Renormalization of SU(2) Yang-Mills theory with flow equations

Alexander Efremov

NNT : 2017SACLX050

THÈSE DE DOCTORAT DE L'UNIVERSITÉ PARIS-SACLAY PRÉPARÉE À L'ECOLE POLYTECHNIQUE

ÉCOLE DOCTORALE N°564
physique de l'Ile-de-France

Spécialité de doctorat : Physique

Par

M. Alexander Efremov

Renormalization of SU(2) Yang-Mills theory with Flow Equations

Thèse présentée et soutenue à Palaiseau, le 27 septembre 2017 :

Composition du Jury :

M. Ch. Kopper	Professeur	École Polytechnique	Directeur de thèse
M. S. Hollands	Professeur	University of Leipzig	Rapporteur
M. M. Salmhofer	Professeur	University of Heidelberg	Rapporteur
M. R. Guida	Docteur	IPhT, CEA	Examinateur
M. T. R. Morris	Professeur	University of Southampton	Examinateur
M. V. Rivasseau	Professeur	Université Paris-Sud	Président
M. N. Wschebor	Professeur	Uruguay University	Examinateur

Acknowledgements

Above all I wish to thank my supervisor Ch. Kopper. My entire work would not have been possible without his unique experience, patience and help.

A special word of gratitude goes to my home laboratory, the Center of Theoretical Physics at the École Polytechnique. In particular I would like to thank the director, academics and staff members.

I am greatly indebted to R. Guida for his help in writing the thesis. His influence on almost every chapter is enormous. I am also greatful to the Mathematical Physics group at the Institute of Theoretical Physics of CEA Saclay for hospitality during the writing.

I wish express my gratitude to all jury members and especially to the reporters for accepting this charge.

I am indepted to the Institute for Theoretical Physics of the University of Leipzig whose facilities and financial support were used during the work on the final version of the manuscript. I am especially greatful to S. Hollands for the time he devoted to study this work, numerous discussions and warm hospitality.

I also thank H. Gies from the University of Jena for his interest in this paper, valuable remarks and questions.

Finally I would like to thank my family for their support in my endeavor.

Contents

1 Introduction

La théorie de la renormalisation basée sur les equations différentielles du flot [Wil71, WH73, Pol84] a permit de construire une approche unifiée de l'analyse de la renormalisabilité pour une grande classe des théories sans utilisation des graphes de Feynman. Cette méthode a été appliquée pour demontrer les bornes supérieures dans l'espace des moments de la theorie ϕ^4 [KM02], elle a été aussi utilisée pour prouver la renormalisabilité de la theorie Yang–Mills SU(2) spontanément brisée [KM09] et pour établir des bornes uniformes sur les fonctions de Schwinger pour la theorie ϕ^4 sans masse [GK11]. Dès les premieres publications, [YM54, FP67a, Sla72, Tay71, LZJ72, tHV72, BRS75, Tyu75, ZJ75] (voir [Lai81] pour plus de références), une variété de résultats sur la renormalisabilité des théories de jauge nonabeliennes a été apparu dans la littérature, dans différents contextes et avec le niveau de rigueur différent. Les papiers sur ce problème dans le contexte des equations du flot (FE) comprennent [RW94, Bec96, BDM95, MD96], et, plus récemment, [FHH16]. Le présent travail traite le problem de la renormalisation perturbative et partage certains aspects avec les articles suivants: de [Bec96] nous empruntons l'idée fructueuse que l'opérateur local décrivant la violation des identités Slavnov–Taylor (STI) pour les fonctions irréductibles (1PI) est contraint par la nilpotence de l'opérateur Slavnov classique [ZJ75, KSZ75]; comme dans [BDM95] (et en contraste avec [FHH16]), nous définissons les corrélateurs marginaux par les conditions aux bords physiques en l'absence de cut-off IR et aux moments non exceptionnels; on s'appuie sur une extension des bornes de [GK11]. Notre résultat principal est une preuve de la renormalisabilité de la théorie Yang–Mills SU(2) qui complète les traveaux précédents en ce qui concerne des caractéristiques suivantes:

i Le contrôle rigoureux du comportement IR et UV des fonctions irréductibles est établi au moyen de bornes uniformes dans l'espace de moments. Les bornes donnent aussi l'existence de la limite IR pour les fonctions 1PI à des moments non exceptionnels et l'existence de la limite UV.

ii La restauration des identités Slavnov–Taylor dans la limite UV est prouvée;

iii Les conditions de renormalisation sont données explicitement.

Il est remarquable que les FE nous permet de construire toutes les fonctions irréductibles $\Gamma^{\vec{\phi}}$ de la théorie en utilisant uniquement les conditions de renormalisation. L'observation clé ici est que les FE, voir (99), peuvent être résolu perturbativement, c'est à dire, à l'aide de l'expansion formelle en nombre de boucles $\Gamma^{\vec{\phi}} = \sum_l \hbar^l \Gamma_l^{\vec{\phi}}$. D'abord on doit considérer la construction des termes irrelevants. On met le cut-off IR Λ égal au cut-off UV Λ_0. Nous obtenons ainsi des termes semi-classiques irrelevants qui doivent tous s'annuler, c'est à dire $\partial^w \Gamma_l^{\Lambda_0\Lambda_0;\vec{\phi}} = 0$. Ici ∂^w signifie les dérivées par rapport aux arguments dans l'espace de moments. Ensuite, nous intégrons le flot par rapport au λ pour trouver ces termes au cut-off Λ arbitraire

$$\partial^w \Gamma_l^{\Lambda\Lambda_0;\vec{\phi}} = \int_{\Lambda_0}^{\Lambda} d\lambda\, \partial^w \partial_\lambda \Gamma_l^{\lambda\Lambda_0;\vec{\phi}} \,. \tag{1}$$

Comme nous l'avons déclaré, les FE peuvent être résolue en utilisant la théorie de la perturbation, donc le côté droite de (1) est donné en termes de fonctions irréductibles avec les ordres en boucle $l' < l$. Les termes relevants doivent être gérés l'un après l'autre, de marginaux aux terms strictement relevants. A partir du point de renormalisation, ces termes sont construits en intégrant les FE à Λ arbitraire et en suite interpolant du point de renormalisation à des moments arbitraires. De la même façon, nous procédons pour construire les fonctions irréductibles en nombre de boucles suivant. Pour rendre cette construction significative, nous demontrerons ici que toutes les étapes ci-dessus sont bien définies en prouvant des bornes sur les fonctions irréductibles. La stratégie de la preuve de ces bournes reste comme dans [GK11].

Pour prouver la restauration de la symétrie BRST [BRS75, Tyu75], nous avons commencé l'analyse de les STI en considérant différents points de renormalisation. Il est devenu évident que différents points nous mènent aux différents types de bornes sur les fonctions irréductibles. Deux candidats à ces bornes ont été trouvés et étudiés. Bien que certains points de renormalisation sont très attirants pour la physique, ils impliquent des structures des bornes difficiles à analyser, voir la section 5. Le choix final a deux vertus importants: la simplicité et la signification physique directe des conditions de renormalisation. Dans ce schéma, les termes qui correspondent aux terms de la théorie classique Yang–Mills sont explicitement renormalises aux points physiques sans le cut-off infrarouge, mais les termes restants sont renormalises en présence d'un régulateur infrarouge, qui est en effet un outil purement mathématique. Il introduit la dependance indésirable du régulateur infrarouge sous-jacent. Cependant, comme indiqué dans la proposition 18, ces conditions moins physiques impliquent l'annulation des fonctions irréductibles à un point physique et par conséquent sont équivalent à des conditions qui auraient pu être imposées sans le cut-off infrarouge.

Notre preuve se réfère explicitement à la théorie Yang–Mills SU(2). Cependant, il pourrait être étendu sans modifications importantes aux autres groupes de Lie compacts semi-simples.

1.1 Preface

Renormalization theory based on the differential Flow Equations [Wil71, WH73, Pol84] has allowed to build a unified approach to the analysis of renormalizability for a wide class of theories without recourse to Feynman graphs. This method was applied to show momentum bounds of massive ϕ^4 theory [KM02], used to prove renormalization of spontaneously broken SU(2) Yang–Mills [KM09] and to establish uniform bounds on Schwinger functions of massless ϕ^4 fields [GK11]. Starting with the milestone works, [YM54, FP67a, Sla72, Tay71, LZJ72, tHV72, BRS75, Tyu75, ZJ75] (see [Lai81] for more references), a variety of results on the renormalizability of nonabelian gauge theories flourished in the literature, in different contexts and with different level of mathematical rigor. Work on this problem in the context of Flow Equations (FE) includes [RW94, Bec96, BDM95, MD96], and, more recently, [FHH16]. The present work deals with perturbative renormalizability and shares certain aspects with some of these articles: from [Bec96] we borrow the fruitful idea that the local operator describing the violation of Slavnov–Taylor identities (STI) for the one-particle irreducible (1PI) functions [ZJ75, KSZ75] is constrained by the nilpotency of the underlying "Slavnov differential operator"; as in [BDM95] (and in contrast with [FHH16]) we define the marginal correlators by physical boundary conditions at vanishing IR cutoff and nonexceptional momenta; we rely on an extension of the bounds of [GK11]. Our main result is a proof of the renormalizability of Yang–Mills theory that complements the previous work in view of the following features:

i Rigorous control of the IR and UV behavior of the one-particle irreducible functions is established by means of uniform bounds in momentum space. In particular, from the bounds follow the existence of the IR and UV limits of 1PI functions at nonexceptional momenta.

ii The vanishing of the STI violation in the UV limit is proven;

iii The renormalization conditions are given explicitly.

It is remarkable that the FE allows us to construct all vertex functions $\Gamma^{\vec{\phi}}$ of the theory using only the renormalization conditions. The key observation here is that the FE, see (99), can be solved using the formal loop expansion $\Gamma^{\vec{\phi}} = \sum_l \hbar^l \Gamma_l^{\vec{\phi}}$. First one should consider the construction of the irrelevant terms. By setting the IR cutoff Λ equal to the UV cutoff Λ_0 we obtain semi-classical irrelevant terms which are all vanishing, i.e. $\partial^w \Gamma_l^{\Lambda_0\Lambda_0;\vec{\phi}} = 0$. Here ∂^w indicates momentum derivatives. Then we integrate the flow wrt Λ to find these terms at arbitrary IR cutoff

$$\partial^w \Gamma_l^{\Lambda\Lambda_0;\vec{\phi}} = \int_{\Lambda_0}^{\Lambda} d\lambda \; \partial^w \partial_\lambda \Gamma_l^{\lambda\Lambda_0;\vec{\phi}} \, . \tag{2}$$

As we have stated the FE can be solved using the theory of perturbation, thus the rhs in (2) is given in terms of vertex functions with loop order $l' < l$. The relevant terms should be handled sequentially from marginal to strictly relevant ones. Starting from a renormalization point these terms are constructed by integrating the FE to arbitrary Λ and by interpolating from the renormalization point to arbitrary momenta. In a similar way we define the vertex functions in the next loop order. To make this construction meaningful we shall show that all above steps are well defined by proving corresponding bounds on the vertex functions. The strategy of the proof of these bounds remains that of [GK11].

To prove the restoration of the BRST symmetry [BRS75, Tyu75], we started the analysis of the STI by considering various renormalization points. It became clear that different points lead to different types of momentum bounds on the vertex functions. Two candidates for such bounds were found and studied. Although some renormalization points look very attractive for physics, they imply complex structures for the bounds, which are difficult to analyze, see section 5. The retained choice exhibits both virtues: simplicity and direct physical significance of the renormalization conditions. In this scheme the terms which have correspondence in the classical Yang–Mills (YM) theory are explicitly fixed at a physical renormalization point with no infrared cutoff, but the remaining terms are fine-tuned in presence of an infrared regulator, which is a pure mathematical tool. It introduces undesired explicit dependence on the underlying infrared regularization. However, as stated in proposition 18, these less physical conditions imply the vanishing of corresponding vertex functions at some physical point and consequently are equivalent to conditions which are imposed without infrared cutoff.

Our proof refers explicitly to SU(2) YM theory. However, it could be extended without important conceptual changes to other semi-simple compact Lie groups.

1.2 Outline of the thesis

We proceed as follows. First we fix the notations, introduce the classical Yang–Mills action, generating functionals and regulators. We define the BRST transformations. Then we derive the FE of the renormalization group, for the connected amputated Schwinger functions and for the one-particle irreducible vertex functions. Finally we study the STI, the Antighost Equation (AGE) and their violation. Remark that in our context gauge invariance is broken through the presence of the cutoffs.

Our proof of renormalizability of Yang-Mills theory is based on momentum bounds for the vertex functions which permit to take the limits $\Lambda \to 0$ (IR-cutoff) and $\Lambda_0 \to \infty$ (UV-cutoff) for nonexceptional external momenta. These bounds are established inductively with the aid of the FE. They are expressed in terms of tree amplitudes and polynomials of logarithms. For our trees we only have to consider vertices of coordination numbers 1 and 3.

In section 2 we present the definitions of the tree structures, and we state the aforementioned bounds. We also have to consider vertex functions with operator insertions which permit to formulate the violation of the STI. Our bounds then permit to show that, for suitable renormalization conditions, the functions describing the violation of the STI vanish in the UV limit.

Section 3 is dedicated to the proof of the bounds of section 2. At loop order l, the rhs of the FE is a sum of chains, i.e. products of vertex functions in lower loop order joined by free propagators. Chains are then closed by a derived propagator and integrated over the circulating loop momentum. Our technique of proof is based on the fact that applying our inductive bounds on these chains we reproduce these bounds in the next loop order. The proof treats irrelevant terms first, then marginal and finally strictly relevant ones. Particular attention has to be paid to the renormalization conditions. Section 3 ends with a proof of UV-convergence for $\Lambda_0 \to \infty$.

In section 4 we prove that the renormalization conditions required in section 3 to prove the bounds of section 2 can actually be imposed. These renormalization conditions are such that they leave us free to fix the physical coupling of the theory and the field normalizations. At the same time they permit us to make vanish the relevant part of all functions describing the violation of the STI. This is required in the previous proof.

In section 5 we present complementary bounds on vertex functions with anti-ghost fields. These bounds are more accurate at small momenta and allow to renormalize all vertex functions at vanishing IR cutoff. Furthermore the important proposition 18, established in section 2.2, is a simple corollary of these bounds

In the Appendices A, B, C, D, respectively, we present some facts on Gaussian measures, examples of chains of vertex functions, an analysis of linear independence of euclidean tensor structures and a large number of elementary bounds on integrals we encounter in the proofs. We also add bounds on the propagators and their derivatives. In the three subsequent appendices E, F, G we analyse the generating functionals of the (inserted) vertex functions, as far as they have relevant content. In the last two appendices H, I we present the list of renormalization points and operator insertions to be considered.

1.3 Notations

$\mathbb{N} = \{0, 1, 2, ...\}$ is the set of nonnegative integers. $|S|$ is the cardinality of a set S. Furthermore, $(a, b, c, ...)$, $\{a, b, c, ...\}$ denote a sequence and a set, respectively. Unless otherwise stated, sequence stands for finite sequence. For shortness we set $[a : b] := \{i \in \mathbb{Z} : a \leqslant i \leqslant b\}$ and $[b] := [1 : b]$. Repeated indices are implicitly summed over, e.g. $A^a t_a := \Sigma_a A^a t_a$. We choose the following basis of the Lie algebra

$$(t_c)^{ab} := -i\epsilon_{abc}, \qquad [t_a, t_b] = i\epsilon_{abc} t_c, \qquad a, b, c \in \{1, 2, 3\}, \tag{3}$$

where ϵ_{abc} is the Levi–Civita symbol, $\epsilon_{123} = 1$. In this manuscript we will deal with tensor fields on $\mathbb{R}^4$ in Cartesian coordinate systems with metric tensor $\delta_{\mu\nu}$. If A, B are two Cartesian tensors of $\mathbb{R}^4$ of rank r with components $A_{\mu_1 \ldots \mu_r}$ and $B_{\mu_1 \ldots \mu_r}$, respectively, then the scalar product (A, B) is the contraction

$$(A, B) := A^*_{\mu_1 \ldots \mu_r} B_{\mu_1 \ldots \mu_r} . \tag{4}$$

Given a Cartesian tensor T, we use the norm

$$|T| := (T,T)^{\frac{1}{2}}. \tag{5}$$

For instance, for $p \in \mathbb{R}^4$, $|p|^2 = \Sigma_\mu p_\mu^2$. Let T, A, and B be Cartesian tensors such that $T_{\vec{\mu}\vec{\nu}} = A_{\vec{\mu}\vec{\rho}} B_{\vec{\rho}\vec{\nu}}$ where $\vec{\mu}$, $\vec{\rho}$, $\vec{\nu}$ are multi-indices, for example $\vec{\mu} := (\mu_1, ..., \mu_n)$. Then using the Cauchy–Schwarz inequality

$$|T|^2 = \sum_{\vec{\mu},\vec{\nu}} |\sum_{\vec{\rho}} A_{\vec{\mu}\vec{\rho}} B_{\vec{\rho}\vec{\nu}}|^2 \leqslant \sum_{\vec{\mu},\vec{\nu}} \sum_{\vec{\rho},\vec{\sigma}} |A_{\vec{\mu}\vec{\rho}}|^2 |B_{\vec{\sigma}\vec{\nu}}|^2 = |A|^2 |B|^2 \,. \tag{6}$$

The integral over $\mathbb{R}^4$ of the product of two functions is denoted by

$$\langle f_1, f_2 \rangle := \int d^4x \, f_1(x) f_2(x) \,, \tag{7}$$

and the Fourier transform of a function is defined by

$$f(p) := \int d^4x \, e^{-ipx} f(x). \tag{8}$$

The convolution of two functions f, g is denoted as below

$$(f * g)(x) = \int d^4y \, f(y) g(x-y). \tag{9}$$

For functions $\phi_i(p_i)$ and $F(p_1, ..., p_k)$ with $p_i \in \mathbb{R}^4$ the symbol $\langle F|\phi_1...\phi_k; p\rangle$ denotes the following integral in momentum space

$$\langle F|\phi_1...\phi_k; p\rangle := \int (2\pi)^4 \delta\Big(\sum_{j=1}^{k} p_j + p\Big) F(p_1, ..., p_k) \phi_1(p_1)...\phi_k(p_k) \prod_{i=1}^{k} \frac{d^4 p_i}{(2\pi)^4} \,. \tag{10}$$

We also use the shorthands

$$\langle \phi_1...\phi_k; p\rangle := \langle 1|\phi_1...\phi_k; p\rangle, \qquad \langle \phi_1...\phi_k\rangle := \langle 1|\phi_1...\phi_k; 0\rangle. \tag{11}$$

A cumulative notation for the elementary fields and corresponding sources is

$$\Phi := \left(A_\mu^a, \quad B^a, \quad c^a, \quad \bar{c}^a\right), \qquad K := \left(j_\mu^a, \quad b^a, \quad \bar{\eta}^a, \quad \eta^a\right), \tag{12}$$

where $c, \bar{c}, \eta, \bar{\eta}$ are generators of an infinite-dimensional anticommuting algebra. Furthermore, we use the following shorthand

$$K \cdot \Phi := \langle j_\mu^a, A_\mu^a\rangle + \langle b^a, B^a\rangle + \langle \bar{\eta}^a, c^a\rangle + \langle \bar{c}^a, \eta^a\rangle. \tag{13}$$

We will have to consider one-particle irreducible functions, also known as vertex functions, whose generating functional is denoted by Γ. These functions are translation-invariant in position space. Their *reduced* Fourier transforms $\Gamma^{\vec{\phi}}$ are defined as follows

$$\Gamma^{\vec{\phi}}(p_1, \ldots, p_{\mathrm{N}-1}) := \int \Big(\prod_{i=1}^{\mathrm{N}-1} d^4x_i \, e^{-ip_ix_i} \Big) \, \Gamma^{\vec{\phi}}(0, x_1, \ldots, x_{\mathrm{N}-1}) \, , \qquad (14)$$

where $\vec{\phi} := (\phi_0, ..., \phi_{\mathrm{N}-1})$ is a sequence of field labels, $\phi_i \in \{A, B, c, \bar{c}\}$, and

$$\Gamma^{\vec{\phi}}(x_0, x_1, \ldots, x_{\mathrm{N}-1}) := \Big(\frac{\delta}{\delta\phi_0(x_0)} \cdots \frac{\delta}{\delta\phi_{\mathrm{N}-1}(x_{\mathrm{N}-1})} \Big) \Gamma \Big|_{\vec{\phi}=0} \, . \qquad (15)$$

The complete Fourier transformed N-point vertex function then satisfies

$$(2\pi)^{4(\mathrm{N}-1)} \Big(\frac{\delta}{\delta\phi_0(-p_0)} \cdots \frac{\delta}{\delta\phi_{\mathrm{N}-1}(-p_{\mathrm{N}-1})} \Big) \Gamma \Big|_{\vec{\phi}=0} = \delta(\sum_{i=0}^{\mathrm{N}-1} p_i) \, \Gamma^{\vec{\phi}} \, , \qquad (16)$$

where $\Gamma^{\vec{\phi}}$ stands for $\Gamma^{\vec{\phi}}(p_1, ..., p_{\mathrm{N}-1})$. The reduced and complete Fourier transforms with $n_\chi \geqslant 1$ composite operator insertions of sources $\vec{\chi} = (\chi_0, ..., \chi_{n_\chi - 1})$ and $\vec{\phi} := (\phi_{n_\chi}, ..., \phi_{n-1})$ are correspondingly related by

$$(2\pi)^{4(n-1)} \Big(\prod_{i=n_\chi}^{n-1} \frac{\delta}{\delta\phi_i(-p_i)} \, \prod_{i=0}^{n_\chi - 1} \frac{\delta}{\delta\chi_i(-p_i)} \Big) \Gamma \Big|_{\substack{\vec{\phi}=0 \\ \vec{\chi}=0}} = \delta(\sum_{i=0}^{n-1} p_i) \, \Gamma^{\vec{\phi}}_{\vec{\chi}} \, , \qquad (17)$$

where the order of the derivatives $\delta/\delta\phi_i$ is the same as before, the derivatives $\delta/\delta\chi_i$ are ordered with left-to-right increasing indices, and $\Gamma^{\vec{\phi}}_{\vec{\chi}}$ stands for $\Gamma^{\vec{\phi}}_{\vec{\chi}}(p_1, ..., p_{n-1})$. Note that $n-1$ is the total number of arguments, e.g. for $\Gamma^{\vec{\phi}}_{\vec{\chi}}(p_1, ..., p_{n-1})$ we have $n = n_\chi + \mathrm{N}$.

It will be useful to keep a bijective relation between momenta and field labels (including possible insertion labels), $p_i \leftrightarrow \phi_i$. Hence, we assume that p_0 is the negative subsum of all other momenta,

$$\mathbb{P}_n := \{\vec{p} \in \mathbb{R}^{4n} : \vec{p} = (p_0, ..., p_{n-1}), \; p_0 = -\sum_{i=1}^{n-1} p_i\}, \quad |\vec{p}|^2 := \sum_{i=0}^{n-1} p_i^2, \quad (18)$$

and, referring to (16), (17), we use the notation

$$\Gamma^{\vec{\phi}}(\vec{p}) = \Gamma^{\vec{\phi}}(p_1, ..., p_{\mathrm{N}-1}), \qquad \vec{p} = (p_0, ..., p_{\mathrm{N}-1}) \in \mathbb{P}_{\mathrm{N}}, \tag{19}$$

$$\Gamma^{\vec{\phi}}_{\vec{\chi}}(\vec{p}) = \Gamma^{\vec{\phi}}_{\vec{\chi}}(p_1, ..., p_{n-1}), \qquad \vec{p} = (p_0, ..., p_{n-1}) \in \mathbb{P}_n. \tag{20}$$

Moreover, we rely on the multi-index formalism for derivatives with respect to momenta. Taking in account that there are no derivatives wrt p_0, we set:

$$\mathbf{w} := (w_{0,1}, w_{0,2}, ..., w_{n-1,4}), \qquad w_{i,\mu} \in \mathbb{N}, \quad w_{0,\mu} := 0, \tag{21}$$

$$w := (w_0, ..., w_{n-1}), \qquad w_i := \sum_{\mu=1}^{4} w_{i,\mu}, \tag{22}$$

$$\mathbb{W}_n := \{w \in \{0\} \times \mathbb{N}^{n-1} : \|w\| \leqslant \bar{w}\}, \qquad \|w\| := \sum_{i=0}^{n-1} w_i, \tag{23}$$

$$\partial^{\mathbf{w}}\Gamma := \prod_{i=0}^{n-1}\prod_{\mu=1}^{4}\Big(\frac{\partial}{\partial p_{i,\mu}}\Big)^{w_{i,\mu}}\Gamma, \qquad \partial^{w}\Gamma := \Big(\prod_{i=0}^{n-1}\partial_{p_i}^{w_i}\Big)\Gamma, \tag{24}$$

where ∂_p^k is the tensor with components $\partial_{p_{\mu_1}}...\partial_{p_{\mu_k}}$, and $\bar{w}$ is an arbitrary integer $\geqslant 4$ fixed throughout the paper.

The following shorthands will be helpful:

$$\dot{\Gamma}^{\Lambda\Lambda_0} := \partial_\Lambda \Gamma^{\Lambda\Lambda_0}, \qquad \partial_{\phi_i}\Gamma^{\vec{\phi}} := \frac{\partial \Gamma^{\vec{\phi}}}{\partial p_i}, \qquad \Gamma^{;w} := \partial^w \Gamma, \tag{25}$$

$$\tilde{\delta}_{\phi(p)} := (2\pi)^4 \frac{\delta}{\delta\phi(-p)}, \qquad \check{\phi}(p) := \phi(-p), \quad \log_+ x := \log\max(1, x). \tag{26}$$

1.4 Classical Action

In this section we first introduce the general notion of classical gauge theory, specializing it to a four-dimensional $SU(2)$ Euclidean classical field theory. Then we briefly present Faddeev's quantization of gauge theories. Finally, we show the general form of a quantum Lagrangian including all possible counterterms compatible with the global symmetries but possibly breaking the gauge invariance.

Denote by $\omega : T_pP \to T_eG$ a connection 1-form on a principal fibre bundle $P \xrightarrow{\pi} P/G$ with a left action $p \mapsto up$ where $u \in G$, $p \in P$ and G is a Lie group. Since the base $P/G = \mathbb{R}^4$ is a flat space P admits a global section

$\sigma : \mathbb{R}^4 \to P$ and hence is a trivial bundle. In other words, there exists a diffeomorphism $P \to G \times \mathbb{R}^4$. Furthermore once a connection ω is given any vector $x \in T_pP$ can be uniquely decomposed into vertical and horizontal components $x = x_v + x_h$, so that $\pi_* x_v = 0$ and $\omega(x_h) = 0$. Suppose now that f is a Lie algebra valued differential k-form on P. The exterior covariant derivative of f is defined by $Df(x_1, ..., x_k) := df(x_{1h}, ..., x_{kh})$, and then the curvature 2-form is given by $\Omega := D\omega$. Using these definitions one can show that $\Omega = d\omega - \omega \wedge \omega$ in the case of a principal bundle whose structure group G is a matrix group. Furthermore, if ϕ is a algebra-valued function on P in the adjoint representation of the Lie group G, i.e. $\phi(up) = u\phi(p)u^{-1}$, then $D\phi = d\phi - [\omega, \phi]$. To prove this identity we need to show that $\forall x \in T_pP$ $(D\phi)(x_v) = (d\phi)(x) - [\omega(x), \phi)]$. By linearity we need to show it for the two cases: $x = x_h$ and $x = x_v$. Since $\omega(x_h) = 0$, the first case is trivial. For the $x = x_v$ we have $d\phi(x_v) = \frac{\partial}{\partial t}\phi(u_t p)|_{t=0} = \frac{\partial}{\partial t} u_t \phi(p) u_t^{-1}|_{t=0}$ where $u_t = e^{igAt}$ with a constant $g \neq 0$ and $iA \in T_eG$. Consequently $d\phi(x_v) = ig[A, \phi(p)]$. By definition $\omega(x_v) = igA$ hence the first and second terms cancel one another.

Let $\sigma : \mathbb{R}^4 \to P$ be a global section. Then we define the gauge potential A and the field strength F as the corresponding pullbacks: $igA := \sigma^*\omega$, $igF := \sigma^*\Omega$. Hence $F = dA - igA \wedge A$. If σ' is another section such that $\sigma' = u\sigma$ for some transition function $u : \mathbb{R}^4 \to G$ then the gauge transformation reads

$$A' = \frac{i}{g}\, u du^{-1} + uAu^{-1} \,. \tag{27}$$

For the purpose of this manuscript we assume that $G = SU(2)$. Using the exponential map $u_t = e^{ig\alpha t}$ where $\alpha : \mathbb{R}^4 \to T_eG$ we obtain the infinitesimal gauge transformation

$$\frac{\partial}{\partial t} A'_t \Big|_{t=0} = d\alpha - ig[A, \alpha] \,. \tag{28}$$

Cartan's criterion states that a Lie algebra is semisimple if and only if the Killing form $K(a, b) := Tr(ab)$ is non-degenerate. Here a, b are elements of the adjoint representation of the Lie algebra. Furthermore a Lie algebra over the field of real numbers $\mathbb{R}$ generates a compact group if and only if the Killing form is negative definite. By our convention the generators t^a are hermitian, see (3). Thus $Tr(F_{\mu\nu}F_{\mu\nu})$ is non-degenerate and positive definite. Here $F = F^a_{\mu\nu} t^a dx^\mu \wedge dx^\nu$. The classical Langrangian density of SU(2) Yang–Mills theory is

$$\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \,. \tag{29}$$

Below we give a contemporary view on the idea proposed by L.D. Faddeev [FP67b] to quantize the YM theory. Using new variables $\boldsymbol{\mathcal{E}}^a_i$, $\boldsymbol{A}^a_i$

$$\boldsymbol{\mathcal{E}}^a_i := \frac{\partial L}{\partial(\partial_0 A^a_i)} = F^a_{0i}\,, \qquad \boldsymbol{A}^a_i := A^a_i \quad \forall i \in \{1,2,3\}\,, \tag{30}$$

the Lagrangian density has the form $\mathcal{L} = \mathcal{E}^a_i \partial_0 \boldsymbol{A}^a_i - H'(A_0, \boldsymbol{A}, \boldsymbol{\mathcal{E}})$ where

$$H'(A_0, \boldsymbol{A}, \boldsymbol{\mathcal{E}}) := H(\boldsymbol{A}, \boldsymbol{\mathcal{E}}) - A^a_0 \mathcal{P}^a(\boldsymbol{A}, \boldsymbol{\mathcal{E}}), \qquad \mathcal{P}^a(\boldsymbol{A}, \boldsymbol{\mathcal{E}}) := D_i \boldsymbol{\mathcal{E}}^a_i\,, \tag{31}$$

$$H(\boldsymbol{A}, \boldsymbol{\mathcal{E}}) := \frac{1}{2}(\boldsymbol{\mathcal{E}}^a_i \boldsymbol{\mathcal{E}}^a_i - \boldsymbol{\mathcal{B}}^a_i \boldsymbol{\mathcal{B}}^a_i)\,, \qquad \boldsymbol{\mathcal{B}}^a_i := \frac{1}{2}\epsilon_{ikl} F^a_{kl}\,. \tag{32}$$

In this representation A^a_0 are the Lagrange multipliers corresponding to the first-class primary constraints $\mathcal{P}^a = 0$, see [Dir64]

$$\{\mathcal{P}^a(x), \mathcal{P}^b(y)\} = g\epsilon^{abc}\mathcal{P}^c \delta(x-y), \tag{33}$$

$$\{H(x), \mathcal{P}^a(y)\} = \frac{1}{2} g\epsilon^{abc} F^b_{ij} F^c_{ij} \delta(x-y) = 0. \tag{34}$$

Here we have no secondary constraints and all A^a_0 are arbitrary. P.A.M. Dirac [Dir64] noticed that the first-class primary constraints generate infinitesimal symmetry transformations. If we have a solution of the equation of motion then because the A^a_0 are arbitrary we can obtain a different solution by taking another functions $A^a_0 \mapsto A^a_0 + \alpha^a$. This leads to a change in the Hamiltonian H' and consequently to the change in the canonical variables

$$\delta \dot{\boldsymbol{A}}^a_k = \alpha^b \{\mathcal{P}^b, \boldsymbol{A}^a_k\} = D_k \alpha^a, \qquad \delta \dot{\boldsymbol{\mathcal{E}}}^a_k = \alpha^b \{\mathcal{P}^b, \boldsymbol{\mathcal{E}}^a_k\} = i^2 g e^{abc} \alpha^b \boldsymbol{\mathcal{E}}^c_k\,. \tag{35}$$

Maskawa and Nakajima, see [MN76], showed that in dynamical systems with second-class constraints it is always possible to find local canonical coordinate variables $(Q, \mathcal{Q})$ and their respective conjugates $(P, \mathcal{P})$ such that the constraints have the form $\mathcal{Q} = 0$, $\mathcal{P} = 0$. For a system with first-class constraints there exist canonical variables $(Q, \mathcal{Q}, \mathcal{Q}')$ and their respective conjugates $(P, \mathcal{P}, \mathcal{P}')$ such that the constraints read $\mathcal{Q}' = 0$, $\mathcal{P} = \mathcal{P}' = 0$. Moreover the equations of motion for the variables Q and P are independent of all other variables. But boundary conditions on the terms $\partial^w_t \mathcal{Q}$ with $w \in [0, n]$ given at $t = t_0$ do not fix these terms at time $t = t_0 + \delta t$, see [GT90]. In the case of the YM theory the sets $\mathcal{P}'$ and $\mathcal{Q}'$ are empty. But the canonical coordinate system $(Q, P, \mathcal{Q}, \mathcal{P})$ is unknown and hence the explicit separation of the physical variables (Q, P) from the constraints $(\mathcal{Q}, \mathcal{P})$ is unfeasible.

Using the existence of such a canonical coordinate system and converting the system with first-class constraints to the system with second-class constraints by imposing the Coulomb gauge-fixing condition $\mathcal{C}^a = \partial_k \boldsymbol{\mathcal{A}}_k^a$ L.D. Faddeev constructed the functional integral for the S-matrix. In this formulation the determinant $|\{\mathcal{C}, \mathcal{P}\}|$ which is nonvanishing for dynamical systems with second-class constraints appears as the Jacobian of the δ-function

$$\delta(\mathcal{Q}) = \delta(\mathcal{C}) \left| \frac{\delta \mathcal{C}}{\delta \mathcal{Q}} \right| = \delta(\mathcal{C}) \, |\{\mathcal{C}, \mathcal{P}\}| \ . \tag{36}$$

The main idea of L.D. Faddeev was to introduce auxiliary ghost fields and to transform this determinant into local gauge-fixing terms, see [FS93]. Furthermore he extended this approach to the general case of gauge-fixing conditions. Following [FP67b, tH71] the semiclassical Lagrangian density in Euclidian space with Lorenz gauge-fixing functional takes the form

$$\mathcal{L}_0^{tot} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2 - \partial_\mu \bar{c}^a (D_\mu c)^a, \tag{37}$$

$$(D_\mu c)^a = \partial_\mu c^a + g \epsilon^{ast} A_\mu^s c^t, \tag{38}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon^{ast} A_\mu^s A_\nu^t, \tag{39}$$

where $\xi > 0$ is the Feynman parameter. The gauge fixing condition restricts the gauge transformations α by the equation $M\alpha = 0$, where $M = \partial_\nu D_\nu$. The kernel of M is said to be a ghost zero-mode and the correspondingly gauge transformed fields A are known as Gribov copies. In a perturbation theory M is invertible, $M^{-1} = \partial^{-2} + ig\partial^{-2}\partial(AM^{-1})$.

We will study the quantum theory in a framework which breaks local gauge invariance due to the presence of momentum space regulators. The regularized propagators in momentum space are defined by the expressions

$$C_{\mu\nu}(p) := \frac{1}{p^2}(\delta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2}), \qquad C_{\mu\nu}^{\Lambda\Lambda_0}(p) := C_{\mu\nu}(p)\, \sigma_{\Lambda\Lambda_0}(p^2)\,, \tag{40}$$

$$S(p) := \frac{1}{p^2}, \qquad S^{\Lambda\Lambda_0}(p) := S(p)\, \sigma_{\Lambda\Lambda_0}(p^2)\,, \tag{41}$$

$$\sigma_{\Lambda\Lambda_0}(s) := \sigma_{\Lambda_0}(s) - \sigma_\Lambda(s), \qquad \sigma_\lambda(s) := \exp(-\frac{s^2}{\lambda^4})\,. \tag{42}$$

The parameters Λ, Λ_0, such that $0 < \Lambda \leqslant \Lambda_0$, are respectively IR and UV cutoffs. For shortness we will also write $C_{\Lambda\Lambda_0}^{-1}$ instead of $(C^{\Lambda\Lambda_0})^{-1}$. In position space we have

$$(C^{\Lambda\Lambda_0}j)_\mu(x) = \int d^4y\, C^{\Lambda\Lambda_0}_{\mu\nu}(x,y)j_\nu(y), \tag{43}$$

$$C^{\Lambda\Lambda_0}_{\mu\nu}(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}C^{\Lambda\Lambda_0}_{\mu\nu}(p). \tag{44}$$

However, the regularized theory still respects global SU(2) symmetry, Euclidean inhomogeneous O(4) and has ghost number zero. We admit all counterterms compatible with these symmetries:

$$\begin{aligned}
\mathcal{L}_{ct} =& r^{0,\bar{c}c\bar{c}c}\bar{c}^b c^b \bar{c}^a c^a + r_1^{0,\bar{c}cAA}\bar{c}^b c^b A^a_\mu A^a_\mu + r_2^{0,\bar{c}cAA}\bar{c}^a c^b A^a_\mu A^b_\mu \\
&+ r_1^{0,AAAA}A^b_\mu A^b_\nu A^a_\mu A^a_\nu + r_2^{0,AAAA}A^b_\nu A^b_\nu A^a_\mu A^a_\mu + 2\epsilon_{abc}r^{0,AAA}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu \\
&- r_1^{0,A\bar{c}c}\epsilon_{abd}(\partial_\mu \bar{c}^a)A^b_\mu c^d - r_2^{0,A\bar{c}c}\epsilon_{abd}\bar{c}^a A^b_\mu \partial_\mu c^d + \Sigma^{0,\bar{c}c}\bar{c}^a\partial^2 c^a \\
&- \frac{1}{2}\Sigma_T^{0,AA}A^a_\mu(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu)A^a_\nu + \frac{1}{2\xi}\Sigma_L^{0,AA}(\partial_\mu A^a_\mu)^2 \\
&+ \frac{1}{2}\delta m^2_{AA}A^a_\mu A^a_\mu - \delta m^2_{\bar{c}c}\bar{c}^a c^a.
\end{aligned} \tag{45}$$

There are eleven marginal and two strictly relevant counterterms. For the marginal terms we use the letters r^0 and in case of two-point functions Σ^0. We use δm^2_{AA} and $\delta m^2_{\bar{c}c}$ to denote the strictly relevant counterterms.

1.5 Generating functionals and Flow equations

In the next two sections we will introduce the essential structural tools required for our proof of renormalizability of nonabelian Yang–Mills theory. These tools are on the one hand the differential Flow Equations of the renormalization group, and on the other hand the (violated) Slavnov–Taylor identities. They are both obtained from the functional integral representation of the theory.

Although the Lagrangian density in (37) contains all physical degrees of freedom to construct 1PI functions this form does not allow to establish the AGE. To preserve unitarity of the S-matrix this equation should be added by hand. Similarly to the gauge theory in Minkowski space we can add an auxiliary field $B = B^a t^a$ to obtain a semi-classical Lagrangian invariant under nilpotent BRST transformations. Nilpotence is a prerequisite for the unitarity of the S-matrix, and in fact the AGE will emerge. To see how this auxiliary field should be introduced it is helpful to consider the following example where we restrict the consideration to the gauge fields A omitting the ghosts and denote by $d\nu_{\Lambda\Lambda_0}(A)$ a Gaussian measure with the covariance $C_{\Lambda\Lambda_0}$ and the Feynman parameter ξ_0.

$$e^{-\frac{1}{2\hbar}\langle j, C_{\xi_0} j\rangle} = \int d\nu_{\Lambda\Lambda_0}(A) e^{\frac{i}{\hbar}\langle j, A\rangle}. \tag{46}$$

To obtain meaningful results we should keep the volume finite. One could put the system on a torus with periodic boundary conditions.

Lemma 1 *Let ξ_1 be a positive real number. Denote by $\tilde{\sigma}$ the regulator $\sigma_{\Lambda\tilde{\Lambda}_0}$ with $\tilde{\Lambda}_0 > \Lambda_0$. Then there exists a normalization function $f(\xi_0, \xi_1)$ such that*

$$e^{-\frac{1}{2\hbar}\langle j, C_{\xi_2} j\rangle} = \lim_{\tilde{\Lambda}_0 \to \Lambda_0} \frac{1}{f(\xi_0, \xi_1)} \int d\nu_{\Lambda\Lambda_0}(A) e^{-\frac{1}{2\hbar\xi_1}\langle \partial A, \tilde{\sigma}^{-1}\partial A\rangle + \frac{i}{\hbar}\langle j, A\rangle}, \tag{47}$$

where $\frac{1}{\xi_2} := \frac{1}{\xi_0} + \frac{1}{\xi_1}$.

The ξ_0-dependence of the normalization factor f stems from the measure $d\nu_{\Lambda\Lambda_0}(A)$.

Proof

$$\chi(\xi_1) := \int d\nu_{\Lambda\Lambda_0}(A) e^{-\frac{1}{2\hbar\xi_1}\langle \partial A, \tilde{\sigma}^{-1}\partial A\rangle + \frac{i}{\hbar}\langle j, A\rangle}. \tag{48}$$

First we want to show that for arbitrary $\tilde{\Lambda}_0 > \Lambda_0$ there exist $f(\xi_0, \xi_1)$ and $\tilde{\xi}_2(\xi_0, \xi_1, p^2)$ such that $\lim\limits_{\xi_1 \to \infty} f = 1$, $\lim\limits_{\xi_1 \to \infty} \tilde{\xi}_2 = \xi_0$ and

$$f(\xi_0, \xi_1) e^{-\frac{1}{2\hbar}\langle j, C_{\tilde{\xi}_2} j\rangle} = \chi(\xi_1). \tag{49}$$

Here dependence of f and $\tilde{\xi}_2$ on Λ, Λ_0 and $\tilde{\Lambda}_0$ is omitted. Let $\chi' := \dfrac{\partial}{\partial \xi_1}\chi$.

$$\begin{aligned}
\chi' = \int d\nu_{\Lambda\Lambda_0} \frac{1}{2\hbar\xi_1^2}\langle \partial A, \tilde{\sigma}^{-1}\partial A\rangle e^{-\frac{1}{2\hbar\xi_1}\langle \partial A, \tilde{\sigma}^{-1}\partial A\rangle + \frac{i}{\hbar}\langle j, A\rangle} \\
= -\int d\nu_{\Lambda\Lambda_0} \frac{1}{2\hbar\xi_1^2}\langle \hbar\partial\frac{\delta}{\delta j}, \tilde{\sigma}^{-1}\hbar\partial\frac{\delta}{\delta j}\rangle e^{-\frac{1}{2\hbar\xi_1}\langle \partial A, \tilde{\sigma}^{-1}\partial A\rangle + \frac{i}{\hbar}\langle j, A\rangle} \\
= -\frac{1}{2\hbar\xi_1^2}\langle \hbar\partial\frac{\delta}{\delta j}, \tilde{\sigma}^{-1}\hbar\partial\frac{\delta}{\delta j}\rangle \chi.
\end{aligned} \tag{50}$$

Then we apply the operator on the rhs in (50) to the lhs of (49)

$$\chi' = \frac{1}{2\hbar\xi_1^2}\Big(\hbar L^4 \int \frac{d^4p}{(2\pi)^4}\tilde{\xi}_2\sigma\tilde{\sigma}^{-1} - \langle j_\mu, \tilde{\xi}_2^2\tilde{\sigma}^{-1}\sigma^2\vec{C}_\mu\vec{C}_\nu j_\nu\rangle\Big)\chi\,, \tag{51}$$

where $\vec{C}_\mu := \frac{p_\mu}{p^2}$, $L^4 := \int d^4x$. On the other hand, application of the derivative to the lhs in (49) yields

$$\chi' = \Big(\frac{f'}{f} - \frac{1}{2\hbar}\langle j_\mu, \tilde{\xi}_2'\sigma\vec{C}_\mu\vec{C}_\nu j_\nu\rangle\Big)\chi. \tag{52}$$

From both equations (51) and (52) we obtain

$$\frac{f'}{f} = \frac{L^4}{2}\int \frac{d^4p}{(2\pi)^4}\frac{\tilde{\xi}_2}{\xi_1^2}\sigma\tilde{\sigma}^{-1}\,, \qquad\qquad \tilde{\xi}_2' = \frac{\tilde{\xi}_2^2}{\xi_1^2}\sigma\tilde{\sigma}^{-1}. \tag{53}$$

The last equation immediately gives us

$$\Big(\frac{1}{\tilde{\xi}_2}\Big)' = -\frac{\sigma\tilde{\sigma}^{-1}}{\xi_1^2} \implies \frac{1}{\tilde{\xi}_2} = \frac{\sigma\tilde{\sigma}^{-1}}{\xi_1} + \frac{1}{\xi_0}. \tag{54}$$

Substituting $\tilde{\xi}_2(\xi,\xi_1)$ into the first equation we have

$$\frac{f'}{f} = \frac{L^4}{2}\int \frac{d^4p}{(2\pi)^4}\frac{\xi_0\sigma\tilde{\sigma}^{-1}}{\xi_1(\xi_1+\xi_0\sigma\tilde{\sigma}^{-1})} = \frac{L^4}{2}\int \frac{d^4p}{(2\pi)^4}\Big(\frac{1}{\xi_1} - \frac{1}{\xi_1+\xi_0\sigma\tilde{\sigma}^{-1}}\Big). \tag{55}$$

Finally, using $f(\xi_0,\infty) = 1$ we obtain

$$f = e^{-\frac{L^4}{2}TR}, \qquad\qquad TR := \int \frac{d^4p}{(2\pi)^4}\log(1 + \frac{\xi_0}{\xi_1}\sigma\tilde{\sigma}^{-1})\,. \tag{56}$$

Finally, it is clear from (54) that $\lim\limits_{\tilde{\Lambda}_0\to\Lambda_0}\tilde{\xi}_2 = \xi_2$. ■

Let us define an auxiliary measure $d\tilde{\nu}_{\xi_1}(B)$

$$e^{-\frac{1}{2\hbar}\langle b, C_B b\rangle} := \int d\tilde{\nu}_{\xi_1}(B)e^{\frac{i}{\hbar}\langle B,b\rangle}, \qquad\qquad C_B := \frac{1}{\xi_1}\sigma_{\Lambda\tilde{\Lambda}_0}\,. \tag{57}$$

Hence we obtain a naive definition of the measure

$$\lim_{\tilde{\Lambda}_0\to\Lambda_0} f^{-1}\int d\nu_{\xi_0}(A)d\tilde{\nu}_{\xi_1}(B)e^{\frac{i}{\hbar}\langle B,\tilde{\sigma}^{-1}\partial A\rangle}e^{\frac{i}{\hbar}\langle j,A\rangle + \frac{i}{\hbar}\langle b,B\rangle}\,. \tag{58}$$

Its Fourier transform is a Gaussian characteristic function

$$
\begin{aligned}
f^{-1} \int d\nu_{\xi_0}(A) d\tilde{\nu}_{\xi_1}(B) e^{\frac{i}{\hbar}\langle B, \tilde{\sigma}^{-1}\partial A\rangle} e^{\frac{i}{\hbar}\langle j, A\rangle + \frac{i}{\hbar}\langle b, B\rangle} \\
= f^{-1} \int d\nu_{\xi_0}(A) d\tilde{\nu}_{\xi_1}(B - \frac{i}{\xi_1}\partial A) e^{-\frac{1}{2\hbar\xi_1}\langle \partial A, \tilde{\sigma}^{-1}\partial A\rangle} e^{\frac{i}{\hbar}\langle j, A\rangle + \frac{i}{\hbar}\langle b, B\rangle} \\
= \int d\nu_{\xi_2}(A) d\tilde{\nu}_{\xi_1}(B) e^{\frac{i}{\hbar}\langle j - \frac{i}{\xi_1}\partial b, A\rangle + \frac{i}{\hbar}\langle b, B\rangle} \\
= e^{-\frac{1}{2}\langle j - \frac{i}{\xi_1}\partial b, C_{\xi_2} j - \frac{i}{\xi_1}\partial b\rangle - \frac{1}{2}\langle b, C_B b\rangle}\,. \qquad (59)
\end{aligned}
$$

To complete the example we consider the semi-classical Lagrangian density which corresponds to (58) with the same interaction as in (37) in the limit $\Lambda \to 0$, $\Lambda_0 \to \infty$

$$
\mathcal{L}_0^{tot} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2\xi_0}(\partial_\mu A^a_\mu)^2 + \frac{\xi_1}{2} B^a B^a - i B^a \partial_\mu A^a_\mu - \partial_\mu \bar{c}^a (D_\mu c)^a\,. \qquad (60)
$$

Let $\beta^2 := 1 + \frac{\xi_1}{\xi_0}$. The BRST transformation for the density $\mathcal{L}_0^{tot}$ is

$$
\delta_\epsilon A = \epsilon D c, \qquad \delta_\epsilon c = \epsilon \frac{1}{2} i g \{c, c\}\,, \qquad (61)
$$

$$
\delta_\epsilon \bar{c} = \epsilon (i\beta B - \frac{1}{\xi_0}\frac{\beta}{1+\beta}\partial A), \qquad \delta_\epsilon B = \epsilon \frac{1}{i\xi_0}\frac{1}{1+\beta}\partial(Dc)\,, \qquad (62)
$$

where ϵ is an element of the Grassmann algebra and $\{c, c\}^d = i\epsilon_{abd} c^a c^b$. Defining the classical operator s by $\delta_\epsilon \Phi = \epsilon s \Phi$ one shows that s is nilpotent. To show the BRST invariance of the Langrangian it is convenient to use the nilpotency of the transformation

$$
\mathcal{L}_0^{tot} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{2} \int d\epsilon\, \delta_\epsilon \left\{ \bar{c}^a \Big(\frac{1+\beta}{\beta} \partial_\mu A^a_\mu + i\xi_2 \beta B^a \Big) \right\}, \qquad (63)
$$

where ξ_2 is as in (47). There are two limit cases:

- $\xi_0 \to \infty$. Thus $\beta = 1$, $\xi_1 = \xi_2$. Let $\xi := \xi_1$. Then (60) has the form

$$
\mathcal{L}_0^{tot} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{\xi}{2} B^a B^a - i B^a \partial_\mu A^a_\mu - \partial_\mu \bar{c}^a (D_\mu c)^a\,. \qquad (64)
$$

  The infinitesimal BRST transformation [BRS75], [Tyu75] is

$$
\delta_\epsilon A = \epsilon\, Dc, \qquad \delta_\epsilon c = \epsilon \frac{1}{2} i g \{c, c\}, \qquad \delta_\epsilon \bar{c} = \epsilon\, iB, \qquad \delta_\epsilon B = 0. \qquad (65)
$$

- $\xi_1 \to \infty$. Here $B = 0$ and $\delta\bar{c} = -\epsilon\frac{1}{\xi_0}\partial_\mu A_\mu$. We recover our initial Lagrangian density (37), but the nilpotence is not conserved.

Unfortunatly, the measure $d\nu_{\xi_0}(A)$ in (58) is not finite in the case $\xi_0 \to \infty$. This does not allow to take (58) as a definition of the measure. But using our result in (59) we can give another definition by means of the characteristic function, see e.g. [DF91],

$$\chi(K) := \int d\nu(\Phi)\, e^{\frac{i}{\hbar}K\cdot\Phi}\,, \tag{66}$$

$$\chi^{\Lambda\Lambda_0}(j,b,\bar{\eta},\eta) := e^{\frac{1}{\hbar}\langle\bar{\eta},S^{\Lambda\Lambda_0}\eta\rangle - \frac{1}{2\hbar}\langle j,C^{\Lambda\Lambda_0}j\rangle - \frac{1}{2\hbar\xi}\langle b,b\rangle}. \tag{67}$$

Here $\Phi \in \mathcal{H}_A \oplus \mathcal{H}_{c\bar{c}}$ where $\mathcal{H}_{c\bar{c}}$ is a super space. Elements of $\mathcal{H}_A$ are called the bosonic part. The integral in (66) is a double integral. The integral over $\mathcal{H}_A$ is defined as the limit of multiple Lebesgue integrals with the measure given by Borel cylinder sets and the exponential Gaussian function, see [GJ87]. In the case of $\mathcal{H}_{c\bar{c}}$ the Lebesgue integration is replaced by the integral over a finite subset of generators of the super algebra, see [Ber66, Sal99].

Definition 2 *Let $d\mu_{\Lambda\Lambda_0}$ be the measure defined by*

$$d\mu_{\Lambda\Lambda_0}(A,B,c,\bar{c}) := d\nu_{\Lambda\Lambda_0}(A, B - i\frac{1}{\xi}\partial A, c, \bar{c}). \tag{68}$$

For $\Phi = (A,B,c,\bar{c})$ and $K = (j,b,\bar{\eta},\eta)$ and an infinitesimal variation $\delta\Phi = (\delta A, 0, \delta c, \delta\bar{c})$, using the properties of Gaussian measures from appendix A, we have

$$\begin{aligned} d\mu_{\Lambda\Lambda_0}(\Phi+\delta\Phi) =& d\mu_{\Lambda\Lambda_0}(\Phi)\Big(1 + \frac{1}{\hbar}\langle\bar{c}, S^{-1}_{\Lambda\Lambda_0}\delta c\rangle + \frac{1}{\hbar}\langle\delta\bar{c}, S^{-1}_{\Lambda\Lambda_0}c\rangle \\ & - \frac{1}{\hbar}\langle A, C^{-1}_{\Lambda\Lambda_0}\delta A\rangle - \frac{1}{\hbar}\langle i\partial(B - i\frac{1}{\xi}\partial A), \delta A\rangle\Big)\,. \end{aligned} \tag{69}$$

Definition 3 *The free partition function $Z_0^{\Lambda\Lambda_0}$ is defined by*

$$Z_0^{\Lambda\Lambda_0}(K) := \int d\mu_{\Lambda\Lambda_0}(\Phi) e^{\frac{1}{\hbar}K\cdot\Phi}. \tag{70}$$

It follows that

$$Z_0^{\Lambda\Lambda_0}(j,b,\bar\eta,\eta) = \chi^{\Lambda\Lambda_0}(i(j - i\frac{1}{\xi}\partial b), i\,b, i\,\bar\eta, i\,\eta), \tag{71}$$

$$Z_0^{\Lambda\Lambda_0}(K) = e^{\frac{1}{2\hbar}\langle K, \mathbf{C}^{\Lambda\Lambda_0} K\rangle}, \tag{72}$$

where $\mathbf{C}^{\Lambda\Lambda_0}$ is a 7x7 matrix,

$$\mathbf{C}^{\Lambda\Lambda_0} := \begin{pmatrix} C_{\mu\nu}^{\Lambda\Lambda_0}, & S^{\Lambda\Lambda_0} p_\mu, & 0, & 0 \\ -S^{\Lambda\Lambda_0} p_\nu, & \frac{1}{\xi}(1-\sigma_{\Lambda\Lambda_0}), & 0, & 0 \\ 0, & 0, & 0, & -S^{\Lambda\Lambda_0} \\ 0, & 0, & S^{\Lambda\Lambda_0}, & 0 \end{pmatrix}. \tag{73}$$

For $\Lambda < \Lambda_0$, $\mathbf{C}^{\Lambda\Lambda_0}$ is invertible:

$$\mathbf{C}_{\Lambda\Lambda_0}^{-1} = \begin{pmatrix} C_{\Lambda\Lambda_0;\mu\nu}^{-1} - \frac{1}{\xi} p_\mu p_\nu, & -p_\mu, & 0, & 0 \\ p_\nu, & \xi, & 0, & 0 \\ 0, & 0, & 0, & S_{\Lambda\Lambda_0}^{-1} \\ 0, & 0, & -S_{\Lambda\Lambda_0}^{-1}, & 0 \end{pmatrix}. \tag{74}$$

We write the bosonic part of $\mathbf{C}_{\Lambda\Lambda_0}^{-1}$ as $\mathbf{P}^T \mathbf{Q}^{\Lambda\Lambda_0} \mathbf{P}$, where $\mathbf{P}$ is a diagonal matrix with $\mathbf{P}_{\mu\mu} = |p|$ for each index $\mu \in \{1, ...4\}$ and $\mathbf{P}_{55} = 1$. The eigenvalues q_α of $\mathbf{Q}^{\Lambda\Lambda_0}$ are:

$$q_{1,2,3} = \sigma_{\Lambda\Lambda_0}^{-1}, \quad q_{4,5} = \frac{\xi_{\Lambda\Lambda_0} \pm (\xi_{\Lambda\Lambda_0}^2 - 4\sigma_{\Lambda\Lambda_0}^{-1})^{\frac{1}{2}}}{2}, \quad \xi_{\Lambda\Lambda_0} := \xi + \frac{1}{\xi}(\sigma_{\Lambda\Lambda_0}^{-1} - 1).$$

The fact that the real part of these eigenvalues is positive is known to be a prerequisite for the existence of a complex measure for the bosonic part of the theory, for analysis of complex mesures see [Hal74, Rud87].

