On some constructions of contact manifolds

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Contents

Chapter 1

Introduction générale

Cette thèse porte sur quelques aspects topologiques de la *géométrie de contact*. Cette dernière est une branche de la géométrie qui étudie les propriétés des *structures de contact*, i.e. des distributions d'hyperplans ξ sur une variété différentielle de dimension impaire V^{2n+1} qui sont définies localement par une 1-forme α telle que $\alpha \wedge d\alpha^n$ est une forme volume (sur son domaine). Plus précisément, on se focalise ici sur les structures de contact coorientées, i.e. sur celles qui sont globalement définies par une 1-forme.

Quelques exemples de variétés de contact fermées viennent naturellement du monde symplectique. Par exemple, si W^{2n} est une variété compacte à bord V^{2n-1}, on peut considérer les formes symplectiques ω sur W qui sont "compatibles" avec V dans le sens suivant : il y a, dans un voisinage de V, un *champ de Liouville* Z (i.e. un champ de vecteurs Z) tel que $\mathcal{L}_Z\omega = \omega$) qui est positivement transverse à V. Dans cette situation, $\xi := \ker(\iota_Z\omega|_V)$ est une structure de contact sur V, et (W,ω) est appelée *remplissage symplectique (fort)* de (V,ξ).

Ceci dit, pas toutes les variétés de contact sont le bord convexe d'une variété symplectique. Des exemples remarquables sont les structures de contact *vrillées*, définies dans [Eli89] pour le cas 3-dimensionnel et généralisées à toutes dimensions impaires dans [BEM15] : leur non-remplissabilité suit des résultats dans [Gro85, Eli90] (voir aussi [Zeh03]) pour le cas de dimension 3 et de [Nie06, BEM15] dans le cas général. Les structures de contact remplissables sont alors une sous-classe des structures non vrillées, qui sont aussi appelées *tendues*.

Les structures de contact vrillées sont une manifestation importante de la nature topologique de la géométrie de contact. En fait, [Eli89, BEM15] montre que le problème géométrique de la construction des structures de contact peut être réduit au problème formel de la construction de leur équivalent homotopique : toute *structure presque de contact* peut être déformée en structure de contact vrillée. De plus, deux structures de contact vrillées sont isotopes si et seulement si elle sont homotopes parmi les structures presque de contact.

Les structures de contact vrillées manifestent donc de la *flexibilité* concernant le problème de construction géométrique. Au contraire, les structures tendues peuvent présenter des comportements *rigides* dans ce contexte : par exemple, deux structures tendues qui sont homotopes parmi les structures presque contact ne sont pas nécessairement isotopes. Ceci dit, des résultats récents suggèrent que, selon le problème considéré, les comportements flexible et rigide peuvent interagir de façon inattendue. Par exemple, [Laz] montre que la cohomologie d'un *domaine de Weinstein flexible* est "codée" dans la variété de contact qu'il remplit, donc que le bord d'un variété de Weinstein a une certaine rigidité.

À cause de ces interactions non triviales, il n'y a pas de séparation nette entre les comportements flexible et rigide flexible en géométrie de contact. Donc, le problème suivant est très intéressant dans ce contexte :

Problème A. *Étudier les interactions entre flexibilité et rigidité sous points de vue différents.*

La réduction du problème géométrique de la construction des structures de contact à son équivalent formel dans [BEM15] n'est pas seulement importante en vue de l'étude des interactions flexibilité-rigidité, mais aussi puisqu'il répond à la question fondamentale de quelles variétés de grandes dimensions admettent des structures de contact (ou, plus précisément, il montre que la topologie algébrique peut donner une réponse complète). Comme la littérature montre (voir par exemple [Lut79, Gei91, Gei97a, GT98, GT01, Bou02, CPP15, Etn12]), ceci a été pour longtemps un problème fondamental en géométrie de contact, qui a motivé en particulier la recherche de constructions explicites de variétés de contact en grandes dimensions. Bien sûr, depuis le travail [BEM15], la question de l'existence des structures de contact n'est plus une motivation pour la recherche de constructions explicites. Ceci dit, la question suivante a sans doute encore un certain intérêt :

Problème B. *Trouver (et étudier) des constructions de variétés de contact avec des propriétés "remarquables", e.g. qui donnent des structures vrillées, tendues, remplissables, etc.*

La Partie I de cette thèse porte principalement sur le Problème A. Plus précisément, il contient un étude de la topologie de l'espace des contactomorphismes, avec une analyse de quelques comportements rigides dans le cas des structures tendues et vrillées en grandes dimensions; ceci sera présenté plus en détail dans la Section 1.1.
La Partie II porte sur le Problème B et contient, plus précisément, une réinterprétation de quelques constructions explicites déjà connues en grandes dimensions et l'étude des propriétés des variétés de contact qui en résultent. Ceci est présenté en détail dans la Section 1.2.

Avant de présenter les résultats contenus dans Parties I, II de cette thèse, le lecteur qui n'a pas beaucoup de familiarité avec la géométrie de contact pourrait lire Section 2, où on rappelle les objets principaux qui apparaîtront dans les énoncés présentés dans Sections 1.1, 1.2.

1.1 La topologie du groupe des contactomorphismes

Dans Partie I de cette thèse, on s'interesse à la topologie de l'espace des contactomorphismes $\mathcal{D}(V,\xi)$ d'une variété de contact (V,ξ), en rapport à celle de l'espace des difféomorphismes $\mathcal{D}(V)$ de la variété différentielle V sous-jacente. Plus précisément, on va étudier les propriétés de l'application $j_*\colon \pi_k(\mathcal{D}(V,\xi)) \to \pi_k(\mathcal{D}(V))$ induite par l'inclusion naturelle $j\colon \mathcal{D}(V,\xi) \to \mathcal{D}(V)$.

Soit $\Xi(V)$ l'espace de toutes les structures de contact sur V. Dans le cas des variétés de contact fermées, l'application naturelle $\mathcal{D}(V) \to \Xi(V)$, donnée par $\phi \mapsto \phi_*\xi$, aide à comprendre les propriétés de j_*, et montre que $\Xi(V)$ joue le rôle d'intermédiaire pour la relation entre la topologie de $\mathcal{D}(V,\xi)$ et celle de $\mathcal{D}(V)$. En fait, la preuve du théorème de Gray implique, modulo un critère général de fibration, que cette application est une fibration localement triviale avec fibre $\mathcal{D}(V,\xi)$; voir [GM17] pour une explication de ce résultat ou [Mas15] pour une preuve détaillée. (On remarque que, dans [GGP04], il est demontré que cette application est une fibration de Serre, ce qui est suffisant pour les considérations suivantes.) Donc, la suite exacte longue en homotopie

$$\ldots \to \pi_{k+1}\left(\Xi\left(V\right)\right) \to \pi_k\left(\mathcal{D}\left(V,\xi\right)\right) \xrightarrow{j_*} \pi_k\left(\mathcal{D}\left(V\right)\right) \to \pi_k\left(\Xi\left(V\right)\right) \to \ldots$$

associé à la fibration donne une relation entre les topologies des trois espaces $\mathcal{D}\left(V\right)$, $\mathcal{D}\left(V,\xi\right)$ et $\Xi\left(V\right)$.

Concernant le cas de dimension 3, la disponibilité de résultats de classifications pour les classes d'isotopie des structures de contact tendues sur des 3-variétés spécifiques V donne quelques résultats explicites sur les groupes d'homotopie en dégrée 0 et 1 sur quelques variétés spécifique. Le lecteur intéressé peut consulter [Gom98, GGP04, Bou06, DG10, GK14, GM17] pour des résultats sur $\pi_1\left(\Xi\left(V\right),\xi\right)$ et [Gir01, GM17] pour des résultats sur $\pi_0\left(\mathcal{D}\left(V,\xi\right)\right)$; bien sûr, les résultats sur ces deux groupes peuvent être reliés entre eux via la suite exacte longue ci-dessus.
Plus précisément, dans [GGP04], les auteurs démontrent l'existence d'un élément d'ordre infini dans le groupe fondamental de l'espace des structures de contact sur les fibrés en tores sur $\mathbb{S}^1$; une preuve indépendante, qui se généralise au cas du tore de dimension 5, est donnée dans [Bou06], en utilisant l'homologie de contact. [GK14] démontre que le groupe fondamental est en effet isomorphe à $\mathbb{Z}$ dans le cas de la variété $\mathbb{T}^3$ et point base la structure de contact standard. Ceci a été ensuite généralisé dans [GM17] (voir aussi [Gir01]), où il est démontré que, pour tout fibré V en $\mathbb{S}^1$ sur une surface fermée Σ et toute structure de contact ξ pour laquelle les fibres sont legendriennes, $\pi_1\left(\Xi\left(V\right),\xi\right)$ est cyclique infini et $\ker(j_*|_{\pi_0})$ est cyclique fini. Dans [Gom98], l'auteur observe que $\mathbb{S}^2\times\mathbb{S}^1$, avec son unique (à isotopie près) structure de contact tendue ξ_{std}, admet une mapping class de contact d'ordre infini; ceci réapparaît dans [DG10], où c'est aussi utilisé pour démontrer que $\ker(j_*|_{\pi_0})$ et $\pi_1(\Xi\left(\mathbb{S}^2\times\mathbb{S}^1\right),\xi_{std})$ sont isomorphes à $\mathbb{Z}$.

La situation en grandes dimensions est moins comprise, à cause du manque de résultats de classification. Les seuls résultats connus à ce moment sont dans [Bou06, MN16, LZ18]. Dans le premier papier, Bourgeois donne des résultats sur quelques groupes d'homotopie $\pi_k\left(\Xi\left(V\right),\xi\right)$, pour des variétés de contact (V,ξ) spécifiques, en utilisant des outils d'homologie de contact. Dans [MN16], Massot et Niederkrüger donnent des exemples de variétés de contact (V,ξ) pour lesquelles $\ker\left(\pi_0\left(\mathcal{D}\left(V,\xi\right)\right)\to\pi_0\left(\mathcal{D}\left(V\right)\right)\right)$ n'est pas trivial; ces exemples utilisent des constructions dans [MNW13], qu'on utilisera aussi dans la suite. Le papier [LZ18], qui porte sur le cas non compact, contient des exemples de plongements de groupes de tresses dans le groupe des contactomorphismes des contactisations de quelques variétés symplectiques non compactes.

Tous les exemples rappelés jusqu'ici sont des variétés de contact *tendues*. On remarque aussi que quelques uns de ces résultats montrent que la rigidité, qui caractérise la classe des structures tendues pour le problème des déformations des structures de contact, entraîne, dans certains cas, de la rigidité aussi pour le problème des déformations des contactomorphismes.

Pour ce qui concerne la classe des structures vrillées, le seul résultat connu à ce moment en dimension 3 est la classification des composantes connexes de l'espace des contactmorphismes pour toutes les structures vrillées sur la 3-sphère. Ce résultat, sans preuve publiée jusqu'à récemment, est attribué à Chekanov dans [EF09, Remark 4.16]. Vogel donne une preuve complète de cette classification dans [Vog18], où il prouve aussi, en utilisant des techniques de topologie de contact en dimension 3, que l'espace des plongements des disques vrillées est non connexe par arcs pour une des structures vrillées sur $\mathbb{S}^3$. Ceci donne en particulier le premier exemple connu de contactomorphisme d'une 3-variété de contact vrillée qui est isotope à l'identité lissement mais pas parmi les contactomorphismes; on rappelle que, d'après [Cer68], chaque difféomorphisme de la 3-sphère qui respecte l'orientation est lissement isotope à l'identité.

Dans Chapitre 3, on s'intéresse au problème de l'existence de sous-groupes cycliques infinis dans $\ker(j_*|_{\pi_0})$, dans le cas tendu. Comme remarqué ci dessus, le seul exemple connu de ce phénomène est donné dans [Gom98, DG10] pour le cas de l'unique (à isotopie près) structure de contact tendue sur $\mathbb{S}^1 \times \mathbb{S}^2$. On généralise ici leur résultat au cas de la structure tendue standard sur $\mathbb{S}^1 \times \mathbb{S}^{2n}$ et, plus généralement, au cas des structures tendues standard sur le produit $\mathcal{D}W \times \mathbb{S}^1$, où $\mathcal{D}W$ est le double $W \cup_{\partial W} \overline{W}$ d'un domaine de Liouville (W^{2n}, λ). Plus précisément, on travaille dans le cadre suivant.

Soit $(F^{2n-2}, \omega_F, Z_F, \psi_F)$ un domaine de Weinstein et on considère sa stabilisation $(F \times \mathbb{C}, \omega_F \oplus \omega_0, Z_F + Z_0, \psi_F + |.|^2_{\mathbb{C}})$, où $\omega_0 = rdr \wedge d\varphi$ et $Z_0 = \frac{1}{2} r\partial_r$, avec coordonnées $z = re^{i\varphi} \in \mathbb{C}$. On suppose que $c > \min \psi_F$ est une valeur régulière de $\psi := \psi_F + |.|^2_{\mathbb{C}}$, et soit W le domain compact $\psi^{-1}((-\infty, c])$. On suppose aussi qu'il y a une structure presque complexe J_F sur F entamée par ω_F et telle que (TF, J_F) est trivial en tant que fibré complexe sur F.
On considère alors le domaine de Weinstein $(F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1, \omega', Z', \psi')$, où, en utilisant des coordonnées $(s, \theta) \in \mathbb{R} \times \mathbb{S}^1$, $\omega' = \omega_F + \omega_0 + 2ds \wedge d\theta$, $Z' = Z + s\partial_s$ et $\psi'(p, z, s, \theta) = \psi(p, z) + s^2$. La préimage $(\psi')^{-1}(c)$, qui est difféomorphe au produit du double $\mathcal{D}W := W \cup_{\partial W} \overline{W}$ de W et de $\mathbb{S}^1$, est naturellement munie de la structure de contact $\xi = \ker \alpha$, où $\alpha = (\iota_{Z'}\omega')|_{\mathcal{D}W \times \mathbb{S}^1}$. De plus, le difféomorphisme de $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$ donné par $(q, z, s, \theta) \mapsto (q, e^{i\theta} z, s, \theta)$ se restreint à un difféomorphisme bien défini Ψ de $\mathcal{D}W \times \mathbb{S}^1$. Dans ce contexte, on prouve le résultat suivant :

Théorème I.A. *Le difféomorphisme Ψ de $\mathcal{D}W \times \mathbb{S}^1$ est lissement isotope à un contactomorphisme Ψ_c de $(\mathcal{D}W \times \mathbb{S}^1, \xi)$ tel que, pour tout entier $k \neq 0$, sa k-ème itérée n'est pas contacto-isotope à l'identité.*

Une application directe du Théorème I.A avec $F = \mathbb{C}^{n-1}$, $\omega_F = 2\sum_{i=1}^{n-1} r_i dr_i \wedge d\varphi_i$, $Z_F = \frac{1}{2}\sum_{i=1}^{n-1} r_i \partial_{r_i}$, $\psi_F(z_1, \ldots, z_{n-1}) = r_1^2 + \ldots r_{n-1}^2$ et $c = 1$, où on utilise des coordonnées polaires $(z_1 = r_1 e^{i\varphi_1}, \ldots, z_{n-1} = r_{n-1} e^{i\varphi_{n-1}})$ sur $F = \mathbb{C}^{n-1}$, donne la généralisation suivante de [Gom98, DG10] à toutes dimensions :

Corollaire I.B. *Soient $(x_1, y_1, \ldots, x_n, y_n, z, \theta)$ des coordonnées sur la variété $\mathbb{R}^{2n+1} \times \mathbb{S}^1$ et ξ la structure de contact tendue sur $V := \mathbb{S}^{2n} \times \mathbb{S}^1$ définie par la restriction de $\lambda = \sum_{i=1}^{n} (x_i dy_i - y_i dx_i) + 2zd\theta$ sur $\mathbb{R}^{2n+1} \times \mathbb{S}^1$ à $\mathbb{S}^{2n} \times \mathbb{S}^1 = \{z^2 + \sum_{i=1}^{n} (x_i^2 + y_i^2) = 1\}$. On considère le difféomorphisme Ψ de $\mathbb{S}^{2n} \times \mathbb{S}^1$ donné par la restriction de*

$$\begin{aligned} \mathbb{R}^{2n+1} \times \mathbb{S}^1 &\to \mathbb{R}^{2n+1} \times \mathbb{S}^1 \\ (x_i, y_i, z, \theta) &\mapsto (\varphi_\theta(x_i, y_i, z), \theta) \end{aligned}$$

où, pour tout $\theta \in \mathbb{S}^1$, $\varphi_\theta \colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ est l'application linéaire qui se restreint à la rotation d'angle θ sur le sous-espace $\mathbb{R}^2$ engendré par (x_n, y_n) et fixe les coordonnées $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z)$.
Alors, Ψ est lissement isotope à un contactomorphisme Ψ_c de (V, ξ) tel que $[\Psi_c^2]$ engendre un sous-groupe cyclique infini de $\ker(\pi_0 \mathcal{D}(V, \xi) \to \pi_0 \mathcal{D}(V))$.

On remarque que tout itérée d'ordre pair de Ψ_c dans Corollaire I.B est effectivement lissement isotope à l'identité : puisque le groupe fondamentale de $SO(m)$ est isomorphe à $\mathbb{Z}_2$ pour tout entier $m \geq 3$, il y a, pour tout $k \in \mathbb{N}$, une isotopie lisse de $\mathbb{S}^{2n} \times \mathbb{S}^1$, préservant (globalement) chaque sous-variété $\mathbb{S}^{2n} \times \{pt\}$, qui relie Ψ^{2k} à l'identité; en particulier, Ψ_c^{2k} est aussi lissement isotope à l'identité.

De façon analogue au Corollaire I.B, une application du Théorème I.A dans le cas $F = T^*\mathbb{T}^n$, $\omega_F = \sum_{i=1}^n dp_i \wedge dq_i$, $Z_F = \frac{1}{2}\sum_{i=1}^n p_i\partial_{p_i}$ et $\psi_F(q_i,p_i) = \sum_{i=1}^n p_i^2$ donne, pour tout entier $n \geq 1$, un autre exemple explicite de (V^{2n+1},ξ) tendue telle que $\ker\left(\pi_0\mathcal{D}\left(V,\xi\right) \to \pi_0\mathcal{D}\left(V\right)\right)$ admet un sous-groupe cyclique d'ordre infini.
En fait, dans ce cas aussi, chaque itérée d'ordre pair de Ψ est lissement isotope à l'identité. Ceci suit des faits que $T^*\mathbb{T}^n \simeq \mathbb{T}^n \times \mathbb{R}^n$, que $\mathcal{D}W \times \mathbb{S}^1 \simeq \mathbb{T}^n \times \mathbb{S}^{n+2} \times \mathbb{S}^1$ et que, pour tout $\theta \in \mathbb{S}^1$, $\Psi\colon\, \mathbb{T}^n \times \mathbb{S}^{n+2} \times \mathbb{S}^1 \to \mathbb{T}^n \times \mathbb{S}^{n+2} \times \mathbb{S}^1$ agit trivialement sur le premier et troisième facteurs et comme une rotation d'angle θ autour d'un axe donné sur chaque $\{pt\} \times \mathbb{S}^{n+2} \times \{\theta\}$; puisque $\pi_1(SO(m)) \simeq \mathbb{Z}_2$ pour tout $m \geq 3$, on peut conclure, comme déjà fait dans le cas du Corollaire I.B, que Ψ_c^{2k} est lissement isotope à l'identité, pour tout $k \neq 1$.
On remarque aussi que le fait que ψ_F sur $T^*\mathbb{T}^n$ ci-dessus n'est pas Morse n'est pas important : en fait, la condition de Morse pour ψ_F ne joue aucun rôle dans la preuve du Théorème I.A.

Dans le Chapitre 4, on étudie la présence d'éléments non-triviaux dans le noyau de l'application $\pi_0\left(\mathcal{D}\left(V,\xi\right)\right) \to \pi_0\left(\mathcal{D}\left(V\right)\right)$ pour quelques variétés de contact vrillées explicites en grandes dimensions, en généralisant le résultat dans [MN16] au cas vrillé. Plus précisément, on commence par démontrer le résultat suivant:

Théorème I.C. *On considère une variété lisse W de dimension $2n \geq 2$ et une structure de contact ξ sur la variété $V := \mathbb{S}^1 \times W$. On suppose que la première classe de Chern $c_1(\xi) \in H^2(V;\mathbb{Z})$ est toroidale et que, pour tout entier $k \geq 2$, la tirée en arrière $\pi_k^*\xi$ de ξ via le revêtement à k-feuilles $\pi_k\colon\, \mathbb{S}^1 \times W \to \mathbb{S}^1 \times W$ donné par $\pi_k(s,p) = (ks,p)$ satisfait $c_1(\pi_k^*\xi) = k \cdot c_1(\xi)$ modulo le sous-module $H^2_{\text{ator}}(V;\mathbb{Z})$ des classes atoroidales.*
Alors, la transformation de contact $f\colon\, (\mathbb{S}^1 \times W, \pi_k^*\xi) \to (\mathbb{S}^1 \times W, \pi_k^*\xi)$ définie par $f(s,p) = (s + \frac{2\pi}{k}, p)$ est isotope à l'identité lissement mais pas parmi les contactomorphismes.

On rappelle qu'une classe $c \in H^2(V;\mathbb{Z})$ est dite *toroidale* s'il existe $f\colon\, \mathbb{T}^2 \to V$ telle que $f^*c \neq 0 \in H^2(\mathbb{T}^2;\mathbb{Z})$, et *atoroidale* sinon.

Remarque. *Le Théorème I.C est aussi vrai (avec preuve similaire) si on remplace*

$(*)$ *$c_1(\xi)$ est toroidale et, pour tout $k \geq 2$, $c_1(\pi_k^*\xi) = k \cdot c_1(\xi) \bmod H^2_{\text{ator}}(V;\mathbb{Z})$,*

par la condition

$(*')$ *$c_1(\xi)$ n'est pas de torsion et, pour tout $k \geq 2$, $c_1(\pi_k^*\xi) = k \cdot c_1(\xi)$.*

On remarque que $a \in H^2(V;\mathbb{Z})$ est toroidal si et seulement si $[a] \in {H^2(V;\mathbb{Z})}\big/{H^2_{\text{ator}}(V;\mathbb{Z})}$ n'est pas torsion, puisque $H^2(\mathbb{T}^2;\mathbb{Z}) \simeq \mathbb{Z}$. En particulier, $(*)$ est equivalent à "$c_1(\xi)$ n'est pas torsion modulo $H^2_{\text{ator}}(V;\mathbb{Z})$ et $c_1(\pi_k^*\xi) = k \cdot c_1(\xi) \bmod H^2_{\text{ator}}(V;\mathbb{Z})$", donc elle est seulement une variation modulo $H^2_{\text{ator}}(V;\mathbb{Z})$ de $(*')$ (et elle n'est pas plus forte ni plus faible que la $(*')$).
On remarque aussi que les structures de contact données par les Propositions I.D et I.F (ainsi que la Proposition I.E.i.) ci-dessous satisfont les Conditions $(*)$ et $(*')$ en même temps; d'autre part, travailler modulo $H^2_{\text{ator}}(V;\mathbb{Z})$, i.e. avec $(*)$, est nécessaire pour la Proposition I.E.ii.
On a donc décidé de tout formuler en termes de $(*)$, même si $(*')$ donnerait (partout sauf dans Proposition I.E.ii.) des preuves un peu plus directes.

On donne ensuite, pour tout $n \geq 1$, un nombre infini de variétés de contact $(\mathbb{S}^1 \times W^{2n}, \xi)$ *explicites* qui satisfont les hypothèses du Théorème I.C :

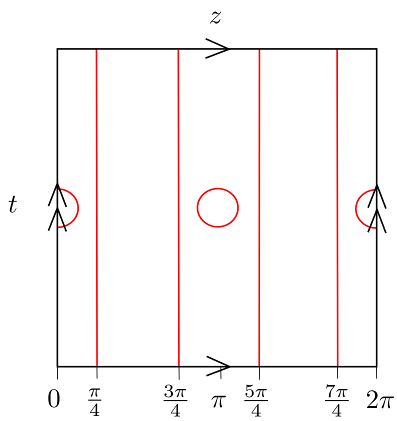

Figure 1.1: Découpage, en rouge, sur le tore $\{\theta_0\} \times \mathbb{T}^2_{(t,s)}$.

Proposition I.D. *Soit $(M^{2n-1}, \alpha_+, \alpha_-)$ une des paires de Liouville construites (en nombre infini) dans [MNW13]. On considère la structure de contact*

$$\eta = \ker\left(\frac{1+\cos(s)}{2}\alpha_+ + \frac{1-\cos(s)}{2}\alpha_- + \sin(s)\,dt\right)$$

sur $V := \mathbb{T}^2_{(s,t)} \times M$ (ici, la notation $\mathbb{T}^2_{(s,t)}$ dénote le choix de coordonnées (s,t) sur $\mathbb{T}^2$) et soit ξ la structure de contact vrillée obtenue par η via un demi twist de Lutz-Mori le long de $\{(0,0)\} \times M$, comme défini dans [MNW13].
Alors, $c_1(\xi) \in H^2(V;\mathbb{Z})$ est toroidale et, pour tout entier $k \geq 2$, on a $c_1(\pi_k^*\xi) = k \cdot c_1(\xi) \mod H^2_{\mathrm{ator}}(V;\mathbb{Z})$, où $\pi_k \colon \mathbb{T}^2_{(s,t)} \times M \to \mathbb{T}^2_{(s,t)} \times M$ est donné par $\pi_k(s,t,q) = (ks,t,q)$.

Exemple. *Si $n = 3$, $(M, \alpha_\pm) = (\mathbb{S}^1, \pm d\theta)$. De plus, si $k = 2$, la structure de contact $\pi_2^*\xi$ sur $V := \mathbb{T}^2 \times M$ est l'unique structure de contact qui est invariante par l'action à gauche par multiplication de $M = \mathbb{S}^1$ sur V, invariant par $f(s,t,\theta) = (s+\pi, t, \theta)$, et telle que chaque tore $\mathbb{T}^2_{(s,t)} \times \{\theta_0\}$ est convexe avec découpage comme dans la Figure 1.1. La Proposition I.D et le Théorème I.C entraînent alors que f n'est pas contacto-isotope à l'identité. On remarque que, même dans ce cas très simple, ce résultat n'était pas présent dans la littérature.*

Si le seul but est celui de donner des exemples, en chaque dimension impaire, d'éléments non triviaux dans le noyau de l'application $\pi_0(\mathcal{D}(V,\xi)) \to \pi_0(\mathcal{D}(V))$, sans vouloir forcement que la variété de contact vrillée sous-jacente (V,ξ) soit très explicite, comme dans le cas de la Proposition I.D, le résultat suivant peut aussi être démontré en utilisant l'existence, démontrée dans [Gir01], des décompositions en livres ouverts qui supportent les structures de contact :

Proposition I.E. *Soit $(V := \mathbb{S}^1 \times W^{2n}, \eta)$ une variété de contact vrillée telle que $c_1(\eta)$ est toroidale et telle que, pour tout $k \geq 2$, la tirée en arrière ξ via le revêtement à k-feuilles $\pi_k \colon \mathbb{S}^1_s \times W \to \mathbb{S}^1_s \times W$, donné par $\pi_k(s,p) = (ks,p)$, satisfait $c_1(\pi_k^*\eta) = k \cdot c_1(\eta) \mod H^2_{\mathrm{ator}}(V;\mathbb{Z})$. Alors, on a que :*

i. La première classe de Chern de chaque structure de contact ξ sur $V \times \mathbb{T}^2$ obtenue via la construction de Bourgeois [Bou02] sur (V,η) satisfait aussi les conditions ci-dessus, par rapport au revêtement $\mu_k := (\pi_k, \mathrm{Id}) \colon V \times \mathbb{T}^2 \to V \times \mathbb{T}^2$.

ii. Soit $\nu \colon V \times \Sigma_g \to V \times \mathbb{T}^2$ induite par un recouvrement $\Sigma_g \to \mathbb{T}^2$ qui est ramifié au dessus de deux points (ici, Σ_g est une surface fermée de genre $g \geq 2$). Alors, la *première classe de Chern de chaque revêtement ramifié ξ_g de ξ sur $V \times \Sigma_g$ satisfait aussi les conditions ci-dessus, par rapport au revêtement $\mu_k^g := (\pi_k, \mathrm{Id})\colon V \times \Sigma_g \to V \times \Sigma_g$. De plus, si η est vrillée et g est assez grand, ξ_g est aussi vrillé.*

Un recouvrement ramifié $\Sigma_g \to \mathbb{T}^2$ comme dans le point ii. ci dessus peut être obtenu, par exemple, en regardant la surface Σ_g comme la 2-sphère avec g anses attachées de façon symétrique par rapport à une rotation de $2\pi/g$ autour de l'axe qui passe par les pôles nord et sud.
Par induction sur la dimension, la Proposition I.E donne, pour tout entier $n \geq 2$, des exemples de $(\mathbb{S}^1 \times W^{2n}, \xi)$ ayant première classe de Chern qui vérifie les propriétés voulue. Concernant le point ii., le lecteur peut consulter [Gei97b] pour une construction des *revêtements ramifiés de contact*, et Section 5.2 pour une définition. On remarque aussi que l'entier g optimal pour garantir le fait que la structure η_g est vrillée est 2, selon une observation due à Niederkrüger (voir l'Observation 9.10).

En utilisant le h-principe de [BEM15], une classe d'exemples (non explicites) encore plus grande peut être obtenue ainsi :

Proposition I.F. *On considère une variété lisse W de dimension $2n$ qui est presque complexe, spin et satisfait $H^1(W;\mathbb{Z}) \neq \{0\}$. Alors, il y a une structure de contact vrillée ξ sur $V := \mathbb{S}^1 \times W$ telle que $c_1(\xi) \in H^2(V;\mathbb{Z})$ est toroidale et $c_1(\pi_k^*\xi) = k \cdot c_1(\xi) \mod H^2_{\mathrm{ator}}(V;\mathbb{Z})$, où $\pi_k\colon \mathbb{S}^1_s \times W \to \mathbb{S}^1_s \times W$ est le revêtement à k-feuilles $\pi_k(s,p) = (ks,p)$.*

Organisation de la Partie I

Le Chapitre 3 est organisé comme suit. Dans la Section 3.1, on décrit une variété de Liouville explicite qui admet $\mathcal{D}W \times \mathbb{S}^1$ comme bord convexe, où $\mathcal{D}W$ est le double $W \cup_{\partial W} \overline{W}$ d'un domaine de Liouville W (ici, $\overline{W}$ dénote W avec orientation opposée). On décrit aussi une façon naturelle de construire des contactomorphismes stricts du bord convexe.

La Section 3.2 introduit la notion de *famille de bases Lagrangiennes* et décrit comment elles donnent un invariant calculable de (classes d'isotopie de contact de) contactomorphismes dans le cas des structures de contact qui sont stablement triviales.

Enfin, la Section 3.3 contient la preuve du Théorème I.A.

Le Chapitre 4 est organisé ainsi. La Section 4.1 contient une preuve par l'absurde du Théorème I.C. En supposant par l'absurde que f est contacto-isotope à l'identité, on construit un contactomorphisme entre deux structures de contact ξ_1 et ξ_2; d'autre côté, l'hypothèse sur la première classe de Chern de ξ entraîne que ξ_1 et ξ_2 ne sont même pas isomorphes en tant que structures presque de contact.

La Section 4.2 montre comment obtenir des exemples de variétés de contact $(\mathbb{S}^1 \times W^{2n}, \xi)$ qui satisfont les hypothèses de Théorème I.C en utilisant les constructions dans [MNW13]. Plus précisément, on rappelle la définition de demi twist de Lutz-Mori et les constructions explicites de paires de Liouville dans [MNW13]. Ensuite, en utilisant une interprétation géométrique des classes de Chern, qu'on rappelle dans l'Appendice A, on décrit les effets d'un demi twist de Lutz-Mori sur les classes de Chern dans ce contexte, et on démontre la Proposition I.D.

Dans la Section 4.3 on décrit comment obtenir des exemples de contactomorphismes qui sont isotopes à l'identité lissement mais pas parmi les contactomorphismes en utilisant l'existence des décompositions adaptés en livres ouverts due à [Gir02] et l'h-principe dans [BEM15]; on prouve en particulier les Propositions I.E et I.F.

1.2 Quelques constructions de variétés de contact revisitées

Les articles [Gei97b] par Geiges et [Bou02] par Bourgeois donnent quelques constructions explicites de structures de contact sur des variétés de grandes dimensions.
Plus précisément, dans le premier papier, en développant des idées dans [Gro86], l'auteur transpose quelques constructions du monde symplectique au cadre de contact, en construisant en particulier les revêtements ramifiés de contact et les sommes fibrées de contact; les réductions de contact sont aussi traitées, mais on ne s'intéressera pas à cette construction dans la suite. (Le lecteur intéressé peut aussi consulter [Gon87] pour le cas des revêtements ramifiés de contact dans le cas de la dimension 3.) Dans le deuxième papier, en prenant inspiration de [Lut79], l'auteur demontre que, étant donné une variété de contact (M^{2n-1}, ξ) et un livre ouvert (B, φ) de M qui porte ξ, il y a une structure de contact η sur $M \times \mathbb{T}^2$ qui est invariant sous l'action naturelle de $\mathbb{T}^2$, qui se restreint à ξ sur chaque sous-variété $M \times \{pt\}$ et qui se déforme naturellement sur la distribution d'hyperplans $\xi \oplus T\mathbb{T}^2$ sur $M \times \mathbb{T}^2$. On remarque que le livre ouvert (B, φ) existe toujours d'après [Gir02].

La motivation principale derrière [Gei97b, Bou02] était le problème de l'existence des structures de contact, i.e. la question de quelles variétés en grandes dimensions admettent des structures de contact. Ce (grand) problème en géométrie de contact a été résolu par [BEM15], et, comme déjà remarqué dans l'introduction générale ci dessus, on s'intéresse maintenant au problème de trouver des exemples *"interessants"* de structures de contact.

Les papiers [Gei97b, Bou02] s'inscrivent bien dans cette perspective, parce qu'ils donnent des variétés de contact plutôt explicites, qu'on peut étudier en détail et qui, sous certains hypothèses, manifestent des propriétés intéressantes, i.e. sont tendues, remplissables, vrillées, etc.
Par exemple, ces deux papiers donnent les premiers procèdures explicites pour construire des variétés de contact en grands dimensions qui sont $\mathcal{PS}$-vrillées (donc vrillées, d'après le travaux postérieurs [CMP15, Hua17]). Le lecteur peut consulter [Pre07] pour le cas de la construction dans [Bou02], [NP10, page 724] pour le cas des revetements ramifiés de contact et [Nie13, Theorem I.5.1], attribué à Presas, pour le cas des sommes fibrées de contact (comparer aussi avec l'Observation 9.10 dans la Section 9.1.2).

Le but de la Partie II est alors de construire des variétés de contact avec des propriétés spécifiques en utilisant [Gei97b, Bou02]. Pour cela, on passe des constructions donnés par Geiges et Bourgeois à des définitions; ensuite, on peut étudier les propriétés de ces structures de contact, indépendamment des choix auxiliaires faites dans leur constructions explicites dans [Gei97b, Bou02].

Concernant les revêtements ramifiés et les sommes fibrées de contact, on remarque que le problème de leur unicité, i.e. la question si les structures de contact obtenues sont indépendantes des choix auxiliaires faites dans leur construction, n'est pas explicitement traité dans [Gei97b]. On propose donc dans cette thèse une définition de revêtements ramifiés et sommes fibrées de contact, qui permet d'obtenir naturellement un énoncé d'unicité (à isotopie près).

On remarque que dans la littérature il y a déjà une définition de revêtement ramifié de contact qui va dans cette direction. Plus précisément, dans [NO07] les auteurs définissent cette notion en termes de déformations en structures de contact qui satisfont une condition additionnelle le long du lieu de ramification. On retire ici cette hypothèse additionnelle et on prouve :

Proposition II.A. *Soit (V^{2n-1}, η) une variété de contact et $\pi\colon \widehat{V} \to V$ un revêtement ramifié avec lieu de ramification (en bas) M. On suppose que $\eta \cap TM$ est une structure de contact sur M. Alors :*

1. *il y a une famille, indexée par $[0,1]$, de distributions d'hyperplans $\widehat{\eta}_t$ sur $\widehat{V}$ telle que $\widehat{\eta}_0 = \pi^*\eta$ et $\widehat{\eta}_t$ est une structure de contact pour tout $t \in (0,1]$;*
2. *si $\widehat{\eta}_t$ et $\widehat{\eta}'_t$ sont comme dans le point 1, alors $\widehat{\eta}_r$ est isotope à $\widehat{\eta}'_s$ pour tout $r, s \in (0,1]$.*

De plus, dans le point 1, $\widehat{\eta}_t$ peut être choisie invariant par les automorphismes (locales) du revêtement π, pour tout $t \in (0,1]$; de façon analogue, l'isotopie du point 2 peut être choisie parmi les structures de contact invariant par les automorphismes (locales) de revêtement, si $\widehat{\eta}_t$ et $\widehat{\eta}'_t$ le sont aussi.

On appelle alors *revêtement ramifié de contact* une structure de contact $\widehat{\eta}$ sur $\widehat{V}$ qui est le point d'arrivée d'un chemin $\widehat{\eta}_t$ comme ci dessus; la Proposition II.A dit exactement que cet objet existe et est bien défini à isotopie près.

À ce point, on peut donner des énoncés précis concernant les propriétés des revêtements ramifiés de contact. Par exemple, on prouve le résultat suivant :

Théorème II.B. *On considère un revêtement ramifié $\pi\colon \widehat{V} \to V$ et une structure de contact ξ sur V, et soit $\widehat{\eta}$ un revêtement ramifié de contact de η. On suppose que (V, η) est faiblement rempli par (W, Ω), de sorte que le lieu de ramification en bas M de π est rempli par une sous-variété symplectique X de (W, Ω). On suppose aussi que π s'étend à un revêtement ramifié $\widehat{\pi}\colon \widehat{W} \to W$, avec lieu de ramification en bas X. Alors, il y a une structure symplectique $\widehat{\Omega}$ sur $\widehat{W}$ qui remplit faiblement $\widehat{\eta}$ sur $\widehat{V} = \partial\widehat{W}$.*

On analyse ensuite la construction de Bourgeois sous différents points de vue.

Premierment, on étudie en détail les liens entre [Bou02] et le papier [Lut79] par Lutz, duquel le premier prends inspiration. En particulier, on rappelle une partie du travail de Lutz dans [Lut79], où il définit une application

$$\left\{ \begin{array}{c} \text{structures de contact} \\ \mathbb{T}^2\text{-invariants sur} \\ \text{le fibré } M \times \mathbb{T}^2 \to M \end{array} \right\} \to \left\{ (\xi, B, \varphi) \;\middle|\; \begin{array}{c} \xi \text{ champ singulier d'hyperplans} \\ \text{sur } M \text{ et } (B, \varphi) \text{ décomposition} \\ \text{en livre ouvert sur } M \end{array} \right\}$$

Ensuite, on remarque que, en utilisant la notion de décomposition en livres ouverts porteurs due à Giroux, les calculs dans [Lut79] montrent (modulo un lemme sur les livres ouverts porteurs par Giroux) que l'image $(\xi_\eta, B_\eta, \varphi_\eta)$ de η via l'application ci dessus est telle que (B_η, φ_η) porte ξ_η, si cette dernière est une structure de contact sur M.

En utilisant ce point de vue des structures de contact $\mathbb{T}^2$-invariant sur le fibré en tores $M \times \mathbb{T}^2 \to M$, on montrera ensuite que, si on part d'une variété de contact (M, ξ) munie d'une décomposition en livre ouvert porteur (B, φ), la composition de la construction de Lutz après celle de Bourgeois redonne la donnée initiale de (M, ξ) et (B, φ).

On analyse aussi la concaténation opposée des deux constructions, et on prouve que, sous certains hypothèses additionnelles, cela donne aussi les données initiales.

Dans une deuxième partie on adopte une perspective qui est, en un certain sens, "orthogonale" à celle ci dessus, en regardant la projection $M \times \mathbb{T}^2 \to \mathbb{T}^2$ au lieu de $M \times \mathbb{T}^2 \to M$. Grâce à la notion de fibré de contact introduite par Lerman dans [Ler04], on peut en fait voir les structures de contact construites par Bourgeois comme des structures de fibrés de contact sur les fibré $M \times \mathbb{T}^2 \to \mathbb{T}^2$, qui admet une connexion plate naturelle. On remarque que cette structure de fibré de contact des exemples dans [Bou02] a déjà été utilisée avec succès dans [Pre07, KN07, NP10] pour obtenir des variétés de contact avec des propriétés remarquables en grandes dimensions.

Dans cette thèse on utilise donc la théorie des fibrés de contact dans [Ler04] pour généraliser la construction de Bourgeois et définir la notion de *structure de contact de Bourgeois*. Plus précisément, sur un fibré $\pi\colon V^{2n+1} \to \Sigma^2$ muni d'une structure de fibré de contact η_0 de référence, chaque fibré de contact η admet un *potentiel* A par rapport à η_0, avec une forme de courbature R_A; dans le cas où le fibré de contact de référence η_0 est plat, on appelle structure de contact de Bourgeois chaque structure de fibré de contact sur $\pi\colon V \to \Sigma$ qui est aussi une structure de contact sur V et satisfait $\frac{1}{\epsilon} R_{\epsilon A} \to 0$ pour $\epsilon \to 0$.

En plus de la nécessité de passer de la procédure de construction dans [Bou02] à une définition, une autre motivation pour l'introduction de cette notion est la suivante : la condition sur la courbure est, d'un côté, suffisamment faible pour être satisfaite par une classe de structures de contact qui contient strictement les résultats de la construction dans [Bou02] et, de l'autre côté, suffisamment forte pour assurer, par exemple, des bonnes propriétés du point de vue des remplissages faibles et des décompositions en livre ouvert porteur.

Concernant la remplissabilité faible, on prouve le résultat suivant :

Proposition II.C. *Soit (M^{2n-1}, ξ) une variété de contact et η une structure de Bourgeois sur le fibré trivial $M \times \mathbb{T}^2 \to \mathbb{T}^2$, qui se restreint à ξ sur $M \times \{pt\} = M$. Si (M, ξ) est faiblement rempli par (X^{2n}, ω), alors $(M \times \mathbb{T}^2, \eta)$ est faiblement rempli par $(X \times \mathbb{T}^2, \omega + \omega_{\mathbb{T}^2})$, où $\omega_{\mathbb{T}^2}$ est une forme volume sur $\mathbb{T}^2$.*

On remarque que le résultat est déjà connu dans le cas de la construction de Bourgeois [Bou02] : l'énoncé et l'idée principale de la preuve sont présentés dans [MNW13, Example 1.1]; voir aussi [LMN18, Theorem A.a] pour une preuve explicite.

Du point de vue des livres ouverts porteurs, les structures de Bourgeois ont implicitement de l'information sur des décompositions en livres ouverts qui portent les structures de contact sur chaque fibre :

Proposition II.D. *Soit η une structure de contact de Bourgeois sur $\pi\colon V \to \Sigma$. Il existe une application ψ_η qui associe à chaque point $b \in \Sigma$ une classe d'isotopie de livres ouverts porteurs sur $(M_b, \xi_b) := \left(\pi^{-1}(b), \eta \cap T\left(\pi^{-1}(b)\right)\right)$. De plus, si $\gamma(t)$, avec $t \in (-\epsilon, \epsilon)$, est un chemin dans un ensemble ouvert U de Σ sur lequel π est trivialisée , i.e. sur lequel π devient la projection sur le premier facteur $\mathrm{pr}_U\colon U \times M \to U$, alors le chemin de classes d'isotopies $\psi_\eta \circ \gamma(t)$ vient d'un chemin de livres ouverts (B_t, φ_t) de $\{\gamma(t)\} \times M$ tel que son image via $\mathrm{pr}_M\colon U \times M \to M$ est une isotopie de livres ouverts sur M.*
Dans le cas des exemples dans [Bou02], via la projection globale $\mathrm{pr}_M\colon M \times \mathbb{T}^2 \to M$, l'application ψ_η donne la classe d'isotopie du livre ouvert (B, φ) utilisé dans la construction.

Autrement dit, si d'un côté la construction de Lutz donne une inverse à celle de Bourgeois du point de vue des structures de contact $\mathbb{T}^2$-invariantes sur le fibré principal $M \times \mathbb{T}^2 \to M$, de l'autre côté ce résultat donne une inverse à la construction de Bourgeois du point de vue des fibrés de contact sur $M \times \mathbb{T}^2 \to \mathbb{T}^2$.

Pour démontrer la Proposition II.D, on donne une reinterprétation des livres ouverts porteurs en termes de paires de champs de vecteurs de contact :

Théorème II.E. *Sur une variété de contact (M^{2n-1}, ξ), chaque paire de champs de vecteurs de contact X, Y, telle que $[X, Y]$ est partout transverse à ξ, donne un livre ouvert explicite de M qui porte ξ. Vice versa, un livre ouvert qui porte ξ donne une paire X, Y comme ci dessus.*

La deuxième partie de ce résultat a été énoncé par Giroux dans des exposés pour le Yashafest de Juin 2007 et pour le AIM workshop de Mai 2012 (voir [Gir12, Claim on page 19]). Un énoncé plus détaillée (ainsi qu'une preuve) du Théorème II.E sera donné dans la Chapter 6. On remarque que ce résultat n'est pas seulement utile à la preuve de la Proposition II.D mais donne aussi un autre point de vue sur les livres ouverts porteurs, ce qui a un intérêt en soi.

Ces réinterprétations et généralisations de [Gei97b, Bou02] amènent à quelques exemples de variétés de contact en grandes dimensions avec des propriétés intéressantes, i.e. qui sont tendues, remplissables ou vrillées. On obtient ainsi deux nouveaux résultats, l'un concernant les variétés tendues virtuellement vrillées et l'autre concernant les plongements des 3-variétés de contact en codimension 2 avec fibré normal trivial.

Concernant le premier résultat, on rappelle que une structure de contact tendue ξ sur M est dite *virtuellement vrillée* si sa tirée en arrière $\widehat{\xi}$ sur un revêtement fini $\widehat{M}$ de M est vrillée. Dans cette thèse, on demontre le résultat suivant :

Théorème II.F. *Les structures virtuellement vrillée existent en toutes dimensions* ≥ 3.

La preuve de ce résultat est par induction sur la dimension. Pour ce qui concerne l'initialisation, l'existence des structures de contact tendues virtuellement vrillées est connue en dimension 3 depuis [Gom98]; le lecteur peut aussi consulter [Gir00, Hon00], qui presentent une classification de ce type de structures de contact sur des 3-variétés particulières. L'étape inductive utilise les Propositions II.C et II.A ci-dessus, i.e. le fait que la construction dans [Bou02] et les revêtements ramifiés de contact préservent la remplissabilité faible, et s'appuie sur l'existence des livres ouverts porteurs démontrée par Giroux [Gir02], sur la construction e Bourgeois [Bou02] et sur le critère des "grandes" voisinages [CMP15, Theorem 3.1].

L'autre application concerne la question suivante : étant donné une variété de contact $(M, \xi = \ker\alpha)$, existe-t-il un $\epsilon > 0$ tel que $\left(M \times D^2_\epsilon, \ker\left(\alpha + r^2 d\varphi\right)\right)$ est tendue? Ici, D^2_ϵ est le disque de rayon ϵ et centré à l'origine de $\mathbb{R}^2$, et (r, φ) sont ses coordonnées polaires.

Ceci est lié au problème de trouver des plongements de contact en codimension 2 avec fibré normal trivial dans des variétés ambiantes tendues. En fait, en codimension 2, le fibré normal est trivial si et seulement si le fibré conforme symplectique normal l'est. Donc, d'après le théorème des voisinages de contact [Gei08, Theorem 2.5.15], si $(M^{2n-1}, \xi = \ker\alpha)$ se plonge dans (V^{2n+1}, η) avec fibré normal trivial, alors il admet un voisinage $\left(M \times D^2_{r_0}, \ker\left(\alpha + r^2 d\varphi\right)\right)$, pour un certain $r_0 > 0$. En particulier, si (V, η) est tendue, ce voisinage l'est aussi.

Historiquement, la première motivation pour s'intéresser à cette question sur la "taille" d'un voisinage d'une sous-variété de contact en codimension 2 est donnée par [NP10], où il est démontré que des voisinages "larges" de sous-variétés de contact vrillée sont une obstruction pour la remplissabilité de la variété de contact ambiante. Comme signalé dans [Nie13], ceci a amené Niederkrüger et Presas à conjecturer que la présence d'un domaine contactomorphe au produit d'un $\mathbb{R}^3$ vrillé et d'un "grand" voisinage dans $\mathbb{R}^{2n}$ avec la structure de Liouville standard aurait pu être la bonne généralisation de la notion de vrillé aux dimensions impaires plus grandes que 3. Après l'introduction dans [BEM15] d'une définition de structure de contact vrillée en toutes dimensions impaires, [CMP15] à confirmé cette conjecture, en démontrant que la présence d'un tel domaine est en fait équivalent à être vrillé. Plus précisément, ceci suit de [CMP15, Theorem 3.1], qui affirme que, si $(M, \xi = \ker\alpha)$ est vrillé, alors $\left(M \times D^2_R, \ker\left(\alpha + r^2 d\varphi\right)\right)$ l'est aussi, si $R > 0$ est suffisamment grand. En particulier, ceci motive la question ci dessus sur l'existence, pour une variété de contact $(M, \xi = \ker\alpha)$ donnée, d'un $\epsilon > 0$ tel que $\left(M \times D^2_\epsilon, \ker\left(\alpha + r^2 d\varphi\right)\right)$ est tendu.

Le problème de trouver des plongements en codimension 2 dans des variétés tendues a

été déjà traité par [CPS16, EF17, EL17]. Plus précisément, [CPS16] démontre que chaque 3-variété de contact fermée vrillée peut être plongée avec fibré normal trivial dans une 5-variété de contact fermée qui admet un remplissage symplectique exact. Dans [EF17], les auteurs montrent comment plonger plusieurs 3-variétés de contact dans la sphère de contact standard de dimension 5. Enfin, dans [EL17] il est démontré que chaque 3-variété de contact peut être plongé dans le (unique) fibré en $\mathbb{S}^3$ non trivial sur $\mathbb{S}^2$, muni d'une structure de contact Stein remplissable.
Dans cette thèse, on demontre le résultat suivant :

Théorème II.G. *Chaque 3-variété de contact fermée (M, ξ) avec $\pi_1(M) \neq \{1\}$ peut être plongée avec fibré normal trivial dans une 5-variété de contact (V^5, η) fermée (hyper)tendue.*

Corollaire II.H. *Pour tout $(M^3, \xi = \ker\alpha)$ avec $\pi_1(M) \neq \{1\}$, il existe $\epsilon > 0$ tel que $\left(M \times D^2_\epsilon, \ker\left(\alpha + r^2 d\varphi\right)\right)$ est tendu.*

On rappelle qu'une structure de contact est dite *hypertendue* si elle admet une forme de contact sans orbites de Reeb fermées contractiles. Toute variété de contact hypertendue est en particulier tendue, d'après [Hof93, AH09, CMP15].

On observe aussi que, d'après la preuve de Perelman de la conjecture de Poincaré, la condition $\pi_1(M) \neq \{1\}$ est équivalente au fait que M n'est pas difféomorphe à $\mathbb{S}^3$. Or, le cas $M = \mathbb{S}^3$ est déjà connu. En fait, le cas des 3-sphères vrillées est traité dans [CPS16, Proposition 11], déjà cité ci dessus, et la 3-sphère tendue standard (qui est l'unique structure de contact tendue sur $\mathbb{S}^3$ à isotopie près d'après [Eli92]), se plonge de façon naturelle dans la 5-sphère standard avec fibré normal trivial.

Les ingrédients principaux qu'on utilise dans la preuve du Théorème II.G sont l'existence des livres ouverts porteurs pour les 3-variétés de contact démontrée par Giroux, et une preuve détaillée de la dynamique du flot de Reeb des formes de contact construites dans [Bou02].
Plus précisément on montre que, si $\pi_1(M) \neq \{1\}$, chaque livre ouvert (B, φ) de M peut être modifié, via des stabilisations positives, de sorte que chaque composante connexe de B soit homotopiquement non triviale (dans M). On démontre ensuite que ceci permet d'obtenir, en utilisant [Bou02], des formes de contact hypertendues sur $M \times \mathbb{T}^2$. Enfin, (M, ξ) se plonge naturellement dans la variété de contact construite par Bourgeois comme une fibre de la fibration $M \times \mathbb{T}^2 \to \mathbb{T}^2$ donnée par la projection sur le deuxième facteur.

Concernant le Corollaire II.H, on remarque qu'il a été récemment généralisé à toutes dimensions, avec des techniques complètement différentes, dans [HMP18].

Organisation de la Partie II

Dans le Chapitre 5, on donne les nouveaux approches annoncés aux revêtements ramifiés de contact et aux sommes fibrées de contact introduites dans [Gei97b], et on démontre en particulier la Proposition II.A et le Théorème II.B.

Le Chapitre 6 décrit la formulation équivalente, basée sur une idée par Giroux [Gir12], de décompositions en livres ouverts porteurs en termes de paires de champs de vecteurs de contact, et contient donc la preuve du Théorème II.E.

Le Chapitre 7 rappelle la construction de Bourgeois dans [Bou02] et décrit l'étude des structures de contact invariantes sur les fibrés principaux en $\mathbb{T}^2$ donné dans [Lut79].

Ensuite, on reformule et généralise dans le Chapitre 8 la construction de Bourgeois en utilisant la notion de fibré de contact introduite dans [Ler04]. En particulier, on donne la définition de structure de contact de Bourgeois et on démontre la Proposition II.D.

Le Chapitre 9 contient ensuite deux applications des outils développés du Chapitre 6 jusqu'au 8. Plus précisement, elle contient l'étude de la remplissabilité faible des structures de contact de Bourgeois, i.e. les preuves de la Proposition II.C et du Théorème II.F, ainsi qu'une analyse de la dynamique du champ de Reeb des formes de contact [Bou02], i.e. les preuves du Théorème II.G et du Corollaire II.H.

Chapter 2

Background

2.1 Basic definitions

In all this section, V denotes a $(2n+1)$-dimensional oriented smooth manifold.

Definition 2.1. A hyperplane field ξ on V is called a *contact structure* if, for each $p \in V$, there is an open neighborhood U of p and $\alpha \in \Omega^1(U)$ such that $\xi|_U = \ker \alpha$ and $d\alpha|_{\xi|_U}$ is a symplectic structure on $\xi|_U$.

One can easily see that this is equivalent to the definition given at the beginning of Chapter 1 in terms of the the volume form $\alpha \wedge d\alpha^n$; both points of view will be adopted in this thesis. Moreover, we will actually focus only on *cooriented* contact structures in the following, i.e. those for which there is a global defining form α, also called *contact form* for the contact structure.

Definition 2.2. Let (V, ξ) and (V', ξ') be two $(2n+1)$-dimensional contact manifold. A diffeomorphism $\varphi\colon V \to V'$ is called *contactomorphism*, and denoted $\varphi\colon (V,\xi) \to (V',\xi')$, if $\varphi^*\xi = \xi$ (i.e. if $\ker(\varphi^*\alpha') = \xi$ for each contact form α' for ξ'). Moreover, a smooth isotopy $(\varphi_t)_{t\in[0,1]}$ from V to V' is called *contact isotopy* if φ_t is a contactomorphism for all $t \in [0,1]$.
Similarly, in the case $\dim(V) < \dim(V')$, an embedding $\varphi\colon V \to V'$ such that $\varphi^*\xi' = \xi$ is called *contact embedding*.

Here's a fundamental criterion to find contactomorphisms between contact structures, which will be used many times in this thesis:

Theorem 2.3 (Gray)**.** *Let V be a closed manifold and $(\xi_t)_{t\in[0,1]}$ a smooth family of contact structures on V, i.e. a family of contact structures defined by a smooth family $(\alpha_t)_{t\in[0,1]}$ of contact forms on V. Then, there is an isotopy $\varphi_t\colon V \to V$ such that $\varphi_t^*\xi_t = \xi_0$.*

For the proof of this result we invite the reader to consult [Gei08, Section 2.2]. As we will use it in the following, we just point out that the isotopy φ_t is generated by the time-dependent vector field X_t such that $X_t \in \xi_t$ and $\iota_{X_t} d\alpha_t|_{\ker\alpha_t} = -(\frac{d}{dt}\alpha_t)|_{\ker\alpha_t}$; notice that this X_t is unique by the contact condition.
We also point out that the hypothesis of V closed is crucial, because Gray's theorem does not hold on open manifolds for instance.

We recall that a choice of a particular contact form α on V gives a unique vector field R_α, called *Reeb vector field* of α, such that $\alpha(R_\alpha) = 1$ and $\iota_{R_\alpha} d\alpha = 0$ everywhere on V.
Reeb vector fields are particular cases of the following:

Definition 2.4. Let ξ be a contact structure on V. A vector field X on V is called *contact vector field* if, for a (hence every) α contact form defining ξ, there is $f\colon V \to \mathbb{R}$ positive such that $\mathcal{L}_X\alpha = f\alpha$.

For this class of vector fields, we have the following correspondence:

Proposition 2.5. *Let α be a contact form on V. Then, contact vector fields for $\xi = \ker\alpha$ are in 1-1 correspondence with functions $H\colon V \to \mathbb{R}$. More precisely, the correspondence is given by:*

- $X \mapsto H_X \coloneqq \alpha(X)$*;*
- $H \mapsto X_H$*, uniquely defined by the conditions* $\alpha(X_H) = H$ *and* $\iota_{X_H}d\alpha = dH(R_\alpha)\alpha - dH$*.*

With the name *contact Hamiltonians*, we will hence refer to a function $H\colon V \to \mathbb{R}$, seen as generating a contact vector field X_H (after having fixed a contact form α). For a proof of Proposition 2.5, we invite the reader to consult [Gei08, Section 2.3].
As it will be useful for Sections 4.1 and 5.3, we also point out the following link between time-dependent contact Hamiltonians and contact isotopies:

Proposition 2.6. *Let α be a contact form on V. Then, there is a 1-1 correspondence between time-dependent contact Hamiltonians H_t and contact isotopies φ_t given by:*

- $H_t \mapsto \varphi_t \coloneqq \psi^t_{X_t}$*, where* $\psi^t_{X_t}$ *is the flow at time t of the time dependent contact vector field* $X_t \coloneqq X_{H_t}$ *(given by Proposition 2.5);*
- $\varphi_t \mapsto H_t \coloneqq \alpha(X_t)$*, where X_t is the vector field generating the isotopy φ_t.*

For a proof of Proposition 2.6, the reader can consult for instance [Gei08, Section 2.2].

2.2 Tight-overtwisted dichotomy

As already mentioned in the short introduction above, in each odd dimension, contact structures are divided in two different subclasses: *tight* and *overtwisted*. The paper [Eli89] introduces this dichotomy in dimension 3, and it is then generalized to all dimensions in [BEM15].

We point out that the exact definition of overtwistedness given in [BEM15] is rather technical, and we will not need it explicitly in this thesis. We hence decided to give here some equivalent conditions that we will actually use in the rest of this manuscript, namely in Chapter 4 and Section 9.1. For this, we need to first recall the following definition:

Definition 2.7 ([Nie06])**.** Let S be a closed smooth manifold of dimension n and λ the standard Liouville form on T^*S. Consider also the contact form $\alpha_{ot} \coloneqq \cos r dz + r\sin r d\theta$ on $\mathbb{R}^3$, where we use cylindrical coordinates (r, θ, z), and let D be the disk $\{z = 0, r \leq \pi\}$ inside $(\mathbb{R}^3, \ker\alpha)$. We call *plastikstufe* with core S, and denote it $\mathcal{PS}$, the submanifold $D \times S \subset \mathbb{R}^3 \times T^*S$ together with the germ of contact structure along it given by $\ker(\alpha + \lambda)$ on $\mathbb{R}^3 \times T^*S$.

