clearly, this condition is satisfied and hence the deviation is profitable whenever $\delta_L + \delta_H < \varepsilon/2$. It is straightforward to find a (δ_L, δ_H) pair that satisfies all the 3 conditions necessary for such a deviation; for example let $\delta_L = \varepsilon/4$ and
$\delta_H = \min\left(\varepsilon/8;\ \frac{1-\alpha}{2\alpha}\varepsilon/4;\ \frac{1}{2}(v_H - p_H - K/\alpha)\right)$. This concludes the proof of Case 2.

Case 3: $Q/k < \min\left(\alpha(v_H - p_H);\ K\right)$

In this case the deviating firm serves $\min\left(K;\ \max\left(0, \alpha(v_H - p_H^D) - Q/k\right)\right)$ high-types. First I show that that this minimum is never K. Rewriting condition (ii) stating that Q is slack, one obtains

$$\alpha(v_H - p_H) \leq 2\frac{Q}{k} - 2\frac{(1-\alpha)(v_L - p_L)}{k} < 2\frac{Q}{k}. \tag{3.4}$$

Now consider the following series of inequalities:

$$\alpha(v_H - p_H^D) - \frac{Q}{k} < \alpha(v_H - p_H) - \frac{Q}{k} < \frac{Q}{k} < K, \tag{3.5}$$

where the first inequality stems from the definition of p_H^D, the second inequality is the result of (3.4) and the last one is from the condition of Case 3. This shows that the deviating firm will serve $\max\left(0, \alpha(v_H - p_H^D) - Q/k\right)$ high-types. Clearly, the deviating firm serves $\alpha(v_H - p_H^D) - Q/k > 0$ high-types if $\delta_H < v_H - p_H - Q/(\alpha k)$.

Then the deviating firm can serve $\min\left(K + Q/k - \alpha(v_H - p_H^D);\ (1-\alpha)(v_L - p_L^D)\right)$ low-types. Next I find conditions on δ_L and δ_H such that $K + Q/k - \alpha(v_H - p_H^D) < (1-\alpha)(v_L - p_L^D)$. This inequality rewrites as

$$\alpha(v_H - p_H^D) + (1-\alpha)(v_L - p_L^D) > K + Q/k.$$

Given that in Case 3 $Q/k < K$, a sufficient condition for the above inequality is

$$\alpha(v_H - p_H^D) + (1-\alpha)(v_L - p_L^D) > 2K,$$

which is satisfied for $\delta_H < \frac{1-\alpha}{\alpha}\delta_L$ (from the proof of Case 2). If this condition holds, the profit of the deviating firm is given by

$$(\alpha(v_H - p_H^D) - Q/k)p_H^D + (K + Q/k - \alpha(v_H - p_H^D))p_L^D.$$

This profit is larger than the one obtained in the candidate equilibrium if

$$(\alpha(v_H - p_H^D) - Q/k)p_H^D + (K + Q/k - \alpha(v_H - p_H^D))p_L^D > \frac{1}{2}\left((1-\alpha)(v_L - p_L)p_L + \alpha(v_H - p_H)p_H\right).$$

Using the definitions of p_L^D, p_H^D and ε; and (i) one can rewrite this inequality as

$$(\alpha(v_H - p_H) - \alpha\delta_H - Q/k)(p_H + \delta_H) + (K + Q/k - \alpha(v_H - p_H) + \alpha\delta_H)(p_L - \delta_L) > Kp_L - \frac{\varepsilon}{2}\alpha(v_H - p_H)$$

which rewrites as

$$\alpha(v_H - p_H)(\delta_L + \delta_H - \frac{\varepsilon}{2}) + (Q/k + \alpha\delta_H)(\varepsilon - \delta_L - \delta_H) - K\delta_L > 0$$

which finally translates to

$$\delta_H(\alpha(v_H - p_H) - Q/k - \alpha\delta_H) + \varepsilon(Q/k + \alpha\delta_H - \frac{1}{2}\alpha(v_H - p_H)) > \delta_L(K + Q/k + \alpha\delta_H - \alpha(v_H - p_H)).$$

Notice that the first term of the left-hand side is positive by the condition previously imposed of δ_H. The second term on the left-hand side is also positive as

$$Q/k + \alpha\delta_H - \frac{1}{2}\alpha(v_H - p_H) \geq 0 \quad \Leftrightarrow \quad \alpha(v_H - p_H) \leq 2Q/k + \alpha\delta_H$$

and the second inequality is guaranteed by (3.4). The right-hand side is also positive by (3.5). An additional condition imposed on δ_L necessary for a profitable deviation is thus

$$\delta_L < \frac{\delta_H(\alpha(v_H - p_H) - Q/k - \alpha\delta_H) + \varepsilon(Q/k + \alpha\delta_H - \frac{1}{2}\alpha(v_H - p_H))}{K + Q/k + \alpha\delta_H - \alpha(v_H - p_H)} \equiv x.$$

Notice that $x \to \varepsilon\frac{Q/k - \frac{1}{2}\alpha(v_H - p_H)}{K + Q/k - \alpha(v_H - p_H)} > 0$ as $\delta_H \to 0$. The limit is strictly positive as (3.4) implies that both the nominator and the denominator are positive. Hence there always exists a (δ_L, δ_H) pair that satisfies all the conditions, which concludes the proof of Case 3 and consequently the proof of Lemma (3.5). □

Proof of Lemma 3.6

I prove the lemma in two steps. Firstly, I show that the two prices must be equal. Secondly, I show that the two pricing being equal implies this price being equal to $\alpha v_H + (1-\alpha)v_L - 2K$.

Notice that $K < Q/k$ means that Q never binds, i.e.,

$$\frac{1}{2}\alpha(v_H - p_H)k + \frac{1}{2}(1-\alpha)(v_L - p_L) < Q.$$

By Lemma 3.5 this means that for (p_L, p_H) to be an equilibrium price-pair,

$$\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) = 2K \tag{3.6}$$

must hold. Also, Lemma 3.5 asserts that $p_L \leq p_H$. To prove equality of prices, I show that for any price-pair $p_L < p_H$ there exists a profitable deviation. Let $\varepsilon = p_H - p_L > 0$. I will show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that the price-pair (pl, p_H^D) with $p_H^D = p_H - \delta$ is a profitable deviation. Given the rationing rule, the deviating firm will sell to $\min\left(\alpha(v_H - p_H^D);\ K\right)$ and serve the remaining capacity to low-types. Depending on the relative size of parameters, I discuss 2 cases.

Case 1: $\alpha(v_H - p_H) \geq K$

In this case $\alpha(v_H - p_H^D) > \alpha(v_H - p_H) \geq K$ so the deviating firm fills its capacity K with high-types, hence its profit is simply Kp_H^D. The deviation is profitable if

$$Kp_H^D > \frac{1}{2}(1-\alpha)(v_L - p_L)p_L + \frac{1}{2}\alpha(v_H - p_H)p_H.$$

Using the definition of p_H^D and 3.6 this rewrites as

$$K(ph - \delta) > Kp_H - \frac{1}{2}(1-\alpha)(v_L - p_L)(p_L - p_H))$$

which by the definition of ε is equivalent to $\delta < \frac{(1-\alpha)(v_L - p_L)}{2K}\varepsilon$.