A useful relation follows from (67), (71):

$$
\begin{aligned}
\hbar\frac{\delta}{\delta b}Z_0^{\Lambda\Lambda_0} &= \hbar\frac{\delta}{\delta b}\chi^{\Lambda\Lambda_0}(i(j-i\frac{1}{\xi}\partial b), ib, i\bar\eta, i\eta) \\
&= \hbar\frac{\delta}{\delta b}e^{-\frac{1}{\hbar}\langle\bar\eta,S^{\Lambda\Lambda_0}\eta\rangle+\frac{1}{2\hbar}\langle j-i\frac{1}{\xi}\partial b,C^{\Lambda\Lambda_0}(j-i\frac{1}{\xi}\partial b)\rangle+\frac{1}{2\hbar\xi}\langle b,b\rangle} \\
&= \frac{1}{\xi}\Big(b+i\partial(j-i\frac{1}{\xi}\partial b)\Big)e^{-\frac{1}{\hbar}\langle\bar\eta,S^{\Lambda\Lambda_0}\eta\rangle+\frac{1}{2\hbar}\langle j-i\frac{1}{\xi}\partial b,C^{\Lambda\Lambda_0}(j-i\frac{1}{\xi}\partial b)\rangle+\frac{1}{2\hbar\xi}\langle b,b\rangle} \\
&= \frac{1}{\xi}\Big(b+i\hbar\partial\frac{\delta}{\delta j}\Big)e^{-\frac{1}{\hbar}\langle\bar\eta,S^{\Lambda\Lambda_0}\eta\rangle+\frac{1}{2\hbar}\langle j-i\frac{1}{\xi}\partial b,C^{\Lambda\Lambda_0}(j-i\frac{1}{\xi}\partial b)\rangle+\frac{1}{2\hbar\xi}\langle b,b\rangle} \\
&= \frac{1}{\xi}\Big(b+i\hbar\partial\frac{\delta}{\delta j}\Big)\chi^{\Lambda\Lambda_0}(i(j-i\frac{1}{\xi}\partial b), ib, i\bar\eta, i\eta) \\
&= \frac{1}{\xi}\Big(b+i\hbar\partial\frac{\delta}{\delta j}\Big)Z_0^{\Lambda\Lambda_0}\,. \qquad (75)
\end{aligned}
$$

Consequently we have

$$\int d\mu_{\Lambda\Lambda_0}(\Phi)\,B\,e^{\frac{1}{\hbar}K\cdot\Phi} = \frac{1}{\xi}\int d\mu_{\Lambda\Lambda_0}(\Phi)\,(b+i\partial A)\,e^{\frac{1}{\hbar}K\cdot\Phi}\,. \qquad (76)$$

Definition 4 *The partition function $Z^{\Lambda\Lambda_0}$ of $SU(2)$ Yang–Mills field theory is given by*

$$Z^{\Lambda\Lambda_0}(K) := \int d\mu_{\Lambda\Lambda_0}(\Phi)\,e^{-\frac{1}{\hbar}L^{\Lambda_0\Lambda_0}}\,e^{\frac{1}{\hbar}K\cdot\Phi}, \qquad L^{\Lambda_0\Lambda_0} := \int d^4x\,\mathcal{L}^{\Lambda_0\Lambda_0}. \qquad (77)$$

The interaction Lagrangian density $\mathcal{L}^{\Lambda_0\Lambda_0} := \mathcal{L}_0^{\Lambda_0\Lambda_0} + \mathcal{L}_{ct}^{\Lambda_0\Lambda_0}$ is given by (45) *and*

$$\mathcal{L}_0^{\Lambda_0\Lambda_0} := g\epsilon_{abc}(\partial_\mu A_\nu^a)A_\mu^b A_\nu^c + \frac{g^2}{4}\epsilon_{cab}\epsilon_{cds}A_\mu^a A_\nu^b A_\mu^d A_\nu^s - g\epsilon_{abc}(\partial_\mu\bar c^a)A_\mu^b c^c\,. \qquad (78)$$

Since we restrict to perturbation theory, all generating functionals are formal series in terms of $\hbar$ and of their source/field arguments. We also emphasize that $\mathcal{L}_0^{\Lambda_0\Lambda_0}$ does not depend on the B field.

Definition 5 *The generating functional of the Connected Schwinger (CS) functions is*

$$e^{\frac{1}{\hbar}W^{\Lambda\Lambda_0}(K)} := Z^{\Lambda\Lambda_0}(K). \qquad (79)$$

The derivation of the FE is usually given considering the generating functional $L^{\Lambda\Lambda_0}$ of the connected amputated Schwinger (CAS) functions.

Definition 6 *The generating functional $L^{\Lambda\Lambda_0}$ of CAS functions is*

$$e^{-\frac{1}{\hbar}L^{\Lambda\Lambda_0}(\Phi)} := \int d\mu_{\Lambda\Lambda_0}(\Phi')\, e^{-\frac{1}{\hbar}L^{\Lambda_0\Lambda_0}(\Phi'+\Phi)}. \tag{80}$$

From definition 3 we have for any polynomial $P(\Phi)$

$$\frac{d}{d\Lambda}\int d\mu_{\Lambda\Lambda_0}(\Phi)P(\Phi) = \frac{\hbar}{2}\int d\mu_{\Lambda\Lambda_0}(\Phi)\langle\frac{\delta}{\delta\Phi}, \hat{\mathbf{1}}\dot{\mathbf{C}}^{\Lambda\Lambda_0}\frac{\delta}{\delta\Phi}\rangle P(\Phi), \tag{81}$$

where

$$(\hat{\mathbf{1}})_{\phi\phi'} = \begin{cases} -\delta_{\phi\phi'} & \text{if } \phi,\phi' \in \{c,\bar{c}\}, \\ \delta_{\phi\phi'} & \textit{otherwise}\,. \end{cases} \tag{82}$$

Using equation (81) one obtains the FE, see e.g. [GJ87], [KKS92],

$$\dot{L}^{\Lambda\Lambda_0}(\Phi) = \frac{\hbar}{2}\langle\frac{\delta}{\delta\Phi}, \hat{\mathbf{1}}\dot{\mathbf{C}}^{\Lambda\Lambda_0}\frac{\delta}{\delta\Phi}\rangle L^{\Lambda\Lambda_0} - \frac{1}{2}\langle\frac{\delta L^{\Lambda\Lambda_0}}{\delta\Phi}, \hat{\mathbf{1}}\dot{\mathbf{C}}^{\Lambda\Lambda_0}\frac{\delta L^{\Lambda\Lambda_0}}{\delta\Phi}\rangle\,. \tag{83}$$

From appendix A on Gaussian measures it follows that

$$d\mu_{\Lambda\Lambda_0}(\Phi - \mathbf{C}^{\Lambda\Lambda_0}\delta\Phi) = d\mu_{\Lambda\Lambda_0}(\Phi)e^{-\frac{1}{2\hbar}\langle\delta\Phi,\mathbf{C}^{\Lambda\Lambda_0}\delta\Phi\rangle}e^{\frac{1}{\hbar}\langle\delta\Phi,\Phi\rangle}. \tag{84}$$

This gives the relation between the generating functionals $W^{\Lambda\Lambda_0}$ and $L^{\Lambda\Lambda_0}$

$$W^{\Lambda\Lambda_0}(K) = \frac{1}{2}\langle K, \mathbf{C}^{\Lambda\Lambda_0}K\rangle - L^{\Lambda\Lambda_0}(\hat{\mathbf{1}}_{\bar{c}}\mathbf{C}^{\Lambda\Lambda_0}K), \tag{85}$$

where

$$(\hat{\mathbf{1}}_{\bar{c}})_{\phi\phi'} = \begin{cases} -1 & \text{if } \phi=\phi'=\bar{c}, \\ \delta_{\phi\phi'} & \textit{otherwise}, \end{cases} \qquad (\hat{\mathbf{1}}_{c})_{\phi\phi'} = \begin{cases} -1 & \text{if } \phi=\phi'=c, \\ \delta_{\phi\phi'} & \textit{otherwise}. \end{cases} \tag{86}$$

Definition 7 (Legendre transform) *For $0<\Lambda<\Lambda_0$, let $K^{\Lambda\Lambda_0}(\underline{\Phi})$ be a solution of the system of equations*

$$\underline{A} - \frac{\delta W^{\Lambda\Lambda_0}}{\delta j}\Big|_{K^{\Lambda\Lambda_0}(\underline{\Phi})} = 0, \qquad \underline{B} - \frac{\delta W^{\Lambda\Lambda_0}}{\delta b}\Big|_{K^{\Lambda\Lambda_0}(\underline{\Phi})} = 0, \tag{87}$$

$$\underline{c} - \frac{\delta W^{\Lambda\Lambda_0}}{\delta \bar{\eta}}\Big|_{K^{\Lambda\Lambda_0}(\underline{\Phi})} = 0, \qquad \underline{\bar{c}} + \frac{\delta W^{\Lambda\Lambda_0}}{\delta \eta}\Big|_{K^{\Lambda\Lambda_0}(\underline{\Phi})} = 0. \tag{88}$$

The effective action is

$$\Gamma^{\Lambda\Lambda_0}(\underline{\Phi}) := K^{\Lambda\Lambda_0}(\underline{\Phi})\cdot\underline{\Phi} - W^{\Lambda\Lambda_0}(K^{\Lambda\Lambda_0}(\underline{\Phi}))\,. \tag{89}$$

A solution of the above system of equations always exists as formal series in $\hbar$ and the fields $\underline{\Phi}$. Moreover this solution is unique.

Definition 8 *For $0 < \Lambda \leqslant \Lambda_0$, let $\Phi^{\Lambda\Lambda_0}(\underline{\Phi})$ be a solution of the equation*

$$\underline{\Phi} = \left(\Phi - \mathbf{C}^{\Lambda\Lambda_0}\frac{\delta L^{\Lambda\Lambda_0}}{\delta\Phi}\right)\Big|_{\Phi^{\Lambda\Lambda_0}(\underline{\Phi})}. \tag{90}$$

The reduced effective action is

$$\mathsf{\Gamma}^{\Lambda\Lambda_0}(\underline{\Phi}) := \left(L^{\Lambda\Lambda_0}(\Phi) - \frac{1}{2}\langle\frac{\delta L^{\Lambda\Lambda_0}}{\delta\Phi}, \hat{\mathbf{1}}\mathbf{C}^{\Lambda\Lambda_0}\frac{\delta L^{\Lambda\Lambda_0}}{\delta\Phi}\rangle\right)\Big|_{\Phi^{\Lambda\Lambda_0}(\underline{\Phi})}. \tag{91}$$

From definitions 7 and 8 it follows that for $0 < \Lambda < \Lambda_0$

$$\begin{aligned}
\mathsf{\Gamma}^{\Lambda\Lambda_0}(\underline{\Phi}) =& \Gamma^{\Lambda\Lambda_0}(\underline{\Phi}) - \frac{1}{2}\langle\underline{\Phi}, \mathbf{C}^{-1}_{\Lambda\Lambda_0}\underline{\Phi}\rangle, &(92)\\
\Phi^{\Lambda\Lambda_0}(\underline{\Phi}) =& \hat{\mathbf{1}}_{\bar{c}}\mathbf{C}^{\Lambda\Lambda_0}K^{\Lambda\Lambda_0}(\underline{\Phi}), &(93)\\
\mathsf{\Gamma}^{\Lambda\Lambda_0}(\underline{\Phi}) =& \left(L^{\Lambda\Lambda_0}(\Phi) - \frac{1}{2}\langle(\underline{\Phi} - \Phi), \mathbf{C}^{-1}_{\Lambda\Lambda_0}(\underline{\Phi} - \Phi)\rangle\right)\Big|_{\Phi^{\Lambda\Lambda_0}(\underline{\Phi})}. &(94)
\end{aligned}$$

Using equation (90) we see that, before the replacement, the rhs of (94), as a function of Φ for fixed $\underline{\Phi}$, has an extremum at $\Phi = \Phi^{\Lambda\Lambda_0}(\underline{\Phi})$. Applying ∂_Λ to equation (94), substituting $\dot{L}$ with the rhs of (83) and using the property of extremum we obtain

$$\dot{\mathsf{\Gamma}}^{\Lambda\Lambda_0}(\underline{\Phi}) = \frac{\hbar}{2}\langle\frac{\delta}{\delta\Phi}, \hat{\mathbf{1}}\dot{\mathbf{C}}^{\Lambda\Lambda_0}\frac{\delta}{\delta\Phi}\rangle L^{\Lambda\Lambda_0}\Big|_{\Phi^{\Lambda\Lambda_0}(\underline{\Phi})}. \tag{95}$$

Defining

$$W^{\Lambda\Lambda_0}_{q,-k} := (2\pi)^4\frac{\delta^2 W^{\Lambda\Lambda_0}}{\delta K(-q)\delta K(k)}\,, \qquad \Gamma^{\Lambda\Lambda_0}_{q,-k} := (2\pi)^4\frac{\delta^2\Gamma^{\Lambda\Lambda_0}}{\delta\underline{\Phi}(-q)\delta\underline{\Phi}(k)}\,, \tag{96}$$

it is easy to see that

$$\int d^4k\, W^{\Lambda\Lambda_0}_{q,-k}\,\hat{\mathbf{1}}_{\bar{c}}\,\Gamma^{\Lambda\Lambda_0}_{k,-p}\,\hat{\mathbf{1}}_c = \delta(q-p)\,. \tag{97}$$

This implies that, using similar notations,

$$L^{\Lambda\Lambda_0}_{q,-p} = \int d^4k\,\mathsf{\Gamma}^{\Lambda\Lambda_0}_{q,-k}\left(\delta(k-p) + \hat{\mathbf{1}}\mathbf{C}^{\Lambda\Lambda_0}(p)\mathsf{\Gamma}^{\Lambda\Lambda_0}_{-p,k}\right)^{-1}\,. \tag{98}$$

Eventually, one obtains the FE for the functional Γ and also for the functional Γ_χ with one operator insertion of source χ

$$\dot{\Gamma} = \frac{\hbar}{2}\langle \dot{\mathbf{C}}\,\delta_\phi\delta_{\bar\phi}\Gamma, \left(1 + \hat{\mathbf{1}}\mathbf{C}\,\delta_{\bar\phi}\delta_\phi\Gamma\right)^{-1}\rangle\,, \tag{99}$$

$$\delta_\chi\dot{\Gamma} = \frac{\hbar}{2}\langle \dot{\mathbf{C}}\left(1 + \delta_{\bar\phi}\delta_\phi\Gamma\,\mathbf{C}\hat{\mathbf{1}}\right)^{-1}\delta_\chi\delta_{\bar\phi}\delta_\phi\Gamma, \left(1 + \hat{\mathbf{1}}\mathbf{C}\,\delta_{\bar\phi}\delta_\phi\Gamma\right)^{-1}\rangle\,, \tag{100}$$

where ϕ, $\bar\phi \in \{\underline{A}, \underline{B}, \underline{c}, \underline{\bar c}\}$, and we omit appropriate sums over field labels. Generalization to $\vec\chi = (\chi_1, ..., \chi_{n_\chi})$ with $n_\chi > 1$ is straightforward. The FE for Γ in modern form has been introduced in [Wet93, BDM93, Mor94, KKS97, KM09]. Flow equations with composite operator insertions have been introduced in [KK92].

The mass dimension of a vertex function $\Gamma_{\vec\chi}^{\vec\phi;w}(\vec p)$ with N fields $\phi_i \in \{A, B, c, \bar c\}$, n_χ insertions of sources χ_i and $\|w\|$ momentum derivatives is $d := 4 - \sum_i^{n_\chi}[\chi_i] - \sum_i^{\mathrm{N}}[\phi_i] - \|w\|$, where $[F]$ stands for the mass dimension of F in position space. We say that such a term is **irrelevant** if $d < 0$, as for example $\Gamma_\gamma^{AAc;w}$ for $[\gamma] = 2$, and **relevant** otherwise. Furthermore, we call a relevant term **marginal** if $d = 0$, as for example Γ^{AAAA}, or **strictly relevant** if $d > 0$.
Expanding in formal power series in $\hbar$ we have

$$\Gamma^{\Lambda\Lambda_0}(\underline{\Phi}) = \sum_{l=0}^{\infty}\hbar^l\Gamma_l^{\Lambda\Lambda_0}(\underline{\Phi})\,. \tag{101}$$

We also note that the FE (99) and (100) admit an inductive structure in the loop number. This property allows us to prove statements by induction, first establishing them at tree-level, then proving that if they hold up to loop order $l - 1 \geqslant 0$ they are also valid at order l.
The proposition that follows proves that vertex functions Γ involving B fields do vanish. We use the notation $B\vec\phi$ to denote sequences of field labels with $\phi_i \in \{A, B, c, \bar c\}$.

Proposition 9 *Assume vanishing renormalization conditions for all relevant terms with at least one B field:*

$$\Gamma_l^{B\vec\phi;0\Lambda_0;w}(\vec q) = 0\,, \tag{102}$$

where $\vec q$ is nonexceptional for marginal terms and vanishing otherwise; for rank-2 marginal terms only the coefficient of $\delta_{\mu\nu}$ in the basis $\{\delta_{\mu\nu}, q_{i\mu}q_{j\nu}\}$ is *set to zero. Then, for all* $B\vec{\phi}$*, l, w,* $\vec{p}$*, and* $0 < \Lambda \leqslant \Lambda_0$*,*

$$\mathsf{\Gamma}_l^{B\vec{\phi};\Lambda\Lambda_0;w}(\vec{p}) = 0\,. \tag{103}$$

Proof We prove the statement by induction, increasing in the loop order, $l-1 \mapsto l$. Given l, we proceed by descending from $\bar{w}$ in the number of derivatives, $\|w\| \mapsto \|w\|-1$. For fixed l and w, all possible terms $\mathsf{\Gamma}_l^{B\vec{\phi};w}$ are considered. By construction, for fixed l and $B\vec{\phi}$, the inductive scheme deals first with the irrelevant terms and continues, if they exist, with the marginal terms, followed by more and more relevant terms. The identity $\mathsf{\Gamma}_0^{B\vec{\phi};\Lambda\Lambda_0;w}(\vec{p}) = 0$ follows from the definition of $\mathsf{\Gamma}$. Assume that the statement of the theorem holds up to loop order $l-1 \geqslant 0$. It follows that at order l the rhs of the FE for vertex functions with B fields vanishes. Using the FE we integrate the irrelevant terms from Λ_0 downward to arbitrary $\Lambda > 0$. The boundary conditions $\mathsf{\Gamma}_l^{B\vec{\phi};\Lambda_0\Lambda_0;w}(\vec{p}) = 0$ and the vanishing of the rhs of the FE imply that all irrelevant terms with B fields vanish at loop order l. The boundary conditions $\mathsf{\Gamma}_l^{B\vec{\phi}_r;0\Lambda_0;w}(\vec{q}) = 0$ and the vanishing of the rhs of the FE imply that all marginal terms vanish at their renormalization point for arbitrary $\Lambda > 0$. The derivatives wrt momenta of marginal terms are irrelevant terms. Consequently, we conclude that the marginal $\mathsf{\Gamma}_l^{B\vec{\phi};\Lambda\Lambda_0;w}(\vec{p})$ vanish for all $\vec{p}$. Similar arguments hold for all strictly relevant terms. ■

In the following we will always adopt the renormalization conditions (102). Consequently, counterterms involving B fields are not generated.

Let us denote by $\tilde{W}$, $\tilde{Z}$ the functional W, Z with b set to zero.

$$\tilde{W}(\tilde{K}) := W(j,0,\bar{\eta},\eta), \quad \tilde{Z}(\tilde{K}) := Z(j,0,\bar{\eta},\eta), \quad \tilde{K} := (j,\bar{\eta},\eta)\,. \tag{104}$$

The covariance matrix $\tilde{\mathbf{C}}$ is obtained from $\mathbf{C}$(73) by removing the column and row which correspond to b.

Proposition 10

$$W^{\Lambda\Lambda_0}(j,b,\bar{\eta},\eta) = \frac{1}{2\xi}\langle b,b\rangle + \tilde{W}^{\Lambda\Lambda_0}(j - i\frac{1}{\xi}\partial b,\bar{\eta},\eta)\,. \tag{105}$$

Proof Using the definition of the partition function $Z^{\Lambda\Lambda_0}$ one computes

$$Z^{\Lambda\Lambda_0}(j+i\frac{1}{\xi}\partial b,b,\bar\eta,\eta)=e^{-\frac{1}{\hbar}\mathbb{L}^{\Lambda_0\Lambda_0}}\chi^{\Lambda\Lambda_0}(ij,ib,i\bar\eta,i\eta),\tag{106}$$

$$\mathbb{L}^{\Lambda_0\Lambda_0}:=L^{\Lambda_0\Lambda_0}(\frac{\delta}{\delta j},\frac{\delta}{\delta\bar\eta},\frac{\delta}{\delta\eta})\,.\tag{107}$$

We define the exponential operator in (106) as a formal series expansion. From definition (67) it follows

$$Z^{\Lambda\Lambda_0}(j+i\frac{1}{\xi}\partial b,b,\bar\eta,\eta)=e^{\frac{1}{2\hbar\xi}\langle b,b\rangle}e^{-\frac{1}{\hbar}\mathbb{L}^{\Lambda_0\Lambda_0}}e^{-\frac{1}{\hbar}\langle\bar\eta,S^{\Lambda\Lambda_0}\eta\rangle+\frac{1}{2\hbar}\langle j,C^{\Lambda\Lambda_0}j\rangle}\,.\tag{108}$$

Observing that the expression multiplying $e^{\frac{1}{2\hbar\xi}\langle b,b\rangle}$ is $\tilde Z^{\Lambda\Lambda_0}(j,\bar\eta,\eta)$ we obtain

$$Z^{\Lambda\Lambda_0}(j,b,\bar\eta,\eta)=e^{\frac{1}{2\hbar\xi}\langle b,b\rangle}\tilde Z^{\Lambda\Lambda_0}(j-i\frac{1}{\xi}\partial b,\bar\eta,\eta)\,.\tag{109}$$

Taking the logarithm finishes the proof. ■

An important consequence of the proposition is the existence of relations between CS functions with and without the B-fields:

Corollary 11

$$\xi W_{\mu}^{\Lambda\Lambda_0;BA}(x,y)=i\frac{\partial W_{\nu\mu}^{\Lambda\Lambda_0;AA}(x,y)}{\partial x_\nu},\tag{110}$$

$$\xi W^{\Lambda\Lambda_0;BB}(x,y)=\delta(x-y)-\frac{1}{\xi}\frac{\partial^2 W_{\mu\nu}^{\Lambda\Lambda_0;AA}(x,y)}{\partial x_\mu\partial y_\nu}.\tag{111}$$

Substitution of $W^{\Lambda\Lambda_0}$ (105) into the definition of $\Gamma^{\Lambda\Lambda_0}$ (89) and integration by parts give

$$\Gamma^{\Lambda\Lambda_0}(\underline{A},\underline{B},\underline{\bar c},\underline{c})=\langle b,\underline{B}-i\frac{1}{\xi}\partial\underline{A}-\frac{1}{2}b\rangle+\tilde\Gamma^{\Lambda\Lambda_0}(\underline{A},\underline{\bar c},\underline{c}),\tag{112}$$

$$\tilde\Gamma^{\Lambda\Lambda_0}(\underline{A},\underline{\bar c},\underline{c}):=\langle j,\underline{A}\rangle+\langle\bar\eta,\underline{c}\rangle+\langle\underline{\bar c},\eta\rangle-\tilde W^{\Lambda\Lambda_0}(j,\bar\eta,\eta)\Big|_{K^{\Lambda\Lambda_0}(\underline{\Phi})}\,.\tag{113}$$

From definition 7 it follows that $b=\xi\underline{B}-i\partial\underline{A}$. Consequently, the above expression becomes

$$\Gamma^{\Lambda\Lambda_0}(\underline{A},\underline{B},\underline{\bar c},\underline{c})=\frac{1}{2\xi}\langle\xi\underline{B}-i\partial\underline{A},\xi\underline{B}-i\partial\underline{A}\rangle+\tilde\Gamma^{\Lambda\Lambda_0}(\underline{A},\underline{\bar c},\underline{c})\,.\tag{114}$$

Differentiation wrt $\underline{A}$ yields an important identity:

Corollary 12

$$\frac{\delta\Gamma^{\Lambda\Lambda_0}}{\delta\underline{A}^a_\mu} = i\partial_\mu(\underline{B}^a - i\frac{1}{\xi}\partial\underline{A}^a) + \frac{\delta\tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta\underline{A}^a_\mu}\,. \tag{115}$$

For $\tilde{\Phi} := (A^a_\mu, c^a, \bar{c}^a)$ the functional $\tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}(\underline{\tilde{\Phi}})$ is defined from $L^{\Lambda\Lambda_0}(\tilde{\Phi})$ in analogy with (91). For $0 < \Lambda < \Lambda_0$, it follows that

$$\tilde{\Gamma}^{\Lambda\Lambda_0}(\underline{\tilde{\Phi}}) = \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}(\underline{\tilde{\Phi}}) + \frac{1}{2}\langle\underline{\tilde{\Phi}}, \tilde{\mathbf{C}}^{-1}_{\Lambda\Lambda_0}\,\underline{\tilde{\Phi}}\rangle. \tag{116}$$

Substitution (92) and (116) into (114) yields $\mathsf{\Gamma}^{\Lambda\Lambda_0}(\underline{\Phi}) = \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}(\underline{\tilde{\Phi}})$. Note also that at $\Lambda = \Lambda_0$ we have $\underline{\Phi} = \Phi$ and

$$\mathsf{\Gamma}^{\Lambda_0\Lambda_0}(\underline{\Phi}) = L^{\Lambda_0\Lambda_0}(\Phi) = L^{\Lambda_0\Lambda_0}(\tilde{\Phi}) = \tilde{\mathsf{\Gamma}}^{\Lambda_0\Lambda_0}(\underline{\tilde{\Phi}}) = \int d^4x\,(\mathcal{L}_0^{\Lambda_0\Lambda_0} + \mathcal{L}_{ct}^{\Lambda_0\Lambda_0})\,. \tag{117}$$

1.6 Violated Slavnov–Taylor identities

We are working in a framework where gauge invariance is broken already in the classical lagrangian due to the gauge fixing term. It has then been realized that invariance of the lagrangian under the BRST transformations ensures the gauge invariance of physical quantities to be calculated from the theory [Nie75, PS85]. On the level of correlation functions (Green's functions in the relativistic theory) this invariance leads to a system of identities between different correlation functions which are called Slavnov Taylor identities (STI) [ZJ75, Sla72, Tay71]. These identities may be used to argue that physical quantities obtained from these functions, as for example the pole of the propagators for all physical fields of the Standard Model, are gauge invariant [GG00].

In our framework gauge invariance is also violated in an even more serious way by the presence of the regulators in (40), (41). We want to show that for a suitable class of renormalization conditions, which does not restrict the freedom in fixing the physical coupling constant and the normalization of the fields, gauge invariance can be recovered in the renormalized theory. This means we want to show that the STI hold once we take the limits $\Lambda \to 0$ and $\Lambda_0 \to \infty$.

The first step is then to write a system of violated STI suitable for our subsequent analysis of their restoration. To do so we thus analyse the behavior of the regularized generating functionals of the correlation functions under

BRST transformations. The infinitesimal BRST transformations can be generated by composite operator insertions for which we also have a freedom of normalization, as encoded by the constants R_i introduced below [KM09].

To derive the *violated* STI we consider the functional $Z_{vst}^{\Lambda\Lambda_0}$ defined with the modified Lagrangian density

$$\mathcal{L}_{vst}^{\Lambda_0\Lambda_0} := \mathcal{L}^{\Lambda_0\Lambda_0} + \gamma\psi^{\Lambda_0} + \omega\Omega^{\Lambda_0}, \qquad L_{vst}^{\Lambda_0\Lambda_0} := \int d^4x\, \mathcal{L}_{vst}^{\Lambda_0\Lambda_0}, \tag{118}$$

where γ, ω are external sources, and

$$\psi^{\Lambda_0} := R_1^{\Lambda_0}\partial c - igR_2^{\Lambda_0}[A,c], \quad \Omega^{\Lambda_0} := \frac{1}{2i} g R_3^{\Lambda_0}\{c,c\}, \quad R_i^{\Lambda_0} = 1 + O(\hbar)\,. \tag{119}$$

The requirement that at zero loop order ψ^{Λ_0} and Ω^{Λ_0} correspond to the classical BRST variation implies that the constants $R_i^{\Lambda_0}$ are equal to one at tree level. At higher orders we admit counterterms for the BRST transformation [KM09]. Performing the change of variables $\Phi \mapsto \Phi + \delta_\epsilon\Phi$ we obtain the identity

$$-\hbar \int \delta_\epsilon\big(d\mu_{\Lambda\Lambda_0} e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0}}\big)\, e^{\frac{1}{\hbar}K\cdot\Phi} = \int d\mu_{\Lambda\Lambda_0} e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0} + \frac{1}{\hbar}K\cdot\Phi} K\cdot\delta_\epsilon(\Phi), \tag{120}$$

where

$$\delta_\epsilon A := \epsilon\,\sigma_{0\Lambda_0} * \psi^{\Lambda_0}, \qquad \delta_\epsilon c := -\epsilon\,\sigma_{0\Lambda_0} * \Omega^{\Lambda_0}, \qquad \delta_\epsilon \bar{c} := \epsilon\,\sigma_{0\Lambda_0} * iB\,. \tag{121}$$

Using formula (69) for an infinitesimal change of variables and substituting the variation $\delta_\epsilon\Phi$ with its explicit form (121) we have

$$-\hbar\,\delta_\epsilon(d\mu_{\Lambda\Lambda_0}(\Phi)) = d\mu_{\Lambda\Lambda_0}(\Phi)\,\epsilon\, I_1^{\Lambda\Lambda_0}(\Phi)\,, \tag{122}$$

$$-\hbar\,\delta_\epsilon(e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0}(\Phi)}) = e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0}(\Phi)}\epsilon\, I_2^{\Lambda_0}(\Phi)\,, \tag{123}$$

where

$$
\begin{aligned}
I_1^{\Lambda\Lambda_0}(\Phi) :=& \langle A, C_{\Lambda\Lambda_0}^{-1}\sigma_{0\Lambda_0} * \psi^{\Lambda_0}\rangle - \langle \bar{c}, S_{\Lambda\Lambda_0}^{-1}\sigma_{0\Lambda_0} * \Omega^{\Lambda_0}\rangle - i\langle B, \sigma_{0\Lambda_0} * S_{\Lambda\Lambda_0}^{-1} c\rangle \\
&+ i\langle \partial(B - i\frac{1}{\xi}\partial A), \sigma_{0\Lambda_0} * \psi^{\Lambda_0}\rangle\,, \qquad (124)\\
I_2^{\Lambda_0}(\Phi) :=& \langle \sigma_{0\Lambda_0} * \frac{\delta L^{\Lambda_0\Lambda_0}}{\delta A}, \psi^{\Lambda_0}\rangle - \langle \sigma_{0\Lambda_0} * \frac{\delta L^{\Lambda_0\Lambda_0}}{\delta c}, \Omega^{\Lambda_0}\rangle \\
&+ i\langle B, \sigma_{0\Lambda_0} * \frac{\delta L^{\Lambda_0\Lambda_0}}{\delta \bar{c}}\rangle + \langle \gamma, Q_{\rho\gamma}^{\Lambda_0}\rangle + \langle \omega, Q_{\rho\omega}^{\Lambda_0}\rangle\,, \qquad (125)\\
Q_{\rho\gamma_\mu^a}^{\Lambda_0} :=& g(R_3^{\Lambda_0} - R_2^{\Lambda_0})\epsilon^{abs}(\sigma_{0\Lambda_0} * \psi^{\Lambda_0})_\mu^b c^s \\
&+ gR_1^{\Lambda_0}R_3^{\Lambda_0}\epsilon^{abs}(c^s(\sigma_{0\Lambda_0} * \partial_\mu c^b) - \sigma_{0\Lambda_0} * (c^s\partial_\mu c^b)) \\
&+ g^2 R_2^{\Lambda_0}R_3^{\Lambda_0}\epsilon^{ase}\epsilon^{ebt}((\sigma_{0\Lambda_0} * (A_\mu^b c^t))c^s - A_\mu^b(\sigma_{0\Lambda_0} * (c^t c^s)))\,, \qquad (126)\\
Q_{\rho\omega^a}^{\Lambda_0} :=& (gR_3^{\Lambda_0})^2 c^s(\sigma_{0\Lambda_0} * (c^a c^s))\,. \qquad (127)
\end{aligned}
$$

The terms $Q_{\rho\gamma}^{\Lambda_0}$ and $Q_{\rho\omega}^{\Lambda_0}$ originate from the variations $\delta_\epsilon(\gamma\psi^{\Lambda_0})$, $\delta_\epsilon(\omega\Omega^{\Lambda_0})$, see (118). Substituting $B \mapsto \frac{1}{\xi}(b + i\partial A)$, see (76), we get

$$
\int d\mu_{\Lambda\Lambda_0}(I_1^{\Lambda\Lambda_0} + I_2^{\Lambda_0})e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0} + \frac{1}{\hbar}K\cdot\Phi} = \int d\mu_{\Lambda\Lambda_0}(J_1^{\Lambda\Lambda_0} + J_2^{\Lambda_0})e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0} + \frac{1}{\hbar}K\cdot\Phi},
$$

where

$$
\begin{aligned}
J_1^{\Lambda\Lambda_0}(\Phi) :=& \langle A, C_{\Lambda\Lambda_0}^{-1}\sigma_{0\Lambda_0} * \psi^{\Lambda_0}\rangle - \langle \bar{c}, S_{\Lambda\Lambda_0}^{-1}\sigma_{0\Lambda_0} * \Omega^{\Lambda_0}\rangle \\
&+ \frac{1}{\xi}\langle \partial A, \sigma_{0\Lambda_0} * S_{\Lambda\Lambda_0}^{-1} c\rangle - i\frac{1}{\xi}\langle b, \sigma_{0\Lambda_0} * (S_{\Lambda\Lambda_0}^{-1} c + \partial\psi^{\Lambda_0})\rangle\,, \qquad (128)\\
J_2^{\Lambda_0}(\Phi) :=& \langle \sigma_{0\Lambda_0} * \frac{\delta L^{\Lambda_0\Lambda_0}}{\delta A}, \psi^{\Lambda_0}\rangle - \langle \sigma_{0\Lambda_0} * \frac{\delta L^{\Lambda_0\Lambda_0}}{\delta c}, \Omega^{\Lambda_0}\rangle \\
&+ i\frac{1}{\xi}\langle b + i\partial A, \sigma_{0\Lambda_0} * \frac{\delta L^{\Lambda_0\Lambda_0}}{\delta \bar{c}}\rangle + \langle \gamma, Q_{\rho\gamma}^{\Lambda_0}\rangle + \langle \omega, Q_{\rho\omega}^{\Lambda_0}\rangle\,. \qquad (129)
\end{aligned}
$$

Introducing the operators $Q_\rho^{\Lambda_0}$ and $Q_\beta^{\Lambda_0}$,

$$
\begin{aligned}
Q_\rho^{\Lambda_0} :=& \frac{\delta L^{\Lambda_0\Lambda_0}}{\delta A_\mu}\sigma_{0\Lambda_0} * \psi_\mu^{\Lambda_0} - \frac{\delta L^{\Lambda_0\Lambda_0}}{\delta c}\sigma_{0\Lambda_0} * \Omega^{\Lambda_0} - \frac{1}{\xi}\frac{\delta L^{\Lambda_0\Lambda_0}}{\delta \bar{c}}\sigma_{0\Lambda_0} * \partial A \\
&+ AC^{-1}\psi^{\Lambda_0} - \bar{c}S^{-1}\Omega^{\Lambda_0} + \frac{1}{\xi}\partial A S^{-1} c\,, \qquad (130)\\
Q_\beta^{\Lambda_0} :=& \sigma_{0\Lambda_0} * \Big(\frac{\delta L^{\Lambda_0\Lambda_0}}{\delta \bar{c}} - \partial\psi^{\Lambda_0}\Big) - S^{-1} c\,, \qquad (131)
\end{aligned}
$$

we have

$$J_1^{\Lambda\Lambda_0} + J_2^{\Lambda_0} = i\frac{1}{\xi}\langle b, Q_\beta^{\Lambda_0}\rangle + \langle \gamma, Q_{\rho\gamma}^{\Lambda_0}\rangle + \langle \omega, Q_{\rho\omega}^{\Lambda_0}\rangle + \int d^4x\, Q_\rho^{\Lambda_0}(x)$$
$$- i\frac{1}{\xi}\langle b, \delta S_{\Lambda\Lambda_0}^{-1} c\rangle + \langle A, \delta C_{\Lambda\Lambda_0}^{-1}\psi^{\Lambda_0}\rangle - \langle \bar{c}, \delta S_{\Lambda\Lambda_0}^{-1}\Omega^{\Lambda_0}\rangle + \frac{1}{\xi}\langle \partial A, \delta S_{\Lambda\Lambda_0}^{-1} c\rangle\,, \quad (132)$$

where $\delta C_{\Lambda\Lambda_0}^{-1} := \delta\sigma_{\Lambda\Lambda_0} C^{-1}$, $\delta S_{\Lambda\Lambda_0}^{-1} := \delta\sigma_{\Lambda\Lambda_0} S^{-1}$, $\delta\sigma_{\Lambda\Lambda_0} := \sigma_\Lambda * \sigma_{\Lambda\Lambda_0}^{-1}$. We may now express the lhs of (120) as

$$\epsilon \int d\mu_{\Lambda\Lambda_0} (J_1^{\Lambda\Lambda_0} + J_2^{\Lambda_0})\, e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0} + \frac{1}{\hbar}K\cdot\Phi} = -\epsilon\, (\mathcal{D}_0^{\Lambda\Lambda_0} + \hbar\mathcal{D}_1^{\Lambda\Lambda_0}) Z_{aux}^{\Lambda\Lambda_0}\Big|_{\rho,\beta=0}, \quad (133)$$

where

$$\mathcal{D}_0^{\Lambda\Lambda_0} := \int d^4x\, \hbar \frac{\delta}{\delta\rho(x)} + i\hbar\frac{1}{\xi}\langle b, \frac{\delta}{\delta\beta} + \delta S_{\Lambda\Lambda_0}^{-1}\frac{\delta}{\delta\bar{\eta}}\rangle, \quad (134)$$
$$\mathcal{D}_1^{\Lambda\Lambda_0} := \langle \frac{\delta}{\delta j}, \delta C_{\Lambda\Lambda_0}^{-1}\hbar\frac{\delta}{\delta\gamma}\rangle + \langle \frac{\delta}{\delta\eta}, \delta S_{\Lambda\Lambda_0}^{-1}\hbar\frac{\delta}{\delta\omega}\rangle - \frac{1}{\xi}\langle \partial\frac{\delta}{\delta j}, \delta S_{\Lambda\Lambda_0}^{-1}\hbar\frac{\delta}{\delta\bar{\eta}}\rangle\,, \quad (135)$$
$$Z_{aux}^{\Lambda\Lambda_0} := \int d\mu_{\Lambda\Lambda_0}(\Phi)\, e^{-\frac{1}{\hbar}L_{aux}^{\Lambda_0\Lambda_0} + \frac{1}{\hbar}K\cdot\Phi}, \quad (136)$$
$$\mathcal{L}_{aux}^{\Lambda_0\Lambda_0} := \mathcal{L}_{vst}^{\Lambda_0\Lambda_0} + \rho Q_\rho^{\Lambda_0} + \rho\gamma_\mu^a Q_{\rho\gamma_\mu^a}^{\Lambda_0} + \rho\omega^a Q_{\rho\omega^a}^{\Lambda_0} + \beta Q_\beta^{\Lambda_0}\,. \quad (137)$$

Defining

$$\mathcal{S} := \langle j, \sigma_{0\Lambda_0} * \hbar\frac{\delta}{\delta\gamma}\rangle + \langle \bar{\eta}, \sigma_{0\Lambda_0} * \hbar\frac{\delta}{\delta\omega}\rangle - i\langle \sigma_{0\Lambda_0} * \hbar\frac{\delta}{\delta b}, \eta\rangle, \quad (138)$$

we write the rhs of (120) in the following form

$$\int d\mu_{\Lambda\Lambda_0}\, e^{-\frac{1}{\hbar}L_{vst}^{\Lambda_0\Lambda_0} + \frac{1}{\hbar}K\cdot\Phi} K\cdot\delta_\epsilon\Phi = -\epsilon\,\mathcal{S} Z_{vst}^{\Lambda\Lambda_0} = -\epsilon\,\mathcal{S} Z_{aux}^{\Lambda\Lambda_0}\Big|_{\rho,\beta=0}. \quad (139)$$

Using equations (133) and (139) we write identity (120) as follows

$$(\mathcal{D}_0^{\Lambda\Lambda_0} + \hbar\mathcal{D}_1^{\Lambda\Lambda_0}) Z_{aux}^{\Lambda\Lambda_0}\Big|_{\rho,\beta=0} = \mathcal{S} Z_{aux}^{\Lambda\Lambda_0}\Big|_{\rho,\beta=0}. \quad (140)$$

We also define the functionals describing the BRST anomalies

$$W_\chi^{\Lambda\Lambda_0} := \frac{\delta W^{\Lambda\Lambda_0}}{\delta\chi}\Big|_{\chi=0}, \qquad W_1^{\Lambda\Lambda_0} := \int d^4x\, W_{\rho(x)}^{\Lambda\Lambda_0}\,, \quad (141)$$

where $\chi \in \{\rho, \beta\}$ and γ, ω are arbitrary and $W^{\Lambda\Lambda_0} = \hbar \log Z^{\Lambda\Lambda_0}_{aux}$. Then substituting (109) into (140) and shifting the source $j \mapsto j + i\frac{1}{\xi}\partial b$ we get two equations

$$i\frac{1}{\xi}\langle b, \tilde{W}^{\Lambda\Lambda_0}_{\beta} + \delta S^{-1}_{\Lambda\Lambda_0}\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\bar{\eta}}\rangle = -i\frac{1}{\xi}\langle b, \sigma_{0\Lambda_0} * (\partial\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\gamma} + \eta)\rangle\,, \tag{142}$$

$$\begin{aligned}\tilde{W}^{\Lambda\Lambda_0}_1 + \langle\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta j}, \delta C^{-1}_{\Lambda\Lambda_0}\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\gamma}\rangle + \langle\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\eta}, \delta S^{-1}_{\Lambda\Lambda_0}\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\omega}\rangle& \\ - \frac{1}{\xi}\langle\partial\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta j}, \delta S^{-1}_{\Lambda\Lambda_0}\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\bar{\eta}}\rangle + \hbar\tilde{\Delta}^{\Lambda\Lambda_0}& \\ = \langle j, \sigma_{0\Lambda_0} * \frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\gamma}\rangle + \langle\bar{\eta}, \sigma_{0\Lambda_0} * \frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\omega}\rangle + \frac{1}{\xi}\langle\sigma_{0\Lambda_0} * \partial\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta j}, \eta\rangle\,,&\end{aligned} \tag{143}$$

where

$$\tilde{\Delta}^{\Lambda\Lambda_0} := \langle\delta C^{-1}_{\Lambda\Lambda_0}\frac{\delta^2\tilde{W}^{\Lambda\Lambda_0}}{\delta j\delta\gamma}\rangle + \langle\delta S^{-1}_{\Lambda\Lambda_0}\frac{\delta^2\tilde{W}^{\Lambda\Lambda_0}}{\delta\eta\delta\omega}\rangle - \frac{1}{\xi}\langle\delta S^{-1}_{\Lambda\Lambda_0}\partial_x\frac{\delta^2\tilde{W}^{\Lambda\Lambda_0}}{\delta j_x\delta\bar{\eta}}\rangle\,. \tag{144}$$

From the definition of the Legendre transform for any source $\chi \in \{\rho, \beta, \gamma, \omega\}$

$$\frac{\delta\tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta\chi} = \frac{\delta\tilde{K}}{\delta\chi}\frac{\delta\,(\tilde{K}\cdot\tilde{\Phi} - \tilde{W}^{\Lambda\Lambda_0})}{\delta\tilde{K}} - \frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\chi} = -\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\chi}\,. \tag{145}$$

Consequently,

$$\tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}_1 = -\tilde{W}^{\Lambda\Lambda_0}_1, \qquad \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}_{\beta} = -\tilde{W}^{\Lambda\Lambda_0}_{\beta}, \tag{146}$$

$$\frac{\delta\tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}}{\delta\gamma} = -\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\gamma}, \qquad \frac{\delta\tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}}{\delta\omega} = -\frac{\delta\tilde{W}^{\Lambda\Lambda_0}}{\delta\omega}, \tag{147}$$

Substituting $\tilde{W}$ in (142), (143) with its Legendre transform $\tilde{\Gamma}$ we obtain

$$
\begin{aligned}
\tilde{\mathsf{\Gamma}}_{\beta}^{\Lambda\Lambda_0} =& \sigma_{0\Lambda_0} * \Big(\frac{\delta \tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta \underline{\bar{c}}} - \partial \frac{\delta \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}}{\delta \gamma} \Big) + \delta S^{-1}_{\Lambda\Lambda_0} \underline{c}\,, \qquad (148) \\
\tilde{\mathsf{\Gamma}}_{1}^{\Lambda\Lambda_0} =& \langle \frac{\delta \tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta \underline{A}}, \sigma_{0\Lambda_0} * \frac{\delta \tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta \gamma} \rangle - \langle \underline{A}, \delta C^{-1}_{\Lambda\Lambda_0} \frac{\delta \tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta \gamma} \rangle \\
& - \langle \frac{\delta \tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta \underline{c}}, \sigma_{0\Lambda_0} * \frac{\delta \tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta \omega} \rangle + \langle \underline{\bar{c}}, \delta S^{-1}_{\Lambda\Lambda_0} * \frac{\delta \tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta \omega} \rangle \\
& - \frac{1}{\xi} \langle \partial \underline{A}, \sigma_{0\Lambda_0} * \frac{\delta \tilde{\Gamma}^{\Lambda\Lambda_0}}{\delta \underline{\bar{c}}} + \delta S^{-1}_{\Lambda\Lambda_0} \underline{c} \rangle + \hbar \tilde{\Delta}^{\Lambda\Lambda_0}\,, \qquad (149) \\
\tilde{\Delta}^{\Lambda\Lambda_0} =& \langle (\sigma_{\Lambda}, 0, 0) \Big(1 + \frac{\delta^2 \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}}{\delta \underline{\tilde{\Phi}} \delta \underline{\tilde{\Phi}}'} \hat{\mathbf{1}} \tilde{\mathbf{C}}^{\Lambda\Lambda_0}_{\underline{\tilde{\Phi}}' \underline{\tilde{\Phi}}} \Big)^{-1} \frac{\delta^2 \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}}{\delta \underline{\tilde{\Phi}} \delta \gamma} \rangle \\
& + \langle (0, \sigma_{\Lambda}, 0) \Big(1 + \frac{\delta^2 \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}}{\delta \underline{\tilde{\Phi}} \delta \underline{\tilde{\Phi}}'} \hat{\mathbf{1}} \tilde{\mathbf{C}}^{\Lambda\Lambda_0}_{\underline{\tilde{\Phi}}' \underline{\tilde{\Phi}}} \Big)^{-1} \frac{\delta^2 \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}}{\delta \underline{\tilde{\Phi}} \delta \omega} \rangle \\
& - \langle (\sigma_{\Lambda} \partial, 0, 0) \Big(1 + \frac{\delta^2 \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}}{\delta \underline{\tilde{\Phi}} \delta \underline{\tilde{\Phi}}'} \hat{\mathbf{1}} \tilde{\mathbf{C}}^{\Lambda\Lambda_0}_{\underline{\tilde{\Phi}}' \underline{\tilde{\Phi}}} \Big)^{-1} \frac{\delta \underline{c}}{\delta \underline{\tilde{\Phi}}} \rangle\,. \qquad (150)
\end{aligned}
$$

In section 3.7 using the bounds of theorem 1 we show that $\lim_{\Lambda \to 0} \tilde{\Delta}^{\Lambda\Lambda_0;\vec{\phi};w} = 0$ at nonexceptional momenta, see (168). To get a more concise form for (148), (149) we define an auxiliary functional

$$
\tilde{\mathcal{F}}^{\Lambda\Lambda_0} := \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0} + \frac{1}{2} \langle \underline{\tilde{\Phi}}, \tilde{\mathbf{C}}^{-1}_{0\Lambda_0} \underline{\tilde{\Phi}} \rangle\,. \qquad (151)
$$

Then (148), (149) yield

$$
\begin{aligned}
\tilde{\mathsf{\Gamma}}_{\beta}^{\Lambda\Lambda_0} =& \sigma_{0\Lambda_0} * \Big(\frac{\delta \tilde{\mathcal{F}}^{\Lambda\Lambda_0}}{\delta \underline{\bar{c}}} - \partial \frac{\delta \tilde{\mathcal{F}}^{\Lambda\Lambda_0}}{\delta \gamma} \Big)\,, \qquad (152) \\
\tilde{\mathsf{\Gamma}}_{1}^{\Lambda\Lambda_0} =& \langle \frac{\delta \tilde{\mathcal{F}}^{\Lambda\Lambda_0}}{\delta \underline{A}}, \sigma_{0\Lambda_0} * \frac{\delta \tilde{\mathcal{F}}^{\Lambda\Lambda_0}}{\delta \gamma} \rangle - \langle \frac{\delta \tilde{\mathcal{F}}^{\Lambda\Lambda_0}}{\delta \underline{c}}, \sigma_{0\Lambda_0} * \frac{\delta \tilde{\mathcal{F}}^{\Lambda\Lambda_0}}{\delta \omega} \rangle \\
& - \frac{1}{\xi} \langle \partial \underline{A}, \sigma_{0\Lambda_0} * \frac{\delta \tilde{\mathcal{F}}^{\Lambda\Lambda_0}}{\delta \underline{\bar{c}}} \rangle + \hbar \tilde{\Delta}^{\Lambda\Lambda_0}\,. \qquad (153)
\end{aligned}
$$

A direct implication of theorem 1, see section 2.2, is that the Cauchy criterion for the functional $\tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0}$ to have the limit $\lim_{\Lambda \to 0} \tilde{\mathsf{\Gamma}}^{\Lambda\Lambda_0} = \tilde{\mathsf{\Gamma}}^{0\Lambda_0}$ is satisfied. Hence $\lim_{\Lambda \to 0} \tilde{\mathcal{F}}^{\Lambda\Lambda_0} = \tilde{\Gamma}^{0\Lambda_0}$, see (92). In the remaining of the section we consider this limit. Our aim is to rewrite the rhs of (153) in a such way that it has no functional derivatives wrt the antighost $\underline{\bar{c}}$. For this purpose we shall combine (152) and (153) into the following expression

$$\tilde{\Gamma}_1^{0\Lambda_0} + \frac{1}{\xi}\langle \partial A, \tilde{\Gamma}_\beta^{0\Lambda_0}\rangle = \langle \frac{\delta\tilde{\Gamma}^{0\Lambda_0}}{\delta \underline{A}} + \frac{1}{\xi}\partial\partial\underline{A}, \sigma_{0\Lambda_0} * \frac{\delta\tilde{\Gamma}^{0\Lambda_0}}{\delta\gamma}\rangle - \langle \frac{\delta\tilde{\Gamma}^{0\Lambda_0}}{\delta\underline{c}}, \sigma_{0\Lambda_0} * \frac{\delta\tilde{\Gamma}^{0\Lambda_0}}{\delta\omega}\rangle. \tag{154}$$

Then defining a new functional $\underline{\Gamma}^{0\Lambda_0}$

$$\underline{\Gamma}^{0\Lambda_0} := i\langle \underline{B}, \bar{\omega}\rangle + \tilde{\underline{\Gamma}}^{0\Lambda_0}, \qquad \tilde{\underline{\Gamma}}^{0\Lambda_0} := \tilde{\Gamma}^{0\Lambda_0} + \frac{1}{2\xi}\langle \underline{A}, \partial\partial\,\underline{A}\rangle\,, \tag{155}$$

and operators

$$\mathcal{S}_A := \langle \frac{\delta\underline{\Gamma}^{0\Lambda_0}}{\delta\underline{A}}, \sigma_{0\Lambda_0} * \frac{\delta}{\delta\gamma}\rangle + \langle \frac{\delta\underline{\Gamma}^{0\Lambda_0}}{\delta\gamma}, \sigma_{0\Lambda_0} * \frac{\delta}{\delta\underline{A}}\rangle, \tag{156}$$

$$\mathcal{S}_{\tilde{c}} := \langle \frac{\delta\underline{\Gamma}^{0\Lambda_0}}{\delta\tilde{c}}, \sigma_{0\Lambda_0} * \frac{\delta}{\delta\bar{\omega}}\rangle + \langle \frac{\delta\underline{\Gamma}^{0\Lambda_0}}{\delta\bar{\omega}}, \sigma_{0\Lambda_0} * \frac{\delta}{\delta\tilde{c}}\rangle, \tag{157}$$

$$\mathcal{S}_c := \langle \frac{\delta\underline{\Gamma}^{0\Lambda_0}}{\delta\underline{c}}, \sigma_{0\Lambda_0} * \frac{\delta}{\delta\omega}\rangle + \langle \frac{\delta\underline{\Gamma}^{0\Lambda_0}}{\delta\omega}, \sigma_{0\Lambda_0} * \frac{\delta}{\delta\underline{c}}\rangle\,, \tag{158}$$

$$\frac{\delta}{\delta\tilde{c}} := \frac{\delta}{\delta\underline{\bar{c}}} - \partial\frac{\delta}{\delta\gamma}\,, \tag{159}$$

we rewrite equations (152) and (154) in the form

$$\langle i\underline{B}, \tilde{\Gamma}_\beta^{0\Lambda_0}\rangle = \frac{1}{2}\mathcal{S}_{\tilde{c}}\underline{\Gamma}^{0\Lambda_0} = \langle i\underline{B}, \sigma_{0\Lambda_0} * \frac{\delta}{\delta\tilde{c}}\tilde{\underline{\Gamma}}^{0\Lambda_0}\rangle, \tag{160}$$

$$\tilde{\mathrm{F}}_1^{0\Lambda_0} = \frac{1}{2}\tilde{S}\tilde{\underline{\Gamma}}^{0\Lambda_0}, \tag{161}$$

where

$$\tilde{S} := \mathcal{S}_A - \mathcal{S}_c, \qquad \tilde{\mathrm{F}}_1^{0\Lambda_0} := \tilde{\Gamma}_1^{0\Lambda_0} + \frac{1}{\xi}\langle \partial A, \tilde{\Gamma}_\beta^{0\Lambda_0}\rangle\,. \tag{162}$$

The introduction of the functional $\tilde{\underline{\Gamma}}$ leads to relation (161) and to the consistency condition given in (164) below. They are important in the analysis of the renormalization conditions for $\tilde{\Gamma}_1^{0\Lambda_0;\vec{\phi};w}$, see section 4.

We have the following algebraic identities

$$(\mathcal{S}_i \mathcal{S}_j + \mathcal{S}_j \mathcal{S}_i)\underline{\Gamma}^{0\Lambda_0} = 0, \qquad \forall i,j \in \{A, c, \tilde{c}\}\,. \tag{163}$$

Consequently $\tilde{S}^2 \tilde{\underline{\Gamma}}^{0\Lambda_0} = 0$. Thus application of the operator $\tilde{S}$ to equation (161) yields

$$\tilde{S}\tilde{\mathrm{F}}_1^{0\Lambda_0} = 0. \tag{164}$$

Using again (163), we also have

$$\frac{1}{2}(\tilde{S}\mathcal{S}_{\tilde{c}} + \mathcal{S}_{\tilde{c}}\tilde{S})\underline{\Gamma}^{0\Lambda_0} = 0, \quad \text{and thus} \quad \tilde{S}\tilde{\mathsf{\Gamma}}_\beta + \sigma_{0\Lambda_0} * \frac{\delta}{\delta \tilde{c}} \tilde{\mathrm{F}}_1^{0\Lambda_0} = 0\,. \tag{165}$$

Finally we set $\gamma, \omega = 0$ in (160), (161) to get the AGE and the STI:

$$\begin{aligned} \tilde{\mathsf{\Gamma}}_\beta^{0\Lambda_0} =& \sigma_{0\Lambda_0} * \Big(\frac{\delta \tilde{\underline{\Gamma}}^{0\Lambda_0}}{\delta \underline{\bar{c}}} - \partial \tilde{\underline{\Gamma}}_\gamma^{0\Lambda_0} \Big) & (AGE)\,, \qquad (166) \\ \tilde{\mathrm{F}}_1^{0\Lambda_0} =& \langle \frac{\delta \tilde{\underline{\Gamma}}^{0\Lambda_0}}{\delta \underline{A}}, \sigma_{0\Lambda_0} * \tilde{\underline{\Gamma}}_\gamma^{0\Lambda_0} \rangle - \langle \frac{\delta \tilde{\underline{\Gamma}}^{0\Lambda_0}}{\delta \underline{c}}, \sigma_{0\Lambda_0} * \tilde{\underline{\Gamma}}_\omega^{0\Lambda_0} \rangle & (STI). \qquad (167) \end{aligned}$$

The goal is to show that we can arrange for boundary conditions such that $\tilde{\mathsf{\Gamma}}_\beta^{0\infty} = 0$ and $\tilde{\mathsf{\Gamma}}_1^{0\infty} = 0$, in the sense of theorems 3, 4.