We can now recall the two formulations of overtwistedness in high dimensions that we will need in the following:

Theorem 2.8 ([CMP15, Hua17])**.** *Let (V, ξ) be a $(2n+1)$-dimensional contact manifold, and $\alpha_{ot} \coloneqq \cos r dz + r\sin r d\varphi$ on $\mathbb{R}^3$, where we use cylindrical coordinates (r, φ, z). Then, the following are equivalent:*

1. *(V, ξ) is overtwisted (in the sense of [BEM15]);*

2. *there is a contact embedding $(\mathbb{R}^3 \times \mathbb{C}^{n-1}, \ker(\alpha_{ot} + \lambda_{std}) \hookrightarrow (V, \xi)$, where λ_{std} is the standard Liouville form $\sum_{i=1}^{n-1} r_i^2 d\varphi$ on $\mathbb{C}^{n-1}$;*

3. *there is a embedding (respecting the germs of contact structures) of a plastikstufe $\mathcal{PS}$ into (V, ξ).*

We point out that [CMP15] contains the equivalence between points 1, 2 and 3, with the additional hypothesis of $\mathcal{PS}$ having *spherical core and trivial rotation*; then, [Hua17] proves that this additional hypothesis on $\mathcal{PS}$ is not needed for the above equivalence.
In Sections 4.3 and 9.1, we will actually use more specifically the following result, which is used in [CMP15] in order to prove the equivalence between points 1 and 2 above:

Theorem 2.9 ([CMP15, Theorem 3.1])**.** *Let $(V^{2n+1}, \xi = \ker \alpha)$ be overtwisted. Then, $(V \times \mathbb{C}, \ker(\alpha + \lambda_{std}))$ is overtwisted too.*

The interest in the distinction between overtwisted and tight is the fact, already pointed out above, that the class of overtwisted contact structures satisfy an h-principle with respect to the problem of constructing contact structures.
In order to give a specific statement, we need to introduce a formal equivalent of contact structures, which we will need also in Section 4.3:

Definition 2.10. An *almost contact structure* is a couple $(\xi, [\omega])$, where ξ is a hyperplane field on V, ω is a symplectic structure on ξ and $[\omega]$ is its conformal class.

We recall that the *conformal class* of ω is defined by the following relation: $\omega_1 \sim \omega_2$ if there is a smooth positive function $f\colon V \to \mathbb{R}$ such that $\omega_2 = f\omega_1$.
Notice that, given a contact structure ξ on V and a contact form α defining it, the conformal class $[d\alpha|_\xi]$ does not depend on the choice of α defining ξ; it is hence called *natural conformal symplectic structure* on ξ and denoted by CS_ξ. In particular, we have a natural inclusion of the space of contact structures in the space of almost contact structures given by $\xi \mapsto (\xi, \mathrm{CS}_\xi)$.

As it will be used in the following chapters, we also point out that each almost contact structure $(\xi, [\omega])$ admit a complex structure J (i.e. a $J \in End(\xi)$ such that $J^2 = -\mathrm{Id}$) which is tamed by $[\omega]$ (i.e. $[\omega]$ is positive on complex lines). Moreover, the space of such J is contractible and, in particular, $(\xi, [\omega])$ have well defined Chern classes; this will be used in Part I.

As anticipated in Chapter 1, overtwisted contact structures satisfy the following h-principle:

Theorem 2.11 ([Eli89, BEM15])**.** *The natural inclusion of the space of overtwisted contact structures in the space of almost contact structures induces a bijection at the π_0 level.*

We point out that [Eli89, BEM15] actually contains a more general parametric h-principle; though, this will not be needed in the following.

2.3 Different shades of fillability

Among tight structures, one can further recognize a special subclass: the *fillable* contact structures.

In the short introduction above, we described the notion of *convex boundary* of a symplectic manifold, which, from the contact point of view, gives the notion of *(strong)*

symplectic filling. As it will be the case in this thesis, depending on the particular situation under consideration, it may be more convenient to replace this notion with a stronger or a weaker one.
On the stronger side of the spectrum, we have the following:

Definition 2.12. A symplectic manifold (W, ω) is called a *Liouville manifold* if there is an exhaustion $W = \cup_{k=0}^{+\infty} W_k$ by compact domains and there is a global Liouville vector field Z for ω which is complete and points outwards along the boundary of each W_k. Moreover, in the case where $W = W_k$ for a certain k, $(W, \lambda := \iota_Z \omega)$ is called a *Liouville (or exact) filling* of $(V = \partial W, \xi = \ker(\lambda|_\partial W))$.

We point out that the fact that $\lambda|_M$ is a positive contact form on M is equivalent to the fact that the Liouville vector field Z of (W, λ) is positively transverse to $V = \partial W$. Moreover, this notion can also be strengthened further:

Definition 2.13. A *Weinstein manifold* is the data of (W, ω, Z, ψ), where (W, ω) is a symplectic $2n$-dimensional manifold, ψ is an exhausting (i.e. proper and bounded from below) Morse function on W and Z is a Liouville vector field for (W, ω) which is gradient-like for ψ. Given a regular value c for ψ, $(M := \psi^{-1}(c), \ker(\iota_Z \omega|_M))$ is said to admit the *Weinstein filling* $(\psi^{-1}((-\infty, c]), \omega, Z, \psi)$.

The complex counterpart of Weinstein structures are the following:

Definition 2.14. A *Stein manifold* is the data of (W, J, ψ), where (W, J) is a complex manifold and $\psi \colon W \to \mathbb{R}$ is exhausting and J-convex.

We recall that $d^{\mathbb{C}}\psi := d\psi \circ J \in \Omega^1(W)$, and that ψ is called *J-convex* if $\omega_\psi := -dd^{\mathbb{C}}\psi$ is a symplectic form on W which is compatible with J, i.e. such that $g_\psi(.,.) := \omega_\psi(., J.)$ is a Riemannian metric on W.

As we will never be interested in the integrability of almost complex structures, we will actually only need the following weaker notion:

Definition 2.15. An *almost Stein manifold* is the data of (W, J, ψ), where (W, J) is an almost complex manifold with boundary, $\psi \colon W \to \mathbb{R}$ is J-convex, ∂W is a regular level set of ψ and J is ω_ψ compatible, where $\omega_\psi := -dd^J\psi$.

Notice that there is a preferred Liouville vector field Z_ψ for ω_ψ, given by the symplectic dual of $-d^J\psi$.

In Section 3.3, we will use the following result on Weinstein structures:

Theorem 2.16. *Let W be a smooth $2n$-dimensional smooth manifold with boundary. Every Weinstein structure (ω, Z, ψ) on W is homotopic, in the space of Weinstein structures with fixed exhausting function ψ, to a Weinstein structure $(\omega_\psi, Z_\psi, \psi)$ coming from an almost Stein structure (J, ψ).*

The above theorem, which can in some sense be considered a "folklore" result in the field, is a weaker form of [CE12, Theorem 1.1], which gives a genuine Stein structure.

In the rest of the manuscript we will also need another version of the fillability condition, which is weaker than all those recalled above.
Let (V^{2n+1}, ξ) be a contact manifold and (W^{2n+2}, ω) a symplectic manifold such that $V = \partial W$. Denote by ω_ξ the restriction of ω to ξ and by CS_ξ the standard conformal symplectic structure on ξ, i.e. the conformal class of $d\alpha|_\xi$, for an arbitrary 1-form α defining ξ. We point out that CS_ξ is well defined because independent of such a choice of α.

Definition 2.17 ([MNW13]). We say that (W, ω) *weakly fills* (V, ξ), or that ω *weakly dominates* ξ, if ω_ξ is symplectic on ξ and $\omega_\xi + \mathrm{CS}_\xi$ is a ray of symplectic structures on ξ.

Another way to formulate this condition is the following: ω weakly dominates $\xi = \ker \alpha$ if and only if $\alpha \wedge (\omega_\xi + \tau d\alpha)^n > 0$ for all $\tau \geq 0$. This will be the formulation we will actually use in the following, as it is more computation-friendly.

2.4 Open books in contact geometry

We recall that an *open book decomposition* on a manifold V is a pair (B, φ), where B is a codimension 2 submanifold and $\varphi\colon V \setminus B \to \mathbb{S}^1$ is a fibration, such that there is a neighborhood $B \times D^2$ of B in V on which φ becomes $B \times \left(D^2 \setminus \{0\}\right) \to \mathbb{S}^1$ given by $(q, r, \theta) \mapsto \theta$, where (r, θ) are polar coordinates on D^2. Moreover, for each $\theta \in \mathbb{S}^1$, the closure $\overline{\varphi^{-1}(\theta)}$ is called a *page* of (B, φ).

Notice that an open book decomposition (B, φ) of V determines a unique element of the mapping class group of the page F, called the *monodromy* of the open book. Following for instance [Gir17], this can be seen as follows.

Consider X a *spinning vector fields* for (B, φ), i.e. a vector field X on V such that:

- $X = 0$ along B and $d\varphi(X) = 1$ on $V \setminus B$;
- X lifts to a smooth vector field on the manifold with boundary obtained from V by a real oriented blowup along B.

Then, the restriction of the flow of X at time 1 gives a diffeomorphism of the page F, relative to B. Moreover, as the set of spinning vector fields for (B, φ) is an affine space, the isotopy class of this diffeomorphism depends only on (B, φ). This mapping class is the *monodromy* of the open book (B, φ).

The link between open book decompositions and contact geometry is captured by the following notion:

Definition 2.18 ([Gir02]). Let (V, ξ) be a contact manifold. An open book decomposition (B, φ) is said to *support* the contact structure ξ if the following conditions are satisfied:

1. B is a contact submanifold of (V, ξ);
2. there is a contact form α defining ξ such that, for each $\theta \in \mathbb{S}^1$, $d\alpha|_{\varphi^{-1}(\theta)}$ is a positive symplectic form on the fiber $\varphi^{-1}(\theta)$;
3. for each $\theta \in \mathbb{S}^1$, the orientation on B induced by ξ coincides with its orientation as boundary of the symplectic manifold $(\varphi^{-1}(\theta), d\alpha|_{\varphi^{-1}(\theta)})$.

Such a contact form α is said to be *adapted* to the open book (B, φ) of V.

The above definition is interesting due to the following (deep) result:

Theorem 2.19 ([Gir02]). *Each contact structure on a closed manifold is supported by an open book decomposition.*

According to [Gir02], such a supporting open book is moreover *unique up to stabilization* in the case of 3-dimensional contact manifolds; as we will need it in Section 9.2, we now give a statement for this uniqueness property.

As explained in [GG06], the following result is essentially due to Stallings:

Theorem 2.20 ([Sta78]). *Let Σ be a compact surface with boundary in a manifold M and δ_0 a properly embedded arc in Σ. Let also $\Sigma' \subset M$ be obtained by plumbing a positive/negative Hopf band to Σ, i.e. if $\Sigma' = \Sigma \cup A^{\pm}$ where $A^{\pm}$ is an annulus in M such that*

1. *the intersection $A^{\pm} \cap \Sigma$ is a tubular neighborhood of δ_0,*
2. *the core curve of $A^{\pm}$ bounds a disk in $M \setminus \Sigma$ and the linking number of the boundary components is ± 1.*

If Σ is a page of an open book decomposition (B, φ) of M, then Σ' is also a page of an open book (B', φ').

Following [Gir02], an open book (B', φ') is said to be a *stabilization* of (B, φ) if it is obtained by (B, φ) via a finite sequence of plumbings of positive Hopf bands. Then, we have the following:

Theorem 2.21 ([Gir02]). *On a 3-dimensional closed manifold, two open book decompositions support the same contact structure if and only if they have isotopic stabilizations.*

We point out that, in this manuscript, and more precisely in Section 9.2, we will actually only need the easier part of this result, i.e. the fact that an open book and all its stabilizations support the same contact structure.

Part I

The topology of the contactomorphism group

Chapter 3

On contact mapping classes of infinite order

The aim of this chapter is to give a proof of Théorème I.A; we start by recalling its setting and statement.

Let $(F^{2n-2}, \omega_F, Z_F, \psi_F)$ be a Weinstein manifold and consider its stabilization $(F \times \mathbb{C}, \omega_F \oplus \omega_0, Z_F + Z_0, \psi_F + |.|_{\mathbb{C}}^2)$, where $\omega_0 = r dr \wedge d\varphi$ and $Z_0 = \frac{1}{2} r \partial_r$ if we use coordinates $z = re^{i\varphi} \in \mathbb{C}$. Suppose that $c > \min \psi_F$ is a regular value of $\psi := \psi_F + |.|_{\mathbb{C}}^2$ and let W be the compact domain $\psi^{-1}((-\infty, c])$. We suppose also that there is an almost complex structure J_F on F tamed by ω_F and such that (TF, J_F) is trivial as complex bundle over F.

Consider now the Weinstein manifold $(F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1, \omega', Z', \psi')$, where, using coordinates $(s, \theta) \in \mathbb{R} \times \mathbb{S}^1$, $\omega' = \omega_F + \omega_0 + 2ds \wedge d\theta$, $Z' = Z + s\partial_s$ and $\psi'(p, z, s, \theta) = \psi(p, z) + s^2$. The preimage $(\psi')^{-1}(c)$, which is diffeomorphic to the product of the double $\mathcal{D}W := W \cup_{\partial W} \overline{W}$ of W and $\mathbb{S}^1$, is naturally equipped with the contact structure $\xi = \ker \alpha$, where $\alpha = \iota_{Z'} \omega'|_{\mathcal{D}W \times \mathbb{S}^1}$. Moreover, the diffeomorphism of $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$ given by $(q, z, s, \theta) \mapsto (q, e^{i\theta} z, s, \theta)$ restricts to a well defined diffeomorphism Ψ of $\mathcal{D}W \times \mathbb{S}^1$.

Théorème I.A. *Le difféomorphisme Ψ de $\mathcal{D}W \times \mathbb{S}^1$ est lissement isotope à un contactomorphisme Ψ_c de $(\mathcal{D}W \times \mathbb{S}^1, \xi)$ tel que, pour tout entier $k \neq 0$, sa k-ème itérée n'est pas contacto-isotope à l'identité.*

This chapter is organized as follows. Section 3.1 describes how, given a Liouville domain W, one can naturally construct an explicit Liouville manifold having $\mathcal{D}W \times \mathbb{S}^1$ as convex boundary, as well as contactomorphisms of the latter; this will then be used in the case of Weinstein domains in the proof of Théorème I.A.

In Section 3.2 we describe a simple invariant, of homotopical nature, for (contact-isotopy classes of) contactomorphisms, introducing the notion of *families of Lagrangian bases*. This invariant will then be used especially in the case of stably trivial contact structures. Then, in Section 3.3 we prove Théorème I.A recalled above.

3.1 Product of doubled Liouville domains and $\mathbb{S}^1$

Let $\widehat{W}^m$ be a smooth manifold and $f\colon \widehat{W} \to \mathbb{R}$ be a proper and bounded from below function which is also a regular equation of a (cooriented) hypersurface $M^{m-1} \subset \widehat{W}$, i.e. a smooth proper function transverse to 0 and such that $M = f^{-1}(0)$ (with coorientation). Denote then by W^m the compact submanifold $f^{-1}((-\infty, 0])$ of $\widehat{W}$.

Definition 3.1. We call f-*double* of W, and denote it by $\mathcal{D}_f W$, the smooth manifold given by $\{(p,s) \in \widehat{W} \times \mathbb{R} \,|\, s^2 + f(p) = 0\}$.

Notice that this set is indeed a smooth submanifold of $W \times \mathbb{R}$ because the function $W \times \mathbb{R} \to \mathbb{R}$ given by $(p,s) \mapsto s^2 + f(p)$ is transverse to 0.
Indeed, one can always find a vector field Z on $\widehat{W}$ which is *boundary-gradient-like* for f, i.e which satisfies $df(Z) \geq 0$ everywhere on $\widehat{W}$ and $df(Z) > 0$ along $M = f^{-1}(0)$: there is a vector field Z' on a neighborhood U of the (cooriented) hypersurface M such that $df(Z') > 0$ on U, and we can choose Z to be Z' multiplied by a non-negative cutoff function χ supported in U. Then, $d(s^2+f)(s\partial_s + Z) = 2s^2 + df(Z) > 0$ along $\mathcal{D}_f W \subset \widehat{W} \times \mathbb{R}$, which shows that $\mathcal{D}_f W$ is a regular hypersurface.

Notation 3.2. If $f\colon \widehat{W} \to \mathbb{R}$, we denote by $f^{\mathcal{D}}\colon \widehat{W} \times \mathbb{R} \to \mathbb{R}$ the function $f^{\mathcal{D}}(p,s) = s^2 + f(p)$; in particular, if f is an equation of the hypersurface $M \subset \widehat{W}$, then $f^{\mathcal{D}}$ defines the hypersurface $\mathcal{D}W \subset \widehat{W} \times \mathbb{R}$, as shown above.
In a similar way, if Z is a vector field on $\widehat{W}$, we denote by $Z^{\mathcal{D}}$ the vector field $Z + s\partial_s$ on $\widehat{W} \times \mathbb{R}_s$; if Z if boundary-gradient-like for f, then so is $Z^{\mathcal{D}}$ for $f^{\mathcal{D}}$.

We have the following uniqueness property of the f-double:

Lemma 3.3. *Let $f_0, f_1\colon \widehat{W} \to \mathbb{R}$ be two regular equations for M and Z be a vector field on $\widehat{W}$ which is boundary-gradient-like for both f_0 and f_1. For each $t \in [0,1]$, denote by f_t the function $tf_1 + (1-t)f_0$. Then the flow of the $[0,1]_t$-parametric vector field*

$$X_t \coloneqq \frac{f_1 - f_0}{df_t^{\mathcal{D}}(Z^{\mathcal{D}})}\, Z^{\mathcal{D}}$$

on $\widehat{W} \times \mathbb{R} \setminus \{s = 0, df_0(Z) = df_1(Z) = 0\}$ gives an isotopy which, at time $t = 1$, restricts to a diffeomorphism from $\mathcal{D}_{f_1} W$ to $\mathcal{D}_{f_0} W$.

Lemma 3.3 tells that $\mathcal{D}_f W$ does not depend on f, up to diffeomorphism. By a slight abuse of notation, we may hence write $\mathcal{D}W$ and simply talk about the *double* of W.

Proof (Lemma 3.3). Notice that if $f_0, f_1\colon \widehat{W} \to \mathbb{R}$ are two regular equations for M (inducing the same coorientation on it), then so is $f_t = tf_1 + (1-t)f_0$, for each $t \in [0,1]$. Moreover, if Z is boundary-gradient-like for both f_0, f_1, then $Z^{\mathcal{D}}$ is also boundary-gradient-like for $f_t^{\mathcal{D}}$, for each $t \in [0,1]$.
In particular, the smooth function $G\colon \widehat{W} \times \mathbb{R} \times [0,1] \to \mathbb{R}$, given by $G(p,s,t) = s^2 + f_t(p)$, is transverse to 0: indeed, $dG(Z^{\mathcal{D}}) > 0$ along $G^{-1}(0) = \bigcup_t \mathcal{D}_{f_t} W \times \{t\}$. Then, $G^{-1}(0)$ is a smooth submanifold of $\widehat{W} \times \mathbb{R} \times [0,1]$, which is moreover contained in $\widehat{W} \times \mathbb{R} \setminus \{s = 0, df_0(Z) = df_1(Z) = 0\}$. In particular, the (well defined on $\operatorname{Im} G$) vector field $-\partial_t + X_t$ is tangent to $G^{-1}(0)$ and the restriction of its flow at time 1 gives a diffeomorphism from $G^{-1}(0) \cap \left(\widehat{W} \times \mathbb{R} \times \{1\}\right) = \mathcal{D}_{f_1} W \times \{1\}$ to $G^{-1}(0) \cap \left(\widehat{W} \times \mathbb{R} \times \{0\}\right) = \mathcal{D}_{f_0} W \times \{0\}$, as wanted. □

Let now $(\widehat{W}^{2n}, \lambda)$ be a Liouville manifold and denote by Z its Liouville vector field. Consider also a smooth proper function $f\colon \widehat{W} \to \mathbb{R}$, bounded from below and such that Z is boundary-gradient-like for f; denote by W the (compact) submanifold $f^{-1}((-\infty, 0])$ of $\widehat{W}$. Notice that $(M, \eta = \ker(\lambda|_M))$ is a contact manifold and that (W, λ) is a Liouville filling of it.

Consider now the Liouville manifold $(\widehat{W} \times \mathbb{R}_s \times \mathbb{S}^1_\theta, \lambda + 2s d\theta)$, where $\mathbb{R}_s$ and $\mathbb{S}^1_\theta$ denote the manifolds $\mathbb{R}$ and $\mathbb{S}^1$ with coordinates s and θ respectively. Notice that the vector field $Z^{\mathcal{D}} = Z + s\partial_s$ and the function can naturally be seen on $\widehat{W} \times \mathbb{R} \times \mathbb{S}^1$. Moreover, $Z^{\mathcal{D}}$ is Liouville for $\lambda + 2sd\theta$ and transverse to $\mathcal{D}_f W \times \mathbb{S}^1 = \{f^{\mathcal{D}} = 0\} \subset \widehat{W} \times \mathbb{R} \times \mathbb{S}^1$; so, $\alpha_f := (\lambda + 2sd\theta)|_{\mathcal{D}_f W \times \mathbb{S}^1}$ is a contact form on $\mathcal{D}_f W \times \mathbb{S}^1$. In analogy with Notation 3.2, we will also denote the Liouville form $\lambda + 2sd\theta$ on $\widehat{W} \times \mathbb{R} \times \mathbb{S}^1$ by $\lambda^{\mathcal{D}}$ in the following.

Lemma 3.4. *Let $f_0, f_1 \colon \widehat{W} \to \mathbb{R}$ be two regular equations for M such that the Liouville vector field Z is boundary-gradient-like for both f_0, f_1 and, for $t \in [0,1]$, denote by f_t the function $tf_1 + (1-t)f_0$. Then, the flow of the $[0,1]_t$-parametric vector field*

$$X_t := \frac{f_1 - f_0}{df_t^{\mathcal{D}}(Z^{\mathcal{D}})} Z^{\mathcal{D}}$$

on $Y := \widehat{W} \times \mathbb{R} \setminus \{s = 0, df_0(Z) = df_1(Z) = 0\}$ induces an isotopy of $Y \times \mathbb{S}^1$ which, at time $t = 1$, restricts to a contactomorphism from $(\mathcal{D}_{f_1} W \times \mathbb{S}^1, \ker(\alpha_{f_1}))$ to $(\mathcal{D}_{f_0} W \times \mathbb{S}^1, \ker(\alpha_{f_0}))$.

We may hence drop the f in the notation and just denote it $(\mathcal{D}W \times \mathbb{S}^1, \ker \alpha)$ from now on.

Proof (Lemma 3.4). According to Lemma 3.3, the only thing to show is that the flow ψ_t of X_t on $Y \times \mathbb{S}^1$ preserves $\ker(\lambda^{\mathcal{D}})$. An explicit computation shows that $\mathcal{L}_{X_t}(\lambda^{\mathcal{D}}) = \frac{f_1 - f_0}{df_t^{\mathcal{D}}(Z^{\mathcal{D}})} \lambda^{\mathcal{D}}$, which implies that $\psi_t^* \lambda^{\mathcal{D}} = h_t \lambda^{\mathcal{D}}$ for a certain function $h_t \colon \widehat{W} \times \mathbb{R} \to \mathbb{R}_{>0}$, as wanted. □

Remark. *In [GS10], Geiges and Stipsicz construct, more generally, contact forms on $(W_1 \cup_M W_2) \times \mathbb{S}^1$, where (W_1, λ_1) and (W_2, λ_2) are Liouville domains with the same (strict) contact boundary (M, α); the contact structure they obtain in the particular case where $W_1 = W_2$ and $\lambda_1 = \lambda_2$ (and ∂W_1 identified with ∂W_2 via the identity) is the same, up to isotopy, as the contact structure on $\mathcal{D}W \times \mathbb{S}^1$ that we described above.*
Even though the construction described here is less general, it has the advantage of involving a natural Liouville filling of the strict contact manifold $(\mathcal{D}W \times \mathbb{S}^1, \alpha)$, which will be useful in Section 3.3; notice, however, that one cannot always expect a presentation involving a symplectic filling for the construction in [GS10]. For instance, in the case $W_1 = D^2$ and $W_2 = \Sigma_g \setminus D^2$, where Σ_g is a closed surface with genus $g \neq 0$, the theory of convex surfaces by Giroux tells that the contact structure on $(W_1 \cup_{\mathbb{S}^1} W_2) \times \mathbb{S}^1$ obtained as in [GS10] is overtwisted: indeed, it is the unique $\mathbb{S}^1$-invariant contact structure on $\Sigma_g \times \mathbb{S}^1$ such that each $\Sigma_g \times \{pt\}$ is a convex surface with dividing set consisting of a homotopically trivial circle.

We now exhibit an explicit natural way to construct (strict) contactomorphisms of $(\mathcal{D}W \times \mathbb{S}^1, \xi := \ker \alpha)$.
Consider an $\mathbb{S}^1$-family of diffeomorphisms $(\varphi_\theta)_{\theta \in \mathbb{S}^1}$ of $\widehat{W}$, each of which preserves both λ and $f \colon \widehat{W} \to \mathbb{R}$; we do not assume that they are the identity on $M = \partial W$. Take then the diffeomorphism $\Psi \colon \mathcal{D}W \times \mathbb{S}^1 \to \mathcal{D}W \times \mathbb{S}^1$ induced by the restriction of $\widehat{\Psi} \colon \widehat{W} \times \mathbb{R} \times \mathbb{S}^1 \to \widehat{W} \times \mathbb{R} \times \mathbb{S}^1$ given by $\widehat{\Psi}(p, s, \theta) = (\varphi_\theta(p), s, \theta)$; notice that this is well defined because φ_θ preserves f.

Lemma 3.5. *The flow ψ_Y^t of the vector field*

$$Y = \frac{\lambda(\frac{\partial \varphi_\theta}{\partial \theta})}{2\, df^{\mathcal{D}}(Z^{\mathcal{D}})} \, (2s\, Z - df(Z)\, \partial_s)$$

gives a smooth isotopy $\Psi \circ \psi_Y^t$ from $\Psi = \Psi \circ \psi_Y^0$ to a contactomorphism $\Psi_c := \Psi \circ \psi_Y^1$ of $(\mathcal{D}W \times \mathbb{S}^1, \xi = \ker \alpha)$.

Notice that Y, given above as a section of the tangent bundle of $\widehat{W} \times \mathbb{R} \times \mathbb{S}^1 \setminus \{s = 0, df_0(Z) = df_1(Z) = 0\}$ is indeed tangent to (the $\mathcal{D}W$ factor of) $\mathcal{D}W \times \mathbb{S}^1$.

Proof (Lemma 3.5). For notational ease, in the following we denote by X_θ the vector field $\frac{\partial \varphi_\theta}{\partial \theta}$ and by h_θ the function $\lambda(X_\theta)$ defined on $\widehat{W}$.

We start by noticing that we have the identity $dh_\theta = -\iota_{X_\theta} d\lambda$: indeed, $\varphi_\theta^* \lambda = \lambda$ for each $\theta \in \mathbb{S}^1$, so that $\mathcal{L}_{X_\theta}\lambda = 0$, which is equivalent to $dh_\theta = -\iota_{X_\theta} d\lambda$. In particular, an evaluation of the above identity on the Liouville vector field Z tells that $dh_\theta(Z) = h_\theta$.

An explicit computation gives that $\widehat{\Psi}^* \lambda^{\mathcal{D}} = \lambda + (2s + h_\theta)\, d\theta$, so that $\Psi^* \left[\lambda^{\mathcal{D}}|_{T(\mathcal{D}W \times \mathbb{S}^1)}\right] = [\lambda + (2s + h_\theta)\, d\theta]\,|_{T(\mathcal{D}W \times \mathbb{S}^1)}$. Denote then, for all $t \in [0,1]$, by $\lambda_t^{\mathcal{D}}$ the 1-form $\lambda + (2s + t h_\theta) d\theta$ on $\widehat{W} \times \mathbb{R} \times \mathbb{S}^1$ and by α_t the 1-form $\lambda_t^{\mathcal{D}}|_{T(\mathcal{D}W \times \mathbb{S}^1)}$ on $\mathcal{D}W \times \mathbb{S}^1$.

We then prove that α_t is a contact form for each $t \in [0,1]$; for this, it's enough to prove that, for each $t \in [0,1]$, $\lambda_t^{\mathcal{D}}$ is a Liouville form and that its Liouville vector field $Z_t^{\mathcal{D}}$ is transverse to $\mathcal{D}W \times \mathbb{S}^1$.
We can compute $d\lambda_t^{\mathcal{D}} = d\lambda + 2ds \wedge d\theta + t dh_\theta \wedge d\theta$; then, it is easy to see that $(d\lambda_t^{\mathcal{D}})^{n+1} = 2(d\lambda)^n \wedge ds \wedge d\theta$, so that $\lambda_t^{\mathcal{D}}$ is indeed a Liouville form on $\widehat{W} \times \mathbb{R} \times \mathbb{S}^1$. Moreover, its vector field Z_t is just $Z^{\mathcal{D}}$: indeed, $\iota_{Z^{\mathcal{D}}} d\lambda_t^{\mathcal{D}} = \iota_Z d\lambda + t dh_\theta(Z) d\theta + 2s d\theta$, which is exactly equal to $\lambda_t^{\mathcal{D}}$, because $dh_\theta(Z) = h_\theta$, as remarked above, and $\iota_Z d\lambda = \lambda$. In particular, $Z_t = Z^{\mathcal{D}}$ is transverse to $\mathcal{D}W \times \mathbb{S}^1$, as wanted.

Now, according to (the proof of) Gray's theorem, the flow of the (a priori time-dependent) vector field X_t such that $\alpha_t(X_t) = 0$ and $\iota_{X_t} d\alpha_t|_{\ker \alpha_t} = -\dot{\alpha}_t|_{\ker \alpha_t}$ gives an isotopy that pulls back $\ker \alpha_t$ to $\ker \alpha_0$. Hence, it's enough to show that the vector field Y in the statement satisfies these two conditions.
An explicit computation gives that $df^{\mathcal{D}}(Y) = 0$ and $\lambda_t^{\mathcal{D}}(Y) = 0$, i.e. that $Y \in \ker \alpha_t = \ker \lambda_t^{\mathcal{D}} \cap T(\mathcal{D}W \times \mathbb{S}^1)$. Moreover, we can compute

$$
\begin{aligned}
\iota_Y d\lambda_t^{\mathcal{D}} &= \frac{h_\theta}{2df^{\mathcal{D}}(Z^{\mathcal{D}})} \left[2s\lambda + 2ts\, dh_\theta(Z)\, d\theta - 2df(Z) d\theta\right] \\
&\overset{(i)}{=} \frac{h_\theta}{2df^{\mathcal{D}}(Z^{\mathcal{D}})} \left[2s\lambda_t^{\mathcal{D}} - 4s^2 d\theta - 2df(Z) d\theta\right] \\
&\overset{(ii)}{=} \frac{2s h_\theta}{2df^{\mathcal{D}}(Z^{\mathcal{D}})} \lambda_t^{\mathcal{D}} - \frac{d}{dt}\lambda_t^{\mathcal{D}}\ ,
\end{aligned}
$$

where for (i) we used that $dh_\theta(Z) = h_\theta$ and for (ii) we used that $df^{\mathcal{D}}(Z^{\mathcal{D}}) = 2s^2 + df(Z)$ and $\frac{d}{dt}\lambda_t^{\mathcal{D}} = h_\theta d\theta$. In particular $\iota_Y d\alpha_t|_{\ker \alpha_t} = -\dot{\alpha}_t|_{\ker \alpha_t}$, as wanted. $\square$

3.2 Families of Lagrangian bases

Let V be a smooth $(2n+1)$-manifold and ξ a contact structure on V. Given a compact manifold Y^m, we call *family of Lagrangian bases* of ξ indexed by Y, and we denote it by $\mathfrak{L}$, the data of a smooth map $\gamma\colon Y \to V$ and, for $j = 1, \ldots, n$, of smooth maps $X_j\colon Y \to \xi$ such that the following diagram commutes

$$
\begin{tikzcd}
 & \xi \arrow[d] \\
Y \arrow[ur, "X_j"] \arrow[r, "\gamma"] & V
\end{tikzcd}
$$

and such that, for each $q \in Y$, the $X_1(q), \ldots, X_n(q)$ are $\mathbb{R}$-linearly independent and generate a Lagrangian subspace of $(\xi_p, (\mathrm{CS}_\xi)_p)$. Here, CS_ξ is the natural conformal symplectic structure on ξ (see the discussion after Definition 2.10); in particular, $(\mathrm{CS}_\xi)_p$

is a conformal class of symplectic alternating forms on ξ_p and, hence, has a well defined class of (isotropic and) Lagrangian subspaces.

We point out that if $f\colon (V,\xi) \to (V,\xi)$ is a contactomorphism, then $f_*\mathfrak{L} := (f \circ \gamma, df(X_1), \dots, df(X_n))$ is also a Y-family of Lagrangian bases of ξ: indeed, f preserves the conformal symplectic structure CS_ξ on ξ.
Moreover, if $f_t\colon (V,\xi) \to (V,\xi)$ is a contact-isotopy from $f_0 = \mathrm{Id}$ to $f_1 = f$, then $(f_t)_*\mathfrak{L}$ is a path of Y-families of Lagrangian bases of ξ from $\mathfrak{L}$ to $f_*\mathfrak{L}$. In other words, we have the following obstruction to contact-isotopies:

Lemma 3.6. *Let $f\colon (V,\xi) \to (V,\xi)$ be a contactomorphism. If there is a Y-family of Lagrangian bases $\mathfrak{L}$ for ξ such that $f_*\mathfrak{L}$ is not homotopic (among families of Lagrangian bases) to $\mathfrak{L}$, then f is not contact-isotopic to the identity.*

Let now J be a complex structure on ξ tamed by CS_ξ. If $\mathfrak{L} = (\gamma, X_1, \dots, X_n)$ is a Y-family of Lagrangian bases for ξ then, for each $q \in Y$, one has $\xi_{\gamma(q)} = \langle X_1(q), \dots, X_n(q)\rangle_{\mathbb{C}}$. Suppose moreover that (ξ, J) is *stably trivial*, i.e. that there is a $k \in \mathbb{N}$ and an isomorphism of complex vector bundles $\Phi\colon (\xi, J) \oplus \varepsilon_V^k \xrightarrow{\sim} \varepsilon_V^{n+k}$ over V; here, ε_V is the trivial complex line bundle $V \times \mathbb{C} \to V$ and ε_V^m denotes the direct sum of ε_V with itself m times. We point out that the property that (ξ, J) is stably trivial is not dependent on a specific choice of J: indeed, the space of complex structures on ξ which are tamed by CS_ξ is contractible, hence (ξ, J) and (ξ, J') are isomorphic as complex vector bundles if J, J' are both tamed by CS_ξ.
Then, if $(e_1, \dots, e_k)$ are the sections of ε_V^k which give, at each point $p \in V$, the canonical bases of the fiber $(\varepsilon_V^k)_p = \mathbb{C}^k$, the image of $\mathfrak{L}_{stab} := (\gamma, X_1, \dots, X_n, e_1 \circ \gamma, \dots, e_k \circ \gamma)$ via Φ gives, pointwisely, a bases of the vector space $\mathbb{C}^{n+k}$ given by the fibers of ε_V^{n+k} over each point of the image of γ. In particular, considering the linear endomorphism of $\mathbb{C}^{n+k}$ obtained by sending the canonical basis to the basis given, pointwisely, by the image of $\mathfrak{L}_{stab}$ via Φ, we then obtain a smooth map $M\colon Y \to GL_{n+k}(\mathbb{C})$.
In the following, we say that the family $\mathfrak{L}_{stab}$ is the $(e_1, \dots, e_k)$-*stabilization* of $\mathfrak{L}$ (sometimes omitting the sections $(e_1, \dots, e_k)$ of ε_V^k if there is no ambiguity) and denote it more concisely by $\mathfrak{L} \oplus (e_1, \dots, e_k)$. We will also say that the map M is the *Y-family of (invertible) matrices* associated (via Φ) to $\mathfrak{L}_{stab}$.

Given a contactomorphism $f\colon (V,\xi) \to (V,\xi)$, the stabilization $(f_*\mathfrak{L})_{stab} = (f \circ \gamma, df(X_1), \dots, df(X_n), e_1 \circ f \circ \gamma, \dots, e_k \circ f \circ \gamma)$ gives, via Φ, another Y-family of invertible matrices, which we denote $f_*M\colon Y \to GL_{n+k}(\mathbb{C})$. As this can also be done parametrically, analogously to Lemma 3.6 above, we obtain:

Lemma 3.7. *Let (V^{2n+1}, ξ) be a contact manifold and J an almost complex structure on ξ such that (ξ, J) is stably trivial, via an isomorphism $\Phi\colon (\xi, J) \oplus \varepsilon_V^k \to \varepsilon_V^{n+k}$ of complex vector bundles over V. Let also $f\colon (V,\xi) \to (V,\xi)$ be a contactomorphism and $\mathfrak{L} = (\gamma, X_1, \dots, X_n)$ be a Y-family of Lagrangian bases for ξ.*
If the Y-family of matrices associated via Φ to the $(e_1, \dots, e_k)$-stabilization $(f_*\mathfrak{L})_{stab}$ is not homotopic, as map $Y \to GL_{n+k}(\mathbb{C})$, to the Y-family of matrices associated via Φ to the $(e_1, \dots, e_k)$-stabilization $\mathfrak{L}_{stab}$, then f is not contact-isotopic to the identity.

3.3 Examples of infinite order contact mapping classes

The aim of this section is to prove Théorème I.A, recalled in the beginning of the chapter; in particular, we will use the notations introduced in its statement.

We start by claiming that we can make the following additional assumption: the Weinstein structure $(F, \omega_F, Z_F, \psi_F)$ comes from an *almost Stein structure* (J_F, ψ_F) (i.e.

ω_F is equal to $-dd^{\mathbb{C}}\psi_F$ and Z_F is equal to the vector field ω_F-dual to $\lambda_F := -d^{\mathbb{C}}\psi_F$; see Definition 2.15), such that moreover (TF, J_F) is trivial as complex vector bundle. We will indeed need this compatibility between the almost complex structure J_F which trivializes TF and the Weinstein structure (ω_F, Z_F, ψ_F) in the proof of Théorème I.A. Here's how these additional assumptions can be arranged.
According to Theorem 2.16, there is an almost Stein structure (J'_F, ψ_F) on F such that the Weinstein structure $(-dd^{\mathbb{C}}\psi_F, Z_F, \psi_F)$ is Weinstein homotopic, with fixed function ψ_F, to the Weinstein structure (ω_F, Z_F, ψ_F) in the statement of Théorème I.A. Because $-dd^{\mathbb{C}}\psi_F$ and ω_F are homotopic as symplectic structures, (a slight adaptation of the proof of) the contractibility of the space of almost complex structures tamed by a given symplectic form also gives that J'_F is homotopic to J_F; (TF, J'_F) is then isomorphic, as complex bundle, to (TF, J_F) and is, in particular, trivial.
Because the Weinstein structure $(\omega_0, Z_0, |.|^2)$ on $\mathbb{C}$ already comes from the almost Stein (actually, Stein) structure $(i, |.|^2)$, we can moreover apply the Weinstein homotopy only on the F-factor of $F \times \mathbb{C}$ in order to ensure the same assumption on the manifold $F \times \mathbb{C}$. Notice that this Weinstein homotopy do not change W, Ψ and (up to isotopy) the contact structure on $\mathcal{D}W \times \mathbb{S}^1$ defined in Théorème I.A, because the homotopy is along the F-factor and with ψ_F fixed.

With a little abuse of notation, we will hence denote J'_F again by J_F; let also $J := J_F \oplus i$ on $F \times \mathbb{C}$. Notice that $J := J_F \oplus i$ can be further extended to $J^{\mathcal{D}}$ on $F \times \mathbb{C} \times \mathbb{R}_s \times \mathbb{S}^1_\theta$ by defining $J^{\mathcal{D}}(\partial_s) := \partial_\theta$ on $T(\mathbb{R}_s \times \mathbb{S}^1_\theta)$. Notice that $(J^{\mathcal{D}}, \psi^{\mathcal{D}} = \psi + s^2)$ is an almost Stein structure on $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$ such that $\omega^{\mathcal{D}} = d\lambda^{\mathcal{D}}$, where $\lambda^{\mathcal{D}} := -d^{\mathbb{C}}\psi^{\mathcal{D}} = \lambda + 2s d\theta$, and $Z^{\mathcal{D}} = Z + s\partial_s$ is the Liouville vector field of $\lambda^{\mathcal{D}}$.

By the hypothesis of Théorème I.A (and the above assumption), there is an isomorphism of complex vector bundles $\nu\colon (TF, J_F) \xrightarrow{\sim} \varepsilon_F^{n-1}$ over F, where ε_F^{n-1} is the trivial complex vector bundle $(F \times \mathbb{C}^{n+1}, J_{std})$. Moreover, ν naturally extends to a trivialization

$$\mu\colon \left(T\left(F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1\right), J^{\mathcal{D}}\right) \xrightarrow{\sim} \varepsilon^{n+1}_{F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1} \tag{3.1}$$

such that, for each $(q, z, s, \theta) \in F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$, one has:

- the following diagram commutes

$$\begin{tikzcd}
(T_qF, J_F) \arrow[rr, "i"] \arrow[d, "\nu_q"] & & \left(T_{(q,z,s,\theta)}\left(F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1\right), J^{\mathcal{D}}\right) \arrow[dd, "\mu_{(q,z,s,\theta)}"] \\
(\varepsilon_F^{n-1})_q = \mathbb{C}^{n-1} \arrow[d, "j"] & & \\
(\varepsilon_F^{n+1})_q = \mathbb{C}^{n+1} \arrow[rr, "\mathrm{Id}"] & & (\varepsilon^{n+1}_{F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1})_{(q,z,s,\theta)} = \mathbb{C}^{n+1}
\end{tikzcd}$$

  where i and j are the natural inclusions given by $T_qF = T_qF \oplus \{(0,0,0)\} \subset T_{(q,z,s,\theta)}\left(F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1\right)$ and $\mathbb{C}^{n-1} = \mathbb{C}^{n-1} \times \{(0,0)\} \subset \mathbb{C}^{n+1}$;

- $\mu_{(q,z,s,\theta)}(\partial_x) = (0, \dots, 0, 1, 0) \in \mathbb{C}^{n+1}$, where we use here coordinates (x, y) on the factor $\mathbb{C}$ of $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$,

- $\mu_{(q,z,s,\theta)}(\partial_s) = (0, \dots, 0, 1) \in \mathbb{C}^{n+1}$, where s is the coordinate on the factor $\mathbb{R}$ of $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$.

Let now $a := \frac{c-\min(\psi_F)}{4} > 0$ (the exact value of this parameter will intervene later in the proof), and consider a non-decreasing smooth cut-off function $\chi\colon \mathbb{R} \to [-1,1]$, equal to 1 on $(2a,+\infty)$, equal to -1 on $(-\infty,-2a)$, and such that $\chi(x) = x$ for $x \in (-a,a)$. Then, the function $f\colon F \times \mathbb{C} \to \mathbb{R}$ defined by $f := \chi(\psi - c)$ is a regular equation of $M = f^{-1}(0) = \psi^{-1}(c)$; in particular, $Z^{\mathcal{D}} = Z + s\partial_s$ on $\widehat{W} \times \mathbb{R}_s$ is transverse to $\mathcal{D}_f W$ too. Notice also that $\mathcal{D}_f W$ is essentially a "flattened" version of $\mathcal{D}_{\psi-c}W$, as Figure 3.1 illustrates.

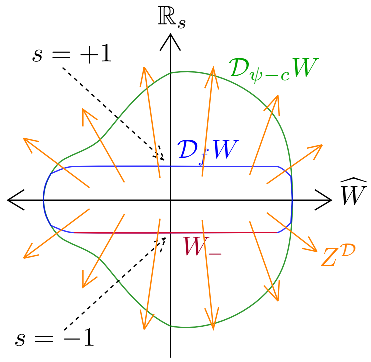

Figure 3.1: $\mathcal{D}_f W$ and $\mathcal{D}_{\psi-c}W$ inside $\widehat{W} \times \mathbb{R}_s$ and the vector field $Z^{\mathcal{D}}$, transverse to both; $W_- := \mathcal{D}_f W \cap \{s = -1\}$ that will appear in Step 1 is also represented.

As we would like to prove Théorème I.A using the equation f instead of $\psi - c$, we need the following:

Lemma 3.8. *If the conclusion of Théorème I.A holds with the special choice of equation f for $\mathcal{D}_f W \subset F \times \mathbb{C} \times \mathbb{R}$, then it holds also for $\mathcal{D}_{\psi-c}W$ defined by $\psi - c$ (i.e. as in the statement of Théorème I.A).*

Proof. Let $f_0 := \psi - c$ and $f_1 := f = \chi(\psi - c)$. According to Lemma 3.4, the flow $\psi^1_{X_t}$ of the vector field $X_t = \frac{f_1 - f_0}{df_t^{\mathcal{D}}(Z^{\mathcal{D}})} Z^{\mathcal{D}}$ gives a contactomorphism from $(\mathcal{D}_{f_0}W \times \mathbb{S}^1, \ker(\alpha_{f_0}))$ to $(\mathcal{D}_{f_1}W \times \mathbb{S}^1, \ker(\alpha_{f_1}))$.
Hence, in order to prove Lemma 3.8, it's enough to show that the diffeomorphism $\psi^1_{X_t} \circ \Psi \circ (\psi^1_{X_t})^{-1}$ of $\mathcal{D}_{f_1}W$ is still induced by the diffeomorphism $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$ given by $(q,z,s,\theta) \mapsto (q, e^{i\theta}z, s, \theta)$. But this is indeed the case, because the flow $\psi^1_{X_t}$ fixes the angular component of the $\mathbb{C}$-factor as well as the $\mathbb{S}^1$-factor of the product $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$, and hence commutes with $(q,z,s,\theta) \mapsto (q, e^{i\theta}z, s, \theta)$. ☐

Let then α, Ψ, Y and Ψ_c be obtained as in Section 3.1 from the Liouville manifold $(F \times \mathbb{C}, \lambda)$, the regular equation f of M and the family $\{\varphi_\theta\}_{\theta \in \mathbb{S}^1}$ of diffeomorphisms of $F \times \mathbb{C}$ which is given by $(p,z) \mapsto (p, e^{i\theta}z)$.
We know from Section 3.1 that, inside the Liouville manifold $(F \times \mathbb{C} \times \mathbb{R}_s \times \mathbb{S}^1_\theta, \lambda^{\mathcal{D}} = \lambda + 2s d\theta)$, the preimage of $(-\infty, 0]$ via $F \times \mathbb{C} \times \mathbb{R}_s \times \mathbb{S}^1_\theta \to \mathbb{R}$, $(p,s,\theta) \mapsto s^2 + f(p)$, gives a Liouville filling of $(\mathcal{D}_f W \times \mathbb{S}^1_\theta, \alpha_f)$.
Moreover, as we are under the hypothesis that the Liouville structure on F comes from an almost Stein structure, the compact manifold $\{f^{\mathcal{D}} \leq 0\} = \{s^2 + \psi \leq c\}$, together with the almost complex structure induced by the ambient almost Stein manifold $(F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1, J^{\mathcal{D}})$, is actually an almost Stein filling of $(\mathcal{D}_f W \times \mathbb{S}^1_\theta, \alpha_f)$. In particular, $\lambda^{\mathcal{D}}$ is equal to $-d^{\mathbb{C}}\psi^{\mathcal{D}}$ on a neighborhood of $\{f^{\mathcal{D}} = 0\} = \{s^2 + \psi = c\}$, which guarantees that the almost complex structure $J^{\mathcal{D}}$ on $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$ restricts to a well defined endomorphism of $\ker \alpha_f$ along $\mathcal{D}_f W \times \mathbb{S}^1_\theta$; this restriction is then automatically tamed by $d\alpha_f|_{\ker \alpha_f}$.

Remark. *If we relax the condition in Théorème I.A of F being a Weinstein manifold to F being a Liouville manifold, we may not be able to find a $J^{\mathcal{D}}$ on $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$ that both restricts to $\ker \alpha_f$ and splits as $J_F \oplus i \oplus J^{\mathcal{D}}|_{T(\mathbb{R}\times\mathbb{S}^1)}$ at the same time; these are both conditions we will need in the following.*

Recall now that $\Psi_c = \Psi \circ \psi_Y^1$, with ψ_Y^t the flow at time t of Y defined in Lemma 3.5. Then, in order to show that, for each $k \neq 0$, Ψ_c^k is not contact isotopic to the identity, we are going to proceed by steps as follows:

1. Let $W_- := \mathcal{D}_f W \cap \{s = -1\} \subset F \times \mathbb{C} \times \mathbb{R}$ (see Figure 3.1); notice that it has non-empty interior by construction of f. We then describe a $\mathbb{S}^1$-family of Lagrangian bases $\mathfrak{L}$ for $\ker(\alpha_f)$ on $W_- \times \mathbb{S}^1$.

2. We remark that, for all $t \geq 0$, $\psi_Y^t(W_- \times \mathbb{S}^1) \subset W_- \times \mathbb{S}^1$, and we describe the behavior of the restriction of Ψ_c, and its iterates, to $W_- \times \mathbb{S}^1$. This allows us to describe, for all $k \geq 1$, the pushforward $(\Psi_c^k)_*\mathfrak{L}$ of $\mathfrak{L}$ via the k-th iterate of Ψ_c.

3. We describe, for each $k \geq 0$, the family of matrices $B_k \colon \mathbb{S}^1 \to GL_{n+1}(\mathbb{C})$ associated, via the trivialization μ, to the stabilization $(\Psi_c^k)_*\mathfrak{L} \oplus Z^{\mathcal{D}}$. We then show that, if $k \geq 1$, B_k is not homotopically trivial as map $\mathbb{S}^1 \to GL_{n+1}(\mathbb{C})$.

According to Lemma 3.7, this proves that, for all $k \geq 1$, the k-th iterate of the contactomorphism Ψ_c is not contact isotopic to the identity. The space of contactomorphism being a group, this implies the same conclusion for all $k \neq 0$.

Step 1 We recall that there is a trivialization $\nu \colon (TF, J_F) \to (F \times \mathbb{C}^{n-1})$; let $(w_1, \ldots, w_{n-1})$ be the inverse image of the sections $(e_1, \ldots, e_{n-1})$ that give, fiber-wisely, the canonical complex basis for ε_F^{n-1}. Then, we have the following:

Lemma 3.9. *There are $q_0 \in F$ and $x_0 \in \mathbb{R}_{>0} \subset \mathbb{C}$ such that $(q_0, x_0, -1) \in W_- \subset F \times \mathbb{C} \times \mathbb{R}$ and $\mathfrak{L} := \left(\gamma, v_1, \ldots, v_{n-1}, \partial_\theta(\theta) + \frac{2}{x_0}\partial_y(x_0)\right)$ is an $\mathbb{S}^1$-family of Lagrangian bases for $\ker(\alpha_f)$, where $\gamma \colon \mathbb{S}^1 \to W_- \times \mathbb{S}^1$ is defined by $\gamma(\theta) = (q_0, x_0, -1, \theta)$, v_j denotes $w_j(q_0) \in T_{q_0}F$ for each $j = 1, \ldots, n-1$, and (x, y) are coordinates on the factor $\mathbb{C}$ of $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$.*

Proof (Lemma 3.9). Let $p_0 = (q_0, x_0) \in M = \psi^{-1}(c - 3a) \subset F \times \mathbb{C}$, where $q_0 \in F$ is a point of (global) minimum for ψ_F and $x_0 \in \mathbb{R}_{>0} \subset \mathbb{C}$ satisfies $\psi_F(q_0) + x_0^2 = c - 3a$.
Recall that $c > \min(\psi_F)$ is the regular value for ψ in the statement of Théorème I.A; also, $a = \frac{c - \min(\psi_F)}{4}$ was introduced in the choice of regular equation f for M and satisfies $c - 3a > \min \psi_F$, so that an $x_0 > 0$ as above actually exists.
Because $W_- = \mathcal{D}W \cap \{s = -1\} \subset F \times \mathbb{C} \times \mathbb{R}_s$, this proves the existence of γ as in Lemma 3.9.

We now have to prove that $\mathfrak{L}$ is an $\mathbb{S}^1$-family of Lagrangian bases for $\ker(\alpha_f)$.
Because q_0 is a (global) minimum of ψ_F, we have $\lambda^{\mathcal{D}} = x_0^2 d\varphi - 2d\theta$ at the point $\gamma(\theta)$ (here, we use coordinates (r, φ) for the factor $\mathbb{C}$ of $F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$). In particular, $\ker(\alpha_f)|_{\gamma(\theta)}$, seen as a sub-bundle of $T_{\gamma(\theta)}\left(F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1\right) = T_{q_0}F \oplus T_{x_0}\mathbb{C} \times T_{-1}\mathbb{R} \times T_\theta\mathbb{S}^1$, is equal to $T_{q_0}F \oplus \mathrm{Span}_{\mathbb{C}}\left(\partial_\theta(\theta) + \frac{2}{x_0}\partial_y(x_0)\right)$ (recall that $\partial_\varphi(x_0) = x_0\partial_y(x_0)$). This means exactly that $\mathfrak{L}$ is a family of Lagrangian bases for $\ker(\alpha_f)$, as wanted. $\square$

Step 2 This step consists in the following two lemmas:

Lemma 3.10. *The contactomorphism Ψ_c of $(\mathcal{D}W \times \mathbb{S}^1, \ker \alpha_f)$ satisfies $\Psi_c(W_- \times \mathbb{S}^1) \subset W_- \times \mathbb{S}^1$. More precisely, for each $k \geq 0$, Ψ_c^k has the following form:*

$$\begin{aligned} \Psi_c^k \colon\ & W_- \times \mathbb{S}^1 \to W_- \times \mathbb{S}^1 \\ & (q, re^{i\varphi}, -1, \theta) \mapsto (Q_k(q,r), R_k(r)e^{i(\varphi+k\theta)}, -1, \theta) \end{aligned}$$

Lemma 3.11. *Let $\gamma, (v_1, \cdots, v_{n-1})$ and $\mathfrak{L}$ be as in Lemma 3.9 and, for each $k \geq 0$, Q_k and R_k as in Lemma 3.10. Then, for each $k \geq 0$, there are a complex basis $(v_1^k, \ldots, v_{n-1}^k)$ of $T_{Q_k(q_0,x_0)}F$ and a real number s_k such that $(\Psi_c^k)_*\mathfrak{L}$ is given by $\big(\Psi_c^k \circ \gamma, v_1^k, \ldots, v_{n-1}^k, \partial_\theta(\theta) + s_k \partial_\varphi(r_k e^{ik\theta})\big)$, where $r_k := R_k(x_0)$.*

Proof (Lemma 3.10). We give a proof by induction on k. The case $k = 0$ is trivial; notice that it's actually enough to show that the lemma holds for $k = 1$, as the inductive step would then become immediate.
Indeed, if both Ψ_c and Ψ_c^k can be written in the form given in the statement of Lemma 3.10, it is immediate to check that the same is true for Ψ_c^{k+1}.

Let's then analyze the case $k = 1$. Recall that $\Psi_c = \Psi \circ \psi_Y^1$, where $\psi_Y^t \colon \mathcal{D}W \times \mathbb{S}^1 \to \mathcal{D}W \times \mathbb{S}^1$ is the flow of Y (given by Lemma 3.5) at time 1.
Notice that the function f is constant and $s = -1$ on $W_- \times \mathbb{S}^1$. Then, Y (which is tangent to $\mathcal{D}_f W \times \mathbb{S}^1$) restricts to $-\frac{r^2}{2} Z_F(q) - \frac{r^3}{4}\partial_r(re^{i\varphi})$ on $W_- \times \mathbb{S}^1$; here, we use polar coordinates $z = re^{i\varphi}$ on the factor $\mathbb{C}$.
In particular, the flow $\psi_Y^t \colon \mathcal{D}W \times \mathbb{S}^1 \to \mathcal{D}W \times \mathbb{S}^1$ of Y at time $t \geq 0$ satisfies $\psi_Y^t(W_- \times \mathbb{S}^1) \subset W_- \times \mathbb{S}^1$: indeed, Y has no component along ∂_s, and its flow preserves the s-coordinate, hence also $W_- \times \mathbb{S}^1 = (\mathcal{D}_f W \times \mathbb{S}^1) \cap \{s = -1\}$.
More precisely, at time $t = 1$, the embedding $\psi_Y^1 \colon W_- \times \mathbb{S}^1 \to W_- \times \mathbb{S}^1$ can be written as $\psi_Y^1(q, re^{i\varphi}, -1, \theta) = (Q(q,r), R(r)e^{i\varphi}, -1, \theta)$, for some functions $Q \colon F \times \mathbb{C} \to F$ and $R \colon \mathbb{C} \to \mathbb{R}$, with Q and R both independent of the angular component φ on $\mathbb{C}$.
Recalling that $\Psi \colon F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1 \to F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$ is given by $\Psi(q, re^{i\varphi}, s, \theta) = (q, re^{i(\varphi+\theta)}, s, \theta)$, we then obtain an expression for $\Psi_c = \Psi \circ \psi_Y^1$ which is exactly as in the statement of Lemma 3.10 (with the choices $Q_1 = Q$ and $R_1 = R$). □

Proof (Lemma 3.11). We are going to describe who $v_1^k, \ldots, v_{n-1}^k$ and s_k in the statement are. For this, we use the expression for Ψ_c^k given in Lemma 3.10.

Notice that $\Psi_c^k \circ \gamma$ is given by $\theta \mapsto (Q(q_0, x_0), R_k(x_0)e^{ik\theta}, -1, \theta) \in W_- \times \mathbb{S}^1 \subset F \times \mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$. An explicit computation also gives $d_{\Psi_c^k \circ \gamma(\theta)} \Psi_c^k(\partial_\theta(\theta)) = \partial_\theta(\theta) + k\partial_\varphi(r_k e^{ik\theta})$ and $d_{\Psi_c^k \circ \gamma(\theta)} \Psi_c^k(\partial_\varphi(x_0)) = \partial_\varphi(r_k e^{ik\theta})$, where $r_k = R_k(x_0)$ as in the statement.
Then, if we choose $s_k := k + \frac{2}{x_0^2}$ and $v_j^k := d_{(q_0,x_0)} G_k(v_j)$ for each $j = 1, \ldots, n-1$, we have that $(\Psi_c^k)_*\mathfrak{L}$ can indeed be written as in the statement of Lemma 3.11. □

Remark 3.12. *The informations in both Lemmas 3.10 and 3.11 could be made much more precise, by computing explicitly the flow ψ_Y^1.*
For instance, the value of s_k given at the end of the proof of Lemma 3.11 is $k + \frac{2}{x_0^2}$. One can easily see that, in order for $\partial_\theta(\theta) + s_k \partial_\varphi(r_k e^{ik\theta})$ to be tangent to $\ker \alpha_f$ (which it has to be, because image of a tangent vector via the differential of a contactomorphism), one needs the equality $k + \frac{2}{x_0^2} = \frac{2}{r_k^2}$. An explicit computation of ψ_Y^1 would have given us an explicit formula for $R_k(r)$ such that $r_k = R_k(x_0)$ satisfies this condition.
To improve the readability, we decided not to include these detailed informations, as the content of the lemmas above is actually all we need for Step 3.

Step 3 The main ingredient of the last step is the following:

Lemma 3.13. *The family of matrices $B_k\colon \mathbb{S}^1 \to GL_{n+1}(\mathbb{C})$ associated via the trivialization μ (defined in Equation (3.1)) to the stabilization*

$$(\Psi_c^k)_*\mathfrak{L} \oplus Z^{\mathcal{D}} = \big(\Psi_c^k \circ \gamma, v_1^k, \dots, v_{n-1}^k, \partial_\theta(\theta) + s_k \partial_\varphi(R_k(x_0)e^{ik\theta}), Z^{\mathcal{D}}(\Psi_c^k \circ \gamma(\theta))\big)$$

is given by matrices of the form

$$B_k(\theta) = \left(\begin{array}{c|cc} B_{0,k} & b_{0,k} & b_{1,k} \\ \hline 0 & b_{2,k}e^{ik\theta} & b_{3,k}e^{ik\theta} \\ 0 & b_{4,k} & b_{5,k} \end{array}\right),$$

where $b_{0,k}, b_{0,k} \in \mathbb{C}^{n-1}$, $b_{2,k}, \dots, b_{5,k} \in \mathbb{C}$ and $B_{0,k} \in GL_{n-1}(\mathbb{C})$.

