Optimal transport and diffusion of currents

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Résumé

Cette thèse concerne l'étude d'équations aux dérivées partielles à la charnière de la physique de la mécanique des milieux continus et de la géométrie différentielle. Le point de départ étant le modèle d'électromagnétisme non-linéaire introduit par Max Born et Leopold Infeld en 1934 comme substitut aux traditionnelles équations linéaires de Maxwell. Ces équations sont remarquables par leurs liens avec la géométrie différentielle (surfaces extrémales dans l'espace de Minkowski) et ont connu un regain d'intérêt dans les années 90 en physique des hautes énergies (cordes et D-branes).

La thèse se compose de quatre chapitres.

La théorie des systèmes paraboliques dégénérés d'EDP non-linéaires est fort peu développée, faute de pouvoir appliquer les principes de comparaison habituels (principe du maximum), malgré leur omniprésence dans de nombreuses applications (physique, mécanique, imagerie numérique, géométrie...). Dans le premier chapitre, on montre comment de tels systèmes peuvent être parfois dérivés, asymptotiquement, à partir de systèmes non-dissipatifs (typiquement des systèmes hyperboliques non-linéaires), par simple changement de variable en temps non-linéaire dégénéré à l'origine (où sont fixées les données initiales). L'avantage de ce point de vue est de pouvoir transférer certaines techniques hyperboliques vers les équations paraboliques, ce qui semble à première vue surprenant, puisque les équations paraboliques ont la réputation d'être plus facile à traiter (ce qui n'est pas vrai, en réalité, dans le cas de systèmes dégénérés). Le chapitre traite, comme prototype, du curve-shortening flow, qui est le plus simple des mouvements par courbure moyenne en co-dimension supérieure à un. Il est montré comment ce modèle peut être dérivé de la théorie des surfaces de dimension deux d'aire extrémale dans l'espace de Minkowski (correspondant aux cordes relativistes classiques) qui peut se ramener à un système hyperbolique. On obtient, presque automatiquement, l'équivalent parabolique des principes d'entropie relative et d'unicité fort-faible qu'il est, en fait, bien plus simple d'établir et de comprendre dans le cadre hyperbolique.

Dans le second chapitre, la même méthode s'applique au système de Born-Infeld proprement dit, ce qui permet d'obtenir, à la limite, un modèle (non répertorié à notre connaissance) de Magnétohydrodynamique (MHD), où on retrouve à la fois une diffusivité non-linéaire dans l'équation d'induction magnétique et une loi de Darcy pour le champ de vitesse. Il est remarquable qu'un système d'apparence aussi lointaine des principes de base de la physique puisse être si directement déduit d'un modèle de physique aussi fondamental et géométrique que celui de Born-Infeld.

Dans le troisième chapitre, un lien est établi entre des systèmes paraboliques et le concept de flot gradient de formes différentielles pour des métriques de transport. Dans le cas des formes volumes, ce concept a eu un succès extraordinaire dans le cadre de la théorie du transport optimal, en particulier après le travail fondateur de Felix Otto et de ses collaborateurs. Ce concept n'en est vraiment qu'à ses débuts: dans ce chapitre, on étudie une variante du curve-shortening flow étudié dans le premier chapitre, qui présente l'avantage d'être intégrable (en un certain sens) et de conduire à des résultats plus précis.

Enfin, dans le quatrième chapitre, on retourne au domaine des EDP hyperboliques en considérant, dans le cas particulier des graphes, les surfaces extrémales de l'espace de Minkowski, de dimension et co-dimension quelconques. On parvient à montrer que les équations peuvent se reformuler sous forme d'un système élargi symétrique du premier ordre (ce qui assure automatiquement le caractère bien posé des équations) d'une structure remarquablement simple (très similaire à l'équation de Burgers) avec non linéarités quadratiques, dont le calcul n'a rien d'évident.

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Contents

Introduction

Optimal transport theory, for which we refer to the books by Villani, Ambrosio-Gigli-Savaré and Santambrogio [52, 53, 2, 47], has become in the last decades an important interdisciplinary field linking together partial differential equations, probability theory, functional analysis, geometric measure theory, differential geometry, continuum mechanics, inverse problems, image processing, economics, etc. The origin of the theory is very old and goes back to 1781 Monge's "mémoire sur les déblais et les remblais". This civil engineering problem amounts to finding the optimal way of moving parcels of earth from a given location to a prescribed destination, the cost to be minimized being the transportation distance in the euclidean space. Monge observed that the optimal transport map should be parallel to the gradient of a potential. He found interesting geometric properties for this potential, related to the theory of envelopes.

Much later, in 1942, Kantorovich related the Monge problem to linear programming and probability theory. He introduced a probabilistic generalization of the concept of optimal transport map and introduced the so-called Monge-Kantorovich problem, for which one can provide a general solution for various cost functions in a general setting. A very detailed account of that theory can be found in the books by Rachev and Rüschendorf [44].

In the late 80s, a fruitful connection was established between optimal transportation and partial differential equations, through the concept of polar factorization of maps by Y. Brenier in connection with the Monge-Ampère and Euler equations, shortly followed by the corresponding regularity results by L. Caffarelli. Then followed the introduction by R. McCann of the key concept of displacement convexity (in order to establish the Brunn-Minkowski inequality). This was the starting point for unexpected connections between optimal transport and differential geometry, for which we refer again to Villani's books [52, 53], including the famous contributions of Otto-Villani, Sturm, Lott-Villani, Ambrosio-Gigli-Savaré. Next, the concept of gradient flow in the framework of optimal transportation, for which we refer to the book by Ambrosio, Gigli and Savaré [2], gave new and deep insights on many parabolic PDEs, after the seminal work of F. Otto on the porous medium equation [32, 41]. Finally, optimal transport techniques have been widely used for numerous concrete applications in natural and computer sciences in the passed 20 years, as well described in the recent book by F. Santambrogio [47].

Various generalizations of the concept of optimal transport can be considered. One of the most interesting possible generalizations is the optimal transportation of closed differential forms, classical optimal transport theory corresponding to the special case of volume forms. In particular, closed differential forms of co-degree one are of special interest. They can be identified to divergence-free vector fields. They can also be seen as 1-currents, where the word current could be understood either in its traditional sense of fluid mechanics and electrodynamics or in its more modern mathematical sense of geometric measure theory. As noticed by Y. Brenier [10], in the case of the 3-dimensional Euclidean space, such a theory is already implicitly present in the somewhat richer nonlinear theory of Electromagnetism introduced by Max Born and Leopold Infeld in 1934 [7]. This is why the Born-Infeld theory, that we will describe in the next subsection, will be a central theme of our investigations.

0.1 The Born-Infeld electromagnetism

The Born-Infeld (BI) equations were originally introduced by Max Born and Leopold Infeld in 1934 [7] as a nonlinear correction to the linear Maxwell equations allowing finite electrostatic fields for point charges. In high energy Physics, D-branes can be modelled according to a generalization of the BI model [42, 30]. In differential geometry, the BI equations are closely related to the study of extremal surfaces in the Minkowski space.

In general, the Born-Infeld theory involves a $n+1$ dimensional Lorentzian space-time manifold of metric $g_{ij}dx^i dx^j$ and vector potentials $A = A_i dx^i$ that are critical points of the fully covariant action

$$\int \sqrt{-\det(g + dA)}\,.$$

In the 4-dimensional Minkowski space of special relativity, by using classical electromagnetic notations, the BI equations form a 6×6 system of conservation laws in the sense of [20], with 2 differential constraints,

$$\partial_t B + \nabla \times \left(\frac{D + B \times (D \times B)}{\sqrt{1 + B^2 + D^2 + |D \times B|^2}} \right) = 0, \quad \nabla \cdot B = 0, \tag{0.1.1}$$

$$\partial_t D + \nabla \times \left(\frac{-B + D \times (D \times B)}{\sqrt{1 + B^2 + D^2 + |D \times B|^2}} \right) = 0, \quad \nabla \cdot D = 0. \tag{0.1.2}$$

where we use the conventional notations for the inner product $\cdot$ and the cross-product $\times$ in $\mathbb{R}^3$, the gradient operator ∇, the curl operator $\nabla\times$ and the electromagnetic field (B, D). By Noether's theorem, this system admits 4 extra conservation laws for the energy density h and Poynting vector P, namely,

$$\partial_t h + \nabla \cdot P = 0, \quad \partial_t P + \nabla \cdot \left(\frac{P \otimes P - B \otimes B - D \otimes D}{h} \right) = \nabla \left(\frac{1}{h} \right), \tag{0.1.3}$$

where

$$P = D \times B, \ \ h = \sqrt{1 + D^2 + B^2 + |D \times B|^2}. \tag{0.1.4}$$

As advocated by Y. Brenier in [10], by viewing h, P as independent variables, the BI system can be augmented as a 10×10 system of hyperbolic conservation laws with an extra conservation law involving a strictly convex entropy, namely

$$h^{-1}(1 + B^2 + D^2 + P^2).$$

This augmented BI system belongs to the nice class of systems of conservation laws with convex entropy, which, under secondary suitable additional conditions, enjoy important properties such as well-posedness of the initial value problem, at least for short times, and "weak-strong" uniqueness principles [20]. It reads:

$$\partial_t B + \nabla \times \left(\frac{D + B \times P}{h} \right) = 0, \quad \nabla \cdot B = 0, \tag{0.1.5}$$

$$\partial_t D + \nabla \times \left(\frac{-B + D \times P}{h} \right) = 0, \quad \nabla \cdot D = 0. \tag{0.1.6}$$

$$\partial_t h + \nabla \cdot P = 0, \quad \partial_t P + \nabla \cdot \left(\frac{P \otimes P - B \otimes B - D \otimes D}{h} \right) = \nabla \left(\frac{1}{h} \right). \tag{0.1.7}$$

In a suitable "high-field" limit [10], this augmented Born-Infeld system degenerates as

$$\partial_t B + \nabla \times \left(\frac{B \times P}{h} \right) = 0, \quad \nabla \cdot B = 0, \tag{0.1.8}$$

$$\partial_t h + \nabla \cdot P = 0, \quad \partial_t P + \nabla \cdot \left(\frac{P \otimes P - B \otimes B}{h} \right) = 0, \tag{0.1.9}$$

which can be interpreted as a system of "optimal transportation" of a divergence-free vector field, namely B, together with a volume form h, by an optimal velocity field, namely $v = P/h$, coupled to B and h, with an (infinitesimal) transportation cost given by hv^2. Thus, it is tempting to think about a theory of optimal-transport gradient flow for 1-currents based on this degenerate version of the (augmented) Born-Infeld equations, in the spirit of the seminal work of F. Otto and his collaborators [32, 41]. As a preliminary step, it turned out to be more fruitful to go back to the full (augmented) Born-Infeld system, which is of hyperbolic nature, and derive from it a parabolized version, which could be interpreted as a paradigm for such a future theory. This is the main purpose of the second chapter of this thesis. In order to get the parabolized version of the augmented Born-Infeld equations, we introduce, in the first chapter of the thesis, a very simple and quite general device to go from the hyperbolic setting to the parabolic one, through a simple quadratic change of time, as explained in the next subsection.

0.2 From conservative to dissipative systems by quadratic change of time

Some interesting examples of dissipative systems can be derived from conservative ones by a simple quadratic change of time. Let us first consider the very simple model of conservative forces in classical mechanics. Under the quadratic change of time $t \to \theta = t^2/2$, the ordinary dynamical system

$$\frac{d^2X}{dt^2} = -\nabla\varphi(X) \tag{0.2.1}$$

becomes

$$-\nabla\varphi(X) = \frac{d}{dt}(\frac{dX}{d\theta}\frac{d\theta}{dt}) = \frac{d}{dt}(t\frac{dX}{d\theta}) = \frac{dX}{d\theta} + t\frac{d\theta}{dt}\frac{d^2X}{d\theta^2} = \frac{dX}{d\theta} + 2\theta\frac{d^2X}{d\theta^2}$$

with two asymptotic regimes as θ becomes either very small or very large:
the gradient flow

$$\frac{dX}{d\theta} = -\nabla\varphi(X), \tag{0.2.2}$$

and the inertial motion

$$\frac{d^2X}{d\theta^2} = 0. \tag{0.2.3}$$

In the second example, we retrieve the Darcy's law and the porous medium equation from the Euler equation of isentropic gases, and, in particular the heat equation from the Euler equation of isothermal gases. These equations read

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) = -\nabla p, \tag{0.2.4}$$

where $(\rho, p, v) \in \mathbb{R}^{1+1+3}$ are the density, pressure and velocity fields of the fluid, p being a given function of ρ (such as $p = \rho$, in the "isothermal" case). This leads, after the quadratic change of time $t \to \theta = t^2/2$, to

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \rho v + 2\theta[\partial_\theta(\rho v) + \nabla \cdot (\rho v \otimes v)] = -\nabla p(\rho).$$

In the regime $\theta >> 1$, we get the asymptotic model of "pressureless" gas dynamics

$$\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \quad \partial_\theta(\rho v) + \nabla \cdot (\rho v \otimes v) = 0, \tag{0.2.5}$$

while, as $\theta << 1$, we recover the Darcy law and the porous medium equation

$$\rho v = -\nabla p(\rho), \quad \partial_\theta \rho = \triangle(p(\rho)), \tag{0.2.6}$$

and, in the isothermal case $p = \rho$, the heat equation

$$\partial_\theta \rho = \triangle \rho. \tag{0.2.7}$$

Our third example is at the interface of Geometry and High Energy Physics. We start with the conservative evolution of classical strings according to the Nambu-Goto action, from which we get, by quadratic change of time, the dissipative geometric model of curve-shortening in $\mathbb{R}^d$, which is the simplest example of mean-curvature flow with co-dimension higher than 1:

$$\partial_\theta X = \frac{1}{|\partial_s X|} \partial_s \left(\frac{\partial_s X}{|\partial_s X|} \right), \tag{0.2.8}$$

where $s \to X(\theta, s)$ describes a time-dependent curve in $\mathbb{R}^d$ and $|\cdot|$ denotes the Euclidean norm. In Chapter 1, the following system of PDEs are obtained

$$\partial_\theta B + \nabla \cdot \left(\frac{B \otimes P - P \otimes B}{\rho} \right) = 0, \quad \nabla \cdot B = 0, \tag{0.2.9}$$

$$P = \nabla \cdot \left(\frac{B \otimes B}{\rho} \right), \quad \rho = |B| \tag{0.2.10}$$

which turns out to be nothing but the "Eulerian version" (in $\mathbb{R}^d$) of the curve-shortening model (1.1.1). Typically, in the case of a single loop X subject to the curve-shortening flow, B would just be the singular vector-valued measure

$$(\theta, x) \to B(\theta, x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(\theta, s)) \partial_s X(\theta, s) ds \in \mathbb{R}^d,$$

for which the system of PDE makes sense since all nonlinearities are homogeneous of degree one. The first chapter of this thesis will be mainly devoted to this geometric example.

Finally, let us consider the case of the 10×10 augmented BI system, which was our original motivation. We obtain, after the quadratic change of the time variable $t \to \theta = t^2/2$, the following asymptotic system as $\theta << 1$:

$$\partial_\theta B + \nabla \times (h^{-1} B \times P) + \nabla \times (h^{-1} \nabla \times (h^{-1} B)) = 0, \tag{0.2.11}$$

$$\partial_\theta h + \nabla \cdot P = 0, \quad P = \nabla \cdot (h^{-1} B \otimes B) + \nabla(h^{-1}). \tag{0.2.12}$$

This system can be interpreted as an unusual, fully dissipative version of standard Magnetohydrodynamics, including a generalized version of the Darcy law, with a fluid of density h, momentum P and pressure $p = -h^{-1}$ (of Chaplygin type), interacting with a magnetic field B. It belongs to the class of non-linear degenerate parabolic PDEs. This can be seen as a dissipative model of Magnetohydrodynamics (MHD) where a fluid of density h and momentum P interacts with a magnetic field B, with several interesting (and intriguing) features:
(i) the first equation, which can be interpreted in MHD terms as the "induction equation" for B, involves a second-order diffusion term typical of MHD: $\nabla \times (h^{-1} \nabla \times (h^{-1} B))$ (with, however, an unusual dependence on h);
(ii) the third equation describes the motion of the fluid of density h and momentum P driven by the magnetic field B and can be interpreted as a (generalized) Darcy law (and not as the usual momentum equation of MHD), just if the fluid was moving in a porous medium (which seems highly unusual in MHD!);
(iii) there are many coefficients which depend on h in a very peculiar way; in particular the Darcy law involves the so-called Chaplygin pressure $p = -h^{-1}$ (with sound speed $\sqrt{dp/dh} = h^{-1}$), which is sometimes used for the modeling of granular flows and also in cosmology, but not (to the best of our knowledge) in standard MHD.
A comprehensive study of this model is the main purpose of the second chapter of this thesis.

0.3 Relative entropy and the concept of dissipative solutions

In chapters 1 and 2 of the thesis, the analysis of various parabolic systems, obtained after a quadratic change of time, will be based on suitable concepts of generalized ("dissipative") solutions related to the relative entropy method, quite well known in the theory of hyperbolic systems of conservation laws [20, 36, 37], kinetic theory [46], parabolic equations [33], and continuum mechanics [23, 28], just to quote few examples. It is also related to the work of P.-L. Lions for the Euler equation of incompressible fluids [39] (see also [12]), L. Ambrosio, N. Gigli, G. Savaré [2] for the heat equation and to the recent study by Y. Brenier in [11] of Moffatt's model of magnetic relaxation.

The dissipative solutions enjoy the so called "weak-strong" uniqueness, which means that any dissipative solutions must coincide with a strong solution emanating from the same initial data as long as the latter exists. In other words, the strong solutions must be unique within the class of weak solutions. For given initial conditions, the set of dissipative solutions is convex and compact (usually with respect to the weak-$*$ topology of mesures). Global existence of such solutions can be proved by using the Faedo-Galerkin method [28].

0.4 Optimal transport for closed $(d-1)-$ forms

Let us finally go back to the concept of optimal transport of closed differential forms of co-degree 1. The theory of optimal transport for differential forms is not yet fully developed but there has been some recent progress, especially for symplectic forms and contact forms [19, 45]. However, to the best of our knowledge, little is known about gradient flows in that context. In the third chapter of this thesis, we present an explicit example of a gradient flow for closed $(d-1)-$forms in the Euclidean space $\mathbb{R}^d$. As already mentioned, such forms can be identified to divergence-free vector fields. For instance, as $d = 3$, any $2-$form β can be written as

$$\beta = B^1 dx^2 \wedge dx^3 + B^2 dx^3 \wedge dx^1 + B^3 dx^1 \wedge dx^2.$$

and

$$d\beta = (\partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3) dx^1 \wedge dx^2 \wedge dx^3 = \nabla \cdot B \, dx^1 \wedge dx^2 \wedge dx^3,$$

So β is a closed form is the same as saying that the vector field $B = (B^1, B^2, B^3)$ is divergence-free. These formulae easily extend to arbitrary dimensions d. For simplicity, we will only discuss about $\mathbb{Z}^d-$periodic forms so that we will use the flat torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ instead of the entire space $\mathbb{R}^d$.

The concept of transport involves time-dependent closed $(d-1)-$forms β_t and vector fields $v(t, x)$. Let ϕ_t be the group of diffeomorphisms generated by $v(t, x)$ such that

$$\frac{d}{dt}\phi_t = v_t \circ \phi_t, \quad \phi_0 = Id.$$

β is transported by v in the way that, at each time t, β_t is the pushforward of β_0 by ϕ_t:

$$\beta_t = (\phi_t)_* \beta_0, \quad \text{or} \quad (\phi_t)^* \beta_t = \beta_0.$$

Therefore, β should satisfy

$$\frac{d}{dt}\beta_t + \mathcal{L}_{v_t}\beta_t = 0,$$

or for the divergence-free vector field B,

$$\partial_t B + \nabla \cdot (B \otimes v - v \otimes B) = 0.$$

The above equation is usually called induction equation in magnetohydrodynamics.

A first example of gradient flow is provided by the Eulerian version of the curve-shortening flow (0.2.9),(0.2.10) studied in the first chapter of the thesis. As explained in the third chapter, this model turns out just to be the gradient flow of the following convex functional

$$\mathcal{F}[B] = \int_{\mathbb{T}^d} |B|,$$

for the (infinitesimal) transportation cost,

$$\|v\|_B = \sqrt{\int_{\mathbb{T}^d} |v|^2 |B|},$$

where v denotes the velocity field that transports B. A second example, still set on the flat torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, reads

$$\partial_t B + \nabla \cdot \left(\frac{B \otimes P - P \otimes B}{\rho} \right) = 0, \quad \nabla \cdot B = 0, \tag{0.4.1}$$

$$\partial_t \rho + \nabla \cdot P = 0, \quad P = \nabla \cdot \left(\frac{B \otimes B}{\rho} \right) \tag{0.4.2}$$

where B is a time dependent divergence-free vector field (i.e. a closed $(d-1)-$form), ρ is a time-dependent companion volume-form, and P stands for ρv, where v is the time-dependent velocity field transporting both ρ and B as differential forms. As will be shown, this system turns out to be the gradient flow of functional

$$\mathcal{F}[\rho, B] = \int_{x \in \mathbb{T}^d} F(\rho(x), B(x)), \quad F(\rho, B) = \frac{|B|^2}{2\rho}, \tag{0.4.3}$$

according to the transportation metric

$$||v||_\rho = \sqrt{\int_{\mathbb{T}^d} v^2 \rho}, \tag{0.4.4}$$

which is just the most usual transport metric for volume-forms [2, 41, 47, 53].

This system is formally integrable and can be viewed as the Eulerian version of the heat equation for curves in the Euclidean space. More precisely, if (B, ρ, P) is of form

$$(B, \rho, P)(t, x) = \int_{a \in \mathcal{A}} \left(\int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t, s, a))(\partial_s X, 1, \partial_t X)(t, s, a) ds \right) d\mu(a) \tag{0.4.5}$$

where $(\mathcal{A}, d\mu)$ is an abstract probability space of labels a, and, for $\mu-$a.e. label a, and every time t, $s \in \mathbb{R}/\mathbb{Z} \to X(t, s, a) \in \mathbb{T}^d$ is a loop subject to the heat equation

$$\partial_t X(t, s, a) = \partial^2_{ss} X(t, s, a), \tag{0.4.6}$$

then (B, ρ, P) is expected to be a solution of (3.1.1,3.1.2), at long as there is no self- or mutual intersection of the different loops. This "integrability" property makes the analysis of this second gradient flow, as done in the third chapter of this thesis, substantially easier than the one needed in the two first chapters.

0.5 Hyperbolicity of the extremal surfaces in Minkowski spaces

In the last chapter of the thesis, we leave the issue of gradient flows for differential forms, and move back to issues closely related to the Born-Infeld theory as revisited by high energy physicists in the 90s in the framework of strings and branes. In the $(1+n+m)-$dimensional Minkowski space $\mathbb{R}^{1+(n+m)}$, let $X(t, x)$ be a time-like $(1+n)-$dimensional surface (called $n-$brane in String Theory [42]), namely,

$$(t, x) \in \overline{\Omega} \subset \mathbb{R} \times \mathbb{R}^n \to X(t, x) = (X^0(t, x), \ldots, X^{n+m}(t, x)) \in \mathbb{R}^{1+(n+m)},$$

where Ω is a bounded open set. This surface is called an extremal surface if X is a critical point, with respect to compactly supported perturbations in the open set Ω, of the following area functional (which corresponds to the Nambu-Goto action in the case $n = 1$)

$$-\iint_\Omega \sqrt{-\det(G_{\mu\nu})} \ , \qquad G_{\mu\nu} = \eta_{MN} \partial_\mu X^M \partial_\nu X^N \ ,$$

where $M, N = 0, 1, \dots, n+m$, $\mu, \nu = 0, 1, \dots, n$, and $\eta = (-1, 1, \dots, 1)$ denotes the Minkowski metric, while G is the induced metric on the $(1+n)-$surface by η. Here $\partial_0 = \partial_t$ and we use the convention that the sum is taken for repeated indices.

By variational principles, the Euler-Lagrange equations gives the well-known equations of extremal surfaces,

$$\partial_\mu \left(\sqrt{-G} G^{\mu\nu} \partial_\nu X^M \right) = 0, \qquad M = 0, 1, \dots, n+m, \tag{0.5.1}$$

where $G^{\mu\nu}$ is the inverse of $G_{\mu\nu}$ and $G = \det(G_{\mu\nu})$. Now, let us consider a special case where the extremal surfaces are graphs of the form (this is usually called the static gauge in High Energy Physics):

$$X^0 = t, \ \ X^i = x^i, \ \ i = 1, \dots, n, \ \ X^{n+\alpha} = X^{n+\alpha}(t, x), \ \ \alpha = 1, \dots, m. \tag{0.5.2}$$

By using the notation that

$$V_\alpha = \partial_t X^{n+\alpha}, \ \ F_{\alpha i} = \partial_i X^{n+\alpha}, \ \ \alpha = 1, \dots, m, \ \ i = 1, \dots, n.$$

$$D_\alpha = \frac{\sqrt{\det(I_n + F^T F)} (I_m + F F^T)^{-1}_{\alpha\beta} V_\beta}{\sqrt{1 - V^T (I_m + F F^T)^{-1} V}}$$

we can find that (0.5.1) is equivalent to solving the following system for a matrix valued function $F = (F_{\alpha i})_{m \times n}$ and a vector valued function $D = (D_\alpha)_{\alpha = 1, 2, \dots, m}$,

$$\partial_t F_{\alpha i} + \partial_i \left(\frac{D_\alpha + F_{\alpha j} P_j}{h} \right) = 0, \quad \partial_t D_\alpha + \partial_i \left(\frac{D_\alpha P_i + \xi'(F)_{\alpha i}}{h} \right) = 0,$$

$$\partial_j F_{\alpha i} = \partial_i F_{\alpha j}, \quad P_i = F_{\alpha i} D_\alpha, \quad h = \sqrt{D^2 + P^2 + \xi(F)}, \quad 1 \le i, j \le n, \ 1 \le \alpha \le m,$$

where

$$\xi(F) = \det \left(I + F^T F \right), \quad \xi'(F)_{\alpha i} = \frac{1}{2} \frac{\partial \xi(F)}{\partial F_{\alpha i}} = \xi(F) (I + F^T F)^{-1}_{ij} F_{\alpha j}.$$

Similar to the Born-Infeld equations, there are other conservation laws for the energy density h and vector P as defined in above, namely,

$$\partial_t h + \nabla \cdot P = 0, \quad \partial_t P_i + \partial_j \left(\frac{P_i P_j}{h} - \frac{\xi(F)(I + F^T F)^{-1}_{ij}}{h} \right) = 0.$$

Viewing h and P as independent variables, the new system have a polyconvex entropy (which means that the entropy can be written as a convex function of the minors of F). Here, for $1 \le k \le r$, and any ordered sequences $1 \le \alpha_1 < \alpha_2 < \dots < \alpha_k \le m$ and $1 \le i_1 < i_2 < \dots < i_k \le n$, let $A = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$, $I = \{i_1, i_2, \dots, i_k\}$, the minor of F with respect to the rows $\alpha_1, \alpha_2, \dots, \alpha_k$ and columns $i_1, i_2, \dots, i_k$ is defined as

$$[F]_{A,I} = \det \left((F_{\alpha_p i_q})_{p,q=1,\dots,k} \right).$$

Now, by viewing these minors $[F]_{A,I}$ as new independent variables, we can further enlarge this system. As it will be shown in Chapter 4, the augmented system is hyperbolic with a convex entropy, linearly degenerate and preserves these algebraic constraints that we abandoned in the process of augmenting the system. We can finally have the following theorem.

Theorem 0.5.1. *In the case of a graph as* (0.5.2)*, the equations of extremal surfaces* (0.5.1) *can be translated into a first order symmetric hyperbolic system of PDEs, which admits the very simple form*

$$\partial_t W + \sum_{j=1}^{n} A_j(W)\partial_{x_j} W = 0, \quad W : (t, x) \in \mathbb{R}^{1+n} \to W(t, x) \in \mathbb{R}^{n+m+\binom{m+n}{n}}, \quad (0.5.3)$$

where each $A_j(W)$ is just a $(n + m + \binom{m+n}{n}) \times (n + m + \binom{m+n}{n})$ symmetric matrix depending linearly on W. Accordingly, this system is automatically well-posed, locally in time, in the Sobolev space $W^{s,2}$ as soon as $s > n/2 + 1$.

The structure of (4.1.3) is reminiscent of the celebrated prototype of all nonlinear hyperbolic PDEs, the so-called inviscid Burgers equation $\partial_t u + u\partial_x u = 0$, where u and x are both just valued in $\mathbb{R}$, with the simplest possible nonlinearity. Of course, to get such a simple structure, the relation to be found between X (valued in $\mathbb{R}^{1+n+m}$) and W (valued in $\mathbb{R}^{n+m+\binom{m+n}{n}}$) must be quite involved. Actually, it will be shown more precisely that the case of extremal surfaces corresponds to a special subset of solutions of (4.1.3) for which W lives in a very special algebraic sub-manifold of $\mathbb{R}^{n+m+\binom{m+n}{n}}$, which is preserved by the dynamics of (4.1.3).

To establish Theorem 4.1.1, the strategy of proof follows the concept of system of conservation laws with "polyconvex" entropy in the sense of Dafermos [20]. The first step is to lift the original system of conservation laws to a (much) larger one which enjoys a convex entropy rather than a polyconvex one. This strategy has been successfully applied in many situations, such as nonlinear Elastodynamis [23, 43], nonlinear Electromagnetism [10, 16, 49], just to quote few examples. In our case, the calculations will crucially start with the classical Cauchy-Binet formula.

At the end of the Chapter 4, as an additional interesting example to the idea of the quadratic change of time recently introduced in [14], we make a connection between the equations of extremal surfaces in Minkowski spaces (0.5.1) and the equations of mean-curvature flows in the Euclidean space, in any dimension and co-dimension.

0.6 Perspectives of the future research

In the joint work with my advisor, Prof. Yann Brenier, we retrieved the curve-shortening flow from the motion of relativistic string by performing the quadratic change of time [14]. The dissipative solutions are discussed in that case. As a natural continuation of the work, a general result regarding to the mean-curvature flow with any co-dimensions can be expected. In fact, by doing the quadratic change of time, we can get the mean-curvature flow equations from the equations of extremal surfaces in Minkowski spaces (or D-Branes). At this moment, it is not clear to me if the concept of dissipative solutions can be applied similarly to the mean-curvature flow or not. This could be a subject to study for the future.

It could also be an interesting subject to study the relative entropy or dissipative solutions in terms of other hyperbolic or parabolic systems such as the augmented system of the extremal surfaces in Minkowski space [24] in connection with nonlinear Elasticity and Magnetohydrodynamics [25]. We also plan to extend the gradient flow approach to a more general setting of optimal transportation of differential forms, beyond the well-established case of volume forms and the case of closed co-dimension 1 forms we already started to investigate in [15].

Chapter 1

From conservative to dissipative systems through quadratic change of time

1.1 Introduction

There are many examples of dissipative systems that can be derived from conservative ones. A classical example is the heat equation (or more generally the so-called "porous medium" equation) that can be derived from the Euler equations of isentropic gases. The derivation can be done in many different ways, for example by adding a very strong friction term or by homogenization techniques or by properly rescaling the time variable by a small parameter (through the so-called "parabolic scaling"). In the thesis, we will focus on a very straightforward idea (that does not seem to be popular, to the best of our knowledge): just perform the quadratic change of time $t \to \theta = t^2/2$. In Section 1.2, we provide several examples, starting with the very simple example of conservative forces in classical mechanics (with the Galileo model of falling bodies as a borderline case). Next, we briefly retrieve from the Euler equation of isentropic gases the Darcy law and the porous medium equation, and, in particular the heat equation from the Euler equation of isothermal gases. Our third example, at the interface of Geometry and High Energy Physics, starts with the conservative evolution of classical strings according to the Nambu-Goto action, from which we get, by quadratic change of time, the dissipative geometric model of curve-shortening in $\mathbb{R}^d$, which is the simplest example of mean-curvature flow with co-dimension higher than 1:

$$\partial_\theta X = \frac{1}{|\partial_s X|}\partial_s\left(\frac{\partial_s X}{|\partial_s X|}\right), \tag{1.1.1}$$

where $s \to X(\theta, s)$ describes a time-dependent curve in $\mathbb{R}^d$ and $|\cdot|$ denotes the Euclidean norm.

In Section 4.1.1, we will finally discuss the system of PDEs

$$\partial_\theta B + \nabla \cdot \left(\frac{B \otimes P - P \otimes B}{\rho}\right) = 0, \quad \nabla \cdot B = 0,$$

$$P = \nabla \cdot \left(\frac{B \otimes B}{\rho}\right), \quad \rho = |B|,$$

which turns out to be nothing but the "Eulerian version" (in $\mathbb{R}^d$) of the curve-shortening model (1.1.1). Typically, in the case of a single loop X subject to the curve-shortening flow, B would just be the singular vector-valued measure

$$(\theta, x) \to B(\theta, x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(\theta, s))\partial_s X(\theta, s) ds \in \mathbb{R}^d,$$

for which the system of PDE makes sense since all nonlinearities are homogeneous of degree one. (See Appendix 1.B for more details.) These equations admit a "non-conservative" version,

$$\partial_\theta b + (v \cdot \nabla) b = (b \cdot \nabla) v + b v^2, \quad v = (b \cdot \nabla) b, \tag{1.1.2}$$

for the reduced variables $b = B/|B|$ and $v = P/|B|$. For the conservative system, we define a concept of "dissipative solutions" related to the work of P.-L. Lions for the Euler equation of incompressible fluids [39] (see also [12]) or to the work of L. Ambrosio, N. Gigli, G. Savaré [2] for the heat equation and, overall, quite similar to the one recently introduced by the first author in [11]. We also refer to the works of A. Tzavaras and collaborators [23], E. Feireisl and collaborators [28] for the concept of "dissipative solutions". The main point of this chapter is to show how to get the formulation right. We start from the Eulerian version of the string equation, for which we can use the "relative entropy" method, quite classical in the theory of hyperbolic systems of conservation laws to get "weak-strong" uniqueness results (see [20]), and, then, we apply the quadratic change of time to get a good concept of dissipative solutions for the curve-shortening flow, namely:

Definition 1.1.1. *Let us fix $T > 0$ and denote $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. We say that (B, P) with*

$$B \in C([0,T], C(\mathbb{T}^d, \mathbb{R}^d)'_{w^*}), \quad P \in C([0,T] \times \mathbb{T}^d, \mathbb{R}^d)'$$

is a dissipative solution of the curve-shortening flow with initial data $B_0 \in C(\mathbb{T}^d, \mathbb{R}^d)'$ if and only if:

i) $B(0) = B_0, \quad \nabla \cdot B = 0$ *in sense of distributions;*

ii) B and P are bounded, respectively in the spaces $C^{1/2}([0,T], (C^1(\mathbb{T}^d))'_{w^*})$ and $C([0,T] \times \mathbb{T}^d, \mathbb{R}^d)'$, by constants depending only on T and $\int_{\mathbb{T}^d} |B_0|$.

iii) For all $\lambda > 0$, $\theta \in [0,T]$, for all smooth trial functions (b^*, v^*, A) valued in $\mathbb{R}^d$, with $\|A\|_\infty \leq \lambda$ and $b^{*2} = 1$, for all $r \geq c^* + \frac{\lambda^2}{2} + \lambda \|v^*\|_\infty$, where c^* is a constant depending explicitly on (b^*, v^*), we have:

$$\begin{aligned} e^{-r\theta} \int \eta(\theta) + \int_0^\theta e^{-r\sigma} \Bigg[\int P \cdot (A - L_3) + \left(r - c^* - \frac{A \cdot (A + 2v^*)}{2} \right) \eta \\ - B \cdot \left(L_2 + b^* \frac{A \cdot (A + 2v^*)}{2} \right) \Bigg] (\sigma) d\sigma \leq \int \eta(0). \end{aligned} \tag{1.1.3}$$

where

$$\eta = |B| - B \cdot b^*, \tag{1.1.4}$$

$$\mathrm{L}_2 = -\partial_\theta b^* - (v^* \cdot \nabla) b^* + (b^* \cdot \nabla) v^* + b^* v^{*2} - b^* (b^* \cdot \nabla)(b^* \cdot v^*), \tag{1.1.5}$$

$$\mathrm{L}_3 = -v^* + (b^* \cdot \nabla) b^*. \tag{1.1.6}$$

Here $C(\mathbb{T}^d, \mathbb{R}^d)'_{w^*}$ is metrizable space, and we can equipe a metric that is consistent with the weak-* topology. The "weak compactness" of such solutions (i.e. any sequence of dissipative solutions has accumulations points, in a suitable weak sense, and each of them is still a dissipative solution) directly follows from

Theorem 1.1.2. *For fixed initial condition B_0, the set of dissipative solutions, if not empty, is convex and compact for the weak-* topology of measures.*

Notice that it is more challenging to prove that the set of dissipative solutions is not empty. The standard strategy is as follows:
i) construct smooth approximate solutions (B^ϵ, P^ϵ) with smooth approximate initial data B_0^ϵ;
ii) show that, the approximate solutions are relatively compact for the weak-* topology of measures, and, for any trial functions (b^*, v^*, A), satisfy inequalities (1.1.3) with some small error terms;
iii) let ϵ go to zero, and prove that the limit (B, P) is a dissipative solution.

To keep the thesis simple, we leave this (important) step for a future work [25], in the more general framework of the Born-Infeld theory [7, 10]. Finally, we establish a "weak-strong" uniqueness principle in the following sense:

Theorem 1.1.3. *Let $(b, v) \in C^1([0,T] \times \mathbb{T}^d; \mathbb{R}^d \times \mathbb{R}^d)$, be a smooth solution of the non-conservative form of the curve-shortening flow (1.1.2) with $b^2 = 1$. Then any dissipative solution satisfies $B = |B|b$ and $P = |B|v$, as soon as $B(0) = |B(0)|b(0)$.*

Notice that this is not a full uniqueness result; only the homogeneous variables $b = B/|B|$ and $v = P/|B|$ get unique and a lot of room is left for the evolution of $|B|$ itself. Thus the concept of dissipative solutions seems to suffer from the same type of ambiguity as the more general concept of Brakke solutions for mean-curvature flows [8].

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1.2 Examples of quadratic change of time

1.2.1 Quadratic change of time of a simple dynamical system

Under the quadratic change of time $t \to \theta = t^2/2$, the ordinary dynamical system

$$\frac{d^2X}{dt^2} = -\nabla\varphi(X)$$

becomes

$$-\nabla\varphi(X) = \frac{d}{dt}(\frac{dX}{d\theta}\frac{d\theta}{dt}) = \frac{d}{dt}(t\frac{dX}{d\theta}) = \frac{dX}{d\theta} + t\frac{d\theta}{dt}\frac{d^2X}{d\theta^2} = \frac{dX}{d\theta} + 2\theta\frac{d^2X}{d\theta^2}$$

with two asymptotic regimes as θ becomes either very small or very large: the "gradient flow"

$$\frac{dX}{d\theta} = -\nabla\varphi(X),$$

and the inertial motion

$$\frac{d^2X}{d\theta^2} = 0.$$

Notice, in the first case, that only the initial position can be chosen freely, since

$$\frac{dX}{dt} = \frac{dX}{d\theta}\frac{d\theta}{dt} = t\frac{dX}{d\theta}$$

necessarily vanishes at $t = 0$. Consistently, the conservation of energy in the original time variable reads

$$\frac{d}{dt}(\frac{1}{2}|\frac{dX}{dt}|^2 + \varphi(X)) = 0$$

and becomes, with the new time variable $\theta = t^2/2$

$$\frac{d}{d\theta}[\varphi(X)] + \theta\frac{d}{d\theta}|\frac{dX}{d\theta}|^2 = -|\frac{dX}{d\theta}|^2,$$

leading to the dissipation of energy

$$\frac{d}{d\theta}[\varphi(X)] = -|\frac{dX}{d\theta}|^2,$$

in the asymptotic gradient flow regime. Furthermore, we may compare the respective solutions $X(t)$ and $Z(\theta)$ of the dynamical system and the gradient flow, with initial conditions

$$X(t=0) = X_0 = Z(\theta = 0), \quad \frac{dX}{dt}(t=0) = 0,$$

just by monitoring the "modulated energy" (or "relative entropy")

$$\frac{1}{2}|\frac{dX}{dt} - t\frac{dZ}{d\theta}|^2 + \varphi(X) - \varphi(Z) - \nabla\varphi(Z)\cdot(X - Z), \tag{1.2.1}$$

provided φ is strongly convex with bounded third derivatives. We get, after elementary calculations,

$$|X(t) - Z(t^2/2)|^2 + |\frac{dX}{dt}(t) - t\frac{dZ}{d\theta}(t^2/2)|^2 \leq Ct^5, \quad \forall t \in [0, T], \tag{1.2.2}$$

where C is a constant that depends only on T, Z and potential φ. (Notice that the smallest expected error is $O(t^6)$ as shown by the example $d = 1$, $\varphi(x) = |x|^2/2$, for which $X(t) = X(0)\cos(t)$, while $Z(\theta) = X(0)\exp(-\theta)$.) More details on the concept of "modulated energy" and the proof of (1.2.2) can be found in Appendix 1.A at the end of this chapter.

Remark: the Galileo experiment

The quadratic change of time $t \to \theta = t^2/2$ remarkably fits with the famous experiment by Galileo, which was the starting point of modern classical mechanics: a rigid ball descends a rigid ramp of constant slope, with zero initial velocity and constant acceleration G, reaching position

$$X = x_0 + \frac{Gt^2}{2}$$

at time t. So, X is just a linear function of the rescaled time θ, $X = x_0 + \theta G$ and we not only get

$$\frac{dX}{d\theta} + 2\theta\frac{d^2X}{d\theta^2} = G,$$

but also *simultaneously*

$$\frac{dX}{d\theta} = G, \quad \frac{d^2X}{d\theta^2} = 0,$$

i.e. *both* gradient flow *and* inertial motion, with respect to the rescaled time θ.

1.2.2 From the Euler equations to the heat equation and the Darcy law

Let us now move to a PDE example and explain how the Darcy law and the "porous medium" equation (and, in particular, the standard heat equation) can be recovered by quadratic change of time from the Euler equations of isentropic compressible fluids. These equations read

$$\partial_t\rho + \nabla\cdot(\rho v) = 0, \quad \partial_t(\rho v) + \nabla\cdot(\rho v\otimes v) = -\nabla p, \tag{1.2.3}$$

where $(\rho, p, v) \in \mathbb{R}^{1+1+3}$ are the density, pressure and velocity fields of the fluid, p being a given function of ρ (such as $p = \rho$, in the "isothermal" case). We set

$$t \to \theta = t^2/2, \quad \rho(t,x) \to \rho(\theta,x), \quad v(t,x) \to v(\theta,x)\frac{d\theta}{dt}. \tag{1.2.4}$$

(Notice the different scaling for v, enforcing $v(t,x)dt \to v(\theta,x)d\theta$.) This leads, after short calculations, to

$$\partial_\theta\rho + \nabla\cdot(\rho v) = 0, \quad \rho v + 2\theta[\partial_\theta(\rho v) + \nabla\cdot(\rho v\otimes v)] = -\nabla p(\rho).$$

In the regime $\theta >> 1$, we get the asymptotic model of "pressureless" gas dynamics

$$\partial_\theta\rho + \nabla\cdot(\rho v) = 0, \quad \partial_\theta(\rho v) + \nabla\cdot(\rho v\otimes v) = 0,$$

while, as $\theta << 1$, we recover the Darcy law and the porous medium equation

$$\rho v = -\nabla p(\rho), \quad \partial_\theta\rho = \triangle(p(\rho)),$$

and, in the isothermal case $p = \rho$, the heat equation

$$\partial_\theta\rho = \triangle\rho.$$

1.2.3 From string motion to curve-shortening

Let us now move to a model at the interface of geometry and high energy physics. We consider a surface

$$(t,s) \in \Omega \subset \mathbb{R}^2 \to (t, X(t,s)) \in \mathbb{R} \times \mathbb{R}^d,$$

parameterized by a sufficiently smooth (at least Lipschitz continuous) function X over a bounded open space-time cylinder Ω. According to classical string theory (see [42], for instance), this surface is a relativistic string if and only if X is a critical point, with respect to all smooth perturbations, compactly supported in Ω, of the "Nambu-Goto Action" defined by

$$\int_\Omega \sqrt{\partial_s X^2(1-\partial_t X^2) + (\partial_t X \cdot \partial_s X)^2} \;\; dtds$$

which is nothing but the area of the surface, in the space $\mathbb{R} \times \mathbb{R}^d$, with respect to the Minkowski metric $(-1, +1, \cdots, +1)$. It is customary to regularize this setting by viewing

$$(t,s) \in \Omega \subset \mathbb{R}^2 \to (t, s, X(t,s)) \in \mathbb{R}^2 \times \mathbb{R}^d,$$

as a graph in the enlarged space $\mathbb{R}^2 \times \mathbb{R}^d$ and considering its area in the enlarged Minkowski space $\mathbb{R}^2 \times \mathbb{R}^d$, with (rescaled) Minkowski metric $(-1, +\epsilon^2, +1, \cdots, +1)$:

$$\int_\Omega \sqrt{(\epsilon^2 + \partial_s X^2)(1-\partial_t X^2) + (\partial_t X \cdot \partial_s X)^2} \;\; dtds.$$

[Of course, we recover the previous setting just as the special (and degenerate) case $\epsilon = 0$.] The variational principle implies that X is a solution to the following first order partial differential system (of hyperbolic type as $\epsilon > 0$):

$$\partial_t(F\partial_t X - G\partial_s X) - \partial_s(G\partial_t X + H\partial_s X) = 0, \tag{1.2.5}$$

where

$$F = \frac{\epsilon^2 + \partial_s X^2}{S}, \quad G = \frac{\partial_t X \cdot \partial_s X}{S}, \quad H = \frac{1-\partial_t X^2}{S},$$
$$S = \sqrt{(\epsilon^2 + \partial_s X^2)(1-\partial_t X^2) + (\partial_t X \cdot \partial_s X)^2}.$$

After performing the quadratic change of time $\theta = t^2/2$, as we did in the previous subsections, while keeping only the zeroth order terms with respect to θ, we easily obtain, as an asymptotic equation, the following nonlinear equation of parabolic type:

$$((\epsilon^2 + \partial_s X^2)\mathbb{I} - \partial_s X \otimes \partial_s X)\partial_\theta X = \sqrt{\epsilon^2 + \partial_s X^2}\; \partial_s\left(\frac{\partial_s X}{\sqrt{\epsilon^2 + \partial_s X^2}}\right). \tag{1.2.6}$$

(Notice that $(\epsilon^2 + \partial_s X^2)\mathbb{I} - \partial_s X \otimes \partial_s X$ is an invertible symmetric matrix with eigenvalues larger or equal to ϵ^2.) In the limit case $\epsilon = 0$, we get

$$(\partial_s X^2 \mathbb{I} - \partial_s X \otimes \partial_s X)\partial_\theta X = |\partial_s X|\partial_s\left(\frac{\partial_s X}{|\partial_s X|}\right) \tag{1.2.7}$$

which becomes an ambiguous evolution equation, since it leaves $\partial_\theta X \cdot \partial_s X$ undetermined. [As a matter of fact, this geometric equation is not modified by any smooth

time-independent change of parameterization of the curve $s \to \sigma(s)$.]
However, we may solve instead the simpler equation

$$\partial_\theta X = \frac{1}{|\partial_s X|}\partial_s(\frac{\partial_s X}{|\partial_s X|}). \tag{1.2.8}$$

Indeed, this is a consistent way of solving (1.2.7) since

$$\partial_\theta X \cdot \partial_s X = \frac{\partial_s X}{|\partial_s X|} \cdot \partial_s(\frac{\partial_s X}{|\partial_s X|}) = \partial_s(\frac{\partial_s X \cdot \partial_s X}{2|\partial_s X|^2}) = 0.$$

Finally, by doing so, we have just recovered the familiar model of "curve-shortening" in the Euclidean space $\mathbb{R}^d$ (see [21], for instance).