2 Momentum bounds

From now on we use the following conventions:

- $0 < \Lambda \leqslant \Lambda_0$, unless otherwise stated.
- M is a fixed mass parameter such that $0 < M \leqslant \Lambda_0$.
- We omit the tilda for all vertex functions and insertions, for example $\tilde{\Gamma}^{\vec{\phi};w} \mapsto \Gamma^{\vec{\phi};w}$, $\tilde{\mathsf{\Gamma}}^{\vec{\phi};w} \mapsto \mathsf{\Gamma}^{\vec{\phi};w}$, $\tilde{\Gamma}^{\vec{\phi};w}_{\chi} \mapsto \Gamma^{\vec{\phi};w}_{\chi}$.
- We use A, c, $\bar{c}$ instead of $\underline{A}$, $\underline{c}$, $\underline{\bar{c}}$, respectively.
- A tensor monomial is a tensor product of Kronecker δ's and momentum variables in $p := (p_1, ..., p_{n-1})$, for example $\delta_{\mu\nu} p_{1\rho} p_{2\sigma}$. Let $\{\delta^s p^k\}$ be the set of all monomials being a product of s Kronecker δ's and k momenta p_i and let $\{\delta^s p^k\}_r$ be the union of the sets $\{\delta^s p^k\}$ such that r equals the rank of the monomials: $r = 2s + k$. For example $\{\delta^2\}_4 = \{\delta_{\mu\nu}\delta_{\rho\sigma}, \delta_{\mu\sigma}\delta_{\rho\nu}, \delta_{\mu\rho}\delta_{\nu\sigma}\}$.
- For $\vec{p} \in \mathbb{P}_n$ we define the η-function by

$$\eta(\vec{p}) := \min_{S \in \wp_{n-1} \setminus \{\emptyset\}} (|\sum_{i \in S} p_i|, M) \,. \tag{168}$$

  where $\wp_{n-1}$ denotes the power set of $[n-1]$ (the sum does not include $p_0 = -\sum_1^{n-1} p_i$). A momentum configuration $\vec{p}$ is said **nonexceptional** iff $\eta(\vec{p}) \neq 0$ and **exceptional** otherwise.
- For a fixed constant c such that $0 < c < 1$ we define

$$\mathbb{M}_n := \{\vec{p} \in \mathbb{P}_n : \eta(\vec{p}) > cM \text{ and } p_i^2 \leqslant M^2 \ \forall i \in [n-1]\} \,. \tag{169}$$

  Every $\vec{p} \in \mathbb{M}_n$ is nonexceptional.
- $\forall n \geqslant 2$, a momentum configuration $\vec{p} \in \mathbb{M}_n$ is **symmetric** iff $\vec{p} \in \mathbb{M}_n^s$,

$$\mathbb{M}_n^s := \{\vec{p} \in \mathbb{M}_n : p_i p_j = \frac{M^2}{n-1}(n\,\delta_{ij} - 1) \ \forall i, j \in [n-1]\} \,. \tag{170}$$

- $\forall n \geqslant 3$, a momentum configuration $\vec{p} \in \mathbb{M}_n$ is **coplanar** iff $\vec{p} \in \mathbb{M}_n^{cp}$,

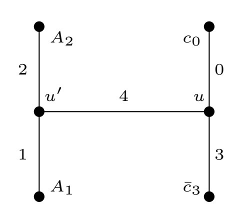
$$\mathbb{M}_n^{cp} := \{\vec{p} \in \mathbb{M}_n : \dim(\mathrm{span}(p_0, ..., p_{n-1})) = 2\} \,. \tag{171}$$

- In the following, a renormalization point is denoted by $\vec{q} \in \mathbb{P}_n$. See appendix H for the list of all relevant terms and their renormalization points.

2.1 Weighted trees

The bounds on the vertex functions presented in section 2.2 are expressed in terms of sets of weighted trees that are introduced by definitions 13, 14 below. As seen from (191), to each edge e of a weighted tree is associated a factor $(|p_e| + \Lambda)^{-\theta(e)}$, p_e being the momentum traversing the edge and $\theta(e)$ being the θ-weight of the edge, expressed as a sum of the ρ and σ-weights of the edge, see (177). The relation (179) expresses the fact that the total θ-weight of a tree is in agreement with power counting. Nonvanishing σ-weights are introduced in order to define viable tree bounds for momentum derived vertex functions. The definition of the σ-weight is inspired by how momentum derivatives are distributed along a tree, taking care of momentum conservation. Before giving the definition of the weighted trees we set up some necessary notations.

- A tree τ is a connected graph with no cycles. The sets of vertices and edges of a tree τ are denoted respectively by $V(\tau)$ and $E(\tau)$, or shortly V, E. In the following, the terms “edge“ and “line“ are equivalent.
- Let V_m be the set of vertices of valence m. Then, $V = \bigcup_{m \geqslant 1} V_m$.
- Let E_1 be the set of edges incident to vertices of valence 1. In other words, E_1 is the set of external edges.
- Each tree has a bijection $\psi : \{0, ..., n-1\} \to V_1$, $i \mapsto v_i$ and a sequence of n field labels $\vec{\varphi} = (\varphi_0, ..., \varphi_{n-1})$, $\varphi_i \in \{A, c, \bar{c}, \gamma, \omega, \beta\}$ and $n = |V_1|$. The field label φ_i defines the type of the vertex $v_i \in V_1$. Let $V_\varphi \subseteq V_1$ be the set of all vertices of type φ, for example V_A. Furthermore, let $E_\varphi \subseteq E_1$ be the set of all edges incident to vertices in V_φ, for example $E_\varkappa$ with $\varkappa \in \{\gamma, \omega\}$.



When needed, the edges are labeled by integers and the vertices by symbols. The edge incident to a vertex $v_i \in V_1$ has the same index i. As an example, for the tree above we have:

$$\begin{array}{llll} V = \{c_0, A_1, A_2, \bar{c}_3, u, u'\}, & V_3 = \{u', u\}, & & \\ V_1 = \{c_0, A_1, A_2, \bar{c}_3\}, & V_A = \{A_1, A_2\}, & V_{\bar{c}} = \{\bar{c}_3\}, & V_c = \{c_0\}, \\ E_1 = \{0, 1, 2, 3\}, & E_A = \{1, 2\}, & E_{\bar{c}} = \{3\}, & E_c = \{0\}. \end{array}$$

- Recalling the definition of $\mathbb{P}_n$ in (18), for every edge $e \in E$ and vertex $v \in V_1$, the momentum assignments p_e, p_v are functions from the set $\mathbb{P}_n$ to $\mathbb{R}^4$, with $n = |V_1|$, defined by the following construction:

  a) label the vertices in V_1 by means of $\psi : i \mapsto v_i$ and set $p_{v_i}(\vec{p}) := p_i$,

  b) apply momentum conservation to all vertices to get $p_e(\vec{p})$.

  We use similar notations for multi-indices: $w_v := w_{\psi^{-1}(v)}$ for $w \in \mathbb{W}_n$ and $v \in V_1$. Given the momentum assignments, a set-valued function K on E is defined by

$$K_e := \{v \in V_1 \backslash \{v_0\} : \frac{\partial p_e}{\partial p_v} \neq 0\}. \tag{172}$$

  For an edge $e \in E$ the momentum p_e should be viewed as a function of all external momenta which flow through the edge. K_e is the set of all vertices corresponding to these momenta. For the tree given above we have $p_{c_0} = -(p_{A_1} + p_{A_2} + p_{\bar{c}_3})$ and

$$K_0 = \{A_1, A_2, \bar{c}_3\}, \qquad K_1 = \{A_1\}, \qquad K_2 = \{A_2\}, \tag{173}$$
$$K_3 = \{\bar{c}_3\}, \qquad K_4 = \{A_1, A_2\}\,. \tag{174}$$

Some additional structure is needed, always in view of the bounds.

- The vertices in V_3 are additionally labeled either as "**regular**" ($\bullet$) or as "**hollow**" ($\circ$). The sets of regular and hollow vertices are respectively denoted by $V_\bullet$ and $V_\circ$, hence $V_3 = V_\bullet \cup V_\circ$. In terms of our bounds, regular vertices do change the ρ-weight of incident edges, while hollow vertices do not, see definition 15. We use hollow vertices for the bounds on 3-point functions and in the proof of the theorems, see for example section 3.2 on the junction of weighted trees and definition 25.

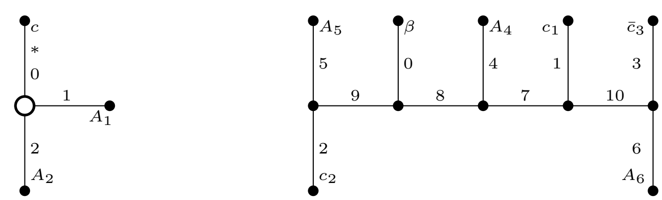
- Edges carry zero or more labels "*". Edges are referred to as "***-edges**" if they have one or more labels "*", and as "**regular edges**" otherwise. The set of all *-edges is denoted by E_*. The *-edges play a special role in our bounds, because to each $e \in E_*$ is associated a supplementary factor $|p_e| + \Lambda$, see (191) and theorem 4.

Definition 13 *Let be given a sequence of $n \geqslant 3$ field labels, $\vec{\varphi} = (\varphi_0, ..., \varphi_{n-1})$, with $\varphi_i \in \{A, c, \bar{c}, \gamma, \omega, \beta\}$. Let $\mathbb{T}_{\vec{\varphi}}$ denote the set of all trees that satisfy the following rules:*

- *There is a bijection $\psi : \{0, ..., n-1\} \to V_1$. Each $v_i \in V_1$ has type φ_i.*
- *$V = V_1 \cup V_3$.*
- *If $n = 3$ then $V_3 = V_\circ$.*
- *$|E_*| \in \mathbb{N}$.*

Definition 14 *In the notations of definition 13, let $\mathcal{T}^{(s)}_{\vec{\varphi}}$ denote the set of all trees in $\mathbb{T}_{\vec{\varphi}}$ with total number of labels "*" equal to s and such that $V_3 = V_\bullet$ whenever $n > 3$. For shortness we set $\mathcal{T}_{\vec{\varphi}} := \mathcal{T}^{(0)}_{\vec{\varphi}}$ and $\mathcal{T}_{1\vec{\varphi}} := \mathcal{T}^{(1)}_{\vec{\varphi}}$.*

As an example, below we show two trees, $\tau_3 \in \mathcal{T}_{1cAA}$ and $\tau_7 \in \mathcal{T}_{\beta cc\bar{c}AAA}$.



By the following two definitions we introduce ρ and σ weights of an edge. ρ and σ-weights are numbers. The ρ-weights of external edges E_1 are assumed to vanish. Using an auxiliary function χ which associates vertices $V_\bullet$ with incident edges $E \backslash E_1$ we define the ρ-weight of an internal edge equal to 2 minus the number of vertices associated with this edge. The σ-weight of an edge coincides with the number of derivatives wrt the external momenta which hit the edge.

Definition 15 *Fix a tree from $\mathbb{T}_{\vec{\varphi}}$. A ρ-weight is a function $\rho : E \to \{0, 1, 2\}$ with the following properties:*

1. $\forall e \in E_1,\ \rho(e) = 0.$

2. There exists a map $\chi : V_\bullet \to E\backslash E_1$ *such that*

a) if $\chi(v) = e$*, then* e *is incident to* v*,*

b) $\forall e \in E\backslash E_1,\ \rho(e) = 2 - |\chi^{-1}(\{e\})|\,.$

Definition 16 *Let be given a tree from* $\mathbb{T}_{\vec{\varphi}}$ *and* $w \in \mathbb{W}_n$*, with* $n = |V_1|$*. A* σ*-weight is a function* $\sigma : E \to \mathbb{N}$ *defined by*

$$\sigma(e) := \sum_{v\in V_1} \sigma_v(e), \tag{175}$$

where $(\sigma_v : E \to \mathbb{N})_{v\in V_1}$ *is a family of functions such that*

$$\sum_{e\in E} \sigma_v(e) = w_v\ , \qquad\qquad \sigma_v(e) = 0 \ \text{if}\ v \notin K_e\ . \tag{176}$$

By definition (23), $w_0 = 0$ for every $w \in \mathbb{W}_n$. Hence, $\sigma_{v_0}(e) = 0\ \forall e \in E$.

Definition 17 *Let be given a tree* $\tau \in \mathbb{T}_{\vec{\varphi}}$ *and* $w \in \mathbb{W}_n$*, with* $n = |V_1|$*. A* θ*-weight is a function* $\theta : E \to \mathbb{N}$ *defined by*

$$\theta(e) := \rho(e) + \sigma(e), \tag{177}$$

where ρ *and* σ *are a* ρ*-weight and a* σ*-weight corresponding to* w*, respectively. The pair* (τ, θ) *is a weighted tree. The total* θ*-weight of* (τ, θ) *is*

$$\theta(\tau) := \sum_{e\in E} \theta(e). \tag{178}$$

The set of all θ*-weights corresponding to given* τ *and* w *is denoted by* Θ^w_τ*.*

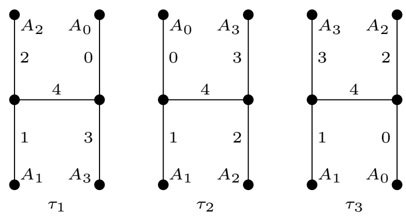
For every tree $\tau \in \mathcal{T}^{(s)}_{\vec{\varphi}}$ with $n \geqslant 4$ the total θ-weight is given by the formula

$$\theta(\tau) = n + \|w\| - 4\,. \tag{179}$$

This relation follows from definitions 14, 15, 16, which give the sum rule $\sum_{e\in E} \theta(e) = \|w\| + 2|E\backslash E_1| - |V_3|$, and from the relations $|E\backslash E_1| - |V_3| + 1 = 0$ and $|V_3| = n - 2$.

As an example we consider three trees $\tau_1, \tau_2, \tau_3 \in \mathcal{T}_{AAAA}$. We give three different weights θ_a, θ_b, θ_c, which all correspond to the derivative wrt the momentum p_1, literally $w_1 = 1$ and $w = (0,1,0,0)$. We find a family of weighted trees $\{(\tau_i, \theta) : \theta \in \Theta^w_{\tau_i}\}_{i\in\{1,2,3\}}$, where

$$\Theta^w_{\tau_3} = \Theta^w_{\tau_1} = \{\theta_a, \theta_b, \theta_c\}, \qquad\qquad \Theta^w_{\tau_2} = \{\theta_a, \theta_c\}. \tag{180}$$



$e \in E$	θ_a	θ_b	θ_c
0	1	0	0
1	0	0	1
2	0	0	0
3	0	0	0
4	0	1	0

2.2 Theorems

We always assume that the renormalization constants are independent of Λ_0 (though weakly Λ_0-dependent "renormalization constants" can also be accommodated for, see [KKS92]). From now on we denote $\vec{\varkappa} := (\varkappa_1, ..., \varkappa_{n_\varkappa})$ with $\varkappa_i \in \{\gamma, \omega\}$, $n_\varkappa \geqslant 0$.

Hypothesis RC1 *We impose on all strictly relevant terms vanishing renormalization conditions at zero momentum and $\Lambda = 0$:*

$$\mathsf{\Gamma}^{0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(0) = 0, \qquad\qquad \textit{if} \quad 2n_\varkappa + \mathrm{N} + \|w\| < 4\,. \tag{181}$$

Hypothesis RC2 *On the following marginal terms we impose renormalization conditions at zero momentum and $\Lambda = M$:*

$$\mathsf{\Gamma}^{M\Lambda_0;c\bar{c}c\bar{c}}(0) = 0, \qquad \mathsf{\Gamma}^{M\Lambda_0;c\bar{c}AA}(0) = 0, \qquad \partial_A \mathsf{\Gamma}^{M\Lambda_0;c\bar{c}A}(0) = 0\,, \tag{182}$$

for the notation see (25).

As it will become clear from proposition 18 the condition $\Lambda = M$ is not essential. Since at the renormalization point $\vec{p} = 0$ these terms do not depend on Λ one could consider the renormalization conditions with $\Lambda \to 0$

$$\lim_{\Lambda\to 0} \mathsf{\Gamma}^{\Lambda\Lambda_0;c\bar{c}c\bar{c}}(0) = 0, \quad \lim_{\Lambda\to 0} \mathsf{\Gamma}^{\Lambda\Lambda_0;c\bar{c}AA}(0) = 0, \quad \lim_{\Lambda\to 0} \partial_A \mathsf{\Gamma}^{\Lambda\Lambda_0;c\bar{c}A}(0) = 0\,. \tag{183}$$

It is important to note that at tree level these terms are incompatible with BRST invariance and already vanish. Remark that Bose–Fermi symmetry and translation invariance imply that $\partial_A \Gamma^{M\Lambda_0;c\bar{c}A}(0) = 0$ iff $\partial_c \Gamma^{M\Lambda_0;A\bar{c}c}(0) = 0$. To prove proposition 18 and theorem 1 all remaining marginal renormalization constants are chosen at $\Lambda = 0$ arbitrarily but in agreement with the global symmetries of the regularized theory: SU(2), Euclidean inhomogeneous O(4), ghost number conservation. For instance, all renormalization conditions must comply with the vanishing of the ghost number violating functions, like $\Gamma^{\Lambda,\Lambda_0;cccc}$ or $\Gamma^{\Lambda,\Lambda_0}_{(\varkappa_0,\varkappa_1)}$ for $\varkappa_i \in \{\gamma, \omega\}$. The list of the remaining marginal renormalization constants follows literally from (721) and appendix F.

Proposition 18 *Assume the validity of hypotheses RC1 and RC2. For all sequences of* $\mathrm{N} \geqslant 3$ *field labels in* $\{A, c, \bar{c}\}$ *with* $\phi_{\mathrm{N}-1} = \bar{c}$, *denoted by* $\vec{\phi}\bar{c}$, *all* $w = (w', 0) \in \mathbb{W}_{\mathrm{N}}$, *all* $(\vec{p}, 0) \in \mathbb{P}_{\mathrm{N}}$, *and all positive* Λ, Λ_0 *s.t.* $\max(\Lambda, M) \leqslant \Lambda_0$,

$$\Gamma^{\Lambda\Lambda_0;\vec{\phi}\bar{c};w}(\vec{p}, 0) = 0\,. \tag{184}$$

Note that in (184) the momentum of the indicated antighost $\bar{c}$ vanishes, and there is no derivative wrt this momentum.

Proof We prove the statement by induction, increasing in the loop order, $l-1 \mapsto l$. Given l, we proceed by descending from $\bar{w}$ in the number of derivatives, $\|w\| \mapsto \|w\| - 1$. For fixed l and w, all possible terms $\Gamma_l^{\vec{\phi}\bar{c};w}$ are considered. By construction, for fixed l and $\vec{\phi}\bar{c}$, the inductive scheme deals first with the irrelevant terms and continues, if they exist, with the marginal terms, followed by more and more relevant terms. Since the momentum of the antighost has been assumed to vanish, the statement holds at loop order $l = 0$. The validity of the statement for all loop orders smaller than l implies that $\dot{\Gamma}_l^{\Lambda\Lambda_0;\vec{\phi}\bar{c};w}(\vec{p}, 0) = 0$. The irrelevant terms have vanishing boundary conditions, hence $\Gamma_l^{\Lambda_0\Lambda_0;\vec{\phi}\bar{c};w}(\vec{p}, 0) = 0$. Integrating the FE from Λ_0 downwards to arbitrary $\Lambda > 0$, we get $\Gamma_l^{\Lambda\Lambda_0;\vec{\phi}\bar{c};w}(\vec{p}, 0) = 0$ for the irrelevant terms. Next we consider the marginal terms. Since the corresponding irrelevant terms have already been shown to vanish at vanishing antighost momentum, we use the Taylor formula to extend (182) to arbitrary momenta $(\vec{p}, 0)$, still preserving the vanishing antighost momentum. Then, we integrate the FE from M to arbitrary $\Lambda > 0$, which completes the proof that $\Gamma_l^{\Lambda\Lambda_0;\vec{\phi}\bar{c};w}(\vec{p}, 0) = 0$ for marginal terms. Similar arguments hold for all the strictly relevant terms $\Gamma_l^{\vec{\phi}\bar{c};w}$. ∎

Corollary 19 *The following counterterms in $L^{\Lambda_0\Lambda_0}$ vanish, see* (45),

$$r_1^{0,\bar{c}cAA} = 0, \qquad r^{0,\bar{c}c\bar{c}c} = 0, \qquad r_2^{0,\bar{c}cAA} = 0, \qquad r_2^{0,A\bar{c}c} = 0. \tag{185}$$

Proof Using (184) we have, for all $p_2, p_3 \in \mathbb{R}^4$ and $\Lambda = \Lambda_0$ (omitted),

$$\Gamma^{c\bar{c}c\bar{c}}(0, p_2, p_3) = 0, \qquad \Gamma^{c\bar{c}AA}(0, p_2, p_3) = 0, \qquad \partial_{p_2}\Gamma^{A\bar{c}c}(0, p_2) = 0\,. \tag{186}$$

Recall that $\Gamma^{\Lambda_0\Lambda_0} = L^{\Lambda_0\Lambda_0}$, see (117). Then the result follows from (45). ■

Corollary 20 *For all $X \in \{\beta, 1\}$, $\vec{\varkappa}$, $\vec{\phi}\bar{c}$, $w = (w', 0)$, $\vec{p}$, and all positive Λ, Λ_0 s.t. $\max(\Lambda, M) \leqslant \Lambda_0$:*

$$\Gamma^{\Lambda\Lambda_0;\vec{\phi}\bar{c};w}_{X\vec{\varkappa}}(\vec{p}, 0) = 0. \tag{187}$$

Proof It follows from the definitions of the inserted functions given in (118), (119), (137) that at tree level $\Gamma^{\Lambda\Lambda_0;\vec{\phi}\bar{c};w}_{X\vec{\varkappa};l=0}(\vec{p}, 0) = 0$. Then using equation (185) one shows that for all these terms we have vanishing boundary conditions, $\Gamma^{\Lambda_0\Lambda_0;\vec{\phi}\bar{c};w}_{X\vec{\varkappa};l}(\vec{p}, 0) = 0\ \forall l$. Assuming that the statement is true at the loop order $l - 1 \geqslant 0$, by induction in l, using (184) and integrating the FE from Λ_0 to arbitrary Λ one shows that it holds at the loop order l. ■

From now on, for simplicity of notation we write $\mathcal{P}_s^{(k)}$ to denote polynomials with nonnegative coefficients and degree s where the superscript is a label to make one polynomial different from another. We define:

$$P_s^{\lambda_1\lambda_2}(\vec{p}) := \mathcal{P}_s^{(0)}\Big(\log_+ \frac{\max(|\vec{p}|, M)}{\lambda_1 + \eta(\vec{p})}\Big) + \mathcal{P}_s^{(1)}\Big(\log_+ \frac{\lambda_2}{M}\Big)\,, \tag{188}$$

$$P_s^{\lambda}(\vec{p}) := P_s^{\lambda\lambda}(\vec{p})\,, \tag{189}$$

$$\Pi^{\lambda}_{\tau,\theta}(\vec{p}) := \prod_{e\in E} (\lambda + |p_e|)^{-\theta(e)}\ , \tag{190}$$

$$Q_\tau^{\Lambda;w}(\vec{p}) := \frac{\prod\limits_{e\in E_*} (\Lambda + |p_e|)}{\prod\limits_{e\in E_\varkappa} (\Lambda + |p_e|)} \begin{cases} \inf\limits_{i\in\mathbb{I}} \sum\limits_{\theta\in\Theta_\tau^{w'(i)}} \Pi^{\Lambda}_{\tau,\theta}(\vec{p}), & |V_1| = 3\,, \\ \sum\limits_{\theta\in\Theta_\tau^{w}} \Pi^{\Lambda}_{\tau,\theta}(\vec{p}), & otherwise\,, \end{cases} \tag{191}$$

where $\tau \in \mathbb{T}_{\vec{\varphi}}$, $w'(i)$ is obtained from w by diminishing w_i by one unit, and, for nonvanishing w, $\mathbb{I} := \{i : w_i > 0\}$. The following sets are also used in theorems 1–4:

$$\mathbb{Y}_n^+ := \{(\Lambda, \Lambda_0) : 0 < \Lambda \leqslant \Lambda_0 \text{ and } \Lambda_0 \geqslant M\} \times \mathbb{P}_n\,, \tag{192}$$

$$\mathbb{Y}_n := \mathbb{Y}_n^+ \bigcup \{(0, \Lambda_0) : \Lambda_0 \geqslant M\} \times \{\vec{p} \in \mathbb{P}_n : \eta(\vec{p}) \neq 0\}. \tag{193}$$

Theorem 1 *There exists a collection of regular vertex functions $\Gamma^{\vec{\phi}}_{\vec{\varkappa};l}$ on $\mathbb{Y}^+_{\mathrm{N}+n_\varkappa}$, complying with the global symmetries of the theory, satisfying the FE and the renormalization conditions given by hypotheses RC1 and RC2, and with irrelevant terms vanishing at $\Lambda = \Lambda_0$. Furthermore, for all $\vec{\phi}$, $\vec{\varkappa}$, all $l \in \mathbb{N}$, $w \in \mathbb{W}_{\mathrm{N}+n_\varkappa}$, the following bounds hold on $\mathbb{Y}^+_{\mathrm{N}+n_\varkappa}$:*

a) $d \geqslant 0$ *or* $\mathrm{N} + n_\varkappa = 2$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant (\Lambda + |\vec{p}|)^d P^\Lambda_r(\vec{p})\,, \tag{194}$$

b) $d < 0$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant \sum_{\tau \in \mathcal{T}_{\vec{\varkappa}\vec{\phi}}} Q^{\Lambda;w}_\tau(\vec{p})\; P^\Lambda_r(\vec{p})\,. \tag{195}$$

Here $d := 4 - 2n_\varkappa - \mathrm{N} - \|w\|$. If $l = 0$ then $r := 0$, otherwise r stands for $r(d, l)$.

$$r(d, l) := \begin{cases} 2l, & d \geqslant 0\,, \\ 2l - 1, & d < 0\,. \end{cases} \tag{196}$$

Theorem 1 shows in particular that the functions $\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}$ are bounded uniformly in Λ_0. To prove convergence in the limit $\Lambda_0 \to \infty$ we establish the following bounds for their derivatives wrt Λ_0.

Theorem 2 *Let be given a collection of vertex functions $\Gamma^{\vec{\phi}}_{\vec{\varkappa};l}$ as in theorem 1. Then, for all $\vec{\phi}$, $\vec{\varkappa}$, all $l \in \mathbb{N}$, $w \in \mathbb{W}_{\mathrm{N}+n_\varkappa}$, the following bounds hold on $\mathbb{Y}^+_{\mathrm{N}+n_\varkappa}$:*

a) $d \geqslant 0$ *or* $\mathrm{N} + n_\varkappa = 2$

$$|\partial_{\Lambda_0} \Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant \frac{\Lambda + M + |\vec{p}|}{\Lambda_0^2} (\Lambda + |p|)^d P^{\Lambda\Lambda_0}_r(\vec{p})\,, \tag{197}$$

b) $d < 0$

$$|\partial_{\Lambda_0} \Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant \frac{\Lambda + M + |\vec{p}|}{\Lambda_0^2} \sum_{\tau \in \mathcal{T}_{\vec{\varkappa}\vec{\phi}}} Q^{\Lambda;w}_{\tau}(\vec{p})\, P^{\Lambda\Lambda_0}_{r}(\vec{p})\,. \tag{198}$$

See theorem 1 for the definition of d and r.

Convergence of the limit $\Lambda \to 0^+$ of the terms $\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(\vec{p})$, $\partial_{\Lambda_0}\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(\vec{p})$ when $\vec{p}$ is nonexceptional (or $d > 0$) follows from the Cauchy criterion

$$|\partial^k_{\Lambda_0} \Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(\vec{p}) - \partial^k_{\Lambda_0} \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(\vec{p})| \leqslant \int_{\Lambda}^{\Lambda'} d\lambda\, |\partial_\lambda \partial^k_{\Lambda_0} \Gamma^{\lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(\vec{p})|\,, \tag{199}$$

and the bounds from theorems 1, 2. Convergence of the limit $\Lambda_0 \to \infty$ of the terms $\Gamma^{0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(\vec{p})$ when $\vec{p}$ is nonexceptional (or $d > 0$) follows from Cauchy criterion and the bounds from theorem 2,

$$|\Gamma^{0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(\vec{p}) - \Gamma^{0\Lambda_0';\vec{\phi};w}_{\vec{\varkappa}}(\vec{p})| \leqslant \int_{\Lambda_0}^{\Lambda_0'} d\lambda_0\, |\partial_{\lambda_0} \Gamma^{0\lambda_0;\vec{\phi};w}_{\vec{\varkappa}}(\vec{p})|\,. \tag{200}$$

In the following we will consider the functions $\Gamma^{0\Lambda_0;\vec{\phi};w}_{1\vec{\varkappa}}$, $\Gamma^{0\Lambda_0;\vec{\phi};w}_{\beta\vec{\varkappa}}$ which appear on the lhs respectively of the ST identities (167) and of the AGE (166). The goal of theorems 3, 4 is to show that $\Gamma^{0\Lambda_0;\vec{\phi};w}_{1\vec{\varkappa}}$ and $\Gamma^{0\Lambda_0;\vec{\phi};w}_{\beta\vec{\varkappa}}$ vanish in the limit $\Lambda_0 \to \infty$, which restores the STI and AGE. The renormalization conditions for these functions at $\Lambda = 0$ are obtained from the rhs of the STI and AGE. In particular for all strictly relevant terms $\Gamma^{\vec{\phi};w}_{1}$ from (153) we have

$$\begin{aligned} |\Gamma^{0\Lambda_0;\vec{\phi};w}_{1;\vec{\varkappa}}(0)| \leqslant \lim_{\Lambda\to 0} \sum_{\substack{\vec{\phi}_1 \oplus \vec{\phi}_2 = \vec{\phi}_\pi \\ \vec{\varkappa}_1 \oplus \vec{\varkappa}_2 = \vec{\varkappa}_{\pi'} \\ w_1 + w_2 = w_\pi}} \Big(& |F^{\Lambda\Lambda_0;A\vec{\phi}_1;w_1}_{\vec{\varkappa}_1}(0)||\Gamma^{\Lambda\Lambda_0;\vec{\phi}_2;w_2}_{\gamma\vec{\varkappa}_2}(0)| \\ & + |F^{\Lambda\Lambda_0;c\vec{\phi}_1;w_1}_{\vec{\varkappa}_1}(0)||\Gamma^{\Lambda\Lambda_0;\vec{\phi}_2;w_2}_{\omega\vec{\varkappa}_2}(0)|\Big) + \frac{1}{\xi} \lim_{\Lambda\to 0} \sum_{\substack{A\oplus\vec{\phi}_2 = \vec{\phi}_\pi \\ 1_A + w_2 = w_\pi}} |F^{\Lambda\Lambda_0;\bar{c}\vec{\phi}_2;w_2}_{\vec{\varkappa}}(0)|, \end{aligned} \tag{201}$$

where the sums run over all partitions and permutations such that

$$\vec{\phi}_1 \oplus \vec{\phi}_2 = (\phi_{\pi(0)}, ..., \phi_{\pi(n-1)}), \qquad \vec{\varkappa}_1 \oplus \vec{\varkappa}_2 = (\varkappa_{\pi'(1)}, ..., \varkappa_{\pi'(n_\varkappa)})\,. \tag{202}$$

Substituting each term in the sum with the bounds of theorem 1 we obtain

$$\lim_{\Lambda\to 0} \Lambda^{3-\mathrm{N}_1-2n_{\varkappa_1}-\|w_1\|}\Lambda^{2-\mathrm{N}_2-2n_{\varkappa_2}-\|w_2\|}\Big(\mathcal{P}_{2l}^{(0)}\Big(\log_+\frac{M}{\Lambda}\Big)+\mathcal{P}_{2l}^{(1)}\Big(\log_+\frac{\Lambda}{M}\Big)\Big)$$
$$=\lim_{\Lambda\to 0}\Lambda^{5-\mathrm{N}-2n_{\varkappa}-\|w\|}\mathcal{P}_{2l}^{(0)}\Big(\log_+\frac{M}{\Lambda}\Big)=0\,. \quad (203)$$

This gives the renormalization conditions for all strictly relevant terms

$$\Gamma_{1;\vec{\varkappa}}^{0\Lambda_0;\vec{\phi};w}(0)=0, \qquad \mathrm{N}+2n_{\varkappa}+\|w\|<5. \quad (204)$$

Substituting the vertex functions on the rhs in (152) with the bounds of theorem 1 we obtain the renormalization conditions for the striclty relevant terms $\Gamma_{\beta}^{\vec{\phi}}$

$$\Gamma_{\beta}^{0\Lambda_0;\vec{\phi};w}(0)=0, \qquad \mathrm{N}+\|w\|<3. \quad (205)$$

In section 4 we show that the required boundary marginal terms $\Gamma_{1\vec{\varkappa}}^{0\Lambda_0;\vec{\phi};w}$, $\Gamma_{\beta\vec{\varkappa}}^{0\Lambda_0;\vec{\phi};w}$ satisfy the bounds of the theorems under the conditions specified in hypothesis RC3.

Hypothesis RC3 *We allow R^{AAA}, r_1^{AA}, $r^{\bar{c}c}$ to be chosen arbitrarily but the remaining marginal renormalization constants must satisfy a set of equations: R_1(425), R_2(444), R_3(447), r_2^{AA}(441), $R_1^{A\bar{c}c}$(431), $R_{1,2}^{AAAA}$(451), see appendix F and* (721) *in appendix E for notations.*

For shortness we also introduce the following definition

$$F_{r\,s}^{\Lambda\Lambda_0}(\vec{p}):=\frac{M+|\vec{p}|+\Lambda}{\Lambda_0}P_{r\,s}^{\Lambda\Lambda_0}(\vec{p})\,, \quad (206)$$

$$P_{r\,s}^{\Lambda\Lambda_0}(\vec{p}):=\Big(1+\Big(\frac{|\vec{p}|}{\Lambda_0}\Big)^{\bar{w}}\Big)\mathcal{P}_s^{(2)}\Big(\frac{|\vec{p}|}{\Lambda+M}\Big)P_r^{\Lambda\Lambda_0}(\vec{p})\,, \quad (207)$$

for $\bar{w}$ see after (24).

Theorem 3 *Let be given a collection of vertex functions $\Gamma_{\vec{\varkappa};l}^{\vec{\phi}}$, regular on $\mathbb{Y}_{\mathrm{N}+n_{\varkappa}}$, complying with the global symmetries of the theory, satisfying the hypotheses RC1, RC2, RC3 and the bounds of theorems 1,2. Let $\Gamma_{\beta,\vec{\varkappa};l}^{\vec{\phi}}$ be a collection of vertex functions with one insertion of the operator $Q_{\beta}^{\Lambda_0}$* (131)*, regular on $\mathbb{Y}_{1+\mathrm{N}+n_{\varkappa}}$, complying with the global symmetries of the theory, satisfying the FE, and s.t. the AGE* (166) *holds. Then, for all $\vec{\phi}$, $\vec{\varkappa}$, all $l\in\mathbb{N}$, $w\in\mathbb{W}_{1+\mathrm{N}+n_{\varkappa}}$, the following bounds hold on $\mathbb{Y}_{1+\mathrm{N}+n_{\varkappa}}$:*

a) $d \geqslant 0$ *or* $\mathrm{N} + n_\varkappa = 1$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\beta\vec{\varkappa};l}(\vec{p})| \leqslant (\Lambda + |\vec{p}|)^d F^{\Lambda\Lambda_0}_{r_\beta\, s_\beta}(\vec{p})\,, \tag{208}$$

b) $d < 0$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\beta\vec{\varkappa};l}(\vec{p})| \leqslant \sum_{\tau \in \mathcal{T}_{\beta\vec{\varkappa}\vec{\phi}}} Q^{\Lambda;w}_{\tau}(\vec{p})\, F^{\Lambda\Lambda_0}_{r_\beta\, s_\beta}(\vec{p})\,. \tag{209}$$

Here $d := 3 - 2n_\varkappa - \mathrm{N} - \|w\|$ *and* $s_\beta := 0$*. If* $l = 0$ *then* $r_\beta := 0$ *otherwise* r_β *stands for* $r_\beta(d,l)$*.*

$$r_\beta(d,l) := \begin{cases} 2l, & d \geqslant 0\,, \\ 2l-1, & d < 0\,. \end{cases} \tag{210}$$

We note that the 1-point vertex function with integrated insertion (146) vanishes, $\Gamma^{\Lambda\Lambda_0;c;w}_{1} = 0$, e.g. due to SU(2) symmetry.

Theorem 4 *Let be given a collection of vertex functions* $\Gamma^{\vec{\phi}}_{\vec{\varkappa};l}$*,* $\Gamma^{\vec{\phi}}_{\beta\vec{\varkappa};l}$ *as in theorem 3. Let* $\Gamma^{\vec{\phi}}_{1,\vec{\varkappa};l}$ *be a collection of vertex functions with one integrated insertion of the appropriate operator among* $Q^{\Lambda_0}_{\rho}$ (130)*,* $Q^{\Lambda_0}_{\rho\gamma}$ (126)*,* $Q^{\Lambda_0}_{\rho\omega}$ (127)*, regular on* $\mathbb{Y}_{\mathrm{N}+n_\varkappa}$*, complying with the global symmetries of the theory, and satisfying the FE. Assume that the STI* (166) *and consistency conditions* (164) *and* (165) *do hold. Then, for all* $\vec{\phi}$*,* $\vec{\varkappa}$*, all* $l \in \mathbb{N}$*,* $w \in \mathbb{W}_{\mathrm{N}+n_\varkappa}$*, the following bounds hold on* $\mathbb{Y}_{\mathrm{N}+n_\varkappa}$*:*

a) $d > 0$ *or* $\mathrm{N} + n_\varkappa = 2$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{1\vec{\varkappa};l}(\vec{p})| \leqslant (\Lambda + |\vec{p}|)^d F^{\Lambda\Lambda_0}_{r_1\, s_1}(\vec{p})\,, \tag{211}$$

b) $d \leqslant 0$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{1\vec{\varkappa};l}(\vec{p})| \leqslant \sum_{\tau \in \mathcal{T}_{1\vec{\varkappa}\vec{\phi}}} Q^{\Lambda;w}_{\tau}(\vec{p})\, F^{\Lambda\Lambda_0}_{r_1\, s_1}(\vec{p})\,. \tag{212}$$

Here $d := 5 - 2n_\varkappa - \mathrm{N} - \|w\|$*. If* $l = 0$ *then* $r_1 := 0$*,* $s_1 := 0$ *otherwise* r_1*,* s_1 *stand respectively for* $r_1(d,l)$*,* $s_1(d,l)$*.*

$$r_1(d,l) := \begin{cases} 3l, & d > 0\,, \\ 3l-1, & d = 0\,, \\ 3l-2, & d < 0\,, \end{cases} \qquad s_1(d,l) := \begin{cases} l, & d \geqslant 0\,, \\ l-1, & d < 0\,. \end{cases} \tag{213}$$

3 Proof of Theorems 1–4

In this section we will prove theorems 1, 2, 3 and 4 in this order. We proceed by induction in the loop order l. We first verify that they hold at tree level $l=0$. Afterwards we assume that they hold true up to loop order $l-1 \geqslant 0$, and we will verify the induction step from $l-1$ to l.

Put $D_X := 4$ for all vertex functions $\Gamma^{\vec{\phi}}_{\vec{\varkappa}}$. For all inserted functions $\Gamma^{\vec{\phi};w}_{X\vec{\varkappa}}$ with $X \in \{\beta, 1\}$ and $\|w\| \leqslant \bar{w}$ let

$$D_X := \begin{cases} 3, & X = \beta\,, \\ 5, & X = 1\,, \end{cases} \qquad\qquad d_X := D_X - 2n_{\varkappa} - \mathrm{N} - \|w\|\,. \tag{214}$$

Note that at zero loop order $\Gamma^{\Lambda\Lambda_0}_{l=0} = \Gamma^{\Lambda_0\Lambda_0}_{l=0}$. Using (117), (146) and the definition of $\mathcal{L}_{aux}$ in (137) one finds that in momentum space

$$\Gamma^{\Lambda_0\Lambda_0;\vec{\phi}}_{1} = Q^{\Lambda_0;\vec{\phi}}_{\rho(0)}\,, \qquad \Gamma^{\Lambda_0\Lambda_0;\vec{\phi}}_{1\varkappa} = Q^{\Lambda_0;\vec{\phi}}_{\rho(0)\varkappa}, \qquad \Gamma^{\Lambda_0\Lambda_0;\vec{\phi}}_{\beta} = Q^{\Lambda_0;\vec{\phi}}_{\beta}\,, \tag{215}$$

where the momentum variable corresponding to the source ρ is set to zero. Everywhere in the following $Q^{\Lambda_0}_{\rho}$ will stand for $Q^{\Lambda_0}_{\rho(0)}$. From definition of $Q^{\Lambda_0}_{\rho}$ in (130) it follows that the vertex functions $Q^{\Lambda_0;\vec{\phi}}_{\rho;l=0}$ with $\mathrm{N}=2$ vanish. The nonvanishing functions $Q^{\Lambda_0;\vec{\phi}}_{\rho;l=0}$, $Q^{\Lambda_0;\vec{\phi}}_{\beta;l=0}$ have the form $h_s(p,q)(1-\sigma_{0\Lambda_0}(p^2))$ where h_s is a homogeneous tensor polynomial of degree $s \leqslant 2$ in the momentum variables $p, q \in \mathbb{R}^4$ which depends at most linearly on the momentum q. From the definitions of $Q^{\Lambda_0}_{\rho\omega}$ (127) and $Q^{\Lambda_0}_{\rho\gamma}$ (126) we obtain that $Q^{\Lambda_0;\vec{\phi}}_{\rho\varkappa;l=0}$ has the form $h_s(p)(\sigma_{0\Lambda_0}((p+q)^2) - \sigma_{0\Lambda_0}(p^2))$ with $s \leqslant 1$. For $\|w\| \leqslant s$ (relevant terms) using inequalities (639), (645), (650) we have

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l=0}(\vec{p})| \leqslant c\,\Lambda_0^{s-\|w\|}\Big(\frac{|\vec{p}|}{\Lambda_0}\Big)^{s+1-\|w\|} \leqslant c\,\frac{|\vec{p}|}{\Lambda_0}(|\vec{p}|+\Lambda)^{s-\|w\|}\,. \tag{216}$$

For $\|w\| > s$ (irrelevant terms) the same inequalities yield

$$\begin{aligned} |\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l=0}(\vec{p})| &\leqslant c\,\Lambda_0^{s-\|w\|}\Big(\frac{|\vec{p}|}{\Lambda_0}+1\Big) \\ &\leqslant c\,(|\vec{p}|+\Lambda)^{s-\|w\|}\,\frac{|\vec{p}|+\Lambda}{\Lambda_0}\Big(\frac{|\vec{p}|}{\Lambda_0}+1\Big)^{\|w\|-s}\,. \end{aligned} \tag{217}$$

Since $s-\|w\|$ is the dimension d_X we have the following bounds

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l=0}(\vec{p})| \leqslant (\Lambda+|\vec{p}|)^{d_X}\,F^{\Lambda\Lambda_0}_{0\,0}(\vec{p}), \qquad\qquad X \in \{\beta, 1\}. \tag{218}$$

Thus the statements of theorems 1–4 hold at loop number $l = 0$. The proof proceeds by induction on l and on the number of derivatives $\|w\|$, ascending in l and, for fixed l, descending in $\|w\|$ from $\bar{w}$ to 0.

3.1 Chains of vertex functions

Definition 21 *A division in m parts of a finite set $\mathbb{I}$ is a sequence $S := (s_j)_{j\in[m]}$ of m disjoint sets $s_j \subseteq \mathbb{I}$, possibly empty, such that $\bigcup_{j\in[m]} s_j = \mathbb{I}$. An ordered partition is a division with all s_j nonempty. Given a division S as above stated, a division of a sequence $\vec{\Psi} = (\Psi_i)_{i\in\mathbb{I}}$ is the sequence of elements $\vec{\Psi}_j := (\Psi_i)_{i\in s_j}$, with $j \in [m]$.*

Definition 22 *Let S be a division in m parts of a finite set $\mathbb{I}$, and $\vec{\Psi} = (\Psi_i)_{i\in\mathbb{I}}$ be a sequence of labels $\Psi_i \in \{A, c, \bar{c}, \gamma, \omega, \beta, 1\}$. Denote by $(\vec{\Psi}_j)$ the division of $\vec{\Psi}$ induced by S. A **chain** of vertex functions is then defined by the expression*

$$\mathcal{F}_S^{\zeta_1\vec{\Psi}\bar{\zeta}_m} := \mathsf{\Gamma}^{\zeta_1\vec{\Psi}_1\bar{\zeta}_1} \prod_{j=2}^{m} \mathbf{C}_{\zeta_j\bar{\zeta}_{j-1}} \mathsf{\Gamma}^{\zeta_j\vec{\Psi}_j\bar{\zeta}_j}\,, \tag{219}$$

where the repeated field labels $\zeta_j, \bar{\zeta}_j$ belong to $\{A, c, \bar{c}\}$ and are summed over (as usual).

Using this definition the FE (99) has the form

$$\dot{\mathsf{\Gamma}}^{\vec{\Psi}} = \frac{\hbar}{2} \sum_S (-)^{\pi_a} \langle \dot{C} \mathcal{F}_S^{A\vec{\Psi}A} + \dot{S}(\mathcal{F}_S^{\bar{c}\vec{\Psi}c} - \mathcal{F}_S^{c\vec{\Psi}\bar{c}}) \rangle\,. \tag{220}$$

The sum above runs over all possible divisions of $[0 : n-1]$, n being the number of components of $\vec{\Psi}$. The symbol π_a denotes the number of transpositions $mod\,2$ of the anticommuting variables $\{c, \bar{c}, \beta, \gamma, 1\}$ in the permutation $i \mapsto \pi(i)$ such that $(\Psi_{\pi(0)}, ..., \Psi_{\pi(n-1)}) = \vec{\Psi}_1 \oplus ... \oplus \vec{\Psi}_m$, where $(a_1, ..., a_p) \oplus (a_{p+1}, ..., a_q) = (a_1, ..., a_q)$.

A preliminary step toward the proof of theorem 1 is to bound $\partial^w(\mathbf{C}_{\zeta\bar{\zeta}}\mathsf{\Gamma}^{\zeta\bar{\zeta}})$ with $\|w\| \leqslant \bar{w}$.

Proposition 23 *For all $0 < k < l$, $0 \leqslant w \leqslant \bar{w}$, $p \in \mathbb{R}^4$*

$$\Big|\Big(\prod_{i=0}^{w} \frac{\partial}{\partial p_{\mu_i}}\Big) \Big(\mathsf{\Gamma}_k^{\zeta\bar{\zeta};\Lambda\Lambda_0}(p) \mathbf{C}_{\zeta\bar{\zeta}}^{\Lambda\Lambda_0}(p)\Big)\Big| \leqslant \frac{P_{2k}^{\Lambda}}{(|p| + \Lambda)^w}\,. \tag{221}$$

Proof Using inequality (6) we see that

$$|\partial^w(\Gamma_k^{\zeta\bar\zeta;\Lambda\Lambda_0}\mathbf{C}_{\zeta\bar\zeta}^{\Lambda\Lambda_0})| \leqslant \sum_{w_1=0}^{w} \frac{w!}{w_1!(w-w_1)!}|\partial^{w_1}\Gamma_k^{\zeta\bar\zeta;\Lambda\Lambda_0}||\partial^{w-w_1}\mathbf{C}_{\zeta\bar\zeta}^{\Lambda\Lambda_0}|\,. \tag{222}$$

Setting $w_2 = w - w_1$ it follows from (692) and the bounds of theorem 1 already proved inductively for $k < l$ that

$$\frac{1}{w_2!}|\partial^{w_1}\Gamma_k^{\zeta\bar\zeta;\Lambda\Lambda_0}||\partial^{w_2}\mathbf{C}_{\zeta\bar\zeta}^{\Lambda\Lambda_0}| \leqslant \frac{c_\xi d^{w_2} P_{2k}^{\Lambda}}{(|p|+\Lambda)^{w_1+w_2}}\,. \tag{223}$$

Because $w \leqslant \bar w$ the constants c_ξ, d^{w_2} may be absorbed in P_{2k}^{Λ}. ■

Definition 24 *Let be given*

1. *a sequence* $\vec\Psi = (\Psi_i)_{i\in\mathbb{I}}$ *as in Definition 22;*
2. *an ordered partition* $S = (s_j)_{j\in[m]}$ *of* $\mathbb{I}$*;*
3. *the sequences of field labels* $(\zeta_j)_{j\in[m]}$, $(\bar\zeta_j)_{j\in[m]}$*;*
4. *a multi-index* $w \in \mathbb{W}_n$ *and a sequence* $\mathsf{w} := (\mathsf{w}_j)_{j\in[m]}$ *such that* $\mathsf{w}_j \in \mathbb{W}_n$ *and* $\sum_{j\in[m]} \mathsf{w}_j = w$.

Then, we define a ***reduced chain*** *of vertex functions as*

$$\Gamma^{\zeta_1\vec\Psi_1\bar\zeta_1;\mathsf{w}_1}\prod_{j=2}^{m}\mathbf{C}_{\zeta_j\bar\zeta_{j-1}}\Gamma^{\zeta_j\vec\Psi_j\bar\zeta_j;\mathsf{w}_j}, \tag{224}$$

where $\Gamma^{\zeta\vec\Psi_j\bar\zeta;\mathsf{w}_j} = \partial^{\mathsf{w}_j}(\Gamma^{\zeta\vec\Psi_j\bar\zeta})$ *are derivatives wrt the external momenta appearing in* $\dot\Gamma^{\vec\Psi}$*, and the sequences* $(\zeta_j)_{j\in[m]}$, $(\bar\zeta_j)_{j\in[m]}$ *are* ***fixed****. Introducing the auxiliary quantities* $\mathring\Upsilon := (\vec\Upsilon_j)_{j\in[m]}$ *and* $\vec\Upsilon_j := \zeta_j\vec\Psi_j\bar\zeta_j$*, we denote (224) by* $\mathcal{S}^{\mathring\Upsilon;\mathsf{w}}$ *or, with some abuse of notations, by* $\mathcal{S}_S^{\zeta_1\vec\Psi\bar\zeta_m;\mathsf{w}}$.

The adjective "reduced" indicates that the chains contain neither Γ^{AA}, $\Gamma^{c\bar c}$ nor derivatives applied to the propagators $\mathbf{C}$.

It follows from inequalities (692), (221) and theorems 1–4 proved in loop order $l-1$ that there exists a common bound for the terms $\Gamma_{l_1+l_2}^{\zeta\vec\Psi\bar\zeta;w_1+w_2+w_3}\mathbf{C}$

and $\Gamma^{\zeta\vec{\Psi}\bar{\zeta};w_1}_{l_1}\partial^{w_2}(\mathbf{C}\Gamma^{\zeta\bar{\zeta}}_{l_2})\partial^{w_3}\mathbf{C}$. Hence to bound $\dot{\Gamma}^{\vec{\Psi}}_{l}$ it is enough to consider a loop integral with a reduced chain

$$\int \frac{d^4k}{(2\pi)^4}\,\dot{\mathbf{C}}_{\zeta\bar{\zeta}}(k)\,\mathcal{S}^{\zeta\vec{\Psi}\bar{\zeta};\mathsf{w}}_{S;l-1}(k,\vec{p}_S,-k), \tag{225}$$

where $\vec{p}_{s_j} = (p_i)_{i\in s_j}$. As an example we give in Appendix B the complete list of chains for $\Gamma^{AAc\bar{c}}$. The appellative "reduced" may be omitted in the following, since it is always clear from the context whether a chain is reduced or not.

3.2 Junction of weighted trees

Given a reduced chain $\mathcal{S}^{\mathring{\Upsilon};\mathsf{w}}$ we define its amplitude $\hat{\mathcal{S}}^{\mathring{\Upsilon};\mathsf{w}}$ by substituting the vertex functions and propagators with their corresponding bounds taken from theorems 1–4 and inequalities (692). Recalling that $\vec{\Upsilon}_j := \zeta_j\vec{\Psi}_j\bar{\zeta}_j$, we then set

$$\hat{\mathcal{S}}^{\Lambda;\mathring{\Upsilon};\mathsf{w}} := \hat{S}^{\Lambda;\vec{\Upsilon}_1;\mathsf{w}_1}\prod_{j=2}^{m}\frac{1}{(\Lambda+|p_{\zeta_j}|)^2}\hat{S}^{\Lambda;\vec{\Upsilon}_j;\mathsf{w}_j} \tag{226}$$

$$\hat{S}^{\Lambda;\vec{\Upsilon}_j;w} := \begin{cases}(\Lambda+|\vec{p}|)^{d_X}, & \textit{case } a,\\ \sum\limits_{\tau\in\mathcal{T}_{\vec{\Upsilon}_j}} Q^{\Lambda;w}_{\tau}, & \textit{case } b,\end{cases}\ . \tag{227}$$

Here the cases a and b refer to the respective parts in theorems 1–4.

The tree structure of the bound is spoiled if there exists an interval $\mathbb{J}^a := [j^a : j^a+m^a-1] \subset [m]$ such that all $\hat{S}^{\Lambda;\vec{\Upsilon}_j;w}$ for $j\in\mathbb{J}^a$ correspond to a strictly relevant contribution, associated to the cases a in the theorems. A workaround for this difficulty will start with the following definition. For every tree $\tau\in\mathbb{T}_{\vec{\varphi}}$, set $E_{1;v} := \{e\in E_1 : e \text{ incident to } v\}$.

Definition 25 *Let $\vec{\varphi}$ be an arbitrary sequence with $\varphi_i\in\{A,c,\bar{c},\beta,\gamma,\omega\}$. A tree $f\in\mathbb{T}_{\vec{\varphi}}$ is a **fragment** if*

a) there exists $s\in\mathbb{N}$ s.t. $|V_\circ|+s$ equals the total number of "" labels,*

b) $\forall v\in V_\circ$: $|E_{1;v}|\geqslant 2$ and $E_{1;v}\cap E_*\neq\emptyset$.

The set of all such fragments is denoted by $\mathcal{F}^{(s)}_{\vec{\varphi}}$. Moreover we set $\mathcal{F}_{\vec{\varphi}} := \mathcal{F}^{(0)}_{\vec{\varphi}}$ and $\mathcal{F}_{1\vec{\varphi}} := \mathcal{F}^{(1)}_{\vec{\varphi}}$.

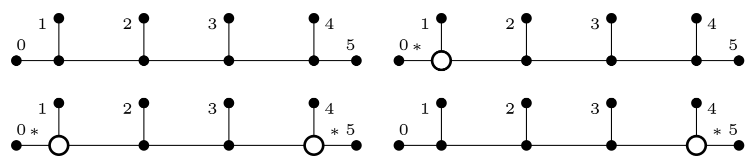
For each $s \in \{0,1\}$, $\mathcal{T}^{(s)}_{\vec{\varphi}} \subset \mathcal{F}^{(s)}_{\vec{\varphi}} \subset \mathbb{T}_{\vec{\varphi}}$.

With the aid of this definition we will now show that the aforementioned contributions with relevant parts $\hat{S}^{\Lambda;\vec{\Upsilon}_j;w}$ can be bounded by fragment amplitudes $Q^{\Lambda;w}_f$, see (191).