Proof (Lemma 3.13). Notice that $\partial_\varphi(r_k e^{ik\theta}) = -r_k \sin(k\theta)\partial_x(r_k e^{ik\theta}) + r_k \cos(k\theta)\partial_y(r_k e^{ik\theta}) = r_k(-\sin(k\theta) + J\cos(k\theta))\partial_x(r_k e^{ik\theta})$ and that $\partial_r(r_k e^{ik\theta}) = \cos(k\theta)\partial_x(r_k e^{ik\theta}) + \sin(k\theta)\partial_y(r_k e^{ik\theta}) = (\cos(k\theta) + J\sin(k\theta))\partial_x(r_k e^{ik\theta})$.
Then, Lemma 3.13 immediately follows from the expression for $(\Psi_c^k)_*\mathfrak{L}$ given in Lemma 3.11 and from $Z^{\mathcal{D}}(\Psi_c^k \circ \gamma(\theta)) = \frac{1}{2}R_k(x_0)\partial_r(R_k(x_0)e^{i(\varphi+k\theta)}) - \partial_s(-1)$. □

Lemma 3.13 tells in particular that B_k is homotopically trivial as map $\mathbb{S}^1 \to GL_{n+1}(\mathbb{C})$ if and only if $k = 0$. Indeed, B_0 is a constant map, and an easy computation tells that $\det(B_k(\theta)) = b_k e^{ik\theta}$, for a certain $b_k \in \mathbb{C} \setminus \{0\}$ (notice that $b_k \neq 0$ necessarily because $B_k(\theta) \in GL_{n+1}(\mathbb{C})$); in particular, $\theta \mapsto \det(B_k(\theta))$ is homotopically non-trivial if $k \geq 1$. This concludes Step 3, hence the proof of Théorème I.A.

Chapter 4

Contact mapping classes on overtwisted manifolds

4.1 A general source of non-trivial contact mapping classes

The aim of this section is to prove Théorème I.C, already stated in Chapter 1:

Théorème I.C. *On considère une variété lisse W de dimension $2n \geq 2$ et une structure de contact ξ sur la variété $V := \mathbb{S}^1 \times W$. On suppose que la première classe de Chern $c_1(\xi) \in H^2(V;\mathbb{Z})$ est toroidale et que, pour tout entier $k \geq 2$, la tirée en arrière $\pi_k^*\xi$ de ξ via le revêtement à k-feuilles $\pi_k\colon\ \mathbb{S}^1 \times W \to \mathbb{S}^1 \times W$ donné par $\pi_k(s,p) = (ks,p)$ satisfait $c_1(\pi_k^*\xi) = k \cdot c_1(\xi)$ modulo le sous-module $H^2_{\mathrm{ator}}(V;\mathbb{Z})$ des classes atoroidales.*
Alors, la transformation de contact $f\colon\ (\mathbb{S}^1 \times W, \pi_k^*\xi) \to (\mathbb{S}^1 \times W, \pi_k^*\xi)$ définie par $f(s,p) = (s + \frac{2\pi}{k}, p)$ est isotope à l'identité lissement mais pas parmi les contactomorphismes.

As each contactomorphism gives in particular an isomorphism of the underlying almost contact structures, Théorème I.C directly follows from the two following lemmas:

Lemma 4.1. *Let $(\mathbb{S}^1 \times W^{2n}, \xi)$ be a contact manifold. For each natural $k \geq 2$, denote by $\pi_k\colon\ \mathbb{S}^1 \times W \to \mathbb{S}^1 \times W$ the k-fold cover $\pi_k(s,p) = (ks,p)$ and by $f\colon\ (\mathbb{S}^1 \times W, \pi_k^*\xi) \to (\mathbb{S}^1 \times W, \pi_k^*\xi)$ the contactomorphism $f(s,p) = (s + \frac{2\pi}{k}, p)$.*
If f is contact-isotopic to the identity, then there is a contactomorphism

$$\phi : (\mathbb{S}^1 \times W, \pi_{kN}^*\xi) \xrightarrow{\sim} (\mathbb{S}^1 \times W, \pi_{kN+1}^*\xi)\ .$$

Lemma 4.2. *Let $(V := \mathbb{S}^1 \times W, \xi)$, π_k and f be as in Lemma 4.1. If moreover $c_1(\xi)$ is toroidal and $c_1(\pi_m^*\xi) = m \cdot c_1(\xi) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$ for every natural $m \geq 2$, then $\pi_m^*\xi$ and $\pi_{m+1}^*\xi$ are not isomorphic as almost contact structures.*

We now prove Lemmas 4.1 and 4.2 above.

Proof (Lemma 4.1). In order to find the desired contactomorphism ϕ, we use an idea that already appeared in Geiges and Gonzalo Perez [GGP04] and in Marinković and Pabiniak [MP16], and which consists in cutting off contact Hamiltonians on a particular cover of the manifold we are working with.

By hypothesis, the contactomorphism $f : (\mathbb{S}^1 \times W, \pi_k^*\xi) \to (\mathbb{S}^1 \times W, \pi_k^*\xi)$ defined by $f(s,p) = (s + \frac{2\pi}{k}, p)$ is contact isotopic to the identity. Call $(F_r)_{r\in[0,1]}$ the isotopy, so that $F_0 = \mathrm{Id}$, $F_1 = f$ and F_r is a contactomorphism for all $r \in [0,1]$.

Take now the universal cover $\mathbb{R}_s$ of the factor $\mathbb{S}^1_s$ of the manifold $\mathbb{S}^1_s \times W$. Then, pull back $\pi_k^*\xi$ to a contact structure η_k on the covering $\mathbb{R}_s \times W$ of $\mathbb{S}^1_s \times W$ and lift the contact isotopy F_r to a contact isotopy Φ_r of $(\mathbb{R}_s \times W, \eta_k)$ starting at the identity. Fix a certain contact form β_k for η_k and denote by $H_r \colon \mathbb{R}_s \times W \to \mathbb{R}$ the path of contact Hamiltonians $\beta_k(Y_r)$ associated to the contact vector field Y_r generating the isotopy Φ_r (see the discussions following Proposition 2.5).

Now, by compactness of W and $[0,1]$, there is an $N > 0$ such that $\Phi(\{0\}_s \times W \times [0,1]_r)$ is contained in $(-2\,(N-1)\,\pi, +\infty)_s \times W$.

Consider then an $\epsilon > 0$ very small and a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(x) = 0$ for $x < -2N\pi + \epsilon$ and $\rho(x) = 1$ for $x > -2\,(N-1)\,\pi - \epsilon$. We can then construct a new contact Hamiltonian: $K_r(s,p) := \rho(s) \cdot H_r(s,p)$, for all $(s,p) \in \mathbb{R}_s \times W$.

We claim that the contact vector field Z_r associated to this new Hamiltonian K_r can be integrated to a contact isotopy $(\Psi_r)_{r\in[0,1]}$ of $(\mathbb{R}_s \times W, \eta_k)$ starting at the identity. Indeed, Z_r is zero for $s < -2N\pi + \epsilon$ and equal to the contact field Y_r for $s > -2\,(N-1)\,\pi - \epsilon$, which means in particular that it is integrable outside of a compact set of $\mathbb{R}_s \times W$ (remark that Y_r is trivially integrable, because it comes from a contact isotopy); this implies integrability on all $\mathbb{R} \times W$. Moreover, $\Psi_r|_{\{0\}\times W} = \Phi_r|_{\{0\}\times W}$ and $\Psi_r|_{\{-2N\pi\}\times W} = \mathrm{Id}\,|_{\{-2N\pi\}\times W}$ for all $r \in [0,1]$.

In particular, Ψ_1 maps $[-2N\pi, 0]\times W$ contactomorphically to $[-2N\pi, \frac{2\pi}{k}]\times W$, where we consider on the domain and on the codomain the structure η_k.

Now, by the periodicity of η_k, we can identify the two boundary components of $[-2N\pi, 0] \times W$ so that the restriction of η_k induces a well defined contact structure on the quotient. More precisely, the quotient contact manifold obtained is $(\mathbb{S}^1_s \times W, \pi_{kN}^*\xi)$. The analogous procedure for the codomain $[-2N\pi, \frac{2\pi}{k}] \times W$ of Ψ_1 gives as quotient the contact manifold $(\mathbb{S}^1_s \times W, \pi_{kN+1}^*\xi)$.

Lastly, because $\Psi_1 : [-2N\pi, 0] \times W \to [-2N\pi, \frac{2\pi}{k}] \times W$ is the identity on a neighborhood of $\{-2N\pi\} \times W$ and a lift of the translation f on a neighborhood of $\{0\} \times W$, it induces on the quotient contact manifolds a well defined contactomorphism

$$\phi : (\mathbb{S}^1_s \times W, \pi_{kN}^*\xi) \xrightarrow{\sim} (\mathbb{S}^1_s \times W, \pi_{kN+1}^*\xi)\ . \qquad \square$$

Proof (Lemma 4.2). Suppose by contradiction that there is an isomorphism of almost contact structures $\psi : (V, \pi_m^*\xi) \xrightarrow{\sim} \left(V, \pi_{m+1}^*\xi\right)$; in particular,

$$\psi_* c_1(\pi_m^*\xi) = c_1(\pi_{m+1}^*\xi)\ . \tag{4.1}$$

Now, the submodule $H^2_{\mathrm{ator}}(V;\mathbb{Z})$ of atoroidal classes is natural, (i.e. it is preserved by pullbacks induced by continuous maps $V \to V$); in particular, ψ_* induces a well defined endomorphism, which is moreover an isomorphism, of the quotient of $H^2(V;\mathbb{Z})$ by $H^2_{\mathrm{ator}}(V;\mathbb{Z})$. We then have $\psi_*(\pi_n^*\xi) = n\psi_* c_1(\xi) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$ for each natural $n \geq 2$, so that Equation 4.1 becomes

$$m\psi_* c_1(\xi) = (m+1)c_1(\xi) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})\ . \tag{4.2}$$

Notice that $N := {H^2(V;\mathbb{Z})}\big/{H^2_{\mathrm{ator}}(V;\mathbb{Z})}$ is a finitely generated $\mathbb{Z}$-module without torsion. In particular, the (well defined) *divisibility* map

$$\begin{aligned} d\colon\ N \setminus \{0\} &\to \mathbb{N} \setminus \{0\} \\ n &\mapsto \{\, k \in \mathbb{N} \,|\, \exists n',\ n = kn' \,\} \end{aligned}$$

satisfies $d(hn) = hd(n)$ and $d(\psi_* n) = d(n)$, for each $n \in N \setminus \{0\}$ and $h \in \mathbb{N} \setminus \{0\}$. Now, because $c_1(\xi)$ is toroidal, we can apply d to both the left and right hand sides of Equation 4.2, thus obtaining the wanted contradiction. □

4.2 Examples from Liouville pairs and half Lutz-Mori twists

In this section we give a proof of Proposition I.D; the main idea is the following. The contact structure η on the manifold $V = \mathbb{S}^1 \times W$ in the statement has trivial Chern classes (better, it is trivializable as complex bundle). We then apply a semi-local modification to η and obtain another contact structure ξ; the explicit nature of this modification (as well as the explicit nature of the original contact manifold (V, η)) allows us to compute the first Chern class of ξ, and to show that it satisfies the wanted conditions.

More precisely, this section is structured in the following way. We recall in Sections 4.2.1 and 4.2.2, respectively, the notion of half Lutz-Mori twist and the construction of Liouville pairs, both appearing in Massot, Niederkrüger and Wendl [MNW13]. We then describe in Section 4.2.3 how half Lutz-Mori twists (along contact submanifolds belonging to one of the Liouville pairs constructed in [MNW13]) affect the Chern classes of the underlying almost contact structure. Finally, Section 4.2.4 contains the proof of Proposition I.D.

4.2.1 The half Lutz-Mori twist

Developing some ideas introduced by Mori in [Mor] in the 5-dimensional case, Massot, Niederkrüger and Wendl introduce in [MNW13] the notion of *Lutz-Mori twist* along a manifold belonging to a *Liouville pair* as a generalization of the well known 3-dimensional Lutz twists.
We give here an explicit description of how to perform the *half* version of the Lutz-Mori twist in particular coordinates in a neighbourhood of the contact submanifold.

Let (V, ξ) be a contact manifold having as a codimension-2 contact submanifold (M, ξ_+) such that α_+ defining ξ_+ belongs to a *Liouville pair*, defined as follows:

Definition 4.3. [MNW13] Let M^{2m+1} be an oriented manifold. We call *Liouville pair on M* a couple of contact forms (α_+, α_-) such that $\pm\alpha_\pm \wedge (d\alpha_\pm)^m > 0$ and such that the form $e^r\alpha_+ + e^{-r}\alpha_-$ is a Liouville form on $M \times \mathbb{R}_r$, i.e. its differential is a symplectic form on $M \times \mathbb{R}$.

We point out that the existence of Liouville pairs on closed manifolds is not trivial; at the moment, the only known examples in high dimension are given by the construction in [MNW13, Section 8], which is nonetheless a source of infinitely many non-homeomorphic manifolds with Liouville pairs in each (odd) dimension. In Section 4.2.2 we will recall the properties of this construction which are needed in order to prove Theorem I.C.

We now want to find particular coordinates near the submanifold (M, ξ_+).
For all $\varepsilon > 0$, denote D^2_ε the (open) disk of radius ε centered at the origin inside $\mathbb{R}^2$. Consider then a smooth map $\Psi : D^2_\varepsilon \setminus \{0\} \to \mathbb{S}^1 \times (0, \varepsilon)$, defined by $\Psi(r, \varphi) = (\varphi, \psi(r))$, where (r, φ) are polar coordinates on D^2_ε and $\psi : (0, \varepsilon) \to (0, \varepsilon)$ is smooth, strictly increasing, equal to r^2 on $(0, \frac{\varepsilon}{3})$ and equal to r on $(\frac{2}{3}\varepsilon, \varepsilon)$.
Consider now the 1-form $\alpha_0 = \frac{1+\cos(s)}{2}\,\alpha_+ + \frac{1-\cos(s)}{2}\,\alpha_- + \sin(s)\, dt$ on $M \times \mathbb{S}^1_t \times (0, \varepsilon)_s$. The fact that (α_+, α_-) is a Liouville pair implies that α_0 is a contact form; see [MNW13,

Proposition 9.1] for the details. If Ψ' denotes the map $(\mathrm{Id}_M, \Psi) : M \times \left(D^2_\varepsilon \setminus \{0\}\right) \to M \times \mathbb{S}^1_t \times (0,\varepsilon)_s$, then the pull-back $(\Psi')^* \alpha_0$ can be written as $\alpha_+ + r^2 d\varphi + \gamma$, with γ smooth on $M \times \left(D^2_\varepsilon \setminus \{0\}\right)$ and

$$\gamma = \frac{\cos\left(r^2\right) - 1}{2}\,\alpha_+ + \frac{1 - \cos\left(r^2\right)}{2}\,\alpha_- + \left[\sin\left(r^2\right) - r^2\right] d\varphi \quad \text{for } r < \frac{\epsilon}{3}\,.$$

Hence, $(\Psi')^* \alpha_0$ naturally extends to a (smooth) contact form α on $M \times D^2_\varepsilon$, which moreover restricts to α_+ on $M \times \{0\} \simeq M$.
Now, each contact submanifold of codimension 2 having topologically trivial normal bundle also has trivial conformal symplectic normal bundle. Hence, by the contact submanifold neighbourhood theorem (see Geiges [Gei08, Theorem 2.5.15]), each contact manifold (V,ξ) containing (M,ξ_+) as a codimension 2 contact submanifold with trivial normal bundle will also contain, for $\varepsilon > 0$ small enough, the above model $\left(M \times D^2_\varepsilon, \xi = \ker(\alpha)\right)$ as a (codimension 0) contact submanifold, in such a way that (M,ξ_+) coincides with $(M \times \{0\}, \xi|_{M\times\{0\}})$.

We now describe how to modify the contact structure in this particular local coordinates around (M,ξ_+) in order to perform the half twist.
Consider another smooth map $\Phi : D^2_\varepsilon \setminus \{0\} \to \mathbb{S}^1 \times (-\pi,\varepsilon)$, defined by $\Phi(r,\varphi) = (\varphi, \phi(r))$, where (r,φ) are again polar coordinates on D^2_ε and $\phi : (0,\epsilon) \to (-\pi,\varepsilon)$ is again smooth, strictly increasing, equal to r on $(\frac{2}{3}\varepsilon, \varepsilon)$, but this time equal to $r^2 - \pi$ on $(0, \frac{\varepsilon}{3})$.

As before, if Φ' denotes the map $(\mathrm{Id}_M, \Phi) : M \times \left(D^2_\varepsilon \setminus \{0\}\right) \to M \times \mathbb{S}^1_t \times (-\pi,\varepsilon)_s$, then the contact form $(\Phi')^* \alpha_0$ naturally extends to a contact form α' on $M \times D^2_\varepsilon$, but this time at $M \times \{0\}$ we have the contact submanifold $(M, \xi_- = \ker(\alpha_-))$.

We remark though that the contact manifolds $(M \times D^2_\varepsilon, \xi = \ker\alpha)$ and $(M \times D^2_\varepsilon, \xi' = \ker\alpha')$ coincide on the subset $\{r \geq \frac{5}{6}\varepsilon\}$ of $M \times D^2_\varepsilon$. If we denote by $\overline{D^2_\delta}$ the closed disk of radius $\delta := \frac{11}{12}\varepsilon$ centered at the origin inside $\mathbb{R}^2$, we can thus replace $(M \times \overline{D^2_\delta}, \xi)$ with $(M \times \overline{D^2_\delta}, \xi')$ inside $(M \times D^2_\varepsilon, \xi) \subset (V,\xi)$; this gives a contact manifold (V,ξ').

Definition 4.4. [MNW13, Remark 9.6] We say that (V,ξ') is obtained from (V,ξ) by a *half Lutz-Mori twist* along the contact submanifold $(M, \xi_+ = \ker(\alpha_+))$ belonging to the Liouville pair (α_+, α_-).

We point out that performing a half Lutz-Mori twist makes the contact manifold overtwisted. Indeed, it is explained in Massot, Niederkrüger and Wendl [MNW13, Remark 9.6] that this half twist always gives a PS-overtwisted manifold, which then is also overtwisted according to [CMP15, Hua17].

4.2.2 Construction of Liouville pairs

We recall here the construction in [MNW13, Section 8], leaving the details that are not important for our purposes.

Consider the product manifold $\mathbb{R}^m \times \mathbb{R}^{m+1}$ with the pair of contact structures ξ_+, ξ_- induced by the following pair of contact forms:

$$\alpha_\pm := \pm e^{t_1+\ldots+t_m} d\theta_0 + e^{-t_1} d\theta_1 + \ldots + e^{-t_m} d\theta_m\ ,$$

where we use coordinates $(t_1,\ldots,t_m)$ on $\mathbb{R}^m$ and $(\theta_0,\ldots,\theta_m)$ on $\mathbb{R}^{m+1}$. A direct computation shows that (α_+,α_-) is a Liouville pair on $\mathbb{R}^m \times \mathbb{R}^{m+1}$.

We now remark that there are two Lie groups acting explicitly on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ by strict contact transformations for both α_+ and α_-.
Indeed, the left action of the group $\mathbb{R}^{m+1}$ on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ given by the translations

$$(\varphi_0,\ldots,\varphi_m)\cdot(t_1,\ldots,t_m,\theta_0,\ldots,\theta_m) := (t_1,\ldots,t_m,\theta_0+\varphi_0,\ldots,\theta_m+\varphi_m) \tag{4.3}$$

and the left action of $\mathbb{R}^m$ given by the law

$$\begin{aligned} &(\tau_1,\dots,\tau_m)\cdot(t_1,\dots,t_m,\theta_0,\dots,\theta_m) \\ &\coloneqq\ (t_1+\tau_1,\dots,t_m+\tau_m,e^{-\tau_1+\dots-\tau_m}\theta_0,e^{\tau_1}\theta_1,\dots,e^{\tau_m}\theta_m) \end{aligned} \tag{4.4}$$

are Lie group left-actions on $\mathbb{R}^m\times\mathbb{R}^{m+1}$ and they both preserve the contact forms α_+ and α_-.

Moreover, these two actions allow us to produce a *compact* contact manifold from $\mathbb{R}^m\times\mathbb{R}^{m+1}$. Indeed, there are lattices Λ,Λ' of $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$ respectively, such that the Λ-action on $\mathbb{R}^m\times\mathbb{R}^{m+1}$ induced by the action of $\mathbb{R}^m$ preserves $\mathbb{R}^m\times\Lambda'$. This implies that, by first taking the quotient of $\mathbb{R}^m\times\mathbb{R}^{m+1}$ by Λ' and then quotienting it by the (well defined by the above property) induced action of Λ, we obtain a compact manifold M.

Finally, this manifold M naturally inherits a Liouville pair, still denoted by (α_+,α_-), from the Liouville pair on the covering $\mathbb{R}^m\times\mathbb{R}^{m+1}$, because $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$ act on $\mathbb{R}^m\times\mathbb{R}^{m+1}$ by strict contactomorphisms for both α_+ and α_-.

We point out that this construction actually gives an infinite number of non homeomorphic manifolds M, hence an infinite number of non isomorphic Liouville pairs, in each odd dimension bigger or equal to 3.
Indeed, the existence of the lattices Λ and Λ' follows from number theory arguments and the manifold M obtained depends on the choice of a totally real field of real numbers $\Bbbk$ with finite dimension over $\mathbb{Q}$. Now, for each dimension ≥ 2 over $\mathbb{Q}$, there are infinitely such fields $\Bbbk$ and the corresponding manifolds are non homeomorphic. See [MNW13, Lemma 8.3] for the details.
As far as Proposition I.D is concerned, this means that we have, in each odd dimension $2n+1\geq 5$, a contact structure satisfying the hypothesis of Theorem I.C on infinitely many different smooth manifolds $\mathbb{T}^2\times M^{2n-1}$; in dimension 3, we obtain one contact structure on $\mathbb{T}^2\times M^1=\mathbb{T}^3$. In both cases, Theorem I.C then gives examples of contactomorphisms smoothly isotopic but not contact isotopic to the identity for the countably many contact structures $(\pi_k^*\xi)_{k\geq 2}$ on each $\mathbb{T}^2\times M$.

4.2.3 Topological effects of the half twists

Using the construction of the previous section, we obtain the following result:

Proposition 4.5. *Let (V^{2m+3},ξ) be a contact manifold containing the (M,ξ_+) of Section 4.2.2 as a codimension 2 contact submanifold with trivial normal bundle. Then, if we denote by ξ' the contact structure on V obtained by performing a half Lutz-Mori twist along the submanifold (M,ξ_+) (where we consider M with the orientation given by ξ_+), we have the following:*

1. *for all $i=2,\dots,m+1$, $c_i(\xi')-c_i(\xi)=0$ in $H^{2i}(V;\mathbb{Z})$;*
2. *$c_1(\xi')-c_1(\xi)=-2\,\mathrm{PD}\,(j_*[M])$ in $H^2(V;\mathbb{Z})$, where $j:M\to V$ is the inclusion, $j_*:H_{2m+1}(M;\mathbb{Z})\to H_{2m+1}(V;\mathbb{Z})$ is the induced map and $\mathrm{PD}(\alpha)$ denotes the Poincaré dual of the homology class $\alpha\in H_*(V;\mathbb{Z})$.*

Remark 4.6. *This result is not in contradiction with Massot, Niederkrüger and Wendl [MNW13, Theorem 9.5], where the authors prove that the contact structures before and after a full Lutz-Mori twist (as defined in [MNW13, Section 9.1]) are homotopic through almost contact structures, hence have the same Chern classes.*
Indeed, the result ξ'' of a full Lutz-Mori twist can be interpreted as a couple of successive

half twists. More precisely, we first perform a half twist along a submanifold (M, ξ_+) to obtain ξ'; this changes the core of the tube where we perform the twist from (M, ξ_+) to (M, ξ_-). We then perform another half twist, this time along the new core (M, ξ_-), to obtain ξ''. Hence, applying Proposition 4.5 twice and using the fact that ξ_- induces an orientation that is opposite to that induced by ξ_+, we get that $c_i(\xi'') = c_i(\xi') = c_i(\xi)$ for all $i = 2, \dots, m+1$ and that $c_1(\xi'') = c_1(\xi') - 2\,\mathrm{PD}\,(j_*[-M]) = c_1(\xi) - 2\,\mathrm{PD}\,(j_*[M]) - 2\,\mathrm{PD}\,(j_*[-M]) = c_1(\xi)$, as we expected from [MNW13, Theorem 9.5].

Chern classes are global invariants of complex vector bundles E over a manifold V. In order to prove Proposition 4.5 above, we then have the following problem: it's not clear how local modifications (i.e. over an open set $\mathcal{U}$ of V) of the complex vector bundle E affect its Chern classes. The solution is hence either to use a relative version of Chern classes or to shift to another point of view more local in nature; we adopt here the second strategy.
More precisely, following [ACMFA07] we describe in Appendix A how each Chern class of E can be interpreted as the Poincaré dual of (a desingularised version of) the locus of points of V where a "generic" set of sections of E is not linearly independent. We point out that this generalizes the classical fact that the top Chern class of E is the Poincaré dual of the zero locus of a section of E which is transverse to the zero section (see [BT82, Property 20.10.6 and Proposition 12.8]).
In particular, we will use the following result, which will be proven in Appendix A, in order to prove Proposition 4.5 above:

Proposition 4.7. *Let E, E' be two complex vector bundles of complex rang r over the same smooth oriented manifold V. Suppose also that there exist two open subsets $\mathcal{O}, \mathcal{U}$ of V, with $\mathcal{O}$ compactly contained in $\mathcal{U}$, such that the following are satisfied:*

1. *There is an isomorphism of vector bundles $\psi : E|_{\mathcal{O}^c} \to E'|_{\mathcal{O}^c}$ over $\mathcal{O}^c := V \setminus \mathcal{O}$.*

2. *The vector bundles $E|_{\mathcal{U}}$ and $E'|_{\mathcal{U}}$ over $\mathcal{U}$ admit complex sub-bundles $F, L \subset E$ and $F', L' \subset E'$ such that $E|_{\mathcal{U}} = F \oplus L$ and $E'|_{\mathcal{U}} = F' \oplus L'$ and satisfying the following conditions:*

   (a) $\psi \circ s_j = s'_j$ over $\mathcal{U} \setminus \mathcal{O}$ for all $j = 1, \dots, r-1$;

   (b) L, L' have complex rank 1, while F, F' have complex rank $r-1$ and are trivialized by two ordered sets of everywhere $\mathbb{C}$-linearly independent sections $(s_1, \dots, s_{r-1})$ and $\left(s'_1, \dots, s'_{r-1}\right)$ of $E|_{\mathcal{U}}$ and $E'|_{\mathcal{U}}$;

   (c) there are two additional sections s_r, s'_r of $E|_{\mathcal{U}}$ and $E'|_{\mathcal{U}}$ respectively, with image contained in L and L' and such that $s_r : \mathcal{U} \to E|_{\mathcal{U}}$ intersects transversely F and $s'_r : \mathcal{U} \to E'|_{\mathcal{U}}$ intersects transversely F' (here F and F' are seen here as submanifolds of $E|_{\mathcal{U}}$ and $E'|_{\mathcal{U}}$);

   (d) $Z := s_r^{-1}(F)$ and $Z' := \left(s'_r\right)^{-1}(F')$, which are oriented smooth manifolds of $\mathcal{U}$ by Hypothesis 2c above, are actually compactly contained in $\mathcal{O}$.

Then, we have the following:

1. *$c_k(E') = c_k(E)$ in $H^{2k}(V; \mathbb{Z})$ for all $2 \leq k \leq r$;*

2. *$c_1(E') - c_1(E) = \mathrm{PD}([Z']) - \mathrm{PD}([Z])$ in $H^2(V; \mathbb{Z})$.*

We are now ready to give a proof of Proposition 4.5:

Proof (Proposition 4.5). Consider the manifold M and the Liouville pair (α_+, α_-) constructed in Section 4.2.2 and take a contact manifold (V, ξ) containing $(M, \xi_+ = \ker(\alpha_+))$ as a contact submanifold of dimension 2.

Denote by (V, ξ') the result of a half Lutz-Mori twist on (V, ξ) along (M, ξ_+). According to Section 4.2.1, we have a tubular neighbourhood $M \times D^2_\epsilon$ of M in which we can arrange to have contact forms α, α' for ξ and ξ' respectively which satisfy the following: if (r, φ) are the polar coordinates on D^2_ϵ and ψ, ϕ are as in Section 4.2.1, then

- $\alpha = \frac{1+\cos(\psi(r))}{2}\,\alpha_+ + \frac{1-\cos(\psi(r))}{2}\,\alpha_- + \sin\left(\psi\left(r\right)\right) d\varphi$,
- $\alpha' = \frac{1+\cos(\phi(r))}{2}\,\alpha_+ + \frac{1-\cos(\phi(r))}{2}\,\alpha_- + \sin\left(\phi\left(r\right)\right) d\varphi$.

Now, we have explicit expressions for α_+ and α_- on the cover $\mathbb{R}^m \times \mathbb{R}^{m+1}$ of M, i.e. $\alpha_\pm := \pm e^{t_1+\ldots+t_m} d\theta_0 + e^{-t_1} d\theta_1 + \ldots + e^{-t_m} d\theta_m$. Thus, on the cover $\mathbb{R}^m \times \mathbb{R}^{m+1} \times D^2_\epsilon$ of the tubular neighbourhood $M \times D^2_\epsilon$ of M inside V, we can write in a more explicit form

(i) $\alpha = e^{\sum_{i=1}^m t_i} \cos\left(\psi\left(r\right)\right) d\theta_0 + \sum_{i=1}^m e^{-t_i} d\theta_i + \sin\left(\psi\left(r\right)\right) d\varphi$,

(ii) $\alpha' = e^{\sum_{i=1}^m t_i} \cos\left(\phi\left(r\right)\right) d\theta_0 + \sum_{i=1}^m e^{-t_i} d\theta_i + \sin\left(\phi\left(r\right)\right) d\varphi$.

Take now the following $2m$ $\mathbb{R}$-linearly independent sections of the pullback of ξ and ξ' to $\mathbb{R}^m \times \mathbb{R}^{m+1} \times D^2_\epsilon$: for $i = 1, \ldots, m$,

- $s_i := \partial_{t_i}$ and $r_i := e^{-\sum_{j=1}^m t_j} \cos\left(\psi\left(r\right)\right) \partial_{\theta_0} - e^{t_i} \partial_{\theta_i} + \sin\left(\psi\left(r\right)\right) \partial_\varphi$ for the pullback of ξ,
- $s'_i := \partial_{t_i}$ and $r'_i := e^{-\sum_{j=1}^m t_j} \cos\left(\phi\left(r\right)\right) \partial_{\theta_0} - e^{t_i} \partial_{\theta_i} + \sin\left(\phi\left(r\right)\right) \partial_\varphi$ for the pullback of ξ'.

Let's also consider the following sections:

- $s_{m+1} := r\partial_r$ and $r_{m+1} := \cos\left(\psi\left(r\right)\right) \partial_\varphi - \sin\left(\psi\left(r\right)\right) e^{-\sum_{i=1}^m t_i} \partial_{\theta_0}$ for the pullback of ξ,
- $s'_{m+1} := r\partial_r$ and $r'_{m+1} := \cos\left(\phi\left(r\right)\right) \partial_\varphi - \sin\left(\phi\left(r\right)\right) e^{-\sum_{i=1}^m t_i} \partial_{\theta_0}$ for the pullback of ξ'.

These last two couples of sections are $\mathbb{R}$-linearly independent whenever s_{m+1} and s'_{m+1} are non-zero.

Moreover, for $i = 1, \ldots, m+1$, s_i, r_i, s'_i and r'_i are invariant under the left-action induced on $\mathbb{R}^m \times \mathbb{R}^{m+1} \times D^2_\epsilon$ by the left-actions of of the Lie groups $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$ on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ described in Equations 4.3 and 4.4 of Section 4.2.2. Hence, they induce well defined sections of ξ and ξ' on $M \times D^2_\epsilon$, which we will still denote using same notations. We also point out that each section coincide with its "primed version" near $M \times D^2_\epsilon$.

We remark now that $\mathrm{Span}_{\mathbb{R}}\left(s_{m+1}(p), r_{m+1}(p)\right)$ and $\mathrm{Span}_{\mathbb{R}}\left(s'_{m+1}(p), r'_{m+1}(p)\right)$, a priori well defined only for $p \in M \times \left(D^2_\epsilon \setminus \{0\}\right)$, actually extend smoothly also over $M \times \{0\}$. Indeed, consider the following couples of sections of ξ and ξ' respectively:

- $S := \frac{1}{r}\left(\cos\left(\varphi\right) s_{m+1} - \sin\left(\varphi\right) r_{m+1}\right)$, $R := \frac{1}{r}\left(\sin\left(\varphi\right) s_{m+1} + \cos\left(\varphi\right) r_{m+1}\right)$;
- $S' := \frac{1}{r}\left(\cos\left(\varphi\right) s'_{m+1} + \sin\left(\varphi\right) r'_{m+1}\right)$, $R' := \frac{1}{r}\left(-\sin\left(\varphi\right) s'_{m+1} + \cos\left(\varphi\right) r'_{m+1}\right)$.

These sections, defined on $M \times \left(D^2_\epsilon \setminus \{0\}\right)$, can be smoothly extended to sections on all $M \times D^2_\epsilon$.
For example, the section S can be rewritten near $r = 0$ as follows:

$$\begin{aligned} S &= \frac{1}{r}\left(\cos(\varphi)\, s_{m+1} \,-\, \sin(\varphi)\, r_{m+1}\right) \\ &= \cos\varphi\,\partial_r \,-\, \frac{\sin\varphi\,\cos r^2}{r}\,\partial_\varphi \,+\, \frac{\sin\varphi\,\sin r^2}{r}\, e^{-\sum_{i=1}^m t_i}\,\partial_{\theta_0} \\ &= \partial_x \,+\, y\,\frac{1-\cos r^2}{r^2}\,(-y\partial_x + x\partial_y) \,+\, y\,\frac{\sin r^2}{r^2}\, e^{-\sum_{i=1}^m t_i}\,\partial_{\theta_0}\ , \end{aligned}$$

and each coefficient extends smoothly to all $M \times D^2_\epsilon$. Analogous computations show that also S', R, R' extend smoothly to $M \times D^2_\epsilon$. We will denote these smooth extensions still by S, R, S', R'.

Moreover, we point out that $(s_1, r_1, \ldots, s_m, r_m, S, R)$ are everywhere $\mathbb{R}$-linear independent sections of ξ, which is hence trivialized by them over $M \times D^2_\epsilon$; the analogue is true for $(s'_1, r'_1, \ldots, s'_m, r'_m, S', R')$. We remark that, unlike the couples (s_{m+1}, r_{m+1}) and (s'_{m+1}, r'_{m+1}), the (S, R) and (S', R') do not coincide near the boundary of $M \times D^2_\epsilon$.

Computing the differentials of α and α' thanks to the above explicit expressions (i) and (ii) for their pullbacks, we see that $d\alpha(s_i, r_i) > 0$ and $d\alpha'(s'_i, r'_i) > 0$ for all $i = 1, \ldots, m$ and that $d\alpha(S, R) > 0$ and $d\alpha'(S', R') > 0$.
Then, the identities $J(s_i) := r_i$ and $J'(s'_i) := r'_i$, for all $i = 1, \ldots, m$, and the identities $J(S) := R$, $J'(S') := R'$ give two complex structures J, J' on ξ and ξ' over $M \times D^2_\epsilon$ which are tamed by $d\alpha$ and $d\alpha'$. In particular, the sections $s_1, \ldots, s_m, S$ are $\mathbb{C}$-linearly independent on ξ and the sections $s'_1, \ldots, s'_m, S'$ are $\mathbb{C}$-linearly independent on ξ'.

We point out that J, J' satisfy also the identities $J(s_{m+1}) = r_{m+1}$ and $J'(s'_{m+1}) = r'_{m+1}$. This shows in particular that J and J' coincide over a neighbourhood of the boundary of $M \times D^2_\epsilon$: indeed, each section coincide with its primed version near the boundary of $M \times D^2_\epsilon$ and the span of $(s_i, r_i)_{i=1}^{m+1}$ and $(s'_i, r'_i)_{i=1}^{m+1}$ are respectively ξ and ξ' on $M \times \left(D^2_\epsilon \setminus \{0\}\right)$.

We can now extend J and J' to complex structures on ξ and ξ' over all V, tamed by contact forms that extend α and α', in such a way that they coincide outside $M \times D^2_\epsilon$. We denote such extensions still with J and J'.

We now claim that we are in the hypothesis of Proposition 4.7 if we choose as open set $\mathcal{O}$ an arbitrary open set compactly contained in $\mathcal{U} := M \times D^2_\epsilon$ and containing the support of the half Lutz-Mori twist.

Indeed, if F, F' are the complex span of $(s_1, \ldots, s_m), (s'_1, \ldots, s'_m)$ and L, L' are the complex lines determined by S, S', then the Hypothesis 1, 2a and 2b are trivially satisfied because ξ and ξ' coincide outside $\mathcal{O}$ and because of the choice of $s_1, \ldots, s_m$ and $s'_1, \ldots, s'_m$.

Let's show that the Hypothesis 2c and 2d are also satisfied in our case.
We claim that $s_{m+1} : M \times D^2_\epsilon \to \xi$ and $s'_{m+1} : M \times D^2_\epsilon \to \xi'$ intersect transversely $F \subset \xi$ and $F' \subset \xi'$ in $M \times \{0\}$ and $-M \times \{0\}$ (i.e. $M \times \{0\}$ but with opposite orientation).

Indeed, using the complex trivialization $(s_1, \ldots, s_m, S)$ for ξ on $\mathcal{U} = M \times D^2_\epsilon$, we can write $s_{m+1} : \mathcal{U} \to \xi = \mathcal{U} \times \mathbb{C}^{m+1}$ as $s_{m+1}(q) = (q, v_1(q), \ldots, v_{m+1}(q))$, with $v_i : \mathcal{U} \to \mathbb{C}$. More precisely, recalling that $JS = R$, that $S = \frac{1}{r}\left(\cos(\varphi)\, s_{m+1} - \sin(\varphi)\, r_{m+1}\right)$ and that $R = \frac{1}{r}\left(\sin(\varphi)\, s_{m+1} + \cos(\varphi)\, r_{m+1}\right)$, for each $q = (m, x, y) \in \mathcal{U} = M \times D^2_\epsilon$, with $m \in M$, we actually have that $v_i(q) = 0$ for all $i = 1, \ldots, m$ and that $v_{m+1}(q) = x + iy$, where $(x, y) \in D^2_\epsilon$ are the Cartesian coordinates. In particular, $d_{(m,0)} v_{m+1}(\partial_x) = \partial_x$ and $d_{(m,0)} v_{m+1}(\partial_y) = \partial_y$, i.e.

$$d_{(m,0)} v_{m+1}|_{\{0_m\} \oplus T_0 D^2_\epsilon} \colon\ \{0_m\} \oplus T_0 D^2_\epsilon \to T_0\mathbb{C}$$

is an orientation preserving isomorphism of vector spaces.
In other words, $s_{m+1} : M \times D^2_\epsilon \to \xi$ intersects transversely $F \subset \xi$ in $M \times \{0\}$, considered as an oriented manifold.

An analogous computation with s'_{m+1} shows that we can write $s'_{m+1} : \mathcal{U} \to \xi' = \mathcal{U} \times \mathbb{C}^{m+1}$ as $s'_{m+1}(q) = \big(q, 0, \dots, 0, v'_{m+1}(q)\big)$, with $v'_{m+1}(q) = x - iy$ for each $q = (m, x, y) \in \mathcal{U} = M \times D^2_\epsilon$. This gives in particular that $d_{(m,0)} v'_{m+1}|_{\{0_m\} \oplus T_0 D^2_\epsilon} : \{0_m\} \oplus T_0 D^2_\epsilon \to T_0\mathbb{C}$ is an orientation reversing isomorphism of vector spaces hence that $s'_{m+1} : M \times D^2_\epsilon \to \xi'$ intersect transversely $F' \subset \xi'$ along the oriented submanifold $-M \times \{0\}$.
At this point, Proposition 4.5 follows from Proposition 4.7. □

4.2.4 Explicit examples of non-trivial contact mapping classes

We now prove Proposition I.D, of which we recall the statement:

Proposition I.D. *Soit $(M^{2n-1}, \alpha_+, \alpha_-)$ une des paires de Liouville construites (en nombre infini) dans [MNW13]. On considère la structure de contact*

$$\eta = \ker\left(\frac{1+\cos(s)}{2}\alpha_+ + \frac{1-\cos(s)}{2}\alpha_- + \sin(s)\, dt\right)$$

sur $V := \mathbb{T}^2_{(s,t)} \times M$ (ici, la notation $\mathbb{T}^2_{(s,t)}$ dénote le choix de coordonnées (s,t) sur $\mathbb{T}^2$) et soit ξ la structure de contact vrillée obtenue par η via un demi twist de Lutz-Mori le long de $\{(0,0)\} \times M$, comme défini dans [MNW13].
Alors, $c_1(\xi) \in H^2(V;\mathbb{Z})$ est toroidale et, pour tout entier $k \geq 2$, on a $c_1(\pi_k^*\xi) = k \cdot c_1(\xi)$ mod $H^2_{\text{ator}}(V;\mathbb{Z})$, où $\pi_k \colon \mathbb{T}^2_{(s,t)} \times M \to \mathbb{T}^2_{(s,t)} \times M$ est donné par $\pi_k(s,t,q) = (ks,t,q)$.

Proof (Proposition I.D). The contact structure η on the manifold $\mathbb{T}^2_{(s,t)} \times M$ admits a trivialization as complex vector bundle given by the following sections and choice of J:

1. $s_i := \partial_{t_i}$ for $i = 1, \dots, m$,
2. $J(s_i) := e^{-\sum_{j=1}^m t_j} \cos(s)\, \partial_{\theta_0} - e^{t_i}\partial_{\theta_i} + \sin(s)\, \partial_t$, for $i = 1, \dots, m$,

where we use locally on M the coordinates $(t_1, \dots, t_m, \theta_0, \dots, \theta_m)$ given by the construction in Section 4.2.2. In particular, all the Chern classes of ξ are zero.
Hence, applying Proposition 4.5 to the couple (ξ, η) we get the following: if we denote by $j : M \to \mathbb{T}^2_{(s,t)} \times M$ the inclusion $j(p) = (0,0,p)$ and by $j_* : H_{2m+1}(M;\mathbb{Z}) \to H_{2m+1}(\mathbb{T}^2 \times M;\mathbb{Z})$ the induced map in homology, then $c_1(\xi) = -2\,\mathrm{PD}\,(j_*[M])$ in $H^2(\mathbb{T}^2 \times M;\mathbb{Z})$.

We now prove that $c_1(\xi)$ is toroidal. Fix a $p \in M$ and consider $f\colon \mathbb{T}^2 \to \mathbb{T}^2 \times M$ given by $f(\theta,\varphi) = (\theta,\varphi,p)$, for every $(\theta,\varphi) \in \mathbb{T}^2$. Because f is transverse to $j(M)$, we have $f^*\,\mathrm{PD}_{\mathbb{T}^2\times M}(j_*[M]) = \mathrm{PD}_{\mathbb{T}^2}\big(\big[f^{-1}(j(M))\big]\big)$; here, the notation PD_X means that we are considering the Poincaré duality on the compact manifold X. Now, $\mathrm{PD}_{\mathbb{T}^2}\big(\big[f^{-1}(j(M))\big]\big) = \mathrm{PD}_{\mathbb{T}^2}([\{(0,0)\}])$ generates $H^2(\mathbb{T}^2;\mathbb{Z}) \simeq \mathbb{Z}$; in other words, $\mathrm{PD}\,(j_*[M])$ is toroidal, and the same is true for $c_1(\xi)$.

The only thing left to show is that $c_1(\pi_k^*\xi) = kc_1(\xi)$ mod $H^2_{\text{ator}}(V;\mathbb{Z})$ for each $k \geq 2$. Because η is a trivial complex vector bundle over $\mathbb{T}^2 \times M$, the same is true for each $\pi_k^*\eta$; in particular, each $\pi_k^*\eta$ has trivial Chern classes. Notice that $\pi_k^*\xi$ can also be seen as obtained from $\pi_k^*\eta$ by performing a half Lutz-Mori twist along each of the k submanifolds $\big\{\big(\frac{2l\pi}{m}, 0\big)\big\} \times M$, where $l = 0, \dots, m-1$. Then, Proposition 4.5 tells that $c_1(\pi_k^*\xi) = kc_1(\xi)$; in particular, $c_1(\pi_k^*\xi) = kc_1(\xi)$ mod $H^2_{\text{ator}}(V;\mathbb{Z})$ too. □

4.3 Other examples of non-trivial contact mapping classes

In this section, we show how to obtain examples of $(\mathbb{S}^1 \times W, \xi)$ as in the hypothesis of Theorem I.C using the existence of adapted open book decompositions due to Giroux [Gir02] and the h-principle of Borman, Eliashberg and Murphy [BEM15].

4.3.1 Examples from adapted open books

We recall the statement of Proposition I.E:

Proposition I.E. *Soit* $(V \coloneqq \mathbb{S}^1 \times W^{2n}, \eta)$ *une variété de contact vrillée telle que* $c_1(\eta)$ *est toroidale et telle que, pour tout* $k \geq 2$*, la tirée en arrière* ξ *via le revêtement à* k*-feuilles* $\pi_k \colon \mathbb{S}^1_s \times W \to \mathbb{S}^1_s \times W$*, donné par* $\pi_k(s,p) = (ks, p)$*, satisfait* $c_1(\pi_k^*\eta) = k \cdot c_1(\eta) \mod H^2_{\mathrm{ator}}(V;\mathbb{Z})$*. Alors, on a que :*

- *i. La première classe de Chern de chaque structure de contact* ξ *sur* $V \times \mathbb{T}^2$ *obtenue via la construction de Bourgeois [Bou02] sur* (V,η) *satisfait aussi les conditions ci-dessus, par rapport au revêtement* $\mu_k \coloneqq (\pi_k, \mathrm{Id}) \colon V \times \mathbb{T}^2 \to V \times \mathbb{T}^2$*.*
- *ii. Soit* $\nu \colon V \times \Sigma_g \to V \times \mathbb{T}^2$ *induite par un recouvrement* $\Sigma_g \to \mathbb{T}^2$ *qui est ramifié au dessus de deux points (ici,* Σ_g *est une surface fermée de genre* $g \geq 2$*). Alors, la première classe de Chern de chaque revêtement ramifié* ξ_g *de* ξ *sur* $V \times \Sigma_g$ *satisfait aussi les conditions ci-dessus, par rapport au revêtement* $\mu_k^g \coloneqq (\pi_k, \mathrm{Id}) \colon V \times \Sigma_g \to V \times \Sigma_g$*. De plus, si* η *est vrillée et* g *est assez grand,* ξ_g *est aussi vrillé.*

In order to give a proof, we need the following lemma which describes the effect of Bourgeois construction [Bou02] and of its branched coverings at the level of almost contact structures (recall Definition 2.10) as well as a sufficient condition for overtwistedness in the case of branched covers:

Lemma 4.8. *Let* (V^{2n-1}, η) *be a contact manifold,* (B, φ) *an open book decomposition supporting* η *and* α *a contact form defining* η *and adapted to the open book. Then, we have the following:*

1. *The Bourgeois construction [Bou02] on* (V,η) *and* (B,φ,α) *gives a contact structure* ξ *on* $V \times \mathbb{T}^2$ *which is homotopic, as an almost contact structure, to the pair* $\big(\eta \oplus T\mathbb{T}^2, d\alpha \oplus \omega_T\big)$*, where* ω_T *is a volume form on* $\mathbb{T}^2$*.*
2. *Any contact branched covering* ξ_g *of* ξ *via a branched covering* $\nu \colon V \times \Sigma_g \to V \times \mathbb{T}^2$*, induced by a covering* $\Sigma_g \to \mathbb{T}^2$ *branched over two points, is homotopic, as an almost contact structure, to* $(\eta \oplus T\Sigma_g, d\alpha \oplus \omega_g)$*, where* ω_g *is a volume form on* Σ_g*.*
3. *Suppose* η *is overtwisted. Then, if* g *is large enough,* ξ_g *is overtwisted too.*

Notice that point 1 above has already been pointed out by Lisi, Marinković and Niederkrüger [LMN18, Remark 2.1].
We now prove, in the order, Proposition I.E and Lemma 4.8:

Proof (Proposition I.E). Denote also by $p \colon V \times \mathbb{T}^2 \to V$, $p_g \colon V \times \Sigma_g \to V$ and $\overline{p}_g \colon V \times \Sigma_g \to \Sigma_g$ the natural projections.
Points 1 and 2 of Lemma 4.8 imply that $c_1(\xi) = p^* c_1(\eta)$ and $c_1(\xi_g) = p_g^* c_1(\eta) + \overline{p}_g^* c_1(T\Sigma_g)$. Recall now that every continuous map $g \colon \mathbb{T}^2 \to \Sigma_g$ has degree 0 (here, we use $g \geq 2$); in particular, for each $f \colon \mathbb{T}^2 \to V \times \Sigma_g$, $f^* \overline{p}_g^* c_1(T\Sigma_g) = (\overline{p}_g \circ f)^* c_1(T\Sigma_g) = 0$ in $H^2(\mathbb{T}^2;\mathbb{Z})$, i.e. $\overline{p}_g^* c_1(T\Sigma_g)$ is atoroidal. We then have that $c_1(\xi) = p^* c_1(\eta) \mod H^2_{\mathrm{ator}}(V \times \mathbb{T}^2;\mathbb{Z})$ and $c_1(\xi_g) = p_g^* c_1(\eta) \mod H^2_{\mathrm{ator}}(V \times \Sigma_g;\mathbb{Z})$.

In order to show that $c_1(\xi)$ and $c_1(\xi_g)$ are toroidal, it's then enough to show that this is true for both $p^*c_1(\eta)$ and $p_g^*c_1(\eta)$.
Because $c_1(\eta)$ is toroidal, there is $f\colon \mathbb{T}^2 \to V$ with $f^*c_1(\eta) \neq 0$. Let then $h\colon \mathbb{T}^2 \to V \times \mathbb{T}^2$, $h(q) = (f(q), *)$, and $h_g\colon \mathbb{T}^2 \to V \times \Sigma_g$, $h_g(q) = (f(q), *)$. As $p \circ h = f$ and $p_g \circ h_g = f$, we have $h^*(p^*c_1(\eta)) = (p \circ h)^*c_1(\eta) = f^*c_1(\eta) \neq 0$ in $H^2(\mathbb{T}^2;\mathbb{Z})$ and, similarly, $h_g^*(p_g^*c_1(\eta)) = f^*c_1(\eta) \neq 0$, i.e. $p^*c_1(\eta)$ and $p_g^*c_1(\eta)$ are both toroidal.

The equalities $c_1(\mu_k^*\xi) = kc_1(\xi) \bmod H^2_{\text{ator}}(V \times \mathbb{T}^2;\mathbb{Z})$ and $c_1((\mu_k^g)^*\xi_g) = kc_1(\xi_g) \bmod H^2_{\text{ator}}(V \times \Sigma_g;\mathbb{Z})$ follow directly from $\pi_k \circ p = p \circ \mu_k$, $\pi_k \circ p_g = p_g \circ \mu_k^g$ and $c_1(\pi_k^*\eta) = kc_1(\eta) \bmod H^2_{\text{ator}}(V;\mathbb{Z})$.

Lastly, if η is overtwisted, point 3 of Lemma 4.8 gives the overtwistedness of ξ_g for g large enough, thus concluding the proof. □

Proof (Lemma 4.8). We start by proving point 1. The Bourgeois construction [Bou02] on (V,η) and (B,φ,α) gives a function $\Phi = (f,g)\colon V \to \mathbb{R}^2$ defining the open book (B,φ) and such that ξ on $V \times \mathbb{T}^2_{(x,y)}$ is defined by $\beta := \alpha + f dx - g dy$. Then, an explicit homotopy of almost contact structures from $(\xi, d\beta|_\xi)$ to $(\eta \oplus T\mathbb{T}^2, d\alpha|_\eta + dx \wedge dy)$ is given by the $[0,1]_t$-family of hyperplane fields ξ_t given by the kernel of $\alpha + (1-t)\,(f dx - g dy)$, together with the symplectic structures given by the restriction of $d\alpha + (1-t)\,[df \wedge dx - dg \wedge dy] + t dx \wedge dy$ to ξ_t.

As far as point 2 is concerned, as explained in [Gei97b], an explicit contact branched covering ξ_g on $V \times \Sigma_g$ is given by the kernel of a differential $1-$form $\nu^*\beta + \epsilon h(r) r^2 d\theta$; here, (r,θ) are radial coordinates on the D^2-factor of a neighborhood $D^2 \times \{p,q\}$ of the branching locus $\{p,q\}$ of the branched covering $\Sigma_g \to \mathbb{T}^2$, $\epsilon > 0$ is very small and $h = h(r)$ is a smooth function with support in $D^2 \times \{p,q\}$, equal to 1 on the branching locus and strictly decreasing in r. As contact branched coverings are unique up to isotopy (see Section 5.2), it's enough to prove that this specific η_g is homotopic to the wanted almost contact structure.
Now, an explicit computation (analogous to the one in Section 8.5) shows that the wanted homotopy is given by the $[0,1]_t$-family of hyperplane fields ξ_g^t defined as the kernel of $\nu^*\alpha + (1-t)\left[\nu^*\left(f dx - g dy\right) + \epsilon h r^2 d\theta\right]$, together with the symplectic structures given by the restriction of $\nu^* d\alpha + (1-t)\left[\nu^*\left(df \wedge dx - dg \wedge dy\right) + \epsilon d\left(hr^2\right) \wedge d\theta\right] + t\omega_g$ to ξ_g^t.

Point 3 will be proven in Section 9.1.2; more precisely, it essentially follows from the following three facts. Firstly, the contact branched covering ξ_g can be chosen (up to isotopy) in such a way that it induces on each fiber of $V \times \Sigma_g \to \Sigma_g$ the original overtwisted contact structure η. Secondly, Niederkrüger and Presas [NP10, page 724] describe how the "size" of a contact neighborhood of each connected component (V,ξ) of the branching set of $V \times \Sigma_g \to V \times \mathbb{T}^2$ is diverging to $+\infty$ as the index g of the branched covering is going to $+\infty$; see also Lemma 9.9. Then, according to Casals, Murphy and Presas [CMP15, Theorem 3.1], topologically trivial contact neighborhoods of overtwisted manifolds in codimension 2 are themselves overtwisted, provided they are sufficiently "large". This concludes the proof of Lemma 4.8. □

4.3.2 Examples from the h-principle

We recall the statement of Proposition I.F:

Proposition I.F. *On considère une variété lisse W de dimension $2n$ qui est presque complexe, spin et satisfait $H^1(W;\mathbb{Z}) \neq \{0\}$. Alors, il y a une structure de contact vrillée ξ sur $V := \mathbb{S}^1 \times W$ telle que $c_1(\xi) \in H^2(V;\mathbb{Z})$ est toroidale et $c_1(\pi_k^*\xi) = k \cdot c_1(\xi)$ mod $H^2_{\text{ator}}(V;\mathbb{Z})$, où $\pi_k\colon \mathbb{S}^1_s \times W \to \mathbb{S}^1_s \times W$ est le revêtement à k-feuilles $\pi_k(s,p) = (ks,p)$.*

The proof is structured as follows. We start from a natural almost contact structure η_0 on $V := \mathbb{S}^1 \times W$ and we modify it to an almost contact structure η with first Chern class $c_1(\eta)$ satisfying the wanted conditions. Then, the h-principle from [BEM15] tells that η can be deformed to an overtwisted contact structure ξ on V; the first Chern class of such a ξ will then satisfy the wanted properties too.

Before entering in the details of the proof of Proposition I.F, we state a lemma from algebraic topology, whose proof is postponed:

Lemma 4.9. *Let (V^{2n+1}, η_0) be an almost contact manifold. For each class $u \in H^2(V;\mathbb{Z})$, there is an almost contact structure η_u on V such that $c_1(\eta_u) = c_1(\eta_0) + 2u$.*

Proof (Proposition I.F). The hyperplane field $\eta_0 = \{0\} \oplus TW$ on $V = \mathbb{S}^1 \times W$ is a (coorientable) almost contact structure thanks to the almost complex structure J_W on W. Moreover, its first Chern class $c_1(\eta_0)$ is equal to $\pi_W^* c_1(W)$, where $\pi_W \colon \mathbb{S}^1 \times W \to W$ is the projection on the second factor.
The hypothesis that W is spin means that the 2nd Stiefel Whitney class $w_2(W) \in H^2(W;\mathbb{Z}_2)$ of W is trivial. Because $w_2(W)$ is the reduction modulo 2 of $c_1(W)$, there is $\lambda \in H^2(W;\mathbb{Z})$ such that $c_1(W) = 2\lambda$. Hence, $c_1(\eta_0) = \pi_W^* c_1(W) = 2\pi_W^*\lambda$.

Consider then a non-trivial $c \in H^1(W;\mathbb{Z})$, that exists by hypothesis, and let v be a generator of $H^1(\mathbb{S}^1;\mathbb{Z})$. Using Kunneth's decomposition theorem, we can see $H^1(\mathbb{S}^1;\mathbb{Z}) \otimes H^1(W;\mathbb{Z})$ as a submodule of $H^2(\mathbb{S}^1 \times W;\mathbb{Z})$. An application of Lemma 4.9 with $u = v \otimes c - \pi_W^*\lambda$ then gives an almost contact structure η on V with $c_1(\eta) = 2v \otimes c$.

Notice that the map π_k^*, induced on $H^2(\mathbb{S}^1 \times W;\mathbb{Z})$ by π_k, acts as multiplication by k on the submodule $H^1(\mathbb{S}^1;\mathbb{Z}) \otimes H^1(W;\mathbb{Z})$ of $H^2(\mathbb{S}^1 \times W;\mathbb{Z})$. In particular, the fact that $c_1(\eta) = 2v \otimes c$ implies that $c_1(\pi_k^*\eta) = kc_1(\eta) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$.

We now claim that $c_1(\eta)$ is toroidal. Indeed, according to the universal coefficient theorem and the Hurewicz theorem, $H^1(W;\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_1(W;\mathbb{Z});\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\pi_1(W);\mathbb{Z})$. In particular, there is $\gamma \colon \mathbb{S}^1 \to W$ such that $\gamma^* c \neq 0 \in H^1(\mathbb{S}^1;\mathbb{Z})$. If we define $f = (\mathrm{Id}, \gamma) \colon \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times W$, we then have $f^* c_1(\eta) = 2v \otimes \gamma^* c \neq 0$ in $H^1(\mathbb{S}^1;\mathbb{Z}) \otimes H^1(\mathbb{S}^1;\mathbb{Z}) \subset H^2(\mathbb{T}^2;\mathbb{Z})$, i.e. $c_1(\eta)$ is toroidal, as wanted.

The h-principle from Borman, Eliashberg and Murphy [BEM15] then gives the wanted contact structure ξ as deformation of η. □

We now give a proof of the lemma used above:

Proof (Lemma 4.9). Bowden, Crowley and Stipsicz [BCS14, Lemma 2.17.(1)] states that if V is a closed connected manifold of dimension $2n+1$ and ζ is a stable almost complex structure on it, then there is an almost contact structure η on V whose stabilization gives ζ. Recall that a *stable almost complex structure* on V is the stable isomorphism class of a complex structure on $TV \oplus \varepsilon_V^k$, where ε_V is the trivial real vector bundle of dimension 1 over V, and the *stabilization of* η is the stable isomorphism class of the complex structure induced by η on $TV \oplus \varepsilon_V$. In particular, in order to prove Lemma 4.9, it's enough to find a stable almost complex structure ζ_u such that $c_1(\zeta_u) = c_1(\eta_0) + 2u$.