Case 2: $\alpha(v_H - p_H) < K$

In this case for $\delta < K/\alpha - v_H + p_H$ we have $\alpha(v_H - p_H^D) < K$. The demand the deviating firm faces is the sum of $\alpha(v_H - p_H^D)$ and $\frac{1}{2}(1-\alpha)(v_L - p_L)$. This is clearly greater than K so the firm serves $\alpha(v_H - p_H^D)$ high-types and $K - \alpha(v_H - p_H^D)$ low-types. This deviation is profitable if

$$\alpha(v_H - p_H^D)p_H^D + (K - \alpha(v_H - p_H^D))p_L > \frac{1}{2}(1-\alpha)(v_L - p_L)p_L + \frac{1}{2}\alpha(v_H - p_H)p_H = Kp_L + \frac{1}{2}\alpha(v_H - p_H)\varepsilon,$$

where the equality comes from using 3.6. Rearranging the terms and using the definition of p_H^D, this translates to

$$\alpha(v_H - p_H)(\frac{\varepsilon}{2} - \delta) + \alpha\delta(\varepsilon - \delta) > 0$$

.

Clearly, if $\delta \leq \varepsilon/2$ then both terms are positive and hence the deviation is profitable. This proves Case 2 and hence we have $p_L = p_H$. Then replacing this condition to 3.6 one obtains

$$\alpha(v_H - p_L) + (1-\alpha)(v_L - p_L) = 2K \iff p_L = p_H = \alpha v_H + (1-\alpha)v_L - 2K,$$

which concludes the proof. □

Proof of Proposition 3.1

I prove this Lemma in 2 steps. First I derive the necessary and sufficient conditions for the existence of an equilibrium where both firms charge the same prices on both markets. Then I show that the additional condition of $K \leq Q/k$ ensures uniqueness. Let p denote the common price on both markets that the firms charge in the candidate equilibrium, i.e.,

$$p \equiv \alpha v_H + (1-\alpha)v_L - 2K.$$

First notice that

$$p < v_L \quad \Leftrightarrow \quad \frac{\alpha(v_H - v_L)}{2} < K \tag{3.7}$$

is necessary for some consumers of both types to be served and

$$K < \frac{\alpha v_H + (1-\alpha)v_L}{2} \tag{3.8}$$

is equivalent to p being positive. By construction, each firm's total demand at the candidate equilibrium price-pair (p, p) equals exactly K. Thus, Lemma 3.4 requires that

$$\alpha(v_H - p)k + (1-\alpha)(v_L - p) \leq 2Q \quad \Leftrightarrow \quad K \leq \frac{2Q - \alpha(1-\alpha)(k-1)(v_H - v_L)}{2(\alpha k + 1 - \alpha)} \tag{3.9}$$

be satisfied for (p, p) to be an equilibrium. Notice that this is exactly one of the conditions stipulated in Proposition 3.1. Next, I will consider the 3 possible types of deviations: $p_H^D < p$, $p_H^D = p$, and $p_H^D > p$. Firstly, any downward deviation on the high-types market, $p_H^D < p$ is unprofitable. Indeed, as K already binds in the candidate equilibrium, decreasing the price charged for the high-types results in serving the same mass of people but at a lower average price. For the same reason, $p_H^D = p$ coupled with $p_L^D < p$ is also always unprofitable.

Next, consider the deviation consisting of keeping the price charged for the high-types unchanged, $p_H^D = p$ together with increasing the price charged for low-types, $p_L^D > p$. Given the rationing rule, both firms will keep serving $\frac{1}{2}\alpha(v_H - p)$ high-types. The deviating firm gets the residual demand of $(1-\alpha)(v_L - p_L^D) - \frac{1}{2}(1-\alpha)(v_L - p)$. The deviating firm will thus face a total demand strictly smaller than K, and its profit is given by

$$\frac{1}{2}\alpha(v_H - p)p + \left((1-\alpha)(v_L - p_L^D) - \frac{1}{2}(1-\alpha)(v_L - p)\right)p_L^D.$$

The no-deviation condition thus writes as

$$\left((1-\alpha)(v_L - p_L^D) - \frac{1}{2}(1-\alpha)(v_L - p)\right)p_L^D \leq \frac{1}{2}(1-\alpha)(v_L - p)p$$

The deviating profit is quadratic in p_L^D, and it equals the candidate equilibrium profit if $p_L^D = p$. Hence the condition for no deviation is simply $p \geq v_L/3$ which rewrites as

$$K \leq \frac{\alpha(v_H - v_L)}{2} + \frac{v_L}{3}. \tag{3.10}$$

The remaining possibilities for deviation consist of increasing the price charged for high-types: $p_H^D > p$. Given the rationing rule, the other firm, which is then cheaper on the high-types market, faces a demand of $\alpha(v_H - p)$ high-types. Given the $\alpha \geq 0.5$ assumption, this demand is always larger than the capacity K, i.e, $\alpha(v_H - p) > K$. Therefore in case of such a deviation, the non-deviating firm always serves K high-types and 0 low-types. Thus for any $p_L^D < v_L$, the deviating firm receives a demand of $(1-\alpha)(v_L - p_L^D)$ low-types and for any $p_H^D > p$ it faces $\max(0;\ \alpha(v_H - p_H^D) - K)$ high-types. Given the rationing rule, the deviating firm first sells to the $\max(0;\ \alpha(v_H - p_H^D) - K) < K$ high-type consumers, where the inequality comes from $p_H^D > p$. I distinguish two cases depending on the price p_H^D the deviating firm chooses.

Case 1: $\alpha(v_H - p_H^D) \leq K$

In this case the deviating firm chooses such a high price for high-types that the residual demand from that group of consumers is 0. This means that the firm only serves to low-types and maximizes its deviating profit of $\min((1-\alpha)(v_L - p_L^D);\ K)p_L^D$. If $K \leq (1-\alpha)v_L/2$ then its optimal deviating price is $p_L^D = v_L - K/(1-\alpha)$ and the resulting profit is Kp_L^D. This profit is lower than Kp, the candidate equilibrium profit, as $\alpha \leq 0.5$ implies $p_L^D < p$. If $K > (1-\alpha)v_L/2$ then the optimal deviating price is $p_L^D = v_L/2$ and the resulting profit is $(1-\alpha)v_L^2/4$. The condition for no deviation from the candidate equilibrium is $(1-\alpha)v_L^2/4 < Kp$, which translates to

$$\begin{aligned}\frac{1}{4}\big(\alpha v_H + (1-\alpha)v_L\big) - \frac{1}{4}\sqrt{(\alpha v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2} < K < \\ \frac{1}{4}\big(\alpha v_H + (1-\alpha)v_L\big) + \frac{1}{4}\sqrt{(\alpha v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2}. \end{aligned} \tag{3.11}$$

Straightforward calculations show that the lower bound on K given by (3.11) is strictly weaker than the lower bound given by (3.7). However, the upper bound in (3.11) may be effective and hence appears in the conditions of Proposition 3.1.

Case 2: $\alpha(v_H - p_H^D) > K$

In Case 2, the firm chooses p_H^D sufficiently low to attract a strictly positive mass of $\alpha(v_H - p_H^D) - K$ high-type consumers. According to the rationing rule, it always serves these consumers first, then it starts serving low-types. For any $p_L^D < v_L$, it receives a demand of $(1-\alpha)(v_L - p_L^D)$ low-types. I further distinguish 2 sub-cases depending on whether the deviating firm chooses a price p_L^D that enables it to fill its remaining capacity or not.