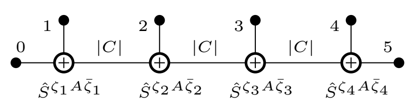
Let be given a subsequence $\mathbb{J}^a := [j^a : j^a + m^a - 1] \subset [m]$ with any number of elements m^a. Let S and $\vec{\Psi}$ be as in definition 24. Define the following restrictions: $S^a = S|_{\mathbb{J}^a}$, $\mathsf{w}^a = \mathsf{w}|_{\mathbb{J}^a}$. Set $w^a := \sum_{j\in\mathbb{J}^a} \mathsf{w}_j$ and $\vec{\Psi}^a := (\Psi_i)$ with $i \in \cup_{j\in\mathbb{J}^a} s_j$. Then, there exists a set of fragments $\mathcal{F}_{\zeta\vec{\Psi}^a\bar{\zeta}}$ such that

$$\hat{\mathcal{S}}^{\Lambda;\zeta\vec{\Psi}^a\bar{\zeta};\mathsf{w}^a}_{S^a} \leqslant 2^{\frac{|S_a|}{2}} \sum_{f\in\mathcal{F}_{\zeta\vec{\Psi}^a\bar{\zeta}}} Q^{\Lambda;w^a}_f \,. \tag{228}$$

Here we give a proof of (228) for an example, generalisation is clear. Consider an amplitude $\hat{\mathcal{S}}^{\zeta_1 A^4 \bar{\zeta}_4;w}$ composed of four elements $\hat{S}^{\zeta_j A \bar{\zeta}_j;w}$ where $j \in \{1,...,4\}$ and $w = 0$. First let us define the following set of fragments $\mathcal{F}$ (here $s = 0$):



In each fragment the $*$-edge corresponds to a factor $\Lambda + |p_e|$ in the corresponding amplitude $Q^{\Lambda;w=0}_{f\in\mathcal{F}}$. One shows the following bound:



$$\leqslant 2^{\frac{4}{2}} \sum_{f\in\mathcal{F}} Q^{\Lambda;w=0}_f . \tag{229}$$

$$\frac{\prod\limits_{v\in V_3} (\Lambda + |\vec{p}_v|)}{\prod\limits_{e\in E\setminus E_1} (\Lambda + |p_e|)^2} \leqslant 2^{\frac{|V_3|}{2}} \sum_{\{\chi\}} \frac{\prod\limits_{v\in V_3} (\Lambda + |p_{\chi(v)}|)}{\prod\limits_{e\in E\setminus E_1} (\Lambda + |p_e|)^2} \,. \tag{230}$$

Here the $\oplus$-vertices stand each for $\hat{S}^{\zeta_j A \bar{\zeta}_j;w=0}$, see (227); $\vec{p}_v$ indicates the set of incoming momenta of the vertex v; p_e denotes the momentum corresponding to an edge e; the sum runs over the set of functions $\chi : V_3 \to E\setminus E_1 \cup \{e_0, e_5\}$ which map every vertex $v \in V_3$ to an edge incident to v; the $|C|$'s stand for the usual bounds on the corresponding propagators.

Let be given a sequence of fragments $\vec{f} := (f_1, ..., f_{m'})$ with $f_j \in \mathcal{F}_{\zeta\vec{\Psi}_j\bar{\zeta}}$ and a sequence $\mathsf{w}' = (\mathsf{w}'_j)_{j\in[m']}$ with $\mathsf{w}'_j \in \mathbb{W}_n$. We define the amplitude $\hat{\mathcal{Q}}^{\Lambda;\mathsf{w}'}_{\vec{f}}$

by

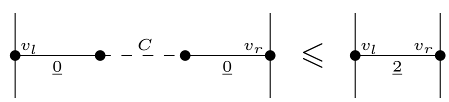
$$\hat{\mathcal{Q}}_{\vec{f}}^{\Lambda;\mathsf{w}'} := Q_{f_1}^{\Lambda;\mathsf{w}'_1} \prod_{j=2}^{m'} \frac{1}{(\Lambda + |p_{\zeta_j}|)^2} Q_{f_j}^{\Lambda;\mathsf{w}'_j} \,. \tag{231}$$

Lemma 26 *Given an amplitude $\hat{\mathcal{Q}}_{\vec{f}}^{\Lambda;\mathsf{w}'}$ as above there exists a fragment $f \in \mathcal{F}_{\zeta\vec{\Psi}\bar{\zeta}}$ such that*

$$\hat{\mathcal{Q}}_{\vec{f}}^{\Lambda;\mathsf{w}'} \leqslant Q_f^{\Lambda;\mathsf{w}'} \,. \tag{232}$$

Proof We proceed by induction in m'. If there are no joints, $m' = 1$, the statement is evident. Assume it is true for some $m' - 1 \geqslant 0$ and consider a sequence of m' fragments. Let $v_l, v_r \in V_3$ be the left and right vertices of a joint.

- $v_l, v_r \in V_3 \backslash V_\circ$.

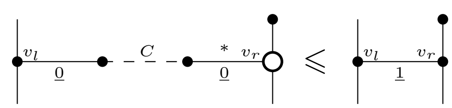
  

  (233)

  $$\Pi_{f_l,\theta_l}^{\Lambda} \frac{1}{(\Lambda + |p|)^2} \Pi_{f_r,\theta_r}^{\Lambda} = \Pi_{f,\theta}^{\Lambda} \tag{234}$$

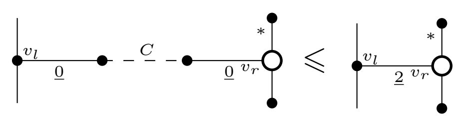
  Here the corresponding external edges has been merged together to form a new internal edge with θ-weight equals to 2. The θ-weight of all other edges is unchanged.

- $v_l \in V_3 \backslash V_\circ, v_r \in V_\circ$ or vice versa.

  (235)

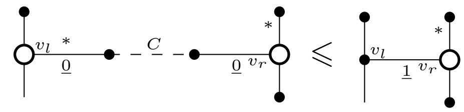
  $$\Pi_{f_l,\theta_l}^{\Lambda} \frac{1}{(\Lambda + |p|)^2} (\Lambda + |p|) \Pi_{f_r,\theta_r}^{\Lambda} = \Pi_{f,\theta}^{\Lambda} \tag{236}$$

  (237)

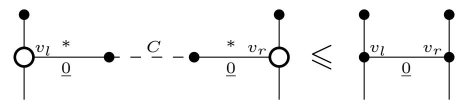
  $$\Pi_{f_l,\theta_l}^{\Lambda} \frac{1}{(\Lambda + |p|)^2} (\Lambda + |p_*|) \Pi_{f_r,\theta_r}^{\Lambda} = (\Lambda + |p_*|) \Pi_{f,\theta}^{\Lambda} \tag{238}$$

- $v_l, v_r \in V_\circ$.



(239)

$$\Pi^{\Lambda}_{f_l,\theta_l}(\Lambda+|p|)\,\frac{1}{(\Lambda+|p|)^2}\,(\Lambda+|p_*|)\Pi^{\Lambda}_{f_r,\theta_r} = (\Lambda+|p_*|)\Pi^{\Lambda}_{f,\theta} \tag{240}$$



(241)

$$\Pi^{\Lambda}_{f_l,\theta_l}(\Lambda+|p|)\,\frac{1}{(\Lambda+|p|)^2}\,(\Lambda+|p|)\Pi^{\Lambda}_{f_r,\theta_r} = \Pi^{\Lambda}_{f,\theta}\,. \tag{242}$$

Hence, by merging two fragments we can decrease the number of joints by one and then apply the induction hypothesis. ■

In the simpler context of ϕ^4 theory, a completely explicit description of the junction of trees can be found in [GK].

According to equations (625), (631) the loop integral in (225) is bounded by the following expression

$$\int \dot{\mathbf{C}}^{\Lambda\Lambda_0}_{\zeta\bar{\zeta}}\hat{\mathcal{S}}^{\Lambda;\mathring{\Upsilon};\mathsf{w}}\,\mathfrak{P}^{\Lambda}_{l-1} \leqslant \Lambda\,\hat{\mathcal{S}}^{\Lambda;\mathring{\Upsilon};\mathsf{w}}\,\mathfrak{P}^{\Lambda}_{l-1}\Big|_{p_\zeta,p_{\bar{\zeta}}=0}\,, \tag{243}$$

$$\mathfrak{P}^{\Lambda}_{l} := \begin{cases} F^{\Lambda\Lambda_0}_{3l\,l}, & X=1,\\ F^{\Lambda\Lambda_0}_{2l\,0}, & X=\beta,\\ P^{\Lambda\Lambda}_{2l}, & otherwise, \end{cases} \tag{244}$$

which follows directly from the definition of r, r_X, s_X given in theorems 1–4.

As can be seen from (243) the loop integration of our bounds leads to a result reproducing the structure of the bounds where the loop momenta have been set to zero. This will be a general feature of the subsequent proof, and we will therefore define the restriction $\mathcal{R}$ as follows:

Definition 27 *For any function f depending on the variable p_ζ, and other variables which we need not specify, we define*

$$\mathcal{R}_\zeta(f(...,p_\zeta,...)) \mapsto f(...,0,...).$$

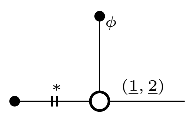
Then we set $\mathcal{R}_{\zeta\bar{\zeta}} := \mathcal{R}_\zeta \circ \mathcal{R}_{\bar{\zeta}}$.

Proposition 28 *For an amplitude* $\hat{\mathcal{S}}_S^{\Lambda;\zeta\vec{\Psi}\bar{\zeta};w}$ *there exists a set of trees such that*

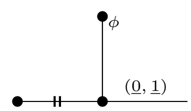
$$\mathcal{R}_{\zeta\bar{\zeta}}(\hat{\mathcal{S}}_S^{\Lambda;\zeta\vec{\Psi}\bar{\zeta};w}) \leqslant 2^{\frac{|S|}{2}} \sum_{\tau\in\mathcal{T}_{\zeta\vec{\Psi}\bar{\zeta}}} \mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\Lambda;w})\,. \tag{245}$$

Proof Using definition (226) and inequalities (228), (232) we obtain that $\hat{\mathcal{S}}_S^{\Lambda;\zeta\vec{\Psi}\bar{\zeta};\mathsf{w}}$ is bounded by a sum of fragment amplitudes. It remains to show that for a fragment $f\in\mathcal{F}_{\zeta\vec{\Psi}\bar{\zeta}}$ there exists a tree $\tau\in\mathcal{T}_{\zeta\vec{\Psi}\bar{\zeta}}$ such that $\mathcal{R}_{\zeta\bar{\zeta}}(Q_f^{\Lambda;w}) \leqslant \mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\Lambda;w})$. We denote by a double bar line the edges $\zeta,\bar{\zeta}$.

- The case $f\in\mathcal{T}_{\zeta\vec{\Psi}\bar{\zeta}}$ is trivial.
- Otherwise $f\notin\mathcal{T}_{\zeta\vec{\Psi}\bar{\zeta}}$. Based on the inequality $\frac{\Lambda}{\Lambda+|p_\phi|}\leqslant 1$ we find



$$\leqslant$$



(246)

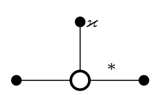
■

At any loop order $l'<l$ using the bound of theorem 1 and the inequality

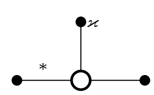
$$1\leqslant \frac{1}{\Lambda+|p_\varkappa|}\left((\Lambda+|p_1|)+(\Lambda+|p_2|)\right), \tag{247}$$

where $p_\varkappa+p_1+p_2=0$, one realizes that for the marginal vertex function $\Gamma^{\Lambda\Lambda_0;\phi_1\phi_2}_{\varkappa;l'}$ the following inequality holds

$$|\Gamma^{\Lambda\Lambda_0;\phi_1\phi_2}_{\varkappa;l'}(\vec{p})| \leqslant \frac{1}{\Lambda+|p_\varkappa|}\Bigg($$



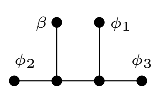
$$+$$



$$\Bigg)P^{\Lambda}_{r_\varkappa(l')}\,. \tag{248}$$

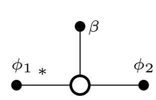
Similarly, substituting the relevant terms $\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\beta;l'}$, $\mathrm{N}+\|w\|\leqslant 3$, at loop number $l'<l$ with the bounds of theorem 3 we have

$$|\Gamma^{\Lambda\Lambda_0;\phi_1\phi_2\phi_3}_{\beta;l'}| \leqslant$$

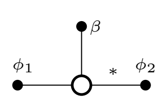


$$F^{\Lambda\Lambda_0}_{r_\beta(l')\,0}\,, \tag{249}$$

$$|\Gamma^{\Lambda\Lambda_0;\phi_1\phi_2}_{\beta;l'}| \leqslant \Bigg($$

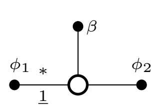


$$+$$



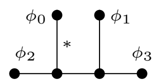
$$\Bigg)F^{\Lambda\Lambda_0}_{r_\beta(l')\,0}\,, \tag{250}$$

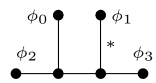
$$|\partial\Gamma^{\Lambda\Lambda_0;\phi_1\phi_2}_{\beta;l'}| \leqslant$$



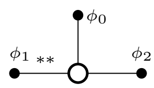
$$F^{\Lambda\Lambda_0}_{r_\beta(l')\,0}\,. \tag{251}$$

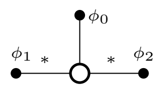
Furthermore using the bounds of theorem 4 which are assumed to be true for any loop order $l' < l$, we obtain the following inequalities for strictly relevant terms $\mathsf{\Gamma}^{\Lambda\Lambda_0\vec{\phi};w}_{1;l'}$, $\mathrm{N} + \|w\| < 5$,

$$|\mathsf{\Gamma}^{\Lambda\Lambda_0;\phi_0\phi_1\phi_2\phi_3}_{1;l'}| \leqslant \Bigg($$


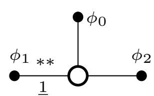
$$+$$


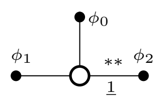
$$+ \dots\Bigg) F^{\Lambda\Lambda_0}_{r_1(l')\,s_1(l')}\,, \qquad (252)$$

$$|\mathsf{\Gamma}^{\Lambda\Lambda_0;\phi_0\phi_1\phi_2}_{1;l'}| \leqslant \Bigg($$


$$+$$


$$+ \dots\Bigg) F^{\Lambda\Lambda_0}_{r_1(l')\,s_1(l')}\,, \qquad (253)$$

$$|\partial\mathsf{\Gamma}^{\Lambda\Lambda_0;\phi_0\phi_1\phi_2}_{1;l'}| \leqslant \Bigg($$


$$+$$


$$\Bigg) F^{\Lambda\Lambda_0}_{r_1(l')\,s_1(l')}\,, \qquad (254)$$

$$|\mathsf{\Gamma}^{\Lambda\Lambda_0;\phi_0\phi_1;w}_{1;l'}\mathbf{C}^{\Lambda\Lambda_0}| \leqslant \frac{\Lambda + |p|}{(\Lambda + |p|)^{\|w\|}} F^{\Lambda\Lambda_0}_{r_1(l')\,s_1(l')}\,, \qquad (255)$$

where the dots stand for omitted fragments obtained by permuting the "*" over the external edges. Substituting the bounds for $\Gamma^{\vec{\phi};w}_{1\varkappa}$ with those for $\partial_{\bar{c}}\Gamma^{\bar{c}\vec{\phi};w}_{1}$ yields a similar result for the relevant terms $\Gamma^{\vec{\phi};w}_{1,\vec{\varkappa}}$ with $n_\varkappa > 0$. Consequently the amplitude $\hat{S}^{\Lambda;\vec{\phi};w}_{1\vec{\varkappa}}$ is bounded by a sum of amplitudes $Q^{\Lambda;w}_{f}$ with $f \in \mathcal{F}_{1\vec{\varphi}}$.

3.3 Irrelevant terms

The irrelevant terms at arbitrary $0 < \Lambda \leqslant \Lambda_0$ are reconstructed by using the FE:

$$\mathsf{\Gamma}^{\Lambda\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l}(\vec{p}) = \mathsf{\Gamma}^{\Lambda_0\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l}(\vec{p}) + \int\limits_{\Lambda_0}^{\Lambda} d\lambda\, \dot{\mathsf{\Gamma}}^{\lambda\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l}(\vec{p})\,. \qquad (256)$$

At $\Lambda = \Lambda_0$ for all loop orders $l > 0$, for all $n_\varkappa \geqslant 0$ and $d_X < 0$ we have

$$|\mathsf{\Gamma}^{\Lambda_0\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l}(\vec{p})| \leqslant \begin{cases} (|\vec{p}| + \Lambda_0)^{d_X} F^{\Lambda_0\Lambda_0}_{r_X\,s_X} & X \in \{\beta, 1\}, \\ 0 & \textit{otherwise}, \end{cases} \qquad (257)$$

where the upper inequality is obtained using the bounds on relevant terms from theorem 1.

To integrate the FE from the boundary Λ_0 to Λ we substitute the chains $\mathcal{S}^{\zeta\phi_0\phi_1\bar{\zeta}}$, $\mathcal{S}^{\zeta\varkappa\phi\bar{\zeta}}$ with the bounds given in theorem 1 and use inequalities (591), (601). Eventually we get

$$|\Gamma_{l}^{\Lambda\Lambda_0;\phi_0\phi_1;w}(p)| \leqslant (\Lambda+|p|)^{2-\|w\|} P_{2l-1}^{\Lambda}, \qquad \|w\|>2\,, \tag{258}$$

$$|\Gamma_{\varkappa;l}^{\Lambda\Lambda_0;\phi;w}(p)| \leqslant (\Lambda+|p|)^{1-\|w\|} P_{2l-1}^{\Lambda}, \qquad \|w\|>1\,. \tag{259}$$

In a similar way substitution of $\mathcal{S}^{\zeta\beta\phi\bar{\zeta}}$, $\mathcal{S}_1^{\zeta\phi_0\phi_1\bar{\zeta}}$ with the bounds of theorems 3, 4 and then integration from Λ_0 to Λ, using (615), give

$$|\Gamma_{\beta;l}^{\Lambda\Lambda_0;\phi;w}(p) - \Gamma_{\beta;l}^{\Lambda_0\Lambda_0;\phi;w}(p)| < \frac{F_{2l-1\,0}^{\Lambda\Lambda_0}}{(\Lambda+|p|)^{\|w\|-2}}, \qquad \|w\|>2\,, \tag{260}$$

$$|\Gamma_{1;l}^{\Lambda\Lambda_0;\phi_0\phi_1;w}(p) - \Gamma_{1;l}^{\Lambda_0\Lambda_0;\phi_0\phi_1;w}(p)| < \frac{F_{3l-2\,l-1}^{\Lambda\Lambda_0}}{(\Lambda+|p|)^{\|w\|-3}}, \qquad \|w\|>3\,. \tag{261}$$

Proposition 29 *There exists $c>0$ such that*

- $\forall \tau \in \mathcal{T}^{(s)}_{\zeta\bar{\zeta}\vec{\phi}}$ *where* $\phi_j \in \{A, c, \bar{c}, \gamma, \omega\}$ *and* $|V_1|>4$,
- $\forall w \in \mathbb{W}_{n+2}$ *where* $w_\zeta = 0$, $w_{\bar{\zeta}} = 0$ *and* $\theta(\tau)>1$,

$\exists \tilde{\tau} \in \mathcal{T}^{(s)}_{\vec{\phi}}$ *and*

$$\mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w}) \leqslant \frac{c}{\lambda(\lambda+\eta(\vec{p}))}\, Q_{\tilde{\tau}}^{\lambda;w}\,. \tag{262}$$

Proof Denote by $v, \bar{v} \in V\backslash V_1$ the vertices incident to $e_\zeta, e_{\bar{\zeta}} \in E_1$. Then

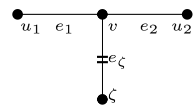
$$w_\zeta = w_{\bar{\zeta}} = 0 \implies \sigma(e_{\bar{\zeta}}) = \sigma(e_\zeta) = 0. \tag{263}$$

Note that once we have proven the statement for the case $E_* = \emptyset$ the generalization to the general case is simple. The case $e_\zeta, e_{\bar{\zeta}} \notin E_*$ is trivial. If $\{e_\zeta, e_{\bar{\zeta}}\} \cap E_* \neq \emptyset$ then we change the multiplicity of "$*$"-labels m_e of an arbitrary edge e of the final tree $\tilde{\tau}$ in this way: $m_e \mapsto \tilde{m}_e = m_e + m_{e_\zeta} + m_{e_{\bar{\zeta}}}$. Since $p_\zeta = 0$ and $p_{\bar{\zeta}} = 0$ it follows that

$$\prod_{e\in E_*(\tau)} (\lambda+|p_e|)^{m_e} \leqslant \prod_{e\in E_*(\tilde{\tau})} (\lambda+|p_e|)^{\tilde{m}_e}. \tag{264}$$

First we assume that $v \neq \bar{v}$. The result of the restriction $\mathcal{R}_\zeta(Q_\tau^{\lambda;w})$ is the amplitude of the tree τ where two edges $e_1 = \{u_1, v\}$, $e_2 = \{u_2, v\}$ carry opposite momenta. Here $\{u_i, v\}$ denotes the edge which links the vertices u_i, v.

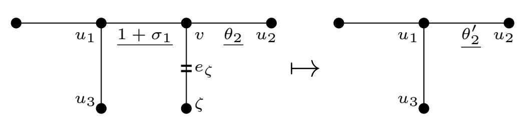
We assume that $\chi(v) = e_1$. The case $\chi(v) = e_2$, if it is possible, is obtained by interchanging e_1 and e_2. Thus, a factor of 2 has to be absorbed in the constant c.



Furthermore we define a subtree τ' of the initial tree τ replacing $e_2 \mapsto e_2'$ where $e_2' = \{u_1, u_2\}$, and removing the vertices v, ζ and the edges e_1, e_ζ. Let us denote this map using the same notation $\mathcal{R}_\zeta : \mathcal{T}^{(s)}_{\zeta\bar{\zeta}\vec{\phi}} \to \mathcal{T}^{(s)}_{\bar{\zeta}\vec{\phi}}$. We mark by prime $'$ the quantities of the tree τ' which are different from the corresponding quantities of the original tree τ, for example θ', E'. Repeating the procedure for the second restriction $\mathcal{R}_{\bar{\zeta}} : \mathcal{T}^{(s)}_{\bar{\zeta}\vec{\phi}} \to \mathcal{T}^{(s)}_{\vec{\phi}}$ we complete the proof. However, a major difference between the two mappings is the case $|V_1| = 4$ which arises as the result of the first mapping for a tree $\tau \in \mathcal{T}^{(s)}_{\zeta\bar{\zeta}\vec{\phi}}$ with $|V_1| = 5$. Moreover the σ-weight of the initial tree τ can be distributed between edges E' and the edge e_1 which will not appear in the final tree τ'. Thus, each restriction gives us also $2^{\bar{w}}$ for the constant c.

$\rho(e_1) = 1$.

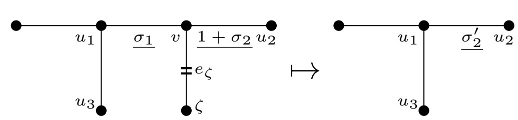
The identity $|p_{e_1}| = |p_{e_2}| = |p_{e_2'}|$ implies



$$\Pi^\lambda_{\tau,\theta} = \frac{1}{\lambda + |p_{e_2'}|} \Pi^\lambda_{\tau',\theta'}\,, \tag{265}$$

where $\sigma'(e_2') = \sigma(e_2) + \sigma(e_1)$ and $\rho'(e_2') = \rho(e_2)$.

$\rho(e_1) = 0$ and $\rho(e_2) > 0$.

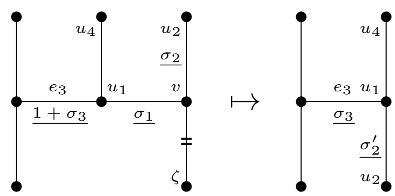


$$\Pi^\lambda_{\tau,\theta} = \frac{1}{\lambda + |p_{e_2'}|} \Pi^\lambda_{\tau',\theta'}\,, \tag{266}$$

where $\sigma'(e_2') = \sigma(e_2) + \sigma(e_1)$, $\rho'(e_2') = \rho(e_2) - 1$ and $\chi'(u_1) = e_2'$.

$\rho(e_1) = \rho(e_2) = 0$ and $|V_1| > 4$.

This implies $e_2 \in E_1$. Because $|V_1| > 4$ the vertex u_1 is incident to an edge $e_3 \in E \backslash E_1$ such that $\rho(e_3) > 0$.



$$\Pi^{\lambda}_{\tau,\theta} = \frac{1}{\lambda + |p_{e_3}|} \Pi^{\lambda}_{\tau',\theta'} \,, \tag{267}$$

where $\sigma'(e_2') = \sigma(e_2) + \sigma(e_1)$, $\rho'(e_3) = \rho(e_3) - 1$ and $\chi'(u_1) = e_3$. Moreover, $e_3 \in E \backslash E_1 \implies e_3 \neq e_{\bar{\zeta}} \implies e_3 \in \tilde{E}$.

$|V_1| = 4$.

Hence $\forall e \in E$ $\rho(e) = 0$. In this case $\theta(e) = \sigma(e)$ and $\sigma(e_{\bar{\zeta}}) = 0$. Then

$$\Pi^{\lambda}_{\tau,\theta} = \frac{1}{\prod\limits_{e \in E'} (\lambda + |p_e|)^{\sigma(e)}} \,, \tag{268}$$

Let $V'_{1;w} := \{v \in V_1' : w_v > 0\}$. $\theta(\tau) > 0 \implies V'_{1;w} \neq \emptyset$. $\forall v \in V'_{1;w}$ $\exists e' \in E' : \sigma_v(e') > 0$. $\sigma_v'(e') := \sigma_v(e') - 1$. One can write (268) in the form

$$\Pi^{\lambda}_{\tau,\theta} = \frac{1}{\lambda + |p_{e'}|} \Pi^{\lambda}_{\tau',\theta'} \leqslant \frac{1}{\lambda + \tilde{\eta}} \inf_{v \in V'_{1;w}} \Pi^{\lambda}_{\tau',\theta'} \,. \tag{269}$$

Finally we consider the case $v = \bar{v}$. We denote by $e \in E \backslash E_1$ the edge incident to v and introduce a vertex $u \in V_3$ adjacent to v. Hence, $\chi(v) = e = \{v, u\}$. If θ_1, θ_2 are two θ-weights where the only difference is that $\chi(u) = e$ for the first and $\chi(u) \neq e$ for the second, then for $|V_1| > 5$

$$\Pi^{\lambda}_{\tau,\theta_1} \leqslant \Pi^{\lambda}_{\tau,\theta_2} = \frac{1}{\lambda} \frac{1}{\lambda + |p_{e \in \tilde{E}}|} \Pi^{\lambda}_{\tilde{\tau},\tilde{\theta}} \,. \tag{270}$$

If $|V_1| = 5$ then using that $\theta(\tau) > 1$ as in (269) we have

$$\Pi^{\lambda}_{\tau,\theta_1} \leqslant \Pi^{\lambda}_{\tau,\theta_2} = \frac{1}{\lambda(\lambda + |p_{e'}|)} \Pi^{\lambda}_{\tilde{\tau},\tilde{\theta}'} \leqslant \frac{1}{\lambda(\lambda + \tilde{\eta})} \inf_{v \in V'_{1;w}} \Pi^{\lambda}_{\tilde{\tau},\tilde{\theta}'} \,. \tag{271}$$

$\blacksquare$

Proposition 30 *Let* $\tau_i \in \mathcal{T}_{\zeta\bar{\zeta}\vec{\phi}}$, $\phi_j \in \{A, c, \bar{c}, \gamma, \omega\}$, $|V_1| > 4$, $k \in \mathbb{N}$ *and* $\theta(\tau_i) > 2$. *Then* $\exists \tau_f \in \mathcal{T}_{\vec{\phi}}$ *such that*

$$\int_{\Lambda}^{\Lambda_0} d\lambda\, \lambda\, \mathcal{R}_{\zeta\bar{\zeta}}(Q_{\tau_i}^{\lambda;w})\, P_k^{\lambda}\Big|_{\eta=0} \leqslant Q_{\tau_f}^{\Lambda;w} P_{k+1}^{\Lambda}. \tag{272}$$

Proof Using proposition 29

$$\mathcal{R}_{\zeta\bar{\zeta}}(Q_{\tau_i}^{\lambda;w}) \leqslant \frac{c}{\lambda(\lambda+\eta_f)} Q_{\tau_f}^{\lambda;w}\,, \tag{273}$$

where all momenta correspond to edges of the final tree τ_f. Since the initial tree has the total weight $\theta(\tau_i) > 2$ we always have at least one denominator in $Q_{\tau_f}^{\lambda;w}$ and thus

$$Q_{\tau_f}^{\lambda;w} \leqslant \frac{\Lambda + |p_a|}{\lambda + |p_a|} Q_{\tau_f}^{\Lambda;w}\,. \tag{274}$$

Then we can bound the corresponding λ-integral as follows, using inequalities (591) and (601):

$$\int_{\Lambda}^{\Lambda_0} \frac{d\lambda\, P_k^{\lambda}|_{\eta=0}}{(\lambda+\eta_f)(\lambda+|p_a|)} \leqslant \frac{P_{k+1}^{\Lambda}}{\Lambda+\eta+|p_a|} \leqslant \frac{P_{k+1}^{\Lambda}}{\Lambda+|p_a|}\,. \tag{275}$$

■

Inequality (272) can be applied to bound $\mathcal{S}^{\Lambda\Lambda_0;\vec{\varkappa}\vec{\phi};w}$. In this case $\tau_i \in \mathcal{T}_{\zeta\bar{\zeta}\vec{\varkappa}\vec{\phi}}$ and the total weight satisfies $\theta(\tau_i) > 2 - n_{\varkappa}$. But in theorem 1 for each edge $e_{\varkappa} \in E_{\varkappa}$ we have $(\Lambda + |p_{\varkappa}|)$ as a denominator which is equivalent to an additional σ-weight of the edge $e_{\varkappa}$. An effective tree $\tilde{\tau}$ with $\tilde{\sigma}(e_{\varkappa}) = \sigma(e_{\varkappa})+1$ has $\theta(\tilde{\tau}) > 2$ and satisfies the conditions of proposition 30. In this case (275) has the form

$$\int_{\Lambda}^{\Lambda_0} \frac{d\lambda\, P_k^{\lambda}|_{\eta=0}}{(\lambda+\eta_f)(\lambda+|p_{\varkappa}|)} \leqslant \frac{P_{k+1}^{\Lambda}}{\Lambda+|p_{\varkappa}|}\,. \tag{276}$$

Using (615) a similar inequality follows for the irrelevant functions $\Gamma_{\beta\vec{\varkappa};l}^{\Lambda\Lambda_0;\vec{\phi};w}$. For any $\tau_i \in \mathcal{T}_{\zeta\bar{\zeta}\beta\vec{\varkappa}\vec{\phi}}$ there exists $\tau_f \in \mathcal{T}_{\beta\vec{\varkappa}\vec{\phi}}$ such that

$$\int_{\Lambda}^{\Lambda_0} d\lambda\, \lambda\, \mathcal{R}_{\zeta\bar{\zeta}}(Q_{\tau_i}^{\lambda;w})\, F_{2(l-1)\,0}^{\lambda\Lambda_0}|_{\eta=0} \leqslant Q_{\tau_f}^{\Lambda;w} F_{2l-1\,0}^{\Lambda\Lambda_0}\,. \tag{277}$$

Before application of (615) to the irrelevant terms $\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{1\vec{\varkappa};l}$ we need a minor change in (275)

$$\int\limits_{\Lambda}^{\Lambda_0} d\lambda \left(\frac{\lambda+|p_*|}{\lambda+|p_e|}\right) \frac{F^{\lambda\Lambda_0}_{k\,s}|_{\eta=0}}{(\lambda+\eta_f)(\lambda+|p_a|)} \leqslant \left(\frac{\Lambda+|p'_*|}{\Lambda+|p_e|}\right) \frac{F^{\Lambda\Lambda_0}_{k+1\,s}}{\Lambda+|p_a|}\,, \tag{278}$$

where p'_* is one out of p_* or p_e in such way that $|p'_*| := \max(|p_*|,|p_e|)$. If $p_* = p_e$ the label "$*$" is moved to edge e. Finally, $\forall \tau_i \in \mathcal{T}_{1\zeta\bar{\zeta}\vec{\varkappa}\vec{\phi}}$

$$\int\limits_{\Lambda}^{\Lambda_0} d\lambda\, \lambda\, \mathcal{R}_{\zeta\bar{\zeta}}(Q^{\lambda;w}_{\tau_i})\; F^{\lambda\Lambda_0}_{3(l-1)\,l-1}|_{\eta=0} \leqslant \sum_{\tau\in\mathcal{T}_{1\vec{\varkappa}\vec{\phi}}} Q^{\Lambda;w}_{\tau} F^{\Lambda\Lambda_0}_{3l-2\,l-1}\,. \tag{279}$$

3.4 Marginal terms

Lemma 31 *Let $\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}$ denote a marginal term, $n_\varkappa \leqslant 1$. Then $\forall \Lambda < \eta(\vec{p})$*

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p}) - \Gamma^{\eta\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p})| \leqslant P^{\Lambda}_{2l-2}(\vec{p}). \tag{280}$$

Proof Note that in theorem 1 for all $l' < l$ the bounds for $\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\varkappa;l'}$ are more restrictive than the ones for $\partial_{\bar{c}}\Gamma^{\Lambda\Lambda_0;\bar{c}\vec{\phi};w}_{l'}$. So we will only treat the case $n_\varkappa = 0$ explicitly. In this case we integrate the FE from η to Λ

$$\begin{aligned}|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{l}(\vec{p}) - \Gamma^{\eta\Lambda_0;\vec{\phi};w}_{l}(\vec{p})| &\leqslant \int\limits_{\Lambda}^{\eta} d\lambda |\dot{\Gamma}^{\lambda\Lambda_0;\vec{\phi}}_{l}(\vec{p})| \\ &\leqslant \sum_{\tau\in\mathcal{T}_{\zeta\bar{\zeta}\vec{\phi}}} \int\limits_{\Lambda}^{\eta} d\lambda\, \lambda\, \mathcal{R}_{\zeta\bar{\zeta}}(Q^{\lambda;w}_{\tau})\, P^{\lambda}_{2l-2}|_{\eta_\tau=0},\end{aligned} \tag{281}$$

where $\theta(\tau) = 2$. For n-point functions with $n > 2$ we can apply proposition 29

$$\mathcal{R}_{\zeta\bar{\zeta}}(Q^{\lambda;w}_{\tau}) \leqslant \frac{c}{\lambda(\lambda+\eta)}. \tag{282}$$

In case of 2-point functions $\forall e \in E$ $\rho(e) = 0$ and thus $\theta(e) = \sigma(e)$. Furthermore $\sigma(e_\zeta) = \sigma(e_{\bar{\zeta}}) = 0$. Denoting by p_a, p_b the momenta corresponding to the edges with nonvanishing σ-weight we again get inequality (282)

$$\mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w}) = \frac{1}{(\lambda+|p_a|)(\lambda+|p_b|)} \leqslant \frac{1}{\lambda(\lambda+\eta)}. \tag{283}$$

Using equations (607),(609) for each term of the sum we obtain

$$\int_\Lambda^\eta d\lambda\, \frac{P_k^\lambda|_{\eta_\tau=0}}{\lambda+\eta} \leqslant P_k^\Lambda, \quad k = 2l-2. \tag{284}$$

■

3.4.1 $\Gamma^{c\bar{c}AA}$ and $\Gamma^{c\bar{c}c\bar{c}}$

The renormalization condition is $\Gamma_l^{M\Lambda_0;\vec{\phi}}(0) = 0$. For $p_2, p_3 \in \mathbb{R}^4$ and $\Lambda' := \max(\Lambda, \eta(\vec{p}))$, equation (184) gives $\Gamma_l^{\Lambda'\Lambda_0;\vec{\phi}}(0_{\bar{c}}, p_2, p_3) = 0$, where the subscript $\bar{c}$ indicates the momentum of the antighost.

$$|\Gamma_l^{\Lambda'\Lambda_0;\vec{\phi}}(\vec{p})| \leqslant \int_0^1 dt\, |p_{\bar{c}}|\, |\partial_{\bar{c}}\Gamma_l^{\Lambda'\Lambda_0;\vec{\phi}}(tp_{\bar{c}}, p_2, p_3)|\,. \tag{285}$$

Substituting $|\partial_{\bar{c}}\Gamma_l^{\Lambda'\Lambda_0;\vec{\phi}}|$ with the bound of theorem 1 and using inequality (623) we obtain

$$|\Gamma_l^{\Lambda'\Lambda_0;\vec{\phi}}(\vec{p})| \leqslant \Big(1+\log_+ \frac{|\vec{p}|}{\Lambda'+\eta}\Big) P_{2l-1}^{\Lambda'\Lambda'}(\vec{p})\Big|_{\eta=0} \leqslant P_{2l}^{\Lambda}(\vec{p})\,. \tag{286}$$

If $\Lambda' = \Lambda$ the proof is finished. Otherwise we use (280).

3.4.2 $\partial_A\Gamma^{c\bar{c}A}$

The renormalization condition is $\partial_A\Gamma_l^{M\Lambda_0;c\bar{c}A}(0) = 0$. For $p_A \in \mathbb{R}^4$ and $\Lambda' = \max(\Lambda, \eta(\vec{p}))$ equation (184) gives $\partial_A\Gamma_l^{\Lambda'\Lambda_0;c\bar{c}A}(0, p_A) = 0$.

$$|\partial_A\Gamma_l^{\Lambda'\Lambda_0;c\bar{c}A}(\vec{p})| \leqslant \int_0^1 dt\, |p_{\bar{c}}|\, |\partial_{\bar{c}}\partial_A\Gamma_l^{\Lambda'\Lambda_0;c\bar{c}A}(tp_{\bar{c}}, p_A)|\,. \tag{287}$$

We substitute $|\partial_{\bar c}\partial_A \Gamma_l^{\Lambda'\Lambda_0;c\bar c A}(\vec p_t)|$ with the bound from theorem 1 with the choice $w'(2) = (0,1,0)$, see after (191) for the definition of w'. Then

$$Q_{c\bar c A}^{\Lambda';(0,1,1)} \in \left\{ \frac{1}{\Lambda' + t|p_{\bar c}|}, \frac{1}{\Lambda' + |tp_{\bar c} + p_A|} \right\}, \tag{288}$$

and using (623) we obtain the inequality

$$|\partial_A \Gamma_l^{\Lambda'\Lambda_0;c\bar c A}(\vec p)| \leqslant P_{2l}^{\Lambda'\Lambda'}(\vec p) \leqslant P_{2l}^{\Lambda}(\vec p)\,. \tag{289}$$

If $\Lambda' = \Lambda$ the proof is finished. If not we use (280).

3.4.3 Renormalization at $\Lambda = 0$

First we consider the marginal terms $\Gamma_{\omega;l}^{cc}$, $\Gamma_{\gamma;l}^{cA}$, Γ_l^{AAAA}, $\partial_{\bar c}\Gamma_l^{c\bar c A}$, $\partial\Gamma_l^{AAA}$, $\partial\partial\Gamma_l^{AA}$, $\partial\partial\Gamma_l^{c\bar c}$, here denoted by $\Gamma_{\vec\varkappa;l}^{\vec\phi;w}$. The marginal terms $\Gamma_{1\vec\varkappa;l}^{\vec\phi}$ and $\Gamma_{\beta;l}^{\vec\phi}$ will be discussed later.

Let $\{\delta^s \underline{e}\}_r$ be a basis at the renormalization point $\vec q$. We define $\Gamma_{\vec\varkappa;l}^{\vec\phi;w}(\vec q)$ in the following way:

$$\Gamma_{\vec\varkappa;l}^{0\Lambda_0;\vec\phi;w}(\vec q) := \sum_{t\in\{\delta^s\}_r} r_t\, t + \sum_{t\alpha\in\{\delta^s\underline{e}^{k>0}\}_r} \zeta_t\, t\,, \tag{290}$$

where the coefficients r_t are fixed by the renormalization conditions, see appendix H and hypothesis RC3, and the remaining coefficients ζ_t are defined using lemma 35. Then, from the bounds on the irrelevant terms and lemma 36, see (560), it follows that $\Gamma_{\vec\varkappa;l}^{\vec\phi;w}(\vec q)$ complies with theorem 1 at loop order l. Let $\Lambda' := \max(\Lambda, \eta(\vec p))$. It is easy to verify the following inequalities

$$\int_0^{\Lambda'} d\lambda \frac{\log_+^k \frac{M}{\lambda}}{\lambda + M} < k! + \log_+ \frac{\Lambda'}{M}, \quad \int_0^{\Lambda'} d\lambda \frac{\log_+^k \frac{\lambda}{M}}{\lambda + M} < 2\Big(1 + \log_+^{k+1} \frac{\Lambda'}{M}\Big)\,. \tag{291}$$

Recalling (243), (245) we obtain the following bound

$$|\dot\Gamma_{\vec\varkappa;l}^{\lambda\Lambda_0;\vec\phi;w}(\vec q)| \leqslant \lambda \frac{\mathcal{P}_{2l-2}^{(0)}(\log_+ \frac{M}{\lambda}) + \mathcal{P}_{2l-2}^{(1)}(\log_+(\frac{\lambda}{M}))}{(\lambda + M)^2}\,. \tag{292}$$

This implies that

$$|\Gamma_{\vec\varkappa;l}^{\Lambda'\Lambda_0;\vec\phi;w}(\vec q) - \Gamma_{\vec\varkappa;l}^{0\Lambda_0;\vec\phi;w}(\vec q)| \leqslant \int_0^{\Lambda'} d\lambda\, |\dot\Gamma_{\vec\varkappa;l}^{\lambda\Lambda_0;\vec\phi;w}(\vec q)| \leqslant \mathcal{P}_{2l-1}^{(1)}\Big(\log_+ \frac{\Lambda}{M}\Big)\,. \tag{293}$$

Using inequality (623) we get

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(0) - \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{q})| \leqslant \sum_{j=1}^{n-1} I_j(\vec{q}) \leqslant P^{\Lambda'}_{2l}(0), \tag{294}$$

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p}) - \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant \sum_{j=1}^{n-1} I_j(\vec{p}) \leqslant P^{\Lambda}_{2l}(\vec{p}), \tag{295}$$

where $I_j(\vec{q})$ is the interpolation along the vector q_j,

$$I_j(\vec{q}) = \int_0^1 dt\, |q_j|\, |\partial_j \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{\mathbf{q}}_j)|, \qquad \vec{\mathbf{q}}_j(t) := \sum_{i=1}^{j-1} \vec{q}_i + t\vec{q}_j\,. \tag{296}$$

Here to each vector $q_i \in \mathbb{R}^4$ is associated $\vec{q}_i \in \mathbb{P}_n$ whose components are $(\vec{q}_i)_k = -q_i\delta_{k,0} + q_i\delta_{k,i}$. Denoting the integration in (296) over a straight line from $\vec{a}_0$ to $\vec{a}_1$ by $\vec{a}_0 \to \vec{a}_1$, the whole path in (294) is

$$0 = \vec{\mathbf{q}}_1(0) \to \vec{\mathbf{q}}_1(1) = \vec{\mathbf{q}}_2(0) \to \ldots \vec{\mathbf{q}}_{n-1}(1) = \vec{q}. \tag{297}$$

To obtain inequalities (294) and (295) we have substituted $|\partial_j \Gamma^{M\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}|$ with the bounds of theorem 1. If $\Lambda = \Lambda'$ we stop here. If not we use (280).

The remaining marginal terms $\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l}$ with $X \in \{\beta, 1\}$ can be treated similarly. Note that the bound from theorem 1 for terms of the type $\partial_{\bar{c}}\Gamma^{\bar{c}\vec{\phi};w}_{1;l}$ is the same as the one for the corresponding terms $\Gamma^{\vec{\phi};w}_{1\varkappa;l}$. Consequently the proof of the bounds for the marginal terms $\Gamma^{ccA}_{1\gamma;l}$, $\Gamma^{ccc}_{1\omega;l}$, $\partial\Gamma^{cc}_{1\gamma;l}$ is the same as the proof for respectively $\partial_{\bar{c}}\Gamma^{\bar{c}ccA}_{1;l}$, $\partial\partial_{\bar{c}}\Gamma^{\bar{c}cc}_{1;l}$ which we shall consider now.

Let us denote by $\Gamma^{0\Lambda_0;\vec{\phi};w}_{X;l}$ any marginal term without insertions γ or ω, and by $\vec{q} \in \mathbb{M}_n$ the corresponding renormalization point as given in appendix H. We now anticipate the important fact that the relevant renormalization constants comply with the bounds, which will be proven in section 4. Then using the bounds on irrelevant terms from theorems 3, 4 and lemma 36, see (560), we obtain

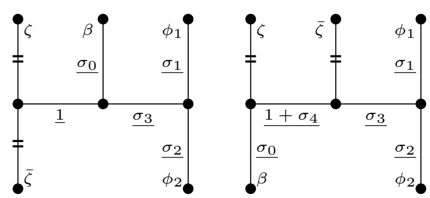
$$|\Gamma^{0\Lambda_0;\vec{\phi};w}_{X;l}(\vec{q})| \leqslant \frac{M}{\Lambda_0}\mathcal{P}^{(1)}_{r_X}\Big(\log_+ \frac{\Lambda_0}{M}\Big)\,, \tag{298}$$

in agreement with theorems 3, 4.

Set $\Lambda' := \max(\Lambda, M)$. To integrate the FE from 0 to Λ' we substitute the chain of vertex functions with the trees from theorems 3, 4. The ghost number of the marginal terms $\Gamma^{\vec{\phi};w}_{\beta\varkappa}$ is nonzero and thus these terms vanish. For the remaining terms $\Gamma^{\vec{\phi}}_{\beta}$ we have

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\beta;l}(\vec{q}) - \Gamma^{0\Lambda_0;\vec{\phi};w}_{\beta;l}(\vec{q})| \leqslant \sum_{\tau\in\mathcal{T}_{\zeta\bar{\zeta}\beta\vec{\phi}}} \int_0^{\Lambda'} d\lambda\,\lambda\,\mathcal{R}_{\zeta\bar{\zeta}}(Q^{\lambda;w}_{\tau})\,F^{\lambda\Lambda_0}_{2l-2\,0}\Big|_{\eta=0}. \qquad (299)$$

The θ-weight of trees $\mathcal{T}_{\zeta\bar{\zeta}\beta\vec{\phi}}$ equals two. Because $\vec{q}\in\mathbb{M}_n$, $\exists c>0$ such that



$$\mathcal{R}_{\zeta\bar{\zeta}}(Q^{\lambda;w}_{\tau}) \leqslant \frac{1}{\lambda}\frac{1}{\lambda+cM}\,,\ \forall\tau\in\mathcal{T}_{\zeta\bar{\zeta}\beta\vec{\phi}}\,. \qquad (300)$$

Using that $0<c<1$ the integrand in (299) is bounded by polynomials

$$\frac{\lambda+M+|\vec{q}|}{\Lambda_0(\lambda+cM)}P^{\lambda\Lambda_0}_{2l-2\,0} \leqslant \frac{M+|\vec{q}|}{c\Lambda_0 M}P^{\lambda\Lambda_0}_{2l-2\,0} \leqslant \frac{1+\sqrt{n}}{c\Lambda_0}P^{\lambda\Lambda_0}_{2l-2\,0}. \qquad (301)$$

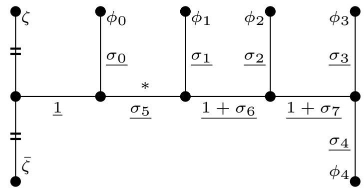
Here $n = \mathrm{N}+1$. Redefining the coefficients of these polynomials the rhs in (299) has the form

$$\frac{\Lambda'}{\Lambda_0}\mathcal{P}^{(a)}_{2l-2}(\log_+\frac{\Lambda_0}{M}) + \frac{1}{\Lambda_0}\int_0^{\Lambda'} d\lambda\,\mathcal{P}^{(b)}_{2l-2}(\log_+\frac{M}{\lambda}) \leqslant \frac{\Lambda'}{\Lambda_0}\mathcal{P}^{(c)}_{2l-2}(\log_+\frac{\Lambda_0}{M})\,, \qquad (302)$$

where we have used inequality (620). For the terms $\Gamma^{\vec{\phi};w}_{1}$ we obtain

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{1;l}(\vec{q}) - \Gamma^{0\Lambda_0;\vec{\phi};w}_{1;l}(\vec{q})| \leqslant \sum_{\tau\in\mathcal{T}_{1\zeta\bar{\zeta}\vec{\phi}}} \int_0^{\Lambda'} d\lambda\,\lambda\,\mathcal{R}_{\zeta\bar{\zeta}}(Q^{\lambda;w}_{\tau})\,F^{\lambda\Lambda_0}_{3l-3\,l-1}\Big|_{\eta=0}. \qquad (303)$$

The θ-weight of trees $\mathcal{T}_{1\zeta\bar{\zeta}\vec{\phi}}$ equals three and $\vec{q}\in\mathbb{M}_n$.



$$\mathcal{R}_{\zeta\bar\zeta}(Q_\tau^{\lambda;w}) \leqslant \frac{1}{\lambda}\frac{\lambda + \max|q_e|}{(\lambda+cM)^2}\,,\ \forall\tau\in\mathcal{T}_{1\zeta\bar\zeta\vec\phi}\,. \tag{304}$$

Using that $|q_e| < |\vec q| \leqslant \sqrt{n}M$ and $0<c<1$ the integrand in (303) is bounded by polynomials

$$\frac{(\lambda+M+|\vec q|)^2}{\Lambda_0(\lambda+cM)^2}P^{\lambda\Lambda_0}_{3l-3\,l-1} \leqslant \frac{(M+|\vec q|)^2}{\Lambda_0(cM)^2}P^{\lambda\Lambda_0}_{3l-3\,l-1} \leqslant 2\frac{1+n}{c^2\Lambda_0}P^{\lambda\Lambda_0}_{3l-3\,l-1}. \tag{305}$$

Redefining the coefficients of these polynomials for the rhs of (303) we have

$$\frac{\Lambda'}{\Lambda_0}\mathcal{P}^{(a)}_{3l-3}(\log_+\frac{\Lambda_0}{M}) + \frac{1}{\Lambda_0}\int_0^{\Lambda'} d\lambda\,\mathcal{P}^{(b)}_{3l-3}(\log_+\frac{M}{\lambda}) \leqslant \frac{\Lambda'}{\Lambda_0}\mathcal{P}^{(c)}_{3l-3}(\log_+\frac{\Lambda_0}{M})\,. \tag{306}$$

Hence at the renormalization point for the rhs of inequalities (299) and (303) we have upper bounds (302) and (306), respectively. Then using $\Lambda' \leqslant \Lambda + M$ we write a bound which complies with theormes 3, 4 at loop order l

$$|\Gamma^{\Lambda'\Lambda_0;\vec\phi;w}_{X,l}(\vec q) - \Gamma^{0\Lambda_0;\vec\phi;w}_{X,l}(\vec q)| \leqslant \frac{\Lambda+M}{\Lambda_0}\mathcal{P}^{(1)}_{r_X}\Big(\log_+\frac{\Lambda_0}{M}\Big). \tag{307}$$

With fixed $\Lambda = \Lambda'$ we sequentially perform two interpolations along the same path pattern which has been used in (296):

$$\vec q = \vec{\mathbf{q}}_{n-1}(1) \to \vec{\mathbf{q}}_{n-1}(0) = 0\,,\quad 0 = \vec{\mathbf{p}}_{n-1}(0) \to \vec{\mathbf{p}}_{n-1}(1) = \vec p\,. \tag{308}$$

First we consider the marginal terms $\Gamma^{\vec\phi;w}_\beta$. Substitution the irrelevant bounds from theorem 3 for each j-th segment of the path $\vec q \to 0$ yields

$$I_j(\vec q) \leqslant \sum_{\tau\in\mathcal{T}_{\beta\vec\phi}}\int_0^1 dt\,|q_j|\,Q_\tau^{\Lambda';w+1_j}(\vec{\mathbf{q}}_j)\,F^{\Lambda'\Lambda}_{2l-1\,0}(\vec{\mathbf{q}}_j)\,,\quad j\in\{1,...,n-1\}\,. \tag{309}$$

Here 1_j stands for a row with the unit in j-th column and filled with zeros for all others. The θ-weight of trees $\mathcal{T}_{\beta\vec\phi}$ equals one. Moreover, any such tree in the sum contains an edge e with nonvanishing σ-weight and such that $v_j \in K_e$, see (176). Let $V_{1;j} = \{v_i : i \geqslant j\}$. The integral in (309) has the form

$$\int_0^1 dt\, |q_j| \frac{\Lambda' + M + |\vec{\mathbf{q}}_j|}{\Lambda_0(\Lambda' + |tq_j + q_\Sigma|)} P^{\Lambda'\Lambda_0}_{2l-1\,0}(\vec{\mathbf{q}}_j), \quad q_\Sigma := \sum_{i \in K_e \setminus V_{1;j}} q_v\,, \tag{310}$$

where $q_\Sigma \in \mathbb{R}^4$ is the momentum flowing through the edge e excluding tq_j. Moreover, since $|\vec{\mathbf{q}}_j| \leqslant |\vec{q}| \leqslant \sqrt{n}M$ and $\Lambda' = \max(\Lambda, M)$ we have

$$P^{\Lambda'\Lambda_0}_{2l-1\,0}(\vec{\mathbf{q}}_j) \leqslant \mathcal{P}^{(1)}_{2l-1}\Big(\log_+ \frac{\Lambda_0}{M}\Big)\,, \tag{311}$$

$$\Lambda' + M + |\vec{\mathbf{q}}_j| \leqslant \Lambda + M(2 + \sqrt{n})\,. \tag{312}$$

Furthermore $|q_j| \leqslant M \leqslant \Lambda'$ and $|q_\Sigma| \geqslant cM$. Then inequality (623) applied to (310) yields

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\beta;l}(0) - \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\beta;l}(\vec{q})| \leqslant \frac{\Lambda + M}{\Lambda_0} P^{\Lambda'\Lambda_0}_{2l-1\,0}(0), \tag{313}$$

The bounds on the integrals I_j over the path $0 \to \vec{p}$ are similar to (309) and obtained by the change of the variables $\vec{\mathbf{q}}_j \mapsto \vec{\mathbf{p}}_j$. In this case

$$P^{\Lambda'\Lambda_0}_{2l-1\,0}(\vec{\mathbf{p}}_j) \leqslant \mathcal{P}^{(0)}_{2l-1}\Big(\log_+ \frac{\max(|\vec{p}|, M)}{\Lambda + M}\Big) + \mathcal{P}^{(1)}_{2l-1}\Big(\log_+ \frac{\Lambda_0}{M}\Big)\,, \tag{314}$$

$$\Lambda' + M + |\vec{\mathbf{p}}_j| \leqslant \Lambda + 2M + |\vec{p}|\,. \tag{315}$$

Because $\eta(\vec{p}) \leqslant M$, we can further simplify (314),

$$P^{\Lambda'\Lambda_0}_{2l-1\,0}(\vec{\mathbf{p}}_j) \leqslant P^{\Lambda\Lambda_0}_{2l-1\,0}(\vec{p}). \tag{316}$$

In this case $|p_\Sigma| \geqslant \eta$. Then using again (623) we have

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\beta;l}(0) - \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\beta;l}(\vec{p})| \leqslant \frac{\Lambda + M + |\vec{p}|}{\Lambda_0} P^{\Lambda\Lambda_0}_{2l\,0}(\vec{p}). \tag{317}$$

In a similar way we proceed for the marginal terms $\Gamma^{\vec{\phi};w}_1$. Integration over the path $\vec{q} \to 0$ gives us

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{1;l}(0) - \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{1;l}(\vec{q})| \leqslant \sum_{j=1}^{n-1} I_j(\vec{q})\,, \tag{318}$$

$$I_j(\vec{q}) = \int_0^1 dt\, |q_j|\, |\partial_j \Gamma_{1;l}^{\Lambda'\Lambda_0;\vec{\phi}}(\vec{\mathbf{q}}_j)|\,. \tag{319}$$

For each j-th segment of the path we substitute the irrelevent term $\partial\Gamma^{\Lambda'\Lambda_0;\vec{\phi}}$ with the bounds of theorem 4

$$I_j(\vec{q}) \leqslant \sum_{\tau\in\mathcal{T}_{1\vec{\phi}}} \int_0^1 dt\, |q_j|\, Q_\tau^{\Lambda';w+1_j}(\vec{\mathbf{q}}_j)\, \frac{\Lambda' + M + |\vec{\mathbf{q}}_j|}{\Lambda_0}\, P_{3l-2\,l-1}^{\Lambda'\Lambda}(\vec{\mathbf{q}}_j)\,. \tag{320}$$

In this case the total θ-weight of trees $\mathcal{T}_{1\vec{\phi}}$ equals two. Moreover, as in (309), each such tree contains an edge e with nonvanishing σ-weight such that $v_j \in K_e$, see (176). As in (310) let $q_\Sigma \in \mathbb{R}^4$ denote the momentum flowing through this edge excluding tq_j. Then the rhs of (320) has the form

$$\int_0^1 dt\, |q_j| \frac{\Lambda' + M + |\vec{\mathbf{q}}_j|}{\Lambda_0(\Lambda' + |tq_j + q_\Sigma|)} \frac{\Lambda' + |\mathbf{q}_*|}{\Lambda' + |\mathbf{q}_{e'}|} P_{3l-2\,l-1}^{\Lambda'\Lambda_0}(\vec{\mathbf{q}}_j), \tag{321}$$

where $\mathbf{q}_*$, $\mathbf{q}_{e'}$ are the momenta corresponding to the *-edge and to some edge $e' \in E$ which is possibly different from the edge e and the *-edge.

$$|\mathbf{q}_*| \leqslant |\vec{\mathbf{q}}| \leqslant |\vec{q}| \leqslant \sqrt{n}M \implies \frac{\Lambda' + |\mathbf{q}_*|}{\Lambda' + |\mathbf{q}_{e'}|} \leqslant 1 + \sqrt{n}\,. \tag{322}$$

Similarly to (311) we have

$$P_{3l-2\,l-1}^{\Lambda'\Lambda_0}(\vec{\mathbf{q}}_j) \leqslant \mathcal{P}_{3l-2}^{(1)}\Big(\log_+ \frac{\Lambda_0}{M}\Big)\,. \tag{323}$$

Substituting inequalities (312), (322), (323) into (321), and redefining the coefficients of the polynomial we obtain

$$I_j(\vec{q}) \leqslant \frac{\Lambda + M}{\Lambda_0} \mathcal{P}_{3l-2}^{(1)}\Big(\log_+ \frac{\Lambda_0}{M}\Big) \int_0^1 dt\, \frac{|q_j|}{(\Lambda' + |tq_j + q_\Sigma|)}\,. \tag{324}$$

Because $|q_j| \leqslant \Lambda'$ then using (623) we eventually get

$$|\Gamma_{1;l}^{\Lambda'\Lambda_0;\vec{\phi};w}(0) - \Gamma_{1;l}^{\Lambda'\Lambda_0;\vec{\phi};w}(\vec{q})| \leqslant \frac{\Lambda + M}{\Lambda_0} P_{3l-2\,0}^{\Lambda'\Lambda_0}(0)\,. \tag{325}$$

A bound on the integral $I_j(\vec{p})$ along the j-th segment of the path $0 \to \vec{p}$ is obtained from (321) by the substitution $\mathbf{q}_j \mapsto \mathbf{p}_j$. However, in this case $\vec{p}$ is arbitrary. As compared to the case with $\Gamma_{\beta}^{\vec{\phi};w}$, there is an additional factor,

$$\frac{\Lambda' + |\mathbf{p}_*|}{\Lambda' + |\mathbf{p}_{e'}|} \leqslant \frac{\Lambda' + |\vec{p}|}{\Lambda'} \leqslant 1 + 2\frac{|\vec{p}|}{\Lambda + M}\,. \tag{326}$$

This factor leads to the polynominal $\mathcal{P}^{(2)}_{s_1}$ in the bounds of theorem 4. From (316) we get $P^{\Lambda'\Lambda_0}_{3l-2}(\vec{\mathbf{p}}_j) \leqslant P^{\Lambda\Lambda_0}_{3l-2}(\vec{p})$, and thus

$$P^{\Lambda'\Lambda_0}_{3l-2\,l-1}(\vec{\mathbf{p}}_j) \leqslant \Big(1 + \frac{|\vec{p}|}{\Lambda_0}\Big)^{\bar{w}} \mathcal{P}^{(2)}_{l-1}\Big(\frac{|\vec{p}|}{\Lambda' + M}\Big) P^{\Lambda\Lambda_0}_{3l-2}(\vec{p}) \leqslant P^{\Lambda\Lambda_0}_{3l-2\,l-1}(\vec{p})\,. \tag{327}$$

Having (315), (326) and (327) the bound on the integrals $I_j(\vec{p})$ has the form

$$I_j(\vec{p}) \leqslant \frac{\Lambda + M + |\vec{p}|}{\Lambda_0} P^{\Lambda\Lambda_0}_{3l-2\,l}(\vec{p}) \int_0^1 dt\, \frac{|p_j|}{\Lambda' + |tp_j + p_\Sigma|}. \tag{328}$$

Noting that $|p_\Sigma| \geqslant \eta$, we use (623) to estimate the integral on the rhs

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{1;l}(0) - \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{1;l}(\vec{p})| \leqslant \frac{\Lambda + M + |\vec{p}|}{\Lambda_0} P^{\Lambda\Lambda_0}_{3l-1\,l}(\vec{p}). \tag{329}$$

For simplicity, one extends the rhs in (313), (325) to a larger bound

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{X;l}(0) - \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{X;l}(\vec{q})| \leqslant \frac{\Lambda + M}{\Lambda_0} P^{\Lambda\Lambda_0}_{r_X}(0). \tag{330}$$

Putting together (317) and (329) we get

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{X;l}(0) - \Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{X;l}(\vec{p})| \leqslant F^{\Lambda\Lambda_0}_{r_X\,s_X}(\vec{p}). \tag{331}$$

Both inequalities, (330) and (331), comply with the bounds of theorem 3, 4.