The existence of such a ζ_u follows, for instance, from [Gei08, Remark 8.1.4], of which we recall here the idea.
There is a bijective correspondence, given by the first Chern class, between isomorphism classes of complex line bundles over V and cohomology classes in $H^2(V;\mathbb{Z})$. Let then L_u be the complex line bundle over V satisfying $c_1(L_u) = u$. Consider a *direct complement* of the dual L_u^* of L_u, i.e. a complex vector bundle E_u over V such that there are $m \in \mathbb{N}_{>0}$ and an isomorphism $\nu \colon L_u^* \oplus_{\mathbb{C}} E_u \simeq (\varepsilon_V^{\mathbb{C}})^m$ of complex vector bundles over V; here, $\varepsilon_V^{\mathbb{C}}$ denotes the complexification of ε_V. We then claim that the complex vector bundle $F_u := \eta_0 \oplus L_u \oplus E_u$ can be used to define the wanted stable complex structure.
The fact that $L_u^* \oplus_{\mathbb{C}} E_u$ is a trivial complex vector bundle implies in particular that

$c_1(E_u) = -c_1(L_u^*) = u$; hence, $c_1(F_u) = c_1(\eta) + u + u = c_1(\eta) + 2u$.
Now, because L_u^* and L_u are isomorphic as real vector bundles, ν induces an isomorphism of real vector bundles $\nu' \colon L_u \oplus_{\mathbb{R}} E_u \simeq \varepsilon_V^{2m}$. Moreover, the choice of a vector field X on V transverse to η_0 gives an isomorphism of real vector bundles $\Psi \colon \eta_0 \oplus \epsilon_V \simeq TV$. We then have an isomorphism θ of real vector bundles over V given by the composition

$$F_u = \eta_0 \oplus L_u \oplus E_u \overset{\mathrm{Id} \oplus \nu'}{\simeq} \eta_0 \oplus \varepsilon_V^{2m} =_{\mathbb{R}} (\eta_0 \oplus \varepsilon_V) \oplus \varepsilon_V^{2m-1} \overset{\Psi \oplus \mathrm{Id}}{\simeq} TV \oplus \varepsilon_V^{2m-1} \ .$$

In particular, the pushforward $\theta_* J$ of the complex structure J on F_u via θ gives the wanted stable almost complex structure ζ_u on V. $\square$

Appendix A

Chern classes as Poincaré duals

Chern classes are global invariants of complex vector bundles E over a manifold V. More precisely, following [ACMFA07] we describe in this appendix how each Chern class of E can be geometrically interpreted as the Poincaré dual of (almost) the locus of points of V where a "generic" set of sections of E is not linearly independent.

Consider a complex vector bundle E of complex rank r over an oriented smooth manifold V. Given k sections $s_1, \dots, s_k$ of E, take the homomorphism of vector bundles $h : V \times \mathbb{C}^k \to E$ defined by $h(p, u_1, \dots, u_k) = \sum_{j=1}^{k} u_j s_j(p)$, where $V \times \mathbb{C}^k$ is the trivial complex vector bundle of rank k over V.
If $\tau : \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \to V$, is the complex vector bundle over V with fiber over $p \in V$ the vector space $\mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}^k, E_p\right)$ of $\mathbb{C}$-linear maps from $\mathbb{C}^k$ to E_p, we can reinterpret the map h as a section s_h of τ given by $s_h(p)(w) := h(p, w)$ for all $w \in \mathbb{C}^k$.

Take now the complex vector bundle $\pi : \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \to V \times \mathbb{CP}^{k-1}$ defined by $\pi(f, d) = (\tau(f), d)$, for every $f \in \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right)$ and $d \in \mathbb{CP}^{k-1}$, and consider the section $\sigma_h : V \times \mathbb{CP}^{k-1} \to \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$ given by $\sigma_h := (s_h, \mathrm{Id}_{\mathbb{CP}^{k-1}})$.

If $\phi : V \times \mathbb{CP}^{k-1} \to V$ and $\widehat{\phi} : \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \to \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right)$ are the projections on the first factor, we then have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \arrow[r, "\widehat{\phi}"] \arrow[d, shift left, "\pi"] & \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \arrow[d, shift right, "\tau"'] \\
V \times \mathbb{CP}^{k-1} \arrow[r, "\phi"] \arrow[u, bend left, "\sigma_h"] & V \arrow[u, bend right, "s_h"']
\end{tikzcd}$$

Now, in the total space $\mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$ of the bundle π we can consider the *blown-up non-injectivity locus*, i.e. the subset

$$\Sigma := \left\{(f, d) \in \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \mid d \subset \ker f\right\} .$$

The adjective *blown-up* comes from the fact that Σ is a version of the *non-injectivity locus*

$$S := \left\{f \in \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \mid \ker(f) \neq \{0\}\right\}$$

where we keep track of the complex lines in the kernel.

Proposition A.1. [ACMFA07, Proposition 4, Proposition 6] Σ *is a smooth oriented submanifold of* $\mathit{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$*, of codimension* $2r$*.*

As we will need it in the following, we give a sketch of proof:

Proof (sketch). Let $\mathrm{pr} : \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \to \mathbb{CP}^{k-1}$ be the projection on the second factor and γ the tautological line bundle over $\mathbb{CP}^{k-1}$; denote then

$$\epsilon_1 := \mathrm{pr}^* \gamma = \left\{ (f, d, v) \in \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \times \mathbb{C}^k \,|\, v \in d \right\} .$$

If $\phi : V \times \mathbb{CP}^{k-1} \to V$ is the projection on the first factor, denote also by ϵ_2 the vector bundle $\pi^* \phi^* E$ over $\mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$, where $\pi : \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \to V \times \mathbb{CP}^{k-1}$ is as above.
Consider then the vector bundle $\Pi : \mathrm{Hom}_{\mathbb{C}}\left(\epsilon_1, \epsilon_2\right) \to \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$ and take the section $\Psi : \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \to \mathrm{Hom}_{\mathbb{C}}\left(\epsilon_1, \epsilon_2\right)$ of Π defined by $\Psi(f, d) = f|_d$.

$$\begin{tikzcd}
 & \epsilon_1 = \mathrm{pr}^* \gamma \arrow[d] & \\
\epsilon_2 = \pi^* \phi^* E \arrow[r] & \mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1} \arrow[d, shift left, "\pi"] \arrow[r, bend right, "\Psi"'] & \mathrm{Hom}_{\mathbb{C}}(\epsilon_1, \epsilon_2) \arrow[l, "\Pi"'] \\
 & V \times \mathbb{CP}^{k-1} \arrow[u, bend left, "\sigma_h"] \arrow[d, "\phi"] & \\
E \arrow[r] & V &
\end{tikzcd}$$

It can be shown that Ψ is transverse to the zero section 0_Π of Π. In particular, $\Sigma = \Psi^{-1}\left(0_\Pi\right)$ is a smooth submanifold of $\mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$.
Finally, Σ is oriented according to Convention A.2 below, thanks to the fact that $\mathrm{Hom}_{\mathbb{C}}\left(\epsilon_1, \epsilon_2\right)$, $\mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$ and 0_Π are naturally oriented: indeed, the first two are complex vector bundles over an oriented base and the third is a section of a vector bundle over an oriented base. $\square$

Convention A.2. *Let X, Y, Z be oriented manifolds and consider $f : X \to Y$ transverse to $Z \subset Y$ at $p \in X$. Take a basis $(v_1, \dots, v_l)$ of $T_p f^{-1}(Z)$, complete it to a positive basis $(v_1, \dots, v_l, u_1, \dots, u_n)$ of $T_p X$ and consider a positive basis $(w_1, \dots, w_m)$ of $T_{f(p)} Z$. Then, $(v_1, \dots, v_l)$ is* positive *if and only if $(w_1, \dots, w_m, d_p f(u_1), \dots, d_p f(u_n))$ is a positive basis of $T_{f(p)} Y$.*

Define now the set

$$Z(h) := \sigma_h^{-1}(\Sigma) = \left\{ (p, d) \in V \times \mathbb{CP}^{k-1} \,|\, d \subset \ker\left(h_p\right) \right\} ,$$

where $h_p : \mathbb{C}^k \to E_p$ is the $\mathbb{C}$-linear map defined by $h_p(.) := h(p, .)$.

Proposition A.3. [ACMFA07, Proposition 5] *For a generic choice of vector bundles map $h : V \times \mathbb{C}^k \to E$, the section $\sigma_h : V \times \mathbb{CP}^{k-1} \to Hom_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$ is transverse to $\Sigma \subset Hom_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$. In particular, $Z(h)$ is a closed oriented submanifold of $V \times \mathbb{CP}^{k-1}$ of codimension $2r$.*

Theorem A.4. [ACMFA07, Theorem 11] *If the section $\sigma_h : V \times \mathbb{CP}^{k-1} \to Hom_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$ is transverse to $\Sigma \subset Hom_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$, then the Chern class $c_{r-k+1}(E)$ is equal to the Poincaré dual of $\phi_* [Z(h)]$, where $\phi : V \times \mathbb{CP}^{k-1} \to V$ is the projection on the first factor.*

Remark A.5. *The statement of the theorem shows the advantage of using the blown-up version Σ of S instead of the non-injectivity locus itself.*
Indeed, while $Z(h)$ is a smooth oriented submanifold for a generic choice of h, hence is has a well defined fundamental class that can be pushed in $H_*(V;\mathbb{Z})$ via ϕ_*, the set $s_h^{-1}(S)$ is only a Whitney stratified submanifold of V (hence not necessarily smooth) for a generic choice of h, and in particular there is no natural way to associate an homology class to $s_h^{-1}(S)$.
We point out though that in the complex analytic setting it is possible to construct a cohomology class directly from $s_h^{-1}(S)$ using the theory of currents: this is done, for example, in Griffiths and Harris [GH78, Section 3.3].

As it will be useful later, we remark that there is also a relative version of Proposition A.3. Indeed, we have the following relative transversality result: if M and N are smooth manifolds and $f : M \to N$ is a smooth map transverse to a submanifold $Z \subset N$ on a neighbourhood of a closed subset $C \subset M$, then f can be C^∞-perturbed to a map $f' : M \to N$ everywhere transverse to Z and such that $f'|_C = f|_C$.
This can be proven, for example, by introducing a little modification in the proof of Lee [Lee13, Theorem 6.36], where it is shown that, for $k \in \mathbb{N}$ big enough, there is a parametric family of functions $F : M \times \mathbb{R}^k \to N$ that is everywhere transverse to Z and such that $F(.,0) = f(.)$. More precisely, the F appearing in the proof of [Lee13, Theorem 6.36] should be defined in our case as $F(p,s) := r\left(f(p) + \chi(p) \cdot e(p) \cdot s\right)$, where $\chi : M \to \mathbb{R}_{\geq 0}$ has support C; the wanted perturbation will then be $F_s := F(.,s)$ for an s given by the parametric transversality theorem (see Lee [Lee13, Theorem 6.35]).
In our setting, using this relative transversality result in the proof of [ACMFA07, Proposition 5] (restated above as Proposition A.3) we can achieve transversality between the map σ_h and the submanifold Σ of $\mathrm{Hom}_{\mathbb{C}}\left(V \times \mathbb{C}^k, E\right) \times \mathbb{CP}^{k-1}$ by C^∞-perturbing h relative to a closed subset $C \subset V \times \mathbb{CP}^{k-1}$ near which σ_h is already transverse to Σ.

We are then ready to prove Proposition 4.7, whose statement we now recall:

Proposition 4.7. *Let E, E' be two complex vector bundles of complex rang r over the same smooth oriented manifold V. Suppose also that there exist two open subsets $\mathcal{O},\mathcal{U}$ of V, with $\mathcal{O}$ compactly contained in $\mathcal{U}$, such that the following are satisfied:*

1. *There is an isomorphism of vector bundles $\psi : E|_{\mathcal{O}^c} \to E'|_{\mathcal{O}^c}$ over $\mathcal{O}^c := V \setminus \mathcal{O}$.*
2. *The vector bundles $E|_{\mathcal{U}}$ and $E'|_{\mathcal{U}}$ over $\mathcal{U}$ admit complex sub-bundles $F, L \subset E$ and $F', L' \subset E'$ such that $E|_{\mathcal{U}} = F \oplus L$ and $E'|_{\mathcal{U}} = F' \oplus L'$ and satisfying the following conditions:*
   - *(a) $\psi \circ s_j = s'_j$ over $\mathcal{U} \setminus \mathcal{O}$ for all $j = 1, \dots, r-1$;*
   - *(b) L, L' have complex rank 1, while F, F' have complex rank $r-1$ and are trivialized by two ordered sets of everywhere $\mathbb{C}$-linearly independent sections $(s_1, \dots, s_{r-1})$ and $\left(s'_1, \dots, s'_{r-1}\right)$ of $E|_{\mathcal{U}}$ and $E'|_{\mathcal{U}}$;*
   - *(c) there are two additional sections s_r, s'_r of $E|_{\mathcal{U}}$ and $E'|_{\mathcal{U}}$ respectively, with image contained in L and L' and such that $s_r : \mathcal{U} \to E|_{\mathcal{U}}$ intersects transversely F and $s'_r : \mathcal{U} \to E'|_{\mathcal{U}}$ intersects transversely F' (here F and F' are seen here as submanifolds of $E|_{\mathcal{U}}$ and $E'|_{\mathcal{U}}$);*
   - *(d) $Z := s_r^{-1}(F)$ and $Z' := (s'_r)^{-1}(F')$, which are oriented smooth manifolds of $\mathcal{U}$ by Hypothesis 2c above, are actually compactly contained in $\mathcal{O}$.*

Then, we have the following:

1. $c_k(E') = c_k(E)$ *in* $H^{2k}(V;\mathbb{Z})$ *for all* $2 \le k \le r$*;*

2. $c_1(E') - c_1(E) = \mathrm{PD}([Z']) - \mathrm{PD}([Z])$ *in* $H^2(V;\mathbb{Z})$*.*

We deduce the above result from Theorem A.4 using the following:

Lemma A.6. *Let E be a complex vector bundle of complex rang r over a smooth oriented manifold V with empty boundary. Let $s_1, \cdots, s_{r-1}$ be $\mathbb{C}$-linearly independent sections of E and denote by F the vector sub-bundle of E generated by them, i.e. the vector bundle with fiber $F_p = \mathrm{Span}_{\mathbb{C}}(s_1(p), \cdots, s_{r-1}(p)) \subset E_p$ over a point $p \in V$. Let also L be a complex line sub-bundle of E such that $E = F \oplus L$ and assume that $s_r : V \to E$ is an additional section with image contained in L and intersecting transversely F (seen as a submanifold of E); denote by M the oriented (by Convention A.2) submanifold $s_r^{-1}(F)$. Then, if $h : V \times \mathbb{C}^r \to E$ is defined by $h(p,u) = \sum_{i=1}^r u_i s_i(p)$ and σ_h is obtained from h as above, we have that:*

1. *$\sigma_h : V \times \mathbb{CP}^{r-1} \to \mathrm{Hom}_{\mathbb{C}}(V \times \mathbb{C}^r, E) \times \mathbb{CP}^{r-1}$ is transverse to the blown-up non-injectivity locus $\Sigma \subset \mathrm{Hom}_{\mathbb{C}}(V \times \mathbb{C}^r, E) \times \mathbb{CP}^{r-1}$ and, in particular, $Z(h) := \sigma_h^{-1}(\Sigma)$ is smooth and naturally oriented;*

2. *the projection on the first factor $\phi : V \times \mathbb{CP}^r \to V$ induces an orientation preserving diffeomorphism $\phi : Z(h) \to M$.*

Proof (proposition 4.7): Consider another open set $\mathcal{V}$ of V, compactly contained in $\mathcal{U}$ and containing the closure of $\mathcal{O}$.
Take the two complex vector bundle homomorphisms $h_{\mathcal{V}} : \mathcal{V} \times \mathbb{C}^r \to E|_{\mathcal{V}}$ and $h'_{\mathcal{V}} : \mathcal{V} \times \mathbb{C}^r \to E'|_{\mathcal{V}}$ defined by $h_{\mathcal{V}}(p, u_1, \ldots, u_r) := \sum_{i=1}^r u_i s_i(p)$ and $h'_{\mathcal{V}}(p, u_1, \ldots, u_r) := \sum_{i=1}^r u_i s'_i(p)$ for all $p \in \mathcal{V}$, $(u_1, \ldots, u_r) \in \mathbb{C}^r$ and $i = 1, \ldots, r$. Extend then $h_{\mathcal{V}}$ and $h'_{\mathcal{V}}$ to two vector bundle homomorphisms $h : V \times \mathbb{C}^r \to E$ and $h' : V \times \mathbb{C}^r \to E$ in such a way that $\psi(h(p,u)) = h'(p,u)$ for all $p \in \mathcal{O}^c$, $u \in \mathbb{C}^r$ and $i = 1, \ldots, r$. Such extensions exist because $\psi \circ s_i = s'_i$ on $\mathcal{U} \setminus \mathcal{O}$ for all $i = 1, \ldots, r$ by Hypothesis 2a.
Given an integer j between 1 and r included, denote respectively by h_j and h'_j the restrictions of h and h' to the sub-bundle $V \times \mathbb{C}^j$ of $V \times \mathbb{C}^r$, where $\mathbb{C}^j$ is the vector subspace of $\mathbb{C}^r$ given by the points $(u_1, \ldots, u_r) \in \mathbb{C}^r$ such that $u_{j+1} = \ldots = u_r = 0$.

Now, σ_{h_j} and $\sigma_{h'_j}$ are transverse to the blown-up non-injectivity locus Σ near the closed set $\overline{\mathcal{O}}$, for all $j = 1, \ldots, r$: indeed, this follows directly from Hypothesis 2b for the case $j = 1, \ldots, r-1$ and from Hypothesis 2c and Lemma A.6 for the case $j = r$ (remark that in Lemma A.6 we do not make compactness assumptions, so we can chose $\mathcal{V}$ as base manifold V in the statement of the lemma). Then, using the relative version of the genericity of the transversality condition, we can perturb h_j, h'_j to g_j, g'_j in such a way that $g_j = h_j$, $g'_j = h'_j$ over $\mathcal{O}$ and that σ_{g_j}'s and $\sigma_{g'_j}$'s are everywhere transverse to Σ. Moreover, because $\psi(h_j(p,.)) = h'_j(p,.)$ for $p \in \mathcal{O}^c$, we can also arrange that $\psi(g_j(p,.)) = g'_j(p,.)$ for $p \in \mathcal{O}^c$: indeed, we can use the same perturbation for h_j and h'_j over $\mathcal{O}^c$ because they coincide there. Lastly, if we choose the perturbation C^0-small, we can arrange to have the submanifolds $Z(g_j)$ and $Z(g'_j)$ compactly contained in $\mathcal{O} \times \mathbb{CP}^{j-1}$.

Now, by construction of the g_j's and the g'_j's, if we write $Z(g_j) = Z_{\mathcal{O}}(g_j) \cup Z_{\mathcal{O}^c}(g_j)$ and $Z(g'_j) = Z_{\mathcal{O}}(g'_j) \cup Z_{\mathcal{O}^c}(g'_j)$, where $Z_{\mathcal{O}}(g_j), Z_{\mathcal{O}}(g'_j) \subset \mathcal{O} \times \mathbb{CP}^{j-1}$ and $Z_{\mathcal{O}^c}(g_j), Z_{\mathcal{O}^c}(g'_j) \subset \mathcal{O}^c \times \mathbb{CP}^{j-1}$, we have that $Z_{\mathcal{O}^c}(g_j) = Z_{\mathcal{O}^c}(g'_j)$ for all $j = 1, \ldots, r$ and $Z_{\mathcal{O}}(g_j) = Z_{\mathcal{O}}(g'_j) = \varnothing$ for $j = 1, \ldots, r-1$. Moreover, if $\mathrm{pr}_V^j : V \times \mathbb{CP}^{j-1} \to V$ is the projection on the first factor for all $j = 1, \ldots, r$, by Lemma A.6 $\phi_r := \mathrm{pr}_V^r|_{Z(g_r)}$ and $\phi'_r := \mathrm{pr}_V^r|_{Z(g'_r)}$ induce orientation preserving diffeomorphisms between $Z_{\mathcal{O}}(g_r)$ and Z and between $Z_{\mathcal{O}}(g'_r)$ and Z' respectively.

By Theorem A.4 and the identities above, we have that for all $j = 1, \dots, r-1$

$$\begin{aligned} c_{r-j+1}(E) &= \mathrm{PD}\left((\phi_j)_* \left[Z(g_j)\right]\right) \\ &= \mathrm{PD}\left((\phi_j)_* \left[Z(g_j) \cap \mathcal{O}^c\right]\right) \\ &= \mathrm{PD}\left((\phi'_j)_* \left[Z(g'_j) \cap \mathcal{O}^c\right]\right) \\ &= \mathrm{PD}\left((\phi_j)_* \left[Z(g'_j)\right]\right) \\ &= c_{r-j+1}(E') \ , \end{aligned}$$

and that

$$\begin{aligned} c_1(E) &= \mathrm{PD}\left((\phi_r)_* \left[Z(g_r)\right]\right) \\ &= \mathrm{PD}\left((\phi_r)_* \left[Z(g_r) \cap \mathcal{O}^c\right]\right) + \mathrm{PD}\left((\phi_r)_* \left[Z(g_r) \cap \mathcal{O}\right]\right) \\ &= \mathrm{PD}\left((\phi_r)_* \left[Z_{\mathcal{O}^c}(g_r)\right]\right) + \mathrm{PD}\left([Z]\right) \ , \\ c_1(E') &= \mathrm{PD}\left((\phi_r)_* \left[Z(g'_r)\right]\right) \\ &= \mathrm{PD}\left((\phi_r)_* \left[Z(g'_r) \cap \mathcal{O}^c\right]\right) + \mathrm{PD}\left((\phi_r)_* \left[Z(g'_r) \cap \mathcal{O}\right]\right) \\ &= \mathrm{PD}\left((\phi_r)_* \left[Z_{\mathcal{O}^c}(g'_r)\right]\right) + \mathrm{PD}\left([Z']\right) \\ &= \mathrm{PD}\left((\phi_r)_* \left[Z_{\mathcal{O}^c}(g_r)\right]\right) + \mathrm{PD}\left([Z']\right) \ , \end{aligned}$$

which give $c_1(E') - c_1(E) = \mathrm{PD}\left([Z']\right) - \mathrm{PD}\left([Z]\right)$. □

Proof (lemma A.6). Because the transversality and the orientation preserving conditions are local, we can restrict our attention to an open set $\mathcal{U}$ on which there is an everywhere non-zero section s of $L|_{\mathcal{U}}$.
In particular, the r-tuple of sections $(s_1, \dots, s_{r-1}, s)$ trivializes $E|_{\mathcal{U}}$, i.e. the map $\mathcal{U} \times \mathbb{C}^r \to E|_{\mathcal{U}}$ given by $(q, w) \mapsto \sum_{i=1}^{r-1} w_i s_i(q) + w_r s(q)$ is an isomorphism of complex vector bundles.

In this local trivialization, we can rewrite s_r as

$$\begin{aligned} s_r : \mathcal{U} &\to \mathcal{U} \times \mathbb{C}^r \\ q &\mapsto (q, \underbrace{0, \dots, 0}_{r-1 \text{ times}}, v(q)) \end{aligned}$$

for a certain $v : \mathcal{U} \to \mathbb{C}$.
Also, the fact that s_r is transverse to F at a certain point $q \in \mathcal{U}$ means that $v : \mathcal{U} \to \mathbb{C}$ is transverse to $\{0\} \subset \mathbb{C}$ at q. Moreover, if we denote by $\nu : \mathcal{U} \times \mathbb{C}^r \to \mathbb{C}^r$ the projection on the second factor, in the open set $\mathcal{U}$ the submanifold $M = s_r^{-1}(F)$ (oriented according to Convention A.2) is actually equal to the oriented manifold $(\nu \circ s_r)^{-1}(0)$; in other words, remarking that $\nu \circ s_r = (0, \dots, 0, v)$, we have that $M \cap \mathcal{U} = v^{-1}(0)$ as oriented manifolds.

Now, let's rewrite σ_h using the chosen trivialization of E over $\mathcal{U}$.

Firstly, the map h becomes $h : \mathcal{U} \times \mathbb{C}^r \to \mathcal{U} \times \mathbb{C}^r$, $h(q, w) = (q, M(q) \cdot w)$, where $\cdot$ denotes the matrix product, $M : \mathcal{U} \to \mathcal{M}_r(\mathbb{C})$ is with values in the space $\mathcal{M}_r(\mathbb{C})$ of square matrices $r \times r$ with complex coefficients and is defined by

$$M = \left(\begin{array}{c|c} \mathrm{I}_{r-1} & 0 \\ \hline 0 & v \end{array} \right) \ , \tag{A.1}$$

with I_{r-1} the identity matrix of dimension $(r-1) \times (r-1)$. In other words, $s_h : \mathcal{U} \to \mathcal{U} \times \mathcal{M}_r(\mathbb{C})$ is given by $s_h(q) = (q, M(q))$.

Moreover, we remark that if $(p,d) \in \mathcal{U} \times \mathbb{CP}^{r-1}$ is such that $\sigma_h(p,d) \in \Sigma$ then $d = [0 : \cdots : 0 : 1]$. We can hence further restrict to the coordinate chart $\mathbb{C}^{r-1} = \left\{ [z_1 : \ldots : z_{r-1} : 1] \in \mathbb{CP}^{r-1} \right\} \subset \mathbb{CP}^{r-1}$ containing the point $[0 : \cdots : 0 : 1]$ and consider σ_h as a map

$$\begin{aligned} \sigma_h : \mathcal{U} \times \mathbb{C}^{r-1} &\to \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \\ (q,z) &\mapsto (q, M(q), z) \end{aligned}$$

where $z = (z_1, \ldots, z_{r-1}) \in \mathbb{C}^{r-1}$.

Now, in order to study the transversality of σ_h with respect to Σ, we have to come back at the construction of Σ as preimage of a transverse intersection.
We then consider the vector bundles ϵ_1, ϵ_2 and the sections $\Psi, 0_\Pi$ of Π as in the (sketch of the) proof of Proposition A.1 and read them in the given trivialization of E over $\mathcal{U}$ and in the chart $\mathbb{C}^{r-1} \subset \mathbb{CP}^{r-1}$. We are then in the following situation:

- ϵ_1, which is globally the product of $\mathrm{Hom}_{\mathbb{C}}(V \times \mathbb{C}^r, E)$ and the tautological line bundle γ over $\mathbb{CP}^{r-1}$, becomes the trivial line vector bundle $\mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathbb{C}$ over $\mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1}$, and the projection map is just the projection on the first three factors: indeed, γ admits over the coordinate chart $\mathbb{C}^{r-1} \subset \mathbb{CP}^{r-1}$ the trivialization $\mathbb{C}^{r-1} \times \mathbb{C} \xrightarrow{\sim} \gamma$ given by $(z_1, \ldots, z_{r-1}, \lambda) \mapsto ([z_1 : \ldots : z_{r-1} : 1], \lambda \overline{z})$, where $\overline{z} := (z_1, \ldots, z_{r-1}, 1)$;

- ϵ_2, defined globally as $\pi^* \phi^* E$, becomes the trivial vector bundle $\mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathbb{C}^r$ over $\mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1}$, again via the projection on the first 3 factors;

- the projection $\Pi : \mathrm{Hom}_{\mathbb{C}}(\epsilon_1, \epsilon_2) \to \mathrm{Hom}_{\mathbb{C}}(V \times \mathbb{C}^r, E) \times \mathbb{CP}^{k-1}$ becomes locally $\Pi : \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1}(\mathbb{C}) \to \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1}$, $(q, A, z, B) \mapsto (q, A, z)$;

- the zero section 0_Π of Π is locally the image of the inclusion

$$\begin{aligned} \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} &\hookrightarrow \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1}(\mathbb{C}) \\ (q, A, z) &\mapsto (q, A, z, 0) \end{aligned}$$

- the section Ψ of Π can be rewritten locally as $\Psi : \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \to \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1}(\mathbb{C})$ and is given by $\Psi(q, A, z) = (q, A, z, A \cdot \overline{z})$, where again $\overline{z} := (z_1, \ldots, z_{r-1}, 1) \in \mathbb{C}^r$, if $z = (z_1, \ldots, z_{r-1})$.

Then, using the expression in Equation A.1 for the matrix $M(q)$, we get

$$\begin{aligned} \Psi \circ \sigma_h : \mathcal{U} \times \mathbb{C}^{r-1} &\to \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1}(\mathbb{C}) \\ (q,z) &\mapsto \left(q, \left(\begin{array}{c|c} \mathrm{I}_{r-1} & 0 \\ \hline 0 & v(q) \end{array} \right), z, \begin{pmatrix} z \\ v(q) \end{pmatrix} \right) \end{aligned}$$

Now, because Ψ is transverse to 0_Π and Σ coincides with the oriented preimage $\Psi^{-1}(0_\Pi)$ (see the sketch of proof of Proposition A.1), we actually have that σ_h is transverse to Σ at $(q, 0) \in \mathcal{U} \times \mathbb{C}^{r-1}$ (recall that if $\sigma_h(q,z)$ is in Σ then $z = 0$) if and only if $\Psi \circ \sigma_h$ is transverse to 0_Π at $(q, 0)$. Moreover, if we have transversality at every point, $Z(h) = \sigma_h^{-1}(\Sigma)$ equals $(\Psi \circ \sigma_h)^{-1}(0_\Pi)$ as oriented manifolds.

If we denote by $\mu : \mathcal{U} \times \mathcal{M}_r(\mathbb{C}) \times \mathbb{C}^{r-1} \times \mathcal{M}_{r,1}(\mathbb{C}) \to \mathcal{M}_{r,1}(\mathbb{C})$ the projection on the last factor, we also get that $\Psi \circ \sigma_h$ is transverse to 0_Π at $(q, 0) \in \mathcal{U} \times \mathbb{C}^{r-1}$ if and only if $\mu \circ \Psi \circ \sigma_h$ is transverse to $\{0\} \subset \mathcal{M}_{r,1}(\mathbb{C})$ and that, in case of transversality at every point, $Z(h) \cap \left(\mathcal{U} \times \mathbb{C}^{r-1}\right) = (\mu \circ \Psi \circ \sigma_h)^{-1}(0)$ as oriented manifolds. In other words, using the fact that

$$\begin{aligned} \mu \circ \Psi \circ \sigma_h : \mathcal{U} \times \mathbb{C}^{r-1} &\to \mathcal{M}_{r,1}(\mathbb{C}) \\ (q, z) &\mapsto \begin{pmatrix} z \\ v(q) \end{pmatrix} \end{aligned}$$

we get that σ_h is transverse to Σ at $(q, 0) \in \mathcal{U} \times \mathbb{C}^{r-1}$ if and only if $v : \mathcal{U} \to \mathbb{C}$ is transverse to $\{0\} \subset \mathbb{C}$ at q and that, if there is transversality everywhere, $Z(h) \cap \left(\mathcal{U} \times \mathbb{C}^{r-1}\right) = v^{-1}(0) \times \{0\} \subset \mathcal{U} \times \mathbb{C}^{r-1}$ as oriented manifolds.

This concludes the proof of lemma A.6, because v is transverse to $0 \in \mathbb{C}$ (as said in the beginning), hence σ_h is transverse to Σ over $\mathcal{U} \times \mathbb{C}^{r-1}$, and $\phi : V \times \mathbb{CP}^{r-1} \to V$ clearly induces an orientation preserving diffeomorphism

$$\phi : Z(h) \cap \left(\mathcal{U} \times \mathbb{C}^{r-1}\right) = v^{-1}(0) \times \{0\} \xrightarrow{\sim} M \cap \mathcal{U} = v^{-1}(0)\ . \qquad \square$$

Part II

Revisiting some known constructions of contact manifolds

Chapter 5

Contact branched coverings and fiber sums

The goal of this chapter is to give definitions of contact branched coverings and contact fiber sums that allow us to naturally obtain uniqueness statements. We will in particular prove Proposition II.A stated in the introduction. We point out that the proofs in this section are mainly a reformulation of those in [Gei97b].

5.1 The smooth case

Branched covers

Let $\widehat{W}^m$, W^m be smooth manifolds with (possibly empty) boundary.

Definition 5.1. A map $\pi : \widehat{W} \to W$ is a *branched covering* if each point $p \in W$ admits a neighborhood U such that for each connected component $\widehat{U}$ of $\pi^{-1}(U)$ we have the following commutative diagram

$$\begin{tikzcd} D^2 \times \Omega \arrow[r, "\widehat{\psi}"] \arrow[d, "\pi_k"'] & \widehat{U} \arrow[d, "\pi_{|\widehat{U}}"] \\ D^2 \times \Omega \arrow[r, "\psi"] & U \end{tikzcd}$$

where Ω is either $\mathbb{R}^{m-2}$ or $\mathbb{R}^{m-3} \times [0, +\infty)$ (depending on whether $p \in W$ is respectively in the interior or on the boundary of W), D^2 is the disk of radius 1 and center 0 in $\mathbb{R}^2$, ψ and $\widehat{\psi}$ are diffeomorphisms, with $\psi(0,0) = p$, and π_k is the map $(z,q) \mapsto (z^k, q)$ for all $z \in D^2$ and all $q \in \Omega$.

If we denote $\widehat{p} = \widehat{\psi}(0,0) \in \widehat{U}$, then k is called the *branching index* at the point $\widehat{p}$. Remark that it is well-defined, i.e. it does not depend on any choice beside $\widehat{p}$. Indeed, $\pi|_{\widehat{U}}$ induces a (unbranched) covering map $\widehat{U} \setminus \widehat{\psi}\left(\{0\} \times \Omega\right) \to U \setminus \psi\left(\{0\} \times \Omega\right)$ and k is its degree, which is independent of the choice of the trivializations.

We denote $\widehat{X}^{m-2}$, and call it *upstairs branching set* the codimension 2 submanifold of $\widehat{W}$ made of the points $\hat{p} \in \widehat{W}$ such that the branching index is > 1, i.e. such that the associated local form π_k has a $k > 1$. We also denote X, and call it *downstairs branching set*, the codimension 2 submanifold of W given by the image of $\widehat{X}$ via π.

The following properties follow directly from the definition:

(a) $\widehat{X}$ and X are *proper* submanifolds of $\widehat{W}$ and W respectively, i.e. $\partial\widehat{X} \subset \partial\widehat{W}$, $\partial X \subset \partial W$ and $\widehat{X} \pitchfork \partial\widehat{W}$, $X \pitchfork \partial W$;

(b) π induces regular coverings $\pi|_{\widehat{W}\setminus\widehat{X}} : \widehat{W} \setminus \widehat{X} \to W \setminus X$ and $\pi|_{\widehat{X}} : \widehat{X} \to X$;

(c) if we denote $V^{m-1} := \partial W$ and $\widehat{V}^{m-1} = \partial\widehat{W}$, then $\pi' := \pi_{|\partial\widehat{W}}$ is a branched covering $\pi' : \widehat{V} \to V$ with branching set $M := \partial X \subset V$; we then denote $\widehat{M}$ the set of points in $\widehat{V}$ where π' has associated local index $k > 1$.

Fiber sums

Let V, M be two oriented smooth manifolds of dimensions $\dim V = m+1$ and $\dim M = m-1$, with M connected. We point out that we do not suppose V connected here.

Let also $j_1 : M \to V$ and $j_2 : M \to V$ be two disjoint orientation-preserving embeddings such that there is a fiber-orientation-reversing isomorphism Φ over M between their normal bundles:

$$\begin{tikzcd}
\mathcal{N}_1 \coloneqq j_1^*\left(TV \middle/ (j_1)_* TM\right) \arrow[rr, "\Phi", "\sim"'] \arrow[dr] & & \mathcal{N}_2 \coloneqq j_2^*\left(TV \middle/ (j_2)_* TM\right) \arrow[dl] \\
& M &
\end{tikzcd}$$

For notational convenience, denote by $j\colon M \to V$ the disjoint union $j_1 \sqcup j_2$.

Definition 5.2. A *fiber sum* of V along j_1, j_2 via Φ is the data (W, H, φ) of a smooth oriented manifold W^{m+1}, a cooriented hypersurface H in W and an orientation-preserving diffeomorphism $\varphi\colon V \setminus j(M) \to W \setminus H$ for which there exist (oriented) tubular neighborhoods $\rho_1\colon \mathcal{N}_1 \to V$ of $j_1(M)$, $\rho_2\colon \mathcal{N}_2 \to V$ of $j_2(M)$ and $\rho\colon \mathcal{N} \to V$ of H, where $\mathcal{N} \coloneqq {}^{TW}/_{TH}$ is the normal bundle of H in W, satisfying the following conditions:

i. ρ_1 and ρ_2 have disjoint images;

ii. for $i = 1, 2$ and $p \in M = 0_M \subset \mathcal{N}_i$, the differential $d_p\rho_i\colon T_p\mathcal{N}_i \to T_{j_i(p)}V$, which can be seen as a map $d_p\rho_i\colon T_pM \oplus (\mathcal{N}_i)_p \to T_{j_i(p)}V$ if we naturally split $T_p\mathcal{N}_i$ as $T_p0_M \oplus (\mathcal{N}_i)_p$, is such that the composition

$$T_{j_i(p)}V \xrightarrow{\mathrm{pr}} j_i^*\left(T_{j_i(p)}V \middle/ (j_i)_* T_pM\right) = (\mathcal{N}_i)_p \xrightarrow{d_p\rho_i|_{(\mathcal{N}_i)_p}} T_{j_i(p)}V$$

is exactly $\mathrm{Id} : T_{j_i(p)}V \to T_{j_i(p)}V$ (here, pr is just the natural projection);

iii. if $\mathcal{N}_1^*, \mathcal{N}_2^*$ and $\mathcal{N}^*$ denote respectively the bundles $\mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}$ deprived of their zero sections, φ induces a diffeomorphism between the image of $\rho_1 \sqcup \rho_2$ in $V \setminus j(M)$ and the image of ρ in $W \setminus H$, and the composition

$$f\colon \mathcal{N}_1^* \sqcup \mathcal{N}_2^* \xrightarrow{\rho_1 \sqcup \rho_2} V \setminus j(M) \xrightarrow{\varphi} \mathrm{Im}\left(\varphi \circ (\rho_1 \sqcup \rho_2)\right) \xrightarrow{\rho^{-1}} \mathcal{N}^*$$

is a diffeomorphism which is $\mathbb{R}_{>0}-$equivariant for the natural actions by multiplication of $\mathbb{R}_{>0}$ on $\mathcal{N}_1^*, \mathcal{N}_2^*$ and $\mathcal{N}^*$, and sends $\mathcal{N}_1^*$ on the positive part of $\mathcal{N}^*$ (H is a cooriented hypersurface, hence $\mathcal{N}$ is divided by the zero section in two connected components, one positive and one negative thanks to the coorientation);

iv. if $\sigma\colon \mathcal{N} \to \mathcal{N}$ is the involution given by $\sigma(v) = -v$ for each $v \in \mathcal{N}$, the composition

$$g\colon \mathcal{N}_1^* \xrightarrow{f|_{\mathcal{N}_1^*}} \mathcal{N} \xrightarrow{\sigma} \mathcal{N} \xrightarrow{f^{-1}} \mathcal{N}_2^*$$

coincides with $\Phi|_{\mathcal{N}_1^*}$

Remark 5.3. *It is a direct consequence of the above definition that if (W, H, φ) is a fiber sum of V along j_1, j_2 via Φ, then we have two diffeomorphisms f_1 and f_2 such that the following diagram is commutative:*

$$\begin{tikzcd} & \mathcal{N}_1^* / \mathbb{R}_{>0} \arrow[dd, "\phi"] \\ H \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & \mathcal{N}_2^* / \mathbb{R}_{>0} \end{tikzcd}$$

where the quotients $\mathcal{N}_1^* / \mathbb{R}_{>0}$ and $\mathcal{N}_2^* / \mathbb{R}_{>0}$ are taken with respect to the natural action by multiplication by a positive real on $\mathcal{N}_1^*$ and $\mathcal{N}_2^*$ respectively and ϕ is the diffeomorphism induced by Φ.

The following result shows that the above definition is non-empty:

Proposition 5.4. *Fiber sums of V along j_1, j_2 via Φ exist.*

Proof. If we fix an auxiliary Riemannian metric on V, Φ induces an isomorphism

$$\begin{aligned} \Phi_S\colon\ \mathrm{S}\mathcal{N}_1 &\to \mathrm{S}\mathcal{N}_2 \\ v &\mapsto \frac{\Phi(v)}{\|\Phi(v)\|} \end{aligned} \tag{5.1}$$

between the two sub-bundles of $\mathcal{N}_1$ and $\mathcal{N}_2$ made of the vectors of norm 1.

For $i = 1, 2$, consider an embedding $\rho_i : \mathcal{N}_i \to V$ which identifies the normal bundle $\mathcal{N}_i$ with a regular tubular neighborhood of $j_i(M)$ inside V, in such a way that their images are disjoint. Now, there is a natural isomorphism $\mathcal{N}_i^* \to \mathrm{S}\mathcal{N}_i \times (0, +\infty)$ of oriented bundles over M, for $i = 1, 2$. We hence obtain embeddings $\tau_i : \mathrm{S}\mathcal{N}_i \times (0, +\infty) \to V$ thanks to ρ_i, for $i = 1, 2$.

Consider then the set $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and the embedding

$$\begin{aligned} \mathrm{S}\mathcal{N}_1 \times \mathbb{R}^* &\overset{\Psi}{\hookrightarrow} V \setminus (j_1(M) \cup j_2(M)) \\ (v, t) &\mapsto \begin{cases} \tau_1(t \cdot v) & \text{for } t > 0 \\ \tau_2(-t \cdot \Phi_S(v)) & \text{for } t < 0 \end{cases} \end{aligned} \tag{5.2}$$

Let W be the smooth manifold obtained as the quotient of the disjoint union

$$(V \setminus j(M)) \bigsqcup \mathrm{S}\mathcal{N}_1 \times \mathbb{R}$$

by the relation $\sim_\Psi$ defined as follows: $p \sim_\Psi (v, t)$ if $p \in V \setminus j(M)$, $(v, t) \in \mathrm{S}\mathcal{N}_1 \times \mathbb{R}^*$ and $p = \Psi(v, t)$. Define also by H the hypersurface $\mathrm{S}\mathcal{N}_1 \times \{0\} \subset \mathrm{S}\mathcal{N}_1 \times \mathbb{R}$ inside W and by φ the diffeomorhism $V \setminus j(M) \to W \setminus H$ given by the natural identification $W \setminus H = V \setminus j(M)$. Then, (W, H, φ) is a fiber sum of V along $j(M)$ via Φ. $\square$

The notion of fiber sum in Definition 5.2 also satisfies the following uniqueness property:

Proposition 5.5. *If (W, H, φ) and $(\overline{W}, \overline{H}, \overline{\varphi})$ are two fiber sums of V along j_1, j_2 via Φ, then there is a diffeomorphism $\Theta\colon W \to \overline{W}$ such that:*

i. $\Theta|_H \colon H \xrightarrow{\sim} \overline{H}$ *is a coorientation preserving diffeomorphism;*

ii. $\theta := \overline{\varphi}^{-1} \circ \Theta \circ \varphi\colon V \setminus j(M) \xrightarrow{\sim} V \setminus j(M)$ *smoothly extends to* $\theta'\colon V \xrightarrow{\sim} V$ *that is isotopic to the identity, via an isotopy fixing* $j(M)$ *pointwise.*

Proposition 5.5 is an immediate consequence of the following:

Lemma 5.6. *If (W, H, φ) and $(\overline{W}, \overline{H}, \overline{\varphi})$ are two fiber sums of V along j_1, j_2 via Φ, then there is $\Theta\colon W \xrightarrow{\sim} \overline{W}$, with $\Theta(H) = \overline{H}$, and an isotopy $F_t : V \xrightarrow{\sim} V$ starting at $F_0 = \mathrm{Id}_V$, fixing $j(M)$ pointwise, and such that the following diagram is commutative:*

$$\begin{array}{ccc} V \setminus j(M) & \xrightarrow{\varphi} & W \setminus H \\ \big\downarrow{\scriptstyle F_1} & & \big\downarrow{\scriptstyle \Theta} \\ V \setminus j(M) & \xrightarrow{\overline{\varphi}} & \overline{W} \setminus \overline{H} \end{array}$$

Proof (Lemma 5.6). Denote by $\rho_1, \rho_2, \rho, \mathcal{N}$ and by $\overline{\rho}_1, \overline{\rho}_2, \overline{\rho}, \overline{\mathcal{N}}$ the embeddings and normal bundle given by Definition 5.2 for (W, H, φ) and $(\overline{W}, \overline{H}, \overline{\varphi})$.

According to the uniqueness theorem for tubular neighborhoods (see for instance [Lan99, Theorem 6.2]), there are vector bundle automorphisms ν_1, ν_2, of respectively $\mathcal{N}_1, \mathcal{N}_2$, and an isotopy $F_t\colon V \to V$ starting at $F_0 = \mathrm{Id}_V$, fixing $j(M)$ pointwise and such that $\overline{\rho}_1 = F_1 \circ \rho_1 \circ \nu_1$ and $\overline{\rho}_2 = F_1 \circ \rho_2 \circ \nu_2$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{N}_1 \sqcup \mathcal{N}_2 & \xrightarrow{\rho_1 \sqcup \rho_2} & V \\ \big\downarrow{\scriptstyle \nu_1 \sqcup \nu_2} & & \big\downarrow{\scriptstyle F_1} \\ \mathcal{N}_1 \sqcup \mathcal{N}_2 & \xrightarrow{\overline{\rho}_1 \sqcup \overline{\rho}_2} & V \end{array}$$

Notice moreover that, by condition ii. in Definition 5.2, we can chose F_1 so that ν_1 and ν_2 are actually the identity isomorphisms of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively.

Consider now the composition

$$\Theta'\colon\ W \setminus H \xrightarrow{\varphi^{-1}} V \setminus j(M) \xrightarrow{F_1} V \setminus j(M) \xrightarrow{\overline{\varphi}} \overline{W} \setminus \overline{H}\ .$$

At this point, it's enough to show that such a Θ' extends on all W to a diffeomorphism $\Theta\colon W \to \overline{W}$ as in the statement.

Let $X := \mathrm{Im}\,(\varphi \circ (\rho_1 \sqcup \rho_2))$, $\overline{X} := \mathrm{Im}\,(\overline{\varphi} \circ (\overline{\rho}_1 \sqcup \overline{\rho}_2))$ and $\mu' := \overline{\rho}^{-1} \circ \Theta'|_X \circ \rho$. We then have the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{N}_1^* \sqcup \mathcal{N}_2^* & \xrightarrow{\rho_1 \sqcup \rho_2} & V \setminus j\,(M) & \xrightarrow{\varphi} & X & \xrightarrow{\rho^{-1}} & \mathcal{N}^* \\ \big\downarrow{\scriptstyle \mathrm{Id}} & & \big\downarrow{\scriptstyle F_1} & & \big\downarrow{\scriptstyle \Theta'|_X} & & \big\downarrow{\scriptstyle \mu'} \\ \mathcal{N}_1^* \sqcup \mathcal{N}_2^* & \xrightarrow{\overline{\rho}_1 \sqcup \overline{\rho}_2} & V \setminus j\,(M) & \xrightarrow{\overline{\varphi}} & \overline{X} & \xrightarrow{\overline{\rho}^{-1}} & \overline{\mathcal{N}}^* \end{array}$$

Now, by Definition 5.2, the upper and the lower rows give two $\mathbb{R}_{>0}-$equivariant diffeomorphisms. Hence, μ' is $\mathbb{R}_{>0}-$equivariant too. Because $\mathcal{N}$ and $\overline{\mathcal{N}}$ are vector spaces of real dimension 1 over H and $\overline{H}$, this means that μ' actually extends to a vector bundle isomorphism $\mu : \mathcal{N} \to \overline{\mathcal{N}}$.

By the commutativity of the above diagram, this implies that Θ' can be extended to a diffeomorphism $\Theta\colon W \xrightarrow{\sim} \overline{W}$, as wanted. $\square$

5.2 Branched coverings of contact manifolds

Suppose $\pi : \widehat{V}^{2n+1} \to V^{2n+1}$ is a branched covering map of manifolds without boundary, branched along the codimension 2 submanifold $M^{2n-1} \subset V$. We denote, like in the previous section, $\widehat{M}^{2n-1}$ the locus of points of $\widehat{V}$ with branching index > 1. Be careful, though, that here $\widehat{V}$, V, $\widehat{M}$, M play, respectively, the roles of $\widehat{W}$, W, $\widehat{X}$, X of Section 5.1, and they all have no boundary.

In this section we also suppose that we are dealing with (V,η) and (M,ξ) contact manifolds, where ξ is the sub−bundle $\eta \cap TM$ of $TV|_M$.

The pullback $\pi^*\eta$ is a well defined hyperplane field on $\widehat{V}$, because if we fix a contact form α for η then $\pi^*\alpha$, which defines $\pi^*\eta$, is nowhere vanishing. Though, it is not a contact form, because at each point $\widehat{p}$ of $\widehat{M}$ we have $\pi^*(\alpha \wedge d\alpha^n)_{|\widehat{p}} = 0$. We point out that, nonetheless, the restriction of $\pi^*\eta$ to $\widehat{M}$ is a honest contact structure on $\widehat{M}$. We then want to show that $\pi^*\eta$ gives a "natural" way to construct contact structures on $\widehat{V}$.

We start by considering a more general setting. Let Y^{2n+1} be a smooth manifold, Z^{2n-1} a codimension-2 submanifold and η an hyperplane field on Y.

Definition 5.7. We say that η is *adjusted* to Z if it is a contact structure on the complement of Z and $\eta \cap TZ$ is a contact structure on Z. If that's the case, we also call *contactization* of η a contact structure ξ such that there is a smooth path $\{\eta_s\}_{s\in[0,1]}$ of hyperplane fields, all adjusted to Z, which starts at $\eta_0 = \eta$, ends at $\eta_1 = \xi$ and such that η_s is a contact structure for all $s \in (0,1]$.

We then have the following general result:

Proposition 5.8. *Let η be an hyperplane field on Y adjusted to Z. Contactizations of η exist and are all isotopic.*

We recall from [ET98, Section 1.1.6] that a *confoliation* on a manifold V is an hyperplane field $\zeta = \ker\alpha$ on V that admits a complex structure $J : \zeta \to \zeta$ *tamed* by $d\alpha|_\zeta$, i.e. such that $d\alpha(X, JX) \geq 0$ for all vector fields X tangent to ζ. In our situation, if η is an hyperplane field on Y adjusted to a codimension 2 submanifold Z, then η is a confoliation. Indeed, we have the following:

Fact 5.9. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of contact structures on a manifold Y^{2n+1} which C^1−converges to a hyperplane field η on Y. Then, $\eta = \ker\alpha$ admits a complex structure J tamed by $d\alpha|_\eta$.*

From now on, we will hence talk directly about confoliations adjusted to a certain codimension 2 submanifold.

Proof (Fact 5.9). A first idea could be to take, for each $n \in \mathbb{N}$, a complex structure J_n on $\eta_n = \ker\alpha_n$ tamed by $d\alpha_n|_{\eta_n}$ (which exists because η_n is a contact structure) and to define J as "the limit" of the sequence $(J_n)_{n\in\mathbb{N}}$. However, such a limit does not necessarily exist for any choice of J_n. The solution is to assure the orthogonality of each of the J_n with respect to an auxiliary riemannian metric g using the polar decomposition of matrices, as follows.

Let $\xi = \ker\beta$ be a contact structure on Y and fix an auxiliary riemannian metric g on Y. Then, by the contact condition, $d\beta$ induces an isomorphism $\widetilde{d\beta}\colon \xi \to \xi^*$, where ξ^* is the vector bundle dual of ξ over M. Now, g also induces an isomorphism $\widetilde{g}\colon \xi^* \to \xi$. We can hence consider $A\colon \xi \to \xi$ given by $A := \widetilde{g} \circ \widetilde{d\beta}$; in other words, $d\beta(X,Y) = g(AX,Y)$

for each couple of sections (X, Y) of ξ. Consider then the (left) polar decomposition RJ of A, i.e. the unique couple of vector bundle-isomorphisms $R, J\colon \xi \to \xi$ with R positive definite and J orthogonal with respect to g. More explicitly, R is the positive definite square root of the symmetric and positive definite AA^*, where A^* is the g−dual of A, and J is the g−orthogonal endomorphism of ξ given by $R^{-1}A$. The positive definiteness of R implies that J is $d\beta|_\xi$−tamed. Moreover, by definition, A is skew−symmetric, i.e. $A^* = -A$. This and the fact that J is g−orthogonal imply that $J^2 = -\operatorname{Id}$, i.e. that J is a complex structure on ξ.

This being said, let's go back to the setting of Fact 5.9. Once fixed a sequence of 1−forms $(\alpha_n)_{n\in\mathbb{N}}$ with α_n defining η_n and that C^1−converges to a 1−form α defining η, consider an auxiliary riemannian metric g on M. Then, according to what we just said, we can define a sequence $(J_n)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $J_n\colon \eta_n \to \eta_n$ is a complex structure on η_n and is orthogonal with respect to g. Then, after obtaining from each J_n an orthogonal $B_n : TM \to TM$ that sends the vector of norm 1 positively orthogonal to η_n to itself, by compactness of the space of orthogonal linear maps $TM \to TM$, we get that there is a sequence of naturals $(n_k)_{k\in\mathbb{N}}$ diverging to $+\infty$ such that B_{n_k} converges (in C^1−topology) to a limit $B\colon TM \to TM$. Now, we recall that $\eta_n \xrightarrow{C^1} \eta$, that $J_n^2 = -\operatorname{Id}|_{\eta_n}$ and that $d\alpha_n(X, JX) > 0$ for each X section of η_n. Hence, B restricts to a well defined $J\colon \eta \to \eta$ such that $J^2 = -\operatorname{Id}|_\eta$ and $d\alpha(X, JX) \geq 0$ for all X section of η, as wanted. □

Proposition 5.8 is a consequence of the following lemma, which deals with the more general situation of any number of parameters:

Lemma 5.10. *Given K a compact set and $(\eta_k)_{k\in K}$ a smooth K−family of confoliations on V adjusted to M, there is a smooth family of confoliations $(\eta_k^s)_{s\in[0,1],\, k\in K}$ such that $(\eta_k^s)_{s\in[0,1]}$ is contactization of η_k, for each $k \in K$. Moreover, if η_k is contact for all k in a closed subset $H \subset K$, then η_k^s can be chosen so that $\eta_k^s = \eta_k$ for all $k \in H$ and $s \in [0, 1]$.*

In the above statement, by a smooth K−family of hyperplane fields we mean the following. Let X be a closed smooth manifold and $K \subset X$ a compact subset. Then, we say that $(\eta_k)_{k\in K}$ is a *smooth K−family* if there is an open set $U \subset X$ containing K and a differential form $\alpha \in \Omega^1(V \times U)$ such, that for all $k \in K$, $\eta_k = \ker(i_k^*\alpha)$, where $i_k\colon V \to V \times U$ is the inclusion $i_k(p) = (p, k)$.

Proof (Proposition 5.8). The existence of contactizations follows directly from Lemma 5.10 with K a point.
Given two contactizations ξ, ξ' of η, we have by definition two associated paths of adjusted confoliations η_t, η'_t, with $t \in [0, 1]$, such that $\eta_0 = \eta'_0 = \eta$, $\eta_1 = \xi$, $\eta'_1 = \xi'$ and η_t, η'_t contact for $t \in (0, 1]$. Then, the path

$$t \mapsto \widehat{\eta}_t := \begin{cases} \eta_{1-2t} & \text{if } t \in [0, 1/2] \\ \eta'_{2t-1} & \text{if } t \in [1/2, 1] \end{cases} \tag{5.3}$$

is a continuous path of adjusted confoliations from $\widehat{\eta}_0 = \xi$ to $\widehat{\eta}_1 = \xi'$. Moreover, up to perturbing it smoothly at $t = 1/2$, we can suppose that $\widehat{\eta}_t$ is smooth in t. Then, applying Lemma 5.10 to $\widehat{\eta}_t$, with $K = [0, 1]$ and $H = \{0, 1\}$, we get a family $(\widehat{\eta}_t^s)_{s\in[0,1],\, t\in[0,1]}$ of adjusted confoliations such that $\widehat{\eta}_0^s = \xi$, $\widehat{\eta}_1^s = \xi'$ for all $s \in [0, 1]$ and such that $\widehat{\eta}_t^s$ is contact for $s > 0$. The subfamily $\widehat{\eta}_t^1$ is then a path of contact structures from ξ to ξ', and it can be turned into an isotopy by Gray's theorem. □

Proof (Lemma 5.10). This proof follows almost step by step the construction and the computations made in [Gei97b, Section 2].

Because the contact condition is an open condition in the space of $1-$forms (with the C^1-topology), there is an open subset U of K which contains H and such that ξ_k is contact for all $k \in U$. We consider then a smooth cut-off function $\rho : K \to [0,1]$, equal to 0 on H and equal to 1 on the complement of U.

Take now an auxiliary Riemannian metric on V and consider the circle bundle $S(\mathcal{N}M)$ given by the vectors of norm 1 in the normal bundle $\mathcal{N}M$ of M inside V. Let γ be a connection form on $S(\mathcal{N}M)$, i.e. a nowhere vanishing $1-$form defining an hyperplane field which is transversal to the fibers of the fibration $S(\mathcal{N}M) \to M$. Then, thanks to the natural retraction $\mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1$, γ can also be seen as a $1-$form on $\mathcal{N}M \setminus M$. Moreover, the form $r^2\gamma$, where r is the radial coordinate in $\mathcal{N}M \setminus M$, smoothly extends over the zero section M to all $\mathcal{N}M$.

We consider then a non-increasing cut-off smooth function $g = g(r)$ which is 1 near $r = 0$ and vanishes for $r > 1$ and we identify $\mathcal{N}M$ with a neighborhood of M inside V. If α_k is a smooth K-family of 1-forms defining ξ_k, set

$$\alpha_k^s := \alpha_k + s\epsilon\rho(k)g(r)r^2\gamma \ .$$

Here ϵ is a positive real constant which will be chosen very small later. Suppose, without loss of generality, that $\epsilon \leq 1$.

Remark that $\xi_k^s := \alpha_k^s$ is a well defined hyperplane field. Moreover, it is adjusted to M, for all values of s, k.

We now want to show that, for an ϵ small enough, ξ_k^s is actually a contact structure on V for all $s > 0$, $k \in K$.

We can compute

$$\begin{aligned}\alpha_k^s \wedge (d\alpha_k^s)^n &= \alpha_k \wedge (d\alpha_k)^n + \\ &+ ns\epsilon \left[rg'(r) + 2g(r)\right] \rho(k)\, \alpha_k \wedge (d\alpha_k)^{n-1} \wedge rdr \wedge \gamma + \\ &+ s\epsilon r^2 g(r)\, \rho(k)\, h \,\mathrm{vol}\end{aligned}$$

where vol is the Riemannian volume form on V and h is a function of $p \in V$, $k \in K$, $s \in [0,1]$, $\epsilon \in \mathbb{R}_{>0}$ and is polynomial in ϵ.

Define now $P_k, Q_k \colon V \to \mathbb{R}$ by the identities $\alpha_k \wedge (d\alpha_k)^n = P_k \,\mathrm{vol}$ and $n\left[rg'(r) + 2g(r)\right] \alpha_k \wedge (d\alpha_k)^{n-1} \wedge rdr \wedge \gamma = Q_k \,\mathrm{vol}$.
Also, define $R_k(\epsilon) := r^2 g(r)\, h(\epsilon, k)$. Then,

$$\alpha_k^s \wedge (d\alpha_k^s)^n = \{P_k + s\epsilon\rho(k)\left[Q_k + R_k(\epsilon)\right]\} \cdot \mathrm{vol} \ .$$

Now, $Q_k > 0$ and $R_k(\epsilon) = 0$ along $\widehat{M}$, for all $k \in K$ and $\epsilon \in [0,1]$ (remark we allow here $\epsilon = 0$). Hence, by compactness of $\widehat{M}$ and $[0,1]$, there is an open neighborhood $\mathcal{O}$ of $\widehat{M}$ inside $\widehat{V}$ such that $Q_k + R_k(\epsilon) > 0$ on $\mathcal{O}$ for all $\epsilon \in [0,1]$.