1.2.4 The Eulerian form of the curve-shortening flow

The string equation (1.2.5) admits a useful "Eulerian" version

$$\partial_t B + \nabla \cdot \left(\frac{B \otimes P - P \otimes B}{\rho}\right) = 0, \tag{1.2.9}$$

$$\nabla \cdot B = 0, \quad \rho = \sqrt{B^2 + P^2}, \tag{1.2.10}$$

$$\partial_t P + \nabla \cdot \left(\frac{P \otimes P}{\rho}\right) = \nabla \cdot \left(\frac{B \otimes B}{\rho}\right) \tag{1.2.11}$$

(which reads, in coordinates,

$$\partial_t B^i + \partial_j(\rho^{-1}(B^i P^j - B^j P^i)) = 0, \quad \partial_i B^i = 0,$$

$$\rho = \sqrt{B_i B^i + P_i P^i}, \quad \partial_t P^i + \partial_j(\rho^{-1}(P^i P^j - B^j B^i)) = 0).$$

[As a matter of fact, defining

$$B(t,x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s))\partial_s X(t,s) ds,$$

(which automatically satisfies $\nabla \cdot B = 0$), and assuming X to be smooth, not self-intersecting, with $\partial_s X$ never vanishing, we get, after elementary calculations (similar to the ones done for the curve-shortening flow in Appendix 1.B, below), that B solves equations (1.2.9,1.2.10,1.2.11) together with

$$P(t,x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s)) \frac{(\partial_s X^2 \mathbb{I} - \partial_s X \otimes \partial_s X)\, \partial_t X(t,s)}{\sqrt{\partial_s X^2(1 - \partial_t X^2) + (\partial_t X \cdot \partial_s X)^2}} ds.]$$

Importantly enough, this system admits an extra conservation law:

$$\partial_t \rho + \nabla \cdot P = \nabla \cdot \left(\frac{(P \cdot B)B}{\rho^2}\right), \quad \rho = \sqrt{B^2 + P^2}, \tag{1.2.12}$$

which describes the local conservation of energy. [This is easy to check. Indeed, using coordinates, we find

$$\partial_t \rho = \frac{B_i \partial_t B^i + P_i \partial_t P^i}{\rho} = \frac{B_i}{\rho}\partial_j \left(\frac{B^j P^i - B^i P^j}{\rho}\right) + \frac{P_i}{\rho}\partial_j \left(\frac{B^j B^i - P^i P^j}{\rho}\right)$$

and notice that the second and fourth terms of the right-hand side combine as:

$$-P^j\partial_j\left(\frac{P^2+B^2}{2\rho^2}\right)-\frac{P^2+B^2}{\rho^2}\partial_jP^j=-\partial_jP^j$$

(since $\rho^2=B^2+P^2$), while the first and third terms give:

$$\partial_j\left(\frac{P^iB^jB_i}{\rho^2}\right)$$

(using $\nabla\cdot B=0$), which leads to the "entropy conservation law" (1.2.12).]

Let us now perform the quadratic change of time:

$$t\to\theta=\frac{t^2}{2},\quad B\to B,\quad P\to\frac{d\theta}{dt}P,$$

which leads, as $\theta<<1$, to the asymptotic system

$$\partial_\theta B+\nabla\cdot\left(\frac{B\otimes P-P\otimes B}{\rho}\right)=0, \tag{1.2.13}$$

$$\nabla\cdot B=0,\quad \rho=|B|, \tag{1.2.14}$$

$$P=\nabla\cdot\left(\frac{B\otimes B}{\rho}\right). \tag{1.2.15}$$

Notice that this implies $B\cdot P=0$, since, in coordinates,

$$B\cdot P=B_i\partial_j\left(\frac{B^iB^j}{\rho}\right)=\rho\frac{B_i}{\rho}B^j\partial_j\left(\frac{B^i}{\rho}\right)=\rho B^j\partial_j\left(\frac{B^2}{2\rho^2}\right)=0$$

(using $\nabla\cdot B=0$ and $|B|=\rho$). We also get the extra equation, derived from (1.2.12),

$$\partial_\theta\rho+\frac{P^2}{\rho}+\nabla\cdot P=0,\quad \rho=|B|. \tag{1.2.16}$$

[Indeed, $\sqrt{B^2+2\theta P^2}=|B|+\frac{\theta P^2}{|B|}+O(\theta^2)$, which leads to

$$\partial_\theta\sqrt{B^2+2\theta P^2}=\partial_\theta|B|+\frac{P^2}{|B|}+O(\theta).$$

We also used $B\cdot P=0$.] Notice that this equation is no longer in conservation form, due to the emergence of the dissipation term P^2/ρ after the quadratic change of time $t\to\theta$. Equations (1.2.13,1.2.16) also provide a priori bounds for any smooth solutions B and P on the flat torus $\mathbb{T}^d$ (that we have already taken into account in Definition 3.1.1):

$$\int_{\mathbb{T}^d}|B(\theta)|\le\int_{\mathbb{T}^d}|B(0)|,\ \forall\theta\in[0,T],\quad \int_0^T\int_{\mathbb{T}^d}\frac{P^2}{|B|}\le\int_{\mathbb{T}^d}|B(0)|,$$

$$\int_0^T\int_{\mathbb{T}^d}|P|\le\sqrt{\int_0^T\int_{\mathbb{T}^d}\frac{P^2}{|B|}}\sqrt{\int_0^T\int_{\mathbb{T}^d}|B|}\le\sqrt{T}\int_{\mathbb{T}^d}|B(0)|,$$

and, for $0 \leq \theta_0 \leq \theta_1 \leq T$, and any smooth vector field $\varphi = \varphi(x)$,

$$\left(\int_{\mathbb{T}^d} \left(B^i(\theta_1) - B^i(\theta_0)\right) \varphi_i\right)^2 = \left(\int_{\theta_0}^{\theta_1} \int_{\mathbb{T}^d} (\partial_j \varphi_i - \partial_i \varphi_j) \frac{P^j B^i}{|B|}\right)^2$$

$$\leq \mathrm{Lip}(\varphi)^2 (\theta_1 - \theta_0) \int_{\theta_0}^{\theta_1} \left(\int_{\mathbb{T}^d} |P|\right)^2$$

(where $\mathrm{Lip}(\varphi)$ denotes the Lipschitz constant of φ),

$$\leq \mathrm{Lip}(\varphi)^2 (\theta_1 - \theta_0) \int_{\theta_0}^{\theta_1} \left(\int_{\mathbb{T}^d} \frac{P^2}{|B|} \int_{\mathbb{T}^d} |B|\right)$$

$$\leq \mathrm{Lip}(\varphi)^2 (\theta_1 - \theta_0) \left(\int_{\mathbb{T}^d} |B(0)|\right)^2,$$

which shows that B is bounded in $C^{1/2}([0,T], (C^1(\mathbb{T}^d))'_{w^*})$ by a constant depending only on $\int_{\mathbb{T}^d} |B(0)|$ and T.

Equations (1.2.13,1.2.14,1.2.15) can also be written in non-conservative form in terms of

$$b = \frac{B}{\rho}, \quad v = \frac{P}{\rho}, \quad \rho = |B|.$$

We already have $b^2 = 1$ and $b \cdot v = 0$. Using coordinates, we first get from (1.2.15)

$$\rho v^i = P^i = \partial_j \left(\frac{B^i B^j}{\rho}\right) = \partial_j (\rho b^i b^j) = \rho b^j \partial_j b^i$$

(since $\partial_j (\rho b^j) = \partial_j B^j = 0$). Next, (1.2.13) becomes

$$\rho(\partial_\theta b^i + v^j \partial_j b^i - b^j \partial_j v^i) = -b^i (\partial_\theta \rho + \partial_j (\rho v^j)) = b^i \rho v^2$$

(thanks to (1.2.16)). So we have obtained

$$\partial_\theta b + (v \cdot \nabla) b = (b \cdot \nabla) v + b v^2, \quad v = (b \cdot \nabla) b, \tag{1.2.17}$$

(which is consistent with $b^2 = 1$ and $b \cdot v = 0$ as can be easily checked). Notice that (1.2.16) can be written according to the non-conservative variables as

$$\partial_\theta \rho + \nabla \cdot (\rho v) = -\rho v^2, \tag{1.2.18}$$

which is a linear equation in ρ.

1.3 Analysis of the Eulerian curve-shortening flow

1.3.1 Relative entropy for the Eulerian equations for strings

We start from the "Eulerian" version (1.2.9,1.2.10,1.2.11) of the string equation (1.2.5). This system belongs to the class of systems of conservation laws

$$\partial_t V + \nabla \cdot \mathcal{F}(V) = 0,$$

where $\mathcal{F}$ is a given function and V is a vector-valued function (for us $V = (B, P)$). If such a system admits an *extra* conservation law

$$\partial_t \mathcal{E}(V) + \nabla \cdot \mathcal{G}(V) = 0,$$

for a pair of functions $\mathcal{E}, \mathcal{G}$, with $\mathcal{E}$ strictly convex, then the system is automatically "hyperbolic" (i.e. well posed, at least for short time), under minor additional conditions [20], and $\mathcal{E}$ is often called an "entropy" for the system (although it should be called "energy" for a large class of applications). The system for strings (1.2.9,1.2.10,1.2.11) admits such an extra conservation law, namely (1.2.12), with $V = (B, P) \to \mathcal{E}(V) = \sqrt{B^2 + P^2}$ as entropy function. [Notice, however, that the entropy ρ is not strictly convex.]

Hyperbolic systems of conservation laws with convex entropy enjoy a "weak-strong uniqueness" principle [20], based on the concept of "relative entropy" (also called "modulated energy" or "Bregman divergence", depending of the frameworks):

$$\eta(V, V^*) = \mathcal{E}(V) - \mathcal{E}(V^*) - \nabla \mathcal{E}(V^*) \cdot (V - V^*),$$

which is just the discrepancy between $\mathcal{E}$ at point V and its linear approximation about a given point V^*. (Observe that, as $\mathcal{E}$ is a convex function with Hessian bounded away from zero and infinity, the relative entropy behaves as $|V - V^*|^2$.) Notice that the relative entropy is as convex as the entropy as a function of V (V^* being kept fixed) since it differs just by an affine term.

In the case of system (1.2.9,1.2.10,1.2.11), the relative entropy density is defined, for $(B, P) \in \mathbb{R}^d \times \mathbb{R}^d$ and $(b^*, v^*) \in \mathbb{R}^d \times \mathbb{R}^d$, by

$$\eta = \frac{|B - \rho b^*|^2 + |P - \rho v^*|^2}{2\rho} = \rho \frac{1 + b^{*2} + v^{*2}}{2} - B \cdot b^* - P \cdot v^*, \quad \rho = \sqrt{B^2 + P^2},$$

which is convex in (B, P). (Notice that, at this stage, we do not assume $b^{*2} + v^{*2} = 1$, which would be natural to define the relative entropy but would lead to contradictions after performing the quadratic change of time as will be done in the next subsection.)

Let us now consider a smooth, $\mathbb{Z}^d-$ periodic in space, solution $(B, P)(t, x)$ of equations (1.2.9,1.2.10,1.2.11) and monitor the evolution, on a fixed time interval $[0, T]$, of the integral of η over $(\mathbb{R}/\mathbb{Z})^d$, for some smooth trial functions:

$$(t, x) \in [0, T] \times (\mathbb{R}/\mathbb{Z})^d \to (b^*(t, x) \in \mathbb{R}^d, \ v^*(t, x) \in \mathbb{R}^d).$$

After tedious and elementary calculations, we find

$$\begin{aligned} \frac{d}{dt} \int \eta = & \int \frac{1}{2\rho} (B_i - \rho b_i^*)(B_j - \rho b_j^*)(\partial_j v_i^* + \partial_i v_j^*) \\ & - \int \frac{1}{2\rho} (P_i - \rho v_i^*)(P_j - \rho v_j^*)(\partial_j v_i^* + \partial_i v_j^*) \\ & - \int \frac{1}{\rho} (B_i - \rho b_i^*)(P_j - \rho v_j^*)(\partial_j b_i^* - \partial_i b_j^*) \\ & + \int \eta \cdot \mathrm{L}_1 + \int B \cdot \mathrm{L}_2 + \int P \cdot \mathrm{L}_3 - \int \frac{(P \cdot B) B}{\rho^2} \cdot \nabla \left(\frac{b^{*2} + v^{*2}}{2} \right) \end{aligned} \tag{1.3.1}$$

where

$$\mathrm{L}_1 = \frac{\zeta^*}{1 + b^{*2} + v^{*2}},$$

$$\zeta^* = D_t^*(b^{*2} + v^{*2}) - 2b^* \cdot \nabla(b^* \cdot v^*), \quad D_t^* = (\partial_t + v^* \cdot \nabla),$$

$$\mathrm{L}_2 = -D_t^* b^* + (b^* \cdot \nabla)v^* + \nabla(b^* \cdot v^*) + b^* \mathrm{L}_1$$

$$\mathrm{L}_3 = -D_t^* v^* + (b^* \cdot \nabla)b^* + v^* \mathrm{L}_1$$

1.3.2 Relative entropy and quadratic change of time

After the quadratic change of time,

$$t \to \theta = \frac{t^2}{2}, \quad \rho \to \rho, \quad B \to B, \quad b^* \to b^*, \quad P \to \theta'(t)P, \quad v^* \to \theta'(t)v^*,$$

we get $\theta'(t)^2 = 2\theta, \quad \theta''(t) = 1$,

$$\begin{aligned}
\frac{d}{d\theta}\int \eta &= \int \frac{1}{2\rho}(B_i - \rho b_i^*)(B_j - \rho b_j^*)(\partial_j v_i^* + \partial_i v_j^*) \\
&\quad - \int \frac{\theta}{\rho}(P_i - \rho v_i^*)(P_j - \rho v_j^*)(\partial_j v_i^* + \partial_i v_j^*) \\
&\quad - \int \frac{1}{\rho}(B_i - \rho b_i^*)(P_j - \rho v_j^*)(\partial_j b_i^* - \partial_i b_j^*) \\
&\quad + \int \eta \cdot \mathrm{L}_1 + \int B \cdot \mathrm{L}_2 + \int P \cdot \mathrm{L}_3 - \int \frac{(P \cdot B)B}{\rho^2} \cdot \nabla \left(\frac{b^{*2} + 2\theta v^{*2}}{2} \right),
\end{aligned}$$

where

$$\mathrm{L}_1 = \left(1 + b^{*2} + 2\theta v^{*2}\right)^{-1} \zeta^*,$$

$$\zeta^* = D_\theta(b^{*2} + 2\theta v^{*2}) - 2b^* \cdot \nabla(b^* \cdot v^*), \quad D_\theta^* = (\partial_\theta + v^* \cdot \nabla),$$

$$\mathrm{L}_2 = -D_\theta^* b^* + (b^* \cdot \nabla)v^* + \nabla(b^* \cdot v^*) + b^* \mathrm{L}_1$$

$$\mathrm{L}_3 = -v^* - 2\theta D_\theta^* v^* + (b^* \cdot \nabla)b^* + 2\theta v^* \mathrm{L}_1.$$

Now, in order to address the Eulerian curve-shortening system, we want to drop the terms of order $O(\theta)$ and limit ourself to the case when $b^{*2} = 1$. However, we have to be very careful about all terms involving ∂_θ. This happens first in the definition of ζ^*, because of the term

$$D_\theta(b^{*2} + 2\theta v^{*2}) = D_\theta(b^{*2}) + 2v^{*2} + O(\theta).$$

So, in the limit $\theta = 0$, with $b^{*2} = 1$, we find

$$\zeta^* = 2v^{*2} - 2b^* \cdot \nabla(b^* \cdot v^*)$$

and therefore,

$$\mathrm{L}_1 = v^{*2} - b^* \cdot \nabla(b^* \cdot v^*).$$

Similarly, we have to take care of

$$\frac{d}{d\theta}\int \eta.$$

where

$$\eta = \frac{1 + b^{*2} + 2\theta v^{*2}}{2}\rho - B \cdot b^* - 2\theta P \cdot v^*$$

and

$$\rho = \sqrt{B^2 + 2\theta P^2} = |B| + \theta \frac{P^2}{|B|} + O(\theta^2)$$

We get

$$\partial_\theta \rho = \partial_\theta |B| + \frac{P^2}{|B|} + O(\theta),$$

$$\partial_\theta \eta = v^{*2}\rho + \partial_\theta (\frac{1 + b^{*2}}{2}\rho) - \partial_\theta (B \cdot b^*) - 2P \cdot v^* + O(\theta)$$

This leads, as $b^{*2} = 1$, to

$$\rho = |B| + O(\theta), \quad \eta = |B| - B \cdot b^* + O(\theta),$$

$$\partial_\theta \eta = v^{*2}\rho + \partial_\theta |B| + \frac{P^2}{|B|} - \partial_\theta (B \cdot b^*) - 2P \cdot v^* + O(\theta)$$

$$= \partial_\theta (|B| - B \cdot b^*) + \frac{(P - |B|v^*)^2}{|B|} + O(\theta) = \partial_\theta (\rho - B \cdot b^*) + \frac{(P - \rho v^*)^2}{\rho} + O(\theta).$$

Finally, after dropping the terms of order $O(\theta)$ and limiting ourself to the case when $b^{*2} = 1$, we have found

$$\mathrm{L}_1 = v^{*2} - b^* \cdot \nabla (b^* \cdot v^*),$$

$$\mathrm{L}_2 = -D_\theta^* b^* + (b^* \cdot \nabla) v^* + \nabla (b^* \cdot v^*) + b^* \mathrm{L}_1$$

$$= -D_\theta^* b^* + (b^* \cdot \nabla) v^* + \nabla (b^* \cdot v^*) + b^* (v^{*2} - b^* \cdot \nabla (b^* \cdot v^*)),$$

$$D_\theta^* = (\partial_\theta + v^* \cdot \nabla), \quad \mathrm{L}_3 = -v^* + (b^* \cdot \nabla) b^*,$$

and, for all smooth trial field b^* such that $b^{*2} = 1$,

$$\begin{aligned} \frac{d}{d\theta} \int \eta + \int \frac{(P - \rho v^*)^2}{\rho} &= \int \frac{1}{2\rho} (B_i - \rho b_i^*)(B_j - \rho b_j^*)(\partial_j v_i^* + \partial_i v_j^*) \\ &- \int \frac{1}{\rho} (B_i - \rho b_i^*)(P_j - \rho v_j^*)(\partial_j b_i^* - \partial_i b_j^*) \\ &+ \int \eta \, \mathrm{L}_1 + \int B \cdot \mathrm{L}_2 + \int P \cdot \mathrm{L}_3, \end{aligned} \tag{1.3.2}$$

where

$$\rho = |B|, \quad \eta = \rho - B \cdot b^* = |B| - B \cdot b^* = \frac{(B - |B| b^*)^2}{2|B|} = \frac{(B - \rho b^*)^2}{2\rho}.$$

Using the Cauchy-Schwarz inequality, we can find a constant c^* depending only on b^* and v^* such that

$$\frac{d}{d\theta} \int \eta + \int \frac{|P - \rho v^*|^2}{2\rho} \le c^* \int \eta \, dx + \int B \cdot \mathrm{L}_2 + \int P \cdot \mathrm{L}_3.$$

This implies, for any constant $r \ge c^*$,

$$(-r + \frac{d}{d\theta}) \int \eta + \int \frac{|P - \rho v^*|^2}{2\rho} + (r - c^*) \int \eta \, dx \le \int B \cdot \mathrm{L}_2 + \int P \cdot \mathrm{L}_3$$

and, after multiplying this inequality by $e^{-r\theta}$ and integrating in time $\sigma \in [0,\theta]$,

$$e^{-r\theta}\int \eta(\theta)+\int_0^\theta e^{-r\sigma}\left((r-c^*)\int \eta+\int \frac{|P-\rho v^*|^2}{2\rho}-R\right)(\sigma)d\sigma \leq \int \eta(0), \quad (1.3.3)$$

where

$$R=\int B\cdot \mathrm{L}_2+\int P\cdot \mathrm{L}_3,$$

$$\mathrm{L}_2=-D^*_\theta b^*+(b^*\cdot\nabla)v^*+b^*v^{*2}-b^*(b^*\cdot\nabla)(b^*\cdot v^*), \quad \mathrm{L}_3=-v^*+(b^*\cdot\nabla)b^*.$$

We can write

$$\begin{aligned}\frac{|P-\rho v^*|^2}{2\rho} &= \sup_A\,(P-\rho v^*)\cdot A-\rho\frac{A^2}{2}\\ &= \sup_A P\cdot A-(\eta+B\cdot b^*)\left(v^*\cdot A+\frac{A^2}{2}\right)\end{aligned}$$

(since $\eta=\rho-B\cdot b^*$) and substitute for inequality (1.3.3) the family of inequalities

$$\begin{aligned}e^{-r\theta}\int \eta(\theta)+\int_0^\theta e^{-r\sigma}\Bigg[\int P\cdot(A-L_3)+\left(r-c^*-\frac{A\cdot(A+2v^*)}{2}\right)\eta\\ -B\cdot\left(L_2+b^*\frac{A\cdot(A+2v^*)}{2}\right)\Bigg](\sigma)d\sigma\leq\int \eta(0). \quad (1.3.4)\end{aligned}$$

Observe that these inequalities are convex in (B,P) as long as r is chosen so that

$$r\geq c^*+\sup_{\theta,x}\frac{A\cdot(A+2v^*)}{2}.$$

However, this creates a problem, since r must depend on A. This is why we input a cut-off parameter $\lambda>0$ and assume that the trial functions A are chosen with $|A(\theta,x)|\leq\lambda$. By doing this, the advantage is that we maintain the convexity of inequality as long as r is chosen big enough only as a function of b^*, v^* and λ, namely:

$$r\geq c^*+\frac{\lambda^2}{2}+\lambda\|v^*\|_\infty.$$

The price to pay is that we cannot fully recover

$$\frac{|P-\rho v^*|^2}{2\rho}$$

by taking the supremum over all A such that $|A|\leq\lambda$, but only the $\lambda-$approximation $K_\lambda(\rho,P-\rho v^*)$, where

$$K_\lambda(\rho,Z)=\sup_{|A|\leq\lambda} Z\cdot A-\rho\frac{A^2}{2}=\frac{|Z|^2}{2\rho}-\frac{(|Z|-\lambda\rho)_+^2}{2\rho}\geq 0.$$

Observe that, by doing so, we keep a good control of the distance between P and ρv^*, since (as can be easily checked)

$$K_\lambda(\rho,P-\rho v^*)\geq\min\left(\frac{(P-\rho v^*)^2}{2\rho},\frac{\lambda|P-\rho v^*|}{2}\right). \quad (1.3.5)$$

So, the supremum of inequalities (1.3.4) over all trial functions A such that $|A| \leq \lambda$, is *equivalent* to

$$e^{-r\theta} \int \eta(\theta) + \int_0^\theta e^{-r\sigma} (\int K_\lambda(\rho, P - \rho v^*) + (r - c^*)\eta - R)(\sigma) d\sigma \leq \int \eta(0). \quad (1.3.6)$$

Now let us consider (B, P) not only as functions but also as vector-valued Borel measures, for which (1.3.4) is still well-defined. The $\lambda-$approximation $K_\lambda(\rho, P-\rho v^*)$ can be interpreted as a function of measures [22] and (1.3.6) is equivalent to (1.3.4) in the sense that,

$$\int_0^\theta e^{-r\sigma} \int K_\lambda(\rho, P - \rho v^*) = \sup_{\substack{A \in C^0 \\ \|A\|_\infty \leq \lambda}} \int_0^\theta e^{-r\sigma} \int (P - \rho v^*) \cdot A - \rho \frac{A^2}{2}.$$

Notice that, due to the convexity of K_λ, we have

$$\int_0^\theta e^{-r\sigma} \int K_\lambda(\rho, P - \rho v^*)(\sigma) d\sigma \geq e^{-r\theta} \int_0^\theta K_\lambda \left(\int \rho(\sigma), \int |P - \rho v^*|(\sigma) \right) d\sigma. \quad (1.3.7)$$

With these calculations, we have recovered the concept of dissipative solutions as given in Definition 3.1.1. Then, the proof of our main results becomes straightforward.

Proof of Theorem 3.1.2

We just have to show that, for fixed initial conditions B_0, the set of dissipative solutions, as defined by Definition 3.1.1, if not empty, is convex and compact for the weak-* topology of measures. The convexity of the set of solutions is almost free. It follows directly from the convexity of inequalities (1.1.3). Let's focus on the compactness. Our goal is to prove that, if $\{(B_n, P_n)\}_{n\in\mathbb{N}}$ is a sequence of dissipative solutions with initial data B_0, then up to a subsequence, it converges in the weak-* topology of measures to a dissipative solution (B, P) with the same initial data. This follows from the inequalities (1.1.3) and suitable bounds that we assume for B_n and P_n. To see this, let's first show that, $\sup_\theta \int |B_n(\theta)|$ is uniformly bounded. (Indeed, let's take $b^* = (1, 0, \dots, 0)$, $v^* = A = 0$ in (1.1.3). Then we have $\int |B_n(\theta)| - B_n^1(\theta) \leq C \int |B_0|$, $\forall \theta \in [0, T]$. Since B_n is bounded in $C^{1/2}([0, T], (C^1(\mathbb{T}^d))'_{w^*})$, there exists a constant C' such that for any n, θ, $|\int B_n^1(\theta) - \int B_0^1| \leq C'$. Thus we get a uniform upper bound of $\int |B_n(\theta)|$.) Therefore, for any $\theta \in [0, T]$, the set $\{B_n(\theta)\}_{n\in\mathbb{N}}$ is relatively compact for the weak-* topology of $C(\mathbb{T}^d, \mathbb{R}^d)'$. Next, we look at the map $[0, T] \to C(\mathbb{T}^d, \mathbb{R}^d)'_{w^*}$, $t \to B(t)$. This map is equicontinuous because of the assumption on B_n. Then, by Arzelà-Ascoli's theorem, there exists $B \in C([0, T], C(\mathbb{T}^d, \mathbb{R}^d)'_{w^*})$, such that, up to a subsequence, $B_n(\theta) \rightharpoonup^* B(\theta)$, $\forall \theta \in [0, T]$. Now, since $\iint |P_n|$ is bounded, there exists $P \in C([0, T] \times \mathbb{T}^d, \mathbb{R}^d)'$, such that $P_n \rightharpoonup^* P$. Then because inequalities (1.1.3) are stable under weak-* convergence, we can prove that the limit (B, P) satisfies all the requirements in Definition 3.1.1, therefore, it is also a dissipative solution with initial data B_0.

Proof of Theorem 3.1.3

Let (b, v) be a smooth solution of the non-conservative form of the curve-shortening flow (1.1.2) with $b^2 = 1$, which directly implies $b \cdot v = 0$. We have to show that any dissipative solution satisfies $B = |B|b$ and $P = |B|v$, as soon as $B(0) = |B(0)|b(0)$. The proof is quite straightforward: we already have $\eta(0) = 0$ since $B(0) = |B(0)|b(0)$. Next, we set $b^* = b$, $v^* = v$, $A = 0$ and fix $\lambda > 0$ in definition (1.1.3). Since we have (1.1.2) and $b \cdot v = 0$, we get $\mathrm{L}_2 = \mathrm{L}_3 = 0$. Since $\eta \geq 0$, the inequality 1.1.3 directly implies $\eta = 0$, $\forall \theta \in [0, T]$, and, therefore $B = |B|b$. Now, let's go back to the inequality 1.1.3 which is already simplified since $\eta = 0$. By taking the supremum over all A such that $\|A\|_\infty \leq \lambda$, we get

$$\int_0^T e^{-r\sigma} \int K_\lambda(\rho, P - |B|v) \leq 0.$$

Using (1.3.7) we deduce

$$\int_0^T K_\lambda \left(\int \rho(\sigma), \int \big|P - |B|v\big|(\sigma) \right) d\sigma = 0,$$

and, therefore, $P = |B|v$ (because of (1.3.5)), which completes the proof.

1.4 Appendix 1.A: Modulated energy and dissipative solutions for ordinary dynamical equations

In this appendix, we explain, in the very elementary case of our dynamical system, the concepts of "modulated energy" (also called "relative entropy") and "dissipative formulation", which will later be used and extended to the dissipative setting.
Here, we crucially assume that the potential φ is convex and, in order to keep the presentation simple, we assume that the spectrum of the symmetric matrix $D^2\varphi(x)$ is uniformly contained in some fixed interval $[r, r^{-1}]$ for some constant $r \in (0, 1/2)$. We further assume that the third derivatives of φ are bounded. The total energy of a curve $t \to X(t)$ is defined by

$$\frac{1}{2}|X'(t)|^2 + \varphi(X(t)) \quad (\text{where } \ X'(t) = \frac{dX}{dt} \ )$$

and is a constant as X is a solution to the dynamical system

$$X''(t) = -\nabla\varphi(X(t)).$$

Given a smooth curve $t \to Y(t)$, we define the "modulated energy" (or "relative entropy") of X at time t with respect to Y by expanding the energy about Y at X:

$$\eta[t, X, Y] = \frac{1}{2}|X'(t) - Y'(t)|^2 + \varphi(X(t)) - \varphi(Y(t)) - \nabla\varphi(Y(t)) \cdot (X(t) - Y(t)).$$

Because of the assumption we made on φ, η is a perfect substitute for the squared distance between (X, X') and (Y, Y'):

$$r \leq \frac{2\eta[t, X, Y]}{|X - Y|^2 + |X' - Y'|^2} \leq r^{-1}.$$

We get

$$\frac{d}{dt}\eta[t, X, Y] = (X' - Y') \cdot (X'' - Y'') + \nabla\varphi(X) \cdot X' - \nabla\varphi(Y) \cdot Y'$$

$$-\nabla\varphi(Y)\cdot(X'-Y')-Y'\cdot D^2\varphi(Y)\cdot(X-Y)$$

$$=(X'-Y')\cdot(X''+\nabla\varphi(X)-Y''-\nabla\varphi(Y))+Y'\cdot(\nabla\varphi(X)-\nabla\varphi(Y)-D^2\varphi(Y)\cdot(X-Y)).$$

We first observe that

$$|\nabla\varphi(X)-\nabla\varphi(Y)-D^2\varphi(Y)\cdot(X-Y)|\leq C|X-Y|^2\leq C\eta[t,X,Y]$$

where, from now on, C is a generic constant that depends only on φ or Y. So,

$$\frac{d}{dt}\eta[t,X,Y]-(X'-Y')\cdot(X''+\nabla\varphi(X)-Y''-\nabla\varphi(Y))\leq C\eta[t,X,Y]$$

and then, after integration in time for $t\in[0,T]$, $T>0$ being an arbitrarily chosen fixed time,

$$\eta[T,X,Y]-\int_0^T(X'(t)-Y'(t))\cdot(\omega_X(t)-\omega_Y(t))e^{(T-t)C}dt\leq\eta[0,X,Y]e^{CT},\quad(1.4.1)$$

where

$$\omega_Z(t)=Z''(t)+\nabla\varphi(Z(t))$$

Let us exploit inequality (1.4.1) in several different ways.
First, we see that for a curve X it is equivalent to be solution of the dynamical system, i.e. $\omega_X=0$ or to satisfy

$$\eta[T,X,Y]+\int_0^T(X'(t)-Y'(t))\cdot\omega_Y(t)e^{(T-t)C}dt\leq\eta[0,X,Y]e^{CT},\quad\forall T>0,\quad(1.4.2)$$

for any smooth curve Y, where C is a constant depending only on Y (up to time T) and φ. Indeed, by taking as Y the unique solution of the dynamical system with initial conditions $Y(0)=X(0)$, $Y'(0)=X'(0)$ provided by the Cauchy-Lipschitz theorem on ODEs, we get both $\omega_Y=0$ and $\eta[0,X,Y]=0$. Thus inequality (1.4.2) just says $\eta[T,X,Y]=0$ for all $T>0$, which means $X=Y$ and, therefore, X is indeed a solution to the dynamical system. Thus, we can take (1.4.2) as an alternative notion of solution, that we call "dissipative solution". This inequality has the advantage to be convex in X, as the initial conditions $X(0),X'(0)$ are fixed, and therefore preserved under weak convergence of (X,X').

Next, we use (1.4.2) to compare a solution X of the dynamical system with zero initial velocity, i.e. $X'(0)=0$, to the solution Z of the gradient flow equation

$$Z'(\theta)+\nabla\varphi(Z(\theta))=0,$$

with initial condition $Z(0)=X(0)$. Indeed, let us set $Y(t)=Z(\theta)$, $\theta=t^2/2$. Then $Y'(t)=tZ'(\theta)$, $Y'(0)=0$, $Y(0)=Z(0)=X(0)$, $Y''(t)=Z'(\theta)+t^2Z''(\theta)$, which implies $\eta[0,X,Y]=0$ and $\omega_Y(t)=t^2Z''(\theta)$. So, (1.4.2) gives

$$\eta[T,X,Y]+\int_0^T(X'(t)-Y'(t))\cdot t^2Z''(t^2/2)e^{(T-t)C}dt\leq 0,$$

which implies (by Cauchy-Schwarz inequality and by definition of η)

$$\eta[T,X,Y]\leq C\int_0^T(\eta[t,X,Y]+t^4)dt,$$

where C is a generic constant depending only on T, φ and Z. By Gronwall's lemma, we conclude that $\eta[t, X, Y] \leq Ct^5$ which implies, by definition of η,

$$|X(t) - Z(t^2/2)|^2 + |\frac{dX}{dt}(t) - t\frac{dZ}{d\theta}(t^2/2)|^2 \leq Ct^5, \quad \forall t \in [0, T],$$

as already claimed, at the beginning of this subsection. (See (1.2.2).) (Notice that the smallest expected error is $O(t^6)$ as shown by the example $d = 1$, $\varphi(x) = |x|^2/2$, for which $X(t) = X(0)\cos(t)$, while $Z(\theta) = X(0)\exp(-\theta)$.)

1.5 Appendix 1.B: Direct recovery of the Eulerian curve-shortening flow

For the sake of completeness, let us check that system (1.2.13,1.2.14,1.2.15) indeed describes the curve-shortening flow in $\mathbb{R}^d$, for a continuum of non intersecting curves. Let us do the calculation in the case of a single smooth time-dependent loop, $s \in \mathbb{R}/\mathbb{Z} \to X(\theta, s)$, that we assume to be non self-intersecting at every fixed time θ, and such that $\partial_s X$ never vanishes. We introduce (as a distribution, or, if one prefers, as a "1−current")

$$B(\theta, x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(\theta, s))\partial_s X(\theta, s)ds,$$

which automatically satisfies $\nabla \cdot B = 0$. Since X is smooth, not self-intersecting, and $\partial_s X$ never vanishes, by assumption, we may find a smooth vector field $v(\theta, x)$ such that

$$\partial_\theta X(\theta, s) = v(\theta, X(\theta, s))$$

that we can interpret as the "Eulerian velocity field" attached to the loop evolution. We also introduce the nonnegative field

$$\rho(\theta, x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(\theta, s))|\partial_s X(\theta, s)|ds$$

which can also be interpreted as $|B(\theta, x)|$ since X is supposed to be non self-intersecting. We get (using indices $i, j, k \in \{1, \cdots, d\}$ with implicit summation on repeated indices)

$$\partial_\theta B^i(\theta, x) = \int_{\mathbb{R}/\mathbb{Z}} [-(\partial_j\delta)(x - X(\theta, s))\partial_\theta X^j(\theta, s)\partial_s X^i(\theta, s)$$

$$+\delta(x - X(\theta, s))\partial^2_{s\theta} X^i(\theta, s)]ds$$

(in distributional sense)

$$= -\int_{\mathbb{R}/\mathbb{Z}} (\partial_j\delta)(x - X(\theta, s))[\partial_\theta X^j(\theta, s)\partial_s X^i(\theta, s) - \partial_\theta X^i(\theta, s)\partial_s X^j(\theta, s)]ds$$

(after integration by part in $s \in \mathbb{R}/\mathbb{Z}$ of the second term)

$$= -\partial_j \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(\theta, s))[\partial_\theta X^j(\theta, s)\partial_s X^i(\theta, s) - \partial_\theta X^i(\theta, s)\partial_s X^j(\theta, s)]ds.$$

So

$$\partial_\theta B(\theta,x) = -\nabla\cdot\int_{\mathbb{R}/\mathbb{Z}} \delta(x-X(\theta,x))(\partial_s X(\theta,s)\otimes\partial_\theta X(\theta,s)-\partial_\theta X(\theta,s)\otimes\partial_s X(\theta,s))ds.$$

Then we can write

$$\partial_\theta B(\theta,x)+\nabla\cdot\int_{\mathbb{R}/\mathbb{Z}} \delta(x-X)(\partial_s X(\theta,s)\otimes v(\theta,X(\theta,s))-v(\theta,X(\theta,s))\otimes\partial_s X(\theta,s))ds = 0$$

which means, exactly, that by the definition of B,

$$\partial_\theta B + \nabla\cdot(B\otimes v - v\otimes B) = 0. \tag{1.5.1}$$

Since X is assumed to be non-intersecting, by definition of v, we may write

$$(|B|v)(\theta,x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x-X(\theta,s))|\partial_s X(\theta,s)|v(\theta,X(\theta,s))ds$$

So far, we have not used equation (1.2.8), namely

$$\partial_\theta X = \frac{1}{|\partial_s X|}\partial_s(\frac{\partial_s X}{|\partial_s X|}),$$

Let us do it now:

$$(\rho v^i)(\theta,x) = (|B|v^i)(\theta,x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x-X(\theta,s))\partial_s(\frac{\partial_s X^i}{|\partial_s X|})ds$$

$$= \int_{\mathbb{R}/\mathbb{Z}} (\partial_j\delta)(x-X(\theta,s))\frac{\partial_s X^j\partial_s X^i}{|\partial_s X|}ds$$

(after integrating by part in $s\in\mathbb{R}/\mathbb{Z}$)

$$= \partial_j\int_{\mathbb{R}/\mathbb{Z}} \delta(x-X(\theta,s))\frac{\partial_s X^j\partial_s X^i}{|\partial_s X|}ds$$

that we can interpret as

$$\rho v = \nabla\cdot\frac{B\otimes B}{|B|} = \nabla\cdot\frac{B\otimes B}{\rho}.$$

Finally we can write (1.5.1) as

$$\partial_\theta B + \nabla\cdot(\frac{B\otimes P - P\otimes B}{\rho}) = 0,\quad \nabla\cdot B = 0,\quad P = \nabla\cdot\frac{B\otimes B}{\rho},\quad \rho = |B|,$$

where P stands for ρv and (B,ρ,P) solves equations (1.2.13,1.2.14,1.2.15). So far, our claim has been justified only in the case of a single loop. We next argue that, due to its homogeneity of degree 1, equations (1.2.13,1.2.14,1.2.15), in spite of their nonlinearity, enjoy a nice superposition principle, in the sense that we may still get a solution by superposing several smooth curves subject to curve-shortening as long as they do not intersect and we may even build smooth solutions by using a continuum of such curves. This concludes the proof of our claim that equations (1.2.13,1.2.14,1.2.15), are the "Eulerian formulation" of the curve-shortening flow. Notice that similar calculations can also be performed to justify the Eulerian version (1.2.9,1.2.10,1.2.11) of the string equation (1.2.5).

Chapter 2

Magnetohydrodynamic regime of the Born-Infeld electromagnetism

2.1 Introduction

There are many examples of dissipative systems that can be derived from conservative ones. The derivation can be done in many different ways, for example by adding a very strong friction term or by homogenization techniques or by properly rescaling the time variable by a small parameter (through the so-called "parabolic scaling"). In the recent work of the author and Y. Brenier [14], we suggested a very straightforward idea: just perform the quadratic change of time $t \to \theta = t^2/2$. Several examples were studied in that paper. One example was the porous medium equation, which can be retrieved from the Euler equation of isentropic gases. Another relevant example, at the interface of Geometry and High Energy Physics, is the dissipative geometric model of curve-shortening flow in $\mathbb{R}^d$ (which is the simplest example of mean-curvature flow with co-dimension higher than 1) that we obtained from the conservative evolution of classical strings according to the Nambu-Goto action. This chapter is a follow-up of [14], where the Born-Infeld model of Electromagnetism is taken as an example, and as a result, we get a dissipative model of Magnetohydrodynamics (MHD) where we have non-linear diffusions in the magnetic induction equation and the Darcy's law for the velocity field.

The Born-Infeld (BI) equations were originally introduced by Max Born and Leopold Infeld in 1934 [7] as a nonlinear correction to the linear Maxwell equations allowing finite electrostatic fields for point charges. In high energy Physics, D-branes can be modelled according to a generalization of the BI model [42, 30]. In differential geometry, the BI equations are closely related to the study of extremal surfaces in the Minkowski space. In the 4-dimensional Minkowski space of special relativity, the BI equations form a 6×6 system of conservation laws in the sense of [20], with 2 differential constraints,

$$\partial_t B + \nabla \times \left(\frac{B \times (D \times B) + D}{h} \right) = 0, \quad \partial_t D + \nabla \times \left(\frac{D \times (D \times B) - B}{h} \right) = 0,$$

$$h = \sqrt{1 + D^2 + B^2 + (D \times B)^2}, \quad \nabla \cdot B = \nabla \cdot D = 0,$$

where we use the conventional notations for the inner product $\cdot$ and the cross-product $\times$ in $\mathbb{R}^3$, the gradient operator ∇, the curl operator $\nabla\times$ and the electromagnetic field (B, D). By Noether's theorem, this system admits 4 extra conservation laws for the energy density h and Poynting vector P, namely,

$$\partial_t h + \nabla \cdot P = 0, \quad \partial_t P + \nabla \cdot \left(\frac{P \otimes P - B \otimes B - D \otimes D}{h} \right) = \nabla \left(\frac{1}{h} \right),$$

where

$$P = D \times B, \;\; h = \sqrt{1 + D^2 + B^2 + |D \times B|^2}.$$

As advocated in [10], by viewing h, P as independent variables, the BI system can be "augmented" as a 10×10 system of hyperbolic conservation laws with an extra conservation law involving a "strictly convex" entropy, namely

$$h^{-1}(1 + B^2 + D^2 + P^2).$$

This augmented BI system belongs to the nice class of systems of conservation laws "with convex entropy", which, under secondary suitable additional conditions, enjoy important properties such as well-posedness of the initial value problem, at least for short times, and "weak-strong" uniqueness principles [20].

For the 10×10 augmented BI system, we obtain, after the quadratic change of the time variable $t \to \theta = t^2/2$, the following asymptotic system as $\theta << 1$:

$$\partial_\theta B + \nabla \times (h^{-1} B \times P) + \nabla \times (h^{-1} \nabla \times (h^{-1} B)) = 0,$$

$$\partial_\theta h + \nabla \cdot P = 0, \quad P = \nabla \cdot (h^{-1} B \otimes B) + \nabla(h^{-1}).$$

This system can be interpreted as an unusual, fully dissipative version of standard Magnetohydrodynamics, including a generalized version of the Darcy law, with a fluid of density h, momentum P and pressure $p = -h^{-1}$ (of Chaplygin type), interacting with a magnetic field B. It belongs to the class of non-linear degenerate parabolic PDEs.

In the rest part of the chapter, we proceed to the analysis of this asymptotic model (that we call "Darcy MHD") obtained after rescaling the 10×10 augmented BI model: (i) in Section 2.3, we define a concept of "dissipative solutions" in a sense inspired by the work of P.-L. Lions for the Euler equation of incompressible fluids [39], the work of L. Ambrosio, N. Gigli, G. Savaré [2] for the heat equation (working in a very general class of metric measured spaces) and quite similar to the one recently introduced by Y. Brenier in [11]; (ii) in Section 2.4, we demonstrate some properties of the dissipative solutions. we establish a "weak-strong" uniqueness principle, in the sense that, for a fixed smooth initial condition, a smooth classical solutions is necessarily unique in the class of dissipative solutions admitting the same initial condition; we prove the "weak compactness" of such solutions (i.e. any sequence of dissipative solutions has accumulations points, in a suitable weak sense, and each of them is still a dissipative solution); (iii) in Section 2.4, we estimate the error between dissipative solutions of the asymptotic system and smooth solutions of the 10 augmented Born-Infeld system; (iv) we finally prove the global existence solution of dissipative solution for any initial condition, without any smoothness assumption. This last point, which is a non-surprising consequence of the weak compactness, nevertheless requires a lengthy and technical proof which is presented in Section 2.6 and Section 2.7.

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2.2 Direct derivation of the diffusion equations

2.2.1 Presentation of the Born-Infeld model

For a $n+1$ dimensional spacetime, the Born-Infeld equations can be obtained by varying the Lagrangian of the following density

$$\mathcal{L}_{BI} = \lambda^2 \left(1 - \sqrt{-\det\left(\eta + \frac{F}{\lambda}\right)}\right)$$

where $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$ is the Minkowski metric tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor with A a vector potential. The parameter $\lambda \in (0, \infty)$ is called the absolute field constant which can be comprehended as the upper limit of the field strength [7]. In the 4 dimensional spacetime, by using the classical electromagnetic field symbols B, D, the BI equations can be written as

$$\partial_t B + \nabla \times \left(\frac{\lambda^2 D + B \times (D \times B)}{\sqrt{\lambda^4 + \lambda^2 B^2 + \lambda^2 D^2 + |D \times B|^2}}\right) = 0, \quad \nabla \cdot B = 0, \tag{2.2.1}$$

$$\partial_t D + \nabla \times \left(\frac{-\lambda^2 B + D \times (D \times B)}{\sqrt{\lambda^4 + \lambda^2 B^2 + \lambda^2 D^2 + |D \times B|^2}}\right) = 0, \quad \nabla \cdot D = 0. \tag{2.2.2}$$

Now, let us introduce some background of the BI model. The BI model was originally introduced by Max Born and Leopold Infeld in 1934 [7] as a nonlinear correction to the linear Maxwell model. Born had already postulated [5] a universal bound λ for any electrostatic field, even generated by a point charge (which is obviously not the case of the Maxwell theory for which the corresponding field is unbounded and not even locally square integrable in space), just as the speed of light is a universal bound for any velocity in special relativity. As $\lambda \to \infty$ the linear Maxwell theory is easily recovered as an approximation of the BI model. Max Born proposed a precise value for λ (based on the mass of the electron) and showed no substantial difference with the Maxwell model until subatomic scales are reached. In this way, the BI model was thought as an alternative to the Maxwell theory to tackle the delicate issue of establishing a consistent quantization of Electromagnetism with λ playing the role of a cut-off parameter. As a matter of fact, the BI model rapidly became obsolete for such a purpose, after the arising of Quantum Electrodynamics (QED), where renormalization techniques were able to cure the problems posed by the unboundedness of the Maxwell field generated by point charges. [Interestingly enough, M. Kiessling has recently revisited QED from a Born-Infeld perspective [34, 35].] Later on, there has been a renewed interest for the BI model in high energy Physics, starting in the 1960s for the modelling of hadrons, with a strong revival in the 1990s, in String Theory. In particular the new concept of D-brane was modelled according to a generalization of the BI model [42, 30].

Another important feature of the BI model is its deep link with differential geometry, already studied in a memoir of the Institut Henri Poincare by Max Born in 1938 [6]. Indeed, the BI equations are closely related to the concept of extremal surfaces in the Minkowski space. As a matter of fact [10, 16], as $\lambda \to 0$, the BI model provides a faithful description of a continuum of classical strings, which are nothing but extremal surfaces moving in the Minkowski space.