If $\Lambda = \Lambda'$ the proof is finished. If not we integrate downwards using the FE and then substituting the chain with the tree bound from theorems 3, 4.

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X;l}(\vec{p}) - \Gamma^{M\Lambda_0;\vec{\phi};w}_{X;l}(\vec{p})| \leqslant \int d\lambda |\dot{\Gamma}^{\Lambda\Lambda_0;\vec{\phi};w}_{X;l}(\vec{p})|\,. \tag{332}$$

For the terms $\mathsf{\Gamma}_{\beta}^{\Lambda\Lambda_0;\vec{\phi};w}$ we have

$$|\mathsf{\Gamma}_{\beta;l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p}) - \mathsf{\Gamma}_{\beta;l}^{M\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \sum_{\tau\in\mathcal{T}_{\zeta\bar{\zeta}\beta\vec{\phi}}} \int_{\Lambda}^{M} d\lambda\,\lambda\,\mathcal{R}_{\zeta\bar{\zeta}}(Q_{\tau}^{\lambda;w})\, F_{2l-2\,0}^{\lambda\Lambda_0}\Big|_{\eta=0}. \tag{333}$$

As in (299) the total θ-weight of trees $\mathcal{T}_{\zeta\bar{\zeta}\beta\vec{\phi}}$ is two. Using proposition 29 we can obtain

$$\mathcal{R}_{\zeta\bar{\zeta}}(Q_{\tau}^{\lambda;w}) \leqslant \frac{c}{\lambda(\lambda+\eta(\vec{p}))}\,. \tag{334}$$

Here compared to (300) the dependence on the momenta appears explicitly. This inequality also holds for all $\tau\in\mathcal{T}_{\zeta\bar{\zeta}\beta\phi}$

$$Q_{\tau}^{\lambda;w} = \frac{1}{(\lambda+|p|)^2} \leqslant \frac{1}{\lambda(\lambda+\eta)}\,. \tag{335}$$

We can further simplify the integrand in (333) by redefining the coefficients of the polynomials

$$F_{2l-2\,0}^{\lambda\Lambda_0}\Big|_{\eta=0} \leqslant \frac{M+|\vec{p}|}{\Lambda_0}\Big(1+\Big(\frac{|\vec{p}|}{\Lambda_0}\Big)^{\bar{w}}\Big) P_{2l-2}^{\lambda\Lambda_0}\Big|_{\eta=0}\,. \tag{336}$$

Consequently, each term of the sum in (333) has the form

$$\frac{M+|\vec{p}|}{\Lambda_0}\Big(1+\Big(\frac{|\vec{p}|}{\Lambda_0}\Big)^{\bar{w}}\Big)\int_{\Lambda}^{M} d\lambda\, \frac{\mathcal{P}_{2l-2}^{(0)}(\log_+ \frac{\max(|\vec{p}|,M)}{\lambda}) + \mathcal{P}_{2l-2}^{(1)}(\log_+ \frac{\Lambda_0}{M})}{\lambda+\eta}\,. \tag{337}$$

To bound this integral we use (611)

$$\begin{aligned}\frac{M+|\vec{p}|}{\Lambda_0}\Big(1+\Big(\frac{|\vec{p}|}{\Lambda_0}\Big)^{\bar{w}}\Big)\Big(\mathcal{P}_{2l-1}^{(0)}\Big(\log_+\frac{\max(|\vec{p}|,M)}{\Lambda+\eta}\Big) \\ + \mathcal{P}_{2l-2}^{(1)}\Big(\log_+\frac{\Lambda_0}{M}\Big)\log_+\frac{\max(|\vec{p}|,M)}{\Lambda+\eta}\Big)\,.\end{aligned} \tag{338}$$

It is clear that $b^{2l-2}a \leqslant b^{2l-1}+a^{2l-1}$. Hence the polynomials on the rhs are bounded by $P_{2l-1}^{\Lambda\Lambda_0}(\vec{p})$ which is a common bound for all trees in the set $\mathcal{T}_{\zeta\bar{\zeta}\beta\vec{\phi}}$. Finally, redefining the coefficients of the polynomials we have

$$|\mathsf{\Gamma}_{\beta;l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p}) - \mathsf{\Gamma}_{\beta;l}^{M\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \frac{M+|\vec{p}|}{\Lambda_0} P_{2l-1\,0}^{\Lambda\Lambda_0}(\vec{p})\,. \tag{339}$$

As we have seen already in (326) the analysis of the marginal terms $\Gamma_1^{\Lambda\Lambda_0;\vec{\phi};w}$ is more involved. We start with an inequality similar to (333)

$$|\Gamma_{1;l}^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p}) - \Gamma_{1;l}^{M\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \sum_{\tau\in\mathcal{T}_{1\zeta\bar{\zeta}\vec{\phi}}} \int_{\Lambda}^{M} d\lambda\,\lambda\,\mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w})\,F_{3l-3\,l-1}^{\lambda\Lambda_0}\Big|_{\eta=0}. \quad (340)$$

Here the total θ-weight of trees $\mathcal{T}_{1\zeta\bar{\zeta}\vec{\phi}}$ is three. For n-point functions with $n \geqslant 3$ using proposition 29 we have

$$\mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w}) \leqslant c\,\frac{\lambda+|\vec{p}|}{\lambda(\lambda+\eta(\vec{p}))(\lambda+|p_e|)}\,. \quad (341)$$

It is easy to check that it is still valid for 2-point functions, see (335). Because in (340) $\Lambda \leqslant \lambda \leqslant M$, for each term in the sum we have

$$\frac{M+|\vec{p}|}{\Lambda_0}\Big(1+\Big(\frac{|\vec{p}|}{\Lambda+M}\Big)^{\bar{w}}\Big)\mathcal{P}_{l-1}^{(2)}\Big(\frac{|\vec{p}|}{\Lambda+M}\Big)\int_{\Lambda}^{M}\frac{d\lambda\,(\lambda+|\vec{p}|)P_{3l-3}^{\lambda\Lambda_0}\Big|_{\eta=0}}{(\lambda+\eta)(\lambda+p_e)}. \quad (342)$$

Then using $|p_{e\in E}| \leqslant |\vec{p}|$ we transform the remaining integral into the form

$$\frac{\Lambda+|\vec{p}|}{\Lambda+|p_e|}\int_{\Lambda}^{M} d\lambda\,\frac{\mathcal{P}_{3l-3}^{(0)}(\log_+\frac{\max(|\vec{p}|,M)}{\lambda})+\mathcal{P}_{3l-3}^{(1)}(\log_+\frac{\Lambda_0}{M})}{\lambda+\eta}, \quad (343)$$

and then apply inequality (611). Therefore for the sum in (340) we get

$$\begin{aligned}\frac{M+|\vec{p}|}{\Lambda_0}(\Lambda+|\vec{p}|)P_{3l-2\,l-1}^{\Lambda\Lambda_0}(\vec{p})\sum_{\tau\in\mathcal{T}_{\vec{\phi}}}\frac{1}{\Lambda+|p_{e_1}|}&\\ \leqslant \frac{M+|\vec{p}|}{\Lambda_0}P_{3l-2\,l-1}^{\Lambda\Lambda_0}(\vec{p})\sum_{\tau\in\mathcal{T}_{1\vec{\phi}}}\frac{\Lambda+|p_*|}{\Lambda+|p_{e_1}|}\,.&\end{aligned} \quad (344)$$

Finally,

$$|\Gamma_{1;l}^{\Lambda\Lambda_0;\vec{\phi};w}(0) - \Gamma_{1;l}^{M\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \frac{M+|\vec{p}|}{\Lambda_0}\sum_{\tau\in\mathcal{T}_{1\vec{\phi}}} Q_\tau^{\Lambda;w}\,P_{3l-2\,l-1}^{\Lambda\Lambda_0}(\vec{p})\,. \quad (345)$$

3.5 Strictly relevant terms

If $n_\varkappa = 0$ the notation $\Gamma^{\vec{\phi};w}_{\vec{\varkappa};l}$ stands for $\Gamma^{c\bar{c}A}_l$, Γ^{AAA}_l, $\partial\Gamma^{AA}_l$, $\partial\Gamma^{c\bar{c}}_l$. In the case $n_\varkappa = 1$, it stands for $\Gamma^{c}_{\gamma;l}$. Moreover we impose $\Gamma^{0\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(0) = 0$ and denote by $\vec{p}$ arbitrary momenta with corresponding $\eta(\vec{p})$. We integrate the FE upwards from 0 to Λ and substitute $|\dot{\Gamma}^{\lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}|$ with the tree bound of theorem 1. Then

$$|\Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant \int_0^\Lambda d\lambda\, |\dot{\Gamma}^{\lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant \int_0^\Lambda d\lambda\, \lambda^{d-1} P^{\lambda\Lambda}_{2l-2}(0) \leqslant \int_0^{\Lambda+\eta} d\lambda\, \lambda^{d-1} P^{\lambda\Lambda}_{2l-2}(0)\,,$$

where $d > 0$. Inequality (620) then gives

$$|\Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant (\Lambda+\eta)^d P^{\Lambda+\eta\ \Lambda}_{2l-2}(0). \tag{346}$$

Substituting $\partial\Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}$ with the bound from theorem 1 we obtain

$$\begin{aligned} |\Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p}) - \Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant &\int_0^1 dt\, |p_i||\partial_i \Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(t\vec{p})| \\ \leqslant &|\vec{p}| \int_0^1 dt\, (\Lambda + t|\vec{p}|)^{d-1} P^{\Lambda}_{2l}(t\vec{p}) \leqslant I + I', \end{aligned} \tag{347}$$

where

$$I := |\vec{p}|(\Lambda+|\vec{p}|)^{d-1} \int_0^1 dt\, \mathcal{P}^{(0)}_{2l}(\log_+ \frac{\max(M,|\vec{p}|)}{\Lambda + t\eta(\vec{p})}), \tag{348}$$

$$I' := (\Lambda+|\vec{p}|)^d\, \mathcal{P}^{(1)}_{2l}(\log_+ \frac{\Lambda}{M}). \tag{349}$$

The calculations given in (620) yield

$$I \leqslant |\vec{p}|(\Lambda+|\vec{p}|)^{d-1}\, \mathcal{P}^{(0)}_{2l}(\log_+ \frac{\max(M,|\vec{p}|)}{\Lambda + \eta(\vec{p})}). \tag{350}$$

This implies

$$|\Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p}) - \Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant (\Lambda+|\vec{p}|)^d P^{\Lambda}_{2l}(\vec{p}). \tag{351}$$

Combining (346) and (351) proves the bounds of theorem 1 to loop order l.

3.5.1 $\partial\Gamma_1^{\phi^3}$, $\Gamma_1^{\phi^4}$ and $\Gamma_{1\gamma}^{cc}$

The goal of this section is to explain the expression for the polynomial degree r_1. We denote by $\Gamma_1^{\vec{\phi};w}$ the following terms: $\partial\Gamma_1^{cAA}$, $\partial\Gamma_1^{cc\bar{c}}$, Γ_1^{cAAA}, $\Gamma_1^{cAc\bar{c}}$ and we impose vanishing renormalization conditions at the origin, $\Gamma_1^{0\Lambda_0;\vec{\phi};w;}(0) = 0$. From the bounds of theorem 4 one realises that the analysis for $\Gamma_{1\gamma}^{cc}$ is similar to $\partial_{\bar{c}}\Gamma_1^{cc\bar{c}}$. For an arbitrary $\vec{p} \in \mathbb{P}_n$ let $\Lambda' := \max(\Lambda, \eta(\vec{p}))$. Using the FE and then substituting the bounds from theorem 4

$$|\Gamma_{1;l}^{\Lambda'\Lambda_0\vec{\phi};w}(0)| \leqslant \sum_{\tau\in\mathcal{T}_{1\zeta\bar{\zeta}\vec{\phi}}} \int_0^{\Lambda'} d\lambda\, Q_\tau^{\lambda;w}, \qquad Q_\tau^{\lambda;w} = \lambda \frac{\lambda(\lambda+M)}{\Lambda_0\lambda^2} P_{3l-3}^{\lambda\Lambda_0}(0)\,. \tag{352}$$

Consequently,

$$|\Gamma_{1;l}^{\Lambda'\Lambda_0\vec{\phi};w}(0)| \leqslant \frac{\Lambda+M}{\Lambda_0}\Big(\Lambda'\mathcal{P}_{3l-3}^{(1)}\Big(\log_+\frac{\Lambda_0}{M}\Big) + \int_0^{\Lambda'} d\lambda\, \mathcal{P}_{3l-3}^{(0)}\Big(\log_+\frac{M}{\lambda}\Big)\Big). \tag{353}$$

To bound the integral on the rhs we use (620),

$$|\Gamma_{1;l}^{\Lambda'\Lambda_0\vec{\phi};w}(0)| \leqslant (\Lambda+\eta)\,\frac{\Lambda+M}{\Lambda_0}\, P_{3(l-1)}^{\Lambda+\eta\;\Lambda_0}(0). \tag{354}$$

We extend $\Gamma_{1;l}^{\Lambda'\Lambda_0;\vec{\phi};w}$ from 0 to $\vec{p}$ using the usual path given in (296)

$$|\Gamma_{1;l}^{\Lambda'\Lambda_0;\vec{\phi};w}(0) - \Gamma_{1;l}^{\Lambda'\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \sum_{j=1}^{n-1} I_j(\vec{p})\,, \tag{355}$$

$$I_j(\vec{p}) = \int_0^1 dt\, |p_j|\, |\partial_j \Gamma_{1;l}^{\Lambda'\Lambda_0;\vec{\phi}}(\vec{\mathbf{p}}_j)|\,. \tag{356}$$

We substitute $\partial\Gamma_{1;l}^{\Lambda'\Lambda_0\vec{\phi};w}$ with the tree bound of theorem 4.

$$I_j(\vec{p}) \leqslant \sum_{\tau\in\mathcal{T}_{1\vec{\phi}}} \int_0^1 dt\, |p_j|\, \frac{\Lambda'+M+|\vec{\mathbf{p}}_j|}{\Lambda_0}\, Q_\tau^{\Lambda';w+1_j}(\vec{\mathbf{p}}_j)\, P_{3l-1\,l}^{\Lambda'\Lambda}(\vec{\mathbf{p}}_j)\,. \tag{357}$$

The total θ-weight of trees $\mathcal{T}_{1\vec{\phi}}$ equals one. Moreover, each such tree contains an edge e with nonvanishing σ-weight such that $v_j \in K_e$, see (176).

$$\int_0^1 dt\,|p_j|\frac{(\Lambda'+M+|\vec{\mathbf{p}}_j|)(\Lambda'+|\mathbf{p}_*|)}{\Lambda_0(\Lambda'+|tp_j+p_\Sigma|)}P^{\Lambda'\Lambda_0}_{3l-1\,l}(\vec{\mathbf{p}}_j), \tag{358}$$

As in (310) let $p_\Sigma \in \mathbb{R}^4$ denote the momentum flowing through this edge excluding tp_j and $\mathbf{p}_*$ is the momentum corresponding to the *-edge. The rational factor in this bound makes these terms different from other strictly relevant terms, for example $\partial\partial\Gamma^{cA}_1$, Γ^{cA}_β or $\partial\Gamma^c_\beta$. Noting that $|\mathbf{p}_*| \leqslant |\vec{\mathbf{p}}_j| \leqslant |\vec{p}|$ and $\Lambda' \leqslant \Lambda+\eta \leqslant \Lambda+|\vec{p}|$ we obtain

$$I_j(\vec{p}) \leqslant \frac{(\Lambda+M+|\vec{p}|)(\Lambda+|\vec{p}|)}{\Lambda_0}\sum_{\tau\in\mathcal{T}_{1\vec{\phi}}}\int_0^1 dt\,\frac{|p_j|\,P^{\Lambda'\Lambda_0}_{3l-1\,l}(\vec{\mathbf{p}}_j)}{\Lambda'+|tp_j+p_\Sigma|}. \tag{359}$$

Furthermore, because $\Lambda' \geqslant \frac{1}{2}(\Lambda+\eta)$

$$P^{\Lambda'\Lambda_0}_{3l-1\,l}(\vec{\mathbf{p}}_j) \leqslant \Big(1+\Big(\frac{|\vec{p}|}{\Lambda_0}\Big)^{\bar{w}}\Big)\mathcal{P}^{(2)}_l\Big(\frac{|\vec{p}|}{\Lambda+M}\Big)P^{\Lambda'\Lambda_0}_{3l-1}(\vec{\mathbf{p}}_j), \tag{360}$$

$$P^{\Lambda'\Lambda_0}_{3l-1}(\vec{\mathbf{p}}_j) \leqslant \mathcal{P}^{(0)}_{3l-1}\Big(\log_+\frac{\max(|\vec{p}|,M)}{\Lambda+\eta}\Big)+\mathcal{P}^{(1)}_{3l-1}\Big(\log_+\frac{\Lambda_0}{M}\Big). \tag{361}$$

Then using inequality (623) and noting that $|p_\Sigma| \geqslant \eta$ we have

$$\int_0^1 dt\,\frac{|p_j|\,P^{\Lambda'\Lambda_0}_{3l-1\,l}(\vec{\mathbf{p}}_j)}{\Lambda'+|tp_j+p_\Sigma|} \leqslant \Big(1+\log_+\frac{\max(|\vec{p}|,M)}{\Lambda+\eta}\Big)F^{\Lambda\Lambda_0}_{3l-1\,l}(\vec{p})\,, \tag{362}$$

and it follows that

$$|\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{1;l}(\vec{p})-\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{1;l}(0)| \leqslant (\Lambda+|\vec{p}|)F^{\Lambda\Lambda_0}_{3l\,l}(\vec{p})\,. \tag{363}$$

If $\Lambda=\Lambda'$ the proof is finished. If not we integrate the FE from η to Λ

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{1;l}(\vec{p})-\Gamma^{\eta\Lambda_0;\vec{\phi};w}_{1;l}(\vec{p})| \leqslant \sum_{\tau\in\mathcal{T}_{1\zeta\bar{\zeta}\vec{\phi}}}\int_\Lambda^\eta d\lambda\,\lambda\,\mathcal{R}_{\zeta\bar{\zeta}}(Q^{\lambda;w}_\tau)\,F^{\lambda\Lambda_0}_{3l-3\,l-1}\Big|_{\eta=0}, \tag{364}$$

where the θ-weight of trees is two. For n-point functions with $n > 2$, proposition 29 yields

$$\mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w}) \leqslant \frac{c\,(\lambda+|p_*|)}{\lambda(\lambda+\eta)} \leqslant \frac{c\,(\eta+|p_*|)}{\lambda(\lambda+\eta)} \leqslant \frac{2c\,|\vec{p}|}{\lambda(\lambda+\eta)}\,. \tag{365}$$

Because $\Lambda \leqslant \lambda \leqslant \eta \leqslant M$ we also have

$$F^{\lambda\Lambda_0}_{3l-3\,l-1}(\vec{p})\Big|_{\eta=0} \leqslant \frac{M+|\vec{p}|}{\Lambda_0} P^{\lambda\Lambda_0}_{3l-3\,l-1}(\vec{p})\Big|_{\eta=0}, \tag{366}$$

$$P^{\lambda\Lambda_0}_{3l-3\,l-1}(\vec{p})\Big|_{\eta=0} \leqslant \Big(1+\Big(\frac{|\vec{p}|}{\Lambda_0}\Big)^{\bar{w}}\Big)\mathcal{P}^{(2)}_{l-1}\Big(\frac{|\vec{p}|}{\Lambda+M}\Big) P^{\lambda\Lambda_0}_{3l-3}(\vec{p})\Big|_{\eta=0}, \tag{367}$$

$$P^{\lambda\Lambda_0}_{3l-3}(\vec{p})\Big|_{\eta=0} \leqslant \mathcal{P}^{(0)}_{3l-3}\Big(\log_+ \frac{\max(|\vec{p}|, M)}{\Lambda+\eta}\Big) + \mathcal{P}^{(1)}_{3l-3}\Big(\log_+ \frac{\Lambda_0}{M}\Big). \tag{368}$$

Then we repeat the steps used to show (280), and using (609) we get

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{1;l}(\vec{p}) - \Gamma^{\eta\Lambda_0;\vec{\phi};w}_{1;l}(\vec{p})| \leqslant |\vec{p}|\, F^{\Lambda\Lambda_0}_{3l-2\,l-1}(\vec{p})\,. \tag{369}$$

3.5.2 $\Gamma^{\vec{\phi};w}_{\beta}$, $\Gamma^{\phi^2;w}_{1}$ and $\Gamma^{\phi^3}_{1}$

In this section we briefly discuss the remaining strictly relevant terms $\Gamma^{\vec{\phi};w}_{\beta}$, $\Gamma^{\phi^3}_{1}$ and $\Gamma^{\phi^2;w}_{1}$ denoting all of them by $\Gamma^{\vec{\phi};w}_{X}$ with $X \in \{\beta, 1\}$. We impose renormalization conditions $\Gamma^{0\Lambda_0;\vec{\phi};w}_{X}(0) = 0$ and integrate the FE from 0 to Λ. We use the bounds of theorems 3,4 and then for an arbitrary $\vec{p}$ extend the integration up to $\Lambda + \eta(\vec{p})$

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X}(0)| \leqslant (\Lambda+\eta)^{d_X} \frac{\Lambda+M}{\Lambda_0} P^{\Lambda+\eta\;\Lambda_0}_{r_X}(0)\,. \tag{370}$$

Integration along the path given in equation (296) and using (620) yields

$$|\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X}(\vec{p}) - \Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X}(0)| \leqslant (\Lambda+|\vec{p}|)^{d_X} F_{s_X}(\vec{p}) P^{\Lambda\Lambda_0}_{s_X}(\vec{p}). \tag{371}$$

3.6 Convergence

We now come to the proof of theorem 2. As was stated after theorem 2 the bounds of this theorem permit to prove convergence of the functions $\Gamma^{\vec{\phi}}$ in the ultraviolet limit $\Lambda_0 \to \infty$ for nonexceptional momenta, using the Cauchy criterion.

We first prove the bounds for $\partial_{\Lambda_0}\Gamma^{\vec{\phi}}$. Then we proceed with the other functions $\partial_{\Lambda_0}\Gamma^{\vec{\phi}}_{\vec{\varkappa}}$ ascending in the number of insertions $n_\varkappa$. We use the same inductive scheme as before, based on the FE.

We start with the irrelevant terms integrating the FE from Λ_0 to Λ, using the boundary conditions $\Gamma_l^{\Lambda_0\Lambda_0;\vec{\phi};w} = 0$ and applying the derivative wrt Λ_0.

$$\partial_{\Lambda_0}\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l} = -\dot{\Gamma}^{\Lambda_0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l} + \int_{\Lambda_0}^{\Lambda} d\lambda\, \partial_{\Lambda_0}\dot{\Gamma}^{\lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}. \tag{372}$$

To bound the first term of the expression we substitute into the FE the irrelevant tree bound

$$|\dot{\Gamma}^{\Lambda_0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}| \leqslant \frac{1}{\Lambda_0}\sum_{\tau\in\mathcal{T}_{\vec{\varkappa}\vec{\phi}}} Q_\tau^{\Lambda_0;w} P^{\Lambda_0\Lambda_0}_{2(l-1)}. \tag{373}$$

If $n_\varkappa = 0$ then $\tau \in \mathcal{T}_{\vec{\phi}}$ and $\theta(\tau) > 0$. Consequently, recalling (190)

$$\Pi^{\Lambda_0}_{\tau,\theta}(\vec{p}) \leqslant \frac{\Lambda + |\vec{p}|}{\Lambda_0}\Pi^{\Lambda}_{\tau,\theta}(\vec{p}), \quad \text{and thus} \quad Q_\tau^{\Lambda_0;w} \leqslant \frac{\Lambda+|\vec{p}|}{\Lambda_0} Q_\tau^{\Lambda;w}. \tag{374}$$

Otherwise, the denominator $\Lambda_0 + |p_e|$ with $e_\varkappa \in E_\varkappa$ gives the inequality

$$\frac{1}{\Lambda_0 + |p_\varkappa|} \leqslant \frac{\Lambda + |\vec{p}|}{\Lambda_0}\frac{1}{\Lambda + |p_\varkappa|}. \tag{375}$$

In both cases this yields

$$|\dot{\Gamma}^{\Lambda_0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}| \leqslant \frac{\Lambda + |\vec{p}|}{\Lambda_0^2}\sum_{\tau\in\mathcal{T}_{\vec{\varkappa}\vec{\phi}}} Q_\tau^{\Lambda;w} P^{\Lambda\Lambda_0}_{2(l-1)}. \tag{376}$$

To analyse the second term we apply $\partial_p^w \partial_{\Lambda_0}$ to the chain of vertex functions given in definition 22. This gives a chain with the element $\partial_{\Lambda_0}((\partial_p^{w_1}\mathbf{C})\Gamma^{;w_2}_{l'})$ $l' < l$ which we bound using (685), (692) and theorem 2.

$$|\partial_{\Lambda_0}((\partial_p^{w_1}\mathbf{C})\Gamma^{\lambda\Lambda_0;\zeta\vec{\phi}\bar{\zeta};w_2}_{\vec{\varkappa};l'})| \leqslant |\partial_{\Lambda_0}\partial_p^{w_1}\mathbf{C}|\,|\Gamma^{\lambda\Lambda_0;\zeta\vec{\phi}\bar{\zeta};w_2}_{\vec{\varkappa};l'}| + |\partial_p^{w_1}\mathbf{C}|\,|\partial_{\Lambda_0}\Gamma^{\lambda\Lambda_0;\zeta\vec{\phi}\bar{\zeta};w_2}_{\vec{\varkappa};l'}|$$

$$\leqslant \frac{c}{(\lambda+|p_\zeta|)^{2+\|w_1\|}}\Big(\frac{\lambda+|p_\zeta|}{\Lambda_0^2}|\Gamma^{\lambda\Lambda_0;\zeta\vec{\phi}\bar{\zeta};w_2}_{\vec{\varkappa};l'}|+|\partial_{\Lambda_0}\Gamma^{\lambda\Lambda_0;\zeta\vec{\phi}\bar{\zeta};w_2}_{\vec{\varkappa};l'}|\Big)$$

$$\leqslant \frac{1}{(\lambda+|p_\zeta|)^2}\frac{\lambda+M+|\vec{p}|}{\Lambda_0^2}\sum_{\tau\in\mathcal{T}_{\vec{\varkappa}\zeta\vec{\phi}\bar{\zeta}}} Q_\tau^{\lambda;w_2+w_1}P^{\lambda\Lambda_0}_{r(l')}(\vec{p})\,. \qquad (377)$$

We proceed as for the proof of inequality (272) substituting expression (275) with the integral

$$\int_\Lambda^{\Lambda_0} d\lambda \frac{(\lambda+M+|\vec{p}|)P^{\lambda\Lambda_0}_{2l-2}|_{\eta_\tau=0}}{\Lambda_0^2(\lambda+\eta_f)(\lambda+|p_a|)} \leqslant \frac{\Lambda+M+|\vec{p}|}{\Lambda_0^2}\frac{1}{\Lambda+|p_a|}P^{\Lambda\Lambda_0}_{2l-1}(\vec{p}). \qquad (378)$$

For any tree $\tau\in\mathcal{T}_{\varkappa\phi\phi}$ the denominator $(\lambda+|p_\varkappa|)$ always appears in the irrelevant bounds $Q_\tau^{\lambda;w}$, even if $\|w\|=1$.

For marginal terms we shall integrate the FE upwards from 0 to Λ. For the terms with the antighost we use renormalization conditions (182)

$$\Gamma_l^{M\Lambda_0;c\bar{c}AA}(0)=0,\qquad \Gamma_l^{M\Lambda_0;c\bar{c}c\bar{c}}(0)=0,\qquad \partial_A\Gamma_l^{M\Lambda_0;c\bar{c}A}(0)=0. \qquad (379)$$

Using equation (184) we obtain that for these terms at $\Lambda'=\max(\Lambda,\eta(\vec{p}))$

$$|\partial_{\Lambda_0}\Gamma_l^{\Lambda'\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \int_0^1 dt\,|p_{\bar{c}}|\,|\partial_{\Lambda_0}\partial_{\bar{c}}\Gamma_l^{\Lambda'\Lambda_0;\vec{\phi};w}(tp_{\bar{c}},...)|$$

$$\leqslant \frac{\Lambda'+M+|\vec{p}|}{\Lambda_0^2}\sum_{\tau\in\mathcal{T}_{\vec{\phi}}}\int_0^1 dt\,|p_{\bar{c}}|\,Q_\tau^{\Lambda';w+1_{\bar{c}}}P^{\Lambda'\Lambda_0}_{2l-1} \leqslant \frac{\Lambda+M+|\vec{p}|}{\Lambda_0^2}P^{\Lambda\Lambda_0}_{2l}\,, \qquad (380)$$

where we have substituted $\partial_{\Lambda_0}\partial_{\bar{c}}\Gamma_l^{\vec{\phi};w}$ with the bound of theorem 2 and applied inequality (623).

The remaining marginal terms Γ^{AAAA}, $\partial_{\bar{c}}\Gamma^{c\bar{c}A}$, $\Gamma^{\phi\phi}_\varkappa$, $\partial\partial\Gamma^{\phi\phi}$, $\partial\Gamma^c_\gamma$, are renormalized at $\Lambda=0$ and nonvanishing momentum $\vec{q}$, chosen in $\mathbb{M}^s_n$ in all cases but Γ^{AAAA} for which $\vec{q}\in\mathbb{M}^{cp}_4$. See appendix H for the list of all relevant terms and their renormalization points. Since the renormalization constants are independent of Λ_0, their derivative wrt Λ_0 vanishes: it follows that the coefficients of δ-tensors in the decomposition of $\partial_{\Lambda_0}\Gamma^{0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{q})$ vanish. Hence using lemma 36 and the bounds on irrelevant terms we have

$$|\partial_{\Lambda_0}\Gamma^{0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{q})| \leqslant \frac{M}{\Lambda_0^2}\mathcal{P}^{(1)}_{2l-1}(\log_+\frac{\Lambda_0}{M}). \qquad (381)$$

We integrate the FE from 0 to Λ' and substitute the chain with the tree bound. Using inequalities (291), (292) it is easy to get the following bound

$$|\partial_{\Lambda_0}\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{q}) - \partial_{\Lambda_0}\Gamma^{0\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{q})| \leqslant \frac{\Lambda+M}{\Lambda_0^2}\mathcal{P}^{(1)}_{2l-1}(\log_+\frac{\Lambda_0}{M}). \tag{382}$$

Integrating back and forth along the path given in equation (296), substituting the irrelevant term $\partial_{\Lambda_0}\partial\Gamma^{\Lambda'\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}$ with its bounds and using inequality (623) we obtain

$$|\partial_{\Lambda_0}\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(0) - \partial_{\Lambda_0}\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{q})| \leqslant \frac{\Lambda+M}{\Lambda_0^2}P^{\Lambda'\Lambda_0}_{2l}(0), \tag{383}$$

$$|\partial_{\Lambda_0}\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(\vec{p}) - \partial_{\Lambda_0}\Gamma^{\Lambda'\Lambda_0;\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant \frac{\Lambda+M+|\vec{p}|}{\Lambda_0^2}P^{\Lambda\Lambda_0}_{2l}(\vec{p}). \tag{384}$$

If $\Lambda=\Lambda'$ the proof of the bounds on marginal terms is complete. Otherwise we integrate the FE downwards from η to Λ and repeat the arguments given to prove inequality (280) with a minor change in the integrand

$$\int_\Lambda^\eta d\lambda\,\frac{\lambda+M+|\vec{p}|}{\Lambda_0^2}\frac{\lambda\,P^{\lambda\Lambda_0}_k|_{\eta_\tau=0}}{(\lambda+|p_a|)(\lambda+|p_b|)} \leqslant \frac{M+|\vec{p}|}{\Lambda_0^2}\int_\Lambda^\eta d\lambda\,\frac{2P^{\lambda\Lambda_0}_k|_{\eta_\tau=0}}{\lambda+|p_b|}. \tag{385}$$

For the strictly relevant terms we integrate the FE from 0 to Λ substituting the vertex functions and propagators with their bounds and extending the upper limit of integration to $\Lambda+\eta$

$$|\partial_{\Lambda_0}\Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant \frac{\Lambda+M}{\Lambda_0^2}\int_0^{\Lambda+\eta} d\lambda\,\lambda^{d-1}P^{\lambda\Lambda_0}_{2l-2}(0), \tag{386}$$

where $d>0$. Using inequality (620) we obtain

$$|\partial_{\Lambda_0}\Gamma^{\Lambda\Lambda_0\vec{\phi};w}_{\vec{\varkappa};l}(0)| \leqslant \frac{\Lambda+M}{\Lambda_0^2}(\Lambda+\eta)^d P^{\Lambda+\eta\;\Lambda_0}_{2l-2}(0). \tag{387}$$

To extend to momentum $\vec{p}$ we proceed as in (347), the only change being an additional factor of $\frac{\Lambda+M+|\vec{p}|}{\Lambda_0^2}$.

3.7 IR limit of $\Gamma_1^{\Lambda\Lambda_0}$

As follows from (153), to show the existence of the limit $\lim_{\Lambda\to 0} \Gamma_{1;l}^{\Lambda\Lambda_0}$ in the loop order l we need to show that $\lim_{\Lambda\to 0} \Delta_{l-1}^{\Lambda\Lambda_0;\vec{\phi};w} = 0$. We expand each of three inverse matrices on the rhs of (150) in power series

$$
\begin{aligned}
\Delta^{\Lambda\Lambda_0} =& \langle(\sigma_\Lambda, 0, 0) \sum_{m=0}^{\infty} \Big(- \frac{\delta^2 \Gamma^{\Lambda\Lambda_0}}{\delta\Phi\delta\Phi'} \hat{\mathbf{1}}\mathbf{C}_{\Phi'\Phi}^{\Lambda\Lambda_0} \Big)^m \frac{\delta^2 \Gamma^{\Lambda\Lambda_0}}{\delta\Phi\delta\gamma} \rangle \\
&+ \langle(0, \sigma_\Lambda, 0) \sum_{m=0}^{\infty} \Big(- \frac{\delta^2 \Gamma^{\Lambda\Lambda_0}}{\delta\Phi\delta\Phi'} \hat{\mathbf{1}}\mathbf{C}_{\Phi'\Phi}^{\Lambda\Lambda_0} \Big)^m \frac{\delta^2 \Gamma^{\Lambda\Lambda_0}}{\delta\Phi\delta\omega} \rangle \\
&- \langle(\sigma_\Lambda \partial, 0, 0) \sum_{m=0}^{\infty} \Big(- \frac{\delta^2 \Gamma^{\Lambda\Lambda_0}}{\delta\Phi\delta\Phi'} \hat{\mathbf{1}}\mathbf{C}_{\Phi'\Phi}^{\Lambda\Lambda_0} \Big)^m \frac{\delta c}{\delta\Phi} \rangle \,,
\end{aligned}
\tag{388}
$$

where for $m > 1$

$$
\Big(\frac{\delta^2 \Gamma^{\Lambda\Lambda_0}}{\delta\Phi\delta\Phi'} \hat{\mathbf{1}}\mathbf{C}_{\Phi'\Phi}^{\Lambda\Lambda_0} \Big)^m = \frac{\delta^2 \Gamma^{\Lambda\Lambda_0}}{\delta\Phi\delta\Phi_1'} \hat{\mathbf{1}}\mathbf{C}_{\Phi_1'\Phi_1}^{\Lambda\Lambda_0} \cdots \frac{\delta^2 \Gamma^{\Lambda\Lambda_0}}{\delta\Phi_{m-1}\delta\Phi'} \hat{\mathbf{1}}\mathbf{C}_{\Phi'\Phi}^{\Lambda\Lambda_0} \,. \tag{389}
$$

In the following we consider a term $\Delta_{\vec{\varkappa}}^{\vec{\phi};w}$. At tree level $\Gamma_{l=0}^{\phi\bar{\phi}} = 0$. It follows that these series at the loop order $l-1$ contain only a finite number of terms. The first and second terms on the rhs of (388) have the following form

$$
\Gamma_{\vec{\varkappa}_1;l_1}^{\Phi_0\Phi_1'\vec{\phi}_1} \hat{\mathbf{1}}\mathbf{C}_{\Phi_1'\Phi_1}^{\Lambda\Lambda_0} \cdots \Gamma_{\vec{\varkappa}_m;l_m}^{\Phi_{m-1}\Phi_m'\vec{\phi}_m} \hat{\mathbf{1}}\mathbf{C}_{\Phi_m'\Phi_m}^{\Lambda\Lambda_0} \Gamma_{\varkappa\vec{\varkappa}_{m+1};l_{m+1}}^{\Phi_m\vec{\phi}_{m+1}} \,. \tag{390}
$$

Here $\sum_{k=0}^{m+1} l_k = l-1$ and introducing permutations π, π'

$$
\vec{\phi}_1 \oplus ...\vec{\phi}_{m+1} = (\phi_{\pi(0)}, ..., \phi_{\pi(\mathrm{N}-1)}), \tag{391}
$$
$$
\vec{\varkappa}_1 \oplus ...\vec{\varkappa}_{m+1} = (\varkappa_{\pi'(1)}, ..., \varkappa_{\pi'(n_\varkappa)}) \,. \tag{392}
$$

Using the notation for a chain of vertex functions, see (219), we write (390) as $\mathcal{F}_S^{\Phi_0\vec{\phi}\varkappa}$, $(\Phi_0, \varkappa) \in \{(A,\gamma),(c,\omega)\}$. The third term on the rhs of (388) is exactly $\mathcal{F}_S^{\Phi\vec{\phi}\Phi'} \hat{\mathbf{1}}\mathbf{C}_{\Phi'\Phi}$. Similarly to (243) using the bounds of theorem 1, the bound on the propagators (692), and inequalities (625), (631) we have the following upper bounds for the loop integrals appearing in (388)

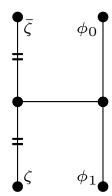
$$\int \sigma_\Lambda \hat{\mathcal{S}}^{\Lambda;\Phi_0\vec{\phi}\varkappa;\mathsf{w}}\, P^\Lambda_{r(l-1)} \leqslant \Lambda^4\, \mathcal{R}_{\Phi_0\varkappa}(\hat{\mathcal{S}}^{\Lambda;\Phi_0\vec{\phi}\varkappa;\mathsf{w}})\, P^\Lambda_{r(l-1)}\Big|_{\eta=0}\,, \tag{393}$$

$$\int \partial\sigma_\Lambda \hat{\mathcal{S}}^{\Lambda;A\vec{\phi}\bar{c};\mathsf{w}} |\mathbf{C}_{\bar{c}c}|\, P^\Lambda_{r(l-1)} \leqslant \Lambda^3\, \mathcal{R}_{A\bar{c}}(\hat{\mathcal{S}}^{\Lambda;A\vec{\phi}\bar{c};\mathsf{w}})\, P^\Lambda_{r(l-1)}\Big|_{\eta=0}\,. \tag{394}$$

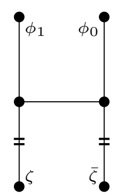
Using inequality (245) for each of the above terms we obtain

$$\left.\begin{array}{r} \Lambda^4\, \mathcal{R}_{\Phi_0\varkappa}(\hat{\mathcal{S}}^{\Lambda;\Phi_0\vec{\phi}\varkappa;\mathsf{w}}) \\ \Lambda^3\, \mathcal{R}_{A\bar{c}}(\hat{\mathcal{S}}^{\Lambda;A\vec{\phi}\bar{c};\mathsf{w}}) \end{array}\right\} \leqslant \Lambda^3 \sum_{\tau\in\mathcal{T}_{\zeta\bar{\zeta}\vec{\phi}}} \mathcal{R}_{\zeta\bar{\zeta}}(Q^w_\tau)\,. \tag{395}$$

The 1-point function Δ^ϕ vanishes due to SU(2) symmetry. For 2-point functions $\Delta^{\phi\phi}$ we have two types of trees



$= 1\,,$

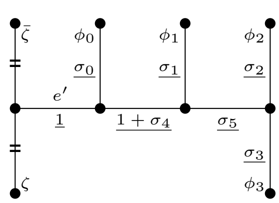


$= 1\,.$ (396)

Here the edges with vanishing external momentum are marked by two crosslines. Consequently for 2-point functions at a nonexceptional momentum, $p \neq 0$,

$$\lim_{\Lambda\to 0} |\Delta^{\Lambda\Lambda_0;\phi\phi}| \leqslant \lim_{\Lambda\to 0} \Lambda^3\, \mathcal{P}_r\Big(\log_+ \frac{\max(|p|, M)}{\Lambda}\Big) = 0\,. \tag{397}$$

Then we consider all remaining terms $\mathrm{N} + 2n_\varkappa + \|w\| \geqslant 3$. Each such tree $\tau \in \mathcal{T}_{\zeta\bar{\zeta}\vec{\varkappa}\vec{\phi}}$ has the total θ-weight $\mathrm{N} + n_\varkappa + \|w\| - 2$. Moreover, it can have at most only one internal edge $e' \in E\backslash E_1$ with vanishing momentum, $p_{e'} = 0$. The ρ-weight of this edge is either zero or one.



$$\mathcal{R}_{\zeta\bar{\zeta}} Q^w_\tau \leqslant \frac{1}{\Lambda(\Lambda+\eta)^{\mathrm{N}+2n_\varkappa+\|w\|-3}}\,. \tag{398}$$

At nonexceptional momenta using the notation of theorem 4 we then obtain

$$\lim_{\Lambda\to 0} |\Delta^{\Lambda\Lambda_0;\vec{\phi};w}_{\vec{\varkappa}}| \leqslant \lim_{\Lambda\to 0} \Big(\frac{\Lambda}{\Lambda+\eta}\Big)^2 (\Lambda+\eta)^d \mathcal{P}_r\Big(\log_+ \frac{\max(|\vec{p}|, M)}{\Lambda}\Big) = 0\,. \tag{399}$$

4 Restoration of the STI

As mentioned before theorems 3, 4 we now consider all nontrivial marginal terms $\Gamma^{\Lambda\Lambda_0;\vec{\phi};w}_{X\vec{\varkappa};l}$ with $X \in \{\beta, 1\}$ at $\Lambda = 0$. We want to show that these terms verify the bounds of theorems 3, 4. In this section, since $\Lambda = 0$, we will omit the parameters Λ, Λ_0 in the notations wherever this is not ambiguous, i.e. we write $\mathsf{\Gamma}^{\vec{\phi}}$ for $\mathsf{\Gamma}^{0\Lambda_0;\vec{\phi}}$.

The subsequent relations are obtained by projecting the AGE (166) and the STI (167) on the respective monomial in the fields (for example, c in section 4.1) to read off the lhs from the rhs, taken at the renormalization point. We will establish appropriate relations in order to make the coefficients of the δ-tensors wrt the monomial basis at the renormalization point satisfy the bounds of these two theorems. In this analysis we make particular use of the consistency conditions, see 4.0.2. In section 4.0.3 we prove the existence of a solution for the above mentioned system of relations that does not depend on the UV cutoff. In the remaining sections we treat the different marginal terms one by one.

4.0.1 Smallness relations

It is helful to introduce the notion of *small* terms, which vanish in the limit $\Lambda_0 \to \infty$. For fixed loop order l and $X \in \{\beta, 1\}$, a homogeneous function $f(\vec{p}, M, \Lambda_0)$ of mass dimension $[f]$ is said *small* on a subset $Y \subset \mathbb{M}_n$, and denoted by $f \overset{X,Y,l}{\sim} 0$, if for all $w \in \mathbb{W}_n$ with $\|w\| \leqslant [f]$, there exists a polynomial $\mathcal{P}^{(1)}_{r_X}$ of degree $r_X([f] - \|w\|, l)$, see theorems 3 and 4, such that the following bound holds for all $\Lambda_0 \geqslant M$ and all $\vec{p} \in Y$, see (169):

$$|\partial^w_p f(\vec{p}, M, \Lambda_0)| \leqslant \frac{M^{1+[f]-\|w\|}}{\Lambda_0} \mathcal{P}^{(1)}_{r_X}(\log_+ \frac{\Lambda_0}{M}). \tag{400}$$

Furthermore, $f \overset{X,Y,l}{\sim} g$ iff $[f] = [g]$ and $f - g \overset{X,Y,l}{\sim} 0$. Because both relations $\overset{1,Y,l}{\sim}$ and $\overset{\beta,Y,l}{\sim}$ only differ by the degree of polynomials we have $f \overset{\beta,Y,l}{\sim} g \implies f \overset{1,Y,l}{\sim} g$. Since the loop order l and the renormalization point $\vec{q}$ are evident from the context, we write:

$$f \overset{Y}{\sim} g \quad \text{for} \quad f \overset{1,Y,l}{\sim} g, \qquad\qquad f \overset{\beta,Y}{\sim} g \quad \text{for} \quad f \overset{\beta,Y,l}{\sim} g, \tag{401}$$

$$f \sim g \quad \text{for} \quad f \overset{1,\{\vec{q}\}}{\sim} g, \qquad\qquad f \overset{\beta}{\sim} g \quad \text{for} \quad f \overset{\beta,\{\vec{q}\}}{\sim} g. \tag{402}$$

Theorem 1 implies that for every vertex function $\Gamma^{\vec{\phi}}_{\vec{\varkappa}}(\vec{p})$ there exists a constant c such that $\forall w \in \mathbb{W}_n$, $\forall \vec{p} \in \mathbb{M}_n$, $\forall \Lambda_0 \geqslant M$

$$|\partial^w \Gamma^{\vec{\phi}}_{\vec{\varkappa}}(\vec{p})| \leqslant c M^{4-2n_{\varkappa}-\mathrm{N}-\|w\|} \,. \tag{403}$$

Using also that

$$|\partial^w(\sigma_{0\Lambda_0} - 1)| < c_w \, \frac{1}{\Lambda_0^w} \,, \quad |\sigma_{0\Lambda_0} - 1| < c_0 \, \frac{M}{\Lambda_0} \,, \tag{404}$$

the terms on rhs of the STI and the AGE satisfy the relations:

$$\Gamma^{\vec{\phi}_1;w_1} \partial^{w_2}(\sigma_{0\Lambda_0} \Gamma^{\vec{\phi}_2}_{\varkappa}) \overset{\beta,Y}{\sim} \Gamma^{\vec{\phi}_1;w_1} \Gamma^{\vec{\phi}_2;w_2}_{\varkappa} \,, \tag{405}$$

$$\partial^w(\sigma_{0\Lambda_0} \Gamma^{\vec{\phi}}) \overset{\beta,Y}{\sim} \Gamma^{\vec{\phi};w} \,, \tag{406}$$

$$\partial^w(p\sigma_{0\Lambda_0} \Gamma^{\vec{\phi}}_{\gamma}) \overset{\beta,Y}{\sim} p\Gamma^{\vec{\phi};w}_{\gamma} \,. \tag{407}$$

This fact will be useful in the calculations underlying the following sections.

4.0.2 Consistency conditions

Here we establish the consistency conditions implied by the nilpotency, see (164) and (165). Below we will rely on the validity of theorem 4 at loop orders $l' < l$ for all terms and at the current order l only for irrelevant terms: these properties are true in our inductive scheme. Recall definitions (159), (169) and (401). Using the AGE (166), the bounds of theorems 1-3, and (165) we get

$$\Big(S\Gamma_\beta\Big)^{\vec{\phi}} \overset{\mathbb{M}_n}{\sim} 0, \quad \text{and thus} \quad \Big(\frac{\delta}{\delta\tilde{c}}\mathrm{F}_1\Big)^{\vec{\phi}} \overset{\mathbb{M}_n}{\sim} 0 \,, \tag{408}$$

where $\vec{\phi} = (\phi_1, ..., \phi_{n-1})$ and $\phi_i \in \{A, c, \bar{c}\}$. Equation (408), theorems 1, 2, and the bounds of theorem 4 for irrelevant terms yield

$$\Big(\frac{\delta}{\delta\tilde{c}}\mathrm{F}_{1,rel}\Big)^{\vec{\phi}} \overset{\mathbb{M}_n}{\sim} 0, \qquad\qquad \vec{\phi} \in \{(c,c),(c,c,A)\} \,. \tag{409}$$

See appendix G for the definition of $\mathrm{F}_{1,rel}$ and of the constants u^{Φ}. In section 4.5 it will be shown that $u^{c\bar{c}cAA}_{1,2,3} \sim 0$. Equation (409) then gives

$$u^{\gamma cc} \sim u_1^{\bar{c}cc} \sim u_2^{\bar{c}cc}, \quad -2u^{\gamma Acc} \sim u_1^{\bar{c}ccA} \sim u_2^{\bar{c}ccA} \sim u_3^{\bar{c}ccA}, \quad u_4^{\bar{c}ccA} \sim 0. \tag{410}$$

Let us exploit (164) to obtain more constraints on the renormalization constants u^Φ. At loop order l

$$S\mathrm{F}_{1;l} = S_0\mathrm{F}_{1;l} + \sum_{l'<l} S_{l-l'}\mathrm{F}_{1;l'}. \tag{411}$$

By induction $(S_{l-l'}\mathrm{F}_{1;l'})^{\vec\phi} \overset{\mathbb{M}_n}{\sim} 0$ for all loop orders $l' < l$. Then equation (164) implies that

$$(S_0\mathrm{F}_{1;l})^{\vec\phi} \overset{\mathbb{M}_n}{\sim} 0, \tag{412}$$

where

$$\begin{aligned} S_0 =& \langle \tilde\delta_{A^d_\sigma}\underline{\Gamma}_0, \sigma_{0\Lambda_0}\tilde\delta_{\gamma^d_\sigma}\rangle + \langle \tilde\delta_{\gamma^d_\sigma}\underline{\Gamma}_0, \sigma_{0\Lambda_0}\tilde\delta_{A^d_\sigma}\rangle \\ &- \langle \tilde\delta_{c^d}\underline{\Gamma}_0, \sigma_{0\Lambda_0}\tilde\delta_{\omega^d}\rangle - \langle \tilde\delta_{\omega^d}\underline{\Gamma}_0, \sigma_{0\Lambda_0}\tilde\delta_{c^d}\rangle, \end{aligned} \tag{413}$$

and recalling notation for $\tilde\delta_\phi$ from (26)

$$\begin{aligned} \tilde\delta_{A^d_\sigma(q)}\underline{\Gamma}_0 =& g\epsilon^{adb}(\langle \check\gamma^a_\sigma \check c^b; q\rangle + \langle ip_\sigma \check{\bar c}^a(p)\check c^b; q\rangle) + A^d_{\sigma'}(q)(\delta_{\sigma'\sigma}q^2 - q_{\sigma'}q_\sigma)\sigma^{-1}_{0\Lambda_0}(q^2) \\ &+ 3\epsilon^{dab}\langle F^{AAA}_{\sigma\mu\nu}(q,\cdot,\cdot)|\check A^a_\mu \check A^b_\nu; q\rangle + 4R^{AAAA}_{\sigma\rho\mu\nu}\langle \check A^d_\rho \check A^a_\mu \check A^a_\nu; q\rangle, \\ \tilde\delta_{\gamma^d_\sigma(q)}\underline{\Gamma}_0 =& iq_\sigma c^d(q) + g\epsilon^{dab}\langle \check A^a_\sigma \check c^b; q\rangle, \\ \tilde\delta_{c^d(q)}\underline{\Gamma}_0 =& g\epsilon^{dab}\langle ip_\mu \check{\bar c}^a(p)\check A^b_\mu + \check\gamma^a_\mu \check A^b_\mu + \check\omega^b \check c^a; q\rangle - iq_\mu\gamma^d_\mu(q) - \bar c^d(q)q^2\sigma^{-1}_{0\Lambda_0}(q^2), \\ \tilde\delta_{\omega^d(q)}\underline{\Gamma}_0 =& \frac{1}{2}g\epsilon^{dab}\langle \check c^a \check c^b; q\rangle. \end{aligned}$$

Here the notation $\langle ip_\sigma \check{\bar c}^a(p)\check c^b; q\rangle$ corresponds to $\langle \phi_1\phi_2; q\rangle$ with $\phi_1(p) = ip_\sigma\check{\bar c}^a(p)$, $\phi_2 = \check c^b$.
For all $\vec\phi$, $\vec\varkappa$ and w such that $\mathrm{N} + 2n_\varkappa + \|w\| = 6$ we have

$$(S_0\mathrm{F}_{1;l})^{\vec\phi;w}_{\vec\varkappa} = (S_0\mathrm{F}_{1,rel;l})^{\vec\phi;w}_{\vec\varkappa} + \sum_\pi (-)^{\pi_a} S^{\vec\phi_1;w_1}_{0,\vec\varkappa_1}\Delta^{\vec\phi_2;w_2}_{\vec\varkappa_2;l}, \tag{414}$$

where $\Delta^{\Lambda\Lambda_0}_l := \mathrm{F}^{\Lambda\Lambda_0}_{1;l} - \mathrm{F}^{\Lambda_0}_{1,rel;l}$, the sum runs over the permutations $\pi = (\pi_\phi, \pi_\varkappa, \pi_w)$ such that $\vec\phi_{\pi_\phi} = \vec\phi_1 \oplus \vec\phi_2$, $\vec\varkappa_{\pi_\varkappa} = \vec\varkappa_1 \oplus \vec\varkappa_2$, $w_{\pi_w} = w_1 + w_2$, and π_a is the number of transpositions $mod\,2$ of anticommuting variables in the permutation π. Using (413), for the terms in the sum on the rhs of (414) we have

$$\begin{aligned} |S^{\vec\phi_1;w_1}_{0,\vec\varkappa_1}\Delta^{\vec\phi_2;w_2}_{\vec\varkappa_2;l}| \leqslant& |\underline{\Gamma}^{A\vec\phi_1;w'_1}_{\vec\varkappa_1;0}, \sigma^{w''_1}_{0\Lambda_0}\Delta^{\vec\phi_2;w_2}_{\vec\varkappa_2\gamma;l}| + |\underline{\Gamma}^{c\vec\phi_1;w'_1}_{\vec\varkappa_1;0}, \sigma^{w''_1}_{0\Lambda_0}\Delta^{\vec\phi_2;w_2}_{\vec\varkappa_2\omega;l}| \\ &+ |\underline{\Gamma}^{\vec\phi_1;w'_1}_{\vec\varkappa_1\gamma;0}, \sigma^{w''_1}_{0\Lambda_0}\Delta^{A\vec\phi_2;w_2}_{\vec\varkappa_2;l}| + |\underline{\Gamma}^{\vec\phi_1;w'_1}_{\vec\varkappa_1\omega;0}, \sigma^{w''_1}\Delta^{c\vec\phi_2;w_2}_{\vec\varkappa_2;l}|. \end{aligned} \tag{415}$$

Let us show that the lhs of (415) is *small* on $\mathbb{M}_n$. Using appendix G and the bounds on irrelevant terms of theorems 3, 4 we see that for all marginal terms

$$\Delta^{\vec{\phi};w}_{\vec{\varkappa};l} \sim 0, \quad \text{and thus} \quad \Delta^{\vec{\phi};w}_{\vec{\varkappa};l} \overset{\mathbb{M}_n}{\sim} 0. \tag{416}$$

The relation on the rhs can be obtained by adapting the interpolation in equations (307)-(331). Define $\mathrm{N}_i := |\vec{\phi}_i|$ and $n_{\varkappa_i} := |\vec{\varkappa}_i|$. Consider the sum of the first and second terms on the rhs. If $2n_{\varkappa_2} + \mathrm{N}_2 + \|w_2\| \geqslant 3$ then the bounds of theorems 1, 3, 4 and (416) imply that the sum is *small*. On the other hand if $2n_{\varkappa_1} + \mathrm{N}_1 + \|w_1\| > 3$ then $\|w_1''\| > 0$ and the bounds of theorems 1, 3 also give that the sum is *small*. The analysis of the sum of the third and fourth term on the rhs is similar. If $2n_{\varkappa_2} + \mathrm{N}_2 + \|w_2\| \geqslant 4$ then the bounds of theorems 1, 3, 4 and (416) imply that the sum is *small*. If $2n_{\varkappa_1} + \mathrm{N}_1 + \|w_1\| > 2$ then $\|w_1''\| > 0$ and using the bounds of theorems 1, 3 we obtain again that the sum is *small*. It follows that the lhs of (415) is *small*. This fact and (412) imply that $(S_0\mathrm{F}_{1,rel;l})^{\vec{\phi};w} \overset{\mathbb{M}_n}{\sim} 0$ for all marginal terms, which leads to the following equations

$$g u^{\gamma cc} \sim -u^{\gamma Acc}, \qquad u_1^{cAA} + u_2^{cAA} \sim \frac{g u^{cA}}{2}, \tag{417}$$

$$u_1^{cAAA} \sim u_2^{cAAA}, \qquad u_3^{cAAA} \sim u_4^{cAAA} \sim u_5^{cAAA}. \tag{418}$$

4.0.3 Existence of a constant solution

By our convention (which is the standard one) the renormalization constants that are solutions of the relations listed in RC3 are supposed not to depend on Λ_0. We give here a proof of this property, which is not evident because these relations contain nontrivial functions of Λ_0, here denoted by $\zeta^{\Lambda_0}_\Phi$. The relations corresponding to the marginal terms $\Gamma^{\vec{\phi};w}_{1\vec{\varkappa}}$, $\Gamma^{\vec{\phi};w}_{\beta\vec{\varkappa}}$ have respectively the general form

$$c^\Phi + \varrho\, C_1^\Phi \varrho + \zeta^{\Lambda_0}_\Phi \sim 0, \qquad \zeta^{\Lambda_0}_\Phi := \varrho\, C_2^\Phi \zeta^{\Lambda_0} + \zeta^{\Lambda_0} C_3^\Phi \zeta^{\Lambda_0}\,, \tag{419}$$

$$c^\Phi + V_1^\Phi \varrho + \zeta^{\Lambda_0}_\Phi \overset{\beta}{\sim} 0, \qquad \zeta^{\Lambda_0}_\Phi := V_2^\Phi \zeta^{\Lambda_0}\,. \tag{420}$$

Here $\varrho = (r_i^{\phi\cdots}, R_i, \Sigma_L^{AA}, \Sigma_T^{AA}, \Sigma^{\bar{c}c})$ denotes the relevant terms for vertex functions, see (721) for the list of $r_i^{\phi\cdots}$ and appendixes E, F for the remaining terms. The sequence ζ^{Λ_0} stands for the irrelevant terms listed in appendixes E, F and for the derivative of $r_i^{\phi\cdots}$, R_i wrt scalar products of momenta. Finally, c^Φ is a constant, $V^\Phi_{1,2}$ are constant vectors, and $C^\Phi_{1,2,3}$ are constant matrices.