P_k is independent of ϵ, s and is non-negative everywhere on $\widehat{V}$ for all k. Moreover, P_k is strictly positive on the complement of $\mathcal{O}$ for all $k \in K$, and even on all $\widehat{V}$ if $k \in U \subset K$ (remember ξ_k is contact if $k \in U$).
Then, $P_k + s\epsilon\rho(k)\left[Q_k + R_k(\epsilon)\right] > 0$ on $\mathcal{O}$, for all $k \in K$ and all $\epsilon \in (0,1]$.

Lastly, for ϵ very small, P_k dominates $s\epsilon\rho(k)\left[Q_k + R_k(\epsilon)\right]$ wherever it is strictly positive, because the latter is bounded above in norm (recall we are working with $\epsilon \leq 1$). Hence, by compactness of $\widehat{V} \setminus \mathcal{O}$, $P_k + s\epsilon\rho(k)\left[Q_k + R_k(\epsilon)\right]$ is also positive on the complement of $\mathcal{O}$ for all $k \in K$, for $\epsilon > 0$ small enough. □

Coming back to the specific case of branched coverings, the hyperplane field $\pi^*\eta$ on $\widehat{V}$ is adjusted to $\widehat{M}$ (hence is in particular a confoliation).

We can thus redefine branched coverings in contact topology as follows:

Definition 5.11. We say that a contact structure on $\widehat{V}$ is a *contact branched covering of* η if it is a contactization of $\pi^*\eta$ and it is invariant under all the diffeomorphisms of $\widehat{V}$ lifting the identity of V.

We point out that, by definition of contactization, if $\widehat{\eta}$ is a contact branched covering of η, the upstairs branching locus $\widehat{M}$ is naturally a contact submanifold in $(\widehat{V}, \widehat{\eta})$. Then, Proposition 5.8 easily implies the following:

Proposition 5.12. *Let $\widehat{V} \to V$ be a smooth branched covering and η a contact structure on V. Then, contact branched coverings of η on $\widehat{V}$ exist and are all isotopic (among contact branched coverings).*

We point out that, in order to deduce this result from Proposition 5.8, the contactization in the statement Proposition 5.8 has to be invariant under deck transformations of π, as requested in Definition 5.11, and the isotopy has to be among invariant contactizations. From the explicit formula in the proof of Lemma 5.10 above, it's clear that both these conditions can be easily arranged.

Let's now recall and prove Proposition II.A, stated in the introduction:

Proposition II.A. *Soit (V^{2n-1}, η) une variété de contact et $\pi\colon \widehat{V} \to V$ un revêtement ramifié avec lieu de ramification (en bas) M. On suppose que $\eta \cap TM$ est une structure de contact sur M. Alors :*

1. *il y a une famille, indexée par $[0,1]$, de distributions d'hyperplans $\widehat{\eta}_t$ sur $\widehat{V}$ telle que $\widehat{\eta}_0 = \pi^*\eta$ et $\widehat{\eta}_t$ est une structure de contact pour tout $t \in (0,1]$;*
2. *si $\widehat{\eta}_t$ et $\widehat{\eta}'_t$ sont comme dans le point 1, alors $\widehat{\eta}_r$ est isotope à $\widehat{\eta}'_s$ pour tout $r, s \in (0,1]$.*

De plus, dans le point 1, $\widehat{\eta}_t$ peut être choisie invariant par les automorphismes (locales) du revêtement π, pour tout $t \in (0,1]$; de façon analogue, l'isotopie du point 2 peut être choisie parmi les structures de contact invariant par les automorphismes (locales) de revêtement, si $\widehat{\eta}_t$ et $\widehat{\eta}'_t$ le sont aussi.

Proof (Proposition II.A). This is a simple consequence of Gray's theorem and the fact that contact branched coverings exist and are unique up to isotopy. Indeed, the $[0,1]$-families of hyperplane fields in points 1 and 2 in the statement are automatically adjusted to the upstairs branching locus for small parameters $t \geq 0$. $\square$

5.3 Fiber sums of contact manifolds

Let's start here with some general definitions and results.
Let V^{2n+1} be a smooth manifold and (M^{2n-1}, ξ) be a contact manifold. Consider now an oriented circle bundle $\pi : S \to M$ and suppose that there is a smooth embedding $S \hookrightarrow V$.

Definition 5.13. A smooth hyperplane field η on V is *adjusted to* S if η is a contact structure away from S and $\eta \cap TS = \pi^*\xi$ as bundles over S. If that's the case, we call *contactization of* η a smooth path of hyperplane fields $(\eta_s)_{s\in[0,1]}$ such that $\eta_0 = \eta$ and such that η_s is a contact structure for all $s \in (0,1]$.

As in Section 5.2, we have the following:

Proposition 5.14. *Let η be a hyperplane field adjusted to S. Contactizations of η exist and are all isotopic.*

We point out that exactly as remarked in Fact 5.9 for the case of hyperplane fields adapted to a codimension 2 submanifold, η is automatically a confoliation on V. We will hence talk directly of confoliation adjusted to S in the following.

In order to prove Proposition 5.14, analogously as in the proof of Proposition 5.8, we use the following:

Lemma 5.15. *Given K a compact set and $(\eta_k)_{k\in K}$ a smooth family of confoliations on V adjusted to S, there is a smooth family of adjusted confoliations $(\eta_k^s)_{s\in[0,1],\,k\in K}$ such that, for all $k \in K$, $\eta_k^0 = \eta_k$ and η_k^s is contact for $s \in (0,1]$. Moreover, if η_k is contact for all k in a closed subset $H \subset K$, then the η_k^s can be chosen so that $\eta_k^s = \eta_k$ for all $k \in H$ and $s \in [0,1]$.*

Proof (Lemma 5.15). Take an open subset U of K which contains H and such that ξ_k is contact for all $k \in U$. Consider a smooth cut-off function $\rho : K \to [0,1]$, equal to 0 exactly on H and equal to 1 on the complement of U.

Now take an auxiliary Riemannian metric on V and consider a connection form γ on the circle bundle $\pi : S \to M$. The hypothesis of S being an oriented circle bundle (together with the fact that M, V are oriented by ξ, η respectively) tells that S is orientable as smooth manifold, and that its normal bundle $\mathcal{N}S$ inside V is trivial. In other words, S has a neighborhood of the form $S \times \mathbb{R}$ inside V. Denote by r a coordinate on the $\mathbb{R}$ factor.

Take now a non-increasing cut-off smooth function $g = g(r)$ which is 1 near $r = 0$ and vanishes for $r > 1$.
Consider then a smooth K-family of 1-forms α_k defining ξ_k and let

$$\alpha_k^s := \alpha_k + s\epsilon\rho(k)g(r)r\gamma \; .$$

Here ϵ is a positive real constant which will be chosen very small later. We can in particular set once and for all $\epsilon < 1$.

We claim that α_k^s defines a contact structure ξ_k^s on V for all $s > 0$. Indeed,

$$\begin{aligned}\alpha_k^s \wedge (d\alpha_k^s)^n &= \alpha_k \wedge (d\alpha_k)^n + \\ &\quad + ns\epsilon\left[rg'(r) + 2g(r)\right]\rho(k)\,\alpha_k \wedge (d\alpha_k)^{n-1} \wedge dr \wedge \gamma + \\ &\quad + s\epsilon r g(r)\,\rho(k)\,h\mathrm{vol}\end{aligned}$$

where vol is the Riemannian volume form on V and h is a function of $p \in V$, $k \in K$, $s \in [0,1]$, $\epsilon \in \mathbb{R}_{>0}$, and is polynomial in ϵ.

At this point, the same arguments as in the proof of Lemma 5.10 show that, for ϵ small enough, the above $(2n+1)-$form is a volume form on all V. □

Let's now consider the case of fiber sums. Exactly as we did in the case of branched coverings, once showed that there is a natural (in the sense of Definition 5.2) notion of smooth fiber sum (W, H, φ), we want to define on it a natural confoliation adapted to H.

Let (V^{2n+1}, η), (M^{2n-1}, ξ) be two contact manifolds and consider two contact embeddings $j_1 : (M, \xi) \to (V, \eta)$ and $j_2 : (M, \xi) \to (V, \eta)$ such that there is a fiber-orientation-reversing isomorphism $\Phi : \mathcal{N}_1 \to \mathcal{N}_2$ of vector bundles over M between the normal bundle $\mathcal{N}_1$ of $j_1(M)$ and the normal bundle $\mathcal{N}_2$ of $j_2(M)$.
Again, let $j := j_1 \sqcup j_2$ for notational convenience.

Definition 5.16. We call *confoliated fiber sum* of (V, η) along the contact embeddings j_1, j_2 via Φ the data (W, H, φ, ζ) of a smooth fiber sum (W, H, φ) of V along j_1, j_2 via Φ and a smooth hyperplane field ζ on W that coincides with $\varphi_* \eta$ on $W \setminus H$ and is adjusted to H.

Here, according to Remark 5.3, H can naturally be seen as the total space of the circle bundle $\mathrm{S}\mathcal{N}_1 \to M$. Hence, ζ is adjusted to H is in the sense of Definition 5.13. In particular, η is a confoliation on V, which explains the nomenclature "confoliated" fiber sum.

Remark 5.17. *If such a confoliation ζ exists, it is uniquely determined by $\varphi_* \eta$, because H is a hypersurface in W, hence $W \setminus H$ is dense in W.*

The following result guarantees that the above definition is non$-$empty:

Proposition 5.18. *Confoliated fiber sums of (V, η) along contact embeddings j_1, j_2 via Φ exist.*

Proof (Proposition 5.18). Let α be a contact form defining ξ on M. Denote also by γ_2 a connection form on the bundle $\mathrm{S}\mathcal{N}_2$ and γ_1 the connection $-\Phi_S^* \gamma_2$ on $\mathrm{S}\mathcal{N}_1$, where Φ_S is as in Equation (5.1).

Lemma 5.19. *There are $\epsilon > 0$ and embeddings $\rho_1 : \mathcal{N}_1 \hookrightarrow V$ and $\rho_2 : \mathcal{N}_2 \hookrightarrow V$ with disjoint images and such that $\rho_1^*(\eta) = \ker\left(\pi_1^* \alpha + \epsilon \arctan\left(r^2\right)\gamma_1\right)$ and $\rho_2^*(\eta) = \ker\left(\pi_2^* \alpha + \epsilon \arctan\left(r^2\right)\gamma_2\right)$, where $\pi_i : \mathcal{N}_i \to M$ is the projection of the vector bundle $\mathcal{N}_i$ onto its base space M.*

Proof (Lemma 5.19). This follows directly from the standard neighborhood theorem for contact submanifolds [Gei08, Theorem 2.5.15], together with the fact that a pair of codimension 2 embeddings of the same contact submanifold have isomorphic conformal symplectic normal bundles if and only if they have isomorphic smooth normal bundles. $\square$

We now call $\tau_i : \mathrm{S}\mathcal{N}_i \times (0, +\infty) \to V$ the embedding induced by ρ_i, for $i = 1, 2$, and we define and embedding Ψ as follows:

$$\begin{aligned} \mathrm{S}\mathcal{N}_1 \times \mathbb{R}^* &\overset{\Psi}{\hookrightarrow} V \setminus j(M) \\ (v, t) &\mapsto \begin{cases} \tau_1\left(\sqrt{\tan\left(t^3\right)} \cdot v\right) & \text{for } t > 0 \\ \tau_2\left(\sqrt{\tan\left(-t^3\right)} \cdot \Phi_S(v)\right) & \text{for } t < 0 \end{cases} \end{aligned} \tag{5.4}$$

We point out that Ψ is an embedding that preserves the orientation (recall $\mathcal{N}_1$ is oriented, hence $\mathrm{S}\mathcal{N}_1$ too).

Using Lemma 5.19 and the fact that $\Phi_S^* \gamma_2 = -\gamma_1$, a direct computation gives that $\Psi^* \eta = \ker\left(\pi_1^* \alpha + \epsilon t^3 \gamma_1\right)$ on $\mathrm{S}\mathcal{N}_1 \times \mathbb{R}^*$.

Now, the 1-form $\gamma := \pi_1^* \alpha + \epsilon t^3 \gamma_1$ on the domain of Ψ naturally extends to a smooth differential form on $\mathrm{S}\mathcal{N}_1 \times \mathbb{R}$, which we will still denote by γ. Moreover, this extension is clearly positively contact away from the submanifold $\mathrm{S}\mathcal{N}_1 \times \{0\}$ and defines a smooth confoliation on all $\mathrm{S}\mathcal{N}_1 \times \mathbb{R}^*$.

Then, denote by W the smooth manifold obtained as the quotient of the disjoint union $(V \setminus j(M)) \sqcup (\mathrm{S}\mathcal{N}_1 \times \mathbb{R})$ by the relation $\sim_\Psi$ defined as follows: $p \sim_\Psi (v, t)$ if $p \in V \setminus j(M)$, $(v, t) \in \mathrm{S}\mathcal{N}_1 \times \mathbb{R}^*$ and $p = \Psi(v, t)$.
Let also H be the hypersurface $\mathrm{S}\mathcal{N}_1 \times \{0\} \subset \mathrm{S}\mathcal{N}_1 \times \mathbb{R}$ inside W, and φ the diffeomorhism $V \setminus j(M) \to W \setminus H$ given by the identification $W \setminus H = V \setminus j(M)$.

Finally, let ζ be the smooth hyperplane (and confoliation) induced by η and $\ker\gamma$ respectively on the subsets $V \setminus j(M)$ and $\mathrm{S}\mathcal{N}_1 \times \mathbb{R}^*$ of W.

Then, (W, H, φ, ζ) is a confoliated fiber sum of V along $j(M)$ via Φ. $\square$

We also have the following uniqueness property:

Proposition 5.20. *If (W, H, φ, ζ) and $(\overline{W}, \overline{H}, \overline{\varphi}, \overline{\zeta})$ are two confoliated fiber sums of V along the contact embeddings j_1, j_2 via Φ, then there is a diffeomorphism $\Theta\colon W \to \overline{W}$ such that:*

i. $\Theta|_H\colon H \xrightarrow{\sim} \overline{H}$ *is a coorientation preserving diffeomorphism;*

ii. $\theta \coloneqq \overline{\varphi}^{-1} \circ \Theta \circ \varphi\colon\ (V \setminus j(M), \eta) \xrightarrow{\sim} (V \setminus j(M), \eta)$ *is a contactomorphism ant it smoothly extends to a contactomorphism* $\theta'\colon\ (V, \eta) \xrightarrow{\sim} (V, \eta)$ *that is contact-isotopic to the identity, via an isotopy fixing $j(M)$ pointwisely;*

iii. $\Theta_*\zeta = \overline{\zeta}$.

We deduce Proposition 5.20 from the following:

Lemma 5.21. *If (W, H, φ) and $(\overline{W}, \overline{H}, \overline{\varphi})$ are two fiber sums of V along $j_1 \sqcup j_2$ via Φ, then there is $\Theta\colon W \xrightarrow{\sim} \overline{W}$, with $\Theta(H) = \overline{H}$, and a contact isotopy $F_t : (V, \eta) \xrightarrow{\sim} (V, \eta)$ starting at $F_0 = \mathrm{Id}_V$, fixing $j(M)$ pointwise, such that the following diagram is commutative:*

$$\begin{tikzcd} V \setminus j(M) \arrow[r, "\varphi"] \arrow[d, "F_1|_{V\setminus j(M)}"'] & W \setminus H, \arrow[d, "\Theta|_{W\setminus H}"] \\ V \setminus j(M) \arrow[r, "\overline{\varphi}"] & \overline{W} \setminus \overline{H} \end{tikzcd}$$

Proof (Proposition 5.20). This follows directly from Lemma 5.21, once noticed that the commutativity of the square in the lemma tells that $\left(\Theta|_{W\setminus H}\right)_* (\varphi_*\eta) = \overline{\varphi}_*\eta$ and that, according to Remark 5.17, ζ and $\overline{\zeta}$ are determined, respectively, by $\varphi_*\eta$ and $\overline{\varphi}_*\eta$. $\square$

Proof (Lemma 5.21). As in the proof of Lemma 5.6, denote by ρ_1,ρ_2,ρ,$\mathcal{N}$ and by $\overline{\rho}_1$,$\overline{\rho}_2$,$\overline{\rho}$,$\overline{\mathcal{N}}$ the embeddings and normal bundle given by Definition 5.2 for respectively (W, H, φ) and $(\overline{W}, \overline{H}, \overline{\varphi})$.

Again according to the uniqueness theorem for tubular neighborhoods (see [Lan99, Theorem 6.2]), there is an isotopy $G_t\colon V \to V$, with $t \in [0,1]$, starting at $G_0 = \mathrm{Id}_V$, fixing $j(M)$ pointwise and such that $\overline{\rho}_1 = G_1 \circ \rho_1$ and $\overline{\rho}_2 = G_1 \circ \rho_2$. Define then $\rho_i^t \coloneqq G_t \circ \rho_i$ for $i = 1, 2$ and $t \in [0,1]$, and denote $R_t \coloneqq \rho_1^t \sqcup \rho_2^t$.

Then, as explained in the proof of [Gei08, Theorem 2.6.12], we can find a family of embeddings $H_t\colon N_1 \sqcup N_2 \to \mathcal{N}_1 \sqcup \mathcal{N}_2$, defined on N_1, N_2 neighborhoods of the zero section in $\mathcal{N}_1, \mathcal{N}_2$ respectively, such that H_0 is the inclusion $i\colon N_1 \sqcup N_2 \to \mathcal{N}_1 \sqcup \mathcal{N}_2$ and $\left(R_t \circ H_t\right)^* \eta = \left(H_t\right)^* \left(R_t^* \eta\right) = i^* \left(\rho_1 \sqcup \rho_2\right)^* \eta$ on $N_1 \sqcup N_2 \subset \mathcal{N}_1 \sqcup \mathcal{N}_2$. More precisely, such a H_t corresponds in the proof of [Gei08, Theorem 2.6.12] to the φ_t obtained from $\phi_t \coloneqq R_t$.

In particular, $R_t \circ H_t\colon N_1 \sqcup N_2 \to V$ pulls back η to $\left(\rho_1 \sqcup \rho_2\right)^* \eta$ and

$$R_t \circ H_t\colon\ \left(N_1 \sqcup N_2, \left(\rho_1 \sqcup \rho_2\right)^* \eta\right) \to (V, \eta)$$

is a path of contact embeddings starting at $R_0 \circ H_0 = (\rho_1 \sqcup \rho_2) \circ i$ and ending at $R_1 \circ H_1 = (\overline{\rho}_1 \sqcup \overline{\rho}_2) \circ H_1$.

Then, using a cut-off of contact Hamiltonians, we can then obtain a contact isotopy $F_t : (V, \eta) \to (V, \eta)$ such that, for smaller neighborhoods U_1, U_2 of the zero section of $\mathcal{N}_1, \mathcal{N}_2$ compactly contained in N_1, N_2, the following diagram is commutative:

$$\begin{tikzcd} U_1 \sqcup U_2 \arrow[r, "H_0=i"] \arrow[d, "\mathrm{Id}"'] & \mathcal{N}_1 \sqcup \mathcal{N}_2 \arrow[r, "R_0=\rho_1\sqcup\rho_2"] & V \arrow[d, "F_1"] \\ U_1 \sqcup U_2 \arrow[r, "H_1"] & \mathcal{N}_1 \sqcup \mathcal{N}_2 \arrow[r, "R_1=\overline{\rho}_1\sqcup\overline{\rho}_2"] & V \end{tikzcd}$$

Now, consider the composition

$$\Theta' \colon\ W \setminus H \xrightarrow{\varphi^{-1}} V \setminus j(M) \xrightarrow{F_1} V \setminus j(M) \xrightarrow{\overline{\varphi}} \overline{W} \setminus \overline{H}\ .$$

It's enough to show that such a Θ' extends on all W to a diffeomorphism $\Theta\colon\ W \to \overline{W}$ as in the statement.

We have the following commutative diagram:

$$\begin{tikzcd} U_1^* \sqcup U_2^* \arrow[r, "H_0=i"] \arrow[d, "\mathrm{Id}"] & \mathcal{N}_1^* \sqcup \mathcal{N}_2^* \arrow[r, "\rho_1\sqcup\rho_2"] & V \setminus j\left(M\right) \arrow[r, "\varphi"] \arrow[d, "F_1|_{V\setminus j(M)}"] & X \arrow[r, "\rho^{-1}"] \arrow[d, "\Theta'|_X"] & \rho^{-1}\left(X\right) \arrow[d, "\mu'"] \\ U_1^* \sqcup U_2^* \arrow[r, "H_1"] & \mathcal{N}_1^* \sqcup \mathcal{N}_2^* \arrow[r, "\overline{\rho}_1\sqcup\overline{\rho}_2"] & V \setminus j\left(M\right) \arrow[r, "\overline{\varphi}"] & \overline{X} \arrow[r, "\overline{\rho}^{-1}"] & \overline{\rho}^{-1}\left(\overline{X}\right) \end{tikzcd}$$

where $X \coloneqq \operatorname{Im}\left(\varphi \circ (\rho_1 \sqcup \rho_2) \circ i\right)$, $\overline{X} \coloneqq \operatorname{Im}\left(\overline{\varphi} \circ (\overline{\rho}_1 \sqcup \overline{\rho}_2) \circ H_1\right)$ and $\mu' \coloneqq \overline{\rho}^{-1} \circ \Theta'|_X \circ \rho$ is defined only on the subset $\rho^{-1}\left(X\right)$ of $\mathcal{N}^*$ and has values in $\overline{\rho}^{-1}\left(\overline{X}\right) \subset \overline{\mathcal{N}}^*$.

Now, by Definition 5.2, the compositions $\rho^{-1} \circ \varphi \circ (\rho_1 \sqcup \rho_2)$ and $\overline{\rho}^{-1} \circ \overline{\varphi} \circ (\overline{\rho}_1 \sqcup \overline{\rho}_2)$ are $\mathbb{R}_{>0}-$equivariant maps. Moreover, we can see from the explicit proof of [Gei08, Theorem 2.6.12], that for $i \in \{1, 2\}$ and all $p \in M \subset U_i$, $d_pH_1 : T_p\left(\mathcal{N}_i\right) \to T_p\left(\mathcal{N}_i\right)$ is of the form

$$\mathrm{Id}_{T_pM} \oplus L_p : T_pM \oplus \mathcal{N}_p \to T_pM \oplus \mathcal{N}_p\ ,$$

where $L : \mathcal{N} \to \mathcal{N}$ is an isomorphism of vector bundles over M. Here, we used the natural splitting $T_p\left(\mathcal{N}_i\right) = T_pM \oplus \mathcal{N}_p$ for $p \in M \subset \mathcal{N}_i$.

By commutativity of the above diagrams, this implies that μ' smoothly extends over H, in such a way that $\mu(H) = \overline{H}$.

Hence, Θ' extends to a diffeomorphism $\Theta\colon\ W \to \overline{W}$ such that $\Theta\left(H\right) = \overline{H}$, which concludes the proof of Lemma 5.21. □

Once showed that we have a natural (and unique, in the sense of Proposition 5.20) confoliation ζ adjusted to the hypersurface H on smooth fiber sum (W, H, φ), we can now define the contact fiber sum in terms of deformations of this confoliation:

Definition 5.22. A *contact fiber sum* on the confoliated fiber sum (W, H, φ, ζ) is a contactization of ζ.

Analogously to Section 5.2, the existence and uniqueness up to isotopy of the contact fiber sum is a direct consequence of Proposition 5.14.

5.4 Weak fillings of contact branched coverings

Let now $\pi : \widehat{W}^{2n+2} \to W^{2n+2}$ be a branched covering of even dimensional manifolds with non-empty boundaries $\widehat{V}^{2n+1} = \partial\widehat{W}$ and $V^{2n+1} = \partial W$.
We also use the same notations as in Section 5.1. In particular, we denote by $\widehat{X}^{2n}$ the upstairs branching set, by X the downstairs branch set, by $M, \widehat{M}$ the boundaries of $X, \widehat{X}$ respectively and by π' the restriction $\pi|_{\widehat{V}} : \widehat{V} \to V$.

Here's a more detailed version of Théorème II.B from Chapter 1:

Theorem 5.23. *Suppose we are in the following situation:*

(a) η is a contact structure on V and $\xi := \eta \cap TM$ is contact on M;

(b) $\widehat{\eta}$ on $\widehat{V}$ is a contact branched covering of (V, η);

(c) ω on W weakly dominates η on V;

(d) X is a symplectic submanifold of (W, ω) and it weakly fills (M, ξ).

Then, $\widehat{W}$ admits a symplectic form $\widehat{\omega}$ that weakly dominates $\widehat{\eta}$ on $\widehat{V}$.

Notice that, because $\pi'|_{\widehat{M}} \colon \widehat{M} \to M$ is a (unbranched) covering map, $\widehat{\xi} := \left(\pi'|_{\widehat{M}}\right)^* \xi = \widehat{\eta} \cap T\widehat{M}$ is a contact structure on $\widehat{M}$.

Proof. Consider the normal bundle of $\widehat{X}$ inside $\widehat{W}$ and see it as a neighborhood $\widehat{\mathcal{U}}$ of $\widehat{X}$. Similarly for a neighborhood $\widehat{\mathcal{M}}$ of $\widehat{M}$ in $\widehat{V}$. In particular, we have a norm function on $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{M}}$, and we can denote by $\widehat{\mathcal{U}}_r, \widehat{\mathcal{M}}_r$ the set of vectors of norm less than r.

Fix now an arbitrary smooth function $f : \widehat{W} \to \mathbb{R}_{\geq 0}$, compactly supported in $\widehat{\mathcal{U}}_1$, depending only on r, non-increasing in it, and equal to 1 on a neighborhood of $\widehat{X}$. We also denote by g its restriction to $\widehat{V} = \partial \widehat{W}$.
We point out that in particular $f'(r) = 0$, hence $g'(r) = 0$, for $r = 0$.

Let now δ be a connection 1-form on the circle bundle $S\widehat{\mathcal{U}}$ given by the vectors of norm 1 in $\widehat{\mathcal{U}}$. Denote also by γ the restriction of δ to the sub-bundle $S\widehat{\mathcal{M}}$ given by the vectors of norm 1 in $\widehat{\mathcal{M}}$; then, γ is in particular a connection form on $S\widehat{\mathcal{M}}$.

The explicit formula in the proof of Lemma 5.10 then shows that, up to isotopy, we can assume the contact branched covering $\widehat{\eta}$ to be the kernel of $\widehat{\alpha}_\epsilon := \pi^*\alpha + \epsilon g(r) r^2 \gamma$, for each ϵ smaller than or equal to a certain constant $\epsilon_0 > 0$.

As far as the symplectic structure on $\widehat{W}$ is concerned, consider the closed 2−form $\widehat{\omega}_\epsilon := \pi^*\omega + \epsilon\, d\left(f(r) r^2 \delta\right)$ on $\widehat{W}$, where $\epsilon > 0$.

Lemma 5.24. *There is $\epsilon_1 > 0$ such that $\widehat{\omega}_\epsilon$ is symplectic for all $0 < \epsilon < \epsilon_1$.*

Proof. We compute

$$\widehat{\omega}_\epsilon = \pi^*\omega + \epsilon\,(2f + rf')\, rdr \wedge \delta + \epsilon f r^2 d\delta \ ,$$

which gives, once fixed a volume form vol on W,

$$\begin{aligned}
\widehat{\omega}_\epsilon^{n+1} &= \left[\pi^*\omega + \epsilon\,(2f + rf')\, rdr \wedge \delta + \epsilon f r^2 d\delta\right]^{n+1} \\
&= \pi^*\omega^{n+1} + (n+1)\,\epsilon\,(2f + rf')\,\pi^*\omega^n \wedge rdr \wedge \delta \\
&\quad + \epsilon r^2 f h \mathrm{vol} \ ,
\end{aligned}$$

where h is a smooth function depending on $p \in \widehat{W}$ and on $\epsilon > 0$.
Using the facts that $\pi^*\omega$ is symplectic away from $\widehat{X}$ and that the restriction of ω to X is symplectic on X, we can now conclude, with arguments analogous to those in the proof of Lemma 5.10, that $\widehat{\omega}_\epsilon^{n+1} > 0$ for ϵ small enough. □

We then want to show that $\widehat{\omega}_\epsilon$ weakly dominates $\widehat{\eta} = \ker(\widehat{\alpha}_\epsilon)$, provided that $\epsilon > 0$ is small enough (and in particular such that $\epsilon < \overline{\epsilon} := \min(\epsilon_0, \epsilon_1)$).
By (the discussion following) Definition 2.17, we need to check that

$$\widehat{\alpha}_\epsilon \wedge \left(\widehat{\omega}_{\epsilon,V} + \tau d\widehat{\alpha}_\epsilon\right)^n > 0\,, \quad \forall \tau \geq 0\ ,$$

where $\widehat{\omega}_{\epsilon,V}$ denotes the pullback of $\widehat{\omega}_\epsilon$ via the inclusion $\widehat{V} \hookrightarrow \widehat{W}$, i.e.

$$\widehat{\omega}_{\epsilon,V} = \pi^*\omega_V + \epsilon d\left(gr^2\gamma\right) = \pi^*\omega_V + \epsilon\left(2g + rg'\right) rdr \wedge \gamma + \epsilon gr^2 d\gamma \ .$$

Once fixed a volume form vol on $\widehat{V}$, using the fact that

$$d\widehat{\alpha}_\epsilon = \pi^*\alpha + \epsilon\left(2g + rg'\right) rdr \wedge \gamma + \epsilon r^2 g d\gamma \ ,$$

we can explicitly compute

$$\begin{aligned}
&\widehat{\alpha}_\epsilon \wedge \left(\widehat{\omega}_V + \tau d\widehat{\alpha}_\epsilon\right)^n \\
&= \left(\pi^*\alpha + \epsilon gr^2\gamma\right) \wedge \left[\pi^*\omega_V + \tau\pi^* d\alpha \right.\\
&\quad \left. + \epsilon\left(1+\tau\right)\left(rg' + 2g\right) rdr \wedge \gamma + \epsilon\left(1+\tau\right) gr^2 d\gamma\right]^n \\
&= \pi^*\left[\alpha \wedge \left(\omega_V + \tau d\alpha\right)^n\right] \\
&\quad + n\epsilon\left(1+\tau\right)\left(rg' + 2g\right)\pi^*\left[\alpha \wedge \left(\omega_V + \tau d\alpha\right)^{n-1}\right] \wedge rdr \wedge \gamma \\
&\quad + \epsilon gr^2 h \mathrm{vol} \ ,
\end{aligned}$$

where h is a smooth function of $\widehat{p} \in \widehat{V}$, ϵ and τ, which is moreover polynomial in ϵ and in τ, with $\deg_\tau h \leq n$.

Denote now by $P_0(\tau)$ and $P_1(\tau)$ the polynomials in τ, with coefficients in the ring of functions $\widehat{V} \to \mathbb{R}$, defined respectively by the identities

$$\begin{aligned}
&P_0(\tau) \cdot \mathrm{vol} = \pi^*\left[\alpha \wedge \left(\omega_V + \tau d\alpha\right)^n\right] \ , \\
&P_1(\tau) \cdot \mathrm{vol} = n\left(1+\tau\right)\left(rg' + 2g\right)\pi^*\left[\alpha \wedge \left(\omega_V + \tau d\alpha\right)^{n-1}\right] \wedge rdr \wedge \gamma \ .
\end{aligned}$$

Similarly, denote by $P_2(\tau, \epsilon)$ the polynomial in τ and ϵ given by $P_2(\tau, \epsilon) = gr^2 h$.

We then have the followings:

Lemma 5.25. *For all $\tau \geq 0$, $P_0(\tau)$ is non-negative everywhere on $\widehat{V}$ and strictly positive away from $\widehat{M}$.*

Proof (Lemma 5.25). This simply follows from the fact that (W, ω) is a weak filling of (V, η) and that $\pi|_{\widehat{V}}$ is a branched covering with (upstairs) branching locus $\widehat{M}$. □

Lemma 5.26. *There are constants $0 < \epsilon_0' < \overline{\epsilon}$ and $r_0 > 0$, such that $P_1(\tau) + P_2(\tau, \epsilon) > 0$ on $\mathcal{M}_{r_0}$ for all $0 \leq \epsilon < \epsilon_0'$ and all $\tau \geq 0$.*

This lemma will be proven after the end of the proof of Theorem 5.23. Notice that we allow $\epsilon = 0$ in the statement of Lemma 5.26.

Now, according to Lemmas 5.25 and 5.26, we have that $\widehat{\alpha}_\epsilon \wedge \left(\widehat{\omega}_V + \tau d\widehat{\alpha}_\epsilon\right)^n$ is a positive volume form on $\mathcal{M}_{r_0}$ for all $0 < \epsilon < \epsilon_0'$ and all $\tau \geq 0$. Remark that here ϵ has to be strictly positive.

At this point, we have the following result, whose proof is also postponed:

Lemma 5.27. *There is $0 < \epsilon_1' < \epsilon_0'$ such that $P_0(\tau) + \epsilon\left[P_1\left(\tau\right) + P_2\left(\tau, \epsilon\right)\right] > 0$ on the complement of $\mathcal{M}_{r_0/2}$, for all $0 \leq \epsilon < \epsilon_1'$ and all $\tau \geq 0$.*

This concludes the proof of Theorem 5.23. □

We now give a proof of Lemmas 5.26 and 5.27 above. They are corollaries of the following:

Lemma 5.28. *Consider a smooth manifold S and a continuous function $p : S \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that, for each $s \in S$, $p_s : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by $p_s(\tau) := p(s, \tau)$ is polynomial in τ. Suppose there is $s_0 \in S$ and a neighborhood U of s_0 such that for all $s \in U$ the followings are satisfied:*

1. $\deg_\tau (p_{s_0}) \geq \deg_\tau (p_s)$;
2. *the leading coefficient of p_{s_0} is positive.*

Then, there is a neighborhood O of s_0 contained in U such that, for all $s \in O$, the minimum m_s of p_s exists and, moreover, depends continuously on s.
In particular, if moreover $m_{s_0} > 0$, then $m_s > 0$ for s sufficiently near to s_0.

Proof (Lemma 5.28). Hypothesis 1 tells us that p_s is of fixed degree $n := \deg_\tau (p_{s_0})$ for all $s \in U$. In particular, the leading coefficient a_n^s of p_s converges to the positive leading coefficient $a_n^{s_0}$ of p_{s_0} for $s \to s_0$. Hence, there is a neighborhood $O \subset U$ of s_0 such that, for all $s \in O$, $a_n^s > {}^{a_n^{s_0}}/_2 > 0$. Then, for $s \in O$, $p_s(\tau) \to +\infty$ for $\tau \to +\infty$. Hence, by the continuity of p_s in τ, p_s admits a global minimum m_s for $s \in O$.

Notice that, up to shrinking O, we can suppose moreover that it has compact closure.

For the dependence of the global minimum m_s on the variable s, we recall the following fact: if X, Y are topological spaces with Y compact, $f \colon X \times Y \to \mathbb{R}$ is continuous and $f_x(.) := f(x, .) \colon Y \to \mathbb{R}$ admits global minimum m_x for all $x \in X$, then m is continuous in $x \in X$. Indeed, for each sequence x_n in X converging to x, there is a subsequence x_{n_k} such that $m(x_{n_k}) \to m(x)$, because, if $y_n \in Y$ denotes a sequence such that $m(x_n) = f(x_n, y_n)$, by compactness of Y there is a subsequence y_{n_k} converging to a certain $y \in Y$, hence $m(x_{n_k}) = f(x_{n_k}, y_{n_k}) \xrightarrow{C^0} f(x, y)$, by continuity of f, and $f(x, y)$ is exactly the global minimum $m(x)$ of f_x.

Going back to the case of Lemma 5.28, in order to apply this discussion we need to show that, for each $s \in O$, the τ_s realizing the identity $p(s, \tau_s) = m_s$ can be chosen in a compact subset K of $\mathbb{R}_{\geq 0}$ which is independent of $s \in O$.
To prove this, we write $p_s(\tau) = a_n^s \tau^n + \sum_{i<n} a_i^s \tau^i$ and we compute, for $s \in O$,

$$
\begin{aligned}
p_s(\tau) &\overset{(a)}{\geq} a_n^s \tau^n - M \sum_{i=1}^{n-1} \tau^i \\
&\overset{(b)}{\geq} \frac{a_n^{s_0}}{2} \tau^n - M \sum_{i=1}^{n-1} \tau^i \\
&\overset{(c)}{\geq} 2a_0^{s_0} \overset{(d)}{\geq} a_0^s \ ,
\end{aligned}
$$

where $M := \max_{s \in O, i<n} (|a_i^s|)$ (recall that O is compact). Here, (a) comes from the triangular inequality and the fact that $a_n^s > 0$ for $s \in O$, (b) comes from the fact that $a_n^s > {}^{a_n^{s_0}}/_2$ for $s \in O$, (c) is true for all $\tau \in [\overline{\tau}, +\infty)$, for a certain $\overline{\tau} > 0$ depending only on s_0, and (d) is true up to shrinking O, because a_0^s converges to $a_0^{s_0}$ for $s \to s_0$.
The above inequality tells that $p_s(\tau) > a_0^s = p_s(0)$ for all $s \in O$ and $\tau \in [\overline{\tau}, +\infty)$. In other words, a τ_s realizing $m_s = p_s(\tau_s)$ has to be in the compact subset $K := [0, \overline{\tau}]$, for all $s \in O$. This is exactly what we needed to prove in order to show that m_s is continuous in $s \in O$, according to the above discussion.

The last part of the statement is now obvious. □

Proof (Lemma 5.26). We would like to use Lemma 5.28, with $S := \widehat{V} \times [0, \overline{\epsilon})$ and $P := P_1 + P_2 : S \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, i.e. $P_{q,\epsilon}(\tau)$ is given by $[P_1(\tau) + P_2(\tau, \epsilon)](q)$ for $(q, \epsilon) \in S =$

$\widehat{V} \times [0, \bar{\epsilon})$. Notice that we allow $\epsilon = 0$ here.
Consider the compact set $K := \widehat{M} \times \{0\}$ in S. If $(q, 0) \in K$, then

$$\begin{aligned} P_{(q,0)} \cdot \mathrm{vol}_{(q,0)} &= \left[P_1(\tau)_q + P_2(\tau, 0)_q \right] \cdot \mathrm{vol}_{(q,0)} \\ &= P_1(\tau)_q \cdot \mathrm{vol}_{(q,0)} \\ &= 2n(1+\tau) \left\{ \pi^* \left[\alpha \wedge (\omega_V + \tau d\alpha)^{n-1} \right] \wedge r dr \wedge \gamma \right\}_q , \end{aligned}$$

which is positive because the restriction of ω to X weakly dominates ξ on $M = \partial X$. In particular, for $(q, 0) \in K$, $P_{(q,0)}$ has positive leading coefficient and $m_{(q,0)} > 0$.

Moreover, for each $(q, 0) \in K$, $\deg_\tau \left(P_{(q,0)} \right) = n \geq \deg_\tau (P_s)$ for all $s \in S = \widehat{V} \times [0, \bar{\epsilon})$.

We are then in the hypothesis of Lemma 5.28, which then tells us, by compactness of K, that there is a neighborhood $\mathcal{U}$ of K in S such that m_s exists and is positive for all $s \in \mathcal{U}$. This is exactly what we wanted because $\mathcal{U}$ contains an open set of the form $\{r < r_0, \epsilon < \epsilon_0'\} \subset S = \widehat{V} \times [0, \bar{\epsilon})$. □

Proof (Lemma 5.27). We use again Lemma 5.28. Here, $S := \mathcal{M}^c_{r_0/2} \times [0, \epsilon_0')$, where $\mathcal{M}^c_{r_0/2}$ is the complement of $\mathcal{M}_{r_0/2}$ in $\widehat{V}$ and r_0, ϵ_0' are given by Lemma 5.26. Also, $P : S \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is here defined as $P_{(p,\epsilon)}(\tau) = P_0(\tau)|_p + \epsilon \left[P_1(\tau) + P_2(\tau, \epsilon) \right](p)$ for $(p, \epsilon) \in S$. Notice that once again we allow $\epsilon = 0$.

Then, if $K := \mathcal{M}^c_{r_0/2} \times \{0\}$, $P_{q,0} = P_1(\tau)_q$ for all $(q, 0) \in K$, hence it is positive by Lemma 5.25. In particular, $P_{q,0}$ has positive leading coefficient and positive minimum $m_{(q,0)}$ for all $(q, 0) \in K$. Moreover, $\deg_\tau \left(P_{(q,0)} \right) = n \geq \deg_\tau (P_{p,\epsilon})$, for all $q, p \in \mathcal{M}^c_{r_0/2}$ and $\epsilon \in [0, \epsilon_0')$.

We are then again in the hypothesis of Lemma 5.28, so that, by compactness of K, we can conclude that $P_{(p,\epsilon)}$ admits minimum $m_{(p,\epsilon)}$, and that it is positive in a neighborhood of K, which is exactly what we wanted. □

Chapter 6

Open books and contact vector fields

We describe here a reinterpretation of adapted open book decompositions in terms of contact vector fields:

Théorème II.E. *Sur une variété de contact (M^{2n-1}, ξ), chaque paire de champs de vecteurs de contact X, Y, telle que $[X, Y]$ est partout transverse à ξ, donne un livre ouvert explicite de M qui porte ξ. Vice versa, un livre ouvert qui porte ξ donne une paire X, Y comme ci dessus.*

A part of this result (i.e. how to go from adapted open books to contact vector fields) has been stated by Giroux during the Yashafest in June 2007 and the AIM workshop in May 2012 (see [Gir12, Claim on page 19]).

The whole chapter is devoted to the proof of Théorème II.E, already stated in Section 1.2. More precisely, in Section 6.1 we describe how to recover the data of an open book decomposition adapted to a certain contact structure ξ from the data of two contact vector fields with Lie bracket everywhere transverse to ξ. In Section 6.2, we show that it is also possible to recover such a couple of contact vector fields from an adapted open book, as claimed in [Gir12].

6.1 From contact vector fields to open books

Proposition 6.1. *Let (M^{2n-1}, ξ) be a closed contact manifold. Suppose X, Y are two contact vector fields with Lie bracket $[X, Y]$ everywhere negatively transverse to ξ. Then, if we denote $X_\theta := \cos\theta \cdot X + \sin\theta \cdot Y$ and $Y_\theta := X_{\theta+\pi/2}$ for $\theta \in \mathbb{S}^1$, we have the following:*

(a) The set $\Sigma_\theta := \{X_\theta \in \xi\}$ is a non-empty regular hypersurface, which is moreover $\xi-$convex.

(b) For $\theta \neq \theta' \mod \pi$, Σ_θ and $\Sigma_{\theta'}$ intersect transversely along a non-empty contact submanifold K of M (independent of the couple (θ, θ')).

(c) For each $\theta \in \mathbb{S}^1$, consider the set

$$F_\theta := \{\, p \in \Sigma_\theta \,|\, Y_\theta(p) \text{ is positively transverse to } \xi_p \,\} \ ,$$

and define $\varphi\colon M \setminus K \to \mathbb{S}^1$ as $\varphi(p) := \theta$ if $p \in F_{-\theta}$. Then, (K, φ) is an open book decomposition of M and is adapted to ξ.

The rest of Section 6.1 is devoted to the proof of the above result, which is a more detailed version of the first part of Théorème II.E. To improve the readability, each lemma in this section will be proved right after the conclusion of the part of the proof in which it is used.

We fix for convenience a contact form α for ξ and we denote by $f, g : M \to \mathbb{R}$ the smooth functions given by $\mathcal{L}_X \alpha = f\alpha$ and $\mathcal{L}_Y \alpha = g\alpha$ respectively: these functions exist because X and Y are contact vector fields. Of course, for the proof of point (c) we will need to change this α conveniently.

We also point out the following:

Remark 6.2. *For all $\theta \in \mathbb{S}^1$, X_θ, Y_θ are contact vector fields and satisfy the identity $[X_\theta, Y_\theta] = [X, Y]$.*

Proof (Remark 6.2). This follows from the fact that $\mathcal{L}_{X_\theta}\alpha = \cos\theta \cdot \mathcal{L}_X\alpha + \sin\theta \cdot \mathcal{L}_Y\alpha = [f\cos\theta + g\sin\theta]\,\alpha$ and from the fact that the Lie bracket is anti-symmetric and bilinear. □

We may now proceed to prove Proposition 6.1.

Proof of point (a). This proof consists of the following two lemmas.

Lemma 6.3. *$\alpha(X_\theta)$ is somewhere zero.*

Lemma 6.4. *$d(\alpha(X_\theta))(Y_\theta) = -\alpha([X, Y])$ along Σ_θ.*

The first one means exactly that $\Sigma_\theta = \{\alpha(X_\theta) = 0\}$ is non-empty.

The second one tells that $\alpha(X_\theta) : M \to \mathbb{R}$ is transverse to $\{0\} \subset \mathbb{R}$, hence Σ_θ is a smooth hypersurface, and that, more precisely, the contact vector field Y_θ is transverse to Σ_θ, i.e. the latter is $\xi-$convex. □

We now prove Lemmas 6.3 and 6.4.

Proof (Lemma 6.3). Suppose by contradiction this is not the case, i.e. $\alpha(X_\theta) > 0$ without loss of generality. If we define $\beta := \frac{1}{\alpha(X_\theta)} \cdot \alpha$, then $X_\theta = R_\beta$.

By Remark 6.2, we have $\beta([X_\theta, Y_\theta]) = \beta([X, Y]) < 0$. On the other hand, we also have $[X_\theta, Y_\theta] = [R_\beta, Y_\theta]$, so that

$$\begin{aligned}
\beta([X_\theta, Y_\theta]) &= \beta([R_\beta, Y_\theta]) \\
&\overset{(i)}{=} [-d\beta(R_\beta, Y_\theta) + d(\beta(Y_\theta))(R_\beta) - d(\beta(R_\beta))(Y_\theta)] \\
&\overset{(ii)}{=} d(\beta(Y_\theta))(R_\beta) \ .
\end{aligned}$$

Here, for the equality (i) we used the fact that $\beta([R_\beta, Y_\theta]) = -d\beta(R_\beta, Y_\theta) + d(\beta(Y_\theta))(R_\beta) - d(\beta(R_\beta))(Y_\theta)$ by the formula for the exterior derivative of differential forms, and for the equality (ii), we used that $d\beta(R_\beta, .) = 0$ and $\beta(R_\beta) = 1$.

Now, $\beta(Y_\theta)$ is a function defined on a closed manifold, hence it has at least one critical point. This contradicts the fact that $\beta([X_\theta, Y_\theta])$ is everywhere negative. □

Proof (Lemma 6.4). Using the formula for the exterior derivative, we compute

$$d\alpha(X_\theta, Y_\theta) = d(\alpha(Y_\theta))(X_\theta) - d(\alpha(X_\theta))(Y_\theta) - \alpha([X_\theta, Y_\theta]) \ . \tag{6.1}$$

Also, by Remark 6.2 there are $f_\theta, g_\theta \colon M \to \mathbb{R}$ such that

$$f_\theta\,\alpha = \mathcal{L}_{X_\theta}\alpha = d\iota_{X_\theta}\alpha + \iota_{X_\theta}d\alpha \quad \text{and} \quad g_\theta\,\alpha = \mathcal{L}_{Y_\theta}\alpha = d\iota_{Y_\theta}\alpha + \iota_{Y_\theta}d\alpha \ . \tag{6.2}$$

Now, evaluating these last two equations respectively on Y_θ and X_θ gives

$$\begin{aligned} d\left(\alpha\left(X_\theta\right)\right)\left(Y_\theta\right) &= f_\theta\, \alpha\left(Y_\theta\right) - d\alpha\left(X_\theta, Y_\theta\right)\ , \\ d\left(\alpha\left(Y_\theta\right)\right)\left(X_\theta\right) &= g_\theta\, \alpha\left(X_\theta\right) + d\alpha\left(X_\theta, Y_\theta\right)\ . \end{aligned} \tag{6.3}$$

Substituting inside Equation (6.1), we get $d\alpha\left(X_\theta, Y_\theta\right) = g_\theta\, \alpha\left(X_\theta\right) + d\alpha\left(X_\theta, Y_\theta\right) - f_\theta\, \alpha\left(Y_\theta\right) + d\alpha\left(X_\theta, Y_\theta\right) - \alpha\left(\left[X_\theta, Y_\theta\right]\right)$, which, using $\alpha\left(X_\theta\right) = 0$ (we are interested at points $p \in \Sigma_\theta$), gives $-d\alpha\left(X_\theta, Y_\theta\right) + f_\theta\, \alpha\left(Y_\theta\right) = -\alpha\left(\left[X_\theta, Y_\theta\right]\right)$. Replacing this identity inside Equation (6.3) gives $d\left(\alpha\left(X_\theta\right)\right)\left(Y_\theta\right) = -\alpha\left(\left[X_\theta, Y_\theta\right]\right)$.

As remarked earlier, $\left[X_\theta, Y_\theta\right] = [X, Y]$, so that $d\left(\alpha\left(X_\theta\right)\right)\left(Y_\theta\right) = -\alpha\left(\left[X_\theta, Y_\theta\right]\right) = -\alpha\left([X, Y]\right)$. □

We point out a direct consequence of Lemma 6.4 and another lemma, which we will both need later:

Corollary 6.5. *$d\left(\alpha\left(Y_\theta\right)\right)\left(X_\theta\right) = \alpha\left([X, Y]\right)$ on all $\Sigma_{\theta+\pi/2} = \{\alpha(Y_\theta) = 0\}$.*
In particular, along $\Sigma_\theta \cap \Sigma_{\theta+\pi/2}$ (which we will show below to be independent of θ and denote by K), we have both $d\left(\alpha\left(X_\theta\right)\right)\left(Y_\theta\right) = -\alpha\left([X, Y]\right)$ and $d\left(\alpha\left(Y_\theta\right)\right)\left(X_\theta\right) = \alpha\left([X, Y]\right)$, which implies also $d\alpha(X_\theta, Y_\theta) = \alpha\left([X, Y]\right) < 0$.

Lemma 6.6. *X_θ is tangent to Σ_θ. Moreover, it is transverse to $\partial F_\theta = \Sigma_\theta \cap \Sigma_{\theta+\pi/2}$ and points outwards from F_θ.*

Proof (Lemma 6.6). X_θ is tangent to Σ_θ because $d\left(\alpha\left(X_\theta\right)\right)\left(X_\theta\right) = 0$; this last identity comes from the evaluation of the left identity in Equation (6.2) on X_θ, at points $p \in \Sigma_\theta$. The second part of the statement follows from the fact that $\alpha\left(Y_\theta\right) = 0$ along $\partial F_\theta = \Sigma_\theta \cap \Sigma_{\theta+\pi/2}$ (by definition of $\Sigma_{\theta+\pi/2}$), and that $d\left(\alpha\left(Y_\theta\right)\right)\left(X_\theta\right) < 0$ along ∂F_θ by Corollary 6.5. Indeed, this means that X_θ points in the region where $\alpha\left(Y_\theta\right) < 0$ along ∂F_θ, being always tangent to Σ_θ, i.e., by definition of F_θ, that it points outwards from F_θ along its boundary. □

Proof of point (b). We start by proving that the intersection $\Sigma_\theta \cap \Sigma'_\theta$ is independent of θ, θ', provided that $\theta \neq \theta' \mod \pi$; for this, we will prove that this intersection coincides with $\nu^{-1}(0)$, where $\nu := \left(\alpha\left(X\right), \alpha\left(Y\right)\right) : \Sigma_\theta \cap \Sigma_{\theta'} \to \mathbb{R}^2$.
By definition, $\Sigma_\theta \cap \Sigma_{\theta'} = \{\alpha\left(X_\theta\right) = 0, \alpha\left(X_{\theta'}\right) = 0\}$. Now, the equations $\alpha\left(X_\theta\right) = \cos\left(\theta\right)\, \alpha\left(X\right) + \sin\left(\theta\right)\, \alpha\left(Y\right)$ and $\alpha\left(X_{\theta'}\right) = \cos\left(\theta'\right)\, \alpha\left(X\right) + \sin\left(\theta'\right)\, \alpha\left(Y\right)$ immediately imply that $\nu^{-1}(0) \subset \Sigma_\theta \cap \Sigma_{\theta'}$. On the other hand, if $p \in \Sigma_\theta \cap \Sigma_{\theta'}$, then, evaluating the same two equations at p, we deduce that $\nu(p)$ has to be proportional to $\left(-\sin\left(\theta\right), \cos\left(\theta\right)\right)$ and to $\left(-\sin\left(\theta'\right), \cos\left(\theta'\right)\right)$; because $\theta \neq \theta' \mod \pi$, this means that $\nu(p) = 0$. In other words, $\Sigma_\theta \cap \Sigma_{\theta'} \subset \nu^{-1}(0)$ too.
We can then denote $\Sigma_\theta \cap \Sigma_{\theta'}$ by K in the following.

We now prove that $K = \Sigma_\theta \cap \Sigma_{\theta+\pi/2}$ is non-empty. Recall that $\Sigma_{\theta+\pi/2} = \{\alpha(Y_\theta) = 0\}$ and that Y_θ is a contact vector field transverse to Σ_θ; in particular, $K = \{\alpha(Y_\theta) = 0\} \cap \Sigma_\theta \subset \Sigma_\theta$ is a dividing set for the characteristic foliation $\Sigma_\theta(\xi)$. Now, according to [Gir91], dividing sets are non-empty 2-codimensional contact submanifolds. □

Proof of point (c). Consider $\phi : M \to \mathbb{R}^2$ given by $\phi(p) = \left(\alpha\left(X\right)_p, -\alpha\left(Y\right)_p\right)$. Let also $\varphi\colon\ M \setminus \phi^{-1}(0) \to \mathbb{S}^1$ be defined by $\varphi := \phi/\|\phi\|$.

Lemma 6.7. *ϕ is transverse to the origin of $\mathbb{R}^2$ and $\phi^{-1}(0) = K$ as subsets of M. Also, φ is a submersion and $\varphi^{-1}(\theta) = F_{-\theta-\pi/2}$ as subsets of M. Moreover, $\varphi^{-1}(\theta)$ is cooriented by the vector $Y_{-\theta-\pi/2}$ and $\phi^{-1}(0)$, naturally oriented as boundary of $\varphi^{-1}(\theta)$ by definition of φ, is also cooriented by the ordered couple of vectors (Y, X).*

The proof of this lemma is postponed.

We now want to show that the couple (K, φ), which is an open book decomposition of M according to Lemma 6.7, is moreover adapted to ξ. Notice that this is enough in order to prove point (c) of Proposition 6.1, because the φ in point (c) is just obtained from the φ of Lemma 6.7 by post-composing with the rotation $\mathbb{S}^1 \to \mathbb{S}^1$ of angle $-\frac{\pi}{2}$: they hence have the same set of pages.

Consider on K, F_θ the orientations such that $\phi^{-1}(0) = K$, $\varphi^{-1}(\theta) = F_{-\theta-\pi/2}$ as oriented manifolds. To show that (K, φ) is adapted to ξ, we then need to verify that $\xi \cap TK$ is a positive contact structure on K and that there is a contact form defining ξ whose differential is a positive symplectic form on each F_θ. To prove this, we use the following result, whose proof is postponed:

Lemma 6.8 (Giroux)**.** *Let (M^{2n-1}, ξ) be a contact manifold. Suppose there are an open book decomposition (K, φ) of M (in particular, K is oriented as boundary of $\varphi^{-1}(\theta)$), a tubular neighborhood $\mathcal{N} = K \times D^2$ of K (here D^2 is the open unit disk in $\mathbb{R}^2$) and a contact form α defining ξ such that:*

(i) φ restricted to $\mathcal{N} \setminus K$ is the angular coordinate of the projection on the second factor $\mathcal{N} = K \times D^2 \to D^2$;

(ii) ξ induces a positive contact structure on each submanifold $K_z := K \times \{z\}$ of $\mathcal{N}$ (notice each K_z is oriented because K is);

(iii) $d\alpha$ induces a positive symplectic form on each fiber of $\varphi_{|M\setminus\mathcal{N}}$.

Then, the open book decomposition (K, φ) is compatible with the contact structure ξ.

Thanks to the above result, the fact that (K, φ) supports ξ follows from the following two lemmas:

Lemma 6.9. *Let Ψ be the map defined by*

$$\begin{aligned} \Psi : K \times D^2_\delta &\to M \\ (p, x, y) &\mapsto \psi^1_{y \cdot X + x \cdot Y}(p) \end{aligned},$$

where ψ^1_Z denotes the time-1 flow of the vector field Z on M and D^2_δ is the 2-disk of radius δ in $\mathbb{R}^2$. Then for $\delta > 0$ sufficiently small, we have the followings:

(i) Ψ is a diffeomorphism onto its image;

(ii) if we denote $\mathcal{N} := \Psi(K \times D^2_\delta)$, then we have the following commutative diagram, where ν is the composition of the projection on $D^2_\delta \setminus \{0\}$ and the natural angle function $D^2_\delta \setminus \{0\} \to \mathbb{S}^1$:

$$\begin{tikzcd} K \times \left(D^2_\delta \setminus \{0\}\right) \arrow[rr, "\Psi"] \arrow[dr, "\nu"'] & & M \setminus K \arrow[dl, "\varphi"] \\ & \mathbb{S}^1 & \end{tikzcd}$$

(iii) each $K_z := \Psi(K \times \{z\})$ is a positive contact submanifold of (M, ξ).

Lemma 6.10. *Let $\mathcal{N}$ be the neighborhood of K given by Lemma 6.9. Then there is a contact form α defining ξ such that:*

(i) α induces a positive contact structure on each submanifold K_z of $\mathcal{N}$;

(ii) $d\alpha$ is a positive symplectic form on the fibers of $\varphi|_{M\setminus\mathcal{N}}$.

This concludes the proof of point (c). □

We now proceed to prove the lemmas used in the above proof.

Proof (Lemma 6.7). Clearly, $\phi^{-1}(0) = \Sigma_0 \cap \Sigma_{\pi/2} = K$ as subsets of M.

Moreover, we can compute $d\phi(X) = d\left(\alpha\left(X\right)\right)(X)\partial_x - d\left(\alpha\left(Y\right)\right)(X)\partial_y$ along K. Now, by Lemma 6.6 and Corollary 6.5, $d\left(\alpha\left(X\right)\right)(X) = 0$ and $d\left(\alpha\left(Y\right)\right)(X) = \alpha\left([X,Y]\right)$ along K, hence $d\phi(X) = -\alpha\left([X,Y]\right)\partial_y$. Similarly, we can compute $d\phi(Y) = -\alpha\left([X,Y]\right)\partial_x$ along K. In other words, ϕ is transverse to the origin of $\mathbb{R}^2$ and the oriented couple (Y,X) gives the positive coorientation of $\phi^{-1}(0)$, hence (Y_θ, X_θ) too.

To study $\varphi^{-1}(\theta)$, we do the following. Suppose $\varphi(p) = \theta$ and write $\phi(p) \in \mathbb{R}^2$ in polar coordinates as $\|\phi(p)\| \cdot (\cos\theta, \sin\theta)$. Then, we can compute

$$\begin{aligned} \alpha\left(X_{-\theta-\pi/2}\right) &= \alpha\left(X\right)\sin\theta + \alpha\left(Y\right)\cos\theta \\ &= \phi_1(p)\sin\theta - \phi_2\cos\theta \\ &= \|\phi(p)\| \cdot (\cos\theta\sin\theta - \sin\theta\cos\theta) \\ &= 0\ , \end{aligned}$$

i.e. we have that $p \in \Sigma_{-\theta-\pi/2}$.
Hence, to show that $p \in F_{-\theta-\pi/2}$, we need to check that $Y_{-\theta-\pi/2}$ is positively transverse to ξ at p, i.e. that $\alpha\left(Y_{-\theta-\pi/2}\right)|_p > 0$. This follows from the following computation:

$$\begin{aligned} \alpha\left(Y_{-\theta-\pi/2}\right)|_p &= \alpha\left(X\right)|_p\cos\theta\ - \alpha\left(Y\right)|_p\sin\theta\ =\ \phi_1(p)\cos\theta + \phi_2\sin\theta \\ &= \|\phi\|\,(p)\left(\cos^2\theta + \sin^2\theta\right)\ =\ \|\phi\|\,(p) > 0\ . \end{aligned}$$

We now check that $\varphi^{-1}(\theta)$ is positively cooriented by $Y_{-\theta-\pi/2}$. For this, we need to check that $d\varphi\left(Y_{-\theta-\pi/2}\right)|_p$ is positive. We can compute

$$\begin{aligned} \|\phi\|\,d\varphi\left(Y_{-\theta-\pi/2}\right)|_p &= \left(\cos\theta\,d\phi_2 - \sin\theta\,d\phi_1\right)\left(Y_{-\theta-\pi/2}\right)|_p \\ &= \left[-\cos\theta\,d\left(\alpha\left(Y\right)\right) - \sin\theta\,d\left(\alpha\left(X\right)\right)\right]\left(Y_{-\theta-\pi/2}\right)|_p \\ &= d\left(\alpha\left(X_{-\theta-\pi/2}\right)\right)\left(Y_{-\theta-\pi/2}\right)|_p \\ &\overset{(*)}{=} -\alpha\left([X,Y]\right)|_p > 0\ , \end{aligned}$$

where $(*)$ comes from Lemma 6.4. This concludes the proof of Lemma 6.7. □

Proof (Lemma 6.8). Let α be a contact form for ξ as in the statement. We notice that Hypothesis (iii) actually means that there is a very small $\epsilon > 0$ such that $d\alpha$ is a symplectic form on each fiber of the restriction of φ to $M \setminus \overline{K \times D^2_{1-\epsilon}}$, where $D^2_{1-\epsilon}$ is the disk of radius $1-\epsilon$ in $\mathbb{R}^2$.