Sub-case 2a: $\alpha(v_H - p_H^D) - K + (1-\alpha)(v_L - p_L^D) \geq K$

In sub-case 2a, the deviating firm chooses a p_L^D low-enough to exhaust its capacity K. Then its profit equals

$$\pi^D = (\alpha(v_H - p_H^D) - K)p_H^D + (2K - \alpha(v_H - p_H^D))p_L^D,$$

the firm maximizes this subject to the condition of Case 2, that of Case 2a and this being an upward deviation on the high-types market, i.e., $p_H^D > p$. π^D is clearly increasing in p_L^D and thus for the best deviation the condition of Case 2a must bind. Therefore one obtains the optimal value of p_L^D as a function of p_H^D and the parameters. Substituting this value into the π^D and maximizing it wrt p_H^D provides the unconstrained optimal prices

$$p_H^D = \frac{(1+\alpha)v_H + (1-\alpha)v_L}{2} - \frac{1+3\alpha)}{2\alpha}K \quad \text{and} \quad p_L^D = v_L + \frac{\alpha}{2}(v_H - v_L) - \frac{3}{2}K.$$

The condition of Case 2 is clearly satisfied, the condition of 2a is satisfied for equality. Therefore whenever $p_H^D > p$, such a deviation is feasible. Moreover, given that π^D is quadratic in p_H^D and $\pi^D(p,p) = Kp$, the deviation is always strictly profitable if $p_H^D > p$. Thus a necessary and sufficient condition for no profitable deviation in this sub-case is $p_H^D \leq p$ which rewrites as

$$\alpha(v_H - v_L) \leq K. \tag{3.12}$$

Sub-case 2b: $\alpha(v_H - p_H^D) - K + (1-\alpha)(v_L - p_L^D) < K$

In sub-case 2b, the deviating firm chooses a p_L^D that is so high that it leaves some idle capacity. Then the deviating firm maximizes its profit

$$\pi^D = (\alpha(v_H - p_H^D) - K)p_H^D + (1-\alpha)(v_L - p_L^D)p_L^D$$

subject to $p_H^D > p$, and the conditions of Case 2 and Case 2b. The unconstrained optimum of

$$p_H^D = \frac{v_H}{2} - \frac{K}{2\alpha} \quad \text{and} \quad p_L^D = v_L/2$$

is feasible and constitute a profitable deviation if

$$\max\left(\frac{\alpha(2\alpha-1)v_H+2\alpha(1-\alpha)v_L}{4\alpha-1};\ \frac{\alpha v_H+(1-\alpha)v_L}{3}\right)<K<\alpha v_H.$$

(3.8) and $\alpha \geq 0.5$ ensure that $K < \alpha v_H$, so the necessary and sufficient conditions for the non-existence of a profitable deviation in Case 2b is

$$K \leq \max\left(\frac{\alpha(2\alpha-1)v_H+2\alpha(1-\alpha)v_L}{4\alpha-1};\ \frac{\alpha v_H+(1-\alpha)v_L}{3}\right). \tag{3.13}$$

(p,p) being an equilibrium requires *all* profitable deviations to be unprofitable, thus conditions (3.7), (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13) must be jointly satisfied. Straightforward comparisons of these conditions provide the final necessary and sufficient condition appearing in Proposition 3.1.

Finally, if $Q/k \geq K$ then Lemma 3.6 applies and ensures that in any equilibrium $p_L = p_H = p$. This establishes the uniqueness of the (p,p) equilibrium and thus concludes the proof of Proposition 3.1. □

Proof of Lemma 3.7

Assume to get a contradiction that $p_H/k = p_L + \varepsilon$ where $\varepsilon > 0$. I will show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that a deviation of $p_H^D = p_H - \delta$ is profitable. Given the rationing rule, $\alpha(v_H - p_H^D)$ high-types aim to buy from the deviating firm. Notice that (i) ensures that K remains slack for the deviating firm: it serves more high-types and less low-types. Thus the deviating firm will be able to serve $\min(\alpha(v_H - p_H^D);\ Q/k)$ high-type consumers.

Case 1: $\alpha(v_H - p_H) \geq Q/k$

In this case $\alpha(v_H - p_H^D) > \alpha(v_H - p_H) \geq Q/k$ so the deviating firm fills its capacity Q with high-types, hence its profit is simply Q/kp_H^D. The deviation is profitable if

$$Q/kp_H^D > \frac{1}{2}(1-\alpha)(v_L - p_L)p_L + \frac{1}{2}\alpha(v_H - p_H)p_H.$$

Using the definition of p_H^D and (ii) this rewrites as

$$Q/k(ph - \delta) > Q/kp_H - \frac{1}{2}(1-\alpha)(v_L - p_L)(p_H/k - p_L))$$

which by the definition of ε is equivalent to

$\delta < \frac{(1-\alpha)(v_L - p_L)}{2Q/k}\varepsilon.$

Case 2: $\alpha(v_H - p_H) < Q/k$

In this case for $\delta < Q/(\alpha k) - v_H + p_H$ we have $\alpha(v_H - p_H^D) < Q/k$. The demand the deviating firm faces is the sum of $\alpha(v_H - p_H^D)k$ and $\frac{1}{2}(1-\alpha)(v_L - p_L)$ which by (ii) is clearly greater than Q, so the firm serves $\alpha(v_H - p_H^D)$ high-types and $Q - \alpha(v_H - p_H^D)k$ low-types. This deviation is profitable if

$$\alpha(v_H - p_H^D)p_H^D + (Q - \alpha(v_H - p_H^D)k)p_L > \frac{1}{2}(1-\alpha)(v_L - p_L)p_L + \frac{1}{2}\alpha(v_H - p_H)p_H = Qp_L + \frac{1}{2}\alpha(v_H - p_H)k\varepsilon,$$

where the equality comes from using (i). Rearranging the terms and using the definition of p_H^D, this translates to

$$\alpha(v_H - p_H)(\frac{k\varepsilon}{2} - \delta) + \alpha\delta(k\varepsilon - \delta) > 0$$

.

Clearly, if $\delta \leq k\varepsilon/2$ then both terms are positive and hence the deviation is profitable. □

Proof of Lemma 3.8

I prove this lemma in two steps. Firstly, I show that the two unit prices must be equal. Secondly, I show that the two unit price being equal implies this price being equal to $\frac{\alpha k v_H + (1-\alpha)v_L - 2Q}{\alpha k^2 + 1 - \alpha}$.

Notice that $Q < K$ means that K never binds, i.e.,

$$\frac{1}{2}\alpha(v_H - p_H) + \frac{1}{2}(1-\alpha)(v_L - p_L) < K.$$

By Lemma 3.5 this means that for (p_L, p_H) to be an equilibrium price-pair,

$$\alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L) = 2Q \tag{3.14}$$

must hold. Also, Lemma 3.7 asserts that $p_L \geq p_H/k$. To prove equality of unit prices, I show that for any price-pair $p_L > p_H/k$ there exists a profitable deviation. Let $\varepsilon = p_L - p_H/k > 0$. I will show that for any $\varepsilon > 0$ there exist $\delta_L > 0$ and $\delta_H > 0$ such that the price-pair (p_L^D, p_H^D) with $p_L^D = p_L - \delta_L$ and $p_H^D = p_H + \delta_H$ form a profitable deviation. Given the rationing rule, the non-deviating firm will face a demand of $\alpha(v_H - p_H)$ high-types. Condition $K > Q$ ensures that K remains slack for the non-deviating firm. Thus it will sell to $\min(\alpha(v_H - p_H);\ Q/k)$ high-types. I discuss the two cases separately.