At loop order l, the terms $\zeta^{\Lambda_0}_{\Phi,l}$ depend only on $\varrho_{l'}$ of loop order $l' < l$: this property holds because each $\zeta^{\Lambda_0}_{\Phi,l}$ is at least linear in the ζ^{Λ_0} and because all the $\zeta^{\Lambda_0}_{l=0}$ vanish. Moreover, at order l for each relation we have a distinct renormalization constant. Consequently, the aforementioned relations have a solution. The existence of a solution ϱ_l independent of Λ_0 follows immediately if the limit $\lim_{\Lambda_0\to\infty} \zeta^{\Lambda_0}_{\Phi,l}$ exists: in this case it is enough to choose a solution of the following equations

$$c^\Phi + \varrho\, C^\Phi_1 \varrho + \zeta^\infty_\Phi = 0, \qquad\qquad c^\Phi + V^\Phi_1 \varrho + \zeta^\infty_\Phi = 0. \tag{421}$$

The convergence of $\zeta^{\Lambda_0}_{\Phi,l}$ relies on the validity of the bounds of theorem 2 up to order l for all irrelevant terms $\Gamma^{\vec{\phi};w}_{\vec{\varkappa}}$ and up to order $l-1$ for all the relevant ones. This property holds because in our inductive scheme at fixed loop order the irrelevant terms are treated before the relevant ones.

4.1 Γ^c_β

The renormalization point is $\vec{q} = (-\bar{q}, \bar{q}) \in \mathbb{M}^s_2$, see (170).

$$\Gamma^{c^a}_{\beta^b}(p) \overset{\beta}{\sim} \sigma_{0\Lambda_0}(p^2)\, \Gamma^{c^a \bar{c}^b}(p) + ip_\mu \Gamma^{c^a}_{\gamma^b_\mu}(p) \overset{\beta}{\sim} -\delta^{ab} f(p^2)\,, \tag{422}$$

$$f(x) := x\,(1 + \Sigma^{\bar{c}c}(x) - R_1(x))\,. \tag{423}$$

For the marginal term we obtain

$$-\frac{\delta^{ab}}{3}\, \Gamma^{c^a;p_\mu p_\nu}_{\beta^b}(p) \overset{\beta}{\sim} 2\delta_{\mu\nu} f'(p^2) + 4p_\mu p_\nu f''(p^2)\,. \tag{424}$$

The coefficient of $\delta_{\mu\nu}$ is *small* at the renormalization point iff

$$f' = 1 - r^{\bar{c}c} - R_1 - \zeta^{\Lambda_0}_{\beta c} \overset{\beta}{\sim} 0, \qquad\qquad \zeta^{\Lambda_0}_{\beta c}(p^2) := p^2 \frac{\partial R_1(p^2)}{\partial p^2}\,. \tag{425}$$

This gives the renormalization condition for R_1.

4.2 Γ^{cA}_{β}

The renormalization point is $\vec{q} = (\bar{k}, \bar{p}, \bar{q}) \in \mathbb{M}^s_3$, see (170). With $k = -p - q$,

$$\Gamma^{c^a A^d_\mu}_{\beta^b}(p,q) \overset{\beta}{\sim} \Gamma^{A^d_\mu c^a \bar{c}^b}(p,k) - ik_\rho \Gamma^{c^a A^d_\mu}_{\gamma^b_\rho}(p,q) \tag{426}$$

$$= i\epsilon^{dab} \mathcal{I}^{0\Lambda_0}_\mu(p,q)\,, \tag{427}$$

$$\mathcal{I}^{0\Lambda_0}_\mu(p,q) = k_\mu R^{A\bar{c}c}_1(k,p) + p_\mu r^{A\bar{c}c}_2(k,p) - gk_\rho F^{\gamma Ac}_{\rho\mu}(q,p)\,. \tag{428}$$

Let $\Delta^{\Lambda\Lambda_0}_{\mu\nu} := \mathcal{I}^{\Lambda\Lambda_0;p_\nu}_\mu - \mathcal{I}^{\Lambda\Lambda_0;q_\nu}_\mu$. At zero external momenta and $\Lambda = M$ we have $\Delta^{M\Lambda_0}_{\mu\nu}(0) = 0$. Then using the bounds of theorem 3 we get

$$|\Delta^{M\Lambda_0}_{\mu\nu}(\vec{q})| \leqslant \int_0^1 dt\, M |\partial\partial\Gamma^{M\Lambda_0;cA}_{\beta}| \overset{\beta}{\sim} 0\,. \tag{429}$$

The term $\Delta^{\Lambda\Lambda_0}_{\mu\nu}$ obeys the FE, see (152). It remains to integrate the FE from 0 to Λ and use inequality (620) to obtain

$$\Delta^{0\Lambda_0}_{\mu\nu}(\vec{q}) - \Delta^{M\Lambda_0}_{\mu\nu}(\vec{q}) \overset{\beta}{\sim} 0, \quad \text{and thus} \quad \Delta^{0\Lambda_0}_{\mu\nu}(\vec{q}) \overset{\beta}{\sim} 0\,. \tag{430}$$

Hence in the monomial basis $\{\delta^s Q^k\}_2$ with $Q = (\bar{p}, \bar{q})$ the δ-component of $\mathcal{I}^{A\bar{c}c;q_\nu}_\mu$ is *small* at the renormalization point if the following condition holds

$$R^{A\bar{c}c}_1 - gR_2 - \zeta^{\Lambda_0}_{\beta cA} \overset{\beta}{\sim} 0\,. \tag{431}$$

This gives the renormalization condition for $R^{A\bar{c}c}_1$.

4.3 Γ^{cAA}_{β} and $\Gamma^{cc\bar{c}}_{\beta}$

The renormalization point is $\vec{q} = (\bar{k}, \bar{l}, \bar{p}, \bar{q}) \in \mathbb{M}^s_4$, see (170).

$$\Gamma^{c^a A^t_\mu A^s_\nu}_{\beta^b}(l,p,q) \overset{\beta}{\sim} \Gamma^{c^a \bar{c}^b A^t_\mu A^s_\nu}(k,p,q) - ik_\rho \Gamma^{c^a A^t_\mu A^s_\nu}_{\gamma^b_\rho}(l,p,q) \tag{432}$$

$$\Gamma^{c^a c^t \bar{c}^s}_{\beta^b}(l,p,q) \overset{\beta}{\sim} \Gamma^{c^a \bar{c}^b c^t \bar{c}^s}(k,p,q) - ik_\rho \Gamma^{c^a c^t \bar{c}^s}_{\gamma^b_\rho}(l,p,q) \tag{433}$$

At $\Lambda = M$ it follows from property (187) that these terms vanish at zero momenta. Denoting the renormalization point by $\vec{q}$, using the bounds of theorem 3 and integrating the FE from M to 0 we obtain

$$|\Gamma_{\beta}^{M\Lambda_0;\vec{\phi}}(\vec{q})| \leqslant \int_0^1 dt\, |\vec{q}||\partial\Gamma_{\beta}^{M\Lambda_0;\vec{\phi}}(t\vec{q})| \overset{\beta}{\sim} 0\,, \tag{434}$$

$$|\Gamma_{\beta}^{0\Lambda_0;\vec{\phi}}(\vec{q}) - \Gamma_{\beta}^{M\Lambda_0;\vec{\phi}}(\vec{q})| \leqslant \frac{M}{\Lambda_0}\mathcal{P}^{(1)}_{2(l-1)}(\log_+ \frac{\Lambda_0}{M}) \overset{\beta}{\sim} 0\,. \tag{435}$$

4.4 $\Gamma_1^{c\bar{c}c\bar{c}c}$, Γ_1^{cAAAA} and $\Gamma_{1\omega}^{ccc}$

These functions do not have nonvanishing marginal terms:

$$\epsilon^{dab}\langle \check{c}^d \check{\bar{c}}^s \check{c}^a \check{\bar{c}}^s \check{c}^b\rangle = 0, \quad \epsilon^{dab}\langle \check{c}^d \check{A}^a_\mu \check{A}^b_\nu \check{A}^s_\mu \check{A}^s_\nu\rangle = 0, \quad \langle \omega^a c^a c^b c^b\rangle = 0\,. \tag{436}$$

4.5 $\Gamma_1^{\bar{c}cAA}$

From equation (187) it follows that for $\Lambda = M$ the function vanishes if the antighost momentum is zero. Using the bounds of theorem 4 first we obtain at the renormalization point $|\Gamma_1^{M\Lambda_0;\bar{c}cAA}(\vec{q})| \sim 0$ where $\vec{q} \in \mathbb{M}^s_5$, and then integrating the FE from M to 0 we show that the term is *small* at $\Lambda = 0$.

4.6 Γ_1^{cA}

The renormalization point is $\vec{q} = (-\bar{q}, \bar{q}) \in \mathbb{M}^s_2$, see (170).

$$\Gamma_1^{c^a A^b_\mu}(p) \sim i\delta^{ab} F^{AA}_{T;\mu\nu}(p) R_1 p_\nu = i\delta^{ab} p_\mu f(p^2)\,, \tag{437}$$

$$f(x) := \frac{1}{\xi} x R_1(x) \Sigma^{AA}_L(x)\,. \tag{438}$$

The marginal term satisfies

$$\Gamma_1^{c^a A^b;ppp}(p) \sim i\delta^{ab}\Big(2f'(p^2)(\Sigma_{t\in\{\delta^2\}_4} t) + 4f''(p^2)(\Sigma_{t\in\{\delta pp\}_4} t) + 8pppp f'''(p^2)\Big)\,.$$

For the coefficient of δ-tensors we have

$$\xi f'(p^2) = R_1(p^2)\Big(\Sigma^{AA}_L(p^2) + p^2\frac{\partial \Sigma^{AA}_L(p^2)}{\partial p^2}\Big) + p^2 \Sigma^{AA}_L(p^2)\frac{\partial R_1(p^2)}{\partial p^2}\,. \tag{439}$$

Recalling the definition of $r^{AA}_{1,2}$ in appendix E,

$$r^{AA}_2(p^2) + r^{AA}_1(p^2) + p^2\frac{\partial r^{AA}_2(p^2)}{\partial p^2} = \frac{1}{\xi}\Big(\Sigma^{AA}_L(p^2) + p^2\frac{\partial \Sigma^{AA}_L(p^2)}{\partial p^2}\Big)\,. \tag{440}$$

We then obtain the following sufficient condition

$$u^{cA} \sim 0 \Longleftrightarrow R_1(r_2^{AA} + r_1^{AA}) + \zeta_{cA}^{\Lambda_0} \sim 0\,, \tag{441}$$

where

$$\zeta_{cA}^{\Lambda_0}(p^2) := p^2\Big(R_1(p^2)\frac{\partial r_2^{AA}(p^2)}{\partial p^2} + \frac{1}{\xi}\Sigma_L^{AA}(p^2)\frac{\partial R_1(p^2)}{\partial p^2}\Big)\,. \tag{442}$$

See appendix G for the definition of u^{cA}. Relation (441) gives us the renormalization condition for r_2^{AA}.

4.7 Γ_1^{cAA}

The renormalization point is $\vec{q} = (\bar{k}, \bar{p}, \bar{q}) \in \mathbb{M}_3^s$, see (170). With $k = -p - q$,

$$\Gamma_1^{c^s A_\mu^a A_\nu^b}(p,q) \sim \Gamma^{A_\rho^t A_\mu^a A_\nu^b}(p,q)\Gamma_{\gamma_\rho^t}^{c^s}(k) + \sum_{Z_2} F_{T;\mu\rho}^{AA}(p)\Gamma_{\gamma_\rho^a}^{c^s A_\nu^b}(k,q)\,, \tag{443}$$

where the sum $\sum_{Z_2}$ runs over all cyclic permutations of $\{(\mu,p,a),(\nu,q,b)\}$. The marginal terms are: $\Gamma_1^{cAA;pp}$, $\Gamma_1^{cAA;pq}$, $\Gamma_1^{cAA;qq}$. Using equation (417) we see that $u_2^{cAA} \sim 0 \implies u_1^{cAA} \sim 0$. Acting with $\partial_p\partial_p$ on both sides of (443) we obtain

$$u_2^{cAA} \sim 0 \qquad \Longleftrightarrow \qquad gR_2\left(1 + r_1^{AA}\right) - 2R_1R^{AAA} + \zeta_{cAA}^{\Lambda_0} \sim 0\,. \tag{444}$$

This gives the renormalization condition for R_2.

4.8 $\Gamma_1^{\bar{c}ccA}$ and $\Gamma_{1\gamma}^{ccA}$

The renormalization point is $\vec{q} = (\bar{l}, \bar{k}, \bar{q}, \bar{p}) \in \mathbb{M}_4^s$, see (170). With $l = -k - q - p$,

$$\begin{aligned}\Gamma_1^{\bar{c}^a c^b c^d A_\mu^s}(k,q,p) \sim\, &\Gamma^{c^d\bar{c}^a A_\mu^s A_\rho^t}(l,p,k)\Gamma_{\gamma_\rho^t}^{c^b}(k) - \Gamma^{c^b\bar{c}^a A_\mu^s A_\rho^t}(l,p,q)\Gamma_{\gamma_\rho^t}^{c^d}(q)\\ &+ \Gamma^{A_\rho^t c^d \bar{c}^a}(q,l)\Gamma_{\gamma_\rho^t}^{A_\mu^s c^b}(p,k) - \Gamma^{A_\rho^t c^b\bar{c}^a}(k,l)\Gamma_{\gamma_\rho^t}^{A_\mu^s c^d}(p,q)\\ &+ F_{T;\mu\rho}^{AA}(p)\Gamma_{\gamma_\rho^s}^{c^b c^d \bar{c}^a}(k,q,l) + \Gamma^{c^t\bar{c}^a A_\mu^s}(l,p)\Gamma_{\omega^t}^{c^b c^d}(k,q)\\ &+ \Gamma^{c^t\bar{c}^a}(l)\sigma_{0\Lambda_0}(l^2)\Gamma_{\omega^t}^{c^b c^d A_\mu^s}(k,q,p)\,.\end{aligned} \tag{445}$$

From equation (410) it follows that $u_4^{\bar{c}ccA} \sim 0$ and

$$u_1^{\bar{c}ccA} \sim 0 \implies u^{\gamma Acc} \sim 0, u_2^{\bar{c}ccA} \sim 0, u_3^{\bar{c}ccA} \sim 0\,. \tag{446}$$

Consequently, we need only one condition

$$u_1^{\bar{c}ccA} \sim 0 \qquad \iff \qquad g(R_2 - R_3)R_1^{A\bar{c}c} + \zeta^{\Lambda_0}_{\bar{c}ccA} \sim 0\,. \tag{447}$$

This gives the renormalization condition for R_3.

4.9 $\Gamma_1^{\bar{c}cc}$ and $\Gamma_{1\gamma}^{cc}$

From equations (410), (417) we have

$$u_1^{\bar{c}cA} \sim 0 \implies u^{\gamma Acc} \sim 0 \implies u^{\gamma cc} \sim 0 \implies u^{\bar{c}cc}_{i\in\{1,2\}} \sim 0. \tag{448}$$

Consequently, the marginal contribution to the functions is *small*.

4.10 Γ_1^{cAAA}

The renormalization point is $\vec{q} = (\bar{l}, \bar{k}, \bar{q}, \bar{p}) \in \mathbb{M}_4^{cp}$, see (171). With $l = -k - q - p$,

$$\begin{aligned}\Gamma_1^{c^s A^t_\mu A^b_\nu A^d_\rho}(p,q,k) \sim &\Gamma^{A^a_\sigma A^t_\mu A^b_\nu A^d_\rho}(p,q,k)\Gamma^{c^s}_{\gamma^a_\sigma}(l) \\ &+ \sum_{Z_3} \Gamma^{A^a_\sigma A^t_\mu A^b_\nu}(p,q)\Gamma^{A^d_\rho c^s}_{\gamma^a_\sigma}(k,l) \\ &+ \sum_{Z_3} F^{AA}_{T;\rho\alpha}(k)\Gamma^{c^s A^t_\mu A^b_\nu}_{\gamma^d_\alpha}(l,p,q),\end{aligned} \tag{449}$$

where F^{AA}_T is defined in (701), and the sum $\sum_{Z_3}$ runs over all cyclic permutations of $\{(d,\rho,k),(b,\nu,q),(t,\mu,p)\}$. From (418) it follows that we need two equations

$$u_1^{cAAA} \sim 0 \qquad \iff \qquad 8R_1R_2^{AAAA} - 4gR_2R^{AAA} + \zeta^{\Lambda_0}_{cAAA,1} \sim 0\,, \tag{450}$$
$$u_3^{cAAA} \sim 0 \qquad \iff \qquad 4R_1R_1^{AAAA} + 2gR_2R^{AAA} + \zeta^{\Lambda_0}_{cAAA,3} \sim 0\,. \tag{451}$$

These equations give the renormalization conditions for $R^{AAAA}_{1,2}$.

5 Vertex functions with antighosts

In this section we present an extension of the bounds of theorem 1 which allows us to bypass proposition 18 and at the same time to impose renormalization conditions on the marginal terms with antighosts in the limit $\Lambda \to 0$. So we are able to impose all renormalization conditions at the physical value $\Lambda = 0$, i.e. in a scheme which is independent of the IR regulator. Moreover proposition 18 which we have used above in (285), (287), (380) follows directly from the bounds of theorem 5.

Hypothesis RC4 *Let $q_i \in \mathbb{R}^4$. We assume that*

$$\Gamma^{0\Lambda_0;c\bar{c}c\bar{c}}(0,q_2,q_3) = 0, \qquad \partial_A \Gamma^{0\Lambda_0;c\bar{c}A}(0,q_2) = 0\,, \tag{452}$$

$$\Gamma^{0\Lambda_0;c\bar{c}AA}(0,q_2,q_3) = 0\,, \tag{453}$$

where $\Gamma^{0\Lambda_0}$ stands for the limit $\lim\limits_{\lambda\to 0} \Gamma^{\lambda\Lambda_0}$.

To construct the bounds we slightly change the definition of the ρ-weight for external edges with ghosts E_c, in this section.

Definition 32 *Fix a tree from $\mathbb{T}_{\vec{\phi}}$. A ρ-weight is a function $\rho : E \to \{0,1,2\}$ with the following properties:*

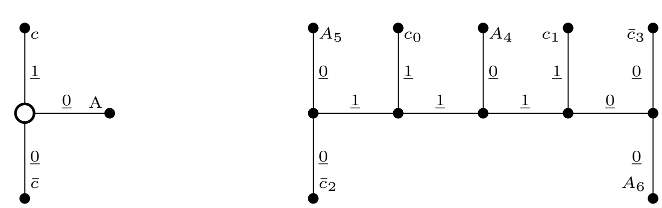
1. $\forall e \in E_1,\ \rho(e) = \begin{cases} 1 & e \in E_c, \\ 0 & \text{otherwise.} \end{cases}$

2. *There exists a map $\chi : V_3 \backslash V_\circ \to E \backslash E_1$ such that*

   a) *if $\chi(v) = e$, then v and e are incident,*

   b) $\forall e \in E\backslash E_1,\ \rho(e) = 2 - |\chi^{-1}(\{e\})|\,.$

Below we give two elements of $\mathcal{T}_{c\bar{c}A}$ and $\mathcal{T}_{cc\bar{c}\bar{c}AAA}$ and the corresponding ρ-weights.



As a consequence of the change of the definition of the ρ-weight we also change the definition of $Q_\tau^{\Lambda;w}$ for trees with ghosts $\tau \in \mathcal{T}_{c\bar{c}\vec{\phi}}$:

$$Q_\tau^{\Lambda;w}(\vec{p}) := \sum_{\bar{w}+\tilde{w}=w} \prod_{v\in V_{\bar{c}}} |p_v|^{1-\bar{w}_v} \begin{cases} 1 & \tilde{w}=0 \text{ and } \tau \in \mathcal{T}_{c\bar{c}A}, \\ \tilde{Q}_\tau^{\Lambda;\tilde{w}}(\vec{p}) & \textit{otherwise}, \end{cases} \tag{454}$$

$$\bar{w}_v \leqslant \begin{cases} 1, & v \in V_{\bar{c}}, \\ 0, & v \in V_1 \backslash V_{\bar{c}}, \end{cases} \tag{455}$$

where $\tilde{Q}_\tau^{\Lambda;\tilde{w}}$ is now given by (191).

Theorem 5 *There exist regular vertex functions $\Gamma_l^{\vec{\phi}}$ with $\vec{\phi} = (c_0, \bar{c}_1, ..., \phi_{n-1})$ complying with the global symmetries of the theory, satisfying the FE and the renormalization conditions given by hypotheses RC1 and RC4, and with irrelevant terms vanishing at $\Lambda = \Lambda_0$. Furthermore, for all $l \in \mathbb{N}$, $w \in \mathbb{W}_n$, the following bounds hold on $\mathbb{Y}_n^+$:*

a)

$$|\Gamma_l^{\Lambda\Lambda_0;c\bar{c};w}(p)| \leqslant \begin{cases} |p|(\Lambda + |p|) P_r^\Lambda(p)\,, & w = 0, \\ (\Lambda + |p|)^{2-\|w\|} P_r^\Lambda(p)\,, & \textit{otherwise}. \end{cases} \tag{456}$$

b)

$$|\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \sum_{\tau\in\mathcal{T}_{\vec{\phi}}} Q_\tau^{\Lambda;w}(\vec{p})\; P_r^\Lambda(\vec{p})\,. \tag{457}$$

See theorem 1 for notations.

In the following sections we give the proof of the theorem.

5.1 Chains and junctions

In analogy with (243) we get for the loop integral

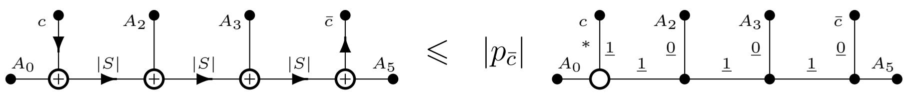
$$\int \dot{\mathbf{C}}_{\zeta\bar{\zeta}}^{\Lambda\Lambda_0} \hat{\mathcal{S}}^{\Lambda;\mathring{\Upsilon};\mathsf{w}}\, P_{2(l-1)}^\Lambda \leqslant \Lambda \left(\mathcal{R}_{A\bar{A}}(\hat{\mathcal{S}}^{\Lambda;\mathring{\Upsilon};\mathsf{w}}) + \Lambda \mathcal{R}_{c\bar{c}}(\hat{\mathcal{S}}^{\Lambda;\mathring{\Upsilon};\mathsf{w}+1_{\bar{c}}})\right) P_{2(l-1)}^\Lambda\,. \tag{458}$$

It follows directly from the bounds of theorem 5 that

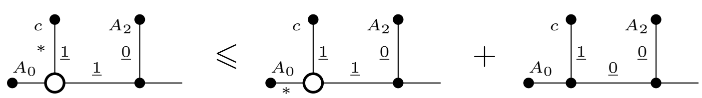
$$\Lambda \mathcal{R}_{c\bar{c}}(\hat{\mathcal{S}}^{\Lambda;c\vec{\Psi}\bar{c};\mathsf{w}+1_{\bar{c}}}) = \mathcal{R}_{AA}(\hat{\mathcal{S}}^{\Lambda;A\vec{\Psi}A;\mathsf{w}}). \tag{459}$$

Consequently we can assume that the edges joined by $\dot{\mathbf{C}}^{\Lambda\Lambda_0}$ are always of bosonic type.

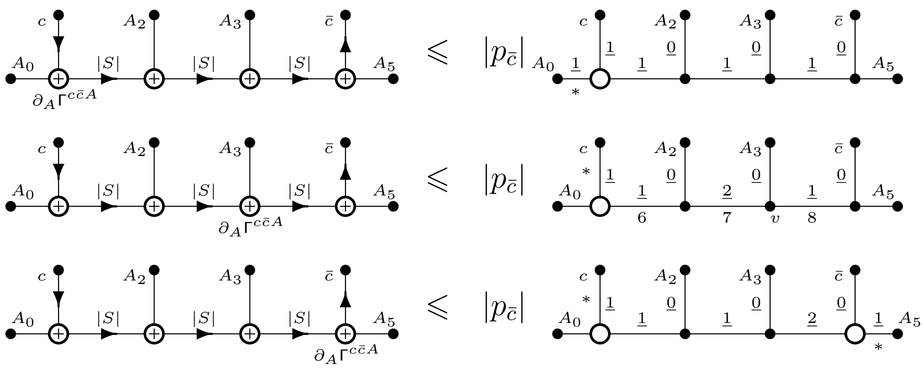
As before, see (226), it is easy to realize that the tree structure is spoiled by three-point vertex functions. Given a subchain of three-point vertex functions $\Gamma^{c\bar{c}A}$ we obtain



Here on the lhs the $\oplus$-vertices stand for the elements $\hat{S}^{cA_j\bar{c}}$ of a reduded chain. The incomming and outgoing arrows denote edges corresponding to ghost and antighost respectively. Moreover in the chain of vertex functions on the left an outgoing edge corresponds to the antighost momentum $|p_e|$ in the numerator of the amplitude $\hat{S}^{cA_j\bar{c}}$. The $|S|$'s should be substituted with the corresponding bounds on the ghost propagator. But compared to (229) the * label appears on an edge $e_c \in E_c$ which is external to the loop integral. Then using the inequality $\Lambda+|p_c| \leqslant (\Lambda+|p_A|)+(\Lambda+|p_{\bar{c}}|)$ it is easy to see that in this case we can bound the chain by the sum of the following fragments.



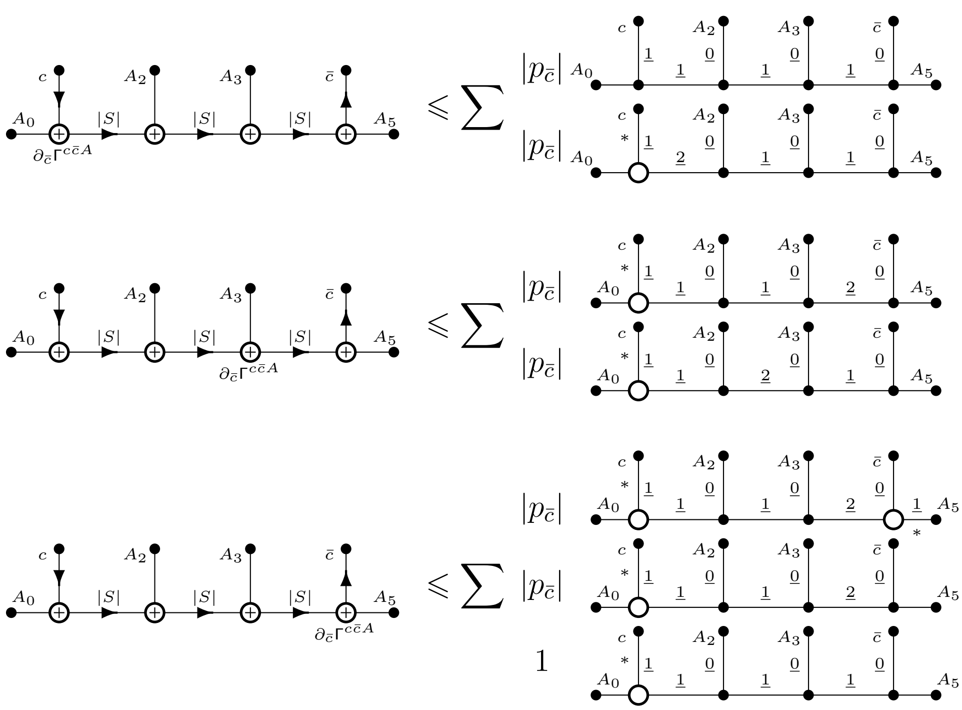
Furthermore due to the diminishing of the multiindex w in the case of 3-point functions we shall verify the compatibility of the bound on the terms $\partial^w \Gamma^{c\bar{c}A}$ with the definition of fragments, see (191) where we pass from $w \to w'$ for $|V_1| = 3$. It is not obvious that a chain of such terms can be bounded by fragments. Certainly the bound on a chain which involves terms with second or higher order of derivatives can be obtained from bounds on chains including terms with first order derivatives by adjusting the σ-weights of corresponding edges. Consequently we shall consider chains containing terms $\partial_A \Gamma^{c\bar{c}A}$ and $\partial_{\bar{c}} \Gamma^{c\bar{c}A}$. Moreover for a given chain such a term can appear in three different places: on the left, in the middle or on the right. First we give the corresponding examples with a $\partial_A \Gamma^{c\bar{c}A}$ term. Generalization of these examples to the case with arbitrary number of vertex functions is then straightforward.



In the second example the distribution of the ρ-weights depends on the position of the edge carrying momentum p_0 wrt to the vertex v:

- If the corresponding derivative can be applied on the left, see (172), then $\chi(v) = e_7$, $\rho(e_7) = 1$, $\rho(e_8) = 1$ and $\sigma(e_7) = 1$.
- Otherwise $\chi(v) = e_8$, $\rho(e_7) = 2$, $\rho(e_8) = 0$ and $\sigma(e_8) = 1$.

The remaining chains with $\partial_{\bar{c}} \Gamma^{c\bar{c}A}$ are given below:



Here in the last example depending on the position of the p_0-edge in the chain, we choose either the first or the second fragment.

5.2 Irrelevant terms

First we consider the irrelevant terms $\Gamma^{c\bar{c};w}$ where $\|w\| > 2$.

$$|\Gamma_l^{\Lambda\Lambda_0;c\bar{c};w}| \leqslant \int_\Lambda^{\Lambda_0} d\lambda \, |\dot{\Gamma}_l^{\lambda\Lambda_0;c\bar{c};w}| \leqslant \sum_{\tau \in \mathcal{T}_{\zeta\bar{\zeta}c\bar{c}}} \int_\Lambda^{\Lambda_0} d\lambda \, \lambda \, \mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w}) \, P_{2(l-1)}^\lambda \Big|_{\eta=0}. \tag{460}$$

Since $|p_c| = |p_{\bar{c}}|$ substituting the bounds from theorem 5 we get

$$\mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w}(\vec{p})) = \frac{|p_{\bar{c}}|}{(\lambda + |p_{\bar{c}}|)^{\|w\|+1}} + \frac{1}{(\lambda + |p_{\bar{c}}|)^{\|w\|}} \leqslant \frac{2}{(\lambda + |p_{\bar{c}}|)^{\|w\|}}. \tag{461}$$

To bound the integral we use inequalities (591) and (601)

$$|\Gamma_l^{\Lambda\Lambda_0;c\bar{c};w}(\vec{p})| \leqslant \frac{P_{2l-1}^\Lambda(\vec{p})}{(\Lambda + |p_{\bar{c}}|)^{\|w\|-2}}. \tag{462}$$

For any tree $\tau \in \mathcal{T}_{\zeta\bar{\zeta}c\bar{c}\vec{\phi}}$ with $|\vec{\phi}| \geqslant 1$ the total weight satisfies $\theta(\tau) > 1$. Thus using the fact that there are no irrelevant counterterms, i.e. the vanishing boundary conditions $\Gamma^{\Lambda_0\Lambda_0;\vec{\phi};w} = 0$, we integrate the FE from Λ_0 to Λ and use proposition 29

$$|\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \sum_{\tau \in \mathcal{T}_{\zeta\bar{\zeta}\vec{\phi}}} \int_\Lambda^{\Lambda_0} d\lambda \, \lambda \, \mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w}) \, P_{2(l-1)}^\lambda \Big|_{\eta=0}. \tag{463}$$

Inequalities (591) and (601) yield

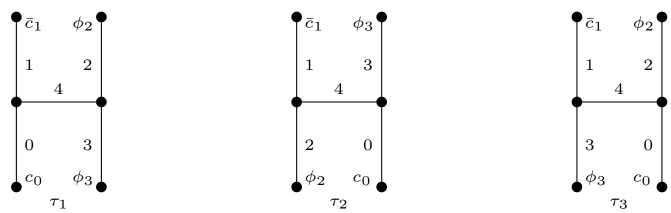
$$|\Gamma_l^{\Lambda\Lambda_0;\vec{\phi};w}(\vec{p})| \leqslant \sum_{\tau \in \mathcal{T}_{\vec{\phi}}} Q_\tau^{\lambda;w} \, P_{2l-1}^\lambda(\vec{p}). \tag{464}$$

5.3 $\Gamma^{c\bar{c}AA}$ and $\Gamma^{c\bar{c}c\bar{c}}$

Using hypothesis RC4 and integrating the FE from 0 to Λ we see that at the renormalization point $\Gamma_l^{\Lambda\Lambda_0;c\bar{c}\phi\phi} = 0$. Then keeping the first momentum argument vanishing we interpolate the two other momenta to arbitrary configuration and obtain $\Gamma_l^{\Lambda\Lambda_0;c\bar{c}\phi\phi}(0,p_2,p_3) = 0 \; \forall p_2, p_3 \in \mathbb{R}^4,\ \Lambda > 0$. Thus in this section we shall prove the bounds of theorem 5 on the remaining interpolation

$$|\Gamma_l^{\Lambda\Lambda_0;c\bar{c}\phi\phi}(\vec{p})| \leqslant \int_0^{p_1} dl \; |\partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}\phi\phi}(l_1,p_2,p_3)|. \tag{465}$$

First we substitute the irrelevant terms $\partial_{\bar{c}}\Gamma_l^{c\bar{c}\phi^2}$ with the bounds of theorem 5.

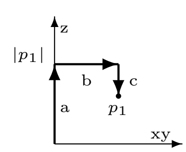


$$Q_{c\bar{c}AA}^{\Lambda;(0,1,0,0)} \leqslant \frac{2}{\Lambda+|l_0|}\Big(1+\sum_q \frac{|l_1|}{\Lambda+|l_1+q|}\Big), \tag{466}$$

$$Q_{c\bar{c}c\bar{c}}^{\Lambda;(0,1,0,0)} \leqslant \frac{|p_3|}{\Lambda+|p_2|}\frac{2}{\Lambda+|l_0|}\Big(1+\sum_q \frac{|l_1|}{\Lambda+|l_1+q|}\Big). \tag{467}$$

Here $\mathcal{T}_{c\bar{c}\phi^2} = \{\tau_1,\tau_2,\tau_3\}$ and the sums run over three elements with $q \in \{p_3, p_2, p_2+p_3\}$.

Next we introduce an orthonormal basis e_x, e_y, e_z such that the vectors p_2, p_3 belong to the xy-plane and $p_1 e_x = 0$, $p_1 e_z \geqslant 0$, $p_1 e_y \geqslant 0$. And then we construct the integration path from three linear segments: a, b, c.



Denote by $I_\alpha^{c\bar{c}\phi\phi}$ the integration over the corresponding segment $\alpha \in \{a,b,c\}$

$$I_\alpha^{c\bar{c}\phi\phi} := \int_\alpha dl \; |\partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}\phi\phi}(l,p_2,p_3)|. \tag{468}$$

Note that $Q^{\Lambda;(0,1,0,0)}_{c\bar{c}c\bar{c}}$ and $Q^{\Lambda;(0,1,0,0)}_{c\bar{c}AA}$ differ only by a rational factor which remains constant on the integration path.

$$Q^{\Lambda;(0,1,0,0)}_{c\bar{c}c\bar{c}} = \frac{|p_3|}{\Lambda + |p_2|} Q^{\Lambda;(0,1,0,0)}_{c\bar{c}AA}. \tag{469}$$

Consequently, without loss we can consider only the $\Gamma^{c\bar{c}AA}$ function. Let $\eta := \eta(\vec{p})$ and $\eta_t := \eta(l_1(t), l_0(t), p_2, p_3)$ where $l_1(t) : [0,1] \to \mathbb{R}$ is an appropriate function on the corresponding segment, see the figure given above, and $l_0(t) = -(l_1(t) + p_2 + p_3)$.
I_a) $\sqrt{2}|l_1(t) + q| \geqslant t|p_1| + |q|$, $\forall q \in \{p_2, p_3, p_2 + p_3\}$. This implies

$$Q^{\Lambda;(0,1,0,0)}_{c\bar{c}AA} \leqslant \frac{2\sqrt{2}(1 + 3\sqrt{2})}{\Lambda + |l_1| + |p_2 + p_3|}\,, \qquad \eta_t \geqslant \frac{t\eta}{\sqrt{2}}. \tag{470}$$

Substituting it into (468) and then using (574), (585) we obtain

$$\begin{aligned} I_a^{c\bar{c}AA} &\leqslant \int_0^1 dt \frac{|p_1|\, P^{\Lambda}_{2l-1}|_{t\eta}}{\Lambda + |p_2 + p_3| + |p_1|t} \leqslant \log\Big(1 + \frac{|p_1|}{\Lambda + |p_1 + p_2|}\Big) P^{\Lambda}_{2l-1} \\ &\leqslant \frac{|p_1|}{\Lambda + |p_0|}\Big(1 + \log_+ \frac{|p_1|}{\Lambda + |p_2 + p_3|}\Big) P^{\Lambda}_{2l-1} \leqslant \frac{|p_1|}{\Lambda + |p_0|} P^{\Lambda}_{2l}\,. \end{aligned} \tag{471}$$

I_b) $|p_1| \leqslant |l_1(t)| \leqslant \sqrt{2}|p_1|$, $|l_1(t) + p_1| \geqslant |p_1|$, $|l_1(t) + p_2| \geqslant |p_1|$ give

$$Q^{\Lambda;(0,1,0,0)}_{c\bar{c}AA} \leqslant \frac{\sqrt{2}(1 + 2\sqrt{2})}{\Lambda + |l_0(t)|} + \frac{2|l_1(t)|}{(\Lambda + |l_0(t)|)^2}. \tag{472}$$

Let $p_{23} = p_2 + p_3$. If $p_1 p_{23} \geqslant 0$ then $l_1 p_{23} \geqslant 0$ and

$$l_0^2(t) \geqslant p_1^2 + p_{23}^2 + 2 l_1 p_{23} \geqslant \frac{(|p_1| + |p_{23}|)^2}{2} \geqslant \frac{p_0^2}{2}\,. \tag{473}$$

If $p_1 p_{23} < 0$ then $p_1 p_{23} \leqslant l_1 p_{23}$ and

$$l_0^2(t) \geqslant p_1^2 + p_{23}^2 + 2 l_1 p_{23} \geqslant p_0^2 + 2(l_1 p_{23} - p_1 p_{23}) \geqslant p_0^2\,. \tag{474}$$

From (473) and (474) it follows that $\sqrt{2}|l_0| \geqslant |p_0|$. Hence $\sqrt{2}\eta_t \geqslant \eta$. Note that $l_0^2(t) = (p_{23} + (t p_1 e_y) e_y)^2 + p_1^2 \geqslant p_1^2$. Consequently we obtain

$$I_b^{c\bar{c}AA} \leqslant \frac{|p_1|}{\Lambda + |p_0|} P^{\Lambda}_{2l-1}. \tag{475}$$

I_c) Let $l_1(t) - p_1 = e_z t\Delta$ where $\Delta \geqslant 0$. Then $\forall q \in \{p_1, p_2, p_1+p_2\}$

$$(l_1(t)+q)^2 = (l_1(t)-p_1+p_1+q)^2 = t^2\Delta^2 + 2t\Delta(e_z p_1) + (p_1+q)^2. \quad (476)$$

This implies $|l_0| \geqslant |p_0|$, $\eta_t \geqslant \eta$ and

$$(l_1(t)+q)^2 \geqslant t^2\Delta^2 + (p_1+q)^2 \geqslant t^2\Delta^2 + \eta^2 \geqslant \frac{(t\Delta+\eta)^2}{2}\,. \quad (477)$$

Then using $l_1 \leqslant \sqrt{2}|p_1|$ and $\Delta \leqslant |p_1|$ we obtain

$$Q^{\Lambda;(0,1,0,0)}_{c\bar{c}AA} \leqslant \frac{2}{\Lambda+|p_0|}\Big(1+\frac{6|p_1|}{\Lambda+\eta+\Delta t}\Big), \quad (478)$$

$$I^{c\bar{c}AA}_c \leqslant \int_0^1 dt\,\Delta\, Q^{\Lambda;(0,1,0,0)}_{c\bar{c}AA} P^{\Lambda}_{2l-1} \leqslant \frac{|p_1|\tilde{P}^{\Lambda}_{2l-1}}{\Lambda+|p_0|}\Big(1+\log_+\frac{\Delta}{\Lambda+\eta}\Big). \quad (479)$$

Finally, we see that

$$I^{c\bar{c}AA}_c \leqslant \frac{|p_1|}{\Lambda+|p_0|} P^{\Lambda}_{2l}. \quad (480)$$

5.4 $\partial_{\bar{c}}\Gamma^{c\bar{c}A}$

Let $\vec{q} \in \mathbb{M}^s_3$ denote the renormalization point. It follows from lemma 33 that for a renormalization point $\underline{q} := (q_1, q_2)$ all the monomial $\{\delta^s \underline{q}^k\}_2$ and $\{\delta^s \underline{q}^k\}_3$ are linearly independent. Consequently each of them forms a basis. The coefficient of the δ-tensor in the monomial decomposition of $\partial_{\bar{c}}\Gamma^{c\bar{c}A}$ is fixed by the renormalization condition and the remaining coefficients by lemma 35 coincide with coefficients of decomposition of the irrelevant terms $\partial_A\partial_{\bar{c}}\Gamma^{c\bar{c}A}$ and $\partial_{\bar{c}}\partial_{\bar{c}}\Gamma^{c\bar{c}A}$. Using the bounds of theorem 5 on irrelevant terms we have

$$|\partial_{\bar{c}}\Gamma^{0\Lambda_0;c\bar{c}A}_{l}(\vec{q})| \leqslant c_1 + |\vec{q}|\left(Q^{0;(0,0,2)}_{c\bar{c}A} + Q^{0;(0,1,1)}_{c\bar{c}A}\right) P^0_{2l-1}(\vec{q}), \quad (481)$$

where

$$Q^{0;(0,1,1)}_{c\bar{c}A} \leqslant \frac{1}{|q_c|} + \frac{|q_{\bar{c}}|}{|q_c|}\left(\frac{1}{|q_c|} + \inf_{j\in\{1,2\}}\frac{1}{|q_j|}\right) = \frac{3}{M}\,, \quad (482)$$

$$Q^{0;(0,0,2)}_{c\bar{c}A} \leqslant \frac{1}{|q_c|} + \frac{|q_{\bar{c}}|}{|q_c|}\left(\frac{1}{|q_{\bar{c}}|} + \frac{1}{|q_c|}\right) = \frac{3}{M}\,, \quad (483)$$

$$P^0_{2l-1}(\vec{q}) \leqslant \mathcal{P}^{(0)}_{2l-1}(\log 3) + \mathcal{P}^{(1)}_{2l-1}(0) \leqslant c_2\,. \quad (484)$$

Thus $\forall l$, $\exists c_3 > 0$ such that $|\partial_{\bar{c}} \Gamma_l^{c\bar{c}A;0\Lambda_0}(\vec{q})| \leqslant c_3$.
Our next step consists in extending the bounds to arbitrary $\Lambda > 0$:

$$|\partial_{\bar{c}} \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q}) - \partial_{\bar{c}} \Gamma_l^{0\Lambda_0;c\bar{c}A}(\vec{q})| \leqslant \int_0^{\Lambda} d\lambda \, |\partial_{\bar{c}} \dot{\Gamma}_l^{\lambda\Lambda_0;c\bar{c}A}(\vec{q})|. \tag{485}$$

Here again we use the FE and substitute the bounds on chains with the tree bounds. On can see that $\exists c_4 > 0$ such that $\forall \tau \in \mathcal{T}_{c\bar{c}A\zeta\bar{\zeta}}$

$$\lambda \, \mathcal{R}_{\zeta\bar{\zeta}}(Q_\tau^{\lambda;w}(\vec{q})) \leqslant \frac{c_4}{\lambda + |q_c|} \left(\frac{|q_{\bar{c}}|}{\lambda + |q_{\bar{c}}|} + \frac{|q_{\bar{c}}|}{\lambda + |q_c|} + 1 \right) \leqslant \frac{3c_4}{\lambda + M} \, . \tag{486}$$

Then using (291) to bound the integral in (485) we obtain

$$|\partial_{\bar{c}} \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q})| \leqslant \mathcal{P}_{2l-1}^{(1)}\Big(\log_+ \frac{\Lambda}{M} \Big). \tag{487}$$

Next we rotate the vectors $\vec{q} \mapsto R\vec{q}$ to satisfy the following conditions:

- $p_1, p_2 \in \mathrm{span}(q_1, q_2)$,
- If $p_2 \neq 0$ then $\exists \alpha_2 > 0$ such that $\alpha_2 q_2 = p_2$,
- If $p_2 = 0$ then $\exists \alpha_1 \geqslant 0$ such that $\alpha_1 q_1 = p_1$,
- $(p_2 \wedge p_1)(q_2 \wedge q_1) \geqslant 0$.

$$|\partial_{\bar{c}} \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(R\vec{q}) - \partial_{\bar{c}} \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q})| \leqslant \int_0^{\pi} d\theta_i |q_i| |\partial_i \partial_{\bar{c}} \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}|. \tag{488}$$

Using the bounds of theorem 5 on the irrelevant terms

$$Q_{c\bar{c}A}^{\Lambda;(0,1,1)}(\vec{q}) \leqslant \frac{1}{\Lambda + |q_c|} + \frac{|q_{\bar{c}}|}{\Lambda + |q_c|} \left(\frac{1}{\Lambda + |q_c|} + \inf_{j \in \{1,2\}} \frac{1}{\Lambda + |q_j|} \right), \tag{489}$$

$$Q_{c\bar{c}A}^{\Lambda;(0,0,2)}(\vec{q}) \leqslant \frac{1}{\Lambda + |q_c|} + \frac{|q_{\bar{c}}|}{\Lambda + |q_c|} \left(\frac{1}{\Lambda + |q_c|} + \frac{1}{\Lambda + |q_{\bar{c}}|} \right), \tag{490}$$

we have

$$|\partial_i \partial_{\bar{c}} \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}| \leqslant \frac{1}{\Lambda + M} \mathcal{P}_{2l-1}^{(1)}(\log_+ \frac{\Lambda}{M}). \tag{491}$$

This means that

$$|\partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(R\vec{q}) - \partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q})| \leqslant \mathcal{P}_{2l-1}^{(1)}(\log_+ \frac{\Lambda}{M}). \qquad (492)$$

From now on we assume that $\vec{q}$ is already oriented as needed and denote by $q_i(t) = \alpha_i(t)q_i$ with $t \in [0,1]$ a linear dilatation such that $\alpha_i(0) = 1$ and $|p_i| = |q_i(1)|$. Then

$$|\partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q}(1)) - \partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q})| \leqslant \sum_{i=1}^{2} \int_0^1 dt\, \dot{\alpha}_i M |\partial_i \partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q}(t))|\,, \qquad (493)$$

where $\dot{\alpha}_i = \partial_t \alpha_i$. To proceed further with the integral on the rhs we need some elementary estimates. It is easy to see that

$$\max(|\vec{q}(t)|^2, M^2) \leqslant \sum_{i=0}^{2} \max(|q_i(t)|^2, M^2) \leqslant 3\max(|\vec{p}|^2, M^2). \qquad (494)$$

And it also follows from (589) of lemma 39 that

$$2\eta(\vec{q}(t)) \geqslant \min(|q_1(t)|, |q_2(t)|, M) \geqslant \min(|p_1|, |p_2|, M) \geqslant \eta(\vec{p}), \qquad (495)$$

$$\frac{1}{\Lambda + |q_c|} \leqslant \frac{2}{\Lambda + |q_i|}, \quad \text{and thus} \quad \frac{|q_{\bar{c}}|}{\Lambda + |q_c|} \leqslant 2. \qquad (496)$$

We substitute these inequalities into the bounds on the irrelevant terms

$$|\partial_i \partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q}(t))| \leqslant \frac{1}{\Lambda + \alpha_i(t)M} P_{2l-1}^{\Lambda}(\vec{p}), \qquad (497)$$

and obtain an upper bound on the integral in (493)

$$\left(1 + \log_+ \frac{|p_i|}{\Lambda + M}\right) P_{2l-1}^{\Lambda}(\vec{p}) \leqslant P_{2l}^{\Lambda}(\vec{p}). \qquad (498)$$

Rotation of q_1 towards p_1 will finish the proof. Let now $q_1(t) = R(t)q_1$ with boundary conditions $q_1(0) = q_1$, $q_1(1) = p_1$ and $q_0(t) := -(q_1(t) + q_2)$.

$$|\partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{p}) - \partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q})| \leqslant \frac{2}{3}\pi \int_0^1 dt\; |q_1| |\partial_{\bar{c}}^2 \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q}(t))|\,. \qquad (499)$$

The bounds on irrelevant terms have a form similar to (490)

$$|\partial_{\bar{c}}^2 \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{q}(t))| \leqslant \Big(\frac{1}{\Lambda+|q_0(t)|}+\frac{|q_1|}{(\Lambda+|q_0(t)|)^2}\Big) P_{2l-1}^{\Lambda}(\vec{q}(t)). \tag{500}$$

If $p_1p_2 \geqslant q_1p_2$ then lemma 39 implies

$$|q_0(t)| \geqslant \frac{|p_1|+|p_2|}{2} \implies |q_0(t)| \geqslant \frac{|p_i|}{2}, \quad \forall i \in \{0,1,2\}. \tag{501}$$

If $p_1p_2 \leqslant q_1p_2$ then

$$q_0^2(t) = p_0^2 + 2(q_1(t)p_2 - p_1p_2) \geqslant p_0^2. \tag{502}$$

Consequently,

$$\eta(\vec{q}(t)) \geqslant \frac{1}{2}\eta(\vec{p}), \quad \text{and thus} \quad P_{2l-1}^{\Lambda}(\vec{q}(t)) \leqslant \tilde{P}_{2l-1}^{\Lambda}(\vec{p})\,. \tag{503}$$

In the case $p_1p_2 \geqslant q_1p_2$ we obtain using (501)

$$\frac{|p_1|}{\Lambda+|q_0(t)|}+\frac{|p_1|^2}{(\Lambda+|q_0(t)|)^2} \leqslant 2+2^2. \tag{504}$$

Otherwise $p_1p_2 \leqslant q_1p_2$.
If $|p_1| \geqslant 2|p_2|$ then $|q_0(t)| \geqslant |p_1|$ and consequently

$$\frac{|p_1|}{\Lambda+|q_0(t)|}+\frac{|p_1|^2}{(\Lambda+|q_0(t)|)^2} \leqslant 1+1\,. \tag{505}$$

If $|p_1| < 2|p_2|$ then denoting the angle between q_1 and p_1 by $\beta : [0,1] \to [\beta_0, 0]$ where $\beta_0 \leqslant \frac{1}{3}\pi$ we have

$$|q_0(t)| \geqslant \frac{1}{\sqrt{2}}\Big(|p_0|+|p_1|\sin\frac{\beta}{2}\Big) \geqslant \frac{1}{\sqrt{2}}\Big(|p_0|+|p_1|\frac{1-t}{2}\Big). \tag{506}$$

Then it follows that

$$\int_0^1 \frac{|p_1|dt}{\Lambda+|q_0(t)|} \leqslant 2^{\frac{3}{2}} \int_0^{|p_1|} \frac{dx}{\Lambda+|p_0|+x} \leqslant 2^{\frac{3}{2}}\Big(\log 2+\log_+\frac{|p_1|}{\Lambda+|p_0|}\Big), \tag{507}$$

$$\int_0^1 \frac{|p_1|^2dt}{(\Lambda+|q_0(t)|)^2} \leqslant 8|p_1| \int_0^{|p_1|} \frac{dx}{(\Lambda+|p_0|+x)^2} \leqslant \frac{8|p_1|}{\Lambda+|p_0|}\,. \tag{508}$$

Using (504), (505), (507), (508) we obtain a bound on the rhs of (499)

$$P_{2l}^{\Lambda}(\vec{p}) + \frac{|p_1|}{\Lambda + |p_0|} P_{2l-1}^{\Lambda}(\vec{p}). \tag{509}$$

Finally we end this section by collecting together intermediate bounds in (487), (492), (498) and (509)

$$|\partial_{\bar{c}} \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{p})| \leqslant P_{2l}^{\Lambda}(\vec{p}) + \frac{|p_1|}{\Lambda + |p_0|} P_{2l-1}^{\Lambda}(\vec{p}). \tag{510}$$

5.5 $\partial_A \Gamma^{c\bar{c}A}$

Using the FE we extend the corresponding renormalization condition of hypothesis RC4 to arbitrary Λ and then interpolate to arbitrary p_2. As in section 5.3 this yields $\partial_A \Gamma_l^{c\bar{c}A;\Lambda\Lambda_0}(0, p_2) = 0$, $\forall p_2 \in \mathbb{R}^4$ and $\Lambda > 0$. Thus the goal of this section is to consider the remaining interpolation

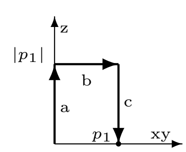
$$|\partial_A \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{p})| \leqslant \int_0^{p_1} dl_1 \, |\partial_{\bar{c}} \partial_A \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(l_1, p_2)| \, . \tag{511}$$

Using the bound on the irrelevant term $\partial_{\bar{c}} \partial_A \Gamma_l^{c\bar{c}A}$, see theorem 5, we have

$$Q_{c\bar{c}A}^{\Lambda;(0,1,1)}(\vec{p}) \leqslant \frac{1}{\Lambda + |p_0|} + \frac{|p_1|}{(\Lambda + |p_0|)^2}, \tag{512}$$

$$|\partial_{\bar{c}} \partial_A \Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}| \leqslant Q_{c\bar{c}A}^{\Lambda;(0,1,1)} P_{2l-1}^{\Lambda} \, . \tag{513}$$

We introduce an orthonormal basis e_x, e_y, e_z such that the vectors p_2, p_1 belong to the xy-plane and the integration path which lies in the zp_1 plane.