The aim is to find a function $f\colon M \to \mathbb{R}_{>0}$ such that $f\alpha$ satisfies the conditions of being compatible with the open book decomposition (K,φ). In other words, we want that $f\alpha$ induces a positive contact form on K (which is trivially satisfied because it induces the same contact structure as α) and such that $d\left(f\alpha\right)$ is a positive volume form on the pages. We search such a function f of the following form: f is a smooth function that depends only on the radius coordinate r on D^2 inside $\mathcal{N}$, non$-$increasing in r, which is equal to 1 on $M \setminus K \times D^2_{1-\epsilon/2}$ and equal to $1 + e^{-kr^2}$ on $K \times \overline{D^2_{1-\epsilon}}$, where $k > 0$ is a

constant yet to determine.
We can then compute

$$\begin{aligned} d\varphi \wedge d\,(f\alpha)^{n-1} &= d\varphi \wedge (\, df \wedge \alpha \,+\, fd\alpha\,)^{n-1} \\ &= f^{n-1} d\varphi \wedge d\alpha^{n-1} \,+\, (n-1) f^{n-2} d\varphi \wedge \frac{\partial f}{\partial r} dr \wedge \alpha \wedge d\alpha^{n-2} \\ &= f^{n-2} \left[f d\varphi \wedge d\alpha^{n-1} - (n-1) \frac{\partial f}{\partial r} dr \wedge d\varphi \wedge \alpha \wedge d\alpha^{n-2} \right] \,. \end{aligned}$$

Now, on $M \setminus K \times D^2_{1-\epsilon/2}$ we have that $f\alpha = \alpha$, hence $d\varphi \wedge d(f\alpha)^{n-1} > 0$ as wanted. We then need to control the situation in $K \times \overline{D^2_{1-\epsilon/2}}$.
Let's start by analyzing what happens on $K \times \overline{D^2_{1-\epsilon}}$. Here, $\frac{\partial f}{\partial r} = -2kre^{-kr^2}$, so that

$$\begin{aligned} f d\varphi \wedge d\alpha^{n-1} - (n-1)\frac{\partial f}{\partial r} dr \wedge d\varphi \wedge \alpha \wedge d\alpha^{n-2} = \\ = e^{-kr^2}\left[d\varphi \wedge d\alpha^{n-1} + 2\,(n-1)\,krdr \wedge d\varphi \wedge \alpha \wedge d\alpha^{n-2}\right] \,. \end{aligned}$$

By Hypothesis (ii), the form $rdr \wedge d\varphi \wedge \alpha \wedge d\alpha^{n-2}$ is strictly positive on $\mathcal{N}$, hence on $K \times \overline{D^2_{1-\epsilon/2}}$, and $d\varphi \wedge d\alpha^{n-1}$ is bounded above in norm, even if we don't know its exact sign. This means that for $k > 0$ big enough, the second form will dominate the first, i.e. their sum will still be positive.

It then remains to study the sign on the open set $K \times \left(D^2_{1-\epsilon/2} \setminus \overline{D^2_{1-\epsilon}} \right)$. Here, the situation is easy because $d\varphi \wedge d\alpha^{n-1}$ is strictly positive and $-\frac{\partial f}{\partial r} dr \wedge d\varphi \wedge \alpha \wedge d\alpha^{n-2}$ is non$-$negative (remember f is a non$-$increasing function of r in this set), so their sum is also strictly positive. $\square$

Proof (Lemma 6.9). We start with point (i). We can explicitly evaluate the differential $d\Psi$ at points of the form $(p, 0, 0)$. On $K \times \{0\}$, we simply have that $d\Psi(\partial_x) = Y$, $d\Psi(\partial_y) = X$ and that $d\Psi(V) = V$ for all vector fields V which are tangent to $K \times \{0\}$. This shows that Ψ is a local diffeomorphism at each point $(p, 0, 0)$. Hence, by compactness, Ψ is also a diffeomophism from $K \times D^2_\delta$ onto its image, provided δ is small enough.

We now prove point (ii). For $\theta \in \mathbb{S}^1$, consider the function $H_\theta : K \times [0, \delta) \to M$ defined as $H_\theta(p, r) = \Psi(p, r\cos\theta, r\sin\theta)$. We then have to show that $\varphi(H_\theta(p, r)) = \theta$. Noticing that $Y_{-\theta} = \sin\theta \cdot X + \cos\theta \cdot Y$, we can rewrite more explicitly $H_\theta(p, r) = \psi^r_{Y_{-\theta}}(p)$, i.e. $H_\theta(., r)$ is the flow of $Y_{-\theta}$ at time r. By Lemma 6.6, $Y_{-\theta} = -X_{-\theta-\pi/2}$ is tangent to $\Sigma_{-\theta-\pi/2}$ and entering in $F_{-\theta-\pi/2}$; in particular, for $r > 0$ we have $\psi^r_{Y_{-\theta}}(p) \in F_{-\theta-\pi/2}$. Now, by Lemma 6.7, $\varphi^{-1}(\theta) = F_{-\theta-\frac{\pi}{2}}$, which implies $\varphi(H_\theta(p, r)) = \theta$, as wanted.

We can now finish with point (iii). Because the contact condition is open, up to shrinking δ, it is enough to prove that $K_0 = \Psi(K \times \{0\})$ is a positive contact submanifold. This actually follows from general results from [Gir91]: indeed, X_θ defines the characteristic foliation of Σ_θ, and K is transverse to it. For completeness' sake, we prove it here also with an explicit computation.
We showed in point (b) of Proposition 6.1 that Y is transverse to K. Due to the symmetry of the situation, the same is true for the contact vector field X. If we consider an arbitrary point $p \in K$ and denote $(K_i)_{i=1,\ldots,2n-3}$ a local base of TK in a neighborhood $\mathcal{N}$ of p in K, we then know that

$$\left[\iota_Y \iota_X \left(\alpha \wedge d\alpha^{n-1}\right)\right](K_1, \ldots, K_{2n-3}) = \alpha \wedge d\alpha^{n-1}\,(X, Y, K_1, \ldots, K_{2n-3}) \neq 0 \,,$$

because $(X, Y, K_1, \dots, K_{2n-3})$ is a basis for TM over $\mathcal{N}$. We can also compute

$$\begin{aligned}\iota_Y \iota_X \left(\alpha \wedge d\alpha^{n-1}\right) &\overset{(i)}{=} -(n-1)\, \iota_Y \left[\alpha \wedge (\iota_X d\alpha) \wedge d\alpha^{n-2}\right] \\ &\overset{(ii)}{=} (n-1)\, d\alpha\,(X, Y)\, \alpha \wedge d\alpha^{n-2} + \\ &\quad - (n-1)(n-2)\alpha \wedge d\alpha\,(X, .) \wedge d\alpha\,(Y\,, .) \wedge d\alpha^{n-3}\end{aligned} \tag{6.4}$$

where equalities (i) and (ii) come from the graded Leibniz rule for the interior product (i.e. the formula $\iota_Z\,(\mu \wedge \nu) = \iota_Z \mu \wedge \nu + (-1)^{\deg \mu} \mu \wedge \iota_Z \nu$ for all differential forms μ, ν and vector fields Z) and from the fact that $\alpha\,(X) = 0$ and $\alpha\,(Y) = 0$ along K.
Now, $d\alpha\,(X, Y) = \alpha\,([X, Y]) \neq 0$ along K by Corollary 6.5, so if we manage to prove that the last term in the second line of Equation (6.4) is zero when evaluated on $(K_i)_i$, we will get that $\alpha \wedge d\alpha^{n-2}\,(K_1, \dots, K_{2n-3}) \neq 0$ too, which is exactly what we wanted.
From Equation (6.2), with $\theta = 0$, we deduce that, for a certain $\mu \in \Omega^1(M)$,

$$d\alpha\,(\,X\,, .\,) \wedge d\alpha\,(\,Y\,, .\,) \;=\; d\,(\alpha\,(X)) \wedge d\,(\alpha\,(Y)) + \alpha \wedge \mu\;.$$

Replacing this in the last summand of the second right hand side of Equation (6.4), we get

$$\alpha \wedge d\alpha\,(\,X\,, .\,) \wedge d\alpha\,(\,Y\,, .\,) \wedge d\alpha^{n-3} \;=\; \alpha \wedge d\,(\alpha\,(X)) \wedge d\,(\alpha\,(Y)) \wedge d\alpha^{n-3}\;.$$

Now, evaluating this on $(K_1, \dots, K_{2n-3})$ gives zero, because $K_i \in TK$ and both $d\,(\alpha\,(X))_p$ and $d\,(\alpha\,(Y))_p$ are zero on $T_pK = T_p\Sigma_0 \cap T_p\Sigma_{\pi/2}$, for every $p \in K$. Then, by Corollary 6.5, the orientation induced on K by the contact structure ξ on M is the one such that the ordered couple of vector fields (Y, X) induces a positive orientation of the normal bundle of K in M. But, according to the computation of $d\phi$ in the proof of Lemma 6.7, this is exactly the case also for $\Psi(K \times \{0\})$ (oriented as image of $K = \phi^{-1}(0)$). This means that K_0 is a positive contact submanifold, as wanted. ☐

Proof (Lemma 6.10). We search a function f such that $\widetilde{\alpha} := f\alpha$ satisfies $d\varphi \wedge d\widetilde{\alpha}^{n-1} > 0$ on $M \setminus Int(\mathcal{N})$.
We can compute

$$\begin{aligned} d\varphi \wedge d\widetilde{\alpha}^{n-1} &= f^{n-1} d\varphi \wedge d\alpha^{n-1} \,+\, (n-1) f^{n-2} d\varphi \wedge df \wedge \alpha \wedge d\alpha^{n-2} \\ &= f^{n-2}\left[f d\varphi \wedge d\alpha^{n-1} - (n-1)\, df \wedge d\varphi \wedge \alpha \wedge d\alpha^{n-2}\right]\;. \end{aligned}$$

Let now $\epsilon > 0$ be such that $\overline{\{\|\phi\| < 2\epsilon\}} \subset \mathcal{N}$ and chose f to be a smooth function, depending only on $\|\phi\|$ and non−increasing in it, equal to $1/\epsilon$ on the set $\{\|\phi\| < \epsilon\}$ and equal to $1/\|\phi\|$ on the set $M \setminus \{\|\phi\| < 2\epsilon\}$.

Let's now analyze $d\varphi \wedge d\widetilde{\alpha}$ on $\mathcal{N}^c$. Here, we have $f = {}^1/_{\|\phi\|}$ and $df = -{}^{d\,\|\phi\|}/_{\|\phi\|^2}$, so that

$$\|\phi\|^{n+1} d\varphi \wedge d\widetilde{\alpha}^{n-1} \;=\; \|\phi\|^2 d\varphi \wedge d\alpha^{n-1} + (n-1)\, \|\phi\|\, d\,\|\phi\| \wedge d\varphi \wedge \alpha \wedge d\alpha^{n-2}\;.$$

Moreover, recalling that $\phi = (\alpha\,(X)\,, -\alpha\,(Y))$, we also have that $\|\phi\|^2 d\varphi = \phi_1 d\phi_2 - \phi_2 d\phi_1 = -\alpha\,(X)\, d\,(\alpha\,(Y)) + \alpha\,(Y)\, d\,(\alpha\,(X))$ and $\|\phi\|\, d\,\|\phi\| \wedge d\varphi = d\phi_1 \wedge d\phi_2 = -d\,(\alpha\,(X)) \wedge d\,(\alpha\,(Y))$, so that

$$\begin{aligned}\|\phi\|^{n+1} d\varphi \wedge d\widetilde{\alpha}^{n-1} &= \left[-\alpha\,(X)\, d\,(\alpha\,(Y)) + \alpha\,(Y)\, d\,(\alpha\,(X))\right] \wedge d\alpha^{n-1} \\ &\quad - (n-1)\, d\,(\alpha\,(X)) \wedge d\,(\alpha\,(Y)) \wedge \alpha \wedge d\alpha^{n-2}\;.\end{aligned} \tag{6.5}$$

Notice now that $\alpha \wedge d\left(\alpha\left(Y\right)\right) \wedge d\alpha^{n-1} = 0$ on M, because $\dim M = 2n-1$. Hence, $\iota_X \left[\alpha \wedge d\left(\alpha\left(Y\right)\right) \wedge d\alpha^{n-1}\right] = 0$ too. In other words, according to the graded Leibniz rule for the interior product,

$$\begin{aligned}\alpha(X) \cdot d\left(\alpha\left(Y\right)\right) \wedge d\alpha^{n-1} - d\left(\alpha\left(Y\right)\right)(X) \cdot \alpha \wedge d\alpha^{n-1} \\ + (n-1)\alpha \wedge d\left(\alpha\left(Y\right)\right) \wedge d\alpha(X,\,.) \wedge d\alpha^{n-2} = 0\ ,\end{aligned}$$

i.e.

$$\begin{aligned}\alpha(X) \cdot d\left(\alpha\left(Y\right)\right) \wedge d\alpha^{n-1} = d\left(\alpha\left(Y\right)\right)(X) \cdot \alpha \wedge d\alpha^{n-1} \\ - (n-1)\alpha \wedge d\left(\alpha\left(Y\right)\right) \wedge d\alpha(X,\,.) \wedge d\alpha^{n-2}\ . \end{aligned} \tag{6.6}$$

Exchanging the roles of X and Y in the above computations, we also get

$$\begin{aligned}\alpha(Y) \cdot d\left(\alpha\left(X\right)\right) \wedge d\alpha^{n-1} = d\left(\alpha\left(X\right)\right)(Y) \cdot \alpha \wedge d\alpha^{n-1} \\ - (n-1)\alpha \wedge d\left(\alpha\left(X\right)\right) \wedge d\alpha(Y,\,.) \wedge d\alpha^{n-2}\ . \end{aligned} \tag{6.7}$$

Now, recall that X, Y are contact vector fields for ξ, i.e. there are functions $f, g\colon M \to \mathbb{R}$ such that

$$\begin{aligned} d\left(\alpha\left(X\right)\right) &= f\alpha - d\alpha(X,\,.)\ , \\ d\left(\alpha\left(Y\right)\right) &= g\alpha - d\alpha(Y,\,.)\ . \end{aligned} \tag{6.8}$$

Then, Equations (6.5) to (6.8), together with the fact that $\alpha \wedge \alpha = 0$, tell that

$$\begin{aligned} \|\phi\|^{n+1}\, d\varphi \wedge d\widetilde{\alpha}^{n-1} = &- d\left(\alpha\left(Y\right)\right)(X) \cdot \alpha \wedge d\alpha^{n-1} \\ &+ d\left(\alpha\left(X\right)\right)(Y) \cdot \alpha \wedge d\alpha^{n-1} \\ &+ (n-1)\alpha \wedge d\alpha(X,\,.) \wedge d\alpha(Y,\,.) \wedge d\alpha^{n-2}\ . \end{aligned} \tag{6.9}$$

Now, again for dimensional reasons, we have $d\alpha^n = 0$ on M, so that $\iota_X \iota_Y d\alpha^n = 0$ too on M. This gives

$$(n-1) d\alpha(X,\,.) \wedge d\alpha(Y,\,.) \wedge d\alpha^{n-2} = d\alpha(X,Y) \cdot d\alpha^{n-1}\ ,$$

so that Equation (6.9) finally becomes

$$\begin{aligned} \|\phi\|^{n+1}\, d\varphi \wedge d\widetilde{\alpha}^{n-1} &= - d\left(\alpha\left(Y\right)\right)(X) \cdot \alpha \wedge d\alpha^{n-1} \\ &\quad + d\left(\alpha\left(X\right)\right)(Y) \cdot \alpha \wedge d\alpha^{n-1} \\ &\quad + d\alpha(X,Y) \cdot \alpha \wedge d\alpha^{n-1} \\ &= -\alpha\left([X,Y]\right) \cdot \alpha \wedge d\alpha^{n-1}\ , \end{aligned}$$

because $d\alpha(X,Y) = d\left(\alpha\left(Y\right)\right)(X) - d\left(\alpha\left(X\right)\right)(Y) - \alpha\left([X,Y]\right)$ according to the exterior derivative formula.

Now, $[X,Y]$ is negatively transverse to ξ by hypothesis, so that the above equation implies that $d\widetilde{\alpha}$ is symplectic on the fibers of $\varphi|_{M\setminus\mathcal{N}}$, as wanted. $\square$

6.2 From open books to contact vector fields

We start by recalling another point of view on open book decompositions in terms of functions with values in $\mathbb{R}^2$. This is presented, for instance, in [Lut79, Page 289] and [Gir17, Page 11].

Consider the standard open book decomposition $\mathcal{OB}$ of $\mathbb{R}^2$, i.e. the one with the origin as binding and with radii starting from the origin as pages.

Definition 6.11. A smooth function $\Psi : M \to \mathbb{R}^2$ is *transverse to* $\mathcal{OB}$ if it is transverse to the binding and to the pages of $\mathcal{OB}$.

Let $K \coloneqq \Psi^{-1}(0)$ and $\varphi \coloneqq \Psi / \|\Psi\| : M \setminus K \to \mathbb{S}^1$. Notice that if Ψ is transverse to $\mathcal{OB}$ then (K, φ) is an open book decomposition of M. Then, we say that Ψ *defines the open book* (K, φ).

Moreover, given two functions $\Psi, \Psi' : M \to \mathbb{R}^2$ transverse to $\mathcal{OB}$, we call them *equivalent*, and write $\Psi \sim \Psi'$, if there is a smooth positive function $f : M \to \mathbb{R}$ such that $\Psi' = f\Psi$.

As already stated for instance in [Lut79, Page 289], open book decompositions are then equivalent to functions transverse to $\mathcal{OB}$ up to equivalence:

Proposition 6.12. *We have a bijection*

$$\left\{ \begin{array}{c} \Psi : M \to \mathbb{R}^2 \\ \textit{transverse to } \mathcal{OB} \end{array} \right\} /\sim \quad \xrightarrow{\sim} \quad \left\{ \begin{array}{c} \textit{open book decompositions} \\ (K, \varphi) \textit{ on } M \end{array} \right\} .$$

$$[\Psi] \longrightarrow (\Psi^{-1}(0), \frac{\Psi}{\|\Psi\|})$$

Proof. The map in the statement is well defined because two functions $M \to \mathbb{R}^2$ transverse to $\mathcal{OB}$ in the same equivalence class clearly define the same open book on M.

Let's show that the map is surjective. Fix a Riemannian metric on M. Then, given an open book decomposition (K, φ), consider the distance function ρ to the submanifold K. Then, (K, φ) is the image of the class of $\Psi = (\rho \cos \varphi, \rho \sin \varphi)$.

Lastly, we show that the map in the statement is injective, i.e. that if Ψ and Ψ' define the same open book (K, φ), then they are equivalent.
Because $\Psi / \|\Psi\| = \varphi = \Psi' / \|\Psi'\|$, a function f such that $\Psi' = f\Psi$ has to be equal to $\|\Psi'\| / \|\Psi\|$ on the complement of K. We then have to check that it smoothly extends at each point $p \in K$.
Now, K admits a neighborhood $K \times D^2$ and local coordinates (p, r, θ), with $p \in K$, $r \in [0, 1)$, $\theta \in \mathbb{S}^1$ such that $\varphi|_{K \times D^2}(p, r, \theta) = \theta$ for $r > 0$. This implies that $\Psi(p, r, \theta) = \lambda(p, r, \theta)(\cos \theta, \sin \theta)$ and $\Psi'(p, r, \theta) = \lambda'(p, r, \theta)(\cos \theta, \sin \theta)$, where $\lambda, \lambda' : K \times D^2 \to \mathbb{R}$ are non-negative functions, strictly positive away from $K \times \{0\}$. Then, $f = \lambda'/\lambda$ as functions $K \times \left(D^2 \setminus \{0\}\right) \to \mathbb{R}$.
Because Ψ, Ψ' are transverse to $0 \in \mathbb{R}^2$, the limits of $d\lambda/dr$ and $d\lambda'/dr$ for $r \to 0$ exist, are both non-zero for all $p \in K$, $\theta \in \mathbb{S}^1$ and do not depend on θ.
Hence, the function $K \to \mathbb{R}$ defined by $d\lambda/dr \cdot (d\lambda'/dr)^{-1}$ smoothly extends $f = \lambda'/\lambda$ over $K \times \{0\}$. ☐

Going back to the contact setting, we have the following converse to Proposition 6.1, which is also a more precise version of the second part of Théorème II.E:

Proposition 6.13 (stated in [Gir12])**.** *Suppose that (B, φ) an open book decomposition of M supporting ξ. Denote by α a contact form defining ξ and such that $d\alpha$ is symplectic on the fibers of φ. Then, there is a smooth function $\phi \colon M \to \mathbb{R}^2$ defining (B, φ) and such that the contact vector fields X and Y, associated via α respectively to the contact Hamiltonians ϕ_1 and $-\phi_2$, have Lie bracket $[X, Y]$ negatively transverse to ξ.*

Proof. This proof is strongly inspired from the computations in [Bou02], which will be recalled in detail in Section 7.2.

Let $\overline{\phi} = (\overline{\phi}_1, \overline{\phi}_2) \colon M \to \mathbb{R}^2$ be a smooth function defining (B, φ).
Consider then $\epsilon > 0$ such that $\alpha \wedge d\alpha^{n-2} \wedge d\overline{\phi_1} \wedge d\overline{\phi}_2$ is positive on $\{\|\overline{\phi}\| < \epsilon\}$. Such an ϵ exists because α induces a contact form on $B = \phi^{-1}(0)$.

Define now $\chi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as follows: $\chi(x)$ is non$-$increasing in x, is equal to x for $x < \epsilon/2$ and equal to 1 for $x \geq \epsilon$. Denote then $f := \chi(\|\phi\|)/\|\phi\|\colon M \to \mathbb{R}_{>0}$ and define $\phi := f\overline{\phi}\colon M \to \mathbb{R}^2$; then, ϕ defines (B, φ) too. Let also $\rho := \|\phi\|$ and $\theta := \phi/\rho\colon M \setminus B \to \mathbb{S}^1$. Notice that $\theta = \varphi$.

Then, we claim that

$$\Omega := n\rho^2 d\theta \wedge d\alpha^{n-1} + n(n-1)\rho d\rho \wedge d\theta \wedge \alpha \wedge d\alpha^{n-2}$$

is a volume form on M. Indeed, the first term is non-negative everywhere and positive away from B, because $d\alpha$ is symplectic on the fibers of $\theta = \varphi$, and the second term is positive along B and non$-$negative everywhere, by construction of f.

This being said, we now denote by X, Y the contact vector fields associated, respectively, to the contact Hamiltonians $\phi_1, -\phi_2$ via the contact form α given in the statement.

Because $\rho d\theta = \phi_1 d\phi_2 - \phi_2 d\phi_1$ and $\rho d\rho \wedge d\theta = d\phi_1 \wedge d\phi_2$, we can then write

$$\begin{aligned}\Omega \;=\; & n\left[-\alpha\left(X\right) d\left(\alpha\left(Y\right)\right) + \alpha\left(Y\right) d\left(\alpha\left(X\right)\right)\right] \wedge d\alpha^{n-1} \\ & - n(n-1) d\left(\alpha\left(X\right)\right) \wedge d\left(\alpha\left(Y\right)\right) \wedge \alpha \wedge d\alpha^{n-2} \; .\end{aligned}$$

Notice then that the right hand side is exactly the same (up to a factor n) as the one of Equation (6.5) in the proof of Lemma 6.10. Hence, the exact same computations made in that proof tell us that

$$\Omega \;=\; -n\alpha\left([X, Y]\right) \cdot \alpha \wedge d\alpha^{n-1} \; .$$

This shows that $[X, Y]$ is negatively transverse to ξ, because Ω is a volume form on M and α is a positive contact form. $\square$

Chapter 7

Lutz' study and Bourgeois' construction of invariant contact manifolds

The aim of this chapter is twofold: we want to reinterpret a part of Lutz' study from [Lut79] using the language of adapted open books introduced in [Gir02] by Giroux, and we want to describe the links between Lutz' work and the Bourgeois construction from [Bou02]. In order to do this, we proceed as follows.

Section 7.1 describes the study from [Lut79] of $\mathbb{T}^2$-invariant contact structures on the total space of a principal $\mathbb{T}^2$-bundle $\pi\colon V^{2n+1} \to M^{2n-1}$. In this context, we recall how Lutz recovers an open book decomposition (B, φ) and a differential $1-$form β on the base manifold M such that its restriction to B is a contact form and, in the case of a flat principal bundle $\pi\colon V \to M$, its differential is symplectic on the fibers of φ. In this part, we recall (and complete the missing details) of the proofs from [Lut79].
Finally, we conclude Section 7.1 by showing, using a lemma on adapted open book decompositions due to Giroux (already stated as Lemma 6.8), that if the induced β happens to be contact on M (which is not always the case), this open book (B, φ) supports the contact structure $\xi = \ker \beta$, at least in the case of flat bundles.

In Section 7.2 we recall the construction presented in [Bou02], where, starting from a contact manifold (M, ξ) and an open book decomposition on M supporting ξ, Bourgeois gives an explicit $\mathbb{T}^2-$invariant contact structure on the particular flat principal $\mathbb{T}^2-$bundle $M \times \mathbb{T}^2$. We will also show that this can be interpreted as a converse of the construction by Lutz recalled in Section 7.1.

7.1 Lutz's study revisited

Let $\pi\colon V \to M$ be a principal $\mathbb{T}^2-$bundle and η be a contact structure on V invariant under the $\mathbb{T}^2$-action $\chi : \mathbb{T}^2 \times V \to V$ on V. Assume that V, M are oriented, that the $\mathbb{T}^2$-action preserves the orientation and that η is positive on V. We then want to give a detailed statement that summarizes some of Lutz' results from [Lut79] in this context their reinterpretation in terms of adapted open books, as introduced by Giroux in [Gir02]. In order to do this, we first need to introduce some notations.

For each $g \in \mathbb{T}^2$, denote by $\chi_g\colon V \to V$ the diffeomorphism given by the action of g on the total space V. If we denote by $\mathfrak{t}$ the Lie algebra of the Lie group $\mathbb{T}^2$, we can then define, for each $v \in \mathfrak{t}$, a vector field X_v such that

$$X_v(p) = \left.\frac{d}{dt}\right|_{t=0} \chi_{\gamma_v(t)}(p) \text{ for each } p \in V \ ,$$

where $\gamma_v : (-\epsilon, \epsilon) \to \mathbb{T}^2$ is a curve passing through $\mathrm{Id} \in \mathbb{T}^2$ and with velocity $v \in \mathfrak{t} = T_{\mathrm{Id}}\mathbb{T}^2$ at $t = 0$.
Notice that $\widetilde{B} \coloneqq \{p \in V \,|\, \forall v \in \mathfrak{t}, X_v \in \eta\}$ is a $\mathbb{T}^2-$invariant subset of V; denote then by B its image in M via the bundle projection $\pi\colon V \to M$.

Now, up to taking an average using a Haar measure on $\mathbb{T}^2$, we can also write $\eta = \ker \alpha$ with $\alpha \in \Omega^1(V)$ invariant under the action of $\mathbb{T}^2$. For the rest of the section we will work with this contact form, but at each step we will remark that all the constructions are actually independent of this particular choice of invariant form and dependent only on η.

Denote by $\mathfrak{t}^*$ the dual of $\mathfrak{t}$ and by $S\mathfrak{t}^*$ its spherization, i.e. the quotient of $\mathfrak{t}^* \setminus \{0\}$ by the natural $\mathbb{R}_{>0}-$action.
Consider now, for each $p \in V$, the element L_p of $\mathfrak{t}^*$ given by $v \mapsto \alpha_p(X_v(p))$. Notice moreover that, by definition of $\widetilde{B}$, L_p is non$-$zero for all $p \in V \setminus \widetilde{B}$. Denote then by $[L_p]$ its class in $S\mathfrak{t}^*$; we point out that $[L_p]$ doesn't depend on the choice of $\mathbb{T}^2-$invariant contact form α for η.

Define then $\widetilde{\varphi}\colon V \setminus \widetilde{B} \to S\mathfrak{t}^*$ by $\widetilde{\varphi}(p) \coloneqq [L_p]$. Notice that $\widetilde{\varphi}$ factors through the projection $\pi\colon V \setminus \widetilde{B} \to M \setminus B$ to a well defined map $\varphi\colon M \setminus B \to S\mathfrak{t}^*$.

Suppose now that the principal $\mathbb{T}^2-$bundle $\pi\colon V \to M$ admits a *principal Ehresmann connection* $\mathcal{F}$, i.e. a codimension 2 distribution on V which is $\mathbb{T}^2-$invariant and transverse to each $\mathbb{T}^2-$orbit. Suppose moreover that $\mathcal{F}$ is a codimension 2 foliation on V.

Lastly, notice that $\mathcal{F}\cap\eta$ is $\mathbb{T}^2-$invariant, because it is the intersection of $\mathbb{T}^2-$invariant distributions. We point out though that it is a priori only a *singular* distribution on V, i.e. it has codimension 2 at each point $p \in V$ such that $\mathcal{F}_p \subset \eta_p$ and codimension 3 at each point $p \in V$ such that $\eta_p \pitchfork \mathcal{F}_p$.
Denote then by ξ the projection of $\mathcal{F} \cap \eta$ to M via the differential of π; this projection is well defined because $\mathcal{F} \cap \eta$ is $\mathbb{T}^2-$invariant.

Remark 7.1. *[Lut79, Page 301] gives an explicit example where $\mathcal{F} \cap \eta$ fails to be a distribution.*
Consider on the principal $\mathbb{T}^2-$bundle $\pi\colon \mathbb{S}^{2n-1} \times \mathbb{T}^2 \to \mathbb{S}^{2n-1}$ the invariant form

$$\alpha = i^*x_1 d\theta_1 + i^*x_2 d\theta_2 + i^*\left(x_3dx_4 - x_4dx_3 + \ldots + x_{2n-1}dx_{2n} - x_{2n}dx_{2n-1}\right) \ ,$$

where (θ_1, θ_2) are coordinates on $\mathbb{T}^2$, $i\colon \mathbb{S}^{2n-1} \to \mathbb{R}^{2n}$ is the natural inclusion and $(x_1, \ldots, x_{2n})$ are coordinates on $\mathbb{R}^{2n}$. Then, via the natural flat connection $\mathcal{F} = T\mathbb{S}^{2n-1} \oplus \{0\}$, $\ker\alpha$ induces on $\mathbb{S}^{2n-1}$ the kernel of

$$\beta \coloneqq i^*\left(x_3dx_4 - x_4dx_3 + \ldots + x_{2n-1}dx_{2n} - x_{2n}dx_{2n-1}\right) \ .$$

This form β is zero on the subset $\left\{(x_1, x_2, 0, \ldots, 0) \in \mathbb{S}^{2n}\right\} \subset \mathbb{S}^{2n}$, hence doesn't define an hyperplane field on $\mathbb{S}^{2n}$.

Then Lutz' work, reinterpreted using the language of adapted open books introduced by Giroux, gives the following:

Theorem 7.2. *In the situation described above, we have the following:*

a. (B, φ) defined above is an open book decomposition of M.

b. *$\xi = \pi_* (\mathcal{F} \cap \eta)$ is a contact structure on M if and only if $(\mathcal{F}, \mathcal{F} \cap \eta)$ is a 2-codimensional contact foliation on V.*

c. *Let $\overline{B}$ be B with opposite orientation and $\overline{\varphi}$ be the composition of φ with an arbitrary orientation−reversing diffeomorphism $S\mathfrak{t}^* \to S\mathfrak{t}^*$. Then, if the conditions in point b. are satisfied, ξ is supported by $(\overline{B}, \overline{\varphi})$.*

We recall that a q-codimensional *contact foliation* on a manifold Y^{2n+1+q} is a couple $(\mathfrak{F}^{2n+1}, \zeta^{2n})$ of a codimension q foliation $\mathfrak{F}$ on Y and a codimension $q+1$ distribution ζ on Y such that, for each leaf L of $\mathfrak{F}$, $(L, \zeta \cap TL)$ is a contact manifold.
We also point out that the particular choice of $\overline{\varphi}$ in point c. of Theorem 7.2 is not important, because if $\overline{\varphi}'\colon M \setminus B \to S\mathfrak{t}^*$ is obtained from $\overline{\varphi}$ by composing with an orientation−preserving diffeomorphism $S\mathfrak{t}^* \to S\mathfrak{t}^*$, then the set of fibers of $\overline{\varphi}$ and the set of fibers of $\overline{\varphi}'$ coincide.
The rest of Section 7.1 is devoted to the proof of the above theorem.

Proof (Theorem 7.2). We start by showing point a., i.e. that (B, φ) is an open book decomposition of M; this is the content of [Lut79, Lemme Fondamental (page 286)].

For computational convenience, we denote by X_1, X_2 the pair of infinitesimal generators of the $\mathbb{T}^2-$ action associated to a choice of coordinates (θ_1, θ_2) on $\mathbb{T}^2$. In other words, for $i = 1, 2$, $X_i := X_{\frac{\partial}{\partial \theta_i}}$, with the notation X_v for $v \in \mathfrak{t}$ introduced above.

Consider then, for $i = 1, 2$, $\widetilde{\phi}_i\colon V \to \mathbb{R}$ defined by $\widetilde{\phi}_i := \alpha(X_i)$.
Notice that, for $i = 1, 2$, $\widetilde{\phi}_i$ is constant along the fibers of $\pi\colon V \to M$, because α and X_i are $\mathbb{T}^2-$invariant. Hence, for $i = 1, 2$, $\widetilde{\phi}_i$ induces a smooth map $\phi_i\colon M \to \mathbb{R}$ such that $\widetilde{\phi}_i = \phi_i \circ \pi$. Denote then $\phi := (\phi_1, \phi_2)\colon M \to \mathbb{R}^2$.

Lemma 7.3. *On all V, we have the following:*

i. *$d\widetilde{\phi}_i = -\iota_{X_i} d\alpha$, for $i = 1, 2$;*

ii. *$d\widetilde{\phi}_j(X_i) = 0$, for $\{i, j\} = \{1, 2\}$;*

iii. *$d\alpha(X_1, X_2) = 0$.*

Proof (Lemma 7.3). By the $\mathbb{T}^2-$invariance of α, we have $\mathcal{L}_{X_i}\alpha = 0$, for $i = 1, 2$. In particular, point i. simply follows from the facts that $\mathcal{L}_{X_i}\alpha = d\iota_{X_i}\alpha + \iota_{X_i} d\alpha$ and that $\widetilde{\phi}_i = \iota_{X_i}\alpha$.
Points ii. and iii. follow directly from point i., from the formula for exterior derivative of differential forms, stating that $d\alpha(X, Y) = X\cdot(\alpha(Y)) - Y\cdot(\alpha(X)) - \alpha([X, Y])$ for all vector fields X, Y on V, and from the fact that $[X_1, X_2] = 0$, because $\mathbb{T}^2$ is commutative. □

Let now $\rho\colon M \to \mathbb{R}_{\geq 0}$ be given by $\rho := \|\phi\|$. Then, we have $B = \rho^{-1}(0)$.
Moreover, $\varphi\colon M \setminus B \to S\mathfrak{t}^* \simeq \mathbb{S}^1$ defined above is just given by $\varphi := {}^{\phi}/_{\rho}$. Here, the identification $S\mathfrak{t}^* \simeq \mathbb{S}^1$ is made through the coordinates (θ_1, θ_2) on $\mathbb{T}^2$.

Let's then prove that ϕ is transverse to the origin of $\mathbb{R}^2$ and that the linear map $d_p\varphi\colon T_p(M \setminus B) \to T_{\varphi(p)}\mathbb{S}^1$ has rank 1 for each $p \in M \setminus B$. This will show that (B, φ) is an open book decomposition of M.

Suppose by contradiction that $d_p\phi\colon T_pM \to T_{\phi(p)}\mathbb{R}^2$ has rank ≤ 1, for a certain $p \in B = \phi^{-1}(0)$. Then, there is $\theta \in \mathbb{S}^1$ such that, for all $v \in T_pM$, $d_p\phi_1(v)\partial_x + d_p\phi_2(v)\partial_y$ is parallel to $(\cos\theta, \sin\theta)$.
We then get that $-\sin\theta\, d_p\phi_1 + \cos\theta\, d_p\phi_2 = 0$ as map $T_pM \to \mathbb{R}$. Consider now the everywhere non-zero vector field $Z := -\sin\theta X_1 + \cos\theta X_2$ on V. We have $\alpha(Z)_{|p} =$

0, because $\phi(p) = 0$, and $d\alpha(Z,.)_{|p} = 0$ by Lemma 7.3 and because $-\sin\theta\, d_p\phi_1 + \cos\theta\, d_p\phi_2 = 0$. By the contact condition, this implies $Z_p = 0$, giving a contradiction.

Take now $p \in M\setminus B$, with $\phi_1(p) \neq 0$ let's say. Then, to show that $d_p\varphi\colon T_p\left(M\setminus B\right) \to T_{\varphi(p)}\mathbb{S}^1$ has rank 1, it's enough to show that the differential of

$$\Psi_2 := \frac{\phi_2}{\rho}\colon\; M\setminus B \xrightarrow{\varphi} \mathbb{S}^1 \xrightarrow{\pi_y|_{\mathbb{S}^1}} \mathbb{R}$$

has rank 1 at p. Here, $\pi_y : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the $y-$axis.

Because

$$d\rho \;=\; \frac{1}{2\rho}d\left(\rho^2\right) \;=\; \frac{1}{\rho}\left(\phi_1 d\phi_1 + \phi_2 d\phi_2\right)\;,$$

we can write

$$d\Psi_2 \;=\; \frac{d\phi_2}{\rho} - \frac{\phi_2}{\rho^2}d\rho \;=\; \frac{d\phi_2}{\rho} - \frac{\phi_2}{\rho^3}\left(\phi_1 d\phi_1 + \phi_2 d\phi_2\right)\;. \tag{7.1}$$

For notational convenience, let $\widetilde{\rho}, \widetilde{\Psi}_2$ denote respectively the compositions $\rho\circ\pi$ and $\Psi_2\circ\pi$.

Suppose now by contradiction that $d\Psi_2 = 0$ and consider the vector field

$$Z := \frac{\widetilde{\Psi}_2\left(\widetilde{\phi}_1 X_1 + \widetilde{\phi}_2 X_2\right)}{\widetilde{\rho}} - X_2$$

defined on $V\setminus\widetilde{B}$.

Then, an easy computation shows that $\alpha(Z) = 0$ on all $V\setminus\widetilde{B}$. Moreover, Equation (7.1) and the hypothesis $d_p\Psi_2 = 0$ imply that $d\alpha\left(Z,.\right)_{|p} = 0$. By the contact condition, we get $Z(p) = 0$. In particular, the coefficient multiplying $X_1(p)$ in $Z(p)$ has to be zero, i.e. $\widetilde{\Psi}_2(p)\widetilde{\phi}_1(p) = 0$. Because $\widetilde{\phi}_1(p)$ is non-zero by assumption, $\widetilde{\Psi}_2(p)$ has to be zero. But this implies $Z(p) = -X_2(p)$, which contradicts the fact that $Z(p) = 0$.
This concludes the proof of the fact that (B,φ) is an open book decomposition of M.

We then proceed to the proof of points b. and c. of Theorem 7.2. We start by showing that, thanks to the $\mathbb{T}^2-$invariant contact form α and a choice of connection form ω defining $\mathcal{F}$, we can induce an explicit $1-$form β on M whose kernel is exactly $\pi_*\left(\eta\cap\mathcal{F}\right)$. This is also explained in [Lut79, Section I.3.6].

We recall that the connection form $\omega \in \Omega^1\left(V,\mathfrak{t}\right)$ defining $\mathcal{F}$ can be written as as $\omega = \omega_1\otimes e_1 + \omega_2\otimes e_2$, with $\omega_1,\omega_2 \in \Omega^1\left(V\right)$ such that $\ker\omega_1\cap\ker\omega_2 = \mathcal{F}$ and (e_1,e_2) the basis of the vector space $\mathfrak{t} = \mathbb{R}^2$ associated to the choice of coordinates (θ_1,θ_2).

Moreover, the fact that $\mathcal{F}$ is a foliation gives, by Frobenius' theorem, the flatness of ω, i.e. tells that its curvature is zero. We recall that the curvature form Ω of ω is the $2-$form on M with values in $\mathfrak{t}$ defined by $d\omega + \frac{1}{2}[\omega,\omega]$. More explicitly, for each couple of vector fields U,V on M and each $p\in M$, $\Omega_p(U_p,V_p) := d\omega_q(X_q,Y_q) + \frac{1}{2}\left[\omega_q(X_q),\omega_q(Y_q)\right]$, where X,Y are lifts of U,V horizontal for $\ker\omega = \mathcal{F}$ and $q\in\pi^{-1}(p)$. This is indeed well defined because the right-hand side is independent of the choice of X,Y,q as above. Notice moreover that $[\omega,\omega] = 0$, because $\mathbb{T}^2$ is abelian, so that $\Omega = d\omega$.
Hence, in our case, the flatness of ω means that $d\omega = d\omega_1\otimes e_1 + d\omega_2\otimes e_2 = 0$.

Now, the $1-$form $\gamma := \alpha - \widetilde{\phi}_1\omega_1 - \widetilde{\phi}_2\omega_2$ on V is $\mathbb{T}^2-$invariant and such that $\gamma(X_1) = \gamma(X_2) = 0$. Hence, there is a $1-$form β on M such that $\gamma = \pi^*\beta$, i.e. such that $\alpha = \widetilde{\phi}_1\omega_1 + \widetilde{\phi}_2\omega_2 + \pi^*\beta$.

Then, we have the following result, whose proof is postponed:

Lemma 7.4 ([Lut79, Proposition (page 296)]). *α is a positive contact form on V if and only if the two following conditions are satisfied:*

i. β induces a negative contact form on B;

ii. for each $\theta \in \mathbb{S}^1$, $d\left(\frac{\beta}{\rho}\right)+\frac{\phi_1}{\rho}\Omega_1+\frac{\phi_2}{\rho}\Omega_2$ induces a negative symplectic form on $\varphi^{-1}(\theta)$.

This tells in particular that, if there is a flat connection ω for $\pi\colon V \to M$, the β obtained on M via ω induces a contact structure on B and the differential of β/ρ is symplectic on the fibers of φ.

Let's suppose, for the moment, that β defines a contact structure on all M.
Then, we want to show point c., i.e. that $\xi = \ker \beta$ is supported by the open book $(\overline{B}, \overline{\varphi})$, where $\overline{B}$ is B with orientation inversed and $\overline{\varphi}$ is φ composed with the orientation reversing diffeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$ given by the restriction of the complex conjugation to $\mathbb{S}^1 \subset \mathbb{C}$. We recall from what said after the statement of Theorem 7.2 that the particular choice of reversing diffeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$ is not important, hence we can choose an arbitrary one.

In order to prove this, we need again Lemma 6.8, already used in Section 6.1. We recall here the statement, adapting the notations to this situation:

Lemma 6.8 (Giroux)**.** *Let (M^{2n-1}, ξ) be a contact manifold. Suppose there are an open book decomposition (K, θ) of M (in particular, B is oriented as the boundary of each fiber of θ), a tubular neighborhood $\mathcal{N} = K \times D^2$ of K (here D^2 is the open unit disk in $\mathbb{R}^2$) and a contact form γ defining ξ such that:*

(i) θ restricted to $\mathcal{N} \setminus K$ is the angular coordinate of the projection on the second factor $\mathcal{N} = K \times D^2 \to D^2$;

(ii) ξ induces a positive contact structure on each submanifold $K_z := K \times \{z\}$ of $\mathcal{N}$ (notice each K_z is oriented because K is);

(iii) $d\gamma$ induces a positive symplectic form on each fiber of $\theta_{|M \setminus \mathcal{N}}$.

Then, the open book decomposition (K, θ) supports the contact structure ξ.

Choose now an $\epsilon > 0$ very small and replace β by $\overline{\beta} := \chi\beta$, with $\chi\colon M \to \mathbb{R}$ a positive function equal to $1/\rho$ on the complement of $\{\rho \geq 2\epsilon\} \subset M$ and equal to $1/\epsilon$ on B. Then, according to Lemma 7.4, $(\overline{B}, \overline{\varphi})$, $\mathcal{N} = \{\rho < 2\epsilon\}$ and $\overline{\beta}$ satisfy the hypothesis of Lemma 6.8, provided that ϵ is small enough. Hence, we get that $(\overline{B}, \overline{\varphi})$ supports ξ, as wanted.

The only thing left in order to finish the proof of Theorem 7.2 is then to show point b., i.e. that $\xi = \pi_*\eta \cap \mathcal{F}$ is a contact structure on M if and only if $(\mathcal{F}, \eta \cap \mathcal{F})$ is a codimension 2 contact foliation on V.

This actually follows from the fact that

$$\beta \wedge d\beta^{n-1} > 0 \text{ on } M \iff \pi^*\left[\beta \wedge d\beta^{n-1}\right]\big|_{\mathcal{F}} > 0 \iff \alpha \wedge d\alpha^{n-1}\big|_{\mathcal{F}} > 0 \,,$$

where we used the facts that the differential $d_p\pi$ is an isomorphism between $\mathcal{F}_p$ and $T_{\pi(p)}M$ and that the connection ω is zero on $\mathcal{F}$. This concludes the proof of Theorem 7.2. □

We now prove Lemma 7.4 used above.

Proof (Lemma 7.4). The form α is a positive contact form on V if and only if $\iota_{X_2}\iota_{X_1}\alpha \wedge d\alpha^n$ induces a positive section of the vector bundle $\Lambda^{2n-1}(\mathcal{F}^*)$ over V, where $\mathcal{F}^*$ is the dual of $\mathcal{F}$. Indeed, $\mathcal{F}$ is a complement of the vector sub-bundle spanned by X_1 and X_2 in TV.

Now, because $\iota_{X_2}\iota_{X_1}\alpha\wedge d\alpha^n$ is $\mathbb{T}^2-$invariant and becomes zero after contraction with X_1 or X_2, it can be written as pullback via π of a differential form δ of maximal degree on M. Hence, $\iota_{X_2}\iota_{X_1}\alpha\wedge d\alpha^n$ is a positive section of $\Lambda^{2n-1}\left(\mathcal{F}^*\right)$ if and only if δ is a volume form on M. Then, what we need to do is find this form δ.

We start by computing

$$\begin{aligned}
\iota_{X_2}\iota_{X_1}\alpha\wedge d\alpha^n &\overset{(a)}{=} \iota_{X_2}\left[\widetilde{\phi}_1 d\alpha^n - n\alpha\wedge\iota_{X_1}d\alpha\wedge d\alpha^{n-1}\right]\\
&\overset{(b)}{=} n\widetilde{\phi}_1\iota_{X_2}d\alpha\wedge d\alpha^{n-1} - n\widetilde{\phi}_2\iota_{X_1}d\alpha\wedge d\alpha^{n-1}\\
&\quad + nd\alpha\left(X_1,X_2\right)\alpha\wedge d\alpha^{n-1} - n(n-1)\alpha\wedge\iota_{X_1}d\alpha\wedge\iota_{X_2}d\alpha\wedge d\alpha^{n-2}\\
&\overset{(c)}{=} -nd\alpha^{n-1}\wedge\pi^*\left(\phi_1 d\phi_2-\phi_2 d\phi_1\right)\\
&\quad - n(n-1)\alpha\wedge d\alpha^{n-2}\wedge\pi^*\left(d\phi_1\wedge d\phi_2\right)\\
&\overset{(d)}{=} -nd\alpha^{n-1}\wedge\pi^*\left(\rho^2 d\varphi\right) - n(n-1)\alpha\wedge d\alpha^{n-2}\wedge\pi^*\left(\rho d\rho\wedge d\varphi\right)\ .
\end{aligned} \tag{7.2}$$

Here, the equalities (a) and (b) come from the graded Leibniz rule for the interior product (i.e. the formula $\iota_Z\left(\mu\wedge\nu\right)=\iota_Z\mu\wedge\nu+(-1)^{\deg\mu}\mu\wedge\iota_Z\nu$ for all differential forms μ,ν and vector fields Z) and from the facts that $\iota_{X_1}\alpha=\widetilde{\phi}_1$ and $\iota_{X_2}\alpha=\widetilde{\phi}_2$. Also, the equality (c) comes from Lemma 7.3 and the (d) follows from the fact that $\phi_1 d\phi_2-\phi_2 d\phi_1=\rho^2 d\varphi$ and $d\phi_1\wedge d\phi_2=\rho d\rho\wedge d\varphi$ by definition of ρ and φ.

Now, along $\widetilde{B}=\pi^{-1}(B)$ we have that

$$\begin{aligned}
\iota_{X_2}\iota_{X_1}\alpha\wedge d\alpha^n|_{\widetilde{B}} &\overset{(a)}{=} -n(n-1)\alpha\wedge d\alpha^{n-2}\wedge\pi^*\left(\rho d\rho\wedge d\varphi\right)|_{\widetilde{B}}\\
&\overset{(b)}{=} -n(n-1)\left(\pi|_{\widetilde{B}}\right)^*\left[\beta\wedge d\beta^{n-2}\wedge\rho d\rho\wedge d\varphi|_B\right]
\end{aligned} \tag{7.3}$$

Here, the equality (a) comes from the fact that $\rho^2 d\varphi=0$ on $B=\phi^{-1}(0)$ (by definition) and the (b) from the fact that, for well chosen differential forms μ_1,μ_2,ν_1,ν_2, we can write

$$\begin{aligned}
\alpha\wedge d\alpha^{n-2} = \pi^*&\left(\beta\wedge d\beta^{n-2}\right)+\widetilde{\phi}_1\mu_1+\widetilde{\phi}_2\mu_2\\
&+d\widetilde{\phi}_1\wedge\nu_1+d\widetilde{\phi}_2\wedge\nu_2\ ,
\end{aligned}$$

so that the only non-zero contribution along $\widetilde{B}$ after wedging with the form $\pi^*\left(d\phi_1\wedge d\phi_2\right)$ is given by the term $\pi^*\left(\beta\wedge d\beta^{n-1}\right)$.

We now analyze the situation on the complement of $\widetilde{B}$.
Let $\widetilde{\varphi}:=\varphi\circ\pi$ and recall that we already introduced the notation $\widetilde{\rho}=\rho\circ\pi$. Then, away from $\widetilde{B}$, we compute

$$\begin{aligned}
\widetilde{\rho}\left[d\left(\frac{\alpha}{\widetilde{\rho}}\right)\right]^{n-1}\wedge d\widetilde{\varphi} &= \widetilde{\rho}\left(\frac{-d\widetilde{\rho}\wedge\alpha+\widetilde{\rho}d\alpha}{\widetilde{\rho}^2}\right)^{n-1}\wedge d\widetilde{\varphi}\\
&= \widetilde{\rho}\,\frac{\widetilde{\rho}^{n-1}d\alpha^{n-1}-(n-1)\widetilde{\rho}^{n-2}d\widetilde{\rho}\wedge\alpha\wedge d\alpha^{n-2}}{\widetilde{\rho}^{2n-2}}\wedge d\widetilde{\varphi}\\
&= \frac{1}{\widetilde{\rho}^n}\left[\widetilde{\rho}^2 d\widetilde{\varphi}\wedge d\alpha^{n-1}+(n-1)\,\alpha\wedge d\alpha^{n-2}\widetilde{\rho}\wedge\left(d\widetilde{\rho}\wedge d\widetilde{\varphi}\right)\right]\ .
\end{aligned}$$

Hence, from Equation (7.2) we get

$$\widetilde{\rho}\left[d\left(\frac{\alpha}{\widetilde{\rho}}\right)\right]^{n-1}\wedge d\widetilde{\varphi} = -\frac{1}{n\widetilde{\rho}^n}\left(\iota_{X_2}\iota_{X_1}\alpha\wedge d\alpha^n\right)\ . \tag{7.4}$$

Now, we point out that on the horizontal space $\mathcal{F}$ of the connection ω we have

$$\begin{aligned} \alpha &= \pi^*\beta \;, \\ d\alpha &= \widetilde{\phi}_1 d\omega_1 + \widetilde{\phi}_2 d\omega_2 + \pi^* d\beta. \end{aligned} \tag{7.5}$$

Indeed, the terms ω_1, ω_2 in α and $d\widetilde{\phi}_1 \wedge \omega_1$ and $d\widetilde{\phi}_2 \wedge \omega_2$ in $d\alpha$ are zero on $\mathcal{F}$.

Then, on $\mathcal{F}$ we also have

$$\begin{aligned} \widetilde{\rho}\left[d\left(\frac{\alpha}{\widetilde{\rho}}\right)\right]^{n-1} \wedge d\widetilde{\varphi} &= \widetilde{\rho}\left(\frac{-d\widetilde{\rho}\wedge\alpha + \widetilde{\rho}d\alpha}{\widetilde{\rho}^2}\right)^{n-1} \wedge d\widetilde{\varphi} \\ &\overset{(a)}{=} \widetilde{\rho}\left(\frac{-d\widetilde{\rho}\wedge\pi^*\beta + \widetilde{\rho}\pi^*d\beta}{\widetilde{\rho}^2} + \frac{\widetilde{\phi}_1}{\widetilde{\rho}}d\omega_1 + \frac{\widetilde{\phi}_2}{\widetilde{\rho}}d\omega_2\right)^{n-1} \wedge d\widetilde{\varphi} \\ &\overset{(b)}{=} \pi^*\left\{\rho\left[d\left(\frac{\beta}{\rho}\right) + \frac{\phi_1}{\rho}\Omega_1 + \frac{\phi_2}{\rho}\Omega_2\right]^{n-1} \wedge d\varphi\right\}. \end{aligned} \tag{7.6}$$

Then, Equation (7.3) together with Equations (7.4) and (7.6) tell us that $\iota_{X_2}\iota_{X_1}\alpha \wedge d\alpha^n$ is a positive section of $\Lambda^{2n-1}(\mathcal{F}^*)$ over V if and only if

- the form $\beta \wedge d\beta^{n-2} \wedge \rho d\rho \wedge d\varphi$ on B is a negative volume form along B,
- the form $\rho\left[d\left(\frac{\beta}{\rho}\right) + \frac{\phi_1}{\rho}\Omega_1 + \frac{\phi_2}{\rho}\Omega_2\right]^{n-1} \wedge d\varphi$ is a negative volume form away from B,

i.e. if and only if

- β induces a negative contact form on B,
- for each $\theta \in \mathbb{S}^1$, $d\left(\frac{\beta}{\rho}\right) + \frac{\phi_1}{\rho}\Omega_1 + \frac{\phi_2}{\rho}\Omega_2$ induces a negative symplectic form on $\varphi^{-1}(\theta)$.

This concludes the proof of Lemma 7.4. □

7.2 The Bourgeois construction

Bourgeois gives in [Bou02] a construction that goes in the opposite direction with respect to that in [Lut79] recalled above. Indeed, using the notion of open book decompositions for contact manifolds (M^{2n-1}, ξ) from [Gir02], he constructs explicit contact structures on the total space of the principal $\mathbb{T}^2$-bundle $\pi\colon M \times \mathbb{T}^2 \to M$. More precisely, he proves the following:

Theorem 7.5 (Bourgeois)**.** *Let (M^{2n-1}, ξ) be a contact manifold and (B, φ) an open book decomposition of M supporting ξ.*

a. There is a smooth map $\phi = (\phi_1, \phi_2)\colon M \to \mathbb{R}^2$ defining the open book (B, φ) and such that $\gamma \wedge d\gamma^{n-2} \wedge d\phi_1 \wedge d\phi_2 \geq 0$ on M, where γ is any contact form defining ξ.

b. If ϕ is as in point a., then for any choice of coordinates (θ_1, θ_2) coordinates on $\mathbb{T}^2$ and for any contact form β defining ξ and adapted to the open book (B, φ), the $1-$form $\alpha := \beta + \phi_1 d\theta_1 - \phi_2 d\theta_2$ is a contact form on $M \times \mathbb{T}^2$.

We point out that the condition $\gamma \wedge d\gamma^{n-2} \wedge d\phi_1 \wedge d\phi_2 \geq 0$ in point a. of Theorem 7.5 is independent of the choice of form γ defining ξ: indeed, it is equivalent to the fact that ξ induces by restriction a contact structure on $\phi^{-1}(z)$, for each regular value z of ϕ. Moreover, the contact form α in point b. is clearly invariant under the natural $\mathbb{T}^2$-action on the principal $\mathbb{T}^2$-bundle $\pi\colon\ M \times \mathbb{T}^2 \to M$.

We now recall the proof of Theorem 7.5 in [Bou02], because it will be useful in the proof of Proposition 7.7.

Proof (Theorem 7.5). Let's start by proving point a.. Chose an auxiliary Riemannian metric on M and fix a tubular neighborhood $B \times D^2$ via the exponential map on the normal bundle of B (recall that the binding has trivial normal bundle by definition of open book decomposition). Up to rescaling the metric, we can suppose that each $\{pt\} \times D^2$ is a geodesic disk of radius 1.

Consider now a map $\tilde{\rho} : B \times D^2 \to \mathbb{R}_{\geq 0}$ which is smooth away from $B \times \{0\}$, equal to r for $r \in (0, \delta/2)$, strictly increasing with respect to the radial coordinate for $r \in (0, \delta)$ and equal to 1 on $r \in [\delta, 1]$. Here, $\delta > 0$ is so small that all $B \times \{pt\}$ in $B \times D^2_\delta$, equipped with the hyperplane field given by the restriction of ξ, are contact submanifolds of (M, ξ). Take then the natural extension $\rho : M \to \mathbb{R}_{\geq 0}$ of $\tilde{\rho}$ given by the constant function 1 defined outside of $B \times D^2$.

Considering the fibration $\varphi : M \setminus B \to \mathbb{S}^1$ as a map with values in $\mathbb{S}^1 \subset \mathbb{C}$, we can now obtain a map $\phi : M \to \mathbb{C}$ as $\phi := \rho \cdot \varphi$ and write $\phi(p) = (\phi_1(p), \phi_2(p)) \in \mathbb{C} = \mathbb{R}^2$ for each $p \in M$.

Then, we claim that ϕ is as in the statement of Theorem 7.5. Indeed, if we denote $\mathcal{N} :=\ B \times D^2_\delta$, then $\|\phi\|$ is 1 on $M \setminus \mathcal{N}$ and each $B \times \{pt\}$ in $\mathcal{N}$ is a fiber of ϕ and a contact submanifold of (M, ξ), so that $\gamma \wedge d\gamma^{n-2} \wedge d\phi_1 \wedge d\phi_2$, with γ defining ξ, is non$-$negative everywhere on M, as wanted.