From a PDE viewpoint, the BI equations belong to the family of nonlinear systems of hyperbolic conservation laws [20], for which the existence and uniqueness of local in time smooth solutions can be proven by standard devices. A rather impressive result was recently established by J. Speck [50] who was able to show the global existence of smooth localized solutions for the original BI system, provided the initial conditions are of small enough amplitude. His proof relies on the null-form method developed by Klainerman and collaborators (in particular for the Einstein equation) combined with dispersive (Strichartz) estimates. This followed an earlier work of Lindblad on the model of extremal surfaces in the Minkowski space which can be seen as a "scalar" version of the BI system [38].

2.2.2 The 10×10 augmented BI system

In 2004, Y. Brenier showed that the structure of the BI system can be widely "simplified" by using the extra conservation laws of energy and momentum provided by the Noether invariance theorem, where the momentum (called Poynting vector) is $P = D \times B$ while the energy density is $h = \sqrt{1 + B^2 + D^2 + P^2}$ [10]. They read (after λ has been normalized to be 1, which is possible by a suitable change of physical units)

$$\partial_t h + \nabla \cdot P = 0, \quad \partial_t P + \left(\frac{P \otimes P - B \otimes B - D \otimes D}{h} \right) = \nabla \left(\frac{1}{h} \right), \qquad (2.2.3)$$

At this point, there are two main possibilities. The first one amounts to add the conservation of momentum (i.e. 3 additional conservation laws) to the 6×6 original BI equations, written in a suitable way, where P is considered as independent from B and D (namely not given by the algebraic relation $P = D \times B$) while h is still $h = \sqrt{1 + B^2 + D^2 + P^2}$. This strategy leads to the 9×9 system and the conservation of energy then reads

$$\partial_t \sqrt{1 + B^2 + D^2 + P^2} + \nabla \cdot P + \nabla \cdot \left(\frac{(D \cdot P)D + (B \cdot P)B - P + D \times B}{1 + B^2 + D^2 + P^2} \right) = 0$$

where the energy is now a strictly convex function of B, D and P. It can be shown [10] that the algebraic constraint $P = D \times B$ is preserved during the evolution of any smooth solution of this system, which implies that, at least for smooth solutions, the 9×9 augmented system is perfectly suitable for the analysis of the BI equations. This idea has been successfully extended to a very large class of nonlinear systems in Electromagnetism by D. Serre [?]. An even more radical strategy was followed and emphasized in [10], where h itself is considered as a new unknown variable, independent from B, D and P, while adding the conservation of both energy and momentum (i.e. 4 conservation laws) to the original 6×6 BI system, written in a suitable way. This leads to the following 10×10 system of conservation law for B, D, P, h:

$$\partial_t h + \nabla \cdot P = 0, \;\; \partial_t B + \nabla \times \left(\frac{B \times P + D}{h} \right) = 0, \;\; \nabla \cdot B = \nabla \cdot D = 0, \qquad (2.2.4)$$

$$\partial_t D + \nabla \times \left(\frac{D \times P - B}{h} \right) = 0, \;\; \partial_t P + \nabla \cdot \left(\frac{P \otimes P - B \otimes B - D \otimes D - I_3}{h} \right) = 0, \tag{2.2.5}$$

Once again, the algebraic constraints, namely

$$P = D \times B, \quad h = \sqrt{1 + B^2 + D^2 + P^2}$$

are preserved during the evolution of smooth solutions. The 10×10 extension has a very nice structure, enjoying invariance under Galilean transforms

$$(t, x, B, D, P, h) \longrightarrow (t, x + Vt, B, D, P - Vh, h)$$

(where $V \in \mathbb{R}^3$ is any fixed constant velocity). This is quite surprising, since the BI model is definitely Lorentzian and not Galilean, but not contradictory since such Galilean transforms are not compatible with the algebraic constraints:

$$P = D \times B, \quad h = \sqrt{1 + B^2 + D^2 + P^2}$$

In [16] it is further observed that, written in non conservation forms, for variables

$$(\tau, b, d, v) = (1/h, B/h, D/h, P/h) \in \mathbb{R}^{10},$$

the 10×10 system reduces to

$$\partial_t b + (v \cdot \nabla) b = (b \cdot \nabla) v - \tau \nabla \times d, \;\; \partial_t d + (v \cdot \nabla) d = (d \cdot \nabla) v + \tau \nabla \times b, \tag{2.2.6}$$

$$\partial_t \tau + (v \cdot \nabla) \tau = \tau \nabla \cdot v, \;\; \partial_t v + (v \cdot \nabla) v = (b \cdot \nabla) b + (d \cdot \nabla) d + \tau \nabla \tau, \tag{2.2.7}$$

which is just a symmetric quadratic system of first order PDEs, automatically well-posed (for short times) in Sobolev spaces, such as $W^{s,2}$ for any $s > 5/2$, without any restriction on the values of (b, d, v, τ) in $\mathbb{R}^{10}$ (including negative values of τ!). Once again, the algebraic constraints, which can be now nicely written as

$$b^2 + d^2 + v^2 + \tau^2 = 1, \quad \tau v = d \times b$$

are preserved during the evolution. Notice that two interesting reductions of this system can be performed. First, it is consistent to set simultaneously $\tau = 0$ and $d = 0$ in the equations, which leads to

$$\partial_t b + (v \cdot \nabla) b = (b \cdot \nabla) v, \;\; \partial_t v + (v \cdot \nabla) v = (b \cdot \nabla) b,$$

while the algebraic constraints become

$$b^2 + v^2 = 1, \quad b \cdot v = 0.$$

This system can be used to describe the evolution of a continuum of classical strings (i.e. extremal 2−surfaces in the 4−dimensional Minkowski space) [16]. A second reduction can be obtained by setting $\tau = 0$, $b = d = 0$ which leads to the inviscid Burgers equation

$$\partial_t v + (v \cdot \nabla) v = 0$$

This equation, as well known, always leads to finite time singularity for all smooth localized initial conditions v, except for the trivial one: $v = 0$ (which, by the way, shows that Speck's result cannot be extended to the 10×10 BI system, without restrictions on the initial conditions).

2.2.3 Quadratic time rescaling of the augmented BI system

Let us perform the following rescaling of the 10×10 augmented BI system (2.2.4)-(2.2.5):

$$t \to \theta = t^2/2, \quad h, B, P, D \to h, B, P\frac{d\theta}{dt}, D\frac{d\theta}{dt}$$

Observe that the symmetry between B and D is broken in this rescaling since D is rescaled in the same way as P but not as B. We obtain, after very simple calculations, the following rescaled equations,

$$\partial_\theta h + \nabla \cdot P = 0, \;\; \partial_\theta B + \nabla \times \left(\frac{B \times P + D}{h}\right) = 0,$$

$$D + 2\theta \left[\partial_\theta D + \nabla \times \left(\frac{D \times P}{h}\right)\right] = \nabla \times \left(\frac{B}{h}\right),$$

$$P + 2\theta \left[\partial_\theta P + \nabla \cdot \left(\frac{P \otimes P - D \otimes D}{h}\right)\right] = \nabla \cdot \left(\frac{B \otimes B}{h}\right) + \nabla(h^{-1}).$$

In the regime $\theta >> 1$, we get a self-consistent system for (D, P, h) (without B!)

$$\partial_\theta h + \nabla \cdot P = 0, \quad \partial_\theta D + \nabla \times \left(\frac{D \times P}{h}\right) = 0,$$

$$\partial_\theta P + \nabla \cdot \left(\frac{P \otimes P - D \otimes D}{h}\right) = 0,$$

which, written in non-conservative variables $(d, v) = (D/h, P/h)$, reduces to

$$\partial_t v + (v \cdot \nabla)v = (d \cdot \nabla)d, \quad \partial_t d + (v \cdot \nabla)d = (d \cdot \nabla)v,$$

that we already saw in the previous subsection as a possible reduction of the (10×10) extended BI system (which describes the motion of a continuum of strings). The regime of higher interest for us is the dissipative one obtained as $\theta << 1$. Neglecting the higher order terms as $\theta << 1$, we first get

$$D = \nabla \times \left(h^{-1}B\right)$$

which allows us to eliminate D and get for (B, P, h) the self-consistent system

$$\partial_\theta B + \nabla \times \left(h^{-1}B \times P\right) + \nabla \times \left(h^{-1}\nabla \times \left(h^{-1}B\right)\right) = 0,$$

$$\partial_\theta h + \nabla \cdot P = 0, \quad P = \nabla \cdot \left(h^{-1}B \otimes B\right) + \nabla \left(h^{-1}\right).$$

This can be seen as a dissipative model of Magnetohydrodynamics (MHD) where a fluid of density h and momentum P interacts with a magnetic field B, with several interesting (and intriguing) features:
(i) the first equation, which can be interpreted in MHD terms as the "induction equation" for B, involves a second-order diffusion term typical of MHD: $\nabla \times (h^{-1}\nabla \times (h^{-1}B))$ (with, however, an unusual dependence on h); (ii) the third equation describes the motion of the fluid of density h and momentum P driven by the magnetic field B and can be interpreted as a (generalized) Darcy law (and not as the usual momentum equation of MHD), just if the fluid was moving in a porous medium (which seems highly unusual in MHD!); (iii) there are many coefficients

which depend on h in a very peculiar way; in particular the Darcy law involves the so-called Chaplygin pressure $p = -h^{-1}$ (with sound speed $\sqrt{dp/dh} = h^{-1}$), which is sometimes used for the modeling of granular flows and also in cosmology, but not (to the best of our knowledge) in standard MHD.

To conclude this subsection, let us emphasize the remarkable structure of the (10×10) extended Born-Infeld system, after quadratic time-rescaling $t \to \theta = t^2/2$, which interpolates between the description of a continuum of strings (as $\theta >> 1$), in the style of high energy physics (however without any quantum feature) and a much more "down to earth" (but highly conjectural) dissipative model of MHD in a porous medium (as $\theta << 1$)!

2.3 Dissipative solution of the diffusion equations

From now on, we focus on the analysis of the following system of diffusion equations (we call Darcy MHD, or DMHD),

$$\partial_t h + \nabla \cdot (hv) = 0, \tag{2.3.1}$$

$$\partial_t B + \nabla \times (B \times v + d) = 0, \tag{2.3.2}$$

$$D = hd = \nabla \times \left(\frac{B}{h}\right), \quad P = hv = \nabla \cdot \left(\frac{B \otimes B}{h}\right) + \nabla \left(h^{-1}\right), \tag{2.3.3}$$

$$\nabla \cdot B = 0. \tag{2.3.4}$$

Written in the non-conservative variables $(\tau, b, d, v) = (1/h, B/h, D/h, P/h)$, the equation reads

$$\partial_t \tau + v \cdot \nabla \tau = \tau \nabla \cdot v, \quad \partial_t b + (v \cdot \nabla) b = (b \cdot \nabla) v - \tau \nabla \times d, \tag{2.3.5}$$

$$d = \tau \nabla \times b, \quad v = (b \cdot \nabla) b + \tau \nabla \tau. \tag{2.3.6}$$

For simplicity, we consider the periodic solutions on $[0, T] \times \mathbb{T}^3$, $T > 0$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

2.3.1 Relative entropy and the idea of dissipative solution

For the moment, ignoring the existence and regularity issues, we assume that there exists a sufficiently smooth solution $(h > 0, B, D, P)$ of the Darcy MHD (2.3.1)-(2.3.4).

First, as introduced in the previous section, the augmented BI equations (2.2.4)-(2.2.5) have a strictly convex entropy, namely,

$$\frac{1 + B^2 + D^2 + P^2}{2h}.$$

By performing the quadratic change of time $t \to \theta = t^2/2$, in the regime $\theta << 1$, the entropy is reduced to

$$\frac{1 + B^2}{2h}.$$

It is natural to consider the above energy for the reduced parabolic system i.e., Darcy MHD. As an easy exercise, we can show that the energy we suggested above is decreasing as time goes on. In fact, we have the following equality,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x \in \mathbb{T}^3} \frac{B^2 + 1}{2h} + \int_{x \in \mathbb{T}^3} \frac{D^2 + P^2}{h} = 0 \tag{2.3.7}$$

(This is easy to check, since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{B^2+1}{2h} &= \int \frac{B\cdot\partial_t B}{h} - \int \frac{B^2+1}{2h^2}\partial_t h \\
&= -\int \nabla\times\left(\frac{B}{h}\right)\cdot(B\times v+d) - \int \nabla\left(\frac{B^2+1}{2h^2}\right)\cdot P \\
&= -\int \left[\nabla\times\left(\frac{B}{h}\right)\right]\cdot\frac{D}{h} - \int\left[\nabla\cdot\left(\frac{B\otimes B+I_3}{h}\right)\right]\cdot\frac{P}{h}
\end{aligned}$$

which, by (2.3.3), gives the dissipative term.)

Now, for any smooth test functions $(b^*, h^*) \in \mathbb{R}^3\times\mathbb{R}^+$, the relative entropy is defined by

$$\frac{1}{2h}\left[(B-hb^*)^2 + (1-hh^{*-1})^2\right]$$

Before going on, let's look at the following lemma which gives us a nice formula for the relative entropy:

Lemma 2.3.1. *For any functions $P, B, D, v^*, b^*, d^* \in C^1([0,T]\times\mathbb{T}^3,\mathbb{R}^3)$, and positive functions $0 < h, h^* \in C^1([0,T]\times\mathbb{T}^3,\mathbb{R})$, suppose (h, B, D, P) is a solution of the Darcy MHD* (2.3.1)-(2.3.4), *then the following equality always holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x\in\mathbb{T}^3}\frac{\left|\widetilde{U}\right|^2}{2h} + \int_{x\in\mathbb{T}^3}\frac{\widetilde{W}^{\mathrm{T}}Q(w^*)\widetilde{W}}{2h} + \int_{x\in\mathbb{T}^3}\widetilde{W}\cdot \mathrm{L}(w^*) = 0 \tag{2.3.8}$$

where

$$\widetilde{U} = \left(1-hh^{*-1}, B-hb^*\right),\quad \widetilde{W} = \left(\widetilde{U}, D-hd^*, P-hv^*\right),\quad w^* = (h^{*-1}, b^*, d^*, v^*),$$

$Q(w^*)$ is a symmetric matrix that has the following expression

$$Q(w^*) = \begin{pmatrix} -2\nabla\cdot v^* & (\nabla\times d^*)^{\mathrm{T}} & -(\nabla\times b^*)^{\mathrm{T}} & 0 \\ \nabla\times d^* & -\nabla v^* - \nabla v^{*\mathrm{T}} & 0 & \nabla b^* - \nabla b^{*\mathrm{T}} \\ -\nabla\times b^* & 0 & 2I_3 & 0 \\ 0 & \nabla b^{*\mathrm{T}} - \nabla b^* & 0 & 2I_3 \end{pmatrix}, \tag{2.3.9}$$

$\mathrm{L}(w^*) = \big(\mathrm{L}_h(w^*), \mathrm{L}_B(w^*), \mathrm{L}_D(w^*), \mathrm{L}_P(w^*)\big)$ *has the following expression*

$$\mathrm{L}_h(w^*) = \partial_t\big(h^{*-1}\big) - h^{*-1}\nabla\cdot v^* + v^*\cdot\nabla\big(h^{*-1}\big), \tag{2.3.10}$$

$$\mathrm{L}_B(w^*) = \partial_t b^* + (v^*\cdot\nabla)b^* - (b^*\cdot\nabla)v^* + h^{*-1}\nabla\times d^*, \tag{2.3.11}$$

$$\mathrm{L}_D(w^*) = d^* - h^{*-1}\nabla\times b^*, \tag{2.3.12}$$

$$\mathrm{L}_P(w^*) = v^* - (b^*\cdot\nabla)b^* - h^{*-1}\nabla\big(h^{*-1}\big). \tag{2.3.13}$$

Moreover, we have $L(w^*) = 0$ if $(h^*, h^*b^*, h^*d^*, h^*v^*)$ is also a solution to the Darcy MHD (2.3.1)-(2.3.4).

With the above lemma and the nice formula of the relative entropy, we can apply the Gronwall's lemma to estimate the growth of the relative entropy. This is the start point of introducing the concept of dissipative solution to study such degenerate parabolic system.

Now, first, we see that the matrix valued function $Q(w^*)$ in (2.3.8) is a symmetric and its right down 6×6 block is always positive definite. Now let us use $I_{n:m}$ to represent the $n\times n$ diagonal matrix whose first m terms are 1 while the rest terms are 0, let I_d be the $d\times d$ identity matrix. Then it is easy to verify that for any $\delta\in(0,2)$, there is a constant $r_0=r_0(w^*,\delta,T)$, such that for all $r\geq r_0$ and $(t,x)\in[0,T]\times\mathbb{T}^3$, we have

$$Q(w^*)+rI_{10:4}\geq(2-\delta)I_{10}>0.$$

For the convenience of writing, let us denote,

$$Q_r(w^*)=Q(w^*)+rI_{10:4}. \tag{2.3.14}$$

Then, (2.3.8) can be written as,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}-r\right)\int\frac{|\widetilde{U}|^2}{2h}+\int\frac{\widetilde{W}^{\mathrm{T}}Q_r(w^*)\widetilde{W}}{2h}+\int\widetilde{W}\cdot\mathrm{L}(w^*)=0. \tag{2.3.15}$$

We integrate it from 0 to t, then we have

$$\int\frac{|\widetilde{U}(t)|^2}{2h(t)}+\int_0^t e^{r(t-s)}\left[\int\frac{\widetilde{W}^{\mathrm{T}}Q_r(w^*)\widetilde{W}}{2h}+\widetilde{W}\cdot\mathrm{L}(w^*)\right]\mathrm{d}s=e^{rt}\int\frac{|\widetilde{U}(0)|^2}{2h(0)}. \tag{2.3.16}$$

Notice that the above equality have a nice structure since the left hand side is in fact a convex functional of (h,B,D,P). It is even possible to extend the meaning of the equality to Borel measures (cf. [22]). In our case, it is quite simple and direct. For any Borel measure $\rho\in C(\mathbb{T}^3,\mathbb{R})'$ and vector-valued Borel measure $U\in C(\mathbb{T}^3,\mathbb{R}^4)'$, we define

$$\Lambda(\rho,U)=\sup\left\{\int_{\mathbb{T}^3}a\rho+A\cdot U,\quad a+\frac{1}{2}|A|^2\leq 0\right\}\in[0,+\infty], \tag{2.3.17}$$

where the supremum is taken over all $(a,A)\in C(\mathbb{T}^d;\mathbb{R}\times\mathbb{R}^4)$. As an easy exercise, we can check that

$$\Lambda(\rho,U)=\begin{cases}\frac{1}{2}\int_{\mathbb{T}^3}|u|^2\rho, & \rho\geq 0,\ U\ll\rho,\ U=u\rho,\ u\in L^2_\rho\\ +\infty, & \text{otherwise}\end{cases} \tag{2.3.18}$$

So we can see that $\Lambda(\rho,U)$ is somehow a generalization of the functional $\int\frac{|U|^2}{2\rho}$ to Borel measures. Similarly, we can define a functional in terms of the space time integral of

$$\int_s^t\int_{\mathbb{T}^3}\frac{W^{\mathrm{T}}QW}{2\rho}$$

More precisely, for any Borel measure $\rho\in C([0,T]\times\mathbb{T}^3,\mathbb{R})'$, vector-valued Borel measure $W\in C([0,T]\times\mathbb{T}^3,\mathbb{R}^{10})'$, and matrix valued function $Q\in C([0,T]\times\mathbb{T}^3,\mathbb{R}^{10\times 10})$ which is always positive definite, we define

$$\widetilde{\Lambda}(\rho,W,Q;s,t)=\sup\left\{\int_s^t\int_{\mathbb{T}^3}a\rho+A\cdot W,\quad a+\frac{1}{2}|\sqrt{Q^{-1}}A|^2\leq 0\right\}\in[0,+\infty], \tag{2.3.19}$$

where the supremum is taken over all $(a, A) \in C([s,t] \times \mathbb{T}^d; \mathbb{R} \times \mathbb{R}^{10})$, $0 \le s < t \le T$. Similarly, we have

$$\widetilde{\Lambda}(\rho, W, Q; s, t) = \begin{cases} \dfrac{1}{2} \displaystyle\int_s^t \int_{\mathbb{T}^3} |\sqrt{Q} w|^2 \rho, & \text{on } [s,t],\ \rho \ge 0,\ W \ll \rho,\ W = w\rho,\ w \in L^2_\rho \\ +\infty, & \text{otherwise} \end{cases} \tag{2.3.20}$$

By using the above defined functional, (2.3.16) can be written as

$$e^{-rt}\Lambda(h(t), \widetilde{U}(t)) + \widetilde{\Lambda}(h, \widetilde{W}, e^{-rs}Q_r(w^*); 0, t) + R(t) = \Lambda(h(0), \widetilde{U}(0)). \tag{2.3.21}$$

where

$$R(t) = \int_0^t \int_{\mathbb{T}^3} e^{-rs}\widetilde{W} \cdot \mathrm{L}(w^*).$$

Now, instead of having an equality, we would like to look for all measure valued solutions such that their relative entropies $\Lambda(h, \widetilde{U})$ are less than the initial data in (2.3.16). This is the idea of introducing the concept of dissipative solution.

2.3.2 Definition of the dissipative solutions

With the help of (2.3.16) and the introducing of $\Lambda(h, U)$. Now we can give a definition of the dissipative solution of (DMHD). Our definition reads,

Definition 2.3.2. *We say that (h, B, D, P) with $h \in C([0,T], C(\mathbb{T}^3, \mathbb{R})'_{w^*})$, $B \in C([0,T], C(\mathbb{T}^3, \mathbb{R}^3)'_{w^*})$, $D, P \in C([0,T] \times \mathbb{T}^3, \mathbb{R}^3)'$, is a dissipative solution of (DMHD)* (2.3.1)-(2.3.4) *with initial data $h_0 \in C(\mathbb{T}^3, \mathbb{R})', B_0 \in C(\mathbb{T}^3, \mathbb{R}^3)'$ if and only if*

(i) *$h(0) = h_0$, $B(0) = B_0$, $\Lambda(h_0, U_0) < \infty$, where $U_0 = (\mathcal{L}, B_0)$, $\mathcal{L}$ is the Lebesgue measure on $\mathbb{T}^3$.*
(ii) *(h, B) is bounded in $C^{0,\frac{1}{2}}([0,T], C(\mathbb{T}^3, \mathbb{R}^4)'_{w^*})$ by some constant that depends only on T and (h_0, B_0).*
(iii) (2.3.1) *and* (2.3.4) *is satisfied in the sense of distributions. More precisely, for all $u \in C^1([0,T] \times \mathbb{T}^3, \mathbb{R})$ and $t \in [0,T]$, we have*

$$\int_0^t \int_{\mathbb{T}^3} \partial_t u\, h + \nabla u \cdot P = \int_{\mathbb{T}^3} u(t)h(t) - \int_{\mathbb{T}^3} u(0)h(0) \tag{2.3.22}$$

$$\int_{\mathbb{T}^3} \nabla u(t) \cdot B(t) = 0 \tag{2.3.23}$$

(iv) *For all $t \in [0,T]$ and all $v^*, b^*, d^* \in C^1([0,T] \times \mathbb{T}^3, \mathbb{R}^3)$, $0 < h^* \in C^1([0,T] \times \mathbb{T}^3, \mathbb{R})$ and all real number $r \ge r_0$, the following inequality always holds*

$$e^{-rt}\Lambda(h(t), \widetilde{U}(t)) + \widetilde{\Lambda}(h, \widetilde{W}, e^{-rs}Q_r(w^*); 0, t) + R(t) \le \Lambda(h(0), \widetilde{U}(0)) \tag{2.3.24}$$

where

$$\widetilde{U} = \left(\mathcal{L} - h^{*-1}h, B - hb^*\right),\ \ \widetilde{W} = \left(\widetilde{U}, D - hd^*, P - hv^*\right)$$

$$w^* = (h^{*-1}, b^*, d^*, v^*),$$

$Q_r(w^*)$ is a symmetric matrix defined by $Q_r(w^*) = Q(w^*) + rI_{10:4}$, where $Q(w^*)$ is defined in (2.3.9)*. r_0 is a constant chosen such that $Q_{r_0}(w^*) \geq I_{10}$ for all $(t,x) \in [0,T] \times \mathbb{T}^3$. $R(t)$ is a functional that depends linearly on $\widetilde{W}$ with the expression*

$$R(t) = \int_0^t \int_{\mathbb{T}^3} e^{-rs}\widetilde{W} \cdot \mathrm{L}(w^*). \tag{2.3.25}$$

where $\mathrm{L}(w^*)$ *is defined in* (2.3.10)-(2.3.13)*.*

Note that in the above definition, $C(\mathbb{T}^3, \mathbb{R})'_{w^*}$ is the dual space of $C(\mathbb{T}^3, \mathbb{R})$ equipped with the weak-$*$ topology. It is a metrizable space, we can define a metric that is consistent with the weak-$*$ topology, for example, we can take

$$d(\rho, \rho') = \sum_{n \geq 0} 2^{-n} \frac{|\langle \rho - \rho', f_n \rangle|}{1 + |\langle \rho - \rho', f_n \rangle|} \tag{2.3.26}$$

where $\{f_n\}_{n\geq 0}$ is a smooth dense subset of the separable space $C(\mathbb{T}^3, \mathbb{R})$, $\langle \cdot, \cdot \rangle$ denote the duality pairing of $C(\mathbb{T}^3, \mathbb{R})$ with its dual space.

2.4 Properties of the dissipative solutions

In this section, we will study some properties of the dissipative solutions that we define in the previous part. We will show that the dissipative solutions satisfy the weak-strong uniqueness, the set of solutions are convex and compact in the weak-$*$ topology, and under what situation, the dissipative solutions become strong solutions.

2.4.1 Consistency with smooth solutions

In this part, let's look at a very interesting question about the dissipative solution. It has been shown that, in Lemma 2.3.1, any strong solution (h, B, D, P) to (DMHD) satisfies the energy dissipative inequality (3.1.8), so it is naturally a dissipative solution. On the contrary, it is generally not true that a dissipative solution is a strong solution. However, if we know that the dissipative solution has some regularity (for example C^1 solutions), then the reverse statement is true. We summarize our result in the following proposition:

Proposition 2.4.1. *Suppose $(h, B, D, P) \in C^1([0,T] \times \mathbb{T}^3, \mathbb{R}^{10})$ is a dissipative solution to (DMHD)* (2.3.1)-(2.3.4) *in the sense of Definition 3.1.1, then is must be a strong solution.*

Proof. The proof follows almost the same computation as in Lemma 2.3.1. First, the equation (2.3.1) and (2.3.4) is naturally satisfied by the definition of dissipative solution. Our goal is to show that (h, B, D, P) also satisfy (2.3.2),(2.3.3) in the strong sense. To prove this, we denote

$$\phi = \partial_t B + \nabla \times \left(\frac{D + B \times P}{h} \right), \quad \psi = D - \nabla \times \left(\frac{B}{h} \right)$$

$$\varphi = P - \nabla \cdot \left(\frac{B \otimes B}{h} \right) - \nabla \left(\frac{1}{h} \right)$$

We only need to prove that $\phi = \psi = \varphi = 0$. In fact, for any test function $v^*, b^*, d^* \in C^1([0,T] \times \mathbb{T}^3, \mathbb{R}^3), h^* > 0 \in C^1([0,T] \times \mathbb{T}^3, \mathbb{R})$, we follow the same computation as in Lemma 2.3.1 (this is shown in the Appendix 2.A), then we can the following equality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\widetilde{U}|^2}{2h} + \int \frac{W^{\mathrm{T}}Q(w^*)W}{2h} + \int \widetilde{W}\cdot \mathrm{L}(w^*) = \int \Big[\phi\cdot(b-b^*)+\psi\cdot(d-d^*)+\varphi\cdot(v-v^*)\Big]$$

Now, let's set $b^* = b - \phi$, $d^* = d - \psi$, $v^* = v - \varphi$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\widetilde{U}|^2}{2h} + \int \frac{W^{\mathrm{T}}Q(w^*)W}{2h} + \int \widetilde{W}\cdot \mathrm{L}(w^*) = \int \big(\phi^2 + \psi^2 + \varphi^2\big)$$

For r big enough, we have

$$e^{-rT}\int \frac{|\widetilde{U}(T)|^2}{2h(T)} + \int_0^T e^{-rs}\left[\int \frac{\widetilde{W}^{\mathrm{T}}Q_r(w^*)\widetilde{W}}{2h} + \widetilde{W}\cdot \mathrm{L}(w^*)\right]\mathrm{d}s - \int \frac{|\widetilde{U}(0)|^2}{2h(0)}$$
$$= \int_0^T \int e^{-rs}\big(\phi^2 + \psi^2 + \varphi^2\big)$$

By the definition of dissipative solution, we have

$$\int_0^T \int e^{-rs}\big(\phi^2 + \psi^2 + \varphi^2\big) \leq 0$$

This implies that $\phi \equiv \varphi \equiv \psi \equiv 0$, which completes the proof.

□

2.4.2 Weak-strong uniqueness and stability result

The weak-strong uniqueness is essentially an important property for a suitable concept of "weak" solution of a given evolution system. By the weak-strong uniqueness, we mean that any weak solution must coincide with a strong solution emanating from the same initial data as long as the latter exists. In other words, the strong solutions must be unique within the class of weak solutions. This kind of problem has been widely studied in various kinds of equations (Navier-Stokes, Euler, etc.), even for measure valued solutions [12]. In our (DMHD), we will show that the dissipative solution also enjoy this kind of property. First, let's us show a stability estimate.

Proposition 2.4.2. *Suppose that $(h^* > 0, B^*, D^*, P^*)$ is a classical (at least C^1) solution of (DMHD) (2.3.1)-(2.3.4) with initial value $(h^*, B^*)|_{t=0} = (h_0^*, B_0^*)$. (h, B, D, P) is a dissipative solution with initial value $(h, B)|_{t=0} = (h_0, B_0)$. Let us denote*

$$\widetilde{U} = \big(\mathcal{L} - hh^{*-1}, B - hb^*\big), \quad \widetilde{W} = \big(\widetilde{U}, D - hd^*, P - hv^*\big)$$

where $b^* = B^*/h^*$, $d^* = D^*/h^*$, $v^* = P^*/h^*$. Then, for any $t \in [0,T]$, there exist a constant C that depends only on the choice of (h^*, B^*, D^*, P^*), the value of $\Lambda(h_0, \widetilde{U}_0)$ and T, such that the following estimates hold

$$\|\widetilde{U}(t)\|_{TV}^2 \leq Ce^{Ct}\Lambda(h_0, \widetilde{U}_0), \quad \|\widetilde{W}\|_{TV^*}^2 \leq Ce^{CT}\Lambda(h_0, \widetilde{U}_0). \tag{2.4.1}$$

Here $\|\cdot\|_{TV}$, $\|\cdot\|_{TV^*}$ respectively represent the total variation of measures on $\mathbb{T}^3$ and $[0,T]\times\mathbb{T}^3$. Furthermore, we have that

$$\|h(t)-h^*(t)\|_{TV}^2,\ \ \|B(t)-B^*(t)\|_{TV}^2 \le Ce^{Ct}\Lambda(h_0,\widetilde{U}_0), \tag{2.4.2}$$

$$\|D-D^*\|_{TV^*}^2,\ \ \|P-P^*\|_{TV^*}^2 \le Ce^{CT}\Lambda(h_0,\widetilde{U}_0). \tag{2.4.3}$$

Proof. The proof is very simple. We just need to take (h^*,b^*,d^*,v^*) defined in the proposition as our test functions and apply it to the energy dissipative inequality (3.1.8). Because (h^*,B^*,D^*,P^*) is a strong solution, so we have $\mathrm{L}(w^*)\equiv 0$, where $w^*=(h^{*-1},b^*,d^*,v^*)$. Let $r_0>0$ be a constant such that $Q_{r_0}(w^*)\ge I_{10}$ for all $(t,x)\in[0,T]\times\mathbb{T}^3$. Then for $r\ge r_0$ and $t\in[0,T]$, (3.1.8) gives

$$e^{-rt}\Lambda(h(t),\widetilde{U}(t))+\widetilde{\Lambda}(h,\widetilde{W},e^{-rs}Q_r(w^*);0,t)\le\Lambda(h_0,\widetilde{U}_0).$$

So we have

$$\Lambda(h(t),\widetilde{U}(t))\le e^{rt}\Lambda(h_0,\widetilde{U}_0),$$

and, since $e^{-rs}Q_r(w^*)\ge e^{-rT}I_{10}$, we have

$$e^{-rT}\widetilde{\Lambda}(h,\widetilde{W},I_{10};0,T)\le\widetilde{\Lambda}(h,\widetilde{W},e^{-rs}Q_r(w^*);0,T)\le\Lambda(h_0,\widetilde{U}_0).$$

Now, since h satisfies (2.3.1) in the sense of distributions, we have

$$\int_{\mathbb{T}^3}h(t)=\int_{\mathbb{T}^3}h_0.$$

Then, by the expression of Λ, $\widetilde{\Lambda}$ in (2.3.18),(2.3.20), and Cauchy-Schwarz inequality, we have

$$\|\widetilde{U}(t)\|_{TV}^2\le 2\Lambda(h(t),\widetilde{U}(t))\int_{\mathbb{T}^3}h(t)\le 2e^{rt}\Lambda(h_0,\widetilde{U}_0)\int_{\mathbb{T}^3}h_0$$

$$\|\widetilde{W}\|_{TV^*}^2\le 2\widetilde{\Lambda}(h,\widetilde{W},I_{10};0,T)\int_0^T\int_{\mathbb{T}^3}h\le 2Te^{rT}\Lambda(h_0,\widetilde{U}_0)\int_{\mathbb{T}^3}h_0$$

Now, we would like to estimate the value of $\|h_0\|_{TV}$, given the value of $\Lambda_0=\Lambda(h_0,\widetilde{U}_0)$ and h_0^*. Since

$$2\Lambda_0\int_{\mathbb{T}^3}h_0\ge\|\widetilde{U}_0\|_{TV}^2\ge\left\|\mathcal{L}-h_0h_0^{*-1}\right\|_{TV}^2\ge\|h_0^*\|_\infty^{-2}\left(\int_{\mathbb{T}^3}|h_0^*-h_0|\right)^2$$

So we have

$$\int_{\mathbb{T}^3}|h_0^*-h_0|\le\|h_0^*\|_\infty\sqrt{2\Lambda_0\int_{\mathbb{T}^3}h_0}$$

Then we have that

$$\int_{\mathbb{T}^3}h_0\le\int_{\mathbb{T}^3}|h_0^*-h_0|+\int_{\mathbb{T}^3}h_0^*\le\|h_0^*\|_\infty\sqrt{2\Lambda_0\int_{\mathbb{T}^3}h_0}+\int_{\mathbb{T}^3}h_0^*$$

$$\le\frac{1}{2}\int_{\mathbb{T}^3}h_0+\|h_0^*\|_\infty^2\Lambda_0+\int_{\mathbb{T}^3}h_0^*$$

So we have the estimate

$$\int_{\mathbb{T}^3} h_0 \leq 2\left(\|h_0^*\|_\infty^2 \Lambda_0 + \int_{\mathbb{T}^3} h_0^*\right)$$

Combining the above results, we can find a constant C which depends only on (h^*, B^*, D^*, P^*), $\Lambda(h_0, \widetilde{U}_0)$ and T, such that the estimate (2.4.1) is satisfied. (2.4.2),(2.4.3) are also easy to prove, since we have

$$\begin{aligned}\|B(t) - B^*(t)\|_{TV} &\leq \|B(t) - h(t)b^*(t)\|_{TV} + \|B^*(t) - h(t)b^*(t)\|_{TV} \\ &\leq \|B(t) - h(t)b^*(t)\|_{TV} + \|B^*(t)\|_\infty \left\|\mathcal{L} - h(t)h^{*-1}(t)\right\|_{TV} \\ &\leq (1 + \|B^*\|_\infty)\|\widetilde{U}(t)\|_{TV}\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|D - D^*\|_{TV^*} &\leq \|D - hd^*\|_{TV^*} + \|D^*\|_\infty \left\|\mathcal{L} - hh^{*-1}\right\|_{TV^*} \\ &\leq (1 + \|D^*\|_\infty)\|\widetilde{W}\|_{TV^*},\end{aligned}$$

$$\|h(t) - h^*(t)\|_{TV} \leq \|h^*\|_\infty \|\widetilde{U}(t)\|_{TV}, \quad \|P - P^*\|_{TV^*} \leq (1 + \|P^*\|_\infty)\|\widetilde{W}\|_{TV^*}$$

Then by (2.4.1), we can quickly get our desired result.

□

The above proposition gives us an estimate of the distance of two different dissipative solution as time evolves. As a direct consequence, we immediately get the weak-strong uniqueness for the dissipative solutions.

Proposition 2.4.3. *Suppose that $(h > 0, B, D, P)$ is a classical (at least C^1) solution to (DMHD)* (2.3.1)-(2.3.4) *with initial value $(h, B)|_{t=0} = (h_0, B_0)$, then it is the unique dissipative solution to (DMHD) with the same initial value.*

Proof. The proof is very simple. Suppose there is another dissipative solution (h', B', D', P'). Since $\widetilde{U}_0 = 0$, the estimates (2.4.2),(2.4.3) in the previous proposition just give us the uniqueness.

□

2.4.3 Weak compactness

From the previous parts, we know that, if we have a smooth dissipative solution, then it should be a strong solution, and, therefore, it should be the unique dissipative solution with the same initial data. However, in general, dissipative solutions are not usually that regular. A natural question is that, what happens to the dissipative solutions that are not smooth? Are they unique? If not, what can we conclude for the set of dissipative solutions? In this part, we will show that the dissipative solutions satisfy the "weak compactness" property (i.e. any sequence of dissipative solutions has accumulations points, in a suitable weak sense, and each of them is still a dissipative solution). We summarize our result in the following theorem:

Theorem 2.4.4. *For any initial data $B_0 \in C(\mathbb{T}^3, \mathbb{R}^3)'$, $h_0 \in C(\mathbb{T}^3, \mathbb{R})'$, satisfying that $\nabla \cdot B_0 = 0$ in the sense of distributions and $\Lambda(h_0, U_0) < \infty$, let $\mathcal{A}$ be the set of all dissipative solutions (h, B, D, P) to (DMHD)* (2.3.1)-(2.3.4) *with initial data (h_0, B_0). Then $\mathcal{A}$ is a non-empty convex compact set in the space $C([0,T], C(\mathbb{T}^3, \mathbb{R} \times \mathbb{R}^3)'_{w^*}) \times C([0,T] \times \mathbb{T}^3, \mathbb{R}^3 \times \mathbb{R}^3)'_{w^*}$.*

Proof. The non-emptiness of $\mathcal{A}$ refers just to the existence of the dissipative solutions. The proof is a little lengthy, we leave the existence proof in Section 2.6 and Section 2.7. Here, let's prove the convexity and compactness. The convexity of $\mathcal{A}$ is quite easy. As we can see in Definition 3.1.1, (2.3.22),(2.3.23) are linear equations. So it is always satisfied under any convex combination of dissipative solutions. Since the functional $\Lambda, \widetilde{\Lambda}$ are convex, so (3.1.8) is also satisfied. So we know that the set $\mathcal{A}$ is convex. Now let's show the compactness. Since $h^* \equiv 1$, $B^* = D^* = P^* = 0$ is a trivial solution, then for any family of dissipative solutions (h_n, B_n, D_n, P_n) with initial data (h_0, B_0), by Proposition 2.4.2, there exist a constant C that depends only on (h_0, B_0) and T, such that

$$\|h_n(t) - 1\|_{TV}, \quad \|B_n(t)\|_{TV}, \quad \|D_n\|_{TV^*}, \quad \|P_n\|_{TV^*} \leq C.$$

Since (h_n, B_n) are uniformly bounded in $C^{0,\frac{1}{2}}([0,T], C(\mathbb{T}^3, \mathbb{R}^4)'_{w^*})$, then up to a subsequence, (h_n, B_n) converge to some function (h, B) in $C([0,T], C(\mathbb{T}^3, \mathbb{R}^4)'_{w^*})$. Also, since (D_n, P_n) are uniformly bounded in $C([0,T] \times \mathbb{T}^3, \mathbb{R}^3 \times \mathbb{R}^3)'$, then up to a subsequence, (D_n, P_n) converge weakly-$*$ to some (D, P) in $C([0,T] \times \mathbb{T}^3, \mathbb{R}^3 \times \mathbb{R}^3)'$. Now we only need to prove that (h, B, D, P) is a dissipative solution. This is easy since (2.3.22),(2.3.23) and (3.1.8) are weakly stable.

□

2.5 Comparison with smooth solutions of the ABI equations

As it has been shown in Section 2.2, we can get our (DMHD) out of the augmented BI equations (2.2.4),(2.2.5) through the quadratic change of the time variable $\theta \to t^2/2$. Now a natural question is that, since the (DMHD) can be seen as an approximation of the ABI equations, how about the solutions of these two systems of equations? Are they close to each other when the initial data are the same? With our concept of the dissipative solution, it is possible to give an answer.

In [16], it is shown that ABI equations can be rewritten as a symmetric hyperbolic system of conservation laws. So smooth solutions exist at least in a short period of time for smooth initial data, see [20]. Now, for any smooth function $h_0 > 0$, B_0, $\nabla \cdot B_0 = 0$, there exist a time interval $[0, t_0]$, such that there exists a smooth solution (h', B', D', P') to the augmented BI system with initial value $(h_0, B_0, 0, 0)$. We will compare the smooth solution (h', B', D', P') with the dissipative solution (h, B, D, P) to the Darcy MHD on $[0, T]$, $T \geq t_0^2/2$, with the same initial value (h_0, B_0). Our estimates are in the following proposition.

Proposition 2.5.1. *Suppose (h', B', D', P') is a smooth solution to the augmented BI equations* (2.2.4),(2.2.5) *on* $[0, t_0]$ *with smooth initial data* $(h_0, B_0, 0, 0)$. (h, B, D, P) *is a dissipative solution to (DMHD)* (2.3.1)-(2.3.4) *on* $[0, T]$, $T \geq t_0^2/2$, *with the same initial data* (h_0, B_0). *Then there exists a constant C that depends only on (h', B', D', P') and t_0, such that for any $t \in [0, t_0]$, we have*

$$\|h'(t) - h(t^2/2)\|_{TV}, \quad \|B'(t) - B(t^2/2)\|_{TV} \leq Ct^3 \tag{2.5.1}$$

$$\big|D'(s) - sD(s^2/2)\big|([0,t] \times \mathbb{T}^3), \quad \big|P'(s) - sP(s^2/2)\big|([0,t] \times \mathbb{T}^3) \leq Ct^4 \tag{2.5.2}$$

Here $|\cdot|$ represents the variation of the vector-valued measures. $sD(s^2/2)$, $sP(s^2/2)$ denote the vector-valued Borel measures on $[0,t_0]\times\mathbb{T}^3$ defined in the way such that, for all $\varphi\in C([0,t_0]\times\mathbb{T}^3,\mathbb{R}^3)$, we have

$$\int_0^{t_0}\int_{\mathbb{T}^3}\varphi(s)\cdot sD(s^2/2)=\int_0^{t_0^2/2}\int_{\mathbb{T}^3}\varphi(\sqrt{2s})\cdot D(s)$$

$$\int_0^{t_0}\int_{\mathbb{T}^3}\varphi(s)\cdot sP(s^2/2)=\int_0^{t_0^2/2}\int_{\mathbb{T}^3}\varphi(\sqrt{2s})\cdot P(s)$$

Proof. First, since (h',B',D',P') is a smooth solution to the augmented BI equations, then the non-conservative variables

$$(\tau',b',d',v')=(1/h',B'/h',D'/h',P'/h')$$

should satisfy the following equations

$$\partial_t b'+(v'\cdot\nabla)b'=(b'\cdot\nabla)v'-\tau'\nabla\times d',\ \ \partial_t d'+(v'\cdot\nabla)d'=(d'\cdot\nabla)v'+\tau'\nabla\times b',$$

$$\partial_t\tau'+(v'\cdot\nabla)\tau'=\tau'\nabla\cdot v',\ \ \partial_t v'+(v'\cdot\nabla)v'=(b'\cdot\nabla)b'+(d'\cdot\nabla)d'+\tau'\nabla\tau',$$

Now let's take our test function h^*,b^*,d^*,v^* defined as following

$$\begin{aligned} h^*(\theta,x)&=h'(\sqrt{2\theta},x) \quad & b^*(\theta,x)&=b'(\sqrt{2\theta},x)\\ d^*(\theta,x)&=\frac{d'(\sqrt{2\theta},x)}{\sqrt{2\theta}} \quad & v^*(\theta,x)&=\frac{v'(\sqrt{2\theta},x)}{\sqrt{2\theta}} \end{aligned}\tag{2.5.3}$$

We should notice that d^*,v^* is well defined and continuous with value $\partial_t d'(0),\partial_t v'(0)$ at time $\theta=0$. Moreover, it is easy to verify that $\partial_t^2 d'(0)=\partial_t^2 v'(0)=0$, so we know that h^*,b^*,d^*,v^* are C^1 functions with $\partial_\theta d^*(0)=\frac{1}{3}\partial_t^3 d'(0),\partial_\theta v^*(0)=\frac{1}{3}\partial_t^3 v'(0)$. Now, let's do the change of time $\theta=t^2/2$, (2.5.3) means

$$\tau'(t,x)=h^{*-1}(\theta,x),\ b'(t,x)=b^*(\theta,x),\ d'(t,x)=td^*(\theta,x),\ v'(t,x)=tv^*(\theta,x)$$

Then our test function should satisfy the following equations,

$$\partial_\theta(h^{*-1})+(v^*\cdot\nabla)h^{*-1}=h^{*-1}\nabla\cdot v^*$$

$$\partial_\theta b^*+(v^*\cdot\nabla)b^*=(b^*\cdot\nabla)v^*-h^{*-1}\nabla\times d^*$$

$$d^*-h^{*-1}\nabla\times b^*=-2\theta\big[\partial_\theta d^*+(v^*\cdot\nabla)d^*-(d^*\cdot\nabla)v^*\big],$$

$$v^*-(b^*\cdot\nabla)b^*-h^{*-1}\nabla(h^{*-1})=-2\theta\big[\partial_\theta v^*+(v^*\cdot\nabla)v^*-(d^*\cdot\nabla)d^*\big],$$

So we have that

$$\mathrm{L}_h(w^*)=\mathrm{L}_B(w^*)=0,\ \ \mathrm{L}_D(w^*)=-2\theta\psi_d^*,\ \ \mathrm{L}_P(w^*)=-2\theta\psi_v^*,$$

where ψ_d^*,ψ_v^* are continuous functions with the following expressions

$$\psi_d^*=\partial_\theta d^*+(v^*\cdot\nabla)d^*-(d^*\cdot\nabla)v^*,$$

$$\psi_v^*=\partial_\theta v^*+(v^*\cdot\nabla)v^*-(d^*\cdot\nabla)d^*.$$

Now, for the dissipative solution (h,B,D,P) to (DMHD), we denote as usual,

$$\widetilde{U}=\big(\mathcal{L}-hh^{*-1},B-hb^*\big),\quad \widetilde{W}=\big(\widetilde{U},D-hd^*,P-hv^*\big)$$

Since the initial value are the same, we have $\widetilde{U}(0)=0$. Now we follow the definition of dissipative solution, there exists a constant $r>0$ such that $Q_r(w^*)\geq I_{10}$ for all $(\theta,x)\in[0,t_0^2/2]\times\mathbb{T}^3$. By (3.1.8), we have, for $\theta\in[0,t_0^2/2]$,

$$e^{-r\theta}\Lambda(h(\theta),\widetilde{U}(\theta))+\widetilde{\Lambda}(h,\widetilde{W},e^{-r\theta'}Q_r(w^*);0,\theta)+R(\theta)\leq 0. \tag{2.5.4}$$

where

$$R(\theta)=-2\theta\int_0^\theta\int_{\mathbb{T}^3}\Big[\psi_d^*\cdot(D-hd^*)+\psi_v^*\cdot(P-hv^*)\Big].$$

By Cauchy-Schwarz inequality, we have

$$\|\widetilde{U}(\theta)\|_{TV}^2\leq 2\Lambda(h(\theta),\widetilde{U}(\theta))\int_{\mathbb{T}^3}h(\theta)=2\Lambda(h(\theta),\widetilde{U}(\theta))\int_{\mathbb{T}^3}h_0$$

$$\Big(\big|\widetilde{W}\big|([0,\theta]\times\mathbb{T}^3)\Big)^2\leq 2\theta e^{r\theta}\widetilde{\Lambda}(h,\widetilde{W},e^{-r\theta'}Q_r(w^*);0,\theta)\int_{\mathbb{T}^3}h_0$$

Now let $M=\|\psi_d^*\|_\infty+\|\psi_v^*\|_\infty$, then we have

$$|R(\theta)|\leq 2\theta\big|\widetilde{W}\big|([0,\theta]\times\mathbb{T}^3)M$$

Therefore, (2.5.4) implies,

$$\theta\|\widetilde{U}(\theta)\|_{TV}^2+\Big(\big|\widetilde{W}\big|([0,\theta]\times\mathbb{T}^3)\Big)^2\leq C\theta^2\big|\widetilde{W}\big|([0,\theta]\times\mathbb{T}^3) \tag{2.5.5}$$

where C is a constant that depends only on M, h_0, r and t_0. Then we have that

$$\big|\widetilde{W}\big|([0,\theta]\times\mathbb{T}^3)\leq C\theta^2,\quad \|\widetilde{U}(\theta)\|_{TV}^2\leq C^2\theta^3.$$

Since

$$\|h(t^2/2)-h'(t)\|_{TV}=\|h(\theta)-h^*(\theta)\|_{TV}\leq\|h^*\|_\infty\|\mathcal{L}-hh^{*-1}\|_{TV}$$

$$\leq\|h^*\|_\infty\|\widetilde{U}(\theta)\|_{TV}\leq 2^{-\frac{3}{2}}\|h^*\|_\infty Ct^3$$

$$\|B(t^2/2)-B'(t)\|_{TV}\leq\|B(\theta)-h(\theta)b^*(\theta)\|_{TV}+\|b^*(\theta)(h(\theta)-h^*(\theta))\|_{TV}$$

$$\leq(1+\|h^*b^*\|_\infty)\|\widetilde{U}(\theta)\|_{TV}\leq 2^{-\frac{3}{2}}(1+\|h^*b^*\|_\infty)Ct^3$$

so we get (2.5.1). Now, for $sD(s^2/2)$, we have

$$\big|D'(s)-sD(s^2/2)\big|([0,t]\times\mathbb{T}^3)=\big|h^*(\theta)d^*(\theta)-D(\theta)\big|([0,\theta]\times\mathbb{T}^3)$$

$$\leq(1+\|h^*d^*\|_\infty)\big|\widetilde{W}\big|([0,\theta]\times\mathbb{T}^3)\leq 2^{-2}(1+\|h^*d^*\|_\infty)Ct^4$$

similarly,

$$\big|P'(s)-sP(s^2/2)\big|([0,t]\times\mathbb{T}^3)\leq 2^{-2}(1+\|h^*v^*\|_\infty)Ct^4$$

so we get (2.5.2).