Denote by I_α with $\alpha \in \{a, b, c\}$ the integral over the corresponging segment. Let $\eta := \eta(\vec{p})$ and $\eta_t := \eta(l_0(t), l_1(t), p_2)$ where $l_1(t) : [0, 1] \to \mathbb{R}^4$ is an

appropriate function on one of these segments and $l_0(t) := -(l_1(t)+p_2)$.
$I_a)$ $\sqrt{2}|l_0(t)| \geqslant |p_2|+|p_1|t$. This implies

$$Q^{\Lambda;(0,1,1)}_{c\bar{c}A} \leqslant \frac{\sqrt{2}+2}{\Lambda+|p_2|+|p_1|t}\,, \qquad\qquad \eta_t \geqslant \frac{t\eta}{\sqrt{2}}. \tag{514}$$

Substituting these inequalities into (511) and using (574), (585) we get

$$\begin{aligned}I_a \leqslant \log_+\Big(1+\frac{|p_1|}{\Lambda+|p_2|}\Big)P^{\Lambda}_{2l-1}\\ \leqslant \frac{|p_1|}{\Lambda+|p_0|}\Big(1+\log_+\frac{|p_1|}{\Lambda+|p_0|}\Big)P^{\Lambda}_{2l-1}\\ \leqslant \frac{|p_1|}{\Lambda+|p_0|}P^{\Lambda}_{2l}.\end{aligned} \tag{515}$$

$I_b)$ $|p_1| \leqslant |l_1(t)| \leqslant \sqrt{2}|p_1|$, $|l_0(t)| \geqslant |p_1|$ and $\eta_t \geqslant \eta$. If $p_1p_2 \geqslant 0$ then

$$l_0^2(t) = l_1^2(t)+p_2^2+2l_1(t)p_2 \geqslant p_1^2+p_2^2 \geqslant \frac{p_0^2}{2}\,. \tag{516}$$

If $p_1p_2<0$ then $p_1p_2 \leqslant l_1(t)p_2$ and

$$l_0^2(t) = l_1^2(t)+p_2^2+2l_1(t)p_2 \geqslant p_0^2+2(l_1(t)p_2-p_1p_2) \geqslant p_0^2\,. \tag{517}$$

From inequalities (516), (517) it follows that $\sqrt{2}|l_0(t)| \geqslant |p_0|$. Consequently

$$I_b \leqslant \Big(\frac{\sqrt{2}|p_1|}{\Lambda+|p_0|}+\frac{2|p_1|^2}{(\Lambda+|p_1|)(\Lambda+|p_0|)}\Big)P^{\Lambda}_{2l-1} \leqslant \frac{|p_1|}{\Lambda+|p_0|}\tilde{P}^{\Lambda}_{2l-1}\,. \tag{518}$$

$I_c)$ $l_0^2(t) = p_0^2+p_1^2t^2 \implies \eta_t \geqslant \eta$. Furthermore $|l_1(t)| \leqslant \sqrt{2}|p_1|$. Hence

$$\frac{|p_1|}{\Lambda+|l_0(t)|} \leqslant \frac{\sqrt{2}|p_1|}{\Lambda+|p_0|+|p_1|t}\,, \tag{519}$$

$$\int_0^1 dt\,\frac{|p_1||l_1(t)|}{(\Lambda+|l_0(t)|)^2} \leqslant \int_0^1 dt\,\frac{2|p_1|^2}{(\Lambda+|p_0|+|p_1|t)^2} \leqslant \frac{2|p_1|}{\Lambda+|p_0|}\,. \tag{520}$$

Finally we obtain

$$I_c \leqslant \frac{|p_1|}{\Lambda+|p_0|}P^{\Lambda}_{2l-1}\,. \tag{521}$$

5.6 $\Gamma^{c\bar{c}A}$

By the same argument as in sections 5.3 and 5.5 we have $\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(0,p_2)=0$, $\forall p_2\in\mathbb{R}^4$ and $\Lambda>0$.

$$|\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(\vec{p})|\leqslant\int\limits_0^{p_1}dl_1\,|\partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(l_1,p_2)|. \tag{522}$$

Here the integration path and notations are the same as in section 5.5. Using the bound in (510) on the marginal term $\partial_{\bar{c}}\Gamma_l^{c\bar{c}A}$ we have

$$|\partial_{\bar{c}}\Gamma_l^{\Lambda\Lambda_0;c\bar{c}A}(l_1,p_2)|\leqslant P_{2l}^{\Lambda}+\frac{|l_1|}{\Lambda+|l_0|}P_{2l-1}^{\Lambda}\,. \tag{523}$$

I_a) $\sqrt{2}|l_0|\geqslant|p_2|+|p_1|t$ and $l_1(t)=|p_1|t$. It follows that

$$\frac{|l_1|}{\Lambda+|l_0|}\leqslant\sqrt{2},\qquad\qquad\eta_t\geqslant\frac{t\eta}{\sqrt{2}}. \tag{524}$$

Using inequality (578) we obtain $I_a\leqslant|p_1|P_{2l}^{\Lambda}$.
I_b) $|p_1|\leqslant|l_1(t)|\leqslant\sqrt{2}|p_1|$, $|l_0|\geqslant|p_1|$ and $\eta_t\geqslant\eta$. Consequently

$$\frac{|l_1|}{\Lambda+|l_0|}\leqslant\sqrt{2}\,,\quad\text{and thus}\quad I_b\leqslant|p_1|P_{2l}^{\Lambda}\,. \tag{525}$$

I_c) $l_0^2=p_0^2+p_1^2t^2\implies\eta_t\geqslant\eta$. Using $l_1\leqslant\sqrt{2}|p_1|$ we get

$$\int\limits_0^1 dt\frac{|p_1||l_1(t)|}{\Lambda+|l_0(t)|}\leqslant\int\limits_0^1 dt\frac{2|p_1|^2}{\Lambda+|p_0|+|p_1|t}\leqslant 2|p_1|\log\Big(1+\frac{|p_1|}{\Lambda+|p_0|}\Big)\,. \tag{526}$$

Finally,

$$I_c\leqslant|p_1|P_{2l}^{\Lambda}+|p_1|\log\Big(1+\frac{|p_1|}{\Lambda+|p_0|}\Big)P_{2l-1}^{\Lambda}\leqslant|p_1|\tilde{P}_{2l}^{\Lambda}. \tag{527}$$

Appendices

A Properties of Gaussian measures

In the following $d\nu = d\nu_C(A)d\nu_S(c,\bar{c})$ is the measure given in (66), (67)

$$d\nu_{C_1+C_2}(A)f(A) = d\nu_{C_1}(A_1)d\nu_{C_2}(A_2)f(A_1+A_2), \tag{528}$$

$$d\nu_{tC}(A)f(A) = d\nu_C(A)f(t^{\frac{1}{2}}A), \tag{529}$$

$$d\nu_C(A-\delta A) = d\nu_C(A)e^{-\frac{1}{2\hbar}\langle\delta A, C^{-1}_{\Lambda\Lambda_0}\delta A\rangle}e^{\frac{1}{\hbar}\langle A, C^{-1}_{\Lambda\Lambda_0}\delta A\rangle}, \tag{530}$$

$$\frac{d}{d\Lambda}d\nu_C f(A) = \frac{1}{2}d\nu_C\langle\frac{\delta}{\delta A}, \hbar\dot{C}^{\Lambda\Lambda_0}\frac{\delta}{\delta A}\rangle f(A), \tag{531}$$

$$d\nu_C\, G(A)\left(AC^{-1}_{\Lambda\Lambda_0} - \hbar\frac{\delta}{\delta A}\right)F(A) = d\nu_C\left(\hbar\frac{\delta}{\delta A}G(A)\right)F(A). \tag{532}$$

When integrating over Grassmann variables one obtains

$$d\nu_{S_1+S_2}(c,\bar{c})f(\bar{c},c) = d\nu_{S_1}(c_1,\bar{c}_1)d\nu_{S_2}(c_2,\bar{c}_2)f(\bar{c}_1+\bar{c}_2, c_1+c_2), \tag{533}$$

$$d\nu_{tS}(c,\bar{c})f(\bar{c},c) = d\nu_S(c,\bar{c})f(t^{\frac{1}{2}}\bar{c}, t^{\frac{1}{2}}c), \tag{534}$$

$$d\nu_S(c-\delta c, \bar{c}-\delta\bar{c}) = d\nu(c,\bar{c})e^{\frac{1}{\hbar}\langle\delta\bar{c}, S^{-1}_{\Lambda\Lambda_0}\delta c\rangle}e^{-\frac{1}{\hbar}(\langle\bar{c}, S^{-1}_{\Lambda\Lambda_0}\delta c\rangle + \langle\delta\bar{c}, S^{-1}_{\Lambda\Lambda_0}c\rangle)}, \tag{535}$$

$$\frac{d}{d\Lambda}d\nu_S f(\bar{c},c) = d\nu_S\langle\frac{\delta}{\delta c}, \hbar\dot{S}^{\Lambda\Lambda_0}\frac{\delta}{\delta\bar{c}}\rangle f(\bar{c},c), \tag{536}$$

$$d\nu_S\,\tilde{G}(\bar{c},c)\left(-\bar{c}S^{-1}_{\Lambda\Lambda_0} + \hbar\frac{\delta_L}{\delta c}\right)\tilde{F}(\bar{c},c) = d\nu_S\left(\hbar\frac{\delta_R}{\delta c}\tilde{G}(\bar{c},c)\right)\tilde{F}(\bar{c},c), \tag{537}$$

$$d\nu_S\,\tilde{G}(\bar{c},c)\left(S^{-1}_{\Lambda\Lambda_0}c + \hbar\frac{\delta_L}{\delta\bar{c}}\right)\tilde{F}(\bar{c},c) = d\nu_S\left(\hbar\frac{\delta_R}{\delta\bar{c}}\tilde{G}(\bar{c},c)\right)\tilde{F}(\bar{c},c). \tag{538}$$

Properties (532), (537), (538) are proved for

$$G = e^{\frac{i}{\hbar}\langle j,A\rangle}, \quad F = e^{\frac{i}{\hbar}\langle j',A\rangle}, \quad \tilde{G} = e^{\frac{i}{\hbar}(\langle\bar{c},\eta\rangle + \langle\bar{\eta},c\rangle)}, \quad \tilde{F} = e^{\frac{i}{\hbar}(\langle\bar{c},\eta'\rangle + \langle\bar{\eta}',c\rangle)}, \tag{539}$$

and extended to polynomials in the fields by functional differentiation.

B Chains of vertex functions

For the purpose of example we give the complete list of reduced chains which appear in the loop integrals for $\dot{\Gamma}^{AAc\bar{c};w}$, together with the corresponding "dotted" propagators. The external fields are underlined. Moreover,

$\sum_{i=0}^{k-1} w_i = w$, k being the number of vertex functions in each chain.

$$\begin{array}{ll}
\dot{C}\Gamma^{A\underline{A}A;w_0} C\Gamma^{A\underline{A}A;w_1} C\Gamma^{A\underline{c}\bar{c};w_2} S\Gamma^{c\underline{\bar{c}}A;w_3}, & \dot{C}\Gamma^{AA\underline{AAc\bar{c}};w}, \\
\dot{S}\Gamma^{c\underline{A}\bar{c};w_0} S\Gamma^{c\underline{A}\bar{c};w_1} S\Gamma^{Ac\underline{\bar{c}};w_2} C\Gamma^{\underline{c}\bar{c}A;w_3}, & \dot{S}\Gamma^{c\bar{c}\underline{AAc\bar{c}};w}, \\
\dot{C}\Gamma^{A\underline{AA}A;w_0} C\Gamma^{A\underline{c}\bar{c};w_1} S\Gamma^{c\underline{\bar{c}}A;w_2}, & \dot{C}\Gamma^{AA\underline{Ac\bar{c}};w_0} C\Gamma^{A\underline{A}A;w_1}, \\
\dot{S}\Gamma^{c\underline{AA}\bar{c};w_0} S\Gamma^{Ac\underline{\bar{c}};w_1} C\Gamma^{\underline{c}\bar{c}A;w_2}, & \dot{S}\Gamma^{\underline{Ac\bar{c}}c\bar{c};w_0} S\Gamma^{\underline{A}c\bar{c};w_1}, \\
\dot{C}\Gamma^{A\underline{Ac}\bar{c};w_0} S\Gamma^{\underline{A}c\bar{c};w_1} S\Gamma^{Ac\underline{\bar{c}};w_2}, & \dot{C}\Gamma^{A\underline{AA}c\underline{\bar{c}};w_0} S\Gamma^{A\underline{c}\bar{c};w_1}, \\
\dot{S}\Gamma^{\bar{c}\underline{Ac}A;w_0} C\Gamma^{A\underline{A}A;w_1} C\Gamma^{Ac\underline{\bar{c}};w_2}, & \dot{S}\Gamma^{A\underline{AA}c\underline{\bar{c}};w_0} C\Gamma^{A\underline{c}\bar{c};w_1}, \\
\dot{C}\Gamma^{A\underline{A\bar{c}}c;w_0} S\Gamma^{A\underline{c}\bar{c};w_1} C\Gamma^{A\underline{A}A;w_2}, & \dot{C}\Gamma^{A\underline{AAc}\bar{c};w_0} S\Gamma^{Ac\underline{\bar{c}};w_1}, \\
\dot{S}\Gamma^{c\underline{A\bar{c}}A;w_0} C\Gamma^{A\underline{c}\bar{c};w_1} S\Gamma^{\underline{A}c\bar{c};w_2}, & \dot{S}\Gamma^{A\underline{AAc}\bar{c};w_0} C\Gamma^{Ac\underline{\bar{c}};w_1}, \\
\dot{C}\Gamma^{A\underline{c\bar{c}}A;w_0} C\Gamma^{A\underline{A}A;w_1} C\Gamma^{A\underline{A}A;w_2}, & \dot{C}\Gamma^{A\underline{A}c\underline{\bar{c}};w_0} S\Gamma^{A\underline{Ac}\bar{c};w_1}, \\
\dot{S}\Gamma^{c\underline{c\bar{c}}\bar{c};w_0} S\Gamma^{\underline{A}c\bar{c};w_1} S\Gamma^{\underline{A}c\bar{c};w_2}, & \dot{S}\Gamma^{A\underline{A}c\underline{\bar{c}};w_0} C\Gamma^{A\underline{Ac}\bar{c};w_1}, \\
\dot{C}\Gamma^{A\underline{AA}A;w_0} C\Gamma^{A\underline{c\bar{c}}A;w_1}, & \dot{S}\Gamma^{\underline{AA}c\bar{c};w_0} S\Gamma^{c\bar{c}\underline{c\bar{c}};w_1}.
\end{array} \tag{540}$$

C Tensors

For the definition of the tensor monomial sets $\{\delta^s q^n\}$, $\{\delta^s q^n\}_r$ see beginning of page 36.

Lemma 33 *Let* $q = (q_1, ..., q_m)$ *where* $q_i \in \mathbb{R}^D$ *are* $m \in \mathbb{N}$ *linearly independent vectors. Then the tensor monomials* $\{\delta^s q^n\}_r$ *of positive rank* $r = 2s + n \leqslant 2(D - m) + 1$ *are linearly independent,*

$$\sum_{t \in \{\delta^s q^n\}_r} c_t \, t = 0 \implies c_t = 0, \ \forall t. \tag{541}$$

Proof Observe that, for $m, r, D \in \mathbb{N}$, the inequality $r \leqslant 2(D - m) + 1$ is equivalent to $m + s \leqslant D$ for all $s, n \in \mathbb{N}$ such that $r = n + 2s$. Let $I := \{1, ..., r\}$. Let $\mathcal{P}_s$ be the set of all divisions of the set I in $m + s$ pairwise-disjoint, possibly-empty sets,

$$I = \left(\cup_{j=1}^{m} V_j \right) \bigcup \left(\cup_{k=1}^{s} S_k \right), \tag{542}$$

such that $S_k = \{s_k^1, s_k^2\}$, $s_k^1 < s_k^2$, and $\min S_1 < ... < \min S_s$. There is a bijection that maps a division $(V_j, S_k) \in \mathcal{P}_s$ to a tensor monomial $t \in \{\delta^s q^n\}$, constructed by the relation

$$t_{\mu_1 \ldots \mu_r} = \prod_{j=1}^{m} \prod_{v_j \in V_j} q_{j;\mu_{v_j}} \prod_{k=1}^{s} \delta_{\mu_{s_k^1} \mu_{s_k^2}}. \tag{543}$$

Let us first prove the statement of the lemma for orthonormal q_j. In an appropriate basis of $\mathbb{R}^D$, their components are

$$q_{j;\mu} = \delta_{j\mu}, \qquad\qquad j \in \{1, ..., m\}, \quad \mu \in \{1, ..., D\}. \tag{544}$$

Let us assume that $\sum_t c_t\, t = 0$, with $t \in \{\delta^s q^n\}_r$. We will proceed by proving inductively that $c_t = 0$ for all $t \in \{\delta^s q^n\}$, from $s = D-m$ down to $s = 0$. Fix $\bar{s} \leq D-m$ and assume that $c_t = 0$ for all t involving more than $\bar{s}$ Kronecker's tensors (which is vacuously true for $\bar{s} = D-m$, due to the rank constraint). Let us prove that $c_{\bar{t}} = 0$ for an arbitrary $\bar{t} \in \{\delta^{\bar{s}} q^{\bar{n}}\}$, which is associated to a division $(\bar{V}_j, \bar{S}_k) \in \mathcal{P}_{\bar{s}}$. Fix the values of the indices $\bar{\mu}_i$, with $i \in I$, by

$$\bar{\mu}_i = \begin{cases} j, & \text{if } \exists j \text{ such that } i \in \bar{V}_j, \\ m+k, & \text{if } \exists k \text{ such that } i \in \bar{S}_k, \end{cases} \tag{545}$$

Note that this choice is possible because $m + \bar{s} \leqslant D$. It is enough to show that whenever $t_{\bar{\mu}_1 \dots \bar{\mu}_r} \neq 0$ for $t \in \{\delta^s q^n\}$ and $s \leqslant \bar{s}$ (i.e. $n \geqslant \bar{n}$) then $s = \bar{s}$ and $t = \bar{t}$: in fact this property, the inductive hypothesis, and the vanishing of the sum $\sum_t c_t\, t$ imply that $c_{\bar{t}} = 0$. To prove the aforementioned property, introduce the division $(V_j, S_k) \in \mathcal{P}_s$ defining the tensor t and, using (544), correspondingly write

$$0 \neq t_{\bar{\mu}_1 \dots \bar{\mu}_r} = \prod_{j=1}^{m} \prod_{v_j \in V_j} \delta_{j \bar{\mu}_{v_j}} \prod_{k=1}^{s} \delta_{\bar{\mu}_{s^1_k} \bar{\mu}_{s^2_k}}. \tag{546}$$

Relations (545) and (546) imply that $V_j \subseteq \bar{V}_j$ for all j, which, together with the inductive condition $n \geqslant \bar{n}$, leads to $n = \bar{n}$ and, because the rank r is fixed, to $s = \bar{s}$. Relations (545), (546), and $s = \bar{s}$ imply that there is an injective map $f : I \to I$ such that $S_j = \bar{S}_{f(j)}$. By definition of the S_j, it then follows that $\min \bar{S}_{f(1)} < ... < \min \bar{S}_{f(1)}$: this is only possible if f is the identity, which concludes the first part of the proof.

Let us now prove the statement for m linearly independent vectors $p_1, ..., p_m$. The sum $\sum_t c_t\, t = 0$, with $t \in \{\delta^s p^n\}_r$, may be rewritten as

$$\sum_{2s+n=r} \sum_{1 \leqslant k_1, \dots, k_n \leqslant m} \sum_{\pi} c_{k_1, \dots, k_n; \pi} \prod_{j=1}^{n} (p_{k_j})_{\mu^\pi_j} \prod_{j'=1}^{s} \delta_{\mu^\pi_{m+2j'-1} \mu^\pi_{m+2j'}} = 0, \tag{547}$$

where $\mu^\pi_j := \mu_{\pi(j)}$ and the sum over π runs over the right coset of permutation groups $S_r \backslash (S_n \times S_s \times S_2^s)$. Expressing the p_k in terms of m orthonormal vectors

$q_{k'}$, $p_k = A_{kk'}q_{k'}$, gives a tensor transformation leading to the coefficients in the $\{\delta^s q^n\}_r$ basis:

$$c'_{k'_1,...,k'_n;\pi} = c_{k_1,...,k_n;\pi} \prod_{j=1}^{n} A_{k_j k'_j}. \tag{548}$$

The validity of equation (541) for the q_k implies that $c'_{k'_1,...,k'_n;\pi} = 0$, which, by invertibility of the matrix $A \in GL(m, \mathbb{R})$, gives $c_{k_1,...,k_n;\pi} = 0$. ■

Note that for $m = n_\phi - 1$ and $D = 4$, the condition for linear independence of the monomials in $\{\delta^s q^k\}_r$ reads

$$r \leqslant 2(4 - (n_\phi - 1)) + 1 = 11 - 2n_\phi. \tag{549}$$

Lemma 34 *Let $q = (q_1, ..., q_m)$ with $q_i \in \mathbb{R}^D$ and m nonnegative integer. The tensor monomials $\{\delta^s q^n\}_r$ of positive rank $r \geqslant 2(D-m+1)$ are linearly dependent.*

Proof It is enough to prove the statement of the lemma for $r = 2(D-m+1)$, $m > 0$ which requires $m \leqslant D$. First we prove the statement of the lemma for orthonormal q_j. There exist $q_{m+1}, ..., q_D$ orthonormal vectors such that in an appropriate basis of $\mathbb{R}^D$ we have $q_{j;\mu} = \delta_{j\mu}$ $\forall j, \mu \in \{1, ..., D\}$. There is an identity

$$\delta_{\mu\nu} = \sum_{i \leqslant m} q_{i,\mu} q_{i,\nu} + \delta^\perp_{\mu\nu}, \qquad \delta^\perp_{\mu\nu} := \sum_{i>m} q_{i,\mu} q_{i,\nu}. \tag{550}$$

Define a tensor of rank $2s$

$$\delta^\perp_{\mu_1,...,\mu_{2s}} := \sum_{\pi \in S_s} (-1)^{N_\pi} \prod_{j=1}^{s} \delta^\perp_{\mu^\pi_{2j-1}\mu_{2j}} \tag{551}$$

where $\mu^\pi_j := \mu_{\pi(j)}$ and N_π stands for the number of transpositions in the permutation π. For $r = 2(D-m+1)$ one always has $\delta^\perp_{\mu_1,...,\mu_r} = 0$. Substitution $\delta^\perp_{\mu\nu}$ with the lhs of (550) implies that $\exists c_t \neq 0$ such that

$$\sum_{t \in \{\delta^s q^n\}_r} c_t\, t = 0, \qquad q := (q_1, ..., q_m)\,. \tag{552}$$

To generalize the result to arbitrary set of linear independent momenta p_j it is enough to substitute $q_{k;\mu}$ with $A_{kk'}p_{k';\mu}$ where $A \in GL(m,\mathbb{R})$ and to define new coefficient c'

$$c'_{k'_1,...,k'_n;\pi} = c_{k_1,...,k_n;\pi} \prod_{j=1}^{n} A_{k_j k'_j}. \tag{553}$$

Invertibility of A implies that $\exists c'_{k'_1,...,k'_n;\pi} \neq 0$. Linear independence of $q_1, ..., q_m$ implies that $q_1 \neq 0$. Then $\forall l \in \mathbb{N}$ one obtains

$$\sum_{t\in\{\delta^s q^n\}_r} c'_t\, t(\otimes q_1)^l = 0. \tag{554}$$

■

The following lemma states a necessary condition for a regular, $O(4)$-invariant tensor field.

Lemma 35 *Let $f(\underline{y})$ be a regular, $O(4)$-invariant tensor field of rank r where $\underline{y} := (y_1, ..., y_m)$ with $y_j \in \mathbb{R}^4$. Assume that the tensor monomials $\{\delta^s \underline{y}^k\}_r$ as well as $\{\delta^s \underline{y}^k\}_{r+1}$ are linearly independent pointwise for all $\underline{y} \in O$ where O is some open set. Then on O we have*

$$f = \sum_{t\in\{\delta^s \underline{y}^k\}_r} f_t\, t\,, \qquad\qquad \partial_j f = \sum_{t'\in\{\delta^s \underline{y}^k\}_{r+1}} f_{j,t'}\, t'\,. \tag{555}$$

Furthermore for every $t \in \{\delta^s \underline{y}^{k>0}\}_r$ there exist j and $t' \in \{\delta^s \underline{y}^k\}_{r+1}$ such that $f_t = f_{j,t'}$ on O.

Proof For shortness we consider only the case $m = 2$. We have

$$f = \sum_{t\in\{\delta^s\}_r} u_t\, t + \sum_{t\in\{\delta^s \underline{y}^{k>0}\}_r} \zeta_t\, t, \tag{556}$$

where u_t, ζ_t are regular functions of the scalar parameters $\mathbb{X} = \{\frac{1}{2}y_1^2, \frac{1}{2}y_2^2, y_1 y_2\}$. Apply the operator ∂_j to both sides of (556). The Leibniz rule gives

$$\partial_j(u_t \delta^s) = \sum_{x\in\mathbb{X}} (\partial_x u_t)\, \delta^s\, \partial_j x, \tag{557}$$

$$\partial_j(\zeta_t \delta^s \underline{y}^k) = \sum_{x\in\mathbb{X}} (\partial_x \zeta_t)\, \delta^s\, \underline{y}^k \partial_j x \;+ \zeta_t \delta^s\, \partial_j \underline{y}^k, \tag{558}$$

where $\partial_j x \in \{0, y_1, y_2\}$, $\underline{y}^k := y_{i_1} ... y_{i_k}$ with $i_l \in \{1, 2\}$, and

$$\partial_j \underline{y}^k = \partial_j \prod_{l=1}^{k} y_{i_l} = \sum_{q=1}^{k} \delta_{j,i_q} \prod_{\substack{l=1 \\ l \neq q}}^{k} y_{i_l} . \tag{559}$$

The nonvanishing tensor monomials arising at a given $j \in \{1, 2\}$ from the rhs of (557) and (558) have rank $r + 1$, are linearly independent by assumption, and are pairwise different. Each coefficient ζ_t of the tensor $\delta^s y_{i_1} ... y_j ... y_{i_k}$ in the decomposition (556) appears also as the coefficient of $\delta^s y_{i_1} ... \delta ... y_{i_k}$ in the decomposition (558) for $\partial_j f$. ■

Lemma 36 below relies on lemma 33, which shows that, for m linearly independent vectors $\underline{e} = (e_1, \cdots, e_m)$, the relation $r+1 \leqslant 9-2m$ is a sufficient condition for the linear independence of the tensor monomials $\{\delta^s \underline{e}^k\}_r$ and $\{\delta^s \underline{e}^k\}_{r+1}$, see lemma 34 for a necessary condition. The renormalization points in appendix H are chosen to comply with the aforementioned relation. The proof of lemma 36 is in the same spirit as the one of the preceding lemma 35.

Lemma 36 *Let F be a regular, $O(4)$-invariant tensor field of rank $r \in \{2, 4\}$ on $\mathbb{P}_n$. Let be given $\vec{q} \in \mathbb{M}_n$ and $m \geqslant 2$ linearly independent vectors $\underline{e} = (e_1, \cdots, e_m)$, such that $\mathrm{span}(\vec{q}) = \mathrm{span}(\underline{e})$. Assume that $r + 1 \leqslant 9 - 2m$. By lemma 33 there exist unique coefficients F_t such that $F(\vec{q}) = \Sigma_{t \in \{\delta^s \underline{e}^k\}_r} F_t \, t$. Furthermore,*

$$|F(\vec{q})| \leqslant c \max \Big(\big(|F_t|\big)_{t \in \{\delta^s\}_r}, (M |\partial_k F(\vec{q})|)_{k \in [n-1]} \Big), \tag{560}$$

$$\Big| \sum_{t \in \{\delta^s \underline{e}^{k>0}\}_r} F_t \, t \Big| \leqslant c \, \max \Big((M |\partial_k F(\vec{q})|)_{k \in [n-1]} \Big) . \tag{561}$$

The bounds hold with the same constant c for all F of equal rank.

Proof The coefficients $\big(|F_t|\big)_{t \in \{\delta^s\}_r}$ in the basis $\{\delta^s \underline{e}^k\}_r$ do not depend on the choice of the vectors $\underline{e}$. Hence it is enough to prove (560) in the case when $e_i e_j = M^2 \delta_{ij}$. For simplicity we assume that $m = 2$, the extension to other m being clear.
By hypothesis, there exists a $(n-1) \times 2$ matrix L such that $q_k = L_{ki} e_i$ and

$$|L| := \sqrt{L_{ki} L_{ki}} = \frac{1}{M} \sqrt{L_{ki} L_{kj} e_i e_j} = \frac{1}{M} \Big(\sum_{k=1}^{n-1} q_k^2 \Big)^{\frac{1}{2}} \leqslant (n-1)^{\frac{1}{2}} . \tag{562}$$

Denote by $\mathbb{E} \subset \mathbb{R}^4$ the linear span of the vectors $q_0, .., q_{n-1}$. The matrix L induces a linear map $L : \mathbb{E}^2 \to \mathbb{R}^{4(n-1)}$, $\underline{y} \mapsto L\underline{y}$, where $\underline{y} = (y_1, y_2)$ and $(L\underline{y})_k = L_{ki} y_i$. We also define an auxiliary function on $\mathbb{E}^2$: $f(\underline{y}) := F(L\underline{y})$. Setting $\partial_{y_i} := \partial/\partial y_i$ and $\partial_k := \partial/\partial p_k$, the Cauchy–Schwarz inequality and (562) imply that,

$$\textstyle\sum_{i=1}^{2} |\partial_{y_i} f(\underline{e})|^2 \leqslant |L|^2 \ \sum_{k=1}^{n-1} |\partial_k F(\vec{q})|^2 \leqslant (n-1)^2 \max_k (|\partial_k F(\vec{q})|^2). \tag{563}$$

For all $y_1, y_2 \in \mathbb{E}$, denote as usual by $\{\delta^s \underline{y}^k\}$ the set of all monomials being a tensor product of k vectors in $\underline{y} = (y_1, y_2)$ and of s Kronecker tensors, and by $\{\delta^s \underline{y}^k\}_r$ the union of all $\{\delta^s \underline{y}^k\}$ such that $2s + k = r$. By (541), whenever $r \leqslant 5$ and y_1, y_2 are linearly independent, the elements of $\{\delta^s \underline{y}^k\}_r$ are linearly independent. In this case, we label the tensor monomials by fixing a family of disjoint sets $A_{k,s}$ and a family of bijections $\alpha \mapsto t_\alpha$ from each $A_{k,s}$ to $\{\delta^s \underline{y}^k\}$. Furthermore, we define the auxiliary sets

$$A^r := \bigcup_{2s+k=r} A_{k,s}\,, \qquad A^r_+ := \bigcup_{2s+k=r,\, k\geqslant 1} A_{k,s}\,, \qquad A^r_0 := A^r \setminus A^r_+\,. \tag{564}$$

For $r \in \{2, 4\}$ and $\underline{y} = (y_1, y_2)$ in an open neighborhood of $\underline{e}$ (in $\mathbb{E}^2$) for which y_1, y_2 are linearly independent, we write the following tensor decomposition:

$$f(\underline{y}) = \sum_{\alpha \in A^r_0} u_\alpha t_\alpha + \sum_{\alpha \in A^r_+} \zeta_\alpha t_\alpha, \tag{565}$$

where u_α, ζ_α are regular functions of the scalar parameters $\mathbb{X} = \{\frac{1}{2} y_1^2, \frac{1}{2} y_2^2, y_1 y_2\}$. Evaluating (565) at $\underline{y} = \underline{e}$ and using the general fact that $|t| = 2^s M^k$ for $t \in \{\delta^s \underline{e}^k\}$ we obtain

$$|f(\underline{e})| \leqslant \sum_{\alpha \in A^r_0} 2^s\, |u_\alpha| + \sum_{\alpha \in A^r_+} |\zeta_\alpha|\, 2^s M^k. \tag{566}$$

We now want to prove the existence of a constant $c_1 > 0$ such that

$$\sum_{\alpha \in A^r_+} |\zeta_\alpha|\, 2^s M^k \ \leqslant\ c_1 M \Big(\sum_{i \in \{1,2\}} |\partial_{y_i} f(\underline{e})|^2 \Big)^{1/2}. \tag{567}$$

Apply the operator ∂_{y_i} to both sides of (565). The Leibniz rule gives

$$\partial_{y_i}(u_\alpha \delta^s) = \sum_{x \in \mathbb{X}} (\partial_x u_\alpha)\, \delta^s\, \partial_{y_i} x, \tag{568}$$

$$\partial_{y_i}(\zeta_\alpha \delta^s y^k) = \sum_{x \in \mathbb{X}} (\partial_x \zeta_\alpha)\, \delta^s\, y^k \partial_{y_i} x \ + \zeta_\alpha \delta^s\, \partial_{y_i} y^k, \tag{569}$$

where $\partial_{y_i} x \in \{0, y_1, y_2\}$, $y^k = y_{i_1} ... y_{i_k}$ with $i_l \in \{1,2\}$, and

$$\partial_{y_i} y^k = \partial_{y_i} \prod_{l=1}^{k} y_{i_l} = \sum_{j=1}^{k} \delta_{i,i_j} \prod_{\substack{l=1 \\ l \neq j}}^{k} y_{i_l}. \tag{570}$$

Fix $\underline{y} = \underline{e}$. The nonvanishing tensor monomials arising at a given $i \in \{1,2\}$ from the rhs of (568) and (569) have rank $r+1 \in \{3,5\}$, are pairwise different, and are a subset of the tensor monomials in A^{r+1}, themselves linearly independent by (541). Denote by $B_i^{r+1} \subset A_+^{r+1}$ the subset labeling the monomials of type $\delta^s \delta_{i,i_j} \prod_{l \neq j} y_{i_l}$ arising from (569) and (570) at the given i. By construction, we can define the maps $\pi_i : B_i^{r+1} \to A_+^r$ with $i \in \{1,2\}$ such that:
i) if $t_{\pi_i(\beta)} \in \{\delta^s \underline{e}^k\}$ then $t_\beta \in \{\delta^{s+1} \underline{e}^{k-1}\}$ (in this case, $|t_{\pi_i(\beta)}| = 2^s M^k$ and $|t_\beta| = 2^{s+1} M^{k-1}$);
ii) for each $\beta \in B_i^{r+1}$ the coefficient of t_β in (569) and that of $t_{\pi_i(\beta)}$ in (565) are the same, namely $\zeta_{\pi_i(\beta)}$;
iii) $\pi_1(B_1^{r+1}) \cup \pi_2(B_2^{r+1}) = A_+^r$.
The following bound holds at $\underline{y} = \underline{e}$ for every tensor $\Psi = \sum_\beta \Psi_\beta t_\beta / |t_\beta|$ with $\beta \in A^{r'}$ and $r' \leqslant 5$, and for every nonempty $B \subset A^{r'}$:

$$|\Psi|^2 = (\Psi, \Psi) = \Psi_\beta^* \, G_{\beta\beta'} \, \Psi_{\beta'} \geqslant \lambda_1 \sum_{\beta \in A^{r'}} |\Psi_\beta|^2 \geqslant \lambda_1 \sum_{\beta \in B} |\Psi_\beta|^2, \tag{571}$$

where $\lambda_1 > 0$ is the smallest eigenvalue of the Gram matrix of components $G_{\beta\beta'} := (t_\beta, t_{\beta'})/(|t_\beta||t_{\beta'}|)$, which is positive definite by (541). Application of (571) to $\Psi_i := \partial_{y_i} f(\underline{e})$ for each $i \in \{1,2\}$, with $r' = r+1$ and $B = B_i^{r+1}$, gives

$$\sum_{i \in \{1,2\}} |\partial_{y_i} f(\underline{e})|^2 \geqslant \lambda_1 \sum_{\substack{i \in \{1,2\} \\ \beta \in B_i^{r+1}}} |\zeta_{\pi_i(\beta)}|^2 |t_\beta|^2 \geqslant \frac{4\lambda_1}{M^2} \sum_{\alpha \in A_+^r} |\zeta_\alpha|^2 \, (2^s M^k)^2, \tag{572}$$

from which follows the bound (567).
Inequalities (566), (572) and (563) lead to the bounds (560), (561), with a constant

$$c = \max \left(2^{\frac{r}{2}} |A_0|, (n-1) \sqrt{\frac{|A_+^r|}{4\lambda_1}} \right). \tag{573}$$

■

D Basic estimates

Lemma 37 *Let* $P, \Lambda > 0$, $\eta \geqslant 0$, $0 < a \leqslant \frac{P}{\Lambda+\eta}$. *The* $\forall k \in \mathbb{N}$ $\exists c$ *such that*

$$\int_0^1 \frac{dx\; a}{1+ax} \log_+^k \frac{P}{\Lambda+\eta x} \leqslant c \log(1+a)\Big(1 + \log_+^k \frac{P}{\Lambda+\eta}\Big). \tag{574}$$

Proof Let $x' = \min(\bar{x}, x_a)$ where

$$\bar{x} := \begin{cases} \sup\limits_{0\leqslant x\leqslant 1} \{x : \frac{P}{\Lambda+\eta x} \geqslant 1\} & k > 0, \\ 1 & k = 0, \end{cases} \qquad x_a := \frac{1}{a}\log(1+a). \tag{575}$$

First we establish a low bound on x_a. Note that $f(a) = 2^a$ is a convex function with $f'' > 0$ and $f(0) = 1$, $f(1) = 2$. Consequently for all $0 < a \leqslant 1$.

$$2^a \leqslant (1+a) \implies \log 2 \leqslant \frac{1}{a}\log(1+a) \implies \frac{1}{2} \leqslant x_a. \tag{576}$$

Furthermore $\forall a > 1$ we have $\log(1+a) > \log(2) > \frac{1}{2}$. Hence

$$x_a \geqslant \begin{cases} \frac{1}{2} & 0 < a \leqslant 1, \\ \frac{1}{2a} & a > 1. \end{cases} \tag{577}$$

We also need the following inequality:

$$\frac{\partial}{\partial x}\left(x \sum_{j=0}^{k} \frac{k!}{j!} \log^j \frac{P}{\Lambda+\eta x}\right) \geqslant \log^k \frac{P}{\Lambda+\eta x}, \quad \forall x \in (0, \bar{x}). \tag{578}$$

Now let I denote the lhs in (574). Using integration by parts and then spliting the domain of integration of the remaining integral into two intervals $[0, x']$ and $[x', \bar{x}]$ we obtain

$$\begin{aligned} I \leqslant \log(1+a)\log_+^k \frac{P}{\Lambda+\eta} &+ ka \int_0^{x'} dx \log^{k-1} \frac{P}{\Lambda+\eta x} \\ &+ \log(1+a) \int_{x'}^{\bar{x}} \frac{dx\, k\, \eta}{\Lambda+\eta x} \log^{k-1} \frac{P}{\Lambda+\eta x} \\ &\leqslant \log(1+a)\Big(k! \sum_{j=0}^{k} \frac{1}{j!} \log^j \frac{P}{\Lambda+\eta x'} + \log^k \frac{P}{\Lambda+\eta x'}\Big). \end{aligned} \tag{579}$$

If $\bar{x} < x_a$ then $x' = \bar{x}$, and (579) yields

$$I \leqslant 2 \log(1+a). \tag{580}$$

Otherwise $x' = x_a$. If $0 < a \leqslant 1$ using (577) we have

$$\begin{aligned}\sum_{j=0}^{k} \frac{1}{j!} \log_+^j \frac{P}{\Lambda + \eta x'} &\leqslant \sum_{j=0}^{k} \frac{1}{j!} \log_+^j \frac{2P}{\Lambda + \eta} \leqslant \sum_{j=0}^{k} \frac{1}{j!} \Big(1 + \log_+ \frac{P}{\Lambda + \eta} \Big)^j \\ &\leqslant \sum_{n,m=0}^{k} \frac{1}{n!m!} \log_+^m \frac{P}{\Lambda + \eta} \leqslant e \sum_{m=0}^{k} \frac{1}{m!} \log_+^m \frac{P}{\Lambda + \eta} \\ &< e \sum_{m=0}^{k} \frac{1}{m!} \Big(\log_+^k \frac{P}{\Lambda + \eta} + 1 \Big) < e^2 \Big(\log_+^k \frac{P}{\Lambda + \eta} + 1 \Big). \end{aligned} \tag{581}$$

If $1 < a \leqslant \frac{P}{\Lambda+\eta}$ then similar to (581) using (577) we obtain

$$\log_+^k \frac{P}{\Lambda + \eta x'} \leqslant 3^k \Big(1 + \log_+^k \frac{P}{\Lambda + \eta} \Big), \tag{582}$$

$$\sum_{j=0}^{k} \frac{1}{j!} \log_+^j \frac{P}{\Lambda + \eta x'} \leqslant e^3 \Big(1 + \log_+^k \frac{P}{\Lambda + \eta} \Big). \tag{583}$$

Finally (579) gives

$$I \leqslant c \log(1+a) \Big(1 + \log_+^k \frac{P}{\Lambda + \eta} \Big), \tag{584}$$

where $c = \max(2, k! e^3 + 3^k, k! e^2 + 1)$. ■

Lemma 38 *Let $p, q \in \mathbb{R}^4$*

$$\frac{|p|}{\Lambda + |p+q|} \left(1 + \log_+ \frac{|p|}{\Lambda + |q|} \right) \geqslant \frac{1}{2} \log \left(1 + \frac{|p|}{\Lambda + |q|} \right). \tag{585}$$

Proof If $\Lambda + |q| \leqslant |p|$ then

$$\frac{|p|}{\Lambda + |p+q|} \geqslant \frac{|p|}{\Lambda + |q| + |p|} \geqslant \frac{1}{2}, \tag{586}$$

$$1 + \log \frac{|p|}{\Lambda + |q|} \geqslant \log \left(1 + \frac{|p|}{\Lambda + |q|} \right). \tag{587}$$

If $\Lambda + |q| > |p|$ then

$$\frac{|p|}{\Lambda + |p+q|} \geqslant \frac{|p|}{\Lambda + |q| + |p|} \geqslant \frac{|p|}{2(\Lambda + |q|)} \geqslant \frac{1}{2} \log \left(1 + \frac{|p|}{\Lambda + |q|} \right). \tag{588}$$

■

Lemma 39 *Let $\{e_i\}_{i=1}^k \in \mathbb{R}^4$ such that $e_i^2 = 1$ and $e_i e_j \geqslant -\frac{1}{k}$ if $i \neq j$. Then $\forall a = \alpha_i e_i$ with $\alpha_i \in \mathbb{R}$*

$$|a| \geqslant \frac{1}{k} \sum_{i=1}^k |\alpha_i|. \tag{589}$$

Proof

$$a^2 \geqslant \frac{1}{k} \sum_{i=1}^k \alpha_i^2 + \frac{1}{k} \Big((k-1) \sum_{i=1}^k \alpha_i^2 - \sum_{i \neq j} \alpha_i \alpha_j \Big) \geqslant \frac{1}{k} \sum_{i=1}^k \alpha_i^2$$
$$+ \frac{1}{k} \Big((k-1) \sum_{i=1}^k \alpha_i^2 - \frac{1}{2} \sum_{i \neq j} (\alpha_i^2 + \alpha_j^2) \Big) = \frac{1}{k} \sum_{i=1}^k \alpha_i^2 \geqslant \frac{1}{k^2} \Big(\sum_{i=1}^k |\alpha_i| \Big)^2. \tag{590}$$

■

Lemma 40 *Let $0 \leqslant q \leqslant p \leqslant P$, $k \in \mathbb{N}$. Then $\exists C_k > 0$ such that*

$$\int_\Lambda^{+\infty} \frac{d\lambda \, \log_+^k \frac{P}{\lambda}}{(\lambda + p)(\lambda + q)} \leqslant C_k \frac{1 + \log_+^{k+1} \frac{P}{\Lambda + q}}{\Lambda + p + q}. \tag{591}$$

Proof Let I^k be the left hand side of the inequality and

$$I^k_{[a,b]} := \int_a^b \frac{d\lambda}{(\lambda + p)(\lambda + q)} \log_+^k \frac{P}{\lambda}. \tag{592}$$

We begin with the case $k \geqslant 1$.

- $\Lambda > P$, $I^k = 0$.

- $q \leqslant \Lambda \leqslant P$, $I^k = I^k_{[\Lambda,P]}$.

$$\begin{aligned} I^k \leqslant & \frac{1}{\Lambda+p}\int_\Lambda^P \frac{d\lambda}{\lambda+q}\log^k\frac{2P}{\lambda+q} \leqslant \frac{2}{k+1}\frac{1}{\Lambda+p+q}\log^{k+1}\frac{2P}{\Lambda+q} \\ & \leqslant \frac{A_k}{\Lambda+p+q}(\log_+^{k+1}\frac{P}{\Lambda+q}+1), \end{aligned} \tag{593}$$

where

$$A_k := \frac{2(\log 2+1)^{k+1}}{k+1}, \tag{594}$$

and we have used the inequality

$$\frac{\log_+ x + \log 2}{(1+\log_+^n x)^{\frac{1}{n}}} < 1 + \log 2. \tag{595}$$

- $\Lambda < q \leqslant P$, $I^k = I^k_{[\Lambda,q]} + I^k_{[q,P]}$.

$$\begin{aligned} I^k_{[q,P]} \leqslant & \frac{1}{\Lambda+p}\int_q^P \frac{d\lambda}{\lambda+q}\log^k\frac{2P}{\lambda+q} \leqslant \frac{1}{\Lambda+p}\int_\Lambda^P \frac{d\lambda}{\lambda+q}\log^k\frac{2P}{\lambda+q} \\ & \leqslant \frac{A_k}{\Lambda+p+q}(\log_+^{k+1}\frac{P}{\Lambda+q}+1), \end{aligned} \tag{596}$$

$$\begin{aligned} I^k_{[\Lambda,q]} \leqslant & \frac{1}{(\Lambda+p)(\Lambda+q)}\int_0^q d\lambda\ \log^k\frac{P}{\lambda} < \frac{qk!}{(\Lambda+p)(\Lambda+q)}\sum_{j=0}^k\frac{1}{j!}\log^j\frac{P}{q} \\ & < \frac{ek!}{\Lambda+p}(\log^k\frac{2P}{\Lambda+q}+1) < \frac{2ek!}{\Lambda+p+q}(\log_+^k\frac{P}{\Lambda+q}+2). \end{aligned} \tag{597}$$

It remains to consider the case $k=0$.

- $\Lambda > p$. This implies $2(\lambda+q) > \lambda+p+2q > \lambda+p$, $\forall\lambda > \Lambda$.

$$I^0 = 2\int_\Lambda^{+\infty}\frac{d\lambda}{(\lambda+p)^2} < \frac{4}{\Lambda+p+q}. \tag{598}$$

- $\Lambda \leqslant p$, $I^0 = I^0_{[\Lambda,p]} + I^0_{[p,+\infty]}$.

$$I^0_{[p,+\infty]} < 2 \int\limits_p^{+\infty} \frac{d\lambda}{(\lambda+p)^2} < \frac{1}{p} < \frac{3}{\Lambda+p+q}, \tag{599}$$

$$\begin{aligned} I^0_{[\Lambda,p]} &< \frac{1}{\Lambda+p} \int\limits_\Lambda^p \frac{d\lambda}{\lambda+q} < \frac{2}{\Lambda+p+q} \log \frac{2P}{\Lambda+q} \\ &< \frac{2}{\Lambda+p+q} (\log 2 + \log_+ \frac{P}{\Lambda+q}). \end{aligned} \tag{600}$$

■

Lemma 41 *Let* $0 \leqslant q \leqslant p$, $\eta := \min(M,q)$, $k \in \mathbb{N}$. *Then* $\exists C_k > 0$ *such that*

$$\int\limits_\Lambda^{+\infty} \frac{d\lambda \ \log_+^k \frac{\lambda}{M}}{(\lambda+p)(\lambda+q)} \leqslant C_k \frac{1 + \log_+^{k+1} \frac{p}{\Lambda+\eta} + \log_+^{k+1} \frac{\Lambda}{M}}{\Lambda+p+q}. \tag{601}$$

Proof Denote by I^k the lhs of equation (601). If $k=0$ then the inequality follows from (591) with $P=p$. Let $k \geqslant 1$ and $\mu = \max(\Lambda, M)$.