Case 1: $\alpha(v_H - p_H) \leq Q/k$

In this case the deviating firm will only sell to low-types as it chooses a higher price for high-types than its competitor. Condition $K > Q$ ensures that the deviating firm can serve $\min((1-\alpha)(v_L - p_L^D);\ Q)$ low-type consumers. However, from (3.14)

$$(1-\alpha)(v_L - p_L^D) > (1-\alpha)(v_L - p_L) = Q + Q - \alpha(v_H - p_H)k \geq Q$$

where the first inequality comes from $\delta_L > 0$ and the second from the condition of Case 1. Thus the deviating firm will be capacity constrained and serves Q low-types. Hence its profit is Qp_L^D. The deviation is then profitable if

$$Qp_L^D > \frac{1}{2}(1-\alpha)(v_L - p_L)p_L + \frac{1}{2}\alpha(v_H - p_H)p_H.$$

Using the definition of p_L^D and (3.14) this rewrites as

$$Q(p_L - \delta_L) > Qp_L + \frac{1}{2}\alpha(v_H - p_H)(p_H - kp_L))$$

which by the definition of ε is equivalent to

$\delta_L < \frac{\alpha(v_H - p_H)k}{2Q}\varepsilon.$

Case 2: $\alpha(v_H - p_H) > Q/k$

In this case the non-deviating firm is constrained and it serves Q/k high-types. The deviating firm is able to sell to $\alpha(v_H - p_H^D) - Q/k$ high-types if

$$0 < \alpha(v_H - p_H^D) - Q/k < Q/k.$$

The first inequality always holds, by the condition of Case 2:

$$\alpha(v_H - p_H^D)k > \alpha(v_H - p_H)k > Q$$

and the second inequqlaity holds for any $\delta_H < v_H - p_H - Q/(\alpha k)$. The deviating firm will be able to fill its remaining capaity space with low-types if

$$\alpha(v_H - p_H^D)k - Q + (1-\alpha)(v_L - p_L^D) > Q$$

which rewrites as

$$\alpha(v_H - p_H - \delta_H)k + (1-\alpha)(v_L - p_L + \delta_L) > 2Q = \alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L)$$

which is satisfied for $\delta_H < \frac{1-\alpha}{\alpha k}\varepsilon$. Hence if this condition holds then the deviating firm will sell to $2Q - \alpha(v_H - p_H^D)k$ low-types. The deviation (p_L^D, p_H^D) is then profitable if

$$(\alpha(v_H - p_H^D) - Q/k)p_H^D + (2Q - \alpha(v_H - p_H^D)k)p_L^D > \frac{1}{2}(1-\alpha)(v_L - p_L)p_L + \frac{1}{2}\alpha(v_H - p_H)p_H.$$

After several transformations this condition rewrites as

$$\alpha(v_H - p_H)(\delta_H + k\delta_L - k\varepsilon/2) + \alpha\delta_H(k\varepsilon - \delta_H - k\delta_L) + Q(\varepsilon - 2\delta_L) > 0.$$

This inequality clearly holds if each term is non-negative for which $\delta_L \leq \varepsilon/2$ and $k\varepsilon/2 \leq \delta_H + k\delta_L < k\varepsilon$ are sufficient conditions. Clearly, the choice of $\delta_L = \varepsilon/2$ and $\delta_H < \min(\varepsilon/2; \frac{1-\alpha}{\alpha k}\varepsilon; v_H - p_H - Q/(\alpha k))$ results in a profitable deviation, which concludes the proof of Lemma 3.8. □

Proof of Proposition 3.2

I prove this Lemma in 2 steps. First I derive the necessary and sufficient conditions for the existence of an equilibrium where both firms charge the same unit prices on both markets.

Then I show that the additional condition of $Q \leq K$ ensures uniqueness. Let p_L denote the common unit price on both markets in the candidate equilibrium:

$$p_L = p_H/k = \frac{\alpha k v_H + (1-\alpha)v_L - 2Q}{\alpha k^2 + 1 - \alpha}$$

First notice that

$$p_H < v_H \quad \Leftrightarrow \quad \frac{(1-\alpha)}{2}(v_L - v_H/k) < Q \tag{3.15}$$

is necessary for some consumers of both types to be served as $p_H < v_H$ implies $p_L < v_L$. Furthermore,

$$Q < \frac{\alpha k v_H + (1-\alpha)v_L}{2} \tag{3.16}$$

is equivalent to p_L and p_H being positive. By construction, each firm's total demand at the candidate equilibrium price-pair (p_L, kp_L) equals exactly Q. Thus, Lemma 3.4 requires that

$$\alpha(v_H - p_H)k + (1-\alpha)(v_L - p_L) \geq 2Q \quad \Leftrightarrow \quad K \geq g(Q) \tag{3.17}$$

be satisfied for (p_L, kp_L) to be an equilibrium. Next, I will consider the 3 possible types of deviations: $p_H^D < p_H$, $p_H^D = p_H$, and $p_H^D > p_H$. Firstly, any downward deviation on the high-types market, $p_H^D < p_H$, is unprofitable. Indeed, as Q already binds in the candidate equilibrium, decreasing the price charged for the high-types results in serving the same mass of people but at a lower average unit price. For the same reason, $p_H^D = p_H$ coupled with $p_L^D < p_L$ is also always unprofitable.

Next, consider the deviation consisting of keeping the price charged for the high-types unchanged, $p_H^D = p_H$ together with increasing the price charged for low-types, $p_L^D > p_L$. The same reasoning as in the proof of the analogous part of Proposition 3.1 results in the no-deviation condition of $p_L \geq v_L/3$ which rewrites as

$$Q \leq \frac{\alpha k v_H}{2} + \frac{2(1-\alpha) - \alpha k^2}{6} v_L. \tag{3.18}$$

The remaining possibilities for deviation consist of increasing the price charged for high-types: $p_H^D > p_H$. I distinguish two main cases according to the level of p_H in the candidate equilibrium.

Case 1: $\alpha(v_H - p_H)k \leq Q \quad \Leftrightarrow \quad Q \leq \frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha}$

As upward deviations are being considered, the non-deviating firm will get a demand of $\alpha(v_H - p_H)k$ high-types. In Case 1, the non-deviating firm is able to serve all this demand, so the residual demand for the deviating firm is 0 high-types. Thus the deviating firm can only serve low-types. Its demand from low-types depend on the price it charges to them, there are 3 possibilities: $p_L^D < p_L$, $p_L^D = p_L$, and $p_L^D > p_L$.

$p_L^D < p_L$ is not profitable: the maximal amount of low-types it can serve is Q, which provides a deviating profit of Qp_L^D which is still below the candidate equilibrium profit of Qp_L.

$p_L^D = p_L$ is not profitable either, as the profit it results in is $\frac{1}{2}(1-\alpha)(v_L - p_L)$, which is clearly below Qp_L, the candidate equilibrium profit.