$\square$

2.6 Faedo-Galerkin approximation

In the following two sections, we will mainly focus on the existence theory of the dissipative solutions. In this section, we consider an approximate system of (DMHD). We want to get a dissipative solution of (DMHD) by the approaching of solutions of the approximate system. In fact, we don't really need to solve the approximate system, we only need to find a sequence of approximate solutions on some finite dimensional spaces, which is quite similar to the Faedo-Galerkin method of E. Feireisl [29, 27, 28]. We consider the following approximate equations

$$\partial_t h + \nabla \cdot (hv) = 0, \tag{2.6.1}$$

$$\partial_t B + \nabla \times (B \times v + d) = 0, \quad \nabla \cdot B = 0 \tag{2.6.2}$$

$$\varepsilon\Big[\partial_t(hd) + \nabla \cdot [h(d \otimes v - v \otimes d)] + (-\triangle)^l d\Big] + hd = \nabla \times b, \tag{2.6.3}$$

$$\varepsilon\Big[\partial_t(hv)+\nabla\cdot(hv\otimes v)-(hd\cdot\nabla)d+(-\triangle)^l v\Big]+hv = \nabla\cdot\left(\frac{B\otimes B}{h}\right)+\nabla\left(h^{-1}\right), \tag{2.6.4}$$

Here, $0 < \varepsilon < 1$, we choose l sufficiently big ($l \geq 8$). The idea of using these equations as approximate system comes from the way we get the Darcy MHD from augmented BI. The time derivatives of d, v here can ensure that the approximate solutions are continuous with respect to time. We introduce the high order derivatives here to get some regularities that will be useful in showing the existence. Very similar to the case of augmented BI, we have the following formula (the proof is quite straightforward, we leave it to interested readers)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\left[\frac{1+B^2}{2h}+\varepsilon\frac{h(v^2+d^2)}{2}\right]+\int_{\mathbb{T}^3} h(v^2+d^2)+\varepsilon\int_{\mathbb{T}^3}\left|\nabla^l v\right|^2+\left|\nabla^l d\right|^2=0 \tag{2.6.5}$$

2.6.1 Classical solution for fixed (d, v)

Now let us consider the solution of (2.6.1) and (2.6.2) when d, v are given smooth functions.

Lemma 2.6.1. *Suppose h_0, B_0 are smooth functions with $\nabla \cdot B_0 = 0,\ h_0 > 0$. Then for any integer $k \geq 1$ and given $d, v \in C([0,T], C^{k+1}(\mathbb{T}^3, \mathbb{R}^3))$, there exists a unique solution $h \in C^1([0,T], C^k(\mathbb{T}^3, \mathbb{R}^3)), B \in C^1([0,T], C^k(\mathbb{T}^3, \mathbb{R}^3))$ to (2.6.1) and (2.6.2) with initial data h_0, B_0.*

Proof. We use the method of characteristics to show the existence of classical solutions. For (2.6.1), the solution can be written explicitly as

$$h(t,x) = h_0\big(\Phi(0,t,x)\big)\exp\left\{-\int_0^t \nabla\cdot v\big(s,\Phi(s,t,x)\big)\mathrm{d}s\right\} \tag{2.6.6}$$

where $\Phi(t,s,x) \in C^1([0,T]\times[0,T]\times\mathbb{T}^3)$ is the unique solution of

$$\begin{cases} \partial_t\Phi(t,s,x) = v\big(t,\Phi(t,s,x)\big) & 0 \leq t \leq T \\ \Phi(s,s,x) = x & 0 \leq s \leq T,\ x \in \mathbb{T}^3 \end{cases} \tag{2.6.7}$$

Because $v \in C([0,T], C^{k+1}(\mathbb{T}^3, \mathbb{R}^3))$, the existence and uniqueness of such solution is obtained directly by Cauchy-Lipschitz Theorem. Moreover, we can proof that the solution $\Phi(t,s,x) \in C^1([0,T], C^1([0,T], C^k(\mathbb{T}^3,\mathbb{R}^3)))$.(Take space derivatives on both side of the equation, the new equation is composed of lower derivatives and is linear for the highest older derivatives. By induction, the Cauchy-Lipschitz Theorem gives a solution and by uniqueness, we can show the solution of (2.6.7) is that sufficiently differentiable.)

For (2.6.2), we also have an explicit expression of the solution

$$B(t,x) = G\big(t, \Phi(0,t,x)\big) \exp\left\{ -\int_0^t \nabla \cdot v\big(s, \Phi(s,t,x)\big) \mathrm{d}s \right\} \tag{2.6.8}$$

where $G(t,x) \in C^1([0,T] \times \mathbb{T}^3)$ is the unique solution of

$$\begin{cases} \partial_t G(t,x) = \nabla v\big(t, \Phi(t,0,x)\big) \cdot G(t,x) & 0 \le t \le T \\ \qquad\qquad -(\nabla \times d)\big(t, \Phi(t,0,x)\big) \exp\left\{ \int_0^t \nabla \cdot v\big(s, \Phi(s,0,x)\big) \mathrm{d}s \right\} & \\ G(0,x) = B_0(x) & x \in \mathbb{T}^3 \end{cases} \tag{2.6.9}$$

By the same reason, $G(t,x) \in C([0,T], C^k(\mathbb{T}^3,\mathbb{R}^3))$. It is not hard verify that (h,B) defined in (2.6.6),(2.6.8) is indeed a solution. At last, let us look into the uniqueness of solutions. Because the equations are linear with respect to h and B, it is easy to show the uniqueness by the L^2 estimates of the difference of two solutions. We can also see it from the following lemma. $\square$

Now for any fixed $z = (d,v) \in C([0,T], C^{k+1}(\mathbb{T}^3,\mathbb{R}^6))$, let us denote $h[z,h_0]$, $B[z,B_0]$ the unique solution of (2.6.1),(2.6.2) with initial value h_0, B_0 at time $t = 0$. Then we have the following lemma.

Lemma 2.6.2. *Suppose $z = (d,v) \in C([0,T], C^{k+1}(\mathbb{T}^3,\mathbb{R}^6))$, $k \ge 1$, h_0, B_0 are smooth functions, $\nabla \cdot B_0 = 0$, $h_0 > 0$. Then we have*
(i) *For all $t \in [0,T]$, $x \in \mathbb{T}^3$,*

$$0 < e^{-\int_0^t \|\nabla \cdot v(s)\|_\infty \mathrm{d}s} \inf_{x\in\mathbb{T}^3} h_0 \le h(t,x) \le e^{\int_0^t \|\nabla \cdot v(s)\|_\infty \mathrm{d}s} \sup_{x\in\mathbb{T}^3} h_0 \tag{2.6.10}$$

(ii) *Suppose $\sup_{t\in[0,T]} \|v(t)\|_{C^{k+1}}$, $\sup_{t\in[0,T]} \|d(t)\|_{C^{k+1}} \le M_{k+1}$, then*

$$\sup_{t\in[0,T]} \big\| h[z,h_0] \big\|_{H^k(\mathbb{T}^3)} \le C_k(T, \|h_0\|_{H^k}, M_{k+1}) \tag{2.6.11}$$

$$\sup_{t\in[0,T]} \big\| B[z,B_0] \big\|_{H^k(\mathbb{T}^3)} \le C_k(T, \|B_0\|_{H^k}, M_{k+1}) \tag{2.6.12}$$

(iii) *For any $\kappa > 0$, any $z, \tilde{z}$ belonging to the set*

$$\mathcal{C}_\kappa = \left\{ z \in C([0,T], C^3(\mathbb{T}^3,\mathbb{R}^6)) \Big| \sup_{t\in[0,T]} \|v(t)\|_{C^3},\ \sup_{t\in[0,T]} \|d(t)\|_{C^3} \le \kappa \right\}$$

we have

$$\sup_{t\in[0,T]} \big\| h[z,h_0] - h[\tilde{z},h_0] \big\|_{H^1(\mathbb{T}^3)} \le c(T, \|h_0\|_{H^2}, \kappa) \sup_{t\in[0,T]} \big\| z(t) - \tilde{z}(t) \big\|_{C^2} \tag{2.6.13}$$

$$\sup_{t\in[0,T]} \big\| B[z,B_0] - B[\tilde{z},B_0] \big\|_{H^1(\mathbb{T}^3)} \le c(T, \|B_0\|_{H^2}, \kappa) \sup_{t\in[0,T]} \big\| z(t) - \tilde{z}(t) \big\|_{C^2} \tag{2.6.14}$$

Proof. (i) This is a direct conclusion from the explicit expression of h in (2.6.6).

(ii) Let's prove (2.6.12). The case for (2.6.11) is simpler. For any $\alpha \in \mathbb{N}^3$, $|\alpha| \leq k$, we have

$$\begin{aligned}
&\partial_t(\partial^\alpha B_i) + \partial_j v_j \partial^\alpha B_i + v_j \partial_j(\partial^\alpha B_i) - \partial_j v_i \partial^\alpha B_j + \epsilon_{ijk}\partial^\alpha(\partial_j d_k) \\
&\qquad + \sum_{\beta<\alpha} c_{\alpha,\beta}\Big[\partial^{\alpha-\beta}(\partial_j v_j)\partial^\beta B_i + \partial^{\alpha-\beta} v_j \partial^\beta(\partial_j B_i) - \partial^{\alpha-\beta}(\partial_j v_i)\partial^\beta B_j\Big] = 0
\end{aligned}$$

Here $c_{\alpha,\beta}$ are constants depending on the choice of α, β, ϵ_{ijk} is the Levi-Civita symbol. From the above equality, we have that

$$\begin{aligned}
&\partial_t \int |\partial^\alpha B_i|^2 + \int \Big[\partial_j v_j |\partial^\alpha B_i|^2 - 2\partial_j v_i \partial^\alpha B_i \partial^\alpha B_j + 2\epsilon_{ijk}\partial^\alpha(\partial_j d_k)\partial^\alpha B_i\Big] \\
&\quad + 2\sum_{\beta<\alpha} c_{\alpha,\beta} \int \Big[\partial^{\alpha-\beta}(\partial_j v_j)\partial^\beta B_i + \partial^{\alpha-\beta} v_j \partial^\beta(\partial_j B_i) - \partial^{\alpha-\beta}(\partial_j v_i)\partial^\beta B_j\Big]\partial^\alpha B_i = 0
\end{aligned}$$

Now, we sum up all the index $|\alpha| \leq k,\ i$, then there exist a constant $c(k)$, such that

$$\partial_t \|B\|^2_{H^k} \leq c(k) M_{k+1}(\|B\|^2_{H^k} + 1) \tag{2.6.15}$$

Therefore, by Gronwall's lemma, we can get the conclusion.

(iii) We only prove the estimate for B. Let $\omega = B[z, B_0] - B[\tilde{z}, B_0]$, then we have that

$$\begin{aligned}
&\partial_t \omega_i + \partial_j v_j \omega_i + v_j \partial_j w_i - \partial_j v_i \omega_j + \epsilon_{ijk}\partial_j(d_k - \tilde{d}_k) \\
&\qquad\qquad + \partial_j(v_j - \tilde{v}_j)\tilde{B}_i + (v_j - \tilde{v}_j)\partial_j \tilde{B}_i - \partial_j(v_i - \tilde{v}_i)\tilde{B}_j = 0
\end{aligned}$$

So we have

$$\begin{aligned}
&\partial_t \int |\omega_i|^2 + \int \partial_j v_j |\omega_i|^2 - 2\int \partial_j v_i \omega_i \omega_j + 2\epsilon_{ijk} \int \partial_j(d_k - \tilde{d}_k)\omega_i \\
&\qquad + 2\int \partial_j(v_j - \tilde{v}_j)\tilde{B}_i \omega_i + 2\int (v_j - \tilde{v}_j)\partial_j \tilde{B}_i \omega_i - 2\int \partial_j(v_i - \tilde{v}_i)\tilde{B}_j \omega_i = 0
\end{aligned}$$

$$\partial_t \|w\|^2_{L^2} \leq c(\|z - \tilde{z}\|_{C^1} + \|v\|_{C^1} + 1)\|w\|^2_{L^2} + c\|z - \tilde{z}\|_{C^1}(\|\tilde{B}\|^2_{H^1} + 1)$$

Now we use the same strategy as in (ii) to compute the estimate for $D\omega$, without entering the details, we finally can get

$$\partial_t \|w\|^2_{H^1} \leq c(\|v - \tilde{v}\|_{C^2} + \|v\|_{C^2} + 1)\|w\|^2_{H^1} + c\|z - \tilde{z}\|_{C^2}(\|\tilde{B}\|^2_{H^2} + 1) \tag{2.6.16}$$

By Gronwall's lemma and(2.6.12), this implies the inequality (2.6.14).

□

2.6.2 The Faedo-Galerkin approximate scheme

Now let us consider the approximate equation (2.6.3) and (2.6.4). We will not try to find a solution. Instead, we will find an approximate solution that satisfy the weak formulation of (2.6.3) and (2.6.4) on a finite dimensional space X_N, very like the Galerkin method. For the torus $\mathbb{T}^3$, we can choose $X_N = [\text{span}\{e_i, \tilde{e}_i\}_{i=1}^N]^3$ equipped with the 2-norm ($\dim X_N = 6N$), where $e_i = \sqrt{2}\sin(2\pi\vec{k}_i \cdot x)$, $\tilde{e}_i = \sqrt{2}\cos(2\pi\vec{k}_i \cdot x)$ and $\{\vec{k}_i\}_{i=1}^{\infty}$ is a permutation of $\mathbb{Z}_+^3$, where

$$\mathbb{Z}_+^3 := \Big\{(n_1, n_2, n_3) \in \mathbb{Z}^3 \mid n_1 \geq 0\Big\} \setminus \Big(\big\{(0, n_2, n_3) \in \mathbb{Z}^3 \mid n_2 < 0\big\} \cup \big\{(0, 0, n_3) \in \mathbb{Z}^3 \mid n_3 \leq 0\big\}\Big).$$

We can see that $\{e_i, \tilde{e}_i\}_{i=1}^{\infty}$ is not only the normalized orthogonal basis of $L^2(\mathbb{T}^3)$, but also the orthogonal basis of $H^k(\mathbb{T}^3),\ k \in \mathbb{N}$. Now we set $X = \bigcup_{n=1}^{\infty} X_n$, then X is dense in $C^k(\mathbb{T}^3, \mathbb{R}^3),\ k \in \mathbb{N}$. Because X_N has finite dimensions ($\dim X_N = 6N$), then there exist a constant $c = c(N, k)$, such that

$$\|v\|_{X_N} = \|v\|_{L^2},\ \|v\|_{C^k} \leq c(N, k)\|v\|_{X_N},\ \forall v \in X_N$$

Now for every strictly positive function $\rho \in L^1(\mathbb{T}^3),\ \rho \geq \underline{\rho} > 0$, we introduce a family of operators

$$\mathcal{M}_N[\rho] :\ X_N \mapsto X_N^*,\ \big\langle \mathcal{M}_N[\rho]v, u \big\rangle = \int_{\mathbb{T}^3} \rho v \cdot u,\ \ \forall u, v \in X_N$$

Then we have the following lemma:

Lemma 2.6.3. *The family of operator $\mathcal{M}_N[\cdot]$ satisfies the following properties*
(i) *For any function $\rho \in L^1(\mathbb{T}^3),\ \mathcal{M}_N[\rho] \in \mathcal{L}(X_N, X_N^*)$.*
(ii) *For any strictly positive function $\rho \in L^1(\mathbb{T}^3),\ \rho \geq \underline{\rho} > 0$, $\mathcal{M}_N[\rho]$ is invertible, $\mathcal{M}_N^{-1}[\rho] \in \mathcal{L}(X_N^*, X_N)$, and*

$$\|\mathcal{M}_N^{-1}[\rho]\|_{\mathcal{L}(X_N^*, X_N)} \leq \underline{\rho}^{-1} \tag{2.6.17}$$

(iii) *For any $\rho_1, \rho_2 \in L^1(\mathbb{T}^3),\ \rho_1, \rho_2 \geq \underline{\rho} > 0$*

$$\|\mathcal{M}_N^{-1}[\rho_1] - \mathcal{M}_N^{-1}[\rho_2]\|_{\mathcal{L}(X_N^*, X_N)} \leq c(N, \underline{\rho})\|\rho_1 - \rho_2\|_{L^1} \tag{2.6.18}$$

Proof. (i) For any $u, v \in X_N$, we have

$$\big|\big\langle \mathcal{M}_N[\rho]v, u \big\rangle\big| \leq \|v \cdot u\|_{L^\infty} \int |\rho| \leq c(N)\|\rho\|_{L^1}\|v\|_{X_N}\|u\|_{X_N} \tag{2.6.19}$$

(ii) For any $v_1, v_2 \in X_N,\ v_1 \neq v_2$, let $u = v_1 - v_2$, then we have

$$\big\langle \mathcal{M}_N[\rho]v_1 - \mathcal{M}_N[\rho]v_2, u \big\rangle = \int \rho|v_1 - v_2|^2 > 0$$

So $\mathcal{M}_N[\rho] :\ X_N \mapsto X_N^*$ is injective. Because X_N has finite dimension, so $\mathcal{M}_N[\rho] : X_N \mapsto X_N^*$ is bijective, thus invertible. For any $\chi \in X_N$, set $v = \mathcal{M}_N^{-1}[\rho]\chi$, then

$$\|\chi\|_{X_N^*}\|v\|_{X_N} \geq \big\langle \chi, v \big\rangle = \int \rho|v|^2 \geq \underline{\rho}\|v\|_{L^2}^2 = \underline{\rho}\|v\|_{X_N}^2$$

$$\Longrightarrow \qquad \|\mathcal{M}_N^{-1}[\rho]\chi\|_{X_N} \leq \underline{\rho}^{-1}\|\chi\|_{X_N^*}$$

(iii) Because of the identity

$$\mathcal{M}_N^{-1}[\rho_1] - \mathcal{M}_N^{-1}[\rho_2] = \mathcal{M}_N^{-1}[\rho_1](\mathcal{M}_N[\rho_2] - \mathcal{M}_N[\rho_1])\mathcal{M}_N^{-1}[\rho_2]$$

Use the inequality in (2.6.17) and (2.6.19), we can get the result. $\square$

With the above properties, by standard fixed point argument, we can get the following theorem:

Theorem 2.6.4. *For any initial data* $0 < h_0 \in C^\infty(\mathbb{T}^3)$*, divergence-free vector field* $B_0 \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$*,* $P_0, D_0 \in L^2(\mathbb{T}^3, \mathbb{R}^3)$ *and any* $T > 0$*, there exists a solution* (h_n, B_n, d_n, v_n) *with* $h_n \in C^1([0,T], C^k(\mathbb{T}^3))$*,* $B_n \in C^1([0,T], C^k(\mathbb{T}^3, \mathbb{R}^3))$*,* $z_n = (d_n, v_n) \in C([0,T], X_n \times X_n)$ *to the equation* (2.6.1)-(2.6.3) *in the following sense:*
(i) (2.6.1) *and* (2.6.2) *are satisfied in the classical sense, that is* $h_n = h[z_n, h_0]$*,* $B_n = B[z_n, B_0]$.
(ii) (2.6.3) *and* (2.6.4) *are satisfied in the weak sense on* X_N*, more specifically, for any* $t \in [0,T]$ *and any* $\psi \in X_N$*,*

$$\int h_n(t) d_n(t) \cdot \psi \mathrm{d}x - \int D_0 \cdot \psi \mathrm{d}x = \int_0^t \int \mathcal{S}_\varepsilon(h_n, B_n, d_n, v_n) \cdot \psi \mathrm{d}x \mathrm{d}s \tag{2.6.20}$$

$$\int h_n(t) v_n(t) \cdot \psi \mathrm{d}x - \int P_0 \cdot \psi \mathrm{d}x = \int_0^t \int \mathcal{N}_\varepsilon(h_n, B_n, d_n, v_n) \cdot \psi \mathrm{d}x \mathrm{d}s \tag{2.6.21}$$

where

$$\mathcal{S}_\varepsilon(h, B, d, v) = -\nabla \cdot [h(d \otimes v - v \otimes d)] - (-\triangle)^l d + \varepsilon^{-1}\big[\nabla \times b - hd\big]$$

$$\mathcal{N}_\varepsilon(h, B, d, v) = (hd \cdot \nabla) d - \nabla \cdot (hv \otimes v) - (-\triangle)^l v + \varepsilon^{-1}\left[\nabla \cdot \left(\frac{B \otimes B}{h}\right) + \nabla\left(h^{-1}\right) - hv\right]$$

Proof. **Step 1: Local Existence**

Let us define a map $J_n[\cdot] : L^2(\mathbb{T}^3, \mathbb{R}^3) \mapsto X_n^*$, such that

$$\big\langle J_n[f], \psi \big\rangle = \int f \cdot \psi, \ \forall \psi \in X_n.$$

J_n can be seen as the orthogonal projection of $L^2(\mathbb{T}^3, \mathbb{R}^3)$ onto X_n^*. We have

$$\|J_n[f]\|_{X_n^*} \leq \|f\|_{L^2}$$

Let us consider the operator $K_n[\cdot] : C([0,T], X_n \times X_n) \mapsto C([0,T], X_n \times X_n)$, which maps $z = (d, v) \mapsto K_n[z] = (K_n[z]^d, K_n[z]^v)$ such that

$$K_n[z]^d(t) = \mathcal{M}_n^{-1}[h[z, h_0](t)] \left(J_n \left[D_0 + \int_0^t \mathcal{S}_\varepsilon(h[z, h_0], B[z, B_0], d, v)(s) \mathrm{d}s \right] \right)$$

$$K_n[z]^v(t) = \mathcal{M}_n^{-1}[h[z, h_0](t)] \left(J_n \left[P_0 + \int_0^t \mathcal{N}_\varepsilon(h[z, h_0], B[z, B_0], d, v)(s) \mathrm{d}s \right] \right)$$

To prove the theorem, it suffices to show that K_n has a fixed point on $C([0,T], X_n \times X_n)$. First, we should prove that K_n is well defined. It is obvious that $K_n[z](t) \in X_n \times X_n$, $\forall t \in [0,T]$. We only need to prove the continuity. In fact, we can prove that $K_n[z](t)$ is Lipschitz continuous in time on the set

$$F_{\kappa,\sigma} = \big\{(d, v) \in C([0,\sigma], X_n \times X_n) \big| \|v(t)\|_{X_n}, \|d(t)\|_{X_n} \leq \kappa, \ \forall t \in [0,\sigma]\big\}, \quad 0 < \sigma \leq T.$$

In fact, for any $s,t\in[0,\sigma]$,

$$\begin{aligned}
&\big\|K_n[z]^v(t)-K_n[z]^v(s)\big\|_{X_n}\\
\le&\left\|\Big(\mathcal{M}_n^{-1}[h[z,h_0](t)]-\mathcal{M}_n^{-1}[h[z,h_0](s)]\Big)\left(J_n\left[P_0+\int_0^s\mathcal{N}_\varepsilon(h,B,d,v)(r)\mathrm{d}r\right]\right)\right\|_{X_n}\\
&+\quad\left\|\mathcal{M}_n^{-1}[h[z,h_0](t)]\left(J_n\left[\int_s^t\mathcal{N}_\varepsilon(h[z,h_0],B[z,B_0],d,v)(r)\mathrm{d}r\right]\right)\right\|_{X_n}
\end{aligned}$$

Now, we let

$$C_0=\max\big\{(\inf h_0)^{-1},\sup h_0,\|h_0\|_{H^4},\|B_0\|_{H^4},\|P_0\|_{L^2},\|D_0\|_{L^2}\big\}\tag{2.6.22}$$

By (2.6.10),(2.6.11),(2.6.12),(2.6.17),(2.6.18), we have

$$\begin{aligned}
\big\|K_n[z]^v(t)-K_n[z]^v(s)\big\|_{X_n}\le\;&c(T,n,\varepsilon,\kappa,C_0)\left[\int\int_s^t\big|\partial_t h[z,h_0](r)\big|\mathrm{d}r\mathrm{d}x+|t-s|\right]\\
\le\;&c(T,n,\varepsilon,\kappa,C_0)|t-s|
\end{aligned}$$

By the same reason,

$$\big\|K_n[z]^d(t)-K_n[z]^d(s)\big\|_{X_n}\le c(T,n,\varepsilon,\kappa,C_0)|t-s|$$

Therefore, we have that

$$K_n[\cdot]:\ F_{\kappa,\sigma}\mapsto W^{1,\infty}([0,\sigma],X_n\times X_n)\subset\subset C([0,\sigma],X_n\times X_n).$$

Moreover, we have

$$\big\|K_n[z]^v(t)\big\|_{X_n}\le\|\mathcal{M}_n^{-1}[h_0]J_n[P_0]\|_{X_n}+c(T,n,\varepsilon,\kappa,C_0)t$$

$$\big\|K_n[z]^d(t)\big\|_{X_n}\le\|\mathcal{M}_n^{-1}[h_0]J_n[D_0]\|_{X_n}+c(T,n,\varepsilon,\kappa,C_0)t$$

Now if we choose $\kappa=\kappa_0$ big enough and $\sigma=\sigma_0$ small enough, for example

$$\kappa_0\ge 2(\inf h_0)^{-1}\big(\|P_0\|_{L^2}+\|D_0\|_{L^2}\big),\quad\sigma_0\le\frac{\kappa_0}{2c(T,n,\varepsilon,\kappa_0,C_0)}\tag{2.6.23}$$

Then we have that $K_n[\cdot]:\ F_{\kappa_0,\sigma_0}\hookrightarrow F_{\kappa_0,\sigma_0}$. By Arzelá-Ascoli theorem, $K_n[F_{\kappa_0,\sigma_0}]$ is a relatively compact subset of F_{κ_0,σ_0}. Now if we can prove that K_n is a continuous map, then by Schauder's Fixed Point Theorem, there exist a fixed point z of K_n on F_{κ_0,σ_0}, such that $z=K_n[z]$. Now, let's show that the map K_n is continuous on F_{κ_0,σ_0}. For $z,\tilde z\in F_{\kappa_0,\sigma_0}$, $\forall t\in[0,\sigma_0]$, we have,

$$\begin{aligned}
&\big\|K_n[z]^v(t)-K_n[\tilde z]^v(t)\big\|_{X_n}\\
\le\;&\left\|\big[\mathcal{M}_n^{-1}[h[z,h_0](t)]-\mathcal{M}_n^{-1}[h[\tilde z,h_0](t)]\big]\left(J_n\left[P_0+\int_0^t\mathcal{N}_\varepsilon(h,B,d,v)(\tau)\mathrm{d}\tau\right]\right)\right\|_{X_n}\\
&+\left\|\mathcal{M}_n^{-1}[h[\tilde z,h_0](t)]\left(J_n\left[\int_0^t\Big(\mathcal{N}_\varepsilon(h,B,d,v)(\tau)-\mathcal{N}_\varepsilon(\tilde h,\tilde B,\tilde d,\tilde v)(\tau)\Big)\mathrm{d}\tau\right]\right)\right\|_{X_n}\\
\le\;&c(T,n,\varepsilon,\kappa_0,C_0)\big\|h[z,h_0](t)-h[\tilde z,h_0](t)\big\|_{L^1}\\
&+c(T,n,\varepsilon,\kappa_0,C_0)\Big\|\mathcal{N}_\varepsilon(h[z,h_0],B[z,B_0],d,v)-\mathcal{N}_\varepsilon(h[\tilde z,h_0],B[\tilde z,B_0],\tilde d,\tilde v)\Big\|_{L^2_{t,x}}
\end{aligned}$$

By (2.6.13) and (2.6.14), we can finally have that

$$\begin{aligned}&\sup_{t\in[0,T]}\big\|K_n[z]^v(t)-K_n[\tilde z]^v(t)\big\|_{X_n}\\ \le\ & c(T,n,\varepsilon,\kappa_0,C_0)\Big(\big\|v-\tilde v\big\|_{C([0,\sigma_0],X_n)}+\big\|d-\tilde d\big\|_{C([0,\sigma_0],X_n)}\Big)\end{aligned}$$

By similar argument, we can show that K_n is continuous on F_{κ_0,σ_0}. Then by Schauder's Fixed Point Theorem, there is a "solution" on $[0,\sigma_0]$.

Step 2: Global Existence

From the above argument, we know that at small time interval $[0,\sigma_0]$ there exist a solution $z\in F_{\kappa_0,\sigma_0}$ s.t. $z=K_n[z]$. Now we want to apply the fixed point argument repeatedly to obtain the existence on the whole time interval $[0,T]$. Now we suppose that on $[0,T_0], T_0<T$, we have a fixed point $z=K_n[z]$, we use $\tilde h_0=h[z,h_0](T_0)$, $\tilde B_0=B[z,B_0](T_0)$, $\tilde D_0=\tilde h_0 d(T_0)$, $\tilde P_0=\tilde h_0 v(T_0)$ as our new initial data, and use the local existence result above to extend the existence interval. This argument can be applied as long as we can prove that there is still a fixed point in F_{κ_0,τ_0} with our new initial data $(\tilde h_0,\tilde B_0,\tilde D_0,\tilde P_0)$ while the constants κ_0,σ_0 chosen in previous part do not change. (They only depend on the initial data $h_0,B_0,D_0,P_0,\varepsilon,T$ and n). Now, we only need to prove that the constant $\tilde C_0$ for the new initial data defined in (2.6.22) has a uniform bound.

Suppose (h_n,B_n,d_n,v_n) is the solution on $[0,T_0]$. Because $d_n,v_n\in C([0,T_0],X_n)$ is Lipschitz continuous, so it is differentiable almost everywhere. We take the derivative on both sides of (2.6.20) and (2.6.21), then we have that, for any $\varphi,\psi\in X_n$, any $g\in C^1(\mathbb{T}^3)$, $\phi\in C^1(\mathbb{T}^3,\mathbb{R}^3)$,

$$\int\partial_t h_n(t)g-\int h_n(t)v_n(t)\cdot\nabla g=0$$

$$\int\partial_t B_n(t)\cdot\phi-\int(B_n\otimes v_n-v_n\otimes B_n):\nabla\phi+\int d_n\cdot(\nabla\times\phi)=0$$

$$\begin{aligned}\int\partial_t\big(h_n(t)d_n(t)\big)\cdot\psi-\int h_n(d_n\otimes v_n-v_n\otimes d_n):\nabla\psi+\int\nabla^l d_n:\nabla^l\psi\\ +\varepsilon^{-1}\int\big[-b_n\cdot(\nabla\times\psi)+h_nd_n\cdot\psi\big]=0\end{aligned}$$

$$\begin{aligned}\int\partial_t\big(h_n(t)v_n(t)\big)\cdot\varphi-\int h_nv_n\otimes v_n:\nabla\varphi-\int\big[(h_nd_n\cdot\nabla)d_n\big]\cdot\varphi\\ +\int\nabla^l v_n:\nabla^l\varphi+\varepsilon^{-1}\int\big[h_n^{-1}(B_n\otimes B_n+I_3):\nabla\varphi+h_nv_n\cdot\varphi\big]=0\end{aligned}$$

Now, let's choose $\phi=b_n(t)=B_n(t)/h_n(t)$, $\psi=\varepsilon d_n(t)$, $\varphi=\varepsilon v_n(t)$ and $g=-\frac12\big(|h_n(t)|^{-2}+|b_n(t)|^2+\varepsilon|v_n(t)|^2+\varepsilon|d_n(t)|^2\big)$, then add the these equations together, we have the following equality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{h_n(t)}{2}\Big(|h_n(t)|^{-2}+|b_n(t)|^2+\varepsilon|v_n(t)|^2+\varepsilon|d_n(t)|^2\Big)\\ +\int h_n(t)(|v_n(t)|^2+|d_n(t)|^2)+\varepsilon\int\big[|\nabla^l v_n(t)|^2+|\nabla^l d_n(t)|^2\big]=0\quad(2.6.24)\end{aligned}$$

We denote $\Lambda_n(t) = \displaystyle\int \frac{h_n(t)}{2}\Big(|h_n(t)|^{-2} + |b_n(t)|^2 + \varepsilon|v_n(t)|^2 + \varepsilon|d_n(t)|^2\Big)$, then from the above equality, we have

$$\sup_{t\in[0,T_0]} \Lambda_n(t),\ \|h_n^{\frac{1}{2}} v_n\|^2_{L^2_{t,x}},\ \|h_n^{\frac{1}{2}} d_n\|^2_{L^2_{t,x}},\ \varepsilon\|\nabla^l v_n\|^2_{L^2_{t,x}},\ \varepsilon\|\nabla^l d_n\|^2_{L^2_{t,x}} \le \Lambda_n(0) \quad (2.6.25)$$

By Cauchy-Schwartz inequality, we can easily get that

$$\|d_n\|_{L^2(L^1)}, \|v_n\|_{L^2(L^1)} \le \sqrt{2}\Lambda_n(0), \quad \|\nabla^l v_n\|_{L^2(L^1)} \le \varepsilon^{-1}\Lambda_n(0)$$

So we get that $\|v_n\|_{L^2(W^{l,1})} \le c(\varepsilon, \Lambda_n(0))$. With $l > 4$, by Sobolev embedding, we have that $\|v_n\|_{L^2(W^{1,\infty})} \le c(\varepsilon, \Lambda_n(0))$. So by the result in Lemma 2.6.2.(i), we have that, there exist a constant $c_0 = c_0(\varepsilon, T, \Lambda_n(0)) > 0$ s.t. for all $t \in [0, T_0],\ x \in \mathbb{T}^3$,

$$0 < c_0 \le h_n(t,x) \le c_0^{-1}$$

So by (2.6.25), we have

$$\sup_{t\in[0,T_0]} \|v_n\|^2_{L^2},\ \sup_{t\in[0,T_0]} \|d_n\|^2_{L^2},\ \sup_{t\in[0,T_0]} \|P_n\|^2_{L^2},\ \sup_{t\in[0,T_0]} \|D_n\|^2_{L^2} \le 2(\varepsilon c_0)^{-1}\Lambda_n(0)$$

Then by Lemma 2.6.2.(ii), we have that

$$\sup_{t\in[0,T_0]} \|h\|_{H^4},\ \sup_{t\in[0,T_0]} \|B\|_{H^4} \le c(T, n, \varepsilon, \Lambda_n(0), \|h_0\|_{H^4}, \|B_0\|_{H^4})$$

So we know that $\tilde{C}_0$ have a uniform bound that depends only on initial data. So we can see from the choice of κ_0, σ_0 in (2.6.23) that by slightly modify the choice of κ_0, σ_0, the fixed point method can be repeatedly applied on the same space F_{κ_0,σ_0}, so we get the global existence on $[0, T]$.

□

2.7 Existence of the dissipative solution

2.7.1 Smooth approximation of initial data

We suppose that our initial data h_0 is a nonnegative Borel measure in $C(\mathbb{T}^3, \mathbb{R})'$, $B_0 \in C(\mathbb{T}^3, \mathbb{R}^3)'$, satisfying $\nabla \cdot B_0 = 0$ in the sense of distributions. Moreover, we suppose $0 < \Lambda(h_0, U_0) < \infty$, where $U_0 = \big(\mathcal{L}, B_0\big)$. Now we will find a family of smooth functions to approach our initial data.

Let us define a positive Schwartz function $\widetilde{\rho}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|x|^2}{2}} \in C^\infty(\mathbb{R}^3, \mathbb{R})$. We have that $\int_{\mathbb{R}^3} \widetilde{\rho}(x)\mathrm{d}x = 1$. For any $0 < \varepsilon < 1$, we define a function ρ_ε on $\mathbb{T}^3$ by

$$\rho_\varepsilon(x) = \sum_{\vec{k}\in\mathbb{Z}^3} \widetilde{\rho}_\varepsilon(x + \vec{k}) = \sum_{\vec{k}\in\mathbb{Z}^3} \frac{1}{\varepsilon^3}\widetilde{\rho}\left(\frac{x+\vec{k}}{\varepsilon}\right). \quad (2.7.1)$$

We can easily check that $\rho_\varepsilon(x)$ is also a smooth positive function on $\mathbb{T}^3$, and we have $\int_{\mathbb{T}^3} \rho_\varepsilon(x)\mathrm{d}x = 1$. Now, for $0 < \varepsilon < 1$, we define

$$h_0^\varepsilon = h_0 * \rho_\varepsilon = \int_{\mathbb{T}^3} \rho_\varepsilon(x-y)\mathrm{d}\, h_0(y), \quad B_0^\varepsilon = B_0 * \rho_\varepsilon \quad (2.7.2)$$

Because $0 < \Lambda(h_0, U_0) < \infty$, then $h_0 \geq 0,\ h_0 \neq 0$. So we have that $h_0^\varepsilon > 0$ for any $0 < \varepsilon < 1$. Besides, it's easily to verify that $B_0^\varepsilon, h_0^\varepsilon$ are smooth functions on $\mathbb{T}^3$ and converge to B_0, h_0 in the weak-$*$ topology of $C(\mathbb{T}^3)'$. Moreover, for any smooth function ϕ on $\mathbb{T}^3$, we have

$$\begin{aligned}\int_{\mathbb{T}^3} \phi \nabla \cdot B_0^\varepsilon &= -\int_{\mathbb{T}^3} \nabla\phi(x) \cdot \left(\int_{\mathbb{T}^3} \rho_\varepsilon(x-y)\mathrm{d}B_0(y)\right) \mathrm{d}x \\ &= \int_{\mathbb{T}^3} \nabla_y \left(\int_{\mathbb{T}^3} \phi(x)\rho_\varepsilon(x-y)\mathrm{d}x\right) \mathrm{d}B_0(y) = 0\end{aligned} \tag{2.7.3}$$

So we know that

$$\nabla \cdot B_0^\varepsilon = 0. \tag{2.7.4}$$

Besides, we can get the following result.

Proposition 2.7.1. *For all* $0 < \varepsilon < 1$, *we have that*

$$\Lambda(h_0^\varepsilon, U_0^\varepsilon) = \int \frac{|B_0^\varepsilon|^2 + 1}{2h_0^\varepsilon} \leq \Lambda(h_0, U_0) \tag{2.7.5}$$

Moreover, we have that

$$\Lambda(h_0^\varepsilon, U_0^\varepsilon) \to \Lambda(h_0, U_0) \quad as \quad \varepsilon \to 0.$$

Proof. We know that

$$\begin{aligned}&\Lambda(h_0^\varepsilon, U_0^\varepsilon) \\ =& \sup_{\substack{a\in C(\mathbb{T}^3,\mathbb{R}), A\in C(\mathbb{T}^3,\mathbb{R}^4) \\ a+\frac{1}{2}|A|^2\leq 0}} \int a(x)\left(\int \rho_\varepsilon(x-y)\mathrm{d}h_0(y)\right)\mathrm{d}x + \int A(x)\left(\int \rho_\varepsilon(x-y)\mathrm{d}U_0(y)\right)\mathrm{d}x \\ =& \sup_{\substack{a\in C(\mathbb{T}^3,\mathbb{R}), A\in C(\mathbb{T}^3,\mathbb{R}^4) \\ a+\frac{1}{2}|A|^2\leq 0}} \int \left(\int a(x)\rho_\varepsilon(x-y)\mathrm{d}x\right)\mathrm{d}h_0(y) + \int \left(\int A(x)\rho_\varepsilon(x-y)\mathrm{d}x\right)\mathrm{d}U_0(y)\end{aligned}$$

By Cauchy-Schwarz inequality, we can easily know that

$$\begin{aligned}\left|\int A(x)\rho_\varepsilon(x-y)\mathrm{d}x\right|^2 &\leq \left(\int |A(x)|^2\rho_\varepsilon(x-y)\mathrm{d}x\right)\left(\int \rho_\varepsilon(x-y)\mathrm{d}x\right) \\ &\leq -2\int a(x)\rho_\varepsilon(x-y)\mathrm{d}x\end{aligned}$$

So we get that

$$\Lambda(h_0^\varepsilon, U_0^\varepsilon) \leq \sup_{\substack{\tilde{a}\in C(\mathbb{T}^3,\mathbb{R}), \tilde{A}\in C(\mathbb{T}^3,\mathbb{R}^4) \\ \tilde{a}+\frac{1}{2}|\tilde{A}|^2\leq 0}} \left\langle h_0, \tilde{a}\right\rangle + \left\langle U_0, \tilde{A}\right\rangle = \Lambda(h_0, U_0)$$

Now because for each fixed continuous function a, A, $\left\langle h_0^\varepsilon, a\right\rangle \to \left\langle h_0, a\right\rangle$, $\left\langle U_0^\varepsilon, A\right\rangle \to \left\langle U_0, B\right\rangle$ as $\varepsilon \to 0$, then we have that

$$\liminf_{\varepsilon\to 0} \Lambda(h_0^\varepsilon, U_0^\varepsilon) \geq \Lambda(h_0, U_0)$$

Combining the above two results, we can get the convergence $\Lambda(h_0^\varepsilon, U_0^\varepsilon) \to \Lambda(h_0, U_0)$ as $\varepsilon \to 0$.

□

2.7.2 Existence of converging sequence

Now let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence such that $0 < \varepsilon_k < 1$, $\lim_{k\to\infty} \varepsilon_k = 0$. By Theorem 2.6.4, for every $n \in \mathbb{N}^*$, there exists a solution $(h_n^{\varepsilon_k}, B_n^{\varepsilon_k}, d_n^{\varepsilon_k}, v_n^{\varepsilon_k})$ on $[0,T]$ that satisfies (2.6.1),(2.6.2),(2.6.20),(2.6.21) with $\varepsilon = \varepsilon_k$ and the initial data $(h_0^{\varepsilon_k},\ B_0^{\varepsilon_k},\ 0,\ 0)$. For simplicity, we denote

$$B_n^{\varepsilon_k} = h_n^{\varepsilon_k} b_n^{\varepsilon_k}, \quad D_n^{\varepsilon_k} = h_n^{\varepsilon_k} d_n^{\varepsilon_k}, \quad P_n^{\varepsilon_k} = h_n^{\varepsilon_k} v_n^{\varepsilon_k},$$

$$\Lambda_n^{\varepsilon_k}(t) = \int \frac{h_n^{\varepsilon_k}}{2}\Big(\big(h_n^{\varepsilon_k}\big)^{-2} + \big|b_n^{\varepsilon_k}\big|^2 + \varepsilon_k \big|d_n^{\varepsilon_k}\big|^2 + \varepsilon_k \big|v_n^{\varepsilon_k}\big|^2\Big), \quad \Lambda_0^{\varepsilon_k} = \int \frac{\big|B_0^{\varepsilon_k}\big|^2 + 1}{2h_0^{\varepsilon_k}}$$

Lemma 2.7.2. *Suppose $(h_n^{\varepsilon_k}, B_n^{\varepsilon_k}, d_n^{\varepsilon_k}, v_n^{\varepsilon_k})$ is the solution in Theorem 2.6.4 with initial data $(h_0^{\varepsilon_k}, B_0^{\varepsilon_k}, 0, 0)$. Then there exist a constant C_0 that depends only on h_0, B_0, such that for all n and ε_k,*

$$\big\|h_n^{\varepsilon_k}\big\|_{L_t^\infty(L_x^1)}, \big\|h_n^{\varepsilon_k} b_n^{\varepsilon_k}\big\|_{L_t^\infty(L_x^1)}, \big\|h_n^{\varepsilon_k} d_n^{\varepsilon_k}\big\|_{L_t^2(L_x^1)}, \big\|h_n^{\varepsilon_k} v_n^{\varepsilon_k}\big\|_{L_t^2(L_x^1)} \le C_0 \tag{2.7.6}$$

$$\sqrt{\varepsilon_k}\big\|\nabla^l d_n^{\varepsilon_k}\big\|_{L^2_{t,x}}, \sqrt{\varepsilon_k}\big\|\nabla^l v_n^{\varepsilon_k}\big\|_{L^2_{t,x}} \le C_0 \tag{2.7.7}$$

Proof. By Lemma 2.6.2, we know that $h_n^{\varepsilon_k}$ is always positive. Since $h_n^{\varepsilon_k}$ solves (2.6.1), we have

$$\int h_n^{\varepsilon_k}(t) = \int h_0^{\varepsilon_k} = \int \Big(\int \rho_{\varepsilon_k}(x-y)\mathrm{d}h_0(y)\Big)\,\mathrm{d}x = \int h_0$$

By (2.6.25) and (2.7.5), we know that

$$\sup_{t\in[0,T]} \int \frac{1+|B_n^{\varepsilon_k}(t)|^2}{2h_n^{\varepsilon_k}(t)} + \int_0^T \int h_n^{\varepsilon_k}(t)(|v_n^{\varepsilon_k}(t)|^2 + |d_n^{\varepsilon_k}(t)|^2)$$
$$+ \varepsilon_k \int_0^T \int \big[|\nabla^l v_n^{\varepsilon_k}(t)|^2 + |\nabla^l d_n^{\varepsilon_k}(t)|^2\big] \le \int \frac{1+|B_0^{\varepsilon_k}|^2}{2h_0^{\varepsilon_k}} \le \Lambda(h_0, U_0) \tag{2.7.8}$$

By Cauchy-Schwartz inequality,

$$\Big(\int \big|B_n^{\varepsilon_k}(t)\big|\Big)^2 \le \Big(\int \frac{|B_n^{\varepsilon_k}(t)|^2}{h_n^{\varepsilon_k}(t)}\Big)\Big(\int h_n^{\varepsilon_k}(t)\Big) \le 4\Lambda(h_0, U_0)\int h_0$$

$$\int_0^T \Big(\int \big|h_n^{\varepsilon_k}(t) v_n^{\varepsilon_k}(t)\big|\Big)^2 \le \int_0^T \int h_n^{\varepsilon_k}(t)\big|v_n^{\varepsilon_k}(t)\big|^2 \int h_n^{\varepsilon_k}(t) \le \Lambda(h_0, U_0)\int h_0$$

We can get the conclusion easily from the above estimates.