- $p \leqslant \Lambda$.

$$I^k \leqslant \int\limits_\mu^{+\infty} d\lambda \frac{2 \log^k \frac{\lambda}{M}}{(\lambda+p)^2} \leqslant \int\limits_\mu^{+\infty} d\lambda f'(\lambda) \leqslant 2^{k+2} k! \sqrt{e} \frac{1 + \log_+^k \frac{\Lambda}{M}}{\Lambda+p+q}, \tag{602}$$

$$f(\lambda) := -\frac{k! 2^{k+1}}{\lambda+p} \sum_{j=0}^k \frac{\log^k \frac{\lambda}{M}}{j! 2^j}. \tag{603}$$

- $\Lambda < p$.

$$J^k := \int\limits_\Lambda^p d\lambda \frac{\log_+^k \frac{\lambda}{M}}{\lambda+q}, \quad I^k \leqslant \frac{2J^k}{\Lambda+p+q} + \int\limits_\mu^{+\infty} d\lambda \frac{2 \log_+^k \frac{\lambda}{M}}{(\lambda+p)^2}. \tag{604}$$

The integral on the rhs of I^k is exactly the same as in the case $p \leqslant \Lambda$. For the integral J^k using the inequality

$$\log_+ \frac{\lambda}{M} \leqslant \log 2 + \log_+ \frac{\lambda}{\Lambda+M} + \log_+ \frac{\Lambda}{M}, \tag{605}$$

we have

$$J^k < 3^{k+1}\left(1+\log_+^{k+1}\frac{p}{\Lambda+\eta}+\log_+^{k+1}\frac{\Lambda}{M}\right). \tag{606}$$

■

Lemma 42 *For* $0<\Lambda<\eta\leqslant M$, $k\in\mathbb{N}$

$$\int\limits_{\Lambda}^{\eta} d\lambda\frac{1}{\lambda+\eta}\log_+^k\frac{\lambda}{M}<1. \tag{607}$$

Proof Denote by I^k the lhs of equation (607). If $k>0$ then $I^k=0$. It remains to consider the integral

$$\int\limits_{\Lambda}^{\eta}\frac{d\lambda}{\lambda+\eta}=\log\frac{2\eta}{\Lambda+\eta}<\log 2. \tag{608}$$

■

Lemma 43 *Let* $0<\Lambda<\eta\leqslant P$, $k\in\mathbb{N}$. *Then* $\exists C_k>0$ *such that*

$$\int\limits_{\Lambda}^{\eta}\frac{d\lambda}{\lambda+\eta}\log_+^k\frac{P}{\lambda}<C_k\Big(1+\log_+^k\frac{P}{\Lambda+\eta}\Big). \tag{609}$$

Proof Denote by $I^k_{[\Lambda,\eta]}$ the lhs of equation (609).

$$I^k_{[\Lambda,\eta]}<I^k_{[0,\eta]}<\frac{1}{\eta}\int\limits_{0}^{\eta}d\lambda\,\log_+^k\frac{P}{\lambda}<2k!e^2\Big(1+\log_+^k\frac{P}{\eta+\Lambda}\Big). \tag{610}$$

■

Lemma 44 *Let* $0<\Lambda<M\leqslant P$, $0\leqslant\eta\leqslant M$, $k\in\mathbb{N}$. *Then* $\exists C_k>0$ *such that*

$$\int\limits_{\Lambda}^{M}\frac{d\lambda}{\lambda+\eta}\log_+^k\frac{P}{\lambda}<C_k\Big(1+\log_+^{k+1}\frac{P}{\Lambda+\eta}\Big). \tag{611}$$

Proof Denote by $I^k_{[\Lambda,M]}$ the lhs of equation (611). If $\Lambda \geqslant \eta$ then

$$I^k_{[\Lambda,M]} \leqslant \int\limits_{\Lambda}^{M} \frac{d\lambda}{\lambda+\eta} \log_+^k \frac{2P}{\lambda+\eta} \leqslant \frac{2^{k+1}}{k+1}\Big(1+\log_+^{k+1} \frac{P}{\Lambda+\eta}\Big). \tag{612}$$

If $\eta \geqslant \Lambda$ then $I^k < I^k_{[0,\eta]} + I^k_{[\eta,M]}$ where

$$I^k_{[0,\eta]} = \int\limits_{0}^{\eta} \frac{d\lambda}{\lambda+\eta} \log_+^k \frac{P}{\lambda} < 2k!e^2\Big(1+\log_+^{k} \frac{P}{\eta+\Lambda}\Big), \tag{613}$$

$$I^k_{[\eta,M]} = \int\limits_{\eta}^{M} \frac{d\lambda}{\lambda+\eta} \log_+^k \frac{2P}{\lambda+\eta} < \frac{2^{k+1}}{k+1}\Big(1+\log_+^{k+1} \frac{P}{\Lambda+\eta}\Big). \tag{614}$$

■

Lemma 45 *For $q \geqslant 0$, $k \in \mathbb{N}$, $P > 0$ there exists a constant $C_k > 0$ such that*

$$\int\limits_{\Lambda}^{\Lambda_0} \frac{d\lambda}{\lambda+q} \log_+^k \frac{P}{\lambda} < C_k\Big(1+\log_+^{k+1} \frac{P}{\Lambda+q} + \log_+ \frac{\Lambda_0}{\Lambda+q}\Big). \tag{615}$$

Proof Denote by $I^k_{[\Lambda,\Lambda_0]}$ the lhs of equation (615). If $k=0$ then

$$I^0_{[\Lambda,\Lambda_0]} < 1+\log_+ \frac{\Lambda_0}{\Lambda+q}. \tag{616}$$

For $k>0$ and $\Lambda \geqslant q$

$$I^k_{[\Lambda,\Lambda_0]} < \frac{2^{k+1}}{k+1}\Big(1+\log_+^{k+1} \frac{P}{\Lambda+q}\Big). \tag{617}$$

If $k>0$ and $\Lambda < q$ then $I^k_{[\Lambda,\Lambda_0]} < I^k_{[0,q]} + I^k_{[q,\Lambda_0]}$ where

$$I^k_{[0,q]} < \frac{1}{q}\int\limits_{0}^{q} d\lambda \, \log_+^k \frac{P}{\lambda} < k!e^2\Big(1+\log_+^{k+1} \frac{P}{\Lambda+q}\Big), \tag{618}$$

$$I^k_{[q,\Lambda_0]} < \frac{2^{k+1}}{k+1}\Big(1+\log_+^{k+1} \frac{P}{\Lambda+q}\Big). \tag{619}$$

■

Lemma 46 *Let $a, d > 0$, $b \geqslant 0$ and $m, k \in \mathbb{N}$. Then $\exists C_{k,m} > 0$ such that*

$$\int_0^a dx\, x^m \log_+^k \frac{d}{b+x} \leqslant C_{k,m} a^{m+1}\Big(1 + \log_+^k \frac{d}{a+b}\Big). \tag{620}$$

Proof By direct calculations it is easy to show that

$$x^m \log_+^k \frac{d}{b+x} \leqslant f', \qquad f := \frac{k! x^{m+1}}{(m+1)^{k+1}} \sum_{j=0}^{k} \frac{(m+1)^j}{j!} \log_+^j \frac{d}{b+x}. \tag{621}$$

Consequently, the lhs of equation (620) is bounded above by $f(a)$,

$$\begin{aligned} f(a) =& \frac{k!}{(m+1)^{k+1}} a^{m+1} \sum_{j=0}^{k} \frac{(m+1)^j}{j!} \log_+^j \frac{d}{a+b} \\ \leqslant& \frac{e^{m+1} k!}{(m+1)^{k+1}} a^{m+1}\Big(1 + \log_+^k \frac{d}{a+b}\Big). \end{aligned} \tag{622}$$

■

Lemma 47 *Let $p, q \in \mathbb{R}^4$, $\Lambda' > 0$, $\Lambda' \geqslant \eta \geqslant 0$. Then*

$$\int_0^1 dt\, \frac{|p|}{\Lambda' + |tp+q|} \leqslant 2\Big(\log 4 + \log_+ \frac{|p|}{\Lambda' + \eta}\Big). \tag{623}$$

Proof Let $I_{[0,1]}$ denote the lhs of equation (623). There exists $t_1 \in [0,1]$ such that $|tp+q| \geqslant |p||t-t_1|$ for all $t \in [0,1]$.

$$\begin{aligned} I_{[0,1]} =& I_{[0,t_1]} + I_{[t_1,1]} \leqslant 2\log \frac{\Lambda' + |p|}{\Lambda'} \leqslant 2\log 4 \max(1, \frac{|p|}{\Lambda' + \eta}) \\ \leqslant& 2\Big(\log 4 + \log_+ \frac{|p|}{\Lambda' + \eta}\Big). \end{aligned} \tag{624}$$

■

Lemma 48 *Let $r > 0$, $w \in \mathbb{N}$ and $x, y \in \mathbb{R}^4$*

$$\frac{e^{-rx^2}}{(1+|x-y|)^w} \leqslant \frac{w! \max(2, 1 + \frac{1}{2r})^w}{(1+|y|)^w}. \tag{625}$$

Proof Choosing the Cartesian coordinate system such that one of the basis vectors e_L is along y we have

$$\frac{e^{-rx^2}}{(1+|x-y|)^w} \leqslant f(t), \qquad f(t) := \frac{e^{-rt^2}}{(1+|t-t_0|)^w}. \tag{626}$$

where t, t_0 are the longitudinal coordinates, $x = x_T + te_L$ and $y = t_0 e_L$, $t_0 = |y|$. If $t \geqslant t_0$ then f is strictly decreasing. If $t \leqslant t_0$ then $f(t) = g(t)$ where

$$g(t) := \frac{e^{-rt^2}}{(t_1-t)^w}, \qquad t_1 := 1 + t_0. \tag{627}$$

If $t < t_1$ then g is either increasing or has a local maximum at t_-.

$$g'(t) \leqslant 0 \text{ if } (\Delta \geqslant 0) \wedge (t_- \leqslant t \leqslant t_+), \qquad t_\mp := \frac{t_1 \mp \sqrt{\Delta}}{2}, \tag{628}$$

$$g'(t) \geqslant 0 \text{ otherwise }, \qquad \Delta := t_1^2 - 2\frac{w}{r}. \tag{629}$$

Consequently, $f(t) \leqslant \max(g(t_-), g(t_0))$ where

$$g(t_-) = \frac{e^{-rt_-^2}}{t_+^w} \leqslant \frac{2^w}{t_1^w} = \frac{2^w}{(1+t_0)^w}, \quad g(t_0) = e^{-rt_0^2} \leqslant \frac{w!(1+\frac{1}{2r})^w}{(1+t_0)^w}. \tag{630}$$

■

Lemma 49 *Let $r > 0$. There is a constant C such that*

$$e^{-r\frac{s^2}{\Lambda^2}} \log_+ \frac{\max(M, \sqrt{p^2+s^2})}{\Lambda} < C + \frac{1}{2}\log_+ \frac{1}{r} + \log_+ \frac{\max(M,p)}{\Lambda}. \tag{631}$$

Proof Using the following inequality

$$\frac{\max(M, \sqrt{p^2+s^2})}{\Lambda} \leqslant 2\max(1, \frac{s}{\Lambda})\max(1, \frac{\max(M,p)}{\Lambda}), \tag{632}$$

we bound the lhs of the statement by

$$\log_+ \frac{\max(M,p)}{\Lambda} + \log 2 + \frac{1}{2}\log_+ \frac{1}{r} + \frac{1}{2}e^{-z}\log_+ z, \quad z := r\frac{s^2}{\Lambda^2}. \tag{633}$$

The inequality $e^{-z}\log_+ z < e^{-1}$ finishes the proof. ■

Lemma 50 *Let $x, y, m \geqslant 0$, $\mathcal{P}^{(0)}(x) = \sum_{k=0}^{n} c_k x^k$ a polynomial of the degree n, $\log_m x := \log_+ \max(x, m)$. Then there exist polynomials $\mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$ of the degree n such that*

$$\mathcal{P}^{(0)}(\log_m \sqrt{y^2 + x^2}) \leqslant \mathcal{P}^{(1)}(\log_m y) + \mathcal{P}^{(2)}(\log_m x). \tag{634}$$

Proof Substitution of the inequalities

$$\max(\sqrt{y^2 + x^2}, m) \leqslant \max(y + x, m) \leqslant \max(y, m) + \max(x, m), \tag{635}$$

$$\max(a + b, 1) \leqslant \max(a, 1) + \max(b, 1) \leqslant 2 \max(a, 1) \max(b, 1), \tag{636}$$

into the definition $\log_+ a := \log \max(a, 1)$ yields

$$\log_m \sqrt{y^2 + x^2} \leqslant \log_m y + \log_m x + \log 2. \tag{637}$$

This gives

$$\mathcal{P}^{(0)}(\log_m \sqrt{x^2 + y^2}) \leqslant \sum_{k=0}^{n} c_k 3^k \left(\log_m^k y + \log_m^k x + 1\right). \tag{638}$$

■

Lemma 51 *For a fixed $s \in \mathbb{N}$ there exists a constant c such that $\forall u \leqslant \bar{w}$ and $\forall x \in \mathbb{R}^4$*

$$|\partial^u \left(x^{\otimes s}(1 - e^{-x^4})\right)| \leqslant c \begin{cases} |x|^{s+1-u}, & u \leqslant s, \\ 1, & \mathit{otherwise}. \end{cases} \tag{639}$$

Proof First we consider the case $u \leqslant s$

$$|\partial^u(x^{\otimes s}(1 - e^{-x^4}))| \leqslant |\partial^u x^{\otimes s}|(1 - e^{-x^4}) + \sum_{u_2 > 0} \frac{u! 4^{u_2}}{u_1! u_2!} |\partial^{u_1} x^{\otimes s + 3u_2}| \, e^{-x^4}. \tag{640}$$

For the derivatives on the right we have

$$|\partial^{u_1} x^{\otimes s + 3u_2}| \leqslant (s + 3u_2)! \, \frac{|x|^{s-u+1}}{(s - u + 1)!} \frac{|x|^{4u_2 - 1}}{(4u_2 - 1)!}. \tag{641}$$

Furthermore, $\forall k \in \mathbb{N}$

$$\frac{|x|^k}{k!}e^{-x^4} < \sum_{n=0}^{\infty} \frac{|x|^n}{n!}e^{-x^4} = e^{-x^4+|x|} < e. \tag{642}$$

Noting that $1 - e^{-x^4} \leqslant |x|$, we obtain the bound in the case $u \leqslant s$

$$\frac{s!|x|^{s+1-u}}{(s-u)!} + \frac{e|x|^{s+1-u}}{(s-u+1)!} \sum_{u_2>0} \frac{u!4^{u_2}(s+3u_2)!}{u_1!u_2!} \leqslant c|x|^{s+1-u}\,. \tag{643}$$

For $u > s$ using (642) we have

$$\sum \frac{u!4^{u_2}(s+3u_2)!}{u_1!u_2!} \frac{|x|^{s+3u_2-u_1}}{(s+3u_2-u_1)!}\, e^{-x^4} \leqslant (s+3u)!5^u e \leqslant c\,. \tag{644}$$

■

Lemma 52 *For a fixed $s \in \mathbb{N}\backslash\{0\}$ there exists a constant c such that for all $u, v \leqslant \bar{w}$ and all $x = (x_1, x_2)$ with $x_i \in \mathbb{R}^4$*

$$|\partial_2^v \partial_1^u \left(x_2 \otimes x_1^{\otimes s-1}(1 - e^{-x_1^4}) \right)| \leqslant c \begin{cases} |x|^{s+1-u-v}, & u+v \leqslant s, \\ |x|+1, & otherwise. \end{cases} \tag{645}$$

Proof First let $v \in \{0, 1\}$. Using lemma 51 we obtain for $0 \leqslant u \leqslant s$

$$|\partial_1^u (x_2 \otimes x_1^{\otimes s-1}(1 - e^{-x_1^4}))| \leqslant c|x|^{s+1-u}\,, \tag{646}$$
$$|\partial_2 \partial_1^u (x_2 \otimes x_1^{\otimes s-1}(1 - e^{-x_1^4}))| \leqslant c|x|^{s-u}\,, \tag{647}$$

and for $u > s$ we also have

$$|\partial_1^u (x_2 \otimes x_1^{\otimes s-1}(1 - e^{-x_1^4}))| \leqslant c|x|\,, \tag{648}$$
$$|\partial_2 \partial_1^u (x_2 \otimes x_1^{\otimes s-1}(1 - e^{-x_1^4}))| \leqslant c\,. \tag{649}$$

Finally, for $v > 1$ $\partial_2^v \partial_1^u (x_2 \otimes x_1^{\otimes s-1}(1 - e^{-x_1^4})) = 0$. ■

Lemma 53 *For $s \in \{0, 1\}$ there exists a constant c such that for all $u, v \leqslant \bar{w}$ and all $x = (x_1, x_2)$ with $x_i \in \mathbb{R}^4$*

$$|\partial_2^v \partial_1^u \left(x_1^{\otimes s}(e^{-x_1^4} - e^{-(x_1+x_2)^4}) \right)| \leqslant c \begin{cases} |x|^{s+1-u-v}, & u+v \leqslant s, \\ |x|+1, & otherwise. \end{cases} \tag{650}$$

Proof First let $s=0$. For $u=v=0$ put $y=x_1+x_2$ and assume $|x_1|\leqslant|y|\leqslant 1$

$$e^{-x_1^4}-e^{-y^4}=e^{-x_1^4}(1-e^{-(y^4-x_1^4)})\leqslant y^4-x_1^4\leqslant|y|\leqslant 2|x|\,. \tag{651}$$

Inequality (642) implies that

$$\partial^w e^{-x^4}<e(3w)!5^4. \tag{652}$$

Consequently for $u+v>0$

$$|\partial_2^v\partial_1^u(e^{-x_1^4}-e^{-(x_1+x_2)^4})|\leqslant|\partial_1^u e^{-x_1^4}|+|\partial_2^v\partial_1^u e^{-(x_1+x_2)^4})|\leqslant c_2\,. \tag{653}$$

Finally, we consider the case $s=1$. Using (651) and (652) we get

$$|x_1(e^{-x_1^4}-e^{-(x_1+x_2)^4})|\leqslant 2|x|^2\,, \tag{654}$$

$$|\partial_2(x_1(e^{-x_1^4}-e^{-x_1+x_2{}^4}))|\leqslant|x||\partial_2(e^{-x_1^4}-e^{-(x_1+x_2)^4})|\leqslant c_3|x|\,, \tag{655}$$

$$|\partial_1(x_1(e^{-x_1^4}-e^{-(x_1+x_2)^4}))|\leqslant c_4|x|\,, \tag{656}$$

and for $u+v>1$

$$|\partial_1^u\partial_2^v(x_1(e^{-x_1^4}-e^{-(x_1+x_2)^4}))|\leqslant c_5+|x|c_6\,. \tag{657}$$

■

Lemma 54 *Let* $0\leqslant\beta\leqslant 1$, $x\geqslant 0$ *and*

$$f(x):=\frac{1}{x}(e^{-\beta x}-e^{-x}) \tag{658}$$

Then $\forall w\in\mathbb{N}$

$$|\partial^w f(x)|<e\frac{w!}{(1+x)^{w+1}}. \tag{659}$$

Proof We have an identity

$$\partial^w f(x)=(-1)^w g_w(x),\qquad\qquad g_w(x):=\int\limits_\beta^1 d\gamma\,\gamma^w e^{-\gamma x}, \tag{660}$$

It follows

$$0 \leqslant g_w(x) \leqslant \int_0^1 d\gamma\, \gamma^w e^{-\gamma(x+1)} e^{\gamma} < e \int_0^1 d\gamma\, \gamma^w e^{-\gamma(x+1)}$$
$$< e\frac{1}{(x+1)^{w+1}} \int_0^\infty dz\, z^w e^{-z} = e\frac{\Gamma(w+1)}{(x+1)^{w+1}}. \tag{661}$$

■

Lemma 55 *Let $0 \leqslant \beta \leqslant 1$, $x \geqslant 0$ and*

$$h(x) := \frac{1}{x}(e^{-\beta x^2} - e^{-x^2}) \tag{662}$$

Then $\forall w \in \mathbb{N}$, $\forall C > 1$

$$|\partial^w h(x)| < w! e(C+1) \frac{(2\sqrt{e}C)^w}{(1+x)^{w+1}}. \tag{663}$$

Proof

$$|\partial^w h(x)| \leqslant |x\partial^w f(x^2)| + |w\partial^{w-1} f(x^2)|, \qquad f(x^2) := \frac{1}{x}h(x). \tag{664}$$

Using an auxiliary variable y

$$\begin{aligned} \frac{\partial^w}{(\partial x)^w} f(x^2) &= \left(\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}\right)^w f(y)|_{y=x^2} \\ &= \sum_{k=0}^{2k\leqslant w} \frac{w!(2k-1)!!}{(w-2k)!(2k)!} \left(2\frac{\partial}{\partial y}\right)^k \left(2x\frac{\partial}{\partial y}\right)^{w-2k} f(y)|_{y=x^2} \\ &= \sum_{k=0}^{2k\leqslant w} \frac{w!}{(w-2k)!k!} (2x)^{w-2k} \left(\frac{\partial}{\partial y}\right)^{w-k} f(y)|_{y=x^2}. \end{aligned} \tag{665}$$

Equation (659) gives

$$|x\partial^w f(x^2)| < w! 2^w e \sum_{k=0}^{2k\leqslant w} \frac{1}{k!} \frac{(w-k)!}{(w-2k)!} 2^{-2k} \frac{x^{w-2k+1}}{(1+x^2)^{w-k+1}}. \tag{666}$$

With an arbitrary constant $C > 1$ we have

$$\frac{x}{1+x^2} < \frac{C}{1+x}, \qquad (1+x^2) \geqslant \frac{1}{2}(1+x)^2, \qquad \frac{(w-k)!}{(w-2k)!} \leqslant w^k. \tag{667}$$

Consequently,

$$|x\partial^w f(x^2)| < \frac{w!Ce(2C)^w}{(1+x)^{w+1}} \sum_{k=0}^{2k\leqslant w} \frac{1}{k!}\left(\frac{w}{2C^2}\right)^k < \frac{w!Ce(2\sqrt{e}C)^w}{(1+x)^{w+1}}. \tag{668}$$

Similarly, we obtain

$$|w\partial^{w-1} f(x^2)| < \frac{2w!e(2\sqrt{e}C)^{w-1}}{(1+x)^{w+1}}. \tag{669}$$

■

Lemma 56 *Let $f(p^2)$ be a scalar function. Then*

$$|\prod_{i=1}^{w} \frac{\partial}{\partial p_{\mu_i}} f(p^2)| \leqslant 2^w \sum_{k=0}^{2k\leqslant w} \frac{w!}{(w-2k)!k!} |p|^{w-2k} | \left(\frac{\partial}{\partial p^2}\right)^{w-k} f(p^2)|. \tag{670}$$

Proof With the aid of an auxiliary variable y

$$\prod_{i=1}^{w} \frac{\partial}{\partial p_{\mu_i}} f(p^2) = \prod_{i=1}^{w} \left(\frac{\partial}{\partial p_{\mu_i}} + 2p_{\mu_i}\frac{\partial}{\partial y}\right) f(y)|_{y=p^2}. \tag{671}$$

A partial derivative wrt p_μ contributes only if it can be paired with $2p_\mu$ term. Consequently, we can compute the right hand side by considering the possible pairs,

$$(\frac{\partial}{\partial p_{\mu_i}} + 2p_{\mu_i}\frac{\partial}{\partial y})(\frac{\partial}{\partial p_{\mu_j}} + 2p_{\mu_j}\frac{\partial}{\partial y}) = 2\delta_{\mu_i\mu_j}\frac{\partial}{\partial y} + 2p_{\mu_i}2p_{\mu_j}\left(\frac{\partial}{\partial y}\right)^2. \tag{672}$$

It gives

$$\sum_{k=0,\pi}^{2k\leqslant w} \frac{1}{(w-2k)!(2k)!} \sum_{\sigma,\pi':=\sigma\pi} \frac{1}{2^k k!} \prod_{i=1}^{k}\left(2\delta_{\pi'_i\pi'_{i+1}}\frac{\partial}{\partial y}\right) \prod_{i=2k+1}^{w}\left(2p_{\pi_i}\frac{\partial}{\partial y}\right), \tag{673}$$

where the outer and inner sums run over $w!$, $(2k)!$ permutations, respectively. Using the inequality $|A_{\vec{\mu}}B_{\vec{\nu}}| \leqslant |A_{\vec{\mu}}||B_{\vec{\nu}}|$ we obtain the upper bound. ■

Lemma 57 *Let $C > 1$*

$$|\prod_{i=1}^{w} \frac{\partial}{\partial p_{\mu_i}} S^{\Lambda\Lambda_0}(p)| < 2w!e(C+1)\frac{(2^2e^{\frac{3}{2}}C^2)^w}{(\Lambda+|p|)^{w+2}}. \tag{674}$$

Proof We change the variable $x_\mu = p_\mu/\Lambda$

$$\prod_{i=1}^{w} \frac{\partial}{\partial p_{\mu_i}} S^{\Lambda\Lambda_0}(p) = \frac{1}{\Lambda^{2w+2}} \prod_{i=1}^{w} \frac{\partial}{\partial x_{\mu_i}} h(x^2). \tag{675}$$

Equations (670), (663) yield

$$|\partial^w h(x^2)| < 2^w e(C+1) \sum_{k=0}^{2k\leqslant w} \frac{w!}{k!} \frac{(w-k)!}{(w-2k)!} \frac{x^{w-2k}(2\sqrt{e}C)^{w-k}}{(1+x^2)^{w-k+1}}. \tag{676}$$

■

Lemma 58 *Let $C > 1$*

$$|\prod_{i=1}^{w} \frac{\partial}{\partial p_{\mu_i}} C^{\Lambda\Lambda_0}_{\mu\nu}(p)| < 2^3 w! e(C+1+(\xi+1)(C^2+3)) \frac{(2^2 e^{\frac{3}{2}} C^2)^w}{(\Lambda+|p|)^{w+2}}. \tag{677}$$

Proof

$$|\partial^w C^{\Lambda\Lambda_0}_{\mu\nu}(p)| \leqslant 4|\partial^w S^{\Lambda\Lambda_0}(p)| + |\xi - 1||\partial^w \frac{p_\mu p_\nu}{p^2} S^{\Lambda\Lambda_0}(p)|. \tag{678}$$

The first term is bounded in (674). Using $x_\mu = p_\mu/\Lambda$ we have an upper bound for the last term

$$x^2|\partial^w f(x^4)| + 2^3 w|x||\partial^{w-1} f(x^4)| + 4^2 w(w-1)|\partial^{w-2} f(x^4)|, \tag{679}$$

where $f(x^4)$ is the same as in (659). Equation 659 gives

$$|\partial^w f(x^2)| < \frac{2w! e(2\sqrt{e}C)^w}{(1+x)^{w+2}}, \qquad |\partial^w f(x^4)| < \frac{2^3 w! e(2^2 e^{\frac{3}{4}} C^2)^w}{(1+x)^{w+4}}. \tag{680}$$

Consequently,

$$|\partial^w x_\mu x_\nu f(x^4)| < 2^3 (C^2+3) e w! \frac{(2^2 C^2 e^{\frac{3}{4}})^w}{(1+x)^{w+2}}. \tag{681}$$

■

Lemma 59 *For all $p \in \mathbb{R}^4$ there exists a constant C such that*

$$|\dot{\mathbf{C}}^{\Lambda\Lambda_0}(p)| \leqslant C \frac{1}{\Lambda^3} e^{-\frac{p^2}{\Lambda^2}}. \tag{682}$$

Proof Using the inequality $xe^{-x^2} \leqslant xe^{1-2x} = xe^{1-x}e^{-x} \leqslant e^{-x}$ we obtain

$$|\dot{S}^{\Lambda\Lambda_0}(p)| = 4\frac{1}{\Lambda^3}xe^{-x^2} \leqslant \frac{4}{\Lambda^3}e^{-x}, \qquad x := \frac{p^2}{\Lambda^2}, \tag{683}$$

$$|\dot{C}^{\Lambda\Lambda_0}_{\mu\nu}(p)| \leqslant 4\frac{1+|\xi-1|}{\Lambda^3}xe^{-x^2}. \tag{684}$$

■

Lemma 60 *For all $p \in \mathbb{R}^4$ there exists a constant C such that*

$$|\partial_p^w \partial_{\Lambda_0} \mathbf{C}^{\Lambda\Lambda_0}(p)| \leqslant C\frac{1}{\Lambda_0(\Lambda_0+|p|)^{2+\|w\|}}. \tag{685}$$

Proof Let $f_{\mu\nu} = e^{-x^4}x_\mu x_\nu$ then using $|\partial_x^w x^2| \leqslant |\partial_x^w x_\mu x_\nu|$ we have

$$|\partial_p^w \partial_{\Lambda_0} C^{\Lambda\Lambda_0}(p)| \leqslant \frac{4(2+|\xi-1|)|\partial_x^w f|}{\Lambda_0^{\|w\|+3}}, \quad |\partial_p^w \partial_{\Lambda_0} S^{\Lambda\Lambda_0}(p)| \leqslant \frac{4|\partial_x^w f|}{\Lambda_0^{\|w\|+3}}, \tag{686}$$

where introducing $C_1 = 2^{\frac{3}{2}\bar{w}}(4\bar{w}+2)!$

$$|\partial_x^w f| \leqslant C_1 e^{-x^4}(|x|^{3\|w\|+2}+1) \leqslant eC_1 e^{-2x^2}(|x|^{3\|w\|+2}+1). \tag{687}$$

Then it is easy to see that for $0 \leqslant m \leqslant 3\bar{w}+2$

$$\max(e^{-x^2}(x^m+1)) \leqslant \max(e^{-x^2}x^m)+1 \leqslant e^{\frac{m}{2}(\log\frac{m}{2}-1)}+1 \leqslant C_2. \tag{688}$$

To go further we need the following inequality for all $y \geqslant 0$ and $k \in \mathbb{N}$

$$g_k(y) \leqslant k!, \qquad g_k(y) := (1+y)^k e^{-y}, \tag{689}$$

which is obtained looking for the maximum $\bar{y}_k = k-1$ and using $g_0(\bar{y}_0) = 1$.

$$g_k(\bar{y}_k) = kg_{k-1}(\bar{y}_k) \leqslant kg_{k-1}(\bar{y}_{k-1}) \implies g_k(y) \leqslant g_k(\bar{y}_k) \leqslant k! \tag{690}$$

Defining $C_3 := e^2 C_1 C_2$ and using inequality (689) we have

$$|\partial_x^w f| \leqslant C_3 e^{-2x} \leqslant C_3 \frac{2!}{(1+|x|)^2}\frac{\|w\|!}{(1+x)^{\|w\|}} \leqslant \frac{2\,C_3\,\bar{w}!}{(1+|x|)^{2+\|w\|}}. \tag{691}$$

■

Lemma 61 *Let $\mathcal{C}^{\Lambda\Lambda_0}$ be one of the propagators $S^{\Lambda\Lambda_0}$ or $C^{\Lambda\Lambda_0}_{\mu\nu}$, as defined in (40), (41). There are positive constants c_0, c_1, d such that for all $w \in \mathbb{N}$, $p \in \mathbb{R}^4$, $0 < \Lambda \leqslant \Lambda_0$, and with $c_\xi := c_0 + \xi c_1$,*

$$|\Big(\prod_{i=1}^{w}\frac{\partial}{\partial p_{\mu_i}}\Big)\mathcal{C}^{\Lambda\Lambda_0}(p)| < \frac{w!\,d^w c_\xi}{(|p|+\Lambda)^{w+2}}, \qquad |\dot{\mathcal{C}}^{\Lambda\Lambda_0}(p)| < \frac{c_\xi}{\Lambda^3}e^{-\frac{p^2}{\Lambda^2}}. \tag{692}$$

Proof The statement follows from (674), (677), (682). ■

E The functional $\Gamma_{n\leqslant 4}$

We expand the generating functional $\tilde{\Gamma}^{0\Lambda_0}(\underline{A},\underline{c},\underline{\bar{c}})$ of (113) as formal series in $\underline{A},\underline{c},\underline{\bar{c}}$. As usual, we adopt the shorthand notation $\Gamma^{0\Lambda_0}(A,\bar{c},c)$ for $\tilde{\Gamma}^{0\Lambda_0}(\underline{A},\underline{c},\underline{\bar{c}})$.

$$\Gamma^{0\Lambda_0} = \Gamma^{0\Lambda_0}_{n\leqslant 4} + \Gamma^{0\Lambda_0}_{n\geqslant 5}, \qquad \Gamma^{0\Lambda_0}_{n\leqslant 4} := \sum_{n=1}^{4} \Gamma^{0\Lambda_0}_{n}, \tag{693}$$

where n counts the number of fields. The functionals $\Gamma^{0\Lambda_0}_n$ with $n \leqslant 4$ contain both relevant and irrelevant terms. We assume hypothesis RC1. In general the tensors $\zeta^{\vec{\phi}}_{\mu_1\dots\mu_r}$ appearing in the form factors $F^{\vec{\phi}}_{\mu_1\dots\mu_r}(\vec{p})$ are elements of $\mathrm{span}(\{\delta^s \underline{p}^{k>0}\}_{r>0})$ where $\underline{p} = (p_1,\dots,p_{n-1})$.

1. **One-point function**
   There are no local terms that preserve Euclidean invariance and global SU(2) symmetry. It follows that $\Gamma_1 = 0$.

2. **Two-point functions**

$$\Gamma^{0\Lambda_0}_2 = \frac{1}{2}\langle F^{AA}_{\mu\nu}\check{A}^a_\mu \check{A}^a_\nu\rangle + \langle F^{\bar{c}c}\check{\bar{c}}^a \check{c}^a\rangle, \tag{694}$$

$$\begin{aligned} F^{AA}_{\mu\nu}(p) &:= (\delta_{\mu\nu}p^2 - p_\mu p_\nu)(\sigma^{-1}_{0\Lambda_0}(p^2) + \Sigma^{AA}_T(p^2)) \\ &\quad + \frac{1}{\xi}p_\mu p_\nu(\sigma^{-1}_{0\Lambda_0}(p^2) + \Sigma^{AA}_L(p^2)), \end{aligned} \tag{695}$$

$$F^{\bar{c}c}(p) := -p^2(\sigma^{-1}_{0\Lambda_0}(p^2) + \Sigma^{\bar{c}c}(p^2))\,. \tag{696}$$

   We assume that the form factors Σ^{AA} and $\Sigma^{\bar{c}c}$ include all loop corrections. Note that for the functional $\underline{\Gamma}^{0\Lambda_0}_2$ we have

$$\underline{\Gamma}^{0\Lambda_0}_2 = \frac{1}{2}\langle \underline{F}^{AA}_{\mu\nu}\check{A}^a_\mu \check{A}^a_\nu\rangle + \langle F^{\bar{c}c}\check{\bar{c}}^a \check{c}^a\rangle, \tag{697}$$

$$\underline{F}^{AA}_{\mu\nu}(p) := F^{AA}_{\mu\nu}(p) - \frac{1}{2\xi}p_\mu p_\nu\,. \tag{698}$$

   With $p^2 = M^2$ substitution of the above definitions into the expressions $\sigma_{0\Lambda_0}F^{AA}$, $\sigma_{0\Lambda_0}F^{\bar{c}c}$ appearing in AGE(166) and STI(167) gives

$$\sigma_{0\Lambda_0}(p^2)\,F^{\bar{c}c}(p) \overset{\beta}{\sim} -p^2(1 + \Sigma^{\bar{c}c}(p^2)), \tag{699}$$

$$\sigma_{0\Lambda_0}(p^2)\, \underline{F}^{AA}_{\mu\nu}(p) \overset{\beta}{\sim} F^{AA}_{T;\mu\nu}(p)\,, \tag{700}$$

$$F^{AA}_{T;\mu\nu}(p) := (\delta_{\mu\nu}p^2 - p_\mu p_\nu)(1+\Sigma^{AA}_T(p^2)) + \frac{1}{\xi}p_\mu p_\nu \Sigma^{AA}_L(p^2). \tag{701}$$

Using (116) for the functional $\mathsf{\Gamma}^{0\Lambda_0}_2$ we have

$$\mathsf{\Gamma}^{0\Lambda_0}_2 = \frac{1}{2}\langle \mathsf{F}^{AA}_{\mu\nu}\check{A}^a_\mu \check{A}^a_\nu\rangle + \langle \mathsf{F}^{\bar{c}c}\check{\bar{c}}^a \check{c}^a\rangle, \tag{702}$$

$$\mathsf{F}^{AA}_{\mu\nu}(p) := (\delta_{\mu\nu}p^2 - p_\mu p_\nu)\Sigma^{AA}_T(p^2) + \frac{1}{\xi}p_\mu p_\nu \Sigma^{AA}_L(p^2)\,, \tag{703}$$

$$\mathsf{F}^{\bar{c}c}(p) := -p^2\Sigma^{\bar{c}c}(p^2)\,. \tag{704}$$

For marginal terms we obtain

$$\mathsf{F}^{AA;p_\rho p_\sigma}_{\mu\nu}(p) = 2\delta_{\mu\nu}\delta_{\rho\sigma}r^{AA}_1 + 2(\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\nu\rho}\delta_{\mu\sigma})r^{AA}_2 + \zeta^{AA}_{\mu\nu\rho\sigma}\,, \tag{705}$$

$$\mathsf{F}^{\bar{c}c;p_\rho p_\sigma}(p) = 2\delta_{\rho\sigma}r^{\bar{c}c}(p^2) + \zeta^{\bar{c}c}_{\rho\sigma}\,, \tag{706}$$

$$r^{AA}_1(p^2) := \Sigma^{AA}_T(p^2) + p^2\frac{\partial \Sigma^{AA}_T(p^2)}{\partial p^2}\,, \tag{707}$$

$$r^{AA}_2(p^2) := \frac{1}{\xi}\Sigma^{AA}_L(p^2) - \Sigma^{AA}_T(p^2)\,, \tag{708}$$

$$r^{\bar{c}c}(p^2) := -\Sigma^{\bar{c}c}(p^2) - p^2\frac{\partial \Sigma^{\bar{c}c}(p^2)}{\partial p^2}\,. \tag{709}$$

3. **Three-point functions**

$$\Gamma^{0\Lambda_0}_3 = \langle \epsilon_{abd}F^{AAA}_{\rho\mu\nu}\check{A}^a_\rho\check{A}^b_\mu\check{A}^d_\nu\rangle + \langle \epsilon_{adb}F^{A\bar{c}c}_\mu \check{A}^a_\mu \check{\bar{c}}^b \check{c}^d\rangle, \tag{710}$$

$$F^{A\bar{c}c}_\mu(k,p,q) := ip_\mu R^{A\bar{c}c}_1(p,q) + iq_\mu r^{A\bar{c}c}_2(p,q), \tag{711}$$

$$R^{A\bar{c}c}_1(p,q) := g + r^{A\bar{c}c}_1(p,q), \tag{712}$$

$$\begin{aligned} F^{AAA}_{\rho\mu\nu}(k,p,q) :=& i\delta_{\mu\nu}(p_\rho - q_\rho)R^{AAA}(p,q) + i\delta_{\mu\nu}k_\rho\zeta^{AAA}_-(p,q) \\ &+ i\zeta^{AAA}_{\rho\mu\nu}(p,q), \end{aligned} \tag{713}$$

$$R^{AAA}(p,q) := \frac{1}{2}g + r^{AAA}(p,q). \tag{714}$$

Here $R^{AAA}(p,q)$ is a symmetric function whereas $\zeta^{AAA}_-(p,q)$ is antisymmetric.

4. **Four-point functions**

$$
\begin{aligned}
\Gamma_4^{0\Lambda_0} =& \langle F_{\sigma\rho\mu\nu}^{AAAA} \check{A}_\sigma^b \check{A}_\rho^b \check{A}_\mu^a \check{A}_\nu^a + F_{1,\mu\nu}^{\bar{c}cAA} \check{\bar{c}}^b \check{c}^b \check{A}_\mu^a \check{A}_\nu^a \\
&+ F_{2,\mu\nu}^{\bar{c}cAA} \check{\bar{c}}^a \check{c}^b \check{A}_\mu^a \check{A}_\nu^b + r^{\bar{c}c\bar{c}c} \check{\bar{c}}^b \check{c}^b \check{\bar{c}}^a \check{c}^a \rangle, \qquad (715)\\
F_{n,\mu\nu}^{\bar{c}cAA} :=& \delta_{\mu\nu} r_n^{\bar{c}cAA} + \zeta_{n,\mu\nu}^{\bar{c}cAA}, \qquad (716)\\
F_{\sigma\rho\mu\nu}^{AAAA} :=& R_{\sigma\rho\mu\nu}^{AAAA} + \zeta_{\sigma\rho\mu\nu}^{AAAA}, \qquad (717)\\
R_{\sigma\rho\mu\nu}^{AAAA} :=& \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\rho\nu}) R_1^{AAAA} + \delta_{\mu\nu}\delta_{\sigma\rho} R_2^{AAAA} \\
&+ \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\rho\nu}) \zeta_-^{AAAA}, \qquad (718)\\
R_1^{AAAA} :=& -\frac{g^2}{4} + r_1^{AAAA}, \qquad (719)\\
R_2^{AAAA} :=& \frac{g^2}{4} + r_2^{AAAA} \qquad (720)
\end{aligned}
$$

Here the terms

$$
\begin{array}{cccccc}
r_1^{AA}, & r_2^{AA}, & r^{\bar{c}c}, & R_1^{A\bar{c}c}, & r_2^{A\bar{c}c}, & R^{AAA}, \\
R_1^{AAAA}, & R_2^{AAAA}, & r_1^{\bar{c}cAA}, & r_2^{\bar{c}cAA}, & r^{\bar{c}c\bar{c}c}, &
\end{array} \qquad (721)
$$

are scalar functions of momenta, Λ_0, and M. All 11 renormalization constants are fine-tuned by imposing appropriate renormalization conditions.

F The functionals $\Gamma_{\gamma;n\leqslant 2}$ and $\Gamma_{\omega;n\leqslant 2}$

With $\Gamma_\varkappa^{0\Lambda_0}(p) := \tilde{\delta}_{\varkappa(p)} \Gamma^{0\Lambda_0}|_{\varkappa=0}$,

$$
\begin{aligned}
\Gamma_{\gamma_\mu^a;n\leqslant 2}^{0\Lambda_0}(p) =& R_1 i p_\mu c^a(p) + g\epsilon^{abs} \langle F_{\mu\nu}^{\gamma Ac} | \check{A}_\nu^b \check{c}^s; p\rangle, \qquad (722)\\
F_{\mu\nu}^{\gamma Ac}(k,q) :=& \delta_{\mu\nu} R_2 + \zeta_{\mu\nu}^{\gamma Ac}(k,q), \qquad (723)\\
\zeta_{\mu\nu}^{\gamma Ac}(k,q) :=& k_\mu q_\nu \zeta_1^{\gamma Ac} + k_\nu q_\mu \zeta_2^{\gamma Ac} + k_\mu k_\nu \zeta_3^{\gamma Ac} + q_\mu q_\nu \zeta_4^{\gamma Ac}, \qquad (724)\\
\Gamma_{\omega^a;n\leqslant 2}^{0\Lambda_0}(p) =& \frac{1}{2} g\epsilon^{abs} \langle R_3 | \check{c}^b \check{c}^s; p\rangle. \qquad (725)
\end{aligned}
$$

Here R_1, R_2, R_3 are scalar functions of momenta, Λ_0, and M.

G The functional $\mathrm{F}_{1,rel}$

In this section we introduce the notation for the renormalization constants for the functional F_1 (162). For this purpose we define the auxiliary functional

$$
\begin{aligned}
\mathrm{F}_{1,rel}^{\Lambda_0} :=&\epsilon^{dab}\langle U_\sigma^{\gamma cc}|\check{\gamma}_\sigma^d\check{c}^a\check{c}^b\rangle + \epsilon^{sab}\epsilon^{sde}u^{\gamma Acc}\langle\check{\gamma}_\sigma^a\check{A}_\sigma^b\check{c}^d\check{c}^e\rangle \\
&+ \langle U_\sigma^{cA}|\check{c}^d\check{A}_\sigma^d\rangle + \epsilon^{dab}\langle U^{\bar{c}cc}|\check{\bar{c}}^d\check{c}^a\check{c}^b\rangle + \epsilon^{dab}\langle U_{\mu\nu}^{cAA}|\check{c}^d\check{A}_\mu^a\check{A}_\nu^b\rangle \\
&+ \langle U_{\rho\mu\nu}^{cAAA}|\check{c}^d\check{A}_\rho^d\check{A}_\mu^a\check{A}_\nu^a\rangle + \langle U_{1;\mu}^{\bar{c}ccA}|\check{\bar{c}}^b\check{c}^b\check{c}^a\check{A}_\mu^a\rangle \\
&+ \langle U_{2;\mu}^{\bar{c}ccA}|\check{\bar{c}}^a\check{c}^b\check{c}^b\check{A}_\mu^a\rangle + \epsilon^{bed}u_1^{c\bar{c}cAA}\langle\check{c}^b\check{\bar{c}}^e\check{c}^d\check{A}_\mu^a\check{A}_\mu^a\rangle \\
&+ \epsilon^{bed}u_2^{c\bar{c}cAA}\langle\check{c}^b\check{\bar{c}}^a\check{c}^d\check{A}_\mu^a\check{A}_\mu^e\rangle + \epsilon^{bed}u_3^{c\bar{c}cAA}\langle\check{c}^a\check{\bar{c}}^e\check{c}^d\check{A}_\mu^a\check{A}_\mu^b\rangle, \qquad (726)
\end{aligned}
$$

where

$$
\begin{aligned}
U_\sigma^{\gamma cc}(l,p,q) :=&i(p+q)_\sigma u^{\gamma cc}, &(727)\\
U_\sigma^{cA}(l,p) :=&ip_\sigma p^2 u^{cA}, &(728)\\
U^{\bar{c}cc}(l,p,q) :=&(p^2+q^2)u_1^{\bar{c}cc} + 2pq u_2^{\bar{c}cc}, &(729)\\
U_{\mu\nu}^{cAA}(l,p,q) :=&(p_\mu p_\nu - q_\mu q_\nu)u_1^{cAA} + \delta_{\mu\nu}(p^2-q^2)u_2^{cAA}, &(730)\\
U_{\rho\mu\nu}^{cAAA}(l,k,p,q) :=&i(u_1^{cAAA}k_\rho + u_2^{cAAA}p_\rho + u_2^{cAAA}q_\rho)\delta_{\mu\nu} \\
&+ i(u_3^{cAAA}k_\nu + u_4^{cAAA}q_\nu + u_5^{cAAA}p_\nu)\delta_{\rho\mu} \\
&+ i(u_3^{cAAA}k_\mu + u_4^{cAAA}p_\mu + u_5^{cAAA}q_\mu)\delta_{\rho\nu}, &(731)\\
U_{1;\mu}^{\bar{c}ccA}(l,k,q,p) :=&ip_\mu u_1^{\bar{c}ccA} + ik_\mu u_2^{\bar{c}ccA} + iq_\mu u_3^{\bar{c}ccA}, &(732)\\
U_{2;\mu}^{\bar{c}ccA}(l,k,q,p) :=&i(k_\mu - q_\mu)u_4^{\bar{c}ccA}, &(733)
\end{aligned}
$$

and the u's are functions of Λ_0 defined by the marginal renormalization conditions

$$\mathrm{F}_{1;\vec{\varkappa}}^{0\Lambda_0;\vec{\phi};w}(\vec{q}) = \mathrm{F}_{1,rel;\vec{\varkappa}}^{\Lambda_0;\vec{\phi};w}(\vec{q}) + \sum_{t\in\{\delta^s\underline{e}^{k>0}\}_r} \zeta_t^{\Lambda_0}\, t\,. \qquad (734)$$

Here $2n_\varkappa + \mathrm{N} + \|w\| = 5$, $\vec{q}$ is the renormalization point defined in section H, $\underline{e} = (e_i)_{i\in[m]}$ is an orthogonal basis for the linear span of $\vec{q}$, r is the tensor rank of $\mathrm{F}_1^{0\Lambda_0;\vec{\phi};w}(\vec{q})$. The $\zeta_t^{\Lambda_0}$ are the uniquely defined coefficients of tensors t. Note that we implicitly set to zero all constants associated to strictly relevant renormalization conditions for F_1. These constants are not needed because, thanks to hypothesis RC1, the RHS of the STI and AGE at the current loop order vanish at zero momenta.

The renormalization points $\vec{q}$ are chosen in agreement with the hypotheses of lemma 36. From lemma 36, theorem 3, and the irrelevant bounds of theorem 4, for the marginal terms one has

$$\mathrm{F}_{1;\vec{\varkappa}}^{0\Lambda_0;\vec{\phi};w}(\vec{q}) \sim \mathrm{F}_{1,rel;\vec{\varkappa}}^{\Lambda_0;\vec{\phi};w}(\vec{q}). \tag{735}$$

H List of the renormalization points

X	$[X]$	n_X	r_{m}	r_X	ren.p.
$\partial\partial\Gamma^{\bar{c}c}$	0	2	7	0	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_2$
$\partial\Gamma^{c}_{\gamma}$	0	2	7	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_2$
$\partial\partial\Gamma^{AA}$	0	2	7	4	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_2$
Γ^{cc}_{ω}	0	3	5	0	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_3$
$\partial_A\Gamma^{c\bar{c}A}$	0	3	5	2	$\Lambda = M,\ \vec{q} = 0$
$\partial_{\bar{c}}\Gamma^{c\bar{c}A}$	0	3	5	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_3$
Γ^{cA}_{γ}	0	3	5	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_3$
Γ^{AA}_{γ}	0	3	5	3	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_3$
$\partial\Gamma^{AAA}$	0	3	5	4	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_3$
$\Gamma^{\bar{c}c\bar{c}c}$	0	4	3	0	$\Lambda = M,\ \vec{q} = 0$
$\Gamma^{\bar{c}cAA}$	0	4	3	2	$\Lambda = M,\ \vec{q} = 0$
Γ^{AAAA}	0	4	3	4*	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^{cp}_4$
$\partial\partial\Gamma^{c}_{\beta}$	0	2	7	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_2$
$\partial\partial\partial\Gamma^{cA}_{1}$	0	2	7	4	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_2$
$\partial\partial\Gamma^{c\bar{c}c}_{1}$	0	3	5	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_3$
$\partial\Gamma^{cc}_{1,\gamma}$	0	3	5	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_3$
$\partial\partial\Gamma^{cAA}_{1}$	0	3	5	4	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_3$
$\Gamma^{c\bar{c}c}_{\beta}$	0	4	3	0	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_4$
$\Gamma^{ccc}_{1,\omega}$	0	4	3	0	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_4$
Γ^{cAA}_{β}	0	4	3	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_4$
$\partial\Gamma^{c\bar{c}cA}_{1}$	0	4	3	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_4$
$\Gamma^{ccA}_{1,\gamma}$	0	4	3	2	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_4$
$\partial\Gamma^{cAAA}_{1}$	0	4	3	4*	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^{cp}_4$
$\Gamma^{c\bar{c}c\bar{c}c}_{1}$	0	5	1	0	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_5$
$\Gamma^{c\bar{c}cAA}_{1}$	0	5	1	2*	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^s_5$
Γ^{cAAAA}_{1}	0	5	1	4*	$\Lambda = 0,\ \vec{q} \in \mathbb{M}^{cp}_5$

X	$[X]$	n_X	r_X	ren.p.
$\partial\Gamma^{\bar{c}c}$	1	2	1	$\Lambda = 0,\ \vec{q} = 0$
$\partial\Gamma^{AA}$	1	2	3	$\Lambda = 0,\ \vec{q} = 0$
$\Gamma^{\bar{c}cA}$	1	3	1	$\Lambda = 0,\ \vec{q} = 0$
Γ^{AAA}	1	3	3	$\Lambda = 0,\ \vec{q} = 0$
$\Gamma^{\bar{c}c}$	2	2	0	$\Lambda = 0,\ \vec{q} = 0$
Γ^{AA}	2	2	2	$\Lambda = 0,\ \vec{q} = 0$
$\partial\Gamma^{c}_{\beta}$	1	2	1	$\Lambda = 0,\ \vec{q} = 0$
$\partial\partial\Gamma^{cA}_{1}$	1	2	3	$\Lambda = 0,\ \vec{q} = 0$
Γ^{cA}_{β}	1	3	1	$\Lambda = 0,\ \vec{q} = 0$
$\partial\Gamma^{c\bar{c}c}_{1}$	1	3	1	$\Lambda = 0,\ \vec{q} = 0$
$\Gamma^{cc}_{1,\gamma}$	1	3	1	$\Lambda = 0,\ \vec{q} = 0$
$\partial\Gamma^{cAA}_{1}$	1	3	3	$\Lambda = 0,\ \vec{q} = 0$
$\Gamma^{c\bar{c}cA}_{1}$	1	4	1	$\Lambda = 0,\ \vec{q} = 0$
Γ^{cAAA}_{1}	1	4	3	$\Lambda = 0,\ \vec{q} = 0$
Γ^{c}_{β}	2	2	0	$\Lambda = 0,\ \vec{q} = 0$
$\partial\Gamma^{cA}_{1}$	2	2	2	$\Lambda = 0,\ \vec{q} = 0$
$\Gamma^{c\bar{c}c}_{1}$	2	3	0	$\Lambda = 0,\ \vec{q} = 0$
Γ^{cAA}_{1}	2	3	2	$\Lambda = 0,\ \vec{q} = 0$
Γ^{cA}_{1}	3	2	1	$\Lambda = 0,\ \vec{q} = 0$

List of all terms preserving the global symmetries, with an arbitrary number of γ, ω insertions, and with at most one β or 1 insertion (not both). Notation: $[X]$ is the mass dimension of X (reduced Fourier transform); r_X is the tensor rank; n_X is the total number of fields and sources (not including 1); ∂ stands for a momentum derivative; "ren.p." stands for "renormalization point". A $*$ in the rank entry means that the condition $r_{\rm m} := 11 - 2n_X \geq r+1$ is violated for a term X: as stated in Lemma 33 the tensor monomials $\{\delta^s \underline{q}^k\}_{r+1}$ are not linearly independent for $\vec{q} = (q_0, \underline{q}) \in \mathbb{M}^s_{n_X}$, hence they are not suitable as a basis for the form-factor decomposition of ∂X. See lemma 36 and sections 4.4, 4.5, 4.10.

I List of insertions

X	[X]	gh(X)	def
ψ^a_μ	2	1	(119)
γ^a_μ	2	-1	
Ω^a	2	2	(119)
ω^a	2	-2	
Q_ρ	5	1	(130)
$Q_{\rho\gamma}$	3	2	(126)
$Q_{\rho\omega}$	3	3	(127)
ρ	-1	-1	
Q_β	3	1	(131)
β	1	-1	

List of operators and sources, and their quantum numbers. Notation: $[X]$ stands for the mass dimension of X in position space; the ghost charge of the ghost field is $\text{gh}(c) := 1$.

References

[BDM93] M. Bonini, M. D'Attanasio, and G. Marchesini. Perturbative renormalization and infrared finiteness in the Wilson renormalization group: the massless scalar case. *Nucl. Phys. B*, 409:441 – 464, 1993.

[BDM95] M. Bonini, M. D'Attanasio, and G. Marchesini. BRS symmetry for Yang-Mills theory with exact renormalization group. *Nucl. Phys. B*, 437:163–186, 1995.

[Bec96] C. Becchi. On the construction of renormalized gauge theories using renormalization group techniques. *arXiv*, 9607188v1, 1996.

[Ber66] F.A. Berezin. *The method of second quantization.* Academic Press, 1966.

[BRS75] C. Becchi, A. Rouet, and R. Stora. Renormalization of the abelian Higgs-Kibble model. *Comm. Math. Phys*, 42(2):127–162, 1975.

[DF91] Yu.L. Dalecky and S.V. Fomin. *Measures and differential equations in infinite-dimensional space.* Springer, 1991.

[Dir64] P.A.M. Dirac. *Lectures on quantum mechanics.* Belfer Graduate School of Science, 1964.

[FHH16] M.B. Fröb, J. Holland, and S. Hollands. All-order bounds for correlation functions of gauge-invariant operators in Yang-Mills theory. *Journal Math. Phys.*, 57(12), 2016.

[FP67a] L.D. Faddeev and V.N. Popov. Feynman diagrams for the Yang-Mills field. *Physics Letters B*, 25:29–30, 1967.

[FP67b] L.D. Faddeev and V.N. Popov. Perturbation theory for gauge-invariant fields. *preprint ITF-67-036 Kiev*, 1967. In russian. English translation: NAL-THY-57.

[FS93] L.D. Faddeev and A.A. Slavnov. *Gauge fields: an introduction to quantum theory.* Westview Press, 2 edition, 1993.

[GG00] P. Gambino and P. A. Grassi. Nielsen identities of the SM and the definition of mass. *Phys. Rev. D*, 62(7):076002, 2000.

[GJ87] J. Glimm and A. Jaffe. *Quantum physics*. Springer, 1987.

[GK] R. Guida and Ch. Kopper. All-order uniform momentum bounds for the massless ϕ_4^4 field theory. *In preparation.*

[GK11] R. Guida and Ch. Kopper. All-order uniform momentum bounds for the massless ϕ^4 theory in four dimensional euclidean space. *arXiv*, 1103.5692, 2011.

[GT90] D.M. Gitman and I.V. Tyutin. *Quantization of Fields with Constraints*. Springer-Verlag, 1990.

[Hal74] P.R. Halmos. *Measure theory*. Springer, 1974.

[KK92] G. Keller and Ch. Kopper. Perturbative renormalization of composite operators via flow equations. *Comm. Math. Phys.*, 148:445–467, 1992.

[KKS92] G. Keller, Ch. Kopper, and M. Salmhofer. Perturbative renormalization and effective lagrangians in ϕ^4 in four-dimensions. *Helv. Phys. Acta*, 65:32–52, 1992.

[KKS97] G. Keller, Ch. Kopper, and C. Schophaus. Perturbative renormalization with flow equations in minkowski space. *Helv.Phys.Acta*, 70:247–274, 1997.

[KM02] Ch. Kopper and F. Meunier. Large momentum bounds from flow equations. *Annales Henri Poincaré*, 3:435–449, 2002.

[KM09] Ch. Kopper and V.F. Müller. Renormalization of spontaneously broken SU(2) Yang-Mills theory with flow equations. *Rev. in Math. Phys.*, 21(6):781–820, 2009.

[KSZ75] H. Kluberg-Stern and J.B. Zuber. Ward identities and some clues to the renormalization of gauge-invariant operators. *Phys. Rev. D*, 12:467–481, Jul 1975.

[Lai81] C.H. Lai, editor. *Gauge theory of weak and electromagnetic interactions*. World Scientific, 1981.

[LZJ72] B.W. Lee and J. Zinn-Justin. Spontaneously broken gauge symmetries. I,II,III. *Phys. Rev. D*, 5:3121–3137, 3137–3155,3155–3160., 1972.

[MD96] T.R. Morris and M. D'Attanasio. Gauge Invariance, the Quantum Action Principle, and the Renormalization Group. *Physics Letters B*, 378:213–221, 1996.

[MN76] T. Maskawa and H. Nakajima. Singular lagrangian and the Dirac-Faddeev method. *Prog. Theor. Phys.*, 56(4):1295–1309, 1976.

[Mor94] T.R. Morris. The exact renormalization group and approximate solutions. *Int. Journ. of Mod. Phys. A*, 09(14):2411–2449, 1994.

[Nie75] N.K. Nielsen. On the gauge dependence of spontaneous symmetry breaking in gauge theories. *Nucl. Phys. B*, 101:173–188, 1975.

[Pol84] J. Polchinski. Renormalization and effective Lagrangians. *Nucl. Phys. B*, 231:269–295, 1984.

[PS85] O. Piguet and K. Sibold. Gauge independence in ordinary Yang-Mills theories. *Nucl. Phys. B*, 253:517–540, 1985.

[Rud87] W. Rudin. *Measure theory.* McGraw-Hill, 1987.

[RW94] M. Reuter and C. Wetterich. Effective average action for gauge theories and exact evolution equations. *Nucl. Phys. B*, 417:181–214, 1994.

[Sal99] M. Salmhofer. *Renormalization.* Springer, 1999.

[Sla72] A.A. Slavnov. Ward identities in gauge theories. *Theo. and Math. Phys.*, 10(2):99–104, 1972.

[Tay71] J.C. Taylor. Ward identities and charge renormalization of the Yang-Mills field. *Nucl. Phys. B*, 33:436–444, 1971.

[tH71] G. 't Hooft. Renormalization of massless Yang-Mills fields. *Nucl. Phys. B*, 33:173–199, 1971.

[tHV72] G. 't Hooft and M. Veltman. Regularization and renormalization of gauge fields. *Nucl. Phys. B*, 44:189–213, 1972.

[Tyu75] I.V. Tyutin. Gauge invariance in field theory and statistical physics in operator formalism. *arXiv*, 0812.0580, 1975.

[Wet93] C. Wetterich. Exanct evolution equation for the effective potential. *Physics Letters B*, 301:90–94, 1993.

[WH73] F. Wegner and A. Houghton. Renormalization group equations for critical phenomena. *Phys. Rev. A*, 8(1):401–412, 1973.

[Wil71] K.G. Wilson. Renormalization group and critical phenomena. *Phys. Rev. B*, 4(9):3174–3183, 1971.

[YM54] C.N. Yang and R.L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.*, 96:191–195, Oct 1954.

[ZJ75] J. Zinn-Justin. *Trends in elementary particle theory*, volume 37. Springer Berlin Heidelberg, 1975. International summer institute on theoretical physics in Bonn 1974.

ÉCOLE DOCTORALE
Sciences chimiques : molécules, matériaux, instrumentation et biosystèmes

Titre : Renormalisation de la théorie de Yang-Mills SU(2) avec les équations du flot du groupe de renormalisation

Mots clés : équation du flot, groupe de renormalisation, théorie quantique des champs, théorie de Yang--Mills

Résumé : L'objectif de ce travail est une construction perturbative rigoureuse de la théorie de la Yang-Mills SU(2) dans l'espace euclidien à quatre dimensions. La technique d'intégration fonctionnelle donne une base mathématique pour établir les équations de flot différentielles du groupe de renormalisation pour l'action efficace. Si l'introduction de régulateurs dans l'espace de moments permet de donner une définition mathématique des fonctions de Schwinger, la difficulté importante de l'approche est le fait que ces régulateurs brisent l'invariance de jauge. Ainsi, le travail principal est alors de prouver à tous les ordres en perturbation l'existence de ces fonctions de correlation et la validité des identités de Slavnov-Taylor pour la théorie renormalisée.

Title : Renormalization of SU(2) Yang-Mills theory with flow equations

Keywords : flow equation, renormalization group, quantum field theory, Yang--Mills theory

Abstract : The goal of this work is a rigorous perturbative construction of the SU(2) Yang-Mills theory in four dimensional Euclidean space. The functional integration technique gives a mathematical basis for establishing the differential Flow Equations of the renormalization group for the effective action. While the introduction of momentum space regulators permits to give a mathematical definition of the Schwinger functions, the important difficulty of the approach is the fact that these regulators break gauge invariance. Thus the main part of the work is to prove at all loop orders the existence of the vertex functions and the restoration of the Slavnov-Taylor identities in the renormalised theory.

Université Paris-Saclay
Espace Technologique / Immeuble Discovery
Route de l’Orme aux Merisiers RD 128 / 91190 Saint-Aubin, France