However, choosing $p_L^D > p_L$ is in some cases a profitable deviation. The deviating firm gets the residual demand of low-types which equals $(1-\alpha)(v_L - p_L^D) - (Q - \alpha(v_H - p_H)k)$ which is always smaller than Q if $p_L^D > p_L$. Its profit is thus given by

$$\pi^D(p_L^D) = ((1-\alpha)(v_L - p_L^D) + \alpha(v_H - p_H)k - Q)p_L^D.$$

This profit is quadratic in p_L^D, moreover, $\pi^D(p_L) = Qp_L$. Therefore whenever the unconstrained optimal p_L^D is above p_L, it is a profitable deviation, and there is no profitable deviation whenever it is below p_L. Hence the necessary and sufficient conditions for the existence of a Case 1 profitable deviation are

$$\frac{(1-\alpha)(\alpha v_H k + (1-\alpha)v_L)}{\alpha k^2 + 3(1-\alpha)} < Q \leq \frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha}, \tag{3.19}$$

where the first condition ensures the unconstrained optimal p_L^D being above p_L, the second is the condition of Case 1.

Case 2: $\alpha(v_H - p_H)k > Q \quad \Leftrightarrow \quad Q > \frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha}$

In Case 2 the non-deviating firm's demand from high-types exceeds its capacity Q, thus according to the rationing rule it serves Q/k high-types and 0 low-types. As a result, the deviating firm faces a demand of $(1-\alpha)(v_L - p_L^D)$ low-types for any $p_L^D < v_L$. Moreover, it has the possibility of choosing a low enough p_H^D to serve some of the residual high-types. I distinguish 2 sub-cases according to the choice of p_H^D.

Sub-case 2a: $p_H^D < v_H - \frac{Q}{k\alpha}$

In Sub-case 2a, the deviating firm chooses a low enough price for high-types that it attracts a strictly positive quantity of $\alpha(v_H - p_H^D) - Q/k$ of them. The demand of these high-types is always below its total capacity Q as $p_H^D > p_H$. Depending on the firm's choice of p_L^D, it can serve $\min((1-\alpha)(v_L - p_L^D);\ 2Q - \alpha(v_H - p_H^D)k)$ low-types.

Firstly, I show that choosing a low enough p_L^D to fill capacity is never a profitable deviation. The profit then writes as

$$\pi^D(p_L^D, p_H^D) = (\alpha(v_H - p_H^D) - Q/k)p_H^D + (2Q - \alpha(v_H - p_H^D))p_L^D.$$

It is clearly increasing in p_L^D thus $(1-\alpha)(v_L - p_L^D) = 2Q - \alpha(v_H - p_H^D)k$ must hold in the optimal deviation, which gives the optimal value of p_L^D as a function of p_H^D. Substituting this value to $\pi^D(p_L^D, p_H^D)$ and maximizing with respect to p_H^D, one obtains the unconstrained optimal value of p_H^D as a function of the parameters. It can be shown that this unconstrained optimal value is always below p_H. This means that there is no profitable deviation as $\pi^D(p_H^D)$ is concave and $\pi^D(p_L, p_H) = Qp_L$.

Secondly, I consider the case when the firm chooses a large p_L^D and leaves some idle capacity. Then the deviating profit writes as

$$\pi^D(p_L^D, p_H^D) = (\alpha(v_H - p_H^D) - Q/k)p_H^D + (1-\alpha)(v_L - p_L^D)p_L^D.$$

The unconstrained optimal prices are given by

$$p_H^D = \frac{v_H}{2} - \frac{Q}{2k\alpha} \quad \text{and} \quad p_L^D = v_L/2.$$

Again, the deviating profit is concave in p_H^D and $\pi^D(p_L, p_H) = Qp_L$, therefore a profitable deviation exists if and only if $p_H^D > p_H$, the conditions of Case 2 and Sub-case 2a are satisfied, and p_L^D is large enough to leave idle capacity. The condition of Case 2a is always satisfied in optimum, the condition of leaving idle capacity rewrites as $Q \geq \frac{\alpha k v_H + (1-\alpha)v_L}{3}$ and $p_H^D > p_H$ rewrites as $Q \geq \frac{\alpha k v_H(\alpha k^2 - 1 + \alpha) + 2\alpha k^2(1-\alpha)v_L}{3\alpha k^2 - 1 + \alpha}$. Therefore there exists a profitable deviation in Sub-case 2a if and only if

$$Q \geq \max\left(\frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha};\ \frac{\alpha k v_H(\alpha k^2 - 1 + \alpha) + 2\alpha k^2(1-\alpha)v_L}{3\alpha k^2 - 1 + \alpha};\ \frac{\alpha k v_H + (1-\alpha)v_L}{3}\right).$$

To simplify this condition, first notice that

$$\frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha} = \frac{\alpha k^2}{\alpha k^2 - 1 + \alpha}(1-\alpha)v_L + \frac{\alpha - 1}{\alpha k^2 - 1 + \alpha}\alpha k v_H < (1-\alpha)v_L,$$

the inequality coming from the fact that it is a linear combination of $(1-\alpha)v_L$ and $\alpha k v_H$, the latter having a negative weight. Furthermore,

$$(1-\alpha)v_L < \frac{(\alpha k^2 - 1 + \alpha)}{3\alpha k^2 - 1 + \alpha}\alpha k v_H + \frac{2\alpha k^2}{3\alpha k^2 - 1 + \alpha}(1-\alpha)v_L = \frac{\alpha k v_H(\alpha k^2 - 1 + \alpha) + 2\alpha k^2(1-\alpha)v_L}{3\alpha k^2 - 1 + \alpha}$$

as the righ-hand side is a convex combination of $(1-\alpha)v_L$ and $\alpha k v_H > (1-\alpha)v_L$. Moreover, the condition of $v_L \leq \frac{2\alpha k v_H}{\alpha k^2 + 1 - \alpha}$ stipulated in Proposition 3.2 implies

$$\frac{\alpha k v_H(\alpha k^2 - 1 + \alpha) + 2\alpha k^2(1-\alpha)v_L}{3\alpha k^2 - 1 + \alpha} > \frac{\alpha k v_H + (1-\alpha)v_L}{3}.$$

Therefore, there is a profitable deviation in Sub-case 2a if and only if

$$Q \geq \frac{\alpha k v_H(\alpha k^2 - 1 + \alpha) + 2\alpha k^2(1-\alpha)v_L}{3\alpha k^2 - 1 + \alpha}. \tag{3.20}$$

Sub-case 2b: $p_H^D > v_H - \frac{Q}{k\alpha}$

In Sub-case 2b, the deviating firm chooses such a high price for high-types that in only serves low-types. Its profit is thus

$$\pi^D(p_L^D) = \min((1-\alpha)(v_L - p_L^D); Q)p_L^D.$$

Firstly, I show that there is no profitable deviation if $Q < (1-\alpha)v_L/2$. In this case the firm's optimal price is $p_L^D = v_L - Q/(1-\alpha)$ and the resulting profit is $v_L Q - Q^2/(1-\alpha)$. Clearly, this deviation is profitable if and only if $v_L - Q/(1-\alpha) > p_L$ which rewrites as $Q \leq \frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha}$ which is the opposite of Case 2's condition.