□

From the above lemma, we know that $(h_n^{\varepsilon_k}, B_n^{\varepsilon_k}, D_n^{\varepsilon_k}, P_n^{\varepsilon_k})$ are bounded in some suitable spaces, so we can extract a converging subsequence.

Lemma 2.7.3. *There exists a subsequence $\{n_i\}_{i=1}^{\infty} \subseteq \mathbb{N}^*$, $h \in C([0,T], C(\mathbb{T}^3,\mathbb{R})'_{w^*})$, $B \in C([0,T], C(\mathbb{T}^3,\mathbb{R}^3)'_{w^*})$, $P, D \in C([0,T]\times\mathbb{T}^3,\mathbb{R}^3)'$, such that*

$$h_{n_i}^{\varepsilon_{n_i}} \to h \textit{ in } C([0,T], C(\mathbb{T}^3,\mathbb{R})'_{w^*}), \quad B_{n_i}^{\varepsilon_{n_i}} \to B \textit{ in } C([0,T], C(\mathbb{T}^3,\mathbb{R}^3)'_{w^*})$$

$$h_{n_i}^{\varepsilon_{n_i}} \xrightarrow{w^*} h \textit{ in } C([0,T]\times\mathbb{T}^3,\mathbb{R})'$$

$$B_{n_i}^{\varepsilon_{n_i}} \xrightarrow{w^*} B,\ P_{n_i}^{\varepsilon_{n_i}} \xrightarrow{w^*} P,\ D_{n_i}^{\varepsilon_{n_i}} \xrightarrow{w^*} D \textit{ in } C([0,T]\times\mathbb{T}^3,\mathbb{R}^3)'$$

Moreover, we have that (h,B) is bounded in $C^{0,\frac{1}{2}}([0,T], C(\mathbb{T}^3,\mathbb{R}^4)'_{w^*})$ by some constant that depends only on T and (h_0,B_0).

Proof. For any smooth function $f \in C^\infty(\mathbb{T}^3, \mathbb{R})$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{T}^3} \big(h_n^{\varepsilon_k}(t,x) - h_n^{\varepsilon_k}(s,x)\big) f(x)\mathrm{d}x\right| \\
=& \left|\int_s^t \int_{\mathbb{T}^3} P_n^{\varepsilon_k}(\sigma,x)\cdot \nabla f(x)\mathrm{d}\sigma\mathrm{d}x\right| \\
\leq& \left(\int_s^t \int_{\mathbb{T}^3} \frac{|P_n^{\varepsilon_k}(\sigma,x)|^2}{h_n^{\varepsilon_k}(\sigma,x)}\mathrm{d}\sigma\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_s^t \int_{\mathbb{T}^3} |\nabla f(x)|^2 h_n^{\varepsilon_k}(\sigma,x)\mathrm{d}\sigma\mathrm{d}x\right)^{\frac{1}{2}} \\
\leq& \|\nabla f\|_\infty \Big(\Lambda(h_0,U_0)\langle h_0, 1\rangle\Big)^{\frac{1}{2}} |t-s|^{\frac{1}{2}}
\end{aligned} \tag{2.7.9}$$

Besides, for any smooth function $\phi \in C(\mathbb{T}^3, \mathbb{R}^3)$

$$\begin{aligned}
&\left|\int_{\mathbb{T}^3} \big(B_n^{\varepsilon_k}(t,x) - B_n^{\varepsilon_k}(s,x)\big)\cdot \phi(x)\mathrm{d}x\right| \\
=& \left|\int_s^t \int_{\mathbb{T}^3} \big(B_n^{\varepsilon_k}\times v_n^{\varepsilon_k} + d_n^{\varepsilon_k}\big)\cdot(\nabla\times\phi)\right| \\
\leq& \sqrt{2}\|\nabla\times\phi\|_\infty \left(\int_s^t \int_{\mathbb{T}^3} \frac{|P_n^{\varepsilon_k}|^2 + |D_n^{\varepsilon_k}|^2}{h_n^{\varepsilon_k}}\right)^{\frac{1}{2}} \left(\int_s^t \int_{\mathbb{T}^3} \frac{|B_n^{\varepsilon_k}|^2+1}{h_n^{\varepsilon_k}}\right)^{\frac{1}{2}} \\
\leq& 2\|\nabla\times\phi\|_\infty \Lambda(h_0,U_0)|t-s|^{\frac{1}{2}}
\end{aligned} \tag{2.7.10}$$

From (2.7.6), we can easily know that, for all ε_k, n and t ,the total variation of $(h_n^{\varepsilon_k}, B_n^{\varepsilon_k})$ is bounded. By Banach-Alaoglu theorem, the closed ball $B_R(0)$ in $C(\mathbb{T}^3,\mathbb{R}^4)'$ is compact with respect to the weak-$*$ topology. From (2.7.9),(2.7.10), we know that $\{(h_n^{\varepsilon_n}, B_n^{\varepsilon_n})\}_{n=1}^\infty$ is uniformly bounded in $C^{0,\frac{1}{2}}([0,T], C(\mathbb{T}^3,\mathbb{R}^4)'_{w^*})$ by some constant that depends only on T and (h_0, B_0). So by Arzelà-Ascoli's theorem, we can extract a subsequence $\{(h_{n_i}^{\varepsilon_{n_i}}, B_{n_i}^{\varepsilon_{n_i}})\}_{i=1}^\infty$ that converge to some measures denoted by (h,B) in $C([0,T], C(\mathbb{T}^3,\mathbb{R}^4)'_{w^*})$, and (h,B) is bounded in $C^{0,\frac{1}{2}}([0,T], C(\mathbb{T}^3,\mathbb{R}^4)'_{w^*})$ by the same constant. Besides, from (2.7.6), we know that the total variations of $P_n^{\varepsilon_n}, D_n^{\varepsilon_n}$ is uniformly bounded in $C([0,T]\times\mathbb{T}^3,\mathbb{R}^3)'$, so we can extract a sub sequence that weakly converge to $P, D \in C([0,T]\times\mathbb{T}^3,\mathbb{R}^3)'$. So we get the conclusion.

□

2.7.3 The limit is a dissipative solution

By Lemma 2.7.3, we can extract a subsequence $(h_{n_i}^{\varepsilon_{n_i}}, B_{n_i}^{\varepsilon_{n_i}}, D_{n_i}^{\varepsilon_{n_i}}, P_{n_i}^{\varepsilon_{n_i}})$ that converge strongly to a function (h,B) in $C([0,T], C(\mathbb{T}^3,\mathbb{R}^4)'_{w^*})$ and weakly-$*$ to D, P in $C([0,T]\times\mathbb{T}^3,\mathbb{R}^3)'$. But it is not clear if these functions are dissipative solutions or not. In the following part, we will prove that (h,B,D,P) satisfies all the requirements in Definition 3.1.1, thus it is indeed a dissipative solution of (DMHD) with the initial data (h_0, B_0).

Firstly, we know that $(h_{n_i}^{\varepsilon_{n_i}}, B_{n_i}^{\varepsilon_{n_i}})|_{t=0} = (h_0^{\varepsilon_{k_i}}, B_0^{\varepsilon_{n_i}})$ converge weakly-$*$ to (h_0, B_0), so $(h,B)|_{t=0} = (h_0, B_0)$.

Secondly, for any $u \in C^1([0,T]\times\mathbb{T}^3,\mathbb{R})$ and $t\in[0,T]$, the limit h, P satisfy (2.3.22). To prove this, for any $\delta > 0$, let's find a smooth non increasing function on $[0,T]$ denoted by Θ_δ, such that $\Theta_\delta(s) = 1,\ s\in[0,t]$, $\Theta_\delta(s) = 0,\ s\in[t+\delta, T]$, $0 \leq \Theta_\delta(s) \leq 1,\ s \in [t, t+\delta]$. Because $(h_{n_i}^{\varepsilon_{n_i}}, P_{n_i}^{\varepsilon_{n_i}})$ satisfies (2.3.1), then we have

$$\begin{aligned}
&\int_0^T \int \Theta_\delta(s)\Big[\partial_s u(s,x) h_{n_i}^{\varepsilon_{n_i}}(s,x) + \nabla u(s,x)\cdot P_{n_i}^{\varepsilon_{n_i}}(s,x)\Big]\mathrm{d}x\mathrm{d}s \\
=& -\int_t^{t+\delta}\int \Theta_\delta'(s) u(s,x) h_{n_i}^{\varepsilon_{n_i}}(s,x)\mathrm{d}x\mathrm{d}s - \int u(0,x) h_{n_i}^{\varepsilon_{n_i}}(0,x)\mathrm{d}x
\end{aligned}$$

Because $\forall s$, $h_{n_i}^{\varepsilon_{n_i}}(s) \xrightarrow{w^*} h(s)$, $P_{n_i}^{\varepsilon_{n_i}} \xrightarrow{w^*} P$ as $i \to \infty$ and the total variation of $h_{n_i}^{\varepsilon_{n_i}}$, $P_{n_i}^{\varepsilon_{n_i}}$ is uniformly bounded, so by the weak-$*$ convergence and Lebesgue's dominated convergence theorem, let $i \to \infty$, we have

$$\int_0^T \int \Theta_\delta\Big(h\partial_s u + P \cdot \nabla u\Big) = -\int_t^{t+\delta} \int \Theta_\delta'(s)u(s)h(s) - \int u(0)h(0)$$

Now because $h \in C([0,T], C(\mathbb{T}^3, \mathbb{R})'_{w^*})$, so $\big\langle h(s), u(s)\big\rangle$ is a continuous function on s, then we let $\delta \to 0$, we have

$$-\int_t^{t+\delta} \Theta_\delta'(s)\big\langle h(s), u(s)\big\rangle \mathrm{d}s \longrightarrow \in u(t)h(t)$$

Because $\Theta_\delta \to \mathbb{1}_{[0,t]}$ for every $s \in [0,T]$, by Lebesgue's dominated convergence theorem, pass the limit $\delta \to 0$ on the left hand side, we finally get (2.3.22).

Moreover, because $\nabla \cdot B_{n_i}^{\varepsilon_{n_i}}(t) = 0$. So for any $\phi \in C^1(\mathbb{T}^3, \mathbb{R})$, we have $\big\langle B_{n_i}^{\varepsilon_{n_i}}(t), \nabla\phi\big\rangle = 0$. By taking the limit, we get $\big\langle B(t), \nabla\phi\big\rangle = 0$. So (2.3.23) is also satisfied.

At last, we will prove that (h, B, D, P) satisfies (3.1.8). We first suppose that for fixed N, $0 < h^* \in C^1([0,T] \times \mathbb{T}^3, \mathbb{R})$, $b^* \in C^1([0,T] \times \mathbb{T}^3, \mathbb{R}^3)$, $v^*, d^* \in C^1([0,T], X_N)$ and r is a big number such that $Q_r(w^*)$ is positive definite for all t, x. Here Q is defined in (2.3.9). Now, let us denote

$$U = \big(\mathcal{L}, B\big),\ U_i = \big(\mathcal{L}, B_{n_i}^{\varepsilon_{n_i}}\big) \in C([0,T], C(\mathbb{T}^3, \mathbb{R}^4)')$$

$$W = \big(U, D, P\big),\ W_i = \big(U_i, D_{n_i}^{\varepsilon_{n_i}}, P_{n_i}^{\varepsilon_{n_i}}\big) \in L^2([0,T], C(\mathbb{T}^3, \mathbb{R}^{10})')$$

$$\widetilde{U} = U - Vh = \left(\mathcal{L} - hh^{*-1}, B - hb^*\right),\ \widetilde{U}_i = U_i - Vh_{n_i}^{\varepsilon_{n_i}}$$

$$\widetilde{W} = W - Fh\mathrm{d}s = \left(\widetilde{U}, D - hd^*, P - hv^*\right),\ \widetilde{W}_i = W_i - Fh_{n_i}^{\varepsilon_{n_i}}$$

$$V = \big(h^{*-1}, b^*\big),\ F = \big(h^{*-1}, b^*, d^*, v^*\big)$$

$$E_i(t) = \int \frac{1}{2h_{n_i}^{\varepsilon_{n_i}}}\Big\{(B_{n_i}^{\varepsilon_{n_i}} - h_{n_i}^{\varepsilon_{n_i}}b^*)^2 + \left(1 - \frac{h_{n_i}^{\varepsilon_{n_i}}}{h^*}\right)^2 + \varepsilon_{n_i}\left[(D_{n_i}^{\varepsilon_{n_i}} - h_{n_i}^{\varepsilon_{n_i}}d^*)^2 + (P_{n_i}^{\varepsilon_{n_i}} - h_{n_i}^{\varepsilon_{n_i}}v^*)^2\right]\Big\}$$

Now, because $(h_{n_i}^{\varepsilon_{n_i}}, B_{n_i}^{\varepsilon_{n_i}}, v_{n_i}^{\varepsilon_{n_i}}, d_{n_i}^{\varepsilon_{n_i}})$ is a some kind of "solution" to (2.6.1)-(2.6.3), we have that for any $\varphi, \psi \in X_{n_i}$, any $g \in C^1(\mathbb{T}^3)$, $\phi \in C^1(\mathbb{T}^3, \mathbb{R}^3)$,

$$\int \partial_t h_{n_i}^{\varepsilon_{n_i}}(t)g - \int h_{n_i}^{\varepsilon_{n_i}}(t)v_{n_i}^{\varepsilon_{n_i}}(t) \cdot \nabla g = 0$$

$$\int \partial_t B_{n_i}^{\varepsilon_{n_i}}(t) \cdot \phi - \int (B_{n_i}^{\varepsilon_{n_i}} \otimes v_{n_i}^{\varepsilon_{n_i}} - v_{n_i}^{\varepsilon_{n_i}} \otimes B_{n_i}^{\varepsilon_{n_i}}) : \nabla\phi + \int d_{n_i}^{\varepsilon_{n_i}} \cdot (\nabla \times \phi) = 0$$

$$\begin{aligned}\int \partial_t\big(h_{n_i}^{\varepsilon_{n_i}}(t)v_{n_i}^{\varepsilon_{n_i}}(t)\big) \cdot \varphi - \int h_{n_i}^{\varepsilon_{n_i}} v_{n_i}^{\varepsilon_{n_i}} \otimes v_{n_i}^{\varepsilon_{n_i}} : \nabla\varphi - \int \left[(h_{n_i}^{\varepsilon_{n_i}} d_{n_i}^{\varepsilon_{n_i}} \cdot \nabla)d_{n_i}^{\varepsilon_{n_i}}\right] \cdot \varphi \\ + \int \nabla^l v_{n_i}^{\varepsilon_{n_i}} : \nabla^l\varphi + \varepsilon_{n_i}^{-1}\int \Big[h_{n_i}^{\varepsilon_{n_i}-1}(B_{n_i}^{\varepsilon_{n_i}} \otimes B_{n_i}^{\varepsilon_{n_i}} + I_3) : \nabla\varphi + h_{n_i}^{\varepsilon_{n_i}} v_{n_i}^{\varepsilon_{n_i}} \cdot \varphi\Big] = 0\end{aligned}$$

$$\begin{aligned}\int \partial_t\big(h_{n_i}^{\varepsilon_{n_i}}(t)d_{n_i}^{\varepsilon_{n_i}}(t)\big) \cdot \psi - \int h_{n_i}^{\varepsilon_{n_i}}(d_{n_i}^{\varepsilon_{n_i}} \otimes v_{n_i}^{\varepsilon_{n_i}} - v_{n_i}^{\varepsilon_{n_i}} \otimes d_{n_i}^{\varepsilon_{n_i}}) : \nabla\psi \\ + \int \nabla^l d_{n_i}^{\varepsilon_{n_i}} : \nabla^l\psi + \varepsilon_{n_i}^{-1}\int \left[-b_{n_i}^{\varepsilon_{n_i}} \cdot (\nabla \times \psi) + h_{n_i}^{\varepsilon_{n_i}} d_{n_i}^{\varepsilon_{n_i}} \cdot \psi\right] = 0\end{aligned}$$

For $n_i \geq N$, we can shoose $\phi = b_{n_i}^{\varepsilon_{n_i}} - b^*$, $\psi = \varepsilon_{n_i}\big(d_{n_i}^{\varepsilon_{n_i}} - d^*\big)$, $\varphi = \varepsilon_{n_i}\big(v_{n_i}^{\varepsilon_{n_i}} - v^*\big)$, and

$$g = \frac{1}{2}\Big[|h^*|^{-2} + |b^*|^2 - |h_{n_i}^{\varepsilon_{n_i}}|^{-2} - |b_{n_i}^{\varepsilon_{n_i}}|^2 + \varepsilon_{n_i}\big(|v^*|^2 + |d^*|^2 - |v_{n_i}^{\varepsilon_{n_i}}|^2 - |d_{n_i}^{\varepsilon_{n_i}}|^2\big)\Big]$$

With the specific chosen test function, we can get that (after a long progress of computation, we skip the tedious part here)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_i(t) + \int \frac{\widetilde{W}_i^{\mathrm{T}} Q(w^*)\widetilde{W}_i}{2h_{n_i}^{\varepsilon_{n_i}}} + \int \widetilde{W}_i \cdot \mathrm{L}(w^*) - \varepsilon_{n_i}\widetilde{R}_i(t)& \\ = -\varepsilon_{n_i}\int \Big(\big|\nabla^l d_{n_i}^{\varepsilon_{n_i}}\big|^2 + \big|\nabla^l v_{n_i}^{\varepsilon_{n_i}}\big|^2\Big) \leq 0 &\quad (2.7.11)\end{aligned}$$

Here

$$\begin{aligned}\widetilde{R}_i(t) =& \int \frac{D_{n_i}^{\varepsilon_{n_i}} \otimes D_{n_i}^{\varepsilon_{n_i}} - P_{n_i}^{\varepsilon_{n_i}} \otimes P_{n_i}^{\varepsilon_{n_i}}}{2h_{n_i}^{\varepsilon_{n_i}}} : \big(\nabla v^* + \nabla v^{*\mathrm{T}}\big) + \int \frac{P_{n_i}^{\varepsilon_{n_i}} \otimes D_{n_i}^{\varepsilon_{n_i}}}{h_{n_i}^{\varepsilon_{n_i}}} : \big(\nabla d^* - \nabla d^{*\mathrm{T}}\big) \\ &+ \int \left(\nabla\left(\frac{|v^*|^2 + |d^*|^2}{2}\right) - \partial_t v^*\right) \cdot P_{n_i}^{\varepsilon_{n_i}} - \int \partial_t d^* \cdot D_{n_i}^{\varepsilon_{n_i}} + \int h_{n_i}^{\varepsilon_{n_i}} \partial_t \left(\frac{|v^*|^2 + |d^*|^2}{2}\right) \\ &+ \int \big[\nabla^l v_{n_i}^{\varepsilon_{n_i}} : \nabla^l v^* + \nabla^l d_{n_i}^{\varepsilon_{n_i}} : \nabla^l d^*\big] - \int D_{n_i}^{\varepsilon_{n_i}} \cdot \nabla\big(d_{n_i}^{\varepsilon_{n_i}} \cdot v^*\big)\end{aligned}$$

Then, for any r such that $Q_r(w^*) > 0$ for all t, x, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - r\right) E_i(t) + \int \frac{\widetilde{W}_i^{\mathrm{T}} Q_r(w^*)\widetilde{W}_i}{2h_{n_i}^{\varepsilon_{n_i}}} + \int \widetilde{W}_i \cdot \mathrm{L}(w^*) - \varepsilon_{n_i}\widetilde{R}_{i,r}(t) \leq 0$$

$$\widetilde{R}_{i,r} = \widetilde{R}_i - r\int \frac{h_{n_i}^{\varepsilon_{n_i}}}{2}\big[(d_{n_i}^{\varepsilon_{n_i}} - d^*)^2 + (v_{n_i}^{\varepsilon_{n_i}} - v^*)^2\big]$$

So we have

$$e^{-rt}E_i(t) + \int_0^t e^{-rs}\left[\int \frac{\widetilde{W}_i^{\mathrm{T}} Q_r(w^*)\widetilde{W}_i}{2h_{n_i}^{\varepsilon_{n_i}}} + \int \widetilde{W}_i \cdot \mathrm{L}(w^*) - \varepsilon_{n_i}\widetilde{R}_{i,r}(s)\right]\mathrm{d}s \leq E_i(0)$$

Because $E_i(t) \geq \Lambda(h_{n_i}^{\varepsilon_{n_i}}(t), \widetilde{U}_i)$, then we have that

$$\begin{aligned}e^{-rt}\Lambda(h_{n_i}^{\varepsilon_{n_i}}(t), \widetilde{U}_i(t)) + \widetilde{\Lambda}(h_{n_i}^{\varepsilon_{n_i}}, \widetilde{W}_i, e^{-rs}Q_r(w^*); 0, t)& \\ + \int_0^t e^{-rs}\left(\int \widetilde{W}_i \cdot \mathrm{L}(w^*) - \varepsilon_{n_i}\widetilde{R}_{i,r}(s)\right)\mathrm{d}s \leq E_i(0) &\quad (2.7.12)\end{aligned}$$

Notice that $h_0^{\varepsilon_{n_i}} \xrightarrow{w^*} h_0$ in $C(\mathbb{T}^3, \mathbb{R})'$, and

$$E_i(0) = \Lambda(h_0^{\varepsilon_{n_i}}, U_0^{\varepsilon_{n_i}}) + \big\langle h_0^{\varepsilon_{n_i}}, \frac{1}{2}|V(0)|^2\big\rangle - \big\langle U_0^{\varepsilon_{n_i}}, V(0)\big\rangle + \varepsilon_{n_i}\big\langle h_0^{\varepsilon_{n_i}}, \frac{|v^*|^2 + |d^*|^2}{2}\big\rangle$$

By Proposition 2.7.1, we know that the right hand side of (2.7.12) $E_i(0) \to \Lambda(h_0, \widetilde{U}_0)$ as $\varepsilon_{n_i} \to 0$. By Lemma 2.7.2, we know that $\sqrt{\varepsilon_{n_i}}\nabla^l v_{n_i}^{\varepsilon_{n_i}}, \sqrt{\varepsilon_{n_i}}\nabla^l d_{n_i}^{\varepsilon_{n_i}}$ are uniformly bounded in $L_{t,x}^2$, thus $\sqrt{\varepsilon_{n_i}} v_{n_i}^{\varepsilon_{n_i}}, \sqrt{\varepsilon_{n_i}} d_{n_i}^{\varepsilon_{n_i}}$ are uniformly bounded in $L^2(W^{1,\infty})$. Moreover $P_{n_i}^{\varepsilon_{n_i}}, D_{n_i}^{\varepsilon_{n_i}}, h_{n_i}^{\varepsilon_{n_i}}$ are uniformly bounded in $L_t^2(L_x^1)$, so we have that

$$\varepsilon_{n_i}\left|\int_0^t e^{r(t-s)}\widetilde{R}_{i,r}(s)\mathrm{d}s\right| \leq \sqrt{\varepsilon_{n_i}}(1 + \sqrt{\varepsilon_{n_i}})(1 + |r|)C$$

Here C only depends on v^*, b^*, d^*, h_0, B_0. So it goes to 0 as $\varepsilon_{n_i} \to 0$. By the weak-* convergent of $P_{n_i}^{\varepsilon_{n_i}}, D_{n_i}^{\varepsilon_{n_i}} \in C'_{t,x}$ and similar method as we did for (2.3.22), we have

$$\int_0^t \int e^{-rs}\widetilde{W}_i \cdot \mathrm{L}(w^*) \to \int_0^t \int e^{-rs}\widetilde{W} \cdot \mathrm{L}(w^*)$$

Besides, we have that,

$$\liminf_{i\to\infty} \Lambda(h_{n_i}^{\varepsilon_{n_i}}(t), \widetilde{U}_i(t)) \geq \Lambda(h(t), \widetilde{U}(t))$$

$$\liminf_{i\to\infty} \widetilde{\Lambda}(h_{n_i}^{\varepsilon_{n_i}}, \widetilde{W}_i, e^{-rs}Q_r(w^*); 0, t) \geq \widetilde{\Lambda}(h, \widetilde{W}, e^{-rs}Q_r(w^*); 0, t)$$

Combining the above results, we take the lower limit on both side of (2.7.12), then we can just get the inequality (3.1.8) for all fixed N, $0 < h^* \in C^1([0,T]\times\mathbb{T}^3, \mathbb{R})$, $b^* \in C^1([0,T]\times\mathbb{T}^3, \mathbb{R}^3)$, $v^*, d^* \in C^1([0,T], X_N)$. Now for any $v^*, d^* \in C^1([0,T]\times\mathbb{T}^3, \mathbb{R}^3)$ and r such that $Q_r(w^*)$ is positive definite, because v^*, d^* is continuous, then there exist $r' < r$ such that $Q_{r'}(w^*)$ is still positive definite. Because $\bigcup_{n=1}^\infty C^1([0,T], X_n)$ is dense in $C^1([0,T]\times\mathbb{T}^3, \mathbb{R}^3)$, So we can find a sequence $\{v_n^*\}, \{d_n^*\} \in C^1([0,T], X_n)$ that converge to v^*, d^* in $C^1([0,T]\times\mathbb{T}^3, \mathbb{R}^3)$ and $Q_r(w_n^*)$ is always positive definite, where $w_n^* = (h^{*-1}, b^*, d_n^*, v_n^*)$. Now let us denote

$$\widetilde{W}_n = \widetilde{W} + \widetilde{F}_n h, \quad \widetilde{F}_n = \big(0, 0, d^* - d_n^*, v^* - v_n^*\big)$$

By Lebesgue's dominated convergence theorem, we have that

$$\int_0^t \int e^{-rs}\widetilde{W}_n \cdot \mathrm{L}(w_n^*) \to \int_0^t \int e^{-rs}\widetilde{W} \cdot \mathrm{L}(w^*)$$

Besides, we have

$$\begin{aligned}\widetilde{\Lambda}(h, \widetilde{W}_n, e^{-rs}Q_r(w_n^*); 0, t) &= \widetilde{\Lambda}(h, \widetilde{W}, e^{-rs}Q_r(w_n^*); 0, t) \\ &\quad + \int_0^t e^{-rs}\big\langle \widetilde{W}, Q_n\widetilde{F}_n\big\rangle + \int_0^t e^{-rs}\big\langle h, \frac{1}{2}|\sqrt{Q_n}\widetilde{F}_n|^2\big\rangle \qquad (2.7.13)\end{aligned}$$

Now we would like to take the limit $n \to \infty$. The following lemma will be useful:

Lemma 2.7.4. *Suppose $Q_n, Q \in C([0,T]\times\mathbb{T}^3, \mathbb{R}^{d^2})$ are positive definite, $\|Q_n - Q\|_\infty \to 0$ as $n\to\infty$, then $\liminf_{n\to\infty} \widetilde{\Lambda}(\rho, W, Q_n; 0, t) \geq \widetilde{\Lambda}(\rho, W, Q; 0, t)$.*

Proof. The proof is quite straightforward. For any $a \in C([0,T]\times\mathbb{T}^3, \mathbb{R}), A \in C([0,T]\times\mathbb{T}^3, \mathbb{R}^d)$ such that $a + \frac{1}{2}|\sqrt{Q^{-1}}A|^2 \leq 0$, we have

$$\int_0^t \int \Big(a\rho + A\cdot W\Big) = \int_0^t \int \Big(\widetilde{a}\rho + A\cdot W\Big) + \int_0^t \int (a - \widetilde{a})\rho$$

where

$$\widetilde{a} = a + \frac{1}{2}|\sqrt{Q^{-1}}A|^2 - \frac{1}{2}|\sqrt{Q_n^{-1}}A|^2$$

Since $\widetilde{a} + \frac{1}{2}|\sqrt{Q_n^{-1}}A|^2 \leq 0$, we have

$$\int_0^t \int \Big(a\rho + A\cdot W\Big) \leq \widetilde{\Lambda}(\rho, W, Q_n; 0, t) + \frac{1}{2}\|\rho\|_{TV}\left\| |\sqrt{Q_n^{-1}}A|^2 - |\sqrt{Q^{-1}}A|^2 \right\|_\infty$$

Take the lower limit on both sides, we have, for any (a, A) s.t. $a + \frac{1}{2}|\sqrt{Q^{-1}}A|^2 \leq 0$,

$$\int_0^t \int \Big(a\rho + A \cdot W\Big) \leq \liminf_{n\to\infty} \widetilde{\Lambda}(\rho, W, Q_n; 0, t)$$

So we have $\widetilde{\Lambda}(\rho, W, Q; 0, t) \leq \liminf_{n\to\infty} \widetilde{\Lambda}(\rho, W, Q_n; 0, t)$. □

Now, by taking the lower limit as $n \to \infty$ in (2.7.13), we can get that (3.1.8) is valid for C^1 functions. So we have completely proved the existence of a dissipative solution. We summarize our result in the following theorem.

Theorem 2.7.5. *Suppose that $B_0 \in C(\mathbb{T}^3, \mathbb{R}^3)'$, $h_0 \in C(\mathbb{T}^3, \mathbb{R})'$, satisfying that $\nabla \cdot B_0 = 0$ in the sense of distributions and $\Lambda(h_0, U_0) < \infty$, where $U_0 = (\mathcal{L}, B_0)$. Then there exists a dissipative solution (h, B, D, P) of (DMHD) with initial value $(h, B)|_{t=0} = (h_0, B_0)$.*

2.8 Appendix 2.A: Proof of Lemma 2.3.1

Let us consider a more general case where (h, B, D, P) only satisfies the continuity equation (2.3.1) and the divergence-free constraint (2.3.4). We denote

$$\phi = \partial_t B + \nabla \times \left(\frac{D + B \times P}{h}\right),$$

$$\psi = D - \nabla \times \left(\frac{B}{h}\right), \quad \varphi = P - \nabla \cdot \left(\frac{B \otimes B}{h}\right) - \nabla \left(\frac{1}{h}\right),$$

Note that ϕ, ψ, φ vanish when (h, B, D, P) is exactly a solution to the Darcy MHD (2.3.1)-(2.3.4). We also use the non-conservative variables, namely,

$$\tau = \frac{1}{h},\ b = \frac{B}{h},\ d = \frac{D}{h},\ v = \frac{P}{h}$$

and, for the convenience of writing, let's denote

$$U = \big(1, B\big),\ \ u^* = (h^{*-1}, b^*),\ \ W = (1, B, D, P),\ \ w^* = (h^{*-1}, b^*, d^*, v^*).$$

To prove the lemma, let's start with computing the time derivative of the energy

$$S(t) = \int_{\mathbb{T}^3} \frac{|U|^2}{2h} = \int_{\mathbb{T}^3} \frac{1 + B^2}{2h}$$

Quite similar to (2.3.7), we have

$$\begin{aligned} S'(t) &= \int b \cdot \partial_t B - \int \frac{1}{2}(\tau^2 + b^2)\partial_t h \\ &= \int b \cdot \big[\phi - \nabla \times (B \times v + d)\big] + \int \frac{1}{2}(\tau^2 + b^2)\nabla \cdot P \\ &= \int b \cdot \phi - \int (B \times v + d) \cdot (\nabla \times b) - \int \big[\tau \nabla \tau + \nabla(b^2/2)\big] \cdot P \\ &= \int b \cdot \phi - \int d \cdot (\nabla \times b) - \int v \cdot \big(\nabla \cdot (b \otimes B) + \nabla \tau\big) \\ &= \int \big(b \cdot \phi + d \cdot \psi + v \cdot \varphi\big) - \int \frac{D^2 + P^2}{h} \end{aligned} \tag{2.8.1}$$

Now, let's look at the relative entropy. Since $\widetilde{U} = U - hu^*$, we have

$$\int \frac{|\widetilde{U}|^2}{2h} = S(t) + \int \frac{h|u^*|^2}{2} - \int U \cdot u^*$$

Therefore, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\widetilde{U}|^2}{2h} &= S'(t) + \int \frac{\partial_t h}{2}|u^*|^2 + \int hu^* \cdot \partial_t u^* - \int \partial_t U \cdot u^* - \int U \cdot \partial_t u^* \\
&= S'(t) - \int \frac{\nabla \cdot P}{2}|u^*|^2 - \int \partial_t B \cdot b^* - \int \widetilde{U} \cdot \partial_t u^* \\
&= S'(t) + \int \frac{P}{2} \cdot \nabla|u^*|^2 - \int b^* \cdot \big[\phi - \nabla \times (B \times v + d)\big] - \int \widetilde{U} \cdot \partial_t u^*
\end{aligned} \tag{2.8.2}$$

Now let's use a small trick to write 0 as,

$$\begin{aligned}
0 &= \int \Big[d^* \cdot (D - \nabla \times b - \psi) + v^* \cdot (P - \nabla \cdot (b \otimes B) - \nabla\tau - \varphi)\Big] \\
&= \int \big(D \cdot d^* + P \cdot v^* - d^* \cdot \psi - v^* \cdot \varphi\big) + \int \left[\sum_{i,j=1}^{3} \frac{\partial_j v_i^*}{h} B_i B_j + \frac{\nabla \cdot v^*}{h} - \frac{\big(\nabla \times d^*\big) \cdot B}{h}\right]
\end{aligned}$$

Then by (2.8.1),(2.8.2), we have,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\widetilde{U}|^2}{2h} &= \sum_{i,j=1}^{3} \int \Big[\frac{\partial_j v_i^*}{h} B_i B_j - \frac{\partial_j b_i^* - \partial_i b_j^*}{h} B_i P_j\Big] - \int \frac{D^2 + P^2}{h} + \int \frac{\nabla \cdot v^*}{h} \\
&\quad + \int \Big[\frac{\big(\nabla \times b^*\big) \cdot D}{h} - \frac{\big(\nabla \times d^*\big) \cdot B}{h}\Big] + \int \Big[D \cdot d^* + P \cdot \Big(v^* + \frac{\nabla|u^*|^2}{2}\Big)\Big] \\
&\quad - \int U \cdot \partial_t u^* + \int \Big[\phi \cdot \big(b - b^*\big) + \psi \cdot \big(d - d^*\big) + \varphi \cdot \big(v - v^*\big)\Big] \\
&= -\int \frac{W^{\mathrm{T}} Q(w^*) W}{2h} - \int \widetilde{U} \cdot \partial_t u^* + \int \Big[D \cdot d^* + P \cdot \Big(v^* + \frac{\nabla|u^*|^2}{2}\Big)\Big] \\
&\quad + \int \Big[\phi \cdot \big(b - b^*\big) + \psi \cdot \big(d - d^*\big) + \varphi \cdot \big(v - v^*\big)\Big]
\end{aligned} \tag{2.8.3}$$

Now since $W = \widetilde{W} + hw^*$, we can rewrite the quadratic like term as

$$\begin{aligned}
\int \frac{W^{\mathrm{T}} Q(w^*) W}{2h} &= \int \frac{\big(\widetilde{W} + hw^*\big)^{\mathrm{T}} Q(w^*) \big(\widetilde{W} + hw^*\big)}{2h} \\
&= \int \left(\frac{\widetilde{W}^{\mathrm{T}} Q(w^*) \widetilde{W}}{2h} + \widetilde{W} \cdot Q(w^*) w^* + \frac{w^{*\mathrm{T}} Q(w^*) w^*}{2} h\right)
\end{aligned}$$

A direct computation gives that

$$Q(w^*)w^* = \mathrm{L}(w^*) - (\partial_t u^*, 0, 0) + \left(\nabla \cdot \big(d^* \times b^* - h^{*-1} v^*\big), -\nabla\big(b^* \cdot v^*\big), d^*, v^* + \frac{1}{2}\nabla|u^*|^2\right)$$

$$\frac{w^{*\mathrm{T}} Q(w^*) w^*}{2} = \frac{v^*}{2} \cdot \nabla|u^*|^2 - b^* \cdot \nabla\big(b^* \cdot v^*\big) + h^{*-1} \nabla \cdot \big(d^* \times b^* - h^{*-1} v^*\big) + d^{*2} + v^{*2}$$

Therefore, since B is divergence free, we have

$$\int \frac{W^{\mathrm{T}} Q(w^*) W}{2h} = \int \left[\frac{\widetilde{W}^{\mathrm{T}} Q(w^*) \widetilde{W}}{2h} + \widetilde{W} \cdot \mathrm{L}(w^*) - \widetilde{U} \cdot \partial_t u^* + D \cdot d^* + P \cdot \Big(v^* + \frac{\nabla|u^*|^2}{2}\Big)\right]$$

So, finally, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\left|\widetilde{U}\right|^2}{2h}+\int\frac{W^{\mathrm{T}}Q(w^*)W}{2h}+\int\widetilde{W}\cdot\mathrm{L}(w^*)=\int\Big[\phi\cdot\big(b-b^*\big)+\psi\cdot\big(d-d^*\big)+\varphi\cdot\big(v-v^*\big)\Big] \tag{2.8.4}$$

Especially, when (h, B, D, P) is a solution of the Darcy MHD (2.3.1)-(2.3.4), i.e., $\psi = \phi = \varphi = 0$, we obtain (2.3.8). Moreover, if $(h^*, h^*b^*, h^*d^*, h^*v^*)$ is also a solution of (2.3.1)-(2.3.4), it is quite easy to verify that $\mathrm{L}_h(w^*), \mathrm{L}_B(w^*), \mathrm{L}_D(w^*), \mathrm{L}_P(w^*)$ respectively correspond to the equation for the non-conservative variables (τ, b, d, v), thus vanish.

Chapter 3

An integrable example of gradient flows based on optimal transport of differential forms

3.1 Introduction

Optimal transport theory has been a powerful tool for the analysis of parabolic equations viewed as gradient flows of volume forms according to suitable transportation metrics [2, 32, 41, 53, 47]. The theory of optimal transport for differential forms is not yet fully developed but there has been some recent progress, especially for symplectic forms and contact forms [19, 45]. However, to the best of our knowledge, little is known about gradient flows in that context. In this chapter we just present an example of gradient flows for closed $(d-1)-$forms in the Euclidean space $\mathbb{R}^d$. Such forms can be identified to divergence-free vector fields. Our example, set on the flat torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, reads

$$\partial_t B + \nabla \cdot \left(\frac{B \otimes P - P \otimes B}{\rho} \right) = 0, \quad \nabla \cdot B = 0, \tag{3.1.1}$$

$$\partial_t \rho + \nabla \cdot P = 0, \quad P = \nabla \cdot \left(\frac{B \otimes B}{\rho} \right) \tag{3.1.2}$$

[or, in coordinates:

$$\partial_t B^i + \partial_j \left(\frac{B^i P^j - P^i B^j}{\rho} \right) = 0, \quad \partial_j B^j = 0,$$

$$\partial_t \rho + \partial_j P^j = 0, \quad P^i = \partial_j \left(\frac{B^i B^j}{\rho} \right)],$$

where B is a time dependent divergence-free vector field (i.e. a closed $(d-1)-$form), ρ is a time-dependent companion volume-form, and P stands for ρv, where v is the time-dependent velocity field transporting both ρ and B as differential forms. As will be shown, this system turns out to be the gradient flow of functional

$$\mathcal{F}[\rho, B] = \int_{x \in \mathbb{T}^d} F(\rho(x), B(x)), \quad F(\rho, B) = \frac{|B|^2}{2\rho}, \tag{3.1.3}$$

according to the transportation metric

$$||v||_\rho = \sqrt{\int_{\mathbb{T}^d} v^2 \rho}, \tag{3.1.4}$$

which is just the most usual transport metric for volume-forms [2, 41, 47, 53]. Notice that $\mathcal{F}$, which is homogeneous of degree one, is a lower semi-continuous functional valued in $[0, +\infty]$ of ρ and B viewed as Borel measures, respectively valued in $\mathbb{R}$ and $\mathbb{R}^d$ (cf. [22]), namely

$$\mathcal{F}[\rho, B] = \sup\left\{\int_{\mathbb{T}^d} \theta\rho + \Theta \cdot B, \quad \theta + \frac{1}{2}|\Theta|^2 \leq 0\right\} \in [0, +\infty],$$

where the supremum is taken over all $(\theta, \Theta) \in C(\mathbb{T}^d; \mathbb{R} \times \mathbb{R}^d)$.

From the PDE viewpoint, system (3.1.1,3.1.2) is of degenerate parabolic type and can also be written, in non-conservative form,

$$\partial_t b + (v \cdot \nabla)b = (b \cdot \nabla)v, \quad v = (b \cdot \nabla)b, \tag{3.1.5}$$

or, equivalently,

$$\partial_t b = b \otimes b : D_x^2 b \tag{3.1.6}$$

[In coordinates:

$$\partial_t b^i + v^j \partial_j b^i = b^j \partial_j v^i, \quad v^i = b^j \partial_j b^i \ \text{ or } \ \partial_t b^i = b^k b^j \partial^2_{jk} b^i],$$

where b, v are the reduced variables $B/\rho, P/\rho$, as will be shown. This system is formally integrable and can be viewed as the Eulerian version of the heat equation for curves in the Euclidean space. More precisely, if (B, ρ, P) is of form

$$(B, \rho, P)(t, x) = \int_{a \in \mathcal{A}} \left(\int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t, s, a))(\partial_s X, 1, \partial_t X)(t, s, a) ds\right) d\mu(a) \tag{3.1.7}$$

where $(\mathcal{A}, d\mu)$ is an abstract probability space of labels a, and, for $\mu-$a.e. label a, and every time t, $s \in \mathbb{R}/\mathbb{Z} \to X(t, s, a) \in \mathbb{T}^d$ is a loop subject to the heat equation

$$\partial_t X(t, s, a) = \partial^2_{ss} X(t, s, a),$$

then (B, ρ, P) is expected to be a solution of (3.1.1,3.1.2), at long as there is no self- or mutual intersection of the different loops. The goal of this chapter is to provide a robust notion of generalized solutions which includes fields (B, ρ, P) of form (3.1.7) as global solutions of system (3.1.1,3.1.2), that we call, from now on, the Eulerian heat equation.

Our definition reads:

Definition 3.1.1. *Let us fix $T > 0$. We say that (B, ρ, P) with*

$$(B, \rho) \in C([0, T], C(\mathbb{T}^d, \mathbb{R}^d \times \mathbb{R})'_{w^*}), \quad P \in C([0, T] \times \mathbb{T}^d, \mathbb{R}^d)',$$

is a dissipative solution of the Eulerian heat equation (3.1.1,3.1.2) with initial data $(B_0, \rho_0 \geq 0) \in C(\mathbb{T}^d; \mathbb{R}^d \times \mathbb{R})'$ if and only if

(i) $(B(0), \rho(0)) = (B_0, \rho_0)$, $\nabla \cdot B = 0$, $\partial_t \rho + \nabla \cdot P = 0$, *in sense of distributions;*
(ii) *For any* $t \in [0, T]$, *and any* $b^*, v^* \in C^1([0,T] \times \mathbb{T}^d, \mathbb{R}^d)$,

$$E(t)e^{-rt} + \int_0^t e^{-rt'} \left[(r - r_0)E(t') + \int_{\mathbb{T}^d} \frac{\widetilde{U}^{\mathrm{T}} Q_{r_0} \widetilde{U}}{2\rho}(t') - R(t') \right] \mathrm{d}t' \leq E(0). \quad (3.1.8)$$

holds true for all $t \in [0, T]$ *and* $r \geq r_0$, *where*

$$\widetilde{U} = (B - \rho b^*, P - \rho v^*), \quad E = \int_{\mathbb{T}^d} \frac{|B - \rho b^*|^2}{2\rho},$$

$$R = \int_{\mathbb{T}^d} \rho\, \mathrm{L}_1 + B \cdot \mathrm{L}_2 + P \cdot \mathrm{L}_3,$$

$$\mathrm{L}_1 = v^{*2} + D_t \left(\frac{b^{*2}}{2} \right) - (b^* \cdot \nabla)(b^* \cdot v^*), \quad D_t = (\partial_t + v^* \cdot \nabla),$$

$$\mathrm{L}_2 = -D_t b^* + (b^* \cdot \nabla) v^*, \quad \mathrm{L}_3 = -v^* + (b^* \cdot \nabla) b^*,$$

and r_0 *is a constant depending on* b^*, v^*, *chosen so that*

$$Q_{r_0} = \begin{pmatrix} -\nabla v^* - \nabla v^{*\mathrm{T}} + r_0 I_d & \nabla b^* - \nabla b^{*\mathrm{T}} \\ \nabla b^{*\mathrm{T}} - \nabla b^* & 2I_d \end{pmatrix}$$

is positive definite.

This concept of solutions turns out to be convex in (B, ρ, P) which is crucial to include fields (B, ρ, P) of form (3.1.7) as global dissipative solutions. It is based on the relative entropy method, quite well known in the theory of hyperbolic systems of conservation laws [20, 36, 37], kinetic theory [46], parabolic equations [33], and continuum mechanics [23, 28], just to quote few examples. It is closely related to Lions' concept of dissipative solutions for the Euler equations of incompressible fluids [39] and related models [9, 12, 54, 11]. It is also related to the way the heat equation is solved for general metric measure spaces in [2]. With such a robust concept, we can see which way fields (B, ρ, P) of form (3.1.7) are, indeed, solutions:

Theorem 3.1.2. *Let*

$$(B_0, \rho_0)(x) = \int_{a \in \mathcal{A}} \left(\int_{\mathbb{R}/\mathbb{Z}} \delta(x - X_0(s,a)) \left(\partial_s X_0(s,a), 1 \right) ds \right) d\mu(a), \quad x \in \mathbb{T}^d,$$

where $(\mathcal{A}, d\mu)$ *is an abstract probability space of labels* a, *and the loops* $s \in \mathbb{R}/\mathbb{Z} \to X_0(s,a) \in \mathbb{T}^d$ *are chosen so that*

$$\begin{aligned} &\int_{x \in \mathbb{T}^d} \frac{|\int_{\mathcal{A} \times \mathbb{R}/\mathbb{Z}} \delta(x - X(0,s,a)) \partial_s X(0,s,a) ds d\mu(a)|^2}{\int_{\mathcal{A} \times \mathbb{R}/\mathbb{Z}} \delta(x - X(0,s,a)) ds d\mu(a)} \\ &= \int_{\mathcal{A} \times \mathbb{R}/\mathbb{Z}} |\partial_s X(0,s,a)|^2 ds d\mu(a) < +\infty. \end{aligned} \quad (3.1.9)$$

Then there is a global dissipative solution (B, ρ, P) *to the Eulerian heat equation (3.1.1,3.1.2), explicitly given by*

$$(B, \rho, P)(t,x) = \int_{a \in \mathcal{A}} \left(\int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s,a)) (\partial_s X, 1, \partial_t X)(t,s,a) ds \right) d\mu(a),$$

where, for $\mu-$*a.e.* a, $X(\cdot, \cdot, a)$ *is solution to the heat equation*

$$\partial_t X(t,s,a) = \partial_{ss}^2 X(t,s,a).$$

Notice that condition (3.1.9) essentially means that, at time $t = 0$, in the definition of (B_0, ρ_0), the loops $s \to X(0, s, a)$ have been chosen without self- or mutual intersections. At this stage, we don't know how this result can be extended to *all* Borel measures B_0, ρ_0 respectively valued in $\mathbb{R}^d$ and $\mathbb{R}_+$ such that

$$\nabla \cdot B_0 = 0, \quad \int_{\mathbb{T}^d} \frac{|B_0|^2}{\rho_0} < +\infty.$$

This is a delicate question of geometric measure theory, closely related to the topics discussed in [1, 51].