We now prove point b. of Theorem 7.5. Let $\alpha = \beta + \phi_1 d\theta_1 - \phi_2 d\theta_2$ as in the statement. Then, we can compute $d\alpha = d\beta + d\phi_1 \wedge d\theta_1 - d\phi_2 \wedge d\theta_2$, which gives

$$\begin{aligned}(d\alpha)^n &= n\,(d\beta)^{n-1} \wedge (d\phi_1 \wedge d\theta_1 - d\phi_2 \wedge d\theta_2) + \\ &\quad - n\,(n-1)\,(d\beta)^{n-2} \wedge d\phi_1 \wedge d\theta_1 \wedge d\phi_2 \wedge d\theta_2\ ,\end{aligned}$$

all the other terms being zero because they contain as a factor $d\phi_1 \wedge d\phi_1$ or $d\phi_2 \wedge d\phi_2$. We then get

$$\begin{aligned}\alpha \wedge (d\alpha)^n &= n\,(d\beta)^{n-1} \wedge (\phi_1 d\phi_2 - \phi_2 d\phi_1) \wedge d\theta_1 \wedge d\theta_2 + \\ &\quad + n\,(n-1)\,\beta \wedge (d\beta)^{n-2} \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_1 \wedge d\theta_2 = \\ &= n\,(d\beta)^{n-1} \wedge \left(\rho^2 d\varphi\right) \wedge d\theta_1 \wedge d\theta_2 + \\ &\quad + n\,(n-1)\,\beta \wedge (d\beta)^{n-2} \wedge (\rho d\rho \wedge d\varphi) \wedge d\theta_1 \wedge d\theta_2\ .\end{aligned}$$

Here, we can conclude with the same argument already used in the proof of Proposition 6.13. Indeed, the first summand is everywhere non-negative and strictly positive away from $B \times \mathbb{T}^2$, because $d\beta$ is a symplectic form on the pages of the open book decomposition chosen (inducing the same orientation as ϕ). The second summand is everywhere non-negative by hypothesis, and it is strictly positive on $B \times \mathbb{T}^2$, because B is a contact submanifold of M (with same orientation as that induced by ϕ). This means that α is a positive contact form on $M \times \mathbb{T}^2$. ☐

Remark 7.6. *If $\phi = (\phi_1, \phi_2)$ satisfies point a. of Theorem 7.5, then, for all $\epsilon > 0$, the same is true for $\epsilon\phi = (\epsilon\phi_1, \epsilon\phi_2)$. In particular, the 1-forms $\alpha_\epsilon := \beta + \epsilon\phi_1 d\theta_1 - \epsilon\phi_2 d\theta_2$ always define positive contact structures by point b. of Theorem 7.5, which are moreover* *all isotopic by Gray's theorem. Notice that* $\alpha_0 = \beta$ *defines the hyperplane field* $\xi \oplus T\mathbb{T}^2$*, which is not a contact structure on* $M \times \mathbb{T}^2$*. Nonetheless, in Chapter 8 we will call it* contact fiber bundle *on* $M \times \mathbb{T}^2$ *and show that it plays an important role in understanding the properties of the construction in Theorem 7.5.*

We are now interested in studying in detail the relations between the Bourgeois construction and Lutz' one, already recalled in Section 7.1.

First of all, we point out that the application of Lutz' procedure after the Bourgeois construction gives back the original data.
More precisely, if we start from (M^{2n-1}, ξ) and (B, φ) supporting ξ, we can apply Theorem 7.5 to obtain the contact manifold $(M \times \mathbb{T}^2, \eta := \ker \alpha)$, with $\alpha = \beta + \phi_1 d\theta_1 - \phi_2 d\theta_2$. Then, if $\omega \in \Omega^1(M \times \mathbb{T}^2, \mathfrak{t})$ denotes the natural flat connection form $d\theta_1 \otimes e_1 + d\theta_2 \otimes e_2$ on $M \times \mathbb{T}^2$, the construction of open book preceding the statement of Theorem 7.2 gives back the original open book (B, φ) and contact structure ξ on M.

Vice versa, we now study the other possible composition, i.e. the application of the Bourgeois construction after the Lutz' one. We then have the following:

Proposition 7.7. *Let* η *be a* $\mathbb{T}^2$*-invariant contact structure on the principal bundle* $\pi\colon M \times \mathbb{T}^2 \to M$ *such that the couple* $(\mathcal{F}, \mathcal{F} \cap \eta)$*, where* $\mathcal{F} := TM \oplus \{0\}$*, is a contact foliation on* $M \times \mathbb{T}^2$*. Let* (B, φ) *and* ξ *be, respectively, the open book and the contact structure on* M *given by Theorem 7.2. Then, for any choice of coordinates* (θ_1, θ_2) *on* $\mathbb{T}^2$*, there is a couple* (β, ϕ) *as in the statement of Theorem 7.5 such that* $\eta = \ker(\alpha)$*, where* $\alpha = \beta + \phi_1 d\theta_1 - \phi_2 d\theta_2$ *on* $M \times \mathbb{T}^2$*, with* $\phi = (\phi_1, \phi_2)$*.*

In other words, provided the $\mathbb{T}^2$-invariant contact structure η we start with satisfies the fact that the push-forward of $\eta \cap (TM \times \{0\})$ via $d\pi$ is a contact structure ξ on M, we can apply the Bourgeois construction after the Lutz' one to get back the initial structure η.

Proof (Proposition 7.7). According to Theorem 7.5, it's enough to show that there are a contact form β defining ξ, a neighborhood $\mathcal{N} = B \times D^2$ of the binding B in M and a map $\phi = (\phi_1, \phi_2)\colon M \to \mathbb{R}^2$ such that

i. β is adapted to (B, φ),

ii. ϕ defines (B, φ) and $\beta \wedge d\beta^{n-2} \wedge d\phi_1 \wedge d\phi_2 \geq 0$ on M,

iii. $\eta = \ker(\beta + \phi_1 d\theta_1 - \phi_2 d\theta_2)$.

As already done in the proof of point a. of Theorem 7.5, we point out that for the point ii. above it's enough to find a map $\phi = (\phi_1, \phi_2)$ and a neighborhood $\mathcal{N} = B \times D^2$ of B in M such that ϕ defines (B, φ), $\|\phi\|$ is constant on $M \setminus \mathcal{N}$ and each $B \times \{pt\}$ in $\mathcal{N} = B \times D^2$ is a fiber of ϕ and a contact submanifold of (M, ξ).

Let's start by writing $\eta = \ker(\alpha')$, with $\alpha' = \beta' + \phi_1' dx - \phi_2' dy$, where β' is a contact form on M and $\phi' := (\phi_1', \phi_2')\colon M \to \mathbb{R}^2$ defines (B, φ). This can be done as in the proof of Theorem 7.2, using here the natural flat connection $d\theta_1 \otimes e_1 + d\theta_2 \otimes e_2$ on $\pi\colon M \times \mathbb{T}^2 \to M$.

Consider now a small normal neighborhood $\mathcal{N} = B \times D^2$ of B in M such that each $B \times \{pt\}$ is a fiber of ϕ' and a contact submanifold of (M, ξ). Such $\mathcal{N}$ exists because ϕ' defines the open book (B, φ) adapted to ξ.

Then, we claim that there are a contact form β'' for ξ on M and a map $\phi'' := (\phi_1'', \phi_2'')\colon M \to \mathbb{R}^2$ such that $\alpha'' := \beta'' + \phi_1'' dx - \phi_2'' dy$ still defines η, $d\beta''$ is symplectic on the fibers of $\varphi|_{M \setminus \mathcal{N}}$, and each $B \times \{pt\}$ in $\mathcal{N}$ is a fiber of ϕ''.
Indeed, let $\epsilon > 0$ be such that $\overline{\{\|\phi'\| < 2\epsilon\}} \subset \mathcal{N}$ and chose f to be a smooth function,

depending only on $\|\phi'\|$, non-increasing in $\|\phi'\|$, equal to $1/\epsilon$ on the set $\{\|\phi'\| < \epsilon\}$ and equal to $1/\|\phi'\|$ on the set $M \setminus \{\|\phi'\| < 2\epsilon\}$. Now, define $\beta'' \coloneqq f\beta'$, $\phi_1'' \coloneqq f\phi_1'$ and $\phi_2'' \coloneqq f\phi_2'$. Then, $\alpha'' = f\alpha'$, i.e. $\eta = \ker(\alpha'')$, and $B \times \{pt\} \subset \mathcal{N}$ are fibers of ϕ''. Moreover, we have $\|\phi''\| = 1$ on all $M \setminus \mathcal{N}$. The computation

$$\begin{aligned}\alpha'' \wedge (d\alpha'')^n &= n\,(d\beta'')^{n-1} \wedge \left(\|\phi''\|^2\, d\varphi\right) \wedge dx \wedge dy + \\ &\quad + n\,(n-1)\,\beta'' \wedge (d\beta'')^{n-2} \wedge [\|\phi''\|\, d\,(\|\phi''\|) \wedge d\varphi] \wedge dx \wedge dy\ ,\end{aligned}$$

analogous to the one recalled in the proof of Theorem 7.5, then tells us that $d\beta''$ has to be symplectic on the fibers of $\varphi|_{M\setminus\mathcal{N}}$, because α'' is a contact form and $\|\phi''\|$ is constant on $M \setminus \mathcal{N}$.

Now, Lemma 6.8 tells us that there is $g\colon\, M \to \mathbb{R}$ such that $\beta \coloneqq g\beta''$, which obviously still defines $\xi = \ker(\beta'')$, is adapted to the open book (B, φ). Moreover, from the explicit proof of Lemma 6.8, we see that g can be chosen to be equal to 1 on the complement of $\mathcal{N}$ and depending only on $\|\phi''\|$.

Define then $\phi_1 \coloneqq g\phi_1''$, $\phi_2 \coloneqq g\phi_2''$, $\alpha \coloneqq \beta + \phi_1 dx - \phi_2 dy$ and $\phi \coloneqq (\phi_1, \phi_2)$. Then, noticing that $\alpha = fg\alpha'$ and that each $B \times \{pt\}$ is also a fiber of ϕ, it is clear that the properties i. to iii. above are satisfied. This concludes the proof of Proposition 7.7. $\square$

Chapter 8

Bourgeois' construction revisited

In Section 7.2, we presented the construction in [Bou02] in terms of invariant structures on the principal $\mathbb{T}^2$-bundle $M \times \mathbb{T}^2 \to M$. Though, the following "orthogonal" point of view can also be adopted: the examples by Bourgeois are *contact fiber bundle structures* on the fiber bundle $M \times \mathbb{T}^2 \to \mathbb{T}^2$.

This change of perspective is motivated by [Pre07, KN07, NP10], where the contact fiber bundle structure of the examples from [Bou02] was used to give other important constructions of high dimensional contact manifolds. Actually, the contact fiber bundle structure, together with the existence of a flat connection, is what remains in the higher genus case, for example taking contact branched coverings of the constructions in [Bou02], as already proposed in [Bou02, Corollary 3].

The aim of this chapter is then to reinterpret and generalize the construction from Bourgeois using the notion of contact fiber bundles, as introduced in [Ler04], under the hypothesis of the existence of a flat connection.

More precisely, in Section 8.1 we recall the definitions and the main properties of contact fiber bundles.
Then, we use this notion to generalize the construction from [Bou02] recalled in Section 7.2. In particular, in Section 8.2 we take a general fibration admitting a flat contact connection and we consider on it two non-trivial subclasses of all its contact connections. The first subclass, described in Section 8.3, is characterized in terms of deformations to the flat contact connection, in a flavor similar to the notion of contactizations introduced in Definitions 5.11 and 5.22. The second subclass, included in the first, is a direct generalization of the examples from [Bou02] in the setting of contact fiber bundles and is presented in Section 8.4, where Proposition II.D from the introduction is also proven.
Lastly, in Section 8.5 we study the stability of the first class under the operation of contact branched covering.

8.1 Generalities

Let B^{2m}, M^{2n-1} and $V^{2m+2n-1}$ be smooth manifolds and $\pi\colon V \to B$ a smooth fiber bundle with fiber M. Denote by M_b the fiber of π over $b \in B$.

Suppose now that V, B are oriented, and give to each fiber M_b the natural orientation as preimage $\pi^{-1}(b)$.

Definition 8.1. [Ler04] A *contact fiber bundle* is a cooriented hyperplane field η on V such that for each fiber M_b of π the intersection $\xi_b := \eta \cap TM_b$ is a positive contact structure on M_b.

For simplicity, in the following we will also denote the data of a contact fiber bundle by (V, B, M, η).

Lemma 8.2. *[Ler04, Lemma 2.3] Let (V, B, M, η) be a contact fiber bundle and α a 1-form on V defining η. The distribution $\mathcal{H}$ defined as the $d\alpha|_\eta$-orthogonal of ξ_b in η is an Ehresmann connection on the bundle (M, V, B) and, at each point $p \in V$, we have $\eta(p) = \xi_{\pi(p)}(p) \oplus \mathcal{H}(p)$. Moreover, its holonomy over a path $\gamma : [0, 1] \to B$ is a contactomorphism between $\xi_{\gamma(0)}$ and $\xi_{\gamma(1)}$.*

In order to avoid issues about the completeness of the parallel transport, we will always assume M compact here.

We point out that the data of ξ_b for all $b \in B$ and $\mathcal{H}$ also allows to restore the hyperplane field η. Hence, we will call contact fiber bundle the data (V, B, M, η) as well as the data $(V, B, M, \{\xi_b\}_{b\in B}, \mathcal{H})$, without any distinction. Also, if v is a vector of T_bB, we denote by $v^\#(p)$ its horizontal lift to $\mathcal{H}(p)$, for all $p \in \pi^{-1}(b)$.

For the rest of Chapter 8, we actually focus on the case of B an oriented smooth surface Σ, as this will be setting for our future considerations.

Moreover, we remark that in Definition 8.1 we do not require η to be necessarily a contact structure on V. On the other hand, this will be the case we are interested in, hence we need a criterion that establishes when this is the case. In order to give a statement, we point out the following.

For a couple (X, Y) of vector fields on Σ, the vector field $[X^\#, Y^\#] - [X, Y]^\#$ on V is actually tangent to the fibers. Indeed, it is mapped to zero via the differential $d\pi$ because $d\pi$ commutes with the Lie bracket and $d\pi(Z^\#) = Z$ for every vector field Z on Σ.

What's more, the restriction of $[X^\#, Y^\#] - [X, Y]^\#$ on M_b depends only on $X(b)$ and $Y(b)$: this can easily be seen with the tensoriality criterion (which is stated precisely in [KMS93, Lemma 7.3] for instance). Hence, for every oriented basis (u, v) of $T_b\Sigma$, we have a well-defined vector field $[u^\#, v^\#] - [u, v]^\#$ on M_p. Then, we have the following:

Proposition 8.3 ([Ler04, Proposition 3.1])**.** *Let (V, Σ, M, η) a contact fiber bundle over an oriented surface Σ. The hyperplane field η is a positive contact structure on V if and only if for every $p \in \Sigma$ and every oriented basis (u, v) of $T_p\Sigma$, the vector field $[u^\#, v^\#] - [u, v]^\#$ on M_p is a negative contact vector field for ξ_p.*

Recall that a contact vector field is called *negative* if it is negatively transverse everywhere to the contact structure.

We also remark that [Ler04, Proposition 3.1] is actually more general, because the base space B is not assumed to be a surface but can have dimension $2n \geq 2$.

8.2 Fibrations with a flat contact connection

In this section we give an adaptation of [Ler04, Remark 3.2 and Theorem 3.6] to the case of *flat contact connections*. For the reader interested in the details, [Ler04, Section 3.2] deals with the case of principal $G-$bundles using the notion of contact moment map, under the implicit assumption of the Lie group G having dimension at least 1. The case of flat contact connection that follows corresponds then to the case of $G = \pi_1(\Sigma)$ of dimension 0, where Σ is the surface which is base of the contact fiber bundle. For this reason, we try to keep our description as self contained as possible.

In the following, we call *fiber bundle with contact fibers*, and denote it (V, Σ, M_ξ), the data of a fiber bundle (V, Σ, M) and a collection of a contact structure ξ_p on each fiber

M_p, depending smoothly on $p \in \Sigma$. In particular, on a fiber bundle with contact fibers (V, Σ, M_ξ), the structure of a contact fiber bundle, with the prescribed contact structures on the fibers, is equivalent to the data of a contact connection.

We focus in this section on those fiber bundles with contact fibers (V, Σ, M_ξ) admitting a structure of contact fiber bundle (V, Σ, M, η_0) with *flat connection*, i.e. such that the associated connection $\mathcal{H}_0$ has *zero curvature*.
We recall that the curvature of an Ehresmann connection $\mathcal{H}$ on a fiber bundle (V, Σ, M) is defined as follows. Via $\mathcal{H}$, each vector field Z on V can be uniquely decomposed as $Z = Z_h + Z_v$, where Z_h is *horizontal*, i.e. everywhere tangent to $\mathcal{H}$, and Z_v is *vertical*, i.e. tangent everywhere to the fibers of the fibration. The curvature R of $\mathcal{H}$ is then the 2-form on V with values in TV, i.e. $R \in \Omega^2(V, TV)$, defined by $R(Z, W) = [Z_h, W_h]_v$ for all vector fields Z, W on V. Frobenius' theorem gives the following equivalence: R is everywhere zero if and only if the connection $\mathcal{H}$ is a foliation on V.

The flatness of $\mathcal{H}_0$ allows us to give a nice presentation of (V, Σ, M, η_0).
Indeed, once fixed a certain fiber (M, ξ) of (V, Σ, M, η_0), we can define a representation $\rho : \pi_1(\Sigma) \to \mathrm{Diff}(M, \xi)$, where $\mathrm{Diff}(M, \xi)$ is the space of contactomorphisms of (M, ξ): for each class $c \in \pi_1(\Sigma)$, we consider the monodromy of the connection $\mathcal{H}_0$ over a (smooth immersed) representative of c. This gives a well defined ρ: indeed, the monodromy doesn't depend on the representative chosen, because $\mathcal{H}_0$ is a foliation, and it is a contactomorphism of the fibers, by Lemma 8.2.

If we denote by $\pi_\Sigma \colon \widetilde{\Sigma} \to \Sigma$ the universal cover of Σ, we get a well defined homomorphism $F : \widetilde{\Sigma} \times M \to V$ of fibrations over Σ given simply by $F(c, q) := \rho_c(q)$. Moreover, the differential of F sends the connection $T\widetilde{\Sigma} \oplus \{0\}$ of $\widetilde{\Sigma} \times M \to \widetilde{\Sigma}$ to the connection $\mathcal{H}_0$ of $\pi : V \to \Sigma$, and the contact structure $\{0\} \oplus \xi$ on the fiber of $\widetilde{\Sigma} \times M$ over $\widehat{p} \in \widetilde{\Sigma}$ to the contact structure ξ_p of the fiber M_p of V over $p = \pi_\Sigma(\widehat{p})$.

Also, if we denote by $\widetilde{\rho}$ the diagonal action of $\pi_1(\Sigma)$ on $\widetilde{\Sigma} \times M$ induced by the natural action on the first factor and by the action ρ on the second factor, F induces an isomorphism $f : \Sigma \times_{\widetilde{\rho}} M \to V$ of fiber bundles over Σ, where $\Sigma \times_{\widetilde{\rho}} M$ is the quotient of $\widetilde{\Sigma} \times M$ by $\widetilde{\rho}$. Moreover, f sends the tangent space of Σ to $\mathcal{H}_0$ and the contact structure $\{0\} \oplus \xi$ on the fiber of $\Sigma \times_{\widetilde{\rho}} M$ over p exactly to the contact structure ξ_p of the fiber M_p of V over p.

On this fiber bundle with contact fibers $(\Sigma \times_{\widetilde{\rho}} M, \Sigma, M_\xi)$, we can moreover explicitly describe all the contact fiber bundles giving the prescribed contact structure on the fibers. For this, we introduce the notion of *potential* of a connection. With a little abuse of notation and thanks to the isomorphism of contact vector bundles found above, we will write $V = \Sigma \times_{\widetilde{\rho}} M$ in the following.

Let's consider a contact connection $\mathcal{H}$ on $(V = \Sigma \times_{\widetilde{\rho}} M, \Sigma, M_\xi)$ and take the 1-form A, with values in the space $\mathfrak{X}_{fib}(V)$ of vector fields on a fiber, which is defined on Σ as follows: take a point $p \in \Sigma$, a vector $v \in T_p\Sigma$ and define $A_v := v^\# - \widehat{v}$, where $v^\#$ is the lift of v to $\mathcal{H}$ and $\widehat{v}$ is the lift of v to $\mathcal{H}_0$. This differential form A is called *potential of $\mathcal{H}$ relative to $\mathcal{H}_0$*. We point out that it clearly allows to recover $\mathcal{H}$ from $\mathcal{H}_0$.
Moreover, A has actually values in the space $\mathfrak{X}^{cont}_{fib}(V)$ of contact vector fields on a fiber: more precisely, for each $v \in T_p\Sigma$ the vector field A_v on the fiber M_p is a contact vector field for the contact manifold (M_p, ξ_p).

Vice versa, each differential form $A \in \Omega^1(\Sigma; \mathfrak{X}^{cont}_{fib}(V))$, such that for each $v \in T_p\Sigma$ the vector field A_v is contact for (M_p, ξ_p), actually defines, together with the flat contact connection $\mathcal{H}_0$, a contact connection $\mathcal{H}$, hence a contact bundle η on the fibration with contact fibers (V, Σ, M_ξ).

Now, between all these contact fiber bundles, we are interested in particular in those

that define a contact structure on the total space V. To describe this subclass in terms of the potential, we have to introduce another two tensors.
The first one is the *exterior derivative* $d_0A \in \Omega^2(\Sigma, \mathfrak{X}^{cont}_{fib}(V))$ of A, which is defined as follows: for all $p \in \Sigma$ and $u, v \in T_p\Sigma$, $d_0A(u,v) := [\widehat{X}, A_Y] - [\widehat{Y}, A_X] - A_{[X,Y]}$, where X, Y are vector fields tangent to Σ defined near p and such that $X(p) = u$ and $Y(p) = v$. We remark that this is indeed well defined because the quantity on the right hand side depends only on u and v: this can easily be checked using the tensoriality criterion.
The second tensor which we will need is the Lie bracket of A with itself, $[A, A] \in \Omega^2(\Sigma, \mathfrak{X}^{cont}_{fib}(V))$, given by $[A, A](u, v) := [A_X, A_Y]$, where u, v, X, Y are as in the definition of d_0A above. Again, the right hand side depends only on u, v, hence this is a well defined tensor.

We can now characterize the potentials that define contact structures:

Proposition 8.4. *On a flat contact fiber bundle $(V, \Sigma, M, \{\xi_p\}_{p\in\Sigma}, \mathcal{H}_0)$ with Σ surface, a contact connection $\mathcal{H}$ with vector potential A gives a contact structure η on the total space if and only if, for all p in Σ and all oriented basis (u, v) of $T_p\Sigma$, the vector field $d_0A(u, v) + [A_u, A_v]$ on M_b is a negative contact vector field for (M_p, ξ_p).*

Proof. This is a direct consequence of Proposition 8.3 and of the following computation: for all X, Y vector fields on Σ such that $X(p) = u$ and $Y(p) = v$,

$$\begin{aligned}
[u^\#, v^\#] - [u, v]^\# &= [\widehat{X} + A_X, \widehat{Y} + A_Y] - \left(\widehat{[X, Y]} + A_{[X,Y]}\right) \\
&= [\widehat{X}, A_Y] - [\widehat{Y}, A_X] - A_{[X,Y]} + [A_X, A_Y] + [\widehat{X}, \widehat{Y}] - \widehat{[X, Y]} \\
&\overset{(*)}{=} d_0A(u, v) + [A, A](u, v) \ ,
\end{aligned}$$

where $(*)$ comes from the definition of $d_0A, [A, A]$ and from the fact that $\mathcal{H}_0$ flat means $[\widehat{X}, \widehat{Y}] = \widehat{[X, Y]}$ for all X, Y vector fields on Σ. $\square$

8.3 Contact deformations of flat contact bundles

Fix for all this section a flat contact fiber bundle (V, Σ, M, η_0).

Definition 8.5. We say that a contact fiber bundle η defining a contact structure on the total space of the fiber bundle (V, Σ, M) is a *contact deformation* of the flat bundle (V, Σ, M, η_0) if there is a smooth family of contact fiber bundles $(\eta_s)_{s\in[0,1]}$ starting at η_0, ending at $\eta_1 := \eta$ and satisfying the followings:

1. for all $p \in \Sigma$ and all $s \in [0, 1]$, η_s defines the same ξ_p on the fiber M_p;
2. η_s defines a contact structure on V for all $s > 0$.

Thanks to Hypothesis 1, we could also rephrase the above notion in terms of a path of contact connections $\mathcal{H}_s$ interpolating between $\mathcal{H}_0$ and $\mathcal{H}$.

We point out that this definition is “non-empty”, i.e. given a flat contact fiber bundle (V, Σ, M, η_0), not all the contact fiber bundles for the same underlying fibration structure (V, Σ, M) and inducing the same contact structures ξ_p on each fiber M_p, are contact deformations of η_0.
For instance consider the contact fiber bundle structure on $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{T}^3$ which is given by the kernel η of $\alpha = d\theta + \cos(\theta)dx - \sin(\theta)dy$, where $\theta \in \mathbb{S}^1$ and (x, y) are coordinates on $\mathbb{T}^2$. This contact fiber bundle structure is a contact deformation of the

flat contact fiber bundle structure given by $\eta_0 = \ker(d\theta)$: the deformation is given by $\alpha_t := d\theta + t\cos(\theta)dx - t\sin(\theta)dy$, with $t \in [0,1]$.
We point out that, by [Gir99, Lemma 10], η admits prelagrangian tori only in the isotopy class of $\{pt\} \times \mathbb{T}^2$. Take now a diffeomorphism ψ of $\mathbb{T}^3$ sending (θ, x, y) to $(\theta + x, x, y)$. Then, $\psi^*\eta$ is still transverse to the $\mathbb{S}^1$ factor, hence it still is a contact fiber bundle on the chosen fibration, and obviously it still defines a contact structure on the total space. However, it has prelagrangian tori in an isotopy class which is different from that of the prelagrangian tori of η. According to [Vog16, Proposition 9.9], this implies that $\phi^*\eta$ cannot be a contact deformation of $\eta_0 = T\mathbb{T}^2 \oplus \{0\} \subset T\left(\mathbb{T}^2 \times \mathbb{S}^1\right)$.

We also remark that, even though the above definition is of a very similar flavor to Definitions 5.11 and 5.22, the objects they define behave differently. For instance, there is no uniqueness up to isotopy for contact deformations.
Indeed, if we take again the fiber bundle $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2$ where we see the fibers as contact manifolds $\left(\mathbb{S}^1, \ker(d\theta)\right)$, then the flat contact bundle defined by $\eta_0 = \ker(d\theta)$ actually admits as contact deformations every contact structure on $\mathbb{T}^3$ defined by $\alpha_n := d\theta + \cos(n\theta)dx - \sin(n\theta)dy$. Though, these are not isotopic one to the other as contact fiber bundles defining contact structures on the total space, because they are not even isomorphic as contact structures on $\mathbb{T}^3$ due to different Giroux torsion (see [Gir99]).

8.4 Bourgeois' contact structures

The aim here is to use what we defined in the previous sections to generalize the construction by Bourgeois recalled in Section 7.2. Let's start by reformulating it with this new terminology.

We start from the trivial fiber bundle $M \times \mathbb{T}^2 \to \mathbb{T}^2$ with fixed contact fiber $(M, \xi = \ker(\alpha))$, and we consider the flat contact fiber bundle structure $\xi \oplus T\mathbb{T}^2$ on the total space of the fibration. Once fixed an open book decomposition (B, φ) adapted to ξ on M and a particular contact form β with differential symplectic on the pages of φ, consider a function $\phi = (\phi_1, \phi_2) : M \to \mathbb{R}^2$ as in the statement of Theorem 7.5. Now take the contact vector fields X and Y on (M, ξ) associated, respectively, to the contact hamiltonians ϕ_1 and $-\phi_2$ via the contact form β, and define $A := -X \otimes dx - Y \otimes dy$, where (x, y) is a choice of coordinates for $\mathbb{T}^2$. A direct computation shows that the contact fiber bundle associated to this potential is exactly the kernel of the contact form $\alpha = \beta + \phi_1 d\theta_1 - \phi_2 d\theta_2$ given by Theorem 7.5. We also remark that, because X and Y are independent from the point of $\mathbb{T}^2$ in the product $M \times \mathbb{T}^2$, the 2-form $d_0 A$ is zero everywhere. Because α is a contact form, Proposition 8.4 then tells us that $[A, A]$ takes values in the space of *negative* contact vector fields of the fibers (M, ξ). In particular, $[X, Y]$ is a negative contact vector field.

We can then generalize the construction by Bourgeois via the following:

Definition 8.6. Let (V, Σ, M, η_0) be a flat contact fiber bundle with associated connection $\mathcal{H}_0$. We call *Bourgeois contact structure* each contact structure on the total space V given by a contact fiber bundle structure η on $V \to \Sigma$ with potential A such that $d_0 A = 0$.

We remark that the curvature R of the connection $\mathcal{H}$ determined by η_0 and A (i.e. the contact connection of η) is just $d_0 A + [A, A]$. In particular, this curvature has two terms which behave differently under rescaling $A \mapsto \epsilon A$, for $\epsilon > 0$: the term $d_0 A$ is rescaling linearly in ϵ, whereas $[A, A]$ is rescaling quadratically in it. Then, if we denote by R_ϵ the curvature associated to the connection $\mathcal{H}_\epsilon$ of potential ϵA with respect to η_0, the condition $d_0 A = 0$ is equivalent to the fact that $\frac{1}{\epsilon} R_\epsilon \to 0$ for $\epsilon \to 0$, which was the condition used to introduce Bourgeois contact structures in Chapter 1.

We now point out that Definition 8.6 is a non-trivial generalization of Bourgeois' construction, i.e. the class of Bourgeois contact structures is not exhausted by the examples on $M \times \mathbb{T}^2$ from [Bou02]:

Proposition 8.7. *There is a flat contact fiber bundle $(V, \mathbb{T}^2, M, \eta_0)$ that admits a Bourgeois contact structure and is non trivial, i.e. not isomorphic (as flat contact fiber bundle) to $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi_M \oplus T\mathbb{T}^2)$.*

We deduce Proposition 8.7 from the following:

Lemma 8.8. *Let (M, ξ) be a contact manifold, G a subgroup of the group of contactomorphisms of (M, ξ), and $\rho : \pi_1(\mathbb{T}^2) \to G$ a group homomorphism. Suppose that there is a G-invariant function $\phi = (\phi_1, \phi_2) \colon M \to \mathbb{R}^2$ defining a ($G-$invariant) open book (B, φ) on M supporting ξ.*
Let's also denote by β a G-invariant contact form for ξ on M such that $d\beta$ is symplectic on the fibers of φ, and by η_0 the flat contact bundle induced on $M \times_{\widetilde{\rho}} \mathbb{T}^2 \to \Sigma$ by the flat contact bundle $\xi \oplus T\mathbb{R}^2$ on $M \times \mathbb{R}^2 \to \Sigma$. Here, $\widetilde{\rho}$ is the action of $\pi_1(\mathbb{T}^2)$ on $M \times \mathbb{R}^2$ given by ρ on the first factor and by the natural action on the universal cover $\mathbb{R}^2 \to \mathbb{T}^2$ on the second factor.
Then, the hyperplane field η on $M \times_{\widetilde{\rho}} \mathbb{T}^2$, induced by $\ker(\beta + \phi_1 d\theta_1 - \phi_2 d\theta_2)$ on $M \times \mathbb{R}^2$, is a Bourgeois contact structure on the flat contact bundle $(M \times_{\widetilde{\rho}} \mathbb{T}^2, \mathbb{T}^2, M, \eta_0)$.

Once noticed that the form $\beta + \phi_1 d\theta_1 - \phi_2 d\theta_2$ on $M \times \mathbb{R}^2$ is invariant under the action $\widetilde{\rho}$, the above lemma can be shown exactly as Theorem 7.5; the proof is hence omitted.

Proof (Proposition 8.7). We recall that van Koert and Niederkrüger exhibited in [KN05] a particular open book decomposition for each Brieskorn manifold $W_k^{2n-1} \subset \mathbb{C}^{n+1}$, with supporting form α_k. In particular, the adapted open book decomposition is defined by a map $\phi \colon W_k^{2n-1} \to \mathbb{R}^2$ which is invariant under the action of a subgroup $SO(n)$ of strict contactomorphisms for the strict contact manifold (W_k^{2n-1}, α_k). More precisely, if $(z_0, \dots, z_n)$ are the coordinates of $\mathbb{C}^{n+1}$, $\mathrm{SO}(n)$ is just the subgroup of linear transformations of $\mathbb{C}^{n+1}$ fixing the first coordinate z_0 and acting on the sub$-$vector $(z_1, \dots, z_n)$ made of the other n coordinates by matrix multiplication; see [KN05] for the details. For simplicity, we denote the couple (W_k^{2n-1}, α_k) by (M, β).

Let now $\rho \colon \pi_1(\mathbb{T}^2) \to \mathrm{SO}(n)$ be defined by $\rho(a, b) = a \cdot f$ for each $(a, b) \in \mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$, where f is any element of $\mathrm{SO}(n)$ of order 2. Then, Lemma 8.8 tells us that the η on $M \times_{\widetilde{\rho}} \mathbb{T}^2$, induced by $\ker(\beta + \phi_1 d\theta_1 - \phi_2 d\theta_2)$ on $M \times \mathbb{R}^2$, is a Bourgeois contact structure on the flat contact bundle $(M \times_{\widetilde{\rho}} \mathbb{T}^2, \mathbb{T}^2, M, \eta_0)$. Here, η_0 is the flat contact bundle induced by $\xi \oplus T\mathbb{R}^2$ on $M \times \mathbb{R}^2 \to \mathbb{R}^2$.

A direct computation shows also that the potential $\widetilde{A}$ associated to the kernel of $\alpha := \beta + \phi_1 dx - \phi_2 dy$ on $M \times \mathbb{R}^2$ with respect to the trivial flat contact bundle $\widetilde{\eta}_0 := \xi \oplus T\mathbb{R}^2$ is given by $\widetilde{A} = -X \otimes dx - Y \otimes dy$, where X, Y are respectively the contact vector fields on (M, ξ) such that $\beta(X) = \phi_1$ and $\beta(Y) = -\phi_2$. In particular, it satisfies $d_0 \widetilde{A} = 0$. Hence, the potential A of $\eta = \ker(\beta)$ on V (with respect to the flat connection η_0 on V induced by $\widetilde{\eta}_0$ on $M \times \mathbb{R}^2$) will also satisfy $d_0 A = 0$. In other words, this is an example of Bourgeois contact structure on the flat contact fiber bundle $(V, \mathbb{T}^2, M, \eta_0)$, as wanted.

The only thing left to show is that $(V, \mathbb{T}^2, M, \eta_0)$ is not isomorphic to the trivial flat contact bundle $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$.
Recall that flat connections define foliations, according to Frobenius' theorem. In particular, the connection $\mathcal{H}_0$ associated to η_0 defines a foliation $\mathcal{F}_0$ by tori $\mathbb{T}^2$ on the total

space V, which is also transverse to the fibers of $\pi\colon V \to \mathbb{T}^2$. Moreover, because of our particular choice of $\rho\colon \pi_1(\mathbb{T}^2) \to \mathrm{SO}(n)$, each leaf L of $\mathcal{F}_0$ intersects every fiber twice. Now, the connection $T\mathbb{T}^2$ on the trivial bundle $p\colon M\times\mathbb{T}^2 \to \mathbb{T}^2$ gives a foliation $\mathcal{F}_1$ with leaves $\{pt\}\times\mathbb{T}^2$, which only intersects each fiber once. In particular, no isomorphism Ψ of fiber bundles (equipped with connections) over $\mathbb{T}^2$ can exist between $(V,\mathbb{T}^2,M,\mathcal{H}_0)$ and $(M\times\mathbb{T}^2,\mathbb{T}^2,M,\{0\}\oplus T\mathbb{T}^2)$: indeed it should send $\mathcal{F}_0$ to $\mathcal{F}_1$, which is not possible because their leaves intersects the fibers (of respectively π and p) a different number of times. $\square$

We point out that the motivation behind Definition 8.6 doesn't only consist in Proposition 8.7 above. Indeed, we now show that the condition $d_0A = 0$ above, while being general enough to be satisfied by a class of contact structures strictly larger than those given by the construction in [Bou02], is also strong enough to ensure some nice properties from the points of view of contact deformations, weak fillability and adapted open book decompositions.

We start by noticing that each Bourgeois contact structure η is in particular a contact deformation of the underlying flat contact bundle η_0.
Indeed, we have the natural path of contact bundle structures $(\eta_t)_{t\in[0,1]}$ that is given by the potential $A_t := t\cdot A$ with respect to $\mathcal{H}_0$, where A is the potential of η. This has the wanted starting point and ending point and gives a contact structure η_t for $t>0$, according to Proposition 8.4, because $d_0A_t = t\cdot d_0A$ is zero and $[A_t,A_t]=t^2[A,A]$ is negatively transverse to $\xi=\eta\cap TM$ for $t>0$.
This property is a direct generalization of the fact that the examples in [Bou02] are actually all contact deformations of the trivial flat contact bundle on $M\times\mathbb{T}^2$ (see Remark 7.6).

The study of weak fillability of Bourgeois contact structures is postponed to Section 9.1.1, where Proposition 9.1 states that if (M,ξ) is weakly fillable then a Bourgeois contact structure η on the flat contact bundle $(M\times\mathbb{T}^2,\mathbb{T}^2,M,\xi\oplus T\mathbb{T}^2)$ is weakly fillable too (see also [MNW13, Example 1.1] and [LMN18, Theorem A.a], that both deal with the particular case of the contact structures obtained as in [Bou02]). This stability of weak fillability is also true in a more general case, as stated in Proposition 9.5.

As far as adapted open book decompositions are concerned, we have the following property: given a Bourgeois contact structure η on the flat contact bundle (V,Σ,M,η_0), we can "naturally" associate to each point b of Σ an open book decomposition of the fiber M_b supporting the contact structure ξ_b. In order to give a precise statement, let's introduce some notations.
Consider a smooth contact bundle η on $X\to Y$, where X is not assumed to be closed. Denote by Λ the space of maps $\Phi\colon X\to\mathbb{R}^2$ such that, for each $y\in Y$:

i. the restriction $\phi_y := \Phi|_{\pi^{-1}(y)}\colon \pi^{-1}(y)\to\mathbb{R}^2$ is transverse to $\{0\}\subset\mathbb{R}^2$,

ii. the map $\frac{\phi_y}{\|\phi_y\|}\colon \pi^{-1}(y)\setminus\phi_y^{-1}(0)\to\mathbb{S}^1$ is a fibration,

iii. $(\phi_y^{-1}(0),\frac{\phi_y}{\|\phi_y\|})$, which is an open book decomposition of $\pi^{-1}(y)$ according to points i., ii. above and to Lemma 6.7, is moreover adapted to the contact structure $\eta\cap T\left(\pi^{-1}(y)\right)$.

Notice that this space Λ comes endowed with a natural $C^\infty-$topology induced by that on the space of functions $X\to\mathbb{R}^2$ in which it is contained. Consider then the quotient ${}^{\Lambda}/_{\sim}$ of Λ by the relation $\sim$ defined as follows: $\Phi_1,\Phi_2\in\Lambda$ are equivalent via $\sim$ if there is a positive function $f: X\to\mathbb{R}$ such that $\Phi_2 = f\Phi_1$. Notice that ${}^{\Lambda}/_{\sim}$ inherits a natural topology as quotient of the topological space Λ. We then call *smooth $Y-$family of open*

books in X (adjusted to η) each element of ${}^{\Lambda}/\sim$.
Remark also that if we have a contact bundle η on a smooth fiber bundle $\pi\colon X \to Y$ and $f\colon Z \to Y$ is a smooth map, we can define the *pullback contact bundle* $f^*\eta$ on the pullback bundle

$$f^*X := \{(z,x) \in Z \times X \,|\, f(z) = \pi(x)\} \xrightarrow{\mathrm{pr}_X} X$$

as the vector sub-bundle $\{W \in T(f^*X) |\, d(\mathrm{pr}_X)(W) \in \eta\}$ of $T(f^*X)$, where $\mathrm{pr}_X\colon Z \times X \to X$ is the projections on the second factor. This is indeed a contact bundle on $f^*X \to Z$ because its trace on each fiber $(\mathrm{pr}_Z)^{-1}(z) \cap f^*X = \{z\} \times \pi^{-1}_{f(z)}$ of $f^*X \to Z$ is exactly $\{0\} \oplus \eta_{f(z)}$; here, $\mathrm{pr}_Z\colon Z \times X \to Z$ is the projections on the first factor.
We now go back to the specific case of Bourgeois contact structure η on a flat contact bundle (V, Σ, M, η_0). Denote, for all $b \in \Sigma$, (M_b, ξ_b) the contact fiber over b, i.e. $M_b := \pi^{-1}(b)$ and $\xi_b := \eta_0 \cap TM_b$, where $\pi\colon V \to M$ is the given fiber bundle. We then say that a couple (K, φ) is a *fiber adapted open book* if there is a point $b \in \Sigma$ such that (K, φ) is an open book decomposition of M_b supporting ξ_b.
Denote lastly by $\mathrm{pr}\colon F\Sigma \to \Sigma$ the frame tangent bundle of Σ, i.e. the (principal) bundle over Σ with fiber over $b \in \Sigma$ given by the set of all oriented basis of $T_b\Sigma$. Then, we can state the following result:

Proposition 8.9. *Given a Bourgeois contact structure η on the flat contact bundle (V, Σ, M, η_0), there is a map*

$$\Psi_\eta\colon F\Sigma \to \{\textit{fiber adapted open book}\}$$

satisfying the following properties:

i. Ψ_η sends, for all $b \in \Sigma$, each positive basis of $T_b\Sigma$ to an open book decomposition of M_b adapted to ξ_b;

ii. for each smooth path $\gamma\colon [0,1] \to F\Sigma$, the composition

$$\Psi_\eta \circ \gamma\colon [0,1] \to \{\textit{fiber adapted open books}\}$$

describes a smooth $[0,1]-$family of open books in $\gamma^* \mathrm{pr}^* V$ adjusted to $\gamma^* \mathrm{pr}^* \eta$.

$$\begin{tikzcd}
(\gamma^* \mathrm{pr}^* V, \gamma^* \mathrm{pr}^* \eta) \arrow[r] \arrow[d] & (\mathrm{pr}^* V, \mathrm{pr}^* \eta) \arrow[r] \arrow[d] & (V, \eta) \arrow[d, "\pi"] \\
I \arrow[r, "\gamma"] & F\Sigma \arrow[r, "\mathrm{pr}"] & \Sigma
\end{tikzcd}$$

From the above result, we can deduce a more precise version of Proposition II.D stated in Section 1.2:

Corollary 8.10. *The map Ψ_η in Proposition 8.9 induces a well defined*

$$\psi_\eta\colon \Sigma \to \{\textit{fiber adapted open books}\} /\sim\ ,$$

where $(K_0, \varphi_0) \sim (K_1, \varphi_1)$ if they are both adapted open books on a same fiber (M_b, ξ_b) and there is an isotopy $(f_t)_{t\in[0,1]}$ of the fiber M_b, starting at $\phi_0 = \mathrm{Id}$, such that $K_1 = f_1(K_0)$, $\varphi_1 = \varphi_0 \circ f_1^{-1}$ and $\left(f_t\left(K_0\right), \varphi_0 \circ f_t^{-1}\right)$ is an open book of M_b adapted to ξ_b. In other words, η uniquely determines an isotopy class of adapted open book decompositions for each fiber (M_b, ξ_b) of (V, Σ, M, η_0).
Moreover, if $\eta = \ker\alpha$ is the Bourgeois contact structure on $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$ given by Theorem 7.5 starting from an open book (B, φ) for (M, ξ), then the corresponding map ψ_η sends each $b \in \mathbb{T}^2$ to an isotopy class of adapted open books on (M_b, ξ_b) that (via the natural identification $(M_b, \xi_b) \simeq (M, \xi)$ given by the projection $M \times \mathbb{T}^2 \to M$) corresponds to the isotopy class of the original open book (B, φ) on (M, ξ).

Proof (Corollary 8.10). Given $b \in \Sigma$, consider an ordered basis (u, v) of $T_b\Sigma$ and define $\psi_\eta(b)$ as the class of $\Psi_\eta(u, v)$ under the relation $\sim$. We then need to show that this is well defined.
Suppose (u', v') is another ordered basis of $T_b\Sigma$; we want to show that $\Psi_\eta(u, v) \sim \Psi_\eta(u', v')$. Choose a curve $\gamma\colon [0, 1] \to F\Sigma$ with image contained in the fiber $\mathrm{pr}^{-1}(b)$ of $\mathrm{pr}\colon F\Sigma \to \Sigma$ and such that $\gamma(0) = (u, v)$ and $\gamma(1) = (u', v')$. Then, according to point ii. of Proposition 8.9, $\Psi_\eta \circ \gamma$ gives a smooth $[0, 1]-$family of open books in $\gamma^* \mathrm{pr}^* V$ adjusted to $\gamma^* \mathrm{pr}^* \eta$. Now, $\gamma^* \mathrm{pr}^* V = [0, 1] \times M_b$ and $\gamma^* \mathrm{pr}^* \eta = T([0, 1]) \oplus \xi_b$, so that we actually have, via the natural projection $[0, 1] \times M_b \to M_b$, a smooth family of open books $(K_t, \varphi_t)_{t\in[0,1]}$ on M_b supporting ξ_b. Because a smooth path of open book decompositions comes from an isotopy as described in the statement, this actually means that (K_0, φ_0) is isotopic to (K_1, φ_1); in other words, $\Psi_\eta(u, v) \sim \Psi_\eta(u', v')$ as wanted.

The last statement about the construction by Bourgeois follows directly from the definition of Ψ_η and from point (c) of Proposition 6.1. Indeed, let $\eta = \ker(\beta + \phi_1 d\theta_2 - \phi_2 d\theta_2)$ be the Bourgeois contact structure on the flat contact bundle $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$ given by Theorem 7.5 starting from open book (B, φ) of M adapted to ξ. As already observed in the beginning of Section 8.4, we can compute that $A_{\partial_{\theta_1}}$ and $A_{\partial_{\theta_2}}$ are respectively the contact vector fields on (M, ξ) of contact hamiltonians $-\phi_1$ and ϕ_2 (via β), with $\phi = (\phi_1, \phi_2)$ defining (B, φ). Then, we can see that point (c) of Proposition 6.1 with $X := A_{\partial_{\theta_1}}$ and $Y = A_{\partial_{\theta_2}}$ gives exactly the open book (B, φ'), where φ' is obtained from φ by composition with the antipodal map $\mathbb{S}^1 \to \mathbb{S}^1$.
In other words, for all $b \in \mathbb{T}^2$, if $(\partial_{\theta_1}, \partial_{\theta_2})$ is the oriented base of $T_b\mathbb{T}^2$ coming from the choice of coordinates $(\theta_1, \theta_2) \in \mathbb{T}^2$ as in the statement of Theorem 7.5, then $\Psi_\eta(\partial_{\theta_1}, \partial_{\theta_2}) = (B, \varphi')$. In particular, $\psi_\eta(b)$ is the isotopy class of (B, φ'), which coincides with that of (B, φ). □

We now derive Proposition 8.9 as a consequence of Proposition 6.1:

Proof (Proposition 8.9). Let's start by defining Ψ_η. Let A be the potential for η relative to the flat contact connection $\mathcal{H}_0$ of η_0. Then, for each $b \in \Sigma$ and each positive basis (u, v) of $T_b\Sigma$, A_u and A_v are two vector fields on M_b which are contact for ξ_b. Moreover, according to Proposition 8.4 (and by definition of Bourgeois contact structure), $[A_u, A_v]$ is negatively transverse to ξ_b; then, Proposition 6.1 gives an open book decomposition $OBD_{(u,v)}$ for M_b supporting ξ_b. Because $OBD_{(u,v)}$ is also a fiber adapted open book, we can define $\Psi_\eta(u, v) := OBD_{(u,v)}$. In particular, it is clear that point i. of Proposition 8.9 is satisfied.

Let's now prove point ii.. Consider $b \in \Sigma$ and a basis (u, v) of $T_b\Sigma$. Let α be a $1-$form defining η on V and denote α_b its restriction to the fiber M_b of $V \to \Sigma$. From the explicit proof of point (c) of Proposition 6.1, we can see that $\Psi_\eta(u, v)$ is the open book defined by the smooth function $\phi_{(u,v)} := (\alpha_b(A_u), -\alpha_b(A_v)) : M_b \to \mathbb{R}^2$.
By definition of pullback smooth bundle and pullback contact bundle, $\Psi_\eta \circ \gamma$ describes then the smooth $[0, 1]-$family of open books in $\gamma^* \mathrm{pr}^* V$ adjusted to $\gamma^* \mathrm{pr}^* \eta$ which is given by the conformal class of $\Psi_\gamma\colon \gamma^* \mathrm{pr}^* V \to \mathbb{R}^2$ defined, for all $(t, p) \in \gamma^* \mathrm{pr}^* V = \{(t, p) \in [0, 1] \times V \,|\, \mathrm{pr} \circ \gamma(t) = \pi(p)\}$, by

$$\Psi_\gamma(t, p) := \left((\mu^*\alpha)_{(t,p)}\left(A_{\gamma_1(t)}(p)\right), (\mu^*\alpha)_{(t,p)}\left(A_{\gamma_2(t)}(p)\right)\right) ,$$

where, for each $t \in [0, 1]$, $\gamma_1(t)$ and $\gamma_2(t)$ are the two vectors of the (ordered) basis $\gamma(t) \in F\Sigma$ and where $\mu\colon \gamma^* \mathrm{pr}^* V \to V$ is just the restriction of the projection $\mathrm{pr}_V\colon [0, 1] \times V \to V$ to $\gamma^* \mathrm{pr}^* V$. Notice that $A_{\gamma_1(t)}(p)$ and $A_{\gamma_2(t)}(p)$ are well defined because $(t, p) \in \gamma^* \mathrm{pr}^* V$. This concludes the proof of point ii. of Proposition 8.9. □

We lastly point out another property of Bourgeois contact structures in the case of trivial flat contact bundle: on the flat contact bundle $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$, where ξ is contact on M, Bourgeois contact structures can also be studied from the point of view of the natural $\mathbb{T}^2$-action on the principal bundle, in the spirit of Chapter 7. Let's give some precise statements.

We start by noticing that the potential A of a contact bundle η, with respect to the natural flat connection $\{0\} \oplus T\mathbb{T}^2 \subset T\left(M \times \mathbb{T}^2\right)$ on $\pi\colon\, M \times \mathbb{T}^2 \to \mathbb{T}^2$, can actually be seen as a 1-form defined on $\mathbb{T}^2$ and with values in the contact vector fields of (M, ξ), thanks to the canonical identification of each fiber of π with M. Then, we have a natural inclusion map

$$i\colon \left\{ \begin{array}{c} \mathbb{T}^2 - \text{invariant contact bundles} \\ \text{on } (M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2), \text{ defining} \\ \text{a contact structure on } M \times \mathbb{T}^2 \end{array} \right\} \to \left\{ \begin{array}{c} \text{Bourgeois} \\ \text{contact structures} \end{array} \right\} .$$

Indeed, a direct computation shows that each $\mathbb{T}^2$-invariant contact bundle η has potential A invariant under the $\mathbb{T}^2$-action, which hence satisfies $d_0 A = 0$.

The other way around, we have the following:

Proposition 8.11. *Let η be a Bourgeois contact structure on the flat contact bundle $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$ and denote by A its potential. The average $\overline{A}$ of A, taken under the natural $\mathbb{T}^2$-action, is the potential of a $\mathbb{T}^2$-invariant Bourgeois contact structure $\overline{\eta}$ on $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$.*

In particular, taking the average of the potential gives a well defined map

$$F\colon \left\{ \begin{array}{c} \textit{Bourgeois} \\ \textit{contact structures} \end{array} \right\} \to \left\{ \begin{array}{c} \mathbb{T}^2 - \textit{invariant contact bundles} \\ \textit{on } (M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2), \textit{ defining} \\ \textit{a contact structure on } M \times \mathbb{T}^2 \end{array} \right\} ,$$

which satisfies $F \circ i = \mathrm{Id}$.

Proof (Proposition 8.11). It is clear that the average $\overline{A}$ satisfies $d_0\overline{A} = 0$, because it is invariant under the $\mathbb{T}^2$-action. By Proposition 8.4, what we need to show is then that $[\overline{A}, \overline{A}]$ is with values in the *negative* contact vector fields for (M, ξ). Let's start by analyzing this condition more explicitly.
Write $A = -X \otimes dx - Y \otimes dy$, with (x, y) coordinates on $\mathbb{T}^2 = \left(\mathbb{R}/2\pi\mathbb{Z}\right)^2$ and X, Y a $\mathbb{T}^2$-family of vector fields on M parametrized smoothly by (x, y). Then, we know that $[X, Y]$ is negatively transverse to ξ everywhere on M, for all $(x, y) \in \mathbb{T}^2$, and we want to show that their averages $\overline{X}, \overline{Y}$ are such that $[\overline{X}, \overline{Y}]$ is also negatively transverse to ξ everywhere on M.

Notice that, if Z, W are $\mathbb{T}^2$-parametric vector fields on M, it is not true in general that the $\mathbb{T}^2$-average of $[Z, W]$ is equal to the Lie bracket of the averages of Z and W. This being said, what we want to show here is that this is actually true for X, Y, thanks to the additional condition $d_0 A = 0$.

Now, X, Y can be seen as smooth functions from $\mathbb{T}^2$ to the space of vector field on M, which has a natural structure of vector space over $\mathbb{R}$. As such, they both admit a complex Fourier series expansion

$$X = \sum_{m,n \in \mathbb{Z}} e^{i(mx+ny)} X_{m,n} \quad \text{and} \quad Y = \sum_{h,k \in \mathbb{Z}} e^{i(hx+ky)} Y_{h,k} \ , \tag{8.1}$$

where, for all $m, n, h, k \in \mathbb{Z}$, $X_{m,n}, Y_{h,k}$ are complex vector fields on M, i.e. sections of the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C} \to M$. Because X, Y are actually real, we have the following condition on the coefficients:

$$\overline{X_{m,n}} = X_{-m,-n} \quad \text{and} \quad \overline{Y_{h,k}} = Y_{-h,-k} \ , \text{ for all } m, n, h, k \in \mathbb{Z}, \tag{8.2}$$

where $\overline{X_{m,n}}$ denotes here the conjugated of $X_{m,n}$ and similarly for $\overline{Y_{h,k}}$.

What's more, the condition $d_0 A = 0$ also gives some information on the Fourier coefficients. We have indeed the following:

Claim 8.12. *$d_0 A = 0$ if and only if $-\frac{\partial}{\partial x} Y + \frac{\partial}{\partial y} X = 0$*

Proof (Claim 8.12). We can explicitly compute

$$\begin{aligned} d_0 A\left(\partial_x, \partial_y\right) &= \left[\widehat{\partial}_x, A_{\partial_y}\right] - \left[\widehat{\partial}_y, A_{\partial_x}\right] - A_{[\partial_x, \partial_y]} \\ &\overset{(i)}{=} -\left[\widehat{\partial}_x, Y\right] + \left[\widehat{\partial}_y, X\right] \\ &\overset{(ii)}{=} -\frac{\partial}{\partial x} Y + \frac{\partial}{\partial y} X \ , \end{aligned}$$

where (i) comes from the fact that ∂_x and ∂_y commute and (ii) follows from the expression in coordinates of the Lie bracket. $\square$

A straightforward computation shows that Claim 8.12 is equivalent to the following condition:

$$m Y_{m,n} = n X_{m,n} \quad \text{for all } m, n \in \mathbb{Z} \ . \tag{8.3}$$

Notice now that the averages of X and Y are, respectively, $X_{0,0}$ and $Y_{0,0}$, which are then in particular real vector fields on M. To avoid confusion with the conjugation, we will hence drop the notation $\overline{X}$ and $\overline{Y}$ for the averages and just denote them by $X_{0,0}$ and $Y_{0,0}$ instead.

Let $[.\,,.]_{\mathbb{C}}$ be the Lie bracket induced on the complex vector space of the sections of $TM \otimes \mathbb{C} \to M$ by the Lie bracket $[.\,,.]$ on the space of tangent vector fields on M. We can then compute:

$$\begin{aligned} [X, Y] &= \left[\sum_{m,n\in\mathbb{Z}} e^{i(mx+ny)} X_{m,n} \, , \, \sum_{h,k\in\mathbb{Z}} e^{i(hx+ky)} Y_{h,k} \right]_{\mathbb{C}} \\ &\overset{(a)}{=} \sum_{m,n\in\mathbb{Z}} \sum_{h,k\in\mathbb{Z}} e^{i[(m+h)x+(n+k)y]} \left[X_{m,n}, Y_{h,k}\right]_{\mathbb{C}} \\ &\overset{(b)}{=} \sum_{r,s\in\mathbb{Z}} e^{i(rx+sy)} \left(\sum_{m,n\in\mathbb{Z}} \left[X_{m,n}, Y_{r-m,s-n}\right]_{\mathbb{C}} \right) \ , \end{aligned}$$

where the equality (a) comes from the fact that the Lie bracket is $\mathbb{C}-$bilinear and is taken on each fiber $M \times \{pt\}$ of $M \times \mathbb{T}^2 \to \mathbb{T}^2$ (where the exponentials are constant), and the equality (b) comes from replacing $r = m + h$ and $s = n + k$.

The above computation shows that $[X, Y]$ has Fourier coefficients

$$[X, Y]_{r,s} = \sum_{m,n\in\mathbb{Z}} \left[X_{m,n}, Y_{r-m,s-n}\right]_{\mathbb{C}} \tag{8.4}$$

for $r, s \in \mathbb{Z}$.

In particular, its average is given by

$$\begin{aligned}
[X,Y]_{0,0} &= \sum_{m,n\in\mathbb{Z}} [X_{m,n}, Y_{-m,-n}]_{\mathbb{C}} \\
&\overset{(a)}{=} [X_{0,0}, Y_{0,0}] + \sum_{m,n\in\mathbb{Z}\setminus\{0\}} [X_{m,n}, Y_{-m,-n}]_{\mathbb{C}} \\
&\overset{(b)}{=} [X_{0,0}, Y_{0,0}] + \sum_{m,n\in\mathbb{Z}\setminus\{0\}} \frac{m}{n}\, [Y_{m,n}, \overline{Y_{m,n}}]_{\mathbb{C}} \\
&\overset{(c)}{=} [X_{0,0}, Y_{0,0}] - 2i \sum_{m,n\in\mathbb{Z}\setminus\{0\}} \frac{m}{n}\, [\Re Y_{m,n}, \Im Y_{m,n}] \\
&\overset{(d)}{=} [X_{0,0}, Y_{0,0}]\ ,
\end{aligned}$$

where $\Re Y_{m,n}$ and $\Im Y_{m,n}$ denote respectively the real and imaginary part of $Y_{m,n}$. Moreover, (a) comes from the fact that $X_{m,n}$ is zero if $m=0, n\neq 0$ and $Y_{-m,-n}$ is zero if $n=0, m\neq 0$ by Equation (8.3), (b) comes from Equations (8.2) and (8.3), (c) comes from the $\mathbb{C}-$bilinearity of $[.,.]_{\mathbb{C}}$ and the anti-symmetry of $[.,.]$ and, finally, (d) comes from the fact that $[X,Y]_{0,0}$ is a (real) tangent vector field, because average of $[X,Y]$, hence has zero imaginary part.

Because $[X,Y]$ is negatively transverse to ξ everywhere on M for all $(x,y)\in\mathbb{T}^2$, its average $[X,Y]_{0,0}=[X_{0,0},Y_{0,0}]$ is also negatively transverse to ξ everywhere on M.

Now, notice that the average $\overline{A}$ of A is exactly $\overline{A}=-X_{0,0}\otimes dx - Y_{0,0}\otimes dy$. Then, the fact that $[X_{0,0},Y_{0,0}]$ is everywhere negatively transverse to ξ means that $[\overline{A},\overline{A}]$ is a $1-$form on $\mathbb{T}^2$ with values in the negative contact vector fields for (M,ξ). This concludes the proof of Proposition 8.11. □

Remark 8.13. *In analogy with the case of Bourgeois contact structures, we could have also considered, on a flat contact fiber bundle (V,Σ,M,η_0), the class of contact structures η on V given by a contact fiber bundle structures having potential A with $[A,A]=0$.*

For such an η, Proposition 8.4 tells us that d_0A is with values in the negative contact vector fields of the fibers. Such a condition, though, is not compatible with the fact that the surface Σ is closed.
Indeed, by explicit computations (analogous to those in the proof of Lemma 9.2 in the following) it can be proved that this condition on d_0A implies the existence of an exact volume form on Σ. Now, the latter can't exist if Σ is closed, according to Stoke's theorem.