Secondly, if $Q \geq (1-\alpha)v_L/2$ then the firm can implement its unconstrained optimal price of $p_L^D = v_L/2$ resulting in a profit of $(1-\alpha)v_L^2/4$. Comparing this profit to the candidate equilibrium profit, and adding the condition of Case 2 reveals two parameter regions in which a profitable deviation exists:

$$Q \geq \max\left(\frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha};\ \frac{(1-\alpha)v_L}{2};\right.$$
$$\left.\frac{1}{4}\big(\alpha k v_H + (1-\alpha)v_L + \sqrt{(\alpha k v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2(\alpha k^2 + 1 - \alpha)}\big)\right) \quad (3.21)$$

and

$$\max\left(\frac{(1-\alpha)v_L}{2};\ \frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha}\right) \leq Q \leq$$
$$\frac{1}{4}\big(\alpha k v_H + (1-\alpha)v_L - \sqrt{(\alpha k v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2(\alpha k^2 + 1 - \alpha)} \quad (3.22)$$

The condition stipulated in Proposition 3.2, $v_L \leq \frac{2\alpha k v_H}{\alpha k^2 + 1 - \alpha}$, considerably simplifies the above conditions. In particular, it implies

$$\frac{(1-\alpha)v_L}{2} > \max\left(\frac{\alpha(1-\alpha)k(v_L k - v_H)}{\alpha k^2 - 1 + \alpha};\right.$$
$$\left.\frac{1}{4}\big(\alpha k v_H + (1-\alpha)v_L - \sqrt{(\alpha k v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2(\alpha k^2 + 1 - \alpha)}\big)\right) \quad (3.23)$$

(3.23) implies in turn that the region described by (3.22) cannot exist. Moreover, using the inequalities

$$\frac{1}{4}\big(\alpha k v_H + (1-\alpha)v_L + \sqrt{(\alpha k v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2(\alpha k^2 + 1 - \alpha)}\big) > \frac{1}{4}\big(\alpha k v_H + (1-\alpha)v_L\big)$$
$$> \frac{(1-\alpha)v_L}{2}$$

together with (3.23) implies that (3.21) simplifies to

$$Q \geq \frac{1}{4}\big(\alpha k v_H + (1-\alpha)v_L + \sqrt{(\alpha k v_H + (1-\alpha)v_L)^2 - 2(1-\alpha)v_L^2(\alpha k^2 + 1 - \alpha)}\big) \quad (3.24)$$

which is thus a necessary and sufficeint condition for the existence of a profitable deviation in Sub-case 2b.

This concludes the discussion of all potential deviations from the (p_L, kp_L) candidate equilibrium. This price-pair constitutes an equilibrium if and only if (3.15), (3.16), (3.17), and (3.18) are jointly satisfied while none of the conditions (3.19), (3.20) and (3.24) hold. These provide exactly the conditions described by Proposition 3.2.

then Lemma 3.8 applies and ensures that in any equilibrium $p_L = p_H/k$. This establishes the uniqueness of the equilibrium and thus concludes the proof of Proposition 3.2. □

Proof of Proposition 3.3

Lemma 3.4 ensures that any equilibrium price pair where both constraints bind must satisfy

$$\alpha(v_H - p_H) + (1-\alpha)(v_L - p_L) = 2K$$

and

$$(v_H - p_H)k + (1-\alpha)(v_L - p_L) = 2Q$$

which pins down the prices as

$$p_L = v_L - \frac{2kK - 2Q}{(1-\alpha)(k-1)} \quad \text{and}$$

$$p_H = v_H - \frac{2Q - 2K}{\alpha(k-1)}.$$

Next I find sufficient conditions that ensure that no deviation from the candidate equilibrium price pair (p_L, p_H) be profitable. I will consider the 3 possible types of deviations: $p_H^D < p_H$, $p_H^D = p_H$, and $p_H^D > p_H$. Firstly, as before, any downward deviation on the high-types market, $p_H^D < p_H$, is unprofitable. As the capacity constraints already bind in the candidate equilibrium, decreasing the price charged for the high-types results in serving the same mass of people but at a lower average unit price. For the same reason, $p_H^D = p_H$ coupled with $p_L^D < p_L$ is also always unprofitable.

Next, consider the deviation consisting of keeping the price charged for the high-types unchanged, $p_H^D = p_H$ together with increasing the price charged for low-types, $p_L^D > p_L$. The same reasoning as in the proof of the analogous part of Proposition 3.1 applies, so there is a profitable deviation if and only if $p_L \leq v_L/3$ which rewrites as

$$K \leq Q/k + \frac{(1-\alpha)(k-1)}{3k} v_L. \tag{3.25}$$

The remaining possibilities for deviation consist of increasing the price charged for high-types: $p_H^D > p_H$. I distinguish two main cases according to the level of p_H in the candidate equilibrium.

Case 1: $\alpha(v_H - p_H)k \leq Q \quad \Leftrightarrow \quad K \geq \frac{k+1}{2k}Q$

As upward deviations are being considered, the non-deviating firm will get a demand of $\alpha(v_H - p_H)k$ high-types. In Case 1, the non-deviating firm is able to serve all this demand, so the residual demand for the deviating firm is 0 high-types. Thus the deviating firm can only serve low-types. Its demand from low-types depends on the price it charges to them, there are 3 possibilities: $p_L^D < p_L$, $p_L^D = p_L$, and $p_L^D > p_L$.

Sub-case 1a: $p_L^D < p_L$

In Sub-case 1a, the deviating firm undercuts its rival on the low market, so it faces a demand of $(1-\alpha)(v_L - p_L^D)$ low-types. Serving only low-types, clearly either its capacity K binds or both of its capacities are slack. Its profit is thus

$$\pi^D(p_L^D) = \min((1-\alpha)(v_L - p_L^D); K)p_L^D.$$

Firstly, I show that there is no profitable deviation if $K < (1-\alpha)v_L/2$. In this case the firm's optimal price is $p_L^D = v_L - K/(1-\alpha)$. The condition of $p_L^D < p_L$ rewrites as $v_L - K/(1-\alpha) < v_L - \frac{2Kk-2Q}{(1-\alpha)(k-1)}$ which is equivalent to $K < \frac{2}{k+1}Q$ which contradicts Case 1's condition of $K \geq \frac{k+1}{2k}Q$.

Secondly, if $K \geq (1-\alpha)v_L/2$ then the firm can implement its unconstrained optimal price of $p_L^D = v_L/2$ resulting in a profit of $(1-\alpha)v_L^2/4$. However, I show that the conditions of Case 1 and Sub-case 1a at the optimal price imply that $K < (1-\alpha)v_L/2$ so this deviation is not feasible. Indeed, $p_L^D = v_L/2 < p_L$ rewrites as

$$K < Q/k + \frac{(1-\alpha)(k-1)v_L}{4k}.$$

The slope of this curve is clearly inferior that the slope of $\frac{k+1}{2k}Q$, so the conditions of Case 1 and Sub-case 1a can only be jointly satisfied if K is smaller than the point where the two curves intersect. They intersect at

$$K = \frac{(1-\alpha)v_L}{2}\frac{(k+1)}{2k} < \frac{(1-\alpha)v_L}{2},$$

therefore $K > \frac{(1-\alpha)v_L}{2}$ is excluded.

Sub-case 1b: $p_L^D = p_L$

This choice clearly cannot constitute a profitable deviation: the deviating firm serves the same mass of low-types for the same price as in the candidate equilibrium, but serves no high-types.