We also get the following "weak-strong" uniqueness result:

Theorem 3.1.3. *Suppose $(b, v) \in C^1([0, T] \times \mathbb{T}^d, \mathbb{R}^d \times \mathbb{R}^d)$ is a solution of the degenerate parabolic system (3.1.5). Let $\rho_0 \geq 0$ given in $C(\mathbb{T}^d, \mathbb{R})'$ chosen so that $\nabla \cdot (\rho_0 b_0) = 0$ and set $B_0 = b_0\rho_0$. Then there is a unique dissipative solution $(B, \rho, P) = (\rho b, \rho, \rho v)$ to the Eulerian heat equation (3.1.1,3.1.2) in the sense of definition 3.1.1 with initial condition (B_0, ρ_0), where ρ is the unique solution of the continuity equation $\partial_t \rho + \nabla \cdot (\rho v) = 0$.*

Miscellaneous remarks

Remark 1

By reducing a complicated degenerate parabolic system to the one-dimensional heat equation for loops, we follow, in a somewhat opposite direction, the path of Evans, Gangbo, Savin [26] who treated a degenerate parabolic system in $\mathbb{R}^d$ with polyconvex entropy as an integrable system, by reducing it, in its Eulerian version, to the scalar equation in $\mathbb{R}^d$.

Remark 2

In a companion paper [14], the authors study the more delicate system

$$\partial_t B + \nabla \cdot \left(\frac{B \otimes P - P \otimes B}{\rho} \right) = 0, \quad \nabla \cdot B = 0, \quad P = \nabla \cdot \left(\frac{B \otimes B}{\rho} \right), \quad \rho = |B|,$$

where the algebraic constraint $\rho = |B|$ substitutes for the continuity equation. Then, the analysis gets substantially more difficult. However, there is still some underlying integrability where the geometric heat equation for curves (or curve-shortening flow, which corresponds to co-dimension $d-1$ mean curvature motion) substitutes for the linear heat equation.

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3.2 Gradient flows for closed $(d-1)$-forms and transportation metrics

Optimal transport theory is largely about giving a Riemannian structure to the space of volume forms $\rho dx^1 \wedge \cdots \wedge dx^d$ in $\mathbb{R}^d$ and many gradient flows can be derived accordingly, following the seminal work of Otto and collaborators [32, 41, 47, 53]. Here we want to extend this theory to the case of closed $d-1$ differential forms in $\mathbb{R}^d$, or, in other words, divergence-free vector fields. For instance, as $d = 3$,

$$B = B^1 dx^2 \wedge dx^3 + B^2 dx^3 \wedge dx^1 + B^3 dx^1 \wedge dx^2,$$

$$0 = dB = (\partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3) dx^1 \wedge dx^2 \wedge dx^3 = \nabla \cdot B \, dx^1 \wedge dx^2 \wedge dx^3,$$

and these formulae easily extend to arbitrary dimensions d. For simplicity, we will only discuss about $\mathbb{Z}^d-$periodic forms so that we will use the flat torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ instead of the entire space $\mathbb{R}^d$.

3.2.1 Elementary closed $(d-1)-$forms and superposition of loops

An elementary example of closed $(d-1)-$forms is given by

$$B(x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(s)) X'(s) ds, \quad x \in \mathbb{T}^d, \tag{3.2.1}$$

where X is a loop, i.e. a Lipschitz map $s \in \mathbb{R}/\mathbb{Z} \to X(s) \in \mathbb{T}^d$ and B should be understood (with an abuse of notation) just as the vector-valued distribution defined by

$$< B^i, \beta_i >= \int_{\mathbb{R}/\mathbb{Z}} \beta(X(s)) \cdot X'(s) ds,$$

for all trial function $\beta \in C^\infty(\mathbb{T}^d; \mathbb{R}^d)$. In particular, $\nabla \cdot B = 0$ immediately follows since

$$< \partial_i B^i, \phi >= - < B^i, \partial_i \phi >= - \int_{\mathbb{R}/\mathbb{Z}} \nabla \phi(X(s)) \cdot X'(s) ds$$

$$= - \int_{\mathbb{R}/\mathbb{Z}} \frac{d}{ds}(\phi(X(s))) ds = 0,$$

for all trial function $\phi \in C^\infty(\mathbb{T}^d; \mathbb{R})$. A much larger class of divergence-free vector fields B can be obtained by superposing loops:

$$B(x) = \int_{a \in \mathcal{A}} \left(\int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(s,a)) \partial_s X(s,a) ds \right) d\mu(a) \tag{3.2.2}$$

where $(\mathcal{A}, d\mu)$ is an abstract probability space of labels a, and, for $\mu-$a.e. label a, $s \in \mathbb{R}/\mathbb{Z} \to X(s,a) \in \mathbb{T}^d$ is a loop. It is an important issue of geometric measure theory to see how large is this class [1, 51]. [A typical "folklore" result being that every divergence-free vector field B, with bounded mass

$$\int_{\mathbb{T}^d} |B| = \sup\{< B, \beta >, \quad \beta \in C^\infty(\mathbb{T}^d; \mathbb{R}^d), \quad \sup |\beta(x)| \le 1 \ \} < +\infty,$$

can be approximated by a superposition of N loops $(X_\alpha, \alpha \in \{1,\cdots,N\})$ in the sense

$$< B^i, \beta_i >= \lim_{N\uparrow\infty} \frac{1}{N} \sum_{\alpha=1}^{N} \int_{\mathbb{R}/\mathbb{Z}} \beta(X_\alpha(s)) \cdot X'_\alpha(s) ds, \quad \forall \beta \in C_0^\infty(\mathbb{T}^d;\mathbb{R}^d),$$

$$\int_{\mathbb{T}^d} |B| = \lim_{N\uparrow\infty} \frac{1}{N} \sum_{\alpha=1}^{N} \int_{\mathbb{R}/\mathbb{Z}} |X'_\alpha(s)| ds.]$$

3.2.2 Transportation of closed $(d-1)-$forms

As for volume forms, the concept of transport involves time-dependent forms $B(t,x)$ and velocity fields $v(t,x) \in \mathbb{R}^d$. To get the transport equation right, we can refer to the case of a moving loop $(t,s) \to X(t,s) \in \mathbb{T}^d$ subject to

$$\partial_t X(t,s) = v(t, X(t,s)).$$

Given a smooth trial function $\beta \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$, we get for the form attached to X

$$B(t,x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s)) \partial_s X(t,s) ds,$$

$$\frac{d}{dt} < B^i(t,\cdot), \beta_i >= \frac{d}{dt} \int_{\mathbb{R}/\mathbb{Z}} \beta_i(X(t,s)) \partial_s X^i(t,s) ds$$

$$= \int_{\mathbb{R}/\mathbb{Z}} (\partial_j \beta_i)(X(t,s)) \partial_t X^j(t,s) \partial_s X^i(t,s) ds + \int_{\mathbb{R}/\mathbb{Z}} \beta_i(X(t,s)) \partial^2_{ts} X^i(t,s) ds$$

$$= \int_{\mathbb{R}/\mathbb{Z}} (\partial_j \beta_i)(X(t,s)) \partial_t X^j(t,s) \partial_s X^i(t,s) ds - \int_{\mathbb{R}/\mathbb{Z}} (\partial_j \beta_i)(X(t,s)) \partial_s X^j(t,s) \partial_t X^i(t,s) ds$$

$$= \int_{\mathbb{R}/\mathbb{Z}} (\partial_j \beta_i)(X(t,s)) \left(v^j(t, X(t,s)) \partial_s X^i(t,s) - \partial_s X^j(t,s) v^i(t, X(t,s)) \right) ds$$

$$=< B^i v^j - B^j v^i, \partial_j \beta_i >= - < \partial_j (B^i v^j - B^j v^i), \beta_i >$$

So, for the transport of B by v, we have found equation

$$\partial_t B + \nabla \cdot (B \otimes v - v \otimes B) = 0,$$

which is linear in B and stays valid, as a matter of fact, for all closed $(d-1)$-differential forms (see [18] for example). In the case $d = 3$, this equation is usually called induction equation in the framework of (ideal) Magnetohydrodynamics, B being interpreted as a magnetic field and v as the velocity field of a conductive fluid. We will retain the name of induction equation for any dimension d. For the sequel of our discussion, it is convenient to attach to each time-dependent form B generated by some loop X, a companion volume form defined as

$$\rho(t,x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s)) ds \geq 0, \quad x \in \mathbb{T}^d,$$

or, equivalently, by duality,

$$< \rho(t,\cdot), \phi >= \int_{\mathbb{R}/\mathbb{Z}} \phi(X(t,s)) ds, \quad \forall \phi \in C_0^\infty(\mathbb{T}^d;\mathbb{R}).$$

[Notice that, in contrast with B, the definition of ρ is affected by a change of parameterization of the loop X with respect to s.] The transportation equation for ρ is nothing but

$$\partial_t\rho + \nabla\cdot(\rho v) = 0,$$

just as in regular optimal transportation theory. This equation is usually called continuity equation in Fluid Mechanics and we will also retain this name for any dimension d.

Transportation cost

Mimicking the case of volume forms, which corresponds to regular optimal transportation, we define a transportation cost by introducing, for each fixed form B, a Hilbert norm, possibly depending on B, for all suitable transporting velocity field $x \in \mathbb{T}^d \to v(x) \in \mathbb{R}^d$. In the case when we attach a volume form ρ to B, the Hilbert norm may depend on both B and ρ and we denote it by

$$v \to ||v||_{B,\rho}.$$

For volume forms, the most popular choice of norm is $v \to \sqrt{\int_{\mathbb{T}^d}|v|^2\rho}$ and we will concentrate on this choice in thesis . Then, the resulting norm depends only on ρ and is simply denoted by $||\cdot||_\rho$.

Steepest descent

Let us give a functional $\mathcal{F}[\rho, B]$ and monitor its steepest descent according to the Hilbert norm $v \to ||v||_{B,\rho}$ on the space of velocity fields v transporting B and ρ. We will concentrate soon on the special case when

$$\mathcal{F}[\rho, B] = \int_{x\in\mathbb{T}^d}\frac{|B(x)|^2}{2\rho(x)},\quad ||v||_\rho = \sqrt{\int_{x\in\mathbb{T}^d}|v(x)|^2\rho(x)dx},$$

which turns out to be, in some sense, the simplest choice, as will be seen later on. Nevertheless, let us start our calculations in the larger framework when

$$\mathcal{F}[\rho, B] = \int_{x\in\mathbb{T}^d} F(\rho(x), B(x)),$$

for some function $F : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, supposed to be smooth away from $\rho = 0$, such as $F(\rho, B) = \frac{B^2}{2\rho}$, for instance. Thanks to the continuity and the induction equations, we get (assuming B, ρ and v to be smooth with $\rho > 0$, using coordinates and denoting by F_ρ and F_B the partial derivatives of F with respect to ρ and B)

$$\frac{d}{dt}\mathcal{F}[\rho, B] = \int_{\mathbb{T}^d}\left(F_\rho\partial_t\rho + F_{B^i}\partial_t B^i\right)$$

$$= -\int_{\mathbb{T}^d}\left[F_\rho\partial_i(\rho v^i) + F_{B^i}\partial_j(B^i v^j - B^j v^i)\right]$$

(using the induction and continuity equations)

$$= \int_{\mathbb{T}^d}\left[\rho\partial_i(F_\rho) - \big(\partial_j(F_{B^i}) - \partial_i(F_{B^j})\big)B^j\right]v^i = -\int_{\mathbb{T}^d} v\cdot\ G$$

where

$$G_i = -\rho\partial_i(F_\rho) + \big(\partial_j(F_{B^i}) - \partial_i(F_{B^j})\big)B^j.$$

So, we have obtained

$$\frac{d}{dt}\mathcal{F}[\rho, B] = -\int_{\mathbb{T}^d} v\cdot\ G \geq -\frac{1}{2}||v||^2_{\rho,B} - \frac{1}{2}||G||^{*}_{\rho,B}{}^2,$$

where $||\cdot||^{*}{}_{\rho,B}$ is the dual norm defined by

$$\frac{1}{2}||g||^{*}_{\rho,B}{}^2 = \sup_w \int_{\mathbb{T}^d} g\cdot w - \frac{1}{2}||w||^2_{\rho,B}.$$

To get the steepest descent according to norm $||\cdot||_{\rho,B}$ it is enough to saturate this inequality so that

$$\frac{1}{2}||G||^{*}_{\rho,B}{}^2 + \frac{1}{2}||v||^2_{\rho,B} = \int G\cdot v,$$

or, in other words, to define v as the derivative with respect to G of half the dual norm squared:

$$v = D_G\left(\frac{1}{2}||G||^{*}_{\rho,B}{}^2\right).$$

The Eulerian version of the heat equation

From now on, we will concentrate on the special case (3.1.3,3.1.4), namely

$$\mathcal{F}[\rho, B] = \int_{\mathbb{T}^d} F(\rho(x), B(x)),\quad F(\rho, B) = \frac{|B|^2}{2\rho},\quad ||v||_\rho = \sqrt{\int_{\mathbb{T}^d} v^2\rho}.$$

According to the previous calculations, we first find

$$F_\rho = -\frac{|B|^2}{2\rho^2},\quad F_{B^i} = \frac{B_i}{\rho},$$

$$\begin{aligned} G_i &= -\rho\partial_i(F_\rho) + \big(\partial_j(F_{B^i}) - \partial_i(F_{B^j})\big)B^j \\ &= \rho\partial_i\left(\frac{B_j}{\rho}\right)\frac{B^j}{\rho} + \partial_j\left(\frac{B^i}{\rho}\right)B^j - \partial_i\left(\frac{B_j}{\rho}\right)B^j = \partial_j\left(\frac{B^iB^j}{\rho}\right) \end{aligned}$$

(using $\partial_j B^j = 0$). Next, we get

$$\frac{1}{2}||G||^{*}_{\rho,B}{}^2 = \int\frac{G^2}{2\rho},$$

and its derivative with respect to G is just G/ρ. Finally, we have obtained the steepest descent of $\mathcal{F}$ with respect to the transportation metric $v\to||v||_\rho$, precisely when

$$v^i = \frac{1}{\rho}\partial_j(\frac{B^iB^j}{\rho}),$$

which, combined with the induction and the continuity equations, leads to the system (3.1.1,3.1.2), namely

$$\partial_t B + \nabla\cdot(\frac{B\otimes P - P\otimes B}{\rho}) = 0,\quad \nabla\cdot B = 0,$$

$$\partial_t \rho + \nabla \cdot P = 0, \quad P = \nabla \cdot \left(\frac{B \otimes B}{\rho} \right),$$

where P stands for ρv (i.e. the momentum, in terms of Fluid Mechanics). When the solution (B, ρ) is smooth with $\rho > 0$ (which is definitely not the case when B is generated by a single loop), this system can be written in "non conservative" form, in terms of the rescaled field $b = B/\rho$. Indeed, from (3.1.1), we get (in coordinates)

$$0 = b^i(\partial_t \rho + \partial_j(\rho v^j)) + \rho \left(\partial_t b^i + v^j \partial_j b^i - b^j \partial_j v^i\right) - v^i \partial_j(\rho b^j)$$
$$= \rho \left(\partial_t b^i + v^j \partial_j b^i - b^j \partial_j v^i\right)$$

and

$$v^i = b^j \partial_j b^i,$$

which leads to the non-conservative version (3.1.5) of (3.1.1,3.1.2), namely

$$\partial_t b + (v \cdot \nabla) b = (b \cdot \nabla) v, \quad v = (b \cdot \nabla) b,$$

in which ρ plays no role. This equation can also be written as the (very) degenerate parabolic system (3.1.6), namely

$$\partial_t b = b \otimes b : D_x^2 b.$$

[Indeed, we get from (3.1.5), in coordinates,

$$\partial_t b^i = -v^j \partial_j b^i + b^j \partial_j v^i = -b^k \partial_k b^j \partial_j b^i + b^j \partial_j (b^k \partial_k b^i)$$
$$= -b^k \partial_k b^j \partial_j b^i + b^j b^k \partial^2_{kj} b^i + b^j \partial_j b^k \partial_k b^i = b^j b^k \partial^2_{kj} b^i.]$$

In a completely different direction, we can interpret (3.1.1,3.1.2) just as a hidden version of the one-dimensional heat equation! Indeed, let us assume that a time dependent loop $(t, s) \to X(t, s)$ solves the linear heat equation

$$\partial_t X(t, s) = \partial^2_{ss} X(t, s). \tag{3.2.3}$$

and never self-intersects during some time interval $[0, T]$, so that we may find two smooth time-dependent vector field v and b such that

$$\partial_t X(t, s) = v(t, X(t, s)), \quad \partial_s X(t, s) = b(t, X(t, s)).$$

Using the chain-rule, we first recover $v = (b \cdot \nabla) b$ directly from (3.2.3) and then we observe that $\partial_t b + (v \cdot \nabla) b = (b \cdot \nabla) v$ is just the compatibility condition for b and v to be partial derivatives of X. Surprisingly enough, we *directly* recover the $non-conservative$ version of (3.1.1,3.1.2). We may recover the conservative form (3.1.1,3.1.2) by reversing the computation we did to get the non-conservative form, after *adding* the field ρ as solution of the continuity equation $\partial_t \rho + \nabla \cdot (\rho v) = 0$, with initial condition

$$\rho(0, x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(0, s)) ds.$$

Indeed, this equation is linear in ρ and admits, since v is supposed to be smooth, a unique distributional solution which must be

$$\rho(t, x) = \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t, s)) ds$$

since $\partial_t X(t, s) = v(t, X(t, s))$. To conclude this subsection, let us rename system (3.1.1,3.1.2) as the Eulerian heat equation.

Remark

At this stage, it seems strange to solve a complicated set of non-linear PDEs such as (3.1.1,3.1.2) while we may, instead, solve the trivial one-dimensional heat equation! However, the derivation of (3.1.1,3.1.2) we just performed is crucially based on the assumptions we made that X is smooth (which is not a problem since X solves the heat equation) and non self-intersecting which, except in some very special situations, is not true globally in time. So we can view (3.1.1,3.1.2) as a way of extending the evolution beyond the first self-intersection time. As a matter of fact, a similar situation is very well known for a collection of particles, labelled by some parameter a, and solving the trivial equation

$$\partial^2_{tt} X(t,a) = 0.$$

Assuming the existence of a smooth velocity field v such that

$$\partial_t X(t,a) = v(t, X(t,a)),$$

we immediately obtain $\partial_t v + (v \cdot \nabla) v = 0$ which is nothing but the multi-dimensional version of the so-called inviscid Burgers equation, or, in other words, the non-conservative version of the pressure-less Euler equations

$$\partial_t P + \nabla \cdot \left(\frac{P \otimes P}{\rho} \right) = 0, \quad \partial_t \rho + \nabla \cdot P = 0. \tag{3.2.4}$$

As well known, in this model, particles of different labels may cross as time goes on (especially in the case when parameter a is continuously distributed). This is why system (3.2.4) is far from being well understood, except in some special case, typically as $d = 1$, where we may use the order of the real line to get a satisfactory formulation (such as in [13]). Let us finally mention, as already done in the introduction, the work by Evans, Gangbo and Savin [26] where the authors are able to solve a complicated degenerate parabolic system with "polyconvex entropy" in $\mathbb{R}^d$ by noticing that its Eulerian version is nothing but the regular scalar heat equation $\mathbb{R}^d$.

3.3 Dissipative solutions to the Eulerian heat equation

Let us consider a loop X solution to the one-dimensional heat equation (3.2.3) and introduce the relative entropy

$$\mathcal{E}(t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{|\partial_s X(t,s) - b^*(t, X(t,s))|^2}{2} ds$$

where $b^* \in C^\infty([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$ is a fixed trial function. We find, after elementary but lengthy computations, (see Appendix 3.A for more details)

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= -\int_{\mathbb{R}/\mathbb{Z}} |\partial_t X - v^*(t,X)|^2 ds \\ &+\frac{1}{2}\int_{\mathbb{R}/\mathbb{Z}} (\partial_s X^i - b^{*i}(t,X))(\partial_s X^j - b^{*j}(t,X))(\partial_j v_i^* + \partial_i v_j^*)(t,X) ds \\ &-\int_{\mathbb{R}/\mathbb{Z}} \big(\partial_s X^i - b^{*i}(t,X)\big)\big(\partial_t X^j - v^{*j}(t,X)\big)(\partial_j b_i^* - \partial_i b_j^*)(t,X) ds \\ &+\int_{\mathbb{R}/\mathbb{Z}} \big(\mathrm{L}_1(t,X) + \partial_s X \cdot \mathrm{L}_2(t,X) + \partial_t X \cdot \mathrm{L}_3(t,X)\big) ds,\end{aligned} \tag{3.3.1}$$

where

$$\mathrm{L}_1 = v^{*2} + D_t\left(\frac{b^{*2}}{2}\right) - (b^*\cdot\nabla)(b^*\cdot v^*), \quad D_t = (\partial_t + v^*\cdot\nabla),$$
$$\mathrm{L}_2 = -D_t b^* + (b^*\cdot\nabla)v^*, \quad \mathrm{L}_3 = -v^* + (b^*\cdot\nabla)b^*.$$

In order to get more compact notations, we introduce

$$\widetilde{W}(t,s) = \big(\partial_s X(t,s) - b^*(t,X(t,s)), \partial_t X(t,s) - v^*(t,X(t,s))\big),$$

$$Q(b^*,v^*) = \begin{pmatrix} -\nabla v^* - \nabla v^{*\mathrm{T}} & \nabla b^* - \nabla b^{*\mathrm{T}} \\ \nabla b^{*\mathrm{T}} - \nabla b^* & 2I_d \end{pmatrix}$$

Then (3.3.1) can be written as

$$\frac{d\mathcal{E}}{dt} + \int_{\mathbb{R}/\mathbb{Z}} \frac{\widetilde{W}^{\mathrm{T}} Q(b^*,v^*)\widetilde{W}}{2}(t,s) ds - \mathcal{R}(t) = 0, \tag{3.3.2}$$

where

$$\mathcal{R}(t) = \int_{\mathbb{R}/\mathbb{Z}} \big(\mathrm{L}_1(t,X(t,s)) + \partial_s X(t,s)\cdot \mathrm{L}_2(t,X(t,s)) + \partial_t X(t,s)\cdot \mathrm{L}_3(t,X(t,s))\big) ds$$

We use $I_{n:m}$ to represent the $n \times n$ diagonal matrix whose first m terms are 1 while the rest terms are 0, let I_d be the $d \times d$ identity matrix. Then we can choose $r_0 \geq 0$ big enough, in terms of the trial functions b^*, v^*, such that

$$Q_{r_0} = Q(b^*,v^*) + r_0 I_{2d:d} \geq I_{2d} > 0.$$

In addition, we observe that

$$\frac{1}{2}(\widetilde{W}^T I_{2d:d}\, \widetilde{W})(t,s) = \frac{1}{2}\big|\partial_s X(t,s) - b^*(t,X(t,s))\big|^2,$$

which is exactly the relative entropy density. Thus, for any constant $r \geq r_0$, we obtain by integrating in time (3.3.2) after multiplication by e^{-rt}:

$$\mathcal{E}(t)e^{-rt} + \int_0^t e^{-rt'}\left[(r-r_0)\mathcal{E}(t') + \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{2}(\widetilde{W}^{\mathrm{T}} Q_{r_0}\widetilde{W})(t',s) ds - \mathcal{R}(t')\right] \mathrm{d}t' = \mathcal{E}(0) \tag{3.3.3}$$

Let us now consider a collection of loops, labelled by $a \in \mathcal{A}$, where $(\mathcal{A}, d\mu)$ is an abstract probability space, and subject to the heat equation

$$\partial_t X(t,s,a) = \partial_{ss}^2 X(t,s,a).$$

We set, for each $a \in \mathcal{A}$,

$$\widetilde{W}(t,s,a) = \big(\partial_s X(t,s,a) - b^*(t, X(t,s,a)), \partial_t X(t,s,a) - v^*(t, X(t,s,a))\big),$$

$$\mathcal{R}(t,a) = \int_{\mathbb{R}/\mathbb{Z}} \big(\mathrm{L}_1(t,X) + \partial_s X \cdot \mathrm{L}_2(t,X) + \partial_t X \cdot \mathrm{L}_3(t,X)\big)(t,s,a)ds,$$

$$\mathcal{E}(t,a) = \int_{\mathbb{R}/\mathbb{Z}} \frac{|\partial_s X(t,s,a) - b^*(t, X(t,s,a))|^2}{2} ds,$$

so that (3.3.3) reads

$$\mathcal{E}(t,a)e^{-rt} - \mathcal{E}(0,a) \tag{3.3.4}$$

$$+ \int_0^t e^{-rt'} \left[(r - r_0)\mathcal{E}(t',a) + \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{2}(\widetilde{W}^{\mathrm{T}} Q_{r_0} \widetilde{W})(t',s,a)ds - \mathcal{R}(t',a)\right] \mathrm{d}t' = 0.$$

Next, we introduce the averaged fields (B, ρ, P):

$$(B,\rho,P)(t,x) = \int_{a\in\mathcal{A}} \left(\int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s,a))(\partial_s X, 1, \partial_t X)(t,s,a)ds \right) d\mu(a)$$

and, also,

$$\widetilde{U}(t,x) = \big(B(t,x) - \rho(t,x)b^*(t,x), P(t,x) - \rho(t,x)v^*(t,x)\big)$$

$$= \int_{a\in\mathcal{A}} \left(\int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s,a))\big(\partial_s X - b^*(t,X), \partial_t X - v^*(t,X)\big)(t,s,a)ds \right) d\mu(a),$$

$$R = \int_{\mathbb{T}^d} \rho\, \mathrm{L}_1 + B \cdot \mathrm{L}_2 + P \cdot \mathrm{L}_3 = \int_{a\in\mathcal{A}} \mathcal{R}(\cdot,a)d\mu(a), \quad E = \int_{\mathbb{T}^d} \frac{|B - \rho b^*|^2}{2\rho}.$$

We see that

$$E(t) = \int_{\mathbb{T}^d} \frac{B^2}{2\rho}(t) + \int_{a\in\mathcal{A}} \int_{\mathbb{R}/\mathbb{Z}} \left(-\partial_s X \cdot b^*(t,X) + \frac{1}{2}|b^*(t,X)|^2 \right)(t,s,a)dsd\mu(a),$$

$$= \int_{a\in\mathcal{A}} \mathcal{E}(t,a)d\mu(a) + \int_{\mathbb{T}^d} \frac{B^2}{2\rho}(t) - \frac{1}{2}\int_{a\in\mathcal{A}} \int_{\mathbb{R}/\mathbb{Z}} |\partial_s X(t,s,a)|^2 dsd\mu(a).$$

By the Cauchy-Schwarz inequality

$$\int_{\mathbb{T}^d} \frac{B^2}{\rho}(t) = \int_{x\in\mathbb{T}^d} \frac{|\int_{a\in\mathcal{A}} \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s,a))\partial_s X(t,s,a)dsd\mu(a)|^2}{\int_{a\in\mathcal{A}} \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(t,s,a))dsd\mu(a)}$$

$$\leq \int_{a\in\mathcal{A}} \int_{\mathcal{R}/\mathcal{Z}} |\partial_s X(t,s,a)|^2 dsd\mu(a),$$

so that

$$E(t) \leq \int_{a\in\mathcal{A}} \mathcal{E}(t,a)d\mu(a).$$

In a similar way,

$$\int_{\mathbb{T}^d} \frac{\widetilde{U}^{\mathrm{T}} Q_{r_0} \widetilde{U}}{\rho}(t) \leq \int_{a\in\mathcal{A}} \int_{\mathbb{R}/\mathbb{Z}} (\widetilde{W}^{\mathrm{T}} Q_{r_0} \widetilde{W})(t,s,a)dsd\mu(a).$$

Thus, when integrating *equality* (3.3.4) in $a \in \mathcal{A}$ with respect to μ, we deduce the following *inequality*

$$E(t)e^{-rt} + \int_0^t e^{-rt'} \left[(r - r_0)E(t') + \int_{\mathbb{T}^d} \frac{\widetilde{U}^{\mathrm{T}} Q_{r_0} \widetilde{U}}{2\rho}(t') - R(t') \right] dt' \leq E(0),$$

provided

$$E(0) = \int_{a \in \mathcal{A}} \mathcal{E}(0, a) d\mu(a),$$

which means that the Cauchy-Schwarz inequality we used saturates at time $t = 0$, i.e.

$$\int_{x \in \mathbb{T}^d} \frac{|\int_{a \in \mathcal{A}} \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(0, s, a)) \partial_s X(0, s, a) ds d\mu(a)|^2}{\int_{a \in \mathcal{A}} \int_{\mathbb{R}/\mathbb{Z}} \delta(x - X(0, s, a)) ds d\mu(a)}$$
$$= \int_{a \in \mathcal{A}} |\partial_s X(0, s, a)|^2 ds d\mu(a).$$

This essentially means, as already explained in the Introduction, that, at time $t = 0$, in the definition of the initial fields $(B, \rho)(t = 0, \cdot)$, the loops $s \to X(0, s, a)$ have been chosen without self- or mutual intersections. The resulting convex *inequality* is precisely the one we have chosen in the Introduction to define dissipative solutions for the Eulerian heat equation (3.1.1,3.1.2), through Definition 3.1.1.

3.3.1 Proof of Theorem 3.1.2

The proof has just been provided, while obtaining the concept of dissipative solutions!

3.3.2 Proof of Theorem 3.1.3

The proof is straightforward. From 3.1.5, we have

$$D_t b = (b \cdot \nabla) v, \quad D_t = (\partial_t + v \cdot \nabla), \quad v = (b \cdot \nabla) b,$$

from which we easily deduce

$$v^2 + D_t(\frac{v^2}{2}) = (b \cdot \nabla)(b \cdot v).$$

Then, it is enough to set $b^* = b$ and $v^* = v$ in definition 3.1.1, to make $L_1 = L_2 = L_3 = 0$, $R = 0$ and $E(0) = 0$. So any dissipative solution (B, ρ, P) in the sense of definition 3.1.1 must satisfy $B = \rho b$ and $P = \rho v$ since $r \geq r_0$ and $Q_{r_0} > 0$, which completes the proof.

3.4 Appendix 3.A: Direct recovery of equation (3.3.1)

Let loop $X(t, s)$ be a solution to the heat equation (3.2.3). For any smooth vector field b^*, the relative entropy

$$\mathcal{E}(t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{|\partial_s X(t, s) - b^*(t, X(t, s))|^2}{2} ds$$

can be written as

$$\mathcal{E}(t)=\int_{\mathbb{R}/\mathbb{Z}}\left(\frac{|\partial_s X|^2}{2}-\partial_s X\cdot b^*(t,X)+\frac{|b^*(t,X)|^2}{2}\right)ds=\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_3(t).$$

For $\mathcal{E}_1(t)=\int|\partial_s X|^2/2$, we have

$$\frac{d}{dt}\mathcal{E}_1(t)=\int\partial_s X\cdot\partial_{ts}X=-\int\partial_{ss}X\cdot\partial_t X=-\int|\partial_t X|^2.$$

(since $\partial_t X=\partial_{ss}X$). For any smooth vector field v^*, we have

$$-\int|\partial_t X|^2=\int-|\partial_t X-v^*(t,X)|^2+|v^*(t,X)|^2-2\partial_t X\cdot v^*(t,X)$$

$$=\int-|\partial_t X-v^*(t,X)|^2+|v^*(t,X)|^2-\partial_t X\cdot v^*(t,X)-\partial_{ss}X\cdot v^*(t,X).$$

In coordinates, we have

$$-\int\partial_{ss}X^i v_i^*(t,X)=\int\partial_s X^i\partial_j v_i^*(t,X)\partial_s X^j$$

$$=\frac{1}{2}\int\big(\partial_s X^i-b^{*i}(t,X)\big)\big(\partial_s X^j-b^{*j}(t,X)\big)(\partial_j v_i^*+\partial_i v_j^*)(t,X)$$

$$+\int\partial_s X^i b^{*j}(t,X)(\partial_j v_i^*+\partial_i v_j^*)(t,X)-\big(b^{*i}b^{*j}\partial_j v_i^*\big)(t,X).$$

So we have,

$$\frac{d}{dt}\mathcal{E}_1(t)=\int-|\partial_t X-v^*(t,X)|^2+\mathrm{L}_1'(t,X)+\partial_s X\cdot\mathrm{L}_2'(t,X)+\partial_t X\cdot\mathrm{L}_3'(t,X)$$

$$+\frac{1}{2}\int\big(\partial_s X^i-b^{*i}(t,X)\big)\big(\partial_s X^j-b^{*j}(t,X)\big)(\partial_j v_i^*+\partial_i v_j^*)(t,X),$$

where

$$\mathrm{L}_1'=|v^*|^2-b^{*i}b^{*j}\partial_j v_i^*,\quad(\mathrm{L}_2')_i=(\partial_j v_i^*+\partial_i v_j^*)b^{*j},\quad(\mathrm{L}_3')_i=-v_i^*.$$

Now let's look at $\mathcal{E}_2(t)=-\int\partial_s X\cdot b^*(t,X)$. In coordinates, we have,

$$\frac{d}{dt}\mathcal{E}_2(t)=\int-\partial_{st}X^i\cdot b_i^*(t,X)-\partial_s X^i(\partial_t b_i^*)(t,X)-\partial_s X^i\partial_t X^j(\partial_j b_i^*)(t,X)$$

$$=\int-\partial_s X^i\partial_t X^j(\partial_j b_i^*-\partial_i b_j^*)(t,X)-\partial_s X^i(\partial_t b_i^*)(t,X)$$

$$=-\int\big(\partial_s X^i-b^{*i}(t,X)\big)\big(\partial_t X^j-v^{*j}(t,X)\big)(\partial_j b_i^*-\partial_i b_j^*)(t,X)$$

$$+\int\mathrm{L}_1''(t,X)+\partial_s X\cdot\mathrm{L}_2''(t,X)+\partial_t X\cdot\mathrm{L}_3''(t,X),$$

where

$$\mathrm{L}_1''=(\partial_j b_i^*-\partial_i b_j^*)b^{*i}v^{*j},\ \ (\mathrm{L}_2'')_i=-\partial_t b_i^*-(\partial_j b_i^*-\partial_i b_j^*)v^{*j},\ \ (\mathrm{L}_3'')_i=(\partial_j b_i^*-\partial_i b_j^*)b^{*j}.$$

For the last term $\mathcal{E}_3(t) = \int |b^*(t,X)|^2/2$, we have,

$$\frac{d}{dt}\mathcal{E}_3(t) = \int \big(\partial_t b_i^*(t,X) + \partial_t X^j \partial_j b_i^*(t,X)\big) b^{*i}(t,X).$$

So, in summary, we have

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \frac{1}{2}\int \big(\partial_s X^i - b^{*i}(t,X)\big)\big(\partial_s X^j - b^{*j}(t,X)\big)(\partial_j v_i^* + \partial_i v_j^*)(t,X)\\ &- \int \big(\partial_s X^i - b^{*i}(t,X)\big)\big(\partial_t X^j - v^{*j}(t,X)\big)(\partial_j b_i^* - \partial_i b_j^*)(t,X)\\ &\int -|\partial_t X - v^*(t,X)|^2 + \mathrm{L}_1(t,X) + \partial_s X\cdot \mathrm{L}_2(t,X) + \partial_t X\cdot \mathrm{L}_3(t,X),\end{aligned}$$

where

$$\mathrm{L}_1 = \mathrm{L}_1' + \mathrm{L}_1'' + \partial_t\left(\frac{b^{*2}}{2}\right) = v^{*2} + D_t\left(\frac{b^{*2}}{2}\right) - (b^*\cdot\nabla)(b^*\cdot v^*),\quad D_t = (\partial_t + v^*\cdot\nabla)$$

$$\mathrm{L}_2 = \mathrm{L}_2' + \mathrm{L}_2'' = -D_t b^* + (b^*\cdot\nabla)v^* + \nabla(b^*\cdot v^*)$$

$$\mathrm{L}_3 = \mathrm{L}_3' + \mathrm{L}_3'' + \nabla\left(\frac{b^{*2}}{2}\right) = -v^* + (b^*\cdot\nabla)b^*.$$

Since

$$\int \partial_s X\cdot\big[\nabla(b^*\cdot v^*)\big](t,X) = \int \partial_s\big(b^*(t,X)\cdot v^*(t,X)\big) = 0,$$

we can remove the gradient term $\nabla(b^*\cdot v^*)$ from L_2 and finally get (3.3.1).

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Chapter 4

Hyperbolicity of the time-like extremal surfaces in Minkowski spaces

4.1 Introduction

In the $(1+n+m)-$dimensional Minkowski space $\mathbb{R}^{1+(n+m)}$, we consider a time-like $(1+n)-$dimensional surface (which usually corresponds to a $n-$brane in String Theory [42]), namely,

$$(t,x) \in \overline{\Omega} \subset \mathbb{R} \times \mathbb{R}^n \to X(t,x) = (X^0(t,x), \dots, X^{n+m}(t,x)) \in \mathbb{R}^{1+(n+m)},$$

where Ω is a bounded open set. This surface is called an extremal surface if X is a critical point, with respect to compactly supported perturbations in the open set Ω, of the following area functional (which is the Nambu-Goto action in the case $n=1$)

$$-\iint_\Omega \sqrt{-\det(G_{\mu\nu})}\ , \qquad G_{\mu\nu} = \eta_{MN}\partial_\mu X^M \partial_\nu X^N\ ,$$

where $M, N = 0, 1, \dots, n+m,\quad \mu, \nu = 0, 1, \dots, n$, and $\eta = (-1, 1, \dots, 1)$ denotes the Minkowski metric, while G is the induced metric on the $(1+n)-$surface by η. Here $\partial_0 = \partial_t$ and we use the convention that the sum is taken for repeated indices.

By variational principles, the Euler-Lagrange equations gives the well-known equations of extremal surfaces,

$$\partial_\mu \left(\sqrt{-G}G^{\mu\nu}\partial_\nu X^M\right) = 0, \qquad M = 0, 1, \dots, n+m, \tag{4.1.1}$$

where $G^{\mu\nu}$ is the inverse of $G_{\mu\nu}$ and $G = \det(G_{\mu\nu})$. In this chapter, we limit ourself to the case of extremal surfaces that are graphs of the form:

$$X^0 = t,\ \ X^i = x^i,\ \ i = 1, \dots, n,\ \ X^{n+\alpha} = X^{n+\alpha}(t,x),\ \ \alpha = 1, \dots, m \tag{4.1.2}$$

The main purpose of this chapter is to prove:

Theorem 4.1.1. *In the case of a graph as (4.1.2), the equations of extremal surfaces (4.1.1) can be translated into a first order symmetric hyperbolic system of PDEs, which admits the very simple form*

$$\partial_t W + \sum_{j=1}^n A_j(W)\partial_{x_j} W = 0, \quad W : (t,x) \in \mathbb{R}^{1+n} \to W(t,x) \in \mathbb{R}^{n+m+\binom{m+n}{n}}, \tag{4.1.3}$$

where each $A_j(W)$ is just a $(n+m+\binom{m+n}{n}) \times (n+m+\binom{m+n}{n})$ symmetric matrix depending linearly on W. Accordingly, this system is automatically well-posed, locally in time, in the Sobolev space $W^{s,2}$ as soon as $s > n/2+1$.

The structure of (4.1.3) is reminiscent of the celebrated prototype of all nonlinear hyperbolic PDEs, the so-called inviscid Burgers equation $\partial_t u + u\partial_x u = 0$, where u and x are both just valued in $\mathbb{R}$, with the simplest possible nonlinearity. Of course, to get such a simple structure, the relation to be found between X (valued in $\mathbb{R}^{1+n+m}$) and W (valued in $\mathbb{R}^{n+m+\binom{m+n}{n}}$) must be quite involved. Actually, it will be shown more precisely that the case of extremal surfaces corresponds to a special subset of solutions of (4.1.3) for which W lives in a very special algebraic sub-manifold of $\mathbb{R}^{n+m+\binom{m+n}{n}}$, which is preserved by the dynamics of (4.1.3).

To establish Theorem 4.1.1, the strategy of proof follows the concept of system of conservation laws with "polyconvex" entropy in the sense of Dafermos [20]. The first step is to lift the original system of conservation laws to a (much) larger one which enjoys a convex entropy rather than a polyconvex one. This strategy has been successfully applied in many situations, such as nonlinear Elastodynamis [23, 43], nonlinear Electromagnetism [10, 16, 49], just to quote few examples. In our case, the calculations will crucially start with the classical Cauchy-Binet formula.

Finally, at the end of the chapter, following the ideas recently introduced in [14], we will make a connection between our result and the theory of mean-curvature flows in the Euclidean space, in any dimension and co-dimension.

Acknowledgements

The author is very grateful to his thesis advisor, Yann Brenier, for introducing the polyconvex system to him and pointing out the possibility of augmenting this system as a hyperbolic system of conservation laws, in the spirit of [10].

4.2 Extremal surface equations for a graph

Let us first write equations (4.1.1) in the case of a graph such as (4.1.2). We denote

$$V_\alpha = \partial_t X^{n+\alpha}, \quad F_{\alpha i} = \partial_i X^{n+\alpha}, \quad \alpha = 1, \dots, m, \quad i = 1, \dots, n.$$

Then the induced metric tensor $G_{\mu\nu}$ can be written as

$$(G_{\mu\nu}) = \begin{pmatrix} -1+|V|^2 & V^T F \\ F^T V & I_n + F^T F \end{pmatrix}.$$

We can easily get that

$$G = \det(G_{\mu\nu}) = -\det(I_n + F^T F)\big(1 - |V|^2 + V^T F (I_n + F^T F)^{-1} F^T V\big).$$

So, in the case of graph, the extremal surface can be solved by varying the follow Lagrangian of the vector V and matrix F,

$$\iint L(V,F), \quad L(V,F) = -\sqrt{-G},$$

under the constraints

$$\partial_t F_{\alpha i} = \partial_i V_\alpha, \quad \partial_i F_{\alpha j} = \partial_j F_{\alpha i}, \quad \alpha = 1, \dots, m, \quad i, j = 1, \dots, n.$$

The resulting system combines the above constraints and

$$\partial_t \left(\frac{\partial L(V,F)}{\partial V_\alpha} \right) + \partial_i \left(\frac{\partial L(V,F)}{\partial F_{\alpha i}} \right) = 0.$$

Now let us denote

$$D_\alpha = \frac{\partial L(V,F)}{\partial V_\alpha} = \frac{\sqrt{\det(I_n + F^T F)}(I_m + FF^T)^{-1}_{\alpha\beta} V_\beta}{\sqrt{1 - V^T (I_m + FF^T)^{-1} V}}$$

and the energy density h by

$$h(D,F) = \sup_V D \cdot V - L(V,F) = \sqrt{\det(I_n + F^T F) + D^T (I_m + FF^T) D}.$$

We have

$$V_\alpha = \frac{\partial h(D,F)}{\partial D_\alpha} = \frac{(I_m + FF^T)_{\alpha\beta} D_\beta}{h}.$$

Therefore, the extremal surface should solve the following system for a matrix valued function $F = (F_{\alpha i})_{m \times n}$ and a vector valued function $D = (D_\alpha)_{\alpha = 1,2,\dots,m}$,

$$\partial_t F_{\alpha i} + \partial_i \left(\frac{D_\alpha + F_{\alpha j} P_j}{h} \right) = 0, \tag{4.2.1}$$

$$\partial_t D_\alpha + \partial_i \left(\frac{D_\alpha P_i + \xi'(F)_{\alpha i}}{h} \right) = 0, \tag{4.2.2}$$

$$\partial_j F_{\alpha i} = \partial_i F_{\alpha j}, \quad 1 \le i, j \le n, \ 1 \le \alpha \le m, \tag{4.2.3}$$

where

$$P_i = F_{\alpha i} D_\alpha, \quad h = \sqrt{D^2 + P^2 + \xi(F)}, \quad 1 \le i, j \le n, \ 1 \le \alpha \le m, \tag{4.2.4}$$

$$\xi(F) = \det\left(I_n + F^T F\right), \quad \xi'(F)_{\alpha i} = \frac{1}{2} \frac{\partial \xi(F)}{\partial F_{\alpha i}} = \xi(F)(I_n + F^T F)^{-1}_{ij} F_{\alpha j}. \tag{4.2.5}$$

In fact, the above equations can be obtained directly from (4.1.1). Interested readers can refer to Appendix 4.A for the details. Moreover, we can find that there are other conservation laws for the energy density h and vector P as defined in the above equations, namely, (see Appendix 4.B for the proof)

$$\partial_t h + \nabla \cdot P = 0, \tag{4.2.6}$$

$$\partial_t P_i + \partial_j \left(\frac{P_i P_j}{h} - \frac{\xi(F)(I_n + F^T F)^{-1}_{ij}}{h} \right) = 0. \tag{4.2.7}$$

Now, let's take h and P as independent variables, then we can find that the system (4.2.1),(4.2.2),(4.2.3),(4.2.6),(4.2.7) admits an additional conservation law for

$$S = \frac{D^2 + P^2 + \xi(F)}{2h},$$

namely,

$$\partial_t S + \nabla \cdot \left(\frac{SP}{h} \right) = \partial_i \left[\frac{\xi(F)(I_n + F^T F)^{-1}_{ij} (P_j - F_{\alpha j} D_\alpha)}{h^2} \right]. \tag{4.2.8}$$

4.3 Lifting of the system

4.3.1 The minors of the matrix F

In previous part, S is generally not a convex function of (h, D, P, F), but a polyconvex function of F, which means that S can be written as convex functions of the minors of F. Let's denote $r = \min\{m, n\}$. For $1 \le k \le r$, and any ordered sequences $1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_k \le m$ and $1 \le i_1 < i_2 < \ldots < i_k \le n$, let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $I = \{i_1, i_2, \ldots, i_k\}$, then the minor of F with respect to the rows $\alpha_1, \alpha_2, \ldots, \alpha_k$ and columns $i_1, i_2, \ldots, i_k$ is defined as

$$[F]_{A,I} = \det\Big((F_{\alpha_p i_q})_{p,q=1,\ldots,k}\Big).$$

For the minors $[F]_{A,I}$, let us first introduce the generalized Cauchy-Binet formula which is very convenient for us to compute the minors of the product of two matrices.