Moreover, even if we allow Σ to have boundary, we do not recover all the informations on the fiber that we have with a Bourgeois contact structure. More precisely, we can't recover in general an (isotopy class of) open book decomposition supporting the contact structure on the fiber.
For instance, consider on the flat contact bundle $(M\times\Sigma,\Sigma,M,\xi_M\oplus T\Sigma)$ the contact fiber bundle structure $\eta=\ker(\alpha+\lambda)$, with $\xi_M=\ker\alpha$ and $d\lambda$ symplectic on Σ (that hence has non$-$empty boundary). Then, an explicit computation shows that $A=-R_\alpha\otimes\lambda$, where R_α is the Reeb vector field of α. In particular, $[A,A]=0$ and $d_0A=-R_\alpha\otimes d\lambda$, and we do not have any way to recover an (isotopy class of) open book decomposition on M from A.

8.5 Branched coverings as contact deformations

We show in this section that the class of contact fiber bundles that are contact deformations of a flat contact fiber bundle is stable under the operation of contact branched coverings. More precisely we have the following:

Proposition 8.14. *Let (V, Σ, M, η_0) be a flat contact fiber bundle and $p : \widehat{\Sigma} \to \Sigma$ a branched covering map that lifts to a branched covering map $\widehat{p} : \widehat{V} \to V$. Consider now the pull-back flat contact fiber bundle $(\widehat{V}, \widehat{\Sigma}, M, \widehat{\eta}_0)$ induced by p, i.e. $\widehat{\eta}_0 := \widehat{p}^*\eta_0$. If η is a contact deformation of η_0, then there is a contact branched covering $\widehat{\eta}$ of η to $\widehat{V}$ that is a contact deformation of $\widehat{\eta}_0$.*

Proof. This follows essentially from the explicit formula for the contact branched covering in the proof of Lemma 5.10.

More precisely, take a smooth family of 1-forms $(\alpha_t)_{t\in[0,1]}$ on V defining the family η_t which, by definition of contact deformation, interpolates between $\eta = \ker(\alpha_1)$ and $\eta_0 = \ker(\alpha_0)$ in such a way that η_t is contact for $t > 0$ and for all t the fibers of the fibration $\pi : V \to \Sigma$ are contact submanifolds, with induced contact structure independent of t.

According to the proof of Lemma 5.10, we can chose $\widehat{\eta}$ on $\widehat{V}$ to be the kernel of $\widehat{\alpha} = \widehat{p}^*\alpha_1 + \epsilon g(r)r^2 d\theta$, with the same notations as in that proof, using the particular choice of closed form $\gamma = d\theta$ as connection on the trivial unit normal bundle of M in V. Define now $(\widehat{\alpha}_t)_{t\in[0,1]}$ by $\widehat{\alpha}_t = \widehat{p}^*\alpha_t + t\epsilon g(r)r^2 d\theta$. We remark that $\ker(\widehat{\alpha}_1) = \widehat{\eta}$ and that $\ker(\widehat{\alpha}_0) = \widehat{\eta}_0$.

We then want to show that this path of 1-forms $\widehat{\alpha}_t$ is the path giving $\widehat{\eta}$ as contact deformation of $\widehat{\eta}_0$.

Now, $\widehat{\alpha}_t$ gives on each fiber a contact structure independent of t, hence the only thing left to show is that $\widehat{\alpha}_t$ actually defines a contact structure for $t > 0$. We can explicitly compute

$$
\begin{aligned}
\widehat{\alpha}_t \wedge d\widehat{\alpha}_t^n &= C^{n+1}\widehat{p}^*\left(\alpha_t \wedge d\alpha_t^n\right) + \\
&+ C^n\, t\left(rg'(r) + 2g(r)\right)\widehat{p}^*\left(\alpha_t \wedge d\alpha_t^{n-1}\right) \wedge r dr \wedge d\theta \\
&+ C^n\, t\, g(r)\, r^2 d\theta \wedge \widehat{p}^* d\alpha_t^n \;.
\end{aligned}
$$

Notice that

$$
\widehat{p}^*\left(\alpha_t \wedge d\alpha_t^{n-1}\right) \wedge r dr \wedge d\theta = \widehat{p}^*\left(\alpha_1 \wedge d\alpha_1^{n-1}\right) \wedge r dr \wedge d\theta \;,
$$

because α_t and α_1 induce the same contact form on each fiber. In particular, $\widehat{p}^*\left(\alpha_t \wedge d\alpha_t^{n-1}\right) \wedge r dr \wedge d\theta$ is bounded below by a positive volume form independent of t.

We can now use the same argument as in the proof of Lemma 5.10 to conclude that the whole sum is positive everywhere for every $t > 0$. □

Chapter 9

Applications

9.1 Virtually overtwisted contact structures in high dimensions

In Section 9.1.1, we prove Proposition II.C from Chapter 1, stating that a Bourgeois contact structure on a fiber bundle with total space $M \times \mathbb{T}^2$ is weakly fillable provided that the same is true for the fiber (M, ξ). Then, Section 9.1.2 contains the proof of Théorème II.F, also stated in Chapter 1, about the existence of virtually overtwisted manifolds in all odd dimensions.

9.1.1 Bourgeois contact structures and weak fillability

Consider (M^{2n-1}, ξ) a contact manifold and take the flat contact bundle $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \eta_0)$, where $\eta_0 = \xi \oplus T\mathbb{T}^2$ and the fibration map $M \times \mathbb{T}^2 \to \mathbb{T}^2$ is just the projection on the second factor.

Proposition 9.1. *Let η be a Bourgeois contact structure on $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \eta_0)$. If $(M,^{2n-1} \xi)$ is weakly filled by (X^{2n}, ω), then $(M \times \mathbb{T}^2, \eta)$ is weakly filled by $(X \times \mathbb{T}^2, \omega + \omega_{\mathbb{T}^2})$, where $\omega_{\mathbb{T}^2}$ is an area form on $\mathbb{T}^2$.*

We point out that the result is already known in the case of η obtained by Bourgeois' construction [Bou02]: the statement and the idea of the proof in that case already appeared in [MNW13, Example 1.1]; see also [LMN18, Theorem A.a] for an explicit proof.

Proof (Proposition 9.1). We start by choosing a convenient contact form for η.
If β is a form on M defining ξ, we claim that we can write $\eta = \ker(\alpha)$, where $\alpha := \beta + \phi_1 d\theta_1 - \phi_2 d\theta_2$, with $\phi_1, \phi_2 \colon M \times \mathbb{T}^2 \to \mathbb{R}$ and (θ_1, θ_2) coordinates on $\mathbb{T}^2$. Indeed, η can be written as $\ker(\gamma + f d\theta_1 + g d\theta_2)$, with $f, g \colon M \times \mathbb{T}^2 \to \mathbb{R}$ and $\gamma \in \Omega^1\left(M \times \mathbb{T}^2\right)$ that is zero on $\{0\} \oplus T\mathbb{T}^2 \subset T\left(M \times \mathbb{T}^2\right)$. Now, because η induces ξ on each fiber, there is a positive function $h \colon M \times \mathbb{T}^2 \to \mathbb{R}_{>0}$ such that $\gamma = h\beta$. Then, the division by h gives a contact form for η of the announced form.
Moreover, we recall from Section 8.4 that if A denotes the potential of η with respect to η_0 then, for each $\epsilon > 0$, the family of potentials $A_\epsilon := \epsilon A$ define a family η_ϵ of Bourgeois contact structure that are all isotopic between Bourgeois contact structures (hence in particular between contact structures).
We then claim that η_ϵ is just the kernel of $\alpha_\epsilon = \beta + \epsilon\phi_1 d\theta_1 - \epsilon\phi_2 d\theta_2$. Indeed, $\epsilon\phi_1$ and

$-\epsilon\phi_2$ are the contact Hamiltonians associated, respectively, to $-(A_\epsilon)_{\partial_x} = -\epsilon A_{\partial_x}$ and $-(A_\epsilon)_{\partial_y} = -\epsilon A_{\partial_y}$.

Hence, up to isotopy we have that $\eta = \ker(\alpha_\epsilon)$, with $\epsilon > 0$ that will be chosen very small in the following.

According to the weak fillability hypothesis for M (and the discussion after Definition 2.17), we have

$$\beta \wedge (\omega_M + \tau d\beta)^{n-1} > 0 \text{ for all } \tau \geq 0 \text{ on } M \ ,$$

where ω_M denotes the restriction of ω to $M = \partial X$. We want to verify that, for $\epsilon > 0$ small enough, we also have

$$\alpha_\epsilon \wedge (\omega_M + \omega_{\mathbb{T}^2} + \tau d\alpha_\epsilon)^n > 0 \text{ for all } \tau \geq 0 \text{ on } M \times \mathbb{T}^2 \ .$$

Lemma 9.2. *Let Ω be an arbitrary volume form on $M \times \mathbb{T}^2$. We then have*

$$\begin{aligned} \alpha_\epsilon \wedge (\omega_M + \omega_{\mathbb{T}^2} + \tau d\alpha_\epsilon)^n \ &= n\beta \wedge (\omega_M + \tau d\beta)^{n-1} \wedge \omega_{\mathbb{T}^2} + \\ &+ \ \epsilon^2 \tau^n \alpha_1 \wedge d\alpha_1^n \ + \ \epsilon^2 h \, \Omega \ \ , \end{aligned}$$

where h is independent of ϵ and polynomial in τ, with $\deg_\tau(h) \leq n-1$.

The proof of this lemma is postponed.

Denote now f and g the functions defined by the equalities $f\,\Omega = n\beta \wedge (\omega_M + \tau d\beta)^{n-1} \wedge \omega_{\mathbb{T}^2}$, $g\,\Omega = \tau^n \alpha_1 \wedge d\alpha_1^n$. Then, what we want to show is that $f + \epsilon^2(g+h) > 0$ on all $M \times \mathbb{T}^2$.

We remark that for each $p \in M \times \mathbb{T}^2$, $f(p)$, $g(p)$ and $h(p)$ are *polynomials* in τ, by explicit computation in the case of f and g, and by Lemma 9.2 in the case of h. Moreover, we have the following properties: for each $p \in M \times \mathbb{T}^2$,

(a) $f(p) > 0$, because (X, ω) weakly fills (M, ξ);

(b) $g(p) > 0$, because α_1 is a contact form for the Bourgeois contact structure η;

(c) $h(p)$ has degree in τ strictly less than $g(p)$, by Lemma 9.2.

We now use the following lemma, whose proof is easy (and omitted):

Lemma 9.3. *Let $P_1, P_2 \in \mathbb{R}[\tau]$ of degree n, with $P_1(\tau) > 0 \ \forall \, \tau \geq 0$ and with P_2 with positive leading coefficient. Then $\exists \, \epsilon_0 > 0$ such that $\forall \, 0 < \epsilon < \epsilon_0$, $P_1 + \epsilon^2 P_2 > 0$ on $\mathbb{R}_{\geq 0}$.*

Now, for each $p \in M \times \mathbb{T}^2$, if we define $P_1 = f(p)$ and $P_2 = g(p) + h(p)$, Lemma 9.3 gives an $\epsilon_p > 0$ such that $f(p) + \epsilon_p(g+h)(p) > 0$. Then, the compactness of $M \times \mathbb{T}^2$ guarantees that there is $\epsilon > 0$ independent of p such that $f + \epsilon(g+h) > 0$, as wanted. $\square$

We now prove Lemma 9.2 used above:

Proof (Lemma 9.2). We can compute

$$d\alpha_\epsilon = d\beta + \epsilon d\phi_1 \wedge d\theta_1 - \epsilon d\phi_2 \wedge d\theta_2 - \left(\frac{\partial \phi_1}{\partial \theta_2} + \frac{\partial \phi_2}{\partial \theta_1} \right) d\theta_1 \wedge d\theta_2 \ . \tag{9.1}$$

Claim 9.4. *$d_0 A = 0$ if and only if $\frac{\partial \phi_1}{\partial \theta_2} + \frac{\partial \phi_2}{\partial \theta_1} = 0$.*

Proof (Claim 9.4). We have $A = -X \otimes d\theta_1 - Y \otimes d\theta_2$, with X, Y the contact vector fields on (M, ξ) with contact Hamiltonians $\phi_1, -\phi_2$ via β, respectively.

By Claim 8.12, we have that $d_0 A = 0$ if and only if $-\frac{\partial}{\partial\theta_1} Y + \frac{\partial}{\partial\theta_2} X$.

Now, because $-\frac{\partial}{\partial\theta_1} Y + \frac{\partial}{\partial\theta_2} X$ is a contact vector field on each fiber (M, ξ), it is zero if and only if its contact Hamiltonian via β is zero, i.e. if and only if

$$0 = -\frac{\partial}{\partial\theta_1}\beta(Y) + \frac{\partial}{\partial\theta_2}\beta(X) = \frac{\partial}{\partial\theta_1}\phi_2 + \frac{\partial}{\partial\theta_2}\phi_1 \ . \qquad \square$$

Because η is a Bourgeois contact structure, Equation (9.1) becomes just

$$d\alpha_\epsilon = d\beta + \epsilon d\phi_1 \wedge d\theta_1 - \epsilon d\phi_2 \wedge d\theta_2 \ .$$

For dimensional reasons, we now get

$$\begin{aligned}\left(\omega_{|_{TM}} + \omega_{\mathbb{T}^2} + \tau d\alpha_\epsilon\right)^n &= \\ &= n\left(\omega_{|_{TM}} + \tau d\beta\right)^{n-1} \wedge \left(\omega_{\mathbb{T}^2} + \tau\epsilon d\phi_1 \wedge d\theta_1 - \tau\epsilon d\phi_2 \wedge d\theta_2\right) + \\ &\quad + \tau^2\epsilon^2 n(n-1)\left(\omega_{|_{TM}} + \tau d\beta\right)^{n-2} \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_1 \wedge d\theta_2 \ .\end{aligned}$$

Hence, we compute

$$\begin{aligned}&\alpha_\epsilon \wedge \left(\omega_{|_{TM}} + \omega_{\mathbb{T}^2} + \tau d\alpha_\epsilon\right)^n = \\ &= \ n(\beta + \epsilon\phi_1 d\theta_1 - \epsilon\phi_2 d\theta_2) \wedge (\omega_M + \tau d\beta)^{n-1} \wedge \\ &\qquad\qquad \wedge \left(\omega_{\mathbb{T}^2} + \tau\epsilon d\phi_1 \wedge d\theta_1 - \tau\epsilon d\phi_2 \wedge d\theta_2\right) + \\ &\quad + \ \tau^2\epsilon^2 n(n-1)\left(\beta + \epsilon\phi_1 d\theta_1 - \epsilon\phi_2 d\theta_2\right) \wedge \left(\omega_{|_{TM}} + \tau d\beta\right)^{n-2} \wedge \\ &\qquad\qquad \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_1 \wedge d\theta_2 = \\ &= \ n\beta \wedge (\omega_M + \tau d\beta)^{n-1} \wedge \omega_{\mathbb{T}^2} \\ &\quad + n\tau\epsilon^2(\phi_1 d\phi_2 - \phi_2 d\phi_1) \wedge (\omega_M + \tau d\beta)^{n-1} \wedge d\theta_1 \wedge d\theta_2 \\ &\quad + \tau^2\epsilon^2 n(n-1)\beta \wedge \left(\omega_{|_{TM}} + \tau d\beta\right)^{n-2} \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_1 \wedge d\theta_2 \ . \end{aligned} \tag{9.2}$$

Now, a similar explicit computation (using again Claim 9.4) shows that

$$\begin{aligned}\alpha_1 \wedge d\alpha_1^n \ &= \ n\left(\phi_1 d\phi_2 - \phi_2 d\phi_1\right) \wedge d\beta^{n-1} \wedge d\theta_1 \wedge d\theta_2 + \\ &\qquad + n(n-1)\beta \wedge d\beta^{n-2} \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_1 \wedge d\theta_2 \ ,\end{aligned}$$

so that the second and third term in the right hand side of the last equality in Equation (9.2) give $\epsilon^2\tau^n\alpha_1 \wedge d\alpha_1^{n-1} + \epsilon^2 h\,\Omega$, where h is as in the statement. This conclude the proof of Lemma 9.2. $\square$

Even if we will not use it in the following, we remark that the local nature of the condition $d_0 A = 0$ and of all the computations in the above proof actually gives the following more general result:

Proposition 9.5. *Let (M^{2n-1}, ξ) be a contact manifold weakly filled by (X^{2n}, ω). Suppose that a representation $\widetilde{\rho}$ of $\pi_1(\Sigma_g)$ in the group of symplectomorphisms of (X, ω) gives, by restriction to the boundary, a representation ρ of $\pi_1(\Sigma_g)$ in the group of contactomorphisms of (M, ξ). Let also η be a Bourgeois contact structure on the flat contact bundle $(M \times_\rho \Sigma_g, \Sigma_g, M, \eta_0)$ (as constructed in Section 8.2). Then, there is a symplectic form Ω on $X \times_{\widetilde{\rho}} \Sigma_g$ that weakly fills η on $M \times_\rho \Sigma_g$.*
More precisely, if $\mathbb{R}^2 \to \Sigma_g$ denotes the universal covering map, Ω can be chosen to be the symplectic form on $X \times_{\widetilde{\rho}} \Sigma_g$ induced by $\omega + \omega^g_{\mathbb{R}^2}$ on $X \times \mathbb{R}^2$ and where $\omega^g_{\mathbb{R}^2}$ is a symplectic form on $\mathbb{R}^2$ invariant by the action of $\pi_1(\Sigma_g)$ on $\mathbb{R}^2$ by deck transformations.

We also point out another result in a similar vein from [LMN18], that deals more precisely with the specific construction from [Bou02]:

Proposition 9.6 ([LMN18, Theorem A.b])**.** *Let (M,ξ) be a contact manifold admitting a supporting open book (B,φ) with monodromy* Id *and Stein pages. Then, for a convenient choice of the smooth function $\phi\colon M \to \mathbb{R}^2$ defining (B,φ), Theorem 7.5 (i.e. Bourgeois' construction) gives a Stein fillable contact structure on $M\times\mathbb{T}^2$.*

Remark that having a supporting open book with monodromy Id and Stein pages is equivalent to being subcritically Stein fillable. Indeed, from such an open book we can easily construct a subcritical Stein filling; the converse is proven by Cieliebak in [Cie02].

Let's now come back to the results we need in order to exhibit examples of virtually overtwisted manifolds in all odd dimensions. Theorem 5.23 and Proposition 9.1 have then the following immediate corollary:

Proposition 9.7. *Consider a branched covering $\Sigma_g \to \mathbb{T}^2$, where Σ_g is the closed genus $g\geq 2$ surface, and the naturally induced branched covering $M\times\Sigma_g \to M\times\mathbb{T}^2$. Let η_g on $M\times\Sigma_g$ be a contact branched covering of a Bourgeois contact structure η on the the contact bundle $(M\times\mathbb{T}^2,\mathbb{T}^2,M,\xi\oplus T\mathbb{T}^2)$, where ξ is a contact structure on the fiber M. Then, if (M,ξ) admits a weak filling (X,ω), there is a symplectic form Ω on $X\times\Sigma_g$ weakly dominating η_g on $M\times\Sigma_g = \partial X\times\Sigma_g$.*

We point out that the explicit proof of Theorem 5.23 actually shows that, up to isotopy, Ω can be chosen to be of the form $\omega+\omega_g$, for a certain area form ω_g on Σ_g.

9.1.2 High dimensional virtually overtwisted manifolds

Let $\pi\colon \Sigma_g \to \mathbb{T}^2$ be a covering map and suppose here that it is branched along two points. Let also $(\mathrm{Id},\pi)\colon M\times\Sigma_g \to M\times\mathbb{T}^2$ be the induced branched covering. We remark that g equals here the branching index along each of the two connected components of the upstairs branching locus of (Id,π).

Proposition 9.8. *Let η be a Bourgeois contact structure on the flat contact bundle $(M\times\mathbb{T}^2,\mathbb{T}^2,M,\eta_0)$ and consider a contact branched covering η_g of η on $M\times\Sigma_g$. If (M,ξ) is weakly fillable and virtually overtwisted, then, for $g\geq 2$ big enough, $(M\times\Sigma_g,\eta_g)$ is weakly fillable and virtually overtwisted.*

Then, starting for example from the base case of a holomorphically fillable virtually overtwisted contact structure on lens spaces, that exist by [Gom98, Proposition 5.1] (see also [Gir00, Theorem 1.1]), and using the construction in [Bou02], a proof by induction on the dimension $2n-1$ of M easily shows that Proposition 9.8 implies the following result, already stated in Section 1.2:

Théorème II.F. *Les structures virtuellement vrillée existent en toutes dimensions ≥ 3.*

Proof (Proposition 9.8). Proposition 9.7 tells us that $(M\times\Sigma_g,\eta_g)$ is weakly fillable for all $g\geq 2$. We then have to show that, for g sufficiently big, this contact manifold admits a finite cover which is overtwisted.

By hypothesis, there is a finite cover $p\colon \overline{M}\to M$ such that $(\overline{M},\overline{\xi}\coloneqq p^*\xi)$ is overtwisted. Consider then the following commutative diagram of smooth maps:

$$\begin{array}{ccc}
\overline{M}\times\Sigma_g & \xrightarrow{(p,\mathrm{Id})} & M\times\Sigma_g \\
{\scriptstyle(\mathrm{Id},\pi)}\downarrow & & \downarrow{\scriptstyle(\mathrm{Id},\pi)} \\
\overline{M}\times\mathbb{T}^2 & \xrightarrow{(p,\mathrm{Id})} & M\times\mathbb{T}^2
\end{array}$$

Consider now $\overline{\eta} := (p, \mathrm{Id})^*\eta$ on $\overline{M} \times \mathbb{T}^2$ and $\zeta_g := (p, \mathrm{Id})^*\eta_g$ on $\overline{M} \times \Sigma_g$. Notice that the restriction of ζ_g to the upstairs branching locus of $(\mathrm{Id}, \pi)\colon \overline{M} \times \Sigma_g \to \overline{M} \times \mathbb{T}^2$ is exactly $\overline{\xi}$.

We claim moreover that $(\overline{M} \times \Sigma_g, \zeta_g)$ is a branched contact covering of $(\overline{M} \times \mathbb{T}^2, \overline{\eta})$. Indeed, we can easily see that ζ_g is a contact deformation of the confoliation $(\mathrm{Id}, \pi)^*\overline{\eta}$ on $\overline{M} \times \Sigma_g$ as follows. If $(\eta_g^t)_{t \in [0,1]}$ is a path of confoliations adapted to the upstairs branching locus of $(\mathrm{Id}, \pi)\colon M \times \Sigma_g \to M \times \mathbb{T}^2$ starting at $\eta_g^0 = (\mathrm{Id}, \pi)^*\eta$, ending at $\eta_g^1 = \eta_g$ and such that η_g^t is contact for $t \in (0, 1]$, then $(p, \mathrm{Id})^*\eta_g^t$ is the path of confoliations on $\overline{M} \times \Sigma_g$ which shows that ζ_g is a contact deformation of $(\mathrm{Id}, \pi)^*\overline{\eta}$.

We now use the following result, whose proof is postponed:

Lemma 9.9 ([NP10]). *For $k \in \mathbb{N}_{>1}$, let $\pi_k\colon \widehat{V}_k \to V^{2n+1}$ be a branched covering map of branching index k. Suppose that all π_k's have same downstairs branching locus M and that the upstairs branching locus $\widehat{M}_k$ of π_k is connected (in particular, $\pi_k|_{\widehat{M}_k}$ induces a diffeomorphism between $\widehat{M}_k$ and M). Suppose also that there is a tubular neighborhood $\mathcal{N} := M \times D^2$ (where D is the $2-$disk centered at 0 and of radius 1) of the downstairs branching locus M over which all the π_k's are trivialized at the same time, i.e. such that $\pi_k\colon M \times D^2 \to M \times D^2$ is just $(p, z) \mapsto (p, z^k)$.*
Let now η be a contact structure on V inducing a contact structure ξ on M and consider a contact branched covering $\widehat{\eta}_k$ of η on the total space of $\widehat{V}_k$.

Then, there is $\epsilon > 0$ such that, for all $k \geq 2$, the upstairs branching locus $(\widehat{M}_k, \widehat{\xi}_k = \ker \pi_k^*\alpha) \overset{\pi_k}{\simeq} (M, \xi = \ker \alpha)$ has a neighborhood of the form $\left(M \times D^2_{\sqrt{k\epsilon}}, \ker\left(\alpha + r^2 d\varphi\right)\right)$ inside $(\widehat{V}_k, \widehat{\eta}_k)$ (here, by D^2_r we denote the open disk centered in 0 and of radius r inside $\mathbb{R}^2$).

In our situation, we have a sequence of branched coverings $\overline{M} \times \Sigma_g$ of $\overline{M} \times \mathbb{T}^2$, together with contact branched coverings ζ_g of $\overline{\eta}$, as in the hypothesis of Lemma 9.9. Then, a direct application of such lemma tells that each of the fibers $(\overline{M}, \overline{\xi} = \ker(\overline{\alpha}))$ that belong to the (upstairs) branching set has a contact neighborhood of the form $\left(\overline{M} \times D^2_{R_g}, \ker\left(\overline{\alpha} + r^2 d\theta\right)\right)$, with $R_g \to +\infty$ for $g \to +\infty$. Because $\overline{\xi}$ on $\overline{M}$ is overtwisted, this implies, according to [CMP15, Theorem 3.1], that if g is big enough then the upstairs branching set has an overtwisted neighborhood, so that $(\overline{M} \times \Sigma_g, \zeta)$ is also overtwisted. In other words, we just proved that, for g big enough, $(M \times \Sigma_g, \eta_g)$ has a finite cover which is overtwisted. □

Let's now prove Lemma 9.9 used above:

Proof (Lemma 9.9). This result is a rephrasing of the discussion made by Niederkrüger and Presas in [NP10, page 724], using our definition of contact branched coverings. Here's a detailed proof.

We start by remarking that, by the uniqueness of the branched contact coverings up to isotopy (Proposition 5.8), it is enough to show that the result is true for a particular choice of contact branched coverings $\widehat{\eta}_k'$'s. Indeed, if $\widehat{\eta}_k'$ contains a contact neighborhood of the upstairs branching locus $(M, \xi = \ker \alpha)$ as in the statement, so does any contact structure isotopic to it via an isotopy fixing M.
Suppose for the moment that we are in the case where $\eta = \ker\left(\alpha + r^2 d\varphi\right)$ on a sub-neighborhood $M \times D^2_\epsilon$ of $\mathcal{N} = M \times D^2$. Then, $\pi_k^*\eta = \ker\left(\alpha + kr^{2k} d\varphi\right)$ on a neighborhood $M \times D^2_{\sqrt[k]{\epsilon}}$ of the upstairs branching locus $\widehat{M} = M$ inside $\widehat{V}$.
Let now $f_k\colon (0, \sqrt[k]{\epsilon}) \to (0, \epsilon)$ be a smooth strictly increasing function such that $f_k(r) = r$ near 0 and $f_k(r) = r^k$ near $\sqrt[k]{\epsilon}$. Then, we can construct a contact branched covering $\widehat{\eta}_k'$ of η by replacing $\pi_k^*\eta = \ker\left(\alpha + kr^{2k} d\varphi\right)$ on the neighborhood $M \times D^2_{\sqrt[k]{\epsilon}}$ with

$\ker\left(\alpha + kf_k^2(r)d\varphi\right)$.
Notice that we have the following contactomorphism:

$$\left(M \times D^2_{\sqrt[k]{\epsilon}}, \ker\left(\alpha + kf_k^2(r)d\varphi\right)\right) \xrightarrow{\sim} \left(M \times D^2_{\sqrt{k}\epsilon}, \ker\left(\alpha + r^2 d\varphi\right)\right)$$
$$(p, r, \varphi) \mapsto (p, \sqrt{k}f_k(r), \varphi)$$

This means that $(\widehat{M}_k, \ker(\pi_k^*\alpha)) \overset{\pi_k}{\simeq} (M, \xi)$ has a contact neighborhood of the form $\left(M \times D^2_{\sqrt{k}\epsilon}, \ker\left(\alpha + r^2\varphi\right)\right)$ inside $(\widehat{V}_k, \widehat{\eta}'_k)$, as wanted.

It only remains to show that we can reduce to the case where η has the wanted form $\ker\left(\alpha + r^2 d\varphi\right)$ on a sub$-$neighborhood $M \times D^2_\epsilon$ of $\mathcal{N} = M \times D^2$, for a certain $\epsilon > 0$ independent of k.
The neighborhood $\mathcal{N} = M \times D^2$ induces (via the choice of a basis of T_0D^2) a normal framing of M inside V. Now, the conformal symplectic normal bundle of the downstairs branching locus (M, ξ) in (V, η) is trivial as symplectic bundle (because M has a trivial normal bundle inside V) and admits moreover a trivialization (as a symplectic bundle) that induces a normal framing of M inside V which is homotopic to that induced by $\mathcal{N}$. The contact neighborhood theorem [Gei08, Theorem 2.5.15] gives then a contact neighborhood $\left(M \times D^2_\delta, \ker\left(\alpha + r^2 d\varphi\right)\right)$ that induces (via the choice of a basis of $T_0 D^2_\delta$) a normal framing of M inside V that is also homotopic to that induced by $\mathcal{N}$.
Hence, by the uniqueness (up to isotopy) of smooth tubular neighborhoods inducing a fixed homotopy class of normal framing (see for instance [Lan99, Theorem 6.2]), there is an isotopy of V that fixes M pointwise and puts η in the wanted normal form $\ker(\alpha + r^2 d\varphi)$ on a sub$-$neighborhood $M \times D^2_\epsilon$ of $\mathcal{N} = M \times D^2$, for a certain $0 < \epsilon < \delta$; notice that this ϵ does not depend on k (indeed, it depends only on $\mathcal{N}$, which does not depend on k). In other words, we have a smooth family of contact structures $(\eta_t)_{t \in [0,1]}$, starting at $\eta_0 = \eta$, ending at a certain η_1 that has the wanted normal form on $M \times D^2_\epsilon$, and such that each η_t restricts to $\eta = \ker\alpha$ on $M = M \times \{0\}$. Then, by Lemma 5.10 (with $K = [0, 1]$), we have a smooth family of contactizations $\widehat{\eta}'_{k,t}$ of $\pi_k^*\eta_t$. Now, the above discussion shows that, for each $k \geq 2$, the upstairs branching locus $(\widehat{M}_k, \ker(\pi_k^*\alpha)) \overset{\pi_k}{=} (M, \xi)$ has a neighborhood $\left(M \times D^2_{\sqrt{k}\epsilon}, \ker\left(\alpha + r^2\varphi\right)\right)$ inside $(\widehat{V}_k, \widehat{\eta}'_{k,1})$. Then, because $\widehat{\eta}'_{k,0}$ and $\widehat{\eta}'_{k,1}$ are isotopic, the same is true inside $(\widehat{V}_k, \widehat{\eta}'_{k,0})$, as wanted. □

We point out that taking $g = 1$ in the statement of Proposition 9.8, i.e. working directly on $M \times \mathbb{T}^2$ without taking a branched covering, is in general not enough to ensure the same conclusion.
For instance, this follows from Section 9.2, where we will show that for each contact manifold (M^3, ξ), with $\pi_1(M) \neq \{1\}$, there is an open book decomposition of M supporting ξ such that the construction in [Bou02] yields a hypertight contact form α on $M \times \mathbb{T}^2$. In particular, even if (M, ξ) is virtually overtwisted, with $(\overline{M}, \overline{\xi})$ an overtwisted finite cover, the pullback $\overline{\alpha}$ of α to $\overline{M} \times \mathbb{T}^2$ will still define a tight contact structure $\overline{\eta} = \ker\overline{\alpha}$. Indeed, if by contradiction $\overline{\eta}$ is overtwisted, according to [CMP15] and [AH09], $\overline{\alpha}$ admits a contractible Reeb orbit in $\overline{M} \times \mathbb{T}^2$, which then projects to a contractible Reeb orbit of α in $M \times \mathbb{T}^2$, contradicting the hypertightness of α.

Notice also that we preferred to take a very big g in Proposition 9.8 in order not to enter too much in technical details and to keep the construction simple, but actually $g = 2$ is already enough:

Observation 9.10 (Niederkrüger)**.** *If (M, ξ) is overtwisted, the contact manifold $(M \times \Sigma_g, \eta_g)$ is overtwisted already for $g = 2$.*

Proof (sketch). Take an arc γ on $\mathbb{T}^2$ going from one (downstairs) branching point of the cover $\Sigma_2 \to \mathbb{T}^2$ to the other, and such that it is radial in the local model (given by Definition 5.1) around the two branching points, in such a way that its double cover δ in Σ_2 is a smooth closed curve. Denote also $p \in \Sigma_2$ one of the two upstairs branching points, and see δ as a loop based at p.

Then, the monodromy of the contact fiber bundle $M \times \Sigma_2 \to \Sigma_2$ over δ is trivial. Indeed, as the proof of Lemma 5.10 shows, the contact branched covering η_2 of a Bourgeois contact structure $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \eta = \ker \beta)$ can be chosen to be defined by a form $\widehat{\beta}$ on $M \times \Sigma_2$ which is invariant under deck transformations of the branched covering $\pi : M \times \Sigma_2 \to M \times \mathbb{T}^2$ and C^∞-close to $\pi^*\beta$. Then, it can be shown that the monodromy of $(M \times \Sigma_2, \Sigma_2, M, \widehat{\eta}_2)$ over δ is obtained as the concatenation of the monodromy f_γ of $\big(M \times \mathbb{T}^2, \mathbb{T}^2, M, \eta = \ker(\beta)\big)$ over γ plus a $C^\infty-$little perturbation h, and the monodromy $(f_\gamma)^{-1}$ over $-\gamma$ plus the inverse h^{-1} of the same perturbation.

Then, using the techniques from [Pre07], we can find an embedded plastikstufe inside $M \times \delta \subset M \times \mathbb{T}^2$. In practice, this PS is obtained by "moving around" an overtwisted disk in $M \times \{p\} \simeq M$ via the monodromy of $(M \times \Sigma_2, \Sigma_2, M, \widehat{\eta}_2)$ along δ. This procedure actually gives an embedded PS because the monodromy along the loop δ is the identity as map $M \times \{p\} \to M \times \{p\}$. At this point, [Hua17] tells us that each PS-overtwisted manifold is also overtwisted. $\square$

9.2 Bourgeois construction and Reeb dynamics

The main aim of this section is to give a proof of Théorème II.G stated in Chapter 1. Let's recall the statement:

Théorème II.G. *Chaque 3-variété de contact fermée (M, ξ) avec $\pi_1(M) \neq \{1\}$ peut être plongée avec fibré normal trivial dans une 5-variété de contact (V^5, η) fermée (hyper)tendue.*

In order to give a proof, in Section 9.2.1 we consider, starting from a contact manifold (M^{2n-1}, ξ) and an open book (B, φ) adapted to ξ, a Bourgeois contact structure η on the flat contact bundle $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$, with fibers (M, ξ), which admits a contact form α with very controlled Reeb vector field. This η is actually one of the examples described in [Bou02]. We then show that the Reeb dynamics of α on $M \times \mathbb{T}^2$ is strictly related to the Reeb dynamics on the binding B of the open book (B, φ). This will give a criterion for the existence of closed contractible Reeb orbits of α on $M \times \mathbb{T}^2$.

Then, we show in Section 9.2.2 how to deduce Théorème II.G as a corollary of this study in the case of $3-$dimensional M.

9.2.1 Bourgeois contact structures and contractible Reeb orbits

Proposition 9.11. *Let (M, ξ) be a $(2n-1)-$dimensional contact manifold and consider an open book decomposition (B, φ) on M supporting ξ. Then, there are a contact form β on M, adapted to the open book (B, φ), and a Bourgeois contact structure η on the flat contact bundle $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$ which admits a contact form α with associated Reeb vector field of the form*

$$R_\alpha = Z + f\,\partial_x - g\,\partial_y \ ,$$

where:

a. Z is a smooth vector field on M such that:

i. on $M \setminus B$, it is tangent to the fibers of φ,

ii. on the binding B, it is equal, up to a non-zero constant factor, to the Reeb vector field R_B of the restriction of β to B;

b. $f, g\colon M \to \mathbb{R}$ are smooth functions such that $f = g = 0$ on B and such that $(f, g)\colon M \to \mathbb{R}^2$ is positively proportional to $(\cos\varphi, \sin\varphi)$ on $M \setminus B$.

Proof (Proposition 9.11). We start by finding the adapted form β in the statement as well as a particular normal neighborhood $\mathcal{N}$ of the binding and a particular smooth map $\phi\colon M \to \mathbb{R}^2$ defining (B, φ).

For this, we use the following result, whose proof can be found, for example, in [DGZ14, Section 3]:

Lemma 9.12 (Giroux)**.** *Let $D^2 \subset \mathbb{R}^2$ be the disk centered at the origin with radius 1 and β be a contact form on $B \times D^2$ with the following properties:*

1. *$\beta_B := \beta|_{TB}$ is a contact form on $B = B \times \{0\}$.*
2. *For each $\varphi \in \mathbb{S}^1$, $d\beta|_{T\Sigma_\varphi}$ is a symplectic form on $\Sigma_\varphi \setminus B$, where*
$$\Sigma_\varphi = \{ (p, r, \varphi) \in B \times D^2 \,|\, p \in B, 0 \le r \le 1 \} .$$
3. *With the orientations of B and Σ_φ induced, respectively, by β_B and $d\beta|_{T\Sigma_\varphi}$, B is oriented as the boundary of Σ_φ.*

Then, for a sufficiently small $\delta > 0$, if we denote by $D^2_\delta \subset \mathbb{R}^2$ the disk centered at the origin and of radius $\delta > 0$, there is an embedding $B \times D^2_\delta \to B \times D^2$ which preserves the angular coordinate φ on the second factor, which is the identity on $B \times \{0\}$ and which pulls back a convenient isotopic modification β' of β (with an isotopy between contact forms that satisfy Hypothesis 1, 2 and 3 above) to a 1-form $h_1(r) \cdot \beta_B + h_2(r) \cdot d\varphi$, such that:

i. $h_1(0) > 0$ and $h_1(r) = h_1(0) + O(r^2)$ for $r \to 0$,

ii. $h_2(r) \sim r^2$ for $r \to 0$,

iii. if $H := h_1^{n-1} \cdot (h_1 h_2' - h_2 h_1')$, then $\frac{H}{r} > 0\ \forall r \ge 0$ (contact condition);

iv. $h_1'(r) < 0$ for $r > 0$, (symplectic condition on Σ_φ).

Then, we choose β, $\mathcal{N}$ and ϕ as follows.

Take a normal neighborhood $B \times D^2$ of the binding B in M such that $\varphi\colon B \times \big(D^2 \setminus \{0\}\big) \to \mathbb{S}^1$ becomes the angular coordinate of $D^2 \setminus \{0\}$: such neighborhood exists by definition of open book decomposition. Also, take a contact form β_0 which defines ξ and is adapted to the open book (B, φ) on M.

Then, Lemma 9.12 gives an isotopic modification β of β_0, still adapted to the same open book, and of the form $h_1\beta_B + h_2 d\varphi$ in the neighborhood $\mathcal{N} := B \times D^2_\delta \subset B \times D^2$.

To define $\phi\colon M \to \mathbb{R}^2$, consider a function $\rho : M \to \mathbb{R}$ which is smooth away from B, equal to the radial coordinate r of D^2_δ on the neighborhood $\{r \le \delta/3\}$ of $B \times \{0\}$ inside $\mathcal{N} = B \times D^2_\delta$, equal to 1 on the complement in M of the open set $\{r < 2\delta/3\} \subset \mathcal{N}$, and depending only on r and strictly increasing in it on the set $\delta/3 < r < 2\delta/3$.

Then, we define $\phi := \rho \cdot (\cos\varphi, \sin\varphi)$. Remark that such a ϕ is indeed well defined and smooth on all M, and defines the open book (B, φ).

We now define two functions $\lambda, \mu \colon M \to \mathbb{R}$ as follows:

$$\lambda \;=\; \begin{cases} \frac{\rho'}{\rho' h_1 - \rho h_1'} & \text{inside } \mathcal{N} \\ 0 & \text{outside } \mathcal{N} \end{cases} \quad \text{and} \quad \mu \;=\; \begin{cases} \frac{-h_1'}{\rho' h_1 - \rho h_1'} & \text{inside } \mathcal{N} \\ 1 & \text{outside } \mathcal{N} \end{cases} .$$

We point out that they are well defined smooth functions on all $M \times \mathbb{T}^2$. Indeed, ρ' smoothly extends as 1 at $r = 0$, $h_1' = O(r)$ near $r = 0$ (by point i. of Lemma 9.12) and the denominator $\rho' h_1 - \rho h_1'$ is positive for $r > 0$ and smoothly extends as $h_1(0)$ at $r = 0$.
Consider then $Z := \lambda R_B$ and $(f, g) := \mu(\cos\varphi, \sin\varphi)$. Here, R_B is seen as as a vector field on $\mathcal{N} = B \times D^2_\delta$ tangent to the first factor and λ has support contained inside $\mathcal{N}$, hence λR_B is well defined on all M. Similarly, f, g are well defined because μ is zero on B.
It is now easy to check that such Z, f, g satisfy points a. and b. of Proposition 9.11.

Lastly, we have to choose a contact form α defining a Bourgeois contact structure η on the flat contact bundle $(M \times \mathbb{T}^2, \mathbb{T}^2, M, \xi \oplus T\mathbb{T}^2)$, as in the statement of Proposition 9.11. Let $\alpha := \beta + \phi_1 dx - \phi_2 dy$, i.e. the one obtained from Theorem 7.5 with the choices of ϕ and $\mathcal{N}$ made above.
We already know that the contact structures obtained from Theorem 7.5 are particular cases of Bourgeois structures. The only thing left to check is that the Reeb vector field of this contact form α can be written as in the statement of Proposition 9.11.

Claim 9.13. *The Reeb vector field for α on $(M \setminus \mathcal{N}) \times \mathbb{T}^2$ is*

$$R_\alpha \;=\; \cos\varphi\, \partial_x \;-\; \sin\varphi\, \partial_y \;.$$

Claim 9.14. *The Reeb vector field for α on $\mathcal{N} \times \mathbb{T}^2$ is*

$$R_\alpha \;=\; \frac{1}{\rho' h_1 - \rho h_1'} \left(\rho'\, R_B \;-\; h_1'\, \cos\varphi\, \partial_x \;+\; h_1'\, \sin\varphi\, \partial_y \right) \;.$$

By the definition of Z, f and g, this concludes the proof of Proposition 9.11. □

We now prove the identities used above:

Proof (Claim 9.13). In this region we simply have $\alpha = \beta + \cos\varphi\, dx - \sin\varphi\, dy$ and $d\alpha = d\beta - \sin\varphi\, d\varphi \wedge dx - \cos\varphi\, d\varphi \wedge dy$. This easily gives $\alpha(R_\alpha) = 1$ and $\iota_{R_\alpha} d\alpha = \cos\varphi\, \sin\varphi\, d\varphi - \sin\varphi\, \cos\varphi\, d\varphi = 0$. □

Proof (Claim 9.14). Here, we have $\alpha = h_1\, \beta_B + h_2\, d\varphi + \rho \cos\varphi\, dx - \rho\, \sin\varphi\, dy$, so that

$$\begin{aligned} d\alpha \;=\; h_1\, d\beta_B \;+\; h_1'\, dr \wedge \beta_B \;+\; h_2'\, dr \wedge d\varphi \;+\; \rho' \cos\varphi\, dr \wedge dx \\ -\; \rho\, \sin\varphi\, d\varphi \wedge dx \;-\; \rho'\, \sin\varphi\, dr \wedge dy \;-\; \rho\, \cos\varphi\, d\varphi \wedge dy \;. \end{aligned}$$

If we denote $V := \rho'\, R_B \;-\; h_1'\, \cos\varphi\, \partial_x \;+\; h_1'\, \sin\varphi\, \partial_y$, we then have $\alpha(V) = h_1\, \rho' \;-\; h_1'\, \rho$, hence $\beta(R_\alpha) = 1$. Moreover, we can compute

$$\begin{aligned} \iota_V d\alpha \;=\; -h_1'\, \rho'\, dr \;+\; h_1' \left[\rho'\, \cos^2\varphi\, dr \;-\; \rho\, \sin\varphi\, \cos\varphi d\varphi \right] \\ +\; h_1' \left[\rho'\, \sin^2\varphi\, dr \;+\; \rho\, \cos\varphi\, \sin\varphi\, d\varphi \right] = 0 \;, \end{aligned}$$

hence $\iota_{R_\alpha} d\alpha \;=\; 0$. □

We then have the following result on the Reeb dynamics:

Corollary 9.15. *Let α on $M \times \mathbb{T}^2$ be the contact structure given by Proposition 9.11. Then, the closed contractible orbits of R_α in $M \times \mathbb{T}^2$ are of the form $\mathcal{O}^B_{q_0} \times \{(x_0, y_0)\}$, where $(x_0, y_0) \in \mathbb{T}^2$ and $\mathcal{O}^B_{q_0}$ is a closed orbit of R_B in B which is contractible in M.*

We remark that even if the orbits are contained in B, we are interested here in their homotopy class as loops in M.

Proof (Corollary 9.15). We start by remarking that if $p_0 \in B \times \mathbb{T}^2 \subset M \times \mathbb{T}^2$ is of the form (q_0, x_0, y_0), with $q_0 \in B$ and $(x_0, y_0) \in \mathbb{T}^2$ then $\mathcal{O}_{p_0} = \mathcal{O}^B_{q_0} \times \{(x_0, y_0)\}$, where $\mathcal{O}^B_{q_0}$ is the orbit of q_0 under the flow of R_B on B. Indeed, along $B \times \mathbb{T}^2$, R_α is just proportional (via a non zero constant) to R_B on the B factor of $B \times \mathbb{T}^2$, so that the orbit of p_0 is tangent to $B \times \{(x_0, y_0)\}$ and actually coincides with $\mathcal{O}^B_{q_0} \times \{(x_0, y_0)\}$.

It's then enough to show that, for each $p_0 \in (M \setminus B) \times \mathbb{T}^2$, if $\mathcal{O}_{p_0}$ is closed then it is not contractible.
Now, $B \times \mathbb{T}^2$ is (globally) invariant under the flow of R_α. Hence, its complement $(M \setminus B) \times \mathbb{T}^2$ is (globally) invariant too; in particular, $\mathcal{O}_{p_0}$ is contained in $(M \setminus B) \times \mathbb{T}^2$. Moreover, the function φ is invariant under the restriction of the flow ϕ_t of R_α to $(M \setminus B) \times \mathbb{T}^2$, because Z is tangent to the fibers of φ on $M \setminus B$, according to Proposition 9.11. In particular, if we write $p_0 = (q_0, x_0, y_0)$, where $q_0 \in M \setminus B$, with let's say $\varphi(q_0) = \varphi_0$, and $(x_0, y_0) \in \mathbb{T}^2$, then $\varphi(\mathcal{O}_{p_0}) = \varphi_0$.
Finally, the component of R_α tangent to the $\mathbb{T}^2-$factor is $f\partial_x - g\partial_y$, which is non$-$zero along all $M \setminus B$, because (f, g) is positively proportional to $(\cos\varphi, \sin\varphi)$ there. In particular, the projection (x_t, y_t) of $\phi_t(p_0)$ via $\pi\colon\ M \times \mathbb{T}^2 \to \mathbb{T}^2$ has velocity $(\dot{x}_t, \dot{y}_t)$ positively proportional to $(\cos\varphi_0, -\sin\varphi_0)$. Hence, in order for $\mathcal{O}_{p_0}$ to be closed, the angle $\varphi_0 \in [0, 2\pi[$ has to be a rational multiple of π and $\{(x_t, y_t) \in \mathbb{T}^2 | t \in \mathbb{R}\}$ has to be the circle on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ passing through (x_0, y_0) and with slope $(\cos\varphi_0, -\sin\varphi_0)$. But this means that $\mathcal{O}_{p_0}$ can't be contractible, because it has a homotopically non trivial projection on $\mathbb{T}^2$. $\square$

9.2.2 Contact embeddings of 3-manifolds

We start with a proposition on (topological) open book decomposition of 3-manifolds:

Proposition 9.16. *Let M be a 3-manifold with non-trivial $\pi_1(M)$. Then, every open book decomposition (K, φ) of M can be transformed, by a sequence of positive stabilizations, to an open book decomposition (K', φ') with binding K' having at most two connected components, each of which is not contractible in M.*

According to Perelman's proof of Poincaré's conjecture (see for instance [Mor05] for a summary), M has trivial fundamental group if and only if it is diffeomorphic to $\mathbb{S}^3$.

Proof. We start by applying a sequence of stabilizations to reduce the number of connected components of the boundary of the pages to one. We can thus suppose that the open book decomposition (K, φ) has connected binding K. Denote by Σ the page of (K, φ).
Let $p \in K$. Notice that if $c := [K] \in \pi_1(M, p)$ is non-trivial, then we have nothing to prove. We can hence suppose that it is trivial.

We now need a presentation of $\pi_1(M, p)$ in terms of the fundamental group of the page and the monodromy of (K, φ). Such a presentation can be found using Van-Kampen theorem as described, for example, in [EO08].

If $\mathbb{F}\langle\mathcal{S}\rangle$ denotes the free group generated by a set $\mathcal{S}$ and $\langle\mathcal{R}\rangle_N$ denotes its normal subgroup generated by the set of relations $\mathcal{R}$, according to [EO08, Section 2.1] $\pi_1(M,p)$ is isomorphic to

$$\mathbb{F}\langle a_i, b_i, c\rangle_{i=1}^{g} \Big/ \langle\, c\prod_{i=1}^{g}[a_i,b_i], a_i\phi_*(a_i^{-1}), b_i\phi_*(b_i^{-1}) \,|\, i=1,\dots,g\,\rangle_N \,. \tag{9.3}$$

Here, $\phi : \Sigma \to \Sigma$ is the monodromy of the open book decomposition and the a_i's and b_i's are the generators of $\pi_1(\Sigma,p)$ given by the classes of the curves α_i's and β_i's depicted in Figure 9.1.
We point out that the hypothesis c trivial in $\pi_1(M,p)$ means that it belongs to the normal subgroup in the above presentation. Hence, we can present $\pi_1(M,p)$ also as

$$\mathbb{F}\langle a_i, b_i, c\rangle_{i=1}^{g} \Big/ \langle\, c\prod_{i=1}^{g}[a_i,b_i], a_i\phi_*(a_i^{-1}), b_i\phi_*(b_i^{-1}),\ c \,|\, i=1,\dots,g\,\rangle_N \,. \tag{9.4}$$

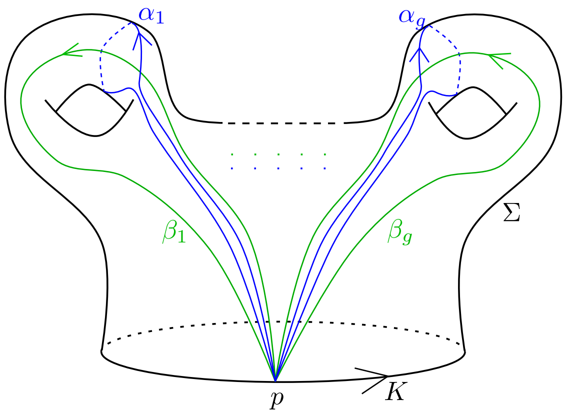

Figure 9.1: Curves on a page Σ of (K,φ) which give the generators of the presentation in Equation (9.3).

Now, by the hypothesis that $\pi_1(M)$ is not trivial, at least one of the generators a_i, b_i of $\pi_1(M,p)$ is non-trivial, let's say b_g. Make then a plumbing of a positive Hopf band, as defined in Theorem 2.20, along a properly embedded arc δ_0 which is in the same class as β_g in $\pi_1(\Sigma,\partial\Sigma)$, as shown in Figure 9.2.

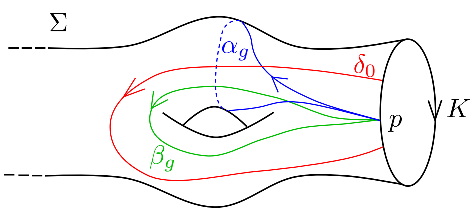

Figure 9.2: Stabilization arc δ_0. Only a part of the surface is shown.

We then obtain an open book decomposition (K',φ') for M, which still supports the original contact structure, according to Theorem 2.21. Notice that the page Σ' is obtained, as embedded surface, from Σ and δ_0 by plumbing a positive Hopf band to Σ. We recall that this means that $\Sigma' = \Sigma \cup A$ where A is an annulus in M such that

1. the intersection $A \cap \Sigma$ is a tubular neighborhood of δ_0,

2. the core curve of A bounds a disk in $M \setminus \Sigma$ and the linking number of the boundary components is 1.

In particular, we know that the core of the annulus A bounds a disk in $M \setminus \Sigma$. Now, the null-homotopy defined by this disk gives a homotopy between one of the two boundary components of Σ', which we can call K_1', and β_g (seen as a curve on Σ' via the natural inclusion $\Sigma \subset \Sigma'$ as surfaces embedded in M). Hence, K_1' is homotopically non-trivial, because $b_g = [\beta_g]$ is non-zero.
Similarly, the fact that the core of A bounds a disk implies that the other connected component of $\partial\Sigma'$, which we can call K_2', is homotopic to $K * \beta_g^{-1}$. Here, $*$ is the concatenation of paths and $.^{-1}$ is the inverse of a path. In particular, K_2' is homotopically non-trivial, because K is trivial and β_g is not. □

Consider now a 3-dimensional contact manifold (M, ξ) and an open book decomposition (Σ, φ) supporting ξ. Corollary 9.15 and Proposition 9.16 have the following direct consequence:

Proposition 9.17. *If $\pi_1(M) \neq \{1\}$, there exists an open book (Σ', φ') supporting ξ, obtained from (Σ, φ) by a sequence of positive stabilizations, such that Proposition 9.11 gives a hypertight Bourgeois contact structure on $M \times \mathbb{T}^2$.*

Let's denote by D^2_R the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^2$ and by (r, φ) the polar coordinates on it.
At this point, we can deduce from Proposition 9.17 the following result, which consists of Théorème II.G and Corollaire II.H stated in Section 1.2:

Theorem 9.18. *Every closed 3-dimensional contact manifold (M, ξ), with $\pi_1(M)$ non-trivial, can be embedded, with trivial conformal symplectic normal bundle, in a hypertight closed 5-dimensional contact manifold (N, η).*
In particular, for each contact form α defining ξ on M, there is an $\epsilon > 0$ such that $\left(M \times D^2_\epsilon, \ker\left(\alpha + r^2 d\varphi\right)\right)$ is tight.

As already remarked in the introduction, we point out that the recent paper [HMP18] deals with the higher dimensional case; more precisely, it contains a generalization of the second part of this result, as well as an analogue (with no control on the codimension) of the first part of it.

Proof. Consider an arbitrary contact 3-manifold (M, ξ) and take one of the hypertight contact manifolds $(M \times \mathbb{T}^2, \eta)$ given by Proposition 9.17.
Each $(M \times \{pt\}, \eta \cap T(M \times \{pt\}))$ is then exactly (M, ξ) and it has topologically trivial normal bundle, hence trivial conformal symplectic normal bundle. Indeed, a symplectic vector bundle of rank 2 is symplectically trivial if and only if it is topologically trivial.

As far as the second part of the statement is concerned, according to the standard neighborhood theorem for contact submanifolds [Gei08, Theorem 2.5.15], the contact submanifold $(M, \xi = \ker(\alpha)) = (M \times \{pt\}, \eta \cap T(M \times \{pt\}))$ of $(M \times \mathbb{T}^2, \eta)$ has a contact neighborhood of the form $\left(M \times D^2_\epsilon, \ker\left(\alpha + r^2 d\varphi\right)\right)$, for a certain real $\epsilon > 0$. Moreover, each hypertight high dimensional contact manifold is in particular tight, according to [AH09, CMP15]; in particular, $(M \times \mathbb{T}^2, \eta)$ is tight. Then, $\left(M \times D^2_\epsilon, \ker\left(\alpha + r^2 d\varphi\right)\right)$ is tight too, because it embeds (in codimension 0) in a tight contact manifold. □

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Titre : Sur quelques constructions de variétés de contact

Mots clés : variété de contact, structure de contact, construction

Résumé : Cette thèse est subdivisée en deux parties.
La première partie porte sur l'étude de la topologie de l'espace des contactomorphismes pour quelques exemples explicites de variétés de contact en grandes dimensions. Plus précisément, en utilisant des constructions et résultats dus à Massot, Niederkrüger et Wendl, on construit, en chaque dimension impaire, une infinité d'exemples de contactomorphismes de variétés de contact vrillées fermées qui sont lissement isotopes mais pas contact-isotopes à l'identité. On donne aussi, en toutes dimensions impaires, des exemples de variétés de contact tendues fermées qui admettent un contactomorphisme tel que tous ses itérées sont lissement isotopes mais pas contacto-isotopes à l'identité; ceci generalise un résultat en dimension 3 dû à Ding et Geiges.
Dans la deuxième partie, on construit des exemples de variétés de contact fermées en grandes dimensions avec des propriétés particulières. Ceci nous amène à l'existence de structures tendues virtuellement vrillées en toutes dimensions impaires, et au fait que chaque variété de contact fermée de dimension 3 se plonge dans une variété de contact tendue fermée de dimension 5 avec fibré normal trivial. Pour cela, on utilise des constructions dues à Bourgeois (sur des produits avec des tores) et à Geiges (sur des revêtements ramifiés). On passe de ces constructions à des définitions; ceci permet de prouver un résultat d'unicité dans le cas des revêtements ramifiés de contact, et d'étudier leurs propriétés globales, en montrant qu'elles ne dépendent d'aucun choix auxiliaire fait dans les procédures. Un deuxième but permis par ces définitions est l'étude des relations entre ces constructions et les notions de livre ouvert porteur, due à Giroux, et de fibré de contact, due à Lerman. Par exemple, on donne une définition de structure de contact de Bourgeois qui est locale, inclue (strictement) les résultats de la construction de Bourgeois et permet de récupérer une classe d'isotopie de livres ouverts porteurs sur les fibres; ceci suit d'une réinterprétation, inspirée par une idée de Giroux, des livres ouverts porteurs en termes de paires de champs de vecteurs de contact.

Title : On some constructions of contact manifolds

Keywords : contact manifold, contact structure, construction

Abstract : This thesis is divided in two parts.
The first part focuses on the study of the topology of the contactomorphism group of some explicit high dimensional contact manifolds. More precisely, using constructions and results by Massot, Niederkrüger and Wendl, we construct (infinitely many) examples in all dimensions of contactomorphisms of closed overtwisted contact manifolds that are smoothly isotopic but not contact-isotopic to the identity. We also give examples of tight high dimensional contact manifolds admitting a contactomorphism whose powers are all smoothly isotopic but not contact-isotopic to the identity; this is a generalization of a result in dimension 3 by Ding and Geiges.
In the second part, we construct examples of higher dimensional contact manifolds with specific properties. This leads us to the existence of tight virtually overtwisted closed contact manifolds in all dimensions and to the fact that every closed contact 3-manifold embeds with trivial normal bundle inside a tight closed contact 5-manifold. This uses known construction procedures by Bourgeois (on products with tori) and Geiges (on branched covering spaces). We pass from these procedures to definitions; this allows to prove a uniqueness statement in the case of contact branched coverings, and to study the global properties (such as tightness and fillability) of the results of both constructions without relying on any auxiliary choice in the procedures. A second goal allowed by these definitions is to study relations between these constructions and the notions of supporting open book, due to Giroux, and of contact fiber bundle, due to Lerman. For instance, we give a definition of Bourgeois contact structures on flat contact fiber bundles which is local, (strictly) includes the results of Bourgeois' construction, and allows to recover an isotopy class of supporting open books on the fibers. This last point relies on a reinterpretation, inspired by an idea by Giroux, of supporting open books in terms of pairs of contact vector fields.

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