Sub-case 1c: $p_L^D > p_L$

In this sub-case the deviating firm sells to none of the high-types and only to the residual amount of $(1-\alpha)(v_L - p_L^D) - Q + \alpha(v_H - p_H)k$ low-types. Either capacity K binds or none of the constraints bind as the deviating firm only sells to low-types. The deviating profit writes as

$$\pi^D(p_L^D) = \min((1-\alpha)(v_L - p_L^D) - Q + \alpha(v_H - p_H)k; K)p_L^D.$$

K binds, i.e., $(1-\alpha)(v_L - p_L^D) - Q + \alpha(v_H - p_H)k > K$ if

$$p_L^D < \overline{p_L^D} \equiv v_L + \frac{(k+1)Q}{(k-1)(1-\alpha)} - \frac{(3k-1)K}{(k-1)(1-\alpha)}$$

The unconstrained profit is given by

$$p_L^{D*} = v_L/2 + \frac{(k+1)Q}{2(k-1)(1-\alpha)} - \frac{kK}{(k-1)(1-\alpha)}.$$

If $p_L^{D*} < \overline{p_L^D}$ then the optimal deviation is $\overline{p_L^D}$, choosing a price to exactly exhaust capacity K. The condition rewrites as

$$K < \frac{(k+1)Q}{4k-2} + \frac{(k-1)(1-\alpha)v_L}{4k-2}.$$

The profit of $\overline{p_L^D}$ is better than the candidate equilibrium profit if

$$(2+\alpha(k-1)(3-k))K < 2Q - \alpha(1-\alpha)(k-1)(v_H - v_L).$$

Thus there is a profitable deviation if either

$$\frac{k+1}{2k}Q < K < \min\left(\frac{(k+1)Q}{4k-2} + \frac{(k-1)(1-\alpha)v_L}{4k-2};\right.$$
$$\left.\frac{2Q-\alpha(1-\alpha)(k-1)(v_H-v_L)}{2+\alpha(k-1)(3-k)}\right) \quad \text{and} \quad k < 1+\sqrt{\frac{2(1-\alpha)}{\alpha}} \tag{3.26}$$

or

$$\max\left(\frac{k+1}{2k}Q;\ \frac{2Q-\alpha(1-\alpha)(k-1)(v_H-v_L)}{2+\alpha(k-1)(3-k)}\right) < K < \frac{(k+1)Q}{4k-2} + \frac{(k-1)(1-\alpha)v_L}{4k-2}$$
$$\text{and} \quad k > 1+\sqrt{\frac{2(1-\alpha)}{\alpha}} \tag{3.27}$$

is satisfied. If $k = 1+\sqrt{\frac{2(1-\alpha)}{\alpha}}$ then the 3 conditions necessary for a profitable deviations are incompatible.

Case 2: $\alpha(v_H - p_H)k > Q \quad \Leftrightarrow \quad K < \frac{k+1}{2k}Q$

In Case 2 the non-deviating firm's demand from high-types exceeds its capacity Q, thus according to the rationing rule it serves Q/k high-types and 0 low-types. Condition (3.2) ensures that Q binds and K is slack if serving only high-types. As a result, the deviating firm faces a demand of $(1-\alpha)(v_L - p_L^D)$ low-types for any $p_L^D < v_L$. Moreover, it has the possibility of choosing a low enough p_H^D to serve some of the residual high-types. I distinguish 2 sub-cases according to the choice of p_H^D.

Sub-case 2a: $p_H^D \geq v_H - \frac{Q}{k\alpha}$

In Sub-case 2a, the deviating firm chooses such a high price for high-types that in only serves low-types. Serving only low-types, clearly either its capacity K binds or both of its capacities are slack. Its profit is thus

$$\pi^D(p_L^D) = \min((1-\alpha)(v_L - p_L^D); K)p_L^D.$$

Firstly, if $K \geq (1-\alpha)v_L/2$ then the firm can implement its unconstrained optimal price of $p_L^D = v_L/2$ resulting in a profit of $(1-\alpha)v_L^2/4$. The candidate equilibrium profit is lower than this deviating profit if

$$\frac{(kK-Q)v_L+(Q-K)v_H}{(k-1)}-\frac{2}{\alpha(1-\alpha)(k-1)^2}(Q^2+(\alpha k^2+1-\alpha)K^2-2KQ(\alpha k+1-\alpha)) < (1-\alpha)v_L^2/4$$

which rewrites as

$$0 < K^2 - K\left(\frac{2(\alpha k+1-\alpha)}{\alpha k^2+1-\alpha}Q+\frac{\alpha(1-\alpha)(k-1)(kv_L-v_H)}{2(\alpha k^2+1-\alpha)}\right)+ \\ \frac{Q^2}{\alpha k^2+1-\alpha}+\frac{\alpha(1-\alpha)^2(k-1)^2v_L^2}{8(\alpha k^2+1-\alpha)}-\frac{\alpha(1-\alpha)(k-1)(v_H-v_L)Q}{2(\alpha k^2+1-\alpha)} \quad (3.28)$$

which corresponds to the area outside of an ellipse in the K-Q capacity space. There is thus a profitable deviation if

$$(1-\alpha)v_L/2 \leq K < \frac{k+1}{2k}Q \quad \text{and (3.28) is also satisfied.} \quad (3.29)$$

Secondly, if $K < (1-\alpha)v_L/2$ then the firm's optimal price is $p_L^D = v_L - K/(1-\alpha)$. This deviation is profitable if

$$Kv_L - K^2/(1-\alpha) > -\frac{2}{\alpha(1-\alpha)(k-1)^2}(Q^2+(\alpha k^2+1-\alpha)K^2-2KQ(\alpha k+1-\alpha))+ \\ \frac{(kK-Q)v_L+(Q-K)v_H}{(k-1)}. \quad (3.30)$$

Similarly to the previous case, geometrically this corresponds to the outside of an ellipse in the K-Q space. Therefore there is a profitable deviation if

$$K < \min\left(\frac{k+1}{2k}Q;\ (1-\alpha)v_L/2\right) \quad \text{and (3.30) is also satisfied.} \quad (3.31)$$

Sub-case 2b: $p_H^D < v_H - \frac{Q}{k\alpha}$

In Sub-case 2b, the deviating firm chooses a low enough price for high-types that it attracts a strictly positive quantity of $\alpha(v_H - p_H^D) - Q/k$ of them. As the deviating firm serves less high-types than in the candidate equilibrium, either K binds and Q is slack or both constraints are slack. In particular, depending on the firm's choice of p_L^D, it can serve $\min((1-\alpha)(v_L - p_L^D);\ K + Q/k - \alpha(v_H - p_H^D))$ low-types.

Firstly, I consider choosing a low enough p_L^D to fill capacity. The profit then writes as

$$\pi^D(p_L^D, p_H^D) = (\alpha(v_H - p_H^D) - Q/k)p_H^D + (K + Q/k - \alpha(v_H - p_H^D))p_L^D.$$

It is clearly increasing in p_L^D thus $(1-\alpha)(v_L - p_L^D) = K + Q/k - \alpha(v_H - p_H^D)$ must hold in the optimal deviation, which gives the optimal value of p_L^D as a function of p_H^D. Substituting this value to $\pi^D(p_L^D, p_H^D)$ and maximizing with respect to p_H^D, one obtains the unconstrained optimal value of p_H^D:

$$p_H^{D*} = \frac{1+\alpha}{2}v_H + \frac{1-\alpha}{2}v_L - K - \frac{Q}{k}\frac{1+\alpha}{2\alpha}.$$

This value always satisfies the condition of Sub-case 2b. For p_H^{D*} to indeed be an upward deviation from p_H, one needs

$$K \le \frac{(3k - \alpha k + 1 + \alpha)Q - \alpha(1-\alpha)(v_H - v_L)(k-1)}{2k(\alpha k - \alpha + 2)}. \tag{3.32}$$

It can be shown that (3.3) is always a stronger condition than $K \le \frac{k+1}{2k}Q$. Therefore there is a profitable deviation in Sub-case 2b with filling capacity K if and only if (3.32) is satisfied.