Lemma 4.3.1. (Cauchy-Binet formula) *Suppose M is a $m \times l$ matrix, N is a $l \times n$ matrix, I is a subset of $\{1, 2, \ldots, m\}$ with $k (\le l)$ elements and J is a subset of $\{1, 2, \ldots, n\}$ with k elements, then*

$$[MN]_{I,J} = \sum_{\substack{K \subseteq \{1,2,\ldots,l\} \\ |K|=k}} [M]_{I,K}[N]_{K,J}. \tag{4.3.1}$$

Now let us look at $\xi(F) = \det\big(I_n + F^T F\big)$, we can show that it is a convex function for the minors $[F]_{A,I}$. In fact, we have,

$$\xi(F) = \det\big(I_n + F^T F\big) = 1 + \sum_{k=1}^{n} \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ |I|=k}} [F^T F]_{I,I}$$

(by the Cauchy-Binet formula)

$$= 1 + \sum_{k=1}^{r} \sum_{|I|,|A|=k} [F^T]_{I,A}[F]_{A,I}.$$

So we have

$$\xi(F) = 1 + \sum_{k=1}^{r} \sum_{|A|,|I|=k} [F]^2_{A,I}. \tag{4.3.2}$$

The above equality shows us that $\xi(F)$ is a polyconvex function of F. By introducing all the minors of F as independent variables, the energy S becomes a strictly convex function of $h, D, P, [F]_{A,I}$. Now we will see that the system can be augmented as a system of conservation laws of $h, D, P, [F]_{A,I}$.

4.3.2 Conservation laws for the minors $[F]_{A,I}$

First, we will see that $[F]_{A,I}$ satisfy similar equations as (4.2.3). For simplicity, we denote $[F]_{A,I} = 1$ if $A = I = \emptyset$.

Proposition 4.3.2. *Suppose F satisfy (4.2.3), then for any $2 \le k \le r+1$, $A' = \{1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_{k-1} \le m\}$, $I = \{1 \le i_1 < i_2 < \ldots < i_k \le n\}$, we have*

$$\sum_{q=1}^{k} (-1)^q \partial_{i_q} \Big([F]_{A', I\setminus\{i_q\}} \Big) = 0. \tag{4.3.3}$$

Proof. This can be showed quite directly, for the left hand side, we have

$$\begin{aligned}
\text{Left} &= \sum_{q=1}^{k} \sum_{\substack{l<q \\ 1\le p\le k-1}} (-1)^{l+p+q} [F]_{A\setminus\{\alpha_p\}, I\setminus\{i_l, i_q\}} \partial_{i_q} F_{\alpha_p i_l} \\
&\quad + \sum_{q=1}^{k} \sum_{\substack{l>q \\ 1\le p\le k-1}} (-1)^{l-1+p+q} [F]_{A\setminus\{\alpha_p\}, I\setminus\{i_l, i_q\}} \partial_{i_q} F_{\alpha_p i_l} \\
&= \sum_{\substack{1\le l<q\le k \\ 1\le p\le k-1}} (-1)^{l+p+q} [F]_{A\setminus\{\alpha_p\}, I\setminus\{i_l, i_q\}} \Big(\partial_{i_q} F_{\alpha_p i_l} - \partial_{i_l} F_{\alpha_p i_p} \Big) \\
&= 0.
\end{aligned}$$

□

With the above proposition, we can get the conservation laws for $[F]_{A,I}$. For $A = \{1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_k \le m\}$, $I = \{1 \le i_1 < i_2 < \ldots < i_k \le n\}$, $1 \le k \le r$, we have

$$\begin{aligned}
\partial_t \big([F]_{A,I}\big) &= \sum_{p,q=1}^{k} (-1)^{p+q} [F]_{A\setminus\{\alpha_p\}, I\setminus\{i_q\}} \partial_t F_{\alpha_p i_q} \\
&= -\sum_{p,q=1}^{k} (-1)^{p+q} [F]_{A\setminus\{\alpha_p\}, I\setminus\{i_q\}} \partial_{i_q} \left(\frac{D_{\alpha_p} + F_{\alpha_p j} P_j}{h} \right) \\
&= -\sum_{p,q=1}^{k} (-1)^{p+q} \partial_{i_q} \left[\frac{[F]_{A\setminus\{\alpha_p\}, I\setminus\{i_q\}} \big(D_{\alpha_p} + F_{\alpha_p j} P_j\big)}{h} \right].
\end{aligned} \tag{4.3.4}$$

4.3.3 The augmented system

Now let us consider the energy density h, the vector field P and the minors $[F]_{A,I}$ as independent variables. The original system (4.2.1)-(4.2.3) can be augmented to the following system of conservation laws. More precisely, for $h > 0$, $D = (D_\alpha)_{\alpha=1,2,\ldots,m}$, $P = (P_i)_{i=1,2,\ldots,n}$, $M_{A,J}$ with $A \subseteq \{1,2,\ldots,m\}$, $I \subseteq \{1,2,\ldots,n\}$, $1 \le |A| = |I| \le r = \min\{m,n\}$, the system are composed of the following equations

$$\partial_t h + \nabla \cdot P = 0, \tag{4.3.5}$$

$$\partial_t D_\alpha + \partial_i \left(\frac{D_\alpha P_i}{h} \right) + \sum_{\substack{A,I,i \\ \alpha\in A, i\in I}} (-1)^{O_A(\alpha)+O_I(i)} \partial_i \left(\frac{M_{A,I} M_{A\setminus\{\alpha\}, I\setminus\{i\}}}{h} \right) = 0, \tag{4.3.6}$$

$$\begin{aligned}
\partial_t P_i + \sum_{\substack{A,I,j \\ j\in I, i\notin I\setminus\{j\}}} (-1)^{O_I(j)+O_{I\setminus\{j\}}(i)} \partial_j \left(\frac{M_{A,(I\setminus\{j\})\bigcup\{i\}} M_{A,I}}{h} \right) \\
+ \partial_j \left(\frac{P_i P_j}{h} \right) - \partial_i \left(\frac{1 + \sum_{A,I} M_{A,I}^2}{h} \right) = 0,
\end{aligned} \tag{4.3.7}$$

$$\partial_t M_{A,I} + \sum_{\substack{i,j \\ i\in I, j\notin I\setminus\{i\}}} (-1)^{O_{I\setminus\{i\}}(j)+O_I(i)}\partial_i\left(\frac{M_{A,(I\setminus\{i\})\bigcup\{j\}}P_j}{h}\right)$$

$$+ \sum_{\substack{\alpha,i \\ \alpha\in A, i\in I}} (-1)^{O_A(\alpha)+O_I(i)}\partial_i\left(\frac{M_{A\setminus\{\alpha\},I\setminus\{i\}}D_\alpha}{h}\right) = 0, \quad (4.3.8)$$

$$\sum_{i\in I}(-1)^{O_I(i)}\partial_i\Big(M_{A',I\setminus\{i\}}\Big) = 0, \quad 2 \le |I| = |A'| + 1 \le r+1. \qquad (4.3.9)$$

Here $O_A(\alpha)$ is the integer number such that α is the $O_A(\alpha)$th smallest element in $A\bigcup\{\alpha\}$. All the sum are taken in the convention that $A \subseteq \{1,\dots,m\}$, $I \subseteq \{1,\dots,n\}$, $1 \le \alpha \le m$, $1 \le i,j \le n$.

Note that there are many different ways to enlarge the original system since the equations can be written in many different ways in terms of minors. Although our above augmented system looks quite complicated, in the following part, we will show that by extending the system in this way is quite useful. Now, let's first show that the augmented system can be reduced to the original system under the algebraic constraints we abandoned we enlarge the system.

Proposition 4.3.3. *We can recover the original system* (4.2.1)-(4.2.3) *from the augmented system* (4.3.5)-(4.3.9)*under the algebraic constrains*

$$P_i = F_{\alpha i}D_\alpha, \quad h = \sqrt{D^2 + P^2 + \xi(F)}, \quad M_{A,I} = [F]_{A,I}.$$

Proof. It suffices to show the following three equalities,

$$\xi'(F)_{\alpha i} = \sum_{\substack{A,I \\ \alpha\in A, i\in I}} (-1)^{O_A(\alpha)+O_I(i)}[F]_{A,I}[F]_{A\setminus\{\alpha\},I\setminus\{i\}}, \qquad (4.3.10)$$

$$\xi(F)(I_n + F^TF)^{-1}_{ij} = (1 + \sum_{A,I}[F]^2_{A,I})\delta_{ij}$$

$$- \sum_{\substack{A,I \\ j\in I, i\notin I\setminus\{j\}}} (-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}[F]_{A,(I\setminus\{j\})\bigcup\{i\}}[F]_{A,I}, \quad (4.3.11)$$

$$\sum_{p=1}^{k}(-1)^{p+q}[F]_{A\setminus\{\alpha_p\},I\setminus\{i_q\}}F_{\alpha_p j} = \begin{cases} (-1)^{O_{(I\setminus\{i_q\})\bigcup\{j\}}(j)+q}[F]_{A,(I\setminus\{i_q\})\bigcup\{j\}} & j\notin I\setminus\{i_q\} \\ 0 & j \in I\setminus\{i_q\} \end{cases}. \qquad (4.3.12)$$

(4.3.12) is obvious because of the Laplace expansion. Now, since

$$\begin{aligned}\xi'(F)_{\alpha i} = &\ \frac{1}{2}\frac{\partial}{\partial F_{\alpha i}}\Big(1 + \sum_{A,I}[F]^2_{A,I}\Big) \ = \ \sum_{A,I}[F]_{A,I}\frac{\partial}{\partial F_{\alpha i}}\Big([F]_{A,I}\Big) \\ = &\ \sum_{\substack{A,I \\ \alpha\in A, i\in I}} (-1)^{O_A(\alpha)+O_I(i)}[F]_{A,I}[F]_{A\setminus\{\alpha\},I\setminus\{i\}},\end{aligned}$$

so (4.3.10) is true. Let's look at (4.3.11). First, we have

$$\begin{aligned}\xi(F)\delta_{ij} - \xi(F)(I_n + F^TF)^{-1}_{ij} = &\ \xi(F)F_{\alpha i}(I_m + FF^T)^{-1}_{\alpha\beta}F_{\beta j} \\ = &\ (-1)^{\alpha+\beta}F_{\alpha i}[I_m + FF^T]_{\{\alpha\}^c,\{\beta\}^c}F_{\beta j}.\end{aligned}$$

Because

$$
\begin{aligned}
&[I_m + FF^T]_{\{\alpha\}^c,\{\beta\}^c} \\
= &\sum_{k=0}^{m-1} \sum_{\substack{|A'|=k \\ \alpha,\beta\notin A'}} (-1)^{O_{A'}(\alpha)+O_{A'}(\beta)} [FF^T]_{(A'\bigcup\{\alpha\})^c,(A'\bigcup\{\beta\})^c} \\
= &\sum_{k=1}^{m} \sum_{\substack{|A|=k \\ \alpha,\beta\in A}} (-1)^{O_A(\alpha)+O_A(\beta)+\alpha+\beta} [FF^T]_{A\setminus\{\alpha\},A\setminus\{\beta\}} \\
= &\sum_{k=1}^{\min\{m,r+1\}} \sum_{\substack{|A|=k,|I'|=k-1 \\ \alpha,\beta\in A}} (-1)^{O_A(\alpha)+O_A(\beta)+\alpha+\beta} [F]_{A\setminus\{\alpha\},I'} [F]_{A\setminus\{\beta\},I'},
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\xi(F)\delta_{ij} - \xi(F)(I_n + F^TF)^{-1}_{ij} \\
= &\sum_{k=1}^{\min\{m,r+1\}} \sum_{\substack{|A|=k,|I'|=k-1 \\ \alpha,\beta\in A}} (-1)^{O_A(\alpha)+O_A(\beta)} [F]_{A\setminus\{\alpha\},I'} [F]_{A\setminus\{\beta\},I'} F_{\alpha i} F_{\beta j} \\
= &\sum_{k=1}^{r} \sum_{\substack{|A|=k,|I'|=k-1 \\ i,j\notin I'}} (-1)^{O_{I'}(i)+O_{I'}(j)} [F]_{A,I'\bigcup\{i\}} [F]_{A,I'\bigcup\{j\}} \\
= &\sum_{\substack{A,I \\ j\in I, i\notin I\setminus\{j\}}} (-1)^{O_I(j)+O_{I\setminus\{j\}}(i)} [F]_{A,(I\setminus\{j\})\bigcup\{i\}} [F]_{A,I}.
\end{aligned}
$$

□

Now we can show that the augmented system have a convex entropy.

Proposition 4.3.4. *The system* (4.3.5)-(4.3.9) *satisfies an additional conservation law for*

$$S(h, D, P, M_{A,I}) = \frac{1 + D^2 + P^2 + \sum_{A,I} M_{A,I}^2}{2h}.$$

More precisely, we have

$$
\begin{aligned}
&\partial_t S + \nabla\cdot\left(\frac{SP}{h}\right) + \sum_{\substack{A,I,i \\ \alpha\in A, i\in I}} (-1)^{O_A(\alpha)+O_I(i)} \partial_i \left(\frac{D_\alpha M_{A\setminus\{\alpha\},I\setminus\{i\}} M_{A,I}}{h^2}\right) \\
+ &\sum_{\substack{A,I,j \\ j\in I, i\notin I\setminus\{j\}}} (-1)^{O_I(j)+O_{I\setminus\{j\}}(i)} \partial_j \left(\frac{P_i M_{A,(I\setminus\{j\})\bigcup\{i\}} M_{A,I}}{h^2}\right) - \partial_j\left(\frac{P_j(1+M_{A,I}^2)}{h^2}\right) = 0.
\end{aligned}
\tag{4.3.13}
$$

We leave the proof in Appendix 4.C.

Remark 4.3.5. *There are many possible ways to augment the original system because of the different ways to write a function of minors. To find the write way to express the equation* (4.2.2) *and* (4.2.7) *such that it has a convex entropy* S *is somehow a little technical.*

4.4 Properties of the augmented system

4.4.1 Propagation speeds and characteristic fields

Let's look at the special case $n = 1$, where our extremal surface is just a relativistic string. In this case, the augmented system coincides with the system of h, P, D, F, where the P is a scalar function and $F = (F_\alpha)_{\alpha=1,\ldots,m}$ becomes a vector. More precisely, the equations in the case $n = 1$ are,

$$\partial_t h + \partial_x P = 0, \quad \partial_t F_{\alpha i} + \partial_x \left(\frac{D_\alpha + F_\alpha P}{h} \right) = 0,$$

$$\partial_t P + \partial_x \left(\frac{P^2 - 1}{h} \right) = 0, \quad \partial_t D_\alpha + \partial_x \left(\frac{D_\alpha P + F_\alpha}{h} \right) = 0.$$

Let us denote $U = (h, P, D_\alpha, F_\alpha)$ then, the system can be written as

$$\partial_t U + A(U) \partial_x U = 0,$$

where

$$A(U) = \frac{1}{h} \begin{pmatrix} 0 & h & 0 & 0 \\ \frac{1-P^2}{h} & 2P & 0 & 0 \\ -\frac{PD+F}{h} & D & PI_m & I_m \\ -\frac{D+PF}{h} & F & I_m & PI_m \end{pmatrix}.$$

We can find that, the propagation speeds are

$$\lambda_+ = \frac{P+1}{h}, \quad \lambda_- = \frac{P-1}{h},$$

with each of them having multiplicity $m + 1$. The characteristic field for λ_+ is composed of

$$v_+^0 = (h, P + 1, D, F), \quad v_+^i = (0, 0, e_i, e_i), \quad i = 1, \ldots, m.$$

Here e_i is the base of $\mathbb{R}^m$. The characteristic field for λ_- is composed of

$$v_-^0 = (h, P - 1, D, F), \quad v_-^i = (0, 0, e_i, -e_i), \quad i = 1, \ldots, m.$$

We can easily check that

$$\frac{\partial \lambda_+(U)}{\partial U} \cdot v_+^i(U) = 0, \quad \frac{\partial \lambda_-(U)}{\partial U} \cdot v_-^i(U) = 0, \quad i = 0, 1, \ldots, m.$$

So the augmented system is linearly degenerate in the sense of the theory of hyperbolic conservation laws [20].

4.4.2 Non-conservative form

Now let's look at the non-conservative form of the augmented system (4.3.5)-(4.3.9). We denote

$$\tau = \frac{1}{h}, \quad d = \frac{D}{h}, \quad v = \frac{P}{h}, \quad m_{A,I} = \frac{M_{A,I}}{h}.$$

For simplicity, we denote $m_{A,I} = \tau$ if $A = I = \emptyset$. We have the following proposition.

Proposition 4.4.1. *Suppose* $(h, D, P, M_{A,I})$ *is a smooth solution of* (4.3.5)-(4.3.9)*, then* $(\tau, d, v, m_{A,I})$ *is the solution of the following symmetric hyperbolic system,*

$$\partial_t \tau + v_j \partial_j \tau - \tau \partial_j v_j = 0, \tag{4.4.1}$$

$$\partial_t d_\alpha + v_i \partial_i d_\alpha + \sum_{A,I,i} \mathbb{1}_{\{\alpha \in A, i \in I\}} (-1)^{O_A(\alpha) + O_I(i)} m_{A \setminus \{\alpha\}, I \setminus \{i\}} \partial_i m_{A,I} = 0, \tag{4.4.2}$$

$$\begin{aligned} \partial_t v_i + \sum_{A,I,j} \mathbb{1}_{\{j \in I, i \notin I \setminus \{j\}\}} (-1)^{O_I(j) + O_{I \setminus \{j\}}(i)} m_{A,(I \setminus \{j\}) \bigcup \{i\}} \partial_j m_{A,I} \\ - \sum_{A,I} m_{A,I} \partial_i m_{A,I} + v_j \partial_j v_i - \tau \partial_i \tau = 0, \end{aligned} \tag{4.4.3}$$

$$\begin{aligned} \partial_t m_{A,I} + v_j \partial_j m_{A,I} + \sum_{i,j} \mathbb{1}_{\{i \in I, j \notin I \setminus \{i\}\}} (-1)^{O_{I \setminus \{i\}}(j) + O_I(i)} m_{A,(I \setminus \{i\}) \bigcup \{j\}} \partial_i v_j \\ - m_{A,I} \partial_j v_j + \sum_{\alpha,i} \mathbb{1}_{\{\alpha \in A, i \in I\}} (-1)^{O_A(\alpha) + O_I(i)} m_{A \setminus \{\alpha\}, I \setminus \{i\}} \partial_i d_\alpha = 0. \end{aligned} \tag{4.4.4}$$

We can prove the above proposition by just using (4.3.9). It is easy to verify that this system is symmetric. If we set $W = (\tau, d_\alpha, v_i, m_{A,I}) \in \mathbb{R}^{n+m+\binom{m+n}{n}}$, then the equations can be written as

$$\partial_t W + \sum_j A_j(W) \partial_j W = 0,$$

where $A_j(W)$ is a symmetric matrix, and more surprisingly, it is a linear function of W. This is exactly the form (4.1.3) announced in the introduction. Notice that this system does not require any restriction on the range of W! In particular the variable τ may admit positive, negative or null values. This is a very remarkable situation, if we compare with more classical nonlinear hyperbolic systems, such as the Euler equations of gas dynamics (where typically τ should admit only positive values).

Now let us prove that the two system are equivalent when initial data satisfies (4.3.9)

Proposition 4.4.2. *Suppose the initial data for* (4.4.1)-(4.4.4) *satisfies* (4.3.9)*, i.e.,*

$$\sum_{i \in I} (-1)^{O_I(i)} \partial_i \Big(\tau^{-1} m_{A', I \setminus \{i\}} \Big) = 0, \quad 2 \le |I| = |A'| + 1 \le r + 1,$$

then the corresponding smooth solutions satisfy (4.3.5)-(4.3.9).

Proof. We only need to proof that the smooth solutions always satisfy (4.3.9) provided that initial data satisfies it. For $2 \le |I| = |A'| + 1 \le r + 1$, let us denote

$$\sigma_{A',I} = \sum_{i \in I} (-1)^{O_I(i)} \partial_i \Big(\tau^{-1} m_{A', I \setminus \{i\}} \Big) = 0,$$

then by (4.4.1),(4.4.4), we have

$$\begin{aligned}
\partial_t\sigma_{A',I} = & \sum_{i\in I}(-1)^{O_I(i)}\partial_i\Big(\tau^{-1}\partial_t m_{A',I\backslash\{i\}} - \tau^{-2}m_{A',I\backslash\{i\}}\partial_t\tau\Big) \\
= & -v_j\partial_j\sigma_{A',I} + \sum_{\substack{\alpha,i \\ \alpha\in A',i\in I}}(-1)^{O_{A'}(\alpha)+O_I(i)}\sigma_{A'\backslash\{\alpha\},I\backslash\{i\}}\partial_i d_\alpha \\
& - \sum_{\substack{i,j \\ i\in I,j\notin I\backslash\{i\}}}(-1)^{O_{I\backslash\{i\}}(j)+O_I(i)}\sigma_{A',(I\backslash\{i\})\bigcup\{j\}}\partial_i v_j.
\end{aligned}$$

Then we have the following estimate,

$$\partial_t\sum_{A',I}\int\sigma^2_{A',I} \le C(\|\nabla v\|_\infty,\|\nabla d\|_\infty)\Big(\sum_{A',I}\int\sigma^2_{A',I}\Big).$$

Since the initial data $\sigma_{A',I}(0)=0$, then by Gronwall's lemma, we have $\sigma_{A',I}\equiv 0$. With these equalities, it is easy to prove the statement just by doing the reverse computation as in the previous proposition.

□

Now let us look at the connection with the original system. It is obvious that the non-conservative form of the augmented system is symmetric, thus, the initial value problem is at least locally well-posed. But for the original system, this kind of property is not obvious. However, we can show that, the augmented system is equivalent to the original system if the initial value satisfy the following constraints

$$P_i = F_{\alpha i}D_\alpha,\quad h=\sqrt{D^2+P^2+\xi(F)},\quad M_{A,I}=[F]_{A,I}, \tag{4.4.5}$$

or, in the non-conservative form,

$$\tau v_i = m_{\alpha i}d_\alpha,\quad 1=d^2_\alpha+v^2_i+\tau^2+m^2_{A,I},\quad m_{A,I}=\tau[F]_{A,I}. \tag{4.4.6}$$

Now let us denote

$$\lambda=\frac{1}{2}(\tau^2+v_i^2+d_\alpha^2+m^2_{A,I}-1),\quad \omega_i=\tau v_i - m_{\alpha i}d_\alpha,$$

$$\varphi^\alpha_{A,I}=\sum_{i\in I}(-1)^{O_A(\alpha)+O_I(i)}m_{A,I\backslash\{i\}}m_{\alpha i} - \mathbb{1}_{\{\alpha\notin A\}}\tau m_{A\bigcup\{\alpha\},I},$$

$$\psi^i_{A,I}=\sum_{\alpha\in A}(-1)^{O_A(\alpha)+O_I(i)}m_{A\backslash\{\alpha\},I}m_{\alpha i} - \mathbb{1}_{\{i\notin I\}}\tau m_{A,I\bigcup\{i\}}.$$

It is obvious that $(\tau, v_i, d_\alpha, m_{A,I})$ satisfy the above constraints (4.4.6) if and only if λ, ω_i, $\varphi^\alpha_{A,I}$,$\psi^i_{A,I}$ vanish for all possible choice of A, I, α, i. Furthermore, we can show that the algebraic constraints (4.4.6) are preserved by the non-conservative system (4.4.1)-(4.4.4). First, we have the following lemma.

Lemma 4.4.3. *If* $(\tau, v_i, d_\alpha, m_{A,I})$ *solves the non-conservative system* (4.4.1)-(4.4.4), *then* λ, ω_i, $\varphi^\alpha_{A,I}$,$\psi^i_{A,I}$ *as defined above satisfy the following equalities,*

$$\partial_t\omega_i=\omega_i\partial_j v_j-\omega_j\partial_i v_j - v_j\partial_j\omega_i+\tau\partial_i\lambda+\sum_{A,I,j}\mathbb{1}_{\{j\in I\}}(-1)^{O_I(j)+O_{I\backslash\{j\}}(i)}\partial_j m_{A,I}\psi^i_{A,I\backslash\{j\}}, \tag{4.4.7}$$

$$\partial_t\lambda = -v_j\partial_j\lambda + \tau\partial_i\omega_i + \sum_{A,|I'|\geq 2,i,j} \mathbb{1}_{\{i,j\in I'\}}(-1)^{O_{I'}(i)+O_{I'}(j)} m_{A,I'\backslash\{j\}}\partial_j\left(\frac{m_{A,I'\backslash\{i\}}\omega_i}{\tau}\right)$$
$$+ \sum_{A',|I|\geq 2,\alpha,i} \mathbb{1}_{\{i\in I\}}(-1)^{O_{A'}(\alpha)+O_I(i)} m_{A',I\backslash\{i\}}\partial_i\left(\frac{\varphi^\alpha_{A',I}d_\alpha}{\tau}\right), \quad (4.4.8)$$

$$\partial_t\varphi^\alpha_{A,I} = 2\varphi^\alpha_{A,I}\partial_j v_j - v_j\partial_j\varphi^\alpha_{A,I} - \sum_{j,k}\mathbb{1}_{\{j\in I,k\notin I\backslash\{j\}\}}(-1)^{O_{I\backslash\{j\}}(k)+O_I(j)}\varphi^\alpha_{A,(I\backslash\{j\})\bigcup\{k\}}\partial_j v_k$$
$$+\sum_{\beta,j}\mathbb{1}_{\{\beta\in A,j\in I\}}(-1)^{O_A(\alpha)+O_A(\beta)+O_{A\backslash\{\beta\}}(\alpha)+O_I(j)}\varphi^\alpha_{A\backslash\{\beta\},I\backslash\{j\}}\partial_j d_\beta, \quad (4.4.9)$$

$$\partial_t\psi^i_{A,I} = 2\psi^i_{A,I}\partial_j v_j - v_j\partial_j\psi^i_{A,I} - \sum_k(-1)^{O_I(i)+O_I(k)}\psi^k_{A,I}\partial_i v_k$$
$$+ \sum_{\beta\in A,j\in I}(-1)^{O_A(\beta)+O_I(j)+O_I(i)+O_{I\backslash\{j\}}(i)}\psi^i_{A\backslash\{\beta\},I\backslash\{j\}}\partial_j d_\beta$$
$$-\sum_{j,k}\mathbb{1}_{\{j\in I,k\notin I\backslash\{j\}\}}(-1)^{O_I(i)+O_I(j)+O_{I\backslash\{j\}}(k)+O_{(I\backslash\{j\})\bigcup\{k\}}(i)}\psi^i_{A,(I\backslash\{j\})\bigcup\{k\}}\partial_j v_k. \quad (4.4.10)$$

The proof of this lemma requires very lengthy and tedious computation. Interesting readers can refer to Appendix 4.D for the details of the proof. By the above lemma, we can show that the algebraic constraints are preserved. We summarise our result in the following proposition.

Proposition 4.4.4. *Supposed $(\tau, v_i, d_\alpha, m_{A,I})$ is a solution to the non-conservative equations* (4.4.1)-(4.4.4) *and the initial data satisfies the constraints*

$$\tau v_i = m_{\alpha i}d_\alpha, \quad 1 = d_\alpha^2 + v_i^2 + \tau^2 + m_{A,I}^2, \quad m_{A,I} = \tau[F]_{A,I},$$

where $F_{\alpha i} = \tau^{-1}m_{\alpha i}$, then the above constraints are always satisfied.

Proof. Let us denote

$$\lambda = \frac{1}{2}(\tau^2 + v_i^2 + d_\alpha^2 + m_{A,I}^2 - 1), \quad \omega_i = \tau v_i - m_{\alpha i}d_\alpha,$$

$$\varphi^\alpha_{A,I} = \sum_{i\in I}(-1)^{O_A(\alpha)+O_I(i)}m_{A,I\backslash\{i\}}m_{\alpha i} - \mathbb{1}_{\{\alpha\notin A\}}\tau m_{A\bigcup\{\alpha\},I},$$

$$\psi^i_{A,I} = \sum_{\alpha\in A}(-1)^{O_A(\alpha)+O_I(i)}m_{A\backslash\{\alpha\},I}m_{\alpha i} - \mathbb{1}_{\{i\notin I\}}\tau m_{A,I\bigcup\{i\}}.$$

It is enough to show that λ, ω_i, $\varphi^\alpha_{A,I}$,$\psi^i_{A,I}$ always vanish. Since $\varphi^\alpha_{A,I}$,$\psi^i_{A,I}$ satisfy (4.4.9) and (4.4.10), which are linear symmetric system of PDEs when we see $(\tau, v_i, d_\alpha, m_{A,I})$ as fixed functions. It is easy to know that 0 is the unique solution when initial data is 0. So we get $\varphi^\alpha_{A,I} = \psi^i_{A,I} = 0$ for all possible choice of A, I, α, i. Therefore, we know that $m_{A,I} = \tau[F]_{A,I}$, where $F_{\alpha i} = \tau^{-1}m_{\alpha i}$. So, by (4.4.7) and (4.4.8), we know that λ, ω_i solves the following linear system of PDEs

$$D_t\lambda = \tau Z_{ij}\partial_j\omega_i + f_i\omega_i \quad (4.4.11)$$

$$D_t\omega_i = \tau\partial_i\lambda + c_{ij}\omega_j \quad (4.4.12)$$

where $D_t = \partial_t + v \cdot \nabla$, $Z_{ij} = \xi(F)(I_n + F^T F)^{-1}_{ij}$ is a positive definite matrix, $f_i = \partial_j Z_{ij}$, $c_{ij} = \delta_{ij} \nabla \cdot v - \partial_i v_j$. This system is of hyperbolic type and looks very like the acoustic waves. Now, since Z_{ij} is positive definite, we can find a positive definite matrix Q such that $Z = Q^2$. Now we do the change of variable $\widetilde{\omega}_i = Q_{ij}\omega_j$, then λ, $\widetilde{\omega}_i$ should solve the following linear symmetric system of PDEs

$$D_t \lambda = \tau Q_{ij} \partial_j \widetilde{\omega}_i + \widetilde{f}_i \widetilde{\omega}_i \tag{4.4.13}$$

$$D_t \widetilde{\omega}_i = \tau Q_{ij} \partial_j \lambda + \widetilde{c}_{ij} \widetilde{\omega}_j \tag{4.4.14}$$

where

$$\widetilde{f}_i = \tau Z_{jk} \partial_k Q^{-1}_{ij} + Q^{-1}_{ij} f_j, \quad \widetilde{c}_{ij} = (D_t Q_{ik} + Q_{il} c_{lk}) Q^{-1}_{kj}$$

By the standard method of analysis of PDEs, it is easy to know that this linear symmetric system has a unique solution. Since $\lambda = \widetilde{\omega}_i = 0$ at $t = 0$, we have $\lambda \equiv \widetilde{\omega}_i \equiv 0$. So we have $\omega_i \equiv 0$, which completes the proof.

□

4.5 Toward mean curvature motions in the Euclidean space

We conclude this chapter by explaining how mean curvature motions in the Euclidean space are related to our study of extremal surfaces in the Minkowski space. This can be done very simply by the elementary quadratic change of time $\theta = t^2/2$ in the extremal surface equations (4.1.1). Let us work in the case where $X^0(t, x) = t$. We do the change of coordinate $\theta = t^2/2$, and in the new coordinate system, the extremal surface is denoted by $X^M(\theta, x)$. The chain rule tells us

$$\partial_t X^0 \equiv 1, \quad \partial_t X^M = \theta' \partial_\theta X^M, \quad M = 1, \dots, m+n.$$

Now for fixed θ, the slice of $X(\theta, x) = (X^1(\theta, x), \dots, X^{m+n}(\theta, x))$ is a n dimensional manifold Σ in $\mathbb{R}^{m+n}$. Let us denote the induced metric on Σ by $g_{ij} = \langle \partial_i X, \partial_j X \rangle$, $i, j = 1, \dots, n$. Denote $g = \det g_{ij}$, g^{ij} the inverse of g_{ij}. Then we can get that

$$G_{00} = -1 + \theta'^2 |\partial_\theta X|^2, \; G_{0i} = G_{i0} = \theta' \langle \partial_\theta X, \partial_i X \rangle = \theta' h_i, \; G_{ij} = g_{ij},$$

$$G = \det \begin{pmatrix} -1 + \theta'^2 |\partial_\theta X|^2 & \theta' h_j \\ \theta' h_i & g_{ij} \end{pmatrix} = -\left[1 + 2\theta \left(h_i h_j g^{ij} - |\partial_\theta X|^2\right)\right] g,$$

$$\sqrt{-G} = \sqrt{g}\left[1 + \theta\left(h_i h_j g^{ij} - |\partial_\theta X|^2\right) + \mathcal{O}(\theta^2)\right],$$

$$G^{00} = -1 + 2\theta\left(h_i h_j g^{ij} - |\partial_\theta X|^2\right) + \mathcal{O}(\theta^2), \; G^{0i} = G^{i0} = \theta' g^{ij} h_j + \mathcal{O}(\theta), \; G^{ij} = g^{ij} + \mathcal{O}(\theta).$$

Therefore, (4.1.1) can be rewritten as

$$\begin{aligned} 0 &= \partial_t\left(\sqrt{-G}G^{00}\right) + \partial_i\left(\sqrt{-G}G^{i0}\right) \\ &= \theta'\left[-\partial_\theta\left(\sqrt{g}\right) + \sqrt{g}\left(h_i h_j g^{ij} - |\partial_\theta X|^2\right) + \partial_i\left(\sqrt{g} g^{ij} h_j\right)\right] + \mathcal{O}(\theta), \end{aligned}$$

$$\begin{aligned} 0 &= \partial_t\left(\sqrt{-G}G^{00}\partial_t X^M\right) + \partial_i\left(\sqrt{-G}G^{i0}\partial_t X^M\right) + \partial_i\left(\sqrt{-G}G^{ij}\partial_j X^M\right) \\ &= -\partial_t\left(\theta'\sqrt{g}\partial_\theta X^M\right) + \partial_i\left(\theta'^2\sqrt{g} g^{ij} h_j \partial_\theta X^M\right) + \partial_i\left(\sqrt{g} g^{ij}\partial_j X^M\right) + \mathcal{O}(\theta) \\ &= -\sqrt{g}\partial_\theta X^M + \partial_i\left(\sqrt{g} g^{ij}\partial_j X^M\right) + \mathcal{O}(\theta). \end{aligned}$$

In the regime $\theta \ll 1$, we have the following equations

$$\partial_\theta\left(\sqrt{g}\right) + \sqrt{g}|\partial_\theta X|^2 = \partial_i\left(\sqrt{g}g^{ij}h_j\right) + \sqrt{g}h_ih_jg^{ij}, \tag{4.5.1}$$

$$\partial_\theta X^M = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j X^M\right), \qquad M = 1, \ldots, m+n. \tag{4.5.2}$$

(4.5.2) is exactly the equation for the n dimensional mean curvature flow in $\mathbb{R}^{m+n}$, and (4.5.1) is just a consequence of (4.5.2).

Remark 4.5.1. *It can be easily shown that* (4.5.2) *is equivalent to the following equation*

$$\partial_\theta X^M = g^{ij}\partial_{ij}X^M - g^{ij}g^{kl}\partial_k X^M \partial_l X^N \partial_{ij}X_N. \tag{4.5.3}$$

Therefore,

$$\begin{aligned} h_i = \partial_\theta X^M \partial_i X_M &= \left(g^{jk}\partial_{jk}X^M - g^{jk}g^{lm}\partial_l X^M \partial_m X^N \partial_{jk}X_N\right)\partial_i X_M \\ &= g^{jk}\partial_{jk}X^M\partial_i X_M - g^{jk}g^{lm}g_{il}\partial_m X^N \partial_{jk}X_N \\ &= 0. \end{aligned}$$

As a consequence, we have

$$\begin{aligned} \partial_\theta\left(\sqrt{g}\right) &= \frac{1}{\sqrt{g}}gg^{ij}\partial_{i\theta}X^M\partial_j X_M \\ &= \partial_i(\sqrt{g}g^{ij}\partial_\theta X^M \partial_j X_M) - \partial_\theta X^M \partial_i\left(\sqrt{g}g^{ij}\partial_j X_M\right) \\ &= -\sqrt{g}|\partial_\theta X|^2, \end{aligned}$$

which is exactly (4.5.1) *since* $h_i = 0$, $i = 1, \ldots, n$.

Therefore, we may expect to perform for the mean-curvature flow the same type of analysis we did for the extremal surfaces, which we intend to do in a future work.

4.6 Appendix 4.A: Direct recovery of the equations for a graph

Let us denote

$$F^\alpha_i = \partial_i X^{p+\alpha}, \;\; V^\alpha = \partial_t X^{p+\alpha}, \;\; \xi_{ij} = \delta_{ij} + F^\alpha_i F_{\alpha j}, \;\; \zeta_{\alpha\beta} = \delta_{\alpha\beta} + F_\alpha{}^i F_{\beta i},$$

and let $\xi^{ij}, \zeta^{\alpha\beta}$ be respectively the inverse of $\xi_{ij}, \zeta_{\alpha\beta}$, $\xi = \det \xi_{ij} = \det \zeta_{\alpha\beta}$, $i, j = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, m$. Since $\xi_{ij}F_\alpha{}^j = F_{\alpha i} + F^\beta_i F_{\beta j}F_\alpha{}^j = F^\beta_i \zeta_{\beta\alpha}$, we have $\xi^{ij}F_{\alpha j} = \zeta^{\alpha\beta}F_{\beta i}$. By using the above notations, the induced metric $G_{\mu\nu}$ has the following expression,

$$(G_{\mu\nu}) = \begin{pmatrix} -1+|V|^2 & F^\alpha_j V_\alpha \\ F^\alpha_i V_\alpha & \xi_{ij} \end{pmatrix}, \; G = -\xi\left(1 - |V|^2 + \xi^{ij}F^\alpha_i F^\beta_j V_\alpha V_\beta\right) = -\xi\left(1 - \zeta^{\alpha\beta}V_\alpha V_\beta\right),$$

$$(G^{\mu\nu}) = G^{-1}\xi \begin{pmatrix} 1 & -\zeta^{\alpha\beta}V_\alpha F_{\beta j} \\ -\zeta^{\alpha\beta}V_\alpha F_{\beta i} & (-1+|V|^2)\xi^{ij} + (\xi^{ik}\xi^{jl} - \xi^{ij}\xi^{kl})F^\alpha_k F^\beta_l V_\alpha V_\beta \end{pmatrix}.$$

Now let's start looking at the equation (4.1.1). The equation for $X^i, i = 1, \ldots, n$, reads

$$\partial_t\left(\frac{\sqrt{\xi}\zeta^{\alpha\beta}V_\alpha F_{\beta i}}{\sqrt{1-\zeta^{\alpha\beta}V_\alpha V_\beta}}\right) - \partial_j\left\{\frac{\sqrt{\xi}\left[(-1+|V|^2)\xi^{ij} + (\xi^{ik}\xi^{jl} - \xi^{ij}\xi^{kl})F^\alpha_k F^\beta_l V_\alpha V_\beta\right]}{\sqrt{1-\zeta^{\alpha\beta}V_\alpha V_\beta}}\right\} = 0.$$

We denote

$$D^\alpha = \frac{-\sqrt{\xi}\zeta^{\alpha\beta}V_\beta}{\sqrt{1-\zeta^{\beta\gamma}V_\beta V_\gamma}}, \quad P_i = f^\alpha_{\ i} D_\alpha, \quad h = \frac{\sqrt{\xi}}{\sqrt{1-\zeta^{\alpha\beta}V_\alpha V_\beta}},$$

then we have

$$V^\alpha = \frac{-D^\alpha - F^\alpha_{\ i}P^i}{\sqrt{\xi+|D|^2+|P|^2}}, \quad h = \sqrt{\xi+|D|^2+|P|^2}.$$

Therefore, the equation can be rewritten as

$$\partial_t P_i + \partial_j \left(\frac{P_i P_j - \xi\xi^{ij}}{h} \right) = 0. \tag{4.6.1}$$

The equation for $X^0 = t$ reads,

$$-\partial_t \left(\frac{\sqrt{\xi}}{\sqrt{1-\zeta^{\alpha\beta}V_\alpha V_\beta}} \right) + \partial_j \left(\frac{\sqrt{\xi}\zeta^{\alpha\beta}V_\alpha F_{\beta j}}{\sqrt{1-\zeta^{\alpha\beta}V_\alpha V_\beta}} \right) = 0,$$

which can be rewritten by using our new notations as

$$\partial_t h + \partial_j P_j = 0. \tag{4.6.2}$$

The equation for $X^{p+\alpha}$, $\alpha = 1, \dots, m$, reads,

$$\begin{aligned} -\partial_t \left(\frac{\sqrt{\xi}(V_\alpha - \zeta^{\beta\gamma}F_\alpha^{\ j}F_{\beta j}V_\gamma)}{\sqrt{1-\zeta^{\alpha\beta}V_\alpha V_\beta}} \right) + \partial_j \left(\frac{\sqrt{\xi}(\zeta^{\beta\gamma}F_\beta^{\ j}V_\gamma V_\alpha - F_{\alpha i}\xi^{ik}F^\beta_{\ k}V_\beta\xi^{jl}F^\gamma_{\ l}V_\gamma)}{\sqrt{1-\zeta^{\alpha\beta}V_\alpha V_\beta}} \right) \\ + \partial_j \left(\sqrt{\xi}\sqrt{1-\zeta^{\alpha\beta}V_\alpha V_\beta}\xi^{ij}F_{\alpha i} \right) = 0, \end{aligned}$$

which can be rewritten as

$$\partial_t D_\alpha + \partial_j \left(\frac{D_\alpha P_j + \xi\xi^{ij}F_{\alpha i}}{h} \right) = 0. \tag{4.6.3}$$

At last, since $\partial_t F_{\alpha i} = \partial_i V_\alpha$, $\partial_i F_{\alpha j} = \partial_j F_{\alpha i}$, we have

$$\partial_t F_{\alpha i} + \partial_i \left(\frac{D_\alpha + F_{\alpha j}P^j}{h} \right) = 0, \quad \partial_i F_{\alpha j} = \partial_j F_{\alpha i}. \tag{4.6.4}$$

(4.6.1)-(4.6.4) are just the equations that we propose.

4.7 Appendix 4.B: Conservation laws for h and P

First, let's prove the equation (4.2.6). Quite directly, we have

$$\begin{aligned} \partial_t h &= \frac{1}{2h}\left(2D_\alpha\partial_t D_\alpha + 2P_i\partial_t P_i + \frac{\partial\xi(F)}{\partial F_{\alpha i}}\partial_t F_{\alpha i} \right) \\ &= \frac{D_\alpha\partial_t D_\alpha + P_i\partial_t(F_{\alpha i}D_\alpha) + \xi'(F)_{\alpha i}\partial_t F_{\alpha i}}{h} \end{aligned}$$

$$= \left(\frac{D_\alpha + F_{\alpha i}P_i}{h}\right)\partial_t D_\alpha + \left(\frac{D_\alpha P_i + \xi'(F)_{\alpha i}}{h}\right)\partial_t F_{\alpha i}$$

$$= -\left(\frac{D_\alpha + F_{\alpha j}P_j}{h}\right)\partial_i\left(\frac{D_\alpha P_i + \xi'(F)_{\alpha i}}{h}\right) - \left(\frac{D_\alpha P_i + \xi'(F)_{\alpha i}}{h}\right)\partial_i\left(\frac{D_\alpha + F_{\alpha j}P_j}{h}\right)$$

$$= -\partial_i\left(\frac{(D_\alpha + F_{\alpha j}P_j)(D_\alpha P_i + \xi'(F)_{\alpha i})}{h^2}\right).$$

Now, since

$$\xi'(F)_{\alpha i}(D_\alpha + F_{\alpha j}P_j) = \xi(F)(I + F^TF)^{-1}_{ik}F_{\alpha k}(D_\alpha + F_{\alpha j}P_j)$$

$$= \xi(F)(I + F^TF)^{-1}_{ik}(P_k + F_{\alpha k}F_{\alpha j}P_j)$$

$$= \xi(F)(I + F^TF)^{-1}_{ik}(I + F^TF)_{kj}P_j = \xi(F)\delta_{ij}P_j = \xi(F)P_i,$$

we have

$$\partial_t h = -\partial_i\left(\frac{(D^2 + P^2 + \xi(F))P_i}{h^2}\right) = -\partial_i P_i,$$

which is just the conservation law for h. Now, let's look at the equation for $P_i = F_{\alpha i}D_\alpha$. We have

$$\partial_t P_i = \partial_t(F_{\alpha i}D_\alpha) = D_\alpha\partial_t F_{\alpha i} + F_{\alpha i}\partial_t D_\alpha.$$

For the first term, we have

$$D_\alpha\partial_t F_{\alpha i} = -D_\alpha\partial_i\left(\frac{D_\alpha + F_{\alpha j}P_j}{h}\right)$$

$$= -\partial_i\left(\frac{D_\alpha^2 + D_\alpha F_{\alpha j}P_j}{h}\right) + \frac{D_\alpha\partial_i D_\alpha + P_j F_{\alpha j}\partial_i D_\alpha}{h}$$

$$= -\partial_i\left(\frac{D^2 + P^2}{h}\right) + \frac{D_\alpha\partial_i D_\alpha + P_j\partial_i(F_{\alpha j}D_\alpha) + \xi'(F)_{\alpha j}\partial_i F_{\alpha j}}{h} - \frac{P_j D_\alpha\partial_i F_{\alpha j} + \xi'(F)_{\alpha j}\partial_i F_{\alpha j}}{h}$$

$$= -\partial_i\left(\frac{h^2 - \xi(F)}{h}\right) + \partial_i h - \left(\frac{D_\alpha P_j + \xi'(F)_{\alpha j}}{h}\right)\partial_i F_{\alpha j}$$

$$= \partial_i\left(\frac{\xi(F)}{h}\right) - \left(\frac{D_\alpha P_j + \xi'(F)_{\alpha j}}{h}\right)\partial_i F_{\alpha j}.$$

The second term reads

$$F_{\alpha i}\partial_t D_\alpha = -F_{\alpha i}\partial_j\left(\frac{D_\alpha P_j + \xi'(F)_{\alpha j}}{h}\right)$$

$$= -\partial_j\left(\frac{(F_{\alpha i}D_\alpha)P_j + F_{\alpha i}\xi'(F)_{\alpha j}}{h}\right) + \left(\frac{D_\alpha P_j + \xi'(F)_{\alpha j}}{h}\right)\partial_j F_{\alpha i}$$

$$= -\partial_j\left(\frac{P_iP_j + F_{\alpha i}\xi'(F)_{\alpha j}}{h}\right) + \left(\frac{D_\alpha P_j + \xi'(F)_{\alpha j}}{h}\right)\partial_i F_{\alpha j}.$$

Thus, we have

$$\partial_t P_i = -\partial_j\left(\frac{P_iP_j + F_{\alpha i}\xi'(F)_{\alpha j} - \xi(F)\delta_{ij}}{h}\right).$$

Now, since

$$(I + F^TF)^{-1}_{ik}(\delta_{jk} + F_{\alpha j}F_{\alpha k}) = \delta_{ij},$$

then we have

$$F_{\alpha i}\xi'(F)_{\alpha j} - \xi(F)\delta_{ij} = \xi(F)((I+F^TF)^{-1}_{ik}F_{\alpha j}F_{\alpha k} - \delta_{ij}) = -\xi(F)(I+F^TF)^{-1}_{ij}.$$

So the conservation laws for vector P can be written as

$$\partial_t P_i + \partial_j\left(\frac{P_iP_j}{h} - \frac{\xi(F)(I+F^TF)^{-1}_{ij}}{h}\right) = 0.$$

4.8 Appendix 4.C: Proof of Proposition 4.3.4

To start with, we have that

$$\partial_t S = \frac{D_\alpha\partial_t D_\alpha + P_i\partial_t P_i + M_{A,I}\partial_t M_{A,I}}{h} - \frac{1+D_\alpha^2+P_i^2+M_{A,I}^2}{2h^2}\partial_t h.$$

Let's look at the first term. We have

$$\frac{D_\alpha\partial_t D_\alpha}{h} = -\frac{D_\alpha}{h}\partial_j\left(\frac{D_\alpha P_j}{h}\right) - \frac{D_\alpha}{h}\sum_{\substack{A,I,i\\ \alpha\in A, i\in I}}(-1)^{O_A(\alpha)+O_I(i)}\partial_i\left(\frac{M_{A,I}M_{A\setminus\{\alpha\},I\setminus\{i\}}}{h}\right)$$

(since (4.3.9), we have $\sum_{i\in I}(-1)^{O_A(\alpha)+O_I(i)}\partial_i M_{A\setminus\{\alpha\},I\setminus\{i\}} = 0$,)

$$= -\frac{D_\alpha^2}{h^2}\partial_j P_j - P_j\partial_j\left(\frac{D_\alpha^2}{2h^2}\right) - \frac{D_\alpha M_{A\setminus\{\alpha\},I\setminus\{i\}}}{h}\sum_{\substack{A,I,i\\ \alpha\in A, i\in I}}(-1)^{O_A(\alpha)+O_I(i)}\partial_i\left(\frac{M_{A,I}}{h}\right).$$

The second term can be written in two parts,

$$\frac{P_i\partial_t P_i}{h} = L_1 + L_2,$$

where

$$L_1 = -\frac{P_i}{h}\sum_{\substack{A,I,j\\ j\in I, i\notin I\setminus\{j\}}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}\partial_j\left(\frac{M_{A,(I\setminus\{j\})\bigcup\{i\}}M_{A,I}}{h}\right),$$

$$L_2 = -\frac{P_i}{h}\partial_j\left(\frac{P_iP_j}{h}\right) + \frac{P_j}{h}\partial_j\left(\frac{1+\sum_{A,I}M_{A,I}^2}{h}\right).$$

Now let's first prove the following equality,

$$\partial_i M_{A,I} = \sum_{j\in I}\mathbb{1}_{\{i\notin I\setminus\{j\}\}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}\partial_j M_{A,(I\setminus\{j\})\bigcup\{i\}}.$$

In fact, since $\mathbb{1}_{\{j\in I, i\notin I\setminus\{j\}\}} = \mathbb{1}_{\{j\in I, i\notin I\}} + \mathbb{1}_{\{j\in I, i=j\}}$, we have

$$\text{Right } = \mathbb{1}_{\{i\notin I\}}\sum_{j\in I}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}\partial_j M_{A,(I\bigcup\{i\})\setminus\{j\}} + \mathbb{1}_{\{i\in I\}}\partial_i M_{A,I}.$$

For $i\neq j$, we can check that

$$O_I(j) + O_{I\setminus\{j\}}(i) \equiv O_I(i) + O_{I\bigcup\{i\}}(j) + 1 \pmod 2.$$

So the right hand side

$$\text{Right} \;= -\mathbb{1}_{\{i\notin I\}}\sum_{j\in I}(-1)^{O_I(i)+O_{I\bigcup\{i\}}(j)}\partial_j M_{A,(I\bigcup\{i\})\setminus\{j\}} + \mathbb{1}_{\{i\in I\}}\partial_i M_{A,I}$$

$$= -\mathbb{1}_{\{i\notin I\}}\sum_{j\in I\bigcup\{i\}}(-1)^{O_I(i)+O_{I\bigcup\{i\}}(j)}\partial_j M_{A,(I\bigcup\{i\})\setminus\{j\}} + \big(\mathbb{1}_{\{i\notin I\}} + \mathbb{1}_{\{i\in I\}}\big)\partial_i M_{A,I}.$$

Because of (4.3.9), we finally get

$$\sum_{j\in I}\mathbb{1}_{\{i\notin I\setminus\{j\}\}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}\partial_j M_{A,(I\setminus\{j\})\bigcup\{i\}} = \partial_i M_{A,I}.$$

So we have

$$L_1 = -\sum_{A,I}\frac{P_iM_{A,I}}{h^2}\partial_i M_{A,I} - \frac{P_iM_{A,(I\setminus\{j\})\bigcup\{i\}}}{h}\sum_{\substack{A,I,j\\ j\in I, i\notin I\setminus\{j\}}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}\partial_j\left(\frac{M_{A,I}}{h}\right).$$

For L_2, we have

$$L_2 = -\frac{P_i^2}{h^2}\partial_j P_j - P_j\partial_j\left(\frac{P_i^2}{2h^2}\right) + \partial_j\left(\frac{P_j(1+M_{A,I}^2)}{h^2}\right) - \frac{1+M_{A,I}^2}{h}\partial_j\left(\frac{P_j}{h}\right).$$

Since

$$-\frac{1+M_{A,I}^2}{h}\partial_j\left(\frac{P_j}{h}\right) = -\frac{1+M_{A,I}^2}{h^2}\partial_j P_j - P_j\partial_j\left(\frac{1+M_{A,I}^2}{2h^2}\right) + \frac{P_jM_{A,I}}{h^2}\partial_j M_{A,I},$$

we have

$$\frac{P_i\partial_t P_i}{h} = -\frac{P_iM_{A,(I\setminus\{j\})\bigcup\{i\}}}{h}\sum_{\substack{A,I,j\\ j\in I, i\notin I\setminus\{j\}}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}\partial_j\left(\frac{M_{A,I}}{h}\right)$$

$$-\frac{1+P_i^2+M_{A,I}^2}{h^2}\partial_j P_j - P_j\partial_j\left(\frac{1+P_i^2+M_{A,I}^2}{2h^2}\right) + \partial_j\left(\frac{P_j(1+M_{A,I}^2)}{h^2}\right).$$

Therefore, we have

$$\frac{D_\alpha\partial_t D_\alpha + P_i\partial_t P_i + M_{A,I}\partial_t M_{A,I}}{h} = -\frac{2S}{h}\partial_j P_j - P_j\partial_j\left(\frac{S}{h}\right) + L_3,$$

where

$$L_3 = \partial_j\left(\frac{P_j(1+M_{A,I}^2)}{h^2}\right) - \sum_{\substack{A,I,j\\ j\in I, i\notin I\setminus\{j\}}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}\partial_j\left(\frac{P_iM_{A,(I\setminus\{j\})\bigcup\{i\}}M_{A,I}}{h^2}\right)$$

$$-\sum_{\substack{A,I,i\\ \alpha\in A, i\in I}}(-1)^{O_A(\alpha)+O_I(i)}\partial_i\left(\frac{D_\alpha M_{A\setminus\{\alpha\},I\setminus\{i\}}M_{A,I}}{h^2}\right).$$

As a result, we have

$$\partial_t S = -\frac{2S}{h}\partial_j P_j - P_j\partial_j\left(\frac{S}{h}\right) + L_3 - \frac{S}{h}\partial_t h = -\partial_j\left(\frac{SP_j}{h}\right) + L_3,$$

which completes the proof.

4.9 Appendix 4.D: Proof of Lemma 4.4.3

Equation for ω_i

First, let's compute $\partial_t \omega_i$. By definition,

$$\partial_t \omega_i = \partial_t \tau v_i + \tau \partial_t v_i - \partial_t m_{\alpha i} d_\alpha - m_{\alpha i} \partial_t d_\alpha.$$

The first two terms are,

$$\partial_t \tau v_i + \tau \partial_t v_i = v_i(\tau \partial_j v_j - v_j \partial_j \tau) + \tau \Big(\sum_{A,I} m_{A,I} \partial_i m_{A,I} - v_j \partial_j v_i + \tau \partial_i \tau \Big) + \Sigma_1$$

$$= (\tau v_i) \partial_j v_j - v_j \partial_j (\tau v_i) + \frac{\tau}{2} \partial_i (\tau^2 + m_{A,I}^2) + \Sigma_1,$$

where

$$\Sigma_1 = -\tau \sum_{A,I,j} \mathbb{1}_{\{j \in I, i \notin I \setminus \{j\}\}} (-1)^{O_I(j) + O_{I \setminus \{j\}}(i)} m_{A,(I \setminus \{j\}) \bigcup \{i\}} \partial_j m_{A,I}.$$

Here we use the equation for $m_{\alpha i}$:

$$\partial_t m_{\alpha i} + v_j \partial_j m_{\alpha i} + m_{\alpha j} \partial_i v_j - m_{\alpha i} \partial_j v_j + \tau \partial_i d_\alpha = 0.$$

The last two terms are,

$$-\partial_t m_{\alpha i} d_\alpha - m_{\alpha i} \partial_t d_\alpha = d_\alpha (v_j \partial_j m_{\alpha i} + m_{\alpha j} \partial_i v_j - m_{\alpha i} \partial_j v_j + \tau \partial_i d_\alpha) + m_{\alpha i} v_j \partial_j d_\alpha + \Sigma_2$$

$$= -(m_{\alpha i} d_\alpha) \partial_j v_j + v_j \partial_j (m_{\alpha i} d_\alpha) + \frac{\tau}{2} \partial_i (d_\alpha^2 + v_j^2) - (\tau v_j - m_{\alpha j} d_\alpha) \partial_i v_j + \Sigma_2,$$

where

$$\Sigma_2 = \sum_{A,I,\alpha,j} \mathbb{1}_{\{\alpha \in A, j \in I\}} (-1)^{O_A(\alpha) + O_I(j)} m_{\alpha i} m_{A \setminus \{\alpha\}, I \setminus \{j\}} \partial_j m_{A,I}.$$

Now, we have

$$\partial_t \omega_i = (\tau v_i - m_{\alpha i} d_\alpha) \partial_j v_j - v_j \partial_j (\tau v_i - m_{\alpha i} d_\alpha) + \frac{\tau}{2} \partial_i (\tau^2 + v_j^2 + d_\alpha^2 + m_{A,I}^2)$$

$$-(\tau v_j - m_{\alpha j} d_\alpha) \partial_i v_j + \Sigma_1 + \Sigma_2$$

$$= \omega_i \partial_j v_j - \omega_j \partial_i v_j - v_j \partial_j \omega_i + \tau \partial_i \lambda + \Sigma_1 + \Sigma_2.$$

It is easy to check that

$$\Sigma_1 + \Sigma_2 = \sum_{A,I,j} \mathbb{1}_{\{j \in I\}} (-1)^{O_I(j) + O_{I \setminus \{j\}}(i)} \partial_j m_{A,I} \psi^i_{A, I \setminus \{j\}}.$$

So ω_i should satisfy the following equation

$$\partial_t \omega_i = \omega_i \partial_j v_j - \omega_j \partial_i v_j - v_j \partial_j \omega_i + \tau \partial_i \lambda + \sum_{A,I,j} \mathbb{1}_{\{j \in I\}} (-1)^{O_I(j) + O_{I \setminus \{j\}}(i)} \partial_j m_{A,I} \psi^i_{A, I \setminus \{j\}}.$$

Equation for λ

Now let's compute $\partial_t\lambda$. We have,

$$\partial_t\lambda = \tau\partial_t\tau + v_i\partial_t v_i + d_\alpha\partial_t d_\alpha + m_{A,I}\partial_t m_{A,I}$$

$$\begin{aligned}&= \tau(\tau\partial_j v_j - v_j\partial_j\tau) + v_i(m_{A,I}\partial_i m_{A,I} - v_j\partial_j v_i + \tau\partial_i\tau) + \Sigma_3 \\ &-d_\alpha v_j\partial_j d_\alpha + \Sigma_4 + m_{A,I}(m_{A,I}\partial_j v_j - v_j\partial_j m_{A,I}) + \Sigma_5 + \Sigma_6 \\ &= -\frac{v_j}{2}\partial_j(\tau^2 + v_i^2 + d_\alpha^2 + m_{A,I}^2) + \tau\partial_i(\tau v_i) + m_{A,I}\partial_i(m_{A,I}v_i) \\ &+\Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6,\end{aligned}$$

where

$$\Sigma_3 = -\sum_{A,I,i,j}\mathbb{1}_{\{j\in I, i\notin I\setminus\{j\}\}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}m_{A,(I\setminus\{j\})\bigcup\{i\}}v_i\partial_j m_{A,I},$$

$$\Sigma_4 = -\sum_{A,I,\alpha,i}\mathbb{1}_{\{\alpha\in A, i\in I\}}(-1)^{O_A(\alpha)+O_I(i)}m_{A\setminus\{\alpha\},I\setminus\{i\}}d_\alpha\partial_i m_{A,I},$$

$$\Sigma_5 = -\sum_{A,I,\alpha,i}\mathbb{1}_{\{\alpha\in A, i\in I\}}(-1)^{O_A(\alpha)+O_I(i)}m_{A\setminus\{\alpha\},I\setminus\{i\}}m_{A,I}\partial_i d_\alpha,$$

$$\Sigma_6 = -\sum_{A,I,i,j}\mathbb{1}_{\{j\in I, i\notin I\setminus\{j\}\}}(-1)^{O_{I\setminus\{j\}}(i)+O_I(j)}m_{A,(I\setminus\{j\})\bigcup\{i\}}m_{A,I}\partial_j v_i.$$

It is easy to see that

$$\Sigma_3 + \Sigma_6 = -\sum_{A,I,i,j}\mathbb{1}_{\{j\in I, i\notin I\setminus\{j\}\}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}m_{A,(I\setminus\{j\})\bigcup\{i\}}\partial_j(m_{A,I}v_i)$$

(since $\mathbb{1}_{\{j\in I, i\notin I\setminus\{j\}\}} = \mathbb{1}_{\{j\in I, i\notin I\}} + \mathbb{1}_{\{j\in I, i=j\}}$)

$$= -\sum_{A,I,i}\mathbb{1}_{\{i\in I\}}m_{A,I}\partial_i(m_{A,I}v_i) - \sum_{A,I,i,j}\mathbb{1}_{\{j\in I, i\notin I\}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}m_{A,(I\setminus\{j\})\bigcup\{i\}}\partial_j(m_{A,I}v_i)$$

(since $\mathbb{1}_{\{i\in I\}} = 1 - \mathbb{1}_{\{i\notin I\}}$)

$$= -m_{A,I}\partial_i(m_{A,I}v_i) + \sum_{A,I,i,j}\mathbb{1}_{\{i\notin I, j=i\}}(-1)^{O_I(i)+O_{I\bigcup\{i\}}(j)}m_{A,(I\bigcup\{i\})\setminus\{j\}}\partial_j(m_{A,I}v_i)$$

$$-\sum_{A,I,i,j}\mathbb{1}_{\{i\notin I, j\in I\}}(-1)^{O_I(j)+O_{I\setminus\{j\}}(i)}m_{A,(I\bigcup\{i\})\setminus\{j\}}\partial_j(m_{A,I}v_i).$$

Now we can easily check that, for any $i\notin I$ and $j\in I$,

$$O_I(i) + O_{I\bigcup\{i\}}(j) \equiv O_I(j) + O_{I\setminus\{j\}}(i) + 1 \pmod 2.$$

[We can prove this equality by discussing in the cases $i<j$ and $i>j$.]
Because $\mathbb{1}_{\{i\notin I, j=i\}} + \mathbb{1}_{\{i\notin I, j\in I\}} = \mathbb{1}_{\{i\notin I, j\in I\bigcup\{i\}\}}$, we have

$$\Sigma_3+\Sigma_6+m_{A,I}\partial_i(m_{A,I}v_i) = \sum_{A,I,i,j}\mathbb{1}_{\{i\notin I, j\in I\bigcup\{i\}\}}(-1)^{O_I(i)+O_{I\bigcup\{i\}}(j)}m_{A,(I\bigcup\{i\})\setminus\{j\}}\partial_j(m_{A,I}v_i)$$

(let $I' = I\bigcup\{i\}$)

$$= \sum_{A,|I'|\geq 2,i,j} \mathbb{1}_{\{i,j\in I'\}}(-1)^{O_{I'}(i)+O_{I'}(j)} m_{A,I'\setminus\{j\}}\partial_j(m_{A,I'\setminus\{i\}}v_i)$$

(and, since $m_{A,I'\setminus\{i\}}v_i = \frac{m_{A,I'\setminus\{i\}}}{\tau}(\omega_i + m_{\alpha i}d_\alpha)$)

$$= \sum_{A,|I'|\geq 2,i,j,\alpha} \mathbb{1}_{\{i,j\in I'\}}(-1)^{O_{I'}(i)+O_{I'}(j)} m_{A,I'\setminus\{j\}}\partial_j\left(\frac{m_{A,I'\setminus\{i\}}\omega_i + m_{A,I'\setminus\{i\}}m_{\alpha i}d_\alpha}{\tau}\right).$$

Also, we have

$$\Sigma_4 + \Sigma_5 = -\sum_{A,I,\alpha,i} \mathbb{1}_{\{\alpha\in A, i\in I\}}(-1)^{O_A(\alpha)+O_I(i)} m_{A\setminus\{\alpha\},I\setminus\{i\}}\partial_i(m_{A,I}d_\alpha)$$

(let $A' = A\setminus\{\alpha\}$)

$$= -\tau\partial_i(m_{\alpha i}d_\alpha) - \sum_{A',|I|\geq 2,\alpha,i} \mathbb{1}_{\{\alpha\notin A', i\in I\}}(-1)^{O_{A'}(\alpha)+O_I(i)} m_{A',I\setminus\{i\}}\partial_i(m_{A'\bigcup\{\alpha\},I}d_\alpha).$$

Since

$$\mathbb{1}_{\{\alpha\notin A'\}}m_{A'\bigcup\{\alpha\},I}d_\alpha = \frac{1}{\tau}\Big(\sum_{j\in I}(-1)^{O_{A'}(\alpha)+O_I(j)} m_{A',I\setminus\{j\}}m_{\alpha j} - \varphi^\alpha_{A',I}\Big)d_\alpha,$$

we have

$$\Sigma_4 + \Sigma_5 = -\tau\partial_i(m_{\alpha i}d_\alpha) + \sum_{A',|I|\geq 2,\alpha,i} \mathbb{1}_{\{i\in I\}}(-1)^{O_{A'}(\alpha)+O_I(i)} m_{A',I\setminus\{i\}}\partial_i\left(\frac{\varphi^\alpha_{A',I}d_\alpha}{\tau}\right)$$

$$- \sum_{A',|I|\geq 2,\alpha,i,j} \mathbb{1}_{\{i,j\in I\}}(-1)^{O_I(i)+O_I(j)} m_{A',I\setminus\{i\}}\partial_i\left(\frac{m_{A',I\setminus\{j\}}m_{\alpha j}d_\alpha}{\tau}\right).$$

We find that the last term is cancel when we add it up with $\Sigma_3 + \Sigma_6$, so we have

$$\partial_t\lambda = -v_j\partial_j\lambda + \tau\partial_i\omega_i + \sum_{A,|I'|\geq 2,i,j,} \mathbb{1}_{\{i,j\in I'\}}(-1)^{O_{I'}(i)+O_{I'}(j)} m_{A,I'\setminus\{j\}}\partial_j\left(\frac{m_{A,I'\setminus\{i\}}\omega_i}{\tau}\right)$$

$$+ \sum_{A',|I|\geq 2,\alpha,i} \mathbb{1}_{\{i\in I\}}(-1)^{O_{A'}(\alpha)+O_I(i)} m_{A',I\setminus\{i\}}\partial_i\left(\frac{\varphi^\alpha_{A',I}d_\alpha}{\tau}\right).$$

Equation for $\varphi^\alpha_{A,I}$

Now let's find the equation for $\varphi^\alpha_{A,I}$. We only consider the case $|A|\geq 2$. We have

$$\partial_t\varphi^\alpha_{A,I} = \partial_t\left(\sum_{i\in I}(-1)^{O_A(\alpha)+O_I(i)} m_{A,I\setminus\{i\}}m_{\alpha i} - \mathbb{1}_{\{\alpha\notin A\}}\tau m_{A\bigcup\{\alpha\},I}\right)$$

$$= \sum_{i\in I}(-1)^{O_A(\alpha)+O_I(i)} m_{\alpha i}\big(m_{A,I\setminus\{i\}}\partial_j v_j - v_j\partial_j m_{A,I\setminus\{i\}}\big) + \Sigma_7 + \Sigma_8$$

$$-\sum_{i\in I}(-1)^{O_A(\alpha)+O_I(i)}m_{A,I\setminus\{i\}}\big(v_j\partial_j m_{\alpha i}+m_{\alpha j}\partial_i v_j-m_{\alpha i}\partial_j v_j+\tau\partial_i d_\alpha\big)$$

$$-\mathbb{1}_{\{\alpha\notin A\}}m_{A\bigcup\{\alpha\},I}(\tau\partial_j v_j-v_j\partial_j\tau)-\mathbb{1}_{\{\alpha\notin A\}}\tau\big(m_{A\bigcup\{\alpha\},I}\partial_j v_j-v_j\partial_j m_{A\bigcup\{\alpha\},I}\big)+\Sigma_9+\Sigma_{10}$$

$$=2\varphi^\alpha_{A,I}\partial_j v_j-v_j\partial_j\varphi^\alpha_{A,I}-\sum_{i\in I}(-1)^{O_A(\alpha)+O_I(i)}m_{A,I\setminus\{i\}}\big(m_{\alpha j}\partial_i v_j+\tau\partial_i d_\alpha\big)+\Sigma_7+\Sigma_8+\Sigma_9+\Sigma_{10},$$

where

$$\Sigma_7=-\sum_{i,j,k}\mathbb{1}_{\{i\neq j\in I,k\notin I\setminus\{i,j\}\}}(-1)^{O_A(\alpha)+O_I(i)+O_{I\setminus\{i\}}(j)+O_{I\setminus\{i,j\}}(k)}m_{A,(I\setminus\{i,j\})\bigcup\{k\}}m_{\alpha i}\partial_j v_k,$$

$$\Sigma_8=-\sum_{\beta,i,j}\mathbb{1}_{\{\beta\in A,i\neq j\in I\}}(-1)^{O_A(\alpha)+O_I(i)+O_A(\beta)+O_{I\setminus\{i\}}(j)}m_{A\setminus\{\beta\},I\setminus\{i,j\}}m_{\alpha i}\partial_j d_\beta,$$

$$\Sigma_9=\sum_{j,k}\mathbb{1}_{\{\alpha\notin A,j\in I,k\notin I\setminus\{j\}\}}(-1)^{O_{I\setminus\{j\}}(k)+O_I(j)}\tau m_{A\bigcup\{\alpha\},(I\setminus\{j\})\bigcup\{k\}}\partial_j v_k,$$

$$\Sigma_{10}=\sum_{\beta,j}\mathbb{1}_{\{\alpha\notin A,\beta\in A\bigcup\{\alpha\},j\in I\}}(-1)^{O_{A\bigcup\{\alpha\}}(\beta)+O_I(j)}\tau m_{(A\bigcup\{\alpha\})\setminus\{\beta\},I\setminus\{j\}}\partial_j d_\beta.$$

Now let's look at Σ_7. First, since

$$\mathbb{1}_{\{i\neq j\in I,k\notin I\setminus\{i,j\}\}}=\mathbb{1}_{\{j\in I,i\in I\setminus\{j\},k\notin I\setminus\{j\}\}}+\mathbb{1}_{\{j\in I,i\in I\setminus\{j\},k=i\}},$$

we have

$$\Sigma_7=-\sum_{i,j,k}\mathbb{1}_{\{j\in I,i\in I\setminus\{j\},k\notin I\setminus\{j\}\}}(-1)^{O_A(\alpha)+O_I(i)+O_{I\setminus\{i\}}(j)+O_{I\setminus\{i,j\}}(k)}m_{A,(I\setminus\{i,j\})\bigcup\{k\}}m_{\alpha i}\partial_j v_k$$

$$+\sum_{j,k}\mathbb{1}_{\{j\in I,k\in I\setminus\{j\}\}}(-1)^{O_A(\alpha)+O_I(j)}m_{A,I\setminus\{j\}}m_{\alpha k}\partial_j v_k.$$

Here we use the fact that, for $k\neq j$,

$$O_I(k)+O_{I\setminus\{k\}}(j)\equiv O_I(j)+O_{I\setminus\{j\}}(k)+1\pmod 2.$$

Further more, for different number i,j,k, we have the following equality

$$O_I(i)+O_{I\setminus\{i\}}(j)+O_{I\setminus\{i,j\}}(k)\equiv O_{I\setminus\{j\}}(k)+O_I(j)+O_{(I\setminus\{j\})\bigcup\{k\}}(i)\pmod 2.$$

Then, we have

$$\Sigma_7=-\sum_{i,j,k}\mathbb{1}_{\{j\in I,i\in I\setminus\{j\},k\notin I\setminus\{j\}\}}(-1)^{O_A(\alpha)+O_{I\setminus\{j\}}(k)+O_I(j)+O_{(I\setminus\{j\})\bigcup\{k\}}(i)}m_{A,(I\setminus\{i,j\})\bigcup\{k\}}m_{\alpha i}\partial_j v_k$$

$$+\sum_{j\in I,k}(-1)^{O_A(\alpha)+O_I(j)}m_{A,I\setminus\{j\}}m_{\alpha k}\partial_j v_k-\sum_{j,k}\mathbb{1}_{\{j\in I,k\notin I\setminus\{j\}\}}(-1)^{O_A(\alpha)+O_I(j)}m_{A,I\setminus\{j\}}m_{\alpha k}\partial_j v_k$$

(since $\mathbb{1}_{\{j\in I,i\in I\setminus\{j\},k\notin I\setminus\{j\}\}}+\mathbb{1}_{\{j\in I,k\notin I\setminus\{j\},i=k\}}=\mathbb{1}_{\{j\in I,k\notin I\setminus\{j\},i\in(I\setminus\{j\})\bigcup\{k\}\}}$)

$$=-\sum_{i,j,k}\mathbb{1}_{\{j\in I,k\notin I\setminus\{j\},i\in(I\setminus\{j\})\bigcup\{k\}\}}(-1)^{O_A(\alpha)+O_{I\setminus\{j\}}(k)+O_I(j)+O_{(I\setminus\{j\})\bigcup\{k\}}(i)}m_{A,((I\setminus\{j\})\bigcup\{k\})\setminus\{i\}}m_{\alpha i}\partial_j v_k$$

$$+\sum_{j\in I,k}(-1)^{O_A(\alpha)+O_I(j)}m_{A,I\setminus\{j\}}m_{\alpha k}\partial_j v_k.$$

Together with Σ_9, we have

$$\Sigma_7 + \Sigma_9 - \sum_{j\in I,k} (-1)^{O_A(\alpha)+O_I(j)} m_{A,I\setminus\{j\}} m_{\alpha k} \partial_j v_k$$

$$= -\sum_{j,k} \mathbb{1}_{\{j\in I, k\notin I\setminus\{j\}\}} (-1)^{O_{I\setminus\{j\}}(k)+O_I(j)} \varphi^\alpha_{A,(I\setminus\{j\})\bigcup\{k\}} \partial_j v_k.$$

Now let's look at Σ_{10}. Since

$$\mathbb{1}_{\{\alpha\notin A, \beta\in A\bigcup\{\alpha\}, j\in I\}} = \mathbb{1}_{\{\alpha\notin A,\beta\in A, j\in I\}} + \mathbb{1}_{\{\alpha\notin A, \beta=\alpha, j\in I\}},$$

and, for any $\alpha \notin A$, $\beta \in A$,

$$O_{A\setminus\{\beta\}}(\alpha) + O_{A\bigcup\{\alpha\}}(\beta) \equiv O_A(\alpha) + O_A(\beta) + 1 \pmod 2,$$

then we have

$$\Sigma_{10} = -\sum_{\beta,j} \mathbb{1}_{\{\alpha\notin A,\beta\in A,j\in I\}} (-1)^{O_A(\alpha)+O_A(\beta)+O_{A\setminus\{\beta\}}(\alpha)+O_I(j)} \tau m_{(A\setminus\{\beta\})\bigcup\{\alpha\}, I\setminus\{j\}} \partial_j d_\beta$$

$$+ \sum_{j\in I} \mathbb{1}_{\{\alpha\notin A\}} (-1)^{O_A(\alpha)+O_I(j)} \tau m_{A,I\setminus\{j\}} \partial_j d_\alpha.$$

Now because $\mathbb{1}_{\{\alpha\notin A,\beta\in A,j\in I\}} = \mathbb{1}_{\{\alpha\notin A\setminus\{\beta\},\beta\in A,j\in I\}} - \mathbb{1}_{\{\alpha=\beta\in A,j\in I\}}$, we have

$$\Sigma_{10} = -\sum_{\beta,j} \mathbb{1}_{\{\alpha\notin A\setminus\{\beta\},\beta\in A,j\in I\}} (-1)^{O_A(\alpha)+O_A(\beta)+O_{A\setminus\{\beta\}}(\alpha)+O_I(j)} \tau m_{(A\setminus\{\beta\})\bigcup\{\alpha\}, I\setminus\{j\}} \partial_j d_\beta$$

$$+ \sum_{j\in I} \left(\mathbb{1}_{\{\alpha\in A\}} + \mathbb{1}_{\{\alpha\notin A\}}\right) (-1)^{O_A(\alpha)+O_I(j)} \tau m_{A,I\setminus\{j\}} \partial_j d_\alpha.$$

Since for any $i \neq j \in I$,

$$O_I(i) + O_{I\setminus\{i\}}(j) \equiv O_I(j) + O_{I\setminus\{j\}}(i) + 1 \pmod 2,$$

then we have

$$\Sigma_8 + \Sigma_{10} - \sum_{j\in I} (-1)^{O_A(\alpha)+O_I(j)} \tau m_{A,I\setminus\{j\}} \partial_j d_\alpha$$

$$= \sum_{\beta,j} \mathbb{1}_{\{\beta\in A,j\in I\}} (-1)^{O_A(\alpha)+O_A(\beta)+O_{A\setminus\{\beta\}}(\alpha)+O_I(j)} \varphi^\alpha_{A\setminus\{\beta\},I\setminus\{j\}} \partial_j d_\beta.$$

In summary, we have

$$\partial_t \varphi^\alpha_{A,I} = 2\varphi^\alpha_{A,I}\partial_j v_j - v_j \partial_j \varphi^\alpha_{A,I} + \sum_{\beta,j} \mathbb{1}_{\{\beta\in A,j\in I\}} (-1)^{O_A(\alpha)+O_A(\beta)+O_{A\setminus\{\beta\}}(\alpha)+O_I(j)} \varphi^\alpha_{A\setminus\{\beta\},I\setminus\{j\}} \partial_j d_\beta$$

$$- \sum_{j,k} \mathbb{1}_{\{j\in I,k\notin I\setminus\{j\}\}} (-1)^{O_{I\setminus\{j\}}(k)+O_I(j)} \varphi^\alpha_{A,(I\setminus\{j\})\bigcup\{k\}} \partial_j v_k.$$

Equation for $\psi^i_{A,I}$

Now we compute $\psi^i_{A,I}$, and we only consider the case when $|I| \geq 2$. We have

$$\partial_t \psi^i_{A,I} = \partial_t\Big(\sum_{\alpha\in A}(-1)^{O_A(\alpha)+O_I(i)} m_{A\setminus\{\alpha\},I} m_{\alpha i} - \mathbb{1}_{\{i\notin I\}}\tau m_{A,I\bigcup\{i\}}\Big)$$

$$= \sum_{\alpha\in A}(-1)^{O_A(\alpha)+O_I(i)} m_{\alpha i}\big(m_{A\setminus\{\alpha\},I}\partial_j v_j - v_j\partial_j m_{A\setminus\{\alpha\},I}\big) + \Sigma_{11} + \Sigma_{12}$$

$$- \sum_{\alpha\in A}(-1)^{O_A(\alpha)+O_I(i)} m_{A\setminus\{\alpha\},I}\big(v_j\partial_j m_{\alpha i} + m_{\alpha j}\partial_i v_j - m_{\alpha i}\partial_j v_j + \tau\partial_i d_\alpha\big)$$

$$-\mathbb{1}_{\{i\notin I\}}\tau\big(m_{A,I\bigcup\{i\}}\partial_j v_j - v_j\partial_j m_{A,I\bigcup\{i\}}\big) + \Sigma_{13} + \Sigma_{14}$$

$$-\mathbb{1}_{\{i\notin I\}} m_{A,I\bigcup\{i\}}(\tau\partial_j v_j - v_j\partial_j\tau)$$

$$= 2\psi^i_{A,I}\partial_j v_j - v_j\partial_j\psi^i_{A,I} + \Sigma_{11} + \Sigma_{12} + \Sigma_{13} + \Sigma_{14}$$

$$- \sum_{\alpha\in A}(-1)^{O_A(\alpha)+O_I(i)} m_{A\setminus\{\alpha\},I}\big(m_{\alpha j}\partial_i v_j + \tau\partial_i d_\alpha\big),$$

where

$$\Sigma_{11} = -\sum_{\alpha,j,k}\mathbb{1}_{\{\alpha\in A, j\in I, k\notin I\setminus\{j\}\}}(-1)^{O_A(\alpha)+O_I(i)+O_I(j)+O_{I\setminus\{j\}}(k)} m_{A\setminus\{\alpha\},(I\setminus\{j\})\bigcup\{k\}} m_{\alpha i}\partial_j v_k,$$

$$\Sigma_{12} = -\sum_{\alpha,\beta,j}\mathbb{1}_{\{\alpha\neq\beta\in A, j\in I\}}(-1)^{O_A(\alpha)+O_I(i)+O_{A\setminus\{\alpha\}}(\beta)+O_I(j)} m_{A\setminus\{\alpha,\beta\},I\setminus\{j\}} m_{\alpha i}\partial_j d_\beta,$$

$$\Sigma_{13} = \sum_{j,k}\mathbb{1}_{\{i\notin I, j\in I\bigcup\{i\}, k\notin(I\bigcup\{i\})\setminus\{j\}\}}(-1)^{O_{(I\bigcup\{i\})\setminus\{j\}}(k)+O_{I\bigcup\{i\}}(j)}\tau m_{A,((I\bigcup\{i\})\setminus\{j\})\bigcup\{k\}}\partial_j v_k,$$

$$\Sigma_{14} = \sum_{\beta,j}\mathbb{1}_{\{\beta\in A, i\notin I, j\in I\bigcup\{i\}\}}(-1)^{O_A(\beta)+O_{I\bigcup\{i\}}(j)}\tau m_{A\setminus\{\beta\},(I\bigcup\{i\})\setminus\{j\}}\partial_j d_\beta.$$

First, let us consider the most complicated part Σ_{13}. Since

$$\mathbb{1}_{\{i\notin I, j\in I\bigcup\{i\}, k\notin(I\bigcup\{i\})\setminus\{j\}\}} = \mathbb{1}_{\{i\notin I, j\in I, k\notin(I\setminus\{j\})\bigcup\{i\}\}} + \mathbb{1}_{\{i\notin I, j=i, k\notin I\}},$$

we have

$$\Sigma_{13} = \sum_k \mathbb{1}_{\{i\notin I\}}(-1)^{O_I(k)+O_I(i)}\tau m_{A,I\bigcup\{k\}} + \Sigma'_{13},$$

where

$$\Sigma'_{13} = \sum_{j,k}\mathbb{1}_{\{i\notin I, j\in I, k\notin(I\setminus\{j\})\bigcup\{i\}\}}(-1)^{O_{(I\setminus\{j\})\bigcup\{i\}}(k)+O_{I\bigcup\{i\}}(j)}\tau m_{A,((I\setminus\{j\})\bigcup\{i\})\bigcup\{k\}}\partial_j v_k.$$

Now for $i\notin I$, $j\in I$, $k\notin I\setminus\{j\}$, $k\neq i$, we can prove that

$$O_{(I\setminus\{j\})\bigcup\{i\}}(k) + O_{I\bigcup\{i\}}(j) \equiv O_I(i) + O_I(j) + O_{I\setminus\{j\}}(k) + O_{(I\setminus\{j\})\bigcup\{k\}}(i) \pmod 2.$$

(We can show this equality by discussing in the 4 cases: $i<\min\{j,k\}$, $i>\max\{j,k\}$, $j<i<k$ and $k<i<j$.) So Σ'_{13} can be written as

$$\Sigma'_{13} = \sum_{j,k}\mathbb{1}_{\{i\notin I, j\in I, k\notin(I\setminus\{j\})\bigcup\{i\}\}}(-1)^S\tau m_{A,((I\setminus\{j\})\bigcup\{i\})\bigcup\{k\}}\partial_j v_k,$$

where $S = O_I(i) + O_I(j) + O_{I\setminus\{j\}}(k) + O_{(I\setminus\{j\})\bigcup\{k\}}(i)$. Now we claim that

$$\mathbb{1}_{\{i\notin I, j\in I, k\notin (I\setminus\{j\})\bigcup\{i\}\}} = \mathbb{1}_{\{i\notin (I\setminus\{j\})\bigcup\{k\}, j\in I, k\notin I\setminus\{j\}\}} - \mathbb{1}_{\{i=j, j\in I, k\notin I\}}.$$

(In fact, we can prove the equality step by step:

$$\mathbb{1}_{\{i\notin I, j\in I, k\notin (I\setminus\{j\})\bigcup\{i\}\}} = \mathbb{1}_{\{i\notin I, j\in I, k\notin I\setminus\{j\}\}} - \mathbb{1}_{\{i\notin I, j\in I, k=i\}}$$

$$\mathbb{1}_{\{i\notin I, j\in I, k\notin I\setminus\{j\}\}} = \mathbb{1}_{\{i\notin I\setminus\{j\}, j\in I, k\notin I\setminus\{j\}\}} - \mathbb{1}_{\{i=j, j\in I, k\notin I\setminus\{j\}\}}$$

$$\mathbb{1}_{\{i\notin I\setminus\{j\}, j\in I, k\notin I\setminus\{j\}\}} = \mathbb{1}_{\{i\notin (I\setminus\{j\})\bigcup\{k\}, j\in I, k\notin I\setminus\{j\}\}} + \mathbb{1}_{\{i=k, j\in I, k\notin I\setminus\{j\}\}}$$

$$\mathbb{1}_{\{i=j, j\in I, k\notin I\setminus\{j\}\}} = \mathbb{1}_{\{i=j, j\in I, k\notin I\}} + \mathbb{1}_{\{i=j=k, j\in I\}}$$

$$\mathbb{1}_{\{i=k, j\in I, k\notin I\setminus\{j\}\}} = \mathbb{1}_{\{i=k, j\in I, k\notin I\}} + \mathbb{1}_{\{i=j=k, j\in I\}}$$

by adding up the above equalities we get the desired result.) Then Σ'_{13} can be written as

$$\Sigma'_{13} = \sum_{j,k} \mathbb{1}_{\{i\notin (I\setminus\{j\})\bigcup\{k\}, j\in I, k\notin I\setminus\{j\}\}} (-1)^S \tau m_{A,((I\setminus\{j\})\bigcup\{k\})\bigcup\{i\}} \partial_j v_k$$

$$- \sum_k \mathbb{1}_{\{i\in I\}} (-1)^{O_{I\setminus\{i\}}(k) + O_{I\bigcup\{k\}}(i)} \tau m_{A,I\bigcup\{k\}}.$$

We can easily verify that for $i \in I$, $k \notin I$,

$$O_{I\setminus\{i\}}(k) + O_{I\bigcup\{k\}}(i) \equiv O_I(k) + O_I(i) + 1 \pmod 2.$$

So we have

$$\Sigma_{13} = \sum_k (-1)^{O_I(k)+O_I(i)} \tau m_{A,I\bigcup\{k\}}$$

$$+ \sum_{j,k} \mathbb{1}_{\{i\notin (I\setminus\{j\})\bigcup\{k\}, j\in I, k\notin I\setminus\{j\}\}} (-1)^S \tau m_{A,((I\setminus\{j\})\bigcup\{k\})\bigcup\{i\}} \partial_j v_k.$$

Then we have

$$\Sigma_{11} + \Sigma_{13} - \sum_{\alpha\in A,k} (-1)^{O_A(\alpha)+O_I(i)} m_{A\setminus\{\alpha\},I} m_{\alpha k} \partial_i v_k$$

$$= -\sum_k (-1)^{O_I(i)+O_I(k)} \psi^k_{A,I} \partial_i v_k - \sum_{j,k} \mathbb{1}_{\{j\in I, k\notin I\setminus\{j\}\}} (-1)^S \psi^i_{A,(I\setminus\{j\})\bigcup\{k\}} \partial_j v_k.$$

Now let's look at Σ_{14}. Since

$$\mathbb{1}_{\{\beta\in A, i\notin I, j\in I\bigcup\{i\}\}} = \mathbb{1}_{\{\beta\in A, i\notin I, j\in I\}} + \mathbb{1}_{\{\beta\in A, i\notin I, j=i\}},$$

we have

$$\Sigma_{14} = \sum_{\beta\in A} \mathbb{1}_{\{i\notin I\}} (-1)^{O_A(\beta)+O_I(i)} \tau m_{A\setminus\{\beta\},I} \partial_i d_\beta + \Sigma'_{14},$$

where

$$\Sigma'_{14} = \sum_{\beta,j} \mathbb{1}_{\{\beta\in A, i\notin I, j\in I\}} (-1)^{O_A(\beta)+O_{I\bigcup\{i\}}(j)} \tau m_{A\setminus\{\beta\},(I\bigcup\{i\})\setminus\{j\}} \partial_j d_\beta.$$

We can check that for $i \notin I$, $j \in I$, we have

$$O_I(i) + O_{I\setminus\{j\}}(i) + O_I(j) + O_{I\bigcup\{i\}}(j) \equiv 1 \pmod 2.$$

So Σ'_{14} can be written as

$$\Sigma'_{14} = -\sum_{\beta,j} \mathbb{1}_{\{\beta\in A, i\notin I, j\in I\}}(-1)^{O_A(\beta)+O_I(i)+O_{I\setminus\{j\}}(i)+O_I(j)}\tau m_{A\setminus\{\beta\},(I\bigcup\{i\})\setminus\{j\}}\partial_j d_\beta.$$

Because

$$\mathbb{1}_{\{\beta\in A, i\notin I, j\in I\}} = \mathbb{1}_{\{\beta\in A, i\notin I\setminus\{j\}, j\in I\}} - \mathbb{1}_{\{\beta\in A, i=j, j\in I\}},$$

then we have

$$\Sigma'_{14} = -\sum_{\beta,j} \mathbb{1}_{\{\beta\in A, i\notin I\setminus\{j\}, j\in I\}}(-1)^{O_A(\beta)+O_I(i)+O_{I\setminus\{j\}}(i)+O_I(j)}\tau m_{A\setminus\{\beta\},(I\bigcup\{i\})\setminus\{j\}}\partial_j d_\beta$$

$$+\sum_{\beta\in A} \mathbb{1}_{\{i\in I\}}(-1)^{O_A(\beta)+O_I(i)}\tau m_{A\setminus\{\beta\},I}\partial_i d_\beta.$$

Since for $\alpha \neq \beta \in A$,

$$O_A(\alpha) + O_{A\setminus\{\alpha\}}(\beta) \equiv O_A(\beta) + O_{A\setminus\{\beta\}}(\alpha) + 1 \pmod 2,$$

we finally have

$$\Sigma_{12} + \Sigma_{14} = \sum_{\beta\in A, j\in I}(-1)^{O_A(\beta)+O_I(j)+O_I(i)+O_{I\setminus\{j\}}(i)}\psi^i_{A\setminus\{\beta\},I\setminus\{j\}}\partial_j d_\beta$$

$$+\sum_{\beta\in A}(-1)^{O_A(\beta)+O_I(i)}\tau m_{A\setminus\{\beta\},I}\partial_i d_\beta.$$

In summary, we have

$$\partial_t\psi^i_{A,I} = 2\psi^i_{A,I}\partial_j v_j - v_j\partial_j\psi^i_{A,I} - \sum_k(-1)^{O_I(i)+O_I(k)}\psi^k_{A,I}\partial_i v_k$$

$$+\sum_{\beta\in A, j\in I}(-1)^{O_A(\beta)+O_I(j)+O_I(i)+O_{I\setminus\{j\}}(i)}\psi^i_{A\setminus\{\beta\},I\setminus\{j\}}\partial_j d_\beta$$

$$-\sum_{j,k}\mathbb{1}_{\{j\in I, k\notin I\setminus\{j\}\}}(-1)^{O_I(i)+O_I(j)+O_{I\setminus\{j\}}(k)+O_{(I\setminus\{j\})\cup\{k\}}(i)}\psi^i_{A,(I\setminus\{j\})\bigcup\{k\}}\partial_j v_k.$$

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Titre : Transport optimal et diffusions de courants

Mots clefs : Transport optimal, équations aux dérivées partielles, magnétohydrodynamique, solution dissipative, méthode d'entropie relative, lois de conservation hyperboliques.

Résumé : Cette thèse concerne l'étude d'équations aux dérivées partielles à la charnière de la physique de la mécanique des milieux continus et de la géométrie différentielle. La thèse se compose de quatre chapitres. Dans le premier chapitre, on montre comment les systèmes paraboliques dégénérés d'EDP non-linéaires peuvent être parfois dérivés à partir de systèmes non-dissipatifs (typiquement des systèmes hyperboliques non-linéaires), par simple changement de variable en temps non-linéaire dégénéré à l'origine. Le chapitre traite, comme prototype, du "curve-shortening flow". On obtient, presque automatiquement, l'équivalent parabolique des principes d'entropie relative et d'unicité fort-faible qu'il est, en fait, bien plus simple d'établir et de comprendre dans le cadre hyperbolique. Dans le second chapitre, la même méthode s'applique au système de Born-Infeld proprement dit, ce qui permet d'obtenir, à la limite, un modèle (non répertorié à notre connaissance) de Magnétohydrodynamique (MHD), où on retrouve à la fois une diffusivité non-linéaire dans l'équation d'induction magnétique et une loi de Darcy pour le champ de vitesse. Dans le troisième chapitre, un lien est établi entre des systèmes paraboliques et le concept de flot gradient de formes différentielles pour des métriques de transport. Enfin, dans le quatrième chapitre, on retourne au domaine des EDP hyperboliques en considérant, dans le cas particulier des graphes, les surfaces extrémales de l'espace de Minkowski, de dimension et co-dimension quelconques. On parvient à montrer que les équations peuvent se reformuler sous forme d'un système élargi symétrique du premier ordre.

Title : Optimal transport and diffusion of currents

Keywords : Optimal transport, partial differential equations, magnetohydrodynamics, dissipative solution, relative entropy method, hyperbolic conservation laws.

Abstract : This thesis concerns about the study of partial differential equations at the hinge of the continuum physics and differential geometry. The thesis is composed of four chapters. In the first chapter, we show how nonlinear degenerate parabolic systems of PDEs can sometimes be derived from non-dissipative systems (typically nonlinear hyperbolic systems), by simple non-linear change of the time variable degenerate at the origin. The chapter deals with the curve-shortening flow as a prototype. We obtain, almost automatically, the parabolic version of the relative entropy method and weak-strong uniqueness, which, in fact, is much simpler to establish and understand in the hyperbolic framework. In the second chapter, the same method applies to the Born-Infeld system itself, which makes it possible to obtain, in the limit, a model (not listed to our knowledge) of Magnetohydrodynamics (MHD) where we have non-linear diffusions in the magnetic induction equation and the Darcy's law for the velocity field. In the third chapter, a link is established between the parabolic systems and the concept of gradient flow of differential forms with suitable transport metrics. Finally, in the fourth chapter, we return to the domain of hyperbolic EDPs considering, in the particular case of graphs, the extremal surfaces of the Minkowski space of any dimension and co-dimension. We can show that the equations can be reformulated in the form of a symmetric first-order enlarged system.

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