Next, consider the choice of a high enough p_L^D so that none of the capacities bind after the deviation. Then the deviating profit writes as

$$\pi^D(p_L^D, p_H^D) = (\alpha(v_H - p_H^D) - Q/k)p_H^D + (1-\alpha)(v_L - p_L^D)p_L^D.$$

For the optimal deviating prices of $p_L^{D*} = v_L/2$ and $p_H^{D*} = \frac{v_H}{2} - \frac{Q}{2k\alpha}$ to be attainable, one needs to ensure that the capacities be indeed idle and that this p_H^{D*} constitute an upward deviation from the candidate equilibrium p_H. As $\pi^D(p_L^D, p_H^D)$ is concave in p_H^D and $\pi^D(p_L, p_H)$ equals the candidate equilibrium profit, these two conditions are indeed necessary and sufficient for (p_L^{D*}, p_H^{D*}) to constitute a profitable deviation. They rewrite as

$$\frac{\alpha v_H + (1-\alpha)v_L}{2} - \frac{Q}{2k} < K < \frac{3k+1}{4k}Q - \frac{\alpha(k-1)}{4}v_H,$$

i.e., there is a profitable deviation in Sub-case 2b with leaving idle capacity if and only if

$$\frac{\alpha v_H + (1-\alpha)v_L}{2} - \frac{Q}{2k} < K < \min\left(\frac{3k+1}{4k}Q - \frac{\alpha(k-1)}{4}v_H;\ \frac{k+1}{2k}Q\right). \tag{3.33}$$

Therefore, for the (p_L, p_H) pair to constitute an equilibrium, one needs conditions (3.2) and

(3.3) to simultaneously hold while none of the following conditions are satisfied: (3.25), (3.26), (3.27), (3.29), (3.31), (3.32), and (3.33). □

Proof of claims in Section 3.5.2

First I show that

$$p_L^{mon} < p_L^K = p_H^K < p_H^{mon}$$

holds for capacity-pairs satisfying the conditions in Proposition 3.1 and either $K \leq \frac{1}{4}(\alpha v_H + (1-\alpha)v_L)$ or $Q \leq \frac{1}{4}(\alpha k v_H + (1-\alpha)v_L)$ or both conditions. The first inequality rewrites as

$$v_L + \frac{\alpha}{2}(v_H - v_L) - 2K < \alpha v_H + (1-\alpha)v_L - 2K \;\Leftrightarrow\; 0 < \frac{1}{2}\alpha(v_H - v_L),$$

whereas the second inequality rewrites as

$$\alpha v_H + (1-\alpha)v_L - 2K < v_H - v_L + \frac{1-\alpha}{2}(v_H - v_L) - 2K \;\Leftrightarrow\; 0 < \frac{1}{2}(1-\alpha)(v_H - v_L),$$

therefore $v_L < v_H$ ensures that the original inequalities hold.

Next, I show that the aggregate consumer surplus is larger in the duopoly than in the monopoly. Replacing the equilibrium prices and the optimal prices to the general formula for consumer surplus, one obtains

$$\begin{aligned} CS^{duop} - CS^{mon} = &\left(\frac{\alpha}{2}\big(2K + (1-\alpha)(v_H - v_L)\big)^2 + \frac{1-\alpha}{2}\big(2K - \alpha(v_H - v_L)\big)^2\right) - \\ &\left(\frac{\alpha}{2}\big(2K + \frac{1}{2}(1-\alpha)(v_H - v_L)\big)^2 + \frac{1-\alpha}{2}\big(2K - \frac{1}{2}\alpha(v_H - v_L)\big)^2\right) = \\ &\frac{3}{8}\alpha(1-\alpha)(v_H - v_L)^2 > 0. \end{aligned}$$

□

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ÉCOLE DOCTORALE
Sciences de l'homme et de la société (SHS)

Titre : Marchés caractérisés par des contraintes de capacité

Mots clés : Contrainte de capacité, Prix, Monopole, Duopole

Résumé : Cette thèse est composée d'une introduction et de trois chapitres. Le premier chapitre étudie le comportement d'un duopole lorsque les deux entreprises sont caractérisées par des contraintes de capacité et produisent un bien différencié à la Hotelling. En limitant l'analyse au cas d'un degré intermédiaire de différentiation horizontale de produits, je démontre l'existence d'au moins un équilibre en stratégie pure pour tous les niveaux de capacités. Le deuxième chapitre analyse un monopole qui est contraint par deux types de contraintes de capacité : un sur les quantités produites, l'autre sur le nombre des consommateurs servis. Je montre l'existence d'une région de paramètres où les deux contraintes de capacité sont saturées, et que dans cette zone de paramètres, le prix n'est pas une fonction monotone du niveau des contraintes de capacité. A une capacité plus grande peut correspondre un prix plus élevé car la composition de la clientèle varie avec la capacité. Pour la même raison, le bien-être agrégé des consommateurs n'augmente pas nécessairement si une des contraintes de capacité est augmentée. Le troisième chapitre étend le modèle de fixation de prix avec deux contraintes de capacité au cas du duopole symétrique. Je caractérise des conditions sous lesquelles un équilibre symétrique existe. Je montre qu'il existe des conditions sous lesquelles la non-monotonicité des prix et du bien-être des consommateurs observés dans le deuxième chapitre est également présente dans le cas du duopole.

Title : Essays on capacity-constrained pricing

Keywords : Capacity constraint, Pricing, Monopoly, Duopoly

Abstract : This Ph.D. thesis is composed of an introduction and three chapters. The first chapter studies duopoly behavior when both firms are characterized by capacity constraints and produce a good which is differentiated a la Hotelling. Assuming a substantial level of product differentiation results in a variety of equilibrium firm behavior and it generates at least one pure-strategy equilibrium for any capacity level. The second chapter studies the price-setting behavior of a monopoly facing two capacity constraints: one on the number of consumers it can serve, the other on the total amount of products it can sell. The characterization of the firm's optimal pricing as a function of its two capacities reveals a rich structure that also gives rise to some surprising results. In particular, I show that prices are non-monotonic in capacity levels. Moreover, there always exists a range of parameters in which weakening one of the capacity constraints decreases consumer surplus. The third chapter extends the analysis of the second chapter to symmetric duopoly pricing under dual capacity constraints, limiting both the total quantity and the number of consumers served. By isolating parameter regions where a symmetric pure-strategy equilibrium exists, I find that several types of equilibria are possible, depending on the model's specifications. Equilibrium prices are non-monotonic in capacity levels if consumers' valuations are sufficiently heterogeneous. Moreover, aggregate consumer surplus is also non-monotonic in some of the equilibria that I identify.

Université Paris-Saclay
Espace Technologique / Immeuble Discovery
Route de l'Orme aux Merisiers RD 128 / 91190 Saint-Aubin, France